VARIATIONAL FORMULAE FOR FUCHSIAN GROUPS OVER FAMILIES OF ALGEBRAIC CURVES

A Thesis Submitted
in Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

by
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May 1998

CERTIFICATE

This is to certify that the Ph. D. thesis submitted by DAKSHINI BHATTACHARYYA, to the University of Madras, entitled VARIATIONAL FORMULAE FOR FUCHSIAN GROUPS OVER FAMILIES OF ALGEBRAIC CURVES , is a record of bonafide work done by her during 1991-1998 under my supervision. The research work presented in this thesis has not formed the basis for the award to the candidate of any degree, diploma, associateship, fellowship or other similar title. It is further certified that the thesis represents independent work on the part of the candidate, and collaboration was necessiated by the nature and scope of the problems dealt with.

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ACKNOWLEDGEMENTS

First and foremost, I take the privillage of acknowledging my sincere gratitude to my teacher Prof. S. Nag, for his guidence. He gave me my thesis problem and helped me to solve the problem.

I thank Prof R. Ramchandran, Director, IMSC, for financial support and research facility.

I thank Prof. R. Balasubramanian for his kind cooperation.

I owe gratitude to Dr D.S. Nagaraj for helping me to solve my problem.

I am grateful to Mr. Surya Ramana for suggesting appropriate reference to solve my problem.

I should thank Dr. R. Radha for her cooperation in all aspects.

I would like to thank Mr Vinodchandran, Mr Swarup Mahalik, and Mr Topobrata Sarkar for their help regarding computer opperation.

I thank Mrs. Usha Otheeswaran for her kind cooperation.

Lastly I would like to thank Library members and office staffs for their cooperation.

Contents

Chapter 1

Introduction

A student of the theory of compact Riemann surfaces can scarce fail to be deeply intrigued by the mystery of the relationship between what is theoretically known to be two equivalent descriptions of Riemann surfaces: namely the description as a complex algebraic curve, and the description via a Fuchsian group. The first is the classical business of complex algebraic geometry, whereas the second is intimately concerned with potential theory on the surface, (since the identification of the universal covering surface as the hyperbolic plane entails the solution of appropriate Dirichlet boundary-value problems for harmonic objects). These two aspects are not easy to relate, and consequently, as is well-known to all practitioners in this field, there is no known passage from the algebraic curve description to the corresponding Fuchsian group.

In this thesis we make a contribution to the problem of understanding the uniformizing Fuchsian groups for a family of plane algebraic curves by determining explicit first variational formulae for the generators of the Fuchsian groups, say G_t, associated to a t-parameter family of compact Riemann surfaces X_t, where the X_t are the Riemann surfaces for the complex algebraic curves arising from a t-parameter family of irreducible polynomials. The main idea of our work is to utilize explicit quasiconformal mappings between algebraic curves, calculate the Beltrami coefficients, and hence utilize the Ahlfors-Bers variational formulae when applied to quasiconformal conjugates of Fuchsian groups.

Let us set up the problem in some detail.

Indeed let $P_t(x,y)$ be irreducible polynomials in x and y whose coefficients depend real or complex analytically on the real or complex variable t, (where suppose that t varies in the open set $|t| < \epsilon$). Thus

$$P_t(x,y) = p_N(x,t)y^N + p_{N-1}(x,t)y^{N-1} + \cdots + p_0(x,t) \tag{1.1}$$

where we have expressed P_t as a polynomial in y with coefficients $p_j(x,t)$ – each $p_j(x,t)$ being a polynomial in x depending analytically on t. To fix ideas we assume that t is a complex parameter in the t-disc, and that the dependence of the coefficients p_j on t is complex analytic. (That is the situation for holomorphically varying families of compact Riemann surfaces which form the subject matter of the important "Kodaira-Spencer families".)

We set up conditions which guarantee that the Riemann surface, X_t, corresponding to the plane algebraic curve $P_t(x,y) = 0$ has a fixed genus, say $g > 1$, for every choice of t in the ϵ-disc above. Of course, and this is the main point, *the complex structure of X_t will, in general, vary with t* – indeed whenever the polynomials P_t are not birationally equivalent to P_0. (We shall give several examples in our work – some drawn from Belyi theory of arithmetic algebraic curves.)

Consequently, we know that by the classical Poincare-Klein-Koebe uniformization theorem, the holomorphic universal covering of each X_t is the Poincare upper half-plane U; thus X_t will be obtained as the quotient of U by the holomorphic action of the deck-transformation group, say G_t. Each G_t is therefore a discrete and fixed-point free subgroup of the holomorphic automorphism group of U: namely, $G_t(\subset Aut(U) = PSL(2,R))$ is a Fuchsian group. It is well-known, (see for instance Farkas-Kra[FK] or Nag[N]), that the Fuchsian group for a Riemann surface is uniquely determined by the Riemann surface up to a conjugacy in $PSL(2,R)$.

Our fundamental query is then to determine the group G_t, at least approximately, (i.e., give variational formulae for generators of these groups), when we are supplied with such a family of polynomials P_t, as well as with the initial (reference) Fuchsian group G_0. The method worked out in this thesis is outlined in the following steps:

Step 1. Consider the nonconstant meromorphic function given by the complex indeterminate $x = x_t$ on each X_t. (When thinking of the transcendental x as a meromorphic function on X_t we call it x_t, in order to emphasize the dependence on t):

$$x_t : X_t \to \textit{Riemann sphere } \mathbf{CP}^1 \tag{1.2}$$

From the theory of algebraic curves and algebraic function fields (see, for example, Siegel [S], or [FK]), we recall that any nonconstant meromorphic function on any compact Riemann surface, X_t, is nothing other than a holomorphic mapping onto $\mathbf{CP}^1$ that exhibits X_t as a *branched covering of* $\mathbf{CP}^1$ *of degree* N, with ramification (=branch) points located on some finite set of the sphere. Note that N is the degree of the polynomial P_t in the y variable.

Step 2. We next demonstrate that the branch points on the Riemann sphere will vary complex analytically with t as the Riemann surface X_t varies. Also, by placing mild restrictions on the t-dependent polynomial $P_t(x, y,)$, we can guarantee that the *monodromy permutations* of the N sheets, as we circuit the branch points on the sphere, *remain the same for all small* t *near* $t = 0$. Thus the topological structure of the branched covering is guaranteed to be exactly the same for each small value of t, as it was at the fiducial point $t = 0$.

To be explicit, let $\{\zeta_1(t), \cdots, \zeta_K(t)\}$ denote the positions on the Riemann sphere of the K branch points for P_t. For convenience assume, (and this entails no loss of generality), that ∞ is not in the branch point set (for any t in $|t| < \epsilon$). Then the Riemann surface X_t may be constructed by the classical process of taking N copies of the "cut-sphere" and appropriately gluing along the cuts. Namely, take N copies of the Riemann sphere and cut each copy along K disjoint lines that connect the K

branch points to the point at infinity; next adjoin these N "sheets" to each other along the lips of the cuts according to the permutations that are dictated by the monodromy around the branch points. See, for instance, Siegel [S], or Knopp [K], for details as well as examples.

Step 3. We now construct a piecewise-affine (and hence quasiconformal) homeomorphism:

$$\phi_t : \mathbf{P}^1 \to \mathbf{P}^1 \tag{1.3}$$

that carries the K branch points for P_0 to the K branch points for P_t, and also carries ∞ to ∞. Since the sheet-joining permutations are invariant while constructing X_t as in constructing X_0, we can demonstrate that *the quasiconformal homeomorphism ϕ_t will lift to a quasiconformal homeomorphism of X_0 onto X_t.* Let us call this lift:

$$\tilde{\phi}_t : X_0 \to X_t \tag{1.4}$$

The upshot is that $\tilde{\phi}_t$ is a quasiconformal marking map from the reference Riemann surface X_0 onto the variable surface X_t, – thus representing X_t as a point of the Teichmüllerspace $T_g = T(X_0)$ by the marked triple $[X_0, \tilde{\varphi}_t, X_t]$. Indeed, the map that associates to each t the point shown above of Teichmüller space is a *holomorphic classifying map* for the Kodaira-Spencer family of holomorphically varying Riemann surfaces X_t.

Step 4. We now assume that the Fuchsian group $G_0 \equiv \Gamma$ (say) corresponding to the base surface X_0 is supplied to us – i.e., $X_0 = U/G_0$. The fundamental facts in the Ahlfors-Bers theory of Teichmüller spaces tell us now (see Chapters 1 and 2 of Nag [N]) that:

$$X_t = U/G_t, \text{ where } G_t = \omega_t \circ G_0 \circ {\omega_t}^{-1} \tag{1.5}$$

Here ω_t denotes the lifting to the universal covering U of the mapping $\tilde{\phi}_t$ which we obtained above. *The upshot is that G_t is obtained, element for element, by conjugating* *the Fuchsian group G_0 by the quasiconformal mapping $\omega_t : U \to U$.* And, as we see, this quasiconformal self-mapping of U descends on the level of the Riemann surfaces to the quasiconformal homeomorphism $\tilde{\phi}_t$.

Step 5. It is crucial to note now that since each map ϕ_t on the Riemann sphere level was a piecewise affine construction, the Beltrami coefficient (i.e., the complex dilatation), say ν_t, of ϕ_t is simply a constant (of modulus less than one) when restricted to the domains of the affine pieces. These pieces are simply a finite set of triangles triangulating a certain rectangle R in the complex plane, (where R is chosen large enough so that the interior of R contains all the K branch points). Moreover, ϕ_t is the identity map outside that rectangle, so that ν_t vanishes identically outside R. For small t,

$$\nu_t = t\nu + o(t) \tag{1.6}$$

with ν being also piecewise constant on these triangles, and 0 outside R.

We are then in a position to compute directly the Beltrami coefficient on the surface X_0, of the lifted quasiconformal mapping $\tilde{\phi}_t$ – and hence compute the complex dilatation, say μ_t, of the quasiconformal map ω_t obtained from $\tilde{\phi}_t$ by lifting all the way to the universal covering. Since ν_t is piecewise constant as explained, we find a rather simple explicit formula for the complex dilatation μ_t as a Γ-automorphic Beltrami coefficient on the upper half plane U.

$$\bar{\partial}\omega_t/\partial\omega_t = \mu_t \in L^\infty(U, \Gamma) \tag{1.7}$$

Step 6: Final Formulae: The variational formulae associated to quasiconformal solutions of the Beltrami equations (see Ahlfors[A] or Sections 1.2.12 and 1.2.14 of Nag[N]) will now provide us with integral formulas (as integrals over U) expressing the first variation of the mappings ω_t in terms of the Beltrami coefficient μ_t. It is this formula that we will transform and manipulate into a very convenient form. Recall here that ω_t varies only real analytically with t, even if the original family of Riemann surfaces varied complex analytically with t (see Section 1.2.14 Nag [N]).

Thus we are seeking the t-derivatives $w_1 = (\partial/\partial t)(\omega_t)$ and $w_1^* = (\partial/\partial \bar{t})(\omega_t)$ at $t = 0$ so that we get the first order expansion:

$$\omega_t(z) = z + tw_1(z) + \bar{t}w_1^*(z) + o(t), z \in U \tag{1.8}$$

Let us state here briefly the chief formula for the variation of the quasiconformal maps ω_t that we obtain in this thesis after suitable manipulations:

$$w_1(z) = \partial\omega_t/\partial t = \frac{z(z-1)}{2\pi\sqrt{-1}} \sum_{k=1}^{N} \int\int_{R \subset \mathbf{CP}^1} \nu(w) V_{k,\Gamma}(w, z) dw \wedge d\bar{w} \tag{1.9}$$

$$w_1^*(z) = \partial\omega_t/\partial \bar{t} = \frac{z(z-1)}{2\pi\sqrt{-1}} \sum_{k=1}^{N} \int\int_{R \subset \mathbf{CP}^1} \overline{\nu(w)} V_{k,\Gamma}(\bar{w}, z) dw \wedge dw \tag{1.10}$$

Here the N "kernel" functions $V_{k,\Gamma}$ are close cousins unto each other, (for $k = 1, 2, \cdots, N$), and each one is determined as a holomorphic function (of two arguments) via summation over the group elements $g \in \Gamma$. [Recall that N denotes the highest power of y appearing in the polynomials P_t.] The exact nature of these kernel functions is determined explicitly in the latter part of the thesis.

Let us note the interesting fact that the above variational formulae involves integrating over the Riemann sphere ($\mathbf{CP}^1$) – this sphere being actually the Riemann sphere (= extended complex plane) which serves as the range of values for the meromorphic function x on X_0. Also, the Beltrami coefficient ν appearing in these formulae arises from the piecewise-affine quasiconformal mappings ϕ_t – as we described in equation (1.6) above. Consequently, *ν is simply piecewise constant on the integration sphere, and the integration is supported on just the finite rectangle R (which contained the ramification points).*

We would like to point out that direct practical implementation of the variational formulae that we have determined in the thesis is quite feasible. We shall explain in the thesis how certain classical Poincaré theta series with respect to the initial Fuchsian group Γ can be brought to bear on this problem of applying these variational formulae in a computer package.

Of course, the knowledge of $w_1(z)$ and $w_1^*(z)$ now allows us to determine the desired t-derivatives of the Fuchsian group elements, $\gamma_t \in G_t$, where

$$\gamma_t = \omega_t \circ \gamma \circ {\omega_t}^{-1} \tag{1.11}$$

with γ being an arbitrary element of the initial Fuchsian group $G_0 \equiv \Gamma$. In other words, *we successfully find the sought-after variational formula:*

$$\gamma_t = \gamma + t\dot{\gamma} + \bar{t}\dot{\gamma}^* + o(t) \tag{1.12}$$

The final formulae for the terms above are:

$$\dot{\gamma} = w_1 \circ \gamma - \gamma' w_1 \tag{1.13}$$

$$\dot{\gamma}^* = w_1^* \circ \gamma - \gamma' w_1^* \tag{1.14}$$

That completes the chief goal of this thesis. See the main theorems in the body of the work for more details about, and for analysis of, the actual expressions that we get.

Remark: Although we have dealt with compact Riemann surfaces and the torsion-free parabolic-free Fuchsian uniformizing group in the introduction above, the theory of Teichmüller spaces works exactly the same for Riemann surfaces of finite conformal type – namely we can allow distinguished points or punctures on the compact Riemann surfaces and correspondingly allow elliptic or parabolic elements in the Fuchsian groups under scrutiny. Those results are exactly parallel and nothing new needs to be said.

Certain Examples: We may choose certain interesting special cases as the base Riemann surface from which to deform and apply our variational formulas. We may take:

(1) the case of the Fermat curves $x^n + y^n = 1$;

(2) certain well-known "modular curves" arising from quotients of the upper half-plane by the congruence subgroups of the elliptic modular group;

(3) Klein's quartic curve of genus 3; etc..

For these algebraic curves we do *know the uniformizing Fuchsian groups* Γ, (with parabolic cusps in the modular curves case). Thus, utilizing these special Riemann surfaces as the fiducial (reference) point, we can make some nice examples and applications for our formulas whenever we perturb the corresponding polynomial equations. Some explicit calculations in especially interesting families are worked out by us in the latter part of the thesis.

In fact, we utilize methods from the modern Belyi theory for arithmetic algebraic curves in the Fermat case. In case of the Fermat curve $x^n + y^n = 1$ for $n \geq 4$ we are able to determine an explicit and complete presentation of its uniformizing Fuchsian group utilizing exactly $2g$ generating Mobius transformations, (where g is the genus of the Fermat curve). This is done by using the theory of the Schreier transversal in the matter of finding generators and relations for subgroups of given finitely presented groups. For exact detail and a tabulation of generators and relations, consult Chapter V of this work. We can apply therefore the whole theory of Chapter I to IV quite explicitly by perturbing the Fermat curve in a parametrized family.

Chapter 2

Invariance of sheet monodromy over families of curves

2.1 The family of polynomials:

Let

$$P_t(x,y) \equiv P(x,y,t) = \sum_{n=0}^{N} \sum_{m=0}^{M} a_{mn}(t) x^m y^n \tag{2.1}$$

be a one-parameter family of polynomials in two complex variables x and y, where the coefficients $a_{mn}(t)$ are assumed to be analytic functions of t, t being within a small disk: $\triangle_\epsilon = \{|t| \mid |t| < \epsilon\}$. We will require that for each t the corresponding polynomial P_t is an *irreducible* member of the polynomial ring $\mathbf{C}[x,y]$. (In the results of this Chapter we shall give sufficient conditions that guarantee that P_t is irreducible if the base polynomial P_0 is so.)

The theory of complex algebraic curves now tells us that each $P_t = 0$ determines *a connected compact Riemann surface*, denoted by X_t, which is the desingularized model of the projective algebraic curve that is defined by the solutions of the (homogenized) polynomial P_t in $\mathbf{CP}^2$. As is standard, there is a surjective holomorphic morphism of X_t onto that projective plane curve, and the variables x and y become interpreted as two nonconstant meromorphic functions on the Riemann surface X_t which together generate (over $\mathbf{C}$) the field of meromorphic functions on X_t.

Recall furthermore that this field of meromorphic functions on X_t, denoted $Mer(X_t)$, is an "algebraic function field in one independent variable". (That "independent variable", which is transcendental over $\mathbf{C}$, may be chosen to be *any* nonconstant meromorphic function on the surface – for instance x itself.) Indeed this means that $Mer(X_t)$ is a finite algebraic extension (by y) of the rational function field $\mathbf{C}(x)$, because y satisfies the algebraic equation $P_t(x, y) = 0$ (whose coefficients are now considered to belong to $\mathbf{C}(x)$).

These facts constitute fundamental material from the theory of complex algebraic curves and their algebraic function fields; good references are: Siegel[S], Farkas-Kra[FK] or Chevalley[C].

The chief study here will concern these compact Riemann surfaces X_t and their uniformizing *Fuchsian groups* G_t. We consider the G_t as discrete groups of real-coefficient Möbius transformations; namely

$$G_t \subset PSL(2, R) = Aut(Upper\ half-plane\ U)$$

G_t acts as a properly discontinuous group of biholomorphic automorphims of the upper half-plane:

$$U = \{z \in \mathbf{C} : Im(z) > 0\}$$

and *the quotient U/G_t is biholomorphically equivalent to the compact Riemann surface X_t.*

Remark: As is well-known, *the Fuchsian group corresponding to a compact Riemann surface varies only real analytically with the complex analytic moduli of the surface*, (see Section 2.5 and also Section 1.3 of Nag[N]). It is therefore rather irrelevant for our study in this work whether t is a real or complex parameter; the main point is that the coefficients of the varying polynomial are real or complex analytic functions of t near $t = 0$. We shall assume, in order to fix ideas, that t is a complex parameter in some suitable t-disc as stated, and that the coefficient functions a_{mn} are holomorphic in t.

Remark: As explained in Chapter I, it is not necessary that the Fuchsian groups we consider be torsion free or that they have no (parabolic) cusps. Indeed, if the Fuchsian groups we are dealing with contain elliptic elements (torsion), and/or have parabolic elements (which correspond to punctures), we shall still be able to have the same formulas valid as long as the uniformized Riemann surface is of finite conformal type, (namely a compact surface minus a finite number of points). In fact, the theory of quasiconformal deformation that we will utilize in this work will go through (see Nag[N]) just the same for such Fuchsian groups.

2.2 Monodromy Invariance Lemma:

We have already explained our method of attack in Chapter I. Note that it is essential *to guarantee that the topological structure of the branched covering of the x-sphere, that describes the compact surface X_t, is invariant for varying values of t.*

Preliminary notations : Given any irreducible polynomial $P(x,y) = 0$ which defines a compact Riemann surface, X, we consider the holomorphic branched covering map:

$$x : X \to x - sphere$$

defined by this meromorphic function x on X. The mapping x is branched (= ramified) over a certain finite set of points on the image sphere - see Siegel[S] or Knopp[K] for details. The complement of these branch points will be called the ordinary points; above each ordinary point there are N distinct solutions for y (where N is the degree of P when considered as a polynomial in y with coefficients from $\mathbf{C}[x]$).

To solve our problems we have to find a correspondence between the ramification (branch) points of $P_t(x,y) = 0$ lying on the x-sphere for different values of t. Also we will need to make a correspondence between the algebraic functions $y_t(x) = y(x,t)$ satisfying $P_t(x, y(x,t)) = 0$ for different values of t, so that the monodromy remains invariant at the corresponding branch points. That will guarantee that the topological structure of the branched covering is kept invariant as t changes.

In order to do this we assume certain restrictions on $P_t(x,y)$:

Assume $\deg P(x,y,t) = D$ for all t. Assume also that there exists r,s such that $r+s=D$ where $0 \le r \le m$, $0 \le s \le N$ and $a_{rs}(0) \ne 0$ i.e degree $P_0(x,y)=D$.

Assume:

(1) $P_0(x,y)$ is irreducible in the polynomial ring $\mathbf{C}[x,y]$.

(2) If degree $P_t(x,y)=D$ then degree $P_0(x,y)=D$.

(3) Suppose P_t is of degree N in the y variable for all small t:

$$\begin{aligned} P_t(x,y) &= P_N(x,t)y^N + P_{N-1}(x,t)y^{N-1} + \ldots + P_0(x,t) \\ \text{where} \quad P_N(x,t) &= a_k(t)x^k + \ldots + a_0(t) \end{aligned}$$

Let $D(t)$ denote the discriminant of $P_N(x,t)$. Then assume that $D(0) \ne 0$ and $a_k(0) \ne 0$.

(4)Let $D(x,t)$ be the discriminant of $P_t(x,y)=0$. Then $D(x,t) = P_N(x,t)Q(x,t)$ where

$$Q(x,t) = Q_0(t)x^r + \ldots + Q_r(t)$$

We assume that $Q_0(0) \ne 0$ and $\tilde{D}(0) \ne 0$ where $\tilde{D}(t)$=discriminant of $Q(x,t)$.

(5)The resultant of $Q(x,t)$ and $P_N(x,t)$ does not vanish at $t=0$.

<u>Lemma:</u> (**Monodromy Invariance Lemma**) Consider the parametrized family of polynomials

$$P_t(x,y) \equiv P(x,y,t) = 0$$

as given in equation (2.1) above. The polynomials $P_t(x,y)$ are subjected to the five conditions already listed above.

Assume

$$P(x,y,\circ) = P_\circ(x,y)$$

is an irreducible polynomial such that $x = 0$ and $x = \infty$ are ordinary points, and the set of ramification points on the x-plane are say located at:

$$\{\zeta_1^0, \ldots, \zeta_k^0\}$$

Then

(i) For all t sufficiently close to 0, the polynomial $P_t(x, y)$ is irreducible and 0 ,∞ are ordinary points.

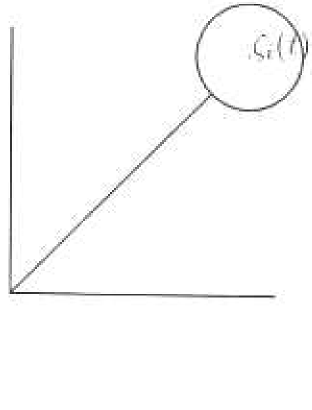
(ii) The ramification points on the x-sphere for $P_t(x, y)$ are holomorphically dependent on t and are given by k holomorphic functions: $\{\zeta_1(t), \ldots, \zeta_k(t)\}$ such that $\zeta_j(0) = \zeta_j^0$ for $0 \leq j \leq k$ and $\zeta_i(t) \neq \zeta_j(t)$ for $i \neq j$ and all t small enough.

(iii) Assume N is the degree of P_t in the y variable (this follows from the stability conditions mentioned above.) Then there exists holomorphic function germs $\{y_1(x, t), \ldots, y_N(x, t)\}$ around $(x, t) = (0, 0) \in \mathbf{C}^2$ such that

$$P_t(x, y_j(x, t)) = 0$$

for all (x, t) sufficiently close to (0.0) and such that N roots of the y equation $P(x, y, t) = 0$ are given by $y_j(x, t)$.

(iv) Analytic continuation of $y_1(x, t)$ for every fixed t, $|\ t\ | \leq \epsilon$ in the x-sphere along the same route (avoiding the branch points) produces the *same permutation* of $\{y_1(x, t), \ldots, y_N(x, t)\}$ – i.e., the monodromy permutations are independent of t.



Proof:

(1) Let

$$P(x,y,t) = \sum_{n=0}^{N}\sum_{m=0}^{M} a_{mn}(t)x^m y^n$$

Let Deg $P_t(x,y) = D$ then by stability condition (2) above degree $P_0(x,y) = D$. Hence there exists r,s such that $0 \leq r \leq M$, $0 \leq s \leq N$, r+s=D and $a_{rs}(0) \neq 0$. So for t small enough $a_{rs}(t) \neq 0$. Let

$$P(x,y,z,t) \equiv P_t(x,y,z)$$

denote the homogeneous polynomial corresponding to the polynomial

$$P(x,y,t) \equiv P_t(x,y)$$

We claim that $P_t(x,y)$ is irreducible if and only if $P_t(x,y,z)$ is irreducible.

Let $P_t(x,y,z)$ be reducible. Then

$$P_t(x,y,z) = Q_t(x,y,z)R_t(x,y,z)$$

where $Q_t(x,y,z)$ and $R_t(x,y,z)$ are homogeneous polynomials in x,y,z. Denote

$$Q_t(x,y,1) = Q_t(x,y)$$

$$\text{and} \quad R_t(x,y,1) = R_t(x,y)$$

$$\Rightarrow \quad P_t(x,y) \equiv P_t(x,y,1) = Q_t(x,y,1)R_t(x,y,1) \equiv Q_t(x,y)R_t(x,y)$$

For definiteness suppose

$$\begin{aligned} Q_t(x,y) &= \text{constant} \\ \Rightarrow Q_t(x,y,z) &= cz^k \text{ for some constant } c \text{ and some integer } k \\ \Rightarrow P_t(x,y,z) & \quad \text{is divisible by } z^k \end{aligned}$$

But this is not true as $a_{rs}(t) \neq 0$ for t small. So $P_t(x,y,z)$ is reducible implies $P_t(x,y)$ is also reducible.

Conversely : Let $P_t(x,y)$ be reducible. Then

$$P_t(x,y) = Q_t(x,y)R_t(x,y)$$

where $Q_t(x,y)$ and $R_t(x,y)$ are nonconstant polynomials in x and y

$$\Rightarrow P_t(x,y,z) = Q_t(x,y,z)R_t(x,y,z)$$

where $Q_t(x,y,z)$ is the homogeneous polynomial corresponding to $Q_t(x,y)$ and similarly for $R_t(x,y,z)$. So $P_t(x,y,z)$ is reducible.

By stability condition (1) we have already assumed that $P_0(x,y)$ is irreducible. Then (by the above claim) $P_0(x,y,z)$ is irreducible. Consider

$$\begin{aligned} Q(x,y,z) &= \sum_{i+j+k=p} b_{ijk}x^i y^j z^k \\ \text{and} \quad R(x,y,z) &= \sum_{i+j+k=q} c_{ijk}x^i y^j z^k \end{aligned}$$

$$\begin{aligned} \text{Then} \quad N(p) &= \sharp(b_{ijk}) = \frac{p(p+3)}{2} \\ N(q) &= \sharp(c_{ijk}) = \frac{q(q+3)}{2} \end{aligned}$$

i.e Q and R are any two homogeneous polynomials of degree p and q respectively. Define a map

$$\begin{aligned} \beta_{pq} : \quad \mathbf{P}^{N(p)-1} \times \mathbf{P}^{N(q)-1} &\longrightarrow \mathbf{P}^{N(D)-1} \\ (< b_{ijk} >, < c_{ijk} >) &\longrightarrow < d_{ijk} > \\ (Q,R) &\longrightarrow QR \end{aligned}$$

where $p+q=D$ and $QR = \sum_{i+j+k=D} d_{ijk}x^i y^j z^k$

$$\begin{array}{ccc} (\mathbf{C}^{N(p)} - \{0\}) \times (\mathbf{C}^{N(q)} - \{0\}) & \xrightarrow{\alpha_{pq}} & \mathbf{C}^{N(D)} - \{0\} \\ \Big\downarrow \pi_p \times \pi_q & & \Big\downarrow \pi \\ \mathbf{P}^{N(p)-1} \times \mathbf{P}^{N(q)-1} & \xrightarrow{\beta_{pq}} & \mathbf{P}^{N(D)-1} \end{array}$$

where

$$\begin{aligned} \alpha_{pq}((b_{ijk}), (c_{ijk})) &= (d_{ijk}) \\ (Q, R) &\longrightarrow QR \end{aligned}$$

Now since α_{pq} is continuous, π, π_p, π_q are open and continuous, Imageβ_{pq} is a compact subset of the Hausdorff space $\mathbf{P}^{N(D)-1}$. Hence Imageβ_{pq} is a closed subset of $\mathbf{P}^{N(D)-1}$. Since $P_0(x, y, z)$ is irreducible $< a_{ijk}(0) > \notin \; Image\beta_{pq}$ for any p, q with $p + q = D$ where $P_t(x, y, z) = \sum_{i+j+k=D} a_{ijk}(t) x^i y^j z^k$. So $< a_{ijk}(0) > \in U_{pq} = \mathbf{P}^{N(D)-1} - Image\beta_{pq}$. Now by continuity for small values of ϵ_{pq}, $< a_{ijk}(t) > \in U_{pq}$ when $t \in \triangle_{\epsilon_{pq}} = \{t : \mid t \mid \leq \epsilon_{pq}\}$. We do this operation for all tuples (p,q) where $p + q = D$. So the number of such β_{pq} is finite. Let us denote them by $\beta_1, \ldots, \beta_k$. Then there exists $\epsilon > 0$ such that $< a_{ijk}(t) > \in \cap_{i=1}^{k} U_i$ when $t \in \triangle_\epsilon = \{t : \mid t \mid \leq \epsilon\}$.

Hence $P_t(x, y, z)$ is irreducible whenever $t \in \triangle_\epsilon$

$\Rightarrow P_t(x, y)$ is irreducible whenever $t \in \triangle_\epsilon$.

