DUALITY TRANSFORMATION OF NON-ABELIAN GAUGE THEORIES

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by

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DECLARATION

I declare that the thesis entitled "Duality Transformation of non-Abelian gauge theories" submitted by me for the Degree of Doctor of Philosophy is the record of work carried out by me during the period from November 1996 to March 2000 under the guidance of Prof. H. S. Sharatchandra and has not formed the basis for the award of any degree, diploma, associateship, fellowship, titles in this or any other University or other similar Institution of Higher Learning.

(Pushan Majumdar)

CERTIFICATE FROM THE SUPERVISOR

I certify that the thesis entitled "Duality Transformation of non-Abelian gauge theories" submitted for the Degree of Doctor of Philosophy by Mr. Pushan Majumdar is the record of research work carried out by him during the period from November 1996 to March 2000 under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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Pushan Majumdar.

SYNOPSIS

Quantum Chromodynamics (QCD) is supposed to be the correct theory for strong interactions. So one should be able to get a complete description of all strong interaction phenomena from it. QCD is an asymptotically free theory. This means that at high energies the coupling of the theory is small. This energy regime can be handled by perturbation theory. On the other hand, at lower energies, the coupling becomes strong and the perturbation theory breaks down. Several other approaches have been tried to tackle the low energy regime of QCD. Among them lattice QCD seems to be very promising. However none of them have been able to explain one of the most fundamental problems of strong interactions, which is confinement of quarks, starting from QCD.

One of the most appealing physical pictures for quark confinement is dual superconductivity. Here the analogy is to type II superconductors. In normal type II superconductors, Meissner effect takes place in the bulk of the superconductor. Some of the magnetic flux which penetrates into the material is squeezed into thin tubes. In the dual superconductor scenario, the QCD vacuum behaves like a dual superconductor. Under similar circumstances, instead of magnetic flux tubes, one would now get electric flux tubes. These flux tubes would begin at a quark and end at an antiquark implying a linearly rising potential between them. This would give a physical picture of confinement. It would then be impossible to separate them into a pair of free quark and antiquark, as that would require infinite energy.

Duality transformation typically maps the strongly coupled region of one theory to the weakly coupled one of another, making it amenable to a perturbative expansion. This has proved very useful for several statistical mechanics systems. For example, 2-dimensional Ising model is self dual and undergoes a second order phase transition. The self duality of the model completely fixes its transition temperature. Also in 2-dimensional XY model, which undergoes a defect driven phase transition, the structure of the defects is revealed by a duality transformation. For field theories, the simplest example is electrodynamics. In this case, duality transformation maps the theory to itself by interchanging the role of electric and magnetic fields. With matter present, usual electrodynamics is not self dual, but with magnetic monopole degrees of freedom put in, it can become self dual again. Many statistical mechanics and field theory systems are topologically non-trivial and have solitons. In these cases, duality transformation generally interchanges the fundamental degrees of freedom with the topological degrees of freedom of either the same or some other model. This way the non-perturbative degrees of freedom get exposed. Non-Abelian gauge theories are topologically non-trivial, and they are believed to have monopole kind of configurations. Therefore if one is able to perform a duality transformation, one hopes to bring out these topological degrees of freedom and identify correctly the relevant degrees of freedom for describing low energy phenomenon in QCD. All these bring out the importance of being able to do a duality transformation in QCD.

The basic difference between electrodynamics and QCD is that the former is an Abelian theory while the latter is non-Abelian. So far all the successes of duality transformation has been for either discrete or U(1) groups, but none for non-Abelian groups. It is believed that once one understands how to deal with the basic non-Abelian theory, dealing with QCD will not be too difficult. Hence in this thesis we look at the problem of duality transformation of the simplest non-Abelian theory which is SU(2) Yang-Mills theory in 2+1 and 3+1 dimensions respectively.

In 2+1 dimensions, a curious analogy exists between SU(2) Yang-Mills theory and the Einstein-Cartan formulation of gravity. Using an auxiliary field one can rewrite the Yang-Mills action so that it is linear in the field strength. Now one can interpret the auxiliary field as the driebein and the field strength as the curvature. The gauge potential plays the role of spin connection. The resulting action now looks like the three dimensional gravity action with an added term that breaks general coordinate invariance. Gauge invariance however is retained. Thus dynamics in 2+1 Yang-Mills is mapped to morphisms of 3-manifolds. In three dimensions pure gravity does not have any local degrees of freedom. In contrast, for Yang-Mills theory, the general coordinate invariance breaking term results in local degrees of freedom. We are now able to identify the dual gluons as local coordinates on the 3-manifold. These are three scalar degrees of freedom as expected for 2+1-dimensional Yang-Mills theory. We identify the monopoles too in a manifestly gauge ivariant manner. They are located at points where the Ricci principal axes become degenerate. Thus both the dual variables and the topological degrees of freedom, have nice geometric interpretations in terms of coordinates and coordinate singularities respectively. When we rewrite the action in terms of the new variables, we get an interaction term which couples the dual gluons with the monopoles naturally.

In 3+1 dimensions, we first try to identify the physical phase space. For that we find a local solution to the non-Abelian Gauss law. Apart from being local, the solution has the additional advantage that it is parametrized by a gauge invariant symmetric matrix. The usual Abelian Hodge decomposition of the potential is extremely useful in handling the Gauss law constraint in electrodynamics. Here we develop techniques to decompose the non-Abelian potential into parts useful for handling the non-Abelian Gauss law and perform duality transformation. This can be thought of as non-Abelian generalization of the usual Hodge decomposition. We show that there are two useful decompositions. One of them is to decompose into the covariant gradient and the covariant curl part. The other one is to rewrite in terms of a magnetic field of some other potential.

In spite of all this, it is not clear that we get all possible solutions of the Gauss law. The reason for this is the existence of Wu - Yang ambiguities. This is a special feature of non-Abelian gauge theories where two gauge inequivalent potentials give rise to the same field strength. Since this affects our parametrization of the solution to the Gauss law, we are forced to analyze this effect. However we find that this is not a generic phenomenon. For most cases only some global solutions exist. So this does not affect our local parametrization very much.

After resolving these issues, we proceed to do duality transformation for the theory. We realize duality transformation as a canonical transformation on the phase space variables of the Yang-Mills theory. We use generating functions for the canonical transformation. This has some distinct advantages. Firstly, since the Jacobian of the transformation is one, we do not pick up any undesirable extra factor in the functional measure. Secondly, the new variables obey their own Gauss law which follows naturally if one uses a gauge invariant generating functional. The dual theory gives the dynamics of the dual gluon. It is however non-local.

Another related but alternative approach is to work in the axial gauge. We note that the dual field is the lagrangian multiplier for the Bianchi identity. For 2+1 dimensions we use axial gauge to completely integrate out the original gauge fields and get the action for the dual gluon. This is non-local. Using auxiliary fields, we restore locality and gauge invariance. The dual gluon is a scalar isotriplet and its topology gives the monopole configurations. We repeat the analysis for 3+1 dimensions.

We conclude with the possible uses of our techniques.

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Chapter 1

Introduction

Quantum Chromodynamics is supposed to be the theory for strong interactions. So one should be able to get a complete description of all strong interaction phenomena from Quantum Chromodynamics (QCD). QCD is an asymptotically free theory. This means that at high energies the coupling of the theory is small. This energy regime can be handled by perturbation theory. On the other hand, at lower energies, the coupling becomes strong and perturbation theory breaks down. Several approaches have been tried to tackle the low energy regime of QCD. Among them lattice QCD seems to be very promising. However gaps remain in our understanding of the mechanism of one of the most fundamental problems of strong interactions, which is confinement of quarks. Reliable techniques of computation of confinement effects have also not yet been developed.

One of the most appealing physical pictures for quark confinement is dual superconductivity. Here the analogy is to type II superconductors. In superconductors, Meissner effect takes place in the bulk of the superconductor. However the magnetic field can penetrate the substrate material by forming tubes of normal conductor within the bulk superconductor. This magnetic flux which penetrates into the material is squeezed into thin tubes. In the dual superconductor scenario, the QCD vacuum behaves like a dual superconductor. Under similar circumstances, instead of magnetic flux tubes, one would now get electric flux tubes. These flux tubes would begin at a quark and end at an antiquark implying an asymptotically linearly rising potential between them. It would then be impossible to separate them into a pair of free quark and antiquark, as that would require infinite energy. This would be a physical picture of confinement.

Duality transformation typically maps the strongly coupled region of one theory to the weakly coupled one of another, making it possible for one to investigate the strongly coupled region of a theory by a perturbative expansion of the dual theory. This has proved very useful for several statistical mechanics systems. For example, 2-dimensional Ising model is self dual and undergoes a second order phase transition. The self duality of the model completely fixes its transition temperature. Also in 2-dimensional XY model, which undergoes a defect driven phase transition, the nature and relevance of the defects is revealed by a duality transformation. For gauge field theories, the simplest example is Maxwell electrodynamics. In this case, duality transformation maps the theory to itself by interchanging the role of electric and magnetic fields. With matter present, usual electrodynamics is not self dual, but with magnetic monopole degrees of freedom put in, it can becomes self dual again. Many statistical mechanics and field theory systems are topologically non-trivial and have solitons. In these cases, duality transformation generally interchanges the fundamental degrees of freedom with the topological degrees of freedom of either the same or some other model. This way, the non-perturbative degrees of freedom get exposed. Non-Abelian gauge theories are topologically non-trivial, and they are believed to have monopole kind of configurations. Therefore if one is able to perform a duality transformation, one hopes to bring out these topological degrees of freedom and identify correctly the relevant degrees of freedom for describing low energy phenomena in QCD. All these bring out the importance of being able to do a duality transformation in QCD.

So far we have pointed out the role of duality transformation in 2-dimensional Ising model, the 2-d XY model and electrodynamics. Other well known systems include the duality between the Sine-Gordon model and massive Thirring model in 1+1 dimensions, 2+1-dimensional U(1) lattice gauge theory and Georgi-Glashow model are related to the couloumb gas problem in 3 dimensions. Duality also plays very important roles in supersymmetric gauge theories and string theories.

Duality transformations have already played crucial roles for understanding many aspects of gauge theories. Indeed the first examples of lattice gauge theories appeared as dual theories of Ising models [1]. Since confinement effects occur in the low energy regime of non-Abelian theories, which are beyond the reach of perturbation theory, duality transformation is especially important for understanding the confinement aspects of gauge theories [2]. It is expected, and in some cases checked, that monopoles play a crucial role for this property.

Quark confinement is well understood in 2+1-dimensional compact U(1) gauge theory. It is a consequence of the existence of a monopole plasma [3][4]. Duality transformation [5] turned out to be very useful in this context. It is of interest to know how far these ideas can be extended to non-Abelian gauge theories. For this reason, duality transformation for 2+1-dimensional lattice Yang-Mills theory was obtained in both hamiltonian [6] and partition function [7] formulations. After duality transformation, SU(2) lattice gauge theory gets related to Ponzano Regge formulation of 3-dimensional gravity.

Duality transformation of an Abelian gauge theory gives the dual potential [8], one which couples minimally to magnetic matter. Therefore it exposes the monopole degrees of freedom. This is brought out in a powerful way in four-dimensional super symmetric gauge theories [9]. Deser and Teitelboim [10] analyzed the possibility of duality invariance of 3+1-dimensional Yang-Mills theory in close analogy to Maxwell theory and concluded that invariance is not realized.

In this thesis we develop techniques for performing duality transformation of 2+1 and 3+1-dimensional Yang-Mills theories. The contents of the thesis is as follows.

In chapter two we review an analogy that exists between gravity and gauge theory. This analogy plays a very important role for SU(2) gauge theory in three dimensions. Since the number of generators of the gauge group and the number of space-time dimensions match, techniques from one can be used in the other almost without any modification.

In the third chapter, we consider duality transformation for 2+1-dimensional (continuum) Yang-Mills theory in close analogy to the case of compact U(1) lattice gauge theory [5]. We reinterpret the Yang-Mills theory as a theory of 3-manifolds, as in gravity, but without diffeomorphism invariance. We use this relation for identifying the dual gluons and their interactions. The dual gluons are related to diffeomorphisms of the 3-manifold. We also identify the monopoles in the dual theory. 't Hooft [11] has advocated the use of a composite Higgs to locate the monopoles. Here we propose to use the orthogonal set of eigenfunctions of a gauge invariant, (symmetric) local, matrix-valued field for this purpose. Isolated points where the eigenvalues are triply degenerate have topological significance and they locate the monopoles. We use the Ricci tensor to construct a new coordinate system for the 3-manifold. The monopoles are located at the singular points of this coordinate system and they have the expected interactions with the dual gluons. We expect that these interactions lead to a mass for the dual gluons and result in confinement as in the U(1) case.

In chapter four, we deal with the problem of gauge field copies. Wu and Yang [14] gave an explicit example of two (gauge inequivalent) Yang-Mills potentials $\vec{A}_i(x) = \{A_i^a(x), a = 1,2,3\}$ generating the same non-Abelian magnetic field

$$\vec{B}_i[A](x) = \epsilon_{ijk}(\partial_j \vec{A}_k + \frac{1}{2}\vec{A}_j \times \vec{A}_k). \tag{1.1}$$

Since then there has been a wide discussion of the phenomenon in the literature [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. We may refer to gauge potentials giving the same non-Abelian magnetic field, as gauge field copies in contrast to gauge equivalent potentials which generate magnetic fields related by a homogeneous gauge transformation. If we require all higher covariant derivatives of B_i^a also match then there are effectively no gauge copies [24]. In each of the space dimensions $d = 1, 2, 3$ this phenomenon has a different manifestation. At present the phenomenon is not understood in its generality for in 3 space-time dimensions. Recently Freedman and Khuri [28] have exhibited several examples of continuous families of gauge field copies in 3 space-time dimensions. Their technique was to use a local map of the gauge field system into a spatial geometry with a second rank symmetric tensor $G_{ij} = B_i^a B_j^a \, detB$ and a connection with torsion constructed from it. We tackle the global version of the problem directly and appeal to the Cauchy-Kowalevsky existence theorems on systems of first order partial differential equations. We conclude that the phenomenon is not generic. We show that for every given magnetic field there corresponds a vector potential and there exists one gauge field copy which is unique upto boundary conditions.

The fifth chapter discusses the solution of the non-Abelian Gauss law and non-Abelian analogs of the Hodge decomposition. Yang-Mills theory has a first class constraint, the non-Abelian Gauss law. This particular constraint is also present in Ashtekar formulation of gravity [29]. Usually this constraint is handled by "fixing a gauge". However it is of interest to obtain a parametrization of the "physical phase space", i.e. the part of the phase space which satisfies the constraint. This would give the physical degrees of freedom. We are interested in a general solution of the Gauss law in terms of local fields. This is of relevance for duality transformation of Yang-Mills theory [30][31][32]. For other approaches to handle the Gauss law, see ref. [33].

In the sixth chapter we consider duality transformation of four-dimensional Yang-Mills theory. The first work to address duality transformation of 3+1-dimensional Yang-Mills theory retaining all the non-Abelian features was by Halpern [34]. Using complete axial gauge fixing, he brought out the crucial role played by the Bianchi identity. The dual theory was a gauge theory with a new gauge potential, though the action was non-local. Another issue closely related to duality transformation is reformulation of the gauge theory dynamics using gauge invariant degrees of freedom. Several authors [35] consider rewriting the functional integral using a gauge covariant second rank tensor. Using the relation of $SO(3)$ lattice gauge theory in 2+1 dimensions with gravity we can formulate the dynamics using local gauge invariant degrees of freedom [37]. Similar situation is true in 3+1 dimensions also [38].

In the previous chapters, gauge invariance was explicitly maintained. However in chapter seven we fix the axial gauge and carry out the duality transformation of Yang-Mills theory in three and four dimensions. The three-dimensional case provides further insight into the duality transformation which we had performed formally in a gauge invariant way in chapter three. The four-dimensional case however is not so clean and we are left with extra auxiliary fields.

Chapter eight contains our conclusions and future directions of work.

Chapter 2

Analogy with gravity

A formulation of gravity which allows one to incorporate spinors is the Einstein-Cartan formulation. In this formulation of gravity, one uses a set of smooth vector fields (vielbeins), as frames for describing things. These frames are parallel transported using the spin connection, which play the role of potentials of gauge theories. The dynamics of the local coordinate and the spin connection is determined by the two Cartan structure equations. In this language, there is a striking similarity between gravity and Yang Mills theory in 2+1 dimensions. Let us now look at this connection a little more closely.

