MEAN VALUE THEOREMS OF DIRICHLET SERIES

A Thesis Submitted

in Partial Fulfillment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

by

R. PADMA

THE INSTITUTE OF MATHEMATICAL SCIENCES,

C.I.T. CAMPUS, THARAMANI,

MADRAS 600 113, INDIA

March 1994

CERTIFICATE

This is to certify that the Ph. D. thesis submitted by R.PADMA, to the University of Madras, entitled MEAN VALUE THEOREMS OF DIRICHLET SERIES, is a record of bonafide work done by her during 1986-1994 under my supervision. The research work presented in this thesis has not formed the basis for the award to the candidate of any degree, diploma, associateship, fellowship or other similar title. It is further certified that the thesis represents independent work on the part of the candidate, and collaboration was necessiated by the nature and scope of the problems dealt with.

March 1994

PROF. R.BALASUBRAMANIAN
The Institute of Mathematical Sciences,
C.I.T. Campus, Tharamani,
Madras 600 113.

ACKNOWLEDGEMENTS

Thanks to **Prof.K.Ramachandra** whose ideas permeate this thesis.

Thanks to **Prof.R.Balasubramanian**, a fitting successor of Prof.Ramachandra's tradition of Analytic Number Theory which I was fortunate to learn from him.

Thanks to **Prof.M.S.Rangachari** who told me that Prof.Balasubramanian is a mathematician worth working with.

Thanks to **Amora** for her interesting conjectures, mathematical and otherwise.

Thanks to **Sandhya** who influenced me by her precision and logical thought.

Thanks are due to **Velammal** and **Venkataraman** with whom I learnt Number Theory.

Thanks to **Radha** for her enthusiastic encouragement.

Thanks to **Dr.H.S.Sharatchandra** who gave his help and advice at the crucial juncture.

Thanks to **Ms.Usha Otheeswaran** and **Ms.Parijatham** who have helped me in TEXing the thesis beyond the usual call of duty.

Finally, thanks to **G.H.Gadiyar** without whose company the thesis would have been completed in half the time.

CONTENTS

INTRODUCTION

The Riemann zeta-function $\zeta(s)$ (where $s = \sigma + it$ is a complex variable) is expressed by the Dirichlet series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ which is absolutely convergent for $\sigma > 1$. It is uniformly convergent in the region $\sigma \geq 1 + \epsilon$ and hence defines an analytic function in $\sigma > 1$. It can be analytically continued to the whole of complex plane but for a simple pole at $s = 1$. The location of complex zeros of $\zeta(s)$ plays an important role in the distribution of prime numbers. If $\pi(x)$ denotes the number of primes upto x and if Θ is the upper bound of the real parts of the zeros of $\zeta(s)$, then

$$\pi(x) = li\, x + O(x^{\Theta} log x)$$

Riemann conjectured that all the complex zeros of $\zeta(s)$ lie on the line $\sigma = \frac{1}{2}$. This is the well known '*Riemann hypothesis*'. By the Euler product representation

$$\zeta(s) = \prod_{p} (1 - \frac{1}{p^s})^{-1}$$

for $\sigma > 1$, where the product runs over all the primes, one knows that $\zeta(s) \neq 0$ in the half plane to the right of 1. It was proved by de la Vallée Poussin that $\zeta(1 + it) \neq 0$ for any t. This is equivalent to the prime number theorem, viz.,

$$\pi(x) \sim \frac{x}{log x} \quad as\, x \to \infty$$

It is not known even if $\zeta(s)$ is zero free to the right of $\sigma = 1 - \epsilon$ for some $\epsilon > 0$.

A slightly weaker hypothesis than that of Riemann is the *hypothesis of Lindelöf* which is expressed by the equation

$$\zeta(\frac{1}{2} + it) \quad = O_{\epsilon}(|t|^{\epsilon})$$

for every positive ϵ. An immediate consequence of this hypothesis is the bound

$$\int_1^T |\zeta(\frac{1}{2}+it)|^{2k}\,dt \quad = O_{\epsilon,k}(T^{1+\epsilon}) \tag{1}$$

for every positive integer k. In fact (1) is equivalent to the Lindelöf hypothesis. The Lindelöf hypothesis is also equivalent to

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = (1+o(1))\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}}T \tag{2}$$

for $\sigma > \frac{1}{2}$ and k = 1,2, ... where

$$d_k(n) = \sum_{n_1\ldots n_k=n} 1$$

is the number of ways n can be written as a product of k factors. (1) is not known to hold for k > 2 and the sharpest results on the lower bound of $\int_T^{2T}|\zeta(\sigma+it)|^{2k}\,dt$ are known for all $\sigma \geq \frac{1}{2}$, see [**B-R**] and Theorem 6.5 of [**Iv**].

A much weaker conjecture than that of Lindelöf is the '*density hypothesis*'. If $N(\sigma,T)$ denotes the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the rectangle $\beta\ \ \sigma \geq 0$, $|\gamma| \leq T$, then the density hypothesis says that

$$N(\sigma,T) \ll T^{2(1-\sigma)}(logT)^C$$

or

$$N(\sigma,T) \ll T^{2(1-\sigma)+\epsilon} \tag{3}$$

uniformly in $\frac{1}{2} \leq \sigma \leq 1$, where $C \geq 0$, and any upper bound of $N(\sigma,T)$ is known as a zero density estimate. The zero density estimates have a large number of applications in number theory. It turns out that in some problems like the estimation of the difference between consecutive primes, results obtainable from the Lindelöf (or even Riemann) hypothesis follow in almost the same degree of sharpness from the density hypothesis. It can be proved that (1) implies (3). In fact, any mean value theorem can be used to give a zero density estimate.

Using their approximate functional equation for $\zeta(s)$, G.H. Hardy and J.E. Littlewood **[H-L1]** obtained the result

$$\frac{1}{T}\int_1^T |\zeta(\frac{1}{2}+it)|^2\,dt \quad \sim logT$$

and

$$\int_{-T}^T |\zeta(\frac{1}{2}+it)|^4\,dt \;=\; O(T(logT)^4)$$

so that (1) is true for k=1 and 2.

Hardy and Littlewood **[H-L2]** further showed that $\zeta^2(s)$ has an approximate functional equation and this was used by Ingham **[In]** to prove that

$$\int_1^T |\zeta(\frac{1}{2}+it)|^4\,dt \;=\; \frac{T}{2\pi^2}(logT)^4 + O(T(logT)^3) \tag{4}$$

Now we will explain what an approximate functional equation is. We know that the Riemann zeta-function $\zeta(s)$ has the representation

$$\zeta(s) \;=\; \sum_{n=1}^{\infty}\frac{1}{n^s} \tag{5}$$

in $\sigma > 1$ and by the functional equation of $\zeta(s)$, namely,

$$\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s) \;=\; \pi^{-(\frac{1-s}{2})}\,\Gamma(\frac{1-s}{2})\,\zeta(1-s),$$

one has the representation

$$\zeta(s) \;=\; \pi^{s-1/2}\,\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\quad\sum_{n=1}^{\infty}\frac{1}{n^{1-s}} \tag{6}$$

if $\sigma < 0$. When $0 \le \sigma \le 1$, neither (5) nor (6) is valid and one can only give an approximate representation combining the two, namely,

$$\zeta(s) = \sum_{n\le x}\frac{1}{n^s} + \pi^{s-1/2}\,\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\sum_{n\le y}\frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(y^{\sigma-1}|t|^{\frac{1}{2}-\sigma}) \tag{7}$$

where $x > H > 0, y > H > 0$, $2\pi xy = |t|$ and the constants implied by the O's depend only on H. The equation (7) is an approximate functional equation of $\zeta(s)$ and is due to Hardy and Littlewood **[H-L1]**.

K. Chandrasekharan and Raghavan Narasimhan in their classical paper **[C-N]** obtained an approximate functional equation for a wide class of zeta functions.

These approximate functional equations have a disadvantage when one tries to estimate the averages of the powers of the moduli of zeta-functions in the critical strip, namely, the lengths of the sums over n depend on t. So it would be nice to have an approximate functional equation which avoids this difficulty. In this connection, there is a result due to K. Ramachandra **[Ram1]** who obtained a new smoothed approximate functional equation for $\zeta^2(s)$ and gave a simple proof of (5) using this and a theorem of H.L. Montgomery and R.C. Vaughan **[M-V]** which is a generalized Hilbert's inequality.

In this thesis, we have proved mean value theorems for a class of Dirichlet series following Ramachandra's method. This method gives a simple proof of known results such as the mean value theorem for $\zeta_K(s)$ due to K. Chandrasekharan and R. Narasimhan **[C-N]** where $\zeta_K(s)$ is the Dedekind zeta function of an algebraic number field K and the mean value theorem for $L_\tau(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$ due to A. Good **[G3]** where $\tau(n)$ is the Ramanujan's τ-function. Our method, in fact, gives an improvement of Good's result.

This thesis consists of two chapters. **Chapter1** is divided into five sections. In **Section1**, we derive an approximate functional equation for a class of Dirichlet series. In **Section2**, we give Ramachandra's proof of the theorem of Montgomery and Vaughan. In **Section3**, we prove our main theorem (Theorem 3.3) and derive the results mentioned above. In **Section4**, we prove mean value theorems for the derivatives of $\zeta(s)$, namely,

$$\int_1^T |\zeta^{(\ell)}(\frac{1}{2}+it)\,\zeta^{(m)}(\frac{1}{2}+it)|^2\,dt = CT(logT)^{2\ell+2m+4} \;+\; O(T(logT)^{2\ell+2m+3})$$

and

$$\int_1^T |\zeta^{(\ell)}(\frac{1}{2}+it)|^2\,dt = \frac{1}{(2\ell+1)}Tlog^{2\ell+1}T + O(Tlog^{2\ell}T),$$

where ℓ and $m \geq 0$ are integers and C is a constant depending on ℓ and m. Here again the proof is similar to that of Theorem3.3. In **Section5**, we prove mean value theorems concerning L-functions $L(s,\chi)$ (where χ is a character modulo q) which are uniform in both q and T. For example,the result of Hinz **[Hi]** on the mean square of the zeta function of a quadratic number field falls in as a special case of Theorem 5.2.

In **Chapter2**, we give a brief survey of zero density estimates.

CHAPTER 1

MEAN VALUE THEOREMS

Section 1. An approximate functional equation

The zeta- function $\zeta(s)$ (where $s = \sigma + it$) of Riemann is represented by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \sigma > 1 \tag{1.1}$$

It satisfies the functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$$

using which one obtains the representation

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s) \quad \text{for } \sigma < 0 \tag{1.2}$$

When $0 \leq \sigma \leq 1$, the representation (1.1) or (1.2) is not valid and one can only give an approximate representation combining the two, and one such is given by

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \sum_{n \leq y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(y^{\sigma-1}|t|^{\frac{1}{2}-\sigma})$$

where $x > H > 0, 2\pi xy = |t|$ and the constants implied by the O's depend only on H. The formula written above is due to Hardy and Littlewood **[H-L1]** and is known as the approximate functional equation of $\zeta(s)$. Hardy and Littlewood applied their approximate functional equation to estimate the order of the magnitude of the mean square and the mean fourth power of $\zeta(s)$ on the "critical line" and proved the results

$$\int_1^T |\zeta(\frac{1}{2} + \mathrm{it})|^2 \, \mathrm{dt} \sim T\log T$$

and

$$\int_1^T |\zeta(\frac{1}{2}+\mathrm{it})|^4\,\mathrm{dt} = O(T(\log T)^4)$$

They [H-L2] further showed that the square of the zeta-function $\zeta^2(s)$ which obviously satisfies the functional equation

$$\pi^{-s}\Gamma^2(\frac{s}{2})\zeta^2(s) = \pi^{-(1-s)}\Gamma^2(\frac{1-s}{2})\zeta^2(1-s)$$

also has an approximate functional equation, namely

$$\zeta^2(s) = \sum_{n\leq\frac{|t|}{2\pi}} \frac{d(n)}{n^s} + \pi^{2s-1}\frac{\Gamma^2(\frac{1-s}{2})}{\Gamma^2(\frac{s}{2})}\sum_{n\leq\frac{|t|}{2\pi}}\frac{d(n)}{n^{1-s}} + O\left(|t|^{\frac{1}{2}-\sigma}\log(|t|+2)\right)$$

where $d(n)$ denotes the number of divisors of n. A.E. Ingham [In] used this result to prove that

$$\int_1^T |\zeta(\frac{1}{2}+\mathrm{it})|^4\,\mathrm{dt} = \frac{T}{2\pi^2}(\log \mathrm{T})^4 + O(\mathrm{T}(\log\mathrm{T})^3) \tag{1.3}$$

K. Chandrasekaran and Raghavan Narasimhan in their paper [C-N] obtained an approximate functional equation for a wide class of zeta-functions, a typical instance is the Dedekind zeta-function $\zeta_K(s)$ of an algebraic number field K, and used it to prove results on the mean square of $\zeta_K(s)$.

The approximate functional equations of these types have a disadvantage when one tries to estimate the averages of the powers of moduli of zeta-functions in the critical strip, the disadvantage being the lengths of the sums over n depend on t and hence it would be nice to have an approximate functional equation which avoids this difficulty.

In this connection, there is a result due to K. Ramachandra [Ram1] who obtained a new approximate functional equation for $\zeta^2(s)$ and used it to give a simple proof of Ingham's result on the mean fourth power of $\zeta(s)$, namely (1.3). His proof essentially uses the fact that $\zeta^2(s)$ satisfies a functional equation. Following his method, we have obtained an approximate functional equation, namely (1.7), for a class of Dirichlet series which satisfy a functional equation.

Before stating the result, we will introduce the notation and furnish the conditions required.

Notation.

We write

$s = \sigma + it$, $w = u + iv$,

$\int_{(a)} = \int_{a-i\infty}^{a+i\infty}$, and $\int_{(a,T)} = \int_{a-iT}^{a+iT}$

Assumptions .

Let $\{\lambda_n\}$ and $\{\lambda_n^*\}$ be two sequences of real numbers such that

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$$

$$0 < \lambda_1^* < \lambda_2^* < \ldots < \lambda_n^* \to \infty.$$

Let the Dirichlet series $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}$ and $F^*(s) = \sum_{n=1}^{\infty} \frac{a_n^*}{\lambda_n^{*s}}$ and have abscissae of absolute convergence 1. Assume that $F(s)$ and $F^*(s)$ satisfy a functional equation of the form

$$F(s) = \psi(s)F^*(1-s) \tag{1.4}$$

where the function $\psi(s)$ is analytic and

$$\psi(s) = O(|s|^{A_0} + 1) \tag{1.5}$$

whenever $\sigma_1 \leq \sigma \leq \sigma_2$. Also suppose that $F(s)$ is of finite order, i.e.,

$$F(s) = O((|s| + 2)^{A_1}) \tag{1.6}$$

in every fixed vertical infinite strip where A_1 is a constant depending only on the strip. Let $F(s)$ have at most a pole at $s = 1$ with residue R_F. Then we have,

Theorem 1.1 Let $\frac{1}{2} \leq \sigma \leq 1$ and $h = 1$ or 2. Then with $0 < \beta < 1$, $h > \eta \geq \sigma + \frac{1}{100}$ to be suitably chosen, and $2 \leq x, y \leq T^{A_2}$, we have

$$\begin{aligned} F(s) &= \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} e^{-(\frac{\lambda_n}{x})^h} + \psi(s) \sum_{\lambda_n^* \leq y} \frac{a_n^*}{\lambda_n^{*1-s}} \\ &\quad - \frac{1}{2\pi i h} \int_{(-\eta)} \psi(s+\omega) \left(\sum_{\lambda_n^* > y} \frac{a_n^*}{\lambda_n^{*1-s-\omega}} \right) \Gamma(\frac{\omega}{h}) x^\omega d\omega \\ &\quad - \frac{1}{2\pi i h} \int_{(\beta)} \psi(s+\omega) \left(\sum_{\lambda_n^* \leq y} \frac{a_n^*}{\lambda_n^{*1-s-\omega}} \right) \Gamma(\frac{\omega}{h}) x^\omega d\omega + O(T^{-10}), \quad (1.7) \end{aligned}$$

where the last term appears only when $F(s)$ has a pole at $s = 1$

Note. When we calculate the mean value estimates on the critical line, we usually split the first sum in (1.7) and write it as follows:

$$\sum_{\lambda_n \leq x} \frac{a_n}{\lambda_n^s} + \sum_{\lambda_n \leq x} \frac{a_n}{\lambda_n^s} (e^{-(\frac{\lambda_n}{x})^h} - 1) + \sum_{\lambda_n > x} \frac{a_n}{\lambda_n^s} e^{-(\frac{\lambda_n}{x})^h}.$$

Proof. By the Mellin inversion of the gamma function, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{(2)} F(s+w) \Gamma(\frac{\omega}{h}) x^w dw &= \frac{1}{2\pi i} \int_{(2)} (\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}) \Gamma(\frac{\omega}{h}) (\frac{x}{\lambda_n})^w dw \\ &= \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} \frac{1}{2\pi i} \int_{(2)} (\frac{\lambda_n}{x})^{-w} \Gamma(\frac{\omega}{h}) dw \\ &= h \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} e^{-(\lambda_n/x)^h} \end{aligned}$$

where the interchange of the order of integration and summation is justified, as

$$|\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} \frac{1}{2\pi i} \int_{(2)} (\frac{x}{\lambda_n})^w \Gamma(\frac{\omega}{h}) dw| \leq x^2 \sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n^{2+\sigma}} \int_{(2)} |\Gamma(\frac{\omega}{h})| dw$$

$$\ll \; x^2,$$

since the series is absolutely convergent and the gamma function is absolutely integrable. Applying the residue theorem to the rectangle with vertices $2 - iT_1$, $2 + iT_1, -\eta + iT_1,$ and $-\eta - iT_1$, we get,

$$\begin{aligned} & \frac{1}{2\pi i}\int_{(2,T_1)} F(s+w)\Gamma(\frac{\omega}{h})x^{\omega}dw \\ = \;& \textit{sum of the residues of } F(s+w)x^w\Gamma(\frac{\omega}{h}) \textit{ at } w=0 \\ & \textit{and (possibly) at } w = 1-s \\ & +\frac{1}{2\pi i}\left\{\int_{(-\eta,T_1)} + \int_{-\eta+iT_1}^{2+iT_1} - \int_{-\eta-iT_1}^{2-iT_1}\right\} F(s+w)x^w\Gamma(\frac{\omega}{h})dw \end{aligned}$$

The last two integrals on the right-hand side are

$$\ll (T+T_1)^{A_1}\, e^{-T_1}x^2/\log x$$

and tend to 0 as T_1 tends to ∞. Here we have used the Stirling's formula for the gamma function and (1.6).

Residue of $F(s+w)x^w\Gamma(\frac{w}{h})$ at $(w=0) \;=\; hF(s)$ and

the residue of $F(s+w)x^w\Gamma(\frac{w}{h})$ at $(w=1-s) \;=\; O(T^{-10})$,

again by Stirling's formula. Hence we have

$$\begin{aligned} \frac{1}{2\pi i}\int_{(2)} F(s+w)x^w\,\Gamma(\frac{\omega}{h})\,dw \;&=\; hF(s) + \frac{1}{2\pi i}\int_{(-\eta)} F(s+w)x^w\,\Gamma(\frac{\omega}{h})dw \\ & \quad +O(T^{-10}) \qquad\qquad (1.8) \end{aligned}$$

where the last term appears only when $F(s)$ has a pole at $s=1$.

By the functional equation (1.4) of $F(s)$, we have

$$F(s+w) = \psi(s+w)F^*(1-s-w)$$

which can be written as

$$= \psi(s+w)\sum_{n=1}^{\infty}\frac{a_n^*}{\lambda_n^{*1-s-w}}$$

by choosing $h > \eta \geq \sigma + \frac{1}{100}$. Using this, we can write

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(-\eta)} F(s+w)\Gamma(\frac{\omega}{h})\, x^w dw \\
&= \frac{1}{2\pi i}\int_{(-\eta)} \psi(s+w)\sum_{n=1}^{\infty}\frac{a_n^*}{\lambda_n^{*1-s-w}}\Gamma(\frac{\omega}{h})x^w dw \\
&= \frac{1}{2\pi i}\int_{(-\eta)} \psi(s+w)\left(\sum_{\lambda_n^*\leq y}\frac{a_n^*}{\lambda_n^{*1-s-w}}\right)\Gamma(\frac{\omega}{h})x^w dw \\
&\quad + \frac{1}{2\pi i}\int_{-(\eta)} \psi(s+w)\sum_{\lambda_n^*>y}\frac{a_n^*}{\lambda_n^{*1-s-w}}\Gamma(\frac{\omega}{h})x^w dw. \qquad (1.9)
\end{aligned}$$

Now we will move the line of integration of the first integral on the right hand side of (1.9) to $u = \beta$. Then, by applying the residue theorem, we get

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(-\eta,T_2)} \psi(s+w)\left(\sum_{\lambda_n^*\leq y}\frac{a_n^*}{\lambda_n^{*1-s-w}}\right)\Gamma(\frac{\omega}{h})x^\omega d\omega \\
&= -h\psi(s)\sum_{\lambda_n^*\leq y}\frac{a_n^*}{\lambda_n^{*1-s-w}} - \frac{1}{2\pi i}\left(\int_{-\eta+iT_2}^{\beta+iT_2} - \int_{-\eta-iT_2}^{\beta-iT_2} - \int_{\beta-iT_2}^{\beta+iT_2}\right) \\
&\quad \left(\psi(s+w)\sum_{\lambda_n^*\leq y}\frac{a_n^*}{\lambda_n^{*1-s-w}}\Gamma(\frac{\omega}{h})x^\omega d\omega\right) \qquad (1.10)
\end{aligned}$$

Arguing as before, we can show that the integrals

$$\int_{-\eta+iT_2}^{\beta+iT_2} \text{ and } \int_{-\eta-iT_2}^{\beta-iT_2}$$

tend to 0 as T_2 tends to ∞. Combining (1.8), (1.9) and (1.10), we get (1.7).

Section 2. Mean Value Theorem for Dirichlet polynomials

The mean value theorem for Dirichlet polynomials is a very useful tool in analytic number theory. The approximate functional equation we obtained in Section 1 combined with this gives a simple proof of the mean value theorems of the powers of the Riemann zeta-function, Dedekind zeta-function, L-series and so on.

Of the mean value theorems for Dirichlet polynomials, the oldest states that, for any $T > 0$ and $a_1, \ldots a_N$ arbitrary complex numbers,

$$\int_0^T |\sum_{n\leq N} \frac{a_n}{n^{it}}|^2 dt = (T + O(N\log N))(\sum_{n\leq N} |a_n|^2)$$

This result was sharpened by Montgomery (Theorem 6.1, [M3]) as follows.

$$\int_0^T |\sum \frac{a_n}{n^{it}}|^2 dt = \left(T + \frac{4\pi}{\sqrt{3}}\theta N\right) \sum_{n\leq N} |a_n|^2$$

where $|\theta| \leq 1$.

This was further sharpened by Montgomery and Vaughan [M-V] as

$$\int_0^T |\sum_{n\leq N} \frac{a_n}{n^{it}}|^2 dt = \sum_{n\leq N} |a_n|^2 (T + O(n)) \tag{2.1}$$

One can take here $N = \infty$ also, provided that both sides are convergent. This result is a particular case of the generalized Hilbert's inequality they proved, namely, if $\lambda_1, \ldots \lambda_N$ are arbitrary complex numbers and $\delta_n = \min_{n\neq m} |\lambda_n - \lambda_m|$, then

$$|\sum_{n\neq m} \frac{a_n \bar{a_m}}{\lambda_n - \lambda_m}| \leq \frac{3\pi}{2} \sum_n |a_n|^2 \delta_n^{-1} \tag{2.2}$$

The case $\lambda_n = n$ is known as the Hilbert's inequality . The inequality (2.2) is closely connected with the large sieve-type inequalities. We don't need this result in this most general set-up; $\lambda_n = \log(n + \alpha)$, where $0 \leq \alpha \leq 1$, is sufficient for our purpose.

K. Ramachandra **[Ram2]** has given a simple proof of (2.2) in this case and the underlying idea of his proof is to reduce to the classical case $\lambda_n = n$.

Theorem 2.1. Suppose $N \geq 2, L_n = \log(n+\alpha), 0 \leq \alpha \leq 1$ is fixed and $n = 1, 2, \ldots N$. Let $a_1, \ldots a_N, b_1, \ldots b_N$ be arbitrary complex numbers. Then we have

$$\left| \sum\sum_{m \neq n} \frac{a_m b_n}{L_m - L_n} \right| \leq C \left(\sum_n n|a_n|^2 \right)^{1/2} \left(\sum_n n|b_n|^2 \right)^{1/2}$$

where C is an absolute numerical constant which is effective.

We need the following.

Lemma 2.1. If $\{q_n\}$ is a sequence of distinct integers, a_n and b_n are complex numbers, then

$$\text{(i)} \qquad \sum\sum_{\substack{n \neq m \\ n, m \leq N}} \frac{a_n \bar{a}_m}{q_n - q_m} \leq \pi \sum_{n \leq N} |a_n|^2 \tag{2.3}$$

and

$$\text{(ii)} \qquad \sum\sum_{n \neq m} \frac{a_n \bar{b}_m}{q_n - q_m} \ll \left(\sum_n |a_n|^2 \right)^{1/2} \left(\sum_n |b_n|^2 \right)^{1/2} \tag{2.4}$$

Proof of (i).

Let

$$E = \sum_{\substack{n \neq m \\ n, m \leq N}} \frac{a_n \bar{a}_m}{q_n - q_m}$$

Clearly E is purely imaginary, i.e., E/i is real. Starting with

$$0 \leq \int_0^1 \int_0^y \left| \sum a_n e(q_n x) \right|^2 dx dy$$

$$= \frac{1}{2}\sum_{n\leq N}|a_n|^2 + \int_0^1 \sum a_n\bar{a}_m \frac{e((q_n-q_m)y)-1}{2\pi i(q_n-q_m)}\,dy$$

$$= \frac{1}{2}\sum_{n\leq N}|a_n|^2 - \frac{E}{2\pi i}, \tag{2.5}$$

where $e(x) = e^{2\pi i x}$ and we have used

$$\int_0^1 e(mx)dx = \begin{cases} 1 & \text{if} \quad m=0 \\ 0 & if \quad m\neq 0, m\epsilon\mathbb{Z} \end{cases}, \tag{2.6}$$

we have

$$\frac{E}{2\pi i} \leq \frac{1}{2}\sum |a_n|^2 \tag{2.7}$$

When $\frac{E}{i} > 0$, (2.5) gives (2.3). If $\frac{E}{i} < 0$, then starting with the sum $\sum a_n e(-q_n x)$, we can prove the same.

