Lie Algebraic Decomposition of Black Hole Partition Functions

By

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Mohammad Shabbir

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Mohammad Shabbir

List of Publications arising from the thesis[1]

Journal

1. "$\widehat{sl(2)}$ decomposition of denominator formulae of some BKM Lie superalgebras"

   Suresh Govindarajan, ***Mohammad Shabbir*** *,Sankaran Viswanath*;

   Nuclear Physics B (2021) 973: 115614

2. "$\widehat{sl(2)}$ decomposition of denominator formulae of some BKM Lie superalgebras–II"

   Suresh Govindarajan, ***Mohammad Shabbir***;

   Nuclear Physics B (2023) 989: 116127

[1]As it is standard in the High Energy Physics Theory (hep-th) community the names of the authors on any paper appear in their alphabetical order.

Conferences and workshops attended

1. "Chennai Strings Meeting 2019" at "Institute of Mathematical Sciences", Chennai from 21-23 November 2019.

2. "Indian Strings Meeting" at "Indian Institute of Science Education and Research(IISER)", Trivandrum from 16-21 December 2018.

3. "Chennai Strings Meeting 2018" at "Institute of Mathematical Sciences", Chennai 5-6 October 2018.

4. "National String Meeting 2017" at "National Institute of Science Education and Research(NISER)", Bhuvaneshwar from 5-10 December 2017.

Presentations

1. *“When Black Holes demand a New Lie Algebra"* virtual talk given at “Indian Strings Meeting 2021" at IIT Roorkee on December 14, 2021.

2. *“When Black Holes demand a New Lie Algebra"* virtual talk given at “Trends in String Theory and Related Topic 2021" at HRI Allahabad on October 09, 2021.

3. *“Dot-Product decomposition of umbral jacobi forms"* virtual talk given at “Chennai String Meeting 2020" at IMSc Chennai on December 17, 2020.

4. *“Affine Lie superalgebras"* talk given at “Chennai String Meeting 2019" at IMSc Chennai on November 23, 2019.

To My Parents

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Contents

Synopsis

Motivation and Introduction

We study two families of Siegel modular forms that are constructed using Jacobi forms connected to twisted/twined elliptic genera of K3. The first family [1] is related to Umbral moonshine and the second family [2] appears in CHL $\mathbb{Z}_N$ orbifolds.

The squares of the Siegel modular forms are the generating function of quarter BPS states in type II string theory compactified on K3 $\times T^2$.

All but three of the examples are the denominator formulae for some Borcherds-Kac Moody Lie superalgebra. *Are there Lie algebras whose denominator formulae are these three Siegel modular forms?* - this is the question that we seek to answer.

In this thesis, we decompose these Siegel modular forms using the characters of an $\widehat{sl(2)}$ subalgebra (and its Borcherds extension) that is present in all cases. The decomposition has led to the appearance of a special type of simple real root in precisely these examples. The implication of this root for the Lie algebra is not known yet.

Black holes in $\mathcal{N} = 4$ string theories

In this chapter we talk about a class of theories with sixteen supercharges, known as $\mathcal{N} = 4$ string theories in four dimensions, where one has a complete understanding of the microscopic degeneracies of supersymmetric black holes. A large number of $\mathcal{N} = 4$ string theories is covered by CHL-models where one takes the orbilfold action on $T^4 \times T^2$

heterotic string theory by the geometric symmetries of the target space. The degeneracy of 1/4-BPS states in any of these models appear as Fourier coefficients of corresponding partition function which is inverse of a genus two Siegel modular form $\Phi^{(N)}$ [3–10], parametrized by the order of orbifolding N, on Siegel upper half plane of complexified chemical potentials (say τ, τ', z).

These CHL models are enriched with the following triality

$$\boxed{\text{CHL}\left(\text{Het}, T^4\times T^2\right)} \xleftrightarrow{\text{string-string}} \boxed{\text{CHL}\left(\text{IIA}, \text{K3}\times T^2\right)} \xleftrightarrow{\text{T-duality}} \boxed{\text{CHL}\left(\text{IIB}, \text{K3}\times T^2\right)}$$

In the context of $K3\times T^2$ a large class of the Siegel modular forms $\Phi^{(N,M)}$ [11–14] appeared as connected to N-twisted / M-twined second-quantized elliptic genera of K3. The partition function here is accounted for the counting of $\mathbb{Z}_M$-twisted 1/4 BPS states in the CHL $\mathbb{Z}_N$-orbifold. When restricting $(N, M) \to (N, 1)$, the $\Phi^{(N,M)}$ becomes $\Phi^{(N)}$.

The partition function in these models could be meromorphic having poles in Siegel upper half plane. Existence of these poles make the spectrum of 1/4 BPS states moduli dependent such that there is a jump in degeneracy whenever the contour crosses a pole. Crossing a pole is amount to the moduli crossing a co-dimension one surface known as walls of marginal stability in the moduli space and this is called 'wall-crossing phenomenon'. The walls of marginal stability depend only on the order of orbifolding not on the twist of dyons.

BKM Lie superalgebras

In this chapter we give a tour from the simplest Lie algebra $su(2)$ to BKM superalgebras with examples. Specified by a Cartan matrix A, one can conveniently define Kac-Moody superalgebra $\mathfrak{g}(A)$ through a set of (anti-)commutation relations due to Chevalley and Serre. Under adjoint representation we get a triangular decomposition

$$\mathfrak{g}(A) = \left[\bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{-\alpha}\right] \oplus \mathfrak{g}_0 \oplus \left[\bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{\alpha}\right] \tag{1}$$

where Δ_+ denotes the set of positive roots.

Our particular interest lies in following denominator and super-denominator identities of such a Lie superalgebra [15]

$$e^{-\rho}\prod_{\alpha\in\Delta^+}(1-\mathrm{sgn}(\mathrm{mult}(\alpha))\,e^{-\alpha})^{\mathrm{mult}(\alpha)} = \sum_{w\in\mathcal{W}}\det(w)\,w(e^{-\rho}\,T) \tag{2}$$

$$e^{-\rho}\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)} = \sum_{w\in\mathcal{W}}\det(w)\,w(e^{-\rho}\,T_s) \tag{3}$$

where ρ is the Weyl vector, $\mathcal{W}$ is the Weyl group and $\mathrm{mult}(\alpha) = \dim\mathfrak{g}_\alpha$ ($\mathrm{mult}(\alpha) = -\dim\mathfrak{g}_\alpha$) for bosonic (fermionic) roots. T (T_s) is the Borcherds correction term for denominator (super-denominator) identity which depends on the set of all distinct, pairwise orthogonal simple imaginary roots. $T = 1$ if there is no simple imaginary root at all (Kac-Moody case) and $T_s = 1$ when there is no bosonic imaginary simple root.

In this chapter we also introduce a class of rank three hyperbolic Cartan matrices of our interest given by $A^{(N)} = (a_{nm})$ where

$$a_{nm} = 2 - \frac{4}{N-4}(\lambda_N^{n-m} + \lambda_N^{m-n} - 2) \tag{4}$$

and λ_N is any solution of the quadaratic equation

$$\lambda^2 - (N-2)\lambda + 1 = 0\,. \tag{5}$$

For $N = 1, 2, 3$, the matrices are finite dimensional with the indices n, m defined modulo $3, 4, 6$ respectively.

$$A^{(1)} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix},\quad A^{(2)} = \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -6 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix},$$

$$A^{(3)} = \begin{pmatrix} 2 & -2 & -10 & -14 & -10 & -2 \\ -2 & 2 & -2 & -10 & -14 & -10 \\ -10 & -2 & 2 & -2 & -10 & -14 \\ -14 & -10 & -2 & 2 & -2 & -10 \\ -10 & -14 & -10 & -2 & 2 & -2 \\ -2 & -10 & -14 & -10 & -2 & 2 \end{pmatrix}.$$

For $N = 4, 5, 6$, the matrices are infinite dimensional with $m, n \in \mathbb{Z}$. For $N = 4$, the Cartan matrix has to be obtained as a limit $N \to 4$ leading to $a_{nm} = 2 - 4(n - m)^2$.

At the end of this chapter we assign the root latices to the $A^{(N)}$ matrices and determine corresponding fundamental Weyl chambers which are hyperbolic polygons for hyperbolic Lie algebras [16].

Borcherds' extension of $A^{(N)}$ algebras

In this chapter we will shed light on the product formulae of $\Phi^{(N,M)}$ [17, 18] from generalized moonshine where product side is proportional to the second-quantized N-twisted / M-twined elliptic genus of K3.

Let us say the square root of Siegel modular form $\Phi^{(N,M)}$, is given by Siegel modular form

$$\Delta^{(N,M)} = \sqrt{\Phi^{(N,M)}}. \tag{6}$$

For $N = 1, 2, 3, 4$, $\Delta^{(N,M)}$ appear in the product side of WKB superdenominator formulae of some Borcherds' extension, say $\mathcal{B}^{(N,M)}(A^{(N)})$, of $A^{(N)}$ algebras [10, 19, 20]

$$\Delta^{(N,M)} = e^{-\rho} \prod_{\alpha \in \Delta^+} \left(1 - e^{-\alpha}\right)^{\text{mult}\alpha} = \sum_{w \in \mathcal{W}} \det(w)\, w\left(e^{-\rho} T^{(N,M)}\right) \tag{7}$$

where the Borcherds' correction term $T^{(N,M)}$ depends on the order of twisting and twining. But the walls of Marginal stability are literally the walls of Weyl chambers associated to the Lie algebras of $A^{(N)}$ for all $N = 1, \ldots, 6$ [13, 14].

According to a no-go theorem due to Gritsenko and Nikulin one can deduce that only an algebra more general than BKM may relate to the partition functions for $N = 5, 6$ as the denominator formula [16].

This got us thinking that there should exist some exotic simple roots for $N = 5, 6$ that does not fit into Borcherds' prescription. So we start exploring the subalgebra $\widehat{sl(2)}$, which is embedded in all $A^{(N)}$ algebras.

Deconstructing the dyonic symmetry

BKM algebras $\mathcal{B}^{(N,M)}\left(A^{(N)}\right)$ can be recognized as the symmetry of the BPS spectrum [20]. In this chapter we look more specifically into the Lie algebraic aspect of the dyonic symmetry by utilizing the fact that all $A^{(N)}$ Cartan matrices contains $\widehat{sl(2)}$ sub-Cartan matrices $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ in a block diagonal form.

$$A^{(1)} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -6 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix},$$

$$A^{(3)} = \begin{pmatrix} 2 & -2 & -10 & -14 & -10 & -2 \\ -2 & 2 & -2 & -10 & -14 & -10 \\ -10 & -2 & 2 & -2 & -10 & -14 \\ -14 & -10 & -2 & 2 & -2 & -10 \\ -10 & -14 & -10 & -2 & 2 & -2 \\ -2 & -10 & -14 & -10 & -2 & 2 \end{pmatrix}, \quad A^{(4,5,6)} \sim \begin{pmatrix} 2 & -2 & \cdots \\ -2 & 2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

This in turn shows $\widehat{sl(2)}$ are subalgebras of $A^{(N)}$ with a natural follow-up that $\mathcal{B}^{(N,M)}\left(A^{(N)}\right)$ admits certain Borcherds' extension of $\widehat{sl(2)}$, say $\mathcal{B}^{(N,M)}\left(\widehat{sl(2)}\right)$, as subalgebras.

In order to spot the exotic roots in $\mathcal{B}^{(N,M)}\left(A^{(5)}\right)$ and $\mathcal{B}^{(N,M)}\left(A^{(6)}\right)$ we need a device to track the simple roots appear in $\Delta^{(5,M)}$ and $\Delta^{(6,M)}$ as denominator formulae. Generally one could do that by decomposing denominator formulae by characters of the subalgebras in a following manner.

First we expand Siegel modular forms as

$$\Delta^{(N,M)} = s^{1/2}\,\varphi^{(N,M)}(\tau,z)\times\left[1+\sum_{m=1}^{\infty} s^{tm}\,\Psi^{(N,M)}_{0,tm}(\tau,z)\right]\ ;\ t=\gcd(M,N) \tag{8}$$

where s is one of the fugacities $\left\{q=e^{2\pi i\tau}\ ,\ r=e^{2\pi iz}\ ,\ s=e^{2\pi i\tau'}\right\}$ and $\varphi^{(N,M)}$ is an index half Jacobi form of same weight as is $\Delta^{(N,M)}$. The leading term $s^{1/2}\,\varphi^{(N,M)}(\tau,z)$ can be understood as the denominator formula for the subalgebra $\mathcal{B}^{(N,M)}\left(\widehat{sl(2)}\right)$. The objects of our interest are Jacobi forms $\Psi^{(N,M)}_{0,tm}(\tau,z)$ of some congruence subgroup Γ which depends on N and M. We then do $\widehat{sl(2)}$-character decomposition of $\Psi^{(N,M)}_{0,tm}(\tau,z)$:

$$\begin{aligned}\Psi^{(N,M)}_{0,tm}(\tau,z) =& g_1(\tau)\chi_{4tm,2tm}(\tau,z)+g_2(\tau)\left[\chi_{4tm,2tm-2}+\chi_{4tm,2tm+2}\right](\tau,z)\\ &+\ldots+g_{tm+1}(\tau)\left[\chi_{4tm,0}+\chi_{4tm,4tm}\right](\tau,z).\end{aligned} \tag{9}$$

Next we, using modular properties of $\Psi^{(N,M)}_{0,tm}$ and normalised characters $\chi_{k,\lambda}$ of $\widehat{sl(2)}$, solve it for g_i's as in power series of q. One can expect then g_i's to form $tm+1$ dimensional vector valued modular form $\mathcal{G}(\tau)=\begin{bmatrix} g_1\\ \vdots\\ g_{tm+1}\end{bmatrix}(\tau)$ of congruence subgroup Γ .

Coming back to $\Delta^{(5,M)}$ and $\Delta^{(6,M)}$ the symplectic automorphism of K3 allowed only $M=1$ leaving us with two examples $\Delta^{(5,1)}$ and $\Delta^{(6,1)}$ as in $\mathbb{Z}_5/\mathbb{Z}_6$-CHL orbifolds, to look for the exotic Lie algebras $\mathcal{B}^{(5,1)}\left(A^{(5)}\right)$ and $\mathcal{B}^{(6,1)}\left(A^{(6)}\right)$.

On top of that we can add one more example, a weight zero Siegel modular form Δ_0 associated to some exotic Lie algebra $\mathcal{B}^{\text{Umb}}\left(A^{(6)}\right)$, from the Umbral moonshine in the sense that four of its predecessor in the family associated to pure A-type Niemeir lattices [21], appear as $\Delta^{(N,N)}$ for $N=1,2,3,4$.

In paper 1&2 [1,2] we deconstruct the Lie algebras of the family $\{\Delta^{(N,N)}|N=1,2,3,4 \text{ and } \Delta_0\}$ associated to Umbral moonshine and of the family $\{\Delta^{(N,1)}|N=2,3,5\}$ associated to CHL orbifolds respectively.

Vector valued modular forms(vvmfs)

In this chapter we explore, for the two families, modular properties of $\mathcal{G}(\tau)$ and try to follow the work of Terry Gannon [22] in order to obtain closed formulae for them.

For the family related to Umbral moonshine we have $t = N$ for $\{\Delta^{(N,N)}|N = 1,2,3,4\}$ and $t = 6$ for Δ_0. We study only five cases associated to $m = 1$ i,e. $\Psi^{(1,1)}_{0,1}$, $\Psi^{(2,2)}_{0,2}$, $\Psi^{(3,3)}_{0,3}$, $\Psi^{(4,4)}_{0,4}$ and $\Psi^{\text{Umb}}_{0,6}$. These are the Jacobi forms under full $SL(2,\mathbb{Z})$.

Let us take the example of $\Psi^{(2,2)}_{0,2}(\tau,z) = -\psi_{0,2}(\tau,z)$ a weight zero and index two Jacobi form. We then proceed in the following steps

Character decomposition:

$$\psi_{0,2}(\tau,z) \;=\; g_1(\tau)\chi_{8,4}(\tau,z) \;+\; g_2(\tau)[\chi_{8,6}(\tau,z) \;+\; \chi_{8,2}](\tau,z) \;+\; g_3(\tau)[\chi_{8,8} \;+\; \chi_{8,0}](\tau,z)$$

where

$$\begin{aligned}
g_1 =& -10q^{\frac{1}{2}}\left(1+3q+9q^2+22q^3+\ldots\right)\\
g_2 =& q^{-\frac{1}{10}}\left(1+3q+18q^2+38q^3+\ldots\right)\\
g_3 =& q^{\frac{1}{10}}\left(3+16q+48q^2+129q^3+\ldots\right)
\end{aligned}$$

Realising vvmf: In the light of definition given by Gannon[2] [22], $\mathcal{G}(\tau) = \begin{bmatrix} g_1\\ g_2\\ g_3\end{bmatrix}(\tau)$ transforms nicely as a vvmf under translation ($\mathcal{T}$) and inversion ($\mathcal{S}$) of τ

$$\mathcal{T}\mathbf{:}\,\mathcal{G}(\tau+1) = T\mathcal{G}(\tau)\;;\; T = \text{diag}\left(e^{i\pi}, e^{-\frac{i\pi}{5}}, e^{\frac{i\pi}{5}}\right)$$

$$\mathcal{S}\mathbf{:}\,\mathcal{G}\left(-\frac{1}{\tau}\right) = S\mathcal{G}(\tau)\;;\; S = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 & 2\\ -1 & \frac{\sqrt{5}-1}{2} & \frac{\sqrt{5}+1}{2}\\ 1 & \frac{\sqrt{5}+1}{2} & \frac{\sqrt{5}-1}{2}\end{bmatrix}$$

[2] Definition 2.2 in "Theory of vector valued modular forms"

such that matrices T and S satisfy

$$S^2 = \mathbb{1}_3 \text{ and } (ST^{-1})^3 = \mathbb{1}_3.$$

Space of weakly holomorphic vvmfs: $\mathcal{G}(\tau)$ can be assigned as one of the free generators that span the full space of weakly holomorphic vvmfs of dimension 3.

The rest of the generators $\mathcal{G}_1(\tau), \mathcal{G}_2(\tau)$ can be determined by allowing the 3×3 matrix $\Xi(\tau) := [\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2](\tau)$ as the solution of matrix differential equation of Fuchsian type proposed by Gannon[3] [22]. We find

$$\mathcal{G}_1(\tau) = \begin{bmatrix} q^{1/2}\left(4590+970020q+46616310q^2+1148613480q^3+...\right) \\ q^{9/10}\left(42483+4119984q+144339894q^2+2963559204q^3+...\right) \\ q^{-9/10}\left(1+27q+46386q^2+3395448q^3+...\right) \end{bmatrix}$$

$$\mathcal{G}_2(\tau) = \begin{bmatrix} q^{-1/2}\left(1+222q+10050q^2+164530q^3+...\right) \\ q^{9/10}\left(-1275-33150q-435150q^2-3915900q^3+...\right) \\ q^{1/10}\left(25+2625q+57900q^2+691650q^3+...\right) \end{bmatrix}.$$

$\Xi(\tau)$ in closed form: The components of every free generator can be expressed in the closed form as Generalized hypergeometric functions.

$$\Xi(\tau) = \begin{bmatrix} -10\,G(\lambda_1+1,\lambda_2-1,\lambda_3;\tau) & 4590\,G(\lambda_1+1,\lambda_2,\lambda_3-1;\tau) & G(\lambda_1,\lambda_2,\lambda_3;\tau) \\ G(\lambda_2,\lambda_1,\lambda_3;\tau) & 42483\,G(\lambda_2+1,\lambda_1,\lambda_3-1;\tau) & -1275\,G(\lambda_2+1,\lambda_1-1,\lambda_3;\tau) \\ 3\,G(\lambda_3+1,\lambda_1,\lambda_2-1;\tau) & G(\lambda_3,\lambda_1,\lambda_2;\tau) & 25\,G(\lambda_3+1,\lambda_1-1,\lambda_2;\tau) \end{bmatrix}$$

[3] Equation (33) in "Theory of vector valued modular forms"

where $(\lambda_1, \lambda_2, \lambda_3) = \left(-\frac{1}{2}, -\frac{1}{10}, -\frac{9}{10}\right)$ and

$$G(a, b, c; \tau) = J(\tau)^{-a} \underbrace{{}_3F_2\left(a, a + \frac{1}{3}, a + \frac{2}{3}; a - b, a - c; \frac{1728}{J(\tau)}\right)}_{\text{Generalized hypergeometric function}}.$$

For the second family $\{\Delta^{(N,1)} | N = 2, 3, 5\}$ related to CHL $\mathbb{Z}_N$ orbifolds, where $t = 1$, we study a class of Jacobi forms $\left\{\Psi^{(N,1)}_{0,m} \mid m = 1, \ldots, N\right\}$ of congruence subgroup $\Gamma^0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z}) \mid b = 0 \bmod N\}$. For each $\Psi^{(N,1)}_{0,m}$ the corresponding $\mathcal{G}$ is a vvfm of congruence subgroup $\Gamma^0(N)$

Conclusion

We first show that all the Lie algebras $A^{(N)}$ and their Borcherds extensions $\mathcal{B}(A^{(N)})$ admit $\widehat{sl(2)}$ and $\mathcal{B}\left(\widehat{sl(2)}\right)$ sub-algebras then we expand the Siegel modular forms in terms of both $\widehat{sl(2)}$ and $\mathcal{B}\left(\widehat{sl(2)}\right)$ characters for the two families of examples.

The $\widehat{sl(2)}$ decomposition provides vector valued modular forms. Many of them are found to be the solutions to a matrix differential equation studied by Terry Gannon [22]. We further have closed formulae for most of them.

We obtain simple real roots of $\mathcal{B}(A^{(N)})$ to various orders(different for both families). A new kind of simple real fermionic root appears that does not fit into Borcherds extension.

Plan of the thesis

- Chapter 1 will provide general introduction and motivation for studying the embedding of $\widehat{sl(2)}$(or affine-$su(2)$) subalgebras in the spectral generating symmetry of the BPS spectrum.
- Chapter 2 will talk about black hole partition functions constructed from twisted/twined elliptic genera of K3 and the walls of marginal stability.
- Chapter 3 will build up Borcherds-Kac-Moody algebras from $su(2)$ algebra the simplest of of all Lie algebras.

- Chapter 4 will be dedicated to the fundamental Weyl chambers of $A^{(N)}$ algebras and Siegel modular forms as denominator formulae associated to Borcherd's extension $\mathcal{B}\left(A^{(N)}\right)$ of $A^{(N)}$ algebras.
- Chapter 5 will analyse the symmetry of the dyon spectrum by studying the embedded subalgebra $\widehat{sl(2)}$ into $\mathcal{B}\left(A^{(N)}\right)$ algebras whose denominator formulae will be decomposed in terms of $\widehat{sl(2)}$-characters.
- Chapter 6 will study the coefficients in $\widehat{sl(2)}$-decomposition of the denominator formulae for the two families. These coefficients form vector valued modular forms of certain congruence subgroups of $sl(2,\mathbb{Z})$ depends on the twine and twist of the elliptic genera of K3. They will be further studied under the work of Terry Gannon.
- Chapter 7 will conclude with a discussion of the results as well as open problems.

List of Figures

Chapter 1

Motivation and Introduction

The use of automorphic forms in the study of Lie algebras first arose in the work of Macdonald. He associated Jacobi forms with denominator formulae for affine Lie algebras and determined the multiplicity of positive roots using this connection [23, 24]. For instance, for $\widehat{sl(2)}$, the product side of the denominator formula is given by[1]

$$s^{1/2}\,\vartheta_1(\tau,z) = s^{1/2}q^{1/8}r^{1/2}\prod_{m=1}^{\infty}(1-q^m)(1-q^m r)(1-q^{m-1}r^{-1})\ . \tag{1.1}$$

Here we make the identifications: $e^{-\alpha_1}\sim qr$, $e^{-\alpha_2}\sim r^{-1}$ and $e^{-\delta}=e^{-\alpha_1-\alpha_2}\sim q$. The right hand side of the above equation then reads (with $e^{-\varrho}\sim s^{1/2}q^{1/8}r^{1/2}$)

$$e^{-\varrho}\prod_{\alpha\in L_+}(1-e^{-\alpha})\ ,$$

where ϱ is the Weyl vector and the set of positive roots L_+ is

$$L_+ = \{m\delta, (m-1)\delta+\alpha_1, (m-1)\delta+\alpha_2 \mid m\in\mathbb{Z}_{>0}\}\ .$$

[1] In this thesis, we follow the notation $q=\exp(2\pi i\tau)$, $r=\exp(2\pi i z)$ and $s=\exp(2\pi i\tau')$.

This product formula shows that one needs to include imaginary roots, of type $m\delta$, in the the set of positive roots.

Borcherds extended this by including situations where imaginary simple roots also appear [25]. This leads to modifications on the sum side of the denominator formulae to account for such simple roots. Today, such Lie algebras are called Borcherds-Kac-Moody (BKM) Lie algebras. Again, automorphic forms play an important role in determining the sum and product side of the denominator formulae which we call the Weyl-Kac-Borcherds denominator formulae. Schematically, one has

$$\Delta = \sum_{w\in W} \det(w) w[e^{-\varrho}\, T] = e^{-\varrho} \prod_{\alpha\in L_+} (1-e^{-\alpha})^{\text{mult}(\alpha)}\ . \tag{1.2}$$

In the above formula, an automorphic form Δ is written as a sum and a product. Where W, ϱ and T are the Weyl group, Weyl vector and the Borcherds correction factor respectively and will be defined later in Chapter-3 , L_+ corresponds to the set of positive roots and $\text{mult}(\alpha)$ is the multiplicity of the root α. The addition of fermionic roots leads to superalgebras and we shall focus on superdenominator formulae where a similar correspondence described above goes through.

In paper [1] we study the connection between five genus two Siegel modular forms (defined in Eq. (4.2))

$$\Delta_{k(N)}(\mathbf{Z})\ ,\ \text{for } N = 1,2,3,4,6 \quad \text{and} \quad k(N) = (6/N) - 1\ , \tag{1.3}$$

and the BKM Lie superalgebras associated with five rank-three Cartan matrices $A^{(N)}$. The Cartan matrices are the inner products of the bosonic real simple roots and are of rank three.

The square of these modular forms appear as the generating function of the refined counting of quarter-BPS states in certain CHL $\mathbb{Z}_N$ orbifolds of the heterotic string compactified on T^6 [13, 14]. Further, the root lattices associated with these Cartan matrices are closely related to the walls of marginal stability where quarter BPS states decay [10,19,20,26,27].