Vielbeins are an orthonormal (with respect to the metric) set of smooth vector fields with one index belonging to the tangent space at that point and the other one being the ordinary space-time index. They obey the following equation.

$$e^{a}{}_{\mu} g^{\mu\nu} e^{a}_{\nu} = \delta^{ab}. \tag{2.1}$$

Note that we are in Euclidean space because we want the holonomy to be in SU(2) and not SU(1,1).

Next we define the spin connections ω_μ^{ab} as

$$\omega_\mu^{ab} = e^{\nu a}(\mathrm{D}_\mu e_\nu)^b. \tag{2.2}$$

The e_μ^a's form a set of basis vectors. Any tensor can be expanded in terms of these. For example the expansion coefficients ω^{cab} of ω_μ^{ab}, called the Ricci rotation coefficients are given by,

$$\omega^{cab} = e^{\mu c} e^{\nu a}(\mathrm{D}_\mu e_\nu)^b. \tag{2.3}$$

Thus we can use the vielbeins or tetrads as they are also called to interchange indices between the tangent space and our ordinary space.

Since the e_μ^a are orthonormal (2.1), and

$$\mathrm{D}_\rho\, g_{\mu\nu} = 0 \tag{2.4}$$

we have

$$\begin{aligned} \omega_\mu^{ab} &= e^{\nu a}\mathrm{D}_\mu e_\nu^b \\ &= -e^{\nu b}\mathrm{D}_\mu e_\nu^a \\ &= -\omega_\mu^{ba} \end{aligned} \tag{2.5}$$

Thus the spin connections are antisymmetric and have only 24 components in 4 dimensions whereas the Christoffel symbols have 40. Finally the curvature tensor can be written in terms of the tetrads as follows.

$$R^{abcd} = R^{\mu\nu\rho\sigma}\, e^a{}_\mu\, e^b{}_\nu\, e^c{}_\rho\, e^d{}_\sigma \tag{2.6}$$

$$= e^a{}_\mu e^b{}_\nu e^c{}_\rho (\mathrm{D}^\mu \mathrm{D}^\nu - \mathrm{D}^\nu \mathrm{D}^\mu)\, e^{\rho d} \tag{2.7}$$

or

$$R^{\mu\nu cd} = e^{c}_{\ \rho}(\mathbf{D}^{\mu}\mathbf{D}^{\nu} - \mathbf{D}^{\nu}\mathbf{D}^{\mu})\, e^{\rho d} \tag{2.8}$$

or

$$R^{cd}_{\ \ \mu\nu} = e^{c}_{\ \rho}(\mathbf{D}_{\mu}\mathbf{D}_{\nu} - \mathbf{D}_{\nu}\mathbf{D}_{\mu})\, e^{\rho d} \tag{2.9}$$

which using the definition of $\omega^{ab}_{\ \ \mu}$ can be written as

$$R^{ab}_{\ \ \nu\rho} = \partial_{\nu}\omega^{ab}_{\ \ \rho} - \partial_{\rho}\omega^{ab}_{\ \ \nu} + [\omega_{\nu}\,,\,\omega_{\rho}]^{ab}. \tag{2.10}$$

In 3+1 dimensions, the Einstein action in terms of the vielbein language is given by

$$S = \int d^{4}x\ \epsilon^{\mu\nu\rho\sigma} e^{a}_{\ \mu} e^{b}_{\ \nu} \{\partial_{\rho}\omega^{ab}_{\ \ \sigma} - \partial_{\sigma}\omega^{ab}_{\ \ \rho} + [\omega_{\rho}\,,\,\omega_{\sigma}]^{ab}\} \tag{2.11}$$

Variation with respect to the tetrads yield the source free Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{2.12}$$

where

$$R = R^{\mu}_{\ \mu} \tag{2.13}$$

and

$$R_{\mu\nu} = g^{\rho\sigma} R_{\mu\rho\nu\sigma} \tag{2.14}$$

with

$$R_{\mu\rho\nu\sigma} = (e_{\rho})^{a}\,(e_{\sigma})^{b} R^{ab}_{\ \ \mu\nu} \tag{2.15}$$

and $R^{ab}_{\ \ \mu\nu}$ is as defined in equation (2.10).

In 3 Euclidean dimensions Einstein action is given by

$$S = \int d^{3}x\ \epsilon^{\mu\nu\rho} e^{a}_{\ \mu} \{\partial_{\nu}\omega^{a}_{\ \rho} - \partial_{\rho}\omega^{a}_{\ \nu} + \epsilon^{abc}\omega^{b}_{\nu}\omega^{c}_{\rho}\} \tag{2.16}$$

Note that here we have written ω with one index. We can however convert it to a 2 indexed notation by

$$\omega_\mu^{ab} = \epsilon^{abc}\omega_\mu^c \tag{2.17}$$

Now let us look at the Yang Mills action in 3 Euclidean dimensions.

$$S = \frac{1}{2g^2}\int d^3x \, tr(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}). \tag{2.18}$$

We can introduce an auxiliary field $e^a{}_\mu$ and write this equation as

$$S = \frac{1}{2}\int d^3x \, (g^2 e^a{}_\mu e^a{}_\mu + i\epsilon^{\mu\nu\rho} e^a{}_\mu F^a{}_{\nu\rho}) \tag{2.19}$$

with an extra integral in the partition function over the auxiliary fields. Note that now the action is linear in the field strength $F^a{}_{\mu\nu}$. This is known as the linearized action. When we try to remove the auxiliary fields by integrating over them, then we shall have to complete the square and then again we will have the quadratic term in $F^a{}_{\mu\nu}$ apart from an unimportant constant factor.

Thus if we add a general coordinate invariance breaking term in the form of $e^a{}_\mu \, e^a{}_\mu$, with a summation over a and μ in the 3-(Euclidean) dimensional Einstein-Cartan action, we get the Yang-Mills action.

Chapter 3

Dual gluons and monopoles in 2+1-dimensional Yang-Mills theory

This chapter considers duality transformation for 2+1-dimensional (continuum) Yang-Mills theory. Since we are in 3 dimensions and the gauge group is SU(2), we have a situation analogus to 3-dimensional gravity. We use this analogy extensively throughout the chapter. Lunev [39] too has suggested a relationship of 2+1-dimensional Yang-Mills theory with gravity. He uses a gauge invariant composite $B_i^a B_j^a$ as a metric, and rewrites the classical Yang-Mills dynamics for it. The corresponding formulation of the quantum theory is somewhat involved. Our metric is in a sense dual of Lunev's choice. As we make formal transformations in the functional integral, the quantum theory is simpler and has a nicer interpretation. There are also approaches that try to relate 3+1-dimensional Yang-Mills theory to a theory of a metric [12]. On the other hand, the dual theory in 3+1-dimensions can also be related to a new SO(3) gauge theory [13].

This chapter is organized as follows. In section 1 we briefly review duality transformation and confinement in 2+1-dimensional compact U(1) lattice gauge theory. In section 2 we obtain the dual description of 2+1-dimensional Yang-Mills theory in close analogy to section 1. We point out the close relationship to gravity and identify the dual gluons and their interactions. In section 3 we provide a new characterization of monopoles using eigenfunctions of the symmetric matrix $B_i^a B_i^b$. In section 4 we use the Ricci tensor to construct a preferred coordinate system for 3-manifolds. We relate the monopoles to singularities of this coordinate system. We also identify their interactions with the dual gluons. Section 5 contains our conclusions.

3.1 Review of confinement in 2+1-dimensional compact U(1) Lattice Gauge Theory

In this section we briefly review duality transformation [5] and confinement [3] in 2+1-dimensional compact U(1) lattice gauge theory. This provides a paradigm for our analysis of 2+1-dimensional Yang-Mills case.

The motivation for U(1) lattice gauge theory comes from the planar spin models. This model has a nearest neighbor interaction between spins which is given by $\sum_{<ij>} \mathbf{s}_i \cdot \mathbf{s}_j$. In terms of the angles of the spins, it can be written as $V(\theta_i - \theta_j) = -K[1 - cos(\theta_i - \theta_j)]$ where i and j are nearest neighbor sites. Since the interaction term is a periodic function, we can expand it in a fourier series.

$$exp\,[V(\theta)] = \sum_{s=-\infty}^{\infty} exp[is\theta + \tilde{V}(s)], \tag{3.1}$$

where the fourier coefficients $\tilde{V}(s)$ are given by

$$exp[\tilde{V}(s)] = \int_0^{2\pi} \frac{d\theta}{2\pi} exp[-is\theta + V(\theta)]. \tag{3.2}$$

In this case $exp[\tilde{V}(s)]$ are related to Bessel functions. However the sum (3.1) converges rather slowly for large arguments of the Bessel functions. To improve their convergence, one can use the Poisson summation formula

$$\sum_{s=-\infty}^{\infty} g(s) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi\; g(\phi) exp[-2\pi i m\phi]. \tag{3.3}$$

Hence eq(3.1) can be written as

$$exp[V(\theta)] = \sum_{m} exp[V_0(\theta - 2\pi m)] \tag{3.4}$$

Now the sum over m enforces the periodicity of $V(\theta)$. Therefore V_0 itself may be a non-periodic function.

Villain considered a modified Hamiltonian (expected to be in the same universality class as the original one) which has

$$exp[V_0(\theta)] = -\frac{1}{2} K\theta^2. \tag{3.5}$$

This is known in the literature as the Villain approximation.

In contrast to the spin model, U(1) lattice gauge theory has the degrees of freedom on the links and to maintain gauge invariance, the action is taken along a closed plaquette. Thus the partition function is

$$Z \equiv \prod_{n,i} \int_0^{2\pi} d\theta_i(n) exp[-\frac{1}{4\kappa^2} \sum_{n,i,j} (1 - cos\theta_{ij}(n))] \tag{3.6}$$

where $\theta_i(n)$ is the angle on the directed link $n \to n+\hat{i}$ and

$$\theta_{ij}(n) = \triangle_i \theta_j(n) - \triangle_j \theta_i(n) \tag{3.7}$$

However the Villain approximation can be performed as in the planar spin model, and we get the Euclidean partition function in the Villain formulation as

$$Z = \sum_{h_{ij}} \prod_{ni} \int_{-\infty}^{\infty} dA_i(n)\ exp\,(-\frac{1}{4\kappa^2}\sum_{nij}[\triangle_i A_j(n) - \triangle_j A_i(n) + h_{ij}(n)]^2). \tag{3.8}$$

Here $A_i(n) \in (-\infty, \infty)$ are non-compact link variables on links joining the sites n and $n+\hat{i}$. $h_{ij}(n) = 0, \pm 1, \pm 2 \ldots$ are integer variables corresponding to the monopole degrees of freedom and are associated with the plaquette $(n\hat{i}\hat{j})$. $\triangle_i$ is the difference operator, $\triangle_i \phi(n) = \phi(n+\hat{i}) - \phi(n)$. We may introduce an auxiliary variable $e_i(n)$ to rewrite Z as

$$\begin{aligned} Z &= \sum_{h_{ij}} \prod_{ni} \int_{-\infty}^{\infty} dA_i(n) \int_{-\infty}^{\infty} de_i(n)\ exp\left(-\sum_{ni}[e_i(n)]^2 \right. \\ &\left. + \frac{2i}{\kappa}\sum_{nij} \epsilon_{ijk}\, e_k(n)[\triangle_i A_j(n) + \frac{1}{2}h_{ij}(n)]\right). \end{aligned} \tag{3.9}$$

Integration over $A_j(n)$ gives the δ function constraint

$$\epsilon_{ijk}\, \triangle_j\, e_k(n) = 0 \tag{3.10}$$

for each n and $\hat{i}$. The solution is $e_i(n) = \triangle_i \phi(n)$. Thus we get the dual form of the partition function

$$Z = \sum_{h_{ij}} \prod_{ni} \int_{-\infty}^{\infty} d\phi(n)\ exp \sum_{n} \left(-[\triangle_i \phi(n)]^2 + \frac{i}{2\kappa}\phi(n)\rho(n)\right), \tag{3.11}$$

where $\rho(n) = \frac{1}{2}\epsilon_{ijk}\triangle_i h_{jk}(n)$. This has the following interpretation. The field ϕ describes the dual photon. (In 2+1 dimensions, the photon has only one transverse degree of freedom and this is captured by the scalar field $\phi(n)$). The monopole number at site n is given by $\rho(n)$. It takes integer values and the dual photon couples locally to it with strength $1/\kappa$.

If we sum over the monopole degrees of freedom, we get a mass term for $\phi(n)$ [3][5]. The reason for this is that the monopole plasma is screening the long range interactions between the monopoles. A Wilson loop for the electric charges in this system would correspond to a dipole sheet in this plasma. This gives an area law and hence a linear confining potential between static electric charges.

The advantage of this formal duality transformation is that it gives a precise separation of the 'spin wave' and the 'topological' degrees of freedom. Therefore it provides a stepping stone for going beyond semi-classical approximations.

We use this approach for 2+1-dimensional Yang-Mills theory in the next section.

3.2 Dual gluons in 2+1-dimensional Yang-Mills theory

In this section we point out the close relationship between Yang-Mills theory and Einstein-Cartan formulation of gravity in 3-dimensional Euclidean space. We use this analogy extensively throughout the chapter.

The Euclidean partition function of 2+1-dimensional Yang-Mills theory is [1]

$$Z = \int \mathcal{D}A_i^a(x)\, exp\left(-\frac{1}{2\kappa^2}\int d^3x B_i^a(x)B_i^a(x)\right) \qquad (3.12)$$

where $\{A_i^a(x),\ \ (i, a = 1, 2, 3)\}$ is the Yang-Mills potential and

$$B_i^a = \frac{1}{2}\epsilon_{ijk}(\partial_j A_k^a - \partial_k A_j^a + \epsilon^{abc}A_j^b A_k^c) \qquad (3.13)$$

is the field strength. As in section 1, we rewrite Z as [7]

$$Z = \int \mathcal{D}A_i^a(x)\, \mathcal{D}e_i^a(x)\, exp\left\{\int d^3x(-\frac{1}{2}[e_i^a(x)]^2 + \frac{i}{\kappa}e_i^a(x)B_i^a(x))\right\}. \qquad (3.14)$$

The second term in the exponent is precisely the Einstein-Cartan action for gravity in 3-(Euclidean) dimensions. $e_i^a(x)$ is the driebein and $\omega_i^{ab} = \epsilon^{abc}A_i^c$ the connection 1-form.

In contrast to section 1, we do not get a δ function constraint on integrating over A_i^a in this case. Since A appears at most quadratically in the exponent, the integration over A may be explicitly performed. This integration is equivalent to solving the classical equations of motion for A as a functional of e and replacing A by this solution :

$$\epsilon_{ijk}(\partial_j\delta^{ac} + \epsilon_{abc}A_j^b[e])e_k^c(x) = 0. \qquad (3.15)$$

Now (3.15) is precisely the condition for a driebein e to be torsion free with respect to the connection 1-form A_i^c.

If we assume the 3×3 matrix e_i^a to be non-singular, then this solution $A[e]$ can be explicitly given [40]. In this case, no information is lost by multiplying

[1]Note that in this chapter we use κ and not g to denote the coupling constant in order to avoid confusion with the determinant of the metric

(3.15) by e_l^a and summing over a. We get,

$$\epsilon_{ijk}e_l^a\partial_j e_k^a + |e|(e^{-1})_b^m\epsilon_{klm}\epsilon_{ijk}A_j^b[e] = 0. \tag{3.16}$$

Defining

$$A_j^b(e^{-1})_{bm} = A_{jm}, \tag{3.17}$$

we get,

$$A_{li}[e] - \delta_{li}A_{mm}[e] = \frac{1}{|e|}\epsilon_{ijk}e_l^a\partial_j e_k^a. \tag{3.18}$$

Taking the trace on both sides,

$$A_{mm}[e] = -\frac{1}{2|e|}\epsilon_{ijk}e_i^a\partial_j e_k^a. \tag{3.19}$$

Finally we obtain

$$A_l^b[e] = \frac{e_i^b}{|e|}\left(\epsilon_{ijk}e_l^a\partial_j e_k^a - \frac{1}{2}\delta_{li}\epsilon_{mjk}e_m^a\partial_j e_k^a\right). \tag{3.20}$$

By a shift of A, $A = A[e] + A'$, the integration over A reduces to

$$\int \mathcal{D}A'\, exp\left(\frac{i}{\kappa}\int A'_{ia}e_{ia,jb}A'_{jb}\right) = \frac{1}{det^{1/2}(e_{ia,jb})} = \frac{1}{det^{3/2}(e_i^a)}, \tag{3.21}$$

where $e_{ia,jb} = \epsilon_{ijk}\epsilon^{abc}e_k^c$.

