NONCLASSICALITY AND PHOTON NUMBER DISTRIBUTIONS IN QUANTUM OPTICS

by

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CERTIFICATE

This is to certify that the Ph.D. thesis titled *Nonclassicality and Photon Number Distributions in Quantum Optics* submitted by *Mary Elizabeth Selvadoray* is a record of bonafide research work done under my supervision. The research work presented in this thesis has not formed the basis for the award to the candidate of any Degree, Diploma, Associateship, Fellowship or other similar titles. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

Prof. R. Simon

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Thesis Supervisor

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ABSTRACT

Nonclassicality and Photon Number Distributions in Quantum Optics

by Mary Elizabeth Selvadoray

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Gaussian states of the radiation field whose density operators are exponentials of quadratics in the boson creation and annihilation operators are important in quantum optics not only because special states like coherent states, squeezed states and thermal states are Gaussian, but also because it is possible to approximate the states generated in many non-linear processes by a gaussian state. They have been studied by many authors [1]. Further nonclassical gaussian states like squeezed states [2] have evoked much interest and have been intensively studied for the last several years. They have also been produced experimentally.

The photon number distribution of a single mode squeezed coherent state shows oscillations, and this phenomenon was explained using the concept of interference in phase space [3]. These oscillations are taken to be a signature of nonclassicality, and they have come to be known as nonclassical oscillations [5]. Recently Dutta et al., [4], in an exhaustive study of the single mode squeezed coherent state, showed that the nonclassical oscillations in the photon number distribution exhibits collapses and revivals similar to those exhibited in the Jaynes-Cummings model.

The above mentioned oscillations have been considered as a qualitative signature

of nonclassicality, though there has been no rigorous proof of the same. Among the quantitative conditions for nonclassicality are squeezing and sub-Poissonian statistics which involve the lower order moments of the diagonal P-distribution. Higher order squeezing criteria have been introduced by Hong and Mandel, and amplitude squared squeezing was introduced by Hillery. Higher order generalization of the Mandel Q-parameter was introduced by Agarwal and Tara.

In this thesis we study among other things nonclassicality as coded in the photon number distribution p_n. Our results show that the scope of the information regarding nonclassicality coded in p_n is considerably more than what seems to have been hitherto appreciated.

Contents of Thesis

This thesis is organized into five Chapters. The first Chapter is introductory in nature. In addition to placing the work in this thesis in perspective, some important concepts central to the study in the rest of the thesis are briefly summarized in this Chapter.

In the second Chapter we study Gaussian States of the radiation field [6]. A comprehensive analysis of the characterization, spectral decomposition, and entropy for general Gaussian states of a system with arbitrary finite number of bosonic degrees of freedom N is presented. The unitary action of the symplectic group $Sp(2N, R)$

on the Hilbert space of an N-mode system and a classic theorem due to Williamson [7] on the normal forms of symmetric matrices under symplectic transformations are exploited. The relationship between the real symmetric parameter-matrix characterizing a Gaussian density operator and the noise matrix characterizing the associated Wigner distribution is derived. This constitutes solution to the Weyl ordering problem [8], and exposes the manner in which the antisymmetric symplectic metric plays the "role" of $i = \sqrt{-1}$. Spectral decomposition of the most general Gaussian density operator is constructed, and it is shown that the eigenvalue spectrum and the entropy are fully determined by the N independent $Sp(2N, R)$ invariants of the noise matrix; the entropy equals the sum of the entropies of N noninteracting harmonic oscillators, each in thermal equilibrium at independent temperatures determined by the $Sp(2N, R)$ invariants of the noise matrix. We also discuss the Fock state representation and photon number distribution of a general Gaussian state.

The third Chapter constitutes a thorough analysis of nonclassicality of a state $\hat{\rho}$ as coded in the photon number distribution $p_n = < n|\hat{\rho}|n >$. Techniques and results from the classical Stieltjes moment problem are exploited for this purpose. A necessary and sufficient condition on the sequence p_n in order that the state is classical is derived. It turns out that oscillation in p_n is not necessarily a signature of nonclassicality! However, it is shown that oscillation in q_n, where $q_n = n!p_n$, n=0,1,2... is a sure signature of nonclassicality. The usual approach to nonclassicality of the photon number distribution is in terms of the normal ordered moments $m_n = \mathrm{tr}(\hat{\rho}\hat{a}^{\dagger n}\hat{a}^n)$. Our approach is local in n in the sense that our conditions involve p_n for three, five,... successive values of p_n and hence is dual to the traditional approach in terms of the moments of p_n. Further, the result of Agarwal and Tara [10] is improved into a necessary and sufficient condition. Finally, using the techniques of Laplace transform, the complete equivalence between the present approach to nonclassicality in terms of local conditions on p_n and the traditional approach in terms of conditions on the moments m_n is established.

The fourth Chapter of this thesis constitutes a study of the two-mode squeezed coherent state with complex squeeze and displacement parameters [11]. A similar state with real parameters, was studied by Caves et al. [5]. In their paper they display plots for the photon number distributions for $\alpha_2 = \alpha_1$ and $\alpha_2 = -\alpha_1$, where α_1 and α_2 are real displacement parameters. The striking difference between the two plots and the work of Dutta et al. [4] in the single mode case with complex squeeze and displacement parameters form the motivation for the study of the two-mode squeezed coherent state with complex parameters. In the above two cases α_1, α_2 are respectively in phase ($\alpha_2 = \alpha_1$) and out of phase ($\alpha_2 = -\alpha_1$), and it is natural to ask if the diametrically opposite behaviours at these two extremes smoothly interpolate as a function of the relative phase between α_1 and α_2.

The two-mode squeezed coherent state [12] is obtained by the application of the two-mode squeeze operator $S(z) = \exp(z^*ab - za^\dagger b^\dagger)$ on the two-mode coherent state, which is got by the two-mode displacement operator $D(\alpha_1, \alpha_2) = D(\alpha_1)\, D(\alpha_2)$, where $D(\alpha_1) = \exp(\alpha_1 a^\dagger - \alpha_1{}^* a)$ and $D(\alpha_2) = \exp(\alpha_2 b^\dagger - \alpha_2{}^* b)$, acting on the two-mode vacuum. Using the symmetry inherent in the system, we find that the two-mode squeeze operator is a rotated version of the product of reciprocal single-mode squeezings. We then exploit an identity relating a sum over product of Hermite polynomials to the associated Laguerre polynomials to get the probability amplitude in terms of a single associated Laguerre polynomial. We find that the photon number distribution possesses a $U(1) \times U(1)$ invariance and use it to argue that the distribution depends only on a particular linear combination χ of the three phases (arising from the two displacement parameters and one squeeze parameter) in the problem. Numerical studies of the distribution are presented and they are shown to exhibit collapses and revivals similar to that in the single mode case. The structure of the distribution between the two extreme cases considered by Caves et al. is studied, leading to a clear picture of the evolution of the distribution from one extreme to the other. We also study the second order coherence properties of the two-mode squeezed coherent state.

In the fifth Chapter the phase distribution and correlation functions associated with the two-mode squeezed coherent state are studied [13], in continuation of the study of the photon distribution for the two mode squeezed coherent state in the previous chapter. We make use of the definition given by Agarwal [14] for the phase distribution and then use the photon number matrix element calculated in the previous chapter to obtain an expression for the phase distribution. We also write down the joint probability distribution for sum and difference phases restricted to a 2π range, following Barnett and Pegg [15]. We find that the phase distribution exhibits phenomena analogous to the bifurcation phenomena predicted by Schleich et al [16] in the phase distribution of a single mode coherent state. We also study the phase distribution in terms of the relative phase χ defined in the previous chapter. In the case of correlations between the phases in the two modes we find that the phase sum correlation has the value $\pi^2/3$ for zero squeezing and zero displacement, which is characteristic of random phase, and vanishes in the large squeezing limit. The variance in the phase sum, on the other hand, is constant at $\pi^2/3$ for zero displacement, for any value of the squeeze parameter. When the displacement is much greater than zero it goes to the random phase variance in the large squeezing limit.

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List of papers included in the thesis

1. *Photon distribution in two-mode squeezed coherent states with complex displacement and squeeze parameters,*

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2. *Gaussian States for multimode systems: Wigner distribution, spectral decomposition and entropy,*

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3. *Phase properties of correlated two-mode squeezed coherent states,*

   Mary Selvadoray and M. Sanjay Kumar, Opt. Comm. **136**, 125 (1997).

4. *Nonclassicality and the concept of local constraints on the photon number distribution,*

   R. Simon, Mary Selvadoray, Arvind and N. Mukunda (submitted for publication).

Contents

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Chapter 1

Introduction

Nonclassical light especially squeezed light [1] has attracted a great deal of attention in recent years in various contexts and many criteria characterizing a nonclassical state have been put forth. This thesis introduces among other things a new set of criteria for characterizing a nonclassical state by obtaining classicality restrictions on the photon number distribution. So as a prelude, we will discuss here a few of the existing criteria for nonclassicality.

Nonclassicality of a state is reflected in the behaviour of the diagonal coherent state quasi-probability distribution (which will be described below) associated with it. There are many (quasi-)probability distributions associated with the density operator which are used in quantum optics[2, 3, 5] and in the second chapter of this thesis, we will be discussing the Wigner distribution for multimode Gaussian states. We will therefore start by reviewing some relevant quasi-probability distributions before proceeding with a discussion of the existing criteria for nonclassicality.

The density operator $\hat{\rho}$ of a system can be written in the following form

$$\chi(\eta) = \mathrm{tr}[\hat{\rho} e^{\eta a^\dagger - \eta^* a}] . \tag{1.1}$$

$\chi(\eta)$ is called the characteristic function. This function can be written in normally ordered and anti-normally ordered forms as

$$\begin{aligned} \chi_N(\eta) &= \mathrm{tr}[\hat{\rho} e^{\eta a^\dagger} e^{-\eta^* a}] , \\ \chi_A(\eta) &= \mathrm{tr}[\hat{\rho} e^{-\eta^* a} e^{\eta a^\dagger}] . \end{aligned} \tag{1.2}$$

If we use the relation

$$e^{A+B} = e^A e^B e^{-[A,B]/2} , \tag{1.3}$$

when $[A, B]$ is a c-number, we can write down relations between various characteristic functions.

$$\chi(\eta) = \chi_N(\eta) \exp(-\frac{1}{2}|\eta|^2) . \tag{1.4}$$

The Fourier transforms of the above characteristic functions find several uses in Quantum Optics. They are the well known quasi-probability distributions - the Wigner function [5], the P-function and the Q-function.

The Wigner distribution was the first quasi-probability distribution to be introduced. It is the Fourier transform of the symmetrically ordered characteristic function $\chi(\eta)$ - i.e.

$$W(\alpha) = \frac{1}{\pi^2} \int d^2\eta \, \exp(\eta^* \alpha - \eta \alpha^*) \, \chi(\eta) . \tag{1.5}$$

In problems involving squeezing it is advantageous to use the Wigner distribution because squeezing transformations act linearly on the arguments of the distribution.

The P-distribution is the Fourier transform of the normally ordered characteristic function -

$$P(\alpha) = \frac{1}{\pi^2}\int d^2\eta \ \exp(\eta^*\alpha - \eta\alpha^*)\chi_N(\eta) . \tag{1.6}$$

In terms of this function, the density operator $\hat{\rho}$ can be written as

$$\hat{\rho} = \int d^2\alpha \ P(\alpha)|\alpha\rangle\langle\alpha| . \tag{1.7}$$

$P(\alpha)$ is not a probability distribution in the general sense of the term. This is because $P(\alpha)$ takes negative values or becomes singular for some states. For these states there can be no classical description and they are called nonclassical states. The P-function is very useful in evaluating normally ordered moments. The Wigner distribution can also be written as a Gaussian convolution of the P-function as

$$W(\alpha) = \frac{2}{\pi}\int d^2\beta \ P(\beta)\exp(-2|\beta - \alpha|^2) .$$

The Fourier transform of the anti-normally ordered characteristic function is called the Q-function -

$$Q(\alpha) = \frac{1}{\pi^2}\int d^2\eta \ \exp(\eta^*\alpha - \eta\alpha^*)\chi_A(\eta) . \tag{1.8}$$

In terms of the density operator $\hat{\rho}$, $Q(\alpha)$ is,

$$Q(\alpha) = \frac{\langle\alpha|\hat{\rho}|\alpha\rangle}{\pi} . \tag{1.9}$$

Since the density operator is positive, $Q(\alpha)$ is positive and this is its advantage over the Wigner and P functions. It is useful in evaulating anti-normally ordered moments.

Quantitative Characterizations of Nonclassicality

As we said earlier, nonclassical states are those for which the P distribution is not a regular probability distribution. Among the characterizations of nonclassical light are the degree of squeezing [4] and the Mandel's Q-parameter [6]. Generalizations of these have been introduced - the higher order squeezing criteria by Hong and Mandel [7] and a generalization of the Mandel Q-parameter by Agarwal and Tara [9]. The Mandel Q parameter for a single mode field is given by

$$Q = (\langle a^{\dagger 2} a^2 \rangle - \langle a^\dagger a \rangle^2)/\langle a^\dagger a \rangle \ , \tag{1.10}$$

where, $a^\dagger$ and a are the creation and annihilation operators. For a nonclassical state Q is negative and the field has sub-Poissonian statistics. The degree of squeezing S is given by

$$S = \langle : (ae^{i\theta} + a^\dagger e^{-i\theta})^2 : \rangle - \langle (ae^{i\theta} + a^\dagger e^{-i\theta}) \rangle^2. \tag{1.11}$$

When there is squeezing S is negative. A state can exhibit either sub-Poissonian statistics or squeezing or both. A negative value of S or Q implies that the P distribution associated with the state cannot be a classical probability distribution. The higher order squeezing criteria introduced by Hong and Mandel and the generalization of the Mandel Q parameter introduced by Agarwal and Tara were prompted by the necessity of quantitatively characterizing nonclassical states for which neither Q nor S is negative.

Higher order squeezing criteria of Hong and Mandel

Here, we give a brief description of the squeezing criteria given by Hong and Mandel. We follow the procedure in reference [7]. Let E^+ and E^- be the positive and negative frequency parts of the electric field, where we expand E^+ and E^- in the familiar manner:

$$E^+(r,t) = \frac{1}{L^{3/2}}\sum_k (\hbar\omega/2\epsilon_0)^{1/2} a_k e^{i(k.r-\omega t)} \quad . \tag{1.12}$$

The commutator is given by

$$[E^+, E^-] = \frac{1}{L^3}\sum_k |(\hbar\omega/2\epsilon_0)| \equiv C \quad . \tag{1.13}$$

The Hermitian quadrature components of the field E_1 and E_2 are

$$\begin{aligned} E_1 &\equiv E^+ e^{i(\omega t-\phi)} + E^- e^{-i(\omega t-\phi)} \quad , \\ E_2 &\equiv E^+ e^{i(\omega t-\phi-\pi/2)} + E^- e^{-i(\omega t-\phi-\pi/2)} \quad . \end{aligned} \tag{1.14}$$

Here ϕ can be chosen to be any angle. The commutator of E_1 and E_2 is

$$[E_1, E_2] = 2iC \quad , \tag{1.15}$$

and the uncertainty relation between them is

$$\langle(\Delta E_1)^2\rangle\langle(\Delta E_2)^2\rangle \geq C^2 \quad . \tag{1.16}$$

For the coherent state the equality sign holds and the value of the uncertainty in each of the quadratures is equal and is equal to C. For a squeezed state though, the uncertainty in one of the quadratures is less than that for a coherent state. If for some phase angle ϕ,

$$\langle(\Delta E_1)^2\rangle \; < \; C \; , \tag{1.17}$$

the state is said to be squeezed in the quadrature E_1. This is second order squeezing. Higher order squeezing is defined as follows: If there exists a phase angle ϕ for which $\langle(\Delta E_1)^{2N}\rangle$ is less than the value it has in a coherent state of the field, then the state is squeezed to the $2N-th$ order in E_1. It is sensible to define this only for even moments, since for odd moments there can be squeezing without the state being quantum-mechanical.

Amplitude-squared squeezing criterion of Hillery

A criterion for nonclassicality related to the above is the amplitude-squared squeezing criterion introduced by Hillery [8]. If a and $a^\dagger$ are the creation and annihilation operators associated with the system then one can introduce operators analogous to the standard quadrature operators as:

$$y_1 \; = \; (a^{\dagger 2} + a^2)/2 \;\; , \; y_2 \; = \; i(a^{\dagger 2} - a^2)/2 \;\; . \tag{1.18}$$

They obey the commutation relations $[y_1, y_2] \; = \; i(2N+1)$ and satisfy the uncertainty relation $\Delta y_1 \Delta y_2 \; \geq \; \langle N + 1/2\rangle$, where N is the number operator. Then, a state is said to be squeezed in y_1 if

$$(\Delta y_1)^2 \; < \; \langle N + 1/2 \rangle \quad . \tag{1.19}$$

Agarwal-Tara criterion for nonclassicality

Let m_n given by

$$m_n \; = \; \langle a^{\dagger n} a^n \rangle \; = \; \int d^2\alpha \; P(\alpha)|\alpha|^{2n} \quad , \tag{1.20}$$

be the factorial moments of the distribution $P(\alpha)$. If one considers the quadratic form F where

$$F \; = \; \sum_{i,j=0}^{2} C_i^* C_j m_{(i+j)} \tag{1.21}$$

then, for a classical distribution F should be positive or the 3×3 matrix $m^{(3)}$ shown below should be positive definite.

$$m^{(3)} \; = \; \begin{pmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{pmatrix} \tag{1.22}$$

The general form of the above statement for a distribution to be classical is that the $n \times n$ matrix $m^{(n)}$ should be positive definite and $m^{(n)}$ is given by

$$m^{(n)} \; = \; \begin{pmatrix} 1 & m_1 & m_2 & \dots & m_{n-1} \\ m_1 & m_2 & m_3 & \dots & m_n \\ \vdots & \vdots & \vdots & & \vdots \\ m_{n-1} & m_n & m_{n+1} & \dots & m_{2n-2} \end{pmatrix} \tag{1.23}$$

One can see that for $n = 2$ the above condition becomes the condition on the Mandel Q-parameter. To obtain a quantity that is bounded by -1 the matrix $\mu^{(n)}$ is introduced and it is defined through the quantity μ_n as follows:

$$\mu_n = \langle (a^\dagger a)^n \rangle \tag{1.24}$$

and $\mu^{(n)}$ is constructed from $m^{(n)}$ by replacing m_n's by μ_n's. The quantity A_3 given by

$$A_3 = \frac{\det m^{(3)}}{\det \mu^{(3)} - \det m^{(3)}} , \tag{1.25}$$

is then a measure of the nonclassicality of a state and is equal to 0 for a coherent state and -1 for a Fock state and for other nonclassical states it will have a value between 0 and -1. Agarwal and Tara have demonstrated for a photon added thermal state that the parameter A_3 is negative in regions where the Mandel Q parameter becomes positive. In the third chapter of this thesis, we will demostrate that *all* photon added states are nonclassical.

Chapter 2

Gaussian states for Multimode systems: Wigner distribution, Spectral Decomposition, and Entropy

2.1 Introduction

In this chapter we study Gaussian states of the radiation field. We are interested in these states because in the later chapters of this thesis we will be studying nonclassicality and a specific nonclassical state, and nonclassical states like squeezed coherent states are Gaussian. It is also true that other special states of light like the coherent state and thermal state are Gaussian and the field generated in a large class of nonlinear optical processes is Gaussian or can be approximated to be so [10, 11]. For example, Agarwal [69], has discussed that the correlated two-mode squeezed displaced thermal state (characterized by a Gaussian Wigner distribution) describes light generated in a wide class of nonlinear optical processes such as four-wave mixers [70], parametric amplifiers and down-converters [71, 7]. The correlated two-mode squeezed coherent state corresponds to the case when losses in the medium are neglected.

Wigner distributions are especially important in the context of squeezed states since these states do not have a well defined P distribution. They are also convenient to use in this context because squeezing transformations act linearly on the arguments of this distribution [13, 18]. The Wigner distribution corresponding to Gaussian states have Gaussian form. Further, many important quantum optical processes are governed by effective Hamiltonians which are quadratic in the canonical operators. But such Hamiltonians are generators of the symplectic group, and it is well known that Gaussian states have a special status in the context of the symplectic group [12, 13]: they are eigenstates of quadratic Hamiltonians which are in turn generators of the symplectic group.

Several aspects of Gaussian states in quantum optics have been studied in the past. These include photon distribution [15], entropy and thermodynamic properties [16, 17], noise characteristics [10], and others [18]. We present in this chapter a comprehensive analysis of multimode Gaussian states including their characterization, spectral decomposition, entropy, and the concise expression that the Weyl ordering [2] rule assumes for these states. We make powerful use of the fact that elements of the symplectic group $Sp(2N, R)$ act *unitarily* on the Hilbert space of an N-mode system [12, 13, 18]. It turns out that given a Gaussian state, these unitary transformations and the Weyl group of phase space displacements can be employed to transform it into a *canonical Gaussian state* whose properties are transparent and well known. The properties of the given Gaussian state will then follow from those of the canonical Gaussian state and the fact that some aspects of the Gaussian states are invariant and others covariant under the $Sp(2N, R)$ and Weyl groups of unitary transformations.

We give a brief description of the method of approach below. In Section 2 we introduce a convenient compact notation, and recall for later reference some results relating to the unitary action of $Sp(2N, R)$ on the density operator and on the Wigner distribution of an N-mode system. In Section 3, we study the single mode case in some detail. In this case the Gaussian density operator is parametrized by a 2×2 real matrix G and the corresponding Wigner distribution by a 2×2 real variance(noise) matrix V, so that the Weyl ordering rule is by definition equivalent to an appropriate invertible relationship between G and V. Such a relationship is presented. The canonical Gaussian state turns out to be the thermal state, so that all single mode Gaussian states are displaced squeezed thermal states. Spectral decomposition for an arbitrary Gaussian density operator shows that the spectrum and entropy of a Gaussian state are fully determined by the determinant of the variance matrix, whereas the eigenmodes are displaced squeezed Fock states. An explicit expression is given for the complex squeeze parameter in terms of the variance matrix V.

The general N-mode case is analysed in Section 4. We find that while a Gaussian distribution in the single mode phase space R^2 is a Wigner distribution if and only if the variance matrix V satisfies the uncertainty principle $\det V \geq 1/4$, the corresponding statement constituting a complete set of uncertainty principles in the N-mode case is much richer [13, 19]. Similarly, while the G and V matrices are multiples of one another in the single mode case, it will be shown that such a relationship is not valid for the N-mode case with $N \geq 2$. This is related to the fact that every symmetric positive definite 2×2 real matrix is a multiple of an $Sp(2, R)$ matrix, a result which does not generalize to $Sp(2N, R)$. We derive the precise relationship between the $2N \times 2N$ matrices G and V forming an expression of the Weyl ordering for N-mode Gaussian states. This relationship turns out to be a concise matrix equation, and constitutes a concrete instance where the standard (antisymmetric) metric underlying the symplectic geometry [20] plays the role of $i = \sqrt{-1}$. Spectral decomposition for the multimode Gaussian density operator shows that the eigenvalue spectrum depends not only on $\det V$, but also on the other $N-1$ independent $Sp(2N, R)$ invariants of the noise matix V. The entropy for an arbitrary N-mode Gaussian state turns out to be the sum of entropies for N single modes, each mode being in thermal equilibrium at an independent temperature determined by these N independent invariants. As in the single mode case, while the spectrum and entropy are $Sp(2N, R)$ invariant, the eigenmodes of the Gaussian state are $Sp(2N, R)$ covariant.

We discuss the physically important case $N = 2$, in section 5 and finish with a discussion of the photon number distribution in Section 6.

2.2 Wigner Distribution and the Symplectic Group

Consider an N-mode system described by boson operators $\hat{a}_j = (\hat{q}_j + i\hat{p}_j)/\sqrt{2}$, $j = 1, 2, \ldots, N$. As is well known, it is often convenient to describe any (pure or mixed) state of the system by the corresponding Wigner distribution [5] in the $2N$-dimensional phase space R^{2N}. It is also useful to arrange the phase space variables $q_1, \ldots, q_N, p_1 \ldots, p_N$ into a real column vector ξ. In the same way we arrange the hermitian canonical operators $\hat{q}_1 \ldots \hat{q}_N, \hat{p}_1 \ldots \hat{p}_N$ into a column $\hat{\xi}$:

$$\xi = \begin{pmatrix} q_1 \\ \vdots \\ q_N \\ p_1 \\ \vdots \\ p_N \end{pmatrix} = \begin{pmatrix} \mathrm{q} \\ \mathrm{p} \end{pmatrix} , \quad \hat{\xi} = \begin{pmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_N \\ \hat{p}_1 \\ \vdots \\ \hat{p}_N \end{pmatrix} = \begin{pmatrix} \hat{\mathrm{q}} \\ \hat{\mathrm{p}} \end{pmatrix} . \tag{2.1}$$

This allows us to express the canonical commutation relations in the compact notation [12, 13, 18]

$$[\hat{\xi}_\alpha, \hat{\xi}_\beta] = i\ \Omega_{\alpha\beta}, \qquad \alpha, \beta = 1, 2, \ldots 2N \quad , \tag{2.2}$$

where Ω is the standard antisymmetric matrix or "metric" fundamental to the symplectic geometry [20]:

$$\Omega = \begin{pmatrix} 0_{N\times N} & 1_{N\times N} \\ -1_{N\times N} & 0_{N\times N} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix} . \tag{2.3}$$

Note that $\Omega^T = \Omega^{-1} = -\Omega$. The Weyl group of unitary phase space displacement operators and their action on the canonical operators can be expressed as

$$D(\xi) = \exp[i(\mathbf{p}.\hat{\mathbf{q}} - \mathbf{q}.\hat{\mathbf{p}})] = \exp(-i\xi^T\Omega\hat{\xi}) ,$$

$$D(\xi)^\dagger \, \hat{\xi}_\alpha \, D(\xi) = \hat{\xi}_\alpha + \xi_\alpha, \quad \alpha = 1, 2, \ldots, 2N. \tag{2.4}$$

Clearly, $D(\xi)^{-1} = D(-\xi) = D(\xi)^\dagger$. Even the BCH formula takes an impressive form in this compact notation:

$$D(\xi) \, D(\xi') = \exp(-\frac{i}{2}\xi^T\Omega \, \xi') \, D(\xi + \xi'). \tag{2.5}$$

A special case of this relation is

$$D(\xi) = D(\mathbf{q}, \mathbf{p}) = \exp(-\frac{i}{2}\mathbf{q}.\mathbf{p}) \, \exp(i\mathbf{p}.\hat{\mathbf{q}}) \, \exp(-i\mathbf{q}.\hat{\mathbf{p}}). \tag{2.6}$$

While coherent excitation is governed by the above displacement operators whose generators (exponents) are linear in the canonical operators, squeezing problems are governed by unitary operators generated by hermitian Hamiltonians which are quadratic in $\hat{\xi}$. That is, by unitary operators of the form

$$\begin{aligned} U &= \exp(-iH), \\ H &= \sum_{\alpha=1}^{2N} \sum_{\beta=1}^{2N} h_{\alpha\beta}\hat{\xi}_\alpha\hat{\xi}_\beta \quad , \end{aligned} \tag{2.7}$$

where $h_{\alpha\beta}$ is real symmetric. With every such unitary operator is associated a $2N \times 2N$ real matrix S defined through

$$U^\dagger \hat{\xi} \, U \;=\; S \, \hat{\xi},$$

$$\text{i.e. } U^\dagger \, \hat{\xi}_\alpha \, U \;=\; S_{\alpha\beta}\hat{\xi}_\beta \;\;;\;\; \alpha = 1, 2, \ldots 2N. \tag{2.8}$$

It follows from the invariance of the commutation relation (2.2) under (2.8) that S is an element of the symplectic group $Sp(2N, R)$ [12, 13, 18]. That is,

$$S^T \Omega \, S = \Omega. \tag{2.9}$$

Since U induces the symplectic transformation S, we will use S to label U as $U(S)$. Strictly speaking, these unitary operators do not form a representation of $Sp(2N, R)$ but rather constitute a faithful representation of the metaplectic group $Mp(2N, R)$, the double cover of $Sp(2n, R)$ [12]. But this does not affect our considerations in the rest of the chapter. Note that $S \in Sp(2N, R)$ implies $S^{-1}, S^T \in Sp(2N, R)$. Further, $\Omega \in Sp(2N, R)$ and $\det S = 1$ for every $S \in Sp(2N, R)$.

Under the unitary evolution $U(S)$, the state vector $|\psi\rangle$ changes as $|\psi\rangle \to |\psi'\rangle = U(S)\,|\psi\rangle$ and hence the density operator $\hat{\rho}$ evolves as $\hat{\rho} \to \hat{\rho}' = U(S)\;\hat{\rho}\;U(S)^\dagger$. Since $\hat{\rho}$ can be written as a function of the canonical operators as $\hat{\rho}(\hat{\xi})$, the above evolution takes the form

$$\begin{aligned} U(S): \;\hat{\rho}(\hat{\xi}) \to \hat{\rho}'(\hat{\xi}) \;&=\; U(S)\;\hat{\rho}(\hat{\xi})\;U(S)^\dagger \\ &=\; \hat{\rho}\;(U(S)\;\hat{\xi}U(S)^\dagger) = \hat{\rho}\;(S^{-1}\hat{\xi}), \end{aligned} \tag{2.10}$$

where we used (2.8) in the last step. The corresponding evolution of the Wigner Distribution is given by [13, 18]

$$U(S): \;W(\xi) \to W'(\xi) = W(S^{-1}\xi). \tag{2.11}$$

We will make effective use of (2.10) and (2.11) in our analysis in the subsequent sections.

We define a $2N$ element column vector $\xi' = S\xi$, then (2.11) can be rewritten in the suggestive form $W'(\xi') = W(\xi)$. Thus we can say following the language of field theory, that the Wigner distribution transforms as a "scalar" field over R^{2N} under the unitary action of $Sp(2N, R)$. We note an important consequence [13, 18] of (2.9), the defining relation of the symplectic group. For any $2N \times 2N$ real symmetric matrix G, and for any $S \in Sp(2N, R)$, let

$$G' = SG\, S^T. \tag{2.12}$$

Since S is not an orthogonal matrix, (2.12) does not constitute a similarity transformation. We will call it a symmetric transformation. Right multiplying both sides of (2.12) by Ω, and making use of $S^T\Omega = \Omega S^{-1}$ which follows from (2.9), we have

$$G'\Omega = SGS^T\Omega = SG\Omega S^{-1}. \tag{2.13}$$

That is, as G undergoes a symmetric transformation $G\Omega$ undergoes a similarity transformation. This can be traced to the fact that for any real symmetric matrix M, the product $M\Omega$ is a generator of the symplectic group and every generator is of this form [12, 13]. We will make repeated use of this fact in Section 4.

Finally, the effects of the displacement operator $D(\xi)$ on the density operator and on the Wigner distribution are easy to compute, and the results will be needed in later sections. From (2.4) we have

$$D(\xi_0)\ \hat{\rho}(\hat{\xi})\ D(\xi_0)^\dagger = \hat{\rho}\ (\hat{\xi} - \xi_0). \tag{2.14}$$

It follows that under $D(\xi_0)$ the Wigner distribution changes as

$$W(\xi) \rightarrow W'(\xi) = W(\xi - \xi_0). \tag{2.15}$$

That is, the Wigner distribution simply undergoes a rigid phase space displacement under the unitary operator $D(\xi_0)$. As will be seen in the subsequent sections the evolution equations (2.14), (2.15) along with (2.10), (2.11) constitute the principal tools of our analysis in this work.

2.3 Single mode Gaussian States

Consider a single mode system described by the boson annihilation operator $\hat{a} = (\hat{q} + \hat{p})/\sqrt{2}$. The relevant symplectic group is $Sp(2, R)$, and in this case the defining condition (2.9) with $N = 1$ becomes equivalent to the condition $\det S = 1$. Thus, $Sp(2, R)$ consists of 2×2 real matrices of unit determinant. That is,
$Sp(2, R) \ = \ SL(2, R)$.