(ii) Let

$$P(x, y, t) = P_N(x, t) y^N + P_{N-1}(x, t) y^{N-1} + \ldots + P_0(x, t)$$

The ramification points are given by $P_N(x, t) = 0$ and $D(x, t) = 0$ where $D(x, t) =$ the discriminant of $P_N(x, t)$. Now

$$P_N(x, t) = a_k(t) x^k + a_{k-1}(t) x^{k-1} + \ldots + a_0(t)$$

Denote $D(t) = discriminant\ of\ P_N(x,t)$. Assume $a_k(0) \neq 0$ and $D(0) \neq 0$. So for t small enough $a_k(t) \neq 0$ and $D(t) \neq 0$. Hence $P_N(x,t) = 0$ has k distinct roots for each such t. Thus

$$P_N(x,t) = G(x_1, x_2, t_1, t_2) + iH(x_1, x_2, t_1, t_2)$$

where $x = x_1 + ix_2$ and $t = t_1 + it_2$.

$$\frac{\partial(G,H)}{\partial(x_1,x_2)} = G_{x_1}H_{x_2} - G_{x_2}H_{x_1}$$

Let x^0 be a root of $P_N(x,0) = 0$ where $x^0 = x_1^0 + ix_2^0$. So

$$\begin{aligned} G(x_1^0, x_2^0, 0, 0) &= 0 \\ H(x_1^0, x_2^0, 0, 0) &= 0 \end{aligned}$$

Now if $\frac{\partial(G,H)}{\partial(x_1,x_2)} \neq 0$ at $x = x^0$ and $t = 0$ then by *implicit function theorem* there exists a neighborhood $\triangle_\epsilon = \{t : \mid t \mid \leq \epsilon\}$ of $t = 0$ and two C^1 functions $x_1(t)$ and $x_2(t)$ defined on $\triangle_\epsilon$ such that

$$G(x_1(t), x_2(t), t_1, t_2) = 0$$

$$H(x_1(t), x_2(t), t_1, t_2) = 0$$

$$\text{for} \quad \mid t \mid \leq \epsilon \quad \text{and } x_1(0) = x_1^0, \ \ x_2(0) = x_2^0$$

$$\text{i.e} \quad P_N(x(t), t) = 0 \quad \text{for } \mid t \mid \leq \epsilon \text{ and } x(0) = x^0$$

Since for $t = 0$, $P_N(x,t)$ is an analytic function of x. So by Cauchy Riemann Equation

$$\begin{aligned} G_{x_1}(x_1, x_2, 0, 0) &= H_{x_2}(x_1, x_2, 0, 0) \\ G_{x_2}(x_1, x_2, 0, 0) &= -H_{x_1}(x_1, x_2, 0, 0) \end{aligned}$$

$$So \ \ G_{x_1}H_{x_2} - H_{x_1}G_{x_2}|_{(x_1^0, x_2^0, 0, 0)} = G_{x_1}^2 + H_{x_1}^2 = \mid P_{N,x_1}(x^0, 0) \mid^2 \neq 0$$

as x^0 is a simple zero of $P_N(x,0)$. Hence there exists $\epsilon > 0$ and c^∞ functions $x_1(t)$, $x_2(t)$ such that

$$\begin{aligned}
& P_N(x(t),t) = 0 \text{ for } |\, t \,|\leq \epsilon \text{ where } x(t) = x_1(t) + ix_2(t) \\
\Rightarrow\ & 0 = \frac{\partial P_N}{\partial t_1} = P_{N,x}x_{t_1} + P_{N,t} \\
& 0 = \frac{\partial P_N}{\partial t_2} = P_{N,x}x_{t_2} + iP_{N,t} \\
\Rightarrow\ & P_{N,x}(x(t),t)(x_{t_1} + ix_{t_2}) = 0 \\
& \text{Since } \quad D(t) \neq 0 \text{ all roots of } P_N(x(t),t) \text{ are distinct} \\
\Rightarrow\ & P_{N,x}(x(t),t) \neq 0 \\
\Rightarrow\ & x_{t_1} + ix_{t_2} = 0 \\
\Rightarrow\ & x(t) \text{ is an analytic function of } t \text{ for } t \text{ small enough.}
\end{aligned}$$

So we can conclude that for $\epsilon > 0$ sufficiently small, there exists k holomorphic functions $\zeta_1(t), \ldots, \zeta_k(t)$ such that

$$\begin{aligned}
P_N(\zeta_i(t),t) &= 0 \text{ for } |\, t \,|\leq \epsilon,\ \ 1 \leq i \leq k \\
\text{and } \ \zeta_i(t) &\neq \zeta_j(t) \text{ for } i \neq j
\end{aligned}$$

Similarly $D(x,t) = P_N(x,t)Q(x,t)$ and we can find r holomorphic functions $\zeta_{k+1}(t), \ldots, \zeta_{k+r}(t)$ such that

$$\begin{aligned}
Q(\zeta_i(t),t) &= 0 \text{ for } |\, t \,|\leq \epsilon,\ \ r+1 \leq i \leq k+r \\
\zeta_i(t) &\neq \zeta_j(t) \quad \text{for } \ i \neq j
\end{aligned}$$

assuming the facts that $Q_0(0) \neq 0$ and $\tilde{D}(0) \neq 0$ where $Q_0(t)$ is the leading coefficient of $Q(x,t)$ and $\tilde{D}(t)$ is the discriminant of $Q(x,t)$. Also assume that the resultant of $P_N(x,t)$ and $Q(x,t)$ does not vanish at $t = 0$ so $\zeta_i(t) \neq \zeta_j(t)$ for $i \neq j$ and for$|\, t \,|\leq \epsilon$. Since $x = 0$ and $x = \infty$ are ordinary points of $P_0(x,y)$ and $\zeta_i(t)$ are holomorphic functions of t so for t small, $x = 0$ and $x = \infty$ are ordinary points of $P_t(x,y) = 0$.

(iii)Assume for $t = 0$, $x = 0$ is an ordinary point. Let

$$P(x,y,t) = P_N(x,t)y^N + P_{N-1}(x,t)y^{N-1} + \ldots + P_0(x,t)$$

So $P_N(0,0) \neq 0$. Set $D(x,t) =$ discriminant of $P(x,y,t) \neq 0$ for $(x,t) = (0,0)$. Then we can find $\epsilon_1 > 0$ small enough such that $P_N(x,t) \neq 0$ and $D(x,t) \neq 0$ for $| x |< \epsilon_1$, $| t |< \epsilon_1$. Hence all the roots of $P_t(x,y) = 0$ are distinct for $| x |< \epsilon_1$, $| t |< \epsilon_1$. Let $\alpha^1, \ldots, \alpha^N$ be distinct roots of $P_0(0,y) = 0$.

$$\begin{aligned}
&\text{Let } P(x,y,t) = G(x_1,x_2,y_1,y_2,t_1,t_2) + iH(x_1,x_2,y_1,y_2,t_1,t_2) \\
&\text{Then } G(0,0,\alpha_1^1,\alpha_2^1,0,0) = 0 \\
&\text{and } H(0,0,\alpha_1^1,\alpha_2^1,0,0) = 0 \\
&\text{where } \alpha^1 = \alpha_1^1 + i\alpha_2^1
\end{aligned}$$

If we can prove that $\frac{\partial(G,H)}{\partial(y_1,y_2)} \neq 0$ for $x = 0$, $t = 0$, then by *implicit function theorem* there exists $\epsilon > 0$ and C^1 functions $y_1(x,t)$, $y_2(x,t)$ such that $0 < \epsilon < \epsilon_1$ and

$$\begin{aligned}
&G(x_1,x_2,y_1(x,t),y_2(x,t),t_1,t_2) = 0 \\
&H(x_1,x_2,y_1(x,t),y_2(x,t),t_1,t_2) = 0 \\
&\text{for } | x |< \epsilon, \ | t |< \epsilon \quad \text{where } x = x_1 + ix_2 \text{ and } t = t_1 + it_2 \\
&\text{and } y_1(0,0) = \alpha_1^1, \ y_2(0,0) = \alpha_2^1
\end{aligned}$$

Let us denote $y(x,t) = y_1(x,t) + iy_2(x,t)$. $P_0(0,y)$ is an holomorphic function of y. So

$$\begin{aligned}
G_{y_1} &= H_{y_2} \\
G_{y_2} &= H_{y_1} \\
\text{at } x = 0, & \quad t = 0 \\
\text{Hence } \frac{\partial(G,H)}{\partial(y_1,y_2)} &= G_{y_1}^2 + H_{y_1}^2
\end{aligned}$$

$$= \ | P_{y_1}(0, y, 0) |^2$$

$\neq \ 0$ since all the roots of $P_0(0, y) = 0$ are all simple and distinct.

We want to show that $y(x, t)$ is a holomorphic function of x and t separately. Hence $y(x, t)$is a holomorphic function of (x, t).

$$P(x, y(x, t), t) = 0 \quad \text{for } | x |< \epsilon, \ | t |< \epsilon$$

$$\Rightarrow \quad 0 = \frac{\partial P}{\partial t_1} = P_y y_{t_1} + P_t$$

$$0 = \frac{\partial P}{\partial t_2} = P_y y_{t_2} + iP_t$$

$$\Rightarrow \quad P_y(x, y(x, t), t)[y_{t_1} + iy_{t_2}] = 0$$

But $P_y(x, y(x, t), t) \neq 0$ for $| x |< \epsilon$, $| t |< \epsilon$, since $P(x, y, t) = 0$ has all its roots distinct for $| x |< \epsilon$, $| t |< \epsilon$

So $y_{t_1} + iy_{t_2} = 0$

$\Rightarrow$ $y(x, t)$ is a holomorphic function of t for $| x |< \epsilon$, $| t |< \epsilon$

Again $P(x, y(x, t), t) = 0$

$$\Rightarrow \quad 0 = \frac{\partial P}{\partial x_1} = P_y y_{x_1} + P_x$$

and $$0 = \frac{\partial P}{\partial x_2} = P_y y_{x_2} + iP_x$$

$$\Rightarrow \quad P_y(y_{x_1} + iy_{x_2}) = 0$$

Now $P_y(x, y(x, t), t) \neq 0$ for $| x |< \epsilon$, $| t |< \epsilon$

$$\Rightarrow \quad y_{x_1} + iy_{x_2} = 0$$

So $y(x, t)$ is a holomorphic function of x for $| x |< \epsilon$, $| t |< \epsilon$.

Hence $y(x, t)$ is a holomorphic function of (x, t) when $| x |< \epsilon$, $| t |< \epsilon$.

(iv) Follow the construction, as in Siegel[S], for each $\zeta_i(0)$ we consider a circle C_i with center at $\zeta_i(0)$ such that any two of them does not intersect and we join the origin to

$\zeta_i(0)$ by a simple curve l_i so that if we cut $\mathbf{CP}^1$ along these curves it remains simply connected. Since ζ_i's are holomorphic function of t we can find a neighborhood of $t = 0$ say, $N = \{t :| t |< \epsilon\}$ such that $\zeta_1(N), \ldots, \zeta_k(N)$ lies inside $C_1, \ldots, C_k$ respectively and each $\zeta_i(N)$ is an open connected subset lying in the interior of C_i $1 \leq i \leq n$. Now for each point x_0 on C_i, $1 \leq i \leq n$ we can find mutually disjoint neighborhood $W_1(x_0), \ldots, W_N(x_0)$ of $\phi_i(x_0, 0)$, $1 \leq i \leq N$ (where $P(x, \phi_i(x_0, 0), 0) = 0$ and $\phi_i(x, 0)$ is an analytic function of x $1 \leq i \leq N$) and an open disc $U(x_0)$ of x_0 and an open disc $V(x_0)$ of $t = 0$ such that $\forall x \in U(x_0)$, $\forall t \in V(x_0)$, $\phi_i(x, t) \in W(x_0)$ and the function germs are analytic on $U(x_0)$ and $U(x_0) \cap \zeta_i(N) = \varphi$ for all i. Again since the points on C_i $1 \leq i \leq n$ form a compact set $D = \cup_{i=1}^k C_i$, the open cover $\{U(x) : x \in D\}$ has a finite subcover. Hence $D \subset \cup_{i=1}^n U(x_i)$. Set $V = \cap_{i=1}^n V(x_i) \cap N$. Note that $\phi_i(x, 0) = y_j(x_0, 0)$ for some j, $1 \leq j \leq N$. Let us consider the monodromy permutation around $\zeta_1(0)$. For simplicity let $y_1(x, 0) \longrightarrow y_2(x, 0) \longrightarrow y_3(x, 0) \longrightarrow y_1(x, 0)$. We shall prove that for each $t \in V$, $y_1(x, t) \longrightarrow y_2(x, t) \longrightarrow y_3(x, t) \longrightarrow y_1(x, t)$.

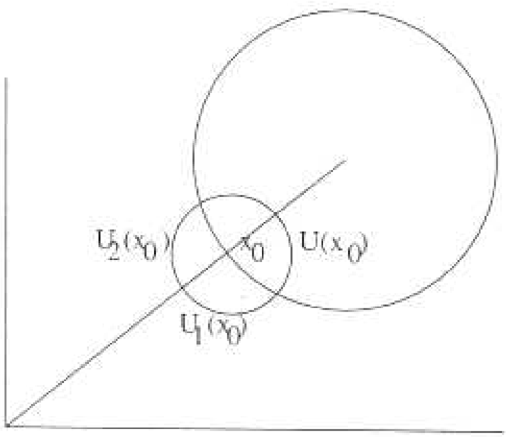


Figure 2.1: This illustrates that $U(x_0) = U_1(x_0) \cup U_2(x_0)$

Let $U(x_0)$ is a neighborhood of x_0 such that $U(x_0) = U_1(x_0) \cup U_2(x_0)$ (See figure

(2.1))

$$
\begin{aligned}
\text{Then}\quad & \forall x \in U_1(x_0),\ \ \forall t \in V,\quad y_1(x,t) \in W_1(x_0)\\
& \forall x \in U_2(x_0),\ \ \forall t \in V,\quad y_3(x,t) \in W_1(x_0)\\
& \text{as}\ \ y_3(x,0) \longrightarrow y_1(x,0) \text{ in the neighborhood of } x = x_0,\\
& \forall x \in U_1(x_0),\ \ \forall t \in V,\quad y_2(x,t) \in W_2(x_0)\\
& \forall x \in U_2(x_0),\ \ \forall t \in V,\quad y_1(x,t) \in W_2(x_0)\\
& \text{as}\ \ y_1(x,0) \longrightarrow y_2(x,0),\\
\text{and}\quad & \forall x \in U_1(x_0),\ \ \forall t \in V,\quad y_3(x,t) \in W_3(x_0)\\
& \forall x \in U_2(x_0),\ \ \forall t \in V,\quad y_2(x,t) \in W_3(x_0)\\
& \text{as}\ \ y_2(x_0) \longrightarrow y_3(x_0).
\end{aligned}
$$

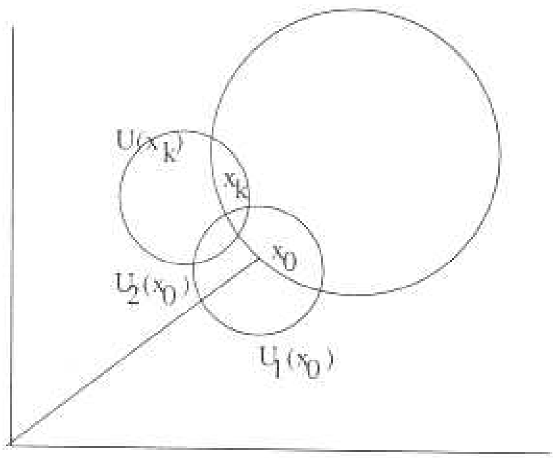


Figure 2.2: This illustrates the following

By construction we can find finite number of points $x_0, \ldots, x_k$ on C_1 and their neighborhood $U(x_0), \ldots, U(x_k)$ (see figure (2.2)) and disjoint open set $W_1(x_i), \ldots, W_N(x_i)$ for each fixed i, $0 \le i \le k$ around $y_j(x_i, 0)$, $1 \le j \le N$ (see figure (2.3)) such that $\forall x \in U(x_i)$, $t \in V$, $y_j(x,t) \in W_j(x_i)$ $1 \leq j \leq N$. Since $y_1(x,0)$ analytically continues to $y_2(x,0)$, $W_1(x_k)$ (i.e the neighborhood of $y_1(x_k,0)$) intersects $W_2(x_0)$ (which is the neighborhood of $y_2(x_0,0)$).

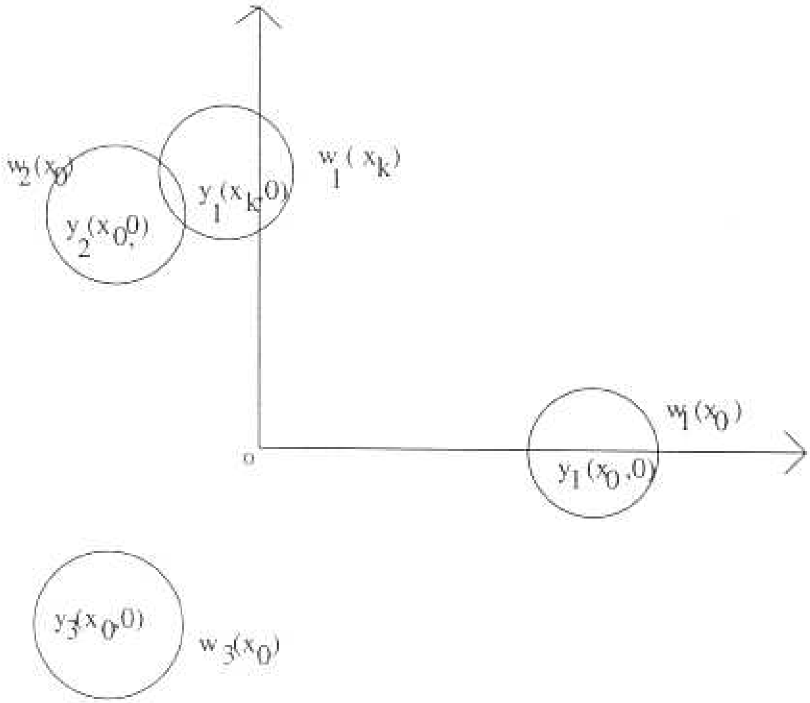


Figure 2.3: This illustrates the above

$\forall x \in U_2(x_0)$, $y_1(x,0) \in W_2(x_0)$

Choose $\tilde{x} \in U(x_k) \cap U_2(x_0)$

$\implies y_1(\tilde{x},0) \in W_2(x_0)$

$\implies y_1(\tilde{x},t) \in W_2(x_0)$ for t small (by continuity of y_1 in t)

as only $\phi_2(\tilde{x},t) \in W_2(x_0)$ $\forall t \in V$

$\implies \phi_2(\tilde{x},t) = y_1(\tilde{x},t)$ for t small

$$\Longrightarrow \phi_2(\tilde{x},t) = y_1(\tilde{x},t)\ \forall t \in V \quad (\text{as } y_1 \text{ and } \phi_2 \text{ are analytic function of } t\ )$$

$$\Longrightarrow y_1(\tilde{x},t) \in W_2(x_0)\ \forall t \in V\ \forall \tilde{x} \in U_2(x_0) \cap U(x_k)$$

$$\Longrightarrow y_1(x,t) \in W_2(x_0)\ \ \forall t \in V\ \ x \in U_2(x_0)$$

$$(\text{as for } t \text{ fixed} \quad y_1(\tilde{x},t) = \phi_2(\tilde{x},t)\ \ \forall x \in U_2(x_0) \cap U(x_k)$$

$$\Longrightarrow y_1(x,t) = \phi_2(x,t)\ \ \forall x \in U_2(x_0)\ \text{ by analyticity in } x)$$

So if we continue $y_1(x,t)$ along l_1 we get $\phi_2(x,t)$. Again only $y_2(x,t) \in W_2(x_0)$ $\forall x \in U_1(x_0)$. Let us fix $t \in V$. If we continue $y_1(x,t)$ across l_1 the function we get say $\tilde{y}(x,t)$ which is a solution of $P(x,y,t) = 0$ (for fixed t) and hence belong to either $W_1(x_0)$ or $W_2(x_0)$ or $W_3(x_0)$

$$\text{Since } \ y_1(x,t) \in W_2(x_0)\ \forall x \in U_2(x_0)$$

$$\text{and } W_2(x_0) \cap W_1(x_0) = \varphi, \quad W_2(x_0) \cap W_3(x_0) = \varphi$$

$$\text{So } \tilde{y}(x,t) \in W_2(x_0)\ \forall x \in U_1(x_0)$$

$$\Longrightarrow \tilde{y}(x,t) = y_2(x,t)\ \forall x \in U_1(x_0)$$

$$\text{as only } \ y_2(x,t) \in W_2(x_0)\ \forall x \in U_1(x_0)\ \forall t \in V$$

Since $t \in V$ is arbitrary $y_1(x,t)$ continues to $y_2(x,t)$ and thus monodromy remains invariant. $\square$

Chapter 3

Construction of quasiconformal marking maps

3.1 The X_t as members of the Teichmüller space $T(X_0)$:

We assume that we are working within the set up of the previous Chapter. Namely, we are scrutinizing the family of compact Riemann surfaces X_t which are the non-singular models of the family of complex algebraic curves, $P_t(x,y) = 0$. Recall that the ramification points on the Riemann sphere for the covering surface X_t, (i.e., the critical value set for the branched covering map x on X_t), are assumed to be located at precisely K points (for each t):

$$(\zeta_1(t), \cdots, \zeta_K(t)).$$

Each $\zeta_j(t)$ was shown to be an analytic function of t around $t = 0$. As implied by the monodromy invariance Lemma, we will assume that the monodromy permutation for the N sheets of X_t around the ramification point $\zeta_j(t)$ may depend on j but *not* on t.

Let g denote the genus of each of the Riemann surfaces X_t.

The aim now is to consider X_0 as the base point for the Teichmüller space $T(X_0) = T_g$, and consequently realize each X_t as a point of this Teichmüller space by constructing an *explicit quasiconformal (q.c.) marking homeomorphism from X_0 onto X_t*:

$$\tilde{\phi}_t : X_0 \longrightarrow X_t \tag{3.1}$$

We shall have ϕ_0 as the identity mapping.

Definition of the Teichmüller space $T(X_0)$: Recall that given any reference Riemann surface X_0, the Teichmüller space consists of the "Teichmüller equivalence classes" of triples of the form $[X_0, \psi, Y]$, where Y is another Riemann surface and ψ is an orientation preserving quasiconformal homeomorphism of X_0 onto Y. ψ is called the "marking homeomorphism" for such a triple.

The *Teichmüller equivalence relation* demands that two such triples, $[X_0, \psi_1, Y_1]$ and $[X_0, \psi_2, Y_2]$, are equivalent if there exists a biholomorphic map, say $H : Y_1 \to Y_2$, such that the quasiconformal self-map ${\psi_2}^{-1} \circ H \circ \psi_1$ of X_0 is homotopic to the identity mapping on X_0. (Since we are interested only in Riemann surfaces that are either compact, or at worst of finite conformal type, we avoid the standard business of "homotopy rel ideal boundary". For these and allied matters, see Nag[N].)

The set of Teichmüller equivalence classes of triples whose first member is X_0 is, by definition, the Teichmüller space $T(X_0)$.

It is well-known (see Ahlfors[A], Nag[N]) that if X_0 is a compact Riemann surface of genus g, then $T(X_0)$ is, in a natural way, a complex analytic manifold of complex dimension 1 if $g = 1$, and of complex dimension $(3g - 3)$ if $g > 1$.

The fundamental universal property for the Teichmüller space asserts that this complex manifold, $T(X_0) = T_g$ *is the universal target space for the holomorphic classifying map* from the base space of any arbitrary holomorphic family of marked Riemann surfaces of genus g. This result is very relevant for the families X_t that we are dealing with, as we will explain in a moment. See Chapter V of Nag[N] for details and proofs for this universal property of the Teichmüller space, and for information regarding the universal family of Riemann surfaces that lives as a holomorphic fiber space over $T(X_0)$.

The upshot therefore is the following: Having obtained the marking homeomorphisms $\tilde{\phi}_t : X_0 \longrightarrow X_t$, we have for each t, a point of the Teichmüller space, $T(X_0)$, given by the equivalence class of the triple: $[X_0, \tilde{\phi}_t, X_t]$. In fact we have thereby constructed a "classifying map":

$$\eta : t \mapsto [X_0, \tilde{\phi}_t, X_t] \tag{3.2}$$

mapping the t-disc $\{|t| < \epsilon\}$ into the Teichmüller space.

Indeed, if the coefficients of the polynomialP_t vary holomorphically with t, we obtain a *holomorphic classifying map* η. (in the sense explained in Chapter V of [N], as cited).

Recall furthermore that the Teichmüller space can be described as a quotient of the space of Beltrami differentials on the reference Riemann surface X_0. In fact, the basic Bers projection:

$$\beta : Bel(X_0) \to T(X_0) \tag{3.3}$$

is known to be a holomorphic submersion of the L^∞ unit ball of Beltrami coefficients on X_0 onto the complex manifold that is Teichmül;ler space. The connection with the previous definition of Teichmüller space above is to associate to a Teichmüller triple $[X_0, \psi, Y]$ the complex dilatation (=Beltrami coefficient) of the marking homeomorphism ψ. (We refer to the book Nag[N] for all this basic material and their proofs.)

Consequently, the computation of the complex dilatation of the marking map $\tilde{\phi}_t$ will will give us *a lifting of the classifying map* η to a map of the t-disc into the ball of Beltrami coefficients:

$$\tilde{\eta} : \{|t| < \epsilon\} \longrightarrow Bel(X_0) \tag{3.4}$$

This computation, and its consequences for the variational formulae that we seek, will be shown in Chapter IV.

3.2 Construction of a piecewise-affine mapping $\phi_t : \mathbf{CP}^1 \to \mathbf{CP}^1$ which carries ramification points of $P_0(x,y)$ to the ramification points of $P_t(x,y)$:

The marking homeomorphism between the compact Riemann surfaces X_0 and X_t will be obtained by lifting a mapping ϕ_t between the Riemann spheres that carries corresponding ramification points to ramification points. Construction of ϕ_t is detailed below:

Recall that ∞ was set up as an ordinary point for the meromorphic function x on each X_t. Hence all the ramification points, $\zeta_i(t)$ $1 \leq i \leq k$ lie in the finite x-plane. Restrict the parameter t in a relatively compact sub-disc around $t = 0$: $t \in \triangle_\epsilon = \{t : \mid t \mid \leq \epsilon\}$. (To save on notation we still call the radius of the sub-disc as ϵ.)

Since the functions ζ_i are analytic in t, we can find a rectangle R containing in its interior all of the points $S = \{\zeta_i(t) : 1 \leq i \leq K,\ t \in \triangle_\epsilon\}$. Outside R we will define ϕ_t to be the identity mapping.

To define ϕ_t inside R we take the first (domain) copy of $\mathbf{CP}^1$ and triangulate R as follows: *we divide R into non-degenerate triangular regions such that each of the points $\zeta_i(0)$ are used as vertices.* Thus the triangulation utilizes a set of vertices containing all the K points $\zeta_i(0)$, as well as some extra points ζ_s for some index set $s = K+1, \cdots, K+L$. (The four vertices of the rectangle R are certainly included amongst these last L vertices. Also note that each triangle utilized is, by requirement, non-degenerate – namely the vertices are always three non-collinear points.)

Now consider another copy of $\mathbf{CP}^1$ (which will serve as the range of the map ϕ_t) and divide the region inside the rectangle R in this second copy into triangular regions in the natural "corresponding" fashion, as detailed next: namely *the vertices of the triangles of this second copy of R consist of the new ramification points $\zeta_i(t)$'s* *in place of the* $\zeta_i(0)$, $1 \leq i \leq K$, *– together with the same extra set of points* ζ_s *(for index set* $s = K+1, \cdots, K+L$*) that were used before.* Note: these last L vertices are left undisturbed. Of course, the edges of the two triangulations correspond exactly since the vertices have the above correspondence. That is, if $(\zeta_i(0), \zeta_j(0), \zeta_k(0))$ form vertices of a triangle in the first copy then$(\zeta_i(t), \zeta_j(t), \zeta_k(t))$ form vertices of the corresponding triangle in the second copy; similarly, if $(\zeta_i(0), \zeta_p, \zeta_q)$ are vertices of a triangle in the first copy then $(\zeta_i(t), \zeta_p, \zeta_q)$ will be the vertices of the corresponding in the second copy, etc..

Remark: Since the initial triangulation is non-degenerate, namely the vertices of any triangle that was utilized were non-collinear, then, by continuity of the functions $\zeta_j(t)$, that non-degeneracy of the corresponding triangulation (on the range copy) remains valid for all small values of t near $t = 0$.

Affine mapping of one triangle onto another: If (z_1, z_2, z_3) are any three non-collinear points in the plane, then recall that their *closed convex hull*, (smallest closed convex set in the plane containing these points), is precisely the triangle T (includes the interior and the edges) with the given points as vertices. From elementary linear geometry one knows that *every point of* T *has a unique representation as a convex combination of the vertex vectors*; namely, each point of T is representable as $\lambda z_1 + \mu z_2 + \nu z_3$, where λ, μ and ν are real numbers in the closed unit interval $[0, 1]$ such that $\lambda + \mu + \nu = 1$.

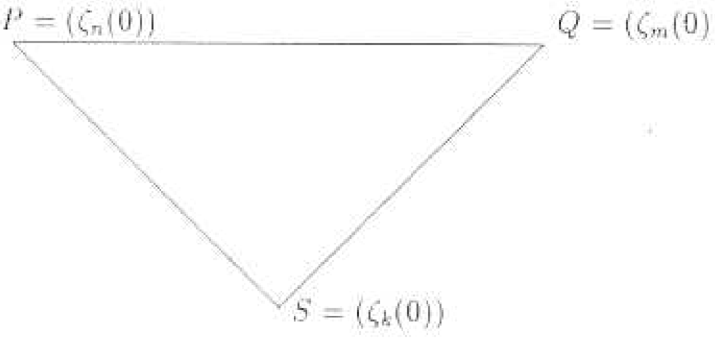
Clearly then, given any other set of three non-collinear vertices (w_1, w_2, w_3) for a second triangle T', there is a natural *affine mapping* of the first triangle onto the second which simply sends the point $\lambda z_1 + \mu z_2 + \nu z_3$ of T to the point $\lambda w_1 + \mu w_2 + \nu w_3$ of T'.

Definition of ϕ_t: We therefore define the desired homeomorphism ϕ_t inside the rectangle R by taking the triangles of the first triangulation, by the above affine mappings, onto the corresponding triangles of the second triangulation. Notice that if two triangles share a common edge, then the affine mappings defined on the two abutting triangles will coincide in their definition along the common edge. That is crucial. Consequently we clearly get a well defined homeomorphism ϕ_t of the rectangle R on itself, and outside R we simply extend ϕ_t by the identity map to the whole Riemann sphere.

It is clear that ϕ_t is a C^∞-diffeomorphism when restricted to the interiors of the triangles used in triangulating R, and also, of course, on the exterior of R.

We shall now show some explicit computations for these piecewise-affine homeomorphisms ϕ_t. Throughout this thesis we make some explicit computations of formulae as far as possible in order to facilitate later explicit work, and also in order to be able to program our variational results with no effort into a computer computational package to calculate the changing Fuchsian groups (which is our goal).