B_i^a is related to the Ricci tensor R_{ik} as follows:

$$R_{ik} = F_{ij}^{ab}e_k^a(e^{-1})_b^j \tag{3.22}$$

where $F_{ij}^{ab} = \epsilon_{ijk}\epsilon^{abc}B_k^c$. Thus an integration over A gives,

$$Z = \int \mathcal{D}g\, exp\left(-\frac{1}{2}g_{ii} + \frac{i}{\kappa}\sqrt{g}R\right) \tag{3.23}$$

where the metric $g_{ij} = e_i^a e_j^a$ and $R = R_{ik} g^{ki}$. Note that $\mathcal{D}g = \mathcal{D}e\, det^{-3/2}(e_i^a)$, as required. The configurations where e is singular is naively a set of measure zero, so that the assumption $|e| \neq 0$ is reasonable.

Equation (3.23) provides a reformulation of 2+1-dimensional Yang-Mills theory (classical or quantum) in terms of gauge invariant degrees of freedom. It is now a theory of metrics on 3-manifolds which however is not diffeomorphism invariant because of the term g_{ii} in the action. As a result, not only the geometry of the 3-manifold, but also the metric g_{ij} of any coordinate system chosen on the manifold is relevant.

For 3-dimensional (Euclidean) gravity, an integration over e (3.14) would give the δ-function constraint $B_i^a = 0$, resulting in a topological field theory [41]. There are no massless gravitons as a consequence. Now however, the diffeomorphisms provide massless degrees of freedom corresponding to gluons. They may be described as follows. The 3-manifolds are described by the metric g_{ij} in the coordinate system x. We may choose a new coordinate system $\phi^A(x)$ $(A = 1, 2, 3)$, with a standard form of the metric $G_{AB}[\phi]$. We have

$$g_{ij}(x) = \frac{\partial \phi^A}{\partial x^i} G_{AB}[\phi] \frac{\partial \phi^B}{\partial x^j}. \tag{3.24}$$

This gives the form of the action as,

$$S = \int d^3x \left[-\left(\frac{\partial \phi^A}{\partial x^i} G_{AB}[\phi] \frac{\partial \phi^B}{\partial x^j} \right) + \frac{i}{2\kappa} \left| \frac{\partial \phi^A}{\partial x^i} \right| \sqrt{G[\phi]} R[\phi] \right], \tag{3.25}$$

where $\left|\frac{\partial \phi^A}{\partial x^i}\right| = det\left(\frac{\partial \phi^A}{\partial x^i}\right)$. We identify $\phi^A(x)$ $(A = 1, 2, 3)$ as the dual gluons. A simple way of seeing this is as follows. Note that the second term comes with a factor $i = \sqrt{-1}$, whereas the first term does not. In this sense it is analogous to the θ-term in QCD which continues to have the factor $i = \sqrt{-1}$ in the Euclidean version. Consider a random phase approximation to Z. The extrema of the phase factor correspond to solutions of the the vacuum Einstein equations. In this case (3 dimensions), this means that the space is flat. Now we may choose the standard form $G_{AB} = \delta_{AB}$. ϕ^A now represent arbitrary curvilinear coordinates for that manifold. Then the first term in (3.25) is just $(\nabla\phi^A)^2$. This describes three massless scalars. As in section 1 they represent the one transverse degree of freedom for each color. Thus the gluons are now described in terms of gauge invariant, local, scalar degrees of freedom.

In the general case $R \neq 0$, consider normal coordinates $\phi^A(x)$ at a given point. The metric has the standard form,

$$G_{AB}[\phi] = \delta_{AB} + R_{ABCD}[\phi]\,\phi^C\phi^D + \dots \qquad (3.26)$$

ϕ^A represents the dual gluons and R the geometric aspects of the manifold. Both are degrees of freedom of 2+1-dimensional Yang-Mills theory. ϕ^A are invariant under the Yang-Mills gauge transformations. Thus equation (3.25) describes Yang-Mills dynamics in terms of gauge invariant degrees of freedom.

3.3 Monopoles

We now identify the monopoles of Yang-Mills theory in terms of the dual variables. Monopoles are related to Yang-Mills configurations $\{A_i^a(x)\}$ with a non-trivial U(1) fiber bundle structure [14]. In such configurations, the monopoles are characterized by points with the following property [42]. Consider a surface enclosing a point and a set of based loops spanning it. Consider eigenvalues of the corresponding Wilson loop operator. As one spans the sphere, the eigenvalue changes continuously from zero to 2π instead of coming back to zero. Thus such points have topological meaning. Moreover a small change in their position can produce a large change in the expectation value of the Wilson loop. Therefore we may expect that such points are relevant for confinement, even though a semi-classical or dilute gas approximation may not be available. Therefore it is important to provide a characterization of these monopoles and their interactions with the dual gluons.

In case of 'tHooft-Polyakov monopole, the location of the monopoles is given by the zeroes of the Higgs field [43]. In pure gauge theory we do not have such an explicit Higgs field. 'tHooft [11] has proposed use of a composite Higgs for this case.

We follow a different procedure here. Consider the eigenvalue equation of the positive symmetric matrix $B_i^a(x)B_i^b(x) = I^{ab}(x)$ for each x.

$$I^{ab}(x)\chi_a^A(x) = \lambda^A(x)\chi_b^A(x). \tag{3.27}$$

The eigenvalues $\lambda^A(x)$, $(A = 1, 2, 3)$ are real and the corresponding three eigenfunctions $\chi_a^A(x)$, $(A = 1, 2, 3)$ form an orthonormal set. The monopoles in any Yang-Mills configuration $A_i^a(x)$ can be located in terms of $\chi_a^A(x)$. We will illustrate this explicitly in case of the Prasad-Sommerfield solution [44]. For this I^{ab} has the tensorial form,

$$I^{ab}(x) = P(r)\delta^{ab} + Q(r)x^a x^b \tag{3.28}$$

with $P(0) \neq 0$ and finite. At $r = 0$, the eigenvalues are triply degenerate. Away from $r = 0$, two eigenvalues are still degenerate, but the third one is distinct from them. The corresponding eigenfunction (labeled A=1, say) is $\chi_a^1(x) = \hat{x}^a$. This precisely has the required behavior for the composite Higgs at the center of the monopole [11].

We may regard $\chi_a^A(x)$ as providing three independent triplets of (normalized) Higgs fields. 'tHooft [45] had used the Higgs field of the Georgi-Glashow model to define an Abelian field strength, using which he characterized the magnetic monopole. In this case, drawing the same analogy, we may construct three Abelian gauge fields,

$$b_i^A(x) = \chi_a^A(x) B_i^a(x) - \frac{1}{3}\epsilon_{ijk}\epsilon^{abc}\chi_a^A D_j\chi_b^A D_k\chi_c^A \tag{3.29}$$

We have

$$b_i^A(x) = \epsilon_{ijk}\partial_j a_k^A - \frac{1}{3}\epsilon_{ijk}\epsilon^{abc}\chi_a^A\partial_j\chi_b^A\partial_k\chi_c^A \tag{3.30}$$

where the three Abelian gauge potentials are given by $a_i^A(x) = \chi_a^A(x) A_i^a(x)$. For each $A = 1, 2, 3$, the second part of the right hand side is the topological current for the Poincare-Hopf index [43]. It is the contribution of the magnetic fields due to the monopoles. These monopoles are located at points where this index is non-zero.

Since, $\chi_a^A = \frac{1}{2}\epsilon_{BC}^A\epsilon_a^{bc}\chi_b^B\chi_c^C$, we may rewrite our Abelian fields as

$$b_i^A(x) = \epsilon_{ijk}(\partial_j a_k^A(x) + \epsilon^{ABC}c_j^B c_k^C) \tag{3.31}$$

where $c_i^A = \epsilon^{ABC}\chi_a^B\partial_i\chi_a^C$ has the form of a 'pure gauge' potential, but is not, because of the singularity in (χ_a^A).

Thus for any configuration $A_i^a(x)$ of the Yang-Mills potential, monopoles are located at the points where the eigenvalues of the symmetric matrix $B_i^a(x)B_i^b(x)$ become triply degenerate. We may use the corresponding eigenfunctions to construct three Abelian gauge fields with respective monopole sources. Instead of I^{ab}, we may also use the gauge invariant symmetric tensor field $B_i^a(x)B_j^a(x)$ and its eigenfunctions $\chi_i^A(x)$. This provides a gauge invariant description of the monopoles.

We may also use the Ricci tensor $R_i^j = R_{ik}(x)g^{kj}(x)$ for this purpose. The three eigenfunctions $\chi_i^A(x)$, $(A = 1, 2, 3)$ (Ricci principal directions [46]) provide three orthogonal vector fields for the 3-manifold. In regions where eigenvalues of R_i^j are degenerate, the choice of the vector fields is not unique. One can make any choice requiring continuity. However *isolated points* where R_i^j is triply degenerate are special, and have topological significance. At such points the vector fields are singular. Thus the monopoles correspond to the singular points of these vector fields. The index of the singular point is the monopole number.

We emphasize that the centers have a topological interpretation which is independent of the way we construct them.

3.4 Interaction of dual gluons with monopoles

Dual gluons are identified with a coordinate system $\phi^A(x)$ $A = 1, 2, 3$ on the 3-manifold eqns.(3.25) (3.26). We now consider special coordinate systems which are singular at the location of the monopole. In case of the Prasad-Sommerfield monopole, they correspond to the spherical coordinates (r, θ, ϕ)

with the monopole at the origin. In the general case, we may construct the coordinate system as follows. At the site of the monopole, one of the eigenfunctions $\chi^1_i(x)$ say, has the radial behavior. Then we may construct the integral curves of this vector field by solving the equations,

$$\frac{dx^1}{\chi^1_1(x)} = \frac{dx^2}{\chi^1_2(x)} = \frac{dx^3}{\chi^1_3(x)}. \tag{3.32}$$

We may choose these integral curves to be the equivalent of the r-coordinate, i.e. we identify these curves with θ =constant, ϕ =constant curves of the new coordinate system. Consider closed surfaces surrounding the monopole which are nowhere tangential to these integral curves. A simple choice is just the spherical surfaces. We may identify them with the surfaces r=constant. (We have not specified the θ, ϕ coordinates completely, but this is not required for our purpose.) We thus have a coordinate system $\chi^A(x)$ whose coordinate singularities correspond to the monopoles. In this coordinate system,

$$\int d^3x\sqrt{g}R = \int d^3x\; (\epsilon_{ijk}\epsilon^{ABC}\partial_i\chi^A\partial_j\chi^B\partial_k\chi^C)\sqrt{\mathcal{G}}(x)R(x) \tag{3.33}$$

where $\mathcal{G}_{ij}$ is the metric in this coordinate system.

Now $\partial_i(\epsilon_{ijk}\partial_j\chi^2\partial_k\chi^3)$ is non-zero at $x = x_0$ and is related to the monopole charge at x_0 as follows. Let

$$\chi^A(x) - \chi^A(x_0) = \rho(x)\hat{\chi}^A(x) \tag{3.34}$$

where

$$\hat{\chi}^A(x)\hat{\chi}^A(x) = 1. \tag{3.35}$$

We see that there is a coupling of the field combination $\sqrt{G(x)}R(x)\rho^3(x)$ to the monopole charge density

$$\partial_i k_i(x) = m_i\delta^3(x_0), \tag{3.36}$$

where

$$k_i(x) = \epsilon_{ijk}\epsilon^{ABC}\hat{\chi}^A\partial_j\hat{\chi}^B\partial_k\hat{\chi}^C. \tag{3.37}$$

Thus a certain combination of the dual gluon $\phi^A(x)$ and the geometric degree of freedom $R(x)$ couples to the monopoles. In analogy to the compact U(1) lattice gauge theory (sec.1), this may be expected to give a mass for the dual gluon and hence confinement. There are other interactions which are not of topological origin and these are to be interpreted as self interactions.

3.5 Conclusion

In this chapter we have argued that the duality transformation for 2+1-dimensional Yang-Mills theory can be carried out in close analogy to the Abelian case. The dual theory has geometric interpretation in terms of 3-manifolds. We identified the dual gluons with the coordinates of the 3-manifolds and monopoles with the coordinate singularities. We expect that this will provide a new approach for understanding quark confinement.

Chapter 4

Gauge field copies

In this chapter, we deal with the problem of gauge field copies. Wu and Yang [14] gave an explicit example of two (gauge inequivalent) Yang-Mills potentials $\vec{A}_i(x) = \{A_i^a(x), a = 1,2,3\}$ generating the same non-Abelian magnetic field

$$\vec{B}_i[A](x) = \epsilon_{ijk}(\partial_j \vec{A}_k + \frac{1}{2}\vec{A}_j \times \vec{A}_k). \tag{4.1}$$

Since then there has been a wide discussion of the phenomenon in the literature [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. We may refer to gauge potentials giving the same non-Abelian magnetic field, as gauge field copies in contrast to gauge equivalent potentials which generate magnetic fields related by a homogeneous gauge transformation. If we require all higher covariant derivatives of B_i^a also match then there are effectively no gauge copies [24].

Deser and Wilczek [17] first pointed out the consistency condition for $\vec{A}_\mu$ and $\vec{A}'_\mu = \vec{A}_\mu + \vec{\Delta}_\mu$ to generate the same field strength. Using the Bianchi identity, they obtained that $\vec{\Delta}_\mu$ had to satisfy the equation

$$[\tilde{\mathbf{F}}_{\mu\nu}, \Delta_\nu] = 0.^1 \tag{4.2}$$

where in 2 dimensions,

$$\tilde{F}^{\mu\nu\, ab} = \frac{1}{2}\epsilon^{\mu\nu}\epsilon^{abc}F^c_{\mu\nu} = M^{a\,b}, \tag{4.3}$$

and in 4 dimensions

$$\tilde{F}^{\mu\nu\, ab} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\epsilon^{abc}F^c_{\rho\sigma} = M^{a\mu\,,b\nu}. \tag{4.4}$$

Treating this as an eigenvalue equation for Δ, we have the condition for existence of non-trivial solutions of Δ is that the determinant of M is zero. In 2 dimensions the determinant corresponding to M vanishes identically and there Δ necessarily has non-trivial solutions. However in 4 dimensions this determinant in generically non-zero and there are hardly any gauge copies.

This sort of analysis however exists only in even dimensions. In 3 Euclidean dimensions, we only get the constraint $\vec{B}_i[A] \times \vec{\Delta}_i = 0$. This equation has many solutions, but this is only a consisitency condition. It does not mean that any $\vec{\Delta}_i$ satisfying this equation gives a gauge copy. Recently Freedman and Khuri [28] have exhibited several examples of continuous families of gauge field copies in d=3. Their technique was to use a local map of the gauge field system into a spatial geometry with a second rank symmetric tensor $G_{ij} = B^a_i B^a_j \, detB$ and a connection with torsion constructed from it.

We adopt a different method and directly ask the question as to how many different solutions (if any), does the system of equations defined by

[1] In the matrix notation

(4.1) have for any specified $\vec{B}_i(x)$. For that we proceed with the analysis using the Cauchy - Kowalevsky existence theorems on systems of partial differential equations. The equations for the gauge field copies are not a priori in the form where this theorem can be applied. However by reorganizing the equations a bit they can be brought to the form so that these theorems can be applied to that system.

4.1 Existence of A for arbitrary B

Let us first state the Cauchy - Kowalevsky existence theorem which we use.