Let $\mathfrak{g}(A^{(N)})$ denote the Kac-Moody algebra associated with the Cartan matrix $A^{(N)}$ [28]. For $N \neq 6$, the Siegel modular form arises as the Weyl-Kac-Borcherds superdenominator formula for a Borcherds-Kac-Moody (BKM) Lie superalgebra that is a Borcherds extension of $\mathfrak{g}(A^{(N)})$ by the addition of imaginary simple roots [29,30]. Let us denote this BKM Lie superalgebra by $\mathcal{B}(A^{(N)})$. For $N = 6$, let $\mathcal{B}(A^{(6)})$ denote an as yet unknown Lie superalgebra whose WKB superdenominator formula is given by $\Delta_{k(6)}(\mathbf{Z})$. All five Siegel modular forms admit a product formula of the form [14] (see Eq. (4.2))

$$\Delta_{k(N)}(\mathbf{Z}) = e^{-2\pi i \mathrm{Tr}(\varrho^{(N)}\,\mathbf{Z})} \prod_{\alpha \in L_+} (1 - e^{-2\pi i \mathrm{Tr}(\alpha\,\mathbf{Z})})^{m(\alpha)} ,$$

where L_+ is the set of positive roots implicitly determined by the product formula and $\varrho^{(N)}$ is the Weyl vector that satisfies $\langle \varrho^{(N)}, \alpha \rangle = -1$ for all simple real roots α. The multiplicities $m(\alpha)$ are determined by the Fourier-Jacobi coefficients of a weight zero, index N, Jacobi form that appears in the context of umbral moonshine [21].

These BKM Lie superalgebras for $N = 1, 2, 3, 4$ naturally fit with the study of Lorentzian Kac-Moody Lie superalgebras associated with rank-three Cartan matrices by Gritsenko and Nikulin [16]. Gritsenko and Nikulin show that there exists no BKM Lie superalgebras associated with hyperbolic root systems with Weyl vector of hyperbolic type. The root lattice associated with $A^{(6)}$ is of this type and hence $\mathcal{B}(A^{(6)})$ cannot be a BKM Lie superalgebra. Our long-term goal is to construct the Lie superalgebra $\mathcal{B}(A^{(6)})$, if it is exists, whose WKB superdenominator formula is given by the Siegel modular form for $N = 6$. Clearly some additional inputs beyond the Borcherds correction term is needed. We do not solve this problem here but take the first step towards this by decomposing the Siegel modular forms in terms of a $\widehat{sl(2)}$ subalgebra of $\mathfrak{g}(A^{(N)})$.

Feingold and Frenkel study a Lorentzian Kac-Moody Lie algebra associated with a rank three Cartan matrix using a $\widehat{sl(2)}$-subalgebra [31]. Inspired by this, we study all the five modular forms in terms of a $\widehat{sl(2)}$ subalgebra present in all five examples. This enables us to carry out a systematic decomposition of the Siegel modular forms in terms of characters of $\widehat{sl(2)}$ and its Borcherds extension that we denote by $\mathcal{B}_N(\widehat{sl(2)})$ – this is a sub-algebra of $\mathcal{B}(A^{(N)})$ for $N \neq 6$.

A summary of the main results of the first paper is as follows.

1. We decompose the Umbral Jacobi forms in terms of characters of a $\widehat{sl(2)}$ Lie algebra. This leads to vector valued modular forms (vvmfs) with well-defined modular properties that we determine.
2. For $N \leq 4$, we show that these vvmfs arise as solutions to a matrix differential equation proposed by Gannon [22]. For $N = 6$, we obtain a closed formula for the Fourier coefficients of the vvmf using a different method.
3. We re-express the decomposition of the Umbral Jacobi form in terms of $\mathcal{B}_N(\widehat{sl(2)})$ characters and assign weights using roots of $\mathcal{B}(A^{(N)})$. Using the covariance properties of the Siegel modular forms, we are able identify Weyl orbits for each of the weights that appear in the Lie algebraic decomposition. This provides a preliminary insight into rewriting the Siegel modular forms as sums of Weyl orbits.

Working on second paper [2], we continue the study of Siegel modular forms that are, in some cases, the denominator formulae for some Borcherds-Kac-Moody (BKM) Lie superalgebras. These Siegel modular forms include examples for which the Lie algebra connection is not yet known. For such examples, the eventual goal is to prove (or disprove) the existence of Lie algebras whose denominator formulae are given by these Siegel modular forms.

In paper [1], we studied a family of Siegel modular forms that are associated with Umbral moonshine [21]. Here we consider Siegel modular forms that are associated with $L_2(11)$-moonshine [14,32]. The squares of these Siegel modular forms are the generating function of quarter BPS states in CHL $\mathbb{Z}_N$ orbifolds (for $N = 1, 2, \ldots, 6$) [3, 10, 14, 19]. The main tool (as in paper [1]) to probe the structure of the Lie algebras are two subalgebras: one is a $\widehat{sl(2)}$ subalgebra and the other is a Borcherds extension of the $\widehat{sl(2)}$ subalgebra. We rewrite the Siegel modular forms in terms of characters of the sub-algebras – it enables us to cleanly track simple roots that appear in the denominator formulae.

In this thesis we focus on the situations when $N = 2, 3, 4, 5$ and 6. These are the modular forms of weight

$$k(N) = \begin{cases} 12/(N+1) - 1\,, & N = 2,3,5 \\ 3/2\,, & N = 4 \\ 1\,, & N = 6 \end{cases}$$

of a level N subgroup of $Sp(4, \mathbb{Z})$. The connection with Mathieu and $L_2(11)$ moonshine leads to a product formula given in Eq. (4.18), for the Siegel modular forms [17, 18, 32]. For the prime cases i.e., $N = 2, 3, 5$, it is consistent with the product formulae given by David et al. [6] in the context of dyon counting. We rewrite the Siegel modular form as follows:

$$\Delta^{(N)}_{k(N)}(\mathbf{Z}) = s^{1/2}\,\phi^{(N)}_{k(N),1/2}(\tau, z) \times [1 + \sum_{m=1}^{\infty} s^m\,\Psi^{(N)}_{0,m}(\tau, z).)]\,, \tag{1.4}$$

The Jacobi forms $\Psi^{(N)}_{0,m}(\tau, z)$ will be the main object of our study. They are Jacobi forms of the congruence subgroup $\Gamma^0(N)$ with weight zero and index m. We obtain explicit formulae for these Jacobi forms in terms of standard modular forms for $m \leq N$. The analogous expansion in paper [1] had non-vanishing terms only for indices that were multiples of N.

We wish to show that the Siegel modular forms $\Delta^{(N)}_{k(N)}(\mathbf{Z})$ are extensions of the Kac-Moody Lie algebra $\mathfrak{g}(A^{(N)})$ obtained from the Cartan matrix, $A^{(N)}$, defined in Eq. (3.22). We call the extension $\mathcal{B}^{CHL}_N(A^{(N)})$ – the CHL refers to the fact that the square of the modular forms are the generating functions of quarter BPS states in CHL $\mathbb{Z}_N$ orbifolds [5, 10, 19]. The Cartan matrices $A^{(N)}$ are obtained from the walls of marginal stability in these models [33]. These have nice behaviour only for $N = 1, 2, \ldots, 6$. The expectation is that for $N \leq 4$, the extension $\mathcal{B}^{CHL}_N(A^{(N)})$ is the usual Borcherds extension of $\mathfrak{g}(A^{(N)})$ which leads to the sum side of the denominator formula given in Eq. (1.2). The Borcherds correction term is shown symbolically as T in this formula – it is the contribution that one obtains by adding imaginary simple roots i.e., roots with negative or zero norm.

A Cartan matrix can also be obtained as the matrix of inner products of simple root vectors which generate a root lattice. In all the six examples, the Cartan matrix has rank three and the root lattice is in Lorentzian space $\mathbb{R}^{2,1}$. A special feature of these lattices is that they admit a lattice Weyl vector $\varrho^{(N)}$ with inner product $\langle \varrho^{(N)}, \alpha \rangle = -1$ where α is a simple root. Such lattices have been studied by Nikulin and the corresponding Lie algebra connection by Gritsenko and Nikulin [16]. An important result from Gritsenko and Nikulin is that the cases of $N \leq 4$ in our examples can admit Borcherds extensions. This is why we expect that $\mathcal{B}_N^{CHL}(A^{(N)})$ are Borcherds extensions. Unlike the examples considered in paper [1], we are unaware of a proof that this is indeed the case for $N \leq 4$.

The reason one hopes that there might be a Lie algebra for $N = 5, 6$ is a physical one. The dyon counting generating function provides us with Siegel modular forms that transform covariantly under the Weyl group of $\mathfrak{g}(A^{(N)})$. We have three examples of this variety, one of which was considered in [1]. We restrict to the case of $N = 5$ for simplicity in this work commenting on some aspects of the $N = 6$ example. Our goal in this work is a modest one. We study two sub-algebras of $\mathcal{B}_N^{CHL}(A^{(N)})$, one is an $\widehat{sl(2)} \in \mathfrak{g}(A^{(N)})$ and another is a Borcherds extension of the $\widehat{sl(2)}$ that we call $\mathcal{B}_N^{CHL}(\widehat{sl(2)})$. Interestingly, these subalgebras are the best examples to understand the idea behind the Borcherds extension. The positive roots of the Lie algebra $\mathcal{B}_N^{CHL}(\widehat{sl(2)})$ that are not in the sub-algebra will organise into a representation of the sub-algebra. This is the motivation for us to look into character decompositions of the $\Psi_{0,m}^{(N)}(\tau, z)$ in terms of $\widehat{sl(2)}$ and $\mathcal{B}_N^{CHL}(\widehat{sl(2)})$.

The goal of the second paper is a modest one. We would like to understand the structure of the irreducible roots that appear in the first N terms. The main result is that we are able to characterize all the roots that appear to this order and they are consistent with our expectations. There are some surprises. For instance, $\Psi_{0,2}^{(3)}(\tau, z)$ vanishes. This is due to perfect cancellations between two different terms. We see the appearance of a real simple fermionic root in the examples of $N = 5, 6$ which has some peculiar properties. This is the first term that does not appear as a Borcherds extension. This is consistent with a no-go theorem of Gritsenko and Nikulin that suggests that modifications be needed for the cases of $A^{(5)}$ and $A^{(6)}$ [16].

The rest of the thesis is organized as follows. In Chapter 2, we give a concise and possibly a linear introduction of the quantum black holes in the CHL models where firstly, our main focus is on the associated quarter-BPS generating functions which can be constructed from the twisted/twined elliptic genera of K3 and finally we introduce the walls of marginal stability in the moduli space. In Chapter 3, we move on to a quick description of the BKM algebras starting from the $su(2)$ algebra. Then we introduce a class of hyperbolic Lie algebras $A^{(N=1,\cdots,6)}$, which we are highly interested in, by defining some of their crucial characteristic aspects. One of those aspects is the fundamental Weyl chambers which can be identified with the walls of marginal stability associated to a given $\mathbb{Z}_N$-CHL orbifold theory. In Chapter 4, we talk about certain Borcherds' extensions of the $A^{(N)}$ algebras , so that the product sides of their generalized Weyl denominator identities appear as square roots of the quarter-BPS generating functions associated to a $\mathbb{Z}_N$-CHL theory upto an appropriate twist in the BPS states. In Chapter 5, considering that all $A^{(N)}$ algebras have $\widehat{sl(2)}$ algebras embedded in them, we explore the Lie algebraic symmetry of the BPS spectrum by decomposing the generating functions or precisely their square roots in terms of $\widehat{sl(2)}$-characters. In Chapter 6, we study the coefficients in $\widehat{sl(2)}$-decomposition of the denominator formulae for the two families. These coefficients form vector valued modular forms of certain congruence subgroups of $sl(2,\mathbb{Z})$ which depend on the twine and twist of the elliptic genera of K3. They are further studied under the work of Terry Gannon for most of the cases. In Chapter 7, we conclude with a discussion of the results as well as open problems. In Appendix A, we try to provide a modest modular background including all sorts of modular forms one may need while going through this thesis. In Appendix B we give a brief introduction of character and super-character formulae in the context of BKM Lie algebra.

Chapter 2

Black holes in $\mathcal{N} = 4$ string theories

2.1 Black holes

A black hole is an object where a great amount of mass is packed into a very small volume which is surrounded by an imaginary surface called event horizon. The event horizon appears completely black to an outside observer because the gravitational pull is so high that nothing can escape it not even light [34].

Black holes, in general, are best described as classical solutions to the equation of motion in Einstein-Maxwell theory where the action is given by [35]

$$\frac{1}{16\pi G_N}\int R\sqrt{g}d^4x - \frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\sqrt{g}d^4x \tag{2.1}$$

for G_N, $F_{\mu\nu}$ and R being the Newton's gravitational constant, the electro-magnetic field tensor and the Ricci scalar of the metric $g_{\mu\nu}$ respectively. We follow the mostly positive convention i.e. metric has signature $(-, +, +, +)$ where indices μ, ν take the values $0, 1, 2, 3$ and $g = -\det(g_{\mu\nu})$.

As a consequence of so called *no-hair theorem* which states "*In four dimensions, all stationary black hole solutions of the Einstein–Maxwell equations can be completely char-*

acterized by their mass M, charge Q, and angular momentum J [36]." we are only left out with the four possibilities

Q \ J	= 0	≠ 0
= 0	I	II
> 0	III	IV

I *Schwarzschild solution* : Non-rotating black holes with no charge.

II *Reissner-Nordström solution* : Non-rotating charged black holes.

III *Kerr solution* : Rotating black holes with no charge.

IV *Kerr-Newman solution* : Rotating charged black holes.

2.1.1 Extremal and non-extremal black holes

An extremal black hole is a black hole with the minimal possible mass that can be compatible with a given charge and angular momentum [37]. All the known spherically symmetric black holes in four dimensions with non-singular horizon can be recognized as either extremal or non-extremal depending on whether there is a two-dimensional Anti-de Sitter(AdS_2) factor or a Rindler like factor present in their near horizon geometry respectively [26, 35].

As an example, Reissner-Nordström(RN) black holes will be the best candidate not to just understand what extremality means but are the ones closely related to our work when we move from a classical to a quantum description in string theoretic perspective as BPS black holes.

The Reissner-Nordström solution in the Einstein- Maxwell theory (2.1), when the units are chosen such that $G_N = \hbar = 1$, is given by [26, 35]

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2+p^2}{4\pi r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2+p^2}{4\pi r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.2}$$

$$F_{rt} = \frac{q}{4\pi r^2}, \qquad F_{\theta\phi} = \frac{p}{4\pi}\sin\theta, \tag{2.3}$$

where q(electric) and p(magnetic) are the charges and M is the mass of the black hole.

From the metric (2.2) the event horizon is determined by solving the following quadratic equation for r

$$g^{tt} = 0 : 1 - \frac{2M}{r} + \frac{q^2+p^2}{4\pi r^2} = 0$$

or

$$r^2 - 2Mr + \frac{q^2+p^2}{4\pi} = 0$$

for the bound $M^2 \geq \frac{q^2+p^2}{4\pi}$ it has two real solutions

$$r_\pm = M \pm \sqrt{M^2 - \frac{q^2+p^2}{4\pi}} \tag{2.4}$$

where $r_+(r_-)$ defines the outer(inner) horizon of the black hole. If $M^2 < \frac{q^2+p^2}{4\pi}$ the two horizons disappear and we have a naked singularity. Theories with supersymmetry usually guarantee that such black holes cannot exist [37] which agrees with more general(including non-supersymmetric theories) conjecture of cosmic censorship [34, 38, 39]. Clearly, for no charge ($q = p = 0$), the RN solution will reduce to Schwarzschild solution where $r_+ = 2M$, $r_- = 0$.

We call the black hole extremal if the mass-charge bound saturates i.e. $M^2 = \frac{q^2+p^2}{4\pi}$ otherwise its a non-extremal black hole. Let us explore the near horizon geometry of the non-extremal RN black hole before we jump into the extremal one. The RN metric can be rewritten in terms of $r_\pm$ as follows

$$ds^2 = -(1 - r_+/r)(1 - r_-/r)dt^2 + \frac{dr^2}{(1 - r_+/r)(1 - r_-/r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.5}$$

to focus on near outer horizon geometry let us define $r = r_+ + \xi$. The metric component $g^{tt} = -(1 - r_+/r)(1 - r_-/r)$ then takes the form

$$g^{tt} = -\left[1 - (1 + \xi/r_+)^{-1}\right]\left[1 - \frac{r_-}{r_+}(1 + \xi/r_+)^{-1}\right] \tag{2.6}$$

near horizon $\xi \ll r_+$ it reduces to

$$g^{tt} = -\frac{\xi}{ar_+}; \quad a = (1 - r_-/r_+)^{-1} \tag{2.7}$$

The full near horizon metric for non-extremal RN black hole approximates to the form

$$ds^2 = -\frac{\xi}{ar_+}dt^2 + \frac{ar_+}{\xi}d\xi^2 + r_+^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.8}$$

If we replace r_+ by $2M$ and put $a = 1$ this simply ends up giving the near horizon Schwarzschild metric. By introducing a new coordinate ρ,

$$\rho^2 = (4ar_+)\xi \quad \text{so that} \quad \frac{ar_+}{\xi}d\xi^2 = d\rho^2$$

the geometry factorizes into 2-dimensional Rindler spacetime and a 2-sphere of radius r_+.

$$ds^2 = \underbrace{-\frac{\rho^2}{4a^2r_+^2}dt^2 + d\rho^2}_{\text{2-dim Rindler spacetime}} + \underbrace{r_+^2(d\theta^2 + \sin^2\theta d\phi^2)}_{\text{2-sphere of radius } r_+}. \tag{2.9}$$

In order to explore the near horizon geometry of an extremal RN black hole, we have to take the two limits $r_- \to r_+$(extremality) and $r \to \frac{r_+ + r_-}{2}$(near horizon). Let us define $r_- = r_+ - 2\lambda$ and $r = \frac{r_+ + r_-}{2} + \delta = r_+ - \lambda + \delta$. The g^{tt} component of the metric (2.5) then looks like

$$g^{tt} = -\left[1 - \left(-\frac{\delta - \lambda}{r_+}\right)^{-1}\right]\left[1 - \left(1 - 2\frac{\lambda}{r_+}\right)\left(-\frac{\delta - \lambda}{r_+}\right)^{-1}\right] \tag{2.10}$$

in the extremal limit ($\lambda \ll r_+$) near horizon ($\delta \ll r_+$) it reduces to

$$g^{tt} = -\left[(\delta/\lambda)^2 - 1\right](\lambda/r_+)^2. \tag{2.11}$$

Full metric (2.5) is then approximates to

$$ds^2 = r_+^2\left[-\left((\delta/\lambda)^2 - 1\right)\ d\left(\lambda t/r_+^2\right)^2 + \frac{d(\delta/\lambda)^2}{(\delta/\lambda)^2 - 1}\right] + r_+^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.12}$$

if we take the limits λ, $\delta \to 0$ so that $\delta/\lambda \to \sigma$ and $\lambda t/r_+^2 \to \tau$ we can see that the near horizon geometry of extremal RN black hole factorizes into a 2-sphere of radius r_+ and a 2-dimensional Anti-de Sitter spacetime parametrized by (σ, τ)

$$ds^2 = \underbrace{r_+^2\left[-\left(\sigma^2 - 1\right)\ d\tau^2 + \frac{d\sigma^2}{\sigma^2 - 1}\right]}_{AdS_2} + \underbrace{r_+^2(d\theta^2 + \sin^2\theta d\phi^2)}_{\text{2-sphere of radius } r_+} \tag{2.13}$$

2.2 Black hole mechanics

Before one can state the laws of black hole mechanics, it is required to identify some governing parameters of black hole evolution. Since the defining characteristic of a black hole is that it has an event horizon, those parameters should essentially related to the properties of the event horizon. There are three important parameters seems to govern the black hole mechanics [35, 40].

1. The radius of the event horizon r_H.
2. The area of the event horizon $A_H = 4\pi r_H^2$.
3. The surface gravity κ of a black hole which heuristically can be assumed as Newto-

nian acceleration at horizon.

Black hole / Parameter	Schwarzschild	Extremal RN
r_H	$2M$	$\sqrt{\frac{q^2+p^2}{4\pi}}$
A_H	$16\pi M^2$	q^2+p^2
κ	$1/(4M)$	0

The work of Bardeen, Carter, Hawking [41], and others shows that the laws of black hole mechanics has a striking resemblance with the three laws of thermodynamics. For a black hole of mass M, spin J and charge Q we summarize the comparison in the table below.

Law	Thermodynamics	Black hole mechanics
Zeroth	Temperature is constant throughout a body at equilibrium. T =constant	Surface gravity is constant on the event horizon. κ =constant
First	Energy is conserved. $dE = Tds + \mu dQ + \Omega dJ$, where μ and Ω are chemical potentials corresponding to charge Q and spin J respectively.	Energy is conserved. $dM = \frac{\kappa}{8\pi}dA + \mu dQ + \Omega dJ$.
Second	Entropy never decrease. $\Delta S \geq 0$	Area never decrease. $\Delta A \geq 0$

Here A is the area of horizon, and κ is the surface gravity, μ is the chemical potential conjugate to Q and Ω is the angular speed conjugate to J.

2.3 Black hole thermodynamics

Classically a black hole is described as a perfect black body with zero absolute temperature otherwise it has to emit radiation (*Stefan–Boltzmann law*). Zero absolute temperature of a classical black hole implies it must always have zero entropy which clearly seems to violate the second law of thermodynamics. This can be understood by simply arguing if some matter with non-zero entropy enters a black hole it causes the total entropy of the universe to decrease. The second law of thermodynamics can be saved only if the black hole also has entropy. Bekenstein argued that the entropy of a black hole must be proportional to its area [42].

Hawking showed that inclusion of quantum effects causes a black hole to radiate [43]. Quantum theory, even in the vacuum, makes sure that particle-antiparticle pairs keep on being created and annihilated constantly. Near the horizon, every once in a while, an antiparticle can fall in and push a particle to leak from the horizon off to infinity. According to Hawking's calculation the spectrum emitted by the black hole is precisely thermal with temperature $T_H = \frac{\kappa}{2\pi}$. This precise relation helps to identify the laws of black hole mechanics with laws of thermodynamics. Using the Hawking temperature and the first law of thermodynamics

$$dM = TdS = \frac{\kappa}{8\pi} dA$$

one gets the famous Bekenstein-Hawking entropy of a black hole

$$S_{BH} = \frac{A_H}{4}$$

Black hole	Schwarzschild	Extremal RN
Hawking's temperature $T_H = \kappa/(2\pi)$	$1/(8\pi M)$	0
Bekenstein-Hawking entropy $S_{BH} = A_H/4$	$4\pi M^2$	$(q^2 + p^2)/4$

2.4 Quantum black holes

The microscopic definition of entropy in conventional statistical mechanics is given by

$$S_{\rm stat} = \ln\, D(n, l, m, \ldots)$$

where $D(n, l, m, \ldots)$ is the number of quantum states available for a given set of macroscopic variables $(n, l, m, \ldots)$ in the underlying theory.

A well described quantum black hole in string theory should be the one where,

1. it shows stability against the Hawking radiation,
2. degeneracies $D(n, l, m, \ldots)$ are well defined and
3. $S_{BH} = S_{\rm stat}$ holds at least for low curvature at the horizon up to leading order term in the expression for $S_{\rm stat}$ obtained by taking the logarithm of the degeneracy of elementary string states.

For certain types of supersymmetric extremal black holes in string theory one can actually provide a microscopic explanation of the Bekenstein-Hawking entropy $A_H/4 = \ln\, D(n, l, m, \ldots)$ in the large charge limit [44]. Where supersymmetry ensures the stability of black hole and extremality helps to define degeneracy as there is no Hawking radiation at zero temperature.

In these black holes usually the degeneracy of microstates is explicitly calculated by counting the number of states of a system of known objects(D-branes, KK monopole etc.) in string theory which carry the same charges as the black hole does.

2.5 A class of $\mathcal{N} = 4$ string theories in four dimensions

A superstring theory $\mathcal{S}$ naturally live in ten dimensional Lorentzian spacetime $\mathcal{M}_{10}$. A four dimensional theory $(\mathcal{S}, C_6)$ can be achieved by compactifying $\mathcal{M}_{10}$ into a product manifold $\mathbb{R}^{1,3} \times C_6$, where C_6 is a compact Calabi-Yau 3-fold. A family of four dimensional theories $(\mathcal{S}, \mathcal{X} \times T^2)$, preserves $\mathcal{N} = 4$ supersymmetry, is related by the following chain of dualities

$$\boxed{\left(\text{Heterotic}, T^4 \times T^2\right)} \xLeftrightarrow{\text{string-string [45]}} \boxed{\left(\text{IIA}, \text{K3} \times T^2\right)} \xLeftrightarrow{\text{T-duality [46]}} \boxed{\left(\text{IIB}, \text{K3} \times T^2\right)}$$

The fact that $\mathcal{X}$CFT(non-linear sigma model with target space $\mathcal{X}$) has a finite automorphism which commutes with all the spacetime supersymmetries and exists at generic points of the moduli space of $\mathcal{X}$CFT, one can obtain a larger class of $\mathcal{N} = 4$ supersymmetric models by orbifolding the original theory $(\mathcal{S}, \mathcal{X} \times T^2)$. These models are called CHL(*Choudhury, Hockney and Lykken*) orbifolds [47–51] .

A CHL-$\mathbb{Z}_N$ orbifold of order $N(= 1, \ldots, 8)$ denoted by CHL$_N$ $(\mathcal{S}, \mathcal{X} \times T^2)$ [52] is obtained by taking a shift of $1/N$ fractional winding along one of the circles of T^2 and also an order N finite automorphism of internal $\mathcal{X}$CFT. The chain of dualities mentioned above, by the adiabatic argument of Vafa-Witten [53], is preserved under CHL orbifolding.

$$\boxed{\text{CHL}_N \left(\text{Het}, T^4 \times T^2\right)} \xLeftrightarrow{\text{string-string}} \boxed{\text{CHL}_N \left(\text{IIA}, \text{K3} \times T^2\right)} \xLeftrightarrow{\text{T-duality}} \boxed{\text{CHL}_N \left(\text{IIB}, \text{K3} \times T^2\right)}$$

This family of CHL models have many scalar fields , known as moduli. The fact that these moduli have identically vanishing potential as a requirement to preserve supersymmetry, they can have arbitrary vacuum expectation value in moduli space. Besides the requirement of $\mathcal{N} = 4$ supersymmetry completely determine the massless field content of the theory in terms of the number of $U(1)$ gauge fields present.