Proof of (ii).

Starting with

$$\int_0^1\int_0^y \sum_n a_n e(q_n x)\sum_n \bar{b}_n e(-q_n x)\,dx\,dy$$
$$= \frac{1}{2}\sum_n a_n\bar{b}_n + \int_0^1 \sum\sum_{n\neq m} a_n\bar{b}_m \frac{e((q_n-q_m)y)-1}{2\pi i(q_n-q_m)}\,dy$$

and using (2.6) we get

$$\frac{1}{2\pi i}\sum\sum_{n\neq m}\frac{a_n\bar{b}_m}{q_n-q_m} = \frac{1}{2}\sum_n a_n\bar{b}_n - \int_0^1\int_0^y \sum_n a_n e(q_n x)\sum_m \bar{b}_m e(-q_m x)dxdy \tag{2.8}$$

Applying Hölder's inequality twice and using (2.5) and (2.7) we get

$$|\int_0^1\int_0^y \sum_n a_n e(q_n x)\sum_m \bar{b}_m e(-q_m x)\,dxdy| \leq \left(\sum_n |a_n|^2\right)^{1/2}\left(\sum_m |b_m|^2\right)^{1/2}. \tag{2.9}$$

Also,

$$\sum_n a_n \bar{b}_n \le \left(\sum_n |a_n|^2\right)^{1/2} \left(\sum_n |b_n|^2\right)^{1/2} \tag{2.10}$$

by Cauchy - Schwarz inequality. Now (2.4) follows from (2.8), (2.9) and (2.10).

Proof of Theorem 2.1. Divide the range $[1, N+1]$ into subintervals $I_j = \left[\frac{N+1}{2^j}, \frac{N+1}{2^{j-1}}\right], j = 1, 2, \ldots$ Denote $m + \alpha$ by m' and let

$$G = \sum\sum_{m \neq n} \frac{a_m \bar{b}_n}{L_m - L_n}$$

Starting with

$$\int_0^1 \int_0^y \left(\sum_m a_m e(L_m x)\right) \left(\sum_n \bar{b}_n e(-L_n x)\right) dx\, dy$$
$$= \frac{1}{2} \sum_m a_m \bar{b}_m - \frac{G}{2\pi i} + \frac{1}{2\pi i} \int_0^1 \sum_{n,m} a_m \bar{b}_n \frac{e(L_m - L_n)y}{L_m - L_n}\, dy$$

we have the inequality

$$|\frac{G}{2\pi i}| \le \frac{1}{2} |\sum a_m \bar{b}_m| + \frac{1}{2\pi} \sum_{k,\ell \ge 1} \left| \sum_{(m',n') \epsilon I_k \times I_\ell} \int_0^1 \frac{a_m \bar{b}_n e(L_m - L_n)y}{L_m - L_n} dy \right|$$

$$+ \left(\int_0^1 \int_{-y}^y |\sum a_m\, e(L_m x)|^2 dx\, dy\right)^{1/2}$$
$$\left(\int_0^1 \int_{-y}^y |\sum \bar{b}_n\, e(-L_n x)|^2 dx\, dy\right)^{1/2}$$

$$= \sum_1 + \sum_2 + (\sum_3)^{1/2} (\sum_4)^{1/2}. \tag{2.11}$$

Since $|\int_0^1 e^{ixt} dx| = |\frac{e^{it} - 1}{t}| \le \frac{2}{|t|}$ for $0 \neq t$ real, for $|k - \ell| \ge 2$, we have

$$\sum_{(m',n') \epsilon I_k \times I_\ell} \int_0^1 a_m \bar{b}_n \frac{e(L_m - L_n)y}{L_m - L_n}\, dy$$

$$
\begin{aligned}
&\ll \sum_{(m',n')\epsilon I_k\times I_\ell} |a_m||b_n| \max_{\substack{(m',n')\\ \epsilon I_k\times I_\ell}} \frac{1}{(L_m-L_n)^2} \\
&\ll \frac{1}{(k-\ell)^2} \sum_{(m',n')\epsilon I_k\times I_\ell} |a_m||b_n| \\
&\le \frac{1}{(k-\ell)^2} \left(\sum_{(m',n')\epsilon I_k\times I_\ell} |a_m|^2\right)^{1/2} \left(\sum_{(m',n')\epsilon I_k\times I_\ell} |b_n|^2\right)^{1/2} \\
&\le \frac{1}{(k-\ell)^2} \frac{N+1}{2^{\frac{\ell+k}{2}}} \left(\sum_{m'\epsilon I_k} |a_m|^2\right)^{1/2} \left(\sum_{n'\epsilon I_\ell} |b_n|^2\right)^{1/2} \\
&\ll \frac{1}{(k-\ell)^2} S_k^{1/2} T_\ell^{1/2}
\end{aligned}
$$

where $S_k = \left(\sum_{n'\epsilon I_k} n|a_n|^2\right)$ and $T_\ell = \sum_{m'\epsilon I_\ell} m|b_m|^2$ Here we have used the Cauchy-Schwarz inequality and

$$
\begin{aligned}
|L_m - L_n| &= |\log m' - \log n'| \\
&\ge \log\tfrac{N+1}{2^k} - \log\tfrac{N+1}{2^{\ell-1}} \\
&= (\log 2)(\ell - k - 1) \\
&\ge \tfrac{1}{4}(\ell - k), \quad \text{if } \ell - k \ge 2
\end{aligned}
$$

Similarly one can prove this when $k - \ell \ge 2$. Applying Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
\sum_{|k-\ell|\ge 2} \frac{1}{(k-\ell)^2} S_k^{1/2} T_\ell^{1/2} &\le \left(\sum_{k\ne\ell} \frac{S_k}{(k-\ell)^2}\right)^{1/2} \left(\sum_{k\ne\ell} T_\ell \frac{1}{(k-\ell)^2}\right)^{1/2} \\
&\ll \left(\sum_n n|a_n|^2\right)^{1/2} \left(\sum_n n|b_n|^2\right)^{1/2},
\end{aligned}
$$

since $\sum_{k\neq\ell} \frac{1}{(k-\ell)^2}$ converges, so that the contribution of $\sum_{|k-\ell|\geq 2}$ to (2.11) is of the desired magnitude. The terms in $\sum_2$ for which $|k-\ell| \leq 1$ may be written as

$$\begin{aligned} &\int_0^1 \sum_{(m',n')\epsilon I_k \times I_\ell} \frac{a_m \bar{b}_n e(L_m - L_n)y}{L_m - L_n} dy \\ = \; & M \int_0^1 \sum_{\substack{(m',n')\epsilon I_k \times I_\ell \\ m\neq n}} \frac{a'_m \bar{b}'_n}{ML_m - ML_n} dy \end{aligned}$$

where $a'_m = a_m e(L_m y), b'_m = b_m e(L_m y)$, and $M = (N+1)2^{6-k}$ and it will be sufficient to majorize the last sum. The reason for introducing M is that if $|k-\ell| \leq 1$, then for $m > n$ and $(m', n') \epsilon I_k \times I_\ell$,

$$\begin{aligned} [ML_m] - [ML_n] &\geq M(L_m - L_n) - 2 \\ &= M\log(1 + \frac{m-n}{n'}) - 2 \\ &\geq \frac{2^{6-k}(N+1)(m-n)}{3(N+1)2^{2-k}} - 2 \\ &\geq \frac{16}{3}(m-n) - 2 \\ &\geq m - n \end{aligned}$$

since $m - n \geq 1, 0 \leq \frac{m-n}{n'} \leq 3$ and $\log(1+x) \geq \frac{x}{3}$ for $0 \leq x \leq 3$. Therefore we have for $|k-\ell| \leq 1$,

$$\begin{aligned} & M \sum_{(m',n')\epsilon I_k \times I_\ell} \frac{a'_m \bar{b}'_n}{ML_m - ML_n} \\ = \; & M \sum_{(m',n')\epsilon I_k \times I_\ell} \frac{a'_m \bar{b}'_n}{ML_m - ML_n} \\ & + M \sum \frac{a'_m \bar{b}'_n [\{ML_m - [ML_m]\} - \{ML_n - [ML_n]\}]}{(ML_m - ML_n)([ML_m] - [ML_n])} \end{aligned}$$

$$= M\sum \frac{a'_m \bar{b}'_n}{[ML_m]-[ML_n]} + O\left(M \sum_{\substack{(m',n')\epsilon I_k X I_\ell \\ m\neq n}} \frac{|a_m||b_n|}{(m-n)^2} \right) \tag{2.12}$$

The O term in (2.12) is

$$\ll M \left(\sum_{\substack{(m',n')\epsilon I_k \times I_\ell \\ m\neq n}} \frac{|a_m|^2}{(m-n)^2} \right)^{1/2} \left(\sum_{\substack{(m',n')\epsilon I_k \times I_\ell \\ m\neq n}} \frac{|b_n|^2}{(m-n)^2} \right)^{1/2}$$

and

$$\sum_{(m',n')\epsilon I_k \times I_\ell} \frac{|a_m|^2}{(m-n)^2} = \sum_{m\epsilon I_k} |a_m|^2 \sum_{\substack{m'\epsilon I_\ell \\ m\neq n}} \frac{1}{(m-n)^2} \ll \sum_{m'\epsilon I_k} |a_m|^2$$

as for a fixed m, $\sum_{\substack{m'\epsilon I_\ell \\ n\neq m}} \frac{1}{(m-n)^2} \ll 1$.

Also,

$$\sum_{m'\epsilon I_k} |a_m|^2 \leq \frac{2^k}{N+1} \sum_{m'\epsilon I_k} m' |a_m|^2$$

Hence the O-term in (2.12) is

$$\ll M . \frac{2^{\frac{k+\ell}{2}}}{N+1} S_k^{1/2} T_\ell^{1/2}$$

$$\ll S_k^{1/2} T_\ell^{1/2}$$

The first term in (2.12) by Lemma 2.1 (ii) is

$$\ll M \left(\sum_{m'\epsilon I_k} |a_m|^2 \right)^{1/2} \left(\sum_{n'\epsilon I_\ell} |b_n|^2 \right)^{1/2}$$

$$\ll M \frac{2^{\frac{k+\ell}{2}}}{N+1} S_k^{1/2} T_\ell^{1/2}$$

$$\leq S_k^{1/2} T_\ell^{1/2}$$

Therefore, the sum of the terms in $\sum_2$ corresponding to $|k-\ell| \leq 1$ is

$$\ll \sum_{\substack{k,\ell \\ |k-\ell|\leq 1}} (S_k T_\ell)^{1/2} \leq \left(\sum_{\substack{k,\ell \\ |k-\ell|\leq 1}} S_k \right)^{1/2} \left(\sum_{\substack{k,\ell \\ |k-\ell|\leq 1}} T_\ell \right)^{1/2}$$

$$\leq 3 \left(\sum_k S_k \right)^{1/2} \left(\sum_\ell T_\ell \right)^{1/2}$$

$$\ll \left(\sum_n n|a_n|^2 \right)^{1/2} \left(\sum_n n|b_n|^2 \right)^{1/2}$$

The method of estimation of $\sum_2$ shows that

$$\sum_3 \ll \sum_n |a_n|^2 + \sum_n n|a_n|^2 \ll \sum_n n|a_n|^2, \text{ and}$$
$$\sum_4 \ll \sum_n n|b_n|^2$$

By Cauchy - Schwarz inequality

$$\sum_1 \leq \left(\sum |a_n|^2\right)^{1/2} \left(\sum |b_n|^2\right)^{1/2}$$

$$\leq \left(\sum n|a_n|^2\right)^{1/2} \left(\sum n|b_n|^2\right)^{1/2}$$

This completes the proof of Theorem (2.1).

Remark. The proof of Theorem 2.1. remains the same when $N = \infty$, only we define $I_j = (2^{j-1}, 2^j)$ this time.

The following theorem is immediate from Theorem 2.1.

Theorem 2.2. If $a_1, \ldots a_N$ are complex numbers, $0 \leq \alpha \leq 1$ is fixed, then

$$\int_0^T |\sum_{n\leq N} \frac{a_n}{(n+\alpha)^{it}}|^2 \, \mathrm{dt} = T \sum_{n\leq N} |a_n|^2 + O\left(\sum_{n\leq N} n|a_n|^2 \right)$$

Proof. Squaring and integrating, we get

$$\int_0^T |\sum_{n\leq N} \frac{a_n}{(n+\alpha)^{it}}|^2 \, \mathrm{dt} =$$

$$T\sum_{n\le N}|a_n|^2+\sum_{m\neq n}\frac{a_n\bar{a}_m}{\log(m+\alpha)-\log(n+\alpha)}\left\{\left(\frac{m+\alpha}{n+\alpha}\right)^{iT}-1\right\}$$

Now apply Theorem 2.1, with $b_n = a_n$ to the 2nd term once directly and once with a_n replaced by $\frac{a_n}{(n+\alpha)^{iT}}$.

As an application of Theorem 2.1, Ramachandra has proved the following results also. These results will be frequently used in Section 5.

Corollary 2.3. Let $q \geq 1$ be an integer. Then, for $1 \leq j \leq q$, we have,

$$\left|\sum\sum_{\substack{n\neq m\\ n\equiv m\equiv j(\text{modq})}}\frac{a_m\bar{b}_n}{\lambda_m-\lambda_n}\right| \leq D_1\left(\frac{|a_j|^2}{\log(\frac{q+j}{j})}+\frac{1}{q}\sum_{\substack{n\equiv j(\text{modq})\\ n>q}}n|a_n|^2\right)^{1/2}$$

$$\left(\frac{|b_j|^2}{\log(\frac{q+j}{j})}+\frac{1}{q}\sum_{\substack{n\equiv j(\text{modq})\\ n>q}}n|b_n|^2\right)^{1/2}$$

where D_1 is an effective numerical constant.

Corollary 2.4 Let $q \geq 1$ be an integer and $\sum_{\chi(\text{modq})}$ denote summation over all Dirichlet characters modulo q. Then we have

$$|\sum_{\chi\text{modq}}\left(\sum_{m\neq n}\frac{a_m\bar{b}_n\chi(m)\bar{\chi}(n)}{\lambda_m-\lambda_n}\right)|$$

$$\leq D_2\phi(q)\left(\sum_{1\leq j\leq q}\frac{|a_j|^2}{\log\frac{2q}{j}}+\frac{1}{q}\sum_{\substack{n>q\\ (n,q)=1}}n|a_n|^2\right)^{1/2}$$

$$\left(\sum_{1\leq j\leq q}\frac{|b_j|^2}{\log\frac{2q}{j}}+\frac{1}{q}\sum_{\substack{n>q\\ (n,q)=1}}n|b_n|^2\right)^{1/2}$$

where D_2 is an effective numerical constant.

Proof of Corollary 2.3 The contribution from the terms for which either $m = j$ or $n = j$ can be estimated thus : we have

$$\int_0^T \left(a_j e^{-i\lambda_j t} \sum_{\substack{n\equiv j(\mathrm{mod}\,q)\\ n>q}} \bar{b}_n e^{-i\lambda_n t} \right) dt = E(T) - E(0)$$

where

$$E(T) = \sum_{\substack{n\equiv j(\mathrm{mod}\,q)\\ n>q}} \frac{a_j \bar{b}_n e^{iT(\lambda_n - \lambda_j)}}{i(\lambda_n - \lambda_j)}$$

Integrating from 0 to 1 with respect to T we have

$$|E(0)| \le \sum \frac{|a_j b_n|}{(\log n/j)^2} + \int_0^1 dT \int_{-T}^T |a_j \sum_{\substack{n\equiv j(\mathrm{mod}\,q)\\ n>q}} \bar{b}_n e^{i\lambda_n t}|\, dt,$$

which by Hölder's inequality is

$$\le \sum \frac{|a_j b_n|}{(\log n/j)^2} + |a_j| \int_0^1 (2T)^{1/2} \left(\int_{-T}^T |\sum \bar{b}_n\, e^{i\lambda_n t}|^2 dt \right)^{1/2} dT$$

$$\le \sum \frac{|a_j b_n|}{(\log n/j)^2} + |a_j| \left(\int_0^1 dT \int_{-T}^T |\sum \bar{b}_n\, e^{i\lambda_n t}|^2 dt \right)^{1/2} dT$$

$$\le \sum \frac{|a_j b_n|}{(\log n/j)^2} + |a_j| \left(\frac{1}{q} \sum n|b_n|^2 + \frac{2D}{q} \sum n|b_n|^2 \right)^{1/2}$$

as

$$\int_0^1 dT \int_{-T}^T |\sum \bar{b}_n\, e^{i\lambda_n t}|^2\, dt$$

$$= \sum_{\substack{n\equiv j(\mathrm{mod}\,q)\\ n>q}} |b_n|^2$$

$$+\int_0^1 dT\{\sum_{\substack{n\neq m\\ n\equiv m(\mathrm{mod q})\\ n,m>q}}\frac{\bar b_n b_m e^{i(\lambda_n-\lambda_m)T}}{i(\lambda_n-\lambda_m)}-\sum\frac{b_n\bar b_m e^{-i(\lambda_n-\lambda_m)T}}{i(\lambda_n-\lambda_m)}\}$$

$$\leq \sum_{\substack{n\equiv j(\mathrm{mod q})\\ n>q}} n|b_n|^2+\frac{2D}{q}\sum_{\substack{n\equiv j(\mathrm{mod q})\\ n>q}} n|b_n|^2,$$

by noting that $n>q$, and by Theorem 2.1,

$$\left|\sum\sum_{\substack{n\equiv n\equiv j(\mathrm{mod q})\\ m,n>q}}\frac{a_m\bar b_n}{\lambda_m-\lambda_n}\right|\leq\frac{D}{q}\left(\sum_{\substack{n>q\\ n\equiv j(\mathrm{mod q})}} n\,|a_n|^2\right)^{1/2}\left(\sum_{\substack{n>q\\ n\equiv j(\mathrm{mod q})}} n\,|b_n|^2\right)^{1/2}$$

as can be seen by writing $\lambda_n-\log q$ in the place of λ_n.

Next

$$\begin{aligned}|a_j|\sum\frac{|b_n|}{(\log\frac{n}{j})^2} &\leq (|a_j|^2)^{1/2}(\sum n|b_n|^2)^{1/2}\left(\sum\frac{1}{n(\log n/j)^4}\right)^{1/2}\\ &\leq \left(\frac{|a_j|^2}{\log\frac{2q}{j}}\right)^{1/2}(\sum n|b_n|^2)^{1/2}\left(\frac{1}{q}\sum_{n=q}^{\infty}\frac{1}{n(\log n)^3}\right)^{1/2}\\ &\ll \left(\frac{|a_j|^2}{\log\frac{2q}{j}}\right)^{1/2}\left(\frac{1}{q}\sum n|b_n|^2\right)^{1/2}\end{aligned}$$

Similar argument applies to $\sum \bar b_j a_n/i(\lambda_n-\lambda_j)$.

This completes the proof of Corollary 2.3.

Proof of Corollary 2.4 . Corollary 2.4 now follows easily from Corollary 2.3.

In view of the relation

$$\sum_{\chi(\mathrm{mod q})}\chi(n)\bar\chi(m)=\begin{cases}\phi(q) & \text{if } n\equiv m(\mathrm{mod q}) \text{ and } (n,q)=1\\ 0 & \text{otherwise}\end{cases}, \tag{2.13}$$

$$\sum_{\chi}\sum_{m\neq n}\frac{a_m\bar{b}_n\chi(m)\bar{\chi}(n)}{\lambda_m-\lambda_n} = \phi(q)\sum_{\substack{m\equiv n(\mathrm{mod} q)\\ m\neq n}}\frac{a_m\bar{b}_n}{\lambda_m-\lambda_n}$$

$$= \phi(q)\sum_{j=1}^{q}\sum_{\substack{m\equiv n\equiv j(\mathrm{mod} q)\\ m\neq n}}\frac{a_m\bar{b}_n}{\lambda_m-\lambda_n}$$

and apply the Corollary 2.3 to the inner sum above.

Corollary 2.5 . If $\sum^{*}_{\chi(\mathrm{mod} q)}$ denotes summation over primitive characters modulo q, then

$$\sum^{*}_{\chi(\mathrm{modq})}\sum\sum_{m\neq n}\frac{a(n)\bar{a}(m)\chi(n)\bar{\chi}(m)}{\log\frac{m}{n}}$$

$$\ll\sum_{k|q}|\mu(q/k)|\left\{\phi(k)\sum_{n\leq\frac{k}{4}}\frac{|a^2(m)|}{|\log k/n|}+\frac{\phi(k)}{k}\sum_{n}n\,|a^2(n)|\right\}$$

$$\ll 2^r\left\{\max_{k|q}\left(k\sum_{n\leq\frac{k}{4}}\frac{|a^2(n)|}{\log k/n}\right)+\sum_{n}n\,|a^2(n)|\right\}$$

Proof. We have

$$\sum^{*}_{\chi(\mathrm{modq})}\sum\sum_{m\neq n}\frac{a(n)\bar{a}(m)\chi(n)\bar{\chi}(m)}{\log\frac{m}{n}}$$

$$=\sum\sum_{\substack{m\neq n\\ (m,q)=(n,q)=1}}\frac{a(n)\bar{a}(m)}{\log m/n}\sum^{*}_{\chi(\mathrm{modq})}\chi(n)\bar{\chi}(m)$$

Now, if we write $q = \prod_{j=1}^{r} p_j^{\ell_j}$, then by the well known properties of primitive characters,

$$\sum_{\chi(\text{modq})}^{*} \chi(n)\bar{\chi}(m) = \prod_{j=1}^{r}\left(\sum_{\chi(\text{mod}p_j^{\ell_j})}^{*} \chi(n)\bar{\chi}(m)\right)$$

$$= \prod_{j=1}^{r}\left(\sum_{t_j=0}^{\ell_j}\mu(p_j^{\ell_j-t_j}) \sum_{\chi(\text{mod}p_j^{t_j})} \chi(n)\bar{\chi}(m)\right)$$

$$= \sum_{k|q}\mu(q/k) \sum_{\chi(\text{modk})} \chi(n)\,\bar{\chi}(m)$$

In view of the relation (2.13), we get

$$\sum_{\chi(\text{modq})}^{*} \sum\sum_{m\neq n} \frac{a(n)\bar{a}(m)\chi(n)\bar{\chi}(m)}{\log n/m}$$

$$= \sum_{k|q}\phi(k)\,\mu(q/k)\left(\sum\sum_{\substack{m\neq n \\ m\equiv n(\text{modk}) \\ (n,k)=1}} \frac{a(n)\bar{a}(m)}{\log n/m}\right)$$

The proof now follows by appealing to Corollary 2.3.

Corollary 2.6

$$i) \quad \sum_{\chi(\text{modq})}^{*} \int_T^{2T} |\sum_{n\leq x} \frac{a(n)\chi(n)}{n^{it}}|^2\, dt = T\sum_{k|q}\mu(k)\phi(q/k)\sum_{n\leq x}|a(n)|^2$$
$$+ O\left(2^r max_{k|q}\, k \sum_{n\leq k/4}\frac{|a^2(n)|}{\log k/n}\right)$$
$$+ O\left(2^r \sum_{n\leq x} n|a(n)|^2\right)$$

$$ii) \quad \sum_{\chi(\text{modq})} \int_T^{2T} |\sum_{n\leq x} \frac{a(n)\chi(n)}{n^{it}}|^2\, dt = \phi(q)T\sum_{n\leq x}|a(n)|^2$$
$$+ O\left(\phi(q)\sum_{j\leq\frac{q}{4}}\frac{|a_j|^2}{\log\frac{2q}{j}} + \frac{\phi(q)}{q}\sum_{\substack{n>q \\ (n,q)=1}} n|a_n|^2\right)$$

Section 3. Mean Value Theorems of Dirichlet Series

In this section, we prove our main results. Our aim is to prove mean value theorems for a certain class of Dirichlet series using the approximate functional equation of Section 1 and the mean value theorem for Dirichlet polynomials stated in Section 2. These results give, for specific choices of the Dirichlet series, already known mean value results. For example, when the Dirichlet series is $\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta^2(s)$, we get Ingham's result, namely,(1.3).

For this purpose, we need an asymptotic formula for a class of arithmetic functions which we present below.

Theorem 3.1. Let $\alpha \geq 1$ be an integer and $\epsilon > 0$ be given. Let a_n be an arithmetic function such that $a_n = O(n^\epsilon)$. If $\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^s} = \zeta(s)^\alpha G(s)$ in $Re.s > 1$ where $G(s)$ is absolutely convergent in $Re.s \geq \frac{3}{4}$, and $h = 1$ or 2, then we have

$$\text{i)} \quad \sum_{n \leq X} |a_n|^2 = \begin{cases} \frac{G(1)}{(\alpha-1)!} X(\log X)^{\alpha-1} + O(X(\log X)^{\alpha-2}), & \text{if } \alpha \geq 2 \\ \\ G(1)X + O(X^{1-\frac{\epsilon}{2}}), & \text{if } \alpha = 1 \,. \end{cases}$$

$$\text{ii)} \quad \sum_{n \leq X} \frac{|a_n|^2}{n} = \frac{G(1)}{\alpha!} (\log X)^\alpha + O((\log X)^{\alpha-1})$$

$$\text{iii)} \quad \sum_{n\le X}\frac{|a_n|^2}{n^\sigma} = \begin{cases} \dfrac{G(1)}{(\alpha-1)!}\dfrac{1}{1-\sigma}X^{1-\sigma}(\log X)^{\alpha-1} \\ \qquad +O_\sigma(X^{1-\sigma}(\log X)^{\alpha-2}) \\ O(X^{1-\sigma}(\log X)^{\alpha-1}) \quad \text{if} \quad \sigma \le 1-\dfrac{1}{10^7} \end{cases}$$

$$\text{iv)} \quad \sum_{n>X}\frac{|a_n|^2}{n^\sigma} = O(X^{1-\sigma}(\log X)^{\alpha-1}) \quad \text{if} \quad \sigma \ge 1+\tfrac{1}{10^7}$$

$$\text{v)} \quad \sum_{n\le X}\frac{|a_n|^2}{n^\sigma}(e^{-2(\frac{n}{X})^h}-1) = O(X^{1-\sigma}(\log X)^{\alpha-1}) \quad \text{if} \quad \sigma \le 2$$

$$\text{vi)} \quad \sum_{n>X}\frac{|a_n|^2}{n^\sigma}e^{-2(\frac{n}{X})^h} = O(X^{1-\sigma}(\log X)^{\alpha-1}) \quad \text{if} \quad \sigma \ge -1+\tfrac{1}{10^7}$$

Proof. Let $\delta = \dfrac{1}{4\alpha+1}, c = 1+\delta$ and $\epsilon < \delta$. We have the inversion formula (see page 376, [**P**]),

$$\sum_{n\le X}|a_n|^2 = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta(s)^\alpha G(s)\frac{X^s}{s}ds + O\left(\frac{X^c}{T(c-1)^\alpha}\right)$$
$$+O\left(\frac{X^{1+\epsilon}}{T}\log 2X\right) + O(X^\epsilon)$$

By moving the line of integration to the line $\sigma = 1-\delta$, we get

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta(s)^\alpha G(s)\frac{X^s}{s}ds$$

$$= \quad \text{Residue of } \zeta(s)^\alpha G(s)\frac{X^s}{s} \text{ at } s = 1$$

$$+\frac{1}{2\pi i}\left\{\int_{1-\delta+iT}^{c+iT} + \int_{1-\delta-iT}^{1-\delta+iT} - \int_{1-\delta-iT}^{c-iT}\right\}\zeta(s)^\alpha G(s)\frac{X^s}{s}ds$$

Using the estimate (see **[P]**)

$$\zeta(s) \ll |t|^\delta, \quad \text{if} \quad \sigma \geq 1-\delta, \quad |t| \geq 2,$$

we see that the first and the third integrals on the right hand side are bounded by $X^{1-\delta/2}$ where we have chosen $T = X^{2\delta}$.