We begin with a restricted class of Gaussian states. The thermal state density operator of the single mode system is given by

$$\begin{aligned}\hat{\rho}_\beta(\hat{\xi}) &= (1 - e^{-\beta})\ \exp(-\beta \hat{a}^\dagger \hat{a}) \\ &= 2\ \sinh(\frac{\beta}{2})\ \exp[-\frac{\beta}{2}(\hat{q}^2 + \hat{p}^2)]\end{aligned}$$

$$= \; 2\,\sinh(\frac{\beta}{2})\;\exp[-\frac{\beta}{2}\hat{\xi}^{\dagger}\hat{\xi}], \tag{2.16}$$

where $\beta = (k_B T)^{-1}$ is the usual thermal parameter, $\hat{\xi}$ is a two element column vector with entries $\hat{q}$, $\hat{p}$ and $\hat{\xi}^T$ is the corresponding two-element row vector. The Wigner distribution of the thermal state $\hat{\rho}_\beta(\hat{\xi})$ is [5, 16, 17]

$$\begin{aligned} W_\beta(\xi) &= \frac{1}{\pi}\;\tanh(\frac{\beta}{2})\;\exp[-\tanh(\frac{\beta}{2})\;\xi^T\xi] \\ &= \frac{1}{\pi}\frac{1}{(2\overline{n}+1)}\;\exp[\frac{-1}{(2\overline{n}+1)}\;\xi^T\xi] \quad , \end{aligned} \tag{2.17}$$

where ξ is a two-element column vector with entries q, p forming coordinates of a point in our phase space R^2, so that $\xi^T\xi \;=\; q^2 + p^2$, and $\overline{n}$ is the mean number of (thermal) photons in the state:

$$\overline{n} = (e^{\beta} - 1)^{-1}, \qquad \frac{\beta}{2} = \text{arc}\,\tanh(\frac{1}{2\overline{n}+1}). \tag{2.18}$$

Since the thermal state density operator $\hat{\rho}_\beta$ is a function of $\hat{a}^{\dagger}\hat{a}$, it is diagonal in the $|n\rangle$ (Fock state) basis and hence has the spectral decomposition

$$\begin{aligned} \hat{\rho}_\beta = \hat{\rho}_{\overline{n}} &= \sum_{n=0}^{\infty}\;(1 - e^{-\beta})\;e^{-\beta n}\;|n\rangle\langle n| \\ &= \sum_{n=0}^{\infty}\;\frac{1}{(1+\overline{n})}\left(\frac{\overline{n}}{1+\overline{n}}\right)^n |n\rangle\langle n| \quad , \end{aligned} \tag{2.19}$$

from which the entropy $\mathcal{S} = -k_B\;\text{tr}(\hat{\rho}\;\ln\hat{\rho})$ is readily computed as

$$\mathcal{S}_\beta = k_B\;\left[\frac{\beta}{e^{\beta}-1}\; - \;\ln(1 - e^{\beta})\right] = k_B\;[(\overline{n}+1)\;\ln(\overline{n}+1) - \overline{n}\;\ln\overline{n}]\;. \tag{2.20}$$

To make further progress we rewrite our thermal state density operator and the associated Wigner distribution in the following suggestive convenient form:

$$\begin{aligned}
\hat{\rho}_\beta(\hat{\xi}) &= \hat{\rho}_{G_0}(\hat{\xi}) = 2\sinh[\frac{(\det\ G_0)^{-\frac{1}{2}}}{2}]\exp[-\frac{1}{2}\hat{\xi}^T G_0^{-1}\hat{\xi}] \quad , \\
W_\beta(\xi) &= W_{V_0}(\xi) = \frac{1}{2\pi}[\det\ V_0]^{-\frac{1}{2}}\exp[-\frac{1}{2}\xi^T V_0^{-1}\xi] \quad ,
\end{aligned} \tag{2.21}$$

where

$$G_0 = \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta^{-1} \end{pmatrix}, \; V_0 = \begin{pmatrix} \frac{1}{2}\coth(\frac{\beta}{2}) & 0 \\ 0 & \frac{1}{2}\coth(\frac{\beta}{2}) \end{pmatrix} . \tag{2.22}$$

The physical significance of the V_0 matrix is that it is the noise matrix (matrix of variances) of the given thermal state. Since $\overline{n} + \frac{1}{2} = (\det\ V_0)^{\frac{1}{2}}$, as can be seen from (2.18), we can rewrite (2.19) and (2.20) in the form

$$\begin{aligned}
\hat{\rho}_{G_0} &= \sum_{n=0}^{\infty} \frac{1}{(\sqrt{\det\ V_0} + \frac{1}{2})}\left(\frac{\sqrt{\det\ V_0} - \frac{1}{2}}{\sqrt{\det\ V_0} + \frac{1}{2}}\right)^n |n\rangle\langle n|, \\
\mathcal{S}_{G_0} &= k_B[(\sqrt{\det\ V_0} + \frac{1}{2})\ln(\sqrt{\det\ V_0} + \frac{1}{2}) - (\sqrt{\det\ V_0} - \frac{1}{2})\ln(\sqrt{\det\ V_0} - \frac{1}{2})] \, , \\
&\sqrt{\det\ V_0} = \frac{1}{2}\coth((\det\ 2G_0)^{-\frac{1}{2}}).
\end{aligned} \tag{2.23}$$

Now consider the transform of $\hat{\rho}_{G_0}(\hat{\xi})$ and $W_{V_0}(\xi)$ under the unitary evolution $U(S)$ corresponding to the 2×2 real unimodular matrix $S \in Sp(2, R)$. According to (2.10), (2.11), the effect of $U(S)$ is to replace $\hat{\xi}$ and ξ respectively in the density operator and the Wigner distribution by $S^{-1}\hat{\xi}$ and $S^{-1}\xi$. Thus under $U(S)$ we obtain from (2.21)

$$\hat{\rho}_{G_0}(\hat{\xi}) \;\rightarrow\; U(S)\hat{\rho}_{G_0}(\hat{\xi})U(S)^\dagger$$

$$= \quad \hat{\rho}_G(\hat{\xi}) = 2\sinh[\frac{(\det\ G)^{-\frac{1}{2}}}{2}]\ \exp[-\frac{1}{2}\hat{\xi}^T G^{-1}\hat{\xi}] \quad ,$$

$$W_{V_0}(\xi) \quad \rightarrow \quad W_V(\xi) = \frac{1}{2\pi}[\det\ V]^{-\frac{1}{2}}\exp[-\frac{1}{2}\hat{\xi}^T V^{-1}\hat{\xi}], \tag{2.24}$$

where

$$G = SG_0S^T, \qquad V = SV_0S^T. \tag{2.25}$$

Note that det G = det G_0 and det V = det V_0, as a consequence of the fact that det $S = 1$, and we have made use of this fact in (2.24). Both G and V are real symmetric positive semi-definite and finite, and the noise matrix V respects the uncertainty principle det $V \geq \frac{1}{4}$. The spectral decomposition of $\hat{\rho}_G(\hat{\xi})$ and the entropy of this state follow from (2.23):

$$\begin{aligned}
\hat{\rho}_G(\hat{\xi}) &= U(S)\hat{\rho}_{G_0}U(S)^\dagger \\
&= \sum_{n=0}^{\infty}\frac{1}{(\sqrt{\det\ V}+\frac{1}{2})}\left(\frac{\sqrt{\det\ V}-\frac{1}{2}}{\sqrt{\det\ V}+\frac{1}{2}}\right)^n |n;S\rangle\langle n;S| \qquad ;
\end{aligned}$$

$$|n;S\rangle = U(S)\ |n\rangle,$$

$$\mathcal{S}_G = k_B[(\sqrt{\det\ V}+\frac{1}{2})\ln(\sqrt{\det\ V}+\frac{1}{2}) - (\sqrt{\det\ V}-\frac{1}{2})\ln(\sqrt{\det\ V}-\frac{1}{2}),$$

$$\sqrt{\det\ V} = \frac{1}{2}\coth\left((\det\ 2G)^{-\frac{1}{2}}\right). \tag{2.26}$$

Thus, the spectrum, and hence the entropy, does not change under the action of $U(S)$; only the eigenstates (modes) change from the Fock states $|n\rangle$ to the squeezed Fock states $|n;S\rangle = U(S)|n\rangle$. That is, the spectrum and the entropy are invariant and the modes are covariant under $Sp(2,R)$.

It is useful to write the relationship (2.23) between the parameter matrix G of a Gaussian density operator and the parameter matrix V of the associated Gaussian Wigner distribution in a convenient form. To this end note from (2.22) that the matrices G_0 and V_0 are multiples of one another, and hence from (2.25) it follows that the transformed real symmetric matrices G and V are multiples of one another as well. That is,

$$(\det\ G)^{-\frac{1}{2}}\, G = (\det\ V)^{-\frac{1}{2}}\, V \quad . \tag{2.27}$$

This is a property characteristic of single mode systems and it will be found in the next section that it is not valid for multimode Gaussian states. In view of (2.26) we can write this relationship in more detail as

$$\begin{aligned} V &= \frac{\frac{1}{2}\coth\left((\det\ 2G)^{-\frac{1}{2}}\right)}{(\det\ G)^{-\frac{1}{2}}}\, G \quad , \\ G &= [\ 2(\det\ V)^{-\frac{1}{2}}\text{arc coth}\,(\det\ 2V)^{-\frac{1}{2}}]^{-1}\, V \quad . \end{aligned} \tag{2.28}$$

We have thus obtained a complete characterization of Gaussian density operators and the associated Wigner distribution obtained by the action of the unitary operators $U(S)$, $S \in Sp(2,R)$, on zero-mean thermal states with $\langle\hat{\xi}\rangle = 0$. Since G and V are the parameter matrices associated with a Gaussian density operator and the associated Wigner distribution function, the expressions (2.28) can be rightly called *the Weyl ordering rule for Gaussian states* [2].

Our approach can be described by the following commutative diagram:

$$
\begin{array}{ccc}
\hat{\rho}_{G_0}(\hat{\xi}) & \xleftrightarrow[\text{Eq. (2.28)}]{\text{Weyl ordering}} & W_{V_0}(\xi) \\
U(S)\big\downarrow & & \big\downarrow S \\
\hat{\rho}_{G}(\hat{\xi}) & \xleftrightarrow[\text{Eq. (2.28)}]{\text{Weyl ordering}} & W_{V}(\xi)
\end{array}
\qquad (2.29)
$$

It turns out that our results derived for Gaussian states related to the thermal state actually apply to all zero-mean Gaussian states, for it is true that every zero-mean Gaussian state can be obtained from an appropriate zero-mean thermal state by the action of $U(S)$ for some $S \in Sp(2, R)$ in the above manner. To see this consider an arbitrary zero-mean Gaussian state. Clearly it has to necessarily be of the form

$$\hat{\rho}_G(\hat{\xi}) = 2\sinh[\frac{(\det\ G)^{-\frac{1}{2}}}{2}]\ \exp[-\frac{1}{2}\hat{\xi}^T G^{-1}\hat{\xi}]\,. \qquad (2.30)$$

Hermiticity of $\hat{\rho}$ demands G to be real symmetric, whereas positive semi-definiteness and traceability of $\hat{\rho}$ demands that G be positive semi-definite and finite. And the prefactor of the exponential in (2.30) is fixed by the condition $\mathrm{tr}\hat{\rho} = 1$. It is clear from the basic rules of Weyl ordering [2] that a zero-mean Gaussian state should have a zero-mean Wigner distribution of the form

$$W_V(\xi) = \frac{1}{2\pi}\frac{(\det\ V)^{-\frac{1}{2}}}{2}\exp[-\frac{1}{2}\hat{\xi}^T V^{-1}\hat{\xi}]\,, \qquad (2.31)$$

for some real symmetric positive definite V, with the condition $\det\ V \geq \frac{1}{4}$ to meet the uncertainty principle. Now since V is positive definite, $(\det\ V)^{-\frac{1}{2}}V$ is a real symmetric matrix with unit determinant, and *hence it is an element of* $Sp(2, R)$. So also is the unique symmetric positive definite square root of $(\det\ V)^{-\frac{1}{2}}V$. That is $S = (\det\ V)^{-\frac{1}{4}}\ V^{\frac{1}{2}} \in Sp(2, R)$. This shows that V is necessarily of the form

$$V = (\det\ V)^{\frac{1}{2}} S^2 = S V_0 S^T ,$$

$$\begin{aligned} V_0 &= \begin{pmatrix} (\det\ V)^{\frac{1}{2}} & 0 \\ 0 & (\det\ V)^{\frac{1}{2}} \end{pmatrix} , \\ S &= S^T = (\det\ V)^{-\frac{1}{4}}\ V^{\frac{1}{2}} \qquad \in Sp(2, R) . \end{aligned} \tag{2.32}$$

Comparison with (2.24) and (2.25) shows that our arbitrary zero-mean Gaussian-Wigner distribution (2.31) with parameter matrix V is indeed the transform of a thermal state Wigner distribution $W_{V_0}(\xi)$ by the unitary evolution $U(S)$ with V_0 and S determined by (2.32). This proves that our results in this section including the connection (2.28) between the parameter matrices G, V are true for arbitrary single mode zero-mean Gaussian states.

In terms of the commutative diagram (2.29), what we have now achieved is to show that this diagram can indeed be reversed:

$$
\begin{array}{ccc}
\hat{\rho}_{G_0}(\hat{\xi}) & \xleftrightarrow[\text{Eq. (2.28)}]{\text{Weyl ordering}} & W_{V_0}(\xi) \\
U(S) \updownarrow & & \updownarrow S \\
\hat{\rho}_{G}(\hat{\xi}) & \xleftrightarrow[\text{Eq. (2.28)}]{\text{Weyl ordering}} & W_{V}(\xi)
\end{array}
\qquad (2.33)
$$

We now turn to an explicit computation and interpretation of $U(S)$, where S is determined by (2.32) in terms of V. To this end note from (2.8) that the unitary operator

$$
\begin{aligned}
U &= U(S) = U(z) = \exp\left[\frac{r}{2}\left(\hat{a}^{\dagger 2} e^{2i\phi} - \hat{a}^2 e^{-2i\phi}\right)\right] , \\
z &= r\, e^{2i\phi} ,
\end{aligned}
\qquad (2.34)
$$

induces the $Sp(2, R)$ transformation

$$
S = \begin{pmatrix} \cosh r + \sinh r \; \cos 2\phi & \sinh r \; \sin 2\phi \\ \sinh r \; \sin 2\phi & \cosh r - \sinh r \; \cos 2\phi \end{pmatrix} . \qquad (2.35)
$$

To determine r, ϕ in terms of the noise matrix V of the given Gaussian Wigner distribution we write V as

$$
V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} , \quad V_{21} = V_{12} . \qquad (2.36)
$$

Now compare (2.36) with (2.32) which reads $V = (\det\; V)^{\frac{1}{2}} S^2$, with S parametrized as in (2.35). This leads to

$$
\begin{aligned}
r &= \frac{1}{2} \text{arc cosh} \left(\frac{\text{tr } V}{2\sqrt{det\ V}} \right) , \\
2\,\phi &= \arg(V_{11} - V_{22} + 2iV_{12}) .
\end{aligned}
\qquad (2.37)
$$

These are the values of the parameters to be used in (2.34) to compute the unitary operator $U(S)$ which acting on the thermal state with variance matrix $V_0 = (\det\ V)^{\frac{1}{2}}$ realizes the given Gaussian state with variance matrix V. Since this $U(S)$ is the familiar squeezing operator [22, 23], we deduce as one consequence that an arbitrary zero-mean Gaussian state is a squeezed thermal state. As another consequence $|n; S\rangle$ given in (2.26) to represent the eigen modes of the Gaussian states with variance matrix V now have the detailed form

$$|n; S\rangle = U(S)\ |n\rangle = \exp\ [\frac{r}{2}\ (\hat{a}^{\dagger 2} e^{2i\phi} - \hat{a}^2 e^{-2i\phi})\ ]|n\rangle \qquad (2.38)$$

where r, ϕ are determined by V through (2.37). Thus we find that *the modes of an arbitrary zero-mean Gaussian state are squeezed Fock states.*

The Gaussian states we have considered so far in this section are zero-mean states with $\langle\hat{\xi}\rangle = 0$. But our analysis can be simply generalized to Gaussian states with nonzero $\langle\hat{\xi}\rangle$. Clearly, the most general such state is obtained from a zero-mean Gaussian state by the action of the unitary displacement operator $D(\xi_0)$ where $\xi_0 = \langle\hat{\xi}\rangle$:

$$\begin{aligned} \hat{\rho}_{G,\xi_0}(\hat{\xi}) &= D(\xi_0)\ \rho_G(\hat{\xi})\ D(\xi_0)^\dagger \\ &= 2\sinh[(\det\ G)^{-\frac{1}{2}}/2]\ \ \exp\ [-\frac{1}{2}(\hat{\xi} - \xi_0)^T G^{-1}(\hat{\xi} - \xi_0)] \quad , \end{aligned} \qquad (2.39)$$

and the corresponding Wigner distribution is

$$W_{V,\xi_0}(\xi) = \frac{1}{2\pi}[\det\ V]^{-\frac{1}{2}} \exp[-\frac{1}{2}(\xi - \xi_0)^T V^{-1}(\xi - \xi_0)] \quad , \qquad (2.40)$$

and this fact follows from (2.15).

It is clear that nonzero value of $\langle\hat{\xi}\rangle$ does not affect the eigenvalue spectrum, and hence the entropy. It changes the modes in an obvious manner, and we have the mode decomposition

$$\begin{aligned}\hat{\rho}_{G,\xi_0}(\hat{\xi}) &= \sum_{n=0}^{\infty} \frac{1}{(\sqrt{\det\ V}+\frac{1}{2})}\left(\frac{\sqrt{\det\ V}-\frac{1}{2}}{\sqrt{\det\ V}+\frac{1}{2}}\right)^n |n;z;\xi_0\rangle\ \langle n;z;\xi_0| \quad , \\ |n;z;\xi_0\rangle &= |n;S;\xi_0\rangle = D(\xi_0)\ \exp\,[\frac{1}{2}(za^{\dagger 2}-z^* a^2)]\ |n\rangle, \qquad (2.41)\end{aligned}$$

with $z = r\ e^{2i\phi}$ determined by V through (2.37). Thus, *the modes are squeezed displaced Fock states.* The displacement is determined by $\xi_0 = \langle\hat{\xi}\rangle$, while the complex squeeze parameter z is determined by the variance matrix. The relationship (2.28) between G and V continue to be valid, and the spectrum and entropy are determined fully in terms of det V. We can summarize the principal results in this section in the form of

Theorem 2.1: The density matrix of the most general Gaussian state of a single mode system is specified by (G,ξ_0) where G is a 2×2 real symmetric positive semi-definite matrix and $\xi_0 = \langle\hat{\xi}\rangle$is a vector in phase space. The corresponding Wigner distribution is a displaced Gaussian specified by (V,ξ_0) where V is the real symmetric positive definite noise matrix with det $V \geq \frac{1}{4}$. The matrices V and G are related through the Weyl ordering rule (2.28). The state is a squeezed displaced thermal state. The complex squeezing parameter z is determined by the variance matrix through (2.37), while ξ_0 equals $\langle\hat{\xi}\rangle$. This state has the eigenmode decomposition given in (2.26). The spectrum, and hence the entropy as seen from (2.26), depends on det V alone.

It should be emphasised in passing that det V is the only invariant in our scheme, and there is quantum restriction only on this object in the form of the uncertainty principle det $V \geq 1/4$. The parameter ξ_0 is free to be any point in R^2.

Much of our detailed analysis in the present section has been carried out in such a form in which it simply generalizes to the multimode case. For this reason our analysis in this section was rather detailed, and for the same reason we can afford to be relatively brief in our analysis of the multimode case in Section 4.

It should be noted, however, that the single mode case governed by $Sp(2, R)$ is special in some aspects. In this case every noise matrix is a constant times an $Sp(2, R)$ matrix. To be specific, given a noise matrix V the matrix $(\det\ V)^{-1/2}V \in Sp(2, R)$. This property does not generalize to the multimode case in any obvious manner. For example, $(\det\ V)^{-1/2N}\ V\ \notin\ Sp(2N, R)$ for a general N-mode noise matrix V. Another aspect which is special to the single mode case is the somewhat related fact given in (2.28) which says that the G matrix parametrizing the Gaussian density operator and the V matrix parametrizing the corresponding Wigner distribution are multiples of one another for every single mode Gaussian state. Finally, in the single mode case the uncertainty principle placed a restriction only on the determinant of the noise matrix, whereas more subtle restrictions on the noise matrix are present in the multimode case.

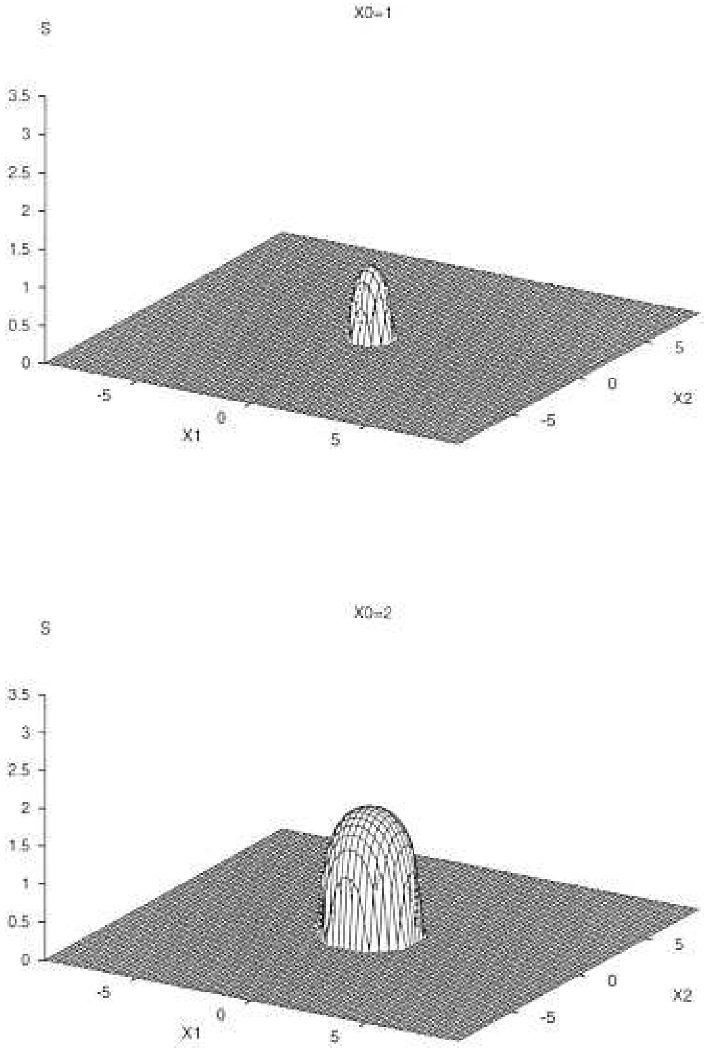


Figure 2.1: The entropy of single mode Gaussian states with the noise matrix parametrized as in (2.41). The behaviour of entropy as a function of x_1, x_2 is shown for $x_0 = 1, 2$. For each value of x_0, the physically allowed values of x_1, x_2 are restricted by the uncertainty principle to the circular region $x_1^2 + x_2^2 \leq x_0^2 - 1/4$. Points on the boundary of this region correspond to pure Gaussian states.

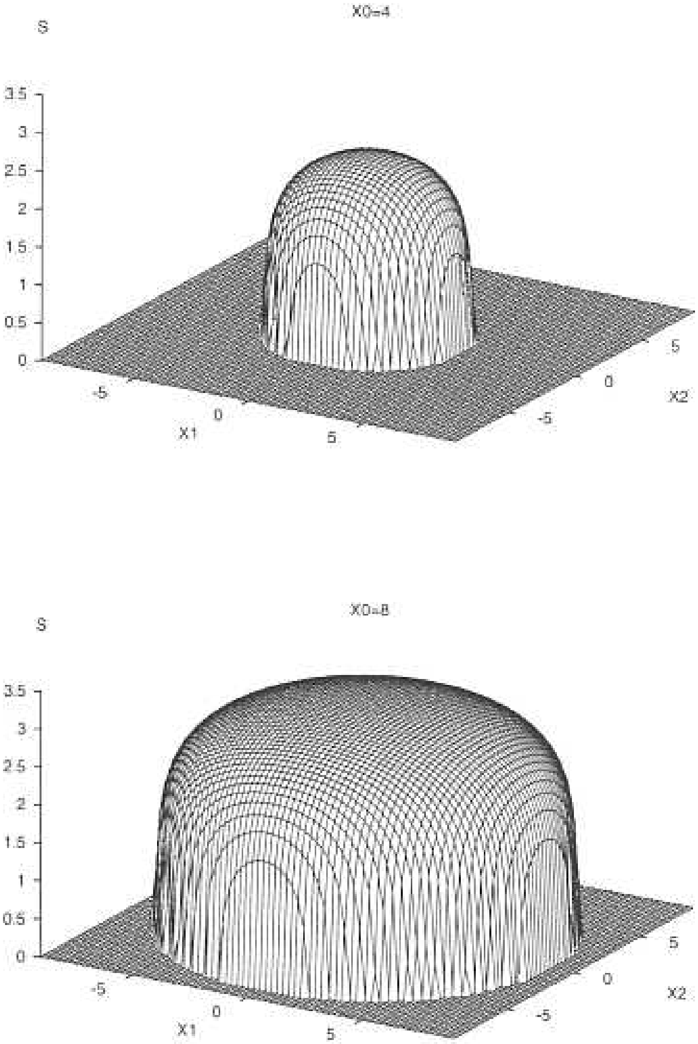


Figure 2.2: Same as figure 2.1, but for $x_0 = 4, 8$

To conclude this section we present in Figs. 2.1 and 2.2 the entropy of a singlemode Gaussian state. Recalling that the entropy in this case is fully determined by the (determinant of the) noise matrix V, we parametrize V as [14]

$$V = \begin{pmatrix} x_0 + x_1 & x_2 \\ x_2 & x_0 - x_1 \end{pmatrix} . \tag{2.42}$$

Then the uncertainty principle reads

$$\det V \;=\; x_0^2 - x_1^2 - x_2^2 \;\geq\; 1/4, \;\; x_0 > 0 \;. \tag{2.43}$$

That is, the physically allowed variance matrices are bounded by the upper sheet of the two-sheeted hyperboloid in a 2+1 dimensional Minkowski space. Thus, for a given value of x_0 the physically allowed values of the parameters x_1, x_2 are constrained by $x_1^2 + x_2^2 \leq x_0^2 - 1/4$. When this inequality is saturated $\det V = 1/4$, and hence from (2.26) the entropy is zero irrespective of the value of x_0, showing that this saturation corresponds to a Gaussian *pure* state [24] i.e. a squeezed coherent state. Finally, $\det V$ is a monotonically increasing function of x_0 and monotonically decreasing function of $|x_1|, |x_2|$ and so also is the entropy, and this is seen in Figs. 2.1 and 2.2.

Finally, while we have tailored our analysis in this section in such a way that it generalizes to the multimode case as already pointed out, the power of the present approach can already be seen in the single mode case by comparing with Ref. [16].

2.4 Multimode Gaussian States

Consider an N-mode system [25, 13, 18, 19] described by annihilation operators $\hat{a}_1, \hat{a}_2 \ldots \hat{a}_N$ and their adjoints $\hat{a}_1^\dagger, \hat{a}_2^\dagger, \ldots, \hat{a}_N^\dagger$ or equivalently by the $2N$ hermitian operators $\hat{\xi}$ defined in (2.1) obeying the commutation relations (2.2). To begin with we consider the special N-mode Gaussian density operator

$$\hat{\rho} = \prod_{j=1}^{N} 2\sinh(\frac{\beta_j}{2})\exp[-\frac{\beta_j}{2}(\hat{q}_j^2 + \hat{p}_j^2)] \ . \tag{2.44}$$

This special state is important, for it will turn out that given any N-mode Gaussian state it can be reduced to this form by a unitary transformation $U(S)$, for some $S \in Sp(2N, R)$. For this reason we will call (2.44) the *canonical Gaussian density operator*. In this state the N modes are individually in thermal equilibrium with temperature parameters $\beta_1, \beta_2, \ldots, \beta_N$, with no coupling between the modes.

The separable-product-Gaussian form of the canonical state (2.44) implies that the corresponding Wigner distribution is separable as well, and we have

$$W(\xi) = \prod_{j=1}^{N} \left(\frac{1}{\pi} \tanh(\frac{\beta_j}{2})\exp[-\tanh(\frac{\beta_j}{2})(q_j^2 + p_j^2)] \right) \tag{2.45}$$

The product form of (2.44) allows us to write down the spectral decomposition of this multimode density operator simply from our result (2.19) for the single mode case, and we have

$$\hat{\rho} = \sum_{n_1,\, n_2, \ldots n_N} \left(\prod_{j=1}^{N} (1 - e^{-\beta_j}) e^{-\beta_j n_j} \right) |n_1,\, n_2, \ldots n_N\rangle\langle n_1,\, n_2, \ldots n_N|. \tag{2.46}$$

where $|n_1, n_2, \ldots n_N\rangle$ are the multimode Fock states with n_j excitations in the j^{th} mode. These states span the N-mode Hilbert space, and the summation in (2.46) is from 0 to ∞ independently for each n_j. From (2.46) and our result (2.20) for the single mode case the expression for the entropy associated with the state (2.44) readily follows:

$$\mathcal{S} = k_B \sum_{j=1}^{\infty} \left[\frac{\beta_j}{e^{\beta_j} - 1} - \ln(1 - e^{-\beta_j}) \right]. \tag{2.47}$$

Generalization of our construction used in the single mode case suggests that we define 2N×2N diagonal matrices G_0 and V_0 through

$$G_0 = \begin{pmatrix} \beta_1^{-1} & & & & & \\ & \ddots & & & & \\ & & \beta_N^{-1} & & & \\ & & & \beta_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \beta_N^{-1} \end{pmatrix}$$

$$\begin{aligned} V_0 &= \begin{pmatrix} \frac{1}{2}\coth(\frac{\beta_1}{2}) & & & & & \\ & \ddots & & & & \\ & & \frac{1}{2}\coth(\frac{\beta_N}{2}) & & & \\ & & & \frac{1}{2}\coth(\frac{\beta_1}{2}) & & \\ & & & & \ddots & \\ & & & & & \frac{1}{2}\coth(\frac{\beta_N}{2}) \end{pmatrix} \\ V_0 &= \frac{1}{2}\coth(\frac{G_0^{-1}}{2}), \end{aligned} \tag{2.48}$$

so that the density operator and the Wigner distribution can be rewritten as

$$\rho_{G_0}(\hat{\xi}) = [\det(2\sinh(G_0^{-1}/2))]^{\frac{1}{2}} \exp[-\frac{1}{2}\hat{\xi}^T G_0^{-1}\hat{\xi}] ,$$
$$W_{V_0}(\xi) = [\det(2\pi V_0)]^{-\frac{1}{2}} \; exp[-\frac{1}{2}\xi^T V_0^{-1}\xi] , \quad (2.49)$$

where $\hat{\xi}$ and ξ are the $2N$-element column vectors defined in (2.1). It should be appreciated that G_0 and V_0 have only N independent entries!

The expressions (2.49) are compact, but they are not yet in a form convenient for the description of the evolution of the density operator and the Wigner distribution under the unitary operators $U(S)$ with $S \in Sp(2N,R)$. To recast them into such a convenient form we use the following

Lemma 1: Let M and σ be $m \times m$ matrices such that $\sigma^2 = 1$, $[\sigma, M] = 0$, and let $f(x)$ and $g(x)$ be any respectively odd and even functions possessing Taylor series expansions about $x = 0$. Then $f(M\sigma) = \sigma f(M)$, and $g(M\sigma) = g(M)$.

Proof of this assertion is elementary. Write $f(M\sigma)$ as a power series in $M\sigma$, and then use the relations $[\sigma, M] = 0$, and $\sigma^2 = 1$. And similarly for $g(M\sigma)$.

To apply this result to our problem, note that the symplectic metric Ω defined in (2.3) satisfies $\Omega^{-1} = \Omega^T = -\Omega$. Let $\sigma = i\Omega$ and $M = G_0$. Clearly $i\Omega$ and G satisfy the hypothesis of the above lemma. Now consider an odd function like $\sinh G_0$. We have

$$\sinh G_0 = \sinh(G_0 \, i\Omega \, i\Omega) = i\Omega \, \sinh(G_0 \, i\Omega) = \Omega^T \, \sin(G_0 \, \Omega), \quad (2.50)$$

where we used the above lemma in the last but one step, and also the facts $\Omega^T = -\Omega$ and $\sinh\, i\theta = i\, \sin\, \theta$ in the last step. By a similar argument we have

$$\coth(\frac{G_0^{-1}}{2}) = \Omega^T \cot(\frac{G_0^{-1}\Omega}{2}) \tag{2.51}$$

This and the fact that $\det \Omega = 1$ allow us to rewrite (2.49) and (2.48) in the form

$$\begin{aligned} \rho_{G_0}(\hat{\xi}) &= [\det(2\sin(G_0^{-1}\Omega/2))]^{\frac{1}{2}} \ \exp[-\frac{1}{2}\hat{\xi}^T G_0^{-1}\hat{\xi}], \\ W_{V_0}(\xi) &= [\det\, (2\pi(V_0))]^{-\frac{1}{2}} \ \exp[-\frac{1}{2}\xi^T V_0^{-1}\xi], \end{aligned} \tag{2.52}$$

where

$$V_0 = \frac{1}{2}\Omega^T \cot(\frac{G_0^{-1}\Omega}{2}). \tag{2.53}$$

There exists an important reason for our choosing to write the connection between G_0 and V_0 of the canonical Gaussian density operator in the form (2.53) rather than as $V_0 = \frac{1}{2}\coth(\frac{G_0^{-1}}{2})$. This will become clear in what follows.

An observation may be made with respect to the identities (2.50) and (2.51). To this end recall first of all the well known canonical procedure [20] by which a given $2N$ dimensional real vector space is converted into an N dimensional complex vector space, with the antisymmetric symplectic metric Ω playing the role of i. Now it is interesting to note that Ω plays the role of i in (2.50) and (2.51) as well. Indeed, rewriting of these as $\sin G_0\Omega = \Omega \sinh G_0$ and $\cot G_0\Omega = \Omega \coth G_0$ renders their analogy with $\sin\, i\theta = i\, \sinh\theta$ and $\cot\, i\theta = i\, \coth\theta$ transparent. Further, an argument similar to the one leading to (2.50) leads to the identity $\cosh G_0 = \cos G_0\Omega$, which is analogous to $\cosh\theta = \cos i\theta$.

The only property of G_0 used in the derivation of these identities is the fact that $[G_0, \Omega] = 0$. Hence it follows that these identities apply to every matrix M which commutes with Ω. Clearly the most general form of such a matrix is

$$M = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \quad , \tag{2.54}$$

where X, Y are arbitrary $N \times N$ matrices. We have thus established the following.

Lemma 2: Let M be any matrix of the form (2.54). Then M respects the following identities

$$\begin{aligned} \cosh M &= \cos M\Omega \quad , \\ \sinh M &= \Omega^T \sin M\Omega \quad , \\ \tanh M &= \Omega^T \tan M\Omega \quad . \end{aligned} \tag{2.55}$$

Having considered the canonical thermal state density operator in some detail, we may now ask: What is the most general zero-mean Gaussian density operator for an N-mode system? As argued in Section 3, it has to be of the form

$$\rho_G(\hat{\xi}) = N(G)\ \exp[-\frac{1}{2}\hat{\xi}^T G^{-1}\hat{\xi}], \tag{2.56}$$

for some real symmetric $2N \times 2N$ matrix G which is positive semi-definite and finite. Clearly, the normalization constant $N(G)$ is determined from the requirement $\mathrm{tr}\hat{\rho} = 1$, and we have

$$N(G) = \left(\mathrm{tr}\ \exp[-\frac{1}{2}\hat{\xi}^T G^{-1}\hat{\xi}] \right)^{-1} . \tag{2.57}$$

The explicit form of $N(G)$ will be determined later.

It is clear that the zero-mean Gaussian state (2.56) has a zero-mean Gaussian Wigner distribution of the form

$$W_V(\xi) = (2\pi \det V)^{-\frac{1}{2}} \exp[-\frac{1}{2}\xi^T V^{-1}\xi], \tag{2.58}$$

for some $2N \times 2N$ real matrix V. The normalization condition

$$\int d^{2N}\xi W(\xi) = 1, \tag{2.59}$$

which corresponds to $\mathrm{tr}\hat{\rho} = 1$, requires V to be positive definite. But positivity alone is not sufficient. The question of necessary and sufficient conditions on a real symmetric positive definite $2N \times 2N$ matrix V so that $W(\xi)$ defined by (2.58) represents a Wigner distribution (i.e. so that the hermitian operator computed from the real $W(\xi)$ by Weyl ordering will be positive semi-definite) is a subtle one first raised by Littlejohn [12], and subsequently answered in [13, 26], and it corresponds to the complete set of uncertainty principles which the multimode noise matrix V has to respect.

Now, the positive definiteness of V allows us to take advantage of a classic theorem due to Williamson [21, 13] which is as follows

Williamson's Theorem: Given any positive definite real symmetric $2N \times 2N$ matrix V, there exists an $S \in Sp(2N, R)$ such that

$$V = S\, V_0\, S^T \quad , \tag{2.60}$$

where V_0 has the special diagonal (Williamson canonical) form

$$V_0 = \begin{pmatrix} \kappa_1 & & & & & \\ & \ddots & & & & \\ & & \kappa_N & & & \\ & & & \kappa_1 & & \\ & & & & \ddots & \\ & & & & & \kappa_N \end{pmatrix} \tag{2.61}$$

$\kappa_1, \kappa_2, \ldots \kappa_N$ being strictly positive.