The theory has $r(\geq 6)$ abelian gauge fields where r depends on the model of compactification being considered and the low energy effective action in the theory is completely fixed upto two derivatives by the supersymmetry and receives no quantum corrections. Eventually, lacking the higher derivative terms, this action does not specify the the details of the underlying string compactification of the theory. In the supergravity approximation, the part of the action containing massless bosonic fields is given by [26]

$$I = \frac{1}{2\pi\alpha'}\int d^4x\sqrt{-\det G}\Big[R_G + \frac{1}{S^2}G^{\mu\nu}(\partial_\mu S\,\partial_\nu S - \frac{1}{2}\partial_\mu a\partial_\nu a) + \frac{1}{8}G^{\mu\nu}\mathrm{Tr}(\partial_\mu \Phi L\partial_\nu \Phi L) - G^{\mu\mu'}G^{\nu\nu'}F^{(i)}_{\mu\nu}(L\Phi L)_{ij}F^{(j)}_{\mu'\nu'} - \frac{a}{S}G^{\mu\mu'}G^{\nu\nu'}F^{(i)}_{\mu\nu}L_{ij}\tilde{F}^{(j)}_{\mu'\nu'}\Big] \tag{2.14}$$

where L is a Lorentzian metric with signature $(6, r-6)$, Φ is a $r\times r$ matrix valued scalar field satisfying $\Phi L\Phi^T = L$ and $\Phi^T = \Phi$, and $F^{(i)}_{\mu\nu}(i = 1,\ldots,r)$ is a vector representing the field strengths of the r gauge fields. The moduli space is given by

$$(\Gamma_1(N)\times SO(6, r-6;\mathbb{Z}))\Big\backslash\left(\frac{SL(2)}{U(1)}\times\frac{SO(6, r-6)}{SO(6)\times SO(r-6)}\right) \tag{2.15}$$

where, in the heterotic picture, $SO(6, r-6;\mathbb{Z})$ is the T-duality group and

$$\Gamma_1(N) = \{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in PSL(2,\mathbb{Z})\mid c = 0 \bmod N\,,\ a = d = 1 \bmod N\}$$

is the S-duality group that is manifest in the equation of motion and is compatible with the charge quantization. The fields that appear at low-energy can be organized into multiplets of these following symmetries.

1. The heterotic dilaton combines with the axion (obtained by dualizing the antisymmetric tensor) to form the complex scalar λ in the complex upper halfplane.
2. The r vector fields transform as an $SO(6, r-6;\mathbb{Z})$ vector under the T-duality group. Thus, the electric(resp. magnetic) charge vector $\vec{Q}$(resp. $\vec{P}$) associated with these vector fields are also vectors (resp. co-vectors) of $SO(6, r-6;\mathbb{Z})$. Further, the

electric and magnetic charges transform as a doublet under the S-duality group $\Gamma_1(N)$.

Under the action of the S-duality group, the axion-dilaton and the charges of dyons transform as follows

$$\lambda \to \frac{a\lambda + b}{c\lambda + d}, \quad \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix}. \tag{2.16}$$

they can further transform under a parity w so that

$$\lambda \overset{w}{\leftrightarrow} \bar{\lambda}, \quad \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} \overset{w}{\leftrightarrow} \begin{pmatrix} \vec{Q} \\ -\vec{P} \end{pmatrix}. \tag{2.17}$$

This inclusion of parity enlarges the modular group $PSL(2,\mathbb{Z})$ to $PGL(2,\mathbb{Z})$ and in turn extends the S-duality group $\Gamma_1(N)$ to $\hat{\Gamma}_1(N)$ [19, 20] which has three generators [10, 20]

$$\gamma^{(N)} = \begin{pmatrix} 1 & -1 \\ N & 1-N \end{pmatrix}, \quad \delta = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.18}$$

One can form three T-duality invariant scalars, $Q^2 = \vec{Q}^T L \vec{Q}$, $P^2 = \vec{P}^T L \vec{P}$ and $Q \cdot P = \vec{Q}^T L \vec{P}$ from the charge vectors. They are quantized such that $\left(\frac{N}{2}Q^2, Q \cdot P, \frac{1}{2}P^2\right) \equiv (n, l, m) \in \mathbb{Z}^3$. It is useful to define this triplet in the form of a symmetric matrix,

$$\mathcal{Q}(n, l, m) := \begin{pmatrix} Q^2 & Q \cdot P \\ Q \cdot P & P^2 \end{pmatrix} = \begin{pmatrix} \frac{2n}{N} & l \\ l & 2m \end{pmatrix}, \tag{2.19}$$

for a given $\gamma \in \hat{\Gamma}_1(N)$, S-duality acts on it as follows

$$\mathcal{Q} \to \gamma \cdot \mathcal{Q} \cdot \gamma^T. \tag{2.20}$$

For a given N there exist other T-duality invariants, due to its discrete nature. One such discrete invariant is the torsion $i = \gcd(\vec{Q} \wedge \vec{P})$ [54]. In our work we will only consider the quarter-BPS dyons with torsion $i = 1$

2.5.1 Spectrum of the particle states in $\mathcal{N} = 4$ supersymmetry

In a supersymmetric theory particle states lie in unitary irreducible representations or supermultiplets of the superalgebra. $\mathcal{N} = 4$ supersymmetry has two 'central charges' Z_1 and Z_2, and without loss of generality one can set the BPS bound so that mass $M \geq Z_1 \geq Z_2$. Thus we have two types of extremal(BPS) states and one of non-extremal(non-BPS) states,

1. $M = Z_1 = Z_2$: 16-dimensional short multiplets which preserve half(eight) of the sixteen supersymmetries. Such states are called half-BPS.
2. $M = Z_1 > Z_2$: 64-dimensional intermediate multiplets which preserve quarter(four) of the sixteen supersymmetries. Such states are called quarter-BPS.
3. $M > Z_1 > Z_2$: 256-dimensional long multiplets which preserve none of the sixteen supersymmetries. Such states are called non-BPS.

One of the consequences of the BPS property is that the nature of the BPS spectrum is 'topological' means it does not change with moduli unless they cross certain surfaces of co-dimension one in the moduli space [55]. Moreover the central charges Z_1, Z_2 depend on charges $(\vec{Q}, \vec{P})$ and asymptotic values of moduli so does the mass $M = Z_1 \geq Z_2$ determined by the BPS states. In that, a BPS state is called purely electric if magnetic charge $\vec{P}$ vanishes but not the electric charge $\vec{Q}$ and purely magnetic if the condition is reversed. Otherwise it is called dyonic when neither of electric and magnetic charges vanishes. If $\vec{Q}$ and $\vec{P}$ are non-parallel, the state is always dyonic and represented by a quarter-BPS multiplet. But if the two charges are parallel, such that one is not integral multiple of the other, one can choose a frame under S-duality transformation (2.16) to make the charge configuration purely electric with $\vec{Q} \neq 0$ and $\vec{P} = 0$. The pure electrically(or magnetically) charged sates are half BPS.

Degeneracy of BPS dyons and a twist

We would like to compute $D(n,l,m)$ the number of microstates which receives contribution only from the BPS state associated to the triplet (n,l,m) of T-duality invariants. Since these BPS states break $4k(k=1,2,\ldots)$ supersymmetries one defines $D(n,l,m)$ not as the absolute degeneracy but as an index called the helicity trace index [56, 57]:

$$B_{2k} = \frac{1}{(2k)!}\mathrm{Tr}\left\{(-1)^F(2h)^{2k}\right\} = \frac{1}{(2k)!}\mathrm{Tr}\left\{(-1)^{2h}(2h)^{2k}\right\} \tag{2.21}$$

where h is the third component of the angular momentum in the rest frame and the trace is taken over states carrying a fixed set of charges. This index has the properties [58] such that it

1. ignores $4k$ fermionic zero modes whose quantization produces Bose-Fermi degenerate states which results in a vanishing Witten index $\mathrm{Tr}\left\{(-1)^F\right\}$.
2. vanishes for more than $4k$ broken supersymmetries.
3. does not vary continuously with the change in moduli of the theory as non-BPS states do not contribute.

Since quarter-BPS states in $\mathcal{N}=4$ supersymmetry break $4\times 3=12$ states their degeneracy count is given by

$$D(n,l,m) = B_6 = \frac{1}{6!}\mathrm{Tr}_{n,l,m}\left\{(-1)^{2h}(2h)^6\right\} \tag{2.22}$$

The twisted indices defined by Sen [11] are

$$B^g_{2k} = \frac{1}{(2k)!}\mathrm{Tr}\left\{g(-1)^{2h}(2h)^{2k}\right\}, \tag{2.23}$$

associated to these indices we may call g-twisted-BPS states. Where g is a discrete symmetry of the theory which requires the moduli to be on special subspaces compatible with this symmetry and the charges should be g-invariant. Degeneracy of a twisted quarter-

BPS state in $\mathcal{N} = 4$ supersymmetry should be defined as

$$D^{(g)}(n,l,m) = B_6^g = \frac{1}{6!}\mathrm{Tr}_{n,l,m}\left\{g(-1)^{2h}(2h)^6\right\}. \tag{2.24}$$

If g is the identity e (no twisting) then $D^{(g)}(n,l,m)$ simply reduces to $D(n,l,m)$.

2.5.2 Twisted Partition function in CHL models

Let us first assume that there is a special sub-moduli space in the theory $(\mathcal{S}, \mathcal{X} \times T^2)$ where $\mathcal{X}$CFT has a $\mathbb{Z}_M \times \mathbb{Z}_N$ discrete symmetry such that the generators g and h(of $\mathbb{Z}_M$ and $\mathbb{Z}_N$ respectively) commute and keep all sixteen supersymmetries invariant. One can therefore define a grand canonical partition function for $\mathbb{Z}_M$-twisted quarter-BPS states in CHL_N $(\mathcal{S}, \mathcal{X} \times T^2)$ orbifold such that

$$Z^{(N,M)}(\mathbf{Z}) \equiv \frac{64}{\Phi^{(N,M)}} = \sum_{\substack{n,l,m \,\in\, \mathbb{Z}\\ m,n \,\geq\, -1}} D^{(M)}(n,l,m)\, q^{n/N} r^l s^m \tag{2.25}$$

where $\mathbf{Z}\left(\begin{smallmatrix}\tau & z\\ z & \tau'\end{smallmatrix}\right)$ and (τ, z, τ') are the chemical potentials with corresponding fugacities ($q = e^{2\pi i\tau}, r = e^{2\pi i z}, s = e^{2\pi i\tau'}$) conjugate to the charge (n,l,m). The factor of 64 reflects the degeneracy of a single quarter-BPS multiplet or one can say $D^{(M)}(n,l,m)/64$ counts the number of quarter-BPS multiplets with charge (n,l,m). One may count the degeneracies of $\mathbb{Z}_M$-twisted quarter-BPS states [5]

$$D^{(M)}(n,l,m) = \frac{(-1)^l}{N}\int_C d\mathbf{Z}\frac{e^{-2\pi i \mathrm{Tr}(Q\cdot\mathbf{Z})}}{\Phi^{(N,M)}(\mathbf{Z})}, \tag{2.26}$$

where $\mathrm{Tr}(Q\cdot\mathbf{Z}) = n\tau/N + lz + m\tau'$ and C is a three-dimensional subspace such that $0 \leq \Re\mathrm{e}(\tau) \leq N$, $0 \leq \Re\mathrm{e}(z) \leq 1$ and $0 \leq \Re\mathrm{e}(\tau') \leq 1$.

2.5.3 Generating functions as Siegel modular forms

A class of generating functions $\Phi^{(N,M)}(\mathbf{Z})$ has been constructed as Siegel modular forms in [12,14] (and references within) for the model CHL_N (IIB, K3 $\times T^2$). For $N \leq M$ they are modular forms of the paramodular group $\Gamma_t(P)$ with level $P = M/t$ and $t = \gcd(M, N)$-see Appendix A. When $N > M$ the modular forms $\Phi^{(N,M)}(\mathbf{Z})$ are given by an S-transformation ($\tau \to -1/\tilde{\tau}$, $z \to z/\tau$, $\tau' \to \tau' - z^2/\tau$) of $\Phi^{(M,N)}(\mathbf{Z})$. A given $\Phi^{(N,M)}(\mathbf{Z})$ can be factorized in to three pieces

$$\Phi^{(N,M)}(\mathbf{Z}) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6} \times \eta^{[g,h]}(\tau) \times \varepsilon^{[g,h]} \tag{2.27}$$

where $\eta^{[g,h]}(\tau)$ is recognized as a class of eta products and $\varepsilon^{[g,h]}$ can be identified as M-twined second quantized elliptic genus of K3/$\mathbb{Z}_N$ which is a multiplicative lift of (N-twisted/M-twined)-elliptic genus of K3 (see A.5 and A.6).

The full $\Phi^{(N,M)}(\mathbf{Z})$ has been computed independently by [10, 13, 14] as an additive lift of the first two pieces $\frac{\theta_1(\tau,z)^2}{\eta(\tau)^6} \times \eta^{[g,h]}(\tau)$, and by [11] where they found each of the three pieces in a product form using a physical derivation from D1-D5 system. In their computation the first term arises from the overall motion of the D1-D5 system in Taub- NUT space; the second term arises from the excitations of KK monopoles and third term arises from the excitations of the D1-branes(wrapping $S^1 \subset T^2$) moving inside the D5-brane(wrapping K3 $\times S^1$).

2.6 Black holes in CHL models

In CHL models, for a spherically symmetric extremal black hole solution carrying arbitrary electric and magnetic charge vectors $\vec{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_r \end{pmatrix}$ and $\vec{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix}$ respectively, the near horizon approximation can be written as

$$\begin{aligned} ds^2 &= \underbrace{\frac{v_1}{16}\left(-\left(\sigma^2-1\right)\,d\tau^2 + \frac{d\sigma^2}{\sigma^2-1}\right)}_{AdS_2} + \underbrace{\frac{v_2}{16}(d\theta^2 + \sin^2\theta d\phi^2)}_{S_2} \\ F_{\sigma\tau} &= \frac{1}{4}e_i\,, \quad F^{(i)}_{\theta\phi} = \frac{p_i}{16\pi}\sin\theta\,, \quad \Phi_{ij} = u_{ij}\,, \quad S = u_s\,, \quad a = u_a \end{aligned} \tag{2.28}$$

2.6.1 Calculating entropy

The fact that strings are extended objects as opposed to point particles it requires the effective action of general relativity to be incorporated with higher order derivative including Riemann tensor and other fields.

$$\mathcal{I} = \frac{1}{16\pi}\int \mathcal{L} \quad \text{where} \quad \mathcal{L} = R + R^2 + R^4 F^4 + \dots$$

Wald derived the first law of thermodynamics using modified action. That led to a versatile expression for the entropy S for a given general action $\mathcal{I}$ including higher order derivatives [59–61]. In the special case of a spherically symmetric black hole this entropy formula can be written as

$$S_{\text{wald}} = -8\pi \int_{horizon} d\theta\phi \frac{\delta \mathcal{I}}{\delta R_{rtrt}} \sqrt{-g_{rr}g_{tt}} \tag{2.29}$$

where action $\mathcal{I}$ is expressed in terms of covariant derivatives of fields by replacing anti-symmetric combinations of covariant derivatives in terms of Riemann tensor and $R_{\mu\nu\alpha\beta}$ are treated as independent variables.

The entropy function formalism

The fact that all known extremal black holes in four dimensions have near horizon isometry $SO(2,1) \times SO(3)$ acting on the $AdS_2 \times S_2$ space. One can exploit this isometry to calculate the Wald entropy easily by using the *entropy function formalism* where it is only needed to solve certain algebraic equations as opposed to rather difficult partial differential equations.

Let us start with a given four dimensional extremal black hole where metric $g_{\mu\nu}$ is coupled to a set of $U(1)$ gauge fields and a number of neutral scalar fields. In *entropy function formalism* we go through the following steps,

1. First evaluate the Lagrangian density in the theory and integrate it over S_2,

2. then take the Legendre transformation of this with respect to the near horizon electric field whose conjugate variable is identified as charge.
3. Eventually extremize the resulting function with respect to the near horizon values of the scalar fields and, the sizes of AdS_2 and S^2.
4. The final result would give the Wald entropy up to a factor of 2π.

For an example let us calculate the Wald entropy for an RN black hole,

$$
\begin{aligned}
ds^2 &= v_1 \left(-\left(\sigma^2 - 1\right)\ d\tau^2 + \frac{d\sigma^2}{\sigma^2 - 1} \right) + v_2 (d\theta^2 + \sin^2\theta d\phi^2) \\
F_{\sigma\tau} &= e, \qquad F_{\theta\phi} = \frac{p}{4\pi} \sin\theta.
\end{aligned}
\tag{2.30}
$$

We integrate the Lagrangian density over S_2 ,in the units so that $G_N = 1$,

$$
\begin{aligned}
f(v_1, v_2, e, p) &\equiv \int d\theta d\phi \sqrt{-g}\mathcal{L} \\
&= 2\pi v_1 v_2 \left[\frac{1}{4\pi}\left(\frac{1}{v_2} - \frac{1}{v_1}\right) + \left(\frac{e}{v_1}\right)^2 - \left(\frac{p}{4\pi v_2}\right)^2 \right].
\end{aligned}
$$

Entropy function ε is given by the 2π times of Legendre transformation of f with respect to e where its conjugate $q = \frac{\partial f}{\partial e}$ is identified as the black hole charge,

$$
\varepsilon(v_1, v_2, e, p, q) = 2\pi(eq - f).
$$

It has an extremum ε^* at

$$
v_1 = v_2 = \frac{q^2 + p^2}{4\pi}, \quad e = \frac{q}{4\pi}.
$$

Substituting this back into ε gives

$$
S_{wald} = \varepsilon^* = \frac{q^2 + p^2}{4} \tag{2.31}
$$

the Wald entropy of extremal RN black hole which we also found out to be the Bekenstein-Hawking entropy for the same black hole.

In order to obtain the entropy of a black hole in CHL models we substitute the metric (2.28) in the action (2.5) and calculate $f(v_1, v_2, e, p)$. Thus using the *entropy function formalism* one can determine the entropy of a quarter-BPS dyonic black hole (2.28) to be

$$S_{BH} = \pi \sqrt{Q^2 P^2 - (Q \cdot P)^2} \tag{2.32}$$

as long as $Q^2P^2 - (Q \cdot P)^2 > 0$, where $\vec{Q} = 2\vec{q}$ and $\vec{P} = \frac{1}{4\pi} L\vec{p}$.
The entropy of an $\mathbb{Z}_M$-twisted quarter-BPS dyonic black hole is simply [11]

$$S_{BH} = \frac{\pi}{M} \sqrt{Q^2 P^2 - (Q \cdot P)^2}. \tag{2.33}$$

For large charge, i.e., $Q^2, P^2, Q \cdot P \gg 0$, Sen [11] has shown that the macroscopic entropy agrees with the microscopic outcome i.e.,

$$S_{BH} = \frac{\pi}{M} \sqrt{Q^2 P^2 - (Q \cdot P)^2} = \ln D^{(M)}. \tag{2.34}$$

2.7 Walls of marginal stability

The Siegel modular forms $\Phi^{(N,M)}(\mathbf{Z})$ described in subsection (2.5.3) have double zeros at $z = 0$ which renders the integrand in expression (2.26) having a couple of poles which further in turn make degeneracies $D^{(M)}(n, l, m)$ contour dependent. In other words whenever contour hits a pole the dyon spectrum changes.

This phenomenon actually turns out to a have an elegant physical interpretation. For a given set of charges $(\vec{Q}, \vec{P})$ there exists, besides single-centered black hole on entire moduli space, two-centered black hole solutions carrying the charge $(\vec{Q}_1, \vec{P}_1)$ on one center and $(\vec{Q}_2, \vec{P}_2)$ on the other so that $(\vec{Q}_1 + \vec{Q}_2, \vec{P}_1 + \vec{P}_2) = (\vec{Q}, \vec{P})$. The distance between the two centers depend on the charge configuration and the moduli fields. As one reaches certain walls(codimension one surfaces) in the moduli space the two centers grow apart to infinity causing two-centered black hole to decay into single-centered. Thus a two-centered black hole remain stable within these walls called *walls of marginal stability*. The shape of the contour C deforms as the moduli are varied. When the contour hits a pole in the contour space the moduli crosses a wall of marginal stability, this is known as wall crossing phenomenon. How the contour depends on moduli, can be seen in [62].

Sen [33] has studied the decay of torsion one quarter-BPS states into a pair of half-BPS states where he identified the walls of marginal stability in the axion-dilaton plane which maps to upper half plane with coordinate λ. One may also look [63, 64].

Let us start with a two centered quarter-BPS black hole which has torsion one total charge configuration $\begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix}$. One of its center carries charge

$$\begin{pmatrix} \vec{Q}_1 \\ \vec{P}_1 \end{pmatrix} = \begin{pmatrix} ad\vec{Q} - bd\vec{P} \\ ca\vec{Q} - cb\vec{P} \end{pmatrix} \tag{2.35}$$

and the other does

$$\begin{pmatrix} \vec{Q}_2 \\ \vec{P}_2 \end{pmatrix} = \begin{pmatrix} -bc\vec{Q} + bd\vec{P} \\ -ac\vec{Q} + ad\vec{P} \end{pmatrix} \tag{2.36}$$

so that they add up to the total charge $\begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix}$.

This two centered black hole will decay , upon crossing a wall of marginal stability, into two half-BPS dyonic black holes if the integers a, b, c, d are such that

1. $ad - bc = 1$,
2. the equivalence relation $(a, b, c, d) \sim (a\sigma^{-1}, b\sigma^{-1}, c\sigma, d\sigma)$ holds for $\sigma \neq 0$,
3. exchanging the two decay products corresponds to the transformation $(a, b, c, d) \rightarrow (c, d, -a, -b)$
4. and the products ad, bd, bc, ac/N are all integers as required by the charge quantization.

The walls of marginal stability are circular arcs in the upper half-plane given by equation,

$$\left(\Re\mathfrak{e}(\lambda) - \frac{ad+bc}{2cd}\right)^2 + \left(\Im\mathfrak{m}(\lambda) + \frac{E}{2cd}\right)^2 = \frac{1}{4c^2d^2}\left(1+E^2\right)\,;\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N) \quad (2.37)$$

where the real function E depends on a, b, c, d, matrix valued moduli Φ, and charge $(\vec{Q}, \vec{P})$.

The arcs intersect the $\Re\mathfrak{e}(\lambda)$ axis at $\frac{a}{c}$ and $\frac{b}{d}$ for a given E. When $E = 0$ the arcs are semi circles centered on the $\Re\mathfrak{e}(\lambda)$ axis with radius $\frac{1}{2cd}$. When either $c = 0$ or $d = 0$, the circles become straight lines tilted at some angle on real axis or perpendicular to the real axis when $E = 0$.

It was discovered that the polygon $\mathcal{P}_N$ enclosed by these walls has a nice description as fundamental Weyl chambers of a certain rank-three hyperbolic Lie algebra $A^{(N)}$ for a given order of orbifolding $N \leq 6$ [10, 13, 14, 19, 20]. This Weyl chamber $\mathcal{P}_N$ has a dihedral symmetry Dih$(\mathcal{P}_N)$ generated by $\gamma^{(N)}$ and δ the two generators of $\hat{\Gamma}_1(N)$ we have mentioned earlier. For $N > 3$ the order of the group reaches infinity - one then needs an infinite number of semi-circles to obtain a closed region. An appropriate Borcherds extension of $A^{(N)}$ can be related to the generating functions $\Phi^{(N,M)}(\mathbf{Z})$ via a generalized Weyl-denominator identity. Although the twisting M does not affect the Weyl chamber $\mathcal{P}_N$ but it does change how the Borcherds extension takes place.

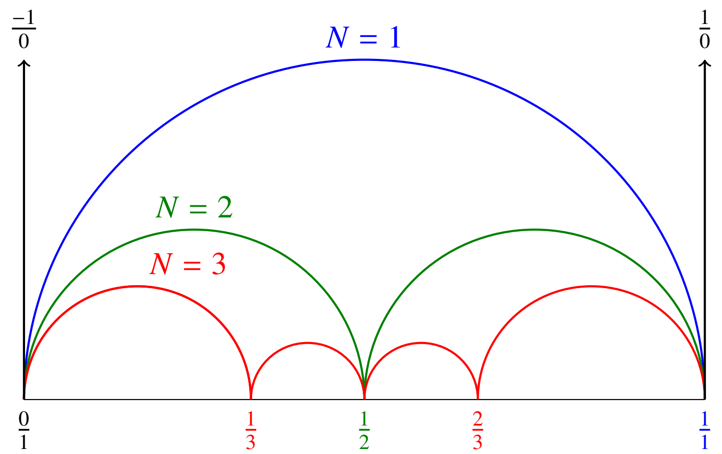


Figure 2.1: Walls of marginal stability for $N = 1, 2, 3$ when $E = 0$[1]

[1] We see figures for general N in the next chapter.

Chapter 3

BKM Lie superalgebras

Even though being the smallest finite dimensional simple Lie algebra, $sl(2, \mathbb{C})$ has a key role to play in the general theory of Kac- Moody algebras. In this chapter we will make a quick excursion from $sl(2, \mathbb{C})$ to BKM superalgebras.

3.1 Kac-Moody algebras as multiple copies of $sl(2, \mathbb{C})$

In particular, one may consider any rank r simple Kac-Moody algebra $\mathfrak{g}$ as a set of r distinct $sl(2, \mathbb{C})$-subalgebras which are intertwined in a non-trivial way. This idea is in fact a linchpin in the representation theory of Kac-Moody algebras [65].

The generators of $sl(2, \mathbb{C})$,

$$e \equiv \frac{\sigma_1 + i\sigma_2}{2}, \; f \equiv \frac{\sigma_1 - i\sigma_2}{2}, \; h \equiv \sigma_3 \tag{3.1}$$

satisfy the commutation relations

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f \tag{3.2}$$

where $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ and $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ are the Pauli matrices. We now intertwine the r copies of $sl(2,\mathbb{C})$ in a way, through the following relations,

$$[h_i, h_j] = 0,$$

$$[e_i, f_j] = \delta_{ij} h_j,$$

$$[h_i, e_j] = a_{ij} e_j,$$

$$[h_i, f_j] = -a_{ij} f_j,$$

so that a_{ij} can be taken as elements of an $r \times r$ square matrix A which has the following properties [28, 65]

1. all diagonal entries a_{ii} are 2 (comes from each copy of $sl(2,\mathbb{C})$),
2. off-diagonal entries a_{ij} are non-positive integers
3. and A is symmetrizable i.e., there exists a diagonal matrix $D = \text{diag}(\epsilon_1, \ldots, \epsilon_r)$, with all $\epsilon_i > 0$, such that the matrix A factorizes according to $A = DB$, where B is a symmetric $r \times r$ matrix.

We can now expect that there exists an abelian Lie algebra $\mathfrak{h}$ such that [28],

1. it has dimension given by $\dim \mathfrak{h} = r + \text{corank}(A)$,
2. it contains h_i from each copy of $sl(2,\mathbb{C})$ i.e., $\{h_1, \ldots, h_r\} \subset \mathfrak{h}$,
3. it is equipped with a non degenerate symmetric real valued bilinear form[1] $(.\ ,\ .)$ so that $(h_i, h_j) = a_{ij}$.