To estimate the second integral, we use the estimate, see **[Rad]**,

$$\zeta(s) \ll \frac{|1+s|^{2+\delta-\sigma}}{|1-s|}, \quad \text{if} \quad -\delta \leq \sigma \leq 1+\delta.$$

We get

$$\frac{1}{2\pi i}\int_{1-\delta-iT}^{1-\delta+iT}\zeta(s)^\alpha G(s)\frac{X^s}{s}\,ds \ll \left(\int_0^2 + \int_2^T\right)\frac{|\zeta(s)|^\alpha|G(s)|X^{1-\delta}}{|s|}dt$$

$$\ll X^{1-\delta} + X^{1-\delta}\int_2^T t^{2\alpha\delta-1}dt$$

$$\ll X^{1-\delta/2}$$

Expanding $\zeta(s)^\alpha G(s)\frac{X^s}{s}$ near $s = 1$, we see that the residue of $\zeta(s)^\alpha G(s)\frac{X^s}{s}$ at the pole at $s = 1$, is equal to

$$G(1)X \quad , \quad \text{if} \quad \alpha = 1 \quad \text{and}$$

$$\frac{G(1)}{(\alpha-1)!}X(\log X)^{\alpha-1} + O\left(X(\log X)^{\alpha-2}\right) \quad , \quad \text{if} \quad \alpha \geq 2.$$

Thus we have proved (i).

Now (ii) - (vi) can be easily deduced from (i) by partial summation.

$$\text{ii)}\quad \sum_{n\le X}\frac{|a_n|^2}{n} = \int_1^X \frac{1}{t}d\left(\sum_{n\le t}|a_n|^2\right)$$

$$= X^{-1}\sum_{n\le X}|a_n|^2 + O(1) + \int_1^X \frac{\sum_{n\le t}|a_n|^2}{t^2}dt$$

$$= \frac{G(1)}{(\alpha-1)!}\int_1^X \frac{(\log t)^{\alpha-1}}{t}dt + O((\log X)^{\alpha-1}) \quad \text{by} \quad (i)$$

$$= \frac{G(1)}{\alpha!}(\log X)^{\alpha} + O((\log X)^{\alpha-1})$$

$$\text{iii)}\quad \sum_{n\le X}\frac{|a_n|^2}{n^\sigma} = \int_1^X \frac{1}{t^\sigma}d\left(\sum_{n\le t}|a_n|^2\right)$$

$$= \frac{G(1)}{(\alpha-1!)}X^{1-\sigma}(\log X)^{\alpha-1} + \frac{G(1)\sigma}{(\alpha-1)!}\int_1^X \frac{(\log t)^{\alpha-1}}{t^\sigma}dt$$

$$+O(X^{1-\sigma}(\log X)^{\alpha-2})$$

$$= \frac{G(1)}{(\alpha-1)!}\frac{1}{1-\sigma}X^{1-\sigma}(\log X)^{\alpha-1} + O_\sigma(X^{1-\sigma}(\log X)^{\alpha-2})$$

$$= O(X^{1-\sigma}(\log X)^{\alpha-1}) \quad \text{if} \quad \sigma \le 1-\frac{1}{10^7}$$

iv) If $\sigma \ge 1+\frac{1}{10^7}$, then

$$\sum_{n>X}\frac{|a_n|^2}{n^\sigma} = \int_X^\infty \frac{1}{t^\sigma}d\left(\sum_{X<n\le t}|a_n|^2\right)$$

$$= \sigma\int_X^\infty\left(\sum_{X<n\le t}|a_n|^2\right)t^{-1-\sigma}dt$$

where we have used (i) which makes $(\sum_{n\le t}|a_n|^2)t^{1-\sigma}$ tend to 0 as $t\to\infty$. Therefore we get

$$\sum_{n>X}\frac{|a_n|^2}{n^\sigma} = O\left(\int_X^\infty \frac{(\log t)^{\alpha-1}}{t^\sigma}dt\right)$$

$$= O(X^{1-\sigma}(\log X)^{\alpha-1})$$

by integrating by parts repeatedly. This proves (iv).

v) To prove (v), we first use the inequality $1-e^{-2(\frac{n}{X})^h}<1$ when $\sigma\le 0.9$ and get

$$\sum_{n\le X}\frac{|a_n|^2}{n^\sigma}(1-e^{-2(\frac{n}{X})^h}) = O(X^{1-\sigma}(\log X)^{\alpha-1})$$

by (iii).

When $0.9\le\sigma\le 2$, we use $1-e^{-2(\frac{n}{X})^h}\le 2(\frac{n}{X})^h$ to get

$$\sum_{n\le X}\frac{|a_n|^2}{n^\sigma}(1-e^{-2(\frac{n}{X})^h}) \ll X^{-h}\sum_{n\le X}|a_n|^2 n^{h-\sigma}$$

$$\ll X^{-\sigma}\sum_{n\le X}|a_n|^2$$

$$\ll (X^{1-\sigma}(\log X)^{\alpha-1})$$

by (i).

vi) Using $e^{-2(\frac{n}{X})^h}\ll(\frac{X}{n})^h$ and (iv) we get (vi).

Corollary 3.2. Let the conditions of the Theorem 3.1 be satisfied. Then

$$i)\quad \sum_{n\le X}|a_n| \quad = \quad O\left(X(\log X)^{\frac{a-1}{2}}\right)$$

$$ii)\quad \sum_{n\le X}\frac{|a_n|}{n} \quad = \quad O\left((\log X)^{\frac{a+1}{2}}\right)$$

$$iii)\quad \sum_{n\le X}\frac{|a_n|}{n^\sigma} \quad = \quad O\left(X^{1-\sigma}(\log X)^{\frac{a-1}{2}}\right) \quad \text{if} \quad \sigma \le 1-\frac{1}{10^7}$$

$$iv)\quad \sum_{n\le X}\frac{|a_n|}{n^\sigma} \quad = \quad O\left(X^{1-\sigma}(\log X)^{\frac{a-1}{2}}\right) \quad \text{if} \quad \sigma \ge 1\pm\frac{1}{10^7}$$

$$v)\quad \sum_{n\le X}\frac{|a_n|}{n^\sigma}(1-e^{-(n/X)^h}) \quad = \quad O\left(X^{1-\sigma}(\log X)^{\frac{a-1}{2}}\right) \quad \text{if} \quad \sigma \le 2$$

$$vi)\quad \sum_{n> X}\frac{|a_n|}{n^\sigma}e^{-(n/X)^h} \quad = \quad O\left(X^{1-\sigma}(\log X)^{\frac{a-1}{2}}\right) \quad \text{if} \quad \sigma \ge -1+\frac{1}{10^7}$$

Proof.

Apply Holder's inequality and use Theorem 3.1 (i) to get (i). Then (ii) to (vi) follow from (i) by partial summation.

Let $F(s)=\sum_{n=1}^{\infty}\frac{a_n}{(Cn)^s}$ and $F^*(s)=\sum_{n=1}^{\infty}\frac{a_n^*}{(Cn)^s}$ be as in Theorem 1.1 and $|a_n|=|a_n^*|$. Assume that the Theorem 3.1 and its Corollary 3.2 hold for the arithmetic functions a_n and hence for a_n^*. Also assume that the function $\psi(s)$ in the functional equation (1.4) of $F(s)$ satisfies

$$T^{A(1-2\sigma)} \ll \psi(s) \ll T^{A(1-2\sigma)} \tag{3.1}$$

uniformly in $\sigma_1 \le \sigma \le \sigma_2$ and $T \le t \le 2T$ where A is a positive constant and

$$|\psi(\frac{1}{2} + it)| = 1 \tag{3.2}$$

Let us write $F_1(s) = C^s F(s), F_1^*(s) = C^s F^*(s)$ and $\psi_1(s) = C^{2s-1}\psi(s)$. Then

$$F_1(s) = \psi_1(s) F_1^*(1-s)$$

and we have the

Theorem 3.3.

Let $s_o = \sigma_o + it$ where $\frac{1}{2} \le \sigma_o \le 1$ is fixed. Then

$$\int_1^T |F_1(\sigma_o + it)|^2 dt = \begin{cases} \dfrac{2G(1)}{\alpha!} T(\log X)^\alpha + O((T+X)(\log X)^{\alpha-1}), \\ \qquad \text{if} \quad \sigma_o = \dfrac{1}{2}, & (3.3) \\ T\displaystyle\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma_o}} + O((T+X)X^{1-2\sigma_o}(\log X)^{\alpha-1}), \\ \qquad \text{if} \quad \dfrac{1}{2} < \sigma_o < 1, & (3.4) \\ T\displaystyle\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^2} + O(\frac{T+X}{X}(\log X)^\alpha), \quad \text{if} \quad \sigma_o = 1 . & (3.5) \end{cases}$$

where $CX = T^A$.

Remarks.

1. The constant A depends on the number of gamma factors appearing in the functional equation of $F(s)$.
2. The O-constant depends on σ_o and α and also on parameters (if any) appearing in $G(s)$.

Proof of Theorem 3.3. The approximate functional equation of $F(s)$ can be got by Theorem 1.1 where we choose $x = y = T^A, h = 2, \eta = 8/5$ and $\beta = 1/4$. Then, with the noatation introduced above, the approximate functional equation of $F_1(s_o)$ is given by

$$\begin{aligned} F_1(s_o) &= \sum_{n=1}^{\infty} \frac{a_n}{n^{s_o}} e^{-(n/X)^2} + \psi(s) \sum_{n\leq X} \frac{a_n^*}{n^{1-s_o}} \\ &- \frac{1}{4\pi i}\int_{(-\frac{8}{5})} \psi_1(s_o+w)\left(\sum_{n>X}\frac{a_n^*}{n^{1-s-w}}\right)\Gamma(\frac{w}{2})X^w dw \\ &- \frac{1}{4\pi i}\int_{(\frac{1}{4})} \psi_1(s_o+w)\sum_{n\leq X}\frac{a_n^*}{n^{1-s-w}}\Gamma(\frac{w}{2})X^w dw + O(T^{-10}) \\ &= J_1(s_o) + \ldots + J_5(s_o) \quad \text{(say)} \end{aligned}$$

where the O-term appears only when $F(s)$ has a pole at $s = 1$.

Then

$$\int_T^{2T} |F_1(\sigma_o + it)|^2 dt = \sum_{k=1}^{4} \int_T^{2T} |J_k(s_o)|^2 dt + O\left(\sum_{\substack{1\leq k,\ell\leq 4\\ k\neq \ell}} \int J_k(s_o)\overline{J_\ell(s_o)}dt\right) + O(1) \qquad (3.6)$$

Since

$$\left|\int_T^{2T} J_k(s_o)\overline{J_\ell(s_o)}dt\right| = \left|\int_T^{2T} \overline{J_k(s_o)}J_\ell(s_o)dt\right|,$$

it is enough to estimate one of the integrals. To estimate each one of the integrals in (3.6), we appeal to Theorem 2.2. Henceforth, we shall apply this theorem to all the integrals of the type $\int_T^{2T}\left|\sum \frac{b_n}{n^{it}}\right|^2 dt$ without referring to it each time.

We need the following lemmas.

Lemma 3.1.

$$\int_T^{2T} |J_1(s_o)|^2 dt = \begin{cases} \frac{G(1)}{\alpha!} T(\log X)^\alpha + O((T+X)(\log X)^{\alpha-1}) \\ \qquad \text{if} \quad \sigma_o = \frac{1}{2} \\ T\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma_o}} + O\left((T+X)X^{1-2\sigma_o}(\log X)^{\alpha-1}\right) \\ \qquad \text{if} \quad \frac{1}{2} < \sigma_o < 1 \\ T\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^2} + O(\frac{T}{X}(\log X)^{\alpha-1}) + O((\log X)^\alpha) \\ \qquad \text{if} \quad \sigma_o = 1 \end{cases} \tag{3.7}$$

$$\int_T^{2T} |J_2(s_o)|^2 dt = \begin{cases} \frac{G(1)}{\alpha!} T(\log X)^\alpha + O((T+X)(\log X)^{\alpha-1}) \\ \qquad \text{if} \quad \sigma_o = \frac{1}{2} \\ O((T+X)(\log X)^{\alpha-1}) \quad \text{if} \quad \sigma_o > \frac{1}{2} \end{cases} \tag{3.8}$$

Proof.

$$\begin{aligned} \int_T^{2T} |J_1(s_o)|^2 dt &= \int_T^{2T} \left|\sum \frac{a_n}{n^{\sigma_o+it}} e^{-(n/X)^2}\right| dt \\ &= \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma_o}} e^{-2(\frac{n}{X})^2} (T + O(n)) \qquad (3.9) \\ &= T\sum_1 + O(\sum_2) \end{aligned}$$

where $\sum_1 = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma_o}} e^{-2(\frac{n}{X})^2}$ and $\sum_2 = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma_o - 1}} e^{-2(\frac{n}{X})^2}$

The sum

$$\sum_1 = \sum_{n\le X}\frac{|a_n|^2}{n^{2\sigma_o}} + \sum_{n\le X}\frac{|a_n|^2}{n^{2\sigma_o}}(e^{-2(\frac{n}{X})^2}-1) + \sum_{n>X}\frac{|a_n|^2}{n^{2\sigma_o}}e^{-2(\frac{n}{X})^2}$$

$$= \sum_{n\le X}\frac{|a_n|^2}{n^{2\sigma_o}} + O(X^{1-2\sigma_o}(\log X)^{\alpha-1}) \tag{3.10}$$

by Theorem 3.1 (v) and (vi)

Again by Theorem 3.1 (ii) and (iv) we have

$$\sum_{n\le X}\frac{|a_n|^2}{n^{2\sigma_o}} = \begin{cases} \dfrac{G(1)}{\alpha!}(\log X)^\alpha + O((\log X)^{\alpha-1}), & \text{if} \quad \sigma_o = \dfrac{1}{2} \\ \displaystyle\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma_o}} + O_{\sigma_o}(X^{1-2\sigma_o}(\log X)^{\alpha-1}), & \text{if} \quad \sigma_o > \dfrac{1}{2}. \end{cases} \tag{3.11}$$

Using $e^{-2(\frac{n}{X})^2} \le 1$ and Theorem 3.1 (ii),(iii) and (vi), we have the sum

$$\sum_2 = \sum_{n\le X}\frac{|a_n|^2}{n^{2\sigma_o-1}}e^{-2(\frac{n}{X})^2} + \sum_{n>X}\frac{|a_n|^2}{n^{2\sigma_o-1}}e^{-2(\frac{n}{X})^2}$$

$$\ll \sum_{n\le X}\frac{|a_n|^2}{n^{2\sigma_o-1}} + X^{2-2\sigma_o}(\log X)^{\alpha-1}) \tag{3.12}$$

and

$$\sum_{n\le X}\frac{|a_n|^2}{n^{2\sigma_o-1}} = \begin{cases} O_{\sigma_o}(X^{2-2\sigma_o}(\log X)^{\alpha-1}) & \text{if} \quad \dfrac{1}{2}\le\sigma_o<1 \\ O((\log X)^\alpha) & \text{if} \quad \sigma_o = 1 \end{cases} \tag{3.13}$$

Now (3.7) follows from (3.9) to (3.13).

From (3.1) and (3.2) we note that'

$$|\psi_1(\frac{1}{2}+it)| = 1$$

and

$$X^{1-2\sigma} \ll \psi_1(\sigma+it) \ll X^{1-2\sigma}$$

uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$ and $T \leq t \leq 2T$.

Hence we have

$$\int_T^{2T} |J_2(s_o)|^2 dt = \begin{cases} \sum_{n\leq X} \frac{|a_n|^2}{n}(T + O(n)) & \text{if} \quad \sigma_o \frac{1}{2} \\ \\ O(X^{2(1-2\sigma_o)}) \sum_{n\leq X} \frac{|a_n|^2}{n^{2-2\sigma_o}}(T + n) & \text{if} \quad \frac{1}{2} < \sigma_o \leq 1 \end{cases}$$

Now (3.8) follows by Theorem 3.1 (iii).

Lemma 3.2.

Let $s = \sigma + it$. Then

$$a) \quad J_1(s) \;=\; O_{\sigma_0} \quad \left(X^{1-\sigma}(\log \mathrm{X})^{\frac{\alpha-1}{2}}\right), \text{ if } \frac{1}{2} - \frac{1}{100} \leq \sigma \leq \sigma_0 < 1$$

$$b) \quad J_2(s) \;=\; O \quad \left(X^{1-\sigma}(\log \mathrm{X})^{\frac{\alpha-1}{2}}\right), \text{ if } \frac{1}{2} - \frac{1}{100} \leq \sigma \leq \frac{3}{2} + \frac{1}{100}.$$

$$c) \quad J_3(s) \;=\; O \quad \left(X^{1-\sigma}(\log \mathrm{X})^{\frac{\alpha-1}{2}}\right), \text{ if } \frac{1}{2} - \frac{1}{100} \leq \sigma \leq \frac{3}{2} + \frac{1}{100}.$$

$$d) \quad J_4(s) \;=\; O \quad \left(X^{1-\sigma}(\log \mathrm{X})^{\frac{\alpha-1}{2}}\right), \text{ if } \frac{1}{2} - \frac{1}{100} \leq \sigma \leq \frac{3}{2} + \frac{1}{100}.$$

Proof. We use Corollary 3.2 for all the estimations.

$$\begin{aligned} a) \quad J_1(s) \quad &= \quad \sum_{n\leq X} \frac{a_n}{n^s} e^{-(n/X)^2} + \sum_{n>X} \frac{a_n}{n^s} e^{-(n/X)^2} \\ &\leq \quad \sum_{n\leq X} \frac{|a_n|}{n^\sigma} + X^2 \sum_{n>X} \frac{|a_n|}{n^{\sigma+2}} \\ &\ll_{\sigma_0} \quad X^{1-\sigma}(\log \mathrm{X})^{\frac{\alpha-1}{2}} \quad \text{if } \frac{1}{2} - \frac{1}{100} \leq \sigma \leq \sigma_0 < 1 \end{aligned}$$

and $J_1(1 + it) \ll (\log \mathrm{X})^{\frac{\alpha+1}{2}}$ by Corollary 3.2 (ii) and (iii).

b) Using (3.2) and Corollary 3.2 (iii), we get

$$J_2(s) \ll X^{1-2\sigma} \sum_{n \le X} \frac{|a_n|}{n^{1-\sigma}}$$

$$\ll X^{1-\sigma}(\mathrm{logX})^{\frac{\alpha-1}{2}}$$

c) We note that the integral in $J_3(s)$ can be broken at $|v| \le \log^3 X$ with a small error as,

$$\frac{1}{4\pi i} \int_{\substack{|v|>\log^3 X \\ u=-8/5}} \psi_1(s+w) \sum_{n>X} \frac{a_n^*}{n^{1-s-w}} \Gamma(\frac{w}{2}) X^w dw$$

$$= \frac{1}{4\pi i} \left(\int_{\log^3 X < |v| \le X} + \int_{|v|>X} \right) \psi_1(s+w) \sum_{n>X} \frac{a_n^*}{n^{1-s-w}} \Gamma(\frac{w}{2}) X^w dw$$

$$\ll X^{1-\sigma}(\log X)^{\frac{\alpha-1}{2}} \int_{\substack{\log^3 X < |v| \le X \\ u=-8/5}} |\Gamma(\frac{w}{2})| dv + X^{\sigma}(\mathrm{logX})^{\frac{\alpha-1}{2}}$$

$$\left(\int_{\substack{|v|>X \\ u=-8/5}} |v|^{1-2\sigma+16/5} |\Gamma(\frac{w}{2})| dv \right)$$

$$\ll X^{-20},$$

by using (3.2) , Corollary 3.2 (iv), and $|\Gamma(w)| \ll e^{-|v|}$

Hence

$$J_3(s) = \frac{1}{4\pi i} \int_{(-\frac{8}{5}, \log^3 X)} \psi_1(s+w) \sum_{n>X} \frac{a_n^*}{n^{1-s-w}} \Gamma(\frac{w}{2}) X^w dw + O(X^{-20}) \qquad (3.14)$$

which by using Corollary 3.2 (iv) again, is

$$\ll X^{1-\sigma} (\mathrm{logX})^{\frac{\alpha-1}{2}}$$

Similarly (d) can be proved.

Lemma 3.3. Let $\frac{1}{2}-\frac{1}{100} \leq \sigma \leq \frac{3}{2}+\frac{1}{100}$. Then

$$i) \quad \int_T^{2T} |J_1(s)|^2\, dt \;=\; O\big((T+X)X^{1-2\sigma}\,(\log X)^{\alpha-1}\big)\ \text{if}\,\sigma \leq \frac{1}{2}-\frac{1}{10^7}$$

$$ii) \quad \int_T^{2T} |J_2(s)|^2\, dt \;=\; O\left((T+X)X^{1-2\sigma}\,(\log X)^{\alpha-1}\right)\ \text{if}\,\sigma \geq \frac{1}{2}+\frac{1}{10^7}$$

$$iii) \quad \int_T^{2T} |J_k(s)|^2\, dt \;=\; O\left((T+X)X^{1-2\sigma}\,(\log X)^{\alpha-1}\right)\ \text{for}\ k=3,4.$$

Proof. We have

$$\int_T^{2T} |J_1(s)|^2\, dt = T\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}}\, e^{-2(\frac{n}{X})^2} + O\left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma-1}}\, e^{-2(\frac{n}{X})^2}\right)$$

By breaking the sums at X and using Theorem 3.1, as was done in Lemma 3.1, we get (i).

Proof of (ii) is straight forward with the use of (3.2) and Theorem 3.1 (iii)

Using (3.14), we have

$$\int_T^{2T} |J_3(s)|^2\, dt \;\ll\; \int_T^{2T} \Big| \int_{(-\frac{8}{5},\log^3 X)} \psi_1(s+w) \sum_{n>X} \frac{a_n^*}{n^{1-s-w}}\, \Gamma\big(\frac{w}{2}\big) X^w\, dw \Big|^2\, dt$$

$$+\, X^{-10}$$

which, by applying Hölder's inequality and then interchanging the order of integration, is

$$\ll\; X^{-16/5} \left(\int_{(-\frac{8}{5},\log^3 X)} \left\{ \int_T^{2T} |\psi_1(s+w) \sum_{n>X} \frac{a_n^*}{n^{1-s-w}}|^2\, dt \right\} |\Gamma(\frac{w}{2})|\, dv \right)$$

$$\left(\int_{(-\frac{8}{5},\log^3 X)} |\Gamma\frac{(w)}{2}|\, dv \right) \;+\, X^{-10}$$

$$\ll\; X^{1-\sigma}(\log X)^{\alpha-1},$$

by using (3.2), Theorem (3.1) (iv) and noting that $\int_{u=-8/5} |\Gamma(w)|\, dv$ is bounded. Similarly one proves (iii) for k = 4.

Completion of the proof of Theorem 3.3.

We have to only estimate the cross terms. For $k \neq 1$,

$$\begin{aligned}\int_T^{2T} J_1(s_o)\overline{J_k(s_o)}\, dt &= \int_T^{2T} J_1(s_o)J_k(\bar{s}_o)\, dt \qquad (*)\\ &= \int_T^{2T} J_1(s_o)\, J_k(2\sigma_o - s_o)\, dt\end{aligned}$$

By moving the line of integration to the line $\sigma_1 = \frac{1}{2} - \frac{1}{10^6}$, and if we write $s' = \sigma + iT$ or $\sigma + 2iT$, then we have,

$$= \int_{\sigma_1+iT}^{\sigma_1+2iT} J_1(s)J_k(2\sigma_o - s)\, dt + O\left(\int_{\sigma_1}^{\sigma_0} J_1(s')J_k(2\sigma_o - s')\, d\sigma\right)$$

$$= O_{\sigma_0}\left((T+X)X^{(1-2\sigma_0)}(\log X)^{\alpha-1}\right)$$

by applying Hölder's inequality and using Lemma 3.3 and Lemma 3.2. The estimation of the other cross terms is also similarly done. Thus (3.6) along with Lemma 3.1. Lemma 3.2 and the above estimations prove Theorem 3.2 for the integral $\int_T^{2T}$. Writing the integral $\int_1^T = \int_{\frac{T}{2}}^T + \int_{\frac{T}{2^2}}^{\frac{T}{2}} + \ldots$ and using the above estimations, we get (3.3), (3.4) and (3.5). This completes the proof of Theorem 3.3.

Note. In (*) we have written $J_k(\bar{s}_0)$ for the convenience of notation. It should be understood that all the complex parameters in J_k (if any) also take conjugate values.

Examples.

For specific choices of $F(s)$, we arrive at the following known mean value results. We choose for $F(s)$ well known functions such as powers of ζ-functions, Dedekind zeta-functions, L-functions and so on.