This assertion is nontrivial, for not every symmetric matrix can be diagonalized by a transformation of the form (2.60), with $S \in Sp(2N, R)$. Note that the Williamson canonical form V_0 has only N independent entries, and is identical in form to the canonical noise matrix (2.48). It should be further noted that κ_j are not the eigenvalues of V, for (2.60) does not represent a similarity transformation. However, as noted in (2.13), under the symmetric transformation (2.60) connecting V and V_0, the matrices $V_0\,\Omega$ and $V\Omega$ are related by a similarity transformation :

$$V\,\Omega = S\,V_0\,\Omega\,S^{-1}. \tag{2.62}$$

It follows that $\kappa_1^2,\ \kappa_2^2, \ldots \kappa_j^2$ are the eigenvalues of $-(V\Omega)^2 = V\Omega V\Omega^T$. This is so because the Williamson normal (canonical) form of $V_0\Omega$ is

$$V_0\Omega = \left(\begin{array}{ccc|ccc} & & & \kappa_1 & & \\ & 0 & & & \ddots & \\ & & & & & \kappa_N \\ \hline -\kappa_1 & & & & & \\ & \ddots & & & 0 & \\ & & -\kappa_N & & & \end{array}\right).$$

Equivalently, these are the eigenvalues of the manifestly symmetric positive definite matrix $V^{\frac{1}{2}}\Omega V\Omega^T V^{\frac{1}{2}}$ (recall that $\Omega^T = -\Omega$) which is obtained from the earlier one through conjugation by $V^{\frac{1}{2}}$. As a consequence V and $V' = SVS^T$ for any $S \in Sp(2N, R)$ have the same set of κ's. In other words the κ's are symplectic invariants. It has been shown in [13, 19, 26] that the complete set of uncertainty principles which a given noise matrix V has to satisfy are

$$\kappa_j \geq \frac{1}{2}, \quad j = 1, 2, \ldots N \tag{2.63}$$

where κ_j are determined from V as described above.

We can now proceed to determine the relationship between G and V, the normalization constant $N(G)$ and the spectral decomposition and entropy of the general Gaussian state (2.56). We begin with the relationship between the matrices G and V and answer the question of Weyl ordering for multimode Gaussian states.

Theorem 2.2: The parameter matrices G and V of a Gaussian density operator and the associated Wigner distribution are related by

$$V = \frac{1}{2}\Omega^T \cot(\frac{G^{-1}\Omega}{2}). \tag{2.64}$$

Proof: Given a Gaussian density operator $\hat{\rho}(\hat{\xi})$ with parameter matrix G, we consider the family of states $U(S)\ \hat{\rho}(\hat{\xi})\ U(S)^\dagger$, with S running over the symplectic group $Sp(2N, R)$. This gives by definition the orbit of $\hat{\rho}$ under $Sp(2N, R)$. We know from (2.10), (2.11) that $U(S)\ \hat{\rho}(\hat{\xi})\ U(S)^\dagger$ and the associated Wigner distribution are obtained from $\hat{\rho}(\hat{\xi})$ and its Wigner distribution $W(\xi)$ simply by replacing $\hat{\xi}$ and ξ by $S^{-1}\,\hat{\xi}$ and $S^{-1}\,\xi$ respectively. But from (2.56) and (2.58) we see that such a replacement is equivalent to

$$\begin{aligned} G &\rightarrow G' = S\ G\ S^T \quad , \\ V &\rightarrow V' = S\ V\ S^T \quad . \end{aligned} \tag{2.65}$$

That is the orbit of $\hat{\rho}$ is determined by one of the (equivalent) orbits (2.65) for the matrix G or V.

As the next step note that the relationship (2.64) is covariant under the symplectic evolution (2.65):

$$\begin{aligned} \frac{1}{2}\Omega^T \cot(G'^{-1}\Omega/2) &= \frac{1}{2}\Omega^T \cot((S^{-1})^T\ G^{-1}\ S^{-1}\ \Omega/2) \\ &= \frac{1}{2}\Omega^T \cot((S^{-1})^T\ G^{-1}\ \Omega\ S^T/2) \\ &= \frac{1}{2}\Omega^T\ (S^{-1})^T \cot((G^{-1}\ \Omega/2)\ S^T \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} S\, \Omega^T \cot((G^{-1}\, \Omega/2)\, S^T \\
&= S\, V\, S^T = V' \quad . \qquad (2.66)
\end{aligned}
$$

The $Sp(2N, R)$ covariance of (2.64) implies that this relationship is true either at all points on the orbit or at no point at all.

Further, since the variance matrix is positive definite, we know by Williamson's theorem that there exists a canonical point of the form (2.48) [i.e. of the form (2.61)] on every orbit. Finally, since we have already proved in (2.53) that the relationship (2.64) is valid at this canonical point of the orbit, it is valid on the entire orbit, since it is covariant. This completes the proof.

The relationship (2.64) is one of the principal results of this Section. We wish to emphasize that this compact expression constitutes a solution to the problem of Weyl ordering (which by definition makes a one-to-one correspondence between density operators and Wigner distributions) in the case of Gaussian N-mode states.

Next, to determine the normalization constant $N(G)$, we use the unitary operator $U(S)$, $S \in Sp(2N, R)$, which takes $\hat{\rho}$ to its canonical form, and note that

$$
\begin{aligned}
N(G)^{-1} &= \mathrm{tr}\, \exp[-\frac{1}{2}\hat{\xi}^T G^{-1} \hat{\xi}] \\
&= \mathrm{tr}\, (U(S)\, \exp[-\frac{1}{2}\hat{\xi}^T G_0^{-1} \hat{\xi}]\, U(S)^\dagger) \\
&= \mathrm{tr} \exp[-\frac{1}{2}\xi^T G_0^{-1} \xi] \\
&= N(G_0)^{-1} \quad . \qquad (2.67)
\end{aligned}
$$

That is, $N(G)$ is constant over the orbit of G. Therefore using (2.52) we have

$$N(G) = N(G_0) = [\det\ (2\sin(G_0^{-1}\Omega\ /2))]^{\frac{1}{2}}\ .$$

But, from the relationship $G\ =\ SG_0S^T$ we have

$$\begin{aligned} \sin(G_0^{-1}\Omega\ /2) &= \sin(\Omega\ S^T\ G^{-1}\ S/2) \\ &= \sin(S^T\ G^{-1}\Omega\ (S^T)^{-1}/2) \\ &= S^T\ \sin(G^{-1}\Omega\ /2)(S^T)^{-1}\quad . \end{aligned} \tag{2.68}$$

Since $\det\ S^T = \det\ S = 1$, for every $S \in Sp(2n, R)$, this implies

$$\det\ (2\sin(G_0^{-1}\Omega\ /2)) = \det(2\sin(G^{-1}\Omega\ /2))\quad ,$$

and hence we have the final result

$$N(G) = [\det\ (2\sin(G^{-1}\Omega\ /2)]^{\frac{1}{2}}\quad . \tag{2.69}$$

Thus, (2.56) can be rewritten in all details as

$$\hat{\rho}_G(\xi) = [\det\ (2\sin(G^{-1}\Omega\ /2)]^{\frac{1}{2}} \exp[-\frac{1}{2}\hat{\xi}^T G^{-1}\hat{\xi}]\quad . \tag{2.70}$$

This completes our discussion of the normalization constant $N(G)$.

We turn our attention now to the spectral decomposition of the general Gaussian density operator $\hat{\rho}_G(\hat{\xi})$ in (2.70) corresponding to the Gaussian Wigner distribution $W_V(\xi)$ in (2.58), with G and V related as in (2.64). Let $S \in\ Sp(2N, R)$ take V to the Williamson canonical form V_0 given in (2.61), which has the same structure as our original canonical form (2.48). It follows from (2.64) that under S, $\hat{\rho}_G(\hat{\xi})$ also goes over to the canonical form $\hat{\rho}_{G_0}(\hat{\xi})$:

$$\hat{\rho}_G(\xi) = U(S)\hat{\rho}_{G_0}(\hat{\xi})\, U(S)^\dagger \quad . \tag{2.71}$$

Recalling that $\kappa_1,\ \kappa_2, \ldots \kappa_N$ are square roots of the $Sp(2N, R)$ invariant eigenvalues of $\Omega V \Omega^T V$, and comparing the canonical forms (2.61) and (2.48) we deduce

$$\begin{aligned} e^{-\beta_j} &= \frac{\kappa_j - \frac{1}{2}}{\kappa_j + \frac{1}{2}} \quad , \\ 1 - e^{-\beta_j} &= \frac{1}{\kappa_j + \frac{1}{2}} \quad . \end{aligned} \tag{2.72}$$

Thus using (2.71) and the spectral decomposition (2.46) for $\hat{\rho}_{G_0}$ we have

$$\hat{\rho}_G = \sum_{n_1,\ n_2 \ldots n_N} \left(\prod_{j=1}^{N} \frac{1}{\kappa_j + \frac{1}{2}} \left(\frac{\kappa_j - \frac{1}{2}}{\kappa_j + \frac{1}{2}}\right)^{n_j} \right) |n_1,\ n_2 \ldots n_N; S\rangle \langle n_1,\ n_2 \ldots n_N; S| \quad ,$$

$$|n_1,\ n_2 \ldots n_N; S\rangle \ = U(S)\ |n_1,\ n_2 \ldots n_N\rangle\ . \tag{2.73}$$

The product form of the eigenvalue spectrum in (2.73) can be exploited to write down the entropy by inspection. From (2.72) we have

$$\begin{aligned} \beta_j &= \ln\left(\frac{\kappa_j + \frac{1}{2}}{\kappa_j - \frac{1}{2}}\right) \quad , \\ e^{\beta_j} - 1 &= (\kappa_j - \frac{1}{2})^{-1} \quad , \end{aligned} \tag{2.74}$$

and therefore using (2.72) and (2.74) in (2.47), and recalling that the spectrum in (2.73) has the product form, the expression for the entropy can be written as

$$\mathcal{S} = \sum_{j=1}^{N} k_B \left[(\kappa_j + \frac{1}{2})\ \ln(\kappa_j + \frac{1}{2})\ -\ (\kappa_j - \frac{1}{2})\ \ln\,(\kappa_j - \frac{1}{2}) \right] \quad . \tag{2.75}$$

As in the single mode case, we find that the spectrum and entropy are invariant and the modes are covariant under $Sp(2N, R)$. That is, the spectrum and entropy depend only on the κ_j's associated with the noise matrix V of the given Gaussian Wigner distribution. Since κ_j's are symplectic invariants, we see that all Gaussian states belonging to the same $Sp(2N, R)$ orbit have the same eigenvalue spectrum, and hence the same value for entropy.

In our analysis above we made effective use of Williamson's theorem which asserts that every zero-mean Gaussian density operator is unitarily equivalent to a canonical Gaussian density operator of the form (2.44), with the β_j's of the canonical form determined by the $Sp(2N, R)$ invariant κ_j's of the noise matrix V of the given state through (2.74). Thus, the invariance of the spectrum and entropy under unitary transformation allowed us to write the spectrum and entropy of a general zero-mean Gaussian state invoking the results (2.46), (2.47) for the canonical state. Since κ_j^2 are the eigenvalues of $-(V\Omega)^2$, they can be determined, for instance, by solving the $Sp(2N, R)$ invariant relations [13, 19]

$$2\sum_{j=1}^{N} \kappa_j^{2l} = (-1)^l \operatorname{tr}(V\Omega)^{2l} \quad , \quad l = 1, 2, \ldots, N. \tag{2.76}$$

That these relations are $Sp(2N, R)$ invariant is a consequence of (2.13).

Extension of our results to Gaussian states with nonzero mean $\langle \hat{\xi} \rangle = \xi_0 \neq 0$ is straightforward. It is transparent from (2.11), (2.15), (2.70) and (2.58) that the density operator and the associated Wigner distribution of such a state have the form

$$\begin{aligned}
\hat{\rho}_{G,\xi_0}(\hat{\xi}) &= D(\xi_0)\, \hat{\rho}_G(\hat{\xi})\, D(\xi_0)^\dagger \\
&= [\det(2\sin(G^{-1}\Omega/2))]^{1/2} \exp[-\frac{1}{2}(\hat{\xi}-\xi_0)^T\, G^{-1}\, (\hat{\xi}-\xi_0)]\,, \quad (2.77) \\
W_{V,\xi_0}(\xi) &= (2\pi \det V)^{-1/2} \exp[-\frac{1}{2}(\xi-\xi_0)^T\, V^{-1}\, (\xi-\xi_0)]\,. \quad (2.78)
\end{aligned}$$

The relationship (2.64)between G and V remains unaltered, as also the particular $S \in Sp(2N,R)$ which takes the given G and V to their Williamson canonical forms G_0, V_0. Further the $Sp(2N,R)$ invariant κ_j's are not affected by the unitary displacement operator $D(\xi_0)$. Thus we have in view of (2.73) the spectral decomposition

$$\hat{\rho}_{G,\xi_0}(\hat{\xi}) = \sum_{n_1,\, n_2 \ldots n_N} \left(\prod_{j=1}^{N} \frac{1}{\kappa_j + \frac{1}{2}} (\frac{\kappa_j - \frac{1}{2}}{\kappa_j + \frac{1}{2}})^{n_j} \right) |n_1, \ldots n_N; S; \xi_0\rangle\, \langle n_1, \ldots n_N; S; \xi_0|\,,$$

$$|n_1,\, n_2 \ldots n_N; S; \xi_0\rangle = D(\xi_0)\, U(S)\, |n_1,\, n_2 \ldots n_N\rangle\,. \quad (2.79)$$

Since ξ_0 does not enter the spectrum, we conclude that the entropy is given by (2.75). We summarize the results in the following form.

Theorem 2.3: The most general Gaussian state of an N-mode system is determined by (G, ξ_0) as in (2.77), where G is a real symmetric positive semi-definite finite $2N \times 2N$ matrix and $\xi_0 \in R^{2N}$. The corresponding Wigner distribution is determined by (V, ξ_0) as in (2.78) where V is the $2N \times 2N$ noise matrix which is real symmetric positive definite and satisfies the $Sp(2N,R)$ invariant uncertainty principles (2.63). The Weyl ordering rule connecting G and V is given by the $Sp(2N,R)$ covariant relation (2.64). This state has the spectral decomposition (2.79) and entropy (2.75).

The spectrum and entropy of the state depend only on the set $(\kappa_1^2, \kappa_2^2, \ldots, \kappa_N^2)$, the $Sp(2N, R)$ invariant eigenvalues of $V\Omega V\Omega^T$, and hence they are constant over any orbit of the semi-direct product of $Sp(2N, R)$ and the N-mode Weyl group. Every Gaussian state is unitarily equivalent to a canonical state of the form (2.44), and is obtained from the latter by the action of an appropriate element of this semi-direct product group.

In the next Section we consider the special case of two-mode systems in some more detail, but it is useful to sketch the principle of our approach in the general N-mode case in the form of a commutative diagram:

$$
\begin{array}{ccc}
\hat{\rho}_{G_0,\xi_0}(\hat{\xi}) & \xleftrightarrow[\text{Eq. (2.64)}]{\text{Weyl ordering}} & W_{V_0,\xi_0}(\xi) \\
U(S)\big\updownarrow & & \big\updownarrow S \\
\hat{\rho}_{G,\xi_0}(\hat{\xi}) & \xleftrightarrow[\text{Eq. (2.64)}]{\text{Weyl ordering}} & W_{V,\xi_0}(\xi)
\end{array}
\qquad (2.80)
$$

Clearly $\kappa_1, \kappa_2, \ldots \kappa_N$ are the only invariants, and there are quantum restrictions only on them in the form of uncertainty principles. There are no restrictions on ξ_0, and ξ_0 can be any point in the N-mode phase space R^{2N}.

2.5 Two-mode system as a special case

In the previous Section we presented the characterization of a general Gaussian density operator and the associated Wigner distribution for an N-mode system. We also derived explicit expressions for the mode decomposition spectrum and for the entropy of a general Gaussian state in terms of the $Sp(2N, R)$ invariants $\kappa_1, \kappa_2, \ldots, \kappa_N$ of the associated noise matrix V. In the present Section we consider the special case of two-mode Gaussian states in some detail.

In the two-mode case governed by $Sp(4, R)$, the Williamson normal form of the noise matrix V is

$$V_0 = SVS^T = \begin{pmatrix} \kappa_1 & & & \\ & \kappa_2 & & \\ & & \kappa_1 & \\ & & & \kappa_2 \end{pmatrix}, \quad S \in Sp(4, R); \quad \kappa_1, \kappa_2 \geq 1/2 \,. \tag{2.81}$$

The spectrum and the entropy are fully determined by (2.73) and (2.75) from a knowledge of κ_1 and κ_2. To determine the positve quantities κ_1 and κ_2, or equivalently κ_1^2 and κ_2^2, note from (2.76) that we have the trace relations

$$2\,(\kappa_1^2 + \kappa_2^2) = -\mathrm{tr}(V\Omega)^2 \,, \tag{2.82}$$

$$2\,(\kappa_1^4 + \kappa_2^4) = +\mathrm{tr}(V\Omega)^4 \,. \tag{2.83}$$

which are $Sp(4, R)$ invariant in view of (2.13). Further, since $\det V$ is $Sp(4, R)$ invariant, we have from (2.81)

$$\kappa_1^2\,\kappa_2^2 = \det V \,. \tag{2.84}$$

Clearly, the above three equations are not independent, and any two of them will suffice to determine κ_1^2, κ_2^2. Indeed we have the relation

$$8 \det V \;+\; 2\mathrm{tr}(V\Omega)^4 \;-\; (\mathrm{tr}(V\Omega)^2)^2 = 0 \,, \tag{2.85}$$

which is just an $Sp(4, R)$ invariant way of expressing the fact that $8\kappa_1^2\kappa_2^2 + 4(\kappa_1^4 + \kappa_2^4) - 4(\kappa_1^2 + \kappa_2^2)^2 = 0$. Choosing $\det V$ and $\mathrm{tr}(V\Omega)^2$ from the above three, we have the following expressions for κ_1^2 and κ_2^2:

$$\kappa_1^2, \kappa_2^2 = \frac{1}{4}\,\mathrm{tr}(V\Omega V\Omega^T) \pm \left[(\frac{1}{4}\mathrm{tr}(VGVG^T))^2 - \det V\right]^{1/2} . \qquad (2.86)$$

Note that the fact that V is positive semi-definite implies that $(\frac{1}{4}\mathrm{tr}(V\Omega V\Omega^T))^2 \geq \det V$. Now the uncertainty principle (2.63) simply reads that the smaller of κ_1^2, κ_2^2 be bounded from below by $1/4$. That is

$$\mathrm{tr}(V\Omega V\Omega^T) - \sqrt{(\mathrm{tr}(V\Omega V\Omega^T))^2 - 16\det V} \geq 1 , \qquad (2.87)$$

which can be rewritten as

$$2\,\mathrm{tr}(V\Omega V\Omega^T) - 16\det V \leq 1 . \qquad (2.88)$$

Finally, using the identity (2.85) the uncertainty principle can be cast in the following equivalent form in terms of $\mathrm{tr}(V\Omega)^2$, $\mathrm{tr}(V\Omega)^4$:

$$8\,\mathrm{tr}(V\Omega)^4 - 4(\mathrm{tr}(V\Omega)^2 - 1/2)^2 \leq 1 . \qquad (2.89)$$

The uncertainty principle (2.88) or (2.89) subsumes the weaker condition $\det V \leq 1/16$, that the above inequalities saturate when the smaller of κ_1, κ_2 equals half, and that when $\kappa_1 = \kappa_2 = 1/2$ the state is necessarily a pure Gaussian state (i.e. a two-mode squeezed coherent state). We will be dealing with the photon statistics and phase distribution of this state in chapters 4 and 5.

To summarize, given a Gaussian real distribution in the four dimensional phase space R^4, it is a bonafide Wigner distribution (i.e. the Hermitian operator computed from it by the Weyl rule is positive semi-definite) if and only if the 4×4 variance matrix V of the given Gaussian distribution respects the uncertainty principle (2.88) (or equivalently (2.89)). Given such a bonafide Gaussian Wigner distribution, the $Sp(4, R)$ invariants κ_1, κ_2 are computed from the noise matrix V through (2.86); the spectrum and the entropy associated with the Gaussian state are given by (2.73) and (2.75) with j running over 1,2.

2.6 Fock Representation and Photon Number Distribution

In this section we discuss the Fock state matrix elements and give an expression for the photon number distribution of the Gaussian density operator. As said earlier, the Fock states $|n_1, n_2, \ldots, n_N\rangle$ form a basis for the N-mode Hilbert space. We would like to evaluate the quantity $\langle n_1, n_2, \ldots, n_N|\hat{\rho}_G|n_1', n_2', \ldots, n_N'\rangle$, the representations of the Gaussian density operator $\hat{\rho}_G$ in the Fock basis. It will be seen that this problem is solved as soon as we have the matrix element $\langle n_1, n_2, \ldots, n_N|U(S)|n_1', n_2', \ldots, n_N'\rangle$ of the metaplectic operator $U(S)$. So let us start by looking at these matrix elements.

Any nonsingular real matrix S has the well known Euler decomposition $S' = RDR'$, where R and R' are rotation matrices and D is a diagonal positive definite matrix. Given S', the Euler factors R, D and R' are determined as follows: R is the rotation matrix that diagonalizes SS^T, R'^T is the rotation that diagonalizes S^TS and the diagonal entries of D are the positive square roots of the eigenvalues of the positive definite matrix SS^T or (S^TS). What is particularly important for us in this context is the following fact: if $S \in Sp(2N, R)$ then the Euler factors R, R' and D are all elements of $Sp(2N, R)$. Now, it is clear from (2.9), that a positive definite diagonal matrix D is in $Sp(2N, R)$ if and only if it is of the special form

$$D(s_1, s_2, \ldots, s_N) = \begin{pmatrix} e^{-s_1} & & & & & \\ & \ddots & & & & \\ & & e^{-s_N} & & & \\ & & & e^{s_1} & & \\ & & & & \ddots & \\ & & & & & e^{s_N} \end{pmatrix}, \qquad (2.90)$$

$$s_j \geq 0 \,, j = 1, 2, \ldots, N \,.$$

Note that the above matrix has only N independent entries. The physical meaning of this element of $Sp(2N, R)$ is clear: it scales $\hat{q}_j$ down by the factor e^{s_j} and $\hat{p}_j$ up by the same factor. In other words D corresponds to *single mode squeezing* in each mode by independent amounts, with the same quadrature (i.e. the position quadrature) squeezed in every single mode. Similarly it follows from (2.9) that a rotation matrix

$R \in SO(2N)$ is in $Sp(2N, R)$ if and only if R has the special form

$$R = \begin{pmatrix} X & -Y \\ Y & -X \end{pmatrix} ,$$

$$(X + iY)^{\dagger}(X + iY) = 1 , \tag{2.91}$$

where the $N \times N$ matrices X, Y are real. That is, rotations in phase space which are also symplectic transformations are in one to one correspondence with elements of the N^2 parameter unitary group $U(N)$:

$$SO(2N) \cap Sp(2N, R) \sim U(N) . \tag{2.92}$$

In particular, not all rotations in phase space are canonical transformations. Interestingly, these are precisely the elements of $Sp(2N, R)$ which commute with the fundamental antisymmetric matrix Ω:

$$SO(2N) \cap Sp(2N, R) = \{S \in Sp(2N, R) \mid S\Omega = \Omega S\} . \tag{2.93}$$

For brevity, we will refer to these phase space rotations as *canonical rotations* and parametrize them as $R(X, Y)$, $X + iY \in U(N)$. One important property of canonical rotations is that the corresponding metaplectic operators $U(R(X, Y))$ preserve the total number of quanta in all the modes put together. This is an immediate consequence of the fact that these unitary operators are generated by Hamiltonians which are linear combinations of $\hat{a}_j^{\dagger}\hat{a}_k$ (product of one creation and one annihilation operator), $j, k = 1, 2, \ldots, N$. That is, they can at most transfer quanta from one mode to the other. In particular we have,

$$\langle n_1, n_2, \ldots, n_N | U(R(X,Y)) | n'_1, n'_2, \ldots, n'_N \rangle = 0, \text{ if} \sum_{j=1}^{N} n_j \neq \sum_{k=1}^{N} n'_k. \qquad (2.94)$$

Free evolutions are canonical rotations of a simple kind. In the case of two modes, a lossless 50-50 beam splitter produces canonical rotations. This example is nontrivial, for every multimode canonical rotation can be built out of a sufficient number of two-mode canonical rotations.

While the canonical rotations mix the modes, they do not "generate" quanta and hence they correspond to passive elements. In contrast the metaplectic unitary operators $U(D(s_1, s_2, \ldots, s_N))$, where $D(s_1, s_2, \ldots, s_N)$ are the positive definite diagonal symplectic matrices referred to earlier, do not mix modes but generate quanta in pairs and correspond to active elements. We have

$$U(D(s_1, s_2, \ldots, s_N)) = \prod_{j=1}^{N} \exp[\frac{1}{2} s_j (\hat{a}^{\dagger^2} - \hat{a}^2)] \ , \qquad (2.95)$$

so that the Fock state matrix elements can be written as

$$\langle n_1, n_2, \ldots, n_N | U(D(s_1, s_2, \ldots, s_N)) | n'_1, n'_2, \ldots, n'_N \rangle = \prod_{j=1}^{N} f_{n_j n_{j'}}(s_j) \ , \qquad (2.96)$$

where,

$$f_{n_j n_{j'}}(s_j) = \langle n_j | \exp[\frac{1}{2} s_j (\hat{a}^{\dagger^2} - \hat{a}^2)] | n'_j \rangle \ . \qquad (2.97)$$

It may be noted that $f_{n_j n_{j'}}(s_j) = 0$ if $n_j - n_{j'}$ is an odd integer.

The Fock state matrix elements for the metaplectic unitary operator $U(S)$ readily follow from the above considerations and the fact that the Euler decomposition for S induces the following Euler decomposition for $U(S)$:

$$\begin{aligned} S &= R(X,Y)D(s_1,s_2,\ldots,s_N)R(X',Y')\,, \\ U(S) &= U(R(X,Y))U(D(s_1,s_2,\ldots,s_N))U(R(X',Y'))\,. \end{aligned} \tag{2.98}$$

Thus,

$$\begin{aligned} \langle n_1,n_2,\ldots,n_N| \quad U(S) \quad |n'_1,n'_2,\ldots,n'_N\rangle &= \langle n_1,n_2,\ldots,n_N|U(R(X,Y))|l_1,l_2,\ldots,l_N\rangle \\ &\times\langle m_1,m_2,\ldots,m_N|U(R(X',Y'))|n'_1,n'_2,\ldots,n'_N\rangle \\ &\times\prod_{j=1}^{N} f_{l_j m_j}(s_j)\,, \end{aligned} \tag{2.99}$$

where summation over the repeated indices $l_1,l_2,\ldots,l_N$ and $m_1,m_2,\ldots,m_N$ is implied and the summation range is 0 to ∞, for each index. As expected, the metaplectic unitary operator $U(S)$ for a generic $S\in Sp(2N,R)$ generates quanta as well as mixes the modes. What is really interesting about the structure of the above matrix elements is the clear seperation between these two aspects: mixing of the modes is entrusted to the canonical rotation part, whereas generation of quanta is entrusted to the single mode squeezing part of $U(S)$.

Use of (2.99) in (2.73) gives the Fock state matrix elements for the Gaussian density operator:

$$\langle n_1,n_2,\ldots,n_N|\hat{\rho}_G|m_1,m_2,\ldots,m_N\rangle =$$

$$\sum_{l_1,l_2,\ldots,l_N}\left(\prod_{j=1}^{N}\frac{1}{\kappa_j+\frac{1}{2}}(\frac{\kappa_j-\frac{1}{2}}{\kappa_j+\frac{1}{2}})^{l_j}\langle n_1,n_2,\ldots,n_N|U(S)|l_1,l_2,\ldots,l_N\rangle\right)$$
$$\times\langle m_1,m_2,\ldots,m_N|U(S)|l_1,l_2,\ldots,l_N\rangle\ . \quad (2.100)$$

Here S is the symplectic transformation which takes $\hat{\rho}_G$ to its canonical form and κ_j's are the symplectic invariants assocaited with the state. The above *universal form* for the matrix elements applies to all Gaussian states. We recall here, the clean separation between the aspects which are invariant and the aspects which are covariant under the metaplectic unitary evolution of $\hat{\rho}_G$: the coefficients of the product of matrix elements on the right hand side are determined entirely by the invariant κ_j's and so they are the same for all Gaussian states in the same $Sp(2N, R)$ orbit; the matrix elements on the right hand side change covariantly as one moves from one state to another in the same $Sp(2N, R)$ orbit.

The diagonal elements give the photon number distribution $p(n_1, n_2, \ldots, n_N)$, i.e. the probability of having n_j photons in the j-th mode, $j = 1, 2, \ldots, N$:

$$p(n_1,n_2,\ldots,n_N)=\langle n_1,n_2,\ldots,n_N|\hat{\rho}_G|n_1,n_2,\ldots,n_N\rangle$$
$$=\sum_{m_1,m_2,\ldots,m_N}\left(\prod_{j=1}^{N}\frac{1}{\kappa_j+\frac{1}{2}}(\frac{\kappa_j-\frac{1}{2}}{\kappa_j+\frac{1}{2}})^{l_j}\right)$$
$$\times|\langle n_1,n_2,\ldots,n_N|U(S)|m_1,m_2,\ldots,m_N\rangle|^2\ . \quad (2.101)$$

2.7 Conclusion

We have presented in this chapter a comprehensive analysis of the most general Gaussian states of a system with an arbitrary finite number of degrees of freedom. The principal tools of our analysis have been the fact that the symplectic group $Sp(2N, R)$ of real linear canonical transformations acts naturally and unitarily on the Hilbert space of such a system, and a theorem due to Williamson on the normal forms of real symmetric $2N \times 2N$ matrices under symmetric $Sp(2N, R)$ transformations.

The N independent $Sp(2N, R)$ invariants $\kappa_1, \ldots, \kappa_N$ of the noise matrix have played a special role. On the one hand these invariants help to characterize the manifold of quantum mechanically allowed Gaussian states in terms of the complete and irreducible set of uncertainty principles (4.20). On the other hand they fully determine several interesting properties of the state like the eigenvalue spectrum and the entropy. The unitary action of $Sp(2N, R)$ and the Williamson normal form jointly helped us in deducing, rather simply, the elegant matrix relation (4.21) which is the solution to the problem of Weyl ordering for multimode Gaussian states.

In Section 4 we considered following (4.21) the orbit of a Gaussian state under the symplectic group $Sp(2N, R)$. Since $\kappa_1, \ldots, \kappa_N$ are invariants over an orbit, they can be used to label the $Sp(2N, R)$ orbits of Gaussian density operators. It should be appreciated that each of the $Sp(2N, R)$ orbits we have thus constructed constitute a manifold of $Sp(2N, R)$ generalized coherent states in the sense of Perelomov [27] notwithstanding the fact that the fiducial Gaussian state and hence all the associated generalized coherent states, can be mixed rather than pure states. When the fiducial state is a pure Gaussian state then all the states in the corresponding family (orbit) of generalized coherent states are pure Gaussian states, and this corresponds to $\kappa_1 = \kappa_2 = \ldots = \kappa_N = 1/2$. It is well known [27] that all the zero mean pure Gaussian states fall into a single orbit of $Sp(2N, R)$. In other words, $Sp(2N, R)$ acts transitively on the family of zero-mean Gaussian pure states. It follows that all the orbits except this special orbit correspond to families of generalized coherent states which are mixed Gaussian states. Finally, our analysis may suggest that the present approach could be profitably used to study evolution of mixed states in the context of atomic or $SU(2)$ coherent states.

Chapter 3

Conditions for Nonclassicality

3.1 Introduction

In this chapter we bring out all the information about the nonclassicality of a state contained in the photon number distribution. In Chapter 1, we discussed some quantitative criteria characterizing a nonclassical state namely, the higher order squeezing criteria of Hong and Mandel [7], the related amplitude squared squeezing criterion of Hillery [8] and the Agarwal-Tara criteria [9]. The former were generalizations of the degree of squeezing (defined in Chapter 1) and the latter was a generalization of the Mandel Q-parameter (also defined in Chapter 1). Other than the quantitative criteria of nonclassicality, one comes across often in literature, a qualitative criterion for nonclassicality. This concerns the photon number distribution of nonclassical states like squeezed states i.e. the photon number distribution of a squeezed state is an oscillating function rather than a smooth one (as is the case with coherent states for example). It has been widely assumed that these oscillations are themselves a signature of the nonclassicality of the state [28, 30, 72] - indeed they have come to be known as *nonclassical oscillations* [29]. This label, as we shall see, is a misleading one because, a manifestly classical state can also have oscillations in the photon number distribution (PND). The virtue of the oscillation criterion is that it is *local* in n, whereas the Mandel Q-parameter and its generalizations are expressed in terms of the moments of p_n, (where p_n is the photon number distribution) and are therefore far from being local in n. In this chapter, we bring out all the information contained in the photon number distribution on the nonclassicality of a state. Some of the results presented here also serve to complete the work initiated by Agarwal and Tara [9].

3.2 Nonclassicality and the sequence $\{p_n\}$

Consider a state of a single mode of radiation, described by the density matrix $\hat{\rho}$. Then p_n, the probability that there are n photons present in the mode, is given by the expression

$$\begin{aligned} p_n &= \mathrm{tr}\,(\hat{\rho}|n\rangle\langle n|) \\ &= \int \frac{d^2\alpha}{\pi} P(\alpha) e^{-\alpha^*\alpha} (\alpha^*\alpha)^n / n! \end{aligned} \tag{3.1}$$

where $|n\rangle$ is the Fock state of the mode with exactly n photons, and $P(\alpha)$ is the diagonal quasi-probability distribution (the P-distribution):

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} P(\alpha)|\alpha\rangle\langle\alpha| \, . \tag{3.2}$$

Since, $p_n \geq 0$ and $\sum_{n=0}^{\infty} p_n = 1$, one interprets p_n as a probability distribution over the discrete set $0, 1, 2, \ldots$. Let us define a "radial" marginal distribution $\Omega(I)$ derived from $P(\alpha)$, by writing $\alpha = I^{1/2}e^{i\theta}$ and averaging over θ:

$$\begin{aligned} \Omega(I) &= \frac{1}{2\pi}\int_0^{2\pi} d\theta P(I^{1/2}e^{i\theta}) , \\ \tilde{\Omega}(I) &= \Omega(I)e^{-I} , \\ p_n &= \int_0^{\infty} dI\tilde{\Omega}(I)\frac{I^n}{n!} . \end{aligned} \tag{3.3}$$

While the P-distribution $P(\alpha)$ is real and has unit integral, it is not necessarily a true probability distribution for every density operator $\hat{\rho}$: it is not pointwise positive in general and can become more singular than the Dirac δ-function. Generically, it is a distribution. Given a state $\hat{\rho}$, if the corresponding $P(\alpha)$ is a true probability distribution then $\hat{\rho}$ is said to be a classical state. A state is nonclassical if it is not classical.