The following relations now define a Lie algebra $\tilde{\mathfrak{g}}(A, \mathfrak{h})$ associated to the matrix A and

[1]Definition: Let F be a field and V be a vector space over F. An F-valued *bilinear form* on V is a function $(.\ ,\ .) : V \times V \to F$ that is linear in each variable when the other one is fixed. That is,

$$(u + v, w) = (u, w) + (v, w),\ \ (cv, w) = c(v, w)$$

and

$$(u, v + w) = (u, v) + (u, w),\ \ (v, cw) = c(v, w),$$

for all $u, v, w \in V$ and $c \in F$. We call $(.\ ,\ .)$ symmetric when $(u, v) = (v, u)$ for all $u, v \in V$ and non-degenerate when $(u, v) = 0$ for all $v \in V$ then $u = 0$.

the abelian Lie algebra $\mathfrak{h}$,

$$\begin{aligned}
&[h, h'] = 0 \ \text{ for } \ h, h' \in \mathfrak{h}\,(\text{since } \mathfrak{h} \text{ is abelian}),\\
&[e_i, f_j] = \delta_{i,j} h_i\,,\\
&[h, e_i] = (h, h_i) e_i\,,\\
&[h, f_i] = -(h, h_i) f_i\,,
\end{aligned} \tag{3.3}$$

The algebra $\tilde{\mathfrak{g}}(A, \mathfrak{h})$ is spanned by the generators of $sl(2, \mathbb{C})$ from each copy, the additional generators from $\mathfrak{h}$ and also by infinite no. of generators in the form of multiple commutators,

$$[e_{i_1}, [e_{i_2}, \ldots, [e_{i_{k-1}}, e_{i_k}] \ldots]], \quad [f_{i_1}, [f_{i_2}, \ldots, [f_{i_{k-1}}, f_{i_k}] \ldots]].$$

So far there is no additional relations among e_i or f_i with themselves in order to truncate the size of these multi-commutators. This, in turn, always renders $\tilde{\mathfrak{g}}(A, \mathfrak{h})$ infinite dimensional.

Theorem 3.1.1. (A reduced Lie algebra [28]) *The algebra $\tilde{\mathfrak{g}}(A, \mathfrak{h})$ has a unique maximal ideal $\mathfrak{i}$ (a unique maximal subalgebra $\mathfrak{i}$ whose Lie bracket with $\tilde{\mathfrak{g}}(A, \mathfrak{h})$ is closed) generated by*

$$\overbrace{[e_i, [e_i, \cdots, [e_i}^{(1-a_{ij})\ times}, e_j] \cdots]] \ \textit{and} \ \overbrace{[f_i, [f_i, \cdots, [f_i}^{(1-a_{ij})\ times}, f_j] \cdots]] \ \textit{for}\ i \neq j\,.$$

One can quotient out $\mathfrak{i}$ from $\tilde{\mathfrak{g}}(A, \mathfrak{h})$ to get a smaller Lie algebra

$$\mathfrak{g}(A, \mathfrak{h}) = \tilde{\mathfrak{g}}(A, \mathfrak{h})/\mathfrak{i} \tag{3.4}$$

this amounts to set generators of $\mathfrak{i}$ *equal to zero i.e.,*

$$\overbrace{[e_i,[e_i,\cdots,[e_i}^{(1-a_{ij})\ times},e_j]\cdots]]=0\,,$$
$$\overbrace{[f_i,[f_i,\cdots,[f_i}^{(1-a_{ij})\ times},f_j]\cdots]]=0\ \textit{for all } i\neq j\,. \tag{3.5}$$

The reduced Lie algebra $\mathfrak{g}(A,\mathfrak{h})$ is called the Kac-Moody Lie algebra and defined by the combined relations (3.3) and (3.5). The associated matrix A and the abelian Lie algebra $\mathfrak{h}$ are now called Cartan matrix and Cartan subalgebra of $\mathfrak{g}(A,\mathfrak{h})$ respectively. Though the Kac-Moody algebra $\mathfrak{g}(A,\mathfrak{h})$ is smaller than $\tilde{\mathfrak{g}}(A,\mathfrak{h})$, it may still end up being infinite dimensional depends on the off-diagonal entries of the matrix A. The tree diagram(3.1) below shows the classifications of Kac-Moody algebras based on the properties of Caratan matrix A.

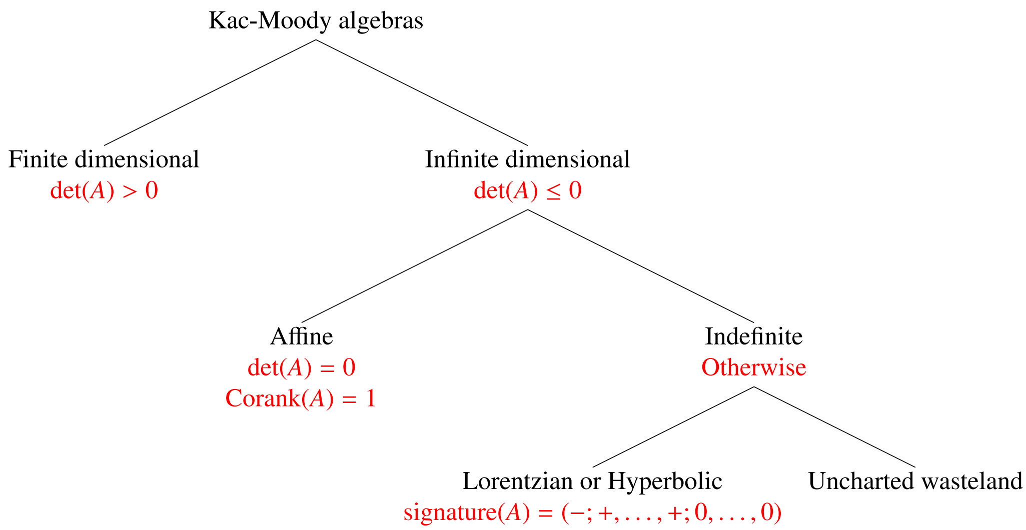


Figure 3.1: Classifications of Kac-Moody algebras

3.2 BKM algebras and their $\mathbb{Z}_2$ grading

The Borcherds-Kac-Moody(BKM) or generalized Kac-Moody(GKM)algebra is a bigger class of Lie algebras introduced by Borcherds by allowing the diagonal entries a_{ii} of the Cartan matrix A not just 2 but also non-positive real numbers which in turn allows off diagonal entries a_{ij} to be non-positive real numbers too.

BKM Lie algebras can be further extended to BKM Lie superalgebras by a $\mathbb{Z}_2$ grading. We start out by recognizing a subset $S \subset I$ of the index set $I = \{1, \ldots, r\}$ of the Cartan matrix $A = (a_{ij})$. In this case the Cartan matrix gets upgraded to Borcherds-Cartan supermatrix which has the following properties,

1. for all $i, j \in I$ and $i \neq j$

   either: $a_{ii} = 2$ and $a_{ij} \in \mathbb{Z}_{\leq 0}$ *(Kac-Moody case)*

   or: $a_{ii} \leq 0$ and $a_{ij} \in \mathbb{R}_{\leq 0}$ *(Borcherds extension)*,

2. A is symmetrizable and

3. $\mathbb{Z}_2$-grading: for all $i \in S$ and $j \notin S$

   if $a_{ii} = 2$ then $a_{ij} \in 2\mathbb{Z}_{\leq 0}$.

3.2.1 Lie superalgebra

A Lie superalgebra $\mathfrak{g}$ together with a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is characterized by the following additional conditions [66]

1. $\mathbb{Z}_2$-grading: $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that

$$[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \subseteq \mathfrak{g}_{\bar{0}}, [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subseteq \mathfrak{g}_{\bar{1}}, [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq \mathfrak{g}_{\bar{0}} \tag{3.6}$$

   where the elements of $\mathfrak{g}_{\bar{0}}$(resp. $\mathfrak{g}_{\bar{1}}$) are called bosonic(resp. fermionic).

2. Graded skew-symmetry:

$$[x, y] = -(-1)^{p(x).p(y)}[y, x] \tag{3.7}$$

   where $x, y \in \mathfrak{g}$ and the parity of x i.e., $p(x) = 0$ or 1 if x is bosonic or fermionic respectively.

3. Graded Jacobi identity:

$$(-1)^{p(x).p(y)}[y, [z, x]] + (-1)^{p(y).p(z)}[z, [x, y]] + (-1)^{p(z).p(x)}[x, [y, z]] = 0 \tag{3.8}$$

   for all $x, y, z \in \mathfrak{g}$.

3.2.2 BKM Lie superalgebras [15]

Associated to a Borcherds-Cartan supermatrix the BKM superalgebra $\mathfrak{g}(A, S, \mathfrak{h})$, is generated by the elements of $\mathfrak{h}$ and other generators $(e_i)_{i\in[1,r]}$, $(f_i)_{i\in[1,r]}$ which satisfy the following relations,

1. Lie superalgebra: $\mathbb{Z}_2$-grading is given by $\mathfrak{g}(A, S) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ where

$$\begin{aligned} \mathfrak{g}_{\bar{0}} &= \langle h_i, e_j, f_j | i \in I, j \in I \setminus S \rangle \\ \mathfrak{g}_{\bar{1}} &= \langle e_i, f_i | i \in S \rangle \end{aligned} \tag{3.9}$$

2. Chevalley-Serre relations:

$$\begin{aligned} &[h, h'] = 0 \ \text{ for } \ h, h' \in \mathfrak{h} \\ &[e_i, f_j] = \delta_{i,j} h_i \\ &[h, e_i] = (h, h_i) e_i \\ &[h, f_i] = (h, h_i) f_i \\ &(\text{ad } e_i)^{1-a_{ij}} e_j = 0 = (\text{ad } f_i)^{1-a_{ij}} f_j, \ \text{for } a_{ii} = 2, \ I \setminus S \ni i \neq j \in I \\ &(\text{ad } e_i)^{1-\frac{a_{ij}}{2}} e_j = 0 = (\text{ad } f_i)^{1-\frac{a_{ij}}{2}} f_j, \ \text{for } a_{ii} = 2, \ S \ni i \neq j \in I \\ &[e_i, e_j] = 0 = [f_i, f_j] \ \text{ if } \ a_{ij} = 0; \ i, j \in I. \end{aligned} \tag{3.10}$$

where the adjoint action is defined as (ad x) y:=[x,y].

In our work we are mostly interested in a class of generalized denominator identities appear in BKM Lie superalgebras, but before we jump into that we should know the following associated definitions.

Simple roots

The bilinear form $(.,.)$ in $\mathfrak{h}$ allowed an induced bilinear form $(.,.)$(using same notation for simplicity) on $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$. Let us begin with a set $\{\alpha_1, \ldots, \alpha_r\} \subset \mathfrak{h}^*$ such that

$(\alpha_i, \alpha_j) = a_{ij}$, these α_i are called simple roots of $\mathfrak{g}(A, \mathfrak{h})$. A simple root α_i is said to be,

1. real if $(\alpha_i, \alpha_i) = 2$ *(Kac-Moody type)*
2. imaginary if $(\alpha_i, \alpha_i) \leq 0$ *(Borcherds type).*

If all simple roots are real the algebra is of Kac-Moody type otherwise it is BKM.

Weyl group

Generated by the real simple roots the lattice $\mathfrak{Q}$ is called the root lattice associated to $\mathfrak{g}(A, \mathfrak{h})$. The reflection of any $\lambda \in \mathfrak{Q}$ with respect to a real simple root α_i is given by

$$r_i(\lambda) = \lambda - 2(\text{the } \alpha_i\text{-component of } \lambda\ ) = \lambda - 2\frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i \tag{3.11}$$

called simple reflection. The subgroup $\mathcal{W}$ of $GL(\mathfrak{Q})$, generated by all simple reflections, is called the Weyl group of $\mathfrak{g}(A, \mathfrak{h})$,

$$\mathcal{W} = \langle r_1, r_2, \ldots, r_n \rangle.$$

Roots

Any element of the Weyl orbit of real simple roots is called root and the full set of roots is given by

$$\Delta = \{w\alpha_1, \ldots, w\alpha_r | w \in \mathcal{W}\}.$$

For every root $\alpha \in \Delta$ there is $-\alpha \in \Delta$, if we call α to be positive $-\alpha$ will be called negative root. Since $\alpha = 0$ is not a root we can partition Δ into two disjoint sets Δ_+ and Δ_- of positive and negative roots respectively.

Root space decomposition

In the adjoint representation, there exist a triangular decomposition of the algebra $\mathfrak{g}(A, \mathfrak{h})$,

$$\mathfrak{g}(A, \mathfrak{h}) = \left[\bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{-\alpha}\right] \oplus \mathfrak{h} \oplus \left[\bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{\alpha}\right]$$

so that $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) \mid [h, x] = \langle\alpha, h\rangle x \text{ for all } h \in \mathfrak{h}\}$ where $\langle ., .\rangle$ is the natural pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$. A root space $\mathfrak{g}_\alpha$ is either contained in the bosonic part $\mathfrak{g}_{\bar{0}}$ or the fermionic part $\mathfrak{g}_{\bar{1}}$ of the superalgebra $\mathfrak{g}(A, S, \mathfrak{h})$ and the associated root α will be addressed as either bosonic or fermionic respectively. In terms of the dimension ($\dim \mathfrak{g}_\alpha$) of a root space $\mathfrak{g}_\alpha$ the multiplicity of the root α is either $\text{mult}(\alpha) = \dim \mathfrak{g}_\alpha$ if the root is bosonic or $\text{mult}(\alpha) = -\dim \mathfrak{g}_\alpha$ if the root is fermionic.

3.2.3 Denominator identities in BKM superalgebras

Denominator identities in BKM superalgebras(also known as Weyl-Kac-Borcherds(WKB) denominator formulae) appear when one apply the WKB character(super-character) formulae to the trivial one dimensional representation of character one. In this special case the denominator and the numerator of the WKB character(super-character) formula, are equal and we get ourselves a WKB denominator(super-denominator) identity. One side of this identity has a product runs over positive roots and on the other side it shows a summation over Weyl group,

$$e^{-\rho} \prod_{\alpha\in\Delta^+} \left(1 - \text{sgn}(\text{mult}(\alpha))\, e^{-\alpha}\right)^{\text{mult}(\alpha)} = \sum_{w\in\mathcal{W}} \det(w)\, w(e^{-\rho}\, T) \tag{3.12}$$

$$e^{-\rho} \prod_{\alpha\in\Delta^+} \left(1 - e^{-\alpha}\right)^{\text{mult}(\alpha)} = \sum_{w\in\mathcal{W}} \det(w)\, w(e^{-\rho}\, T_s) \tag{3.13}$$

where $\rho \in \mathfrak{h}^*$ is the Weyl vector defined by $(\rho, \alpha_i) = -\frac{1}{2}(\alpha_i, \alpha_i)$. The Borcherds correction term for denominator (super-denominator) identity is given by

$$
\begin{aligned}
T &= \sum_{\mu} (-1)^{\mathrm{ht}(\mu)} e^{(\mu)} \\
T_s &= \sum_{\mu_s} (-1)^{\mathrm{ht}(\mu_s)} e^{(\mu)}
\end{aligned}
\tag{3.14}
$$

where μ(or μ_s) runs over all possible sums of distinct, simple imaginary roots of all type(or bosonic type) which in addition are also pairwise orthogonal with respect to the bilinear form induced on $\mathfrak{h}^*$. The notation $\mathrm{ht}\,\mu$ implies the height of a given sum μ.

Alternatively let us denote by $\mathcal{B}$ and $\mathcal{F}$ the subsets of bosonic and fermionic roots(resp.) in the set of pairwise orthogonal, simple imaginary roots, then

$$
\begin{aligned}
T &= \sum_{\substack{B \in P(\mathcal{B}),\\ F \in P(\mathcal{F})}} (-1)^{\mathrm{ht}[S_{B,F}]} e^{-S_{B,F}} \\
T_s &= \sum_{B \in P(\mathcal{B})} (-1)^{|B|}\, e^{-\sum_{\beta \in B} \beta}
\end{aligned}
\tag{3.15}
$$

where $P(\mathcal{B})$ and $P(\mathcal{F})$ are power sets of $\mathcal{B}$ and $\mathcal{F}$ respectively and the sum of the roots

$$
S_{B,F} = \sum_{\beta \in B} \beta + \sum_{\phi \in F} k_\phi \phi \tag{3.16}
$$

for any $B \in P(\mathcal{B})$ and $F \in P(\mathcal{F})$ such that $k_\phi \in \mathbf{Z}_{>0}$ and $k_\phi \geq 2$ implies ϕ is null i.e., $(\phi, \phi) = 0$. The height of $S_{B,F}$ is

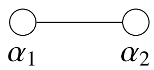
$$
\mathrm{ht}[S_{B,F}] = |B| + \sum_{\phi \in F} k_\phi, \tag{3.17}
$$

where $|B|$ is the cardinality of B. If there is no simple imaginary roots $\mathcal{B} = \emptyset = \mathcal{F}$ then $T = T_s = 1$(Kac-Moody case) and $T_s = 1$ if there is no bosonic imaginary simple root $\mathcal{B} = \emptyset$.

Examples

$su(3)$:

Simple roots : $\{\alpha_1, \alpha_2\}$ both are real and bosonic,



Cartan matrix : $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$

Weyl group : $\mathcal{W} = \{1, r_1, r_2, r_1r_2, r_2r_1, r_1r_2r_1\} = S_3.$

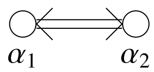
Root multiplicity : all roots have multiplicity one.

Roots : $\Delta = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\}.$

Weyl denominator formula : $e^{-\rho}\left(1 - e^{-\alpha_1}\right)\left(1 - e^{-\alpha_2}\right)\left(1 - e^{-\alpha_1-\alpha_2}\right) = \sum\limits_{w \in S_3} \det(w) e^{w(-\rho)}.$

$\widehat{sl(2)}$ or Affine-$sl(2, \mathbb{C})$:

Simple roots : $\{\alpha_1, \alpha_2\}$ are real and bosonic,



Cartan matrix : $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$ $\det(A) = 0.$

Weyl group : $\mathcal{W} = \{1, (r_1r_2)^n, (r_2r_1r_2)^n | n \in \mathbb{Z}\}.$

Roots : $\widehat{\Delta} = \{\pm\alpha_2,\ \pm\alpha_2 + n\delta,\ n\delta | n \in \mathbb{Z}, n \neq 0\}$ where $\delta = \alpha_1 + \alpha_2$ is an imaginary root since $(\delta, \delta) = 0$.

Root multiplicity : all roots have multiplicity one.

Weyl denominator formula : Under the identifications $e^{-\alpha_1} \sim qr$, $e^{-\alpha_2} \sim r^{-1}$ and $e^{-\delta} \sim q$

$$r^{\frac{1}{2}} q^{\frac{1}{8}} \prod_{n \geq 0} \left(1 - r^{-1} q^n\right) \prod_{n>0} (1 - rq^n)(1 - q^n) = \theta_1(\tau, z) \tag{3.18}$$

this the famous Jacobi Triple product identity.

$\mathcal{B}_k(\widehat{sl(2)})$: A Borcherds extension of $\widehat{sl(2)}$

In this example along with the real simple roots of $\widehat{sl(2)}$ we identify, as imaginary simple roots, the k distinct copies of each of the null roots $\delta, 2\delta, \ldots$ from $\widehat{sl(2)}$. This way we introduce a new BKM algebra $\mathcal{B}_k(\widehat{sl(2)})$. This process renders new Borcherds-Cartan matrix infinite dimensional with an embedded sub-Cartan matrix of $\widehat{sl(2)}$.

Simple roots : $\{\alpha_1, \alpha_2\} \cup \left\{ \underbrace{\color{red}{\delta}, \color{green}{\delta}, \ldots, \color{blue}{\delta}}_{k \text{ times}}; \underbrace{\color{red}{2\delta}, \color{green}{2\delta}, \ldots, \color{blue}{2\delta}}_{k \text{ times}}; \ldots \right\}$ these imaginary simple roots are pairwise orthogonal and each δ is distinct from the other hence the colour coding.

WKB denominator formula : Since there is no fermionic imaginary root the Borcherds' correction T is given by

$$\begin{aligned}
T &= \sum_{B \in P(\mathcal{B})} (-1)^{|B|} \, e^{-\sum\limits_{\beta \in B} \beta} \\
&= \sum_{n_1, n_2, \ldots \geq 0}^{k} (-1)^{n_1 + n_2 + \ldots} e^{-n_1 \delta} \binom{k}{n_1} e^{-n_2 2\delta} \binom{k}{n_2} \ldots \\
&= \left(1 - e^{-\delta}\right)^k \left(1 - e^{-2\delta}\right)^k \ldots = \prod_{n>0} \left(1 - e^{-n\delta}\right)^k \\
&= \prod_{n>0} (1 - q^n)^k
\end{aligned} \tag{3.19}$$

multiply this to the denominator identity of $\widehat{sl}(2)$

$$r^{\frac{1}{2}}q^{\frac{1}{8}}\prod_{n>0}\left(1-r^{-1}q^{n-1}\right)\left(1-rq^{n}\right)\left(1-q^{n}\right)^{k+1}=\sum_{w\in\mathcal{W}}\det(w)w\left(e^{-\rho}T\right) \tag{3.20}$$

using the Jacobi triple product identity (3.18)

$$\theta_1(\tau,z)\eta(\tau)^k=\sum_{w\in\mathcal{W}}\det(w)w\left(e^{-\rho}T\right) \tag{3.21}$$

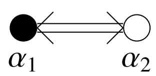
this is the WKB denominator identity for $\mathcal{B}_k(\widehat{sl}(2))$.

Twisted-$\widehat{sl(2)}$ algebras : Inclusion of fermionic roots

Case 1: One of the two simple roots of $\widehat{sl}(2)$ is fermionic

$A^{(4)}(0,2)$ [67] is a twisted affine Lie superalgebra associated to $\widehat{sl}(2)$ when one of the simple roots is fermionic.

Simple roots : $\left\{\alpha_1,\alpha_2\equiv\frac{1}{2}\delta-\alpha_1\right\}$ are real and α_1 is fermionic



Cartan matrix : Highest root $\theta=2\alpha_1$ and $(\theta,\theta)=2$ so $(\alpha_1,\alpha_1)=\frac{1}{2}$ this normalization leads to the Cartan super-matrix $\begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix}$.

Weyl group : $\mathcal{W}=\{1,(r_1r_2)^n,(r_2r_1r_2)^n|n\in\mathbb{Z}\}$.

Positive roots : $\Delta^+ = \Delta^+_{\bar{0}} \cup \Delta^+_{\bar{1}}$ where

$$\Delta^+_{\bar{0}} = \left\{\frac{1}{2}(n+1)\delta; \frac{1}{2}(n+\frac{1}{2})\delta \pm \alpha_1; 2n\delta + 2\alpha_1, 2(n+1)\delta - 2\alpha_1 \mid n \geq 0\right\} \quad \textit{(bosonic)}$$
$$\Delta^+_{\bar{1}} = \left\{\frac{1}{2}(n+\frac{1}{2})\delta; n\delta + \alpha_1, (n+1)\delta - \alpha_1 \mid n \geq 0\right\} \quad \textit{(fermionic)}$$

Root multiplicity :

1. Multiplicity all fermionic (real or imaginary) roots is -1.
2. Multiplicity of all real bosonic roots is 1.
3. Multiplicity of imaginary bosonic root of type $n\delta$ is

$$\text{mult}(n\delta) = \begin{cases} 1, & n \in \text{even} \\ -1, & n \in \text{half odd} \\ 2, & n \in \text{odd} \end{cases} .$$

Weyl denominator formula : (product side)

$$\prod_{n\geq 0} \frac{(1-e^{-2n\delta-2\alpha_1})(1-e^{-2(n+1)\delta+2\alpha_1})(1-e^{-(n+\frac{1}{2}\delta-\alpha_1})(1-e^{-(n+\frac{1}{2})\delta+\alpha_1})(1-e^{-(n+1)\delta})^{\text{mult}(\delta(n+1))}}{(1+e^{-n\delta-\alpha_1})(1+e^{-(n+1)\delta+\alpha_1})(1+e^{-(n+\frac{1}{2})\delta})}$$

using realization $e^{-\frac{1}{2}\delta} = q$, $e^{\alpha_1} = r$

$$= \prod_{n\geq 0} \frac{(1-r^{-2}(q^{2n})^2)(1-r^2(q^{2(n+1)})^2)(1-r^{-1}q^{2n+1})(1-rq^{2n+1})(1-q^{2(n+1)})^{\text{mult }(n+1)}}{(1+r^{-1}q^{2n})(1+rq^{2(n+1)})(1+q^{2n+1})}$$
$$= \prod_{n\geq 0} \frac{(1-r^{-1}q^{2n})(1-rq^{2(n+1)}))(1-r^{-1}q^{2n+1})(1-rq^{2n+1})(1-q^{2(n+1)})}{(1+q^{2n+1})} \prod_{n\in 2\mathbb{Z}_{\geq 0}} (1-q^{2(n+1)})$$
$$= \prod_{n\geq 0} \frac{(1-r^{-1}q^{n})(1-rq^{n+1}))(1-q^{2(n+1)})(1-(q^{2n+1})^2)}{(1+q^{2n+1})}$$
$$= \prod_{n\geq 0} (1-r^{-1}q^n)(1-rq^{n+1}))(1-q^{n+1})$$
$$= r^{-\frac{1}{2}} q^{-\frac{1}{8}} \theta_1(r,q)$$

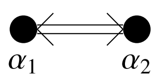
Weyl super-denominator formula : (product side)

$$\prod_{n\geq 0}\frac{(1-e^{-2n\delta-2\alpha_1})(1-e^{-2(n+1)\delta+2\alpha_1})(1-e^{-(n+\frac{1}{2}\delta-\alpha_1)})(1-e^{-(n+\frac{1}{2})\delta+\alpha_1})(1-e^{-(n+1)\delta})^{\mathrm{mult}((n+1)\delta)}}{(1-e^{-n\delta-\alpha_1})(1-e^{-(n+1)\delta+\alpha_1})(1-e^{-(n+\frac{1}{2})\delta})}$$
$$=\prod_{n\geq 0}\frac{(1-r^{-2}(q^{2n})^2)(1-r^2(q^{2(n+1)})^2)(1-r^{-1}q^{2n+1})(1-rq^{2n+1})(1-q^{2(n+1)})^{\mathrm{mult}(n+1)\delta}}{(1-r^{-1}q^{2n})(1-rq^{2(n+1)})(1-q^{2n+1})}$$
$$=\prod_{n\geq 0}\frac{(1+r^{-1}q^{2n})(1+rq^{2(n+1)})(1-r^{-1}q^{2n+1})(1-rq^{2n+1})(1-q^{2(n+1)})}{(1-q^{2n+1})}\prod_{n\in 2\mathbb{Z}_{\geq 0}}(1-q^{2(n+1)})$$
$$=\prod_{n\geq 0}(1-(q^2)^{n+1})(1+r^{-1}(q^2)^n)(1+r(q^2)^{n+1})(1-r^{-1}(q^2)^{n+\frac{1}{2}})(1-r(q^2)^{n+\frac{1}{2}})\prod_{n\geq 0}\frac{(1-(q^{2n+1})^2)}{(1-q^{2n+1})}$$
$$=\prod_{n\geq 0}(1-(q^2)^{n+1})(1+r^{-1}(q^2)^n)(1+r(q^2)^{n+1})\prod_{n\,\in\,\mathbb{N}+\frac{1}{2}}(1-r^{-1}(q^2)^n)(1-r(q^2)^n)\prod_{n\geq 0}(1+q^{2n+1})$$
$$=r^{-\frac{1}{2}}q^{-\frac{1}{4}}\theta_2(r,q^2)\,\frac{\theta_4(r,q^2)}{\varphi(q^2)}\prod_{n\geq 0}(1+q^{2n+1})\,;\quad (\varphi(q)\text{ is the Euler function}).$$

Case 2: Both of the simple roots of $\widehat{sl(2)}$ are fermionic

$C^{(2)}(2)$ [67] is a twisted affine Lie superalgebra associated to $\widehat{sl(2)}$ when both of the simple roots are fermionic.