1. $\zeta(s)$, the Riemann-zeta function and its square.

Corollary 3.4. We have,

1) $\int_1^T |\zeta(\frac{1}{2}+it)|^2\,dt = T\log T + O(T)$

and

2) $\int_1^T |\zeta(\frac{1}{2}+it)|^4\,dt = \frac{T}{2\pi^2}(\log T)^4 + O(T(\log T)^3)$

(1) was first proved by Hardy and Littlewood [**H-L1**] using their approximate functional equation. This result follows from Theorem 3.3 immediately by taking $F(s)=\zeta(s)$. (2) was first proved by A.E. Ingham [**In**] using the Hardy - Littlewood's approximate functional equation for $\zeta^2(s)$. Later K. Ramachandra [**Ram1**] gave a simple proof of this by using another approximate functional equation of $\zeta^2(s)$ and a theorem of Montgomery and Vaughan (Theorem 2.2). The proof of our main theorem is modelled along this proof.

E.C. Titchmarsh [T] proved the weaker formula

$$\int_0^T |\zeta(\frac{1}{2}+it)|^4\,dt = (1+o(1))\frac{T}{2\pi^2}(\log T)^4\,(T\to\infty) \tag{3.15}$$

This follows from the investigation of the integrals

$$I(T) = \int_0^T |\zeta(\sigma+it)|^{2k}\,dt,$$

$$J(\delta) = \int_0^\infty |\zeta(\sigma+it)|^{2k}\,e^{-\delta t}\,dt.$$

where $k\geq 1$ is a fixed integer, $T\to\infty$ and $\delta\to 0+\frac{1}{2}\leq\sigma<1$ is fixed. A simple Tauberian argument shows that, for $C,D>0, I(T)\sim CT\log^D T$ is equivalent to $J(\delta)\sim C\delta^{-1}(\log\delta^{-1})^D$ and Titchmarsh deduces (3.15) from

$$\int_0^\infty |\zeta(\frac{1}{2}+it)|^4\,e^{-\delta t}\,dt = (1+o(1))\frac{1}{2\pi^2}\delta^{-1}(\log\delta^{-1})^4\,(\delta\to 0+)$$

Proof of Corollary 3.4.

Let $F_1(s) = \zeta^2(s)$, $F_1^*(s) = \zeta^2(s)$. The functional equation of $\zeta^2(s)$ is given by

$$\zeta^2(s) = \psi(s)\zeta^2(1-s)$$

where $\psi(s) = \pi^{2s-1}\Gamma^2(\frac{1-s}{2})/\Gamma^2(s/2)$.

We note that $|\psi(\frac{1}{2}+it)| = 1$ and by Stirling's formula, (3.1) is satisfied with $A = 1$

Since $\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$ if $\sigma > 1$, we have $a_n = d(n)$ in Theorem 3.3 and hence

$$\sum_{n=1}^{\infty} \frac{|a_n^2|}{n^s} = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)} \quad \text{if } \sigma > 1$$

This can be easily checked by writing the Euler product of $\sum_{n=1}^{\infty} \frac{d^2(n)}{n^s}$

So we have $\sigma_0 = \frac{1}{2}, \alpha = 4, A = 1, G(s) = \frac{1}{\zeta(2s)}$ and $X = T$ in Theorem (3.3) which gives

$$\int_1^T |\zeta(\frac{1}{2}+it)|^4\,dt = \frac{G(1)}{12}T(\log\mathrm{T})^4 + O(T(\log\mathrm{T})^3)$$

But we know that $G(1) = (\zeta(2))^{-1} = \frac{6}{\pi^2}$ and this proves 2) of Corollary 3.4

D.R. Heath-Brown [HB2] has improved this result substantially by showing that

$$\int_O^T |\zeta(\frac{1}{2}+it)|^4\,dt = T\sum_{k=0}^{4} c_k(\log\mathrm{T})^{4-k}\, O(T^{7/8+\epsilon})$$

where $c_0 = \frac{1}{2\pi^2}$ and the other constants are computable. As is to be expected, the proof of this is long and difficult. With the method we have adopted here, we can not hope to get a result of the type above, as at most a factor of log T only can be saved as one uses the theorem of Montgomery and Vaughan.

2. Higher powers of $\zeta(s)$.

Corollary 3.5. If $1 - \frac{1}{k} < \sigma_0 < 1$, where $k \geq 2$, then

$$\int_1^T |\zeta(\sigma_0 + it)|^{2k}\, dt = T\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma_0}} + O\left(T^{k(1-\sigma_0)} (\log T)^{k^2-1}\right)$$

Proof. Take $F(s) = \zeta(s)^k$ in Theorem 3.3. Then $F^*(s) = F(s)$, and the functional equation of $\zeta(s)^k$ is given by

$$\zeta(s)^k = \psi(s)(\zeta(1-s))^k$$

where $\psi(s) = \pi^{\frac{k}{2}(2s-1)}\,\Gamma^k(\frac{1-s}{2})/\Gamma^k(\frac{s}{2})$

We know that

$$\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \quad \text{if } \sigma > 1$$

and

$$\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^s} = \zeta(s)^{k^2} G(s) \quad \text{if } \sigma > 1$$

where $G(s)$ is absolutely convergent for $\sigma > \frac{1}{2}$.

We note that $\alpha = k^2, A = k/2$ (by Stirling's formula) and hence $X = T^{\frac{k}{2}}$ in Theorem 3.3.

Now Corollary 3.5 follows from (3.4)

3. The Dedekind-zeta function.

The Dedekind zeta-function $\zeta_K(s)$ over a number field K of degree n is defined to be

$$\zeta_K(s) = \sum_{a} \frac{1}{(Na)^s} \quad \text{if} \quad \sigma > 1$$

where the summation runs over all the integral ideals of K. We can also write it as

$$\zeta_K(s) = \sum_{m=1}^{\infty} \frac{a_K(m)}{m^s}$$

where $a_K(m)$ is the number of integral ideals of norm m. $\zeta_K(s)$ satisfies the functional equation

$$\zeta_K(s) = \psi(s)\zeta_K(1-s), \tag{3.15}$$

where $\psi(s) = B^{1-2s}\Gamma^{r_1}(\frac{1-s}{2})\Gamma^{r_2}(1-s)/\Gamma^{r_1}(\frac{s}{2})\Gamma^{r_2}(s)$ with $B = 2^{r_2}\pi^{n/2}(d(K))^{-1/2}$, where r_1 and $2r_2$ are the number of real and complex embeddings of K and $d(K)$ is the discriminant of K. We have

Corollary 3.6.

$$\int_1^T |\zeta_K(\sigma_o+it)|^2 dt = \begin{cases} c_1 B^{4\sigma_o-2}T^{n(1-2\sigma_o)+1} \sum_{m=1}^{\infty} \frac{a_K^2(m)}{m^{2-2\sigma_o}} \\ \quad +O(T^{n(1-2\sigma_o)+n\sigma_o}(\log T)^{n-1}), \quad \text{if} \quad 0 \le \sigma < \frac{1}{n} \\ O(T^{n(1-\sigma_o)}(\log T)^n), \quad \text{if} \quad \frac{1}{n} \le \sigma \le 1-\frac{1}{n} \\ T\sum_{n=1}^{\infty} \frac{a_K^2(m)}{m^{2\sigma_o}} + O(T^{n(1-\sigma_o)}(\log T)^{n-1}), \\ \qquad \text{if} \quad \sigma_o > 1-\frac{1}{n} \end{cases}$$

Proof. K.Chandrasekaran and R.Narasimhan [C-N] proved that

$$\sum_{m\le x} a_K^2(m) = O(x(\log x)^{n-1}).$$

and if the field K is Galois, then there exists a constant $C = C(K)$ such that

$$\sum_{m\le x} a_K^2(m) \sim Cx(log x)^{n-1}$$

Hence we have $\alpha = n$, $A = \frac{n}{2}$ and $X = T^{n/2}$ in Theorem 3.3 and thus

$$\int_1^T |\zeta_K(\sigma_o + it)|^2 dt = \begin{cases} T\sum_{n=1}^{\infty} \frac{a_K^2(m)}{m^{2\sigma_o}} + O(T^{n(1-\sigma_o)}(\log T)^{n-1}), \\ \qquad \text{if} \quad 1 - \frac{1}{n} < \sigma_o < 1 & (3.16) \\ O(T^{n(1-\sigma_o)}(\log T)^{n-1}) \quad \text{if} \quad \frac{1}{2} < \sigma_o \leq 1 - \frac{1}{n} & (3.17) \\ O(T^{n/2}(\log T)^n) \quad \text{if} \quad \sigma_o = \frac{1}{2} \end{cases}$$

Using the functional equation of $\zeta_K(s)$, we can find the mean value of $|\zeta_K(s_o)|^2$ when $\sigma_o \leq \frac{1}{2}$.

The functional equation (3.15) of $\zeta_K(s)$ gives

$$|\zeta_K(\sigma_o + it)|^2 = C_1 B^{4\sigma_o - 2} t^{n(1-2\sigma_o)}(1 + O(\frac{1}{t}))|\zeta_K(1 - \sigma_o + it)|^2, t > 0\,,$$

(since $\zeta_K(s)$ assumes conjugate values at conjugate points). If we write

$$f(T) \;=\; \int_1^T |\zeta_K(1 - \sigma_o + it)|^2 dt,$$

then from (3.16) and (3.17), we have

$$f(T) = \begin{cases} T\sum \frac{a^2(m)}{m^{2-2\sigma_o}} + O(T^{n\sigma_o}(\log T)^{n-1}) \quad \text{if} \quad 0 \leq \sigma_o < \frac{1}{n} \\ O(T^{n\sigma_o})(\log T)^{n-1}) \quad \text{if} \quad \frac{1}{n} \leq \sigma_o < \frac{1}{2} \end{cases}$$

Thus we obtain

$$\begin{aligned} \int_1^T |\zeta_K(\sigma_o + it)|^2 dt \;&=\; C.B^{4\sigma_o - 2} \int_1^T t^{n(1-2\sigma_o)} |\zeta_K(1 - \sigma_o + it)|^2 dt \\ &\quad + O\left(\int_1^T t^{n(1-2\sigma_o)-1} |\zeta_K(1 - \sigma_o + it)|^2 dt\right) \end{aligned} \qquad (3.18)$$

The first term on the right hand side of (3.18)

$$= C_1 B^{4\sigma_o - 2} \left\{ T^{n(1-2\sigma_o)} f(T) + n(2\sigma_o - 1) \int_1^T t^{n(1-2\sigma_o)-1} f(t) dt \right\}$$

$$= \begin{cases} C_1 B^{4\sigma_o-2} T^{n(1-2\sigma_o)+1} \sum_{m=1}^{\infty} \frac{a_K^2(m)}{m^{2-2\sigma_o}} + O(T^{n(1-\sigma_o)}(\log T)^{n-1}) & \text{if} \quad 0 \le \sigma < \frac{1}{n} \\ O(T^{n(1-\sigma_o)}(\log T)^{n-1}) \quad \text{if} \quad \frac{1}{n} \le \sigma_o < \frac{1}{2} \end{cases} \tag{3.19}$$

while the second term is

$$= O(T^{n(1-2\sigma_o)-1} f(T) + (1 + 2n\sigma_o - n) \int_1^T f(t) t^{n(1-2\sigma_o)-2} dt$$

$$= O(T^{n(1-\sigma_o)}(\log T)^{n-1}) \quad \text{if} \quad 0 \le \sigma < \frac{1}{2} \tag{3.20}$$

Thus (3.16) to (3.20) prove Corollary 3.6.

Remark.

K.Chandrasekaran and Raghavan Narasimhan [C-N] obtained the mean square of Dedekind zeta-function using their approximate functional equation and their result is given below :

$$\int_1^T |\zeta_K(\sigma_o + it)|^2 dt = \begin{cases} c_1 B^{4\sigma_o-2} T^{n(1-2\sigma_o)+1} \sum_{m=1}^{\infty} \frac{a_K^2(m)}{m^{2-2\sigma_o}} \\ +O(T^{n(1-2\sigma_o)+\frac{n\sigma_o}{2}+\frac{1}{2}}(\log T)^{n/2}), \quad \text{if} \quad \sigma_o > 1 - \frac{1}{n}, \\ O(T^{n(1-\sigma_o)}) \log^n T, \quad \text{if} \quad \frac{1}{n} \le \sigma_o \le 1 - \frac{1}{n}, \\ T \sum_{m=1}^{\infty} \frac{a_K^2(m)}{m^{2\sigma_o}} + O(T^{\frac{1}{2}n(1-\sigma_o)+\frac{1}{2}} \log^{n/2} T), \\ \qquad \text{if } \sigma_o > 1 - \frac{1}{n}. \end{cases}$$

Note that our result gives a better error term when $\sigma_o > 1 - \frac{1}{n}$ and when $\sigma_o < \frac{1}{n}$.

4. Cusp forms of level N.

Let k and N be positive integers. Let f be a cusp form of the type $(-k, N)$ in the sense of Hecke [**He**] with the Fourier series

$$f(z) = \sum a_n e^{\frac{2\pi i n z}{N}}, Im.z > 0,$$

and $L_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, where $s = \sigma + it, \sigma > \frac{k+1}{2}$ denote the corresponding L-series. Then we have

Corollary 3.7.

$$\int_1^T |L_f(\sigma + it)|^2 dt = \begin{cases} 2AkT \log T + O(T) & \text{if} \quad \sigma = \frac{k}{2}, \\ T \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} + O(T^{k+1-2\sigma}) & \text{if} \quad \frac{k}{2} < \sigma < \frac{k+1}{2}, \\ T \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} + O(\log T) & \text{if} \quad \sigma = \frac{k+1}{2}, \end{cases}$$

as $T \to \infty$. Here,

$$A = \frac{12(4\pi)^{k-1}}{j(N)N^k\Gamma(k+1)} \int\int_D |f(u+iv)|^2 v^{k-2} du dv,$$

where D is the fundamental domain of the principal inhomogeneous congruence subgroup of level N and $j(N)$ is the index of this group in the inhomogeneous full modular group.

Proof.

We know that $L_f(s)$ satisfies the functional euqation

$$L_f(s) = \psi_s(s) \; L_f(k - s) \tag{3.21}$$

where $\psi(s) = (\frac{2\pi}{N})^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}$

Now, define $a_1(n) = \frac{a_n}{n^{\frac{k-1}{2}}}$. Let $L_{f_1}(s)$ denote the corresponding L-series. Then we know from (3.21) that $L_{f_1}(s)$ satisfies the function equation

$$L_{f_1}(s) = \psi_1(s)\, L_{f_1}(1-s)$$

where

$$\psi_1(s) = (\frac{2\pi}{N})^{2s-1}\frac{\Gamma(\frac{k+1}{2}-s)}{\Gamma(\frac{k-1}{2}+s)} \tag{3.22}$$

We use the asymptotic formula due to Rankin [**R**]

$$\sum_{n\leq x} |a_n|^2 = Ax^k + O(x^{k-2/5}) \tag{3.23}$$

which gives by partial summation,

$$\sum_{n\leq x} |a_1(n)|^2 = Akx + O(x^{3/5})$$

Hence we have $\alpha = 1$ in Theorem 3.1 (i) and the error term is much smaller than the main term. From (3.22), we note that $|\psi_1(\frac{1}{2}+it)| = 1$ and $A = 1$, and hence $X = T$ in Theorem 3.3. Hence we have

$$\int_1^T |L_{f_1}(\sigma_o + it)|^2 dt = \begin{cases} 2AkT\log T + O(T) & \text{if} \quad \sigma_o = \frac{1}{2} \\ T\sum_{n=1}^{\infty}\frac{a_1^2(n)}{n^{2\sigma_o}} + O(T^{2(1-\sigma_o)}), & \text{if} \quad \frac{1}{2} < \sigma_o < 1 \\ T\sum_{n=1}^{\infty}\frac{a_1^2(n)}{n^2} + O(\log T) & \text{if} \quad \sigma_o = 1 \end{cases}$$

By noting that $L_f(s) = L_{f_1}(s - \frac{k-1}{2})$, we see that the Corollary 3.7 is proved.

Remark 1.

This result was first proved by A.Good [G3]. When $\sigma = \frac{k+1}{2}$, he obtained the following, namely,

$$\int_1^T |L_f(\sigma+it)|^2 dt = T\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} + O(\log^2 T).$$

Note that we could save a factor of $\log T$ in the error term.

Remark 2.

As a special case of this result, we can take $a_n = \tau(n)$, the Ramanujan's τ-function. We have $N = 1$, $k = 12$ in this case, and we get

$$\int_1^T |L_\tau(\sigma+it)|^2 dt = \begin{cases} 24AT\log T + O(T) & \text{if} \quad \sigma = 6, \\ T\sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^{2\sigma}} + O(T^{13-2\sigma}), & \text{if} \quad 6 < \sigma < 6\frac{1}{2}, \\ T\sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^{2\sigma}} + O(\log T) & \text{if} \quad \sigma = 6\frac{1}{2}. \end{cases}$$

Section 4. Mean Value Theorems for the Derivatives

In this section, we prove mean value theorems for the derivatives of the zeta-function.

Notation.

For $k \geq 0$ an integer, let $g^{(k)}(s)$ denote the k-th derivative of $g(s)$ where $g^{(o)}(s) = g(s)$. Let $\ell, m \geq 0$ be integers and fixed through out this section.

For $A, B, P \geq 0$ integers, we define

$$\begin{aligned}
a(n, A, B) &= (-1)^{A+B} \sum_{n_1 n_2 = n} \log^A n_1 \log^B n_2 \\
a(n) &= a(n, \ell, m) \\
b(n, T) &= \sum_{n_1 n_2 = n} \log^\ell \frac{T}{n_1} \log^m \frac{T}{n_2} \\
b^1(n, T) &= \sum_{n_1 n_2 = n} \log^\ell (T n_1) \log^m (T n_2) \\
b_1(n, t, p) &= \sum_{k_1 = 0}^{p} (-1)^{k_1} k_1 \binom{p}{k_1} \log^{p-k_1} n \log^{k_1} t
\end{aligned}$$

Define for $a, b \geq 0$ integers,

$$\begin{aligned}
f(x, a, b) &= x^a (1-x)^b + x^b (1-x)^a \\
&= \textstyle\sum_r C_{r,a,b} x^r
\end{aligned}$$

We know that

$$\zeta^{(\ell)}(s) = (-1)^\ell \sum_{n=1}^{\infty} \frac{\log^\ell n}{n^s} \quad \text{if} \quad Re.s > 1$$

and can be analytically continued to the whole complex plane but for a pole at $s = 1$. From the functional equation( ) of $\zeta(s)$ we see that $\zeta^{(\ell)}(s)$ satisfies the functional equation

$$\zeta^{(\ell)}(s) = \sum_{k=0}^{\ell} (-1)^{\ell-k} \binom{\ell}{k} \psi^{(k)}(s) \zeta^{(\ell-k)}(1-s) \tag{4.1}$$

Hence

$$\zeta^{(\ell)}(s)\zeta^{(m)}(s) = \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{m}(-1)^{\ell+m-k_1-k_2}\begin{pmatrix}\ell\\k_1\end{pmatrix}\begin{pmatrix}\ell\\k_2\end{pmatrix}\psi^{(k_1)}(s)\psi^{(k_2)}(s)$$

$$\times(\zeta^{(\ell-k_1)}(1-s)\zeta^{(\ell-k_2)}(1-s))$$

Using this, we obtain an approximate functional equation for $\zeta^{(\ell)}(s)\zeta^{(m)}(s)$ of the type (1.7) and since the proof is similar, we omit the proof and state only the result.

Theorem 4.1.

$$\zeta^{(\ell)}(\frac{1}{2}+it)\zeta^{(m)}(\frac{1}{2}+it) =$$

$$\sum_{n\le T}\frac{a(n)}{n^{\frac{1}{2}+it}} + \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{m}(-1)^{\ell+m-k_1-k_2}\begin{pmatrix}\ell\\k_1\end{pmatrix}\begin{pmatrix}m\\k_2\end{pmatrix}\psi^{(k_1)}(\frac{1}{2}+it)$$

$$(\psi^{(k_2)}(\frac{1}{2}+it)\zeta^{(\ell-k_1)}(\frac{1}{2}-it)\zeta^{(m-k_2)}(\frac{1}{2}-it))$$

$$\left(\sum_{n\le T}\frac{a(n,\ell-k_1,m-k_2)}{n^{\frac{1}{2}-it}}\right) + \sum_{n\le T}\frac{a(n)}{n^{\frac{1}{2}+it}}\,(e^{-n/T}-1)$$

$$+\sum_{n>T}\frac{a(n)}{n^{\frac{1}{2}+it}}\,e^{-n/T} - \frac{1}{2\pi i}\int_{(-\frac{3}{4})}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{m}(-1)^{\ell+m-k_1-k_2}\begin{pmatrix}\ell\\k_1\end{pmatrix}\begin{pmatrix}m\\k_2\end{pmatrix}$$

$$\times(\psi^{(k_1)}(\frac{1}{2}+it+\omega)\psi^{(k_2)}(\frac{1}{2}+it+\omega)(\sum_{n>T}\frac{a(n,\ell-k_1,m-k_2)}{n^{\frac{1}{2}-it-\omega}})\Gamma(\omega)T^{\omega}dw)$$

$$-\frac{1}{2\pi i}\int_{(\frac{1}{4})}\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{m}(-1)^{\ell+m-k_1-k_2}\begin{pmatrix}\ell\\k_1\end{pmatrix}\begin{pmatrix}m\\k_2\end{pmatrix}\psi^{(k_1)}(\frac{1}{2}+it+\omega)$$

$$\psi^{(k_2)}(\frac{1}{2}+it+\omega)(\sum_{n\le T}\frac{a(n,\ell-k_1,m-k_2)}{n^{1/2-it-\omega}})\Gamma(\omega)T^{\omega}dw$$

$$+O(T^{-10})$$

$$= \sum_{k=1}^{7} J_k(\frac{1}{2}+it) \quad \text{(say)} \tag{4.2}$$

For the proof of Theorem 4.1, we use

$$\zeta^{(k)}(\sigma+it) \ll \begin{cases} |t|^{1/2-\sigma+\epsilon}, & \text{if} \quad \sigma\le 0 \\ |t|^{1/2(1-\sigma)+\epsilon}, & \text{if} \quad 0\le\sigma\le 1 \\ |t|^{\epsilon}, & \text{if} \quad \sigma\ge 1 \end{cases}$$

and $\psi^{(k)}(\sigma+it) \ll |t|^{1/2-\sigma}\log^k|t|$ for $|t|\ge 2$ and $k\ge 0$.

Using Theorem 4.1 and theorems in Section 2, we prove the following :

Theorem 4.2.

$$\int_1^T |\zeta^{(\ell)}(\frac{1}{2}+it)\zeta^{(m)}(\frac{1}{2}+it)|^2 = CT\log^{2\ell+2m+4}T + O(T\log^{2\ell+2m+3}T) \tag{4.3}$$

where $C=C(\ell,m)=C_1+C_2$ and

$$C_1 = \frac{(\zeta(2))^{-1}}{8(2\ell+2m+4)}\int_0^1\left[\int_y^1\left\{\left(\frac{x}{2}\right)^{\ell}\left(1-\frac{x}{2}\right)^{m}+\left(\frac{x}{2}\right)^{m}\left(1-\frac{x}{2}\right)^{\ell}\right\}dx\right]^2 dy \tag{4.4}$$

and

$$C_2 = \frac{(\zeta(2))^{-1}}{8}\int_0^1 t^3\int_0^1\left[\int_y^1\left\{\left(1-\frac{tx}{2}\right)^{\ell}(1-t(1-\frac{x}{2}))^{m}+\left(1-\frac{tx}{2}\right)^{m}\left(1-t(1-\frac{x}{2})\right)^{\ell}\right\}dx\right]^2 dydt \tag{4.5}$$

Using (4.2), we get

$$\int_T^{2T}|\zeta^{(\ell)}(\frac{1}{2}+it)\zeta^{(m)}(\frac{1}{2}+it)|^2dt = \int_T^{2T}(|J_1|^2+|J_2|^2)dt$$

$$+ \sum_{k=2}^{6} \int_T^{2T} (J_1 \bar{J}_k + \bar{J}_1 J_k) dt$$

$$+ \sum_{\substack{k=1 \\ k \neq 2}}^{6} \int_T^{2T} (J_2 \bar{J}_k + \bar{J}_2 J_k) dt$$

$$+ \sum_{k,k'=3}^{6} \int_T^{2T} J_k \bar{J}_{k'} dt \; + \; O(1) \quad (4.6)$$

where we have used the notation $J_k = J_k(\frac{1}{2} + it)$. As in the case of Theorem 3.3, we expect the integrals $\int_T^{2T} |J_1|^2 dt$ and $\int_T^{2T} |J_2|^2 dt$ to contribute to the main term and all the other integrals go to the error term. We need various lemmas to prove this.

Lemma 4.1.

$$\psi^{(k)}(s) = \psi(s) \left\{ \left(-\log \frac{|t|}{2\pi} \right)^k + O\left(\frac{\log^{k-1} |t|}{|t|} \right) \right\}$$

where $|t| \geq 1$ and $|\sigma| \ll 1$.

Proof.

We follow the proof of Gonek [G2].

The proof is by induction on k. The case $k = 0$ is obviously true. Now suppose the lemma is proved for $k = 0, 1, \ldots, \mu - 1$. We differentiate the identity

$$\psi'(s) \;=\; \psi(s) \frac{\psi'(s)}{\psi(s)}$$

and obtain

$$\psi^{(\mu)}(s) \;=\; \sum_{k=0}^{\mu-1} \binom{\mu - 1}{k} \psi^{(k)}(s) \left(\frac{\psi'}{\psi} \right)^{(\mu-1-k)} (s) \tag{4.7}$$

By Cauchy's estimate for the derivatives of an analytic function applied to a small disc centered at s, we find that

$$\frac{\psi'}{\psi}(s) \;=\; -\log \frac{|t|}{2\pi} \;+\; O\left(\frac{1}{|t|} \right) \tag{4.8}$$

and

$$\left(\frac{\psi'}{\psi}\right)^{(k)}(s) = O\left(\frac{1}{|t|}\right) \tag{4.9}$$

for $|t| \geq 1$, $|\sigma| \ll 1$ and $k \geq 1$.