Let $\zeta_i(t) = (x_i(t), y_i(t))\ \forall i = 1, \ldots K$



Let $\zeta_m(0) = P, \zeta_n(0) = Q, \zeta_k(0) = S$ form one of the triangles in the triangulation of R in the domain plane. Then, as we said above, any point $z \in \triangle PQS$ can be written as follows (with $0 \leq \lambda, \mu \leq 1$):

$$\begin{aligned} x + iy = z &= \lambda\zeta_n(0) + \mu\zeta_m(0) + (1 - \lambda - \mu)\zeta_k(0) \\ &= \lambda(x_n(0), y_n(0)) + \mu(x_m(0), y_m(0)) + (1 - (\lambda + \mu))(x_k(0), y_k(0)) \\ \Longrightarrow x &= \lambda(x_n(0) - x_k(0)) + \mu(x_m(0) - x_k(0)) + x_k(0) \end{aligned}$$

$$
\begin{aligned}
\text{And } y &= \lambda(y_n(0) - y_k(0)) + \mu(y_m(0) - y_k(0)) + y_k(0) \\
x - x_k(0) &= \lambda(x_n(0) - x_k(0)) + \mu(x_m(0) - x_k(0)) \\
y - y_k(0) &= \lambda(y_n(0) - y_k(0)) + \mu(y_m(0) - y_k(0)) \\
\implies \lambda &= \frac{(x - x_k(0))(y_m(0) - y_k(0)) - (y - y_k(0))(x_m(0) - x_k(0))}{(x_n(0) - x_k(0))(y_m(0) - y_k(0)) - (y_n(0) - y_k(0))(x_m(0) - x_k(0))} \\
&= \frac{x(y_m(0) - y_k(0)) - y(x_m(0) - x_k(0)) - x_k(0)y_m(0) + y_k(0)x_m(0)}{y_m(0)(x_n(0) - x_k(0)) - y_n(0)(x_m(0) - x_k(0)) + y_k(0)(x_m(0) - x_n(0))} \\
\implies \mu &= \frac{(x - x_k(0))(y_n(0) - y_k(0)) - (y - y_k(0))(x_n(0) - x_k(0))}{(x_m(0) - x_k(0))(y_n(0) - y_k(0)) - (y_m(0) - y_k(0))(x_n(0) - x_k(0))} \\
&= \frac{x(y_n(0) - y_k(0)) - y(x_n(0) - x_k(0)) - x_k(0)y_n(0) + y_k(0)x_n(0)}{y_n(0)(x_m(0) - x_k(0)) - y_m(0)(x_n(0) - x_k(0)) + y_k(0)(x_n(0) - x_m(0))} \\
\text{Define } c(t) &= (x_m(t) - x_k(t))(y_n(t) - y_k(t)) - (y_m(t) - y_k(t))(x_n(t) - x_k(t)) \\
u + iv = w &= \phi_t(z) \\
&= \lambda(x_n(t), y_n(t)) + \mu(x_m(t), y_m(t)) + (1 - (\lambda + \mu))(x_k(t), y_k(t)) \\
\implies u &= \lambda(x_n(t) - x_k(t)) + \mu(x_m(t) - x_k(t)) + x_k(t) \\
\text{and } v &= \lambda(y_n(t) - y_k(t)) + \mu(y_m(t) - y_k(t)) + y_k(t) \\
u &= \lambda(x_n(t) - x_k(t)) + \mu(x_m(t) - x_k(t)) + x_k(t) \\
&= \frac{1}{c}[-x(y_m(0) - y_k(0)) + y(x_m(0) - x_k(0)) + x_k(0)y_m(0) \\
&- y_k(0)x_m(0)](x_n(t) - x_k(t)) + \frac{1}{c}[x(y_n(0) - y_k(0)) - y(x_n(0) \\
&- x_k(0)) - x_k(0)y_n(0) + y_k(0)x_n(0)](x_m(t) - x_k(t)) + x_k(t) \\
v &= \lambda(y_n(t) - y_k(t)) + \mu(y_m(t) - y_k(t)) + y_k(t) \\
&= \frac{1}{c}[-x(y_m(0) - y_k(0)) + y(x_m(0) - x_k(0)) + x_k(0)y_m(0) \\
&- y_k(0)x_m(0)](y_n(t) - y_k(t)) + \frac{1}{c}[x(y_n(0) - y_k(0)) - y(x_n(0) \\
&- x_k(0)) - x_k(0)y_n(0) + x_n(0)y_k(0)](y_m(t) - y_k(t)) + y_k(t)
\end{aligned}
$$

Thus we have now shown the explicit type of formulae ("piecewise") for the piecewise-affine mappings ϕ_t.

Lemma 1: Each homeomorphism ϕ_t is orientation preserving.

Proof: We may do this by explicit computation of the Jacobian determinant of the mapping. We observe that:

$$\begin{aligned}
u_x(x,y) &= \frac{1}{c}[-(y_m(0)-y_k(0))(x_n(t)-x_k(t))+(y_n(0)-y_k(0))(x_m(t)-x_k(t))] \\
u_y(x,y) &= \frac{1}{c}[(x_m(0)-x_k(0))(x_n(t)-x_k(t))-(x_n(0)-x_k(0))(x_m(t)-x_k(t))] \\
v_x(x,y) &= \frac{1}{c}[(y_n(0)-y_k(0))(y_m(t)-y_k(t))-(y_m(0)-y_k(0))(y_n(t)-y_k(t))] \\
v_y(x,y) &= \frac{1}{c}[(y_n(t)-y_k(t))(x_m(0)-x_k(0))-(x_n(0))-x_k(0))(y_m(t)-y_k(t))] \\
u_xv_y &= \frac{1}{c^2}[(x_m(t)-x_k(t))(x_m(0)-x_k(0))(y_n(t)-y_k(t))(y_n(0)-y_k(0)) \\
&- (x_m(0)-x_k(0))(x_n(t)-x_k(t))(y_m(0)-y_k(0))(y_n(t)-y_k(t)) \\
&- (x_n(0)-x_k(0))(x_m(t)-x_k(t))(y_n(0)-y_k(0))(y_m(t)-y_k(t)) \\
&+ (x_n(0)-x_k(0))(x_n(t)-x_k(t))(y_m(0)-y_k(0))(y_m(t)-y_k(t))] \\
u_yv_x &= \frac{1}{c^2}[(x_m(0)-x_k(0))(x_n(t)-x_k(t))(y_m(t)-y_k(t))(y_n(0)-y_k(0)) \\
&- (x_n(0)-x_k(0))(x_m(t)-x_k(t))(y_n(0)-y_k(0))(y_m(t)-y_k(t)) \\
&- (x_m(0)-x_k(0))(x_n(t)-x_k(t))(y_m(0)-y_k(0))(y_n(t)-y_k(t)) \\
&+ (x_n(0)-x_k(0))(x_m(t)-x_k(t))(y_m(0)-y_k(0))(y_n(t)-y_k(t))] \\
u_xv_y-u_yv_x &= \frac{1}{c^2}[(x_m(0)-x_k(0))(y_n(0)-y_k(0))\{(x_m(t)-x_k(t))(y_n(t)-y_k(t)) \\
&- (y_m(t)-y_k(t))(x_n(t)-x_k(t))\}-(x_n(0)-x_k(0))(y_m(0)-y_k(0))\{(x_m(t) \\
&- x_k(t))(y_n(t)-y_k(t))-(x_n(t)-x_k(t))(y_m(t)-y_k(t))\}] \\
&= \frac{1}{c^2}\{(x_m(0)-x_k(0))(y_n(0)-y_k(0))-(x_n(0)-x_k(0))(y_m(0)-y_k(0))\} \\
&\quad \{(x_m(t)-x_k(t))(y_n(t)-y_k(t))-(x_n(t)-x_k(t))(y_m(t)-y_k(t))\} \\
&> 0
\end{aligned}$$

This last inequality follows since the expressions: $(x_m(t)-x_k(t))(y_n(t)-y_k(t))-(y_m(t)-y_k(t))(x_n(t)-x_k(t))$ and $(x_m(0)-x_k(0)(y_n(0)-y_k(0))-(y_m(0)-y_k(0))(x_n(0)-$

$x_k(0)$) have the same sign for t small enough.

Thus the piecewise affine mappings that fit together to produce ϕ_t are all orientation preserving, and clearly then ϕ_t is an orientation preserving homeomorphism of the Riemann sphere to itself, with the desired action at the ramification points. Hence the proof of the lemma follows. □

Lemma 2: ϕ_t is quasiconformal for each t in the ϵ disc. The Beltrami coefficient of ϕ_t, is a complex constant (of modulus less than unity) when restricted to the interior of each triangle in the initial triangulation of the rectangle R. Of course, the Beltrami coefficient is identically zero in the exterior of R.

Proof: Indeed, the complex dilatation of any mapping that is affine (on a region) is obviously a constant on that region. Since the map restricted to each triangle was seen to be orientation preserving, that complex dilatation constant, for each triangle, is necessarily less than 1 in absolute value.

We note here the following useful algebraic formula for the modulus square of any Beltrami coefficient (for any map $w = \phi(z)$):

$$\begin{aligned}
\left|\frac{\phi_{\bar z}}{\phi_z}\right|^2 &= \left|\frac{\frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})(u+iv)}{\frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(u+iv)}\right|^2 \\
&= \left|\frac{(u_x - v_y) + i(u_y + v_x)}{(u_x + v_y) + i(v_x - u_y)}\right|^2 \\
&= \frac{(u_x - v_y)^2 + (u_y + v_x)^2}{(u_x + v_y)^2 + (v_x - u_y)^2} \\
&= \frac{{u_x}^2 + {u_y}^2 + {v_x}^2 + {v_y}^2 - 2(u_x v_y - u_y v_x)}{{u_x}^2 + {u_y}^2 + {v_x}^2 + {v_y}^2 + 2(u_x v_y - u_y v_x)}
\end{aligned}$$

We typically compute over $z \in \triangle PQS$ (notation as before), setting $w = \phi_t(z)$, and we get:

$$\begin{aligned}
\left|\frac{\phi_{t,\bar z}}{\phi_{t,z}}\right|^2 &= \text{constant [since, by affine-ness, } u_x, u_y, v_x, v_y \text{ depend only on } t \text{ and not} \\
&\qquad \text{on } z = x + iy] \\
&< 1 \text{ as } (u_x v_y - u_y v_x > 0), \text{ over } \triangle PQS
\end{aligned}$$

Outside the rectangle R, the mapping is the identity – hence $\frac{\phi_{t,\bar{z}}}{\phi_{t,z}} = 0$ outside R. Since there are only a finite number of decomposing triangles inside R, and the above computation applies to any of them, the map ϕ_t is seen to be a quasiconformal homeomorphism (for each $t \in \triangle_\epsilon$) with the properties stated.This completes the proof. □

3.3 Lifting of $\phi_t : \mathbf{CP}^1 \longrightarrow \mathbf{CP}^1$ to $\tilde{\phi}_t : X_0 \longrightarrow X_t$:

Consider the following diagram of Riemann surfaces with the vertical arrows being, as we know, holomorphic branched coverings:

$$\begin{array}{ccc} X_0 & \xrightarrow{\tilde{\phi}_t} & X_t \\ \Big\downarrow{\scriptstyle x} & & \Big\downarrow{\scriptstyle x_t} \\ \mathbf{CP}^1 & \xrightarrow{\phi_t} & \mathbf{CP}^1 \end{array}$$

Proposition : There exists a quasiconformal, orientation preserving homeomorphism:

$$\tilde{\phi}_t : X_0 \longrightarrow X_t$$

lifting the map $\phi_t : \mathbf{CP}^1 \to \mathbf{CP}^1$ and making the above diagram commute. (Note that $\tilde{\phi}_0$ is the identity.)

Proof: In fact, in order to deal with unbranched covering spaces, we define the following punctured Riemann surfaces:

$$X_0' = x^{-1}\{\mathbf{CP}^1 - \textit{ all critical values of } x\}$$

and

$$X_t' = x_t^{-1}\{\mathbf{CP}^1 - \textit{ all critical values of } x_t\}$$

Restricted to $X_0^/$ and $X_t^/$, the vertical mappings are now smooth (=unbranched) covering projections. Observe that the ϕ_t was designed so as to map the critical values of x onto those of x_t. Now we can apply the standard lifting criterion for maps from the theory of covering spaces to demonstrate that ϕ_t lifts. Consequently, at the level of fundamental groups we need to look at the image of the action on π_1 of $(\phi_t \circ x)$ as compared with that of x_t. (See, for instance, Theorem 5.1, pg 128, of Massey[M] for the statement of the usual lifting criterion.)

Since the monodromy permutation at any critical point say $\zeta_m(0)$ is the same as that around the perturbed critical point $\zeta_m(t)$, and since $\phi_t(\zeta_m(0)) = \zeta_m(t)$, we see that:

$$\pi_1(\phi_t \circ x)\pi_1(X_0^/, w_0) = \pi_1(x_t)\pi_1(X_t^/, \beta_0)$$

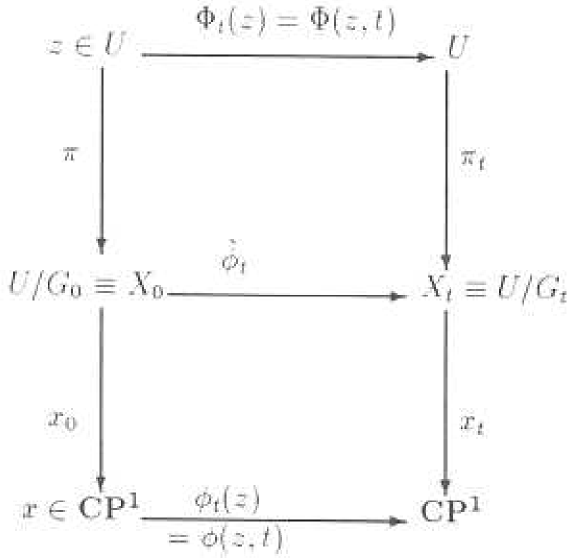
(where $w_0 \in X_0^/$ and $x(w_0) = z_0$ and $\beta_0 \in X_t^/$ such that $x_t(\beta_0) = \phi_t(z_0)$).

Clearly then the lifting criterion is satisfied, and hence the homeomorphism ϕ_t lifts to a homeomorphism $\tilde{\phi}_t$, as desired. Certainly the lift is quasiconformal since the vertical mappings are holomorphic. This completes the proof of the proposition.

□

Finally then, for our applications to the variation of Fuchsian groups we may *lift all the way to the universal covering upper half-planes* and obtain the quasiconformal homeomorphism $\Phi_t(z) = \Phi(z,t)$ from U to U, obtained by lifting the mapping to $\tilde{\phi}_t : X_0 \longrightarrow X_t$.

Thus we have determined $\Phi_t(z)$ so that the following diagram commutes:


$\Phi_t(z) = \Phi(z, t)$
$z \in U$
U
π
π_t
$\dot{\phi}_t$
$U/G_0 \equiv X_0$
$X_t \equiv U/G_t$
x_0
x_t
$x \in \mathbf{CP}^1$
$\phi_t(z)$
$= \phi(z, t)$
$\mathbf{CP}^1$

Chapter 4

Variational formulae for the Fuchsian groups of varying curve

4.1 The fundamental variational term:

Let $\mu_t(z)$ denote a one-parameter family of Beltrami coefficients on the upper half-plane depending real or complex analytically on the (real or complex) parameter t near $t = 0$. Suppose also that $\mu_0(z) \equiv 0$. Thus the situation is set up to be the following:

$$\mu_t(z) = \mu(z, t); \quad z \in U, \quad t \in \{|t| < \epsilon\} \tag{4.1}$$

We assume that each μ_t belongs to $L^\infty(U)_1$, (which denotes the L^∞ open unit ball of essentially bounded complex measurable functions on U).

For our context – arising from families of algebraic curves whose coefficients were assumed to depend holomorphically on t – we may as well assume that the map $t \mapsto \mu_t$ of the t-domain into $L^\infty(U)_1$ is complex-analytic (i.e., this map is holomorphic as a map into a complex Banach space).

By the classic *Ahlfors-Bers generalized Riemann mapping theorem*, we recall that *for every given $\mu(z)$ in $L^\infty(U)_1$, there is a quasiconformal homeomorphism, $w = w_\mu$: $U \to U$, whose complex dilatation ($\equiv$ Beltrami coefficient $= w_{\bar{z}}/w_z$), is equal to μ a.e. on U.*

Moreover, this quasiconformal self-homeomorphism of U is uniquely determined by μ up to post-composition by an arbitrary Möbius transformation (from $PSL(2, R) =$

$HolAut(U)$) that preserves U. Remember also that every quasiconformal homeomorphism of U extends continuously to the boundary of U (in the Riemann sphere) as a quasisymmetric homeomorphism of ∂U. Thus we may uniquely normalize the solution w_μ so that it fixes each of the following three boundary points of U: $\{0, 1, \infty\}$. For this material one may see the books Ahlfors[A], Lehto-Virtanen[LV].

Remark: It is important to emphasize that even if μ varies holomorphically with parameters t, the corresponding quasiconformal self-mapping of the upper half-plane U can only vary real analytically. Indeed, then $w_{\mu_t}(z)$ is a *real analytic* function of t, for every fixed $z \in U$. See 1.2.14 in Nag[N] for this matter. The fact that the variation is only real analytic is fundamentally relevant to our work here - since we are going to deal with the variation of the Fuchsian group (equivalently, the Fuchsian projective structure), for the varying Riemann surface.

The perturbation formula for quasiconformal mappings: We come now to the main formula that we shall apply. If $\mu_0 \equiv 0$, and if for small t the Beltrami coefficient is given by:

$$\mu_t(z) = t\hat{\nu}(z) + o(t), \ \textit{where} \ \hat{\nu} \in L^\infty(U), \tag{4.2}$$

then one has an important integral formula expressing the solutions of the family of Beltrami equations, as a perturbation of the identity homeomorphism:

$$w_{\mu_t}(z) = z + tw_1(z) + o(t), z \in U.$$

Indeed, the crucial first variation term, $w_1 = \dot{w}$, for *real* t is given by:

$$w_1(z) = -\frac{1}{\pi}\int\int_U [\hat{\nu}(\zeta)R(\zeta, z) + \overline{\hat{\nu}(\zeta)}R(\zeta, z)]d\xi d\eta$$

$$R(\zeta, z) = \frac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)}, \ \textit{and} \ \zeta = \xi + i\eta$$

This perturbation formula, (see Ahlfors[A], or Section 1.2.13, 1.2.14, as well as page 175, equation (1.21) of Nag[N]), will be fundamental for us. We shall apply it to the family of quasiconformal mappings Φ_t (Chapter III) standing for the family w_{μ_t}.

Since in our set up t is a *complex* parameter we may as well deduce the form of the variational terms for general t complex – which follows by simply applying the real t formula above appropriately. We show this:

If t is complex write in polar form: $t = |t|e^{i\alpha}$ then put $\tau = e^{-i\alpha}t = |t|$. Then we may apply the real parameter formula to the τ variable. In fact, set:

$$\tilde{\nu}(z) = e^{i\alpha}\hat{\nu}(z)$$

Therefore:

$$\begin{aligned}
w_{\mu_t}(z) &= z + \tau\tilde{w}_1(z) + o(\tau) \\
&= z + e^{-i\alpha}t\tilde{w}_1(z) + o(t) \\
&= z + tw_1(z) + o(t) \\
w_1(z) &= e^{-i\alpha}\tilde{w}_1(z) \\
\tilde{w}_1(z) &= -\frac{1}{\pi}\int\int_U [\tilde{\nu}(\zeta)R(\zeta, z) + \overline{\tilde{\nu}(\zeta)}R(\bar{\zeta}, z)]d\xi d\eta \\
w_1(z) &= -\frac{1}{\pi}\int\int_U [e^{-i\alpha}\tilde{\nu}(\zeta)R(\zeta, z) + e^{-i\alpha}\overline{\tilde{\nu}(\zeta)}R(\bar{\zeta}, z)]d\xi d\eta \\
&= -\frac{1}{\pi}\int\int_U [\hat{\nu}(\zeta)R(\zeta, z) + e^{-2i\alpha}\overline{\hat{\nu}(\zeta)}R(\bar{\zeta}, z)]d\xi d\eta
\end{aligned}$$

where $\alpha = arg(t)$. But $te^{-2i\alpha}$ is the conjugate of t. Therefore, this last formula says that for complex t we have the final important formulae:

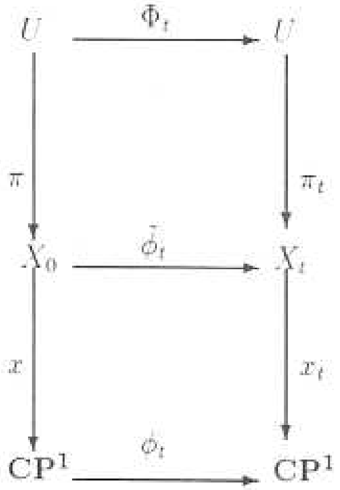
$$w_{\mu_t}(z) = z + tw_1(z) + \bar{t}w_1^*(z) + o(t), \quad z \in U \tag{4.3}$$

where:

$$(w_1(z), w_1^*(z)) = (-\frac{1}{\pi}\int\int_U [\hat{\nu}(\zeta)R(\zeta, z)]d\xi d\eta, -\frac{1}{\pi}\int\int_U [\overline{\hat{\nu}(\zeta)}R(\bar{\zeta}, z)]d\xi d\eta) \tag{4.4}$$

Equation (4.4) will be manipulated to produce the chief formulae of Chapter IV.

Recall then the following *main commutative diagram* lifting successively the ϕ_t, first to the compact surfaces, and then to the respective universal coverings:



Let $\Gamma \equiv G_0 \subset PSL(2, \mathbf{R})$ denote the uniformizing Fuchsian group acting as deck transformations for the covering π. Then there is a biholomorphic equivalence:

$$X_0 = U/G_0 \tag{4.5}$$

It follows from the standard Ahlfors-Bers deformation theory of Fuchsian groups (see Nag[N]) that the quasiconformal homeomorphism Φ_t is *compatible with the Fuchsian group* G_0, in the sense that $g_t = \Phi_t \circ g \circ \Phi_t^{-1}$ is again a Möbius transformation in $PSL(2, \mathbf{R})$ for every $g \in G_0$, and the new Fuchsian group (which evidently remains abstractly isomorphic to G_0) is the Fuchsian group:

$$G_t = \Phi_t \circ G_0 \circ \Phi_t^{-1} \tag{4.6}$$

This is the group of deck transformations for the covering π_t, so that X_t is biholomorphically equivalent to U/G_t. We shall write

$$g_t = \Phi_t \circ g \circ \Phi_t^{-1} \in G_t \tag{4.7}$$

for any fixed $g \in G_0 \equiv \Gamma$.

In this notation, the central problem of our work is to determine explicit and applicable formulae for the variation of g_t *– or, equivalently, to compute the* t*-derivative:*

$\dot{g}_t$ *at* $t = 0$. *As* g *varies over any generating set of elements for the group* G_0, *we shall then obtain, up to first order approximation, a corresponding set of generating elements for the deformed groups* G_t.

We shall naturally assume that all the information regarding the case for $X_0 = U/G_0$ – namely knowledge concerning the left vertical edge in the above commuting diagram – is given to us.

4.2 The Beltrami coefficient μ_t of Φ_t:

Notational set up: Let us, for notational convenience, denote as x_* the meromorphic function on U given by $x \circ \pi$. (this is, of course, a holomorphic branched covering of the Riemann sphere by the upper half plane). Clearly, x_* is automorphic with respect to the Fuchsian group Γ, since x_* descends onto the surface X_0 as the meromorphic function x thereon. *In particular, let us note the well-known fact that this function,* x_*, *can be expressed in terms of the standard Poincare theta-series on* U *with respect to the group* Γ. See Remarks in IV.5 below.

Now recall from the previous Chapter that the mapping ϕ_t was, by our very definition, a piecewise affine quasiconformal mapping. So the Beltrami coefficient of ϕ_t was a complex constant on each triangle of the triangulation of the domain rectangle R. (The Beltrami coefficient need only be specified almost everywhere – therefore we will ignore it on the edges and vertices of the triangulation.)

Moreover we know that the vertices of the triangulation (in the image plane) depend holomorphically on t – since the ramification points $\zeta_j(t)$ were holomorphic functions of t. Here is the main proposition we require:

Proposition : The Beltrami coefficient of Φ_t is:

$$\mu_t(z) = t\hat{\nu}(z) + o(t), z \in U, \hat{\nu}(z) \in L^\infty(U).$$

where:

$$\hat{\nu}(z) = \nu(w)\frac{\overline{(x \circ \pi)'(z)}}{(x \circ \pi)'(z)}, \textit{ where } w = (x \circ \pi)(z) = x_*(z) \in \mathbf{CP}^1 \tag{4.8}$$

Here the Beltrami coefficient for the piecewise-affine mappings ϕ_t on the Riemann w-sphere has been expanded up to first order in t as below:

$$\frac{\phi_{t,\bar{w}}(w)}{\phi_{t,w}(w)} = t\nu(w) + o(t), \nu \in L^\infty(\mathbf{CP}^1) \tag{4.9}$$

Further note that $\nu(w)$ is a constant on each triangle of the first (domain) triangulation of R, and it is zero for all w outside R.

Note: The Γ invariant Beltrami coefficient $\hat{\nu}$ above, represents the *tangent vector* to the one parameter family of Beltrami coefficients μ_t which arise from the one parameter family of quasiconformal mappings Φ_t.

Proof: From the above commutative diagram for the liftings we have:

$$(x_t \circ \pi_t) \circ \Phi_t = \phi_t \circ (x \circ \pi) \tag{4.10}$$

Taking the ∂ and $\bar{\partial}$ derivatives in (4.10), and remembering that all the vertical maps are *holomorphic* coverings (possibly branched as we know), we obtain the Beltrami coefficient of Φ_t on U:

$$\mu_t(z) \equiv \frac{\Phi_{t,\bar{z}}}{\Phi_{t,z}} = \frac{\phi_{t,\bar{w}}(w)}{\phi_{t,w}(w)}\frac{\overline{(x \circ \pi)'(z)}}{(x \circ \pi)'(z)}, \; z \in U, \tag{4.11}$$

Clearly then the statements in the Proposition follow because the $\phi_t(w)$ are a family of piecewise affine quasiconformal homeomorphisms on the w-sphere which vary holomorphically in t. Thus, remembering that ϕ_0 is the identity, we see that the Beltrami coefficients of the family ϕ_t indeed must have an expression as in (4.9) with $\nu(w)$ being piecewise-constant. □

Remark on the holomorphic dependence on t of the Beltrami coefficient of ϕ_t: Our claim is that the Beltrami coefficient of ϕ_t is a *holomorphic* function of the parameter

t in the neighbourhood of $t = 0$. Note that the map

$$t \mapsto \frac{\phi_{t,\bar{z}}}{\phi_{t,z}}$$

takes values in the complex Banach space $L^\infty(\mathbf{CP}^1)$, – and the holomorphy is as a map into this Banach space. (In what follows, it is well to remember that L^∞ Beltrami differentials need only be defined almost everywhere.)

Now, the Beltrami coefficient of ϕ_0 is identically zero because that quasiconformal map is the identity mapping; consequently, we will have the following basic first order expansion of these Beltrami coefficients in the t parameter:

$$\frac{\phi_{t,\bar{z}}}{\phi_{t,z}} = t\nu(z) + o(t)$$

By the known piecewise-affine structure of each of the maps ϕ_t we are further guaranteed that ν *is a piecewise constant function on the Riemann sphere*, – a complex constant on (the interior of) each triangle of the triangulation of (the "domain copy of") the rectangle R, and is zero outside R.

The above assertions will clearly all follow from the fundamental observation that each of the mappings

$$t \mapsto \phi_t(z)$$

are *holomorphic* functions of t, for each fixed z in $\mathbf{CP}^1$. That follows easily since each of the ramification points $\zeta_j(t)$, $j = 1, \cdots, K$, is a holomorphic function of t. Indeed, if z lies, say, in a typical triangle of the triangulation of R, with vertices of the triangle being at : $P = \zeta_m(0), Q = \zeta_n(0), S = \zeta_k(0)$, as before, and if the real convex combination coefficients determining the location of z are given so that: $z = \lambda P + \mu Q + \nu S$, $\lambda + \mu + \nu = 1$ then:

$$\phi_t(z) = \lambda\zeta_m(t) + \mu\zeta_n(t) + \nu\zeta_k(t)$$

which is of course holomorphic in t. If the triangle (or edge) containing z has vertices that are among the extra ones that do not move with t, then clearly also the assertion of holomorphy for $\phi_t(z)$ remains valid.

Beltrami coefficients automorphic with respect to Γ: We must remember from the general theory (see Section 1.3.3 of Nag[N]) one further fundamental fact: Since the quasiconformal maps Φ_t are compatible with Γ their Beltrami coefficients are $(-1,1)$ forms on U w.r.t. Γ. (We called them Γ- invariant Beltrami coefficients.)

Indeed, if μ is the complex dilatation of a quasiconformal mapping that conjugates Γ into any group of Möbius transformations, then

$$(\mu \circ g)(\bar{g'}/g') = \mu, a.e., \ for \ all \ g \in \Gamma \tag{4.12}$$

We denote the Banach space of complex valued L^∞ functions on U that satisfy equation (4.12) for every $g \in \Gamma$, by the notation: $L^\infty(U, \Gamma)$. See page 49 of [N]. *Thus, μ_t belongs to the open unit ball of this Banach space for all small t , and also therefore $\dot{\nu}$ belongs to this Banach space of automorphic objects.*

4.3 The variational formula for Φ_t:

We come to the chief application of the perturbation formula (equation (4.4)) in our specific context of varying algebraic curves.

Let F denote a closed fundamental domain, with boundary of two-dimensional measure zero, for the action of Γ on U; (for instance, we may choose as F any standard Dirichlet fundamental polygon for the Fuchsian group Γ). Thus π maps F onto X_0, and π is one-to-one when restricted to the interior of F.

Recall that x was itself a meromorphic function of degree N on the compact Riemann surface X_0, (see Chapters II and III). Consequently, when restricted to the interior of F the mapping x_* is a N-to-1 branched holomorphic covering map onto the Riemann sphere – missing only a set of areal measure zero. Since this is a finite covering space situation (aside from a measure zero set of branch points which we may discard to start with), we may choose a decomposition of F into N regions:

$$F = D_1 \cup D_2 \cup \cdots \cup D_N \tag{4.13}$$

here the D_j are mutually disjoint domains, (except for boundary contact, as usual in choice of fundamental regions), partitioning F, with the basic property that each D_j maps, via x_*, in a one-to-one fashion onto the entire Riemann sphere (missing atmost a measure zero subset). (Recall that the compact Riemann surface X_0 was described as an N-sheeted branched cover of the sphere – by the degree N meromorphic function x.)

A Kernel function associated to Γ*:* We introduce as an useful matter of notation, the following function of two variables: $z \in U$, $\tau \in \mathbf{C}$ (not lying on the Γ orbit of z):

$$K_\Gamma(z,\tau) = \sum_{g\in\Gamma} \frac{[g'(z)]^2}{g(z)(g(z)-1)(g(z)-\tau)}. \tag{4.14}$$

We are now in a position to state a main result:

<u>**Theorem: On Variation of Φ_t:**</u> *The lifted quasiconformal maps Φ_t on U satisfy the following first order expansion for small t:*

$$\Phi_t(z) = z + t w_1(z) + \bar{t} w_1^*(z) + o(t), z \in U \tag{4.15}$$

where:

$$w_1(z) = \frac{z(z-1)}{2\pi\sqrt{-1}} \sum_{k=1}^{N} \int\int_{\mathbf{CP1}} \{\nu(w) K_\Gamma(x_{*,k}^{-1}(w), z)[\frac{\partial x_{*,k}^{-1}}{\partial w}(w)]^2\} dw \wedge d\bar{w}$$

$$w_1^*(z) = \frac{z(z-1)}{2\pi\sqrt{-1}} \sum_{k=1}^{N} \int\int_{\mathbf{CP1}} \{\overline{\nu(w)} K_\Gamma(\overline{x_{*,k}^{-1}(w)}, z)[\overline{\frac{\partial x_{*,k}^{-1}}{\partial w}(w)}]^2\} dw \wedge d\bar{w}$$

Here we have denoted by $x_{*,k}$ the restriction of the projection $x_* = x \circ \pi$ (which is a meromorphic and Γ-automorphic function on U), to the region $D_k \subset F$, $k =$

$1, \cdots, N$. Here ν denotes the function on the w-sphere appearing in formula (4.9) of the Proposition in Section IV.2 above. (Recall that ν is simply a constant assigned on each triangle in the triangulation of R, with ν being identically zero outside R.)