Simple roots : $\left\{\alpha_1, \alpha_2 \equiv \frac{1}{2}\delta - \alpha_1\right\}$ are real and fermionic



Cartan matrix : Highest root $\theta = 2\alpha_1$ and $(\theta,\theta) = 2$ so $(\alpha_1,\alpha_1) = \frac{1}{2}$ this normalization leads to the Cartan super-matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

Weyl group : $\mathcal{W} = \{1, (r_1r_2)^n, (r_2r_1r_2)^n | n \in \mathbb{Z}\}$.

Positive roots : $\Delta^+ = \Delta^+_{\bar{0}} \cup \Delta^+_{\bar{1}}$ where

$$\Delta^+_{\bar{0}} = \left\{ n\delta + 2\alpha_1; (n+1)\delta - 2\alpha_1; \frac{1}{2}(n+1)\delta \mid n \geq 0 \right\} \quad \textit{(bosonic)}$$
$$\Delta^+_{\bar{1}} = \left\{ \frac{1}{2}n\delta + \alpha_1; \frac{1}{2}(n+1)\delta - \alpha_1 \mid n \geq 0 \right\} \quad \textit{(fermionic)}$$

Root multiplicity : Multiplicities of all bosonic and fermionic roots are 1 and −1 respectively. *Weyl denominator formula :* (product side)

$$\prod_{n\geq 0} \frac{(1 - e^{-n\delta - 2\alpha_1})(1 - e^{-(n+1)\delta + 2\alpha_1})(1 - e^{-\frac{1}{2}(n+1)\delta})}{(1 + e^{-\frac{1}{2}n\delta - \alpha_1})(1 + e^{-\frac{1}{2}(n+1)\delta + \alpha_1})}$$

using realization $e^{-\frac{1}{2}\delta} = q$, $e^{\alpha} = r$

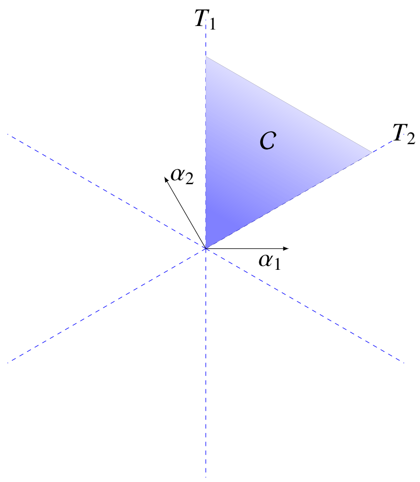
$$= \prod_{n\geq 0} \frac{(1 - r^{-2}q^{2n})(1 - r^2 q^{2(n+1)})(1 - q^{n+1})}{(1 + r^{-1}q^n)(1 + rq^{n+1})}$$
$$= \prod_{n\geq 0} (1 - r^{-1}q^n)(1 - rq^{n+1})(1 - q^{n+1})$$
$$= ir^{-\frac{1}{2}} q^{-\frac{1}{8}} \theta_1(r, q)$$

Weyl super-denominator formula : (product side)

$$\prod_{n\geq 0} \frac{(1 - e^{-n\delta - 2\alpha_1})(1 - e^{-(n+1)\delta + 2\alpha_1})(1 - e^{-\frac{1}{2}(n+1)\delta})}{(1 - e^{-\frac{1}{2}n\delta - \alpha_1})(1 - e^{-\frac{1}{2}(n+1)\delta + \alpha_1})}$$
$$= \prod_{n\geq 0} \frac{(1 - r^{-2}q^{2n})(1 - r^2 q^{2(n+1)})(1 - q^{n+1})}{(1 - r^{-1}q^n)(1 - rq^{n+1})}$$
$$= \prod_{n\geq 0} (1 + r^{-1}q^n)(1 + rq^{n+1})(1 - q^{n+1})$$
$$= r^{-\frac{1}{2}} q^{-\frac{1}{8}} \theta_2(r, q)$$

3.3 Fundamental Weyl chamber

The fundamental Weyl chamber of a Lie algebra $C \subset \sum_i \mathbb{R}\alpha_i$ is the region enclosed by hyperplanes perpendicular to the real simple roots such that for any $\gamma \in C$, $(\gamma, \alpha_i) > 0$ for all real simple roots α_i.



Example:
The shaded region is the fundamental Weyl chamber for the simple roots $\{\alpha_1, \alpha_2\}$ of $su(3)$ Lie algebra associated to

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Figure 3.2: Fundamental Weyl chamber of $su(3)$

The bilinear form $(.,.)$ in hyperbolic Lie algebras has Lorentzian signature and it vanishes for an element at the lightcone in h^*. As a special property of hyperbolic Lie algebras the fundamental chamber C always lie in the past lightcone though one may choose to treat $\{-\alpha_i\}$ as simple roots instead of $\{\alpha_i\}$ to bring the fundamental chamber into future lightcone. In all our example of hyperbolic Lie algebras we consider future lightcone convention where definition of Weyl vector is used $(\rho, \alpha_i) = -\frac{1}{2}(\alpha_i, \alpha_i)$ as opposed to the standard definition where minus sign is missing.

3.4 A class of rank-three hyperbolic Lie algebras

In our work we are interested in a class of rank-three Cartan matrices of hyperbolic type, given by

$$A^{(N)} = (a_{nm}) \tag{3.22}$$

where

$$a_{nm} = 2 - \frac{4}{N-4}(\lambda_N^{n-m} + \lambda_N^{m-n} - 2) \tag{3.23}$$

and λ_N is any solution of the quadaratic equation

$$\lambda^2 - (N-2)\lambda + 1 = 0\ . \tag{3.24}$$

For $N = 1, 2, 3$, the matrices are finite dimensional with the indices n, m defined modulo $3, 4, 6$ respectively.

$$A^{(1)} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix},\ A^{(2)} = \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -6 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} 2 & -2 & -10 & -14 & -10 & -2 \\ -2 & 2 & -2 & -10 & -14 & -10 \\ -10 & -2 & 2 & -2 & -10 & -14 \\ -14 & -10 & -2 & 2 & -2 & -10 \\ -10 & -14 & -10 & -2 & 2 & -2 \\ -2 & -10 & -14 & -10 & -2 & 2 \end{pmatrix},\ A^{(4,5,6)} = \begin{pmatrix} 2 & -2 & \cdots \\ -2 & 2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{3.25}$$

For $N = 4, 5, 6$, the matrices are infinite dimensional with $m, n \in \mathbb{Z}$. For $N = 4$, the Cartan matrix has to be obtained as a limit $N \to 4$ leading to $a_{nm} = 2 - 4(n-m)^2$. Let us denote by $\mathfrak{g}(A^{(N)})$ the Kac-Moody Lie algebra associated to $A^{(N)}$

3.4.1 Root Lattices, hyperbolic polygons and Weyl groups for $\mathfrak{g}(A^{(N)})$

Consider the following two matrices.

$$\alpha_1 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \alpha_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\ . \tag{3.26}$$

For $m \in \mathbb{Z}$, define

$$\alpha_{2m+a} = \left(\gamma^{(N)}\right)^m \cdot \alpha_a \cdot \left((\gamma^{(N)})^T\right)^m \text{ for } a = 1, 2, \tag{3.27}$$

where

$$\gamma^{(N)} = \begin{pmatrix} 1 & -1 \\ N & 1-N \end{pmatrix} .$$

Using the above definition, observe that

$$\alpha_0 = \begin{pmatrix} 2N-2 & 2N-1 \\ 2N-1 & 2N \end{pmatrix} \text{ and } \alpha_3 = \begin{pmatrix} 0 & 1 \\ 1 & 2N \end{pmatrix}. \tag{3.28}$$

Let $\mathbf{X}_N$ denote the ordered sequence of independent matrices α_i generated in this fashion. One has

$$\mathbf{X}_N = (\alpha_i) \text{ for } i \in \mathcal{S}_N = \begin{cases} (1, 2, 3 \bmod 3) , & N = 1 \\ (0, 1, 2, 3 \bmod 4) , & N = 2 \\ (0, 1, 2, 3, 4, 5 \bmod 6) , & N = 3 \\ \mathbb{Z} , & N = 4, 6 . \end{cases} . \tag{3.29}$$

We shall call the elements of $\mathbf{X}_N$ roots in anticipation of the facts that they are indeed the real simple roots of a Lie algebra.

The elements of $\mathbf{X}_N$ are in one-to-one correspondence with edges of a polygon, $\mathcal{M}_N$, in the hyperpoblic upper half plane with vertices at rational points in the real line which is the boundary of the upper half plane. Let $(\frac{b}{a}, \frac{d}{c})$ denote adjacent vertices of the hyperbolic polygon. The root corresponding to the edge connecting these two vertices is given by the map [19]

$$\left(\frac{b}{a}, \frac{d}{c}\right) \longrightarrow \alpha = \begin{pmatrix} 2bd & ad+bc \\ ad+bc & 2ac \end{pmatrix} . \tag{3.30}$$

We illustrate this for $N = 1, 2, 3$ in Figure 2.1 and in general for some roots in Figure 3.3.

For $N = 6$ consider the following additional roots.

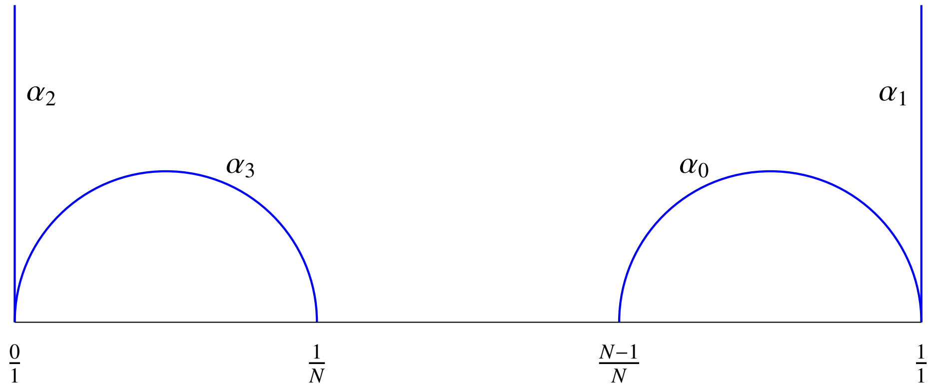


Figure 3.3: The four roots $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ represented as semi-circles in the upper half plane.

$$\left(\tfrac{2}{3}, \tfrac{1}{2}\right) \longleftrightarrow \tilde{\alpha}_1 = \begin{pmatrix} 2 & 5 \\ 5 & 12 \end{pmatrix} \quad , \quad \left(\tfrac{1}{2}, \tfrac{2}{3}\right) \longleftrightarrow \tilde{\alpha}_2 = \begin{pmatrix} 4 & 7 \\ 7 & 12 \end{pmatrix} \quad . \tag{3.31}$$

For $m \in \mathbb{Z}$, define

$$\tilde{\alpha}_{2m+a} = \left(\gamma^{(6)}\right)^m \cdot \tilde{\alpha}_a \cdot \left((\gamma^{(6)})^T\right)^m \text{ for } a = 1, 2. \tag{3.32}$$

Define $\tilde{\mathbf{X}}_6 = (\tilde{\alpha}_i)$ for $i \in \mathbb{Z}$. The two sets of (infinite) roots combine to give the hyperbolic polygon $\mathcal{M}_6$ (see Figure 3.4).

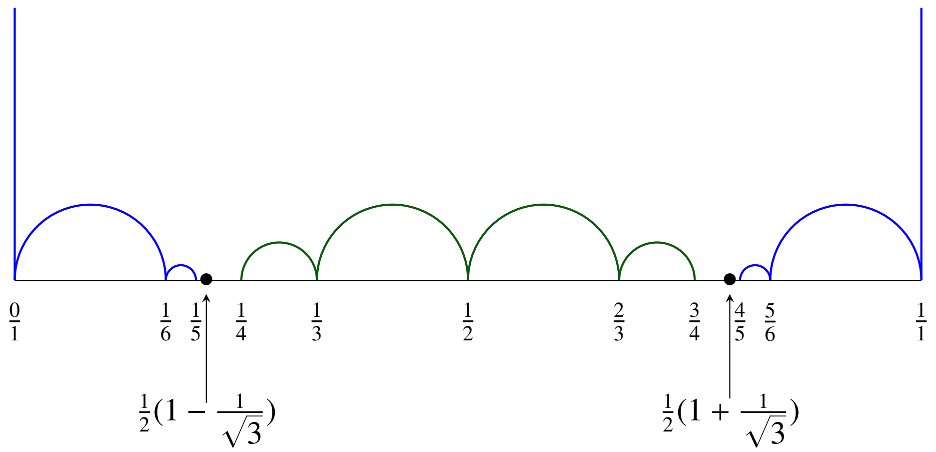


Figure 3.4: We show some of the roots in $\mathbf{X}_6$ (in blue) and $\tilde{\mathbf{X}}_6$ (in green) are represented by the hyperbolic polygon $\mathcal{M}_6$. The two dark circles indicate the limit points.

The operation δ acts on the roots as follows:

$$\alpha \to \delta \cdot \alpha \cdot \delta^T \quad \text{with } \delta = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{3.33}$$

and α is any root. δ^2 acts as the identity operator on the roots. It acts on the elements of $\mathbf{X}_N$ as the following involution.

$$\alpha_m \overset{\delta}{\longleftrightarrow} \alpha_{3-m} \,. \tag{3.34}$$

The group $\mathrm{Dih}(\mathcal{M}_N) := \langle \gamma^{(N)}, \delta \rangle$ acts as a dihedral symmetry of the hyperbolic polygon, $\mathcal{M}_N$.

A real symmetric 2×2 matrix can be considered as a vector in $\mathbb{R}^{2,1}$ as follows [19].

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \longleftrightarrow v = \begin{pmatrix} t+y & x \\ x & t-y \end{pmatrix}$$

with norm $(v, v) = -2\det(v) = 2(x^2 + y^2 - t^2)$. The Cartan matrix for the real simple roots is given by the matrix of inner products of the roots of $\mathbf{X}_N$, i.e., $A^{(N)} := (a_{nm})$, with $n, m \in \mathcal{S}_N$. A closed formula for the Cartan matrices is given by Eq. (3.22). For $N = 6$, there are additional roots that appear in $\tilde{X}_6$, The following inner products hold

$$(\alpha_n, \alpha_m) = (\widetilde{\alpha}_m, \widetilde{\alpha}_n) = (\widetilde{\alpha}_m, \alpha_n) + 12 \,. \tag{3.35}$$

The following relation holds

$$\alpha_m + (N-1)\,\alpha_{m+2} = (N-1)\,\alpha_{m+1} + \alpha_{m+3} \quad \forall m \in \mathcal{S}_N \,. \tag{3.36}$$

This is consistent with the matrices $A^{(N)}$ having rank three. We will focus on the following four real simple roots $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. Using this norm, the matrix of inner products of these roots is

$$A^{(N)}_{\text{trun}} = \begin{pmatrix} 2 & -2 & (2-4N) & (-2+8N-4N^2) \\ -2 & 2 & -2 & (2-4N) \\ (2-4N) & -2 & 2 & -2 \\ (-2+8N-4N^2) & (2-4N) & -2 & 2 \end{pmatrix} \tag{3.37}$$

When $N = 1$, the first three columns and rows give $A^{(1)}$ as $\alpha_0 = \alpha_3$. When $N = 2$, $A^{(2)}_{\text{trun}} = A^{(2)}$ as there are only four real simple roots. In all other cases, we obtain a truncation of the Cartan matrix $A^{(N)}$.

The Weyl Group and its extension

Let w_i denote the elementary Weyl reflection by the simple root $\alpha_i \in \mathbf{X}_N$ and $\tilde{w}_i$ the Weyl reflection by the simple root $\tilde{\alpha}_i \in \tilde{\mathbf{X}}_6$. Let β be a root. Then

$$w_i(\beta) = \beta - 2\,\frac{(\alpha_i,\beta)}{(\alpha_i,\alpha_i)}\,\alpha_i \,.$$

Let $W = W(A^{(N)})$ denote the Weyl group generated by elementary Weyl reflections associated with all simple roots in $\mathbf{X}_N$. Let us call the group

$$\mathcal{W} := W(A^{(N)}) \rtimes \text{Dih}(\mathcal{M}_N)\,,$$

the *extended* Weyl group. The extended Weyl group is generated by $\langle \gamma^{(N)}, \delta, w_2 \rangle$.

3.4.2 Root Lattices with Weyl vector

Consider the vectors contained in $\mathbf{X}_N$ for $N \le 4$. For $N = 6$, the vectors are given by $\mathbf{X}_N \cup \widetilde{\mathbf{X}}_N$. These generate lattices in $\mathbb{R}^{2,1}$. For $N \le 4$, the lattice is given by

$$\mathcal{L}^{(N)} = \oplus_{m\in\mathcal{S}_N}\mathbb{Z}\,\alpha_m\,, \quad \text{with } \alpha_m \in \mathbf{X}_N\,. \tag{3.38}$$

and for $N = 6$

$$\mathcal{L}^{(6)} = \left(\oplus_{m\in\mathbb{Z}} \mathbb{Z}\,\alpha_m \right) \oplus \left(\oplus_{m\in\mathbb{Z}} \mathbb{Z}\,\widetilde{\alpha}_m \right), \quad \text{with } \alpha_m \in \mathbf{X}_6 \text{ and } \tilde{\alpha}_m \in \tilde{\mathbf{X}}_6\ . \tag{3.39}$$

The $\mathcal{L}^{(N)}$ are all rank-three Lorentzian lattice with lattice Weyl vector

$$\varrho^{(N)} = \begin{pmatrix} 1/N & 1/2 \\ 1/2 & 1 \end{pmatrix} \text{ with } \left(\varrho^{(N)}, \varrho^{(N)}\right) = \frac{1}{2} - \frac{2}{N}\ . \tag{3.40}$$

The Weyl vector has the following properties:

1. The norm of $\varrho^{(N)}$ is $\left(\frac{N-4}{2N}\right)$. Thus, the norm is time-like for $N < 4$, light-like for $N = 4$ and space-like for $N = 6$.
2. The inner product of the Weyl vector with real simple roots are:

$$(\varrho^{(N)}, \alpha_m) = -1 \quad \forall\, \alpha_m \in \mathbf{X}_N\ .$$

   and for $N = 6$, additionally one has

$$(\varrho^{(6)}, \widetilde{\alpha}_m) = +1 \quad \forall\, \widetilde{\alpha}_m \in \widetilde{\mathbf{X}}_6\ .$$

3. The generators of $\mathrm{Sym}(\mathcal{M}_N)$ act on the Weyl vector as follows:

$$\gamma^{(N)} : \varrho^{(N)} \to \varrho^{(N)} \text{ and } \delta : \varrho^{(N)} \to \varrho^{(N)},$$

   Thus $\mathrm{Dih}(\mathcal{M}_N)$ preserves the Weyl vector.

The rank-three hyperbolic root lattices $\mathcal{L}^{(N)}$ with lattice Weyl vector $\varrho^{(N)}$ appear in Nikulin's classification of hyperbolic root systems of rank three [68]. For $N \leq 4$, the lattice Weyl vector is of elliptic type, while for $N = 4$ it is of parabolic type and for $N = 6$, it is of hyperbolic type. The type is determined by the norm of the lattice Weyl vector.

3.4.3 Embedding $\widehat{sl(2)}$ in the $\mathfrak{g}(A^{(N)})$

Defining $\widehat{sl(2)}$

Let (e, h, f) be the generators of the $sl(2)$ Lie algebra. The non-zero Lie brackets are:

$$[e, f] = h \quad , \quad [h, e] = 2e \quad , \quad [h, f] = -2f \ ,$$

and the (normalised) Killing form is $\langle e, f\rangle = 1$ and $\langle h, h\rangle = 2$.

The affine Lie algebra $\widehat{sl(2)}$ is defined by

$$\widehat{sl(2)} = sl(2) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\,\hat{k} \oplus \mathbb{C}\,d \ ,$$

where $\hat{k}$ is the central extension and $d = -td/dt$. The Lie algebra (with $x \in sl(2)$)

$$\begin{aligned}[x \otimes t^n, y \otimes t^m] &= [x, y] \otimes t^{n+m} + n \,\langle x, y\rangle\, \hat{k}\, \delta_{n+m,0} \\ [d, x \otimes t^n] &= -n\, x \otimes t^n\end{aligned}$$

The Cartan sub-algebra is $(h \otimes 1, \hat{k}, d)$ with inner product such that $\langle h \otimes 1, h \otimes 1\rangle = 2$ and $\langle \hat{k}, d\rangle = -1$.

We would like to embed $\widehat{sl(2)}$ into the Kac-Moody Lie algebra $\mathfrak{g}(A^{(N)})$ with symmetric Cartan matrix given by $A^{(N)}$. Consider the Chevalley generators $\{e_i, f_i, h_i \mid i = 1, 2, 3\}$ corresponding to the real simple roots $\alpha_1, \alpha_2, \alpha_3$ of $\mathfrak{g}(A^{(N)})$. These satisfy

$$[h_i, h_j] = 0 \ , \ [h_i, e_j] = a_{ji}\, e_i \ , \ [e_i, f_i] = h_i \ ,$$

where (a_{ij}) is the Cartan matrix for these three roots

$$\begin{pmatrix} 2 & -2 & (2-4N) \\ -2 & 2 & -2 \\ (2-4N) & -2 & 2 \end{pmatrix} .$$

The Lie subalgebra of $\mathfrak{g}(A^{(N)})$ generated by $e_1, f_1, e_2, f_2, h_1, h_2$ and h_3 is isomorphic to $\widehat{sl(2)}$. Following Feingold and Frenkel [31], we make the identification

$$e \otimes 1 = e_2 \ , \ f \otimes 1 = f_2 \ , \ f \otimes t = e_1 \ , \ e \otimes t^{-1} = f_1 \ ,$$

For the Cartan subalgebra of $\widehat{sl(2)}$, using the above identification, we obtain

$$h_1 = [e_1, f_1] = -h \otimes 1 + \hat{k} \ , \ h_2 = [e_2, f_2] = h \otimes 1 \ , \ h_3 = -h \otimes 1 + 4N\, d \ .$$

The inverse is

$$h \otimes 1 = h_2 \ , \ \hat{k} = h_1 + h_2 \ , \ d = \frac{1}{4N}(h_2 + h_3) \ .$$

Chapter 4

Borcherds' extension of $A^{(N)}$ algebras

In previous chapter, shedding some light on the work of [19], we have seen that how the fundamental Weyl chambers associated to $A^{(N)}$- hyperbolic algebras(for $N = 1, \ldots, 6$) can be identified with the walls of marginal stability in CHL_N $(\mathrm{IIB}, \mathrm{K3} \times T^2)$ models irrespective of the twisting of quarter-BPS states. The answer to the question "How does the twisting make itself evident in the framework of Lie algebras associated to $A^{(N)}$?" lies in the Borcherds' extensions of these $A^{(N)}$-algebras(for $N = 1, \ldots, 4$). For each twisting of order $M(\leq 4)$ for a given CHL_N $(\mathrm{IIB}, \mathrm{K3} \times T^2)$ theory a BKM superalgebra $\mathcal{B}^{(N,M)}(A^{(N)})$ seems to make an appearance in the form of WKB superdenominator formulae (3.13) [10,19,20],

$$\Delta^{(N,M)} \equiv \sqrt{\Phi^{(N,M)}} = e^{-\rho} \prod_{\alpha \in \Delta^+} \left(1 - e^{-\alpha}\right)^{\mathrm{mult}\alpha} = \sum_{w \in \mathcal{W}} \det(w)\, w\left(e^{-\rho} T^{(N,M)}\right) \tag{4.1}$$

where Borcherds' correction $T^{(N,M)}$ is evident from the product side in the form of eta-products however for special cases of $N = M$ it can be easily explained from the sum side too [16, 69].

According to a no-go theorem due to Gritsenko and Nikulin [16], $\mathcal{B}^{(N,M)}$ (for $N = 5, 6$) can not be a BKM algebra but there is no denial to the existence of some more general Lie algebra whose denominator identities contain the the product side given by

$$\Delta^{(N,M)} = e^{-\rho} \prod_{\alpha\in\Delta^+} (1 - e^{-\alpha})^{\text{mult}\alpha} \text{ for } N = 5, 6.$$

Our work started out as to explore the possibility of such Lie algebras, in order to do that we need as many examples as possible to look into. The table below shows the compatible orders of orbifolding (N) and twisting (M) of the quarter-BPS states in CHL_N (IIB, K3 $\times T^2$) theories for $1 \le N, M \le 6$ [14].

N \ M	1	2	3	4	5	6
1	✓	✓	✓	✓	✓	✓
2	✓	✓	✓	✓	✗	✗
3	✓	✓	✓	✗	✗	✗
4	✓	✓	✗	✓	✗	✗
5	✓	✗	✗	✗	✗	✗
6	✓	✗	✗	✗	✗	✗

This table provides two examples ($N = 5, 6$) in CHL_N (IIB, K3 $\times T^2$) for untwisted ($M = 1$) quarter-BPS states.

In the context of generalized Mathieu moonshine, the generating functions $\Phi^{(N,M)}$ are the multiplicative lift of N-twisted/M-twined elliptic genera of K3 [14]. One can relate these generating functions, for the diagonal entries of the above table, to the Umbral moonshine by considering that the associated elliptic genera are the weight zero weak Jacobi forms (Umbral Jacobi forms (A.23)) $\psi_{0,N}(\tau, z)$ appear in [21] for pure A-type Neimeier lattices. In addition to these four Umbral Jacobi forms there is one more Umbral Jacobi form $\psi_{0,6}$ not associated to any elliptic genus of K3 whose multiplicative lift gives a weight zero Siegel modular form Φ_0. The square root of the Siegel form Φ_0 can serve as another example granting us the product side of a denominator identity associated to some Lie algebra of A^6 . Eventually we have three examples to work on, two of them belong to the category of $\mathbb{Z}_N$-CHL orbifolds and one appears in Umbral moonshine [14].