Also,

$$\psi(1-s) = O(|t|^{\frac{1}{2}-\sigma}) \quad \text{for} \quad |t| \geq 1 \tag{4.10}$$

Now the lemma follows from (4.7) to (4.10) and the induction hypothesis.

Lemma 4.2.

i) $$\sum_{n\leq x} \frac{\phi(n)}{n^2} \log^k n = \frac{1}{\zeta(2)} \frac{\log^{k+1} x}{k+1} + O(\log^k x).$$

ii) $$\sum_{n\leq x} \frac{\log^k n}{n} = \frac{\log^{k+1} x}{k+1} + O(\log^k x).$$

Proof.

We know that

$$\sum_{n\leq x} \frac{\phi(n)}{n^2} = \frac{1}{\zeta(2)} \log x + O(1)$$

and

$$\sum_{n\leq x} \frac{1}{n} = \log x + O(1)$$

(i) and (ii) now follow trivially by partial summation.

Lemma 4.3.

If A,B,C,D are non-negative integers, then there exists a constant $C_3 = C_3(A, B, C, D)$ such that

i)

$$(-1)^{A+B+C+D} \textstyle\sum_{n\leq T} a(n, A, B) a(n, C, D) = C_3 T \log^{A+B+C+D+3} T + O(T \log^{A+B+C+D+2} T)$$

where

$$C_3 = \frac{1}{8\zeta(2)} \int_0^1 \left\{ \int_y^1 f(\frac{x}{2}, A, B) dx \right\} \left\{ \int_y^1 f(\frac{x}{2}, C, D) dx \right\} dy \tag{4.11}$$

and

ii) $$(-1)^{A+B+C+D} \sum_{n \le T} \frac{a(n,A,B)a(n,C,D)}{n}$$

$$= \frac{C_3}{A+B+C+D+4} \log^{A+B+C+D+4} T$$

$$+ O(\log^{A+B+C+D+3} T)$$

To prove Lemma 4.3, we need the following result.

Lemma 4.4. If L, M, N, P are non-negative integers, then

$$\sum_{n \le T} \sum_{\substack{d_1, d_2 | n \\ d_1, d_2 \le \sqrt{n}}} \log^L d_1 \log^M \frac{n}{d_1} \log^N d_2 \log^P \frac{n}{d_2}$$

$$= T \sum_{d_1, d_2 \le \sqrt{T}} \frac{1}{[d_1, d_2]} \log^L d_1 \log^M \frac{T}{d_1} \log^N d_2 \log^P \frac{T}{d_2}$$

$$+ O(T \log^{L+M+N+P+2} T) \tag{4.12}$$

Proof. The sum

$$\sum_{n \le T} \sum_{\substack{d_1, d_2 | n \\ d_1, d_2 \le \sqrt{n}}} \log^L d_1 \log^M \frac{n}{d_1} \log^N d_2 \log^P \frac{n}{d_2}$$

$$= \sum_{d_1, d_2 \le \sqrt{T}} \log^L d_1 \log^N d_2 \sum_{\substack{n \equiv 0([d_1, d_2]) \\ \max(d_1^2, d_2^2) \le n \le T}} \log^M \frac{n}{d_1} \log^P \frac{n}{d_2}$$

$$= \sum_{d_1, d_2 \le \sqrt{T}} \log^L d_1 \log^N d_2 \left\{ \left(\sum_{\substack{n \equiv 0([d_1, d_2]) \\ n \le T}} - \sum_{\substack{n \equiv 0([d_1, d_2]) \\ n \le \max(d_1^2, d_2^2)}} \right) \log^M \frac{n}{d_1} \log^P \frac{n}{d_2} \right\}$$

$$= S_1 - S_2 \quad \text{say.}$$

We will first show that $S_2 = O(T\log^{L+M+N+P+2}T)$

By taking trivial estimates, we get

$$S_2 \ll \sum_{d_1\leq\sqrt{T}} \log^{L+M+N+P} d_1 \sum_{d_2\leq d_1} \sum_{\substack{n\equiv 0([d_1,d_2])\\ n\leq d_1^2}} 1$$

$$\log^{L+M+N+P} T \sum_{d_1\leq\sqrt{T}} \sum_{d_2\leq d_1} \left\{\frac{d_1^2}{[d_1,d_2]}+1\right\}$$

Writing $[d_1,d_2] = \dfrac{d_1 d_2}{(d_1,d_2)}$ and $(d_1,d_2) = \sum_{\substack{j|d_1\\ j|d_2}} \phi(j)$, we get

$$S_2 \ll \log^{L+M+N+P} T \sum_{j\leq\sqrt{T}} \phi(j) \sum_{d_1,d_2\leq\frac{\sqrt{T}}{j}} \frac{d_1}{d_2} + T\log^{L+M+N+P} T$$

$$\ll T\log^{L+M+N+P+1} T \sum_{j\leq\sqrt{T}} \frac{\phi(j)}{j^2},$$

by using $\sum_{n\leq x}\dfrac{1}{n} \ll \log x$ and $\sum_{n\leq x} n \ll x^2$, and thus

$$S_2 \ll \log^{L+M+N+P+2} T,$$

as $\displaystyle\sum_{j\leq\sqrt{T}} \frac{\phi(j)}{j^2} \ll \log T.$

Let us write $d = [d_1,d_2]$ and $n = dn'$. Then,

$$\sum_{\substack{n\equiv 0(\mathrm{mod}\, d)\\ n\leq T}} \log^M \frac{n}{d_1} \log^P \frac{n}{d_2} = \sum_{n'\leq\frac{T}{d}} \log^M \frac{dn'}{d_1} \log^P \frac{dn'}{d_2}$$

$$= \sum_{r=0}^{M}\sum_{s=0}^{P} \binom{M}{r}\binom{P}{s} \log^{M-r}\frac{d}{d_1} \log^{P-s}\frac{d}{d_2}$$

$$\times \Big(\sum_{n'\leq\frac{T}{d}} \log^{r+s} n'\Big)$$

which by using

$$\sum_{d\leq x} \log^a d = x\log^a x + O(x\log^{a-1} x)$$

is equal to

$$\frac{T}{d}\sum_{r=0}^{M}\sum_{s=0}^{P}\binom{M}{r}\binom{P}{s}\log^{M-r}\frac{d}{d_1}\log^{P-s}\frac{d}{d_2}\left\{\log^{r+s}\frac{T}{d}+O(\log^{r+s-1}\frac{T}{d})\right\}$$

$$= \frac{T}{d}\left\{\sum_{r=0}^{M}\binom{M}{r}\log^{M-r}\frac{d}{d_1}\log^{r}\frac{T}{d}\right\}\left\{\sum_{s=0}^{P}\binom{P}{s}\log^{P-s}\frac{d}{d_2}\log^{s}\frac{T}{d}\right\}$$

$$+O(\frac{T}{d}\log^{M+P-1}T)$$

$$= \frac{T}{d}\log^{M}\frac{T}{d_1}\log^{P}\frac{T}{d_2}+O(\frac{T}{d}\log^{M+P-1}T)$$

where for the error term, we have used simply the estimates $\log\frac{d}{d_i}\leq\log T$ and $\log\frac{T}{d}\leq\log T$. Hence,

$$S_1 = T\sum_{d_1,d_2\leq\sqrt{T}}\frac{1}{d}\log^{L}d_1\log^{N}d_2\left\{\log^{M}\frac{T}{d_1}\log^{P}\frac{T}{d_2}+O(\log^{M+P-1}T)\right\}$$

Now, writing again $d=\frac{d_1d_2}{(d_1,d_2)}$ and $(d_1,d_2) = \sum_{\substack{j|d_1\\ j|d_2}}\phi(j)$, we have the error term is equal to

$$O(T\log^{L+M+N+P-1}T\sum_{j\leq\sqrt{T}}\frac{\phi(j)}{j^2}\sum_{\substack{d_1',d_2'\\ \leq\sqrt{T}/j}}\frac{1}{d_1'd_2'})$$

$$=O(T\log^{L+M+N+P+2}T).$$

Thus we have proved Lemma 4.4.

Proof of Lemma 4.3. We write

$$(-1)^{A+B+C+D}\sum_{n\leq T}a(n,A,B)a(n,C,D)$$

$$= \sum_{n\le T} \left\{ \sum_{\substack{n_1|n \\ n_1\le\sqrt{n}}} \log^A n_1 \log^B \frac{n}{n_1} + \log^B n_1 \log^A \frac{n}{n_1} \right\}$$

$$\left\{ \sum_{\substack{d|n \\ d\le\sqrt{n}}} \log^C d \log^D \frac{n}{d} + \log^D d \log^C \frac{n}{d} \right\}$$

$$= T \sum_{d,n_1\le\sqrt{T}} \frac{1}{[n_1,d]} \left\{ \log^A n_1 \log^B \frac{T}{n_1} + \log^B n_1 \log^A \frac{T}{n_1} \right\}$$

$$\left\{ \log^C d \log^D \frac{T}{d} + \log^D d \log^C \frac{T}{d} \right\}$$

$$+O(T \log^{A+B+C+D+2} T) \tag{4.13}$$

by Lemma 4.4. Writing

$$\begin{aligned}
&\log^A n_1 \log^B \frac{T}{n_1} + \log^B n_1 \log^A \frac{T}{n_1} \\
&= \log^B T \log^A n_1 (1 - \frac{\log n_1}{\log T})^B + \log^A T \log^B n_1 (1 - \frac{\log n_1}{\log T})^A \\
&= \log^{A+B} T \left\{ (\frac{\log n_1}{\log T})^A (1 - \frac{\log n_1}{\log T})^B + (\frac{\log n_1}{\log T})^B (1 - \frac{\log n_1}{\log T})^A \right\} \\
&= \log^{A+B} T \sum_r C_{r,A,B} (\frac{\log n_1}{\log T})^r \\
&= \sum_r C_{r,A,B} \log^r n_1 \log^{A+B-r} T
\end{aligned}$$

and from (4.13), we have,

$$\begin{aligned}
&(-1)^{A+B+C+D} \sum_{n\le T} a(n,A,B) a(n,C,D) \\
&= T \sum_{r,r'} C_{r,A,B} C_{r',C,D} \log^{A+B+C+D-r-r'} T \left(\sum_{n_1,d\le\sqrt{T}} \frac{1}{[n_1,d]} \log^r n_1 \log^{r'} d \right) \\
&+O(T \log^{A+B+C+D+2} T)
\end{aligned} \tag{4.14}$$

We will now estimate the sum

$$\sum_{n_1,d\le\sqrt{T}} \frac{1}{[n_1,d]}\log^r n_1 \log^{r'} d = \sum_{n_1,d\le\sqrt{T}} \frac{\log^r n_1 \log^{r'} d}{n_1 d}(n_1,d)$$
$$= \sum_{j\le\sqrt{T}} \frac{\phi(j)}{j^2} \sum_{d,n_1\le\sqrt{T}} \frac{\log^r(n_1 j)\log^{r'}(dj)}{n_1 d}$$

By Lemma 4.2 (ii), we have

$$\sum_{n_1\le\frac{\sqrt{T}}{j}} \frac{\log^r n_1 j}{n_1} = \sum_{k=0}^{r}\binom{r}{k}\log^{r-k} j(\frac{\log^{k+1}\frac{\sqrt{T}}{j}}{k+1}) + O(\log^r T)$$

$$= \frac{1}{r+1}\sum_{k=0}^{r}\binom{r+1}{k+1}\log^{r-k} j \log^{k+1}\frac{\sqrt{T}}{j} + O(\log^r T)$$

$$= \frac{1}{r+1}\left\{\log^{r+1}\sqrt{T} - \log^{r+1} j\right\} + O(\log^r T)$$

Hence

$$\sum_{n_1,d\le\sqrt{T}} \frac{1}{[n_1,d]}\log^r n_1 \log^{r'} d$$
$$= \frac{1}{(r+1)(r'+1)}\sum_{j\le\sqrt{T}}\frac{\phi(j)}{j^2}\left\{\log^{r+1}\sqrt{T} - \log^{r+1} j\right\}\left\{\log^{r'+1}\sqrt{T} - \log^{r'+1} j\right\}$$
$$= \frac{1}{\zeta(2)(r+1)(r'+1)}\log^{r+r'+3}\sqrt{T}\left\{1 - \frac{1}{r+2} - \frac{1}{r'+2} + \frac{1}{r+r'+3}\right\}$$

$$+O(\log^{r+r'+2} T) \qquad (4.15)$$

by Lemma 4.2(i) Thus from (4.13), (4.14) and (4.15) we have

$$(-1^{A+B+C+D}\sum_{n\le T} a(n,A,B)a(n,C,D)$$
$$= \frac{T}{8\zeta(2)}\log^{A+B+C+D+3} T\sum_{r,r'}\frac{C_{r,A,B}C_{r',C,D}}{(r+1)(r'+1)2^{r+r'}}$$
$$\left\{1 - \frac{1}{r+2} - \frac{1}{r'+2} + \frac{1}{r+r'+3}\right\} + O(T\log^{A+B+C+D+2} T).$$

We can write the above double sum as

$$\begin{aligned}
&\sum_{r,r'} \frac{C_{r,A,B}C_{r',C,D}}{(r+1)(r'+1)2^{r+r'}} \int_0^1 (1-y^{r+1})(1-y^{r'+1})dy \\
&= \int_0^1 \left(\sum_r \frac{C_{r,A,B}(1-y^{r+1})}{(r+1)2^r}\right)\left(\sum_{r'} \frac{C_{r',C,D}(1-y^{r'+1})}{(r'+1)2^{r'}}\right) dy \\
&= \int_0^1 \left(\int_y^1 \sum_r C_{r,A,B}(\frac{x}{2})^r dx\right)\left(\int_y^1 \sum_{r'} C_{r',C,D}(\frac{x}{2})^{r'} dx\right) dy \\
&= \int_0^1 \left(\int_y^1 f(\frac{x}{2},A,B)dx\right)\left(\int_y^1 f(\frac{x}{2},C,D)\,dx\right) dy
\end{aligned}$$

Thus we have proved (i). (ii) follows now from (i) by partial summation. Now we are ready to evaluate the main terms in Theorem 4.2

Lemma 4.5.

i) $$\int_T^{2T} |J_1|^2 dt = C_1 T \log^{2\ell+2m+4} T + O(T(\log^{2\ell+2m+3} T)$$

ii) $$\int_T^{2T} |J_2|^2 dt = C_2 T log^{2\ell+2m+4} T + O(T(\log^{2\ell+2m+3} T)$$

where C_1 and C_2 are given by (4.4) and (4.5) respectively .

Proof. By Theorem 2.2, we have

$$\begin{aligned}
\int_T^{2T} |J_1|^2\, dt &= \sum_{n\le T} \frac{|a_n|^2}{n}(T+O(n)) \\
&= C_1 T \log^{2\ell+2m+4} T + O(T\log^{2\ell+2m+3} T)
\end{aligned}$$

by taking $A=C=\ell$ and $B=D=m$ in Lemma 4.3.

Next, to prove ii), we use Lemma 4.1 to write

$$\int_T^{2T} |J_2|^2\, dt = \sum_{k_1,k_1'=0}^{\ell} \sum_{k_2,k_2'=0}^{m} (-1)^{k_1+k_2+k_1'+k_2'} \begin{pmatrix} \ell \\ k_1 \end{pmatrix}\begin{pmatrix} \ell \\ k_1' \end{pmatrix}\begin{pmatrix} m \\ k_2 \end{pmatrix}\begin{pmatrix} m \\ k_2' \end{pmatrix}$$

$$\sum_{n,n'\le T} \frac{a(n,\ell-k_1,m-k_2)\,a(n',\ell-k_1',m-k_2')}{(nn')^{\frac{1}{2}}}$$

$$\int_T^{2T} \left\{ \frac{(-\log\frac{t}{2\pi})^{k_1+k_2+k_1'+k_2'}}{(\frac{n}{n'})^{it}} + O\left(\frac{(\log^{k_1+k_2+k_1'+k_2'-1} t}{t} \right) \right\} dt \tag{4.16}$$

The terms corresponding to $n = n'$ give

$$\sum_{k_1,k_1'=0}^{\ell} \sum_{k_2,k_2'=0}^{m} \binom{\ell}{k_1}\binom{m}{k_2}\binom{\ell}{k_1'}\binom{m}{k_2'}$$

$$\sum_{n\leq T} \frac{a(n,\ell-k_1,m-k_2)a(n,\ell-k_1',m-k_2')}{n}$$

$$\left\{T(\log^{k_1+k_2+k_1'+k_2'} T + O(T(\log^{k_1+k_2+k_1'+k_2'-1} T)\right\}$$

$$= \sum_{k_1,k_1'=0}^{\ell} \sum_{k_2,k_2'=0}^{m} (-1)^{k_1+k_2+k_1'+k_2'} \binom{\ell}{k_1}\binom{m}{k_2}\binom{\ell}{k_1'}\binom{m}{k_2'}$$

$$\frac{C(\ell-k_1,m-k_2,\ell-k_1',m-k_2')}{2\ell+2m-k_1-k_2-k_1'-k_2'+4} \frac{T}{8\zeta(2)} \log^{2\ell+2m+4} T$$

$$+O(T(\log^{2\ell+2m+3} T) \tag{4.17}$$

Now we will show that the 4-sum in (4.17) is equal to $8\zeta(2)C_2$.

Writing $\dfrac{1}{2\ell+2m-k_1-k_2-k_1'-k_2'+4}$ as $\int_0^1 t^{2\ell+2m-k_1-k_2-k_1'-k_2'+3}\,dt$, the 4-sum is equal to

$$\int_0^1 t^3\,dt \int_0^1 \left[\int_y^1 \left\{ \sum_{k_1=0}^{\ell} (-1)^{k_1} \binom{\ell}{k_1} (\frac{tx}{2})^{\ell-k_1} \sum_{k_2=0}^{m} (-1^{k_2} \binom{m}{k_2} (t(1-\frac{x}{2}))^{m-k_2} \right.\right.$$

$$\left.\left. + \sum_{k_1=0}^{\ell} (-1)^{k_1} \binom{\ell}{k_1} (t(1-\frac{x}{2}))^{m-k_1} \sum_{k_2=0}^{m} (-1)^{k_2} \binom{m}{k_2} (\frac{tx}{2})^{m-k_2} \right\} dx \right]$$

$$\times \left[\int_y^1 \left\{ \sum_{k_1'=0}^{\ell} (-1)^{k_1'} \binom{\ell}{k_1'} (\frac{tx}{2})^{\ell-k_1'} \sum_{k_2'=0}^{m} (-1)^{k_2} \binom{m}{k_2'} (t(1-\frac{x}{2}))^{m-k_2'} \right.\right.$$

$$\left.\left. + \sum_{k_1'=0}^{\ell} (-1)^{k_1'} \binom{\ell}{k_1'} (t(1-\frac{x}{2}))^{\ell-k_1'} \sum_{k_2'=0}^{m} (-1)^{k_2'} \binom{m}{k_2'} (\frac{tx}{2})^{m-k_2'} \right\} dx \right] dy$$

$$= \int_0^1 t^3\,dt \int_0^1 \left[\int_y^1 \left\{(1-\tfrac{tx}{2})^\ell (1-t(1-\tfrac{x}{2}))^m \right.\right.$$

$$\left.\left. +(1-\tfrac{tx}{2})^m (1-t(1-\tfrac{x}{2}))^\ell \right\} dx\right]^2 dy$$

$$= 8\zeta(2)C_2$$

When $n \neq n'$, we have

$$\int_T^{2T} \frac{(\log t)^k}{(\frac{n}{n'})^{it}}\,dt = -\frac{1}{i}\frac{\log^k 2T}{(\frac{n}{n'})^{2iT}\log\frac{n}{n'}} + \frac{1}{i}\frac{\log^k T}{(\frac{n}{n'})^{iT}\log\frac{n}{n'}}$$

$$+\frac{k}{i}\int_T^{2T} \frac{\log^{k-1} t}{t(\frac{n}{n'})^{it}\log\frac{n}{n'}}\,dt \tag{4.18}$$

Thus, when $t = T$ or $2T$, by Theorem 2.1, the sum

$$\sum_{k_1,k_1'=0}^{\ell} \sum_{k_2,k_2'=0}^{m} \sum_{\substack{n,n'\\ n\neq n'}} \frac{a(n,\ell-k_1,m-k_2)a(n',\ell-k_1',m-k_2')}{(nn')^{\frac{1}{2}}(\frac{n}{n'})^{it}\log\frac{n}{n'}}$$

$$\binom{\ell}{k_1}\binom{m}{k_2}\binom{\ell}{k_1'}\binom{m}{k_2'} \log^{k_1+k_2+k_1'+k_2'} t$$

$$= \sum_{\substack{n\neq n'\\ n,n'\leq T}} \frac{b(n,t)b(n',t)}{n^{\frac{1}{2}+it}n'^{\frac{1}{2}-it}\log\frac{n}{n'}}$$

$$\ll \sum_{n\leq T} b^2(n,t),$$

$$\ll T\log^{2\ell+2m+3} T$$

by the trivial estimate.

Again when $n \neq n'$, the sums in (4.16) corresponding to the integral in (4.18) can be written as

$$i\int_T^{2T}\frac{1}{t\log t}\sum_{\substack{n\neq n'\\ n,n'\leq T}}\left\{\sum_{n,n_2=n} b_1(n_1,t,\ell)\log^m\frac{n_2}{t})\right.$$

$$\left.\sum_{n_1'n_2'=n'}\log^\ell\frac{n_1'}{t}\log^m\frac{n_2'}{t}+\text{similar terms}\right\}\frac{1}{(\frac{n}{n'})^{it}\log\frac{n}{n'}}$$

$$\ll\ T\log^{2\ell+2m+2}T$$

by Theorem 2.1 and the inequality $|b_1(n,t,\ell)|\leq \ell b'(n,t)$ If we take the trivial estimate for the O-term in (4.16), we get

$$\log^{2\ell+2m-1}T\left(\sum_{n\leq T}\frac{d(n)}{n^{\frac{1}{2}}}\right)^2\ll T\log^{2\ell+2m+1}T$$

Thus we have shown that

$$\int_T^{2T}|J_2|^2\,dt\ =\ C_2T\log^{2\ell+2m+4}T+O(T\log^{2\ell+2m+3}T)$$

To estimate the remaining terms in (4.6) we need the following results.

bf Lemma 4.6. Let $s=\sigma+it$ and $|\sigma-\frac{1}{2}|\leq\frac{1}{100}$ and $T\leq t\leq 2T$ then $J_k(s)=O(T^{1-\sigma}\log^{\ell+m+1}T)$ for $k=1,\ldots..,6$

Proof.

$$\begin{aligned}|J_1(s)|\ &\leq\ \sum_{n\leq T}\frac{|a_n|}{n^\sigma}\\ &\leq\ \log^{\ell+m}T\sum_{n\leq T}\frac{d(n)}{n^\sigma}\\ &\ll\ T^{1-\sigma}\log^{\ell+m+1}T\end{aligned}$$

.

$$\begin{aligned}|J_2(s)|\ &\leq\ |\psi^2(s)|\sum_{n\leq T}\frac{b'(n,T)}{n^{1-\sigma}}\\ &\ll\ T^{1-2\sigma}\log^{\ell+m}T\sum_{n\leq T}\frac{d(n)}{n^{1-\sigma}},\end{aligned}$$

by Lemma 4.1 and taking trivial estimate for $b(n', T)$, and thus,

$$|J_2(s)| \ll T^{1-\sigma} \log^{\ell+m+1} T.$$

$J_3(s)$ and $J_4(s)$ can be similarly estimated. For estimating $J_5(s)$ and $J_6(s)$ see Lemma 3.2

Lemma 4.7. Let $s = \sigma + it$ and $|\sigma - \frac{1}{2}| \leq \frac{1}{100}$. Then

i) $\int_T^{2T} |J_1(s)|^2 \, dt = O(T^{2(1-\sigma)} \log^{2\ell+2m+3} T)$, if $\sigma \leq \frac{1}{2} - \frac{1}{10^7}$.

ii) $\int_T^{2T} |J_2(s)|^2 \, dt = O(T^{2(1-\sigma)} \log^{2\ell+2m+3} T)$, if $\sigma \geq \frac{1}{2} + \frac{1}{10^7}$.

iii) $\int_T^{2T} |J_k(s)|^2 \, dt = O(T^{2(1-\sigma)} \log^{2\ell+2m+3} T)$, for $k = 3$ *to* 6.

Proof. We sketch the proof of (ii). For proving (i) and (iii), look at the proof of Lemma 3.3

If $\sigma \geq \frac{1}{2} + \frac{1}{10^7}$, then

$$\int_T^{2T} |J_2(s)|^2 \, dt = \int_T^{2T} \sum_{k_1, k_1'=0}^{\ell} \sum_{k_2, k_2'=0}^{m} (-1)^{k_1+k_2+k_1'+k_2'}$$
$$\binom{\ell}{k_1}\binom{m}{k_2}\binom{\ell}{k_1'}\binom{m}{k_2'} \psi^{(k_1)}(s)\,\overline{\psi^{(k_1')}(s)}\,\psi^{(k_2)}(s)\,\overline{\psi^{(k_2')}(s)}$$
$$\sum_{n,n' \leq T} \frac{a(n, \ell - k_1, m - k_2)\, a(n', \ell - k_1', m - k_2')}{n^{1-s}\, n'^{1-\bar{s}}} \, dt$$

which by observing that $\overline{\psi^{(k)}(s)} = \psi^{(k)}(\bar{s}) = \psi^{(k)}(2\sigma - s)$ and by applying Lemma 4.1

$$= \sum_{k_1, k_1'=0}^{\ell} \sum_{k_2, k_2'=0}^{m} \binom{\ell}{k_1}\binom{m}{k_2}\binom{\ell}{k_1'}\binom{m}{k_2'}$$
$$\sum_{n,n' \leq T} \frac{a(n, \ell - k_1, m - k_2)\, a(n', \ell - k_1', m - k_2')}{(nn')^{1-\sigma}}$$

$$\times \int_T^{2T} \left\{ \frac{\psi^2(s)\psi^2(2\sigma - s)(\log\frac{t}{2\pi})^{k_1+k_2+k_1'+k_2'}}{(\frac{n}{n'})^{it}} + O\left(T^{2(1-2\sigma)}\frac{(\log t^{k_1+k_2+k_1'+k_2'})}{t}\right)\right\} dt \tag{4.19}$$

when $n = n'$ the first integral on the right hand side becomes

$$\int_T^{2T} \psi^2(s)\psi^2(2\sigma - s)(\log\frac{t}{2\pi})^{k_1+k_2+k_1'+k_2'}\,dt \sim \int_T^{2T} t^{2(1-2\sigma)}\,(\log t)^{k_1+k_2+k_1'+k_2'}\,dt,$$

by Stirling's formula and thus

$$= O\left(T^{2(1-2\sigma)+1}\log^{k_1+k_2+k_1'+k_2'} T\right)$$

Hence by Lemma 4.3 and Theorem 3.1 we have the first integral combined with all the sums in (4.19)

$$= O\left(T^{2(1-\sigma)}\log^{2\ell+2m+3} T\right)$$

when $n \neq n'$, we proceed as in Lemma 4.5 and obtain $O\left(T^{2(1-\sigma)}\log^{2\ell+2m+3} T\right)$

For the proof of Theorem 4.2, follow the proof of Theorem 3.3 and use Lemmas 4.5, 4.6 and 4.7.