Let a set of partial equations be given in the form

$$\frac{\partial z_i}{\partial x_1} = \sum_{j=1}^{m} \sum_{r=2}^{n} G_{ijr} \frac{\partial z_j}{\partial x_r} + G_i \tag{4.5}$$

for values $i = 1, ..., m$, being m equations in m dependent variables ; the coefficients G_{ijr} and the quantities G_i are functions of all the variables, dependent and independent. Let $c_1, ..., c_m, a_1, ..., a_n$ be a set of values of $z_1, ...z_m, x_1, ..., x_n$ respectively, in the vicinity of which all the functions G_{ijr} and G_i are regular ; and let $\phi_1, ..., \phi_m$ be a be a set of functions of $x_2, ..., x_n$, which acquire values $c_1, ..., c_m$ respectively when $x_2 = a_2, ..., x_n = a_n$, which are regular in the vicinity of these values of $x_2, ..., x_n$, and which are otherwise arbitrary. Then *a system of integrals of the equations can be determined, which are regular functions of* $x_1, ..., x_n$ *in the vicinity of the values* $x_1 = a_1, x_2 = a_2, ..., x_n = a_n$, *and which acquire the values* $\phi_1, ..., \phi_m$ *when* $x_1 = a_1$ *; moreover, the system of integrals determined in accordance with these conditions, is the only system of integrals that can be determined as* *regular functions.*

Our system of equations is

$$\vec{B}_1 = \partial_2\vec{A}_3 - \partial_3\vec{A}_2 + \vec{A}_2 \times \vec{A}_3 \tag{4.6}$$

$$\vec{B}_2 = \partial_3\vec{A}_1 - \partial_1\vec{A}_3 + \vec{A}_3 \times \vec{A}_1 \tag{4.7}$$

$$\vec{B}_3 = \partial_1\vec{A}_2 - \partial_2\vec{A}_1 + \vec{A}_1 \times \vec{A}_2 \tag{4.8}$$

where $\vec{B}_1, \vec{B}_2$ and $\vec{B}_3$ are treated as given variables and we want to solve for $\vec{A}_1, \vec{A}_2$ and $\vec{A}_3$. With this definition of the $B's$, the bianchi identity $D_iB_i = 0$ follows automatically. However the existence theorem cannot be applied directly to this set of equations. For that we rewrite the equations in a different way. Consider

$$\partial_3\vec{A}_2 = \partial_2\vec{A}_3 + \vec{A}_2 \times \vec{A}_3 - \vec{B}_1 \tag{4.9}$$

$$\partial_3\vec{A}_1 = \partial_1\vec{A}_3 - \vec{A}_3 \times \vec{A}_1 + \vec{B}_2. \tag{4.10}$$

The existence theorem implies that we have solution for $\vec{A}_1$ and $\vec{A}_2$ for any specified $\vec{B}_1, \vec{B}_2$ and $\vec{A}_3$. But $\vec{A}_1$ and $\vec{A}_2$ so obtained have to satisfy equation (4.8). Is this always possible with some choice of $\vec{A}_3$, and if yes, is the choice of $\vec{A}_3$ unique? To address this question, we presume that the initial data on $x_3 = 0$ satisfies equations (4.6)-(4.8). This is always possible for any given $\vec{B}_i(x)$ as follows from our discussion in the 1+1-dimensional case. Then equation (4.8) may be equivalently replaced by another equation obtained by applying ∂_3 on it and using (4.6)-(4.7). This is just the Bianchi identity. We write it in the form

$$\vec{A}_3 \times \vec{B}_3 = -\partial_3\vec{B}_3 - \partial_2\vec{B}_2 - \vec{A}_2 \times \vec{B}_2 - \partial_1\vec{B}_1 - \vec{A}_1 \times \vec{B}_1 \tag{4.11}$$

Now let us decompose $\vec{A}_3$ in directions parallel and perpendicular to $\vec{B}_3$,

$$\vec{A}_3 = \alpha \vec{B}_3 + \vec{A}_{3\perp}. \tag{4.12}$$

In the generic case, where $|\vec{B}| \neq 0$, equation (4.11) determines $\vec{A}_{3\perp}$ entirely. Taking the cross product of (4.11) with $\vec{B}_3$, we get,

$$\vec{A}_3 = \alpha \vec{B}_3 - \frac{1}{|\vec{B}_3|^2} \vec{B}_3 \times \left[(\vec{A}_2 \times \vec{B}_2) + (\vec{A}_1 \times \vec{B}_1) + (\partial_i \vec{B}_i) \right]. \tag{4.13}$$

where α can be arbitrarily chosen.

We now address the question whether α can also be determined. Taking the dot product of (4.11) with $\vec{B}_3$, we get,

$$\vec{B}_3 \cdot \partial_i \vec{B}_i + (\vec{B}_3 \times \vec{B}_1) \cdot \vec{A}_1 + (\vec{B}_3 \times \vec{B}_2) \cdot \vec{A}_2 = 0. \tag{4.14}$$

This is a constraint which $\vec{A}_1$ and $\vec{A}_2$ have to satisfy. It is satisfied on $x_3 = 0$. In order that it is satisfied at all x_3, we require

$$\begin{aligned} -(\partial_1 \vec{A}_3 - \vec{A}_3 \times \vec{A}_1 + \vec{B}_2) \cdot (\vec{B}_1 \times \vec{B}_3) - \vec{A}_1 \cdot \partial_3 (\vec{B}_1 \times \vec{B}_3) & \\ -(\partial_2 \vec{A}_3 + \vec{A}_2 \times \vec{A}_3 - \vec{B}_1) \cdot (\vec{B}_2 \times \vec{B}_3) - \vec{A}_2 \cdot \partial_3 (\vec{B}_2 \times \vec{B}_3) & \\ -\partial_3 (\partial_i \vec{B}_i) \cdot \vec{B}_3 - (\partial_i \vec{B}_i) \cdot (\partial_3 \vec{B}_3) &= 0 \end{aligned} \tag{4.15}$$

Now we can substitute the expression for $\vec{A}_3$ from (4.13). Note that in this substitution, the derivatives do not act on α since in that case we get terms $\vec{B}_3 \cdot \vec{B}_1 \times \vec{B}_3$ and $\vec{B}_3 \cdot \vec{B}_2 \times \vec{B}_3$ which vanish. Generically the equation for α is invertible and this explicitly gives us α as functions of $\vec{A}_1, \vec{A}_2$ and $\vec{B}_i$.

Thus in the generic case, we can solve for $\vec{A}_3$ as local functions of $\vec{A}_1, \vec{A}_2$ and $\vec{B}_i$'s. Substituting this in equations (4.9-10), we can apply the theorem to get $\vec{A}_1, \vec{A}_2$ and hence $\vec{A}_3$ as unique functionals of $\vec{B}_i(x)$.

Alternately we could consider the system of equations

$$\partial_3 \vec{A}_2 = \partial_2 \vec{A}_3 + \vec{A}_2 \times \vec{A}_3 - \vec{B}_1 \quad (4.16)$$

$$\partial_3 \vec{A}_1 = \partial_1 \vec{A}_3 - \vec{A}_3 \times \vec{A}_1 + \vec{B}_2 \quad (4.17)$$

$$\begin{aligned}\partial_3(\vec{A}_3 \times \vec{B}_3) = & -(\partial_1 \vec{A}_3 - \vec{A}_3 \times \vec{A}_1 + \vec{B}_2) \times \vec{B}_1 - \vec{A}_1 \times \partial_3 \vec{B}_1 - \partial_3(\partial_i \vec{B}_i) \\ & -(\partial_2 \vec{A}_3 + \vec{A}_2 \times \vec{A}_3 - \vec{B}_1) \times \vec{B}_2 - \vec{A}_2 \times \partial_3 \vec{B}_2 \quad (4.18)\end{aligned}$$

$$\partial_3(\vec{A}_3 \cdot \vec{B}_3) = \partial_3(|\vec{B}_3|^2 \alpha(\vec{A}_1, \vec{A}_2, \vec{B}_i)). \quad (4.19)$$

Here in the last equation $\alpha(\vec{A}_1, \vec{A}_2, \vec{B}_i)$ is to be replaced by the expression obtained for α from equation (4.15) and $\partial_3 \vec{A}_1$ and $\partial_3 \vec{A}_2$ are to be replaced using (4.16) and (4.17). This system of equations is in the form where the Cauchy-Kowalevsky theorem can be applied and this system uniquely determines all the unknown variables once the initial data is specified. The first two equations contain the six unknowns $\vec{A}_1$ and $\vec{A}_2$. The third one contains the two components of $\vec{A}_3$ transverse to $\vec{B}_3$ and the fourth one has the component of $\vec{A}_3$ parallel to $\vec{B}_3$. Thus all the nine degrees of freedom are uniquely determined.

4.2 Existence of continuous family of gauge copies

In this section we address the question if there exists any continuous family of potentials which generate the same magnetic field. Let $\vec{A}_i$ and $\vec{A}_i + \epsilon \vec{e}_i$ generate the same magnetic field, where ϵ is a small parameter. Then $\vec{e}_i$

satisfies the equation

$$\epsilon_{ijk}(\partial_j \vec{e}_k + \vec{A}_j \times \vec{e}_k) = 0. \tag{4.20}$$

We also have a consistency condition by taking the covariant derivative of this equation. That is given by

$$\vec{B}_k \times \vec{e}_k = 0 \tag{4.21}$$

Let us rewrite the equations in a more convenient way. We take our system of equations as

$$\partial_3 \vec{e}_2 = \partial_2 \vec{e}_3 + \vec{A}_2 \times \vec{e}_3 - \vec{A}_3 \times \vec{e}_2 \tag{4.22}$$

$$\partial_3 \vec{e}_1 = \partial_1 \vec{e}_3 + \vec{A}_1 \times \vec{e}_3 - \vec{A}_3 \times \vec{e}_1 \tag{4.23}$$

and the consistency condition (4.21). This set is equivalent to the set of equations (4.20). As in the previous case, we first look at the consistency condition. Let us decompose $\vec{e}_3$ as

$$\vec{e}_3 = \beta \vec{B}_3 + \vec{e}_{3\perp} \tag{4.24}$$

Again (4.21) fixes for us $\vec{e}_{3\perp}$ in terms of the magnetic fields.

$$\vec{e}_{3\perp} |\vec{B}_3| = \vec{B}_3 \times \vec{e}_3 = -\vec{B}_I \times \vec{e}_I \tag{4.25}$$

where I goes over $1, 2$. Thus $\vec{e}_3$ is given by

$$\vec{e}_3 = \beta \vec{B}_3 - \frac{1}{|\vec{B}_3|} \vec{B}_I \times \vec{e}_I \tag{4.26}$$

Now we can substitute this form of $\vec{e}_3$ in the equations (4.22-23). Assuming that the potentials have already been determined in the previous step, we would obtain $\vec{e}_1$ and $\vec{e}_2$ as unique functions of β and the magnetic fields. However this $\vec{e}_1$ and $\vec{e}_2$ have to satisfy the consistency conditions

$$\vec{B}_3 \cdot \vec{B}_I \times \vec{e}_I = 0 \tag{4.27}$$

where again I goes over 1, 2. Taking ∂_3 of equation (4.27), we get, using (4.22) and (4.23)

$$D_3(\vec{B}_3 \times \vec{B}_I) \cdot \vec{e}_I + \vec{B}_I \cdot \vec{B}_3 \times D_I \vec{e}_3 = 0 \tag{4.28}$$

Putting in the expression of $\vec{e}_3$, we get an equation for β

$$D_3(\vec{B}_3 \times \vec{B}_I) \cdot \vec{e}_I + (\vec{B}_I \times \vec{B}_3) \cdot (D_I \vec{B}_3)\beta - (\vec{B}_I \times \vec{B}_3) \cdot D_I[\frac{1}{|\vec{B}_3|}(\vec{B}_J \times \vec{e}_J)] = 0 \tag{4.29}$$

This equation can be generically inverted to solve for β as a function of $\vec{e}_1, \vec{e}_2, \vec{A}_1, \vec{A}_2$ and $\vec{B}_i$.

Formally we could have also looked at the set of equations

$$\begin{aligned} \partial_3 \vec{e}_2 &= \partial_2 \vec{e}_3 + \vec{A}_2 \times \vec{e}_3 - \vec{A}_3 \times \vec{e}_2 && (4.30) \\ \partial_3 \vec{e}_1 &= \partial_1 \vec{e}_3 + \vec{A}_1 \times \vec{e}_3 - \vec{A}_3 \times \vec{e}_1 && (4.31) \\ \partial_3(\vec{B}_3 \times \vec{e}_3) &= -(\partial_3 \vec{B}_2) \times \vec{e}_2 - (\partial_3 \vec{B}_1) \times \vec{e}_1 \\ &\quad -\vec{B}_2 \times (\partial_2 \vec{e}_3 + \vec{A}_2 \times \vec{e}_3 - \vec{A}_3 \times \vec{e}_2) \\ &\quad -\vec{B}_1 \times (\partial_1 \vec{e}_3 + \vec{A}_1 \times \vec{e}_3 - \vec{A}_3 \times \vec{e}_1) && (4.32) \\ \partial_3(\vec{B}_3 \cdot \vec{e}_3) &= \partial_3 \beta(\vec{e}_1, \vec{e}_2, \vec{A}_1, \vec{A}_2, \vec{B}_i). && (4.33) \end{aligned}$$

In the last equation, β has to be replaced by its solution from (4.29) and $\partial_3 e_I$ is to be substituted from (4.30) and (4.31).

Applying the Cauchy-Kowalevsky theorem to this set of equations, we get a unique smooth solution for $\vec{e}_1, \vec{e}_2$ and $\vec{e}_3$ exactly as in the case for the potentials. If we choose $e_i^a = 0$ on the surface $x_3 = 0$ as the initial data, then $e_i^a = 0$ everywhere. Thus with the gauge potential specified on a 2-dimensional surface, there are no gauge field copies (in the generic case).

4.3 An explicit calculation

We now illustrate these results by an explicit calculation for the special case $A_i^a = \delta_i^a$. In momentum space, the equation looks like

$$\epsilon_{ijk}(-ip_j\delta^{ac} + \epsilon_{abc}\delta_j^b)e_k^c(p) = 0 \tag{4.34}$$

or

$$(-i\epsilon_{ijk}p_j\delta^{ac} + \delta_i^a\delta_k^c - \delta_i^c\delta_k^a)e_k^c(p) = 0 \tag{4.35}$$

In three dimensions we can choose three orthogonal vectors. We choose three such vectors as $(\vec{p}, \vec{n}, \vec{m})$ where $\vec{p}$ coincides with the $\vec{p}$ which appears in the equation and $\vec{n}$ and $\vec{m}$ are unit vectors . We also orient $(\vec{p}, \vec{n}, \vec{m})$ such that $\vec{p}\times\vec{m} = |\vec{p}|\vec{n}$ and $\vec{p}\times\vec{n} = -|\vec{p}|\vec{m}$. Next we write a general solution for e_k^c in terms of the dyad basis as

$$\begin{aligned} e_{kc} &= a_1\, n_c m_k + a_2\, n_k m_c + a_3\, n_k n_c + a_4\, m_k m_c \\ &\quad + a_5\, p_c m_k + a_6\, p_k m_c + a_7\, p_c n_k + a_8\, p_k n_c + a_9\, p_k p_c, \end{aligned} \tag{4.36}$$

where a_i's are unknown coefficients to be determined.

Substituting the solution in the equation, we get various relations among the coefficients. a_5, a_6, a_7, a_8 and a_9 turn out to be zero identically. In addition we get

$$-i|\vec{p}|a_1 = -i|\vec{p}|^3 a_2 = a_3 = |\vec{p}|^2 a_4. \tag{4.37}$$

Therefore, we get a non-zero solution only if

$$|\vec{p}| = 1, \tag{4.38}$$

in which case,

$$-ia_1 = -ia_2 = a_3 = a_4 = a \tag{4.39}$$

Thus the general solution is

$$e_{ib}(x) = \int d\Omega\, a(\Omega) e^{i\hat{p}\cdot x} (\hat{m} + i\hat{n})_i (\hat{m} - i\hat{n})_b \tag{4.40}$$

Here the integration is over all directions of the vector $\hat{p}$. The solutions have an arbitrary function $a(\Omega)$. We may fix $a(\Omega)$ by using initial data on $x_3 = 0$ surface. This may be interpreted as the arbitrary choice of $\vec{e}_i(x)$ at the boundary. However if we require $\vec{e}_i(x)$ vanishes rapidly at infinity, there may not be any solutions. Thus gauge copies would be absent in this case.

A similar exercise can be carried out for any constant vector potential and gives an identical result.

4.4 Conclusions

In this chapter we have looked at two problems regarding the existence of non-Abelian vector potentials. First we asked the question if there exists a vector potential for any arbitrary magnetic field. We found that there are many choices of $\vec{A}_i(x)$ on the $x_3 = 0$ surface which reproduces $\vec{B}_i(x)$ on the surface. (This is the gauge field ambiguity in 1+1 dimensions.) For each such boundary condition on $\vec{A}_i(x)$ we have seen (in the generic case) that there is a unique potential $\vec{A}_i(x)$ which reproduces the given magnetic field everywhere. The non-Abelian Bianchi identity does not constrain the non-Abelian magnetic fields in contrast to the abelian case. The ambiguity in the choice of the potentials is (in the generic case) only due to the ambiguity in $\vec{A}_i(x)$ on the $x_3 = 0$ surface. Thus it is related to the gauge copy problem in 1+1 dimensions.

Chapter 5

General solution of the non-Abelian Gauss law and non-Abelian analogs of the Hodge decomposition

We need to know the solutions of the constraint equations of a theory for mapping out its phase space. For Maxwell electrodynamics, the solution of the constraint equation, which is the Gauss law, leads to a dual description of the theory with a dual vector potential. The construction of this potential is facilitated by the Hodge decomposition. However there is no such known decomposition for the non-Abelian case. In this chapter we address the two related questions. Solution of the non-Abelian Gauss law and non-Abelian analogs of the Hodge decomposition.