It is clear that $\Omega(I)$ will be a true probability distribution if $P(\alpha)$ is. It follows that if $\Omega(I)$ is not a true probability distribution, then $P(\alpha)$ is not. But there exists the interesting possibility that $P(\alpha)$ is not a true probability distribution, but $\Omega(I)$ is. An illustration of this situation is the one-parameter family of states that results when a coherent state $|\alpha_0\rangle$ evolves through a nonlinear Kerr medium for a time interval t. Indeed, for a suitable value of t the state that results is the Yurke-Stoler state [31]

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\alpha_0\rangle + e^{\pm i\pi/2}|-\alpha_0\rangle) . \tag{3.4}$$

This, being the superposition of two coherent states, has a $P(\alpha)$ more singular than a tempered distribution; nevertheless, when angle-averaged it leads to the same $\Omega(I)$ as for the coherent state $|\alpha_0\rangle$: $\Omega(I) = \delta(I - |\alpha_0|^2)$. The Hamiltonian of the Kerr medium being a function of $a^\dagger a$ leaves the diagonals $\langle n|\hat{\rho}|n\rangle$ unaffected, but changes only the phase of the coherences $\langle m|\hat{\rho}|n\rangle$. Thus, one is led to a three-fold classification of the quantum states of a radiation mode: classical if $P(\alpha)$, and hence $\Omega(I)$, is a true probability; semiclassical if $P(\alpha)$ is not, but $\Omega(I)$ is a true probability; and strongly nonclassical if $\Omega(I)$, and hence $P(\alpha)$, fails to be a probability distribution. It is clear that measurements of $\{p_n\}$, as also measurements of any set of phase-insensitive quantities involving only $a^{\dagger^m} a^m$, $m = 0, 1, 2, \ldots$ cannot distinguish between the classical and semiclassical cases. With such measurements one can at best conclude whether the given state is strongly nonclassical or not.

Since we are interested only in the phase insensitive aspects here, we will refer to both the classical and semiclassical states as "classical" and the strongly nonclassical states as "nonclassical" That is, we will call a state $\hat{\rho}$ classical or nonclassical depending on whether the associated angle averaged distribution $\Omega(I)$ (equivalently $\tilde{\Omega}(I)$) is or is not a true probability distribution.

Although it is clear that the sequence $\{p_n\}$ cannot possibly capture all the information contained in $P(\alpha)$ (i.e. contained in $\hat{\rho}$, for its definition involves only the diagonal elements $\langle n|\hat{\rho}|n\rangle$ and ignores all the coherences $\langle m|\hat{\rho}|n\rangle$, $m \neq n$), one can easily show that $\{p_n\}$ and $\Omega(I)$ determine each other uniquely.

Let us define the generating function $\Lambda(K)$ through

$$\begin{aligned} \Lambda(K) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} K^n p_n \, , \\ p_n &= (-1)^n \frac{d^n}{dK^n} \Lambda(K)|_{K=0} \, . \end{aligned} \tag{3.5}$$

Clearly $\Lambda(K)$ converges for all real values of K and is related to $\tilde{\Omega}(I)$ through

$$\begin{aligned} \Lambda(K) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} K^n \int_0^{\infty} dI \tilde{\Omega}(I) \frac{I^n}{n!} \\ &= \int_0^{\infty} dI \tilde{\Omega}(I) J_0(2\sqrt{IK}) \, . \end{aligned} \tag{3.6}$$

This Fourier-Bessel transform can be inverted using the identity

$$\int_0^{\infty} dK \, J_0(2\sqrt{IK}) J_0(2\sqrt{I'K}) = \delta(I - I') \, , \tag{3.7}$$

and we have

$$\tilde{\Omega}(I) = \int_0^{\infty} dK \Lambda(K) J_0(2\sqrt{IK}) \, . \tag{3.8}$$

Thus the radial distribution not only determines, but also is determined, by the sequence $\{p_n\}$.

The most prominent among the signatures of nonclassicality of the PND is the classic condition on the Mandel Q-parameter [6] which involves the lowest two moments: the state is nonclassical (specifically, it exhibits sub-Poissonian statistics) if $Q(\hat{\rho}) < 0$.

More recently Agarwal and Tara [9] formulated an infinite sequence ofsuccessive conditions, violation of any one of which will amount to the state $\hat{\rho}$ being nonclassical. The lowest of this sequence is, ofcourse, the familiar condition on the Mandel Q -parameter. Recalling the definition of the Mandel Q-parameter given in the introduction, we rewrite it as

$$\begin{aligned} Q &= (\langle n^2\rangle - \langle n\rangle^2 - \langle n\rangle)/\langle n\rangle; \\ \langle n\rangle &= \sum_{n=0}^{\infty} np_n = \int_0^{\infty} dI\tilde{\Omega}(I)e^{I}I\,, \\ \langle n^2\rangle &= \sum_{n=0}^{\infty} n^2 p_n = \int_0^{\infty} dI\hat{\Omega}(I)e^{I}I(I+1); \end{aligned} \tag{3.9}$$

and keep in mind the definition of nonclassicality with respect to the Mandel Q parameter given in the introduction. One can see that there are (uncountably many) states $\hat{\rho}$ for which Q is undefined, since either or both $\langle n^2\rangle$ and $\langle n\rangle$ can be divergent. This happens because Q involves p_n for *all* n, and the rate of decrease of p_n as $n \to \infty$ may not be fast enough to prevent the above mentioned quantities from becoming divergent. The factorial moments of the photon number distribution also face the same problem: i.e. they involve p_n for all but a finite number of values of n, and are undefined for the vast majority of states $\hat{\rho}$, as can be seen below.

$$\begin{aligned} m_n &= \langle \hat{a}^{\dagger n}\hat{a}^{n}\rangle \\ &= \sum_{l=n}^{\infty} \frac{l!}{(l-n)!} p_n \\ &= \int_0^{\infty} dI\hat{\Omega}(I)e^{I}I^{m}\,, n = 0, 1, 2\ldots \end{aligned} \tag{3.10}$$

In terms of these the Mandel Q parameter is

$$Q \;=\; (m_2 - m_1^2)/m_1$$

It is well known that [38] a classical PND implies:

$$m_{n'}m_n \leq m_{n'+n} \leq (m_{2n'}m_{2n})^{1/2} \; ; n', n = 0, 1, 2, \ldots$$

However, since $\{m_n\}$ also (like Q) involve p_n for all but a finite number of values of n they are undefined for a majority of states $\hat{\rho}$.

3.3 Classicality and Local conditions on $\{p_n\}$

In this section we derive a set of necessary conditions on $\{p_n\}$ in order that the associated state is classical. These conditions will turn out to be local in n, as against the usual conditions which are in terms of the moments of $\{p_n\}$ and hence are nonlocal in n.

It turns out to be convenient to define a sequence $\{q_n\}$ in the place of $\{p_n\}$ through

$$q_n \;=\; n!p_n \quad , \; n = 0, 1, 2, \ldots \tag{3.11}$$

Thus a Poissonian distribution $\{p_n\}$ is fully characterized by the fact that q_n/q_{n+1} is independent of n. It follows from (3.3) that $\{q_n\}$ is simply the sequence of moments of the distribution $\tilde{\Omega}(I) = \Omega(I)e^{-I}$:

$$q_n \;=\; \int_0^\infty dI\tilde{\Omega}(I)I^n \equiv \langle I^n \rangle_{\tilde{\Omega}} \quad . \tag{3.12}$$

Now suppose we are given a classical state so that $\Omega(I) \geq 0$, $0 \leq I < \infty$, and consider the polynomial $f(I) = I^n(I+x)^2$. Since $f(I)$ is manifestly nonnegative for any real value of x, nonnegativity of $\tilde{\Omega}(I)$ implies

$$\langle f(I)\rangle_{\tilde{\Omega}} = x^2 q_n - 2q_{n+1}x + q_{n+2} \geq 0\ , \tag{3.13}$$

for all real x. That is,

$$q_n q_{n+2} \geq q_{n+1}^2\ . \tag{3.14}$$

Written in terms of $\{p_n\}$, this condition reads

$$p_n p_{n+2} \geq (\frac{n+1}{n+2}) p_{n+1}^2\ ,\ n = 0, 1, 2 \ldots \tag{3.15}$$

These are the *local conditions* to be satisfied by the photon distribution $\{p_n\}$ of any classical state.

We plot in the following figures the quantity $C(n) = \frac{(n+2)}{(n+1)}[p_n p_{n+2}/p_{n+1}^2]$ for the thermal state, photon added thermal state and photon added coherent state.

Several interesting conclusions can be drawn from these local conditions. Firstly, since $\{q_n\}$ is a geometric sequence for a Poissonian distribution, we see that the local conditions are saturated by a Poissonian distribution, for every value of n. Thus we can interpret these conditions as saying that for any classical state the sequence $\{p_n\}$ is *locally Poissonian or super-Poissonian* at each n.

Secondly, suppose that the given state is such that $p_{n_0} = 0$ (and hence $q_{n_0} = 0$) for some integer $n_0 \geq 0$, and assume that the state is classical. The choice $n = n_0$

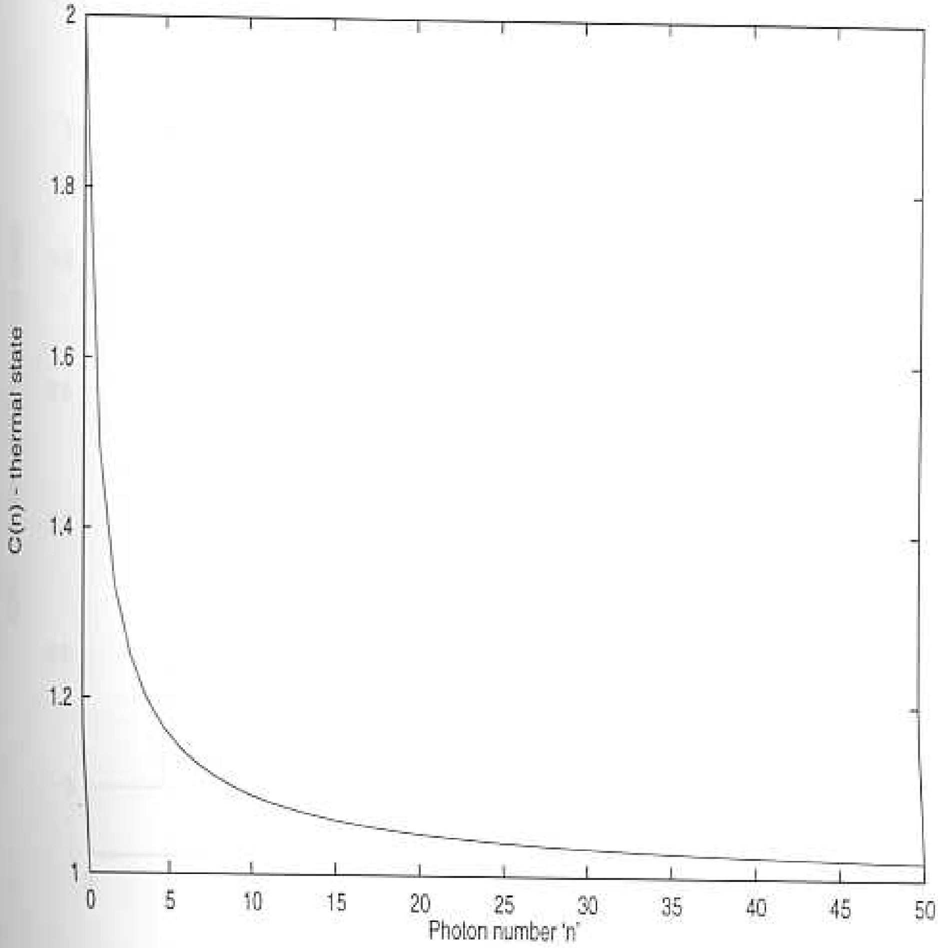


Figure 3.1: A plot of $C(n)$ for the thermal state

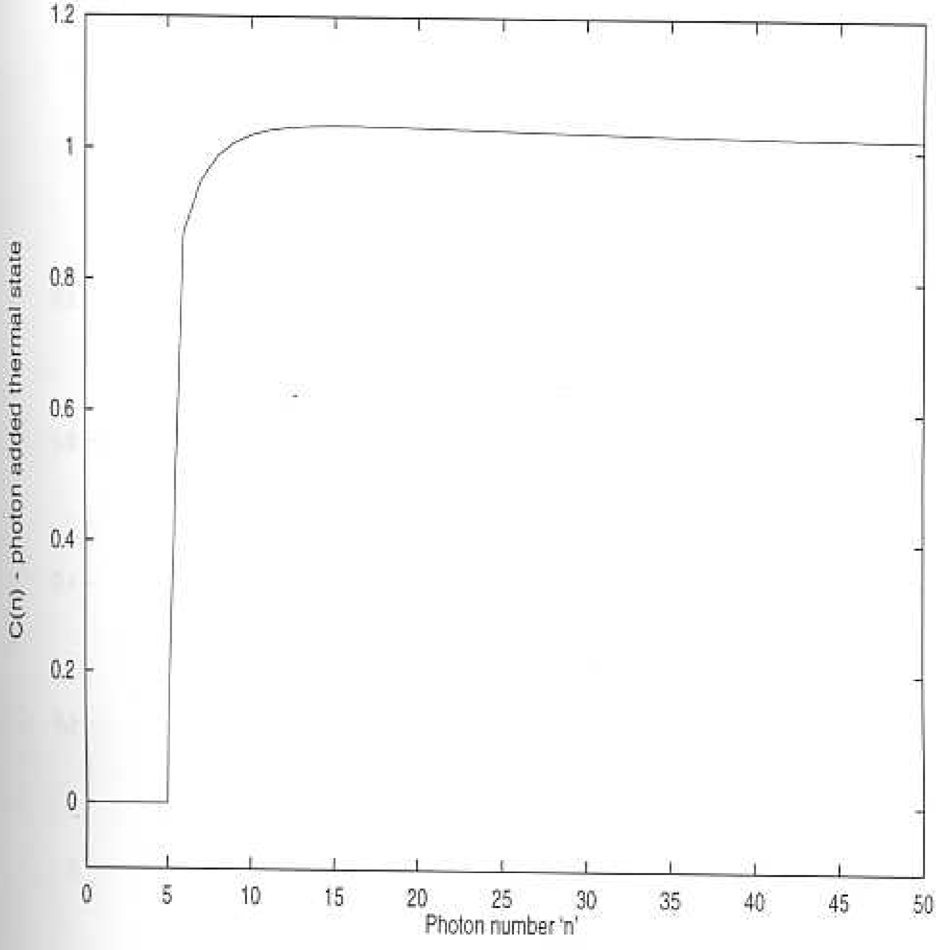


Figure 3.2: A plot of $C(n)$ for the photon added thermal state. The number of photons is $m = 5$.

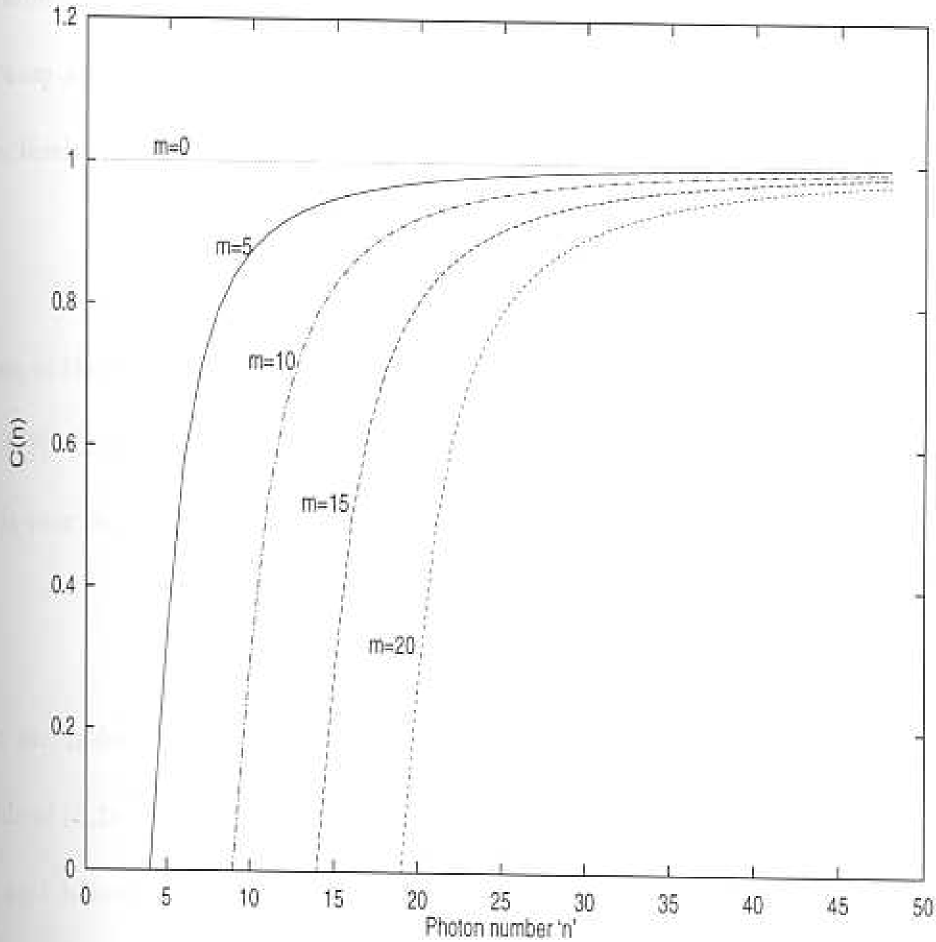


Figure 3.3: A plot of $C(n)$ for the photon added coherent state, for various values of added photons.

in the local condition implies that $q_{n_0+1} = 0$. Continuing this process we find that $q_{n_0} = 0$ implies $q_n = 0$ for all $n > n_0$. On the other hand the choice $n + 2 = n_0$ implies $q_{n_0-1} = 0$, unless $n_0 = 1$. Thus for a classical state either p_n is nonzero for every value of n, or $p_n = 0$ for all $n > 0$. In other words, *a classical state other than the vacuum state, cannot be orthogonal to any Fock state.*

Nonclassicality of a class of states defined through

$$\hat{\rho} = N a^{\dagger m} \hat{\rho_{th}} a^m \tag{3.16}$$

where $\hat{\rho_{th}}$ is the thermal state with inverse temperature parameter β has been studied in detail [9]. In view of our local condition, the nonclassicality of these states for every $m \geq 0$ is now manifest from its very definition, for

$$p_n = \langle n|\hat{\rho}|n\rangle = 0 \ , \tag{3.17}$$

for $n < m$. Indeed, we can arrive at a stronger conclusion: replace $\hat{\rho_{th}}$ on the right hand side of (3.16) by an arbitrary state $\hat{\rho}'$; the resulting "photon added" state satisfies (3.17), and hence is nonclassical. That is, *all photon added states are nonclassical.* Nonclassicality of photon added coherent states has already been studied in [32].

Finally it is of interest to characterize the extent to which oscillations as a function of n can occur in the sequence $\{p_n\}$ of a classical state. Ever since the important work of Schleich and Wheeler [28] on interference in phase space, the statement that oscillations in $\{p_n\}$ are a signature of nonclassicality [30, 72, 40] has become a widely

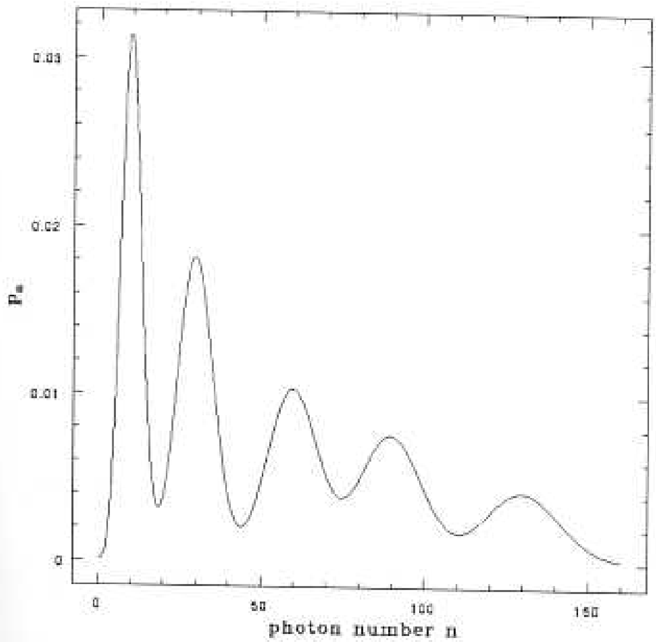


Figure 3.4: p_n for an incoherent superposition of coherent states. The values of the displacements are $\alpha^2 = 10, 30, 60, 90, 130$ and the corresponding values of the superposition parameter are $\lambda = 0.25, 0.25, 0.2, 0.18, 0.12$.

accepted one. Indeed oscillations in $\{p_n\}$ are known as nonclassical oscillations [29]. The striking virtue of this characterization is that it is *local* in n as against the other characterizations based on the moments of $\{p_n\}$. We have shown in Fig. 3.4. the sequence $\{p_n\}$, for a suitably chosen incoherent superposition of coherent states. The state is classical by construction, yet it exhibits oscillations in $\{p_n\}$ showing that the above characterization needs quantification while retaining its attractive feature of being local in n. Our local conditions can be viewed as a quantification of this type. The inequality (3.14) says that $\{q_n\}$ for a classical state cannot have a local

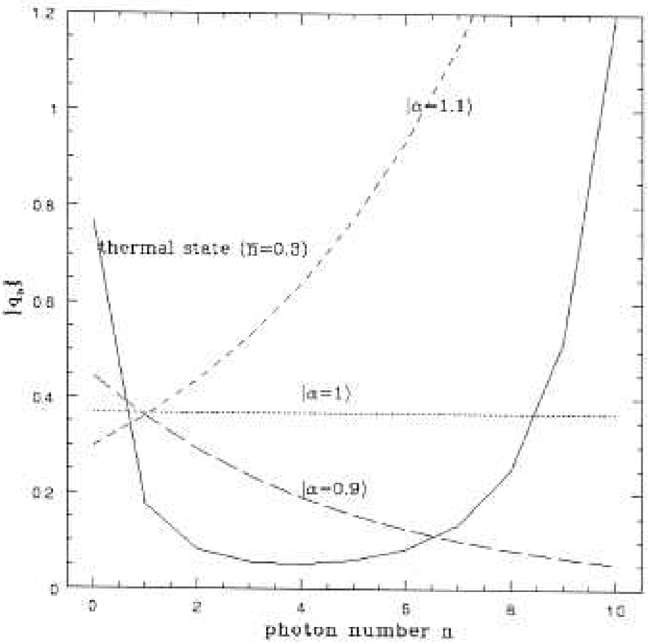


Figure 3.5: The four patterns allowed for q_n

maximum (if it had a local maximum, the inequality will be violated by taking $n+1$ to correspond to the maximum). Thus, for a classical state $\{q_n\}$ cannot exhibit any oscillation. In other words, oscillations in $\{q_n\}$ are a sure sign of nonclassicality.

Since $\{q_n\}$ for a classical state is forbidden from having a local maximum, it can have atmost one local minimum. Thus there are only four generic patterns for the behaviour of $\{q_n\}$ as a function of n: monotone increasing, constant in n, monotone decreasing, or a graph with one local minimum as shown in Fig. 3.5.

In view of the factor $\frac{n+1}{n+2}$ on the right hand side of the inequality (3.15) coming from the factorials, it will appear that some amount of oscillation is allowed in $\{p_n\}$, even for a classical state. This is the kind of oscillation shown in Fig. 3.4. Note that the period of oscillation (difference between the value of n at two, successive maxima of $\{p_n\}$) in Fig. 3.4. is substantially greater than two as against the oscillation occurring in a squeezed state or cat state where the period is two. For period two oscillations the inequality (3.15) places substantial restrictions on the amplitude. Indeed the next higher order local condition to be derived in Section 4 makes these restrictions even more stringent. Further the factor $\frac{n+1}{n+2}$ approaches unity as n becomes large; thus the restriction on the amplitude of the period two oscillation is stronger at higher values of n. While placing these restrictions on the amplitude of oscillations in the $\{p_n\}$ of a classical state, our local condition does not altogether forbid such "local" oscillations as can be seen from Fig. 3.6.

We now apply our local condition to a state obtained as the superposition of two coherent states:

$$\begin{aligned} |\Psi\rangle &= N[|\alpha_0\rangle + e^{i\theta}|-\alpha_0\rangle] \;, \\ N &= [2(1+\cos\theta e^{-2\alpha_0^*\alpha_0})]^{-1/2} \;. \end{aligned} \tag{3.18}$$

Here θ is the relative phase (in the Pancharatnam sense [34]) between the two components of the superposition. The above superposition includes as special cases the Yurke-Stoler states [31] (with $\theta = \pm\pi/2$) and the cat states (with $\theta = 0, \pi$). We have

$$p_n = |\langle n|\Psi\rangle|^2 = e^{-\alpha_0^*\alpha_0}\frac{(\alpha_0^*\alpha_0)^n}{n!}\frac{[1+(-1)^n\cos\theta]}{[1+\cos\theta e^{-2\alpha_0^*\alpha_0}]} \;, \tag{3.19}$$

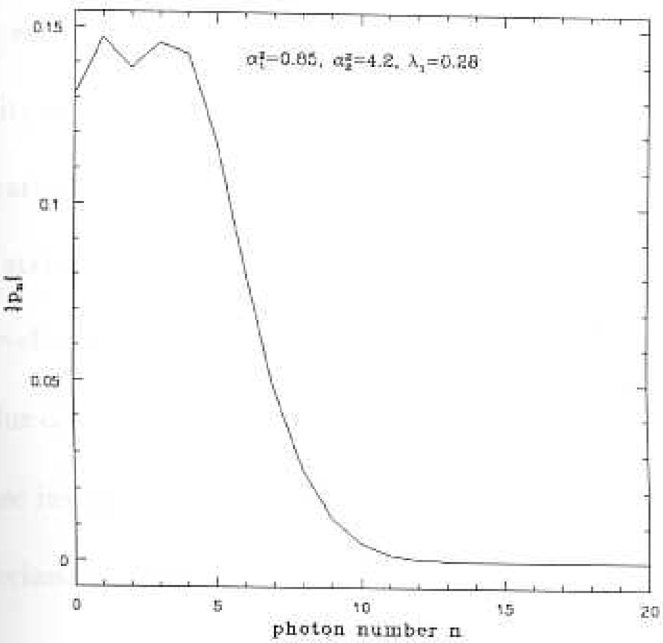


Figure 3.6: Local oscillations in the state $\lambda_1|\alpha_1^2 = 0.85\rangle + (1-\lambda_1)|\alpha_2^2 = 4.2\rangle$, $\lambda_1 = 0.28$.

so that

$$\frac{q_n q_{n+2}}{q_{n+1}^2} = \frac{(1+(-1)^n \cos\theta)^2}{(1+(-1)^{n+1}\cos\theta)^2} \equiv f_n(\theta)\ . \tag{3.20}$$

Clearly, $f_n(\theta) < 1$ for odd values of n if $-\pi/2 < \theta < \pi/2$, and $f_n(\theta) < 1$ for even values of n if $-3\pi/2 < \theta < -\pi/2$. Thus, the state $|\Psi\rangle$ violates the local condition and hence is nonclassical for all values of $\theta \neq \pm\pi/2$. For $\theta = \pm\pi/2$, the local conditions are saturated for every value of n; and we have a Poissonian $\{p_n\}$. It is well known that nonclassicality of the Yurke-Stoler states can be exhibited only through phase sensitive considerations.

That all pure states other than the coherent states are nonclassical at least at the phase sensitive level is common knowledge [33]. What is really interesting is the fact that for every value of $\theta \neq \pm\pi/2$, the nonclassicality of the superposition state $|\Psi\rangle$ is coded in the phase insensitive photon distribution $\{p_n\}$ and that our local condition captures this nonclassicality!

3.4 The Stieltjes Moment Problem and Necessary and Sufficient Conditions for Nonclassicality

We showed in the last section that positivity of $\tilde{\Omega}(I)$ implies the local condition on the sequence $\{q_n\}$. Central to the derivation of these local conditions was the appreciation of the fact that $\{q_n\}$ are the *moments* of $\tilde{\Omega}(I)$. Note that $\tilde{\Omega}(I)$ is not normalized, but this poses no problem for it can be normalized simply by multiplying it by q_0^{-1}.

Then it can be treated mathematically as though it were a probability distribution over $[0, \infty)$. In this section we exhibit the necessary and sufficient conditions on the sequence $\{q_n\}$ in order that the associated state $\hat{\rho}$ is classical.

The reconstruction of a probability distribution from its moment sequence constitutes the classical moment problem on which there exists enormous amount of literature [35]. When the probability distribution is over the semi-infinite real line $[0, \infty)$ one calls it the Stieltjes moment problem. The Hamburger moment problem corresponds to the case where the probability distribution is over the entire real line $(-\infty, \infty)$, and the Hausdorff or the *little* moment problem corresponds to the finite interval $[0, 1]$. Since the argument of the "radial" quasi-probability $\tilde{\Omega}(I)$ is nonnegative, our problem of deriving necessary and sufficient conditions on its moment sequence $\{q_n\}$ in order to ensure that $\tilde{\Omega}(I)$ is a true probability distribution is indeed a Stieltjes moment problem.

The solution of this classical problem is well known. To exhibit this solution, we arrange the moment sequence $\{q_n\}$ into two matrices $L^{(N)}$, $\tilde{L}^{(N)}$ defined by

$$L^{(N)}_{mn} = q_{m+n}\,, \quad \tilde{L}^{(N)}_{mn} = q_{m+n+1}\,, \quad m, n = 0, 1, 2, \ldots N. \tag{3.21}$$

That is,

$$L^{(N)} = \begin{pmatrix} q_0 & q_1 & q_2 & \cdots & q_N \\ q_1 & q_2 & q_3 & \cdots & q_{N+1} \\ \vdots & \vdots & \vdots & & \vdots \\ q_N & q_{N+1} & q_{N+2} & \cdots & q_{2N} \end{pmatrix} ,$$

$$\tilde{L}^{(N)} = \begin{pmatrix} q_1 & q_2 & q_3 & \cdots & q_{N+1} \\ q_2 & q_3 & q_4 & \cdots & q_{N+2} \\ \vdots & \vdots & \vdots & & \vdots \\ q_{N+1} & q_{N+2} & q_{N+3} & \cdots & q_{2N+1} \end{pmatrix} . \tag{3.22}$$

Theorem 3.1: The necessary and sufficient condition on the photon number distribution sequence $\{q_n = n!p_n\}$,

$$q_n = \int_0^\infty dI\hat{\Omega}(I)I^n = \langle I^n\rangle_{\hat{\Omega}} , \tag{3.23}$$

in order that the associated quasi-probability distribution $q_0^{-1}\hat{\Omega}(I)$ is a true probability distribution over $[0, \infty)$ is that these matrices be nonnegative:

$$L^{(N)} \geq 0 , \quad \tilde{L}^{(N)} \geq 0, \quad N = 0, 1, 2 \ldots \tag{3.24}$$

Proof: The proof of the above theorem makes use of the following two facts. The first fact is that a distribution $\hat{\Omega}(I)$ over the domain $I \in [0, \infty)$ is nonnegative if and only if it gives nonnegative expectation values for every polynomial $f(I)$ pointwise nonnegative over $I \in [0, \infty)$ [36]:

$$\langle f(I) \rangle_{\tilde{\Omega}} = \int_0^\infty dI \tilde{\Omega}(I) f(I) \geq 0 \,, \quad \forall f(I) \geq 0 \,. \tag{3.25}$$

The second fact is that any polynomial $f(I)$, nonnegative over $[0, \infty)$, can be written in terms of two square polynomials as [37]

$$f(I) = [f_1(I)]^2 + I[f_2(I)]^2 \,. \tag{3.26}$$

To prove the necessity part of the theorem, suppose that $\tilde{\Omega}(I)$ is nonnegative. Consider the polynomial $f_1(I) = \sum_{n=0}^{N} c_n I^n$, where c_n are arbitrary real coefficients. By (3.25) we have $\langle [f_1(I)]^2 \rangle_{\tilde{\Omega}} \geq 0$. That is,

$$\begin{aligned} \langle [f_1(I)]^2 \rangle_{\tilde{\Omega}} &= \sum_{m,n=0}^{N} c_m c_n \langle I^{m+n} \rangle_{\tilde{\Omega}} \\ &= \sum_{m,n=0}^{N} c_m c_n q_{m+n} \\ &= \sum_{m,n=0}^{N} c_m c_n L_{mn}^{(N)} \geq 0 \,. \end{aligned} \tag{3.27}$$

This means $L \geq 0$, for every N. Similarly, writing $f_2(I) = \sum_{n=0}^{N} d_n I^n$ and taking the expectation of the nonnegative polynomial $I[f_2(I)]^2$ we have, in view of (3.25),

$$\begin{aligned} \langle I[f_2(I)]^2 \rangle_{\tilde{\Omega}} &= \sum_{m,n=0}^{N} d_m d_n \langle I^{m+n+1} \rangle_{\tilde{\Omega}} \\ &= \sum_{m,n=0}^{N} d_m d_n q_{m+n+1} \\ &= \sum_{m,n=0}^{N} d_m d_n \tilde{L}_{mn}^{(N)} \geq 0 \,. \end{aligned} \tag{3.28}$$

That is, $\tilde{L}^{(N)} \geq 0$, for every N. Thus nonnegativity of $\tilde{\Omega}(I)$ implies (3.24).

Assume, conversely that (3.24) is obeyed. Given any nonnegative polynomial $f(I)$, writing $f(I)$ in the form (3.26) we find that $\langle f(I)\rangle_{\tilde{\Omega}} \geq 0$, for every nonnegative polynomial. This implies, in view of the first fact stated at the beginning of the proof, that $\tilde{\Omega}(I)$ is nonnegative. This completes the proof of the theorem.