Corollary 4.3.

a) $\int_1^T |\zeta(\frac{1}{2}+it)|^4\,dt = \frac{T}{2\pi^2}\log^4 T + O(T\log^3 T)$

b) $\int_1^T |\zeta(\frac{1}{2}+it)\zeta'(\frac{1}{2}+it|^2\,dt = \frac{2}{15\pi^2}T\log^6 T + O(T\log^5 T)$

c) $\int_1^T |\zeta'(\frac{1}{2}+it)|^4\,dt = \frac{61}{1680\pi^2}T\log^8 T + O(T\log^7 T)$

d) $\int_1^T |\zeta(\frac{1}{2}+it)\zeta''(\frac{1}{2}+it|^2\,dt = \frac{3}{56\pi^2}T\log^8 T + O(T\log^7 T)$

By the same method, one can also prove

Theorem 4.4.

$$\int_1^T |\zeta^{(\ell)}(\frac{1}{2}+it)|^2 dt = \frac{T}{2\ell+1}\log^{2\ell+1} T + O(T\log^{2\ell} T)$$

This result was first proved by Ingham [**In**]. He proved a more general result, namely,

$$\int_1^T \zeta^{(\ell)}(\frac{1}{2}+it)\zeta^{(m)}(\frac{1}{2}-it)\, dt \sim \frac{T}{\ell+m+1}\log^{\ell+m+1} T$$

Proof.

Let $X=\sqrt{T}, a(n,k)=(-1)^k \log^k n$ and $a_n = a(n,\ell)$

Then, by using the functional equation (4.1) of $\zeta^{(\ell)}(s)$, we get

$$\begin{aligned}
\zeta^{(\ell)}(\frac{1}{2}+it) &= \sum_{n\le X}\frac{a_n}{n^{\frac{1}{2}+it}} + \sum_{k=0}^{\ell}(-1)^{\ell-k}\begin{pmatrix}\ell\\ k\end{pmatrix}\psi^{(k)}(\frac{1}{2}+it)\sum_{n\le X}\frac{a(n,\ell-k)}{n^{\frac{1}{2}-it}}\\
&+\sum_{n\le X}\frac{a_n}{n^{\frac{1}{2}+it}}(e^{-n/X}-1)+\sum_{n>X}\frac{a_n}{n^{\frac{1}{2}+it}}e^{-n/X}\\
&-\frac{1}{2\pi i}\int_{(-\frac{3}{4})}\sum_{k=0}^{\ell}(-1)^{\ell-k}\begin{pmatrix}\ell\\ k\end{pmatrix}\psi^{(k)}(\frac{1}{2}+it+w)\sum_{n>X}\frac{a(n,\ell-k)}{n^{\frac{1}{2}-it-w}}\Gamma(w)X^w dw\\
&-\frac{1}{2\pi i}\int_{(\frac{1}{4})}\sum_{k=0}^{\ell}(-1)^{\ell-k}\begin{pmatrix}\ell\\ k\end{pmatrix}\psi^{(k)}(\frac{1}{2}+it+w)\sum_{n\le X}\frac{a(n,\ell-k)}{n^{\frac{1}{2}-it-w}}\Gamma(w)X^w dw\\
&+O(T^{-10})\\
&=\sum_{k=1}^{7} J_k \quad \text{(say)}
\end{aligned}$$

Then

$$\begin{aligned}
\int_T^{2T}|\zeta^{(\ell)}(\frac{1}{2}+it)|^2 dt &= \int_T^{2T}(|J_1|^2+|J_2|^2)dt + \sum_{k=2}^{6}\int_T^{2T}\Big(J_1\bar{J}_k\\
&+\bar{J}_1 J_k\Big)\, dt + \sum_{\substack{k=1\\ k\ne 2}}^{6}\int_T^{2T}(J_2\bar{J}_k+\bar{J}_2 J_k)dt\\
&+\sum_{k,k'=3}^{6}\int_T^{2T} J_k\bar{J}_{k'}dt + O(1)
\end{aligned}$$

We will only evaluate the main term. We have, by Theorem 2.2,

$$\int_T^{2T} |J_1|^2 dt = \sum_{n\le\sqrt{T}} \frac{\log^{2\ell} n}{n}(T+O(n))$$

$$= \frac{1}{2^{2\ell+1}(2\ell+1)} T\log^{2\ell+1} T + O(T\log^{2\ell+1})$$

by Lemma 4.2 (ii)

Using Lemma 4.1, we get,

$$\int_T^{2T} |J_2|^2 dt = \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell} (-1)^{2\ell-k_1-k_2} \binom{\ell}{k_1}\binom{\ell}{k_2} \sum_{n,n'\le\sqrt{T}} \frac{a(n,\ell-k_1)a(n',\ell-k_2)}{(nn')^{\frac{1}{2}}(\frac{n'}{n})^{it}}$$

$$\times\left(\int_T^{2T}\left\{(-1)^{k_1+k_2}\frac{\log^{k_1+k_2}\frac{t}{2\pi}}{(\frac{n'}{n})^{it}} + O\left(\frac{\log^{k_1+k_2-1} t}{t}\right)\right\} dt\right)$$

The terms corresponding to $n = n'$ contribute to the main term. Using Lemma 4.2 (ii), we get

$$\sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}(-1)^{k_1+k_2}\binom{\ell}{k_1}\binom{\ell}{k_2}\sum_{n\le\sqrt{T}}\frac{\log^{2\ell-k_1-k_2} n}{n} T\log^{k_1+k_2} T + O(T\log^{2\ell} T)$$

$$\begin{aligned}
&= T\log^{2\ell+1} T \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}(-1)^{k_1+k_2}\frac{\binom{\ell}{k_1}\binom{\ell}{k_2}}{2^{2\ell-k_1-k_2+1}(2\ell-k_1-k_2+1)} \\
&\quad +O(T\log^{2\ell} T) \\
&= \frac{1}{2}T\log^{2\ell+1} T\int_0^1 \sum_{k_1=0}^{\ell}\sum_{k_2=0}^{\ell}(-1)^{k_1+k_2}\binom{\ell}{k_1}\binom{\ell}{k_2}\left(\frac{x}{2}\right)^{2\ell-k_1-k_2} dx \\
&\quad +O(T\log^{2\ell} T) \\
&= \left\{\frac{1}{2}\int_0^1 (1-\frac{x}{2})^{2\ell} dx\right\} T\log^{2\ell+1} T + O(T\log^{2\ell} T) \\
&= \left\{\frac{1}{2\ell+1} - \frac{1}{2^{2\ell+1}(2\ell+1)}\right\} T\log^{2\ell+1} T + O(T\log^{2\ell} T)
\end{aligned}$$

Arguing as in Lemma 4.5, one can show that the terms corresponding to $n \neq n'$ contribute $O(T\log^{2\ell} T)$.

Thus, $\int_T^{2T}(|J_1|^2 + |J_2|^2)dt = \frac{T}{2\ell+1}\log^{2\ell+1} T + O(T\log^{2\ell} T)$.

In [C], J.B. Conrey proved that

$$\int_1^T A_1A_2(\frac{1}{2}+it)\,A_3A_4(\frac{1}{2}-it)\,dt \sim C(P_1,P_2,P_3,P_4)TL^4/(\pi^2/6) \tag{4.20}$$

where $A_i(s) = P_i(-\frac{1}{L}\frac{d}{ds})\zeta(s), 1 \le i \le 4$, with polynomials P_i and $L = \log\frac{T}{2\pi}$. He used Motohashi's method to obtain approximate functional equations for A_i and applied Ingham's method to prove this result.

Theorem 4.2 and Theorem 4.4 are special cases of this result and we have used Ramachandra's approximate functional equation and the theorem of Montgomery and Vaughan. Our method can give only mean square estimates. We cannot get a result of the type (4.20) using this method.

In (4.20) $C(P_1,P_2,P_3,P_4)$ is given by

$$\iiiint_{\substack{\alpha+\beta+\gamma+\delta\le 1\\ 0\le\alpha,\beta,\gamma,\delta\le 1}} (P_1(\alpha+\beta)P_2(\gamma+\delta)P_3(\alpha+\gamma)P_4(\beta+\delta)$$
$$+P_1(1-\alpha-\beta)P_2(1-\gamma-\delta)P_3(1-\alpha-\gamma)P_4(1-\beta-\delta))\,d\alpha\,d\beta\,d\gamma d\delta$$
$$= S_1 + S_2 \quad \text{(say)}. \tag{4.21}$$

When $P_1(x) = P_3(x) = x^\ell$ and $P_2(x) = P_4(x) = x^m$, we get Theorem 4.2. To prove this, we have to show that $C_1' + C_2' = S_1 + S_2$, where $C_i' = \zeta(2)C_i, i = 1,2$. In fact, we can prove that $C_1' = S_1$ and $C_2' = S_2$. By making the change of variable

$$\alpha+\beta = xt, \gamma+\delta = zt, \alpha+\beta+\gamma+\delta = t \quad \text{and} \quad \delta = yt$$

in S_1 and S_2, we find that the Jacobian of the transformation is t^3 and the conditions $0 \le \alpha,\beta,\gamma,\delta \le 1$ imply that

$$1 \le x+y+z \le 1+\frac{1}{t},$$

$$1-\frac{1}{t} \leq y+z \leq 1,$$
$$1-\frac{1}{t} \leq y+x \leq 1,$$

and

$$0 \leq t \leq 1.$$

Further we note that x, y and z are non-negative, as $\alpha+\beta = xt, \gamma+\delta = zt$ and $\delta = yt$ are non-negative.

Hence we have

$$0 \leq y+z \leq 1 \text{ and } 0 \leq y+x \leq 1.$$

These two conditions imply that

$$\begin{aligned} x+y+z &\leq 2-y \\ &\leq 1+\tfrac{1}{t} \end{aligned}$$

as $0 \leq t \leq 1$ and $y \geq 0$.

Hence we have

$$\begin{aligned} S_1 &= \int_0^1 \int_0^1 \iint_{\substack{x+y\leq 1 \\ y+z\leq 1}\ \substack{x+y+z\geq 1 \\ y\geq 0}} t^{2\ell+2m+3} x^\ell (1-x)^m z^\ell (1-z)^m \, dxdzdydt \\ &= \int_0^1 t^{2\ell+2m+3} \left(\int_0^1 \int_0^1 x^\ell (1-x)^m z^\ell (1-z)^m \int_{\max(0,1-x-z)}^{\min(1-x,1-z)} dydzdx \right) dt \\ &= \int_0^1 t^{2\ell+2m+3} \left(\int_0^1 \int_0^1 (xz)^\ell (1-x)^m (1-z)^m \right. \\ &\qquad \left. \min(x, z, 1-x, 1-z) dxdz \right) dt \qquad (4.22) \end{aligned}$$

Now, let us consider

$$\begin{aligned} C_1' &= \frac{1}{2}\int_0^1 t^{2\ell+2m+3} \int_0^1 \left(\int_{\frac{y}{2}}^{1-y/2} x^\ell(1-x)^m dx\right)\left(\int_{\frac{z}{2}}^{1-\frac{z}{2}} z^\ell(1-z)^m dz\right) dydt \\ &= \frac{1}{2}\int_0^1 t^{2\ell+2m+3} \int_0^1 \int_0^1 (xz)^\ell(1-x)^m(1-z)^m \\ &\qquad \left(\int_0^{2\min(x,z,1-x,1-z)} dy\right) dxdzdt \\ &= S_1 \end{aligned}$$

from (4.22)

Similarly we can prove that $C_2' = S_2$.

Also, when $A_2 = A_4 = 1$ and $P_1(x) = P_3(x) = (-1)^\ell x^\ell$, we get

$$\int_1^T |\zeta^{(\ell)}(\frac{1}{2}+it)|^2 dt \sim c(P_1, P_3)T\log^{2\ell+1} T$$

with

$$\begin{aligned} c(P_1, P_3) &= \int_0^1 P_1(x)P_3(x)dx \\ &= \frac{1}{2\ell+1}. \end{aligned}$$

which is Theorem 4.4.

Section 5. Mean value Theorems for L-functions

If q is a positive integer and χ is a character modulo q, then the function $L(s,\chi)$ is defined to be $\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$ for $Re.s>1$. We know that this series is absolutely convergent in $Re.s>1$ and is uniformly convergent in $Re.s\geq 1+\in$ and hence defines an analytic function in the half-plane to the right of 1. If χ is not a principal character modulo q, then this series is convergent for $Re.s>0$ and can be continued analytically to the whole complex plane. If χ is a principal character modulo q, then $L(s,\chi)$ can be continued analytically everywhere except for a simple pole at $s=1$. Further we know that if χ is a primitive character modulo q, then $L(s,\chi)$ satisfies the functional equation

$$L(s,\chi)=\psi(s,\chi)L(1-s,\bar{\chi}),$$

where

$$\psi(s,\chi)=\epsilon(\chi)\left(\frac{q}{\pi}\right)^{s-\frac{1}{2}}\Gamma\left(\frac{1-s+a}{2}\right)/\Gamma\left(\frac{s+a}{2}\right)$$

where

$$a = 0 \text{ and } \epsilon(\chi) = \tau(\chi)/q^{1/2}, \text{ if } \chi(-1)=1$$

$$a = 1 \text{ and } \epsilon(\chi) = \tau(\chi)/iq^{1/2}, \text{ if } \chi(-1)=-1,$$

where $\tau(\chi)$ is the character sum corresponding to χ.

In this section, we want to prove mean-value results concerning $L(s,\chi)$ which are uniform both in q and T. We find that the Theorem 3.1 and Theorem 3.3. can be applied directly to prove these results, provided that we estimate the function $G(s)$ appearing in Theorem 3.1 and its derivatives explicitly in terms of q and T.

Let χ_1 and χ_2 be primitive characters modulo q_1 and q_2 respectively. We write

$s = \sigma + it$ and

$$L(s,\chi_1)L(s,\chi_2) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{if} \quad \sigma > 1$$

where

$$a_n = \sum_{d|n} \chi_1(d)\chi_2(\frac{n}{d})$$

We know that $L(s,\chi_1)L(s,\chi_2)$ satisfies the functional equation

$$L(s,\chi_1)L(s,\chi_2) = \psi(s)L(1-s,\bar{\chi}_1)L(1-s,\bar{\chi}_2)$$

where

$$\psi(s) = \psi(s,\chi_1)\psi(s,\chi_2)\,.$$

By Stirling's formula, we have

$$(q_1q_2)^{\frac{1}{2}-\sigma}T^{1-2\sigma} \ll \psi(s) \ll (q_1q_2)^{\frac{1}{2}-\sigma}T^{1-2\sigma}\,,$$

if $\sigma_1 \le \sigma \le \sigma_2$ and $T \le t \le 2T$. If we choose $\lambda_n = \lambda_n^* = cn$ where $c = (q_1q_2)^{-1/2}$, $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}$, $F^*(s) = \frac{\bar{a}_n}{\lambda_n^{*s}}$, $x = y = T, h = 1, \eta = \frac{3}{4}$ and $\beta = \frac{1}{4}$ in Theorem 1.1, then by multiplying through out by c^s we get the approximate functional equation of $L(s,\chi_1)L(s,\chi_2)$ and we write it as

$$\begin{aligned} L(s,\chi_1)L(s,\chi_2) \;&=\; \sum_{n\le X} \frac{a_n}{n^s} \;+\; \psi(s)\sum_{n\le X} \frac{\bar{a}_n}{n^{1-s}} \;+\; \sum_{n\le X} \frac{a_n}{n^s}(e^{-n/X}-1) \\ &+\; \sum_{n>X} \frac{a_n}{n^s}e^{-n/X} - \frac{1}{2\pi i}\int_{(-\frac{3}{4})} \psi(s+w)\Bigg(\sum_{n>X} \frac{\bar{a}_n}{n^{1-s-w}}\Bigg)\Gamma(w)X^w dw \\ &-\; \frac{1}{2\pi i}\int_{(\frac{1}{4})} \psi(s+w)\Bigg(\sum_{n\le X} \frac{\bar{a}_n}{n^{1-s-w}}\Bigg)\Gamma(w)X^w dw \\ &+\; O(T^{-10}\log q_i) \end{aligned}$$

$$= \sum_{k=1}^{7} J_k(s,\chi_1,\chi_2) \quad \text{(say)} \tag{5.1}$$

where $X = c^{-1}T = \sqrt{q_1q_2}T$ and the last term appears only when q_1 or q_2 is equal to 1. Now, let $q = [q_1, q_2]$, the l.c.m. of q_1 and q_2 and χ_o denote the principal character modulo q. Then we prove the following results.

Theorem 5.1. If $\chi_1\bar{\chi}_2 = \chi_o(\text{mod} q)$, then

$$\int_1^T |L(\frac{1}{2}+it,\chi_1)L(\frac{1}{2}+it,\chi_2)|^2 dt =$$
$$\frac{1}{2\pi^2}\prod_{p|q}(1-\frac{1}{p})^2\prod_{p|q_1}(1+\frac{1}{p})^{-1}\prod_{p|q_2}(1+\frac{1}{p})^{-1}\prod_{p|(q_1,q_2)}(1-\frac{1}{p^2})$$
$$\left(T\log^4(\sqrt{q_1q_2}T)\right) + O\left(\sqrt{q_1q_2}T\log^3(\sqrt{q_1q_2}T)\right).$$

Theorem 5.2. If $\chi_1\bar{\chi}_2 \neq \chi_o(\text{mod} q)$, then

$$\int_1^T |L(\frac{1}{2}+it,\chi_1)L(\frac{1}{2}+it,\chi_2)|^2 dt =$$
$$\frac{6}{\pi^2}\prod_{p|q}(1+\frac{1}{p})^{-1}\prod_{p|(q_1,q_2)}(1-\frac{1}{p})|L(1,\chi_1\bar{\chi}_2)|^2 T\log^2(\sqrt{q_1q_2}T)$$
$$+O\left(\sqrt{q_1q_2}T\log(\sqrt{q_1q_2}T)|L(1,\chi_1\bar{\chi}_2)|^2\right).$$

The following Corollary is immediate from Theorem 5.1.

Corollary 5.3. If χ is a primitive character modulo d, then

$$\int_1^T |L(\frac{1}{2}+it,\chi)|^4 dt = \frac{1}{2\pi^2}\prod_{p|d}(1-\frac{1}{p})^3(1+\frac{1}{p})^{-1}T\log^4 dT + O\left(dT\log^3 dT\right).$$

From the proofs of Theorems 3.1 and 3.3, we know that it is sufficient to prove the following lemmas.

Lemma 5.1. If $\chi_1\bar{\chi}_2 = \chi_o(\mathrm{mod} q)$, then

i) $\sum_{n\leq y} |a_n|^2 = O(y\log^3 y)$

ii) $$\sum_{n\leq X} \frac{|a_n|^2}{n} = \frac{1}{4\prod^2}\prod_{p|q}(1-\frac{1}{p})^2\prod_{p|q_1}(1+\frac{1}{p})^{-1}\prod_{p|q_2}(1+\frac{1}{p})^{-1}\prod_{p|(q_1,q_2)}(1-\frac{1}{p^2})$$

$$(T\log^4 X) + O\left(\log^3 X\log\log 3q\right)$$

Lemma 5.2. If $\chi_1\bar{\chi}_2 \neq \chi_o(\mathrm{mod} q)$, then

i) $\sum_{n\leq y} |a_n|^2 = O(y(\log y)|L(1,\chi_1\bar{\chi}_2)|^2)$

ii) $$\sum_{n\leq X} \frac{|a_n|^2}{n} = \frac{3}{\pi^2}\prod_{p|q_1}(1+\frac{1}{p})^{-1}\prod_{p|(q_1,q_2)}(1-\frac{1}{p})|L(1,\chi_1\bar{\chi}_2)|^2\log^2 X$$

$$+O\left(\sqrt{q_1 q_2}\log X\right)$$

Since a_n's are multiplicative, we have

$$\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^s} = \prod_p(1+\frac{|a_p|^2}{p^s}+\frac{|a_{p^2}|^2}{p^{2s}}+\ldots) \quad \text{if} \quad \sigma>1$$

$$= \prod_{p\nmid q}(1-\frac{1}{p^s})^{-2}(1-\frac{1}{p^{2s}})(1-\frac{\chi_1\bar{\chi}_2(p)}{p^s})^{-1}(1-\frac{\bar{\chi}_1\chi_2(p)}{p^s})^{-1}$$

$$(\prod_{p|q}(1-\frac{1}{p^s})^{-1}\prod_{p|(q_1,q_2)}(1-\frac{1}{p^s})) \tag{5.2}$$

$$= \frac{\zeta^2(s)}{\zeta(2s)}L(s,\chi_1\bar{\chi}_2)L(s,\bar{\chi}_1\chi_2)\prod_{p|q_1}(1+\frac{1}{p^s})^{-1}$$

$$\prod_{p|q_2}(1+\frac{1}{p^s})^{-1}\prod_{p|(q_1,q_2)}(1-\frac{1}{p^{2s}})$$

Proof of Lemma 5.1. (i) is trivial by noting that $|a_n| \le d(n)$. Since $\chi_1\bar{\chi}_2 = \chi_o \pmod q$, we have by (5.2) that

$$\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)} G_1(s) \quad \text{if} \quad \sigma > 1$$

where

$$G_1(s) = \prod_{p|q}(1-\frac{1}{p^s})^2 \prod_{p|q_1}(1+\frac{1}{p^s})^{-1} \prod_{p|q_2}(1+\frac{1}{p^s})^{-1} \prod_{p|(q_1,q_2)}(1-\frac{1}{p^{2s}})$$

By Perron's formula, we have

$$\sum_{n\le X} \frac{|a_n|^2}{n} = \frac{1}{2\pi i}\int_{c-iT_1}^{c+iT_1} \frac{\zeta^4(s+1)}{\zeta(2s+2)} G_1(s+1)\frac{X^s}{s}ds + O(\frac{X^c}{T_1}\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{1+c}|\log\frac{X}{n}|})$$

where we choose $c = \dfrac{1}{\log X}$ and $T_1 = X^{\frac{1}{8}}$.
Since $|a_n| \le d(n)$, we can easily show that the error term is $O(1)$. If we move the line of integration to $\sigma = -1/4$, then we pick up a pole of the integrand at $s = 0$, and the residue at $s = 0$ is equal to

$$\frac{G_1(1)}{4!\zeta(2)}\log^4 X + O(\log^3 X G_1'(1) + \log^2 X G_1''(1) + \log X G_1'''(1) + G_1^{(iv)}(1)) \qquad (5.3)$$

By taking lagarithmic differentiation, we get

$$G_1'(s) = G_1(s)\left\{2\sum_{p|q}\frac{\log p}{p^s-1} + \sum_{p|q}\frac{\log p}{p^s+1} + \sum_{p|(q_1,q_2)}\frac{\log p}{p^s-1}\right\}$$

and hence

$$G_1'(1) \ll \log\log 3q\,.$$

Differentiating $G_1'(s)$ successively, we can show that

$$G_1^{(k)}(1) \ll (\log\log 3q)^k \quad \text{for} \quad k = 2,3,4\,.$$

Since $q_1q_2 \geq q$, the error term in (5.3) is equal to $O(\log^3 X \log\log 3q)$. This completes the proof of Lemma 5.1.

Proof of Lemma 5.2. If $\chi_1\bar{\chi}_2 \neq \chi_o \pmod q$, then from (5.2), we have

$$\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)} L(s,\chi_1\bar{\chi}_2)L(s,\bar{\chi}_1\chi_2)G_2(s), \quad \text{if} \quad Re.s > 1$$

where

$$G_2(s) = \prod_{p|q_1}(1+\frac{1}{p^s})^{-1}\prod_{p|q_2}(1+\frac{1}{p^s})^{-1}\prod_{p|(q_1,q_2)}(1-\frac{1}{p^{2s}})$$

We write

$$L(s,\chi_1\bar{\chi}_2)L(s,\bar{\chi}_1\chi_2)G_2(s)\zeta(2s)^{-1} = \sum_{n=1}^{\infty}\frac{b(n)}{n^s}$$

and this series is convergent in $Re.s > \frac{1}{2}$. Hence,

$$\begin{aligned}
\sum_{n\leq y}|a_n|^2 &= \sum_{\ell m\leq y} d(\ell)b(m) \\
&= \sum_{m\leq y} b(m)\sum_{\ell\leq \frac{y}{m}} d(\ell) \\
&= y\log y\sum_{m\leq y}\frac{b(m)}{m} - y\sum_{m\leq y}\frac{b(m)}{m}\log m \\
&\quad + (2\gamma-1)y\sum_{m\leq y}\frac{b(m)}{m} + O(y^{1/2}\sum_{m\leq y}\frac{|b(m)|}{m^{1/2}})
\end{aligned}$$

It is easy to see from the Euler product of $\sum_{n=1}^{\infty}\frac{b(n)}{n^s}$ that $|b(n)| \leq d(n)$ and hence the error term becomes $O(y\log y)$.

Also, since the series $\sum_{m=1}^{\infty}\frac{b(m)}{m^s}$ is convergent, we have

$$\left|\sum_{m\leq y}\frac{b(m)}{m}\right| \ll \left|\sum_{m=1}^{\infty}\frac{b(m)}{m}\right| \ll |L(1,\chi_1\bar{\chi}_2)|^2$$

and

$$\left|\sum_{n\leq y}\frac{b(m)\log m}{m}\right| \ll (\log\ y)|L(1,\chi_1\bar{\chi}_2)|^2$$

by Abel's summation.

Thus we have

$$\sum_{n\le y} |a_n|^2 = O(y(\log y)\, |L(1,\chi_1\bar\chi_2)|^2)$$

and this proves (i).