An important, related question is that given a field strength can one write down a non-Abelian potential from which the field strength can be derived. There has been extensive discussion about this in the literature. Halpern [34] attempted to construct A from F in 1+1, 2+1 and 3+1 dimensions in the completeiy fixed axial gauge. Weiss [47] pointed out that in 1+1 dimensions every field strength tensor can be derived from a potential. In fact if $\vec{F}_{\mu\nu} \neq 0$, then there is a huge ambiguity in the choice of potentials as we saw in the previous chapter. But this result does not generalize to $d > 2$.

In 1+1 dimensions, $\vec{F}_{\mu\nu}$ has only one component say $\vec{E}(x,t)$. With the definitions

$$A_0^{a[U]}(x,t) = 0 \tag{5.1}$$

and

$$A_1^{a[U]}(x,t) = \int_{t_0}^{t} d\tau\, U^{ab}(x,\tau) E^b(x,\tau) \tag{5.2}$$

we get

$$F_{01}^{a[U]}(x,t) = U^{ab} E^b(x,t) \tag{5.3}$$

Choosing U to be identity, we get the potential in the gauge $A_0^a = 0$ at all t and $A_1^a = 0$ at $t = t_0$. This is a complete fixation of the axial gauge in 1+1 dimensions.

In all these cases, the potential was obtained as a non-local integral over the field strength. In this chapter we will concentrate on local solutions of the non-Abelian Gauss law and local expressions of the non-Abelian Hodge decomposition.

5.1 Solution of the non-Abelian Gauss law

Yang-Mills theory has the conjugate variables $\vec{A}_i(x)$ and $\vec{E}_i(x)$, where $\vec{A}_i(x) = A_i^a(x), (i, a = 1, 2, 3)$. $\vec{A}_i(x)$ is the Yang-Mills potential and $\vec{E}_i(x)$ is the non-Abelian electric field. There is a first class constraint, the non-Abelian Gauss law,

$$\partial_i \vec{E}_i + \vec{A}_i \times \vec{E}_i = 0. \tag{5.4}$$

This constraint also appears in Ashtekar formulation of gravity [29] and its solution is of importance there too.

In this chapter, we are interested in a general solution of (5.4) in terms of local fields. This is relevant for duality transformation of Yang-Mills theory [30][31][32]. Note that in the Abelian case the Gauss law constraint $\partial_i E_i = 0$ has the solution $E_i = \epsilon_{ijk}\partial_j C_k$. Here C_k turns out to be the dual gauge potential which couples minimally to magnetic matter.

In analogy to the Abelian case, we consider the ansatz

$$E_i = \epsilon_{ijk}[D_j[A], C_k]. \tag{5.5}$$

Here it is useful to adopt the matrix notation; $A_i = \vec{A}_i \cdot \vec{\sigma}/2$ etc., where σ^a are the Pauli matrices and $D_j[A] = \mathbf{1}\partial_j + A_i$. In this notation the non-Abelian Gauss law becomes $[D_i[A], E_i] = 0$. Substituting (5.5)in (5.4) and using the Jacobi identity, we get,

$$[B_i[A], C_i] = 0 \tag{5.6}$$

where sum over i is implied. Here

$$\vec{B}_i[A](x) = \epsilon_{ijk}(\partial_j \vec{A}_k + \frac{1}{2}\vec{A}_j \times \vec{A}_k) \tag{5.7}$$

is the non-Abelian magnetic field. We consider a generic case where the 3×3 matrix B_i^a is invertible in a certain region of space. Then we may use B_i^a to "lower" the index a in C_i^a:

$$C_i^a = C_{ij} B_j^a. \tag{5.8}$$

From (5.6), we get,

$$|B|(B^{-1})_a^k(\epsilon_{ijk} C_{ij}) = 0, \tag{5.9}$$

where

$$|B| \equiv det(B_i^a). \tag{5.10}$$

Therefore equation (5.6) is satisfied if and only if C_{ij} is an arbitrary symmetric matrix. Thus we have obtained a class of solutions

$$E_i = \epsilon_{ijk}(B_l[A]\partial_j C_{lk} + [D_j[A], B_l[A]]C_{lk}). \tag{5.11}$$

This presents the solution as a covariant curl in close analogy to the Abelian case.

The symmetric tensor field C_{ij} has six degrees of freedom at each x. Therefore it appears that the solution in terms of the gauge invariant field C_{ij} is a general solution. An exception to this case would be when two fields C and C' give the same solution of the non-Abelian Gauss law. Such a situation occurs if there is a field e_k satisfying

$$\epsilon_{ijk}[D_j[A], e_k] = 0, \tag{5.12}$$

where $e_k = (C - C')_k$. This is precisely the equation for a driebein e_i^a to be torsion free with respect to the connection one form $\omega_i^{ab} = \epsilon^{abc} A_i^c$. This situation has been analyzed in detail in chapter 4. For any A_i^a there is only one e_i^a fixed by the boundary condition. This does not affect our local parametrization very much.

Thus the gauge invariant second rank tensor C_{ij}, in equation (5.8) effectively describes the physical degrees of freedom of Yang-Mills theory.

5.2 Non-Abelian analog of the Hodge decomposition

Given an E_i satisfying the non-Abelian Gauss law (5.4), construction of C is as follows. Applying covariant curl on both sides of (5.5), we get,

$$\epsilon_{ijk}[D_j[A], E_k] = -[D_j[A], [D_j[A], C_i]] + [D_j[A], [D_i[A], C_j]] \qquad (5.13)$$

Thus we get a second order equation for C in terms of E. The solution involves inversion of the covariant laplacian in analogy to the Abelian case. Therefore we may expect the solution to exist.

5.2.1 Covariant gradient and curl

We may use the above results for obtaining the non-Abelian analogs of the Hodge decomposition. Consider any isotriplet vector field $V_i^a(x)$. We first consider a decomposition of V_i as a sum of a covariant curl and a covariant gradient with respect to any specified Yang-Mills potential $A_i^a(x)$. Consider

$$\vec{\mathcal{E}}_i = \vec{V}_i - D_i[A]D^{-2}[A](D_j[A]\vec{V}_j). \qquad (5.14)$$

Here we are presuming that the covariant laplacian $D^2[A]$ has no zero eigenvalues and is therefore invertible. This would be true for fields vanishing rapidly at infinity on $\mathbf{R}^3$. Thus any V_i^a has a unique decomposition

$$\vec{V}_i = D_i[A]\vec{\chi} + \vec{\mathcal{E}}_i \tag{5.15}$$

where

$$D_i[A]\vec{\mathcal{E}}_i = 0. \tag{5.16}$$

For $\mathcal{E}_i$ we have a general decomposition as in (5.5). Thus we have a decomposition of V_i into covariant curl and covariant gradient.

The above procedure may also be generalized when the covariant laplacian $D^2[A]$ has null eigen-vectors, for example on compact manifolds. In this case a "harmonic form" is also required for the decomposition. We may expect this harmonic part to have a cohomological interpretation.

5.2.2 Interpolation

Next we consider a different non-Abelian analog of the Hodge decomposition. We seek a decomposition,

$$\vec{V}_i = \vec{B}_i[C] + D_i[C]\vec{\phi} \tag{5.17}$$

in terms of the non-Abelian magnetic field and covariant gradient with respect to a new gauge potential C. Note that in contrast to the previous case, $B_i[C]$ is non-linear in C. Therefore even if the decomposition exists, the reconstruction of C is not easy. In case a specified background field A is "close" to C,

$$\vec{V}_i - \vec{B}_i[A] \simeq \epsilon_{ijk}[D_j[A], \Delta C_k] + D_i[A]\phi \tag{5.18}$$

where $\Delta\vec{C}_k = \vec{C}_k - \vec{A}_k$. Thus the previous decomposition may be regarded as a special case of this when the given vector field V_i is "close" to $B_i[A]$ for the specified Yang-Mills potential A_i.

We first note that $B_i[C]$ and $D_i[C]\phi$ represent independent degrees of freedom of V_i just as curl and gradient. As a consequence of the bianchi identity, the inner product

$$\int d^3x \vec{B}_i[C] \cdot D_i[C]\vec{\phi} = \int dS_i \, \vec{\phi} \cdot \vec{B}_i \tag{5.19}$$

gives the first Chern class. For fields vanishing rapidly at infinity, this is zero. Moreover the equation

$$\vec{B}_i[C] = D_i[C]\vec{\phi} \tag{5.20}$$

is precisely the Bogomolnyi equation. All solutions of this are known by ADHM construction and are labeled by positions and (isospin) orientations of (anti-)monopoles. In case we require fields vanish faster than r^{-1} at infinity, then there are effectively no solutions. Thus $B_i[C]$ and $D_i[C]\phi$ represent distinct degrees of freedom.

In order to construct C and ϕ, we consider an interpolation procedure. Consider

$$\lambda\vec{V}_i = \vec{B}_i[C(\lambda)] + D_i[C(\lambda)]\vec{\phi}(\lambda) \tag{5.21}$$

where

$$C(\lambda) = \sum_{n=1}^{\infty} \lambda^n C^{(n)} \text{ and } \phi(\lambda) = \sum_{n=1}^{\infty} \lambda^n \phi^{(n)} \tag{5.22}$$

Terms linear in λ give

$$\vec{V}_i = \epsilon_{ijk}\partial_j \vec{C}_k^{(1)} + \partial_i \vec{\phi}^{(1)}, \tag{5.23}$$

which is just the usual Hodge decomposition of V_i. We know the decomposition exists and is unique. $C_k^{(1)}$ and $C_k^{(1)} + \partial_k \Lambda$, both give the same decomposition. Terms quadratic in λ gives

$$\epsilon_{ijk}\partial_j \vec{C}_k^{(2)} + \partial_i \vec{\phi}^{(2)} = -\epsilon_{ijk}\vec{C}_j^{(1)} \times \vec{C}_k^{(1)} - \vec{C}_i^{(1)} \times \vec{\phi}^{(1)}. \tag{5.24}$$

$C^{(1)}$ and $\phi^{(1)}$ are already known. Hence $\phi^{(2)}$ and $C^{(2)}$ are determined. Again the gradient part of $C^{(2)}$ is arbitrary. This way all the $C^{(n)}$'s and $\phi^{(n)}$'s are determined successively. If we impose a "gauge condition" such as

$$\partial_i \vec{C}_i^{(n)} = 0, \tag{5.25}$$

$C^{(n)}$ and $\phi^{(n)}$ are unique at each stage. This interpolation procedure makes the connection to the usual Hodge decomposition explicit and provides a plausible technique for reconstructing C and ϕ. We do not address the question of convergence in (5.22), but provide an alternate procedure for reconstruction of C and ϕ below.

A way of avoiding interpolation is as follows. Consider,

$$\vec{\mathcal{E}}_i[C] = \vec{V}_i - D_i[C]D^{-2}[C]D_j[C]\vec{V}_j \tag{5.26}$$

which satisfies $D_i[C]\vec{\mathcal{E}}_i[C] = 0$. We have to choose C, such that

$$\vec{e}_i[C,\theta] \times \vec{\mathcal{E}}_i[C] = 0 \tag{5.27}$$

for every driebein $e_i[C,\theta]$ which is torsion free with respect to the connection one form C. Then, $\mathcal{E}_i[C]$ is the non-Abelian magnetic field $B_i[C]$ and we have the decomposition (5.17).

5.3 Remark

We have shown in chapter 4 that any non-Abelian vector field $\vec{b}_i(x)$ may be solved in terms of the non-Abelian vector potential

$$\vec{b}_i(x) = \vec{B}_i[C](x). \tag{5.28}$$

Thus the covariant gradient term in equation (5.17) is not necessary. This is in stark contrast to the abelian case.

5.4 Conclusion

We have obtained a general solution of the non-Abelian Gauss law in close analogy to the Poincare lemma. We have used it to address the non-Abelian analog of the Hodge decomposition. These are useful for duality transformation of Yang-Mills theory [30][31][32].

Chapter 6

Duality transformation for 3+1-dimensional Yang-Mills theory

In 1977 Montonen and Olive [51] conjectured that just like Maxwell electrodynamics, Yang-Mills theories might also have a duality symmetry. This was first investigated by Deser and Teitelboim [10]. Maxwell's equations

$$\partial_\mu F^{\mu\nu} = 0 \tag{6.1}$$

and

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{6.2}$$

are invariant under the transformations

$$\delta \vec{E}(x) = \beta \vec{B}(x) \tag{6.3}$$

and

$$\delta\vec{B}(x) = -\beta\vec{E}(x) \tag{6.4}$$

The action is also invariant under this infinitesimal transformation.

However in the formally analogus Yang-Mills case, duality transformation cannot even be consistently implemented. No transformation of the variables exist which leave the action invariant and reduces to a duality rotation on shell. If one demands the existence of a set of variations δA_μ, which on the mass shell should give

$$\delta\vec{F}^{\mu\nu} = \beta\vec{\tilde{F}}^{\mu\nu} \tag{6.5}$$

and

$$\delta\vec{\tilde{F}}^{\mu\nu} = -\beta\vec{F}^{\mu\nu} \tag{6.6}$$

one obtains the equations

$$\delta\vec{A}_\mu \times \vec{F}^{\mu\nu} = 0 \tag{6.7}$$

and

$$\delta\vec{A}_\mu \times \vec{\tilde{F}}^{\mu\nu} = 0. \tag{6.8}$$

For non-trivial solutions of δA_μ, one should have the consistency condition $detM = 0$ where M is given by

$$M^{a\mu,b\nu} = \epsilon^{abc}F^{\mu\nu\,c}. \tag{6.9}$$

But we have already seen this equation in the context of gauge field copies and we know that this determinant is generically non-zero in 3+1 dimensions. Thus Deser and Teitelboim concluded that duality in the sense of Maxwell electrodynamics is not present for Yang-Mills theory.

Another significant contribution was made in this area by M.B.Halpern [34]. His definition for the dual potential, at least for QED was

$$\tilde{F}^{\mu\nu}(A) = F^{\mu\nu}(\tilde{A}) \tag{6.10}$$

where $\tilde{F}^{\mu\nu}$ is the dual field strength tensor. He obtained the dual gauge potential by inverting the dual field strength. Working in the axial gauge with no residual degrees of freedom he used the Bianchi identities crucially as consistency conditions. The dual potentials had Higgs type couplings and coupled to the monopole currents. However his analysis was non-local and the dual potentials were functionals of the field strength.

In this chapter we bring in new techniques which are useful for duality transformation of non-Abelian gauge theories. Though we use the language of functional integrals, our procedure can be stated directly for classical Yang-Mills theory. We adopt the Hamiltonian formalism. This is the most direct method for duality transformation in Maxwell's theory as reviewed in the first section. This brings out the crucial role played by the Gauss law and the Hodge decomposition in duality transformation which we developed in the previous chapter. This approach automatically gives the dual theory as a SO(3) gauge theory, with a non-Abelian dual gauge field.

We also use a generating function of a canonical transformation to perform the duality transformation (6.2.2). We find that it is an extremely powerful technique for handling non-Abelian theories. It is very helpful for obtaining the implication of the non-Abelian Gauss law for the dual theory. It turns out that it is natural to treat the dual gauge field as a background gauge field of the Yang-Mills theory and vice-versa. (We use rescaled fields such that the gauge transformations do not involve the coupling constants.) Choosing the generating function to be invariant under a common gauge transformation, the Gauss law constraint simply goes over to a similar constraint in the dual theory (section 6.2.3). Another important issue is the gauge copy problem [50, 28], i.e. gauge inequivalent potentials which give the same non-Abelian magnetic field. In analogy to the Abelian case, we would like to replace $\vec{E}_i$, the non-Abelian electric field by $\vec{B}_i[C]$, the non-Abelian magnetic field of the dual gauge potential C. But if gauge copies are present, then this naive replacement runs into problems. We have argued in chapter 4 that there is only boundary degrees of freedom for the gauge field copies. As a consequence the number of degrees of freedom provided by $\vec{B}_i[C]$ are sufficient.

We explore the possibility of self duality of 3+1-dimensional Yang-Mills theory in section 3 and conclude that it is absent. All the canonical transformations that we consider lead to a dual theory which is non-local.

We summarize our results in sec 4.