Note furthermore, that since x_* is a meromorphic function on U, we may replace in the above formula the derivative of its inverse by the reciprocal of its own derivative, as shown below:

$$\frac{\partial x_{*,k}^{-1}}{\partial w}(w) = 1/\frac{dx_{*,k}}{dz}(z), \; w = x_*(z), \; z \in D_k;$$

These derivatives can therefore be calculated from the expression for x_* which will be available in terms of the standard Poincare theta series on U with respect to Γ.

During the course of the proof we shall show that all integrals and summations appearing in sight are absolutely convergent. For facts regarding Poincare theta series and their utilization in expressing meromorphic functions on U/Γ, see Kra[Kr], and the Remarks in IV.4 after the proof.

<u>Proof:</u> We shall have to manipulate the variational formula (4.4) which said:

$$w_1(z) = \frac{1}{2i\pi}\int\int_U [\hat{\nu}(w)R(w,z) + \overline{\hat{\nu}(w)}R(\bar{w},z)]dw \wedge d\bar{w}$$

with $R(w,z) = \frac{z(z-1)}{w(w-1)(w-z)}$.

By general theory quoted above, the integrals involved in (4.4) are necessarily absolutely convergent. (Indeed, it is sufficient to observe that:

$$-\tau(\tau-1)\int\int_U \frac{\hat{\nu}(z)}{z(z-1)(z-\tau)}dxdy$$
$$= -\frac{1}{\pi}\int\int_U \frac{\hat{\nu}(z)}{(z-\tau)}dxdy \;\; + \;\; (\tau-1)\int\int_U \frac{\hat{\nu}(z)}{z}dxdy - \tau\int\int_U \frac{\hat{\nu}(z)}{(z-1)}dxdy$$

But $\int\int_U \frac{\hat{\nu}(z)}{(z-\tau)}dxdy$, $\int\int_U \frac{\hat{\nu}(z)}{z}dxdy$, and $\int\int_U \frac{\hat{\nu}(z)}{(z-1)}dxdy$ are each absolutely convergent (see that by passing to polar coordinates, for example). We have therefore proved that the required type of integral over U, namely: $\int\int_U \frac{\hat{\nu}(z)}{z(z-1)(z-\tau)}dxdy$ is indeed absolutely convergent, as desired.)

To obtain the final result for w_1 and w_1^*, there are several chief ideas which we first explain in words:

i.) Write each of the two integrals over U as a sum of integrals over all the tiles in the Γ-tessellation of U – obtained by decomposing U as the union of the fundamental domain F and its translates: i.e., $U = \cup_{g\in\Gamma}(g(F))$.

ii.) Utilizing the Γ-automorphic nature of the Beltrami coefficient $\hat{\nu}$ (see equation (4.12) above), and making a change of variables by $w = g(z)$, we can transform the integral over $g(F)$ to an integral again over F itself.

iii.) Consequently, the original expression for w_1 becomes simply an integration over F of a certain expression on F, after interchanging summation and integration. (The validity of the interchange is guaranteed by the absolute convergence of the result, together with the dominated convergence theorem. The main details of this critical interchange of sum and integral are spelled out in the remarks attached at the end of the proof.)

iv.) Finally we decompose F itself into the N pieces $D_1, ..., D_N$ (as explained with equation (4.13) above) – and hence we may eliminate $\hat{\nu}$ by replacing it with occurrences of ν itself, and thus express the final result as integrations over the Riemann sphere $\mathbf{CP}^1$, as desired.

Let us now get down to the main business of showing the exact nature of how these transformations come about in the expression for w_1. First of all note:

$$\int\int_U \frac{\hat{\nu}(w)}{w(w-1)(w-\tau)}dw\wedge d\bar{w}$$

$$= \sum_{g\in\Gamma}\int\int_{g(F)} \frac{\hat{\nu}(w)}{w(w-1)(w-\tau)}dw\wedge d\bar{w}, \qquad F = \text{fundamental region of } \Gamma \text{ in } U$$

Perform a change of variables on $g(F)$ by $w = u + iv = g(z)$

$$= \sum_{g\in\Gamma} \int\int_F \frac{\hat{\nu}(z)\frac{g'(z)}{\overline{g'(z)}} \mid g'(z) \mid^2}{g(z)(g(z)-1)(g(z)-\tau)} dz \wedge d\bar{z}$$
$$= \sum_{g\in\Gamma} \int\int_F \frac{\hat{\nu}(z)[g'(z)]^2}{g(z)(g(z)-1)(g(z)-\tau)} dz \wedge d\bar{z}$$

For convergence arguments we note that

$$\text{since } \sum_{g\in\Gamma} \int\int_F \frac{\mid \hat{\nu}(z) \mid \frac{|g'(z)|}{|\overline{g'(z)}|} \mid g'(z) \mid^2}{\mid g(z) \mid\mid g(z)-1 \mid\mid g(z)-\tau \mid} dxdy$$
$$= \sum_{g\in\Gamma} \int\int_{g(F)} \frac{\mid \hat{\nu}(w) \mid}{\mid w \mid\mid w-1 \mid\mid w-\tau \mid} dudv < \infty$$

this demonstrates that the series:

$$\sum_{g\in\Gamma} \int\int_F \frac{\hat{\nu}(z)[g'(z)]^2}{g(z)(g(z)-1)(g(z)-\tau)} dz \wedge d\bar{z} = \sum_{g\in\Gamma} \int\int_F \psi_g(z) dz \wedge d\bar{z} \qquad (4.16)$$

is absolutely convergent. Note that, for convenience, we have written ψ_g here for the following frequently recurring expression:

$$\psi_g(z) = \frac{\hat{\nu}(z)[g'(z)]^2}{g(z)(g(z)-1)(g(z)-\tau)}$$

We shall show by a measure-theoretic Lemma in the remarks appended to the bottom of this proof, that we are allowed to change summation and integration in the summation (4.16). We shall utilize crucially this interchange immediately in what follows. Returning therefore to the actual expression for the variational term w_1, we now obtain:

$$w_1(z) \quad = \quad -\frac{1}{2i\pi} \int\int_U \hat{\nu}(\zeta) R(\zeta, z) d\zeta \wedge d\bar{\zeta}$$

$$= \frac{z(z-1)}{2\pi i} \int\int_U \frac{\hat{\nu}(\zeta)}{\zeta(\zeta-1)(\zeta-z)} d\zeta \wedge d\bar{\zeta}$$

$$= \frac{z(z-1)}{2\pi i} \quad \sum_{g\in\Gamma} \int\int_F \frac{\hat{\nu}(\zeta)[g'(\zeta)]^2}{[g(\zeta)][g(\zeta)-1][g(\zeta)-z]} d\zeta \wedge d\bar{\zeta}$$

$$= \frac{z(z-1)}{2\pi i} \quad \int\int_F \quad \sum_{g\in\Gamma} \frac{\hat{\nu}(\zeta)[g'(\zeta)]^2}{g(\zeta)(g(\zeta)-1)(g(\zeta)-z)} d\zeta \wedge d\bar{\zeta}$$

$$= \frac{z(z-1)}{2\pi i} \quad \int\int_F \quad \hat{\nu}(\zeta) K_\Gamma(\zeta, z) d\zeta \wedge d\bar{\zeta}$$

Similarly

$$w_1^*(z) = \frac{z(z-1)}{2\pi i} \int\int_F \overline{\hat{\nu}(\zeta)} K_\Gamma(\bar{\zeta}, z) d\zeta \wedge d\bar{\zeta}$$

That completes the manipulation of the formula to a point that already has points of interest; we have carried out steps i, ii, iii, - and now we are integrating over F (i.e., over X_0), rather than over U.

The final steps are for carrying out the program outlined in point number (iv) above. This goes as detailed below:

Let

$$(x \circ \pi)(D_i) = \mathbf{CP^1}, \; and \; denote \; x \circ \pi|_{D_i} = x_{*,i}$$

for each $i = 1, \cdots, N$. Setting $(x \circ \pi)(\zeta) = w$, $\zeta \in U$ and $w \in \mathbf{CP^1}$, and using the relation (equation (4.8)) between $\hat{\nu}$ and ν, we will have:

$$w_1(z) \quad =$$

$$\frac{z(z-1)}{2\pi i} \quad \sum_{i=1}^{N} \quad \int\int_{\mathbf{CP^1}} [\sum_{g\in\Gamma} (\frac{\nu(w) \frac{\overline{[\frac{dw}{d\zeta}]}}{[\frac{dw}{d\zeta}]} [g'(x_{*,i}^{-1}(w))]^2}{g(x_{*,i}^{-1}(w))(g(x_{*,i}^{-1}(w))-1)(g(x_{*,i}^{-1}(w))-z)})] \frac{dw \wedge d\bar{w}}{|\frac{dw}{d\zeta}|^2}$$

$$= \frac{z(z-1)}{2\pi i} \quad \sum_{i=1}^{N} \quad \int\int_{\mathbf{CP^1}} [\sum_{g\in\Gamma} \frac{\nu(w)[g'(x_{*,i}^{-1}(w))]^2 [\frac{\partial x_{*,i}^{-1}}{\partial w}(w)]^2}{g(x_{*,i}^{-1}(w))(g(x_{*,i}^{-1}(w))-1)(g(x_{*,i}^{-1}(w))-z)}] dw \wedge d\bar{w}$$

$$= \frac{z(z-1)}{2\pi i} \sum_{i=1}^{N} \int\int_{\mathbf{CP}^1} \left[\nu(w) K_\Gamma(x_{*,i}^{-1}(w), z)[\frac{\partial x_{*,i}^{-1}}{\partial w}(w)]^2\right] dw \wedge d\bar{w}$$

Similarly

$$w_1^*(z) = \frac{z(z-1)}{2\pi i} \sum_{i=1}^{N} \int\int \left[\overline{\nu(w)} K_\Gamma(\overline{x_{*,i}^{-1}(w)}, z)[\overline{\frac{\partial x_{*,i}^{-1}}{\partial w}(w)}]^2\right] dw \wedge d\bar{w}$$

That last is exactly the expression desired and claimed in the Theorem. We are through.

□

Some remarks about the above arguments: When we interchanged summation and integration above in the series (4.16), we needed some straightforward facts from the theory of measure and integration. For instance, our purposes are adequately served by the following result (See Rudin[R]):

Lemma: Suppose $\{f_n\}$ is a sequence of complex measurable functions defined almost everywhere on a complete measure space (X, μ) such that

$$\sum_1^\infty \int_X |f_n| \, d\mu < \infty$$

Then the series $f(x) = \sum_1^\infty f_n(x)$ converges absolutely for almost all x, and $f \in L^1(\mu)$; moreover, the summation and integration can be interchanged, namely:

$$\int_X f d\mu = \sum_1^\infty \int_X f_n d\mu$$

□

Let us apply this Lemma with $f_n = \psi_g$, as g ranges over the countable group Γ. Now, we saw in the argument preceding equation (4.16) that the series (4.16) was absolutely convergent. Hence if we set $\psi(z) = \sum_{g\in\Gamma} \psi_g(z)$ then this series converges absolutely almost everywhere on U by the above Lemma, and indeed ψ will belong to

$L^1(\mu)$ (where μ denotes two-dimensional Lebesgue measure). Moreover the Lemma guarantees that we are justified in interchanging sum and integral, as we have done in the course of the manipulations in the proof.

Another small point concerns the measurability of the various functions we have in the integrands. It is sufficient again to consider the case of the ψ_g, and prove that $\psi_g(z)$ is defined a.e. on U, and is measurable. That is very simple. In fact, it is very easy to see that ψ_g is measurable on the set $U' = U - (x_*)^{-1}\{\textit{union of the edges in the triangulation and } \infty\}$; but U' differs from U only by a set of null (areal) measure. (because, x_* is a holomorphic mapping of U onto the Riemann sphere.) That suffices.

4.4 Variational formula for Fuchsian group elements:

We have arrived at the desired Main Theorem:

<u>**Theorem: On Variation of G_t:**</u> *Let X_t be as before the varying family of compact Riemann surfaces corresponding to the family of algebraic curves $P_t = 0$. Let $G_0 = \Gamma$ denote a Fuchsian group uniformizing X_0, and consequently, with notations as throughout above, G_t is the corresponding Fuchsian group for the Riemann surface X_t. Then, for any $\gamma \in \Gamma$, the variational formula for*

$$\gamma_t = \Phi_t \circ \gamma \circ \Phi_t^{-1}$$

is:

$$\gamma_t = \gamma + t\dot{\gamma} + \bar{t}\dot{\gamma}^* + o(t) \tag{4.17}$$

where the formulae for the terms above are:

$$\dot{\gamma} = w_1 \circ \gamma - \gamma' w_1$$

$$\dot{\gamma}^* = w_1^* \circ \gamma - \gamma' w_1^*$$

with w_1 and w_1^* being as stated in the main Theorem of Section IV.3 above.

Proof: This is a simple matter of applying the chain rules in order to calculate the t and $t-bar$-derivatives of γ_t at $t=0$. A direct computation for complex t (see also for instance formula (1.8), p. 170, in Section 3.1.1 of Nag[N] for the real t situation) gives the result as stated. □

Remark on generators of G_t: If we choose a set of generators for the initial Fuchsian group G_0, then the above Theorem lets us determine, up to first order in t, a corresponding set of generators for the deformed Fuchsian groups G_t. That is exactly the sort of result we desire in any explicit calculation with a given family of algebraic curves. Note that a standard set of generators $\{A_1, \cdots, A_g, B_1, \cdots, B_g\}$ for the fundamental group of a compact Riemann surface of genus g will produce, by deformation again just such a standard set of generators for the new Fuchsian groups. (The relations are obviously preserved under conjugation by Φ_t.)

4.5 Computational aspects of our formulae and remarks on Poincaré theta series:

We would like to make some remarks on the practical implementation of the variational formulae that we have determined in the theorems of Sections IV.3 and IV.4.

The chief formulae for the variation of the quasiconformal maps Φ_t were written out in Theorem of Section IV.3. The basic nature of this formula is more easily captured by giving separate names to the N kernel functions that multiply the piecewise constant Beltrami differential, ν. In fact, we may rewrite the main formulae we derived for w_1 and w_1^* as follows:

$$w_1(z) = \partial\Phi_t/\partial t = \frac{z(z-1)}{2\pi\sqrt{-1}} \sum_{k=1}^{N} \int\int_{R\subset \mathbf{CP^1}} \nu(w) V_{k,\Gamma}(w,z)\, dw \wedge d\bar{w} \qquad (4.18)$$

$$w_1^*(z) = \partial\Phi_t/\partial\bar{t} = \frac{z(z-1)}{2\pi\sqrt{-1}} \sum_{k=1}^{N} \int\int_{R\subset \mathbf{CP^1}} \overline{\nu(w)} V_{k,\Gamma}(\overline{w}, z) dw \wedge d\bar{w} \qquad (4.19)$$

Here the N "kernel" functions $V_{k,\Gamma}$ are close cousins unto each other, (for $k = 1, 2, \cdots, N$), and each one is determined as a holomorphic function (of two arguments) via summation over the group elements $g \in \Gamma$. [Recall that N was the degree of the polynomials P_t as polynomials in the y variable.] In fact,

$$V_{k,\Gamma}(w, z) = K_\Gamma(x_{*,k}^{-1}(w), z)[\frac{dx_{*,k}^{-1}}{dw}(w)]^2 \qquad (4.20)$$

(Compare formula (4.14) for $K_\Gamma(w, z)$.)

But we know further that the function x_* is itself computable as the ratio of Poincare theta series – which are again summations over the same Fuchsian group Γ that is given to us. We take a moment to recall this basic material and indicate the relevance to our situation:

The Poincare theta series of weight q acting on f is::

$$\theta_q(f)(z) = \theta_{q,\Gamma}(f)(z) = \sum_{g\in\Gamma} f(g(z))(g'(z))^q, \quad z \in U, \qquad (4.21)$$

where $q \geq 2$ is an integer, and f is a meromorphic function (e.g., a rational function) on U.

In fact, by the standard completeness theorems for theta series (see Kra [Kr], and Section 1.4 of Nag[N]) we know that as f ranges over rational functions on U we can thus manufacture all possible meromorphic q-forms on U/Γ. But any meromorphic function on $X_0 = U/\Gamma$ is expressible as the ratio of suitable meromorphic q-forms (in fact we may simply take $q = 2$) – consequently, knowing the zeros and poles of the function x_* we can manufacture explicit expression for it as a ratio of two Poincare theta series

$$x_* = \frac{\theta_{q,\Gamma}(f_1)}{\theta_{q,\Gamma}(f_2)} \qquad (4.22)$$

For actual ways to set up theta series with prescribed singularities on the Riemann surface, see the book by Behnke and Thullen "Complex Analysis", - where indeed a proof of the classical Riemann-Roch theorem for compact Riemann surfaces is obtained using completeness results for these theta series.

Therefore, the upshot is, that a computer calculation of the variational formulae above is not unfeasible.

Chapter 5

Variation from the Fermat curves $x^p + y^p = 1$

5.1 Consideration of family of curves:

Consider the following one parameter family of curves namely

$$P_t(x,y) \equiv x^p + y^p + tx^{p-1} + \ldots + tx - (1-t) = 0 \tag{5.1}$$

where t lies in a small disc around the origin of the complex plane $\mathbf{C}$ and p is any integer even or odd (p need not be prime) such that $p \geq 4$.

Note that $P_0(x,y) \equiv P(x,y,0) \equiv x^p + y^p - 1 = 0$ that is the well-known Fermat Curve.

<u>**Lemma1:**</u> The polynomial $P_t(x,y)$ given above can also be represented by

$$P_t(x,y) = y^p + (x-(1-t))(x - e^{\frac{2\pi i}{p}}) \ldots (x - e^{\frac{2\pi i}{p}(p-1)}) \tag{5.2}$$

<u>Proof:</u> Observe that

$$x^p - 1 = (x-1)(x - e^{\frac{2\pi i}{p}}) \ldots (x - e^{\frac{2\pi i}{p}(p-1)}) \tag{5.3}$$

Equating coefficients of x^{p-1} of (5.3),

$$\sum_{k=1}^{p-1} e^{\frac{2\pi i}{p}k} + 1 = 0 \Longrightarrow \sum_{k=1}^{p-1} e^{\frac{2\pi i}{p}k} = -1$$

Equating coefficient of x^{p-2} of (5.3)

$$\begin{aligned}\sum_{k=1}^{p-1} e^{\frac{2\pi\imath}{p}k} + \sum \sum_{\{r,s\}\, r\neq s\ 1\leq r\leq(p-1)\ 1\leq s\leq(p-1)} e^{\frac{2\pi\imath}{p}(r+s)} &= 0\\ \Longrightarrow \sum \sum_{\{r,s\}\, r\neq s\ 1\leq r\leq(p-1)\ 1\leq s\leq(p-1)} e^{\frac{2\pi\imath}{p}(r+s)} &= (-1)^2\end{aligned}$$

Suppose by induction we have by equating coefficients of x^{p-k}

$$\sum_{r_i\neq r_j\ 1\leq i,j\leq k\ 1\leq r_i\leq(p-1)\ (r_1,\ldots,r_k)} e^{\frac{2\pi\imath}{p}(r_1+\ldots+r_k)} = (-1)^k$$

where $\{r_1,\ldots,r_k\}$ denote unordered pair.

Then by equating coefficient of $x^{p-(k+1)}$ we have

$$\begin{aligned}\sum_{\{r_1\ldots r_{k+1}\}\ r_i\neq r_j\ 1\leq i,j\leq(k+1)\ 1\leq r_i\leq(p-1)} e^{\frac{2\pi\imath}{p}(r_1+\ldots+r_{k+1})} + \sum_{\{r_1\ldots r_k\}\ r_i\neq r_j 1\leq i,j\leq k} e^{\frac{2\pi\imath}{p}(r_1+\ldots r_k)} = 0\\ \Longrightarrow \sum_{\{r_1,\ldots,r_k\}\ r_i\neq r_j 1\leq r_i\leq(p-1)\ 1\leq i,j\leq k+1} e^{\frac{2\pi\imath}{p}(r_1+\ldots+r_{k+1})} = (-1)^{k+1}\end{aligned}$$

So we have

$$\sum_{\{r_1,\ldots,r_k\}\ r_i\neq r_j\ 1\leq i\leq k\ 1\leq r_i\leq(p-1)} e^{\frac{2\pi\imath}{p}(r_1+\ldots+r_k)} = (-1)^k \quad 1\leq k\leq(p-1) \tag{5.4}$$

Now consider

$$\begin{aligned}&(x-(1-t))(x-e^{\frac{2\pi\imath}{p}})\ldots(x-e^{\frac{2\pi\imath}{p}(p-1)})\\ =\ &(x-\tau)(x-e^{\frac{2\pi\imath}{p}})\ldots(x-e^{\frac{2\pi\imath}{p}(p-1)}) \text{ Putting } \tau = 1-t\\ =\ &x^p - (\tau + \sum_{k=1}^{p-1} e^{\frac{2\pi\imath}{p}k})x^{p-1} + (\tau\sum_{k=1}^{p-1} e^{\frac{2\pi\imath}{p}k} + \sum_{\{r,s\}\ r\neq s\ 1\leq r,s\leq(p-1)} e^{\frac{2\pi\imath}{p}(r+s)})x^{p-2}\\ +\ &\ldots + (-1)^k(\tau \sum_{\{r_1\ldots r_{k-1}\} r_i\neq r_j\ 1\leq r_i\leq(p-1)} e^{\frac{2\pi\imath}{p}(r_1+\ldots r_{k-1})} +\\ &\sum_{\{r_1\ldots r_k\}\ r_i\neq r_j\ 1\leq r_i\leq(p-1)} e^{\frac{2\pi\imath}{p}(r_1+\ldots r_k)})x^{p-k} + \ldots +\end{aligned}$$

$$
\begin{aligned}
&(-1)^{p-1} \quad (\tau \sum_{\{r_1 \ldots r_{p-2}\}\ r_i \neq r_j\ 1 \leq r_i \leq (p-1)} \epsilon^{\frac{2\pi\iota}{p}(r_1 + \ldots + r_{p-2})} + \epsilon^{\frac{2\pi\iota}{p}(1 + \ldots + (p-1))})x + \\
&(-1)^p \quad \tau \Pi_{k=1}^{p-1} \epsilon^{\frac{2\pi\iota}{p}k} \\
= \quad & x^p - (\tau - 1)x^{p-1} + (-\tau + 1)x^{p-2} + \ldots + (-1)^k(\tau(-1)^{k-1} + (-1)^k)x^{p-k} + \\
& \ldots + (-1)^{p-1}(\tau(-1)^{p-2} + (-1)^{p-1})x + \tau(-1)^p(-1)^{p-1} \text{ using (5.4)} \\
= \quad & x^p + (1-\tau)x^{p-1} + (1-\tau)x^{p-2} + \ldots + (1-\tau)x^{p-k} + \\
& \ldots + (1-\tau)x - \tau \\
= \quad & x^p + tx^{p-1} + tx^{p-2} + \ldots + tx^{p-k} + \ldots + tx - (1-t) \text{ [putting } \tau = 1-t] \\
= \quad & P_t(x, y)
\end{aligned}
$$

Hence the lemma follows. □

5.2 Irreducibility of Fermat Curve:

We claim that $P_0(x, y)$ is irreducible . Recall the following theorem. (For the following theorem and the allied matter see Lang[L].)

<u>Theorem 1:</u> **(Eisenstein's criterion)** Let R be a Principal Ideal Domain. Let

$$f = y^n + a_{n-1}y^{n-1} + \ldots + a_1 y + a_0$$

be a polynomial (with leading coefficient 1) of degree n in y with coefficients from R (namely f is in $R[y]$)

Let p in R be a prime element. Assume

(*i*) p divides a_j $\forall j \leq n-1$

(*ii*) p^2 does not divide a_0

Then f is irreducible in $F[y]$ and hence in $R[y]$ where $F = Q(R)$ where $Q(R)$ is the quotient field of R. □

Now applying this theorem in the case of Fermat curve.

Let $R = \mathbf{C}[x]$ is a Principal Ideal Domain. $f \in R[y]$ is irreducible if one can find a prime p in $\mathbf{C}[x]$ as is needed in the theorem. For $f(x,y) = y^p + x^p - 1$ we have $a_j = 0$ for $j = 1, \ldots, n-1$ and $a_0 = x^p - 1$. Take $p = x - 1$ is prime in R. Then p divides a_0 but p^2 does not divide a_0 since 1 is simple root of a_0. Hence Fermat curve $y^p + x^p - 1$ is irreducible for all p.

Hence $P_t(x,y)$ is irreducible for small values of t (which follows from *monodromy invariance lemma* as degree of $P_t(x,y)$ is p in y). Also 0 and ∞ are ordinary points.

5.3 Determination of ramification points and function germs of $P_t(x,y)$:

Observe that

$$\begin{aligned} P_0(x,y) &= y^p + x^p - 1 \\ P_{0,y}(x,y) &= py^{p-1} \end{aligned}$$

Discriminant of $P_0(x,y)$

$$= \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & (x^p-1) & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 & (x^p-1) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & (x^p-1) \\ p & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & p & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & p & 0 & 0 & \ldots & 0 \end{vmatrix}$$

$$= (-1)^{(p+2)}(x^p - 1)\begin{vmatrix} 0 & 1 & \dots & 0 & 0 & (x^p-1) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & (x^p-1) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & (x^p-1) \\ p & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & p & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & p & 0 & 0 & \dots & 0 \end{vmatrix}$$

[expanding with respect to $(p+1)$th column]

$$= (-1)^{(p+2)(p-1)}(x^p - 1)^{(p-1)}\begin{vmatrix} p & 0 & \dots & \dots & 0 \\ 0 & p & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & p \end{vmatrix}$$

[the $p \times p$ matrix]

$$= (-1)^{(p+2)(p-1)}(x^p - 1)^{p-1}p^p$$

$$= (x^p - 1)^{p-1}p^p \quad \text{as either } (p-1) \text{ or } (p+2) \text{ is even}$$

which vanishes only at the pth roots of unity.

Similarly the discriminant of $P_t(x, y)$

$$= [x^p + tx^{p-1} + tx^{p-2} + \dots + tx - (1-t)]^{p-1}p^p$$

$$= (x - (1-t))^{p-1}(x - e^{\frac{2\pi i}{p}})^{p-1}\dots(x - e^{\frac{2\pi i}{p}(p-1)})^{p-1}p^p \qquad (5.5)$$

Let X_0 denote the Riemann Surface corresponding to the polynomial equation $P_0(x, y) = 0$ and X_t corresponds to $P_t(x, y) = 0$.

Here the ramification points are given by p holomorphic functions of t namely

$$\zeta_1(t) = 1 - t \tag{5.6}$$

$$\zeta_{k+1}(t) = e^{\frac{2\pi i}{p}k} \quad 1 \leq k \leq p-1 \tag{5.7}$$

and the function germs are given by

$$y_j(x,t) = \sqrt[p]{-(x-(1-t))(x-e^{\frac{2\pi i}{p}})\dots(x-e^{\frac{2\pi i}{p}(p-1)})}e^{\frac{2\pi i}{p}j} \tag{5.8}$$
$$0 \leq j \leq p-1$$

such that

$$P(x, y_j(x,t), t) = 0 \tag{5.9}$$

Also it is obvious that monodromy remains invariant at the corresponding ramification point.

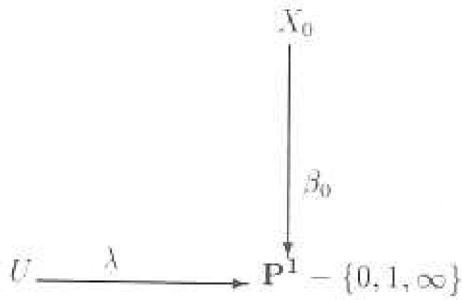
One thing is needed to mention all conditions needed to prove *monodromy invariance lemma* is satisfied excepting the fact that the ramification points are not multiple points of the discriminant of $P_t(x,y)$ but that is needed only to show that ramification point are holomorphic functions of t. as we have used *implicit function theorem* to show the Jacobian does not vanish. But here it is so obvious that we need not use *implicit function theorem*. Hence the whole theory established before is applicable in this case once we can find out the Fuchsian group of $P_0(x,y)$.

5.4 Determination of Fuchsian group G_0 of the Fermat curve $P_0(x,y) = x^p + y^p - 1 = 0$:

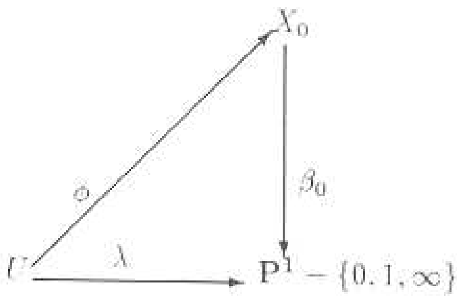
In order to find out the Fuchsian group G_0 of the polynomial $P_0(x,y) = 0$, we have to do the following

Lemma2: A compact Riemann surface X is defined over $\overline{Q}$ if and only if $X \equiv \overline{U/H}$ for some subgroup H of finite index in $\Gamma(2)$. (See Jones and Singerman [JS2])

Proof: Let X be defined over $\overline{Q}$. Then there is a Belyi function $\beta : X \longrightarrow \mathbf{P^1}$ (that is a Belyi function ramified only over 0, 1, ∞). Let $X_0 = \beta^{-1}\{\mathbf{P^1} - \{0,1,\infty\}\}$ and $\beta_0 = \beta|_{X_0}$.



Let $U = \{z : imz > 0\}$=upper half plane. Now the well known λ-function $U \stackrel{\lambda}{\longrightarrow} \mathbf{P^1} - \{0,1,\infty\}$ is a universal covering projection of $\mathbf{P^1} - \{0,1,\infty\}$ having $\Gamma(2)$ as the deck transformation group. So $\Gamma(2) \equiv \pi_1\{\mathbf{P^1} - \{0,1,\infty\}\}$ where π_1 is the fundamental group of $\mathbf{P^1} - \{0,1,\infty\}$.



Since X_0 has U as its universal covering space we have a universal covering map $\phi : U \longrightarrow X_0$ such that the adjoining diagram commute and $X_0 \equiv U/H$ where H is a finite index subgroup of $\Gamma(2)$. In fact $[\Gamma(2) : H] = N = deg\beta_0$. Let $i : U/H \longrightarrow X_0$ be an isomorphism. Then it extends to an isomorphism $\bar{i} : \overline{U/H} \longrightarrow X$. Also $\lambda : U/\Gamma(2) \longrightarrow \mathbf{P^1} - \{0,1,\infty\}$ is an isomorphism which also extends to an isomorphism $\overline{\lambda} : \overline{U/\Gamma(2)} \longrightarrow \mathbf{P}^1$ such that the following diagram commute

$$\begin{array}{ccc} \overline{U/H} & \xrightarrow{\ \bar{i}\ } & X \\ \Big\downarrow \nu & & \Big\downarrow \beta \\ \overline{U/\Gamma(2)} & \xrightarrow{\ \overline{\lambda}\ } & \mathbf{P}^1 \end{array}$$

Conversely: Let $X \equiv \overline{U/H}$ where H is a finite index subgroup of $\Gamma(2)$. Then $U/H \longrightarrow^{\nu} U/\Gamma(2)$ the natural projection map (unramified covering projection) can be extended to $\bar{\nu} : \overline{U/H} \longrightarrow \overline{U/\Gamma(2)}$. Let $X \longrightarrow^{\phi} \overline{U/H}$ and $\overline{U/\Gamma(2)} \longrightarrow^{\psi} \mathbf{P}^1$ be isomorphism. Then $X \longrightarrow^{\beta} \mathbf{P}^1$ given by $\beta = \psi \circ \bar{\nu} \circ \phi$ is ramified only over $\{0, 1, \infty\}$ and hence a Belyi function.