To prove (ii), we proceed as in Lemma 5.1 (ii) and move the line of integration to $Re.s = -\frac{1}{10}$ and use the estimate

$$|L(1,\chi_s\bar\chi_2)| \ll (q|t|)^\delta \quad \text{if} \quad \sigma \ge 1-\delta \quad \text{and} \quad |t| \ge 2\,.$$

The residue of the integrand at the pole at $s = 0$ is equal to

$$\frac{1}{2\zeta(2)} G_2(1)|L(1,\chi_1\bar\chi_2)|^2 \log^2 X \; + \; O(\log X|L'(1,\chi_1\bar\chi_2)||L(1,\bar\chi_1\chi_2)|)$$

$$+O(\log X|L(1,\chi_1\bar\chi_2)|^2 \log\log 3q) + O(|L''(1,\chi_1\bar\chi_2)||L(1,\bar\chi_1\chi_2)|)$$

$$+O(|L(1,\chi_1\bar\chi_2)|^2(\log\log 3q)^2)$$

The integral

$$\int_{-T_1}^{T_1} \frac{\zeta^2(\frac{9}{10}+it)L(\frac{9}{10}+it,\chi_1\bar\chi_2)L(\frac{9}{10}+it,\bar\chi_1\chi_2)}{\zeta(\frac{9}{5}+2it)} \; \frac{G_2(\frac{9}{10}+it)}{(\frac{9}{10}+it)} X^{-\frac{1}{10}+it} dt$$

$$\ll \; q^{1/5}T_1^{2/5}X^{\frac{-1}{10}} \exp(C(\log q)^{1/10})$$

$$\ll \; 1$$

by choosing $T_1 = X^{1/8}$

This proves (ii).

Proof of Theorem 5.1 and 5.2

From (5.1), we have

$$\int_T^{2T} |L(\frac{1}{2}+it,\chi_1)L(\frac{1}{2}+it,\chi_2)|^2 dt = 2\int_T^{2T} |J_1|^2 dt$$

$$+O(\sum_{k=3}^{6}\int_T^{2T}(J_1+J_2)\bar{J}_k dt) + O(\sum_{k=3}^{6}\int_T^{2T}|J_k|^2 dt) + O(1)$$

Following the proof of Theorem 3.3 and using (i) of Lemmas 5.1 and 5.2, we can easily prove that each error term =

$O(\sqrt{q_1q_2}T\log^3\sqrt{q_1q_2}T)$ if $\chi_1\bar{\chi}_2 = \chi_o(\text{mod} q)$

and

$O(\sqrt{q_1q_2}T(\log\sqrt{q_1q_2}T)|L(1,\chi_1\bar{\chi}_2)|^2)$ if $\chi_1\bar{\chi}_2 \neq \chi_o(\text{mod} q)$.

The main term

$$2\int_T^{2T}|J_1|^2 dt = 2\sum_{n\leq X}\frac{|a_n|^2}{n} + O(\sum_{n\leq X}|a_n|^2)$$

and now we appeal to (ii) of Lemmas 5.1 and 5.2. This completes the proof of Theorems 5.1 and 5.2.

As a special case of Theorem 5.2, we arrive at the following asymptotic formula for the mean square of the Dedekind-zeta function of a quadratic number field.

Corollary 5.4

If K is a quadratic number field with discriminant D, and if $\zeta_K(s)$ denotes the zeta-function of K, then

$$\int_1^T |\zeta_K(\frac{1}{2}+it)|^2 dt = \frac{6}{\pi^2}\prod_{p|D}(1+\frac{1}{p})^{-1}|R_K|^2 T\log^2(\sqrt{|D|}T)$$

$$+ O(\sqrt{|D|}T\log(\sqrt{|D|}T)|R_K|^2)$$

where R_K is the residue of $\zeta_K(s)$ at the pole at $s = 1$.

Proof . If K is a quadratic field, we know that $\zeta_K(s)$ has the representation

$$\zeta_K(s) = \zeta(s)L(s,\chi)$$

where $L(s,\chi)$ is the L-series for the character χ with modulus $|D|$. (See for example [**B-S**]). Further, we know that χ is primitive mod $|D|$. Also $\zeta_K(s)$ has a pole at $s = 1$ with the residue $R_K = L(1,\chi)$.

Now the Corollary follows by appealing to Theorem 5.2.

Deriving Corollary 5.4 as a special case of Theorem 5.2 has improved the result of Hinz [**Hi**] who also used Ramachandra's method to get an asymptotic formula for $\int_1^T |\zeta_K(\frac{1}{2}+it)|^2 dt$, and his result is given below.

$$\int_1^T |\zeta_K(\frac{1}{2}+it,\chi)|^2 dt = \frac{6}{\pi^2}\prod_{p|D}(1+\frac{1}{p})^{-1}|R_k|^2 T\log^2 T + O(D^{1+\epsilon}T\log T)$$

for any $\epsilon > 0$.

Influenced by D.R. Heath-Brown's result on the mean fourth power of the zeta-function, Wolfgang Müller [**Mü**] has substantially improved this result by showing that

$$\int_1^T |\zeta_K(\frac{1}{2}+it)|^2 dt = a_2 T\log^2 T + a_1 T\log T + a_0 T + O_\epsilon(|d|^{35/16+\epsilon}T^{7/8+\epsilon})$$

where a_0, a_1 and a_2 can be calculated explicitly.

Theorems 5.1 and 5.2 were stated in a slightly weaker form in K.Ramachandra [**Ram1**], as follows :

If χ_1 and χ_2 are any two characters modulo q_1 and q_2 then

$$\int_1^T |L(\frac{1}{2}+it,\chi_1)L(\frac{1}{2}+it,\chi_2)|^2 dt = \begin{cases} C_1 T(\log T)^4 + O(T(\log T)^3) \\ \qquad \text{if} \chi_1\bar{\chi}_2 = \chi_0(mod q) \\ C_2 T(\log T)^2 + O(T(\log T)) \\ \qquad \text{if} \chi_1\bar{\chi}_2 \neq \chi_0(mod q) \end{cases}$$

where the constants C_1 and C_2 and those implied the O's depend on q_1 and q_2.

In the remaining part of this section, we want to discuss the averages $\sum^*_{\chi(mod q)} \int_1^T |L(\frac{1}{2}+it,\chi)|^4 dt$ and $\sum^*_{\chi(mod q)} \int_1^T |L(\frac{1}{2}+it,\chi)|^4 dt$ where $\sum^*_{\chi \ mod \ q}$ denotes the summation over primitive characters modulo q and $\sum_{\chi \ mod \ q}$ denotes the summation over all characters modulo q. An asymptotic formula for the former sum has been obtained V.V.Rane [**R**], the proof of which is based on the method of Ramachandra. Hence we state this result and give only a sketch of the proof. For the details, one is referred to Rane's paper. Regarding the latter sum, we obtain only an upper bound, see Theorem 5.7. We derive this easily by establishing an upper bound for $\sum^*_{\chi} \int_1^T |L(\frac{1}{2}+it,\chi)|^4 dt$, i.e., we prove that

$$\sum_{\chi(mod d)}^{*} \int_1^T |L(\frac{1}{2}+it,\chi)|^4 dt = O(\phi(d) T \log^4 dT)$$

.

This result has already been proved by Montgomery [M3]. Montgomery's proof makes use of Lavrik's approximate functional equation of $L(s,\chi)$ while we will be using that of Ramachandra to prove the same.

Before stating the results, we introduce the notation.

N denotes the number of primitive characters modulo q. We assume $N \geq 1$. It should be noted that $N \geq 1$ if and only if either q is odd or $4|q$. We write $q = p_1^{\ell_1} \dots p_r^{\ell_r}$ where $p_i's$ are primes and r = number of distinct prime factors of q.

Also,

$$N = \prod_{i=1}^{r} \theta(p_i^{\ell_i})$$

where

$$\theta(p_i^{\ell_i}) = \begin{cases} p_i - 2 & \text{for } \ell_i = 1 \\ p_i^{\ell_i - 2}(p_i - 2)^2, & \text{for } \ell_i \geq 2 \end{cases}$$

for $i = 1, 2, \dots r$.

By the multiplicativity property of the primitive characters we can show that $N = \sum_{d|q} \mu(d)\phi(q/d)$

Now we state the theorem.

Theorem 5.5.

$$\sum_{\chi (mod\ q)}^{*} \int_1^T |L(\frac{1}{2} + it, \chi)|^4 dt = \frac{N}{2\pi^2} \prod_{p|q} (1 - \frac{1}{p})^3 (1 + \frac{1}{p})^{-1} T \log^4 qT$$
$$+ O\left(2^r qT \log^3 qT \log\log 3q\right)$$

The proof of this theorem depends heavily on Corollary 2.6 (i) which is ,in turn, a consequence of Theorem 2.1.

Taking $q_1 = q_2 = q$ and $\chi_1 = \chi_2 = \chi$ in (5.1), we get the approximate functional equation of $L^2(s, \chi)$ and

$$L^2(s, \chi) = \sum_{k=1}^{7} J_k(s, \chi)$$

where $X = qT$, $a_n = d(n)\chi(n)$ and $J_7(s, \chi)$ appears only when $q = 1$. We write $J_k(\frac{1}{2} + it, \chi) = J_k(\chi)$. Squaring and integrating $|L^2(\frac{1}{2} + it, \chi)|$, and then summing over all the primitive characters χ (mod q), we get

$$\begin{aligned}\textstyle\sum_\chi^* \int_T^{2T} |L(\frac{1}{2}+it,\chi|^4 dt &= 2\textstyle\sum_\chi^* \int_T^{2T} |J_1(\chi)|^2 dt \\ &+O(\textstyle\sum_\chi^* \int_T^{2T} J_1(\chi)\overline{J_2(\chi)}dt) \\ &+O(\textstyle\sum_\chi^* \int_T^{2T} (J_1(\chi)+J_2(\chi))(\sum_{k=3}^6 \overline{J_k(\chi)})dt) \\ &+O(\textstyle\sum_\chi^* \int_T^{2T} \sum_{k=3}^6 |J_k(\chi)|^2 dt) + O(1)\end{aligned} \tag{5.3}$$

The main term is calculated in the following lemma.

Lemma 5.3.

$$\begin{aligned}\sum_\chi^* \int |J_1^2(\chi)| dt &= NT \sum_{\substack{n\le X\\(n,q)=1}} \frac{d^2(n)}{n} + O(2^r qT\log^3 X \log\log 3q) \\ &= \frac{N}{2\pi^2}\prod_{p|q}(1-\frac{1}{p})^3(1+\frac{1}{p})^{-1}T\log^4 X \\ &\quad +O(2^r qT\log^3 X\log\log 3q)\end{aligned}$$

Proof . Using Corollary 2.6 (i), we get

$$\begin{aligned}\sum_\chi^* \int_T^{2T} |J_1^2(\chi)| dt &= NT \sum_{\substack{n\le X\\(n,q)=1}} \frac{d^2(n)}{n} + O(2^r q \max_{k|q} \sum_{n\le \frac{k}{4}} \frac{d^2(n)}{n\log\frac{k}{n}}) \\ &\quad +O(2^r \sum_{n\le X} d^2(n))\end{aligned}$$

The second error term is easily seen to be $O(2^r X\log^3 X)$

We now prove

$$\sum_{n\le\frac{k}{4}} \frac{d^2(n)}{n\log\frac{k}{n}} \ll \log^3 k\log\log 3k\,.$$

This will give the first error term to be

$$\ll 2^r q \log^3 q \log\log 3k\,.$$

Writing $K = [\frac{\log k}{\log 2}]$, we have

$$\begin{aligned}
\sum_{n\le\frac{k}{4}} \frac{d^2(n)}{n\log\frac{k}{n}} &= \sum_{0\le m\le K-1}\left(\sum_{2^m<n\le 2^{m+1}} \frac{d^2(n)}{n\log\frac{k}{n}}\right)\\
&\le \sum_{0\le m\le k-1} \frac{1}{2^m(\log k/2^{m+1})}\Big(\sum_{2^m<n\le 2^{m+1}} d^2(n)\Big)\\
&\ll \sum_{0\le m\le K} \frac{m^3}{\frac{\log K}{\log 2}-m}
\end{aligned}$$

$$\ll \log^3 k \log\log 3k\,.$$

By Lemma 5.1 (ii), we have

$$\sum_{\substack{n\le X\\(n,q)=1}} \frac{d^2(n)}{n} = \frac{1}{4\pi^2}\prod_{p|q}(1-\frac{1}{p})^3(1+\frac{1}{p})^{-1}\log^4 X + O(\log^3 X\log\log 3q)$$

This completes the proof of Lemma 5.3.

Using Corollary 2.4, we can prove the following

Lemma 5.4.

i) $\sum_\chi^* \int_T^{2T} |J_k^2(s,\chi)|dt \ll 2^r X^{2-2\sigma}\log^3 X$ if $|\sigma-1/2|\le\frac{1}{100}$, for $k=3$ to 6.

ii) $\sum_\chi^* \int_T^{2T} |J_1^2(s,\chi)|dt \ll 2^r X^{2-2\sigma}\log^3 X$ if $\sigma\le\frac{1}{2}-\frac{1}{10^7}$ and

iii) $\sum_\chi^* \int_T^{2T} |J_2^2(s,\chi)|dt \ll 2^r X^{2-2\sigma}\log^3 X$ if $\sigma\ge\frac{1}{2}+\frac{1}{10^7}$.

Proof. We will prove only (ii).

By Corollary 2.4 (i), we have

$$\sum_{\chi}^{*}\int_{T}^{2T}|J_1^2(s,\chi)|dt = NT\sum\frac{d^2(n)}{n^{2\sigma}} + O(2^r\max_{k|q} k\sum_{n\leq\frac{k}{4}}\frac{d^2(n)}{n^{2\sigma}\log\frac{k}{n}})$$

$$+ O(2^r\sum_{n\leq X}\frac{d^2(n)}{n^{2\sigma-1}})$$

The first and third terms are respectively,

$$O(NTX^{1-2\sigma}\log^3 X) \quad \text{and} \quad O(2^r X^{2-2\sigma}\log^3 X) \quad \text{if} \quad \sigma\leq\frac{1}{2}-\frac{1}{10^7}.$$

and the second term is

$$\ll 2^r q^{2-2\sigma}\log^3 q \quad \text{as} \quad \sigma\leq\frac{1}{2}-\frac{1}{10^7}.$$

Thus Lemma 5.4 (ii) is proved.

For the proof of the other parts, see [**R**].

Lemma 5.5. Let $s=\sigma+it$ with $t=T$ or $2T$. Then

i) $\int_{\sigma_o}^{\frac{1}{2}} d\sigma \sum_{\chi}^{*}|J_1^2(s,\chi)| \ll 2^r X^{2-2\sigma}\log^3 X\log\log 3q$
where $\sigma_o=\frac{1}{2}-\frac{1}{10^7}$.

ii) $\int_{\frac{1}{2}}^{\sigma_1} d\sigma \sum_{\chi}^{*}|J_2^2(s,\chi)| \ll 2^r X^{2-2\sigma_1}\log^3 X\log\log 3q$
where $\sigma_1=\frac{1}{2}+\frac{1}{10^7}$.

iii) $\sum_{\chi}^{*}|J_k^2(s,\chi)| \ll 2^r X\log^3 X$
for $\frac{1}{2}-\frac{1}{10^7}\leq\sigma\leq\frac{1}{2}+\frac{1}{10^7}$ and $3\leq k\leq 6$.

iv) $\int_{\sigma_o}^{\frac{1}{2}} d\sigma \sum_{\chi}^{*}|J_1(s,\chi)\overline{J_k(s,\chi)}| \ll 2^r X\log^3 X\log\log 3q$

v) $\int_{\frac{1}{2}}^{\sigma_1} d\sigma \sum_{\chi}^{*}|J_2(s,\chi)\overline{J_k(s,\chi)}| \ll 2^r X\log^3 X\log\log 3q$
for $3\leq k\leq 6$.

Proof . See [**R**].

Proof of Theorem 5.5 . The last error term in (5.3) is equal to $O(2^r qT\log^3 X)$ by Lemma 5.4(i).

By moving the line of integration to $\sigma_o = \frac{1}{2} - \frac{1}{10^7}$, the first O-term

$$\begin{aligned} &= \sum_{\chi}^{*} \int_T^{2T} J_1(\chi)\overline{J_2(\chi)}dt \\ &= O\left(\sum_{\chi}^{*} \int_{\substack{T \\ \sigma=\sigma_o}}^{2T} J_1(s,\chi)J_2(1-s,\bar{\chi})dt\right) \\ &\quad + O\left(\sum_{\chi}^{*} \int_{\sigma_o}^{1/2} J_1(s,\chi)J_2(1-s,\bar{\chi})dt\right) \\ &= O(2^r X\log^3 X\log\log 3q) \end{aligned}$$

by Lemma 5.4 (ii), (iii) and Lemma 5.5 (iv). The second error term is also similarly calculated. We move the line of integration to the left of 1/2 whenever the integrand has $J_1(\chi)$ in it and we move to the right of 1/2 whenever $J_2(\chi)$ is present in the integrand.

This completes the proof of Theorem 5.5.

Next, we prove

Theorem 5.6.

$$\sum_{\chi \bmod q}^{*} \int_1^T \int |L(\frac{1}{2}+it,\chi)|^4 dt = O(\phi(q)T\log^4 qT)$$

Proof .

Since we want only an upper bound, we use the inequality

$$\sum_{\chi(\bmod q)}^{*} \int_T^{2T} |J_k(\chi)|^2 dt \le \sum_{\chi(\bmod q)} \int_T^{2T} |J_k(\chi)|^2 dt$$

and then apply Corollary 2.6 (ii).

Also, we don't have to worry about moving the line of integration away from $\sigma = 1/2$. In fact, we can prove the following :

i) $\sum_\chi \int_T^{2T} |J_1^2(\chi)| dt \ll \phi(q)T(\log qT)^4$

ii) $\sum_\chi \int_T^{2T} |J_2^2(\chi)| dt \ll \phi(q)T(\log qT)^4$ and

iii) $\sum_\chi \int_T^{2T} |J_k^2(\chi)| dt \ll \phi(q)T\log^3 qT$ *for* $k = 3$ *to* 6.

We prove (i), and (iii) for k = 5.

Proof of (i). By Corollary 2.4 (ii),

$$\sum_\chi \int_T^{2T} |J_1^2(\chi)| \ll \phi(q)T\sum_{n\le X}\frac{d^2(n)}{n} + \phi(q)\sum_{j\le \frac{q}{4}}\frac{d^2(j)}{j\log\frac{2q}{j}}$$
$$+\frac{\phi(q)}{q}\sum_{\substack{n>q\\(n,q)=1}} d^2(n)$$

The first and the third terms are respectively

$$O(\phi(q)T\log^4 qT) \quad \text{and} \quad O(\phi(q)T\log^3 qT)$$

where as the second term is $= O(\phi(q)\log^3 q\log\log 3q)$ (see the proof of Lemma 5.3).

Proof of (iii) for $k = 5$. We can break the integral

$$\frac{1}{2\pi i}\int_{(-\frac{3}{4})}\psi(s+w,\chi)(\sum_{n>X}\frac{\bar\chi(n)d(n)}{n^{1-s-w}})\Gamma(w)X^w dw$$

at $|v| \le \log^3 X$ with a small error.

Hence, if $s = \sigma + it = \frac{1}{2} + it$, then

$\sum_\chi \int_T^{2T} |J_5(\chi)|^2 dt$

$$\ll \sum_\chi \int_T^{2T}\left|\frac{1}{2\pi}\int_{(-\frac{3}{4},\log^3 X)}\psi(s+w,\chi)(\sum_{n>X}\frac{d(n)\bar\chi(n)}{n^{1-s-w}})\Gamma(w)X^w dw\right|^2 dt$$

which, by applying Hölder's inequality and interchanging the order of integration is

$$\ll X^{2(1-2\sigma)+\frac{3}{2}}\int_{(-\frac{3}{4},log^3 X)}|\Gamma(w)|dv\sum_\chi\int_T^{2T}\left|\sum_{n>X}\frac{d(n)\bar\chi(n)}{n^{1-s-w}}\right|^2 dt$$
$$\ll X^{3/2}\left\{\phi(q)T\sum_{n>X}\frac{d^2(n)}{n^{2-2\sigma+\frac{3}{2}}} + \frac{\phi(q)}{q}(\sum_{n>X}\frac{d^2(n)}{n^{1-2\sigma+3/2}})\right\}$$
$$\ll \phi(q)T\log^3 qT.$$

We can now easily derive an upper bound for $\sum_{\chi(\text{mod}q)} \int_1^T |L(\frac{1}{2}+it,\chi)|^4 dt$ from Theorem 5.6.

Theorem 5.7.

$$\sum_{\chi(\text{mod}q)} \int_1^T |L(\frac{1}{2}+it,\chi)|^4 dt = O(qT\log^4 qT\exp(c\sqrt{\log q}))$$

where c is an absolute constant.

Proof . If χ (mod q) is a character induced by a primitive character χ^* (mod d) for some d dividing q, then we can write the L-series $L(s,\chi)$ in terms of $L(s,\chi^*)$ as

$$L(s,\chi) = \prod_{p|q}(1-\frac{\chi^*(p)}{p^s})L(s,\chi^*).$$

Hence,

$$\begin{aligned}
&\sum_{\chi(\text{mod}q)} \int_1^T |L(\frac{1}{2}+it,\chi)|^4 dt \\
= \ & \sum_{d|q} \sideset{}{^*}\sum_{\chi^*(\text{mod}d)} \int_1^T |\prod_{p|q}(1-\frac{\chi^*(p)}{p^{\frac{1}{2}+it}})L(\frac{1}{2}+it,\chi^*)|^4 dt \\
\leq \ & \prod_{p|q}(1+\frac{1}{p^{\frac{1}{2}}})^4 \sum_{d|q} \sideset{}{^*}\sum_{\chi^*(\text{mod}d)} |L(\frac{1}{2}+it,\chi^*)|^4 dt \\
\ll \ & \prod_{p|q}(1+\frac{1}{p^{\frac{1}{2}}})^4 \sum_{d|q} \phi(d)T\log^4 dT \\
\ll \ & qT\log^4 qT\exp(c\sqrt{\log q})
\end{aligned}$$

where we have applied Theorem 5.6 to the sum $\sum^*_{\chi^*(\text{mod}d)}$ and used the identity $\sum_{d|q}\phi(d) = q$.

CHAPTER 2

ZERO-DENSITY ESTIMATES

In this Chapter, we give a survey of results concerning the zero-density estimates of a certain class of Dirichlet series.

Section 1. Main Theorems

Notation and assumptions.

Let $F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, $a(1) \neq 0$ be a Dirichlet series that converges absolutely for Re.$(s) > 1$ and that can be continued to a function analytic on $\mathrm{Re}(s) > -1$, except for a finite number of poles in the strip $0 < \mathrm{Re}(s) \leq 1$. Let $N(\sigma, T)$ be the number of zeros, ρ, of $F(s)$ with $1 \geq \mathrm{Re}(\rho) \geq \sigma$ and $|\mathrm{Im}(\rho)| \leq T$, where $\sigma \geq \frac{1}{2}$ and $T \geq 1$.

Let $G(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$ be a Dirichlet series that also converges absolutely for $Re(s) > 1$. Let $\Delta(s) = \prod_{j=1}^{N} \Gamma(\alpha_j s + \beta_j)$, where $\alpha_j > 0$ and β_j are complex, $1 \leq j \leq N$. We assume that there exist real numbers C and θ, with $C > 0$ and a complex number δ such that $F(s)$ and $G(s)$ satisfy the functional equation

$$\Delta(s)F(s) = C^{\theta s+\delta}\Delta(1-s)G(1-s) \tag{1.1}$$

We shall assume the following estimates on the co-efficients of $F(s)$ and $G(s)$.

$$\sum_{n \leq x} |a(n)|^2 \ll x \log^{M_1} x \tag{1.2}$$

and

$$\sum_{n \leq x} |b(n)|^2 \ll x \log^{M_2} x \tag{1.3}$$

Let $a^{*-1}(n)$ be the Dirichlet convolution inverse of $a(n)$, i.e.,

$$(a * a^{*^{-1}})(n) = \sum_{d/n} a(d) a^{*^{-1}}(\frac{n}{d}) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1 \end{cases}$$

This exists since $a(1) \neq 0$. We assume

$$x \log^{M_3} x \ll \sum_{n \leq x} |a^{*^{-1}}(n)|^2 \ll x \log^{M_3} x \tag{1.4}$$

Note that if $a(n) \geq 0$, then $|a^{*-1}(n)| \leq a(n)$, and so the upper estimate follows from (1.2) with $M_3 = M_1$. Let $W \geq 1$ and

$$C(n) = C(n, W) = \sum_{\substack{d \| n \\ d \leq W}} a(d) a^{*-1}(\frac{n}{d})$$

Then $C(1, W) = 1$ and $C(n, W) = 0$ for $1 < n \leq W$. We assume

$$\sum_{n \leq x} |C(n)|^2 \ll x \log^{M_4} x \tag{1.5}$$

Don Redmond **[Red]** obtained the following density results.

Theorem 1.1. Let $\frac{1}{2} \leq \sigma \leq 1$ and let $k \geq 2$ be an integer. If we assume (1.4), (1.5) and that there exist constants $\mu(k)$ and $\nu(k)$ such that

$$\int_{-T}^{T} |F(\frac{1}{2} + it)|^k dt \ll T^{\mu(k)} \log^{\nu(k)} T, \tag{1.6}$$

as $T \to \infty$, then, as $T \to \infty$,

$$N(\sigma, T) \ll \left(T^{2(1-\sigma)} + T^{2(k+2\mu(k))(1-\sigma)/(k+4-4\sigma)} \right) \log^{M_1(k)} T,$$

where

$$M_1(k) = \max(M_4 + 10, 3 + (2\nu(k) + (M_3 + 5)k)/(k + 2)).$$

Theorem 1.2. If we assume (1.4), (1.5) and (1.6), then for

$$\sigma \geq (8\mu(k)+3k-4)/(8\mu(k)+4k-4)$$

we have

$$N(\sigma,T) \ll T^{(4\mu(k)+k)(1-\sigma)/(4-k+(2k-4)\sigma)} \log^{M_2(k)} T,$$

as $T \to \infty$, where $M_2(k) = \max(M_4+6, \nu(k)+3, M_3+6)$.