6.1 Gauss law and duality transformation in Maxwell's theory

Consider the free Maxwell theory. The extended phase space has the canonical variables, the vector potential A_i and the electric field E_i, $i = 1, 2, 3$ with the Poisson bracket

$$[A_i(x), E_j(y)]_{PB} = \delta_{ij}\delta(x - y). \tag{6.11}$$

The Hamiltonian density is,

$$H(x) = \frac{1}{2}(E_i^2(x) + B^2[A]_i(x)) \tag{6.12}$$

where the magnetic field $B[A]_i = \epsilon_{ijk}\partial_j A_k$. A_i and $A_i + \partial_i \Lambda$ give rise to same $B[A]_i$. The physical phase space is the subspace given by the Gauss law constraint,

$$\partial_i E_i = 0. \tag{6.13}$$

A very easy way of obtaining the dual theory is to solve the Gauss law constraint. We have the general solution,

$$E_i = \epsilon_{ijk}\partial_j C_k \tag{6.14}$$

We can compute the Poisson bracket of the new variable C with the old variables as follows. We have the Poisson bracket

$$[B_i(x), E_j(y)]_{PB} = -\epsilon_{ijk}\partial_k \delta(x - y). \tag{6.15}$$

Substituting the above ansatz for E we get as a consistent solution the non-zero Poisson bracket

$$[B_i(x), C_j(y)]_{PB} = \delta_{ij}\delta(x - y). \tag{6.16}$$

Thus we have the new canonical pair (C, $\mathcal{E} = B[A]$) in contrast to the old set (A, E). In terms of this new pair the Hamiltonian takes the form

$$H(x) = \frac{1}{2}(\mathcal{E}_i^2(x) + B_i^2[C](x)). \tag{6.17}$$

Thus we have made a canonical transformation from the pair (A, E) to (C, B) and the Hamiltonian has the same form in terms the new variables. The analogy is complete since C is also a gauge field (the dual gauge field), with $C_i(x)$ and $C_i(x) + \partial_i\lambda(x)$ giving rise to the same $B[C]$. This is the dual local gauge transformation. Also the new extended phase space has the dual Gauss law constraint

$$\partial_i \mathcal{E}_i = 0. \tag{6.18}$$

The old vector potential A couples minimally to the electric currents. In contrast the new vector potential couples minimally to the magnetic current as can be verified by introducing sources. Thus the dual symmetry is complete.

The duality transformation can be viewed as a canonical transformation induced by the generating function

$$S(A, C) \equiv \langle C|B[A]\rangle = \int \epsilon_{ijk} C_i \partial_j A_k \tag{6.19}$$

of the old and the new coordinates A and C respectively. We have the symmetry

$$\langle C|B[A]\rangle = -\langle A|B[C]\rangle. \tag{6.20}$$

This is a very convenient technique for obtaining the new momentum and for computing the Poisson brackets of the old and the the new variables. We get the old and new momenta to be,

$$E_i = \frac{\delta S}{\delta A_i} = \epsilon_{ijk}\partial_j C_k = B[C], \tag{6.21}$$

and

$$\mathcal{E}_i = -\frac{\delta S}{\delta C_i} = B[A]_i \tag{6.22}$$

respectively. The generating function is invariant under the old gauge transformation. This gives the identity, that for any λ

$$\int \partial_i \lambda \frac{\delta S}{\delta A_i} = 0. \tag{6.23}$$

As λ is arbitrary, it follows

$$\partial_i \frac{\delta S}{\delta A_i} = 0, \tag{6.24}$$

which is the Gauss law constraint. This is a very convenient way of making the duality transformation preserving the Gauss law constraints. The generating function is also invariant under the new gauge transformation which implies the new Gauss law $\partial_i \mathcal{E}_i = 0$.

We extend and generalize these techniques for non-Abelian gauge theories.

6.2 Techniques for duality transformation

In this section we introduce various techniques useful for the duality transformation of non-Abelian gauge theories.

6.2.1 Functional integral with phase space variables

The Euclidean functional integral for 3+1-dimensional Yang-Mills theory is formally

$$Z = \int \mathcal{D}A^a_\mu \, exp\{-\frac{1}{4g^2}\int \vec{F}_{\mu\nu} \cdot \vec{F}_{\mu\nu}\} \tag{6.25}$$

where

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + \vec{A}_\mu \times \vec{A}_\nu \tag{6.26}$$

With this choice the gauge transformation does not involve the coupling constant. We could as well have started with the Minkowski space functional integral. However the Euclidean version makes the role of the non-Abelian Gauss law even more transparent.

Introducing an auxiliary field E_i^a, (6.25) becomes

$$Z = \int \mathcal{D}A_0^a \mathcal{D}A_i^a \mathcal{D}E_i^a \; exp \int \{ (\frac{-g^2}{2} \vec{E}_i \cdot \vec{E}_i - \frac{1}{2g^2} \vec{B}_i[A] \cdot \vec{B}_i[A]) + i\vec{E}_i \cdot (\partial_0 \vec{A}_i - D_i[A]\vec{A}_0) \} \tag{6.27}$$

where

$$D_i[A] = \partial_i + \vec{A}_i \times \tag{6.28}$$

is the covariant derivative and

$$\vec{B}_i[A] = \frac{1}{2}\epsilon_{ijk}(\partial_j \vec{A}_k - \partial_k \vec{A}_j + \vec{A}_j \times \vec{A}_k) \tag{6.29}$$

is the non-Abelian magnetic field. Integration over A_0 gives

$$Z = \int \mathcal{D}A_i^a \mathcal{D}E_i^a \; \delta(D_i[A]E_i) \; exp\{ \int (-\mathcal{H} + i\vec{E}_i \cdot \partial_0 \vec{A}_i) \}. \tag{6.30}$$

Using the Feynman time slicing procedure, it is clear that A_i, E_i are the conjugate variables of the phase space and

$$\mathcal{H} = \frac{1}{2}(g^2 E^2 + \frac{1}{g^2} B^2) \tag{6.31}$$

is the hamiltonian density. There are also three first class constraints, the non-Abelian Gauss law :

$$D_i[A]\vec{E}_i = 0. \tag{6.32}$$

6.2.2 Duality transformation via a canonical transformation

In close analogy to the Abelian case, we consider a change of variables from E to C.

$$\vec{E}_i = \epsilon_{ijk} D_j[A] \vec{C}_k. \tag{6.33}$$

Naively C_i^a is the canonical conjugate of the non-Abelian electric field E_i^a. This can be checked directly. Note that

$$[E_m^d(x), B_i^a(y)]_{PB} = \epsilon_{ijm}(\delta^{da}\partial_j + \epsilon^{dab}A_j^b)\delta(x-y). \tag{6.34}$$

Using (6.33), the left hand side is

$$\epsilon_{ijm}(\delta^{de}\partial_j + \epsilon^{deb}A_j^b)[C_m^e(x), B_i^a(y)]_{PB}. \tag{6.35}$$

This is consistent with

$$[C_m^e(x), B_i^a(y)]_{PB} = \delta^{ea}\delta_{mi}\delta(x-y). \tag{6.36}$$

An easy way to see this is by using the generator of canonical transformations

$$S(A,C) = \int C_i^a B_i^a[A] \tag{6.37}$$

Then $E_i^a = \frac{\delta S}{\delta A_i^a} = \epsilon_{ijk}(D_j[A]C_k)^a$ and the new momentum conjugate to the new variable C_i^a is

$$\mathcal{E}_i^a = -\frac{\delta S}{\delta C_i^a} = -B_i^a[A]. \tag{6.38}$$

The great advantage of realizing duality transformation via a canonical transformation is that the phase space measure in the functional integral is invariant.

$$\mathcal{D}A\mathcal{D}E = \mathcal{D}C\mathcal{D}\mathcal{E} \tag{6.39}$$

Also

$$\sum p_i\dot{q}_i = \sum P_i\dot{Q}_i \tag{6.40}$$

and

$$H'(P,Q) = H(p(P,Q), q(P,Q)) \tag{6.41}$$

under a canonical transformation $(q,p) \to (Q,P)$. Therefore it is easy to express the exponent in equation (6.30) also in terms of the new variables.

6.2.3 New Gauss law from the old Gauss law

In order to satisfy the Gauss law constraint (6.32), we need

$$\vec{B}_i[A] \times \vec{C}_i = 0, \tag{6.42}$$

where sum over i is implied. Here we have used

$$\epsilon_{ijk} D_j[A] D_k[A] C_i = \vec{B}_i[A] \times C_i. \tag{6.43}$$

Now

$$D_i[C]\vec{\mathcal{E}}_i = -(\vec{C} - \vec{A})_i \times \vec{B}_i[A] \tag{6.44}$$

as

$$D_i[C] = D_i[A] + (\vec{C} - \vec{A})_i \times \tag{6.45}$$

and we have the Bianchi identity

$$D_i[A] B_i[A] = 0. \tag{6.46}$$

This immediately indicates that it is better to change the ansatz (6.33) to

$$\vec{E}_i = \epsilon_{ijk} D_j[A](\vec{C} - \vec{A})_k \tag{6.47}$$

This corresponds to the generating function

$$S(A, C) = \int (\vec{C} - \vec{A})_i \vec{B}_i[A] \tag{6.48}$$

With this choice the old Gauss law (6.32) simply goes over to the new Gauss law

$$D_i[C]\vec{\mathcal{E}}_i = 0. \tag{6.49}$$

Such a feature is very useful for the duality transformation. It can be easily realized in general as shown below.

In ansatz (6.33), C transforms homogeneously (as an isotriplet vector field) under the A-gauge transformation, whereas A transforms inhomogeneously.

$$\delta A_i = D_i[A]\Lambda \tag{6.50}$$

In contrast, in ansatz (6.47) C transforms as a gauge field under A-gauge transformations. Note that if C and A both transform as gauge fields, $\alpha C + (1-\alpha)A$ also transforms like a gauge field for any choice of a real parameter α. Also $(C - A)$ transforms homogeneously, i.e. as a matter field in the adjoint representation. Consider a canonical transformation $S(A, C)$ which is gauge invariant under these common gauge transformations as in equation (6.48). Some choices of terms in $S(A, C)$ are

$$\begin{array}{lll}
(a) & \epsilon_{ijk}(\vec{A}_i \cdot \partial_j \vec{A}_k + \frac{1}{3}\vec{A}_i \cdot \vec{A}_j \times \vec{A}_k) & \equiv \mathcal{CS}[A] \\
(b) & \epsilon_{ijk}(\vec{C}_i \cdot \partial_j \vec{C}_k + \frac{1}{3}\vec{C}_i \cdot \vec{C}_j \times \vec{C}_k) & \equiv \mathcal{CS}[C] \\
(c) & (\vec{C} - \vec{A})_i \cdot \vec{B}_i[A] & \\
(d) & \epsilon_{ijk}\frac{1}{3!}(\vec{C} - \vec{A})_i \cdot (\vec{C} - \vec{A})_j \times (\vec{C} - \vec{A})_k & \equiv det(C - A).
\end{array} \tag{6.51}$$

Here $\mathcal{CS}$ is the Chern-Simons density. Since

$$\frac{\delta \mathcal{CS}[A]}{\delta A_i} = B_i[A], \tag{6.52}$$

it contributes a piece which is independent of C to E_i. Note that the functional integral (6.30) is insensitive to shifts

$$E_i \to E_i + \alpha B_i[A] \tag{6.53}$$

where α is an arbitrary real parameter. First of all, the Gauss law condition

$$D_i[A]\vec{E}_i = 0 \tag{6.54}$$

does not change as a consequence of the Bianchi identity (6.46). Next, the term $E_i\dot{A}_i$ changes by

$$\alpha B_i[A]\dot{A}_i = \alpha\frac{\partial}{\partial t}\mathcal{CS}[A]. \tag{6.55}$$

This being a total derivative, does not matter. (This conclusion is not correct when instanton number [52] is non-zero.) This invariance is reflected in the possible addition of $\mathcal{CS}[A]$ (6.51 a) to the generating function $S[A,C]$

Invariance of $S(A,C)$ under simultaneous gauge transformation of A (6.50) and C, where,

$$\delta\vec{C}_i = D_i[C]\vec{\Lambda} \tag{6.56}$$

implies

$$\int\left\{(\mathcal{D}_i[A]\Lambda)^a\frac{\delta S}{\delta A_i^a} + (\mathcal{D}_i[C]\Lambda)^a\frac{\delta S}{\delta C_i^a}\right\} = 0 \tag{6.57}$$

As this is true for any arbitrary choice of Λ, we get,

$$D_i[A]\vec{E}_i = D_i[C]\vec{\mathcal{E}}_i \tag{6.58}$$

so that the old Gauss law constraint implies the new Gauss law constraint. Another advantage of such a choice of $S(A,C)$ is that the dual field C appears as a background gauge field for A and vice-versa.

The new gauss law may be realized through an auxiliary field C_0 which would play the role played by A_0 in (6.27). This naturally leads to the action functional formulation of the dual theory, once we integrate over $\mathcal{E}_i$:

$$Z = \int \mathcal{D}C_0\mathcal{D}C_i\mathcal{D}\mathcal{E}_i\, exp\int\left\{-H'[C,\mathcal{E}] + i(\partial_0\vec{C} - D_i[C]\vec{C}_0)\cdot\vec{\mathcal{E}}_i\right\}$$

$$= \int \mathcal{D}C_0\mathcal{D}C_i \, exp\left(-S[C_0, C_i]\right) \tag{6.59}$$

where $S[C_0, C_i]$ is gauge invariant under the full gauge transformation, $\delta\vec{C}_\mu = D_\mu[C]\vec{\Lambda}$.

6.2.4 Degrees of freedom

The constraint equation (6.42) can be handled in a different way. In the generic case where $det\, B \equiv |B|$, the determinant of the 3×3 matrix $B_i^a(i, a = 1, 2, 3)$ is non-zero, it is easy to solve this constraint on C [40]. Use B_i^a to "lower" the color index in C_i^a.

$$C_i^a = C_{ij}B_j^a. \tag{6.60}$$

Equation (6.42) is satisfied if and only if C_{ij} is a symmetric tensor. This corresponds to the choice

$$S(A, C) = \int C_{ij}b_{ij} \tag{6.61}$$

where C_{ij} would be the new coordinates and $b_{ij} = \vec{B}_i[A] \cdot \vec{B}_j[A]$, the new conjugate momenta.

Thus the "physical" phase space of Yang-Mills theory may be described in terms of the conjugate pair C_{ij}, b_{ij} which are gauge invariant symmetric second rank tensors. Each of these have six degrees of freedom at each x which appears to match the required degrees of freedom. The situation could have been more involved because of the Wu-Yang ambiguities [50]. But as was analyzed in chapter 4, this is not a generic phenomenon. The equation

$$\epsilon_{ijk}D_j[A]e_k = 0. \tag{6.62}$$

essentially has a unique solution. Therefore we can write

$$\vec{E}_i = \epsilon_{ijk} D_j[A](\vec{C}_k - \vec{A}_k) \tag{6.63}$$

Alternately we can use the decomposition of the form [40]

$$\vec{E}_i = \vec{B}_i[C] \tag{6.64}$$

This seems to be closest to the choice in the Abelian case which had duality invariance. Note that

$$\vec{B}_i[C] = \vec{B}_i[A] + \epsilon_{ijk} D_j[A](\vec{C} - \vec{A})_k + \frac{1}{2}\epsilon_{ijk}(\vec{C} - \vec{A})_j \times (\vec{C} - \vec{A})_k \tag{6.65}$$

which corresponds to an expansion of $B_i[C]$ about a "background gauge field" A with $(\vec{C} - \vec{A})$ as the quantum fluctuation. If E_i satisfies the Gauss law (6.32), so does $E_i - B_i[A]$. Therefore the ansatz (6.47) and (6.64) essentially differ through the last term on the right hand side of (6.65). This is obtained by including the term $det(C_A)$ (6.51 d) in the generating functional of the canonical transformation.

The choice (6.64) is appealing for many reasons. We have,

$$\int \frac{1}{2} E_i^2 = \int \left(\frac{1}{2} B_i^2[C]\right) \tag{6.66}$$

also

$$\frac{\delta S}{\delta \vec{A}_i} \partial_0 \vec{A}_i + \frac{\delta S}{\delta \vec{C}_i} \partial_0 \vec{C}_i = \partial_0 S, \tag{6.67}$$

a total derivative, so that,

$$\int \vec{E}_i \partial_0 \vec{A}_i = \int \vec{\mathcal{E}}_i \cdot \partial_0 \vec{C}_i \tag{6.68}$$

Therefore the exponent in (6.27) can be expressed easily in terms of the new variables as before.

6.3 Duality Transformation

In Maxwell theory we had duality invariance because $E_i = B_i[C]$ and $\mathcal{E}_i = -B_i[A]$. Such a simple interchange does not work for the non-Abelian case as seen from equations (6.38) and (6.47). Note that if we add $\mathcal{CS}[A]$, equation (6.51) to the generating function (6.48), we can make

$$\vec{E}_i = \vec{B}_i[A] + \epsilon_{ijk} D_j[A](\vec{C} - \vec{A})_k. \tag{6.69}$$

As seen from (6.65) the quadratic term in $(C - A)$ is missing.