□

Given a Belyi function $\beta : X \longrightarrow \mathbf{P}^1$ we can determine the corresponding subgroup H of $\Gamma(2)$ such that $X \equiv \overline{U/H}$ and $[\Gamma(2) : H] = N = deg\beta$. Choose a point $p \in \mathbf{P}^1 - \{0, 1, \infty\}$. Let $\Omega = \beta^{-1}\{p\}$ so $|\Omega| = N$. Let

$$\Omega = \{q_1, q_2, \ldots, q_N\} \tag{5.10}$$

Let $\pi = \pi_1(\mathbf{P}^1 - \{0, 1, \infty\}, p)$, the first fundamental group of $\mathbf{P}^1 - \{0, 1, \infty\}$ based at p.

Now we have a permutation representation $\theta : \pi \longrightarrow S^N$ where S^N is the permutation group of the set Ω given by

$$\theta[\gamma] = g_\gamma \tag{5.11}$$

where $g_\gamma(q_i) = q_j$ where q_j is the end point of the lifting $\tilde{\gamma}$ (of γ) beginning at q_i. Obviously

$$\theta(\pi) = G(\beta) = \text{monodromy group of } \beta \tag{5.12}$$

and acts transitively on Ω, since X is connected.

$$\beta_0 : X_0 \longrightarrow \mathbf{P^1} - \{0,1,\infty\} \ (\beta|_{X_0} = \beta_0 \ and \ X_0 = \beta^{-1}(\mathbf{P^1} - \{0,1,\infty\}, p))$$

Hence

$$\beta_0^* : \pi_1(X_0, q) \longrightarrow \pi_1(\mathbf{P^1} - \{0,1,\infty\}, p)$$

is a monomorphism (where $q = q_i \ \ 1 \le i \le N$) and

$$H \equiv \beta_0^*(\pi_1(X_0, q)) \tag{5.13}$$

So it suffices to find out $\beta_0^*(\pi_1(X_0, q)))$ and $\beta_0^*(\pi_1(X_0, q))$ consists of all those elements of $\pi = \pi_1(\mathbf{P^1} - \{0,1,\infty\}, p)$ whose image under θ carries q to itself.

Now H is the image $\beta_0^*(\pi_1(X_0, q))$ under the isomorphism $\pi = \pi_1(\mathbf{P^1} - \{0,1,\infty\}, p) \longrightarrow^i \Gamma(2)$.

Note that π is generated by classes $\sigma_i = [\gamma_i]$ of suitably chosen loops γ_i around $0, 1, \infty$ respectively such that $\sigma_0\sigma_1\sigma_\infty = 1$ and

$$\Gamma(2) =< S_0, S_1, S_\infty | S_0 S_1 S_\infty = 1 > \tag{5.14}$$

Also $\pi \longrightarrow^i \Gamma(2) \ \ i(\sigma_i) = S_i \ \ i = 0, 1, \infty$ is an isomorphism.

Let us recall $\theta : \pi \longrightarrow S^N$ and denote $\theta(\sigma_i) = g_i$. Let $g_i^{l_i} = 1, \ \ i = 0, 1, \infty$. Construct the group

$$\Delta = \Delta(l_0, l_1, l_\infty) =< T_0, T_1, T_\infty | T_0^{l_0} = T_1^{l_1} = T_\infty^{l_\infty} = T_0 T_1 T_\infty = 1 > \tag{5.15}$$

that is the usual triangle group.

If

$$\frac{1}{l_0} + \frac{1}{l_1} + \frac{1}{l_\infty} < 1$$

then $U/\Delta \longrightarrow \mathbf{P^1}$ there is an isomorphism and Δ is a discrete subgroup acting discontinuously on U.

$$If\ \frac{1}{l_0} + \frac{1}{l_1} + \frac{1}{l_\infty} = 1$$

then $\triangle$ acts on $\mathbf{C}$ discontinuously.

$$If\ \frac{1}{l_0} + \frac{1}{l_1} + \frac{1}{l_\infty} > 1$$

then $\triangle$ acts on $\mathbf{P^1}$ discontinuously.

<u>**Lemma 3:**</u> A compact Riemann surface is defined over $\overline{Q}$ if and only if $X \equiv \chi/K$ where K is a finite indexed subgroup of $\triangle$ and $\chi = \mathbf{P^1}$ or $\mathbf{C}$ or U. (See Jones and Singerman [JS2])

<u>Proof:</u> Let X is defined over $\overline{Q}$. Let $\beta : X \longrightarrow \mathbf{P^1}$ be a Belyi function. Then $X \equiv \overline{U/H}$ where H is a finite indexed subgroup of $\Gamma(2)$. Let $\theta : \pi \longrightarrow S^N$ be a permutation representation where $[\Gamma(2) : H] = deg\beta = N$, $\theta(\sigma_i) = g_i \quad i = 0, 1, \infty$, $\pi = \pi_1(\mathbf{P^1} - \{0, 1, \infty\}) \equiv \Gamma(2)$. Let $g_i^{l_i} = 1$ where $i = 0, 1, \infty$. Construct $\triangle =< T_0, T_1, T_\infty | T_0^{l_0} = T_1^{l_1} = T_\infty^{l_\infty} = T_0 T_1 T_\infty = 1 >$. Then $\pi \longrightarrow^{\psi} \triangle$ be the epimorphism such that $\sigma_i \mapsto^{\psi} T_i$. Let the image of H under ψ be K. Assume

$$\frac{1}{l_0} + \frac{1}{l_1} + \frac{1}{l_\infty} < 1$$

We claim that

$$U/K \equiv X$$

One can show that if

$$\Gamma(2) = H \cup g_1 H \cup \ldots \cup g_k H$$

then

$$\triangle = K \cup \psi(g_1) K \cup \ldots \cup \psi(g_k) K$$

$$\begin{array}{ccc} U/K - \pi^{-1}\{vertices\} & \xrightarrow{\tilde{\phi}} & U/H \\ \downarrow \pi & & \downarrow \beta_0 \\ U/\triangle - \phi^{-1}\{0,1,\infty\} & \xrightarrow{\phi} & \mathbf{P}^1 - \{0,1,\infty\} \end{array}$$

$$(\phi \circ \pi)_* \pi_1\{U/K - \pi^{-1}\{vertices\}\} = (\beta_0)_* \pi_1(U/H)$$

Hence ϕ lifts to an isomorphism $\tilde{\phi}$ above and which can be extended to an isomorphism

$$U/K \longrightarrow \overline{U/H}$$

Recall that

$$\Gamma(2) \longrightarrow^{\psi} \triangle$$

and

$$\Gamma(2) = g_0 H \cup g_1 H \cup \ldots \cup g_N H, \quad g_0 = 1$$

Let $T \in \triangle$ then $T = \psi(g)$ $g \in \Gamma(2)$. Also $g = g_i h$ $h \in H$ for some i. So $T = \psi(g) = \psi(g_i)\psi(h)$ and $\psi(h) \in \psi(H) = K$. Again $Ker\psi \subset Ker\theta \subset H$.

$$h \in \psi(g_i)K \cap \psi(g_j)K$$

$$\Longrightarrow \quad h = \psi(g_i)k_i = \psi(g_j)k_j$$

$$\text{Now} \quad k_i = \psi(h_i), \quad k_j = \psi(h_j), \quad h_i, h_j \in H_i$$

$$\Longrightarrow \quad g_i h_i (g_j h_j)^{-1} \in Ker\psi \subset H$$

$$\Longrightarrow \quad g_i h_i H = g_j h_j H$$

$$\Longrightarrow \quad g_i H = g_j H$$

$$\Longrightarrow \quad g_i = g_j$$

$$\Longrightarrow \quad \psi(g_i) = \psi(g_j)$$

Conversely : The natural map

$$U/K \longrightarrow U/\triangle$$

is ramified only over the vertices. Also recall that

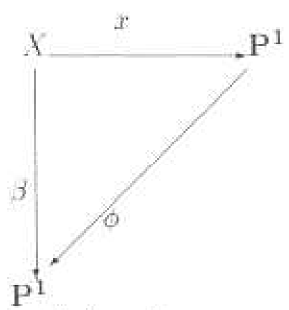
$$U/\triangle \equiv \mathbf{P}^1$$

Hence the result follows. □

Consider the Fermat curve :

$$x^p + y^p = 1$$

Let $X \equiv X_0$ be the Riemann surface of the curve. Consider the function $\beta : X \longrightarrow \mathbf{P}^1$ where $\beta(x, y) = x^p$.



$\phi(z) = z^p$

The adjoining diagram commutes. Now x is ramified over $e^{\frac{2\pi i}{p}k}$ where $k = 0, \ldots, p-1$ and ϕ is ramified over 0 and ∞

$$\phi(e^{\frac{2\pi i}{p}k}) = 1$$

Hence β is ramified over 0, 1, ∞. Hence β is a Belyi function. Note that ϕ is p sheeted and x is p sheeted. Hence

$$deg\,\beta = p^2$$

Now we shall determine monodromy around 0, 1, ∞ under the map β. Choose a small circle around 0. Points on the circle C are given by $z = re^{2\pi i t}$ $0 \leq t \leq 1$. Now lift of C by ϕ are p curves $C_1, \ldots, C_p$ where

$$C_k = \{r^{\frac{1}{p}} e^{\frac{2\pi i}{p}(t+k)} : 0 \leq t \leq 1\} \quad k = 0, \ldots, p-1$$

So via ϕ monodromy around 0 is given by

$$(r^{\frac{1}{p}}, r^{\frac{1}{p}} e^{\frac{2\pi i}{p}}, r^{\frac{1}{p}} e^{\frac{4\pi i}{p}}, \ldots, r^{\frac{1}{p}} e^{\frac{2\pi i}{p}(p-1)}) \tag{5.16}$$

By x,

C_0 lifts to a curve $\tilde{C}_0$ which begins at $(r^{\frac{1}{p}}, \sqrt[p]{1-r})$ and ends at $(r^{\frac{1}{p}} e^{\frac{2\pi i}{p}}, \sqrt[p]{1-r})$.
C_1 lifts to a curve $\tilde{C}_1$ which begins at $(r^{\frac{1}{p}} e^{\frac{2\pi i}{p}}, \sqrt[p]{1-r})$ and ends at $(r^{\frac{1}{p}} e^{\frac{4\pi i}{p}}, \sqrt[p]{1-r})$
and so on.
C_{p-1} lifts to a curve $\tilde{C}_{p-1}$ which begins at $(r^{\frac{1}{p}} e^{\frac{2\pi i}{p}(p-1)}, \sqrt[p]{r-1})$ and ends at $(r^{\frac{1}{p}}, \sqrt[p]{1-r})$.
Giving rise to a monodromy

$$\alpha_0 = \{(r^{\frac{1}{p}}, \sqrt[p]{1-r}), (r^{\frac{1}{p}} e^{\frac{2\pi i}{p}}, \sqrt[p]{1-r}), \ldots, (r^{\frac{1}{p}} e^{\frac{2\pi i}{p}(p-1)}, \sqrt[p]{1-r})\} \tag{5.17}$$

Similarly

$$\alpha_k = \{(r^{\frac{1}{p}}, e^{\frac{2\pi i}{p}k} \sqrt[p]{1-r}), (r^{\frac{1}{p}} e^{\frac{2\pi i}{p}}, e^{\frac{2\pi i}{p}k} \sqrt[p]{1-r}), \ldots, (r^{\frac{1}{p}} e^{\frac{2\pi}{p}(p-1)}, e^{\frac{2\pi i}{p}k} \sqrt[p]{1-r})\} \tag{5.18}$$

for $0 \leq k \leq (p-1)$.
Thus

$$\begin{aligned} g_0 &= \text{monodromy around } 0 \text{ of } \beta \\ &= \alpha_0 \ldots \alpha_{p-1} \end{aligned} \tag{5.19}$$

that is product of p disjoint cycles each of length p.

Next we shall determine monodromy around 1. Then $C = \{1 + re^{i\theta} \quad 0 \le \theta \le 2\pi\}$ determines a circle around 1. Via ϕ, C has p lifts $C_0, \ldots, C_{p-1}$ where C_k is a curve around $e^{\frac{2\pi i}{p}k}$ $0 \le k \le (p-1)$. Firstly consider C_0, it consists of points $\{(1+re^{i\theta})^{\frac{1}{p}} \quad 0 \le \theta \le 2\pi\}$. Via x, C_0 has p lifts. Consider the function germ $y_1(x,0) = \sqrt[p]{1-x^p}$ and continue it along C_0 that is equivalent to continue the function germ $y = \sqrt[p]{1-x}$ along C which carries $\sqrt[p]{-r}$ to $e^{\frac{2\pi i}{p}}\sqrt[p]{-r}$.

Hence C_0 has a lift $\tilde{C_{10}}$ via x which starts at $\{(1+r)^{\frac{1}{p}}, \sqrt[p]{-r}\}$ and ends at $\{(1+r)^{\frac{1}{p}}, e^{\frac{2\pi i}{p}}\sqrt[p]{-r}\}$.

Again if we continue the function germ $y_2(x,0) = e^{\frac{2\pi i}{p}}\sqrt[p]{1-x^p}$ and continue it along C_0 that is equivalent to continue $e^{\frac{2\pi i}{p}}\sqrt[p]{1-x}$ along C which carries $e^{\frac{2\pi i}{p}}\sqrt[p]{-r}$ to $e^{\frac{4\pi i}{p}}\sqrt[p]{-r}$.

Thus we have a lift of C_0 say $\tilde{C_{20}}$ which begins at $((1+r)^{\frac{1}{p}}, e^{\frac{2\pi i}{p}}\sqrt[p]{-r})$ and ends at $((1+r)^{\frac{1}{p}}, e^{\frac{4\pi i}{p}}\sqrt[p]{-r})$.

Proceeding in this way we continue the function germ $y_k(x,0) = e^{\frac{2\pi i}{p}(k-1)}\sqrt[p]{1-x^p}$ along C_0 that is equivalent to continue $e^{\frac{2\pi i}{p}(k-1)}\sqrt[p]{1-x}$ along C which carries $e^{\frac{2\pi i}{p}(k-1)}\sqrt[p]{-r}$ to $e^{\frac{2\pi i}{p}k}\sqrt[p]{-r}$.

Thus we have a lift of C_0 say $\tilde{C_{k0}}$ which begins at $((1+r)^{\frac{1}{p}}, e^{\frac{2\pi i}{p}(k-1)}\sqrt[p]{-r})$ and ends at $((1+r)^{\frac{1}{p}}, e^{\frac{2\pi i}{p}k}\sqrt[p]{-r}) \quad 1 \le k \le p$.

Giving rise to a monodromy

$$\beta_0 = \{((1+r)^{\frac{1}{p}}, \sqrt[p]{-r}), ((1+r)^{\frac{1}{p}}, e^{\frac{2\pi i}{p}}\sqrt[p]{-r}), \ldots, ((1+r)^{\frac{1}{p}}, e^{\frac{2\pi i}{p}(p-1)}\sqrt[p]{-r})\} \quad (5.20)$$

Similarly

$$\beta_k = \{((1+r)^{\frac{1}{p}}e^{\frac{2\pi i}{p}k}, \sqrt[p]{-r}), ((1+r)^{\frac{1}{p}}e^{\frac{2\pi i}{p}k}, e^{\frac{2\pi i}{p}}\sqrt[p]{-r}), \ldots, ((1+r)^{\frac{1}{p}}e^{\frac{2\pi i}{p}k}, e^{\frac{2\pi i}{p}(p-1)}\sqrt[p]{-r})\}$$

which is true for $0 \le k \le (p-1)$.

Thus monodromy around 1 is given by

$$g_1 = \beta_0\beta_1 \ldots \beta_{p-1} \quad (5.21)$$

that is the product of p disjoint cycles each of length p.

In fact

$$g_0(x,y) = (xe^{\frac{2\pi i}{p}}, y) \tag{5.22}$$

$$g_1(x,y) = (x, e^{\frac{2\pi i}{p}}y) \tag{5.23}$$

Since $g_0 g_1 g_\infty = 1$ which follows from the permutation representation $\theta : \pi \longrightarrow S^N$ where $\theta[\gamma] = g_\gamma$ and the fact that

$$\pi =< \sigma_0, \sigma_1, \sigma_\infty : \sigma_0\sigma_1\sigma_\infty = 1 >$$

we have

$$g_\infty(x,y) = (xe^{-\frac{2\pi i}{p}}, ye^{-\frac{2\pi i}{p}}) \tag{5.24}$$

Note that

$$g_0 g_1 = g_1 g_0$$

$$\begin{aligned} & g_0^m g_1^k(x,y) = (x,y) \\ \Longrightarrow\ & (e^{\frac{2\pi i}{p}m}x, e^{\frac{2\pi i}{p}k}y) = (x,y) \\ \Longrightarrow\ & m = tp, \quad k = sp \\ \Longrightarrow\ & g_0^m g_1^k = 1 \end{aligned}$$

The above may not be true if $x = 0$ or $y = 0$ that is $x = 0$ or $x = e^{\frac{2\pi i}{p}k}$ $0 \leq k \leq p-1$ but because of the choice of the point $P \in \mathbf{P}^1 - \{0, 1, \infty\}$ this possibility will not occur. Hence $g_0^m g_1^k$ keeps a point q_i above any point $P \in \mathbf{P}^1 - \{0, 1, \infty\}$ fixed implies $g_0^m g_1^k = 1$.

Now we apply the above theory with $X \equiv X_o$ corresponds to the compact Riemann surface of $x^p + y^p = 1$. $\beta : X \longrightarrow \mathbf{P}^1$ is given by $\beta = x^p$. Choose a point $P \in \mathbf{P}^1 - \{0, 1, \infty\}$ then $\Omega = \beta^{-1}(P) = \{q_1, \ldots, q_{p^2}\}$ (recall equation (5.10)). Also $q_i =$

$(x_i, y_i) \Rightarrow x_i \neq 0$ and $x_i \neq e^{\frac{2\pi i}{p}k}$ $0 \leq k \leq p-1$ for otherwise $\beta(q_i) = 0$ or 1. Recall that $\theta : \Gamma(2) = \pi_1(\mathbf{P}^1 - \{0, 1, \infty\}) \longrightarrow S^{p^2}$, the permutation group of Ω given by $\theta(S_i) = g_i$ where $\Gamma(2) =< S_0, S_1, S_\infty : S_0 S_1 S_\infty = 1 >$. Here we have identified $\Gamma(2)$ with $\pi = \pi_1(\mathbf{P}^1 - \{0, 1, \infty\})$ (compare with (5.14) where the identification is done by an isomorphism $\pi \longrightarrow^i \Gamma(2)$). Since $S_0 S_1 S_\infty = 1$ we have $g_0 g_1 g_\infty = 1$. We have just observed $g_0^p = g_1^p = g_\infty^p = 1$. H (that is a subgroup of $\Gamma(2)$ recall the equation (5.14)) consists of all those elements S of $\Gamma(2)$ such that $\theta(S)$ keeps some q_i fixed. Hence by the discussion of previous paragraph $\theta(S) = 1$. Conversely $\theta(S) = 1$ implies $\theta(S)$ keeps that q_i fixed, hence $S \in H$.

$$\text{Thus } S \in H \Leftrightarrow \theta(S) = 1 \tag{5.25}$$

Again recall the epimorphism $\psi : \Gamma(2) \longrightarrow \triangle[p, p, p]$ given by $\psi(S_i) = T_i$ where $\triangle = \triangle[p, p, p] =< T_0, T_1, T_\infty : T_0^p = T_1^p = T_\infty^p = T_0 T_1 T_\infty = 1 >$. Set $\psi(H) = K$ (recall Lemma (3) $X \equiv U/K$).

<u>Theorem 2:</u> Let $X \equiv X_0$ represent the Riemann Surface corresponding to the polynomial equation $x^p + y^p = 1$. Then

$$X \equiv X_0 \equiv U/[\triangle, \triangle] \tag{5.26}$$

<u>Proof:</u> We follow the theory developed above. By Lemma 3 we have $X \equiv U/K$ where K is a finite indexed subgroup of $\triangle$. Also recall by Lemma 2 $X \equiv \overline{U/H}$ (where H is a finite indexed subgroup of $\Gamma(2)$) and the mappings $\theta : \Gamma(2) \longrightarrow S^{p^2}$ and $\psi : \Gamma(2) \longrightarrow \triangle[p, p, p]$ of the above paragraph. We claim that $K = [\triangle, \triangle]$.

$$T \in K \Longrightarrow T = \psi(S)$$

for some $S \in H$. Hence by (5.25) we have $\theta(S) = 1$.

We can write

$$T = T_0^{r_1} T_1^{m_1} T_0^{r_2} T_1^{m_2} \dots T_0^{r_k} T_1^{m_k}$$

where $0 \leq \sum r_i < p$, $0 \leq \sum m_i < p$. Hence

$$\theta(S) = g = g_0^{r_1} g_1^{m_1} g_0^{r_2} g_1^{m_2} \ldots g_0^{r_k} g_1^{m_k}$$

since

$$g_0^p = g_1^p = g_\infty^p = g_0 g_1 g_\infty = 1$$

Hence

$$\theta(S) = g_0^{r_1+\ldots+r_k} g_1^{m_1+\ldots+m_k}$$

as $g_0 g_1 = g_1 g_0$. Now by (5.25) we have $\theta(S) = 1$ as $S \in H$. Hence

$$\sum_{i=1}^{k} r_i = 0, \quad \sum_{i=1}^{k} m_i = 0$$

Thus

$$\begin{aligned}
T &= T_0^{r_1} T_1^{m_1} T_0^{r_2} T_1^{m_2} \ldots T_0^{r_k} T_1^{m_k} \\
&= (T_0^{r_1} T_1^{m_1} T_0^{-r_1} T_1^{-m_1}) T_1^{m_1} T_0^{r_1+r_2} \ldots T_0^{r_k} T_1^{m_k} \\
&= \tilde{T} T_1^{m_1} T_0^{r_1+r_2} T_1^{m_2} \ldots T_0^{r_k} T_1^{m_k} \\
&= \tilde{T} T_1^{n_1} T_0^{p_2} T_1^{n_2} \ldots T_0^{p_k} T_1^{n_k}
\end{aligned}$$

So the number of $T_1^{n_i}$ and $T_0^{p_i}$ has reduced with $\sum n_i = 0$, $\sum p_i = 0$ and $\tilde{T} \in [\triangle, \triangle]$. Hence by induction $T \in [\triangle, \triangle]$ as $\sum r_i = 0$ and $\sum m_i = 0$.

$$Hence \;\; T \in [\triangle, \triangle]$$

Thus

$$K \subset [\triangle, \triangle]$$

<u>*Conversely :*</u> $T \in [\triangle, \triangle]$ shows that $T = \Pi_{i=1}^{k} A_i B_i A_i^{-1} B_i^{-1}$ where $A_i, B_i \in \triangle$ $1 \leq i \leq k$. Hence if $T_i^{k_i}$ is in T then $T_i^{-k_i}$ is in the word T. Hence

$$T = T_0^{r_1} T_1^{m_1} \ldots T_0^{r_k} T_1^{m_k}$$

where $\sum_{i=1}^{k} r_i = sp$ and $\sum_{i=1}^{k} m_i = np$ for some integers s and n. Now $T = \psi(S)$ where

$$S = S_0^{r_1} S_1^{m_1} S_0^{r_2} \ldots S_0^{r_k} S_1^{m_k}$$

Hence

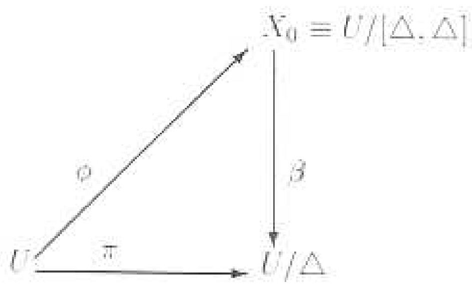
$$\begin{aligned} g = \theta(S) &= g_0^{r_1} g_1^{m_1} g_0^{r_2} \cdots g_0^{r_k} g_1^{m_k} \\ &= g_0^{\sum_{i=1}^{k} r_i} g_1^{\sum_{i=1}^{k} m_i} \\ &= g_0^{sp} g_1^{np} \\ &= 1 \end{aligned}$$

Hence $S \in H$ giving $T = \psi(S) \in K$.

Hence $[\triangle, \triangle] \subset K$.

Thus $K = [\triangle, \triangle]$ and this completes our proof. □

We claim that $[\triangle, \triangle]$ *is the uniformizing Fuchsian group (that is* $[\triangle, \triangle]$ *is torsion free) of* $X \equiv X_0$.



Take a branch β_1 of β and continue $\beta_1^{-1} \circ \pi$ analytically. Since U is simply connected that can be done uniquely. So let ϕ be the lift. Then as β is $z \mapsto z^p$ above each point over $0, 1, \infty$ and π is also so, we have ϕ as unramified covering projection and hence universal covering projection. So $[\triangle, \triangle]$ is the deck transformation group of the map $\psi \circ \phi$ where $\psi : X_0 \longrightarrow U/[\triangle, \triangle]$ is an isomorphism. *Hence* $[\triangle, \triangle]$ *is torsion free.*

5.5 Calculation of genus of X_0:

We claim that the genus g of X_0, that is the Riemann surface corresponding to the polynomial equation $x^p + y^p = 1$, is given by

$$g = \frac{1}{2}(p-1)(p-2) \tag{5.27}$$

We can calculate the genus of X_0 with the help of β. Consider a triangulation of $\mathbf{P}^1$ which consists of three vertices $\{0, 1, \infty\}$, three edges namely the segments of $\mathbf{R}$ joining 0 to 1, 1 to ∞ and ∞ to 0 and two faces namely U, L (that is the upper and lower half plane). Then lift this triangulation by β. We have

$$number\ of\ vertices = 3p$$

(since the number of points over 0 is p, the number of points over 1 is p and the number of points over ∞ is p).

$$number\ of\ edges = 3p^2$$

(as β is of degree p^2 and β is ramified over 0, 1, ∞).

$$number\ of\ faces = 2p^2$$

(as β is p^2 sheeted).

Let g be the genus of $X = X_0$. Then

$$\begin{aligned} 2-2g &= 3p - 3p^2 + 2p^2 \\ &= 3p - p^2 \\ \Rightarrow 2g &= p^2 - 3p + 2 \\ \Rightarrow g &= \frac{1}{2}(p-1)(p-2) \end{aligned}$$

It is easy to verify that X_t, the Riemann surface corresponding to $P_t(x, y) = 0$, is not biholomorphically equivalent to X_0, that is the Riemann surface corresponding to $P_0(x,y) = 0$, for small values of t. There does not exists an automorphism of $\mathbf{P}^1$ carrying the branch points of $P_t(x,y)$ to the branch points of $P_0(x,y)$. Indeed the cross ratio

$$(1-t, e^{\frac{2\pi i}{p}}, e^{\frac{4\pi i}{p}}, e^{\frac{6\pi i}{p}}) = \frac{[1-t-e^{\frac{2\pi i}{p}}][e^{\frac{4\pi i}{p}} - e^{\frac{6\pi i}{p}}]}{[1-t-e^{\frac{6\pi i}{p}}][e^{\frac{4\pi i}{p}} - e^{\frac{2\pi i}{p}}]}$$

depends on t.

5.6 Calculation of $\nu(z)$:

We claim that

$$\nu(z) = \frac{[-1 + i\cot\frac{2\pi}{p}]}{2}$$ inside the triangular region with vertices origin, 1, and $e^{\frac{2\pi i}{p}}$.

$$= \frac{[-1 + i\cot\frac{2\pi}{p}(p-1)]}{2}$$ inside the triangular region with vertices origin, 1 and $e^{\frac{2\pi i}{p}(p-1)}$.

$$= \frac{1}{2}[1 + i\frac{(2-\cos\frac{2\pi}{p})}{\sin\frac{2\pi}{p}}]$$ inside the triangular region with vertices 1, 2, and $e^{\frac{2\pi i}{p}}$.

$$= \frac{1}{2}[1 + i\frac{(2-\cos\frac{2\pi}{p}(p-1))}{\sin\frac{2\pi}{p}(p-1)}]$$ inside the triangular region with vertices 1, 2, and $e^{\frac{2\pi i}{p}(p-1)}$.

$= 0$ elsewhere.

Let us consider a triangulation of $\mathbf{P}^1$ with only four triangles with vertex 1 namely the triangle with vertices origin, 1 ,$e^{\frac{2\pi i}{p}}$; the triangle with vertices origin, 1, $e^{\frac{2\pi i}{p}(p-1)}$;

the triangle with vertices 1, $e^{\frac{2\pi i}{p}}$, 2 and the triangle with vertices 1, $e^{\frac{2\pi i}{p}(p-1)}$, 2. We shall calculate $\nu(z)$ (Recall IV.2) in these four triangles. Obviously $\nu(z)$ is zero outside these four triangular region as the map ϕ_t is identity there.

The triangle with vertices origin, $\zeta_1(0) = 1$ and $\zeta_2(0) = e^{\frac{2\pi i}{p}}$ is mapped by ϕ_t linearly to the triangle with vertices origin, $\zeta_1(t) = 1 - t$ and $\zeta_2(t) = e^{\frac{2\pi i}{p}}$. Any point (x, y) in the first triangle can be represented as

$$x + iy = z = \lambda e^{\frac{2\pi i}{p}} + \mu + (1 - \lambda - \mu).0$$

$$\text{where } 0 \le \lambda \le 1, \quad 0 \le \mu \le 1, \quad 0 \le \lambda + \mu \le 1$$

$$\Rightarrow \quad x = \lambda \cos \frac{2\pi}{p} + \mu, \quad y = \lambda \sin \frac{2\pi}{p}$$

$$\Rightarrow \quad \lambda = \frac{y}{\sin \frac{2\pi}{p}} \tag{5.28}$$

$$\text{and} \quad \mu = \frac{x \sin \frac{2\pi}{p} - y \cos \frac{2\pi}{p}}{\sin \frac{2\pi}{p}} \tag{5.29}$$

Again any point (u, v) in the second triangle can be represented as

$$u + iv = \lambda e^{\frac{2\pi i}{p}} + \mu(1 - t) + (1 - \lambda - \mu).0$$

$$\text{where } 0 \le \lambda \le 1, \quad 0 \le \mu \le 1, \quad 0 \le \lambda + \mu \le 1$$

$$\Rightarrow \quad u = \lambda \cos \frac{2\pi}{p} + \mu(1 - t_1) \tag{5.30}$$

$$\text{and} \quad v = \lambda \sin \frac{2\pi}{p} - \mu t_2 \quad \text{where } t = t_1 + it_2 \tag{5.31}$$

Hence $\phi_t((x, y))$ will be mapped to $u(x, y) + iv(x, y)$ where

$$u(x, y) = \frac{y}{\sin \frac{2\pi}{p}} \cos \frac{2\pi}{p} + \frac{(x \sin \frac{2\pi}{p} - y \cos \frac{2\pi}{p})}{\sin \frac{2\pi}{p}} (1 - t_1) \tag{5.32}$$

$$v(x, y) = \frac{y}{\sin \frac{2\pi}{p}} \sin \frac{2\pi}{p} - \frac{(x \sin \frac{2\pi}{p} - y \cos \frac{2\pi}{p})}{\sin \frac{2\pi}{p}} t_2 \tag{5.33}$$

So we have

$$u_x(x, y) = 1 - t_1$$

$$\begin{aligned} u_y(x,y) &= \cot\frac{2\pi}{p} - \cot\frac{2\pi}{p}(1-t_1) \\ &= t_1\cot\frac{2\pi}{p} \\ v_x(x,y) &= -t_2 \\ v_y(x,y) &= 1 + t_2\cot\frac{2\pi}{p} \end{aligned}$$

$$\frac{\phi_{t,\bar{z}}}{\phi_{t,z}} = \frac{(u_x - v_y) + i(u_y + v_x)}{(u_x + v_y) + i(v_x - u_y)} = \frac{\beta(t)}{\alpha(t)}$$

where

$$\begin{aligned} \beta(t) &= 1 - t_1 - (1 + t_2\cot\frac{2\pi}{p}) + i(t_1\cot\frac{2\pi}{p} - t_2) \\ &= -(t_1 + t_2\cot\frac{2\pi}{p}) + i(t_1\cot\frac{2\pi}{p} - t_2) \qquad (5.34) \\ \alpha(t) &= 1 - t_1 + 1 + t_2\cot\frac{2\pi}{p} + i(-t_2 - t_1\cot\frac{2\pi}{p}) \\ &= 2 - t_1 + t_2\cot\frac{2\pi}{p} + i(-t_2 - t_1\cot\frac{2\pi}{p}) \qquad (5.35) \end{aligned}$$

Observe that

$$\begin{aligned} (u_x - v_y)_{t_1} &= -1 = (u_y + v_x)_{t_2} \\ (u_x - v_y)_{t_2} &= -\cot\frac{2\pi}{p} = -(u_y + v_x)_{t_1} \end{aligned}$$

Hence $\beta(t)$ is holomorphic as $(u_x - v_y) + i(u_y + v_x)$ satisfies Cauchy Riemann equation. Also $\beta(0) = 0$.