Corollary 1.3. Uniformly on $\frac{1}{2} \leq \sigma \leq 1$ we have, as $T \to \infty$,

$$N(\sigma,T) \ll \left(T^{(k+2\mu(k))(8\mu(k)+4k-4)(1-\sigma)/(4k\mu(k)+2k^2)} \log^{M_3(k)} T\right),$$

where

$$M_3(k) = \max(M_1(k),(M_2(k))$$

In the proofs of Theorems 1.1 and 1.2, Redmond adapted the method of Montgomery [**M1**]. In [**S**], Sokolovskii used Ingham's method of convexity theorems to give estimates for $N(\sigma,T)$ for the same class of Dirichlet series. He assumes (1.2), (1.4) and (1.5) and the essential tool for him is an estimate for $F(\frac{1}{2}+it)$. In his proof, Don Redomond has replaced this estimate by (1.6).

Since the method of proving Theorems 1.1 and 1.2 is a standard one, we give the proof of Theorem 1.1. For the proof of Theorem 1.2, we refer the reader to Redmond's paper. We need the following lemma.

Lemma 1.1. Let M be given and $\{a_n\}, 1 \leq n \leq M$ be complex numbers. For $1 \leq r \leq R$, let $s_r = \sigma_r + it_r$ be arbitrary complex numbers. Let

$$\begin{aligned} \tau &= \min\{t_a - t_b : 1 \leq a < b \leq R\}, \\ S &= 1+\max\{t_r : 1 \leq r \leq R\} - \min\{t_r : 1 \leq r \leq R\} \\ \omega &= \min\{\sigma_r : 1 \leq r \leq R\}. \end{aligned}$$

Then

$$\sum_{r=1}^{R}\left|\sum_{n=1}^{M} a_n n^{-s_r}\right|^2 \ll (S+M)(1+\tau^{-1}\log^2 M)\log^4 M \sum_{n=1}^{M}\frac{|a_n|^2}{n^{2\omega}}$$

This is a version of Theorem 2 of [M1].

Proof of Theorem 1.1.

Let $M(s) = M(s,W) = \sum_{n\le W} \frac{a^{*^{-1}}(n)}{n^s}$

Then

$$F(s)M(s) = \sum_{n=1}^{\infty}\frac{C(n,W)}{n^s}.$$

If $r_1 > 1$, then by a standard integration formula, we have, if $U > 1$,

$$e^{-\frac{1}{U}} + \sum_{n>W}\frac{C(n,W)e^{-n/U}}{n^s} = \sum_{n=1}^{\infty}\frac{C(n)e^{-n/U}}{n^s}$$

$$= \frac{1}{2\pi i}\int_{(r_1)} F(s+z)M(s+z)U^z\Gamma(z)dZ\,.$$

We assume $W \le U \le T^{C_1}$.

Let $s = \sigma + it$, where $\frac{1}{2} < \sigma \le 1$, and move the contour to $Re(z) = \frac{1}{2} - \sigma$. Then we pick up the poles of the integrand, by the residue theorem, which are the poles of $F(s+z)$ in $0 < \sigma \le 1$ and the pole at $z = 0$ of $\Gamma(z)$. Since $F(s)$ satisfies the functional equation, and both $F(s)$ and $G(s)$ are absolutely convergent for $\sigma < 1$, it follows by a standard Phragman-Lindelöf argument that, if Q is sufficiently small,

$$\sum_{n=1}^{\infty}\frac{C(n)}{n^s}e^{-n/U} = \frac{1}{2\pi}\int_{(\frac{1}{2}-\sigma)} F(s+z)M(s+z)U^z\Gamma(z)dz$$

$$+F(s)M(s) + \sum_{\lambda\ne 0}\frac{1}{2\pi}\int_{|z-(\lambda-s)|=Q} F(s+z)M(s+z)U^z\Gamma(z)dz \qquad (1.7)$$

where the sum over λ denotes a sum over the poles of the integrand.

Using the properties of the gamma function, it is easy to show that

$$\frac{1}{2\pi}\int_{|z-(\lambda-s)|=Q} F(s+z)M(s+z)U^{z}\Gamma(z)dz \;=\; o(1)\,, \tag{1.8}$$

as $T \to \infty$.

It is shown in Theorem.10 of **[B]** that $F(s)$ has $\ll T\log T$ zeros in the rectangle $0 \leq \mathrm{Re}(s) \leq 1, |\mathrm{Im}(s)| \leq T$. Thus there are $\ll \log^3 T$ zeros with $|\gamma| \leq \log^2 T$. Thus, if we add to the final estimate an $O(\log^3 T)$ term to account for the neglected zeros, we can assume $|\gamma| \geq \log^2 T$. Thus, by (1.7) and (1.8), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{C(n)}{n^s}e^{-n/U} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\frac{1}{2}+i(t+u))M(\frac{1}{2}+i(t+u))U^{\frac{1}{2}-\sigma-iu}\\ &\quad \left(\Gamma(\frac{1}{2}-\sigma+iu)du\right)+F(s)M(s)+o(1) \qquad (1.9)\end{aligned}$$

Since $F(s)$ satisfies the functional equation (1.1) and is absolutely convergent for $\mathrm{Re}(s) > 1$, we know that $F(s)$ is of finite order, and by (1.5) we know that $|C(n)| \leq c_2.n^{c_3}$. Hence, we can show that

$$\int_{\pm\frac{\log^2 T}{2}}^{\infty} F(\frac{1}{2}+i(t+u))M(\frac{1}{2}+i(t+u))U^{\frac{1}{2}-\sigma+iu}\Gamma(\frac{1}{2}-\sigma+iu)du \;=\; o(1) \tag{1.10}$$

and

$$\sum_{n>U^2}\frac{C(n)}{n^s}e^{-n/U} \;=\; o(1) \tag{1.11}$$

as $T \to \infty$, if U tends to ∞ with T.

Thus by (1.9) - (1.11), we have

$$\begin{aligned}&e^{-\frac{1}{U}}\sum_{W\subset n\leq U^2}\frac{C(n)}{n^s}e^{-n/U}\\ &= F(s)M(s)+\frac{1}{2\pi i}\int_{(\frac{1}{2}-\sigma,\log^2 T/2}F(s+z)M(s+z)U^{z}\Gamma(z)dz\\ &\quad +o(1) \qquad (1.12)\end{aligned}$$

as $T \to \infty$.

Let $\rho = \beta + i\gamma$ be a zero of $F(s)$. Then, from 1.12, we have either

$$\left| \sum_{W<n\leq U^2} \frac{C(n)}{n^\rho} e^{-n/U} \right| \gg 1, \tag{1.13}$$

$$\left| \int_{(\frac{1}{2}-\sigma, \frac{\log^2 T}{2})} F(\rho+z) M(\rho+z) U^z \Gamma(z) dz \right| \gg 1 \tag{1.14}$$

or both.

Of the zeros ρ with $\beta \geq \sigma$ and $|\gamma| \leq T$, we take a subset R of them so that if ρ_1, ρ_2 are two zeros, then

$$|\gamma_1 - \gamma_2| \geq 2\log^2 T \tag{1.15}$$

Also, by Theorem 3 of [B], we have

$$N(\frac{1}{2}, t+1) - N(\frac{1}{2}, t) \ll \log T$$

for $|t| \leq T$. Thus we may choose the subset R of zeros so that

$$N(\sigma, T) \ll (R+1)\log^3 T.$$

Finally, let R_1 and R_2 be the number of zeros such that (1.13) and (1.14), respectively, hold. Then $R \leq R_1 + R_2$.

Estimation of R_1

If (1.13) holds, then there is a Y such that $W < Y \leq U^2$ and

$$\left| \sum_{n=Y}^{2Y} \frac{C(n)}{n^\rho} e^{-n/U} \right| \gg (\log U)^{-1}$$

for $\gg R_1(\log U)^{-1}$ zeros for which (1.13) holds. If $\rho_j, 1 \le j \le R$, are the zeros under consideration, then by Lemma 1.1,

$$R_1(\log U)^{-3} \ll \sum_{j=1}^{R_1} |\sum_{n=Y}^{2Y} \frac{C(n)}{n^{\rho_j}} e^{-n/U}|^2$$

$$\ll (T+2Y)(1+\tau^{-1}\log^2 2Y)(\log^4 2Y) \sum_{n=Y}^{2Y} \frac{|C(n)|^2}{n^{2\sigma}} e^{-\frac{Y}{U}},$$

where $\tau = \min|\gamma_i - \gamma_j| \ge 2\log^2 T$, by (1.15). Thus by (1.5), we have

$$R_1 \ll (T+Y)e^{-Y/U}\, Y^{1-2\sigma}(\log T)^{M_3+7} \tag{1.16}$$

If $f(Y) = Y^p e^{-Y/U}$ for $Y > 0$, then it is easy to show that $f(Y)$ attains its maximum at $Y = pU$. Thus, from (1.16), we have

$$R_1 \ll (TW^{1-2\sigma} + U^{2-2\sigma})(\log T)^{M_3+7} \tag{1.17}$$

Estimation of R_2:

Suppose (1.14) holds and let $\rho_j, 1 \le j \le R_2$, be the zeros under consideration. For these values, let t_j be such that $|t_j - \gamma_j| \le \log^2 T/2$ and $|F(\frac{1}{2}+it_j)M(\frac{1}{2}+it_j)|$ is maximal. Assume that $\beta \ge \frac{1}{2} + \frac{1}{\log T}$. Then

$$\int_{-\infty}^{\infty} |\Gamma(\frac{1}{2} - \beta + iu)|\, du \quad \ll \log T$$

Thus

$$|F(\frac{1}{2}+it_j)\, M(\frac{1}{2}+it_j)| \gg U^{\sigma-\frac{1}{2}}(\log T)^{-1} \tag{1.18}$$

If ρ_a and ρ_b are zeros with $| \le a \le b \le R_2$ and t_a and t_b are the corresponding values of t, then, by the triangle inequality, the definition of t_j and (1.15). We have $|t_a - t_b| \ge \log^2 T$.

For any integer $k \geq 2$, we have

$$\sum_{j=1}^{R_2} |F(\frac{1}{2} + it_j)|^k \ll \int_{-T}^{T} |F(\frac{1}{2} + it)|^k \, dt.$$

Then, by Lemma 1.1, (1.4) and (1.18), we have

$$R_2 U^{\frac{k}{k+2}(2\sigma-1)} (\log T)^{-\frac{2k}{k+2}} \ll \sum_{j=1}^{R_2} \left(|F(\frac{1}{2} + it_j) M(\frac{1}{2} + it_j)| \right)^{\frac{k}{k+2}}$$

which by Hölder's inequality,

$$\ll \quad \left(\sum_{j=1}^{R_2} |f(\tfrac{1}{2} + it_j)|^k \right)^{\frac{2}{k+2}} \left(\sum_{j=1}^{R_2} |M(\tfrac{1}{2} + it_j)|^2 \right)^{\frac{2}{k+2}}$$

$$\ll \quad \left(T^{2\mu(k)/k+2} (\log T)^{2\nu(k)/k+2} \right) \left\{ (T + W) (\log W)^{M_3+5} \right\}^{\frac{k}{k+2}}$$

Thus

$$R_2 \ll T^{2\mu(k)/k+2} (T + W)^{k/k+2} U^{\frac{k}{k+2}(1-2\sigma)} (\log T)^{(2\nu(k)+(M_3+5)k)/k+2} \tag{1.19}$$

Thus by (1.17) and (1.19), we have

$$\begin{aligned} N(\sigma, T) &\ll (R + 1) \log^3 T \\ &\ll (R_1 + R_2 + 1) \log^3 T \\ &\ll (TW^{1-2\sigma} + U^{2-2\sigma}) (\log T)^{M_4+10} \\ &+ (T + W)^{k/k+2} T^{2\mu(k)/(k+2)} U^{(1-2\sigma)k/k+2} \\ &\qquad (\log T)^{3+(2\nu(k)+(M_3+5)k)/(k+2)} \end{aligned}$$

If we choose $W = T$ and $U = T^{(k+2\mu(k))/(k+4-4\sigma)}$ we have

$$N(\sigma, T) \ll T^{2(1-\sigma)} + T^{2(k+2\mu(k))(1-\sigma)/(k+4-4\sigma)} (\log T)^{M_1(k)}.$$

This completes the proof of Theorem 1.1.

Section 2. Examples

We now give a few examples of Dirichlet series to which the theorems mentioned in Section 1 can be applied.

1. The Riemann-zeta function.

Here $F(s) = G(s) = \zeta(s), a(n) = b(n) = 1, a^{*^{-1}}(n) = \mu(n)$, the Möbius function, $\Delta(s) = \Gamma(s/2), C = \bar{\pi}, \theta = 1$ and $\delta = -\frac{1}{2}$. Thus we can take $M_1 = M_2 = M_3 = 0$ and $M_4 = 3$ as $|C(n)| \leq d(n)$. By Corollary 3.4 of Chapter 1, we have $\mu(2) = \mu(4) = \nu(2) = 1$ and $\nu(4) = 4$.

Thus for $k = 2$, we have from Theorem 1.1,

$$N(\sigma, T) \ll T^{4(1-\sigma)/(3-2\sigma)} \log^{13} T \tag{2.1}$$

and for $k = 4$, we have

$$N(\sigma, T) \ll T^{3(1-\sigma)/(2-\sigma)} \log^{13} T$$

The first result is due to Titchmarsh and the second is due to Ingham .

By the Corollary 1.3, these results may be improved to

$$N(\sigma, T) \ll T^{3(1-\sigma)} \log^{13} T$$

and

$$N(\sigma, T) \ll T^{5(1-\sigma)/2} \log^{13} T .$$

respectively, for $\frac{1}{2} \leq \sigma \leq 1$. The second result is due to Montgomery **[M2]**.

Using the "large values" method of Halasz, Huxley **[Hu]** proved that

$$N(\sigma, T) \ll T^{3(1-\sigma)/(3\sigma-1)} \log^{44} T . \tag{2.2}$$

uniformly in $\frac{3}{4} \leq \sigma \leq 1$.

This, along with Ingham's estimate (2.1) gives

$$N(\sigma,T) \ll T^{\frac{12}{5}(1-\sigma)}\log^{44} T. \tag{2.3}$$

uniformly in $1/2 \le \sigma \le 1$.

Also, the result (2.2) implies that

$$N(\sigma,T) \ll T^{2(1-\sigma)}\log^{44} T. \tag{2.3}$$

for $\frac{5}{6} \le \sigma \le 1$, i.e., the density hypothesis holds in the range $\frac{5}{6} \le \sigma \le 1$.

We know that the zero density estimates of $\zeta(s)$ have a number of applications to prime number theory, and one such is the estimation of the difference between consecutive primes. It is well known that any result

$$N(\sigma,T) \ll T^{\lambda(1-\sigma)}\log^{B} T.$$

uniform in $1/2 \le \sigma \le 1$ implies

$$p_{n+1} - p_n < p_n^{\delta}$$

for all sufficiently large n, whenever

$$\delta > 1 - \frac{1}{\lambda}.$$

From (2.2), we have $\delta > \frac{7}{12}$.

The Dedekind-zeta function.

Let K be an algebraic number field of degree n and $\zeta_K(s)$ denote the Dedekind-zeta function of K. Then $\zeta_K(s) = \sum_{m=1}^{\infty} \frac{a_K(m)}{m^s}$ for $Re.s > 1$ where $a_K(m)$ is the number of integral ideals of norm m. Then $a_K(m) \le d_n(m)$ and by Corollary (3.6) of Chapter 1, we have

$$\int_{-T}^{T} |\zeta_K(\frac{1}{2}+it)|^2 dt \ll T^{n/2}\log^n T$$

Also, since $a_K(m) \geq 0$, we see that $|a_K^{*-1}(m)| \leq a_K(m)$ and we have

$$|C(m)| = \sum_{\substack{d \leq w \\ d|m}} |a_K(d)||a_K^{*-1}(\tfrac{m}{d})|$$

$$\leq d_{2n}(m)$$

Thus

$$\sum_{m \leq x} |C(m)|^2 \leq \sum_{m \leq x} d_{2n}^2(m) \ll x \log^{4n^2-1} x$$

Thus we have here $M_1 = M_2 = M_3 = n - 1$, $M_4 = 4n^2 - 1$, $\mu(2) = n/2$ and $\nu(2) = n$.

Thus by Theorem 1.1 and Corollary 1.3 we have

$$N(\sigma, T) \ll T^{(n+2)(1-\sigma)/(3-2\sigma)} \log^{4n^2+9} T$$

and

$$N(\sigma, T) \ll T^{(n+1)(1-\sigma)} \log^{4n^2+9} T$$

respectively, for $\frac{1}{2} \leq \sigma \leq 1$.

D.R.Heath-Brown has improved these results by showing that if $n \geq 3$, then, for any $\epsilon > 0$,

$$N(\sigma, T) \ll T^{(n+\epsilon)(1-\sigma)}$$

When $n = 2$, he showed that then exists a constant C for any $\epsilon > 0$ such that

$$N(\sigma, T) \ll \begin{cases} T^{(2+6\epsilon)(1-\sigma)}, & \text{for } \frac{1}{2} \leq \sigma \leq \frac{1}{2} + \epsilon, \text{ if } 0 < \epsilon < \frac{1}{6} \\ T^{4(1-\sigma)/(3-2\sigma)}(\log T)^C, & \text{for } \frac{1}{2} + \epsilon \leq \sigma \leq \frac{3}{4} \\ T^{2(1-\sigma)/\sigma}(\log T)^C, & \text{for } \frac{3}{4} \leq \sigma \leq 1 - \epsilon, \\ T^{(2+\epsilon)(1-\sigma)(\log T)^C}, & \text{for } 1 - \epsilon \leq \sigma \leq 1 \text{ and } \epsilon < \frac{1}{10} \end{cases}$$

The proofs of these results depend on the techniques of Montgomery together with the later developments of Jutila. In addition, he uses the mean-value theorem for $|\zeta_K(s)|^2$ at $\sigma = 1 - \frac{1}{n}$ (see Corollary 3.6 of Chapter 1).

These results can be used to give an asymptotic formula for the number of prime ideals whose norms lie in interval :

Suppose that $1 \geq a \geq 1 - \frac{1}{k}$ if $n \geq 3$ and $1 \geq a > \frac{5}{8}$ if $n = 2$. If $\pi_K(x)$ denotes the number of prime ideals with norm $\leq x$, then

$$\pi_K(x+Y) - \pi_K(x) \sim \frac{Y}{\log x} \text{ as } x \to \infty$$

uniformly for $x^a \leq y \leq x$. In particular, when $x \geq x(a)$, there exists a prime ideal of K with $x < N\wp \leq x + x^a$.

3. Cusp forms of weight k with Euler product.

Let

$$L_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

be a cusp form of weight k with Euler product. We know that $L_f(s)$ satisfies the function equation (3.21) of Chapter 1.

Let $f_1(n) = \frac{f(n)}{n^{(k-1)/2}}$ and $L_{f_1}(s) = \sum_{n=1}^{\infty} \frac{f_1(n)}{n^s}$

Then we know that $L_{f_1}(s)$ satisfies the functional equation (3.22). Here we have $a(n) = b(n) = f_1(n), \Delta(s) = \Gamma(s + \frac{(k+1)}{2}), C = 2\pi, \theta = 2$ and $\delta = -1$.

Also, by (3.23) of Chapter 1, we have

$$M_1 = M_2 = 0 .$$

Goldstein[G1] has shown that, for every prime p,

$$f^{*^{-1}}(p^j) = \begin{cases} 1 & , \ j=0 \\ -f(p) & , \ j=1 \\ p^{k-1} & , \ j=2 \\ 0 & , \ j>3 \end{cases}$$

and is defined on integers by multiplicativity. From this, it is easy to show that

$$x^k \ll \sum_{n\leq x} |a^{*^{-1}}(n)|^2 \ll x^k$$

Thus $M_3 = 0$.

By the Ramanujan - Petersson conjecture (which was proved by Deligne)

$$|f(n)| \leq d(n) n^{\frac{k-1}{2}} .$$

From this, it is easy to show that

$$\sum_{n\leq x} |C_1(n)|^2 \ll x \log^{15} x$$

By Corollary 3.7 of Chapter 1, we have $\mu(2) = \nu(2) = 1$. This gives, by Theorem 1.1,

$$N(\sigma, T) \ll T^{4(1-\sigma)/(3-2\sigma)} \log^{25} T$$

By Corollary 1.3, we have

$$N(\sigma, T) \ll T^{3(1-\sigma)} \log^{25} T \tag{2.4}$$

for $\frac{1}{2} \leq \sigma \leq 1$.

If we translate (2.4) back to the Cusp form $L_f(s)$, we have

$$N(\sigma,T) \ll T^{3((k+1)/2-\sigma)} \log^{25} T .$$

4. ***L*-functions.**

Suppose that $q \geq 1$, $\sum_{\chi (mod\ q)}$ denotes the summation over all characters modulo q, and $N(\sigma,T,\chi)$ denotes the number of zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ in the rectangle $\sigma \leq \beta \leq 1$ and $|\gamma| \leq T$, then

$$\sum_{\chi} N(\sigma,T,\chi) \ll (qT)^{\frac{3(1-\sigma)}{2-\sigma}} (\log qT)^9$$

for $\frac{1}{2} \leq \sigma \leq 4/5$, and

$$\sum_{\chi} N(\sigma,T,\chi) \ll (qT)^{\frac{2(1-\sigma)}{\sigma}} (\log qT)^{14}$$

for $\frac{1}{2} \leq \sigma \leq 1$.

The proof of this can be found in [**M3**].Redmond's proof of Theorem 1.1 and Theorem 1.2 is modelled along the proof of this result.

REFERENCES

[B-R] R. Balasubramanian and K. Ramachandra, *Proof of some conjectures on the mean values of Titchmarsh series - I.*, Hardy Ramanujan Journal, Vol.13 (1990) 1-20.

[B] B.C.Berndt, *On the zeros of a certain class of Dirichlet series*, I, Illinois J.Math., vol.14 (1970),244-258.

[B-S] Z.I.Borevich and I.R.Shafarevich, *Number Theory*,Academic Press.

[C-N] K. Chandrasekharan and Raghavan Narasimhan, *The approximate functional equation for a class of zeta functions*, Math. Annalen 152 (1963) 30-64.

[C] J.B.Conrey, *The fourth moment of derivatives of the Riemann zeta-function*, Quart.J.Math., Oxford(2),39 (1988),21-36.

[G1] L.J.Goldstein, *A necessary and sufficient condition for the Riemann hypothesis for zeta-functions attached to the eigen functions of the Hecke operators*, Acta Arith., vol.15(1969),205-215.

[G2] S.M.Gonek, *Mean values of the Riemann zeta-function and its derivatives*, Invent.Math.75, (1984), 123-141.

[G3] A. Good, *Approximative Funktionalgleichungen und Mittelwertsätze für Dirichletreihen, die Spitzen formen assoziiert sind*, Comment. Math. Helvetici 50 (1975) 327-361.

[H-L1] G.H. Hardy and J.E. Littlewood , *The approximate functional equation in the theory of the zeta function with applications to the divisor problems of Dirichlet and Piltz*, Proc. London Math. Soc. 21 (1923) 39-74.

[H-L2] G.H. Hardy and J.E. Littlewood, *The approximate functional equation for $\zeta(s)$ and $\zeta^2(s)$*, Ibid 29 (1929) 81-97.

[HB1] D.R.Heath-Brown, *On the density of the zeros of the Dedekind zeta-function*, Acta Arith., vol.33(1977),169-181.

[HB2] D.R.Heath-Brown, *The fourth power moment of the Riemann zeta-function*, Proc.London Math.Soc., 38(3),(1979), 385-422..

[He] Hecke,*Über Modulfunctionen und die Dirichletschen Reihen mit Eulerscher Prodüktentwicklung I*, Math.Ann. 114,(1937), 1-28.

[Hi] J.G.Hinz, *A mean value theorem for the Dedekind zeta - function of a quadratic number field*, Monatsh.Math. 87,229-239 (1979).

[Hu] M.N.Huxley, *On the diffeence between consecutive primes*, Invent.Math.15,155-164 (1972).

[In] A.E. Ingham, *Mean value theorems in the theory of Riemann zeta-function*, Ibid 27 (1928) 273 - 300.

[Iv] A. Ivic, *Lectures on Mean values of the Riemann zeta function*, Tata Institute of Fundamental Research, Bombay, 1991.

[M1] H.L.Montgomery, *Mean and large values of Dirichlet polynomials*, Invent.Math. vol.8 (1969) pp.334-345.

[M2] H.L.Montgomery, *Zeros of L-functions*,Invent. Math. vol.8 (1969) pp.346-354.

[M3] H.L. Montgomery, *Topics in multiplicative Number Theory*, Lecture Notes in Math.227 (Springer, Berlin, 1971).

[M-V] H.L. Montgomery and R.C.Vaughan, *Hilbert's inequality*, J. London Math. Soc.(2), 8 (1974) 73-82.

[Mü] Müller, Wolfgang, *The mean square of the Dedekind zeta-function in quadratic number fields*, Math.Proc.Cambridge Phil.Soc.(106) (1989), no.3, 403-417.

[P] K.Prachar, *Primzahlvertilung*, Berlin-Göttingen- Heidelberg, Springer, 1957.

[Rad] H.Rademacher,*On the Phragmen-Lindelöf theorem and some applications.*, Math.Z. 72,192-204 (1959).

[Ram1] K. Ramachandra, *Applications of a theorem of Montgomery and Vaughan to the zeta function*, J. London Math. Soc. 10 (2) (1975) 482-486.

[Ram2] K.Ramachandra, *Remarks on a theorem of Montgomery and Vaughan*, J.Number Theory 11,465-471 (1980).

[R] V.V.Rane, *A note on the mean value of L-series*, Proc.Indian Acad.Sci. (math.Sci) vol.89,273-286 (1980).

[Ran] R.A.Rankin, *Contributions to the theory of Ramanujan's function* $\tau(n)$ *and similar arithmetical functions* II, Proc. Cambridge Phil.Soc.(35) (1939),357-372.

[Red] D.Redmond, *A density theorem for a class of Dirichlet series*, Illinios J.Math., vol.24 (1980),pp.296-312.

[S] A.V.Sokolovskii, *Density theorems for a class of zeta- functions*, Izv.Akad.Nauk. Uzbek, vol.10(1966) (3),pp.33-40.

[T] E.C.Titchmarsh, *The Theory of Riemann-zeta function*, Oxford University Press.