Assume that we have a classical state so that (3.24) is satisfied. Since (3.24) is equivalent to

$$\begin{aligned} d_N &= \det L^{(N)} \geq 0 \ , \\ \tilde{d}_N &= \det \tilde{L}^{(N)} \geq 0 \ , \quad N = 0, 1, 2, \ldots \end{aligned} \tag{3.29}$$

there arise two possibilities. Either $d_N > 0$, $\tilde{d}_N > 0$ for all N or $d_N > 0$, $\tilde{d}_N > 0$ for $N \leq k$ and $d_N = \tilde{d}_N = 0$ for $N > k$. In the latter case the support of $\tilde{\Omega}(I)$ (the set of values of I for which $\tilde{\Omega}(I)$ is nonzero) is a finite set of order k. It is an infinite set in the former case. This can be intuitively understood along the following lines. Suppose $\tilde{\Omega}(I) = \delta(I - I_0)$. Then $q_n = I_0^n$ so that $L_{mn}^{(N)} = q_{m+n} = I_0^m I_0^n$ and $\tilde{L}_{mn}^{(N)} = I_0 I_0^m I_0^n$. That is, $L^{(N)}$ and $\tilde{L}^{(N)}$ are (essentially) projection matrices. Thus when the support of $\tilde{\Omega}(I)$ is finite $L^{(N)}$ (as also $\tilde{L}^{(N)}$) is the sum of k projections. Since $I_0 = 0$ contributes a projection only to the matrix $L^{(N)}$ but not to $\tilde{L}^{(N)}$ one has the following refinement: if support of $\tilde{\Omega}(I)$ is of order k and $I = 0$ is contained in the support, then $d_N > 0$ ($\tilde{d}_N > 0$) if and only if $N \leq k$ ($N \leq k - 1$).

It is useful to remark that the first fact we have used in the proof of the theorem is common for all the three types of moment problems. What changes from one moment problem to the other is the second fact dealing with the decomposition of nonnegative polynomials in terms of square polynomials thus enabling us to write (3.25) as simple matrix conditions. For instance, for the Hamburger moment problem on the entire real line, we have, in place of (3.26), the statement that every polynomial nonnegative over the real line can be written as the sum of two square polynomials. Thus, in the Hamburger case we have to deal only with the matrix $L^{(N)}$, and the condition $L^{(N)} \geq 0$ for every N is both necessary and sufficient.

It is immediate to relate our local condition to the above theorem. Nonnegativity of $L^{(N)}$, $\tilde{L}^{(N)}$ demands as a necessary condition, nonnegativity of the determinant of the diagonal 2×2 blocks of $L^{(N)}$, $\tilde{L}^{(N)}$. This is precisely what our local condition (3.14) is. It is also clear why our local condition (3.14) is only a necessary condition: positivity of the diagonal 2×2 blocks of $L^{(N)}$, $\tilde{L}^{(N)}$ does not capture in its entirity the positivity of $L^{(N)}$ and $\tilde{L}^{(N)}$ given in (3.24).

We now derive the next level of local conditions using our necessary and sufficient conditions (3.24). Given the sequence $\{q_n\}$ we define

$$x_n = \frac{q_n q_{n+2}}{q_{n+1}^2}, \quad n = 0, 1, 2, \ldots \tag{3.30}$$

Then our first order local condition (3.14) which involves q_n for three successive values of n simply reads that $x_n \geq 1$, $\forall n$, for any classical state. As one may anticipate, the second order local condition which we now derive involves q_n for five successive values of n or equivalently, x_n for three successive values of n.

A necessary condition for the nonnegativity of $L^{(N)}$, $\tilde{L}^{(N)}$ is that their diagonal 3×3 blocks be nonnegative definite. That is

$$A_n = \begin{pmatrix} q_n & q_{n+1} & q_{n+2} \\ q_{n+1} & q_{n+2} & q_{n+3} \\ q_{n+2} & q_{n+3} & q_{n+4} \end{pmatrix} \geq 0 \,, \quad n = 0, 1, 2, \ldots \tag{3.31}$$

Now define the positive diagonal matrix

$$S = \begin{pmatrix} q_n^{-1/2} & 0 & 0 \\ 0 & q_n^{1/2} q_{n+1}^{-1} & 0 \\ 0 & 0 & q_n^{3/2} q_{n+2}^{-2} \end{pmatrix} . \tag{3.32}$$

We have

$$B_n = S A_n S = \begin{pmatrix} 1 & 1 & x_1 \\ 1 & x_1 & x_1^2 x_2 \\ x_1 & x_1^2 x_2 & x_1^3 x_2^2 x_3 \end{pmatrix} . \tag{3.33}$$

It is clear that $A_n \geq 0$ if an only if $B_n \geq 0$. Since $x_1 \geq 1$, the condition $B_n \geq 0$ is equivalent to the requirement that $\det B_n \geq 0$. But

$$\det B_n = x_n^3 [x_{n+1}^2 (x_n - 1)(x_{n+2} - 1) - (x_{n+1} - 1)^2] \,, \tag{3.34}$$

and hence we conclude that for a classical state $\{x_n\}$ defined through (3.30) has to necessarily satisfy

$$(x_n - 1)(x_{n+2} - 1) \geq \left(\frac{x_{n+1} - 1}{x_{n+1}}\right)^2 , \quad n = 0, 1, 2, \ldots \tag{3.35}$$

These are our *second order local conditions* on $\{q_n\}$ or, equivalently, on $\{p_n\}$. They involve three successive x_n's and hence five successive p_n's. Just like the first order conditions, these are only necessary conditions for classicality.

To close this section we present an interesting implication of these conditions. As already noted, if $\{p_n\}$ is Poissonian then the corresponding $\{q_n\}$ is a geometric sequence saturating the first order local condition for every n, and rendering $x_n = 1$ identically. We now ask whether it is possible to have a classical state for which $q_n q_{n+2} = q_{n+1}^2$ for some values of n whereas $q_n q_{n+2} > q_{n+1}^2$ for other values of n. Such classical states, if they exist, can be said to be *locally Poissonian* at these former values of n.

Suppose a classical state is locally Poissonian at some $n = n_0$. That is, $x_{n_0} = 1$. Then two applications of (3.35), once with $n_0 = n$ and then with $n_0 = n + 2$, shows that the state will cease to be classical unless $x_{n_0+1} = 1$ and $x_{n_0-1} = 1$. Continuing this process we find that $x_n = 1$ for all n. Thus, there exists no classical state which is locally Poissonian: A classical state is either Poissonian ($x_n = 1$ for all n) or is everywhere locally super-Poissonian ($x_n > 1$ for all n).

In the light of this result we can now strengthen our first order condition (3.15) to make it a strict inequality:

For a classical state, either

$$p_n p_{n+2} = \left(\frac{n+1}{n+2}\right) p_{n+1}^2 \ (Poissonian),$$

or

$$p_n p_{n+2} > (\frac{n+1}{n+2}) p_{n+1}^2 \,, \quad n = 0, 1, 2, \ldots \tag{3.36}$$

This is a refinement of our first order local condition achieved in the light of the second order condition.

3.5 Dual approach based on $\mathrm{tr}(a^{\dagger n} a^n \hat{\rho})$

In this Section we present an approach to nonclassicality of a state $\hat{\rho}$ based on the normal ordered moments $\mathrm{tr}(a^{\dagger n} a^n \hat{\rho})$ i.e. the sequence $\{m_n\}$. This approach will be seen to be along the lines of Agarwal and Tara [9]. However, the conditions we derive for nonclassicality are both *necessary* and *sufficient*.

Suppose we have a state $\hat{\rho}$ whose normal ordered moments (i.e. factorial moments of p_n) m_n are known. Our problem is to find necessary and sufficient conditions on the sequence $\{m_n\}$ in order that the state $\hat{\rho}$ is classical. Writing m_n in terms of the P-distribution $P(\alpha)$, and writing $\alpha = I^{1/2} e^{i\theta}$, we have

$$\begin{aligned} m_n &= \int \frac{d^2\alpha}{\pi} P(\alpha) \alpha^{*n} \alpha^n \\ &= \int_0^\infty dI \Omega(I) I^n = \langle I^n \rangle_\Omega \,. \end{aligned} \tag{3.37}$$

That is $\{m_n\}$ is the moments sequence of $\Omega(I)$, in exactly the same manner in which the sequence $\{q_n\}$ was related to $\tilde{\Omega}(I)$.

It is now clear that our present problem, is again a Stieltjes moment problem similar to the earlier problem in Section 3. The state $\hat{\rho}$ being classical is equivalent to $\Omega(I)$ being a true probability distribution. With this identification the solution to our present problem is immediate. Form two matrices $M^{(N)}$ and $\tilde{M}^{(N)}$ using the moment sequence $\{m_n\}$:

$$M^{(N)} = \begin{pmatrix} m_0 & m_1 & m_2 & \cdots & m_N \\ m_1 & m_2 & m_3 & \cdots & m_{N+1} \\ \vdots & \vdots & \vdots & & \vdots \\ m_N & m_{N+1} & m_{N+2} & \cdots & m_{2N} \end{pmatrix}, \tag{3.38}$$

$$\tilde{M}^{(N)} = \begin{pmatrix} m_1 & m_2 & m_3 & \cdots & m_{N+1} \\ m_2 & m_3 & m_4 & \cdots & m_{N+2} \\ \vdots & \vdots & \vdots & & \vdots \\ m_{N+1} & m_{N+2} & m_{N+3} & \cdots & m_{2N+1} \end{pmatrix}. \tag{3.39}$$

Theorem 3.2: The necessary and sufficient condition that the state $\hat{\rho}$ with normal ordered moment sequence $\{m_n\}$ given by (3.37) be classical is that

$$M^{(N)} \geq 0, \;\; \tilde{M}^{(N)} \geq 0, \;\; N = 0, 1, 2, \ldots \tag{3.40}$$

The proof of this theorem is exactly parallel to the one in the previous section: the role of $\tilde{\Omega}(I)$ there is now played by $\Omega(I)$, that of $\{q_n\}$ by $\{m_n\}$, and the role of $L^{(N)}$, $\tilde{L}^{(N)}$ by the matrices $M^{(N)}$, $\tilde{M}^{(N)}$.

It may be noted that the present theorem completes the work initiated by Agarwal and Tara [9] by improving their necessary condition ($M \geq 0$) for classicality into a necessary and sufficient condition. Thus, the constraints on the moments arising from the requirement $M^{(N)} \geq 0$ are the same as in their work: with $N = 1$ we have $m_0 m_2 \geq m_1^2$, which is the same as requiring the Mandel Q-parameter to be nonegative; with $N = 2$ we obtain the additional condition that $\det M^{(2)} \geq 0$; and so on. However the constraints on the moments arising from the positivity requirement on $\tilde{M}^{(N)}$ are new: with $N = 0$ we have $m_1 \geq 0$, with $N = 1$ we have $m_1 m_3 \geq m_2^2$, and so on.

To conclude this section we re-examine the class of superposition states $|\Psi\rangle$, defined in (3.18), within the present approach. The sequence of normal ordered moments are easily calculated:

$$\begin{aligned} m_n &= \langle\Psi|a^{\dagger n}a^n|\Psi\rangle \\ &= (\alpha_0^*\alpha_0)^n \left(\frac{1+(-1)^n \cos\theta e^{-2\alpha_0^*\alpha_0}}{1+\cos\theta e^{-2\alpha_0^*\alpha_0}}\right) \end{aligned} \tag{3.41}$$

and we have

$$M^{(N)} = A \begin{pmatrix} 1 & \sigma & 1 & \sigma & \cdots \\ \sigma & 1 & \sigma & 1 & \cdots \\ 1 & \sigma & 1 & \sigma & \cdots \\ \sigma & 1 & \sigma & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} A\ , \tag{3.42}$$

$$\tilde{M}^{(N)} = B \begin{pmatrix} \sigma & 1 & \sigma & 1 & \cdots \\ 1 & \sigma & 1 & \sigma & \cdots \\ \sigma & 1 & \sigma & 1 & \cdots \\ 1 & \sigma & 1 & \sigma & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} B \ . \tag{3.43}$$

The positive diagonal matrices A, B are given by

$$A = \mathrm{diag}(1, \alpha_0^*\alpha_0, (\alpha_0^*\alpha_0)^2, (\alpha_0^*\alpha_0)^3, ...) \ ,$$

$$B = \mathrm{diag}((\alpha_0^*\alpha_0)^{1/2}, (\alpha_0^*\alpha_0)^{3/2}, (\alpha_0^*\alpha_0)^5/2....) \ , \tag{3.44}$$

and

$$\sigma = \frac{1 - \cos\theta e^{-2\alpha^*\alpha}}{1 + \cos\theta e^{-2\alpha^*\alpha}} \ . \tag{3.45}$$

It is clear that both $M^{(N)}$ and $\tilde{M}^{(N)}$ are matrices of rank = 2. It is further clear from the structure of these matrices that $M^{(N)} \geq 1$ if and only if $\sigma \leq 1$ and $\tilde{M}^{(N)} \geq 1$ if and only if $\sigma \geq 1$. Now, when $-\pi/2 < \theta < \pi/2$ we have $\sigma < 1$ and our superposition state violates the nonclassicality condition $\tilde{M}^{(N)} \geq 0$ in (3.40) whereas for $-3\pi/2 < \theta < -\pi/2$ we have $\sigma > 1$ and the state violates the condition $M^{(N)} \geq 0$. Thus, the dual approach fully recovers the conclusions of the approach based on $\tilde{\Omega}(I)$: our superposition state exhibits phase insensitive nonclassicality for all values of θ other than $\theta = \pm\pi/2$.

It is important to note again that the condition $\tilde{M}^{(N)} \geq 0$ in (3.40) is needed over and above the condition $M^{(N)} \geq 0$ to form a necessary and sufficient set of conditions for classicality. This is ultimately due to the fact that our moment problem is a Stieltjes problem: had it been a Hamburger problem, the condition $M^{(N)} \geq 0$ would have been both necessary and sufficient!

3.6 Connection between the two approaches

We have presented two approaches to the problem of phase-insensitive nonclassicality of a state $\hat{\rho}$: one based on the sequence $q_n = n!\mathrm{tr}(\hat{\rho}|n\rangle\langle n|)$ which is the moment sequence of $\tilde{\Omega}(I)$, and a dual approach based on $m_n = \mathrm{tr}(a^{\dagger n}a^n\hat{\rho})$, the moment sequence of $\Omega(I)$. In each case we obtained necessary and sufficient conditions for nonclassicality. In both cases we exploited the fact that the underlying problem was a moment problem of the Stieltjes type. In this Section we bring out explicitly the connection between these dual approaches and establish their equivalence, in the case when all m_n are defined.

The fact that $\tilde{\Omega}(I) = \Omega(I)e^{-I}$ suggests the use of the Laplace transform. Let $\Phi(s)$, $\tilde{\Phi}(s)$ be the Laplace transform of $\Omega(I)$, $\tilde{\Omega}(I)$ respectively:

$$\begin{aligned} \Phi(s) &= \int_0^\infty dI\Omega(I)e^{-sI} \quad , \\ \tilde{\Phi}(s) &= \int_0^\infty dI\tilde{\Omega}(I)e^{-sI} = \Phi(s+1) \quad . \end{aligned} \tag{3.46}$$

We will now exploit the fact that the moments of a distribution are simply related to the derivatives of its Laplace transform evaluated at the origin:

$$\begin{aligned} m_n &= \int_0^\infty dI\Omega(I)I^n &= (-1)^n \frac{d^n\Phi(s)}{ds^n}|_{s=0} \ , \\ q_n &= \int_0^\infty dI\Omega(I)e^{-I}I^n &= (-1)^n \frac{d^n\tilde{\Phi}(s)}{ds^n}|_{s=0} \\ & &= (-1)^n \frac{d^n\Phi(s)}{ds^n}|_{s=1} \ . \end{aligned} \tag{3.47}$$

Making two Taylor series expansions of $\Phi(s)$, once about $s = 0$ and then about $s = 1$, equating the two expansions, and making use of (3.47), we have

$$\sum_{k=0}^{\infty}(-1)^k \frac{m_k}{k!} s^k = \sum_{l=0}^{\infty}(-1)^l \frac{q_l}{l!}(s-1)^l \ . \tag{3.48}$$

The n^{th} derivative of (3.48) at $s = 0$ gives m_n in terms of $\{q_k\}$,

$$m_n = \sum_{k=0}^{\infty} \frac{q_{n+k}}{k!} \ , \tag{3.49}$$

whereas the n^{th} derivative evaluated at $s = 1$ gives the inverse relation

$$q_n = \sum_{k=0}^{\infty}(-1)^k \frac{m_{n+k}}{k!} \ . \tag{3.50}$$

Writing the sequence $\{q_n\}$ as a column vector $\mathbf{q}$ and $\{m_n\}$ as the corresponding vector $\mathbf{m}$ we have the matrix equations

$$\mathbf{m} = S\mathbf{q} \ , \mathbf{q} = S^{-1}\mathbf{m} \ , \tag{3.51}$$

where

$$S = \begin{pmatrix} 1 & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \cdots \\ 0 & 1 & \frac{1}{1!} & \frac{1}{2!} & \cdots \\ 0 & 0 & 1 & \frac{1}{1!} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} ,$$

$$S^{-1} = \begin{pmatrix} 1 & \frac{-1}{1!} & \frac{1}{2!} & \frac{-1}{3!} & \cdots \\ 0 & 1 & \frac{-1}{1!} & \frac{1}{2!} & \cdots \\ 0 & 0 & 1 & \frac{-1}{1!} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} . \qquad (3.52)$$

We have displayed S, S^{-1} to exhibit the fact that the structure of these matrices is unaffected if the first row and first column are deleted. We shall have occasion to return to this important feature.

While $\mathbf{q}$ and $\mathbf{m}$ considered as (infinite-dimensional) column vectors are connected by the matrix S, the corresponding infinite-dimensional matrices $L = L^{(\infty)}$ and $M = M^{(\infty)}$ are connected through a symmetric transformation through $S^{1/2}$. We have

$$\begin{aligned} (S^{1/2})_{jk} &= \frac{1}{2^{j-k}(k-j)!} \ , \text{ if } k \geq j \ , \\ &= 0 \ , \text{ if } k < j; \\ (S^{-1/2})_{jk} &= \frac{-1^{k-j}}{2^{j-k}(k-j)!} \ , \text{ if } k \geq j \ , \\ &= 0 \ , \text{ if } k < j. \end{aligned} \qquad (3.53)$$

That $S^{1/2}$ and $S^{-1/2}$ so defined are indeed inverses of one another simply follows from the familiar properties of binomial coefficients. Using the same properties it may be verified that

$$M = S^{1/2}L(S^{1/2})^T \ , \ \ L = S^{-1/2}M(S^{-1/2})^T \ . \tag{3.54}$$

This proves that $M \geq 0$ if and only if $L \geq 0$.

Let $\tilde{\mathbf{q}}$ be the moment sequence derived from $\mathbf{q}$ by simply dropping q_0. Clearly, $\tilde{L} = \tilde{L}^\infty$ is in the same relation to $\tilde{\mathbf{q}}$ as L is to $\mathbf{q}$. Similarly, if we form $\tilde{\mathbf{m}}$ from the sequence $\mathbf{m}$ by simply dropping m_0, then $\tilde{M} = \tilde{M}^{(\infty)}$ will be seen to be in the same relation to $\tilde{\mathbf{m}}$ as M is to $\mathbf{m}$. Now recalling the fact that S, S^{-1}, $S^{1/2}$, $S^{-1/2}$ have the interesting property that they are invariant under dropping of the first row and first column, we conclude

$$\tilde{\mathbf{m}} = \mathbf{S}\tilde{\mathbf{q}} \ , \ \ \tilde{\mathbf{q}} = \mathbf{S}^{-1}\tilde{\mathbf{m}} \ . \tag{3.55}$$

It follows that

$$\tilde{M} = S^{1/2}\tilde{L}(S^{1/2})^T \ , \ \ \tilde{L} = S^{-1/2}\tilde{M}(S^{-1/2})^T \ . \tag{3.56}$$

This proves that $\tilde{M} \geq 0$ if and only if $\tilde{L} \geq 0$. We have shown in (3.54) and (3.56) that $L \geq 0$, $M \geq 0$ if and only if $\tilde{L} \geq 0$, $\tilde{M} \geq 0$, thus we have established the equivalence of the two approaches: the two approaches are dual to one another and, in particular, theorem 3.1 is equivalent to theorem 3.2.

The above analysis shows that if we drop the first k terms from the moment sequence of a bonafide Stieltjes probability distribution (in the semi-infinite interval) the result is again a bonafide Stieltjes moment sequence (of some other valid probability density). This is a distinguishing feature of the Stieltjes moment problem. It is not difficult to see that the corresponding statement will be true for the Hamburger moment problem (on the entire real line) only if *even* number of initial terms are dropped from the moment sequence.

3.7 Conclusion

We have given here a quantitative treatment of oscillations in the photon number distribution. We have also given a complete analysis of phase insensitive measurements of the single mode quantized radiation field and developed necessary and sufficient conditions for nonclassicality of the field. We now go on to study a manifestly nonclassical state - the two-mode squeezed coherent state.

Chapter 4

A study of the Photon Statistics in Two-Mode Squeezed Coherent States with Complex Displacement and Squeeze Parameters

4.1 Introduction

In the previous chapter we derived conditions on the sequence p_n (the photon number distribution) for a state to be classical. In this chapter we compute the photon number distribution of a manifestly nonclassical state - the two-mode squeezed coherent state. Photon number distributions of various nonclassical states of light have been studied by several authors[39, 15]. Interest in such studies was triggered in part atleast by the work of Schleich and Wheeler [28] which was mentioned in the previous chapter in connection with oscillations in the photon number distribution being taken as a signature of nonclassicality. More recently Dutta *et al*[40] studied the single mode squeezed coherent state with *complex* squeeze and displacement parameters and found that in addition to the oscillations found by Schleich and Wheeler, the photon number distribution of the single mode squeezed coherent state exhibits collapses and revivals similar to the oscillations familiar from the Jaynes-Cummings model [41]. It is to be noted that these oscillations materialise only when the parameters which are involved are taken to be complex. The two-mode analogue of the above state was studied by Caves *et al* [29] for real squeeze and displacement paramters and they found that the distributions for $\alpha_1 = \alpha_2$ (parallel) and for $\alpha_1 = -\alpha_2$ (antiparallel) were strikingly different. (Here α_1 and α_2 are the displacements of the two modes). Motivated by the results of Dutta *et al* where such a dramatic difference in the quality of oscillations was found when the complex nature of the relevant parameters was taken into account and by the work of Caves *et al* where the distributions for parallel and antiparallel cases were entirely different, we study the two-mode squeezed coherent state with complex squeeze and displacement parameters.

The contents of this chapter are organized as follows. In section 2 we develop an expression for the photon number distribution in an arbitrary two-mode squeezed coherent state with complex squeeze and displacement parameters. The analysis of Caves *et al* is based on normal ordering techniques. Our approach is symmetry based: we exploit the SU(2) dynamical symmetry underlying two-mode systems. This allows us to view the two-mode squeeze operator as a rotated version of the product of reciprocal single mode squeezings. Thus the probability amplitude for the photon distribution becomes a linear combination of the product of the well known single mode Yuen matrix elements [22] given in terms of Hermite polynomials, the coefficients of the linear combination being determined by the matrix elements of a particular SU(2) rotation. Finally an identity relating associated Laguerre polynomials helps us to write the probability amplitude in terms of a single associated Laguerre polynomial. It is of interest to note that this identity itself is an immediate consequence of the SU(2) structure. Conformity of our final result with that of Caves *et al* is noted.

In section 3 we bring out the fact that this two-mode photon distribution possesses a $U(1) \times U(1)$ invariance property. As a consequence, even though our problem has three phases (one each arising from the two displacement parameters and the third from the squeeze parameter), the photon distribution depends only on one $U(1) \times U(1)$-invariant linear combination χ of these phases. We bring out also a Gouy phase [42] in the manner in which this invariant χ influences the argument of the associated Laguerre polynomial.

Some examples of photon distribution are studied numerically in Section 4. Our principal aim is to bring out the sensitivity of the photon distribution to the $U(1) \times U(1)$ invariant χ. It will be seen that while our results are in conformity with the results of Caves *et al* for those values of χ which correspond to their studies, there are new interesting features for other values.

In Section 5 we study some properties which turn out to be invariant to the phases. Second order coherence functions are briefly considered in Section 6, and it is shown that they exhibit nonclassical behaviour in some range of χ.

4.2 Photon Distribution

The general two mode squeezed coherent state is unitarily related to $|vac\rangle = |0,0\rangle$, the ground state of the two-mode system described by annihilation operators a and b, in the following familiar manner:

$$
\begin{aligned}
|z;\alpha_1,\alpha_2\rangle &= D(\alpha_1,\alpha_2)\ S(z)|0,0\rangle \quad , \\
S(z) = \exp(z^* ab - z a^\dagger b^\dagger) \quad &, \quad D(\alpha_1,\alpha_2) = D(\alpha_1)\ D(\alpha_2) \quad , \\
D(\alpha_1) = \exp(\alpha_1 a^\dagger - \alpha_1{}^* a) \quad &, \quad D(\alpha_2) = \exp(\alpha_2 b^\dagger - \alpha_2{}^* b) \quad .
\end{aligned}
\tag{4.1}
$$

Here z is a complex two-mode squeeze parameter and α_1, α_2 are complex displacement (coherent excitation) parameters. Detailed analysis of two-mode squeezed coherent states has been made by several authors [43, 24]. In the above definition we have allowed, (following Caves et al[29]), the squeeze operator to act on vacuum before displacing the resulting two-mode squeezed vacuum. Sometimes it will be more convenient to order these operations the other way in the definition of the squeezed coherent state. Both definitions are equivalent, and we have the following identity:

$$
\begin{aligned}
&|z;\alpha_1,\alpha_2\rangle = S(z)\ D(\tilde{\alpha}_1,\tilde{\alpha}_2)|0,0\rangle \quad ; \\
&\tilde{\alpha_1} = \alpha_1\mu + \alpha_2{}^*\nu \quad , \quad \tilde{\alpha_2} = \alpha_2\mu + \alpha_1{}^*\nu \quad ; \\
&z = re^{2i\phi} \quad ; \quad \mu = \cosh r \quad , \quad \nu = e^{2i\phi}\sinh r \quad .
\end{aligned}
\tag{4.2}
$$

The photon distribution $p(n_1, n_2)$ in the two-mode squeezed coherent state $|z; \alpha_1, \alpha_2\rangle$ is given by

$$\begin{aligned} p(n_1, n_2) &= |c(n_1, n_2)|^2 \quad , \\ c(n_1, n_2) &= \langle n_1, n_2 | z;\ \alpha_1, \alpha_2\rangle = \langle n_1, n_2 | S(z)\ D(\tilde{\alpha}_1, \tilde{\alpha}_2)|0, 0\rangle. \quad , \end{aligned} \tag{4.3}$$

where $|n_1, n_2\rangle$ are the familiar Fock states of the two-mode system. We will compute $p(n_1, n_2)$ in several steps.

As the first step we exploit the dynamical SU(2) symmetry underlying the two-mode system. Two boson realization of the SU(2) symmetry is originally due to Schwinger[44] and has more recently played an important role in quantum optics [45, 46]. The basis of all these applications is the easily verified fact that the hermitian operators J_1, J_2, J_3 defined through

$$J_1 = \frac{a^\dagger b + b^\dagger a}{2}, \quad J_2 = -i\frac{a^\dagger b - b^\dagger a}{2}, \quad J_3 = \frac{a^\dagger a - b^\dagger b}{2}, \tag{4.4}$$

satisfy the SU(2) algebra $[J_k, J_l] = i\epsilon_{klm} J_m$. This fact becomes obvious if one notes that $J_k = \frac{1}{2}\xi^\dagger \sigma_k \xi$, where ξ is a two element column vector with entries a, b and σ_k are the Pauli matrices. With the help of these SU(2) generators, we can write our two-mode squeeze operator $S(z)$ as

$$S(z) = \exp(-i\frac{\pi}{2}J_2)\ S_a(z)\ S_b(-z)\ \exp(i\frac{\pi}{2}J_2)\ , \tag{4.5}$$

where $S_a(z)$, $S_b(-z)$ are the single mode squeeze operators

$$S_a(z) = \exp[\frac{1}{2}(z^* a^2 - z a^{\dagger 2})],$$

$$S_b(-z) = \exp[-\frac{1}{2}(z^* b^2 - z b^{\dagger 2})]. \tag{4.6}$$

Since $\exp[-i\frac{\pi}{2}J_2]$ produces a $\frac{\pi}{4}$ rotation in the mode space, the important identity (4.5) shows that our two-mode squeeze operator $S(z)$ is indeed a rotated version of product of single mode squeeze operators producing *reciprocal* squeezing.

When the identity (4.5) is used in (4.3) we obtain

$$c(n_1, n_2) = \langle n_1, n_2 | e^{-i\frac{\pi}{2}J_2}\; S_a(z)\; S_b(-z)\; e^{i\frac{\pi}{2}J_2}\; D(\tilde{\alpha}_1, \tilde{\alpha}_2)|0,0\rangle. \tag{4.7}$$

Since

$$\exp(i\frac{\pi}{2}J_2) D(\tilde{\alpha}_1, \tilde{\alpha}_2) \exp(-i\frac{\pi}{2}J_2) = D\left(\frac{\tilde{\alpha}_1 + \tilde{\alpha}_2}{\sqrt{2}}, \frac{\tilde{\alpha}_2 - \tilde{\alpha}_1}{\sqrt{2}}\right), \tag{4.8}$$

and since $\exp(i\frac{\pi}{2}J_2)$ acts as identity operator on $|0,0\rangle$, we have the useful relation

$$e^{i\frac{\pi}{2}J_2} D(\tilde{\alpha}_1, \tilde{\alpha}_2)|0,0\rangle = \left|\frac{\tilde{\alpha}_1 + \tilde{\alpha}_2}{\sqrt{2}}, \frac{\tilde{\alpha}_2 - \tilde{\alpha}_1}{\sqrt{2}}\right\rangle. \tag{4.9}$$

This allows us to rewrite (4.7) as

$$c(n_1, n_2) = \sum_{n_1', n_2'} \langle n_1, n_2 | e^{-i\frac{\pi}{2}J_2} | n_1', n_2'\rangle \times \langle n_1', n_2' | S_a(z)\; S_b(-z) | \frac{\tilde{\alpha}_1 + \tilde{\alpha}_2}{\sqrt{2}}, \frac{\tilde{\alpha}_2 - \tilde{\alpha}_1}{\sqrt{2}}\rangle. \tag{4.10}$$

As the next step, we recognize that the matrix elements entering (4.10) are well known from other contexts. The expression $\langle n_1, n_2 | e^{-i\frac{\pi}{2}J_2} | n_1', n_2'\rangle$ are the Wigner matrix elements [47] familiar from the quantum theory of angular momentum:

$$\langle n_1, n_2 | e^{-i\frac{\pi}{2}J_2} | n_1', n_2'\rangle = d^j_{mm'}(\frac{\pi}{2}) = d^j_{m'm}(-\frac{\pi}{2}) \quad , \tag{4.11}$$

where

$j = (n_1' + n_2')/2 = (n_1 + n_2)/2,\ m = (n_1 - n_2)/2,\ m' = (n_1' - n_2')/2;$

$$d^j_{m'm}(-\frac{\pi}{2}) = (-1)^{m'-m} d^j_{m'm}(\frac{\pi}{2})$$

$$= (-1)^{m'-m} 2^{-j} \sum_\mu (-1)^{\mu - m' + m} \frac{[(j+m')!(j-m')!(j+m)!(j-m)!]^{\frac{1}{2}}}{(j-m'-\mu)!(j+m+\mu)!\mu!(m'-m+\mu)!} \quad (4.12)$$

The other expression in (4.10) is the product of the Yuen matrix elements of single mode squeeze operators between coherent states and Fock states [22, 40]

$$\langle n_1', n_2'|S_a(z)S_b(-z)|\frac{\tilde{\alpha}_1 + \tilde{\alpha}_2}{\sqrt{2}}, \frac{\tilde{\alpha}_2 - \tilde{\alpha}_1}{\sqrt{2}}\rangle$$
$$= \langle n_1'|S_a(z)|\frac{\tilde{\alpha}_1 + \tilde{\alpha}_2}{\sqrt{2}}\rangle\langle n_2'|S_b(-z)|\frac{\tilde{\alpha}_2 - \tilde{\alpha}_1}{\sqrt{2}}\rangle$$

$$\langle n_1'|S_a(z)|\frac{\tilde{\alpha}_1 + \tilde{\alpha}_2}{\sqrt{2}}\rangle \ = \ (n_1'!\mu)^{\frac{-1}{2}}(\frac{\nu}{2\mu})^{\frac{n_1'}{2}} H_n[(\tilde{\alpha}_1 + \tilde{\alpha}_2)(4\mu\nu)^{\frac{-1}{2}}]$$
$$\times \ \exp\,[-\frac{1}{4}|(\tilde{\alpha}_1 + \tilde{\alpha}_2)|^2 + \frac{\nu^*}{4\mu}(\tilde{\alpha}_1 + \tilde{\alpha}_2)^2]. \quad (4.13)$$

And $\langle n_2'|S_b(-z)|\frac{1}{\sqrt{2}}(\tilde{\alpha}_2 - \tilde{\alpha}_1)\rangle$ has an expression similar to (4.13) with $(\tilde{\alpha}_1 + \tilde{\alpha}_2)$ replaced by $(\tilde{\alpha}_2 - \tilde{\alpha}_1)$ and ν by $-\nu$.