Again

$$(u_x + v_y)_{t_1} = -1 = (v_x - u_y)_{t_2}$$

$$(u_x + v_y)_{t_2} = \cot\frac{2\pi}{p} = -(v_x - u_y)_{t_2}$$

Hence $(u_x + v_y)$ and $(v_x - u_y)$ satisfy Cauchy Riemann Equation. Hence $\alpha(t)$ is holomorphic. Also $\alpha(0) = 2$.

$$\frac{\beta(t)}{\alpha(t)} = \frac{\beta(0)}{\alpha(0)} + t(\frac{\beta}{\alpha})'(0) + o(t)$$

$$= t(\frac{\beta}{\alpha})'(0) + o(t) \text{ as } \beta(0) = 0$$

$$\begin{aligned}
\text{where } (\frac{\beta}{\alpha})'(0) &= \frac{\alpha(0)\beta'(0) - \beta(0)\alpha'(0)}{\alpha(0)^2} \\
&= \frac{\beta'(0)}{\alpha(0)} \text{ as } \beta(0) = 0 \\
\beta'(t) &= (u_x - v_y)_{t_1} + i(u_y + v_x)_{t_1} \\
&= -1 + i\cot\frac{2\pi}{p} \\
\text{Thus } (\frac{\beta}{\alpha})'(0) &= \frac{-1 + i\cot\frac{2\pi}{p}}{2} \text{ as } \alpha(0) = 2
\end{aligned}$$

So

$$\frac{\beta(t)}{\alpha(t)} = t\frac{[-1 + i\cot\frac{2\pi}{p}]}{2} + o(t) = t\nu(z) + o(t)$$

Hence

$$\nu(z) = \frac{[-1 + i\cot\frac{2\pi}{p}]}{2} \tag{5.36}$$

inside the triangle with vertices origin, 1 and $e^{\frac{2\pi i}{p}}$.

Similarly

$$\nu(z) = \frac{[-1 + i\cot\frac{2\pi}{p}(p-1)]}{2} \tag{5.37}$$

inside the triangle with vertices origin, $\zeta_1(0) = 1$ and $\zeta_p(0) = e^{\frac{2\pi i}{p}(p-1)}$ since it is mapped linearly to the triangle with vertices origin, $\zeta_1(t) = 1 - t$ and $\zeta_p(t) = e^{\frac{2\pi i}{p}(p-1)}$.

Consider the triangle with vertices $\zeta_1(0) = 1$, $\zeta_2(0) = e^{\frac{2\pi i}{p}}$ and $\zeta_{p+1}(0) = 2$ and ϕ_t carries it linearly to the triangle with vertices $\zeta_1(t) = 1 - t$, $\zeta_2(t) = e^{\frac{2\pi i}{p}}$ and $\zeta_{p+1}(t) = 2$. Any point in the first triangle is represented as

$$x + iy = z = \lambda.1 + \mu.2 + (1 - \lambda - \mu)e^{\frac{2\pi i}{p}}$$

$$\begin{aligned}
&\text{where } \lambda \geq 0, \ \mu \geq 0, \ 0 \leq \lambda + \mu \leq 1 \\
&\Rightarrow \quad x = \lambda + 2\mu + (1 - (\lambda + \mu))\cos\frac{2\pi}{p} \\
&\Rightarrow \quad y = (1 - \lambda - \mu)\sin\frac{2\pi}{p}
\end{aligned}$$

Hence

$$\begin{aligned} x - \cos\frac{2\pi}{p} &= \lambda(1 - \cos\frac{2\pi}{p}) + \mu(2 - \cos\frac{2\pi}{p}) \\ y - \sin\frac{2\pi}{p} &= -\lambda \sin\frac{2\pi}{p} - \mu\sin\frac{2\pi}{p} \end{aligned}$$

$$\begin{aligned} \lambda &= \frac{(x - \cos\frac{2\pi}{p})\sin\frac{2\pi}{p} + (y - \sin\frac{2\pi}{p})(2 - \cos\frac{2\pi}{p})}{(1 - \cos\frac{2\pi}{p})\sin\frac{2\pi}{p} - \sin\frac{2\pi}{p}(2 - \cos\frac{2\pi}{p})} \\ &= \frac{-x\sin\frac{2\pi}{p} - y(2 - \cos\frac{2\pi}{p}) + 2\sin\frac{2\pi}{p}}{\sin\frac{2\pi}{p}} \\ \mu &= \frac{(x - \cos\frac{2\pi}{p})\sin\frac{2\pi}{p} + (y - \sin\frac{2\pi}{p})(1 - \cos\frac{2\pi}{p})}{(2 - \cos\frac{2\pi}{p})\sin\frac{2\pi}{p} - \sin\frac{2\pi}{p}(1 - \cos\frac{2\pi}{p})} \\ &= \frac{x\sin\frac{2\pi}{p} + y(1 - \cos\frac{2\pi}{p}) - \sin\frac{2\pi}{p}}{\sin\frac{2\pi}{p}} \end{aligned}$$

So

$$u + iv = \phi_t(z) = \lambda(1 - t) + \mu 2 + (1 - \lambda - \mu)e^{\frac{2\pi i}{p}}$$

$$\text{where} \quad t = t_1 + it_2$$

Hence

$$\begin{aligned} u &= \lambda(1 - t_1 - \cos\frac{2\pi}{p}) + \mu(2 - \cos\frac{2\pi}{p}) + \cos\frac{2\pi}{p} \\ &= \frac{[-x\sin\frac{2\pi}{p} - y(2 - \cos\frac{2\pi}{p}) + 2\sin\frac{2\pi}{p}]}{\sin\frac{2\pi}{p}}(1 - t_1 - \cos\frac{2\pi}{p}) + \\ &\quad \frac{[x\sin\frac{2\pi}{p} + y(1 - \cos\frac{2\pi}{p}) - \sin\frac{2\pi}{p}]}{\sin\frac{2\pi}{p}}(2 - \cos\frac{2\pi}{p}) + \cos\frac{2\pi}{p} \end{aligned}$$

Similarly

$$\begin{aligned} v &= -\lambda t_2 + (1 - \lambda - \mu)\sin\frac{2\pi}{p} \\ &= \lambda(-t_2 - \sin\frac{2\pi}{p}) - \mu\sin\frac{2\pi}{p} + \sin\frac{2\pi}{p} \end{aligned}$$

$$= \frac{[-x\sin\frac{2\pi}{p} - y(2-\cos\frac{2\pi}{p}) + 2\sin\frac{2\pi}{p}]}{\sin\frac{2\pi}{p}}(-t_2 - \sin\frac{2\pi}{p}) - \frac{[x\sin\frac{2\pi}{p} + y(1-\cos\frac{2\pi}{p}) - \sin\frac{2\pi}{p}]}{\sin\frac{2\pi}{p}}\sin\frac{2\pi}{p} + \sin\frac{2\pi}{p}$$

So

$$\begin{aligned}
u_x &= \frac{[-\sin\frac{2\pi}{p}(1 - t_1 - \cos\frac{2\pi}{p}) + \sin\frac{2\pi}{p}(2-\cos\frac{2\pi}{p})]}{\sin\frac{2\pi}{p}} \\
&= -1 + t_1 + 2 \\
&= 1 + t_1 \\
u_y &= \frac{[-(2-\cos\frac{2\pi}{p})(1 - t_1 - \cos\frac{2\pi}{p}) + (1-\cos\frac{2\pi}{p})(2-\cos\frac{2\pi}{p})]}{\sin\frac{2\pi}{p}} \\
&= \frac{t_1}{\sin\frac{2\pi}{p}}[2 - \cos\frac{2\pi}{p}] \\
v_x &= t_2 \\
v_y &= 1 + t_2\frac{[2-\cos\frac{2\pi}{p}]}{\sin\frac{2\pi}{p}}
\end{aligned}$$

Thus

$$\begin{aligned}
u_x - v_y &= t_1 - t_2\frac{(2-\cos\frac{2\pi}{p})}{\sin\frac{2\pi}{p}} \\
u_y + v_x &= t_2 + t_1\frac{(2-\cos\frac{2\pi}{p})}{\sin\frac{2\pi}{p}} \\
(u_x - v_y)_{t_1} &= (u_y + v_x)_{t_2} \\
(u_x - v_y)_{t_2} &= -(u_y + v_x)_{t_1}
\end{aligned}$$

Hence $\beta(t) = (u_x - v_y) + i(u_y + v_x)$ is a holomorphic function of t.

$$\begin{aligned}
u_x + v_y &= 2 + t_1 + t_2\frac{[2-\cos\frac{2\pi}{p}]}{\sin\frac{2\pi}{p}} \\
v_x - u_y &= t_2 - t_1\frac{(2-\cos\frac{2\pi}{p})}{\sin\frac{2\pi}{p}}
\end{aligned}$$

Similarly $\alpha(t) = (u_x + v_y) + i(v_x - u_y)$ is a holomorphic function of t.

$$\frac{\phi_{t,\bar z}}{\phi_{t,z}} = \frac{(u_x - v_y) + i(u_y + v_x)}{(u_x + v_y) + i(v_x - u_y)} = \frac{\beta(t)}{\alpha(t)}$$

Observe that $\beta(0) = 0$ and $\alpha(0) = 2$. Hence $\frac{\beta(t)}{\alpha(t)}$ is a holomorphic function of t in the neighborhood of $t = 0$.

$$\frac{\phi_{t,\bar{z}}}{\phi_{t,z}} = \frac{\beta(t)}{\alpha(t)} = t(\frac{\beta}{\alpha})'|_{t=0} + o(t)$$

$$\begin{aligned} \nu(z) &= (\frac{\beta}{\alpha})'|_{t=0} = \frac{\beta'(0)}{\alpha(0)} \\ &= \frac{1}{2}[(u_x - v_y)_{t_1} + i(u_y + v_x)_{t_1}] \\ &= \frac{1}{2}[1 + i\frac{(2 - \cos\frac{2\pi}{p})}{\sin\frac{2\pi}{p}}] \end{aligned} \quad (5.38)$$

Similarly

$$\nu(z) = \frac{1}{2}[1 + i\frac{(2 - \cos\frac{2\pi}{p}(p-1))}{\sin\frac{2\pi}{p}(p-1)}] \quad (5.39)$$

inside the triangular region with vertices 1 , $e^{\frac{2\pi i}{p}(p-1)}$ and 2.

5.7 Determination of elements of the triangle group $\triangle[p,p,p]$:

Theorem 3: Let

$$S = \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \quad T = \frac{1}{r}\begin{bmatrix} \frac{\rho}{\mu} - \frac{\mu}{\rho} & \mu - \frac{1}{\mu} \\ \frac{1}{\mu} - \mu & \mu\rho - \frac{1}{\mu\rho} \end{bmatrix}$$

where $\lambda = e^{-\frac{i\pi}{p}}$ and $\mu = e^{\frac{i\pi}{p}}$ and ρ is the positive root of

$$\rho^2[\mu + \frac{1}{\mu} + \frac{\mu}{\lambda} + \frac{\lambda}{\mu}] = \mu + \frac{1}{\mu} + 2$$

and $r = \frac{1}{\rho} - \rho$.

Let S^*, T^* be respectively the Mobius transformations with matrices S, T. Then

$$\triangle[p,p,p] = < S^*, T^* : (S^*)^p = (T^*)^p = (S^*T^*)^p = 1 >$$

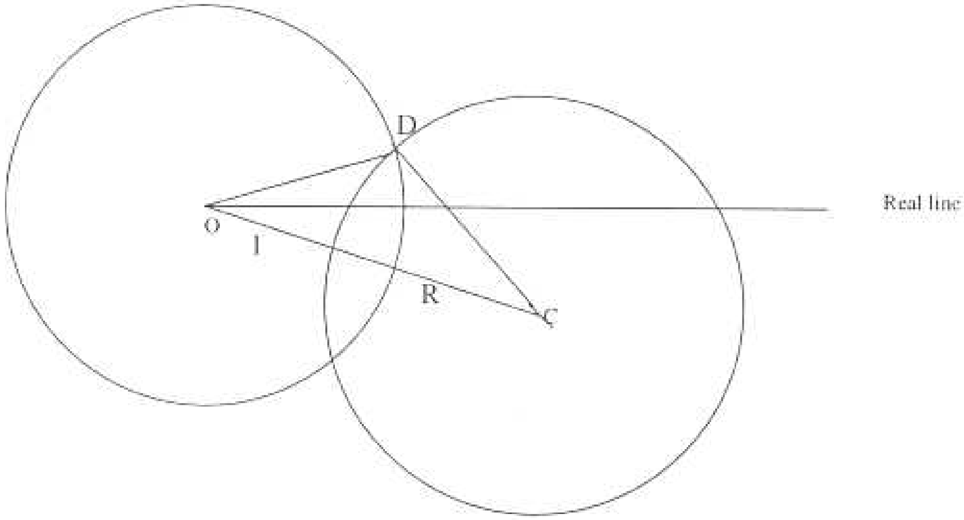


Figure 5.1: This illustrates the following

Furthermore p is the exact order of S^*, T^* and S^*T^*.

Proof:

Let us consider a circle centered at C and radius R. Then OC is the straight line joining the origin that is the center of the unit circle and C. Let the distance between C and origin be $(l + R)$. We claim that the circle centered at C is orthogonal to the unit circle if and only if $1 = l(l + 2R)$ and $(l + R) \geq 1$.

$$\begin{aligned} 1 &= l(l+2R) \\ \Leftrightarrow 1 &= (OC - PC)(OC + PC) \\ \Leftrightarrow 1 &= OC^2 - PC^2 \end{aligned}$$

where P is a point on the circle centered at C.

Suppose $1 = l(l + 2R)$ then $l < 1$ as $l + 2R \geq l$. Then the two circles must intersect at the points D and D' say. Hence $1 = OD^2 = (OD')^2$. Now by last equation $1 = OC^2 - DC^2$. So we have $OD^2 = OC^2 - DC^2$ giving the $\triangle ODC$ is right-angled at D and similarly $\triangle D'OC$ is right-angled at D'. Hence the two circles are orthogonal to each other.

Conversely if the two triangles are orthogonal to each other then there is a point D on both the circles such that $\triangle ODC$ is right-angled at D. So the center C must be outside the unit disk (for OC and DC are perpendicular) and $1 = OD^2 = OC^2 - DC^2$. Thus by above $1 = l(l+2R)$ and $l + R \geq 1$.

Let the line OC makes an angle $-\alpha$ with the positive direction of the real axis. Hence the equation of the circle is

$$\mid z - (l+R)e^{-i\alpha} \mid^2 = R^2 \tag{5.40}$$

In order to find the equation of an orthogonal circle we need to know three unknown real constants. The unknown constants are l, R, α So we can find a circle orthogonal to the unit circle which passes through $(\tilde{\rho}, 0)$ and $\tilde{\rho}e^{-i\frac{\pi}{p}}$ and whose tangent at $(\tilde{\rho}, 0)$ makes an angle $\frac{\pi}{p}$ with the positive direction of x-axis where $\tilde{\rho}$ have to be determined afterwards as we have the constraint $1 = l(l+2R)$. Next that is after determining $l, R, \alpha, \tilde{\rho}$ we shall find out the tangent at $\tilde{\rho}e^{-i\frac{\pi}{p}}$ makes an angle $\frac{\pi}{p}$ with the ray joining the origin and the point $\tilde{\rho}e^{-\frac{\pi}{p}}$.

$$1 = l(l+2R) \tag{5.41}$$

$$\begin{aligned}
& \mid z - (l+R)e^{-i\alpha} \mid^2 = R^2 \\
\Rightarrow\ & (z - (l+R)e^{-i\alpha})(\bar{z} - (l+R)e^{i\alpha}) = R^2 \\
\Rightarrow\ & \mid z \mid^2 - (l+R)\bar{z}e^{-i\alpha} - z(l+R)e^{i\alpha} + (l+R)^2 = R^2 \\
\Rightarrow\ & \mid z \mid^2 - (l+R)(ze^{i\alpha} + \bar{z}e^{-i\alpha}) + l^2 + 2lR = 0 \\
\Rightarrow\ & x^2 + y^2 - 2(l+R)(x\cos\alpha - y\sin\alpha) + 1 = 0 \text{ by (5.41)} \qquad (5.42) \\
\Rightarrow\ & 2x + 2y\frac{dy}{dx} - 2(l+R)(\cos\alpha - \frac{dy}{dx}\sin\alpha) = 0 \\
\Rightarrow\ & \frac{dy}{dx} = \frac{(l+R)\cos\alpha - x}{y + (l+R)\sin\alpha} \qquad (5.43)
\end{aligned}$$

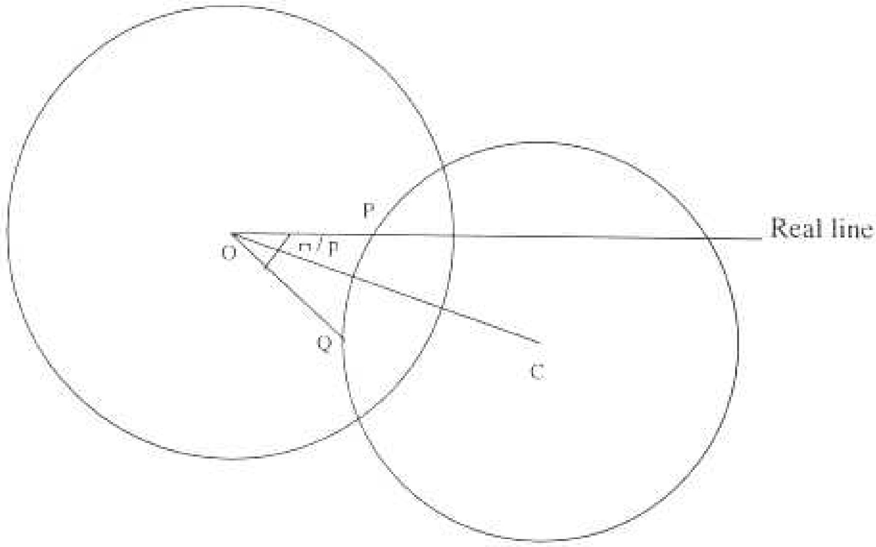


Figure 5.2: This illustrates the following

In order that the orthogonal circle passes through $P = (\tilde{\rho}, 0)$ we have from (5.42)

$$\begin{aligned} &\tilde{\rho}^2 - (l+R)\tilde{\rho}(2\cos\alpha) + 1 = 0 \\ \Rightarrow \quad &\cos\alpha = \frac{\tilde{\rho}^2 + 1}{2\tilde{\rho}(l+R)} \end{aligned} \tag{5.44}$$

In order that the orthogonal circle passes through $Q = \tilde{\rho}e^{-i\frac{\pi}{p}}$ we have

$$\begin{aligned} &\tilde{\rho}^2 - (l+R)[\tilde{\rho}e^{-i\frac{\pi}{p}}e^{i\alpha} + \tilde{\rho}e^{i\frac{\pi}{p}}e^{-i\alpha}] + 1 = 0 \\ \Rightarrow \quad &\tilde{\rho}^2 - \tilde{\rho}(l+R)[e^{i(\alpha-\frac{\pi}{p})} + e^{-i(\alpha-\frac{\pi}{p})}] + 1 = 0 \\ \Rightarrow \quad &\tilde{\rho}^2 - 2\tilde{\rho}(l+R)\cos(\alpha - \frac{\pi}{p}) + 1 = 0 \\ \Rightarrow \quad &\tilde{\rho}^2 - 2\tilde{\rho}(l+R)[\cos\alpha\cos\frac{\pi}{p} + \sin\alpha\sin\frac{\pi}{p}] + 1 = 0 \end{aligned} \tag{5.45}$$

In order that the tangent at $(\tilde{\rho}, 0)$ makes an angle $\frac{\pi}{p}$ with the positive direction of x axis we have from (5.43)

$$\tan\frac{\pi}{p} = \frac{(l+R)\cos\alpha - \tilde{\rho}}{(l+R)\sin\alpha}$$

$$\Rightarrow \quad (l+R)\sin\alpha\tan\frac{\pi}{p} = (l+R)\cos\alpha - \tilde{\rho}$$

$$\Rightarrow \quad (l+R)\sin\alpha\tan\frac{\pi}{p} = \frac{\tilde{\rho}^2+1}{2\tilde{\rho}} - \tilde{\rho} \text{ by (5.44)}$$

$$\Rightarrow \quad (l+R)\sin\alpha\tan\frac{\pi}{p} = \frac{1-\tilde{\rho}^2}{2\tilde{\rho}} \tag{5.46}$$

From (5.45) we have

$$\tilde{\rho}^2 - [(\tilde{\rho}^2+1)\cos\frac{\pi}{p} + (1-\tilde{\rho}^2)\cos\frac{\pi}{p}] + 1 = 0 \text{ by (5.44) and (5.46)}$$

$$\Rightarrow \quad \tilde{\rho}^2 - 2\cos\frac{\pi}{p} + 1 = 0$$

$$\Rightarrow \quad \tilde{\rho}^2 = -1 + 2\cos\frac{\pi}{p} \tag{5.47}$$

So $0 < \tilde{\rho}^2 < 1$ for $p \geq 4$. Hence such a $\tilde{\rho}$ exists (that is $(\tilde{\rho}, 0)$ lies inside the unit disk).

$$\tan\alpha\tan\frac{\pi}{p} = \frac{1-\tilde{\rho}^2}{1+\tilde{\rho}^2} \text{ by (5.44) and (5.46)}$$

$$\tan\alpha = \frac{1-\cos\frac{\pi}{p}}{\cos\frac{\pi}{p}}\cot\frac{\pi}{p}$$

$$\Rightarrow \quad \tan\alpha = \frac{1-\cos\frac{\pi}{p}}{\sin\frac{\pi}{p}}$$

$$\Rightarrow \quad \tan\alpha = \tan\frac{\pi}{2p}$$

$$\Rightarrow \quad \alpha = \frac{\pi}{2p}, \quad \pi + \frac{\pi}{2p} \tag{5.48}$$

For convenience we take $\tilde{\rho} > 0$. Now (5.48) gives α is $\frac{\pi}{2p}$ or $\pi + \frac{\pi}{2p}$. But α in third quadrant gives $\cos\alpha$ negative. Hence by (5.44) $\tilde{\rho}$ is negative which is a contradiction. So we have chosen $\tilde{\rho}$ to be positive and α to be $\frac{\pi}{2p}$ (The other choice could be $\tilde{\rho}$ to be negative and α to be $\pi + \frac{\pi}{2p}$ that is a symmetrically opposite thing).

So we have determined $\tilde{\rho}$ and α. Now $(l+R)$ can be determined from (5.46). Using $l(l+2R) = 1$ we can determine l and R

Denote $(l+R)$ by d for convenience.

$$d = (l+R) \;=\; \frac{\tilde{\rho}^2+1}{2\tilde{\rho}\cos\frac{\pi}{2p}} > 0 \text{ (as } \tilde{\rho} > 0 \text{ and } \alpha = \tfrac{\pi}{2p}) \tag{5.49}$$

$$\begin{aligned}
\Rightarrow d^2 = (l+R)^2 &= [\frac{1+\tilde{\rho}^2}{2\tilde{\rho}\cos\frac{\pi}{2p}}]^2 \\
&= \frac{4\cos^2\frac{\pi}{p}}{4\tilde{\rho}^2\cos^2\frac{\pi}{2p}} \text{ by (5.47)} \\
&= \frac{\cos^2\frac{\pi}{p}}{\tilde{\rho}^2\cos^2\frac{\pi}{2p}}
\end{aligned}$$

We claim that $d^2 > 1$ .For otherwise

$$\begin{aligned}
& \cos^2\frac{\pi}{p} \leq \tilde{\rho}^2\cos^2\frac{\pi}{2p} \\
\Rightarrow\ & (2\cos^2\frac{\pi}{2p} - 1)^2 \leq (2\cos\frac{\pi}{p} - 1)\cos^2\frac{\pi}{2p} \\
\Rightarrow\ & 4\cos^4\frac{\pi}{2p} + 1 - 4\cos^2\frac{\pi}{2p} \leq [2(2\cos^2\frac{\pi}{2p} - 1) - 1]\cos^2\frac{\pi}{2p} \\
\Rightarrow\ & 4\cos^4\frac{\pi}{2p} + 1 - 4\cos^2\frac{\pi}{2p} \leq [4\cos^2\frac{\pi}{2p} - 3]\cos^2\frac{\pi}{2p} \\
\Rightarrow\ & 1 - 4\cos^2\frac{\pi}{2p} \leq -3\cos^2\frac{\pi}{2p} \\
\Rightarrow\ & \cos^2\frac{\pi}{2p} \geq 1
\end{aligned}$$

which is a contradiction. Hence $d > 1$ (as $d > 0$ and $d^2 > 1$).

$$\begin{aligned}
& l(l+2R) = 1 \\
\Rightarrow\ & l[2(l+R) - l] = 1 \\
\Rightarrow\ & l^2 - 2l(l+R) + 1 = 0 \\
\Rightarrow\ & l^2 - 2ld + 1 = 0 \ \text{ as } l + R = d \text{ by (5.49)}
\end{aligned}$$

Let γ and β be two roots of the above equation.

$$\begin{aligned}
\gamma + \beta &= 2d \\
\gamma\beta &= 1 \\
(\gamma - \beta)^2 &= (\gamma + \beta)^2 - 4\gamma\beta \\
&= 4d^2 - 4
\end{aligned}$$

$$= 4(d^2 - 1) > 0 \quad \text{as } d > 1$$

$$\text{So } \gamma - \beta = \pm 2\sqrt{d^2 - 1} = \text{real} \quad \text{as } d > 1$$

So

$$\gamma = d + \sqrt{d^2 - 1}$$

and

$$\beta = d - \sqrt{d^2 - 1}$$

If we take $l = \gamma > d$ then as $l + R = d$ we have $R < 0$. On the other hand if we choose $l = \beta < d$ then $R > 0$.

So with the choice of $l = d - \sqrt{d^2 - 1}$, $R = \sqrt{d^2 - 1}$, $\alpha = \frac{\pi}{2p}$ and $\tilde{\rho}$ is the positive square-root of $2\cos\frac{\pi}{p} - 1$ where $d = \frac{\cos\frac{\pi}{p}}{\tilde{\rho}\cos\frac{\pi}{2p}}$ we can obtain an orthogonal circle which passes through the points $(\tilde{\rho}, 0)$ and $\tilde{\rho}e^{-\frac{i\pi}{p}}$ and whose tangent at $(\tilde{\rho}, 0)$ makes an angle $\frac{\pi}{p}$ with the positive direction of the x axis.

We claim that the tangent to the orthogonal circle at $\tilde{\rho}e^{-i\frac{\pi}{p}}$ makes an angle $\pi - \frac{2\pi}{p}$ with the positive direction of the x-axis that is the same as the ray joining the origin and $\tilde{\rho}e^{-i\frac{\pi}{p}}$ with the tangent at $\tilde{\rho}e^{-i\frac{\pi}{p}}$ makes an angle $\frac{\pi}{p}$.

$$\begin{aligned}
\frac{dy}{dx}\Big|_{\tilde{\rho}e^{-i\frac{\pi}{p}}} &= \frac{(l+R)\cos\alpha - \tilde{\rho}\cos\frac{\pi}{p}}{-\tilde{\rho}\sin\frac{\pi}{p} + (l+R)\sin\alpha} \quad \text{by (5.43)} \\
&= \frac{\frac{\tilde{\rho}^2+1}{2\tilde{\rho}} - \tilde{\rho}\cos\frac{\pi}{p}}{-\tilde{\rho}\sin\frac{\pi}{p} + \frac{1-\tilde{\rho}^2}{2\tilde{\rho}}\cot\frac{\pi}{p}} \quad \text{by (5.44) and (5.46)} \\
&= \frac{(\tilde{\rho}^2 + 1 - 2\tilde{\rho}^2\cos\frac{\pi}{p})}{-2\tilde{\rho}^2\sin\frac{\pi}{p} + (1 - \tilde{\rho}^2)\cot\frac{\pi}{p}} \\
&= \frac{(\tilde{\rho}^2 + 1 - 2\tilde{\rho}^2\cos\frac{\pi}{p})\sin\frac{\pi}{p}}{-2\tilde{\rho}^2\sin^2\frac{\pi}{p} + (1 - \tilde{\rho}^2)\cos\frac{\pi}{p}} \\
&= \frac{[1 + \tilde{\rho}^2(1 - 2\cos\frac{\pi}{p})]\sin\frac{\pi}{p}}{-\tilde{\rho}^2(\cos\frac{\pi}{p} + 2\sin^2\frac{\pi}{p}) + \cos\frac{\pi}{p}}
\end{aligned}$$

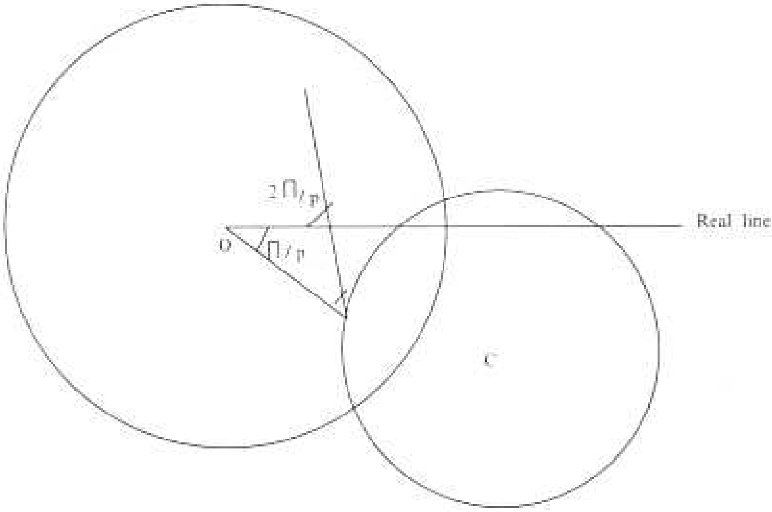


Figure 5.3: This illustrates the following

$$
\begin{aligned}
&= \frac{[1-(1-2\cos\frac{\pi}{p})^2]\sin\frac{\pi}{p}}{(1-2\cos\frac{\pi}{p})(2+\cos\frac{\pi}{p}-2\cos^2\frac{\pi}{p})+\cos\frac{\pi}{p}} \quad \text{by (5.47)} \\
&= \frac{[-4\cos^2\frac{\pi}{p}+4\cos\frac{\pi}{p}]\sin\frac{\pi}{p}}{(1-2\cos\frac{\pi}{p})(2-2\cos^2\frac{\pi}{p}+\cos\frac{\pi}{p})+\cos\frac{\pi}{p}} \\
&= \frac{2\sin\frac{2\pi}{p}(1-\cos\frac{\pi}{p})}{2-2\cos\frac{\pi}{p}-4\cos^2\frac{\pi}{p}+4\cos^3\frac{\pi}{p}} \\
&= \frac{2\sin\frac{2\pi}{p}(1-\cos\frac{\pi}{p})}{2(1-\cos\frac{\pi}{p})(1-2\cos^2\frac{\pi}{p})} \\
&= \frac{\sin\frac{2\pi}{p}}{1-2\cos^2\frac{\pi}{p}} \\
&= \frac{\sin\frac{2\pi}{p}}{-\cos\frac{2\pi}{p}} \\
&= -\tan\frac{2\pi}{p} \\
\tan\phi &= -\tan\frac{2\pi}{p} = \tan(\pi-\frac{2\pi}{p}) \\
\phi &= \pi-\frac{2\pi}{p}
\end{aligned}
$$

where ϕ is the angle which the tangent at $\hat{\rho}e^{-i\frac{\pi}{p}}$ makes with the positive direction of the x-axis.

$$\phi = \pi - \frac{2\pi}{p}$$

implies the tangent makes at $\hat{\rho}e^{-i\frac{\pi}{p}}$ an angle $\frac{\pi}{p}$ with the ray joining 0 to $\hat{\rho}e^{-i\frac{\pi}{p}}$.