Let us denote by $\Delta_{k(N)} = \begin{cases} \Delta^{(N,N)}, & N = 1, \ldots, 4 \\ \sqrt{\Phi_0}, & N = 6 \end{cases}$ the five Siegel forms appear in the context of Umbral moonshine and by $\Delta^{(N)}_{k(N)}$ the Siegel forms $\{\Delta^{(N,1)}\}$ appear in $\mathbb{Z}_N$-CHL orbifolds , where $k(N)$ is the weight depends on N.

4.1 Siegel forms in Umbral moonshine [1]

4.1.1 The construction of the Siegel modular forms

The following theorem enables one to construct Siegel modular forms in the form of a product formula. All the five Siegel modular forms that we study arise in this fashion as we will show.

Theorem 4.1.1 (Gritsenko-Nikulin [30]). *Let ψ be a nearly holomorphic Jacobi form*[1] *of weight 0 and index t with integral Fourier coefficients.*

$$\psi(\tau, z) = \sum_{n,\ell} c(n, \ell)\, q^n r^\ell \quad , c(n, \ell) \in \mathbb{Z} \, .$$

Then the product

$$B_\psi(\mathbf{Z}) = q^A r^B s^C \prod_{\substack{n,\ell,m \in \mathbb{Z} \\ (n,\ell,m)>0}} (1 - q^n r^\ell s^{tm})^{c(nm,\ell)} \, ,$$

where $(n, l, m) > 0$ implies: if $m > 0$, then $n \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$; if $m = 0$ and $n > 0$, then $\ell \in \mathbb{Z}$; if $m = n = 0$, then $\ell < 0$ and

$$A = \frac{1}{24} \sum_{\ell \in \mathbb{Z}} c(0, \ell), \;\; B = \frac{1}{2} \sum_{\ell \in \mathbb{Z}_{>0}} \ell\, c(0, \ell), \;\; C = \frac{1}{4} \sum_{\ell \in \mathbb{Z}} \ell^2\, c(0, \ell) \, ,$$

defines a meromorphic Siegel modular form of weight

$$k = \frac{1}{2} \sum_{\ell \in \mathbb{Z}} c(0, 0)$$

[1] We call a holomorphic function $\phi(\tau, z)$ a nearly holomorphic Jacobi form of weight k and index t if it satisfies the functional equation of the definition and there exists $n \in \mathbb{N}$ such that $\Delta(\tau)^n \phi(\tau, z)$ is a Jacobi form [30].

with respect to $\mathbf{\Gamma}_t^+$ possibly with character. The character is determined by the zeroth Fourier-Jacobi coefficient (i.e., the coefficient of s^C) of $B_\psi(\mathbf{Z})$ which is a Jacobi form of weight k and index C of the Jacobi subgroup of $\mathbf{\Gamma}_t^+$.

The five Siegel modular forms of interest are defined as follows. For $N = 1, 2, 3, 4, 6$, let

$$\Delta_{k(N)}(\mathbf{Z}) := B_{\psi_{0,N}}(\mathbf{Z}) \,, \tag{4.2}$$

where $\psi_{0,N}(\tau, z)$ are the Umbral Jacobi forms defined in Eq. (A.23). Using the Fourier expansion given there, we obtain $A = 1/2N$, $B = C = 1/2$ and $k(N) = (6/N) - 1$. The zeroth Fourier-Jacobi coefficient is

$$\phi_{k(N),1/2}(\tau, z) = \vartheta_1(\tau, z)\eta(\tau)^{2k(N)-1} \,. \tag{4.3}$$

The Jacobi forms transform with character $v_\eta^{\frac{12}{N}} \times v_H$, where v_η is the character associated with the Dedekind eta function and v_H is defined in Eq. (A.38).

Properties of the Siegel modular forms:

$$\begin{aligned} \Delta_{k(N)}(\gamma^{(N)} \cdot \mathbf{Z}) &= +\Delta_{k(N)}(\mathbf{Z}) \,, \\ \Delta_{k(N)}(\delta \cdot \mathbf{Z}) &= +\Delta_{k(N)}(\mathbf{Z}) \,, \\ \Delta_{k(N)}(w_2 \cdot \mathbf{Z}) &= -\Delta_{k(N)}(\mathbf{Z}) \,. \end{aligned} \tag{4.4}$$

where

$$\gamma^{(N)} \cdot \mathbf{Z} := ((\gamma^{(N)})^T)^{-1}\mathbf{Z}\,(\gamma^{(N)})^{-1} \text{ and } \delta \cdot \mathbf{Z} := (\delta^T)^{-1}\,\mathbf{Z}\,(\delta)^{-1} \,.$$

These properties imply that the Siegel modular forms transform covariantly under the extended Weyl group. The proof of these properties are given, for instance, in [13, 14].

4.2 Deconstructing the Denominator Formula

It is known that for $N = 1,2,3,4$ that the Siegel modular forms $\Delta_{k(N)}(\mathbf{Z})$ are the modular forms associated with the Weyl-Kac-Borcherds denominator formulae for Borcherds extensions of the Kac-Moody Lie algebra $\mathfrak{g}(A^{(N)})$ [13, 29, 30]. Eq. (4.4) shows that the Siegel modular forms transform as expected from a formula for the denominator of the Weyl character formula. Further, it is easy to see from explicit formulae that one has

$$\Delta_{k(N)}(\mathbf{Z}) = \sum_{w \in W} \det(w)\, w\left(e^{-\varrho^{(N)}}\right) + \cdots^{2} \tag{4.5}$$

The simple roots (α_1, α_2) generate a $\widehat{sl(2)}$ sub-algebra. We study the Siegel modular forms in terms of this $\widehat{sl(2)}$ sub-algebra as well as a Borcherds extension of it that we call $\mathcal{B}_N(\widehat{sl(2)})$.

4.2.1 Characters

As before, let $W = W(A^{(N)})$ denote the Weyl group of $\mathfrak{g}(A^{(N)})$; this is generated by the elementary reflections w_i for all simple roots $i \in \mathbf{X}_N$. Let $\widehat{W}$ be the subgroup generated by the reflections w_1, w_2, corresponding to the simple roots α_1 and α_2. This is isomorphic to the Weyl group of $\widehat{sl(2)}$, with

$$\widehat{W} := \{(w_1 \cdot w_2)^k, w_2 \cdot (w_1 \cdot w_2)^k \mid k \in \mathbb{Z}\}$$

We have

$$W = \widehat{W} \cup (\widehat{W} \cdot w_0) \cup (\widehat{W} \cdot w_3) \cup (w_0 \cdot \widehat{W}) \cup (w_3 \cdot \widehat{W}) \cup \cdots$$

[2] The ellipsis denotes the terms in the Weyl group W which are not of the form given by the two listed terms.

We will be focusing mostly on the first three terms. The numerator of the Weyl character formula for a highest weight state with weight $\tilde{\Lambda}$ of $\mathcal{B}_N(\widehat{sl(2)})$ takes the form

$$\begin{aligned}\text{num}(\tilde{\Lambda}) &= \sum_{w\in W} \det(w) w \left(e^{-\varrho^{(N)}-\Lambda}\, T_{\tilde{\Lambda}}\right) \\ &= \sum_{w\in \widehat{W}} \det(w) w \left(e^{-\varrho^{(N)}-\tilde{\Lambda}}\, T_{\tilde{\Lambda}}\right) - \sum_{w\in \widehat{W}} \det(w) w \left(e^{-w_3(\varrho^{(N)}+\tilde{\Lambda})}\, T_{\tilde{\Lambda}}\right) + \cdots \end{aligned} \tag{4.6}$$

Note that num(0) is the denominator formula. We have included imaginary simple roots in the above formula as $T_{\tilde{\Lambda}}$.

Let $\delta = (\alpha_1 + \alpha_2)$. A straightforward computation gives the following formulae.

$$\begin{aligned}(w_1 \cdot w_2)(\varrho^{(N)}) &= \varrho^{(N)} + 3\delta - 2\alpha_2 \\ (w_1 \cdot w_2)(\delta) &= \delta\ , \\ (w_1 \cdot w_2)(\alpha_2) &= -2\delta + \alpha_2 \\ (w_1 \cdot w_2)(\alpha_3) &= \alpha_3 + (4N+2)\delta - 4N\alpha_2 \\ (w_1 \cdot w_2)(\alpha_0) &= \alpha_0 + (4N-1)\delta - \alpha_2\end{aligned}$$

Using this computation, we can show that for $\tilde{\Lambda} = a\delta + b\alpha_2 + c\alpha_3$, one has (with $m = (4Nc+2)$ and $\ell = (2c - 2b)$)

$$\begin{aligned}(w_1 \cdot w_2)^k[\varrho^{(N)} + \tilde{\Lambda}] - (\varrho + \tilde{\Lambda}) &= \left[(mk^2 + (\ell+1)k)\delta + (-km)\alpha_2\right] \\ w_2 \cdot (w_1 \cdot w_2)^k[\varrho^{(N)} + \tilde{\Lambda}] - (\varrho + \tilde{\Lambda}) &= \left[(mk^2 + (\ell+1)k)\delta + (km + \ell + 1)\alpha_2\right]\end{aligned}$$

After making the following identifications:

$$e^{-\delta} \sim q = \exp(2\pi i\tau),\ e^{-\alpha_2} \sim r^{-1} = \exp\left(-2\pi i z\right) \text{ and } e^{-\alpha_3} \sim s^N r\ ,$$

we obtain the master formula

$$\frac{\sum_{w\in\widehat{W}} \det(w) w\left(e^{-\varrho^{(N)}-\tilde{\Lambda}}\right)}{\left(e^{-\varrho^{(N)}-\tilde{\Lambda}}\right)} = q^{-\frac{(\ell+1))^2}{4m}} r^{-\frac{(\ell+1)}{2})} \left(\theta_{m,\ell+1}(\tau,z) - \theta_{m,-\ell-1}(\tau,z)\right) \tag{4.7}$$

where

$$\theta_{m,a}(\tau,z) := \sum_{k\in\mathbb{Z}} q^{m(k+\frac{a}{2m})^2} r^{m(k+\frac{a}{2m})} . \tag{4.8}$$

The denominator formula for $A^{(N)}$:

It is known that for $N = 1,2,3,4$, the genus two Siegel modular form $\Delta_{k(N)}(\mathbf{Z})$ arise as the WKB denominator formula for an extension of the Kac-Moody Lie algebra $\mathfrak{g}(A^{(N)})$ by the addition of imaginary simple roots. For $N = 1,2,3,4,6$, it is known that the modular forms admit the following expansion [13, 14]:

$$\Delta_{k(N)}(\mathbf{Z}) = s^{1/2}\phi_{k(N),1/2}(\tau,z)[1 - s^N\, \psi_{0,N}(\tau,z) + O(s^{2N})] , \tag{4.9}$$

where $\phi_{k(N),1/2}(\tau,z)$ is defined in Eq. (4.3) and $\psi_{0,N}(\tau,z)$ is an Umbral Jacobi form defined in Eq. (A.23)(see [21] for its connection to Umbral Moonshine).

We interpret $\phi_{k(N),1/2}(\tau,z)$ as the WKB denominator formula for the Borcherds extension $\mathcal{B}_N(\widehat{sl(2)})$(see the example of $\mathcal{B}_k(\widehat{sl(2)})$ in section 3.2.3). Then, using the following result that follows from Eq. (4.7)

$$\sum_{w\in\widetilde{W}} \det(w) w\left(e^{-\varrho^{(N)}-a\delta}\right) = q^a\,[q^{1/N} r^{1/2} s^{1/2}]\, q^{-1/8} r^{-1/2} \left(\theta_{2,-1}(\tau,z) - \theta_{2,1}(\tau,z)\right) ,$$

we see that

$$s^{1/2}\,\phi_{k(N),1/2}(\tau,z) = \sum_{w\in\widehat{W}} \det(w) w\left(e^{-\varrho^{(N)}}\,\widetilde{T}\right) \tag{4.10}$$

with the Borcherds correction defined in Eq. (B.2) given by $\widetilde{T} = \prod_{p=1}^{\infty}(1 - e^{-p\delta})^{2k(N)-1}$. The Weyl character formula for the weight vector $\widetilde{\Lambda} = a\delta + b\alpha_2 + c\alpha_3$ of $\mathcal{B}_N(\widehat{sl(2)})$ with

$\langle\widetilde{\Lambda}, \delta\rangle < 0$ is obtained by applying the formula for the supercharacter given in Eq. (B.3).[3] We obtain

$$\widetilde{\chi}_{\widetilde{\Lambda}}(\tau, z) = \frac{q^{a-\frac{(\lambda+1))^2}{4(\hat{k}+2)}+\frac{1}{8}}}{\varphi(\tau)^{2k(N)-1}} \chi_{\hat{k},\lambda}(\tau, z) , \tag{4.11}$$

where $\varphi(\tau) = \prod_{m=1}^{\infty}(1 - q^m)$, $\hat{k} = 4Nc$, $\lambda = (2c - 2b)$ and the normalized $\widehat{sl(2)}$ character $\chi_{\hat{k},\lambda}(\tau, z)$ is defined by

$$\chi_{\hat{k},\lambda}(\tau, z) = \frac{\theta_{\hat{k}+2,\lambda+1}(\tau, z) - \theta_{\hat{k}+2,-\lambda-1}(\tau, z)}{\theta_{2,1}(\tau, z) - \theta_{2,-1}(\tau, z)} \text{ for } \hat{k}, \lambda \in \mathbb{Z}_{\geq 0} \text{ and } \lambda \leq \hat{k} . \tag{4.12}$$

Remark: The $\mathbb{Z}_2$ outer automorphism of the $\widehat{sl(2)}$ Lie algebra corresponding to $\alpha_1 \leftrightarrow \alpha_2$ is represented by the generator $\widehat{\delta}$ (see Eq. (3.33)) of the Dihedral group, $\text{Dih}(\mathcal{M}_N)$. This exchanges the characters $\chi_{4N,j}$ and $\chi_{4N,4N-j}$.

Some examples of interest

Applying the master formula Eq. (4.7) to the real simple roots $\widetilde{\Lambda} = \alpha_0$ and $\widetilde{\Lambda} = \alpha_3$ as well as the imaginary simple roots (with zero norm and multiplicity $[2k(N) - 1]$) $\widetilde{\Lambda} = (\alpha_0 + \alpha_1)$ and $\widetilde{\Lambda} = (\alpha_2 + \alpha_3)$, we obtain (after dropping a pre-factor of s^N that is present in all terms)

$$\begin{aligned}
\widetilde{\chi}_{\alpha_3} &= \frac{q^{(N-4)/(8N+4)}(\theta_{4N+2,3} - \theta_{4N+2,-3})}{(\theta_{2,1} - \theta_{2,-1})\,\varphi(\tau)^{2k(N)-1}} = \frac{q^{(N-4)/(8N+4)}\chi_{4N,2}}{\varphi(\tau)^{2k(N)-1}} \\
\widetilde{\chi}_{\alpha_0} &= \frac{q^{(N-4)/(8N+4)}(\theta_{4N+2,4N-3} - \theta_{4N+2,-4N+3})}{(\theta_{2,1} - \theta_{2,-1})\,\varphi(\tau)^{2k(N)-1}} = \frac{q^{(N-4)/(8N+4)}\chi_{4N,4N-2}}{\varphi(\tau)^{2k(N)-1}} \\
\widetilde{\chi}_{\alpha_0+\alpha_1} &= \frac{q^{N/(8N+4)}(\theta_{4N+2,4N+1} - \theta_{4N+2,-4N-1})}{(\theta_{2,1} - \theta_{2,-1})\,\varphi(\tau)^{2k(N)-1}} = \frac{q^{N/(8N+4)}\chi_{4N,4N}}{\varphi(\tau)^{2k(N)-1}} \\
\widetilde{\chi}_{\alpha_2+\alpha_3} &= \frac{q^{N/(8N+4)}(\theta_{4N,1} - \theta_{4N,-1})}{(\theta_{2,1} - \theta_{2,-1})\,\varphi(\tau)^{2k(N)-1}} = \frac{q^{N/(8N+4)}\chi_{4N,0}}{\varphi(\tau)^{2k(N)-1}}
\end{aligned}$$

In the sequel, we will refer to $\widetilde{\chi}_{\alpha_3}$ as $\widetilde{\chi}_{4N,2}$ and so on. This extends the labels that we use for $\widehat{sl(2)}$ characters to characters of $\mathcal{B}_N(\widehat{sl(2)})$.

[3]The condition $\langle\widetilde{\Lambda}, \delta\rangle < 0$ ensures that there is no Borcherds correction term in the numerator of the character formula.

4.3 Siegel forms in $\mathbb{Z}_N$ CHL orbifolds [2]

Let $A = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. Then, $\mathfrak{g}(A)$ is the $\widehat{sl(2)}$ Kac-Moody Lie algebra with simple roots (α_1, α_2) and $\delta = \alpha_1 + \alpha_2$ is an imaginary root with zero norm. We will consider a family of Borcherds corrections that appear in this work. For $N = 1, 2, 3, 5$, consider a situation where one has $12/(N+1)$ distinct imaginary simple roots of weight $\frac{1}{N}(\delta, 2\delta, 3\delta, \ldots)$ and $(12/(N+1)) - 3$ imaginary simple roots of weight $(\delta, 2\delta, 3\delta, \ldots)$. The Borcherds correction factor due to these imaginary simple roots takes the form

$$T_N(\delta) = \prod_{j=1}^{\infty} \left(1 - e^{-\frac{j\delta}{N}}\right)^{\frac{12}{N+1}} \left(1 - e^{-j\delta}\right)^{-3+\frac{12}{N+1}} .$$

For $N = 5$ a negative power appears in the second term in the infinite product. The imaginary simple roots in this case correspond to isotropic(zero norm) fermionic simple roots and we consider a superdenominator formula to account for this. Identifying $e^{-\delta} \sim q = \exp(2\pi i\tau)$, we obtain a function of τ. Let

$$T_N(\tau) = \prod_{j=1}^{\infty} \left(1 - q^{j/N}\right)^{\frac{12}{N+1}} \left(1 - q^{j}\right)^{-3+\frac{12}{N+1}} . \tag{4.13}$$

Up to an overall power of q, $T_N(\tau)$ can be expressed in terms of products of the Dedekind eta function. The automorphic form, that is denoted by Δ in Eq. (1.2), for these examples is given by the Jacobi form $\phi_{k(N),1/2}(\tau, z)$ defined in Eq. (4.15). We will refer to these Borcherds-Kac-Moody Lie algebras by $\mathcal{B}_N^{CHL}(\widehat{sl(2)})$. As can be seen, there can be several inequivalent Borcherds extensions of a Kac-Moody Lie algebra.

4.3.1 The $\mathcal{B}^{CHL}(A^{(N)})$ Lie algebras

Let $\mathcal{B}^{CHL}(A^{(N)})$ denote an extension of the $\mathfrak{g}(A^{(N)})$ whose denominator formula is given by the Siegel modular forms, $\Delta_{k(N)}^{(N)}(\mathbf{Z})$ which we define next[4]. Then the BKM Lie algebras

[4]Here $\mathbf{Z} = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ is a point in the Siegel upper half space, $\mathbb{H}_2$. See appendix A.3.

$\mathcal{B}_N^{CHL}(\widehat{sl(2)})$ are naturally sub-algebras of $\mathcal{B}^{CHL}(A^{(N)})$.

A connection with Mathieu and $L_2(11)$ moonshine leads to the following formula for a Siegel modular form [17, 18, 32]. Let $g \in L_2(11)_B$ be an element of order $N \leq 6$. A second-quantized version of moonshine gives the following formula for $\Delta_{k(N)}^{(N)}(\mathbf{Z})$.

$$\Delta_{k(N)}^{(N)}(\mathbf{Z}) = s^{1/2}\phi_{k(N),1/2}(\tau, z) \exp\left[-\sum_{m=1}^{\infty} s^m \psi_{0,1}^{(N)[1,g]}(\tau, z) | T(m) \right] \tag{4.14}$$

where the Hecke-like operator $T(m)$ is defined as follows[5]

$$\psi_{0,1}^{(N)[1,g]}(\tau, z)|T(m) := \frac{1}{m} \sum_{ad=m} \sum_{b=0}^{d-1} \psi_{0,1}^{(N)[g^{-b},g]}\left(\tfrac{a\tau+b}{d}, az\right)$$

and

$$\phi_{k(N),1/2}(\tau, z) = \frac{\theta_1(\tau, z)}{\eta(\tau)^3}\, \eta^{[1,g]}(\tau) \tag{4.15}$$

are index half Jacobi forms with the eta products $\eta^{[1,g]}(\tau)$ defined in Table 4.1. It has been shown in ref. [14] that this leads to a Borcherds-type product formula for $\Delta_{k(N)}^{(N)}(\mathbf{Z})$. Consider the Fourier expansion

$$\psi_{0,1}^{[g^b,g^d]}(\tau, z) = \sum_{n\in\mathbb{Z}, n\geq 0} \sum_{\ell\in\mathbb{Z}} c^{[b,d]}(n, \ell)\, q^{\frac{n}{N}} r^{\ell} , \tag{4.16}$$

where g is of order N, $q = e^{2\pi i\tau}$ and $r = e^{2\pi iz}$.

Define $\tilde{c}^{[\alpha,d]}(n, \ell)$ as follows (with $\omega_N = \exp(2\pi i/N)$)

$$\tilde{c}^{[\alpha,d]}(n, \ell) = \frac{1}{N} \sum_{b=0}^{N-1} (\omega_N)^{-\alpha b}\, c^{[b,d]}(n, \ell) . \tag{4.17}$$

Then one has the product formula that provides the product side of the denominator for-

[5]Here $\psi_{0,1}^{(N)[g^s,g^r]}$ is half the g^r-twisted elliptic genus of $K3$ twined by the element g^s. In other words, the trace is over the Hilbert space twisted by g^r with insertion of g^s ('twined') in the trace.

N	1	2	3	4	5	6
Cycle shape	1^{12}	$1^4 2^4$	$1^3 3^3$	-	$1^2 5^2$	$1^1 2^1 3^1 6^1$
$k(N)$	5	3	2	3/2	1	1
$\eta^{[1,g]}(\tau)$	$\eta(\tau)^{12}$	$\eta(\tau)^4\eta(\tau/2)^4$	$\eta(\tau)^3\eta(\tau/3)^3$	$\eta(\tau)^2\eta(\tau/2)\eta(\tau/4)^2$	$\eta(\tau)^2\eta(\tau/5)^2$	$\eta(\tau)\eta(\tau/2)\eta(\tau/3)\eta(\tau/6)$

Table 4.1: Eta products

mula that defines $\mathcal{B}^{CHL}(A^{(N)})$.

$$\Delta_{k(N)}^{(N)}(\mathbf{Z}) = q^{1/2N}r^{1/2}s^{1/2} \times \prod_{m=0}^{\infty}\prod_{\alpha=0}^{N-1}\prod_{\substack{n\in\mathbb{Z}+\frac{\alpha}{N}\\ n\geq 0}}\prod_{\ell\in\mathbb{Z}}(1-q^n r^\ell s^m)^{\tilde{c}^{[\alpha,m]}(nmN,\ell)}\ , \tag{4.18}$$

with $s = e^{2\pi i\tau'}$. The modularity of the above formula is not manifest. However, it follows from a result in ref. [70] that it is a Siegel modular form of a level N subgroup of $Sp(4,\mathbb{Z})$.

The sum side of the Weyl denominator formula is usually obtained from an additive lift [14]. There is a construction of Cléry and Gritsenko that leads to closely related Siegel modular form (at level N) starting from a index half Jacobi form [71]. It has been shown in [32] that the expansion of this Siegel modular form about another cusp (given by the S-transform) matches with the product formula given in Eq. (4.18) to fairly high order. Combined with modularity, it is enough to prove that the two formulae are equivalent. It is not a clean formula in the sense that a closed formula was not given but the transformation rules for the Hecke operator were worked out on a case-by-case basis.

4.3.2 Covariance under the extended Weyl group

The extended Weyl group of the root system $\mathbf{X}_N$ is generated by three types of generators [10, 14, 19]

1. The Weyl group W of $\mathfrak{g}(A^{(N)})$ is generated by *all* elementary Weyl reflections, s_m, due to the simple roots α_m for all m in $\mathcal{S}_N$,
2. the generator $\gamma^{(N)}$, and
3. the generator $\widehat{\delta} = \left(\begin{smallmatrix}-1 & 1\\ 0 & 1\end{smallmatrix}\right)$ which acts on roots via the action $\alpha \to \widehat{\delta}\cdot\alpha\cdot\widehat{\delta}^T$. It acts on

the simple roots in $\mathbf{X}_N$ as an involution:

$$\widehat{\delta} : \alpha_m \leftrightarrow \alpha_{3-m} \; .$$

The action of the generators of the extended Weyl group can be translated into an action on upper half space with coordinates $\mathbf{Z}$. With this in hand, one can show, using the modular properties of the Siegel modular forms, that

$$\begin{aligned} \Delta^{(N)}_{k(N)}(s_m \cdot \mathbf{Z}) &= -\Delta^{(N)}_{k(N)}(\mathbf{Z}) \; , \\ \Delta^{(N)}_{k(N)}(\gamma^{(N)} \cdot \mathbf{Z}) &= +\Delta^{(N)}_{k(N)}(\mathbf{Z}) \; , \\ \Delta^{(N)}_{k(N)}(\widehat{\delta} \cdot \mathbf{Z}) &= +\Delta^{(N)}_{k(N)}(\mathbf{Z}) \; . \end{aligned}$$

These properties show that the Siegel modular forms have the necessary covariance under the extended Weyl group.

Chapter 5

Deconstructing the dyonic symmetry

BKM algebras $\mathcal{B}^{(N,M)}\left(A^{(N)}\right)$ can be recognized as the symmetry of the BPS spectrum [20]. In this chapter we look more specifically into the Lie algebraic aspect of the dyonic symmetry by utilizing the fact that all $A^{(N)}$ Cartan matrices contains $\widehat{sl(2)}$ sub-Cartan matrices $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ in a block diagonal form (3.4.3).

This in turn shows $\widehat{sl(2)}$ are subalgebras of $A^{(N)}$ with a natural follow-up that $\mathcal{B}^{(N,M)}\left(A^{(N)}\right)$ admits certain Borcherds' extension of $\widehat{sl(2)}$, say $\mathcal{B}^{(N,M)}\left(\widehat{sl(2)}\right)$, as subalgebras.