For our final step, we need the important identity [46, 48]

$$\sum_{m'=-j}^{+j} \ \Omega^j_{m'm}\ [2^{2j}\ (j+m')!\ (j-m')!]^{\frac{-1}{2}}\ H_{j+m'}(x)\ H_{j-m'}(y)$$
$$= \quad \exp[-i\frac{\pi}{2}(2(j-|m|)-(j-m))] \quad [\frac{(j-|m|)!}{(j+|m|)!}]^{\frac{1}{2}}$$
$$\times \quad (x^2+y^2)^{2|m|}\ L^{2|m|}_{j-|m|}(x^2+y^2) \quad e^{2im\theta}\ ,$$

(4.14)

$$\Omega^j_{m'm} = i^{m'-m} d^j_{m'm}(\frac{\pi}{2}). \tag{4.15}$$

The above identity has played an important role in constructing the normal mode spectrum of the twisted Gaussian Schell model beam in classical optics [46]. Its derivation is straightforward and brings out the power of viewing SU(2) described in (4.4) as the dynamical symmetry of the two dimensional isotropic oscillator in the $x-y$ plane. Eigenstates of such an oscillator can be constructed by either diagonalizing J_3 in which case the eigenstates will be products of Hermite polynomials in x and y, or by diagonalizing J_2 which generates rotations in the $x-y$ plane in which case the eigenstates will be the rotationally covariant associated Laguerre polynomials in x^2+y^2. The identity (4.15) is a consequence of the fact that J_2 and J_3 are related through conjugation by $\exp[-i\frac{\pi}{2}J_1]$. In fact $\Omega^j_{m'm}$ are the matrix elements of $\exp[-i\frac{\pi}{2}J_1]$ and the factor $i^{m'-m}$ in the relationship (4.15) arises from the fact that

$$\exp[-i\frac{\pi}{2}J_1] = \exp[i\frac{\pi}{2}J_3] \ \ \exp[-i\frac{\pi}{2}J_2] \ \ \exp[-i\frac{\pi}{2}J_3] \tag{4.16}$$

It is important to appreciate that the identity (4.15) connecting the Hermite polynomials and the associated Laguerre polynomials is valid not only for real x, y but also for complex values of x, y with ρ, θ defined, in either case, through $x + iy = \rho e^{i\theta}$ so that $\rho^2 = x^2 + y^2$, $e^{2i\theta} = (x^2 - y^2 + 2ixy)/(x^2 + y^2)$

Using the expressions (4.12) and (4.13) in (4.10) and making use of the identity (4.15) we have our final expression

$$c(n_1, n_2) \equiv c(j+m, j-m)$$

$$= \exp[-i\frac{\pi}{2}(j-|m|)] \left[\frac{(j-|m|)!}{(j+|m|)!}\right]^{\frac{1}{2}} [\tilde{\alpha}_1\tilde{\alpha}_2/(\mu\nu)]^{|m|} \mu^{-1} (\nu/\mu)^j$$
$$\times L^{2|m|}_{j-|m|}\left(\frac{\tilde{\alpha}_1\tilde{\alpha}_2}{\mu\nu}\right) (\frac{\tilde{\alpha}_1}{\tilde{\alpha}_2})^m \exp\left[\frac{-|\tilde{\alpha}_1|^2-|\tilde{\alpha}_2|^2}{2}\right] \exp\left[\frac{\nu^*\tilde{\alpha}_1\tilde{\alpha}_2}{\mu}\right] \qquad (4.17)$$

The double sum over n'_1, n'_2 in (4.10) reduced to a single sum over $m' = (n'_1 - n'_2)/2$ owing to the fact that the rotation matrix element $\langle n_1, n_2|e^{-i\frac{\pi}{2}J_2}|n'_1, n'_2\rangle$ in (4.10) is nonzero only when $n_1 + n_2 = n'_1 + n'_2$, thus enabling us to use the identity (4.15).

To relate our final expression to that of Caves *et al*, we note that $p = j - |m|$ is the smaller of n_1, n_2 and $q = j + |m|$ is the larger. Thus substituting for μ, ν from (4.2), we can rewrite (4.17) as

$$c(n_1, n_2) = (-1)^p \sqrt{\frac{p!}{q!}}\, \tilde{\alpha}_1^{n_1-p}\; \tilde{\alpha}_2^{n_2-p}\, \frac{(\tanh r)^p}{\cosh r}\, (e^{2i\phi})^p$$
$$\times L^{q-p}_p \left(\frac{2\tilde{\alpha}_1\tilde{\alpha}_2}{\sinh 2r}\, e^{-2i\phi}\right) \exp\left[\frac{-(\alpha_1^*\tilde{\alpha}_1 + \alpha_2^*\tilde{\alpha}_2)}{2\cosh r}\right] . \qquad (4.18)$$

It is seen that (4.18) for real values of the parameters indeed reproduces equation (2.14) of Caves *et al*, since their $\mu_j = \tilde{\alpha}_j/\cosh r$.

4.3 $U(1) \times U(1)$ Invariance and Gouy Effect

We have three complex parameters in the problem. These are $z = |z|e^{2i\phi}$, $\alpha_1 = |\alpha_1|e^{i\phi_1}$ and $\alpha_2 = |\alpha_2|e^{i\phi_2}$. However symmetry considerations should convince one that the phases ϕ_1 , ϕ_2 , ϕ will not enter the photon distribution independently. To see this, let us write $c(n_1, n_2)$ in more detail as

$$c(n_1, n_2; z; \alpha_1, \alpha_2) = \langle n_1, n_2|z; \alpha_1, \alpha_2\rangle. \qquad (4.19)$$

Now note that

$$\exp(i\zeta_1 a^\dagger a)\ \exp(i\zeta_2 b^\dagger b)\ |z;\ \alpha_1,\alpha_2\rangle = |ze^{i(\zeta_1+\zeta_2)};\ \alpha_1 e^{i\zeta_1},\ \alpha_2 e^{i\zeta_2}\rangle \quad .$$

Projecting onto the Fock state $|n_1, n_2\rangle$ we have

$$c(n_1,n_2;\ ze^{i(\zeta_1+\zeta_2)};\ \alpha_1 e^{i\zeta_1},\alpha_2 e^{i\zeta_2}) = e^{i(n_1\zeta_1+n_2\zeta_2)}c(n_1,n_2;\ z;\ \alpha_1,\alpha_2) \quad ,$$

We see that under the $U(1)\times U(1)$ transformations generated by $\exp(i\zeta_1 a^\dagger a)$, $\exp(i\zeta_2 b^\dagger b)$, the probability amplitude $c(n_1,n_2;z;\alpha_1,\alpha_2)$ defined in (4.18) changes only by a phase. Since the photon distribution is given by the square of the absolute value of this amplitude, we see that it has $U(1)\times U(1)$ invariance:

$$p(n_1,n_2;ze^{i(\zeta_1+\zeta_2)},\alpha_1 e^{i\zeta_1},\alpha_2 e^{i\zeta_2}) = p(n_1,n_2;z;\alpha_1,\alpha_2) \qquad (4.20)$$

This $U(1)\times U(1)$ invariance is analogous to the $U(1)$ invariance in the single mode case [40] and implies that our photon distribution will depend on the three phases ϕ_1,ϕ_2 and ϕ only through the $U(1)\times U(1)$-invariant combination $\phi_1+\phi_2-2\phi$.

We can verify that the photon distribution described by the probability amplitude given in (4.18) indeed possesses this $U(1)\times U(1)$ invariance. To this end note that under the transformation

$$\alpha_j \to \alpha_j e^{i\zeta_j} \quad , \quad z \to ze^{i(\zeta_1+\zeta_2)}, \qquad (4.21)$$

we have $\theta_j \to \theta_j + \zeta_j$, $\nu \to \nu e^{i(\zeta_1+\zeta_2)}$ and $2\phi \to 2\phi+\zeta_1+\zeta_2$. Further, it is clear from (4.2) that $\tilde{\alpha}_j \to \tilde{\alpha}_j e^{i\zeta_j}$ under (4.21). Thus,

$$\tilde{\alpha}_1^{n_1-p}\ \ \tilde{\alpha}_2^{n_2-p}(e^{2i\phi})^p \to \tilde{\alpha}_1^{n_1-p}\ \ \tilde{\alpha}_2^{n_2-p}\ \ (e^{2i\phi})^p\ \ e^{i(n_1\zeta_1+n_2\zeta_2)}.$$

To complete the verification we will now show that the argument of the associated Laguerre polynomial as well as the exponent in the last factor in (4.18) are functions of only the $U(1)\times U(1)$ invariant combination $\chi = \phi_1+\phi_2-2\phi$. From (4.2) connecting the α's to the $\tilde{\alpha}$'s we have

$$\alpha_1^*\tilde{\alpha}_1 + \alpha_2^*\tilde{\alpha}_2 = (|\alpha_1|^2 + |\alpha_2|^2)\cosh r + 2|\alpha_1\alpha_2|\sinh r\ \cos\chi - i2|\alpha_1\alpha_2|\sinh r\ \sin\chi\ . \tag{4.22}$$

We further deduce from (4.2)

$$\tilde{\alpha}_1\tilde{\alpha}_2 e^{-2i\phi} = \frac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2)\sinh 2r + 2|\alpha_1\alpha_2|(\cosh 2r\ \cos\chi + i\ \sin\chi) \tag{4.23}$$

Thus, the expression (4.17) has the behaviour required by (4.20) under the $U(1)\times U(1)$ transformation (4.21), showing explicitly that our photon distribution is indeed $U(1)\times U(1)$ invariant.

Having appreciated this fact we switch for brevity to use of $p(n_1,n_2)$, rather than $p(n_1,n_2;z;\alpha_1,\alpha_2)$, to denote the photon distribution.

Our analysis in the foregoing paragraphs shows that there are only two ways in which χ, the $U(1)\times U(1)$ invariant combination of the phases of α_1,α_2 and z, enters the photon number distribution: through the exponent as in (4.22), and through the associated Laguerre polynomial as in (4.23). The former one is independent of n_1,n_2 and hence contributes to the overall amplitude of the distribution. That is, it just ensures the fact that $p(n_1,n_2)$ summed over n_1,n_2 is normalized to unity. Thus it need not be pursued any further. The role of χ in the latter however, is nontrivial.

We will see in the next section that the dependence of the argument of the associated Laguerre polynomial on χ leads to a sensitive χ-dependance of $p(n_1, n_2)$. But here we wish to note the interesting manner in which the phase of the argument of the associated Laguerre polynomial depends on χ. To this end let Φ be the phase of the argument of the associated Laguerre polynomial in (4.18):

$$L_p^{q-p}(\frac{2\tilde{\alpha}_1\tilde{\alpha}_2}{\sinh 2r}e^{-2i\phi}) = L_p^{q-p}(\frac{2|\tilde{\alpha}_1\tilde{\alpha}_2|}{\sinh 2r}e^{i\Phi}). \tag{4.24}$$

From (4.23) we see that

$$\Phi = \arctan\left[\frac{|\alpha_1\alpha_2|\sin\chi}{\frac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2)\sinh 2r + |\alpha_1\alpha_2|\cosh 2r \ \cos\chi}\right]. \tag{4.25}$$

We show in figure 4.1. the behaviour of Φ as a function of χ for the case $|\alpha_1| = |\alpha_2|$. It is seen that while Φ is linear in χ for r=0, with increasing value of the squeeze parameter r, Φ becomes a highly nonlinear function of χ. This is the Gouy effect for two-mode squeezed coherent states. Gouy effect for (focussed) light beams has been known for a long time, [49, 50] and recently Gouy effect for single-mode squeezed light has also been studied [51].

We note in passing that if either α_1 or α_2 equals zero, then the argument of the associated Laguerre polynomial becomes real positive irrespective of the phase of the squeeze parameter z.

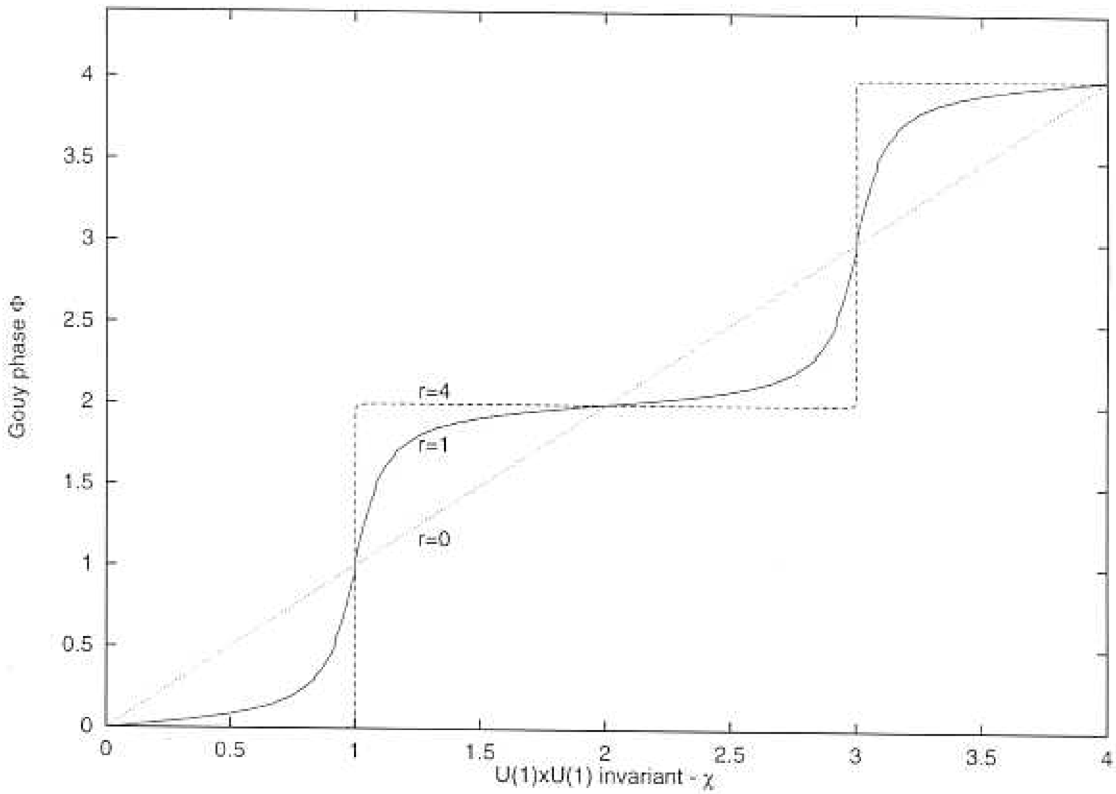


Figure 4.1: The Gouy phase Φ for different values of the $U(1) \times U(1)$ invariant phase χ. Both Φ and χ are in units of π and $\alpha_1 = \alpha_2 = 7$.

4.4 Examples of Photon Distributions

We have given in (4.18) the probability amplitude $c(n_1, n_2)$ for the two-mode squeezed coherent state; square of the absolute value of this expression gives the photon distribution $p(n_1, n_2)$. We are primarily interested in the effect of the phases $\phi_1, \phi_2, 2\phi$ of the complex parameters α_1, α_2, z. We have already shown that these phases enter the photon distribution only through the argument of the associated Laguerre polynomial; and that too in the $U(1) \times U(1)$ invariant combination $\chi = \phi_1 + \phi_2 - 2\phi$. We give in figure 4.2 the distribution $p(n_1, n_2)$ for fixed $|\alpha_1| = |\alpha_2|$ and fixed r, and selected values of χ in the range $0 \leq \chi < \pi$.

It should be appreciated that the effective range of χ, as far as $p(n_1, n_2)$ in (4.18) is concerned, is $0 \leq \chi \leq \pi$ rather than the full $0 \leq \chi \leq 2\pi$. This comes about from the fact that $p(n_1, n_2)$ is invariant under $\chi \to 2\pi - \chi$.

It is easy to see that Fig.1b and Fig.2b of Caves *et al* correspond to $\chi = 0$ and π respectively. And for these values of χ our results in figure 4.2 are clearly in agreement with theirs. But from $\chi = 0$ to $\chi = \pi$ the distribution "evolves" in an interesting manner. As χ is increased from zero, the ripple perpendicular to the diagonal starts breaking. With increasing value of χ these breaks increase in number, the period parallel to the diagonal increases and the distribution pulls itself towards $(n_1, n_2) = (0, 0)$. With further increase, the strength of the distribution falls rapidly as one moves away from the diagonal so that when $\chi = 180°$ is reached one is left

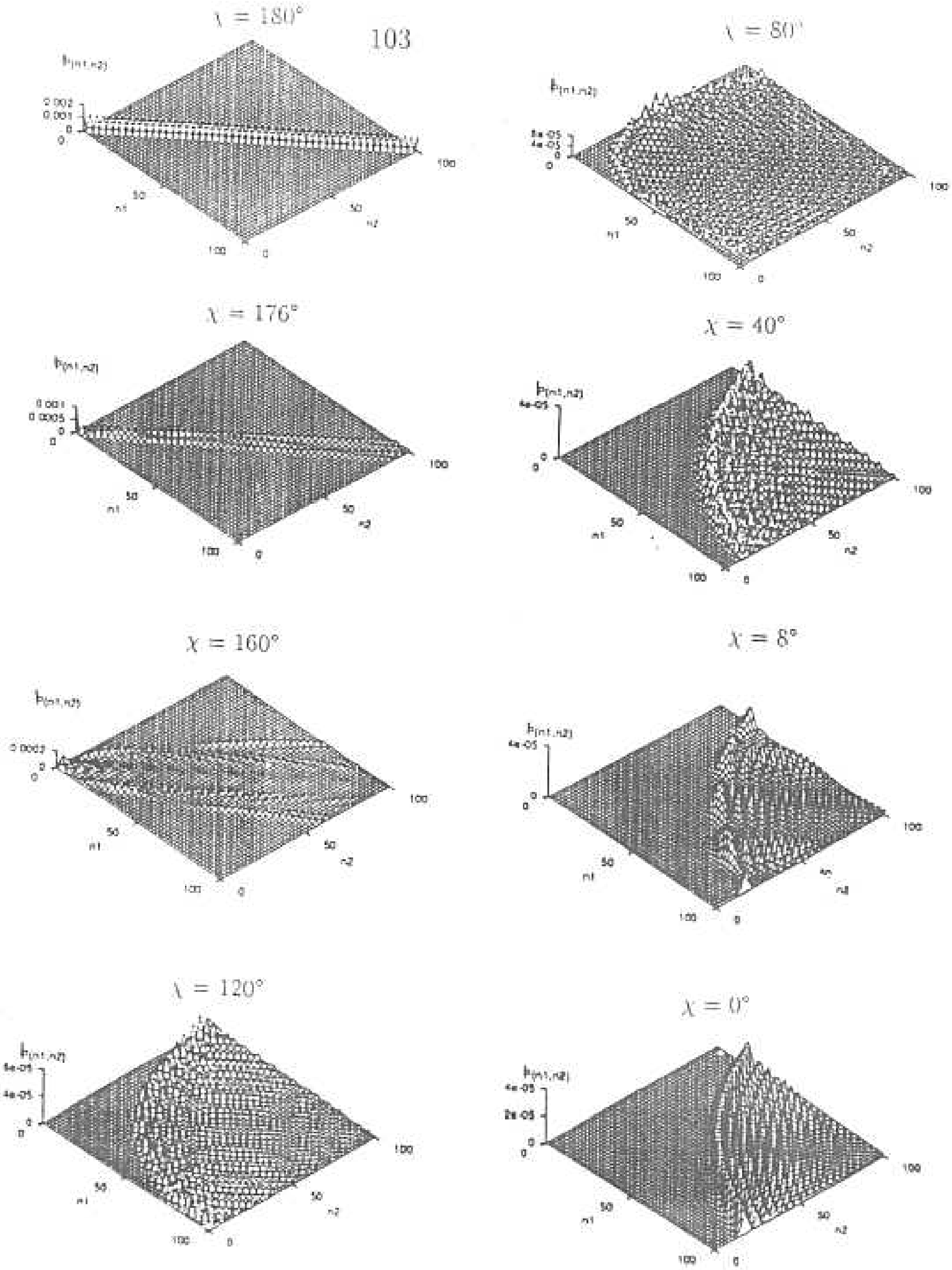


Figure 4.2: Photon number distribution $p(n_1, n_2)$ as a function of n_1 and n_2 for $\alpha_1 = \alpha_2 = 7.00$ and r=4.00. The distribution is concentrated along the diagonal for $\chi = 180°$. As χ decreases oscillations along and perpendicular to the diagonal pick up and saturate around $\chi = 120°$. Thereafter there is a gradual collapse of the oscillations perpendicular to the diagonal which evolves with decreasing χ towards the parabola like ripple structure at $\chi = 0°$.

with essentially a diagonal distribution. Thus, our figure 4.2 gives insight into the manner in which the photon distribution interpolates between the two extreme limits studied in [29].

To gain further understanding of the photon distribution, we probe the diagonal distribution $p(n,n)$ in some detail. In figure 4.3 we present $p(n,n)$ for the same values of $|\alpha_1| = |\alpha_2|$ and r as figure 4.2, and for various values of χ. Collapses and revivals in the oscillation may be noticed.

This result is reminiscent of the findings of Dutta *et al* for the single mode case. The major departure from the single mode case is that in the present case the collapses and revivals are persistent for a wider range of the parameter χ. In particular, they survive even in the limit $\chi = 0°$.

It may be noticed that the oscillations in $p(n,n)$ are most rapid at $\chi = 0°$; and the period of oscillation steadily increases as χ goes to the limit $180°$ where the diagonal distribution becomes essentially a constant. The region near $\chi = 180°$ is further explored in figure 4.4.

It is of interest to analyse the photon distribution in n_1 for fixed n_2. This corresponds to state reduction which has received considerable interest recently [52]. In figure 4.5, we show $p(n_1) \equiv p(n_1, n_2)$ for constant n_2 (i.e. the distribution as a function of n_1 for fixed n_2) for the same values of parameters $|\alpha_1| = |\alpha_2|$ and r, as in figures 4.2-4.4 and for selected values of χ. Again, collapses and revivals can be no-

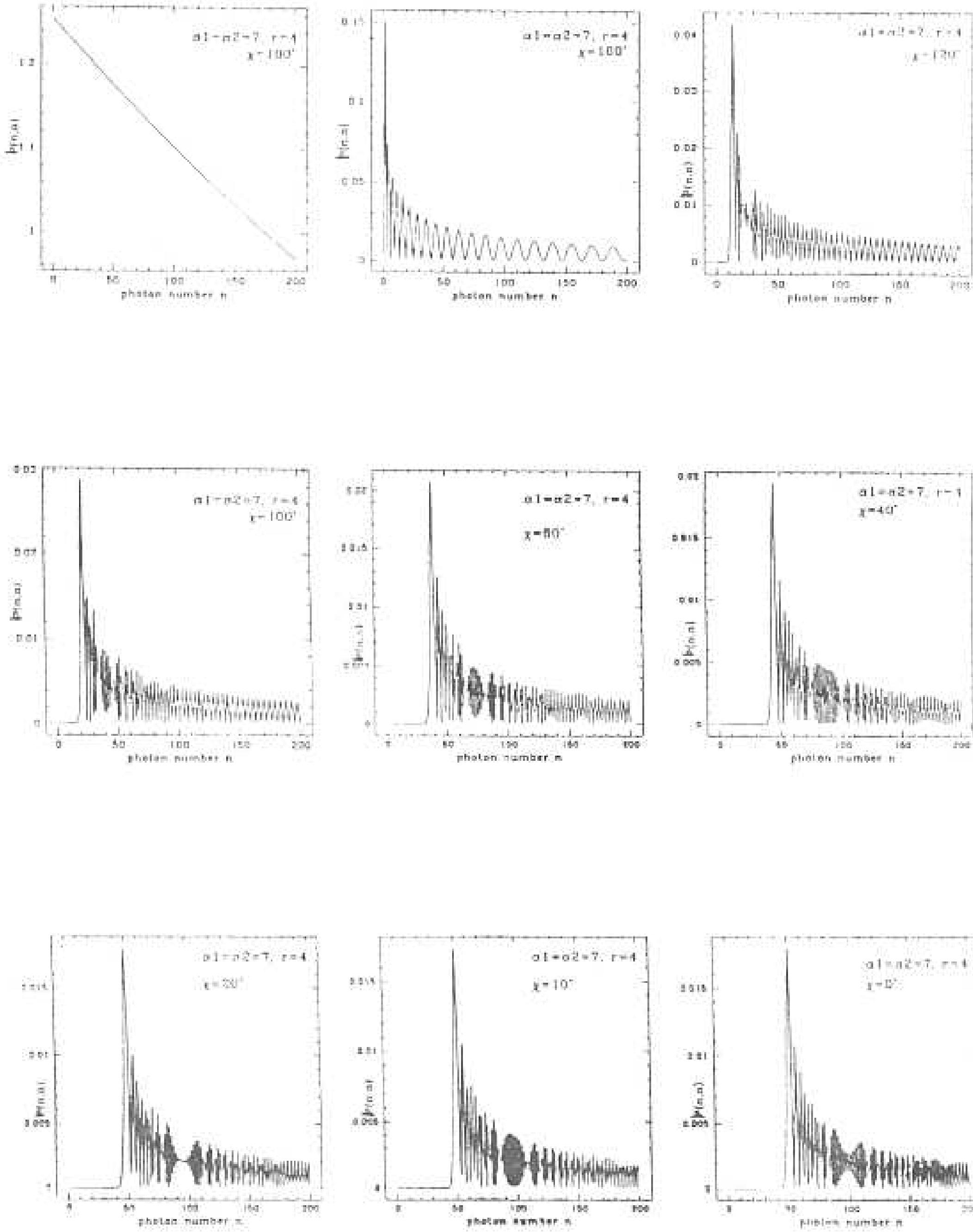


Figure 4.3: Diagonal photon number distribution $p(n,n)$ as a function of n with $\alpha_1 = \alpha_2 = 7$ and $r = 4$. Collapses and revivals are seen in the distribution, and these are reminiscent of the ones in the Jaynes-Cummings model. Figure shows $p(n,n)$ in units of 10^{-3}.

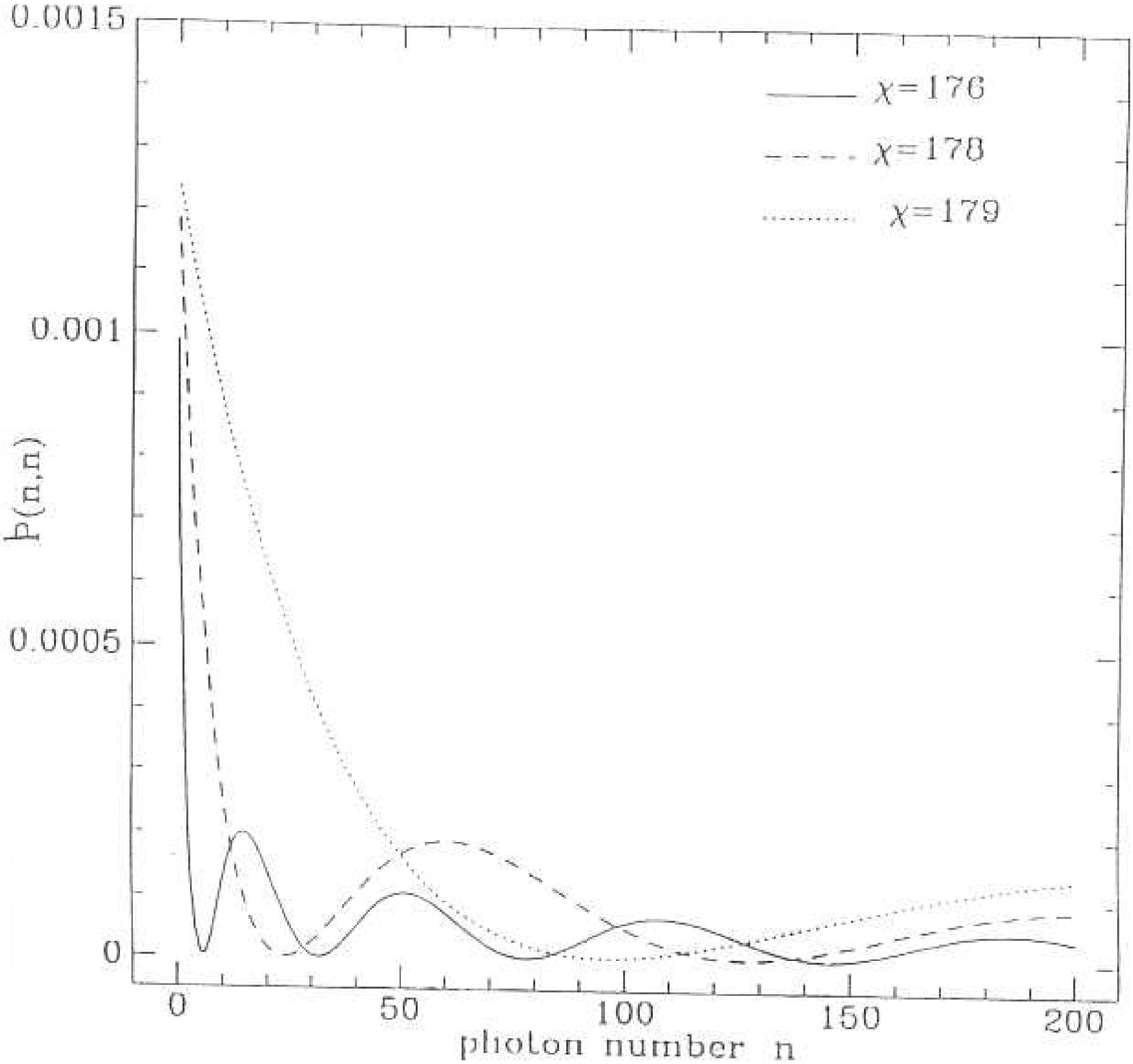


Figure 4.4: A closer look at the diagonal distribution in the region around $\chi = 180°$. It can be seen that the amplitude and the period of the oscillations decrease as χ decreases.

ticed. But the structure of this phenomenon is now quite different from the diagonal case and much richer.

4.5 Properties Insensitive to Phase

We have shown in the preceding two sections that the photon distribution of the two-mode squeezed coherent state $|z;\alpha_1,\alpha_2\rangle$ is quite sensitive to the $U(1)\times U(1)$-invariant combination of the phases of the squeeze and displacement parameters. But there are properties of this state which are insensitive to the phases of these operators. We present in this Section examples of two such properties.

The first such property we consider is the total energy E in the state $|z;\alpha_1,\alpha_2\rangle$. This is given by the expectation value of $(a^\dagger a + b^\dagger b)$. The computation is straightforward:

$$\begin{aligned} E &= \langle z;\alpha_1,\alpha_2| \ (a^\dagger a + b^\dagger b) \ |z;\alpha_1,\alpha_2\rangle \\ &= \langle vac|S^\dagger(z) \ D^\dagger(\alpha_1,\alpha_2) \ (a^\dagger a + b^\dagger b) \ D(\alpha_1,\alpha_2) \ S(z)|vac\rangle \end{aligned} \tag{4.26}$$

The contribution from the $a^\dagger a$ term is

$$\begin{aligned} &\langle vac| \ S^\dagger(z) \ (a^\dagger + \alpha_1^*) \ (a+\alpha_1) \ S(z) \ |vac\rangle \\ &= |\alpha_1|^2 + \langle vac|S^\dagger(z) \ a^\dagger a \ S(z)|vac\rangle \\ &= |\alpha_1|^2 + \sinh^2 r \quad , \end{aligned} \tag{4.27}$$

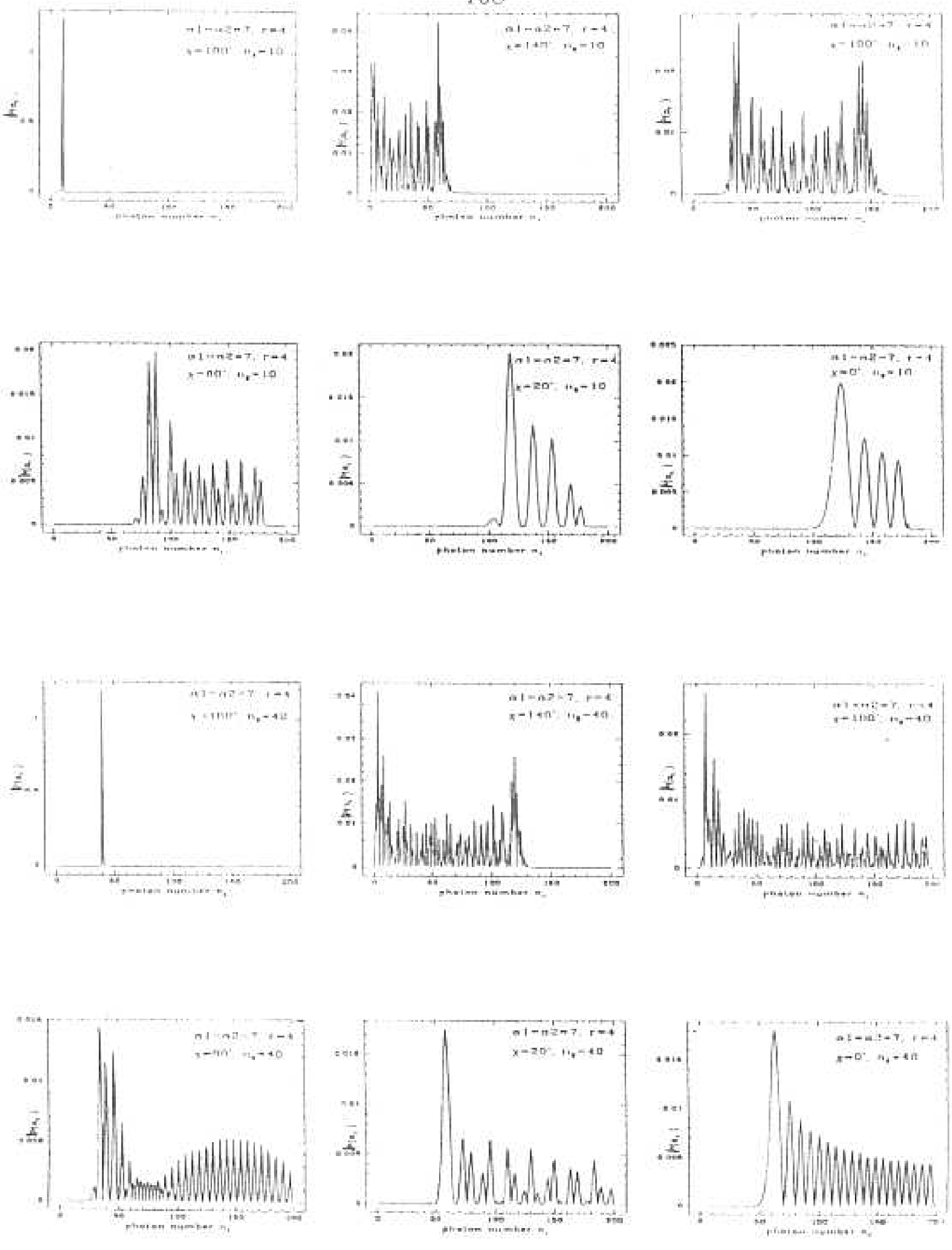


Figure 4.5: Off diagonal distribution $p(n_1) \equiv p(n_1, n_2)$ for fixed values of n_2. There are collapses and revivals similar to the diagonal case in figure 4.3 but the structure is richer. Figure shows $p(n_1)$ in units of 10^{-3} and the parameter values are $\alpha_1 = \alpha_2 = 7$, $r = 4$.

where we make use of the fact

$$S^{\dagger}(z) \;\; a \;\; S(z) = a\cosh r - b^{\dagger}e^{2i\phi}\sinh r \quad . \tag{4.28}$$

The expression (4.27) is similar to the one in the single mode case, but the $\sinh^2 r$ term comes from the expectation value of $b^{\dagger}b$. It is easy to see that the contribution from the $b^{\dagger}b$ term in (4.27) equals $|\alpha_2|^2 + \sinh^2 r$. Hence, the energy of the state $|z;\alpha_1,\alpha_2\rangle$ is[29]

$$E = |\alpha_1|^2 + |\alpha_2|^2 + 2\sinh^2 r. \tag{4.29}$$

Thus the total energy content of $|z;\alpha_1,\alpha_2\rangle$ is insensitive even to the invariant combination χ, even though the photon distribution itself is phase sensitive! That is, changing the value of χ simply redistributes the photons in the various two-mode Fock states without changing the total number of photons.