So there is an orthogonal circle which passes through $(\hat{\rho}, 0)$ and $\hat{\rho}e^{-i\frac{\pi}{p}}$ and whose tangent at $(\hat{\rho}, 0)$ makes an angle $\frac{\pi}{p}$ with the part of real axis joining the origin and $(\hat{\rho}, 0)$ also its tangent at $\hat{\rho}e^{-i\frac{\pi}{p}}$ makes an angle $\frac{\pi}{p}$ with the ray joining origin with $\hat{\rho}e^{-i\frac{\pi}{p}}$. So we have a non-euclidean triangle whose vertices are the origin , $(\hat{\rho}, 0)$ and $\hat{\rho}e^{-i\frac{\pi}{p}}$ and edges are part of the real axis joining origin with $(\hat{\rho}, 0)$, the part of the radius of the unit circle joining origin and $\hat{\rho}e^{-i\frac{\pi}{p}}$ and the arc of the orthogonal circle joining $(\hat{\rho}, 0)$ and $\hat{\rho}e^{-i\frac{\pi}{p}}$ (which lies inside the unit disk). So each angle of the non-euclidean triangle is $\frac{\pi}{p}$.

Hence the triangle group $\triangle[p, p, p]$ is generated by elliptic element S^* with fixed points 0 and ∞ of order p , T^* with fixed points $\hat{\rho}$ and $\frac{1}{\hat{\rho}}$ of order p, $(ST)^*$ with fixed pints $\hat{\rho}e^{-i\frac{\pi}{p}} and \frac{1}{\hat{\rho}}e^{-i\frac{\pi}{p}}$ of order p. That is

$$\triangle[p, p, p] = < S^*, T^* : S^{*p} = T^{*p} = (ST)^{*p} = 1 >$$

The elliptic transformation S^* of order p with fixed points 0 and ∞ is obviously

$$\begin{aligned} w &= e^{-\frac{2\pi i}{p}} z \\ &= \frac{\lambda}{\frac{1}{\lambda}} z \quad \text{where } \lambda = e^{-i\frac{\pi}{p}} \end{aligned} \tag{5.50}$$

Hence the matrix associated with the transformation is

$$S = \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \tag{5.51}$$

$$Certainly \quad S^p = 1$$

The elliptic transformation T^* of order p with fixed points $\tilde{\rho}$ and $\frac{1}{\tilde{\rho}}$ is given by the following

$$
\begin{aligned}
& \frac{w-\tilde{\rho}}{w-\frac{1}{\tilde{\rho}}} = e^{-\frac{2\pi i}{p}} \frac{z-\tilde{\rho}}{z-\frac{1}{\tilde{\rho}}} \\
\Rightarrow\quad & \frac{w-\tilde{\rho}}{\tilde{\rho}w-1} = \frac{1}{\mu^2}\frac{z-\tilde{\rho}}{\tilde{\rho}z-1} \quad \text{where } \mu = e^{\frac{i\pi}{p}} \\
\Rightarrow\quad & \mu^2(w-\tilde{\rho})(\tilde{\rho}z-1) = (\tilde{\rho}w-1)(z-\tilde{\rho}) \\
\Rightarrow\quad & w[\mu^2(\tilde{\rho}z-1) - \tilde{\rho}(z-\tilde{\rho})] = \mu^2\tilde{\rho}(\tilde{\rho}z-1) - (z-\tilde{\rho}) \\
\Rightarrow\quad & w = \frac{z[\mu^2\tilde{\rho}^2-1] + [\tilde{\rho}-\mu^2\tilde{\rho}]}{z[\mu^2\tilde{\rho}-\tilde{\rho}] + [\tilde{\rho}^2-\mu^2]} \\
\Rightarrow\quad & w = \frac{z[\mu\tilde{\rho}-\frac{1}{\mu\tilde{\rho}}] + [\frac{1}{\mu}-\mu]}{z[\mu-\frac{1}{\mu}] + [\frac{\tilde{\rho}}{\mu}-\frac{\mu}{\tilde{\rho}}]} \\
\Rightarrow\quad & w = \frac{z[\frac{1}{\mu\tilde{\rho}}-\mu\tilde{\rho}] + [\mu-\frac{1}{\mu}]}{z[\frac{1}{\mu}-\mu] + [\frac{\mu}{\tilde{\rho}}-\frac{\tilde{\rho}}{\mu}]}
\end{aligned}
\tag{5.52}
$$

Observe that

$$
\begin{aligned}
& (\frac{1}{\mu\tilde{\rho}} - \mu\tilde{\rho})(\frac{\mu}{\tilde{\rho}} - \frac{\tilde{\rho}}{\mu}) + (\mu - \frac{1}{\mu})^2 \\
= \quad & \frac{1}{\tilde{\rho}^2} - \mu^2 - \frac{1}{\mu^2} + \tilde{\rho}^2 + \mu^2 + \frac{1}{\mu^2} - 2 \\
= \quad & (\tilde{\rho} - \frac{1}{\tilde{\rho}})^2
\end{aligned}
$$

$$\text{Setting} \quad r = \tilde{\rho} - \frac{1}{\tilde{\rho}} \tag{5.53}$$

we have the matrix T associated with T^*

$$
\begin{aligned}
T \;&=\; \frac{1}{r}\begin{bmatrix} \frac{1}{\mu\tilde{\rho}} - \mu\tilde{\rho} & \mu - \frac{1}{\mu} \\ \frac{1}{\mu} - \mu & \frac{\mu}{\tilde{\rho}} - \frac{\tilde{\rho}}{\mu} \end{bmatrix} \\
&=\; \frac{1}{r}\begin{bmatrix} \frac{\rho}{\mu} - \frac{\mu}{\rho} & \mu - \frac{1}{\mu} \\ \frac{1}{\mu} - \mu & \mu\rho - \frac{1}{\mu\rho} \end{bmatrix}
\end{aligned}
\tag{5.54}
$$

$$\text{where } \rho = \frac{1}{\hat{\rho}} \tag{5.55}$$

$$\text{Hence } r = \frac{1}{\rho} - \rho \tag{5.56}$$

Also observe that

$$\rho^2[\mu + \frac{1}{\mu} + \frac{\mu}{\lambda} + \frac{\lambda}{\mu}] = \mu + \frac{1}{\mu} + 2 \tag{5.57}$$

$$\Leftrightarrow \quad \rho^2[e^{i\frac{\pi}{p}} + e^{-i\frac{\pi}{p}} + e^{\frac{2i\pi}{p}} + e^{-\frac{2i\pi}{p}}] = e^{i\frac{\pi}{p}} + e^{-i\frac{\pi}{p}} + 2$$

$$\text{as } \mu = e^{i\frac{\pi}{p}} \quad \lambda = e^{-i\frac{\pi}{p}}$$

$$\Leftrightarrow \quad \rho^2[2\cos\frac{\pi}{p} + 2\cos\frac{2\pi}{p}] = 2\cos\frac{\pi}{p} + 2$$

$$\Leftrightarrow \quad \rho^2 = \frac{1 + \cos\frac{\pi}{p}}{\cos\frac{\pi}{p} + \cos\frac{2\pi}{p}}$$

$$\Leftrightarrow \quad \rho^2 = \frac{1 + \cos\frac{\pi}{p}}{2\cos^2\frac{\pi}{p} + \cos\frac{\pi}{p} - 1}$$

$$\Leftrightarrow \quad \rho^2 = \frac{1 + \cos\frac{\pi}{p}}{(1 + \cos\frac{\pi}{p})(2\cos\frac{\pi}{p} - 1)}$$

$$\Leftrightarrow \quad \rho^2 = \frac{1}{2\cos\frac{\pi}{p} - 1} = \frac{1}{\hat{\rho}^2} \quad \text{by (5.47)} \tag{5.58}$$

The elliptic transformation of order p with fixed points $\frac{1}{\rho}e^{-i\frac{\pi}{p}} = \hat{\rho}e^{-i\frac{\pi}{p}}$ and $\rho e^{-i\frac{\pi}{p}} = \frac{1}{\hat{\rho}}e^{-\frac{i\pi}{p}}$ is given by

$$\frac{w - \frac{1}{\rho}e^{-i\frac{\pi}{p}}}{w - \rho e^{-i\frac{\pi}{p}}} = \mu^2 \frac{z - \frac{1}{\rho}e^{-\frac{\pi i}{p}}}{z - \rho e^{-\frac{\pi i}{p}}}$$

$$\Rightarrow \quad (z - \rho e^{-i\frac{\pi}{p}})(w - \frac{1}{\rho}e^{-i\frac{\pi}{p}}) = \mu^2(z - \frac{1}{\rho}e^{-i\frac{\pi}{p}})(w - \rho e^{-i\frac{\pi}{p}})$$

$$\Rightarrow \quad w = \frac{z e^{-i\frac{\pi}{p}}(\frac{1}{\rho} - \rho\mu^2) - e^{-2i\frac{\pi}{p}}(1 - \mu^2)}{z(1 - \mu^2) - e^{-\frac{\pi i}{p}}(\rho - \frac{\mu^2}{\rho})}$$

$$\Rightarrow \quad w = \frac{z\lambda(\frac{1}{\rho} - \rho\mu^2) + \lambda(\mu - \frac{1}{\mu})}{\frac{1}{\lambda}z(\frac{1}{\mu} - \mu) + \frac{1}{\lambda}(-\frac{\rho}{\mu^2} + \frac{1}{\rho})}$$

$$\text{as } \lambda = \frac{1}{\mu}, \quad \lambda = e^{-i\frac{\pi}{p}}$$

$$\Rightarrow \quad w = \frac{\lambda z(\frac{\rho}{\mu} - \frac{\mu}{\rho}) + \lambda(\mu - \frac{1}{\mu})}{\frac{1}{\lambda}z(\frac{1}{\mu} - \mu) + \frac{1}{\lambda}(\rho\mu - \frac{1}{\mu\rho})} \tag{5.59}$$

The last equality holds because we claim that

$$1 - \rho^2\mu^2 = \frac{\rho^2}{\mu} - \mu \tag{5.60}$$

$$\text{and} \quad 1 - \lambda^2\rho^2 = \mu\rho^2 - \frac{1}{\mu} \tag{5.61}$$

From (5.57) we have

$$\rho^2[\mu + \frac{1}{\mu} + \frac{\mu}{\lambda} + \frac{\lambda}{\mu}] = \mu + \frac{1}{\mu} + 2$$

$$\Rightarrow \quad \mu\rho^2 + \frac{\rho^2}{\mu} + \mu^2\rho^2 + \frac{\rho^2}{\mu^2} = \mu + \frac{1}{\mu} + 2 \text{ as } \mu = \tfrac{1}{\lambda}$$

$$\Rightarrow \quad \frac{\rho^2}{\mu} - \mu = (1 - \mu^2\rho^2) + 1 + \frac{1}{\mu} - \mu\rho^2 - \frac{\rho^2}{\mu^2}$$

(5.60) will be proved if we can show that

$$1 + \frac{1}{\mu} - \mu\rho^2 - \frac{\rho^2}{\mu^2} = 0 \tag{5.62}$$

$$\begin{aligned}
& 1 + \frac{1}{\mu} - \mu\rho^2 - \frac{\rho^2}{\mu^2} \\
= \;& 1 + e^{-i\frac{\pi}{p}} - \rho^2 e^{i\frac{\pi}{p}} - \rho^2 e^{-\frac{2i\pi}{p}} \\
= \;& [1 + \cos\frac{\pi}{p} - \rho^2(\cos\frac{\pi}{p} + \cos\frac{2\pi}{p})] + i[-\sin\frac{\pi}{p} - \rho^2(\sin\frac{\pi}{p} - \sin\frac{2\pi}{p})] \\
= \;& [1 + \cos\frac{\pi}{p} - \rho^2(2\cos^2\frac{\pi}{p} + \cos\frac{\pi}{p} - 1)] + i[-\sin\frac{\pi}{p} + \sin\frac{\pi}{p}] \text{by (5.58)} \\
= \;& [(1 + \cos\frac{\pi}{p}) - \rho^2(2\cos\frac{\pi}{p} - 1)(1 + \cos\frac{\pi}{p})] \\
= \;& [(1 + \cos\frac{\pi}{p}) - (1 + \cos\frac{\pi}{p})] \quad \text{by (5.58)} \\
= \;& 0
\end{aligned}$$

Hence (5.60) is proved.

Again from (5.57) we have

$$\rho^2[\mu + \frac{1}{\mu} + \frac{\mu}{\lambda} + \frac{\lambda}{\mu}] = \mu + \frac{1}{\mu} + 2$$
$$\Rightarrow \rho^2[\mu + \frac{1}{\mu} + \mu^2 + \lambda^2] = \mu + \frac{1}{\mu} + 2$$
$$\Rightarrow \mu\rho^2 - \frac{1}{\mu} = (1 - \lambda^2\rho^2) + \mu + 1 - \rho^2(\frac{1}{\mu} + \mu^2)$$

So (5.61) will be proved if we can show that

$$1 + \mu - \rho^2(\mu^2 + \frac{1}{\mu}) = 0$$

$$\begin{aligned} & 1 + \mu - \rho^2(\mu^2 + \frac{1}{\mu}) \\ = & \overline{[1 + \frac{1}{\mu} - \rho^2(\frac{1}{\mu^2} + \mu)]} \text{ as } \mu = e^{i\frac{\pi}{p}} \\ = & \overline{[1 + \frac{1}{\mu} - \mu\rho^2 - \frac{\rho^2}{\mu^2}]} \\ = & 0 \text{ by } (5.62) \end{aligned}$$

Hence (5.61) is proved.

Now from (5.59) the matrix associated with the elliptic transformation of order p with fixed points $\frac{1}{\rho}e^{-i\frac{\pi}{p}} = \bar{\rho}e^{-i\frac{\pi}{p}}$ and $\rho e^{-i(\frac{\pi}{p}} = \frac{1}{\bar{\rho}}e^{-i\frac{\pi}{p}}$ is given by

$$\begin{aligned} W &= \begin{bmatrix} \lambda(\frac{\rho}{\mu} - \frac{\mu}{\rho}) & \lambda(\mu - \frac{1}{\mu}) \\ \frac{1}{\lambda}(\frac{1}{\mu} - \mu) & \frac{1}{\lambda}(\mu\rho - \frac{1}{\mu\rho}) \end{bmatrix} \qquad (5.63) \\ &= rST \text{ So } \det W = r \text{ as } \det S = \det T = 1 \end{aligned}$$

As

$$ST = \frac{1}{r}\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}\begin{bmatrix} \frac{\rho}{\mu} - \frac{\mu}{\rho} & \mu - \frac{1}{\mu} \\ \frac{1}{\mu} - \mu & \mu\rho - \frac{1}{\mu\rho} \end{bmatrix}$$

$$= \frac{1}{r}\begin{bmatrix} \lambda(\frac{\rho}{\mu}-\frac{\mu}{\rho}) & \lambda(\mu-\frac{1}{\mu}) \\ \frac{1}{\lambda}(\frac{1}{\mu}-\mu) & \frac{1}{\lambda}(\mu\rho-\frac{1}{\mu\rho}) \end{bmatrix}$$

Hence

$$(ST)^p = 1$$

So

$$\triangle = \triangle[p,p,p] =< S^*, T^* : S^{*p} = T^{*p} = (ST)^{*p} = 1 > \tag{5.64}$$

where S^* and T^* are the transformations associated with the matrices S, T (given below) respectively, where

$$S = \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \quad T = \frac{1}{r}\begin{bmatrix} \frac{\rho}{\mu}-\frac{\mu}{\rho} & \mu-\frac{1}{\mu} \\ \frac{1}{\mu}-\mu & \mu\rho-\frac{1}{\mu\rho} \end{bmatrix} \tag{5.65}$$

$$\mu = \frac{1}{\lambda} \quad \mu = e^{i\frac{\pi}{p}} \quad \lambda = e^{-i\frac{\pi}{p}} \tag{5.66}$$

and ρ is the positive root of

$$\rho^2 = \frac{1}{2\cos\frac{\pi}{p} - 1} \tag{5.67}$$

$$\text{or equivalently} \quad \rho^2[\mu + \frac{1}{\mu} + \frac{\mu}{\lambda} + \frac{\lambda}{\mu}] = \mu + \frac{1}{\mu} + 2 \qquad \Box \tag{5.68}$$

5.8 Determination of elements of the commutator subgroup $[\triangle, \triangle]$ of the triangle-group $\triangle = \triangle[p,p,p]$:

We claim that $S^i(T^jST^{-j}S^{-1})S^{-i}$ where $0 \le i \le (p-2)$ and $1 \le j \le (p-2)$ generates $[\triangle, \triangle]$ that is the commutator subgroup of $\triangle = \triangle[p,p,p]$.

Let us recall some basic facts of Algebra (For these and allied matters see Johnson [J])

The Schreier transversal

Let H denote a fixed subgroup of a free group $F = F(X)$ where $X = \{x_i\}$ is the set of free generators of F. Recall that a right coset of H in F is a subset of F of the form $Hw = \{hw : h \in H\}$ for fixed $w \in F$ and that any two cosets are either equal or disjoint:given $u, v \in F$ then $Hu = Hv$ or $Hu \cap Hv = \varphi$.

The cosets of H thus partitions F into disjoint sets and by choosing one element from each coset,we obtain a (right) transversal U for H in F. For any $w \in F$, $Hw \cap U$ thus consists of a single element which will be denoted by $\bar{w}$.

Definition: A subset S of F has *Schreier property* if it contains all initial segments of all its elements that is

$$w = x_1 x_2 \dots x_n \in S \Rightarrow x_1 x_2 \dots x_{n-1} \in S$$

where $l(w) = n \geq 1$.

A Schreier transversal for H in F is a (right) transversal for H with Schreier property.

Note that every Schreier subset and thus every Schreier transversal, contains the empty word e.

The next step is to find the generators for H.

This is done in terms of Schreier transversal U for H in F and the function

$$\begin{array}{ccc} F & \longrightarrow & U \\ w & \mapsto & \overline{w} \end{array}$$

defined by $Hw \cap U = \{\overline{w}\}$.

Also note that $\overline{\overline{w}} = \overline{w}$, $Hw = H\overline{w}$ $\forall w \in F$.

Lemma 4: The elements of the set

$$A = \{ux\overline{ux}^{-1} : u \in U, \ \ x \in X^{\pm}\}$$

generates H.(For proof see Johnson [J]) $\square$

Lemma 5:

$$\begin{aligned} B &= \{ux\overline{ux}^{-1} : u \in U,\ x \in X,\ ux \notin U\} \\ \hat{B} &= \{ux\overline{ux}^{-1} : u \in U,\ x \in X^{-1},\ ux \notin U\} \\ B^{-1} &= \{b^{-1} : b \in B\} \end{aligned}$$

Then $B^{-1} = \hat{B}$ and $A - \{e\} = B \cup B^{-1}$. (For proof see Johnson[J]) $\square$

Lemma 6: Given a subgroup K of a free group F. Let R be a subset of F whose normal closure $\overline{R}$ in F lies in K. Then $\overline{R}$ is the normal closure in K of the set

$$\hat{R} = \{uru^{-1} : u \in U,\ r \in R\}$$

where U is any transversal for K in F.(For proof see Johnson [J]) $\square$

Proposition 1: Let H be a subgroup of $G = < X|R >$. Then $H = < B|\hat{S} >$ where $B = \{ux\overline{ux}^{-1} : u \in U,\ x \in X\}$ and $\hat{S}$ is the element of $\hat{R} = \{uru^{-1} : u \in U,\ r \in R\}$ expressed in terms of elements of B. U is the Schreier transversal of H in G.

Proof: Consider $G = < X|R >$ and H is a subgroup of G. Let $F = F(X)$ that is the free group generated by the set X.

$$\nu = nat : F(X) \longrightarrow G$$

with $K = \nu^{-1}(H)$. Let U be the Schreier transversal for K in F. Then the set B (of Lemma 5) freely generates K (by the above results). Let $\overline{R}$ be the normal closure of R in F.Then $\overline{R}$ is generated by wrw^{-1}, $w \in F$. But $\nu(wrw^{-1}) = 1$. So $wrw^{-1} \in K$. Hence $\overline{R} \subset K$. Hence by lemma 6, $\overline{R}$ is the normal closure in K of $\hat{R} = \{uru^{-1} : u \in U\ r \in R\}$. Let $\hat{S}$ denote the set of elements of $\hat{R}$ written as words in $B^{\pm}$ where $B = \{ux\overline{ux}^{-1} : u \in U, x \in X\}$. Hence $< B|\hat{S} >$ is a representation of

$K/\overline{R}$ which is isomorphic to H. Now if we can show that pull back of a transversal is a transversal then the proposition is proved. Let

$$G = Hg_1 \cup \ldots \cup Hg_n$$

where $g_i = x_1^{i_1} \ldots x_r^{i_r}$ is a reduced word in $X^{\pm}$. Then we claim that

$$F = Kg_1 \cup \ldots \cup Kg_n$$

$$\begin{aligned}
x \in F &\Rightarrow \nu(x) = g \in G \text{ for some } g \in G \\
&\Rightarrow g = hg_i \text{ for some } h \in H \quad 1 \leq i \leq n \\
&\Rightarrow \nu(xg_i^{-1}) = \nu(x)\nu(g_i)^{-1} = gg_i^{-1} = h \in H \\
&\Rightarrow xg_i^{-1} \in K \\
&\Rightarrow Kx = Kg_i
\end{aligned}$$

Hence $x \in Kg_i$. Again

$$\begin{aligned}
Kg_i \cap Kg_j \neq \varphi &\Rightarrow g_i g_j^{-1} \in K \\
&\Rightarrow \nu(g_i g_j^{-1}) \in H \\
&\Rightarrow g_i g_j^{-1} \in H \\
&\Rightarrow Hg_i = Hg_j
\end{aligned}$$

which is a contradiction.

Hence if $\{g_1, \ldots g_n\} = U$ form a Schreier transversal for H in G that is all initial words of elements of U is in U and $G = \cup_{i=1}^{n} Hg_i$ is a complete coset representation of H in G then U will suffice our purpose. Recall that

$\{ux\overline{ux}^{-1} : u \in U \ x \in X\} = B$ generates K as $\nu(K) = H$. Also it does not differ whether we calculate $\overline{ux}$ in G or in F. Indeed

$$Hux = Hg_i \Rightarrow Kux = Kg_i$$

as $\nu(uxg_i^{-1}) \in H$. □

We need the following

(i) A Schreier transversal U for H in $G =< X|R >$.

(ii) Generators B in H.

(iii) Defining relators $\hat{R} = \hat{R}(X)$ for H.

(iv) Defining relators $\hat{S} = \hat{S}(B)$ for H.

Once (i) is found out

$$B = \{ux\overline{ux}^{-1} : u \in U, \quad x \in X\}$$

$$\hat{R} = \{uru^{-1} : u \in U, \quad r \in R\}$$

$\hat{S}$ is the elements of $\hat{R}$ expressed in terms of elements of B.Then $< B|\hat{S} >$ is a presentation of H.

Recall some algebra of finitely generated abelian groups. Let $G^{/}$ or $[G, G]$ denote the derived subgroup or the commutator subgroup of G that is the subgroup generated by the set $\{ghg^{-1}h^{-1} = [g, h] \mid g, h \in G\}$. Then $G^{/}$ is a normal subgroup of G and $G_{ab} = G/G^{/}$ is abelian in fact this is the largest abelian factor group of G.

Proposition 2: If $G =< X|R >$ then $G_{ab} =< X|R, C >$ where $X = \{x_1, \ldots x_r\}$ $C = \{[x_i, x_j] | 1 \leq i < j \leq r\}$ $r \in \mathbf{Z}$.(For proof see Johnson [J]) □

In case of $\triangle = \triangle[p, p, p]$ given by

$$\triangle =< S, T | S^p = T^p = (ST)^p = 1 >$$

$$\begin{aligned}\triangle_{ab} &= < S,T|S^p,T^p,(ST)^p,STS^{-1}T^{-1} > \\ &= < S,T|S^p,T^p,S^pT^p,STS^{-1}T^{-1} >\end{aligned}$$

Let us denote the free abelian group on X by $A = A(X)$ and continue to write it additively.

Proposition 3: If X generates an abelian group G then there is an epimorphism $\theta : A(X) \longrightarrow G$ fixing X element-wise. Every abelian group is a homomorphic image of some free abelian group.(For proof see Johnson [J]) □

Proposition 4: If $A = A(X)$ is free abelian group of rank r and B is a subgroup of A then B is free abelian of rank atmost r.(For proof see Johnson [J]) □

Example: In case of $\triangle = \triangle[p,p,p]$ $\triangle_{ab} \equiv \frac{A}{B}$ where $A = A(S,T)$ is a free abelian group of rank 2 and $B =< pS, pT, pS + pT >$.

Let $A = A(X)$ is a free abelian group of rank r and B is finitely generated by a set $Y = \{y_1, \ldots, y_s\}$ of $\mathbf{Z}$ -linear combination of elements of X. Let

$$y_k = \sum_{i=1}^{r} m_{ki}x_i \quad 1 \leq k \leq s$$

Then $M = (m_{ki})$ is the coefficient matrix.

Proposition 5: The subgroup $B =< Y >$ of $A(X)$ is determined by the $s \times r$ coefficient matrix $M = (m_{ki})$ given above. Changing generators of X and Y corresponds to post and pre -multiplication of M respectively by invertible matrices over $\mathbf{Z}$.

Conversely: If T and Q^{-1} are unimodular, the coefficient matrix TMQ^{-1} determines the same subgroup B of $A(X)$ as does M(and thus the same factor group $\frac{A(X)}{B}$).(For proof see Johnson [J]) □

In case of $\triangle$ the coefficient matrix is

$$\begin{bmatrix} p & 0 \\ 0 & p \\ p & p \end{bmatrix}$$

Invariant factor theorem for matrices

Just as in ordinary linear algebra, pre and post multiplication by invertible matrices corresponds to performing elementary row and column operations.

P: Permuting rows.

M: Multiplying a row by a unit (± 1).

A: Adding to a row a scalar multiple of another row and similarly for columns.

We now describe an algorithm for reducing any $r \times s$ matrix M over $\mathbf{Z}$ to canonical form

$$D = diag(d_1, \ldots, d_k) \quad k = min(r, s)$$

$d_i \in \mathbf{Z} \cup \{0\}$ $1 \leq i \leq k$,d_i / d_{i+1} $1 \leq i \leq k-1$.

1. Pick an entry d of M of minimal positive modulus (if no such exists, then M is the zero matrix which is already in canonical form) and remove it to $(1,1)$ place by P operations.

2. Use an M operation to ensure $d > 0$.

3. Use an A operation to replace the $(2,1)$ entry by b, $0 \leq b < d$ (If $s = 1$ go at once to step 7).

4. If $b = 0$, go to step 6, and if not transpose rows 1 and 2 and revert to step 3,with b in place of d.

5. Repeat steps 3 and 4 until the $(2,1)$ entry is zero (whereupon the new d in $(1,1)$ th place will be the highest common factor of $(1,1)$ and $(2,1)$th entries at the start of step 3).

6.Perform steps $3-5$ on the remaining rows until the only nonzero entry in the first column is the d in $(1,1)$ place.

7.Perform steps $3-6$ on columns until every entry in the first row is zero except for d in the $(1,1)$ place.

8. If d divides every entry in the matrix, go to step 11. If not use a P operation and an A operation to get b into $(2,1)$ place with d does not divide b.

9. Repeat steps $3-5$ until the only nonzero entry in the first column is d in the place $(1,1)$, then return to step 7 with new d.

10. Repeat steps 8 and 9 the $(1,1)$ entry $d = d_1$ divides every other entry in the matrix.

11. Apply steps $1-10$ to the matrix obtained by removing the first row and column to obtain a d_2 in the place $(1,1)$ that divides every other entry (and is divisible by d_1)

12.Repeat steps 11 until either rows or columns or nonzero entries run out.

$$\begin{aligned} G_{ab} &= \frac{A(X)}{B} \\ &= \frac{A\{x_1,\ldots,x_r\}}{<d_1x_1, d_2x_2,\ldots d_kx_k>} \\ &\equiv \mathbf{Z}_{d_1}\times\mathbf{Z}_{d_2}\times\ldots\times\mathbf{Z}_{d_k}\times\mathbf{Z}\times\ldots\mathbf{Z} \end{aligned}$$

(where $r-k$ copies of $\mathbf{Z}$).

Now apply the above algorithm to

$$\text{Step I} \quad \begin{bmatrix} p & 0 \\ 0 & p \\ p & p \end{bmatrix}$$

Step II $\begin{bmatrix} p & 0 \\ 0 & p \\ 0 & p \end{bmatrix}$

[obtained by subtracting first row from the third row.]

Step III $\begin{bmatrix} p & 0 \\ 0 & p \\ 0 & 0 \end{bmatrix}$

[obtained by subtracting 2nd row from the 3rd row.]

Thus we have

$$\begin{aligned} \triangle_{ab} = \triangle/[\triangle,\triangle] &\equiv \frac{A\{S,T\}}{< pS, pT >} \\ &\equiv \mathbf{Z}_p \times \mathbf{Z}_p \end{aligned} \tag{5.69}$$

Note that

$$U = \{S^i T^j : 0 \leq i \leq p-1, \;\; 0 \leq j \leq p-1\}$$

is a complete coset representation of $[\triangle,\triangle]$ in $\triangle$.