In order to spot the exotic roots in $\mathcal{B}^{(N,M)}\left(A^{(5)}\right)$ and $\mathcal{B}^{(N,M)}\left(A^{(6)}\right)$ we need a device to track the simple roots appear in $\Delta^{(5,M)}$ and $\Delta^{(6,M)}$ as denominator formulae. Generally one could do that by decomposing denominator formulae by characters of the subalgebras in a following manner.

We have seen, in previous chapter, the following expansion of general Siegel forms of type $\Delta^{(N,M)}$

$$\Delta^{(N,M)} = s^{1/2}\,\varphi^{(N,M)}(\tau,z)\times\left[1+\sum_{m=1}^{\infty} s^{tm}\,\Psi^{(N,M)}_{0,tm}(\tau,z)\right]\ ;\ t=\gcd(M,N) \tag{5.1}$$

where $\varphi^{(N,M)}$ is an index half Jacobi form of same weight as is $\Delta^{(N,M)}$. The leading term $s^{1/2}\,\varphi^{(N,M)}(\tau,z)$ can be understood as the denominator formula for the subalgebra

$\mathcal{B}^{(N,M)}\left(\widehat{sl(2)}\right)$. The objects of our interest are Jacobi forms $\Psi^{(N,M)}_{0,tm}(\tau,z)$ of some congruence subgroup Γ which depends on N and M. We then do $\widehat{sl(2)}$-character decomposition of $\Psi^{(N,M)}_{0,tm}(\tau,z)$:

$$\begin{aligned}\Psi^{(N,M)}_{0,tm}(\tau,z) =& g_1(\tau)\chi_{4tm,2tm}(\tau,z) + g_2(\tau)\left[\chi_{4tm,2tm-2} + \chi_{4tm,2tm+2}\right](\tau,z) \\ &+ \ldots + g_{tm+1}(\tau)\left[\chi_{4tm,0} + \chi_{4tm,4tm}\right](\tau,z).\end{aligned} \tag{5.2}$$

Next we, using modular properties of $\Psi^{(N,M)}_{0,tm}$ and normalised characters $\chi_{k,\lambda}$ of $\widehat{sl(2)}$, solve it for g_i's as in power series of q. One can expect then g_i's to form $tm+1$ dimensional vector valued modular form

$$\mathcal{G}(\tau) = \begin{bmatrix} g_1 \\ \vdots \\ g_{tm+1}\end{bmatrix}(\tau) \tag{5.3}$$

of congruence subgroup Γ .

5.1 Umbral Jacobi forms [1]

5.1.1 Lie algebra decompositions of Umbral Jacobi Forms

For the cases of interest, we wish to decompose the Umbral Jacobi Forms in terms of characters of $\widehat{sl(2)}$ and $\mathcal{B}_N(\widehat{sl(2)})$. The decomposition takes the form

$$\psi_{0,N}(\tau,z) = \sum_{j=1}^{2N} g_j(\tau)\chi_{4N,2N+2-2j}(\tau,z)\ , \tag{5.4}$$

$$= \sum_{j=1}^{2N} f_j(\tau)\widetilde{\chi}_{4N,2N+2-2j}(\tau,z)\ , \tag{5.5}$$

Further, one observes that $g_j(\tau) = g_{2N-j}(\tau)$. This follows from the $\mathbb{Z}_2$ outer automorphism under which $\alpha_1 \leftrightarrow \alpha_2$ and $\alpha_0 \leftrightarrow \alpha_3$. Thus one has $(N+1)$ independent functions that we organize into a vector $\mathbf{g} := (g_1, g_2, \ldots, g_{N+1})^T$. Using the modular properties of the normalized characters and the Umbral Jacobi form, we can show that $\mathbf{g}(\tau)$ is a weight zero vector-valued modular form (vvmf) with the following modular properties:

$$\mathbf{g}(\tau+1) = T\cdot\mathbf{g}(\tau) \quad , \quad \mathbf{g}\left(-1/\tau\right) = S\cdot\mathbf{g}(\tau)\,. \tag{5.6}$$

The above transformations define the matrices T and S.

$N = 1$

In this case, $\alpha_0 = \alpha_3$. Using the above results, we obtain the following expansion:

$$\psi_{0,1}(\tau,z) = f_1(\tau)\,\widetilde{\chi}_{\alpha_3}(\tau,z) + f_2(\tau)\big[\widetilde{\chi}_{\alpha_1+\alpha_3}(\tau,z) + \widetilde{\chi}_{\alpha_2+\alpha_3}(\tau,z)\big] \tag{5.7}$$

with $f_1(\tau) = 1 - 92q + 54q^2 - 85q^3 + \cdots$ and $f_2(\tau) = 9(1 + 10q + 11q^2 - 73q^3 + \cdots)$. The terms that appear at order q and higher in f_1 and f_2 are due to imaginary simple roots with negative norm. The expansion in terms of $\widehat{sl(2)}$ characters is given by

$$\psi_{0,1}(\tau,z) = g_1(\tau)\,\chi_{4,2}(\tau,z) + g_2(\tau)\big[\chi_{4,0}(\tau,z) + \chi_{4,4}(\tau,z)\big] \tag{5.8}$$

One has

$$\mathbf{g}(\tau) = \begin{pmatrix} q^{-\frac{1}{4}}\left(1 - 84q - 729q^2 - 4366q^3 - 19935q^4 - 77274q^5 - 264610q^6 + \cdots\right) \\ 9\,q^{-\frac{11}{12}}\left(q + 19q^2 + 155q^3 + 821q^4 + 3541q^5 + 13082q^6 + \cdots\right) \end{pmatrix} \tag{5.9}$$

For the above rank-two vvmf, we obtain the following T and S matrices using the known transformations of the normalized characters given in Eq. (A.35)

$$T = \begin{bmatrix} e^{-\frac{i\pi}{2}} & 0 \\ 0 & e^{\frac{i\pi}{6}} \end{bmatrix} \quad , \quad S = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\,. \tag{5.10}$$

$N = 2$

The Umbral Jacobi form at lambency 3(seeA.4) has the following decomposition in terms of $\widehat{sl(2)}$ characters:

$$\begin{aligned}\psi_{0,2}(\tau, z) &= g_1(\tau)\chi_{8,4}(\tau, z) + g_2(\tau)\left[\chi_{8,2}(\tau, z) + \chi_{8,6}(\tau, z)\right] + g_3(\tau)\left[\chi_{8,0}(\tau, z) + \chi_{8,8}(\tau, z)\right] \\ &= f_1(\tau)(\tilde{\chi}_{\alpha_1+\alpha_3} + \tilde{\chi}_{\alpha_0+\alpha_2}) + f_2(\tau)(\tilde{\chi}_{\alpha_3} + \tilde{\chi}_{\alpha_0}) + f_3(\tau)(\tilde{\chi}_{\alpha_2+\alpha_3} + \tilde{\chi}_{\alpha_0+\alpha_1}) \, ,\end{aligned}$$

where

$$f_1(\tau) = \frac{g_1(\tau)\varphi(\tau)^3}{2q^{1/2}} \quad , \quad f_2(\tau) = \frac{g_2(\tau)\varphi(\tau)^3}{q^{-1/10}} \quad , \quad f_3(\tau) = \frac{g_3(\tau)\varphi(\tau)^3}{q^{1/10}} \quad .$$

Note that since the Cartan matrix $A^{(2)}$ has rank three, one has the identity $\alpha_0+\alpha_2 = \alpha_1+\alpha_3$. Using the symmetry generated by $\widehat{\delta}$, we write the coefficient of f_1 as $(\tilde{\chi}_{\alpha_1+\alpha_3} + \tilde{\chi}_{\alpha_0+\alpha_2})$ anticipating that these two terms will be distinct if we make the Cartan matrix invertible. The weight vector $(\alpha_0 + \alpha_2)$ is associated with an imaginary simple root of negative norm.

The vvmf $\mathbf{g}(\tau)$ has rank three and the first few terms in the Fourier expansion are:

$$\mathbf{g}(\tau) = \begin{pmatrix} -10q^{-\frac{1}{2}}\left(q + 3q^2 + 9q^3 + 22q^4 + 51q^5 + 105q^6 + \cdots\right) \\ q^{-\frac{1}{10}}\left(1 + 3q + 18q^2 + 38q^3 + 99q^4 + 207q^5 + 438q^6 + \cdots\right) \\ q^{-\frac{9}{10}}\left(3q + 16q^2 + 48q^3 + 129q^4 + 294q^5 + 642q^6 + \cdots\right) \end{pmatrix} . \tag{5.11}$$

For the above rank-three vvmf, using Eq. (A.35) we obtain the following T and S matrices:

$$T = \begin{bmatrix} e^{i\pi} & 0 & 0 \\ 0 & e^{-\frac{i\pi}{5}} & 0 \\ 0 & 0 & e^{\frac{i\pi}{5}} \end{bmatrix} \quad , \quad S = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 & 2 \\ -1 & \frac{\sqrt{5}-1}{2} & \frac{\sqrt{5}+1}{2} \\ 1 & \frac{\sqrt{5}+1}{2} & \frac{\sqrt{5}-1}{2} \end{bmatrix} . \tag{5.12}$$

$N = 3$

The Umbral Jacobi Form at lambency 4 has the following decomposition in terms of $\widehat{sl(2)}$ characters:

$$\psi_{0,3}(\tau, z) = g_1(\tau)\chi_{12,6}(\tau, z) + g_2(\tau)\left[\chi_{12,4}(\tau, z) + \chi_{12,8}(\tau, z)\right] \\ + g_3(\tau)\left[\chi_{12,2}(\tau, z) + \chi_{12,10}(\tau, z)\right] + g_4(\tau)\left[\chi_{12,12}(\tau, z) + \chi_{12,0}(\tau, z)\right]. \quad (5.13)$$

Define

$$f_1 = \frac{g_1(\tau)\varphi(\tau)}{2q^{1/4}} \quad , \quad f_2 = \frac{g_2(\tau)\varphi(\tau)}{q^{19/28}} \quad , \quad f_3 = \frac{g_2(\tau)\varphi(\tau)}{q^{-1/28}} \quad , \quad f_4 = \frac{g_3(\tau)\varphi(\tau)}{q^{3/28}} \quad .$$

Then the decomposition in terms of $\mathcal{B}_3(\widehat{sl(2)})$ characters is given by the following labelled table.

Label	1	2	3	4
Weights	$\alpha_1 + 5\alpha_2 + \alpha_3$	$\alpha_1 + \alpha_3$	α_3	$\alpha_2 + \alpha_3$
	$\alpha_0 + 5\alpha_1 + \alpha_2$	$\alpha_0 + \alpha_2$	α_0	$\alpha_0 + \alpha_1$
Norm	-6	-16	2	0

Labels 1 and 2 are associated with imaginary simple roots. The real simple roots α_0 and α_1 appear with label 3 and the zero-norm imaginary roots $(\alpha_0 + \alpha_1)$ and $(\alpha_2 + \alpha_3)$ appear with label 4.

The vvmf $\mathbf{g}(\tau)$ has rank four and the first few terms in the Fourier expansion are:

$$\mathbf{g}(\tau) = \begin{pmatrix} -2q^{-\frac{3}{4}}\left(q + q^2 + 2q^3 + 3q^4 + 5q^5 + 7q^6 + 11q^7 + \cdots\right) \\ -q^{-\frac{9}{28}}\left(q + q^3 + 2q^4 + 3q^5 + 3q^6 + 6q^7 + \cdots\right) \\ q^{-\frac{1}{28}}\left(1 + 2q + 3q^2 + 5q^3 + 8q^4 + 11q^5 + 17q^6 + \cdots\right) \\ q^{-\frac{25}{28}}\left(q + q^2 + 3q^3 + 3q^4 + 6q^5 + 8q^6 + 13q^7 + \cdots\right) \end{pmatrix}. \quad (5.14)$$

For the above rank-three vvmf, we obtain the following T and S matrices:

$$T = \text{Diag}(e^{\frac{i\pi}{2}}, e^{i\pi\left(\frac{19}{14}\right)}, e^{-\frac{i\pi}{14}}, e^{i\pi\left(\frac{3}{14}\right)}) \quad ,$$
$$S = \frac{1}{\sqrt{7}} \begin{bmatrix} -1 & 2 & -2 & 2 \\ 1 & -2\sin(\frac{3\pi}{14}) & -2\sin(\frac{\pi}{14}) & 2\cos(\frac{\pi}{7}) \\ -1 & -2\sin(\frac{\pi}{14}) & 2\cos(\frac{\pi}{7}) & 2\sin(\frac{3\pi}{14}) \\ 1 & 2\cos(\frac{\pi}{7}) & 2\sin(\frac{3\pi}{14}) & 2\sin(\frac{\pi}{14}) \end{bmatrix} . \tag{5.15}$$

$N = 4$

The Umbral Jacobi form at lambency 5 has the following decomposition in terms of $\widehat{sl(2)}$ characters:

$$\psi_{0,4}(\tau, z) = -\chi_{16,8}(\tau, z) + (\chi_{16,2}(\tau, z) + \chi_{16,14}(\tau, z)). \tag{5.16}$$

The character $\chi_{16,8}(\tau, z)$ is identified with a bosonic simple root corresponding to $qr^{-4}s^4$ of zero norm. Recall that $\mathcal{B}_4(\widehat{sl(2)}) = \widehat{sl(2)}$ as there were no imaginary simple roots added to $\widehat{sl(2)}$. So this is the first appearance of an imaginary simple root. The above linear combination is invariant under T as it only contains integral powers of q. Under the S transform, it maps itself up to the phase associated with the index. Hence, there is only one overall constant which we fix by using explicit formulae.

We also see that the characters $\chi_{16,0}(\tau, z)$ and $\chi_{16,16}(\tau, z)$ do *not* appear in the above expansion. This is consistent with the fact that there were no imaginary simple roots added to $\widehat{sl(2)}$ in this case. So we do not expect imaginary simple roots $(\alpha_i + \alpha_{i+1})$ for $i = 0, 1, 2$ to be present. This is borne out in the $\widehat{sl(2)}$ decomposition of $\psi_{0,4}$. However, we find a new imaginary simple root of zero norm appearing with character $\chi_{16,8}(\tau, z)$. It is represented by the matrix

$$\begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix} \longleftrightarrow (\alpha_1 + 6\alpha_2 + \alpha_3) .$$

The imaginary root $(\alpha_2 + 6\alpha_1 + \alpha_0)$ with zero norm also appears in the expansion of

$\chi_{16,8}(\tau, z)$.

$N = 6$

As mentioned earlier, we do not have a BKM Lie superalgebra associated with this Cartan matrix. We do know that the Siegel modular form transforms suitably under the extended Weyl group. In particular, one can see that the Umbral Jacobi form is invariant under $\widehat{W}$ and $\widehat{\delta}$. The $\widehat{sl(2)}$ characters may be viewed as providing a basis for expanding the Jacobi form. The Umbral Jacobi form at lambency 7 has the following decomposition in terms of $\widehat{sl(2)}$ characters.

$$\begin{aligned}\psi_{0,6}(\tau, z) = g_1(\tau)\chi_{24,12}(\tau, z) + g_2(\tau)\left[\chi_{24,10}(\tau, z) + \chi_{24,14}(\tau, z)\right] \\ + g_3(\tau)\left[\chi_{24,8}(\tau, z) + \chi_{24,16}(\tau, z)\right] + g_4(\tau)\left[\chi_{24,6}(\tau, z) + \chi_{24,18}(\tau, z)\right] \\ + g_5(\tau)\left[\chi_{24,4}(\tau, z) + \chi_{24,20}(\tau, z)\right] + g_6(\tau)\left[\chi_{24,2}(\tau, z) + \chi_{24,22}(\tau, z)\right] \\ + g_7(\tau)\left[\chi_{24,0}(\tau, z) + \chi_{24,24}(\tau, z)\right]\end{aligned} \quad (5.17)$$

We rewrite the above formula in terms of $\mathcal{B}_6(\widehat{sl(2)})$ characters using the following definition.

$$f_1 = \frac{g_1(\tau)}{q^{1/2}\varphi(\tau)}\ ,\ f_2 = \frac{g_2(\tau)}{q^{-1/26}\varphi(\tau)}\ ,\ f_3 = \frac{g_2(\tau)}{q^{9/26}\varphi(\tau)}\ ,\ f_4 = \frac{g_3(\tau)}{q^{43/26}\varphi(\tau)}\ ,$$
$$f_5 = \frac{g_2(\tau)}{q^{23/26}\varphi(\tau)}\ ,\ f_6 = \frac{g_2(\tau)}{q^{1/26}\varphi(\tau)}\ ,\ f_7 = \frac{g_3(\tau)}{q^{3/26}\eta(\tau)}\ . \quad (5.18)$$

In the labelled table below, we give the roots associated with the weight for each of the seven characters.

Label	1	2	3	4	5	6	7
Weights	$\widetilde{\alpha}_1 + \alpha_1$ $= \widetilde{\alpha}_2 + \alpha_2$	$\widetilde{\alpha}_1$ $\widetilde{\alpha}_2$	$\widetilde{\alpha}_1 + \alpha_2$ $\widetilde{\alpha}_2 + \alpha_1$	$2\alpha_1 + \alpha_3$ $\alpha_0 + 2\alpha_2$	$\alpha_1 + \alpha_3$ $\alpha_0 + \alpha_2$	α_3 α_0	$\alpha_2 + \alpha_3$ $\alpha_0 + \alpha_1$
Norm	-24	2	-16	-78	-40	2	0

We assume that there is indeed a Lie superalgebra $\mathcal{B}(A^{(6)})$ and make the following statements based on that assumption. The real simple roots $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ are associated with label 2. Labels 1, 3, 4 and 5 correspond to imaginary simple roots with negative norm.

The vvmf $\mathbf{g}(\tau)$ has rank seven and the first few terms in the Fourier expansion are:

$$\mathbf{g}(\tau) = \begin{pmatrix} 2q^{-\frac{1}{2}}\left(q + q^2 + 2q^3 + 3q^4 + 5q^5 + 7q^6 + 11q^7 + \cdots\right) \\ -q^{-\frac{1}{26}}\left(1 + q + 3q^2 + 4q^3 + 7q^4 + 10q^5 + 16q^6 + \cdots\right) \\ q^{-17/26}\left(q + q^2 + 2q^3 + 3q^4 + 6q^5 + 7q^6 + 12q^7 + \cdots\right) \\ -q^{-9/26}\left(q^2 + q^4 + q^5 + 2q^6 + 2q^7 + \cdots\right) \\ -q^{-3/26}\left(q + q^2 + 2q^3 + 2q^4 + 4q^5 + 6q^6 + 9q^7 + \cdots\right) \\ q^{-25/26}\left(q + q^2 + 2q^3 + 4q^4 + 6q^5 + 9q^6 + 14q^7 + \cdots\right) \\ -q^{-23/26}\left(q + 2q^2 + 3q^3 + 5q^4 + 8q^5 + 12q^6 + 18q^7 + \cdots\right) \end{pmatrix} \tag{5.19}$$

For the above rank-seven vvmf, using Eq. (A.35) we obtain the following T and S matrices:

$$T = \text{Diag}\left(e^{i\pi},\ e^{-\frac{i\pi}{13}},\ e^{-i\pi\left(\frac{17}{13}\right)},\ e^{-i\pi\left(\frac{9}{13}\right)},\ e^{-i\pi\left(\frac{3}{13}\right)},\ e^{\frac{i\pi}{13}},\ e^{i\pi\left(\frac{3}{13}\right)}\right)$$

$$S = \frac{1}{\sqrt{13}}\begin{bmatrix} 1 & -2 & 2 & -2 & 2 & -2 & 2 \\ -1 & 2\cos\left(\frac{2\pi}{13}\right) & -2\sin\left(\frac{5\pi}{26}\right) & 2\sin\left(\frac{\pi}{26}\right) & 2\sin\left(\frac{3\pi}{26}\right) & -2\cos\left(\frac{3\pi}{13}\right) & 2\cos\left(\frac{\pi}{13}\right) \\ 1 & -2\sin\left(\frac{5\pi}{26}\right) & -2\sin\left(\frac{3\pi}{26}\right) & 2\cos\left(\frac{\pi}{13}\right) & -2\cos\left(\frac{3\pi}{13}\right) & -2\sin\left(\frac{\pi}{26}\right) & 2\cos\left(\frac{2\pi}{13}\right) \\ -1 & 2\sin\left(\frac{\pi}{26}\right) & 2\cos\left(\frac{\pi}{13}\right) & -2\sin\left(\frac{3\pi}{26}\right) & -2\cos\left(\frac{2\pi}{13}\right) & 2\sin\left(\frac{5\pi}{26}\right) & 2\cos\left(\frac{3\pi}{13}\right) \\ 1 & 2\sin\left(\frac{3\pi}{26}\right) & -2\cos\left(\frac{3\pi}{13}\right) & -2\cos\left(\frac{2\pi}{13}\right) & 2\sin\left(\frac{\pi}{26}\right) & 2\cos\left(\frac{\pi}{13}\right) & 2\sin\left(\frac{5\pi}{26}\right) \\ -1 & -2\cos\left(\frac{3\pi}{13}\right) & -2\sin\left(\frac{\pi}{26}\right) & 2\sin\left(\frac{5\pi}{26}\right) & 2\cos\left(\frac{\pi}{13}\right) & 2\cos\left(\frac{2\pi}{13}\right) & 2\sin\left(\frac{3\pi}{26}\right) \\ 1 & 2\cos\left(\frac{\pi}{13}\right) & 2\cos\left(\frac{2\pi}{13}\right) & 2\cos\left(\frac{3\pi}{13}\right) & 2\sin\left(\frac{5\pi}{26}\right) & 2\sin\left(\frac{3\pi}{26}\right) & 2\sin\left(\frac{\pi}{26}\right) \end{bmatrix}$$

5.2 CHL $\mathbb{Z}_N$ orbifolds [2]

5.2.1 Deconstructing the Lie algebra

The Siegel modular form defined in Eq. (4.14) can be expanded as a power series in the variable s. The leading term in the expansion is $s^{1/2}\,\phi^{(N)}_{k(N),1/2}(\tau, z)$ which is the denominator formula for the sub-algebra $\mathcal{B}_N^{CHL}(\widehat{sl(2)})$.

$$\Delta^{(N)}_{k(N)}(\mathbf{Z}) = s^{1/2}\,\phi^{(N)}_{k(N),1/2}(\tau,z)[1+\sum_{m=1}^{\infty} s^m\,\Psi^{(N)}_{0,m}(\tau,z)]\ , \tag{5.20}$$

The above equations define the weight zero and index m Jacobi forms $\Psi^{(N)}_{0,m}(\tau,z)$. Explicit formulae for the Jacobi forms can be obtained by expanding the exponential in Eq. (4.14). For instance, one obtains

$$\Psi^{(N)}_{0,1}(\tau,z) = -\psi^{(N)[1,g]}_{0,1}(\tau,z)\ , \tag{5.21}$$

$$\Psi^{(N)}_{0,2}(\tau,z) = -\frac{1}{2}\left(\psi^{(N)[1,g]}_{0,1}(\tau,z)|T(2) - (\psi^{(N)[1,g]}_{0,1}(\tau,z))^2\right)\ . \tag{5.22}$$

We will be studying the first N terms in the expansion. They can be rewritten in terms of standard modular forms thereby enabling us to have formulae that can be directly used. A weak Jacobi form of $\Gamma_0(N)$, ξ_m, of weight zero and index m can be expanded as follows:

$$\xi_m(\tau,z) = \sum_{j=0}^{m} \alpha_j(\tau)\, A(\tau,z)^{m-j}B(\tau,z)^j\ ,$$

where $\alpha_j(\tau)$ are weight $2j$ modular forms of $\Gamma_0(N)$ and $A(\tau,z)$, $B(\tau,z)$ are defined in Eq. (A.22). However the $\Psi^{(N)}_{0,m}(\tau,z)$ are Jacobi forms of $\Gamma^0(N)$. Thus, we identify ξ_m with their transform $\Psi^{(N)}_{0,m}(\tau,z)|S$ as they are modular forms of $\Gamma_0(N)$. This method is useful as the generators of the ring of modular forms of $\Gamma_0(N)$ are well-known [72]. We give the generators for the cases of interest in appendix A.3.

Details of the examples

We now present explicit formulae for the Jacobi forms $\Psi^{(N)}_{0,m}(\tau,z)|S$ for $N = 2,3,5$ and $m = 1,\ldots,N$.

$N = 2$

The Weyl-Kac-Borcherds denominator formula is given by the weight three Siegel modular form of a level 2 subgroup of $Sp(4,\mathbb{Z})$.

$$\Delta_3^{(2)}(\mathbf{Z}) = s^{1/2}\,\phi^{(2)}_{3,1/2}(\tau,z)\big[1 + s\,\Psi^{(2)}_{0,1}(\tau,z) + s^2\,\Psi^{(2)}_{0,2}(\tau,z) + O(s^3)\big]\,, \tag{5.23}$$

where

$$\begin{aligned}
\phi^{(2)}_{3,1/2}(\tau,z) &= \theta_1(\tau,z)\,\eta(\tau)^4\eta(\tau/2)^4 \\
\Psi^{(2)}_{0,1}(\tau,z) &= \tfrac{1}{3}A(\tau,z) - \tfrac{1}{3}E_2^{(2)}(\tau/2)\,B(\tau,z) \\
\Psi^{(2)}_{0,2}(\tau,z) &= -\tfrac{1}{72}A(\tau,z)^2 - \tfrac{1}{18}E_2^{(2)}(\tau/2)A(\tau,z)B(\tau,z) \\
&\quad + \left(\tfrac{29}{288}E_2^{(2)}(\tau/2)^2 - \tfrac{1}{32}E_4(\tau/2)\right)B(\tau,z)^2
\end{aligned}$$

are Jacobi forms of $\Gamma^0(2)$. We expect to observe two real simple roots in $\Psi^{(2)}_{0,2}(\tau,z)$.