The next quantity we consider is the reduced density operator for mode 1. Caves *et al* compute this through the P-distribution. Our computation is based on the equivalent two-mode Wigner Distribution $W(\xi_1,\xi_2)$. The advantage of the Wigner distribution over the P-distribution arises from the fact that squeezing transformations simply act as linear transformations on the arguments of this distribution. Displacement operators act as rigid translations as in the P-distribution case. Using these facts and the fact that the Wigner distribution for $|vac\rangle$ is given by $W(\xi_1,\xi_2) = \frac{4}{\pi^2}\exp[-2(|\xi_1|^2 + |\xi_2|^2)]$, the Wigner distribution for the state $|z;\alpha_1,\alpha_2\rangle$

is easily computed to be

$$W(\xi_1, \xi_2) = \frac{4}{\pi^2} \exp[-2[(|\xi_1 - \alpha_1|^2 + |\xi_2 - \alpha_2|^2) \cosh 2r$$
$$+ \sinh 2r((\xi_1 - \alpha_1)(\xi_2 - \alpha_2)e^{-2i\phi} + (\xi_1^* - \alpha_1^*)(\xi_2^* - \alpha_2^*)e^{2i\phi})]]. \quad (4.30)$$

While the two-mode squeezed coherent state $|z; \alpha_1, \alpha_2\rangle$ has such a nice Gaussian Wigner distribution, it is well known that this state, being nonclassical, has no P-distribution function in the familiar sense of the term function.

The single mode Wigner distribution corresponding to the reduced density operator for mode 1 is now obtained by taking the marginal $\int d^2\xi_2 W(\xi_1, \xi_2)$ and we have

$$W(\xi_1) = \int d^2\xi_2 \, W(\xi_1, \xi_2) = \frac{2}{\pi \cosh 2r} \exp\left[-\frac{2|\xi_1 - \alpha_1|^2}{\cosh 2r}\right] \quad , \quad (4.31)$$

which corresponds to a displaced, but not squeezed, thermal state. We see that the phase of the squeeze operator does not enter this reduced Wigner distribution. In fact the P−distribution corresponding to (4.31) can be written down by inspection. We have

$$P(\xi_1) = \frac{1}{\pi \sinh^2 r} \exp\left[-\frac{2|\xi_1 - \alpha_1|^2}{\sinh^2 r}\right] \quad (4.32)$$

which coincides with the result of Caves *et al.* consistent with its insensitiveness to phase.

4.6 Second Order Coherence Function

In the last Section, we considered examples of properties of $|z;\alpha_1,\alpha_2\rangle$ which are insensitive to the phase of the squeeze parameter. We now turn briefly to some coherence properties which turn out to be sensitive to the phase.

We consider the Glauber coherence functions $g_{ab}^{(2)}(0)$ and $g_p^{(2)}(0)$. These are defined through

$$g_{ab}^{(2)}(0) = 1 + \frac{\langle \hat{n}_a \hat{n}_b\rangle - \langle \hat{n}_a\rangle\langle \hat{n}_b\rangle}{\langle \hat{n}_a\rangle\langle \hat{n}_b\rangle} \tag{4.33}$$

$$g_p^{(2)}(0) = 1 + \frac{\langle(\Delta(\hat{n}_a + \hat{n}_b))^2\rangle - \langle(\hat{n}_a + \hat{n}_b)\rangle}{\langle(\hat{n}_a + \hat{n}_b)\rangle}, \tag{4.34}$$

Motivation for these definitions can be found in Gilles and Knight [53]. Since these functions depend only on the photon distribution, it is clear that they can depend on the phases of z, α_1, α_2 atmost through the $U(1) \times U(1)$ invariant combination χ.

Classical values of these functions are bounded from below by unity.

It is seen from figure 4.6 and figure 4.7 that these coherence functions take non-classical values for some range of values of r whenever $\chi < 90°$.

4.7 Conclusion

We have studied the photon distribution in two-mode squeezed coherent states with complex squeeze and displacement parameters. The entire analysis was guided, often

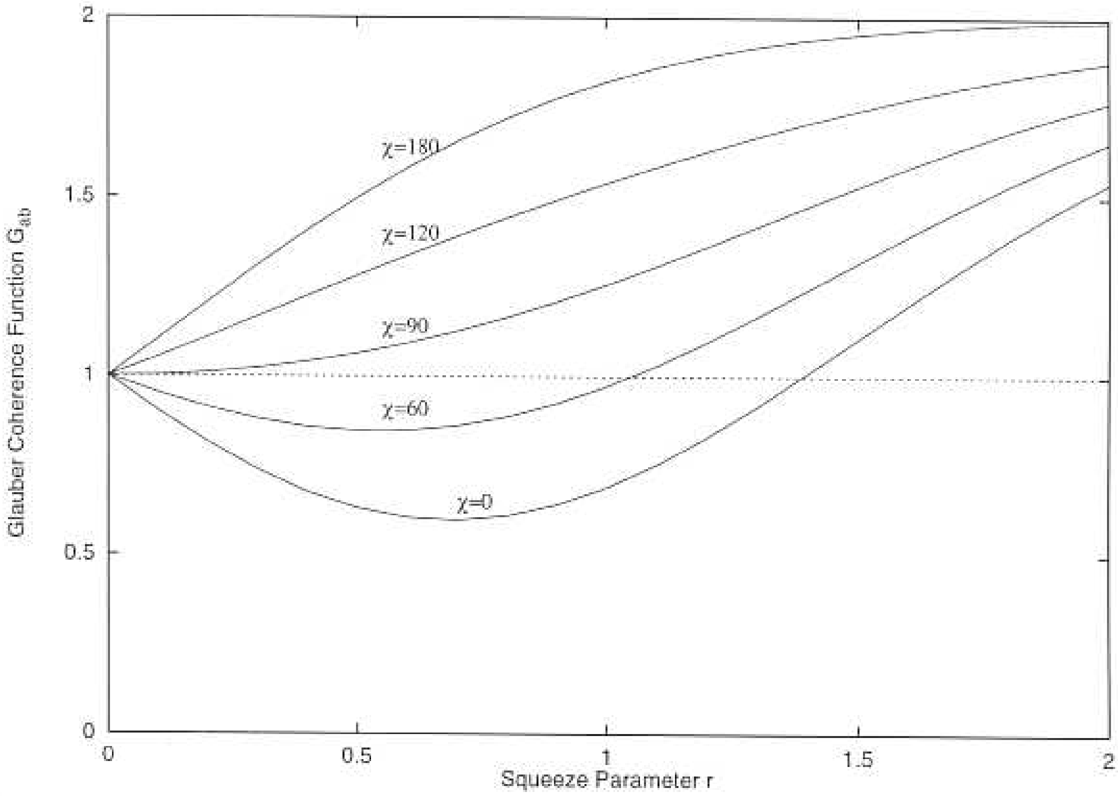


Figure 4.6: The Glauber coherence function G_{ab} as a function of the squeeze parameter r. Nonclassical behaviour is seen for $\chi < 90°$. Here $\alpha_1 = 1$ and $\alpha_2 = 2$.

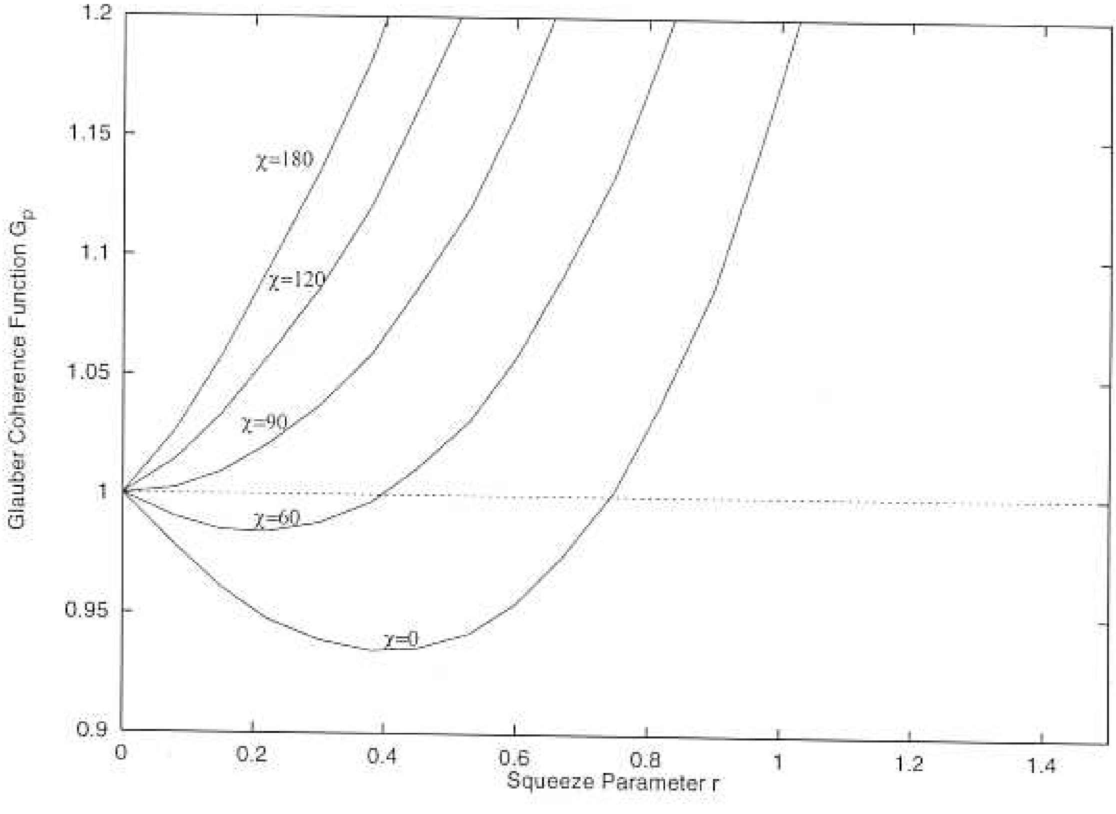


Figure 4.7: The Glauber coherence function G_p as a function of the squeeze parameter r. Nonclassical behaviour is seen for $\chi < 90°$. Here $\alpha_1 = 1$ and $\alpha_2 = 2$.

explicitly and sometimes implicitly, by an appreciation of the SU(2) structure underlying two-mode systems as a dynamical symmetry. Thus, realization of the fact that the two-mode squeeze operator is essentially the product of two correlated (in fact, reciprocal) single mode squeeze operators allowed us to write the probability amplitude for photon distribution using the well known Yuen results for the matrix elements in the single mode case, and the matrix elements of a particular SU(2) rotation. Finally, the SU(2) identity given in (4.15) enabled us to write the photon distribution in the compact closed form (4.18).

The $U(1) \times U(1)$-invariance of the photon distribution helped in simplifying the analysis, particularly in the case of numerical studies. That is, even though there were three phases in the problem to begin with, it turned out that there is only one nontrivial phase (the $U(1) \times U(1)$-invariant linear combination χ) which we have to consider as far as photon distribution is concerned. Our numerical analysis concentrated on the effect of this phase on various properties.

In all our examples we have taken $|\alpha_1| = |\alpha_2|$. We will conclude with some general observations on the situation when $|\alpha_1| \neq |\alpha_2|$.

The defining relations (4.2) can be written as

$$\begin{aligned} \tilde{\alpha}_1 &= e^{i\phi_1}(|\alpha_1|\mu + |\alpha_2\nu|e^{-i\chi}), \\ \tilde{\alpha}_2 &= e^{i\phi_2}(|\alpha_2|\mu + |\alpha_1\nu|e^{-i\chi}). \end{aligned} \tag{4.35}$$

It is now transparent that $|\alpha_1| = |\alpha_2|$ implies $|\tilde{\alpha}_1| = |\tilde{\alpha}_2|$. It may further be noted that

q, p in (4.18) are invariant under interchange of n_1 and n_2. Thus, it follows from (4.18) that $p(n_1, n_2) = p(n_2, n_1)$ whenever $|\alpha_1| = |\alpha_2|$. That is, the photon distribution is invariant under reflection about the diagonal $n_1 = n_2$. This property is manifest in figure 4.2.

If $|\alpha_1| \neq |\alpha_2|$, then $p(n_1, n_2)$ will be expected to become asymmetric with respect to the diagonal. From (4.18) we see that the only source of asymmetry in n_1, n_2 is the factor $|\tilde{\alpha}_1|^{2(n_1-p)} \; |\tilde{\alpha}_2|^{2(n_2-p)}$.

Since

$$\frac{|\tilde{\alpha}_1|^2}{|\tilde{\alpha}_2|^2} = \frac{1-(|\alpha_2|^2 - |\alpha_1|^2)/A}{1+(|\alpha_2|^2 - |\alpha_1|^2)/A},$$
$$A = (|\alpha_1|^2 + |\alpha_2|^2)\cosh 2r + |\alpha_1\alpha_2| \cos\chi \;\; \sinh 2r, \tag{4.36}$$

as can be seen from (4.35), one will expect the asymmetry to become less and less prominent with increasing value of the squeeze parameter $r = |z|$.

Chapter 5

A study of the Phase Statistics in Correlated Two-Mode Squeezed Coherent States

5.1 Introduction

In this chapter we complete the study of the two-mode squeezed coherent state initiated in the previous chapter, by analyzing its phase properties. The study of the phase properties of states of the radiation field exhibiting nonclassical behaviour has attracted a great deal of attention in recent years [54]-[60]. In particular effort has been directed towards re-examining the phase properties of various states from the point of view of a Hermitian phase operator defined by Pegg and Barnett [55] in a truncated Hilbert space. There has been a lot of work relating to what may be the most correct description of the phase properties of light [61], [62]. The 'operational' (trigonometric) phase operators defined by Mandel and coworkers [63] appear to be the only ones which agree with their particular experiments. This is not surprising as these operators were formulated as a specific description of the quantity measured by these experiments. However, this cannot be described by a Hermitian phase operator. To measure phase properties derived from the Hermitian phase operator [55], differently designed experiments are required. As expected, measurement of the properties of the Hermitian phase operator shows good agreement between theory and experiment [64].

Out of the various nonclassical states the correlated two-mode states are of particular interest - for example the pair coherent states [65]-[67] and the entangled states [68] generated in two-photon down conversion. Another example of a correlated two-mode state which has been studied quite extensively is the correlated two-mode squeezed coherent state [43, 24].

In the previous chapter we studied [72] the photon distribution of the correlated two-mode squeezed coherent states exploiting a $U(1) \times U(1)$ invariance in the problem, and brought out in detail the sensitive dependence of the photon distribution on the $U(1) \times U(1)$ - invariant relative phase between the complex squeeze and displacement parameters. That was a generalization of the earlier results of Caves *et al* [29] which were restricted to real squeeze and displacement parameters. Here we study the phase statistics of the same state.

We follow the approach of Agarwal *et al* [60] and investigate the phase properties of the correlated two-mode squeezed coherent states in terms of the phase distribution. We also write down the joint probability distribution for sum and difference phases restricted to a 2π range, following Barnett and Pegg [73]. Our aim is two-fold. Firstly we would like to look for features in the phase distribution of the correlated two-mode squeezed coherent state that may be considered to be analogous to the bifurcation phenomena predicted by Schleich *et al* [56] in the phase distribution of a single mode squeezed coherent state for large squeezing. Secondly, we would like to study the dependence of the phase distribution on the invariant relative phase between the complex squeeze and displacement parameters.

The organization of the chapter is as follows: In Section 2 we recall the definition of the phase distribution given by Agarwal *et al* [60]. We make use of the photon number matrix element of the correlated two-mode squeezed coherent state calculated in the previous chapter to obtain an expression for the phase distribution. We then write down the explicit formula for the probability distribution for sum and difference phases restricted to a 2π range. In Section 3 we discuss some special cases of the phase distribution. We discuss numerical results for the phase distributions for various values of the complex squeeze and displacement parameters. In particular we bring out the sensitive dependence of the phase distributions on the relative phase. In Section 4 we study the correlations between the phases of the two modes.

5.2 Phase distribution

We follow Agarwal *et al* [60] and define the phase distribution of a state $|\Psi\rangle$ of a two-mode field as

$$P(\theta_1, \theta_2) \;=\; \frac{1}{(2\pi)^2}\, |\langle\theta_1, \theta_2|\Psi\rangle|^2 \;,\; -\pi \le \theta_1, \theta_2 \le \pi \tag{5.1}$$

where the two-mode phase state $|\theta_1, \theta_2\rangle$ is defined to be the eigenstate of the two-mode Susskind-Glogower [74] phase operator $\hat{e} \;=\; (1 + \hat{a}^\dagger \hat{a})^{-1/2}\, \hat{a}\, (1 + \hat{b}^\dagger \hat{b})^{-1/2}\, \hat{b}$. Thus

$$|\theta_1, \theta_2\rangle \;=\; \sum_{n_1, n_2 = 0}^{\infty} e^{i(n_1\theta_1 + n_2\theta_2)}\, |n_1, n_2\rangle \;. \tag{5.2}$$

Note that the states $|\theta_1, \theta_2\rangle$ are non-normalizable and non-orthogonal. However they form a complete set. Thus

$$\frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2\, |\theta_1, \theta_2\rangle\langle\theta_1, \theta_2| \;=\; I \;. \tag{5.3}$$

The phase distribution is by definition positive, and is normalized:

$$\int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2\, P(\theta_1, \theta_2) \;=\; 1 \;. \tag{5.4}$$

It may be noted that the phase distribution calculated by (5.1) coincides with that calculated via the definition of a Hermitian phase operator in a truncated Hilbert space [55].

The correlated two-mode squeezed coherent state is unitarily related to $|vac\rangle = |0, 0\rangle$, the ground state of the two-mode system described by the annihilation operators a and b in the following manner:

$$
\begin{aligned}
|z;\alpha_1,\alpha_2\rangle &= D(\alpha_1)D(\alpha_2)S(z)|0,0\rangle, \\
S(z) &= \exp(za^\dagger b^\dagger - z^* ab), \\
D(\alpha_1) &= \exp(\alpha_1 a^\dagger - \alpha_1^* a), \; D(\alpha_2) = \exp(\alpha_2 b^\dagger - \alpha_2^* b).
\end{aligned} \tag{5.5}
$$

Here z is a complex two-mode squeeze parameter and α_1, α_2 are complex displacement (coherent excitation) parameters. Note that the $S(z)$ we have here is equivalent to $S(-z)$ in the previous chapter. We have made this change to facilitate computation in the numerical studies, later in this chapter. An alternative but equivalent definition of the above state can be given where the order of the squeeze and displacement operators is interchanged. Thus

$$
\begin{aligned}
|z;\alpha_1,\alpha_2\rangle &= S(z)D(\tilde{\alpha}_1)D(\tilde{\alpha}_2)|0,0\rangle \; ; \\
\tilde{\alpha}_1 &= \alpha_1\mu - \alpha^*{}_2\nu \; , \; \tilde{\alpha}_2 \;=\; \alpha_2\mu - \alpha^*{}_1\nu \; ; \\
z = re^{2i\phi} \; ; \; \mu &= \cosh r \; , \; \nu \;=\; e^{2i\phi}\sinh r \; .
\end{aligned} \tag{5.6}
$$

Hence the phase distribution associated with the state (5.5) or (5.6) would be, according to the definition (5.1),

$$
P(\theta_1,\theta_2) = \frac{1}{(2\pi)^2} |\langle\theta_1,\theta_2|z;\alpha_1,\alpha_2\rangle|^2 \; . \tag{5.7}
$$

The phase probability amplitude $\langle\theta_1,\theta_2|z;\alpha_1,\alpha_2\rangle$ can be expressed in terms of the photon number amplitudes:

$$\langle \theta_1, \theta_2 | z; \alpha_1, \alpha_2 \rangle = \sum_{n_1,n_2=0}^{\infty} e^{-i(n_1\theta_1+n_2\theta_2)} \langle n_1, n_2 | z; \alpha_1, \alpha_2 \rangle . \quad (5.8)$$

The structure of the photon number matrix element $\langle n_1, n_2 | z; \alpha_1, \alpha_2 \rangle$ has been worked out for real squeeze and displacement parameters by Caves *et al* [29] by using normal ordering techniques. The same for the general case of complex squeeze and displacement parameters was calculated in the previous chapter by using an identity relating the two-mode squeeze operator $S(z)$ to the product of single mode operators producing reciprocal squeezing. This is given by

$$\begin{aligned}
\langle n_1, n_2 | z; \alpha_1, \alpha_2 \rangle &= e^{i(n_1\phi_1+n_2\phi_2)} \, c(n_1, n_2), \\
c(n_1, n_2) &= \frac{1}{\cosh r} e^{-\frac{1}{2}(|\alpha_1|\mu_1+|\alpha_2|\mu_2)} \sqrt{\frac{n!}{N!}} \, (\mu_1)^{n_1-n} \, (\mu_2)^{n_2-n} \\
&\times \, (\tanh r)^n \, e^{2in\chi} \, L_n^{N-n} \left(-\frac{\mu_1\mu_2}{\tanh r} e^{-2i\chi} \right) , \\
\mu_1 &= |\alpha_1| - |\alpha_2| \, \tanh r \, e^{2i\chi}, \mu_2 = |\alpha_2| - |\alpha_1| \, \tanh r \, e^{2i\chi}, \\
\chi &= \phi - \frac{1}{2}(\phi_1 + \phi_2), \\
n &= min(n_1, n_2), \; N = max(n_1, n_2).
\end{aligned} \quad (5.9)$$

Here ϕ_1 and ϕ_2 are the phases of the displacement parameters, viz., $\alpha_1 = |\alpha_1| e^{i\phi_1}$, $\alpha_2 = |\alpha_2| e^{i\phi_2}$. Hence it follows from (5.7) and (5.8) that the phase distribution of the two-mode squeezed coherent state will be given by

$$P(\theta_1, \theta_2) = \frac{1}{(2\pi)^2} \left| \sum_{n_1,n_2=0}^{\infty} e^{-i(n_1(\theta_1-\phi_1)+n_2(\theta_2-\phi_2))} \, c(n_1, n_2) \right|^2 . \quad (5.10)$$

It is clear from the expression (5.9) for the photon number matrix element $\langle n_1, n_2|z;\alpha_1,\alpha_2\rangle$, that it depends, apart from the phase factor $e^{i(n_1\phi_1+n_2\phi_2)}$, only on the invariant linear combination of the phases, viz., $\chi = \phi - (\phi_1 + \phi_2)/2$. It may be recalled that this property is what ensured the $U(1)\times U(1)$ invariance (invariance under free time evolution) of the photon distribution of the state $|z;\alpha_1,\alpha_2\rangle$ discussed in the previous chapter.

Now let us see what happens to the phase distribution (5.10) under free time evolution. One can see that under free time evolution generated by the transformation $\exp(-i\omega t\,\hat{a}^\dagger\hat{a} - i\omega t\,\hat{b}^\dagger\hat{b})$ the phase probability amplitude undergoes the transformation $\langle n_1, n_2|z;\alpha_1,\alpha_2\rangle \;\rightarrow\; \langle\theta_1 - \omega_1 t,\ \theta_2 - \omega_2 t|z;\alpha_1,\alpha_2\rangle$.

Thus it can be seen from the above result and (5.10) that the phase distribution undergoes a rigid translation in the (θ_1,θ_2) plane under free time evolution. So if we look at the phase distribution in a frame where the origin is chosen to be $(\phi_1(t),\phi_2(t)) = (\phi_1 + \omega_1 t, \phi_2 + \omega_2 t)$, then, as is manifest from the structure of the phase distribution (5.10), $P(\theta_1,\theta_2)$ depends only on the invariant combination of angles, viz., $\chi = \phi - \frac{1}{2}(\phi_1 + \phi_2)$. With this understanding we will henceforth take α_1 and α_2 to be real (i.e., $\phi_1 = \phi_2 = 0$) without loss of generality.

As has been pointed out by Barnett and Pegg [73], though using a 4π range for the sum and difference phases is legitimate, one encounters problems with interpretation. Therefore, following their recipe we write down an explicit formula for a joint probability for the sum and difference phases, which is 2π-periodic in these phases. To this end we define

$$\begin{aligned} \theta_\pm &= \theta_1 \pm \theta_2; \\ \theta_1 &= \frac{1}{2}(\theta_+ + \theta_-), \quad \theta_2 = \frac{1}{2}(\theta_+ - \theta_-); \\ d\theta_1 d\theta_2 &= \frac{1}{2} d\theta_+ d\theta_-; \end{aligned} \tag{5.11}$$

and

$$P_{4\pi}(\theta_+, \theta_-) = \frac{1}{4} P(\theta_1, \theta_2) = \frac{1}{4} P\left(\frac{\theta_+ + \theta_-}{2}, \frac{\theta_+ - \theta_-}{2}\right). \tag{5.12}$$

Clearly, the phase distribution $P_{4\pi}(\theta_+, \theta_-)$ is periodic with period 4π in θ_+ as well as in θ_-, with fundamental region spanned by the interval $[-2\pi, 2\pi]$ in both these variables as indicated in figure 5.1 by the tilted square formed by dotted lines.

Recall that $P(\theta_1, \theta_2)$ is 2π-periodic in θ_1 as well as in θ_2, with the undotted square in figure 5.1 as the fundamental domain. We may add that the distribution $P_{4\pi}(\theta_+, \theta_-)$ does not take independent values at all points in the dotted square; for instance, it takes identical values over the two solid triangles ABC and NOP.

In order to construct a 2π periodic phase distribution $P_{2\pi}(\theta_+, \theta_-)$ from the 4π periodic distribution $P_{4\pi}(\theta_+, \theta_-)$, we use the elementary fact that if $f(x)$ is L periodic (i.e. $f(x) = f(x + L) \neq f(x + L/2)$), then $f(x) + f(x + L/2)$ is $L/2$-periodic:

$$\begin{aligned} P_{2\pi}(\theta_+, \theta_-) &= P_{4\pi}(\theta_+, \theta_-) + P_{4\pi}(\theta_+ + 2\pi, \theta_-) \\ &\quad + P_{4\pi}(\theta_+, \theta_- + 2\pi) + P_{4\pi}(\theta_+ + 2\pi, \theta_- + 2\pi). \end{aligned} \tag{5.13}$$

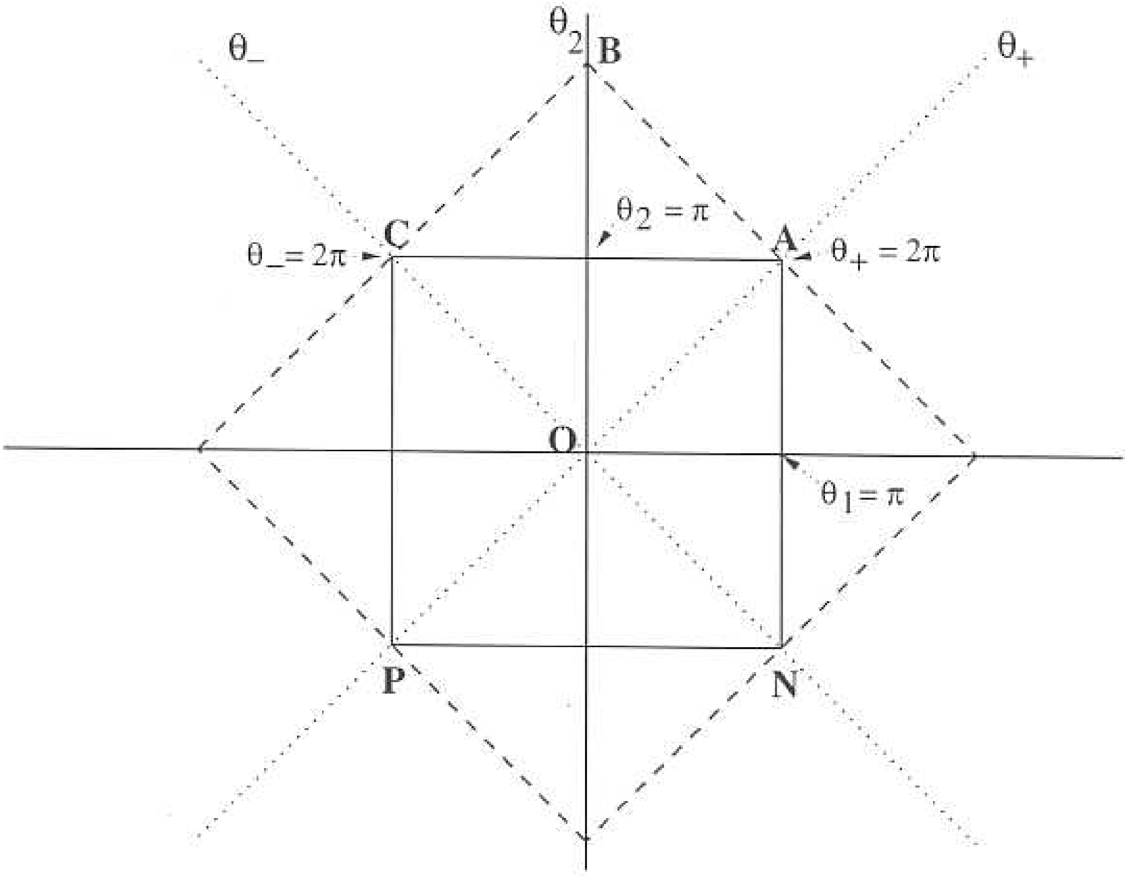


Figure 5.1: Showing the relationship between the variables (θ_1, θ_2) and (θ_+, θ_-) and the fundamental domains associated with the distributions $P(\theta_1, \theta_2)$, $P_{4\pi}(\theta_+, \theta_-)$ and $P_{2\pi}(\theta_+, \theta_-)$.

In going from $P_{4\pi}(\theta_+,\theta_-)$ to $P_{2\pi}(\theta_+,\theta_-)$, there is some loss of information as can be seen from the fact that the square OABC in figure 5.1 is the fundamental domain for $P_{2\pi}(\theta_+,\theta_-)$. Using the 2π periodicity of the original $P(\theta_1,\theta_2)$ we obtain the final expression for $P_{2\pi}(\theta_+,\theta_-)$:

$$P_{2\pi}(\theta_+,\theta_-) = \frac{1}{2}\left[P\left(\frac{\theta_+ + \theta_-}{2},\frac{\theta_+ - \theta_-}{2}\right) + P\left(\frac{\theta_+ + \theta_-}{2}+\pi,\frac{\theta_+ - \theta_-}{2}+\pi\right)\right] \quad (5.14)$$

5.3 Examples of phase distributions

In this Section we consider some special cases of the distribution $P(\theta_1,\theta_2)$ of a two-mode squeezed coherent state. We discuss numerical results for the phase distributions for various values of the complex squeeze and displacement parameters.

For the case of $r=0$, the phase distribution $P(\theta_1,\theta_2)$ given by Eqs. (5.9), (5.10) reduces to that of a two-mode coherent state

$$P(\theta_1,\theta_2) = \frac{1}{(2\pi)^2}\left|\sum_{n_1,n_2=0}^{\infty} e^{-i(n_1\theta_1+n_2\theta_2)}e^{-\frac{1}{2}(\alpha_1^2+\alpha_2^2)}\frac{\alpha_1^{n_1}}{\sqrt{n_1!}}\frac{\alpha_2^{n_2}}{\sqrt{n_2!}}\right|^2 . \quad (5.15)$$

In the other extreme case when $\alpha_1=\alpha_2=0$, the result (5.9), (5.10) for the phase distribution reduces to a closed form expression

$$P(\theta_1,\theta_2) = \frac{1}{(2\pi)^2}[\cosh 2r - \sinh 2r\ \cos(\theta_1+\theta_2-2\phi)]^{-1} , \quad (5.16)$$

corresponding to the two-mode squeezed vacuum. Note that in this case the phase distribution depends only on the sum of the two phases θ_1,θ_2. Thus the sum of the phases θ_+ is locked to the phase of the complex squeeze parameter z. One may note here a similarity between this phase distribution and the phase distributions of the pair coherent state and the entangled state, to the extent that in the latter two cases as well $P(\theta_1, \theta_2)$ depends only on the phase sum θ_+. However the analytical expressions for $P(\theta_1, \theta_2)$ corresponding to these states are entirely different [60].

An important feature of the phase distribution $P(\theta_1, \theta_2)$ given by Eqs.(5.9), (5.10) is that for $\alpha_1 = \alpha_2$, one has $P(\theta_1, \theta_2) = P(\theta_2, \theta_1)$. This follows from the fact that under the interchange of α_1 and α_2, $c(n_1, n_2)$ in Eq. (5.9) $\to c(n_2, n_1)$. So the distribution is symmetric about a diagonal in the (θ_1, θ_2) plane. This point should be borne in mind since in all the numerical results that we will discuss we have taken $\alpha_1 = \alpha_2$.

In figure 5.2 we have plotted the phase distribution $P(\theta_1, \theta_2)$ for a two-mode coherent state given by (5.15). As one would expect [60] it has a Gaussian form centred at the origin.

In figure 5.3(a) we plot the phase distribution of the two-mode squeezed state given by Eq. (5.16) for $\alpha_1 = \alpha_2 = 1$ and the phase of the squeeze parameter $\phi = 0$. Note that the distribution is symmetric with respect to the diagonals of the (θ_1, θ_2) phase window. For a different value of ϕ, as is clear from Eq. (5.16), the phase distribution merely undergoes a translation in the (θ_1, θ_2) plane. Hence, within the phase window the distribution appears to be peaked along two straight lines parallel to the main diagonal (see figure 5.3(b)).