We now weaken our requirement. It is sufficient if,

$$g^2 E^2 + \frac{1}{g^2} B^2[A] = g^2 B^2[C] + \frac{1}{g^2}\mathcal{E}^2 \tag{6.70}$$

If we use a generating function $S(A, C)$, we require

$$-g^2\left(\frac{\delta S}{\delta A_i}\right)^2 + \frac{1}{g^2}\left(\frac{\delta S}{\delta C_i}\right)^2 = -g^2 B^2[C] + \frac{1}{g^2} B^2[A]. \tag{6.71}$$

Consider the $g = 1$ case. Now equation (6.71) can be rewritten as

$$\begin{aligned} \frac{\delta S}{\delta\left(\frac{A+C}{2}\right)_i}\frac{\delta S}{\delta\left(\frac{A-C}{2}\right)_i} &= \epsilon_{ijk} D_j\left[\frac{A+C}{2}\right]\left(\frac{\vec{A}-\vec{C}}{2}\right)_k \cdot \left\{\vec{B}_i\left[\frac{A+C}{2}\right]\right. \\ &\quad \left. + \frac{1}{2}\epsilon_{ijk}\left(\frac{\vec{A}-\vec{C}}{2}\right)_j \times \left(\frac{\vec{A}-\vec{C}}{2}\right)_k\right\} \end{aligned} \tag{6.72}$$

using equation (6.65) for the background gauge field $(\frac{C+A}{2})$. It is amusing to note that the generating function

$$S\left(\frac{A+C}{2}, \frac{A-C}{2}\right) = \left(\frac{\vec{A}-\vec{C}}{2}\right)_i \cdot \vec{B}_i\left[\frac{A+C}{2}\right] + det\left(\frac{A-C}{2}\right) \tag{6.73}$$

gives the right hand side of the above equation, but with the opposite sign. Self duality is achieved in the Abelian case by using

$$S = \mathcal{CS}\left(\frac{C+A}{2}\right) - \mathcal{CS}\left(\frac{C-A}{2}\right). \tag{6.74}$$

The non-Abelian case should have something similar and not (6.73). Unfortunately there is no S satisfying (6.72). As a consequence self duality is ruled out.

We consider generating functions

$$\begin{aligned} S(A,C) &= \alpha_1 \mathcal{CS}(A) + \alpha_2 \mathcal{CS}(C) + \alpha_3 (\vec{A}-\vec{C})_i \cdot \vec{B}_i[A] \\ &\quad + \frac{\alpha_4}{2}\epsilon_{ijk}(\vec{A}-\vec{C})_i \cdot D_j[A](\vec{A}-\vec{C})_k + \alpha_5 det(A-C). \end{aligned} \tag{6.75}$$

where $\alpha_1, \ldots \alpha_5$ are arbitrary real parameters for the present. Now we get

$$\vec{E}_i = \beta_1 \vec{B}_i[A] + \beta_2 \epsilon_{ijk} D_j[A](\vec{A}-\vec{C})_k + \frac{\beta_3}{2}\epsilon_{ijk}(\vec{A}-\vec{C})_j \times (\vec{A}-\vec{C})_k \tag{6.76}$$

$$\vec{\mathcal{E}}_i = \gamma_1 \vec{B}_i[A] + \gamma_2 \epsilon_{ijk} D_j[C](\vec{A}-\vec{C})_k + \frac{\gamma_3}{2}\epsilon_{ijk}(\vec{A}-\vec{C})_j \times (\vec{A}-\vec{C})_k \tag{6.77}$$

where $\beta_1 = \alpha_1 + \alpha_3$; $\beta_2 = \alpha_3 + \alpha_4$; $\beta_3 = \alpha_4 + \alpha_5$; and $\gamma_1 = -\alpha_2 + \alpha_3$; $\gamma_2 = \alpha_4$; $\gamma_3 = \alpha_5$. For no choice of the parameters $\alpha_1, \ldots \alpha_5$ do we get a local Hamiltonian in the dual variables. We illustrate this for a specific choice, $\alpha_1, \alpha_4, \alpha_5 = 0$ and $\alpha_3 = 1$. We get $\vec{\mathcal{E}}_i = \vec{B}_i[A]$ but $\vec{E}_i = \vec{B}_i[C] - \frac{1}{2}\epsilon_{ijk}(\vec{A} - \vec{C})_j \times (\vec{A}-\vec{C})_k$. Therefore the dual action becomes

$$g^2\{B_i[C] - \frac{1}{2}\epsilon_{ijk}(\vec{A}-\vec{C})_j \times (\vec{A}-\vec{C})_k\}^2 + \frac{1}{g^2}\mathcal{E}^2. \tag{6.78}$$

$(A-C)$ may be regarded as a non-local functional of the dual variables $(C, \mathcal{E})$; solution of

$$\epsilon_{ijk} D_j[C](\vec{A}-\vec{C})_k + \frac{1}{2}\epsilon_{ijk}(\vec{A}-\vec{C})_j \times (\vec{A}-\vec{C})_k = \vec{\mathcal{E}}_i - \vec{B}_i[C] \tag{6.79}$$

We consider the specific choice

$$S[A,C] = \int (\vec{A}-\vec{C})_i \cdot \vec{B}_i[C] \tag{6.80}$$

in some detail. Here κ is a real parameter. Now

$$\vec{E}_i = \vec{B}_i[C] \tag{6.81}$$

$$-\vec{\mathcal{E}}_i = \vec{B}_i[C] + \frac{1}{2}\epsilon_{ijk} D_j[C](\vec{A}-\vec{C})_k \tag{6.82}$$

$-\vec{\mathcal{E}}_i$ can also be written as $\vec{B}_i[A] - \frac{1}{2}\epsilon_{ijk}(\vec{A}-\vec{C})_j \times (\vec{A}-\vec{C})_k$. The hamiltonian is

$$H = \int \left(\frac{1}{2} g^2 \vec{E}_i^2 + \frac{1}{2g^2}\vec{B}_i^2[A]\right) \tag{6.83}$$

$$= \int \frac{1}{2} g^2 \vec{B}_i^2[C] + \frac{1}{2g^2}\left(\vec{\mathcal{E}}_i + \frac{1}{2}\epsilon_{ijk}(\vec{A}-\vec{C})_j \times (\vec{A}-\vec{C})_k\right)^2 \tag{6.84}$$

$$= \int \frac{1}{2} g^2 \vec{B}_i^2[C] + \frac{1}{2g^2}\left(\vec{\mathcal{E}}_i + \frac{1}{2}\epsilon_{ijk}(\vec{A}-\vec{C})_j \times (\vec{A}-\vec{C})_k\right)^2. \tag{6.85}$$

Where we have used $D_i[C](\frac{1}{2}\epsilon_{ijk}(\vec{A}-\vec{C})_j \times (\vec{A}-\vec{C})_k) = (\vec{\mathcal{E}}_k - \vec{B}_k[C] - \frac{1}{2}\epsilon_{klm}(\vec{A}-\vec{C})_l \times (\vec{A}-\vec{C})_m) \times (\vec{A}-\vec{C})_k = (\vec{\mathcal{E}}_k - \vec{B}_k[C]) \times (\vec{A}-\vec{C})_k$. Equation (6.85) gives the Hamiltonian in the dual variables. (Note that the new gauge coupling is g^{-1}). Since $(\vec{A}-\vec{C})_i$ is a non-local functional of dual variables, the hamiltonian is also non-local.

Consider a modified Yang-Mills Hamiltonian

$$H = \int \left(\frac{1}{2} g^2 \vec{E}_i^2 + \frac{1}{2g^2}\vec{\mathcal{E}}_i^2\right) \tag{6.86}$$

where it is presumed that $\mathcal{E}_i = -\frac{\delta S}{\delta C_i}$ is expressed in terms of (A, E). This theory would be self dual, if the generating function $S(A, C)$ is symmetric under the interchange $A \leftrightarrow C$. A simple way of realizing this is to have S (regarded as a functional of $(A+C)$ and $(A-C)$), even in $(A-C)$. For all choices of S we have considered, the theory is non-local.

6.4 Conclusion

In this chapter we have constructed a dual form of the 3+1 Yang-Mills theory. We have argued that the functional integral using phase space variables is best suited for the purpose. Now the duality transformation can be realized as a canonical transformation. This provides a powerful tool, because the action and the measure in the dual variables as also the implications of the Gauss law constraint for the dual theory are easily written. The dual theory is also a SO(3) gauge theory. The dual theory, though a SO(3) gauge theory, is a non-local theory. However Yang-Mills theory with a non-local action is self dual Our techniques for obtaining the dual theory may provide a firm basis for the computations of the confining properties in the dual QCD approach of Baker, Ball and Zachariasen [53].

Chapter 7

In the axial gauge

Axial gauges are often used in non-Abelian gauge theories as they are more physical than covariant gauges. Quite often this simplifies calculations. Also in this gauge, the Faddeev-Popov ghosts decouple from the theory. However there are problems too, as spurious poles appear in the gauge boson propagator in this gauge and it is not yet clear what is the correct prescription to handle these poles.

An axial gauge is defined by imposing on the potential $\vec{A}_\mu(x)$ the condition $\vec{n}^a_\mu \vec{A}^a_\mu = 0$ (no sum over a) where $\vec{n}$ is some fixed vector for each $a = 1, 2, 3$. Axial gauges can be of 3 types : i) temporal $\vec{A}_0 = 0$, (i.e. $n^a_\mu = 1, 0, 0, 0$) ii) spacelike $\vec{A}_3 = 0$ (i.e. $n^a_\mu = 0, 0, 0, 1$) and iii) lightcone gauges $n^2 = 0$. In this chapter we will work in the spacelike axial gauge which is $\vec{A}_3 = 0$. Deser and Teitelboim, [10] had looked at the question of duality transformation of Yang-Mills theories in the axial gauge. They concluded that duality symmetry in the same sense as in Maxwell's electrodynamics does not exist for Yang-Mills theories. We look at a new way of doing duality transformation, which is by using delta function constraints. We integrate out the vector potentials $\vec{A}_\mu$ and rewrite the partition function in terms of some other fields which can be regarded as dual fields. However, unlike electrodynamics, the theory is not self-dual.

In section one, we look at the Abelian case. Sections two and three deal with the non-Abelian case in 3 and 4 Euclidean dimensions respectively.

7.1 Abelian case

Let us first look at the Abelian case in three dimensions. The partition function can be written as,

$$Z = \int \mathcal{D}b_i\, \mathcal{D}A_i\, \delta(b_i - B_i[A])\, exp\left[-\frac{1}{2g^2}\int b_i^2\right]. \tag{7.1}$$

Integrating over the delta function, we recover the usual partition function. Now let us introduce one more auxiliary field C_i and raise the delta function to the exponent. Then we get,

$$Z = \int \mathcal{D}C_i\, \mathcal{D}b_i\, \mathcal{D}A_i\, exp\left[i\int C_i b_i\right]\, exp\left[-i\int \epsilon_{ijk}C_i\partial_j A_k\right]\, exp\left[-\frac{1}{2}\int b_i^2\right]. \tag{7.2}$$

Now we are in a position to integrate out A_i. As it occurs linearly in the exponent, we get a delta function constraint after the integration.

$$Z = \int \mathcal{D}C_i\, \mathcal{D}b_i\, \delta(\epsilon_{ijk}\partial_j C_i)\, exp\left[i\int C_i b_i\right]\, exp\left[-\frac{1}{2}\int b_i^2\right]. \tag{7.3}$$

We can now solve the constraint by putting $c_i = \partial_i\phi$. Rewriting the partition function in terms of ϕ, we pick up an extra jacobian and the resulting form of the partition function is

$$Z = \int \mathcal{D}\phi\, \mathcal{D}b_i\, det|\partial_i|\, exp\left[i\int(\partial_i\phi)b_i\right]\, exp\left[-\frac{1}{2}\int b_i^2\right]. \qquad (7.4)$$

To get the dual form of the partition function, we just have to integrate out b_i which is easy to do since the integration is a gaussian one. Thus finally we get the dual form of the partition function as

$$Z = \int \mathcal{D}\phi\, det|\partial_i|\, exp\left[-\frac{1}{2}\int(\partial_i\phi)^2\right]. \qquad (7.5)$$

Duality transformation for the Abelian theory can be carried out much more easily than above. But the above procedure has the advantage that it can be directly generalized to the non-Abelian case.

7.2 Non-Abelian case : 3 dimensions

Let us now come to the non-Abelian case. We first consider three dimensions and the axial gauge: $\vec{A}_3 = 0$. This gauge in some sense makes the theory closest to the Abelian case. The term non-linear in the potential is present in only one component of the magnetic field:

$$\vec{B}_1 = -\partial_3\vec{A}_2 \qquad (7.6)$$

$$\vec{B}_2 = \partial_3\vec{A}_1 \qquad (7.7)$$

$$\vec{B}_3 = \partial_1\vec{A}_2 - \partial_2\vec{A}_1 + \vec{A}_1 \times \vec{A}_2. \qquad (7.8)$$

In exact analogy to the Abelian case, we now write the partition function (upto an unimportant constant) as

$$Z = \int \mathcal{D}\vec{b}_1\, \mathcal{D}\vec{b}_2\, \mathcal{D}\vec{b}_3 \int \mathcal{D}\vec{A}_1\, \mathcal{D}\vec{A}_2\, \delta(\partial_3\vec{b}_3 - \vec{B}_3[A])\, \delta(\vec{b}_1 + \partial_3\vec{A}_2)\, \delta(\vec{b}_2 - \partial_3\vec{A}_1)$$

$$\times\, exp\left[-\frac{1}{2}\int \vec{b}_i^{\,2}\right]. \tag{7.9}$$

Again introducing the auxiliary fields $\vec{c}_1, \vec{c}_2$ and $\vec{\phi}$ to exponentiate the constraints, we get

$$\begin{aligned} Z &= \int \mathcal{D}\vec{b}_i\, \mathcal{D}\vec{c}_1\, \mathcal{D}\vec{c}_2\, \mathcal{D}\vec{\phi}\, \mathcal{D}\vec{A}_1\, \mathcal{D}\vec{A}_2\, exp\left[-\frac{1}{2}\int \vec{b}_i^{\,2}\right] exp\left[i\int \vec{c}_1\cdot(\vec{b}_1+\partial_3\vec{A}_2)\right] \\ &\quad \times exp\left[i\int \vec{c}_2\cdot(\vec{b}_2-\partial_3\vec{A}_1)\right] exp.\left[i\int \vec{\phi}\cdot(\partial_3\vec{b}_3-\vec{B}_3[A])\right]. \end{aligned} \tag{7.10}$$

Note here that due to the asymmetry in the expression for the magnetic fields, introduced by the gauge conditions, we use a different auxiliary field for $\vec{B}_3$. Next after expanding the exponent and doing some integrations by parts, we get,

$$\begin{aligned} Z &= \int_{b,c,\phi,A} exp\left[i\int\{-(\partial_3\vec{\phi})\cdot\vec{b}_3+(\partial_1\vec{\phi})\cdot\vec{A}_2-(\partial_2\vec{\phi})\cdot\vec{A}_1-\vec{\phi}\cdot\vec{A}_1\times\vec{A}_2\right. \\ &\quad \left.+\vec{c}_1\cdot\vec{b}_1-(\partial_3\vec{c}_1)\cdot\vec{A}_2+\vec{c}_2\cdot\vec{b}_2+(\partial_3\vec{c}_2)\cdot\vec{A}_1\}\right] exp\left[-\frac{1}{2}\int \vec{b}_i^{\,2}\right]. \end{aligned} \tag{7.11}$$

Now we are ready to do the integral over $\vec{b}_i$'s. Carrying out the gaussian integral, we get, apart from some normalization factors,

$$\begin{aligned} Z &= \int_{c,\phi,A} exp\left[-\frac{1}{2}\int\left\{(\partial_3\vec{\phi})^2+\vec{c}_1^{\,2}+\vec{c}_2^{\,2}\right\}\right] \qquad (7.12) \\ &\quad exp\left[i\int\left\{(\partial_1\vec{\phi})\cdot\vec{A}_2-(\partial_2\vec{\phi})\cdot\vec{A}_1-(\vec{\phi}\times\vec{A}_1)\cdot\vec{A}_2-(\partial_3\vec{c}_1)\cdot\vec{A}_2+(\partial_3\vec{c}_2)\cdot\vec{A}_1\right\}\right]. \end{aligned}$$

Next we integrate over $\vec{A}_2$ to get a delta function constraint.