Indeed any element in $[\triangle,\triangle]$ is generated by $ABA^{-1}B^{-1}$ where $A, B \in \triangle$ and

$$A = S^{i_1}T^{j_1} \ldots S^{i_k}T^{j_k}$$

$$B = S^{m_1}T^{n_1} \ldots S^{m_r}T^{n_r}$$

So in $ABA^{-1}B^{-1}$ the sum of powers of S(similarly sum of powers of T)is zero. Hence if $W \in [\triangle,\triangle]$, then

$$W = S^{p_1}T^{q_1} \ldots S^{p_s}T^{q_s}$$

where $\sum_{i=1}^{s} p_i = kp$ and $\sum_{i=1}^{s} q_i = np$ (as the only cancelation done contains $S^p = T^p = (ST)^p = 1$ all involves p-th power of S and T).

Observe that

$$S^iT^j(S^kT^l)^{-1} = S^iT^{j-l}S^{-k} \notin [\triangle,\triangle]$$

as neither $i-k=mp$ nor $j-l=np$ for $0 \le i,k \le p-1;\ 0 \le j,l \le p-1$.
So U contains p^2 elements no two of which denote the same coset representation of $[\triangle,\triangle]$ in $\triangle$. Also $\mid \triangle/[\triangle,\triangle] \mid = p^2$ by (5.69).

Hence U gives a complete coset representation of $[\triangle,\triangle]$ in $\triangle$. Also any initial word of any element of U is in U. Hence U represent a Schreier Transversal of $[\triangle,\triangle]$ in $\triangle$.

Hence $\{ux\overline{ux}^{-1} : u \in U,\ \ x \in X = \{S,T\}\ \}$ generates $[\triangle,\triangle]$ and $\{uru^{-1} : u \in U,\ \ r \in R = \{S^p, T^p, (ST)^p\}\}$ gives relations of $[\triangle,\triangle]$.

Observe that

$$S^iT^jS \equiv S^{i+1}T^j \quad mod[\triangle,\triangle]$$

because if we write $A = S^iT^j$ and $B = S$ then the above is the same as

$$AB \equiv BA \ \ mod[\triangle,\triangle]$$

and that is true as $ABA^{-1}B^{-1} \in [\triangle,\triangle]$.

Hence

$$\begin{aligned}
\overline{S^iT^jS} = \overline{S^{i+1}T^j} &= S^{i+1}T^j \text{ if } 0 \le i \le p-2\ ,\ 0 \le j \le p-1 \\
&= S^{i+1-p}T^j \text{ if } i = p-1\ ,\ 0 \le j \le p-1 \\
\overline{S^iT^jT} = \overline{S^iT^{j+1}} &= S^iT^{j+1} \text{ if } 0 \le i \le p-1\ ,\ 0 \le j \le p-2 \\
&= S^iT^{j+1-p} \text{ if } 0 \le i \le p-1\ ,\ j = p-1
\end{aligned}$$

TABLE 1: Observe the following table where rows are indicated by elements of U and columns are indicated by elements of $X = \{S, T\}$ and the (u, x) entry is $ux\overline{ux}^{-1}$ with $u \in U$, $x \in X$.

TABLE 1

	S	T
1	1	1
S	1	1
...	...	...
S^i	1	1
...	...	...
S^{p-1}	S^p	1
T	$TST^{-1}S^{-1}$	1
ST	$STST^{-1}S^{-2}$	1
...	...	...
S^iT	$S^iTST^{-1}S^{-1-i}$	1
...	...	...
$S^{p-2}T$	$S^{p-2}TST^{-1}S^{-p+1}$	1
$S^{p-1}T$	$S^{p-1}TST^{-1}$	1
T^2	$T^2ST^{-2}S^{-1}$	1
ST^2	$ST^2ST^{-2}S^{-2}$	1
...	...	...
S^iT^2	$S^iT^2ST^{-2}S^{-1-i}$	1
...	...	...
$S^{p-2}T^2$	$S^{p-2}T^2ST^{-2}S^{-p+1}$	1
$S^{p-1}T^2$	$S^{p-1}T^2ST^{-2}$	1

T^j	$T^jST^{-j}S^{-1}$	1
ST^j	$ST^jST^{-j}S^{-2}$	1
...	...	...
S^iT^j	$S^iT^jST^{-j}S^{-i-1}$	1
...	...	...
$S^{p-2}T^j$	$S^{p-2}T^jST^{-j}S^{-p+1}$	1
$S^{p-1}T^j$	$S^{p-1}T^jST^{-j}$	1
T^{p-2}	$T^{p-2}ST^{2-p}S^{-1}$	1
ST^{p-2}	$ST^{p-2}ST^{2-p}S^{-2}$	1
...	...	...
S^iT^{p-2}	$S^iT^{p-2}ST^{2-p}S^{-i-1}$	1
...	...	...
$S^{p-2}T^{p-2}$	$S^{p-2}T^{p-2}ST^{2-p}S^{1-p}$	1
$S^{p-1}T^{p-2}$	$S^{p-1}T^{p-2}ST^{2-p}$	1
T^{p-1}	$T^{p-1}ST^{1-p}S^{-1}$	T^p
ST^{p-1}	$ST^{p-1}ST^{1-p}S^{-2}$	ST^pS^{-1}
...	...	...
S^iT^{p-1}	$S^iT^{p-1}ST^{1-p}S^{-i-1}$	$S^iT^pS^{-i}$
...	...	...
$S^{p-2}T^{p-1}$	$S^{p-2}T^{p-1}ST^{1-p}S^{-p+1}$	$S^{p-2}T^pS^{2-p}$
$S^{p-1}T^{p-1}$	$S^{p-1}T^{p-1}ST^{1-p}$	$S^{p-1}T^pS^{1-p}$

END OF TABLE 1

Table 2: In the following table the rows are indicated by elements of U and the columns are indicated by element of $R = \{S^p, T^p, (ST)^p\}$ and the (u, r) entry is uru^{-1} where $u \in U$ and $r \in R$.

TABLE 2

	S^p	T^p	$(ST)^p$
1	S^p	T^p	$(ST)^p$
S	S^p	ST^pS^{-1}	$S(ST)^pS^{-1}$
...	...	...	...
S^i	S^p	$S^iT^pS^{-i}$	$S^i(ST)^pS^{-i}$
...	...	...	...
S^{p-1}	S^p	$S^{p-1}T^pS^{1-p}$	$S^{p-1}(ST)^pS^{1-p}$
T	TS^pT^{-1}	T^p	$T(ST)^pT^{-1}$
ST	$STS^pT^{-1}S^{-1}$	ST^pS^{-1}	$(ST)^p$
...	...	...	...
S^iT	$S^iTS^pT^{-1}S^{-i}$	$S^iT^pS^{-i}$	$S^iT(ST)^pT^{-1}S^{-i}$
...	...	...	...
$S^{p-2}T$	$S^{p-2}TS^pT^{-1}S^{2-p}$	$S^{p-2}T^pS^{2-p}$	$S^{p-2}T(ST)^pT^{-1}S^{2-p}$
$S^{p-1}T$	$S^{p-1}TS^pT^{-1}S^{1-p}$	$S^{p-1}T^pS^{1-p}$	$S^{p-1}T(ST)^pT^{-1}S^{1-p}$
T^2	$T^2S^pT^{-2}$	T^p	$T^2(ST)^pT^{-2}$
ST^2	$ST^2S^pT^{-2}S^{-1}$	ST^pS^{-1}	$ST^2(ST)^pT^{-2}S^{-1}$
...	...	...	...
S^iT^2	$S^iT^2S^pT^{-2}S^{-i}$	$S^iT^pS^{-i}$	$S^iT^2(ST)^pT^{-2}S^{-i}$
...	...	...	...
$S^{p-2}T^2$	$S^{p-2}T^2S^pT^{-2}S^{2-p}$	$S^{p-2}T^pS^{2-p}$	$S^{p-2}T^2(ST)^pT^{-2}S^{2-p}$
$S^{p-1}T^2$	$S^{p-1}T^2S^pT^{-2}S^{1-p}$	$S^{p-1}T^pS^{1-p}$	$S^{p-1}T^2(ST)^pT^{-2}S^{1-p}$
T^j	$T^jS^pT^{-j}$	T^p	$T^j(ST)^pT^{-j}$
ST^j	$ST^jS^pT^{-j}S^{-1}$	ST^pS^{-1}	$ST^j(ST)^pT^{-j}S^{-1}$
...	...	...	...

S^iT^j	$S^iT^jS^pT^{-j}S^{-i}$	$S^iT^pS^{-i}$	$S^iT^j(ST)^pT^{-j}S^{-i}$
...	...	...	...
$S^{p-2}T^j$	$S^{p-2}T^jS^pT^{-j}S^{2-p}$	$S^{p-2}T^pS^{2-p}$	$S^{p-2}T^j(ST)^pT^{-j}S^{2-p}$
$S^{p-1}T^j$	$S^{p-1}T^jS^pT^{-j}S^{1-p}$	$S^{p-1}T^pS^{1-p}$	$S^{p-1}T^j(ST)^pT^{-j}S^{1-p}$
T^{p-2}	$T^{p-2}S^pT^{2-p}$	T^p	$T^{p-2}(ST)^pT^{2-p}$
ST^{p-2}	$ST^{p-2}S^pT^{2-p}S^{-1}$	ST^pS^{-1}	$ST^{p-2}(ST)^pT^{2-p}S^{-1}$
...	...	...	...
S^iT^{p-2}	$S^iT^{p-2}S^pT^{2-p}S^{-i}$	$S^iT^pS^{-i}$	$S^iT^{p-2}(ST)^pT^{2-p}S^{-i}$
...	...	...	...
$S^{p-2}T^{p-2}$	$S^{p-2}T^{p-2}S^pT^{2-p}S^{2-p}$	$S^{p-2}T^pS^{2-p}$	$S^{p-2}T^{p-2}(ST)^pT^{2-p}S^{2-p}$
$S^{p-1}T^{p-2}$	$S^{p-1}T^{p-2}S^pT^{2-p}S^{1-p}$	$S^{p-1}T^pS^{1-p}$	$S^{p-1}T^{p-2}(ST)^pT^{2-p}S^{1-p}$
T^{p-1}	$T^{p-1}S^pT^{1-p}$	T^p	$T^{p-1}(ST)^pT^{1-p}$
ST^{p-1}	$ST^{p-1}S^pT^{1-p}S^{-1}$	ST^pS^{-1}	$ST^{p-1}(ST)^pT^{1-p}S^{-1}$
...	...	...	...
S^iT^{p-1}	$S^iT^{p-1}S^pT^{1-p}S^{-i}$	$S^iT^pS^{-i}$	$S^iT^{p-1}(ST)^pT^{1-p}S^{-i}$
...	...	...	...
$S^{p-2}T^{p-1}$	$S^{p-2}T^{p-1}S^pT^{1-p}S^{2-p}$	$S^{p-2}T^pS^{2-p}$	$S^{p-2}T^{p-1}(ST)^pT^{1-p}S^{2-p}$
$S^{p-1}T^{p-1}$	$S^{p-1}T^{p-1}S^pT^{1-p}S^{1-p}$	$S^{p-1}T^pS^{1-p}$	$S^{p-1}T^{p-1}(ST)^pT^{1-p}S^{1-p}$

END OF TABLE 2

Table 3: We denote the generators of $[\triangle, \triangle]$ by b_i's that is the elements in the last two columns of the first table and express uru^{-1} $u \in U$ and $r \in R$ that is elements in the second third and fourth columns in the second table in terms of b_i's. Denote by $b_0 = S^p$ and $b_{kp+i} = S^{i-1}T^{k+1}ST^{-1-k}S^{2-i}$ where $1 \leq i \leq p$ and $0 \leq k \leq p-2$ and $b_{i+p(p-1)} = S^{i-1}T^pS^{-i+1}$ where $1 \leq i \leq p$. Observe that in each set of p generators that is $\{b_{kp+i} : 1 \leq i \leq p\}$ for any fixed k the exponent of T is $k+1$. We can write

the first table in terms of b_i's

TABLE 3

	S	T
1	-	-
S	-	-
...	...	...
S^i	-	-
...	...	...
S^{p-1}	b_0	-
T	b_1	-
ST	b_2	-
...	...	...
S^iT	b_{i+1}	-
...	...	...
$S^{p-2}T$	b_{p-1}	-
$S^{p-1}T$	b_p	-
T^2	b_{p+1}	-
ST^2	b_{p+2}	-
...	...	...
S^iT^2	b_{p+i+1}	-
...	...	...
$S^{p-2}T^2$	b_{p+p-1}	-
$S^{p-1}T^2$	b_{p+p}	-
T^j	$b_{(j-1)p+1}$	-

ST^j	$b_{(j-1)p+2}$	-
...	...	...
S^iT^j	$b_{(j-1)p+i+1}$	-
...	...	...
$S^{p-2}T^j$	$b_{(j-1)p+p-1}$	-
$S^{p-1}T^j$	$b_{(j-1)p+p}$	-
T^{p-2}	$b_{(p-3)p+1}$	-
ST^{p-2}	$b_{(p-3)p+2}$	-
...	...	...
S^iT^{p-2}	$b_{(p-3)p+i+1}$	-
...	...	...
$S^{p-2}T^{p-2}$	$b_{(p-3)p+p-1}$	-
$S^{p-1}T^{p-2}$	$b_{(p-3)p+p}$	-
T^{p-1}	$b_{(p-2)p+1}$	$b_{(p-1)p+1}$
ST^{p-1}	$b_{(p-2)p+2}$	$b_{(p-1)p+2}$
...	...	...
S^iT^{p-1}	$b_{(p-2)p+i+1}$	$b_{(p-1)p+i+1}$
...	...	...
$S^{p-2}T^{p-1}$	$b_{(p-2)p+p-1}$	$b_{(p-1)p+p-1}$
$S^{p-1}T^{p-1}$	$b_{(p-2)p+p}$	$b_{(p-1)p+p}$

END OF TABLE 3

Table 4: We shall determine how we can express the elements in the column indicated by S^p in the second table in terms of b_is.

The element $T^jS^pT^{-j}$ has started with T^jS and only b_i that starts with T^jS is $T^jST^{-j}S^{-1}$ that is $b_{(j-1)p+1}$, and there is no other element which starts with T^jS (as exponent of T is j only among the group $\{b_{(j-1)p+k}\ 1 \leq k \leq p\}$ and each is obtained from the previous one by conjugating with S). The next element we need must start with ST^jS and only such element is $b_{(j-1)p+2} = Sb_{(j-1)p+1}S^{-1}$. Their product is $b_{(j-1)p+1}b_{(j-1)p+2} = T^jS^2T^{-j}S^{-2}$ and adjoining one $b_{(j-1)p+k}$ successively in the product we increase power of S by 1.

$$b_{(j-1)p+1}\dots b_{(j-1)p+p-1} = T^jS^{p-1}T^{-j}S^{1-p}$$

$$b_{(j-1)p+1}\dots b_{(j-1)p+p} = T^jS^{p-1}ST^{-j} = T^jS^pT^{-j}$$

Similarly we have

$$S^iT^jS^pT^{-j}S^{-i} = b_{(j-1)p+i+1}b_{(j-1)p+i+2}\dots b_{(j-1)p+i}$$

$0 \leq i \leq p-1$

We could have determined this also by observing $b_{(j-1)p+i+1} = Sb_{(j-1)p+i}S^{-1}$ with $1 \leq j \leq p-1$ and $1 \leq i \leq p-1$, $b_{(j-1)p+p} = Sb_{(j-1)p+p-1}S^{-1}b_0$, $b_{(j-1)p+1} = b_0^{-1}Sb_{(j-1)p+p}S^{-1}$ and $S^iT^jS^pT^{-j}S^{-i} = S^i(T^jS^pT^{-j})S^{-i}$.

Thus we have expressed the relations in the column indicated by S^p in the second table in terms of b_is.

The relations in the column indicated by T^p in the second table, are nothing but $b_{(p-1)p+j}$ $1 \leq j \leq p$. Since they are both generators and relations we can treat them as identity or we can discard them.

Lastly we shall express the words in the column indicated by $(ST)^p$ in the second table in terms of b_is.

Consider $T^j(ST)^pT^{-j}$. This starts with T^jST. So the only element which starts with T^jS is $b_{(j-1)p+1} = T^jST^{-j}S^{-1}$. The next element we need must start with $ST^{j+1}S$ so that its product with $b_{(j-1)p+1}$ will give T^jSTS. Since the exponential of

T in $ST^{j+1}S$ is $j+1$ the element must be among $\{b_{jp+k} : 1 \leq k \leq p\}$ and the element is $b_{jp+2} = ST^{j+1}ST^{-j-1}S^{-2}$ giving

$$b_{(j-1)p+1}b_{jp+2} = T^j(ST)ST^{-j-1}S^{-2}$$

The next element we need must start with $S^2T^{j+2}S$ so that the product with $b_{(j-1)p+1}b_{jp+2}$ will start with $T^j(ST)(ST)S$ and the element must be $b_{(j+1)p+3} = S^2T^{j+2}ST^{-2-j}S^{-3}$. So

$$b_{(j-1)p+1}b_{jp+2}b_{(j+1)p+3} = T^j(ST)^2ST^{-j-2}S^{-3}$$

So each time we need an element whose exponential in T increases by 1 from the previous one and exponential in S also increases by 1 from the previous one and addition of such an appropriate b_i in the product will give rise to an (ST). Thus

$$b_{(j-1)p+1}b_{jp+2}\dots b_{(j-3)p+p-1} = T^j(ST)^{p-2}ST^{-j+2}S^{1-p}$$

and the next element we need must be $b_{(j-2)p+p} = S^{p-1}T^{j-1}ST^{1-j}$. So

$$\begin{aligned} b_{(j-1)p+1}b_{jp+2}\dots b_{(j-3)p+p-1}b_{(j-2)p+p} &= T^j(ST)^{p-1}ST^{1-j} \\ &= T^j(ST)^{p-1}ST^{1-j}T^jT^{-j} \\ &= T^j(ST)^pT^{-j} \end{aligned}$$

Proceeding in the above way we get the following table by writing the second table in terms of b_is.

TABLE 4

	S^p	T^p	$(ST)^p$
1	b_0	$b_{p(p-1)+1}$	$b_2b_{p+3}\dots b_{p(p-2)+p}b_{p(p-1)+1}$

S	b_0	$b_{p(p-1)+2}$	$b_3 b_{p+4} \dots b_{p(p-2)+1} b_{p(p-1)+2}$
$\dots$	$\dots$	$\dots$	$\dots$
S^i	b_0	$b_{p(p-1)+i+1}$	$b_{i+2} b_{p+i+3} \dots b_{p(p-2)+i} b_{p(p-1)+i+1}$
$\dots$	$\dots$	$\dots$	$\dots$
S^{p-1}	b_0	$b_{p(p-1)+p}$	$b_0 b_1 b_{p+2} \dots b_{p(p-2)+p-1} b_{p(p-1)+p}$
T	$b_1 b_2 \dots b_p$	$b_{p(p-1)+1}$	$b_1 b_{p+2} \dots b_{p(p-2)+p-1} b_{p(p-1)+p}$
ST	$b_2 b_3 \dots b_p b_1$	$b_{p(p-1)+2}$	$b_2 b_{p+3} \dots b_{p(p-2)+p} b_{p(p-1)+1}$
$\dots$	$\dots$	$\dots$	$\dots$
$S^i T$	$b_{i+1} b_{i+2} \dots b_{i-1} b_i$	$b_{p(p-1)+i+1}$	$b_{i+1} b_{p+i+2} \dots b_{p(p-2)+i-1} b_{p(p-1)+i}$
$\dots$	$\dots$	$\dots$	$\dots$
$S^{p-2} T$	$b_{p-1} b_p \dots b_{p-3} b_{p-2}$	$b_{p(p-1)+p-1}$	$b_{p-1} b_{p+p} \dots b_{p(p-2)+p-3} b_{p(p-1)+p-2}$
$S^{p-1} T$	$b_p b_1 \dots b_{p-2} b_{p-1}$	$b_{p(p-1)+p}$	$b_p b_{p+1} \dots b_{p(p-2)+p-2} b_{p(p-1)+p-1}$
T^2	$b_{p+1} b_{p+2} \dots b_{p+p}$	$b_{p(p-1)+1}$	$b_{p+1} b_{2p+2} \dots b_{p(p-1)+p-1} b_p$
ST^2	$b_{p+2} b_{p+3} \dots b_{p+1}$	$b_{p(p-1)+2}$	$b_{p+2} b_{2p+3} \dots b_{p(p-1)+p} b_1$
$\dots$	$\dots$	$\dots$	$\dots$
$S^i T^2$	$b_{p+i+1} b_{p+i+2} \dots b_{p+i}$	$b_{p(p-1)+i+1}$	$b_{p+i+1} b_{2p+i+2} \dots b_{p(p-1)+i-1} b_i$
$\dots$	$\dots$	$\dots$	$\dots$
$S^{p-2} T^2$	$b_{p+p-1} b_{p+p} \dots b_{p+p-2}$	$b_{p(p-1)+p-1}$	$b_{p+p-1} b_{2p+p} \dots b_{p(p-1)+p-3} b_{p(p-1)+p-2}$
$S^{p-1} T^2$	$b_{p+p} b_{p+1} \dots b_{p+p-1}$	$b_{p(p-1)+p}$	$b_{p+p} b_{2p+1} \dots b_{p(p-1)+p-2} b_{p(p-1)+p-1}$
T^j	$b_{(j-1)p+1} \dots b_{(j-1)p+p}$	$b_{p(p-1)+1}$	$b_{(j-1)p+1} b_{jp+2} \dots b_{(j-2)p+p}$
ST^j	$b_{(j-1)p+2} \dots b_{(j-1)p+1}$	$b_{p(p-1)+2}$	$b_{(j-1)p+2} b_{jp+3} \dots b_{(j-2)p+1}$
$\dots$	$\dots$	$\dots$	$\dots$
$S^i T^j$	$b_{(j-1)p+i+1} \dots b_{(j-1)p+i}$	$b_{p(p-1)+i+1}$	$b_{(j-1)p+i+1} b_{jp+i+2} \dots b_{(j-2)p+i}$
$\dots$	$\dots$	$\dots$	$\dots$
$S^{p-2} T^j$	$b_{(j-1)p+p-1} \dots b_{(j-1)p+p-2}$	$b_{p(p-1)+p-1}$	$b_{(j-1)p+p-1} b_{jp+p} \dots b_{(j-2)p+p-2}$

$S^{p-1}T^{j}$	$b_{(j-1)p+p} \dots b_{(j-1)p+p-1}$	$b_{p(p-1)+p}$	$b_{(j-1)p+p}b_{jp+1} \dots b_{(j-2)p+p-1}$
T^{p-2}	$b_{p(p-3)+1} \dots b_{p(p-3)+p}$	$b_{p(p-1)+1}$	$b_{p(p-3)+1}b_{p(p-2)+2} \dots b_{p(p-4)+p}$
ST^{p-2}	$b_{p(p-3)+2} \dots b_{p(p-3)+1}$	$b_{p(p-1)+2}$	$b_{p(p-3)+2}b_{p(p-2)+3} \dots b_{p(p-4)+1}$
$\dots$	$\dots$	$\dots$	$\dots$
$S^{i}T^{p-2}$	$b_{p(p-3)+i+1} \dots b_{p(p-3)+i}$	$b_{p(p-1)+i+1}$	$b_{p(p-3)+i+1}b_{p(p-2)+i+2} \dots b_{p(p-4)+i}$
$\dots$	$\dots$	$\dots$	$\dots$
$S^{p-2}T^{p-2}$	$b_{p(p-3)+p-1} \dots b_{p(p-3)+p-2}$	$b_{p(p-1)+p-1}$	$b_{p(p-3)+p-1}b_{p(p-2)+p} \dots b_{p(p-4)+p-2}$
$S^{p-1}T^{p-2}$	$b_{p(p-3)+p} \dots b_{p(p-3)+p-1}$	$b_{p(p-1)+p}$	$b_{p(p-3)+p}b_{p(p-2)+1} \dots b_{p(p-4)+p-1}$
T^{p-1}	$b_{p(p-2)+1} \dots b_{p(p-2)+p}$	$b_{p(p-1)+1}$	$b_{p(p-2)+1}b_{p(p-1)+2} \dots b_{p(p-3)+p}$
ST^{p-1}	$b_{p(p-2)+2} \dots b_{p(p-2)+1}$	$b_{p(p-1)+2}$	$b_{p(p-2)+2}b_{p(p-1)+3} \dots b_{p(p-3)+1}$
$\dots$	$\dots$	$\dots$	$\dots$
$S^{i}T^{p-1}$	$b_{p(p-2)+i+1} \dots b_{p(p-2)+i}$	$b_{p(p-1)+i+1}$	$b_{p(p-2)+i+1}b_{p(p-1)+i+2} \dots b_{p(p-3)+i}$
$\dots$	$\dots$	$\dots$	$\dots$
$S^{p-2}T^{p-1}$	$b_{p(p-2)+p-1} \dots b_{p(p-2)+p-2}$	$b_{p(p-1)+p-1}$	$b_{p(p-2)+p-1}b_{p(p-1)+p} \dots b_{p(p-3)+p-2}$
$S^{p-1}T^{p-1}$	$b_{p(p-2)+p} \dots b_{p(p-2)+p-1}$	$b_{p(p-1)+p}$	$b_{p(p-2)+p}b_{p(p-1)+1} \dots b_{p(p-3)+p-1}$

END OF TABLE 4

Observe that

$$
\begin{aligned}
& b_{(j-1)p+i+1}b_{jp+i+2} \dots b_{(j-3)p+i-1}b_{(j-2)p+i} = 1 \\
\Leftrightarrow \quad & b_{(j-2)p+i}b_{(j-1)p+i+1}b_{jp+i+2} \dots b_{(j-3)p+i-1} = 1 \qquad (5.70) \\
& 2 \leq j \leq p-1
\end{aligned}
$$

For example consider the two sets of relations in the column indicated by $(ST)^p$ namely $\{b_1b_{p+2} \dots b_{p(p-1)+p}, b_2b_{p+3} \dots b_{p(p-1)+1}, \dots, b_{i+1}b_{p+i+2} \dots b_{p(p-1)+i}, \dots, b_pb_{p+1} \dots b_{p(p-1)+p-1}\}$ and $\{b_{p+1}b_{2p+2} \dots b_{p(p-1)+p-1}b_p, b_{p+2}b_{2p+3} \dots b_{p(p-1)+p}b_1, \dots, b_{p+i+1}b_{2p+i+2} \dots b_{p(p-1)+i-1}b_i, \dots, b_{p+p}b_{2p+1} \dots b_{p(p-1)+p-2}b_{p-1}\}$ .By the above observation the second set is equivalent to $\{b_pb_{p+1}b_{2p+2} \dots b_{p(p-1)+p-1},$

$b_1 b_{p+2} b_{2p+3} \dots b_{(p-1)p+p}, \dots, b_i b_{p+i+1} b_{2p+i+2} \dots b_{p(p-1)+i-1}, \dots,$
$b_{p-1} b_{p+p} b_{2p+1} \dots b_{p(p-1)+p-2}\}$ which is nothing but the first set.

Similarly for the relations in the column indicated by S^p we observe that for a fixed j,

$$b_{(j-1)p+i+1} b_{(j-1)p+i+2} \dots b_{(j-1)p+i-1} b_{(j-1)p+i} = 1 \tag{5.71}$$
$$\Leftrightarrow \quad b_{(j-1)p+i} b_{(j-1)p+i+1} b_{(j-1)p+i+2} \dots b_{(j-1)p+i-1} = 1$$

(obtained by conjugating both sides by $b_{(j-1)p+i}$ of (80)).

Hence we have by Proposition 1
$[\triangle, \triangle] =< b_i \quad 0 \leq i \leq p(p-1) + p \quad | \quad b_0, b_{p(p-1)+i} \quad 1 \leq i \leq p, \Pi_{i=1}^{p} b_{jp+i} \quad 0 \leq j \leq$
$p-2, b_2 b_{p+3} \dots b_{(p-2)p+p}, b_{p(p-1)+1}, b_3 b_{p+4} \dots b_{p(p-2)+1} b_{p(p-1)+2},$
$\dots, b_i b_{p+i+1} \dots b_{p(p-2)+i-2} b_{p(p-1)+i-1}, \dots, b_p b_{p+1} b_{2p+2} \dots b_{p(p-2)+p-2} b_{p(p-1)+p-1},$
$b_1 b_{p+2} \dots b_{p(p-2)+p-1} b_{p(p-1)+p} >$
$=< b_{jp+i} \quad 1 \leq i \leq p-1 \quad, 0 \leq j \leq p-3 \quad | \quad \Pi_{i=1}^{p} b_{jp+i} \quad 0 \leq j \leq (p-2),$
$b_2 b_{p+3} \dots b_{p(p-2)+p}, b_3 b_{p+4} \dots b_{p(p-2)+1}, \dots, b_i b_{p+i+1} \dots b_{p(p-2)+i-2},$
$\dots, b_p b_{p+1} \dots b_{p(p-2)+p-2}, b_1 b_{p+2} \dots b_{p(p-2)+p-1} >$

So the generators of $[\triangle, \triangle]$ are

$$S^{i-1}(T^{j+1} S T^{-j-1} S^{-1}) S^{-i+1} = b_{jp+i}$$

with $1 \leq i \leq p-1$ and $0 \leq j \leq p-3$ or equivalently

$$S^i (T^j S T^{-j} S^{-1}) S^{-i}$$

where $0 \leq i \leq p-2$ and $1 \leq j \leq p-2$.

Our final result regarding the Fuchsian group for X_0 is :

Thus X_0 which is the compact Riemann Surface corresponding to the polynomial equation $P_0(x,y) = x^p + y^p - 1 = 0$ is biholomorphically equivalent to $U/[\triangle, \triangle]$ where $\triangle = \triangle[p,p,p] =< S,T : S^p = T^p = (ST)^p = 1 >$ and $[\triangle,\triangle]$ (which is the commutator subgroup of $\triangle$) is generated by $2g = (p-1)(p-2)$ generators (where g is the genus of the Riemann Surface X_0). The generators are $S^i(T^jST^{-j}S^{-1})S^{-i}$ $0 \leq i \leq p-2$ $1 \leq j \leq p-2$ and S and T are given by the following two matrices

$$S = \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \qquad T = \frac{1}{r}\begin{bmatrix} \frac{\rho}{\mu} - \frac{\mu}{\rho} & \mu - \frac{1}{\mu} \\ \frac{1}{\mu} - \mu & \mu\rho - \frac{1}{\mu\rho} \end{bmatrix}$$

where $r = \frac{1}{\rho} - \rho$ and ρ is the positive square root of

$$\rho^2[\mu + \frac{1}{\mu} + \frac{\mu}{\lambda} + \frac{\lambda}{\mu}] = \mu + \frac{1}{\mu} + 2$$

and $\mu = e^{\frac{\pi i}{p}}$ and $\lambda = e^{-\frac{\pi i}{p}}$. The tables 1 to 4 give explicit and complete set of generators and relations for $[\triangle,\triangle]$

It is now clear that one can directly apply the theorems developed in Chapter IV to the present case, in order to obtain the actual generators (up to first order in t) of the deformed Fuchsian groups that represent the deformations of the Fermat curves.

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