In figure 5.4 we plot the phase distribution given by Eq.(5.16) as a function of

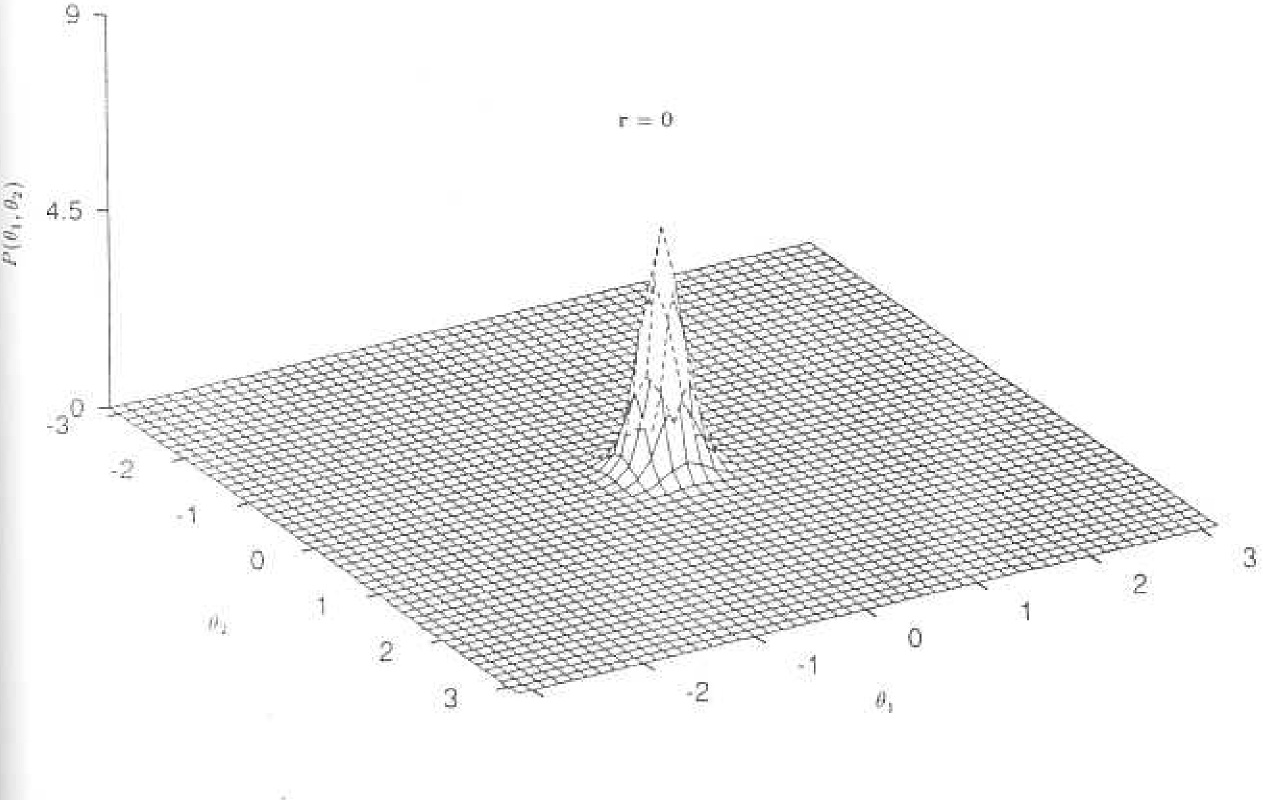
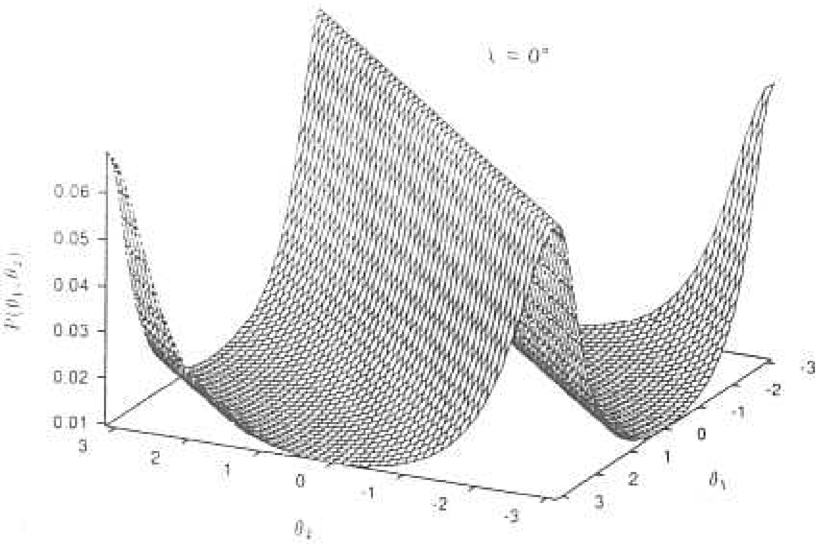


Figure 5.2: Phase distribution of the two-mode coherent state. The values of the displacement parameters are $\alpha_1 = \alpha_2 = 3.0$.



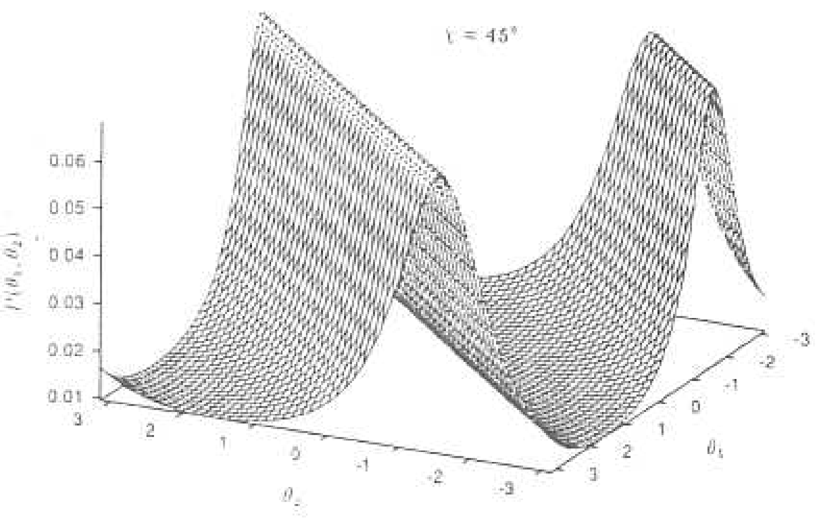


Figure 5.3: Phase distribution of the correlated two-mode squeezed state. The strength of squeezing is $r = 1.0$. The phase of the squeeze parameter is $\phi = 0°(a)$, $\phi = 45°(b)$.

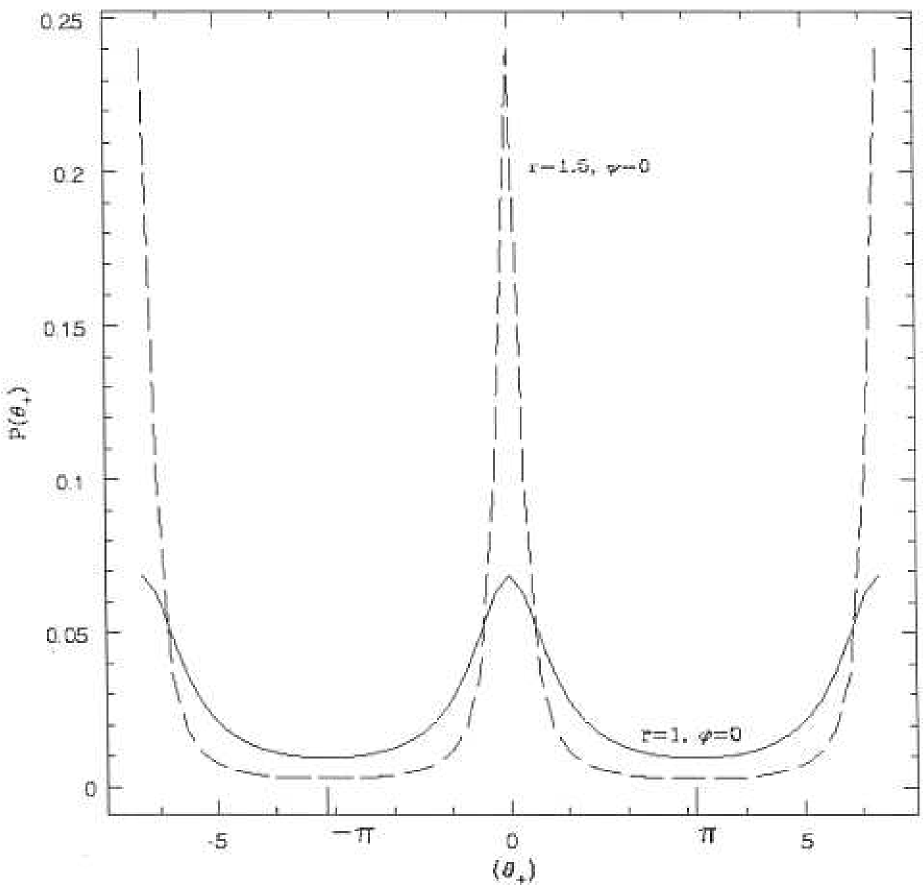


Figure 5.4: Phase distribution of the two-mode squeezed state plotted as a function of $\theta_1 + \theta_2$ for various values of r. The various plots correspond to $r = 1.0$ (solid line), 1.5 (dashed line). The phase of the squeeze parameter is $\phi = 0°$.

the phase sum θ_+ for various values of the strength of squeezing r. This amounts to looking at the section of the full distribution in a plane perpendicular to the symmetry axis. Note that the distribution in figure 5.4 becomes narrower as r increases.

In figure 5.5 we illustrate the sensitive dependence of the phase distribution of the two-mode squeezed coherent state (given by Eqs.(5.9-5.10)) on the invariant relative phase χ. For $\chi = 0$ the phase distribution is peaked along a diagonal in the (θ_1, θ_2) phase window. However as χ increases the phase distribution undergoes dramatic changes. For smaller values of χ ($\chi = 10°, 30°$), the distribution tends to bend around the axis of symmetry. For larger values of χ ($\chi = 70°, 90°$), the distribution shows some additional peaks. It is interesting to compare the figures corresponding to relative phases χ and $180° - \chi$. As can be seen from figure 5.5 these are mirror images of each other. This is a consequence of the fact that χ appears in the expression for $P(\theta_1, \theta_2)$ [Eqs.(5.9), (5.10)] only in the form $e^{2i\chi}$. For the same reason the distributions for relative phases χ and $180° + \chi$ on the other hand will be identical.

The phase distributions shown in figure 5.2 and figure 5.3 should be viewed as two extreme cases of the phase distribution of the general two-mode squeezed coherent state $|z; \alpha_1, \alpha_2\rangle$ shown in figure 5.5. The latter could be thought of as interpolating between these two extreme cases. This situation may be compared with that in the case of a single mode squeezed coherent state. As Schleich *et al* [56] have shown the phase distribution $P(\theta)$ of a single mode squeezed coherent state interpolates between

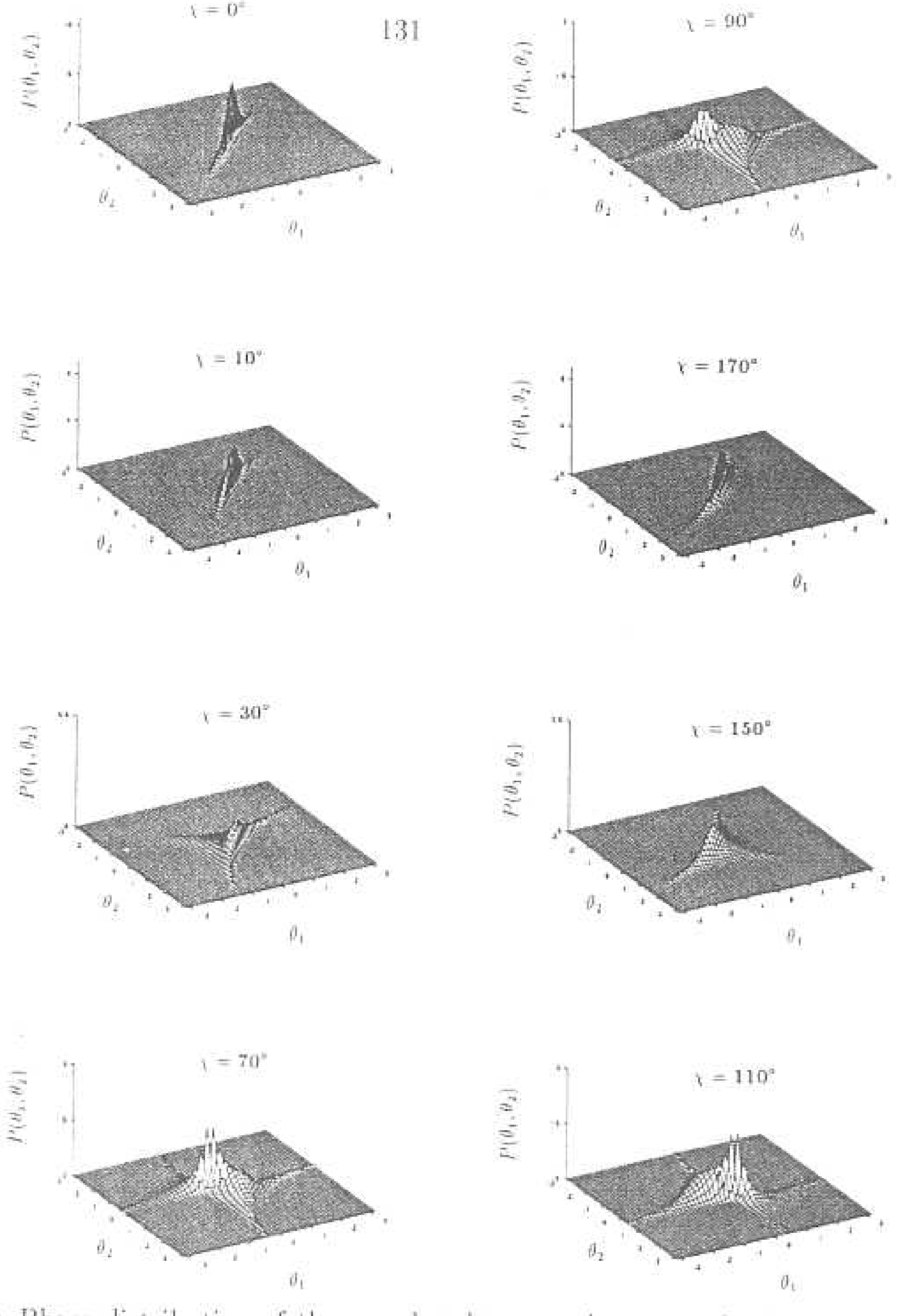


Figure 5.5: Phase distribution of the correlated two-mode squeezed coherent state. The values of the various parameters are: $\alpha_1 = \alpha_2 = 3.0$, $r = 1.5$. Different plots correspond to different values of the relative phase $\chi = 0°, 10°, 30°, 70°, 90°, 110°, 150°, 170°$.

the phase distribution of a coherent state (Gaussian centred at $\theta = 0$) and a double-peaked structure, the peaks centred at $\theta = \pm\pi/2$. We would like to point out that the phenomenon of the phase distribution of the two-mode squeezed state $P(\theta_1, \theta_2)$ figure (5.3) tending to peak along a diagonal in the (θ_1, θ_2) plane may be considered to be analogous to the above mentioned 'bifurcation' that the phase distribution of the single-mode squeezed state exhibits.

In figure 5.6 and figure 5.7 we have shown, for two different values of the relative phase χ, what happens to the distribution $P(\theta_1, \theta_2)$ as the squeezing strength r increases while keeping the other parameter values fixed. As can be seen from figure 5.6 and figure 5.7 the distribution exhibits with increasing r, a tendency towards becoming concentrated along the main diagonal in the (θ_1, θ_2) phase window. We expect that as r increases further (i.e., $r \gg \alpha_1, \alpha_2$) the phase distributions in figure 5.5 and figure 5.6 approach that of the two-mode squeezed vacuum (5.16).

5.4 Correlations between the phases

In this Section we make use of the joint probability distribution $P(\theta_1, \theta_2)$ to calculate the correlation between the phases of the two modes. The correlation is defined by

$$C_{12} = \langle\theta_1\theta_2\rangle - \langle\theta_1\rangle\langle\theta_2\rangle . \tag{5.17}$$

We saw in the previous chapter (and also [29]) that the marginal Wigner distribution corresponding to one of the modes obtained by tracing over the phase space

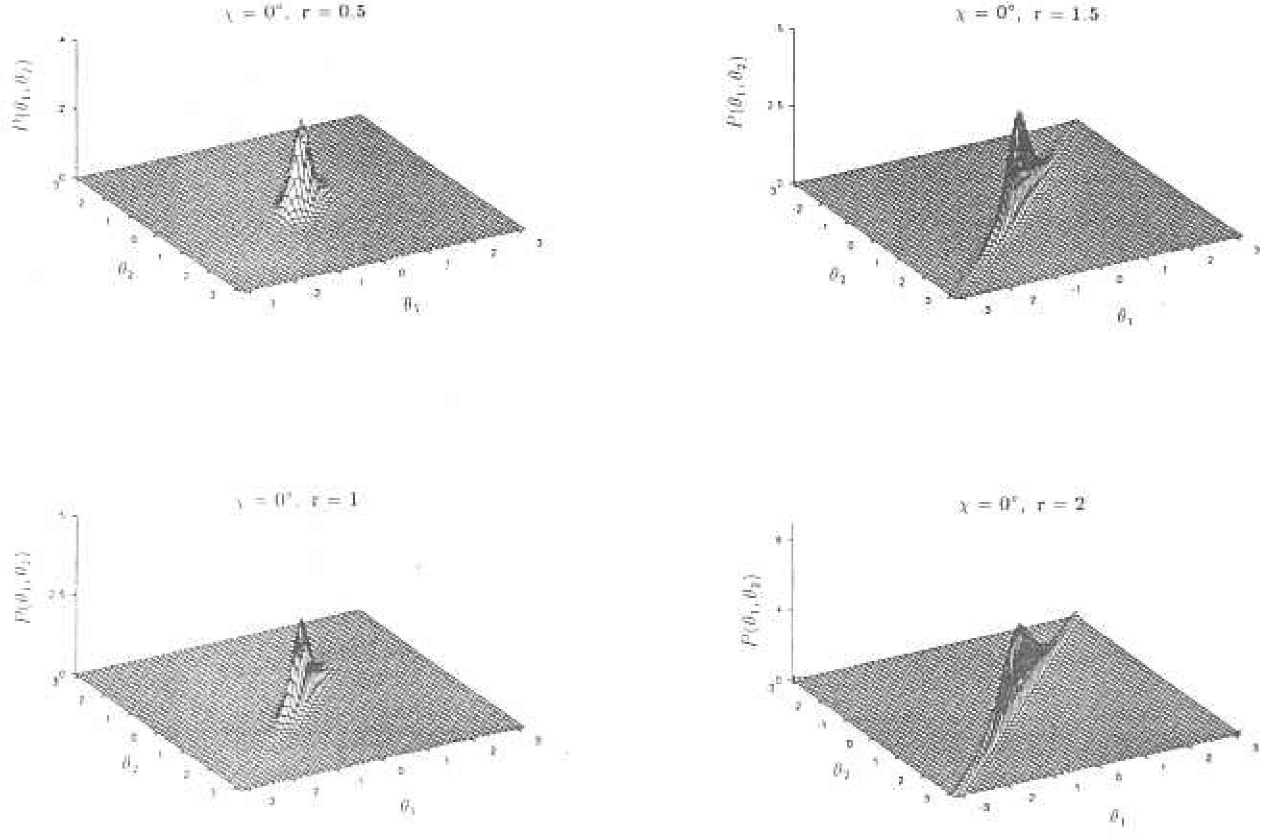


Figure 5.6: Phase distribution of the correlated two-mode squeezed coherent state. The values of the various parameters are: $\alpha_1=\alpha_2=2.0$, $\chi=0°$. Different plots correspond to different values of the squeezing strength $r=0.5$, 1.0, 1.5, 2.0.

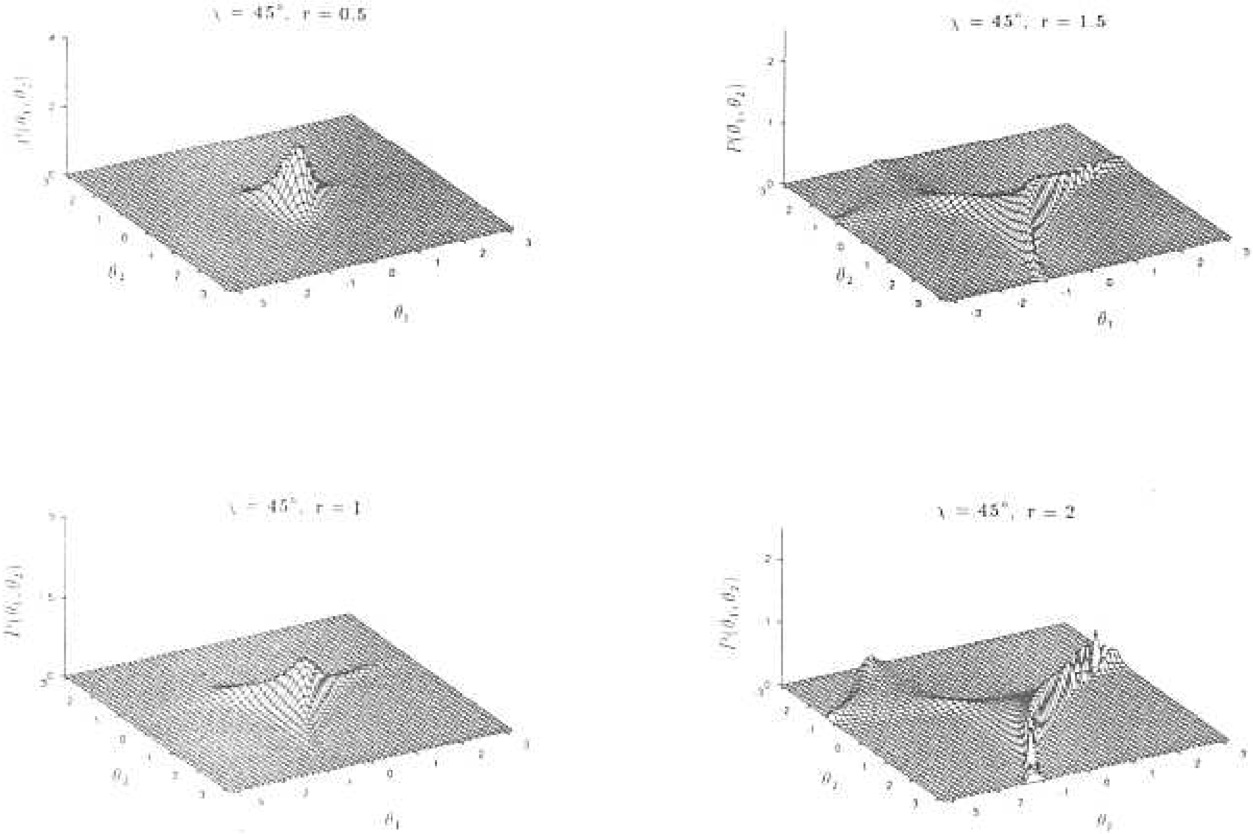


Figure 5.7: Same as in figure 5.6 but now for a different value of the relative phase, $\chi = 45°$.

coordinates of the other mode has the structure of the Wigner distribution of a displaced thermal state

$$W(\xi_1) = \int d^2\xi_2 W(\xi_1,\xi_2) = \frac{2}{\pi(1+2\bar{n})}\exp\left(-\frac{2|\xi_1-\alpha_1|^2}{1+2\bar{n}}\right), \tag{5.18}$$

where the mean photon number of the thermal state is $\bar{n} = \sinh^2 r$. The phase distribution associated with this mode is given by

$$P(\theta_1) = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{1}{1+\bar{n}}\left(\frac{\bar{n}}{1+\bar{n}}\right)^k \left|\sum_{n=0}^{\infty} e^{-in\theta_1}\langle n|D(\alpha_1)|k\rangle\right|^2, \tag{5.19}$$

where the matrix element of $D(\alpha_1)$ in the number basis, which is well-known [25] is given by

$$\begin{aligned}\langle n|D(\alpha_1)|k\rangle &= \sqrt{\frac{k!}{n!}}\alpha_1^{n-k}L_k^{n-k}(|\alpha_1|^2)e^{-\frac{1}{2}|\alpha_1|^2}\ ,\ n\geq k\\ &= \sqrt{\frac{n!}{k!}}(-\alpha_1^*)^{k-n}L_n^{k-n}(|\alpha_1|^2)e^{-\frac{1}{2}|\alpha_1|^2}\ ,\ n<k\ .\end{aligned} \tag{5.20}$$

Since we have taken α_1 to be real, it can be seen from (5.19, 5.20) that $P(\theta_1) = P(-\theta_1)$. Hence it follows that $\langle\theta_1\rangle = 0$. By a similar argument we have $\langle\theta_2\rangle = 0$.

Hence the correlation C_{12} will be given by

$$C_{12} = \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2\ \theta_1\theta_2 P(\theta_1,\theta_2)\ , \tag{5.21}$$

where $P(\theta_1,\theta_2)$ is given by (5.9, 5.10).

In figure 5.8 we plot the correlation C_{12} as a function of the relative phase χ for various values of the squeezing strength r. It is interesting to note that C_{12} changes

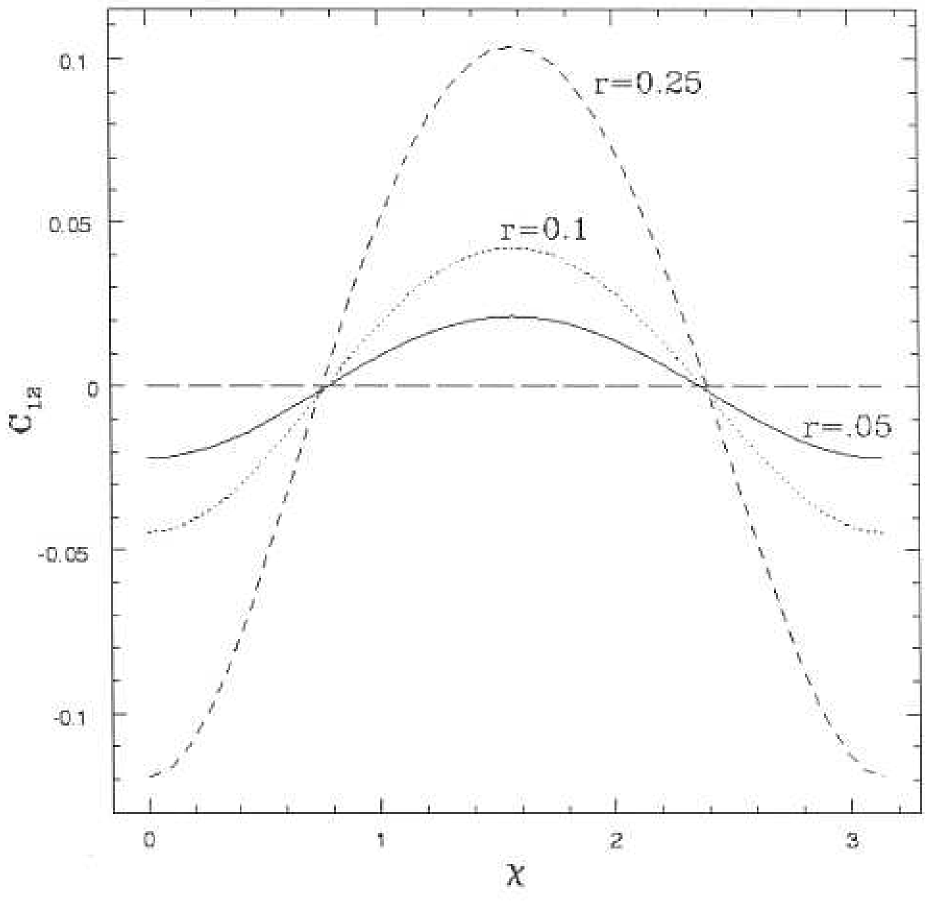


Figure 5.8: The phase correlation C_{12} plotted as a function of the relative phase χ for various values of the squeezing strength r. Here α_1=α_2=1.0 and different plots correspond to different values of the squeezing strength r = 0, 0.05, 0.1 and 0.25.

sign twice as χ is swept from $0°$ to $180°$. Further, as one would expect the correlation is zero in the absence of squeezing and as the squeezing strength r increases, the magnitude of the correlation $|C_{12}|$ also increases. It may be noted that C_{12} becoming negative in some parameter region is indicative of the nonclassical nature of light in a two-mode squeezed coherent state.

It will be interesting also to look at the variances in the sum and difference of the phases of the two modes. We can define

$$\begin{aligned} C_+ &= \langle(\theta_1+\theta_2)^2\rangle - (\langle\theta_1\rangle+\langle\theta_2\rangle)^2 , \\ C_- &= \langle(\theta_1-\theta_2)^2\rangle - (\langle\theta_1\rangle-\langle\theta_2\rangle)^2 . \end{aligned} \tag{5.22}$$

In figures 5.9 and 5.10 we have plotted the variances C_+ and C_- as a function of the squeezing strength r for various values of the displacement parameters $\alpha_1 = \alpha_2$ and for relative phase $\chi = 0$, using the 2π periodic distribution (5.14). One can see from the figures that C_+ tends to zero, i.e., the phase sum becomes less and less uncertain as either the displacement or the squeezing strength increases. In particular, for the case $\alpha_1 = \alpha_2 = 0$, it corresponds to figure 3 in Barnett and Pegg [73], whereas C_- tends to $\frac{\pi^2}{3}$ as α tends to zero, consistent with their results.

5.5 Conclusion

We have studied the phase properties of the correlated two-mode squeezed coherent states. We have calculated and illustrated graphically the phase distribution, the

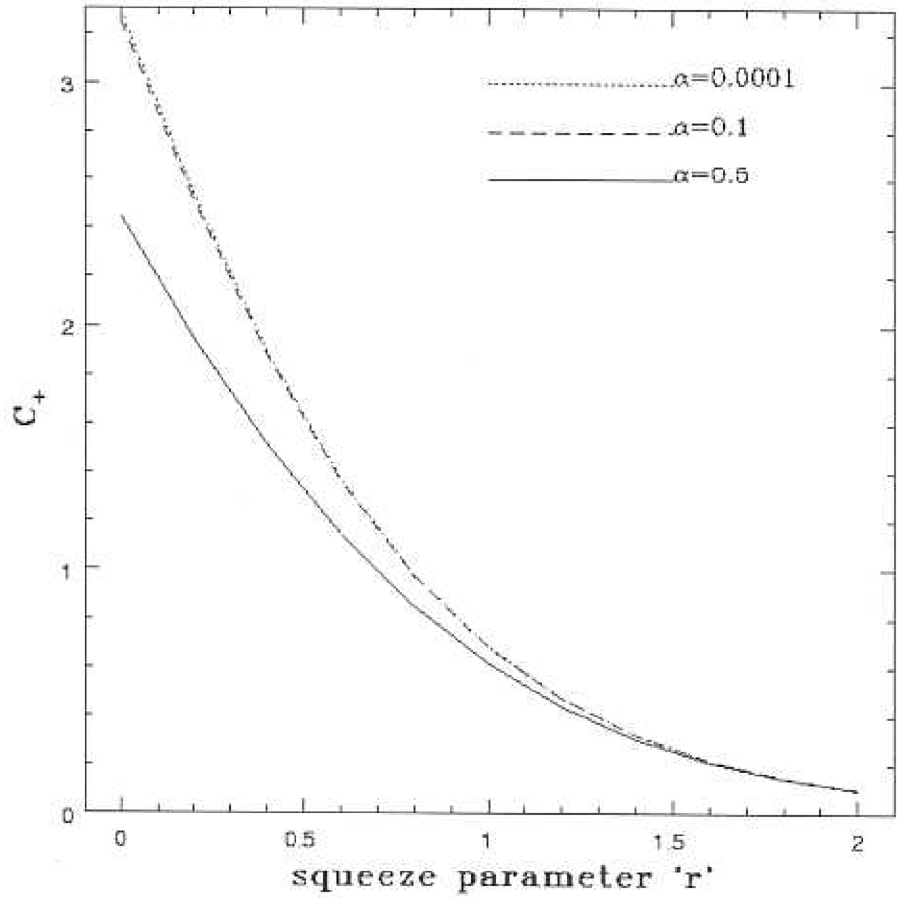


Figure 5.9: The variance C_+ of the sum θ_+ of the phases of the two modes plotted as a function of the squeezing strength r for various values of α_1=α_2 =0 (dotted line), 0.1 (dashed line), 0.5 (solid line). The value of the relative phase is $\chi = 0$.

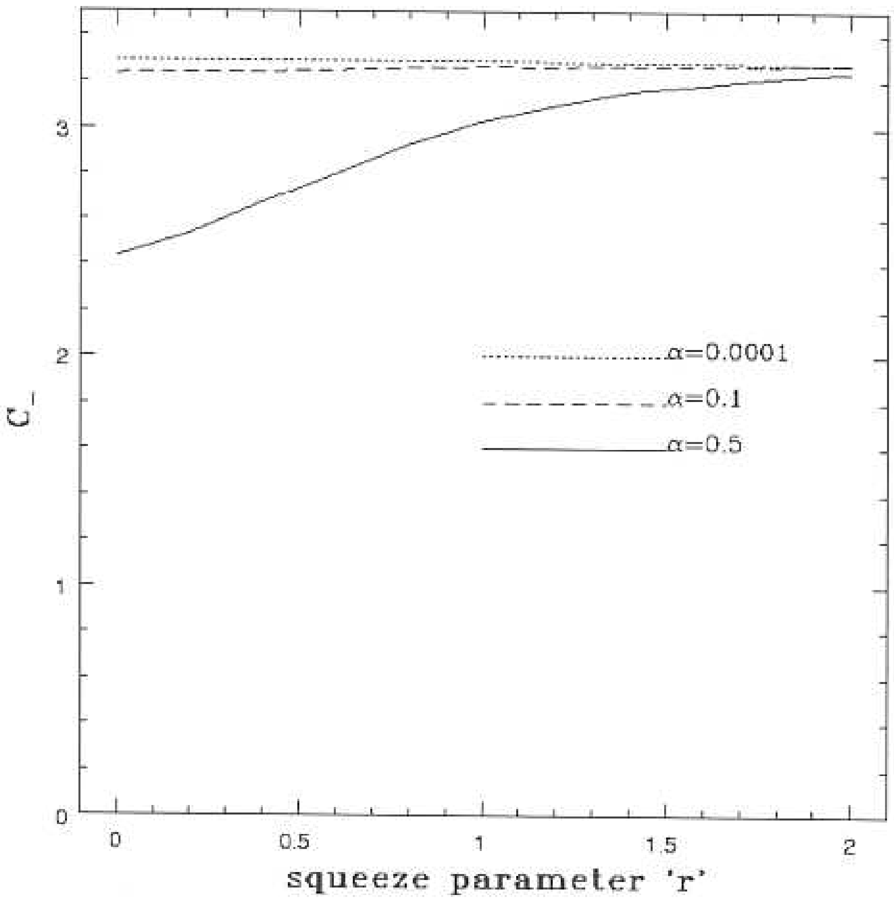


Figure 5.10: The variance C_- of the difference θ_- of the phases of the two modes plotted as a function of the squeezing strength r for various values of $\alpha_1 = \alpha_2 = 0$ (dotted line), 0.1 (dashed line), 0.5 (solid line). The value of the relative phase is $\chi = 0$.

correlation between the phases of the two modes, as well as the variances in the sum and difference of the phases of the two modes. The sensitive dependence of these quantities on the relative phase between the squeeze and displacement parameters has been brought out. We have demonstrated that the phase distribution of a two-mode squeezed coherent state exhibits phenomena analogous to the bifurcation phenomena predicted by Schleich *et al* [56] in the phase distribution of a single mode squeezed coherent state.

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