$$\begin{aligned} Z &= \int_{c,\phi,A_1} exp\left[-\frac{1}{2}\int\left\{(\partial_3\vec{\phi})^2+\vec{c}_1^{\,2}+\vec{c}_2^{\,2}\right\}\right] \qquad (7.13) \\ &\quad \times exp\left[i\int\left\{-(\partial_2\vec{\phi})\cdot\vec{A}_1+(\partial_3\vec{c}_2)\cdot\vec{A}_1\right\}\right]\delta(\partial_1\vec{\phi}-\vec{\phi}\times\vec{A}_1-\partial_3\vec{c}_1). \end{aligned}$$

This constraint equation can be handled in many ways. We want to get a dual form of the partition function by integrating out A. So we can solve for

$\vec{A}_1$ in terms of the other fields from this constraint equation. The equation can be written as

$$\partial_1\vec{\phi} - \partial_3\vec{c}_1 = \vec{\phi} \times \vec{A}_1 \tag{7.14}$$

Taking the cross product of the constraint equation with $\vec{\phi}$, we get,

$$\vec{\phi} \times \partial_1\vec{\phi} - \vec{\phi} \times \partial_3\vec{c}_1 = \vec{\phi} \times (\vec{\phi} \times \vec{A}_1). \tag{7.15}$$

Now let us decompose $\vec{A}_1$ along the direction of $\vec{\phi}$ and perpendicular to it as $\vec{A}_1 = \alpha\vec{\phi} + \vec{A}_\perp$. Plugging back this decomposition in the above equation, we get,

$$\vec{\phi} \times \partial_1\vec{\phi} - \vec{\phi} \times \partial_3\vec{c}_1 = \alpha\vec{\phi}\left|\vec{\phi}\right|^2 - \vec{A}_1\left|\vec{\phi}\right|^2. \tag{7.16}$$

Thus we get $\vec{A}_1$ as

$$\vec{A}_1 = \frac{1}{|\vec{\phi}|^2}(-\vec{\phi} \times \partial_1\vec{\phi} + \vec{\phi} \times \partial_3\vec{c}_1 + \alpha\vec{\phi}\left|\vec{\phi}\right|^2). \tag{7.17}$$

Putting this back in the partition function, we finally get the form of the partition function as

$$\begin{aligned} Z &= \int_{c,\phi} exp\left[-\frac{1}{2}\int(\partial_3\vec{\phi})^2 + \vec{c}_1{}^2 + \vec{c}_2{}^2\right] \\ &\quad \times exp\left[i\int\{-(\partial_2\vec{\phi}) + (\partial_3\vec{c}_2)\} \cdot \left\{-\frac{\vec{\phi} \times \partial_1\vec{\phi}}{|\vec{\phi}|^2} + \frac{\vec{\phi} \times \partial_3\vec{c}_1}{|\vec{\phi}|^2} + \alpha\vec{\phi}\right\}\right]. \end{aligned} \tag{7.18}$$

Note that a topological term $\vec{\phi} \cdot \partial_1\vec{\phi} \times \partial_2\vec{\phi}$ comes out automatically.

So far $\vec{c}_1$ and $\vec{c}_2$ have been completely unspecified. But we can express $\vec{c}_1$ as a sum of derivative of $\vec{\phi}$ and another vector which can be expressed in terms of $\vec{\phi}$ itself. We can choose $\vec{c}_1$ as $\partial_1\vec{\phi} + \vec{\Lambda}_1$. Similarly we can choose $\vec{c}_2$ to be $\partial_2\vec{\phi} + \vec{\Lambda}_2$. Since $\vec{\phi}$ provides a coordinate basis at every point in the space, $\vec{\Lambda}_1$ and $\vec{\Lambda}_2$ can be expressed in terms of the components of $\vec{\phi}$. Then we have a field theory for $\vec{\phi}$ with a kinetic term and complicated self interactions. This can be thought of as a gauge fixed version of what is carried out geometrically in chapter 3. A difference of course is here the variables are not gauge invariant, but transform as vectors under gauge transformations. This case however gives us some flavor of how the interaction terms, which were not determined in chapter 3, may look like.

7.3 Non-Abelian case : 3+1 dimensions

Let us now look at the 3+1-dimensional case in the axial gauge. Our starting point is the usual partition function

$$Z = \int \mathcal{D}A_\mu \, exp\left[-\frac{1}{4g^2}\int \vec{F}^{\,2}_{\mu\nu}[A]\right] \tag{7.19}$$

where $\vec{F}_{\mu\nu}$ is defined as $\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + \vec{A}_\mu \times \vec{A}_\nu$. Introducing an auxiliary field $\vec{E}_i$ (not as a function of A) to linearize the $\vec{A}_0$ factors, we can rewrite the functional integral as

$$Z = \int \mathcal{D}\vec{E}_i \mathcal{D}\vec{A}_0 \mathcal{D}\vec{A}_i \, exp \int \left[i(\partial_0 \vec{A}_i - \partial_i \vec{A}_0 + \vec{A}_0 \times \vec{A}_i)\cdot \vec{E}_i - \frac{1}{2}(g^2 \vec{E}_i^2 + \frac{1}{g^2}\vec{B}_i^2)\right] \tag{7.20}$$

where $\vec{B}_i$ is now given by $\vec{B}_i = \frac{1}{2}\epsilon_{ijk}\vec{F}_{jk}$ $(i,j,k = 1,2,3)$. Doing the $\vec{A}_0$ integral, we get,

$$Z = \int \mathcal{D}\vec{E}_i \mathcal{D}\vec{A}_i \, \delta(\vec{D}_i[A]\vec{E}_i) \, exp \int \left[i(\partial_0 \vec{A}_i)\cdot \vec{E}_i - \frac{1}{2}(g^2 \vec{E}_i^2 + \frac{1}{g^2}\vec{B}_i^2)\right]. \tag{7.21}$$

From here we proceed exactly as in the 3-dimensional case. Introducing the auxiliary fields $\vec{b}_i$, we write the partition function as

$$
\begin{aligned}
Z &= \int \mathcal{D}\vec{E}_i\, \mathcal{D}\vec{A}_0\, \mathcal{D}\vec{A}_i \;\; \delta(\vec{D}_i[A]\vec{E}_i)\, \delta(\vec{b}_i - \vec{B}_i[A]) \\
&\quad \times exp\left[\int \left\{ i(\partial_0 \vec{A}_i) \cdot \vec{E}_i - \frac{1}{2}(g^2 \vec{E}_i^2 + \frac{1}{g^2}\vec{b}_i{}^2) \right\}\right]. \qquad (7.22)
\end{aligned}
$$

Again raising the delta function constraints to the exponent by introducing further auxiliary fields $\vec{\phi}$ and $\vec{\psi}_i$, we get

$$
\begin{aligned}
Z &= \int_{E,A,b,\phi,\psi} exp\left[i \int \left\{ \dot{\vec{A}}_i \cdot \vec{E}_i + \vec{\phi} \cdot (\vec{D}_i[A]\vec{E}_i) + \vec{\psi}_i \cdot (\vec{b}_i - \vec{B}_i[A]) \right\} \right] \\
&\quad \times exp\left[- \int \left\{ \frac{1}{2}(g^2 \vec{E}^2 + \frac{1}{g^2}\vec{b}^2) \right\} \right]. \qquad (7.23)
\end{aligned}
$$

Expanding the exponent and recalling that $\vec{A}_3 = 0$, we get the exponent as

$$
\begin{aligned}
& i \int (\dot{\vec{A}}_1 \cdot \vec{E}_1 + \dot{\vec{A}}_2 \cdot \vec{E}_2) + \int \tfrac{1}{2}(g^2 \vec{E}_i^2 + \tfrac{1}{g^2}\vec{b}_i^2) + i \int \vec{\phi} \cdot (\partial_i \vec{E}_i) \\
& \quad + i \int \vec{\phi} \cdot (\vec{A}_1 \times \vec{E}_1 + \vec{A}_2 \times \vec{E}_2) + i \int \vec{\psi}_i \cdot \vec{b}_i \\
& - i \int (-\vec{\psi}_1 \cdot (\partial_3 \vec{A}_2) + \vec{\psi}_2 \cdot (\partial_3 \vec{A}_1) + \vec{\psi}_3 \cdot (\partial_1 \vec{A}_2 - \partial_2 \vec{A}_1 + \vec{A}_1 \times \vec{A}_2)) \quad .
\end{aligned}
$$

Our aim is to integrate out A and rewrite the theory in terms of the other fields. For that purpose, let us write the terms containing A. After doing a few integration by parts, we get

$$
\begin{aligned}
& i \int \Big[-\dot{\vec{E}}_1 \cdot \vec{A}_1 - \dot{\vec{E}}_2 \cdot \vec{A}_2 - \vec{\phi} \times (\vec{E}_1 \cdot \vec{A}_1 + \vec{E}_2 \cdot \vec{A}_2) - (\partial_3 \vec{\psi}_1) \cdot \vec{A}_2 \\
& \quad + (\partial_3 \vec{\psi}_2) \cdot \vec{A}_1 + (\partial_1 \vec{\psi}_3) \cdot \vec{A}_2 - (\partial_2 \vec{\psi}_3) \cdot \vec{A}_1 - \vec{\psi}_3 \times \vec{A}_1 \cdot \vec{A}_2 \Big].
\end{aligned}
$$

Doing the $\vec{A}_2$ integral, we get the constraint equation,

$$
-\dot{\vec{E}}_2 - \vec{\phi} \times \vec{E}_2 - \partial_3 \vec{\psi}_1 + \partial_1 \vec{\psi}_3 - \vec{\psi}_3 \times \vec{A}_1 = 0 \qquad (7.24)
$$

Again we are interested in getting rid of $\vec{A}_1$. So we will use the constraint equation to solve for $\vec{A}_1$. Let us rewrite the constraint equation as

$$-\partial_0 \vec{E}_2 - \vec{\phi} \times \vec{E}_2 - \partial_3 \vec{\psi}_1 + \partial_1 \vec{\psi}_3 = \vec{\psi}_3 \times \vec{A}_1 \tag{7.25}$$

Taking the cross product with $\vec{\psi}_3$, we have

$$-\vec{\psi}_3 \times \partial_0 \vec{E}_2 - \vec{\psi}_3 \times (\vec{\phi} \times \vec{E}_2) - \vec{\psi}_3 \times \partial_3 \vec{\psi}_1 + \vec{\psi}_3 \times \partial_1 \vec{\psi}_3 = \vec{\psi}_3 (\vec{\psi}_3 \cdot \vec{A}_1) - \vec{A}_1 \left|\vec{\psi}_3\right|^2 . \tag{7.26}$$

Again decomposing $\vec{A}_1$ as parallel to $\vec{\psi}_3$ and $\perp$ to it, we write $\vec{A}_1$ as $\vec{A}_1 = \alpha\vec{\psi}_3 + \vec{A}_\perp$. Then we get

$$-\vec{\psi}_3 \times (D_0 \vec{E}_2) - \vec{\psi}_3 \times (\partial_3 \vec{\psi}_1 - \partial_1 \vec{\psi}_3) = |\vec{\psi}_3|^2 (\alpha \vec{\psi}_3 - \vec{A}_1) \tag{7.27}$$

where D_0 is given by $\partial_0 + \vec{\phi}\times$. Thus we get $\vec{A}_1$ to be

$$\vec{A}_1 = \frac{\vec{\psi}_3 \times (D_0 \vec{E}_2)}{|\vec{\psi}_3|^2} + \frac{\vec{\psi}_3 \times (\partial_3 \vec{\psi}_1 - \partial_1 \vec{\psi}_3)}{|\vec{\psi}_3|^2} - \alpha \vec{\psi}_3. \tag{7.28}$$

Plugging this back, we get the partition function to be

$$\begin{aligned} Z \;=\; & \int_{E,b,\phi,\psi} exp \int \left[-\frac{1}{2}(g^2 \vec{E}_i{}^2 + \frac{1}{g^2}\vec{b}_i{}^2) + i(\vec{\phi} \cdot \partial_i \vec{E}_i + \vec{\psi}_i \cdot \vec{b}_i) \right. && (7.29) \\ & \left. + i \left\{ -D_0 \vec{E}_1 + \partial_3 \vec{\psi}_2 - \partial_2 \vec{\psi}_3 \right\} \cdot \left\{ \frac{\vec{\psi}_3 \times (D_0 \vec{E}_2)}{|\vec{\psi}_3|^2} + \frac{\vec{\psi}_3 \times (\partial_3 \vec{\psi}_1 - \partial_1 \vec{\psi}_3)}{|\vec{\psi}_3|^2} - \alpha \vec{\psi}_3 \right\} \right] . \end{aligned}$$

Now if we want, we can integrate out $\vec{b}_i$ to get the partition function in terms of $\vec{E}_i$ and $\vec{\psi}_i$ as

$$\begin{aligned} Z \;=\; & \int_{E,\phi,\psi} exp \int \left[-\frac{1}{2} g^2 (\vec{E}_i{}^2 + \vec{\psi}_i{}^2) + i \, \vec{\phi} \cdot (\partial_i \vec{E}_i) \right. && (7.30) \\ & \left. + i \left\{ -D_0 \vec{E}_1 + \partial_3 \vec{\psi}_2 - \partial_2 \vec{\psi}_3 \right\} \cdot \left\{ \frac{\vec{\psi}_3 \times (D_0 \vec{E}_2)}{|\vec{\psi}_3|^2} + \frac{\vec{\psi}_3 \times (\partial_3 \vec{\psi}_1 - \partial_1 \vec{\psi}_3)}{|\vec{\psi}_3|^2} - \alpha \vec{\psi}_3 \right\} \right] . \end{aligned}$$

Note that here again the topological term $\vec{\psi}_3 \cdot \partial_1 \vec{\psi}_3 \times \partial_2 \vec{\psi}_3$ appears automatically with $\vec{\psi}_3$ playing the role of $\vec{\phi}$ of the 3-dimensional case. Here again we have too many degrees of freedom left over. All of them cannot be independent.

7.4 Conclusion

In this chapter we have seen how one integrate out the gauge fields and re-express the theory in terms of other fields in 2+1 and 3+1 dimensions. In addition the process has naturally led to the appearance of underlying topological degrees of freedom.

Chapter 8

Discussion and future work

In this thesis we have developed techniques to perform duality transformations of non-Abelian gauge theories.

In 2+1 dimensions, exploiting the analogy that exists between SU(2) Yang-Mills theory and Einstein-Cartan formulation of gravity, we interpret the auxiliary field as the driebein and the field strength as the curvature. The gauge potential plays the role of spin connection. The resulting action looks like the three dimensional gravity action with an added term that breaks general coordinate invariance. Gauge invariance however is retained. Dual gluons are identified as local coordinates on the 3-manifold. Monopoles are located at points where the Ricci principal axes become degenerate. In terms of the new variables, we get an interaction term which couples the dual gluons with the monopoles naturally.

We have proposed a gauge invariant way of identifying monopoles in 2+1 SU(2) Yang-Mills theory. This geometric picture is rederived and further substantiated when we perform the duality transformation in the axial gauge. That gives us explicit form of the interactions in terms of the auxiliary fields. It is of interest to use this in numerical simulations and then comparing it with the existing ways of detecting monopoles using the Abelian projection.

In 3+1 dimensions, we first identified the physical phase space. For that we found a local solution to the non-Abelian Gauss law. The solution was parametrized by a gauge invariant symmetric matrix. Then we developed techniques to decompose the non-Abelian potential into parts useful for handling the non-Abelian Gauss law and perform duality transformation. An important conclusion from this exercise is that any generic non-Abelian field can be written as the magnetic field of a dual vector potential.

We also looked at the Wu-Yang ambiguities in three dimensions. We have found that there are many choices of the vector potential on a surface which reproduces the magnetic field on the surface. (This is the gauge field ambiguity in 1+1 dimensions.) For each such boundary condition, (in the generic case) there is a unique potential which reproduces the given magnetic field everywhere. The non-Abelian Bianchi identity does not constrain the non-Abelian magnetic fields in contrast to the Abelian case. The ambiguity in the choice of the potentials is (in the generic case) only due to the ambiguity in the potential on a surface. Thus it is related to the gauge copy problem in 1+1 dimensions.

The duality transformation in 3+1 dimensions is realized as a canonical transformation on the phase space variables of the Yang-Mills theory. Using generating functions for the canonical transformation has some distinct advantages. Firstly, since the Jacobian of the transformation is one, one does not pick up any undesirable extra factor in the functional measure. Secondly, the new variables obey their own Gauss law which follow naturally if one uses a gauge invariant generating functional. The dual theory gives the dynamics of the dual gluon.

Even though we have been able to write down a dual version of 3+1-dimensional SU(2) Yang-Mills theory, the resulting hamiltonian has turned out to be non-local and is difficult to handle. Quite possibly, this is bound to happen if we use local quantities as the transformed variables. On the other hand it is quite probable that with a suitable choice of non-local variables, one might obtain a tractable dual theory.

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