$N = 3$

The Weyl-Kac-Borcherds denominator formula is given by the weight two Siegel modular form of a level 3 subgroup of $Sp(4,\mathbb{Z})$.

$$\Delta_2^{(3)}(\mathbf{Z}) = s^{1/2}\,\phi^{(3)}_{2,1/2}(\tau,z)[1 + s\,\Psi^{(3)}_{0,1}(\tau,z) + s^2\,\Psi^{(3)}_{0,2}(\tau,z) + s^3\,\Psi^{(3)}_{0,3}(\tau,z) + O(s^4)]\,, \tag{5.24}$$

where

$$
\begin{aligned}
\phi^{(3)}_{2,1/2}(\tau,z) &= \theta_1(\tau,z)\,\eta(\tau)^3\eta(\tau/3)^3 \\
\Psi^{(2)}_{0,1}(\tau,z) &= \tfrac{1}{4}A(\tau,z) - \tfrac{1}{4}E^{(3)}_2(\tau/3)\,B(\tau,z) \\
\Psi^{(3)}_{0,2}(\tau,z) &= 0 \\
\Psi^{(3)}_{0,3}(\tau,z) &= \tfrac{1}{864}A(\tau,z)^3 - \tfrac{1}{96}E^{(3)}_2(\tau/3)A(\tau,z)^2B(\tau,z) \\
&\quad + \left(\tfrac{25}{1296}E^{(3)}_2(\tau/3)^2 - \tfrac{5}{2592}E_4(\tau/3)\right)A(\tau,z)B(\tau,z)^2 \\
&\quad + (-\tfrac{145}{11664}E^{(3)}_2(\tau/3)^3 + \tfrac{85}{23328}E^{(3)}_2(\tau/3)E_4(\tau/3) + \tfrac{1}{1458}E_6(\tau/3))B(\tau,z)^3
\end{aligned}
$$

are Jacobi forms of $\Gamma^0(3)$. It is interesting to observe that $\Psi^{(3)}_{0,2}(\tau,z) = 0$. This arises from a cancellation of multiple terms. The expectation is that there would have been no real simple roots and imaginary simple roots in this term. The vanishing says that there are no imaginary simple roots with negative norm. It could also be that there is a Bose-Fermi cancellation i.e., there are equal numbers of bosonic and fermionic roots. We expect to see two real simple roots in $\Psi^{(3)}_{0,3}(\tau,z)$ which is non-vanishing.

$N = 5$

The Weyl-Kac-Borcherds denominator formula is given by the weight one Siegel modular form of a level 5 subgroup of $Sp(4,\mathbb{Z})$.

$$
\Delta^{(5)}_1(\mathbf{Z}) = s^{1/2}\,\phi^{(5)}_{1,1/2}(\tau,z)\big[1 + \sum_{m=1}^{5} s^m\,\Psi^{(5)}_{0,m}(\tau,z) + O(s^6)\big]\ , \qquad (5.25)
$$

$$
\begin{aligned}
\phi^{(5)}_{1,1/2}(\tau,z) &= \theta_1(\tau,z)\,\eta(\tau/5)^2\eta(\tau)^2 \\
\Psi^{(5)}_{0,1}(\tau,z) &= \tfrac{1}{5}A(\tau,z) - \tfrac{1}{5}E^{(5)}_2(\tau/5)\,B(\tau,z)
\end{aligned}
$$

We have shortened $A(\tau,z), B(\tau,z)$ to A, B to make equations more compact.

$$\begin{aligned}
\Psi^{(5)}_{0,2}(\tau,z) &= -\tfrac{1}{144}A^2 - \tfrac{1}{72}E^{(5)}_2(\tau/5)AB \\
&\quad + \left(-\tfrac{53}{7200}E^{(5)}_2(\tau/5)^2 + \tfrac{1}{2400}E_4(\tau/5) - \tfrac{19}{200}\eta(\tau/5)^4\eta(\tau)^4\right) B^2 \\
\Psi^{(5)}_{0,3}(\tau,z) &= \tfrac{1}{864}A^3 - \tfrac{1}{288}E^{(5)}_2(\tau/5)A^2B \\
&\quad + \left(\tfrac{17}{4800}E^{(5)}_2(\tau/5)^2 - \tfrac{1}{14400}E_4(\tau/5) + \tfrac{19}{1200}\eta(\tau/5)^4\eta(\tau)^4\right) A\,B^2 \\
&\quad + \left(-\tfrac{53}{43200}E^{(5)}_2(\tau/5)^3 + \tfrac{1}{14400}E^{(5)}_2(\tau/5)E_4(\tau/5) - \tfrac{19}{1200}E^{(5)}_2(\tau/5)\eta(\tau/5)^4\eta(\tau)^4\right) B^3 \\
\Psi^{(5)}_{0,4}(\tau,z) &= \tfrac{1}{20736}A^4 - \tfrac{1}{5184}E^{(5)}_2(\tau/5)A^3\,B \\
&\quad + \left(\tfrac{17}{57600}E^{(5)}_2(\tau/5)^2 - \tfrac{1}{172800}E_4(\tau/5) + \tfrac{19}{14400}\eta(\tau/5)^4\eta(\tau)^4\right) A^2\,B^2 \\
&\quad + \left(-\tfrac{53}{259200}E^{(5)}_2(\tau/5)^3 + \tfrac{1}{86400}E^{(5)}_2(\tau/5)E_4(\tau/5) - \tfrac{19}{7200}E^{(5)}_2(\tau/5)\eta(\tau/5)^4\eta(\tau)^4\right) A\,B^3 \\
&\quad + \Big(\tfrac{2117}{25920000}E^{(5)}_2(\tau/5)^4 - \tfrac{1}{28800}E^{(5)}_2(\tau/5)^2E_4(\tau/5) + \tfrac{2641}{360000}E^{(5)}_2(\tau/5)^2\eta(\tau/5)^4\eta(\tau)^4 \\
&\qquad\qquad + \tfrac{11}{8640000}E_4(\tau/5)^2 + \tfrac{779}{60000}\eta(\tau/5)^8\eta(\tau)^8\Big)\,B^4
\end{aligned}$$

are Jacobi forms of $\Gamma^0(5)$. We have not given an explicit formula for $\Psi^{(5)}_{0,5}(\tau,z)$ as the formula is big and unilluminating.

Characters of $\widehat{sl(2)}$ and $\mathcal{B}_N(\widehat{sl(2)})$

We will represent from now on the real simple roots α_0 and α_3 (3.28) with an superscript '(N)' i.e., $\alpha_0 \equiv \alpha^{(N)}_0$ and $\alpha_3 \equiv \alpha^{(N)}_3$. We will track these roots as well as the zero-norm imaginary simple roots

$$\delta'_N := (\alpha^{(N)}_3 + \alpha_2) \quad \text{and} \quad \delta''_N := (\alpha^{(N)}_0 + \alpha_1)\,. \tag{5.26}$$

The subscript N is to emphasise that they change with N unlike the zero-norm imaginary simple root $\delta = (\alpha_1 + \alpha_2)$.

For weights $\tilde{\Lambda} = a\delta + b\alpha_2 + c\delta'_N$ satisfying the condition $\langle\tilde{\Lambda}, \delta\rangle < 0$, the character of

$\mathcal{B}_N(\widehat{sl(2)})$ when $a = 0$ is given by(see section 4.2.1)

$$\widetilde{\chi}_{k,\ell} = q^{\frac{1}{8}-\frac{(\ell+1)^2}{4(k+2)}} \frac{\chi_{k,\ell}}{T_N(\tau)} , \tag{5.27}$$

with $k = 4Nc$, $\ell = -2b$ and $T_N(\tau)$ being the Borcherds correction factor (4.13). The weights are such that $a \in \frac{1}{N}\mathbb{Z}_{\geq 0}$ and $c \in \frac{1}{N}\mathbb{Z}_{>0}$. The character with $a \neq 0$ is then $q^a\, \widetilde{\chi}_{k,\ell}$.

5.2.2 VVMFs from $\widehat{sl(2)}$ decomposition

The Jacobi forms $\Psi^{(N)}_{0,m}$ can be expanded in terms of characters of $\widehat{sl(2)}$ and those of the Borcherds extension $\mathcal{B}_N(\widehat{sl(2)})$ The decomposition takes the form

$$\begin{aligned}\Psi^{(N)}_{0,m}(\tau,z) &= \sum_{j=-m}^{m} g^{N,m}_{j+1}(\tau)\,\chi_{4m,2m+2j}(\tau,z) \,, \\ &= \sum_{j=-m}^{m} f^{N,m}_{j+1}(\tau)\,\widetilde{\chi}_{4m,2m+2j}(\tau,z) \,.\end{aligned}$$

Due to the $\mathbb{Z}_2$ outer automorphism under which $\alpha_1 \leftrightarrow \alpha_2$ and $\alpha^{(N)}_0 \leftrightarrow \alpha^{(N)}_3$, one observes that $g^{N,m}_{j+1}(\tau) = g^{N,m}_{-j+1}(\tau)$ and $\Psi^{(N)}_{0,m}(\tau,z)$ can be rewritten as

$$\Psi^{(N)}_{0,m}(\tau,z) = \sum_{j=0}^{m} g^{N,m}_{j+1}(\tau)\,\left[\chi_{4m,2m-2j}(\tau,z) + \chi_{4m,2m+2j}(\tau,z)\right] \,, \tag{5.28}$$

$$= \sum_{j=0}^{m} f^{N,m}_{j+1}(\tau)\,\left[\widetilde{\chi}_{4m,2m-2j}(\tau,z) + \widetilde{\chi}_{4m,2m+2j}(\tau,z)\right] \,. \tag{5.29}$$

Thus one has $(m+1)$ independent functions that we organize into a vector $\mathbf{g} := (g_1, g_2, \ldots, g_{m+1})^T$. These are rank $(m+1)$ vector valued modular forms of $\Gamma^0(N)$.

Remark: The multiplicities of roots are given the coefficients of the $f^{N,m}_j(\tau)$ which can be obtained from the $g^{N,m}_j(\tau)$ using Eq. (5.27).

5.2.3 The vvmfs

Below we give the $\widehat{sl(2)}$ decompositions for $N = 2, 3, 5$ and the cases, did not appear in the paper [2], for composite $N = 4, 6$ as well . The format is as follows: (i) The vvmf $\mathbf{g}^{N,m}$ has rank $(m + 1)$; (ii) The first entry of $\mathbf{g}^{N,m}$ is associated with the $\widehat{sl(2)}$ character $\chi_{4m,2m}$; (iii) subsequent entries involve a pair of characters related by the $\mathbb{Z}_2$ automorphism and appear as $\chi_{4m,2m-2j}$ and $\chi_{4m,2m+2j}$ for $j = 1, \ldots, m$; (iv) the power of q shown is the one associated with $k = 2m$, $\ell = (2m - 2j)$ i.e., $q^{\frac{1}{8}-\frac{(\ell+1)^2}{4(k+2)}}$ as can be read off from Eq. (5.27). We carefully track all roots with positive and zero norm that appear in the character expansion.

The weight $\tilde{\Lambda}$ defined in the context of equation (5.27) can be written as

$$\tilde{\Lambda} = a\delta + \left(\frac{k}{4N} - \frac{l}{2}\right)\alpha_2 + \frac{k}{4N}\alpha_3^{(N)} ,$$

following this, a root $\tilde{\Lambda}^{(a)}_{4m,2m-2j}$ associated to a term like '$q^{\frac{1}{8}-\frac{(\ell+1)^2}{4(k+2)}}\left\{q^a\chi_{4m,2m-2j}(\tau,z)\right\}$' appears in '$g^{N,m}_{j+1}\chi_{4m,2m-2j}(\tau,z)$' can be given by

$$\tilde{\Lambda}^{(a)}_{4m,2m-2j} = a\delta + \left(\frac{m}{N} - m + j\right)\alpha_2 + \frac{m}{N}\alpha_3^{(N)} , \quad \text{for } j = 0, 1, \ldots, m$$

since $(1/N - 1)\alpha_2 + \alpha_3^{(N)}/N = \alpha_3^{(1)}$, this simply transforms to

$$\tilde{\Lambda}^{(a)}_{4m,2m-2j} = \begin{cases} a\delta + j\alpha_2 + m\alpha_3^{(1)} , & \text{if } j = 0, 1, \ldots, m-1 \\ a\delta + \frac{m}{N}(\alpha_2 + \alpha_3^{(N)}) = a\delta + \frac{m}{N}\delta'_N , & \text{if } j = m \end{cases} \tag{5.30}$$

similarly a root $\tilde{\Lambda}^{(a)}_{4m,2m+2j}$ associated to a term like '$q^{\frac{1}{8}-\frac{(\ell+1)^2}{4(k+2)}}\left\{q^a\chi_{4m,2m+2j}(\tau,z)\right\}$' appears in

‘$g^{N,m}_{j+1}\chi_{4m,2m+2j}(\tau,z)$’ can be given by

$$\tilde{\Lambda}^{(a)}_{4m,2m+2j} = \begin{cases} a\delta + j\alpha_1 + m\alpha_0^{(1)}\,, & \text{if } j = 0,1,\ldots,m-1 \\ a\delta + \frac{m}{N}(\alpha_1 + \alpha_0^{(N)}) = a\delta + \frac{m}{N}\delta''_N\,, & \text{if } j = m \end{cases} \tag{5.31}$$

$N = 2$

The coefficients of the Fourier series $T_2(\tau)$ give the mutiplicity of the imaginary simple roots δ'_2 and δ''_2. The coefficient of q^y gives the multiplicity of the roots $y\,\delta'_2$ and $y\,\delta''_2$. One has

$$T_2(\tau) = 1 - \mathbf{4}\,q^{1/2} + \mathbf{1}\,q + O(q^{3/2}))\,.$$

We will see that the expansions below are consistent with these numbers.

$$\mathbf{g}^{2,1}(\tau) = \begin{pmatrix} q^{-1/4}(8q^{1/2} + 40q + 128q^{3/2} + 368q^2 + 936q^{5/2} + 2176q^3 + \cdots) \\ q^{1/12}(-\mathbf{4} - 24q^{1/2} - 88q - 264q^{3/2} - 692q^2 - 1656q^{5/2} + \cdots) \end{pmatrix}$$

The leading term in the first row corresponds to the imaginary simple roots $(\alpha_3^{(1)} + \frac{1}{2}\delta)$ and $(\alpha_0^{(1)} + \frac{1}{2}\delta)$ as the constant piece is vanishing. This is consistent with simple real roots $\alpha_3^{(1)}$ and $\alpha_0^{(1)}$ not being present. In the second row, the leading term has multiplicity -4 and corresponds to the imaginary roots $\frac{1}{2}\delta'_2$ and $\frac{1}{2}\delta''_2$. All other terms correspond to imaginary simple roots with negative norm.

$$\mathbf{g}^{2,2}(\tau) = \begin{pmatrix} q^{-1/2}(-4q^{1/2} + 2q - 16q^{3/2} - 2q^2 - 56q^{5/2} + 2q^3 - 144q^{7/2} + \cdots) \\ q^{-1/10}(-\mathbf{1} + 4q^{1/2} + q + 8q^{3/2} - 2q^2 + 24q^{5/2} + 2q^3 + 64q^{7/2} + \cdots) \\ q^{1/10}(\mathbf{1} + 8q^{1/2} + 28q^{3/2} + 80q^{5/2} - q^3 + \cdots) \end{pmatrix}$$

The leading term in the second row above is the multiplicity of the real simple roots $\alpha_0^{(2)}$ and $\alpha_3^{(2)}$. They have multiplicity 1 and the minus sign comes from $\det(w)$ in the denominator formulae. The Lie algebra $\mathfrak{g}(A^{(2)})$ has four real simple roots. Thus, there are no more simple real roots to track. In the third/last row, the leading term has multiplicity +1 and corresponds to the imaginary roots δ_2' and δ_2''.

Definition 5.2.1. *Let $\mathcal{I}$ denote the set of imaginary simple roots with negative norm whose multiplicities are given by the Fourier expansions of $f_j^{N,m}(\tau)$ for $j = 1, \ldots, (m+1)$ and $m = 1, \ldots, N$.*

These are not the complete set of imaginary simple roots as more appear when $m > N$.

$N = 3$

The coefficients of the Fourier series $T_3(\tau)$ give the multiplicity of the imaginary simple roots proportional to δ_3' and δ_3''. The coefficient of q^y gives the multiplicity of the roots $y\,\delta_3'$ and $y\,\delta_3''$. One has

$$T_3(\tau) = 1 \mathbf{-3}\, q^{1/3} + \mathbf{0}\, q^{2/3} \mathbf{-5}\, q + O(q^{4/3}) \,.$$

$$\mathbf{g}^{3,1}(\tau) = \frac{3\eta(\tau)^3}{\eta(\tau/3)^3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} q^{-1/4}\,(3\,q^{1/3} + O(q^{2/3}) \\ q^{1/12}\,(\mathbf{-3} + O(q^{1/3}) \end{pmatrix} \tag{5.32}$$

The leading term in the first row corresponds to the imaginary simple roots $(\alpha_3^{(1)} + \frac{1}{3}\delta)$ and $(\alpha_0^{(1)} + \frac{1}{3}\delta)$ as the constant piece is vanishing. This is consistent with simple real roots $\alpha_3^{(1)}$ and $\alpha_0^{(1)}$ not being present. In the second row, the leading term has multiplicity -3 and corresponds to the imaginary roots $\frac{1}{3}\delta_3'$ and $\frac{1}{3}\delta_3''$. All other terms correspond to imaginary simple roots with negative norm.

$$\mathbf{g}^{3,3}(\tau) = \begin{pmatrix} q^{-3/4}(14q + 42q^{4/3} + 126q^{5/3} + 308q^2 + 714q^{7/3} + 1512q^{8/3} + \cdots) \\ q^{-9/28}(-3q^{1/3} - 9q^{2/3} - 38q^1 - 99q^{4/3} - 252q^{5/3} - 549q^2 + \cdots) \\ q^{-1/28}(\mathbf{-1} - 3q^{1/3} - 9q^{2/3} - 35q - 75q^{4/3} - 180q^{5/3} - 372q^2 + \cdots) \\ q^{3/28}(\mathbf{5} + 24q^{1/3} + 72q^{2/3} + 191q + 453q^{4/3} + 999q^{5/3} + \cdots) \end{pmatrix}$$

$N = 4$

The coefficients of the Fourier series $T_4(\tau)$ give the multiplicity of the imaginary simple roots proportional to δ'_4 and δ''_4. The coefficient of q^y gives the multiplicity of the roots $y\,\delta'_4$ and $y\,\delta''_4$. One has

$$\begin{aligned} T_4(\tau) &= \prod_{j=1}^{\infty} \left(1 - q^{j/2}\right)\left(1 - q^{j/4}\right)^2 \left(1 - q^{j}\right)^{-1} \\ &= 1 - \mathbf{2}\, q^{1/4} - \mathbf{2}\, q^{1/2} + \mathbf{4}\, q^{3/4} + \mathbf{2}\, q + O(q^{3/2})\,. \end{aligned}$$

$$\mathbf{g}^{4,1}(\tau) = \begin{pmatrix} q^{-1/4}(2q^{1/4} + 4q^{1/2} + 8q^{3/4} + 20q + 36q^{5/4} + 64q^{3/2} + \cdots) \\ q^{1/12}(-\mathbf{2} - 6q^{1/4} - 12q^{1/2} - 24q^{3/4} - 44q - 78q^{5/4} - 132q^{3/2} + \cdots) \end{pmatrix}$$

$$\mathbf{g}^{4,2}(\tau) = \begin{pmatrix} q^{-1/2}(2q^{1/2} + 4q + 8q^{3/2} + 16q^{2} + 28q^{5/2} + 44q^{3} + \cdots) \\ q^{-1/10}(-2q^{1/2} - 2q - 4q^{3/2} - 8q^{2} - 12q^{5/2} + \cdots) \\ q^{1/10}(-\mathbf{2} - 4q^{1/2} - 8q - 14q^{3/2} - 24q^{2} - 40q^{5/2} + \cdots) \end{pmatrix}$$

$$\mathbf{g}^{4,3}(\tau) = \begin{pmatrix} q^{-3/4}(4q^{3/4} + 8q + 24q^{5/4} + 56q^{3/2} + 120q^{7/4} + 232q^{2} + \cdots) \\ q^{-9/28}(-4q^{1/2} - 12q^{3/4} - 28q - 60q^{5/4} - 120q^{3/2} - 228q^{7/4} - 416q^{2} + \cdots) \\ q^{-1/28}(-4q^{1/4} - 4q^{1/2} - 12q^{3/4} - 24q - 48q^{5/4} - 88q^{3/2} + \cdots) \\ q^{3/28}(\mathbf{4} + 12q^{1/4} + 32q^{1/2} + 68q^{3/4} + 144q + 276q^{5/4} + \cdots) \end{pmatrix}$$

$$\mathbf{g}^{4,4}(\tau) = \begin{pmatrix} q^{-1}(3q + 4q^{5/4} + 8q^{3/2} + 16q^{7/4} + 28q^{2} + 48q^{9/4} + \cdots) \\ q^{-5/9}(-2q - 2q^{5/4} - 4q^{3/2} - 8q^{7/4} - 12q^{2} - 20q^{9/4} + \cdots) \\ q^{-2/9}(-2q^{1/4} - 4q^{1/2} - 8q^{3/4} - 16q - 28q^{5/4} - 48q^{3/2} + \cdots) \\ (-\mathbf{1} + 2q^{1/4} + 4q^{1/2} + 8q^{3/4} + 14q + 24q^{5/4} + \cdots) \\ q^{1/9}(\mathbf{2} + 4q^{1/4} + 8q^{1/2} + 16q^{3/4} + 28q + 46q^{5/4} + \cdots) \end{pmatrix}$$

The leading term in the first row of $\mathbf{g}^{4,1}(\tau)$ corresponds to the imaginary simple roots $(\alpha_3^{(1)} + \frac{1}{4}\delta)$ and $(\alpha_0^{(1)} + \frac{1}{4}\delta)$ as the constant piece is vanishing. This is consistent with simple real roots $\alpha_3^{(1)}$ and $\alpha_0^{(1)}$ not being present. In the second row of $\mathbf{g}^{4,1}(\tau)$, the leading term has multiplicity -2 and corresponds to the imaginary roots $\frac{1}{4}\delta'_2$ and $\frac{1}{4}\delta''_2$. All other terms correspond to imaginary simple roots with negative norm.

The leading term in the third row of $\mathbf{g}^{4,2}(\tau)$ has multiplicity -2 and corresponds to the imaginary roots $\frac{1}{2}\delta'_2$ and $\frac{1}{2}\delta''_2$.

For $N = 2, 3, 4$; for the terms that we have studied we are able to see that the denominator term can be written as

$$\Delta^{(N)}_{k(N)}(\mathbf{Z}) = \sum_{w\in W} \det(w) w\Big[(e^{-\varrho}(T_N(\delta) + (T_N(\delta'_N) - 1) + (T_N(\delta''_N) - 1) + \sum_{a\in\mathcal{I}} m(a)\, e^{-a} + \cdots)\Big] \quad (5.33)$$

where the set $\mathcal{I}$ is as defined in Definition 5.2.1. The ellipsis refers to contributions from higher orders. Additional terms may be added by incorporating the action of the symmetry $\gamma^{(N)}$ to make the right hand side manifestly invariant under the extended Weyl group. The symmetry under the action of $\widehat{\delta}$ is already present. Terms such as these fit into the Borcherds extension of $\mathfrak{g}(A^{(N)})$.

$N = 5$

The coefficients of the Fourier series $T_5(\tau)$ give the multiplicity of the imaginary simple roots δ'_5 and δ''_5. The coefficient of q^y gives the multiplicity of the roots $y\,\delta'_5$ and $y\,\delta''_5$. One has

$$T_5(\tau) = 1 - \mathbf{2}\, q^{1/5} - \mathbf{1}\, q^{2/5} + \mathbf{2}\, q^{3/5} + \mathbf{1}\, q^{4/5} + \mathbf{3}\, q + O(q^{6/5})\,.$$

These appear as the leading coefficient in the bottom row of each vvmf $\mathbf{g}^{5,m}$ for $m = 1, \ldots, 5$.

$$\mathbf{g}^{5,1}(\tau) = \begin{pmatrix} q^{-1/4}(q^{1/5} + 3q^{2/5} + 4q^{3/5} + 7q^{4/5} + 17q + 24q^{6/5} + 44q^{7/5} + \cdots) \\ q^{1/12}(-\mathbf{2} - 3q^{1/5} - 9q^{2/5} - 12q^{3/5} - 21q^{4/5} - 35q + \cdots) \end{pmatrix}$$

For many purposes, it is useful to consider the leading terms in each row. In particular it is easy to extract the weight vector by inspection. In the first row, it is $q^{1/5}\tilde{\chi}_{4,2}$ whose weight vector is $(\frac{\delta}{5} - \alpha_2 + \delta'_5) = \frac{\delta}{5} + \alpha_3^{(1)}$ which has norm $2(1-4/5) = 2/5$. This is a real fermionic root. Let us call this root β. Note that $\beta = 2\varrho^{(5)}$. One can show that

$$\langle \varrho^{(5)}, \beta^\vee \rangle = +1 \; . \tag{5.34}$$

where the co-root $\beta^\vee := \frac{2\beta}{\langle\beta,\beta\rangle}$. Note that this has the 'wrong' sign (in our convention) where simple real roots such as α_1 are such that $\langle \varrho^{(5)}, \alpha_1^\vee \rangle = -1$.

$$\mathbf{g}^{5,2}(\tau) = \begin{pmatrix} q^{-1/2}(q^{2/5} + q^{3/5} + 2q^{4/5} + q - 2q^{6/5} + 7q^{7/5} + 4q^{8/5} + 8q^{9/5} + \cdots) \\ q^{-1/10}(q^{1/5} - 2q^{2/5} - q^{3/5} - 3q^{4/5} + q + 5q^{6/5} - 8q^{7/5} - 3q^{8/5} + \cdots) \\ q^{1/10}(\mathbf{-1} - 3q^{1/5} + q^{2/5} - 2q^{3/5} - q^{4/5} - 5q - 12q^{6/5} + \cdots) \end{pmatrix}$$

$$\mathbf{g}^{5,3}(\tau) = \begin{pmatrix} q^{-3/4}(q^{3/5} + 4q^{4/5} + 9q + 14q^{6/5} + 33q^{7/5} + 52q^{8/5} + 126q^{9/5} + \cdots) \\ q^{-9/28}(-q^{2/5} - 3q^{3/5} - 15q^{4/5} - 25q - 37q^{6/5} - 74q^{7/5} - 106q^{8/5} + \cdots) \\ q^{-1/28}(-3q^{1/5} - 4q^{2/5} - 11q^{3/5} - 2q^{4/5} - 18q - 38q^{6/5} - 59q^{7/5} + \cdots) \\ q^{3/28}(\mathbf{2} + 9q^{1/5} + 17q^{2/5} + 41q^{3/5} + 53q^{4/5} + 110q + 201q^{6/5} + \cdots) \end{pmatrix}$$

$$\mathbf{g}^{5,4}(\tau) = \begin{pmatrix} q^{-1}(q^{4/5} + 2q + 5q^{6/5} + 8q^{7/5} - 2q^{8/5} + 16q^{9/5} + 13q^2 + 68q^{11/5} + \cdots) \\ q^{-5/9}(2q^{3/5} - q^{4/5} - 11q - 11q^{6/5} + 24q^{7/5} - 11q^{8/5} + 11q^{9/5} + \cdots) \\ q^{-2/9}(-q^{2/5} - 10q^{3/5} - 7q^{4/5} - 18q - 18q^{7/5} - 103q^{8/5} - 59q^{9/5} + \cdots) \\ (-2q^{1/5} - q^{2/5} + 14q^{3/5} + 5q^{4/5} + 19q - 14q^{6/5} + 6q^{7/5} + 123q^{8/5} + \cdots) \\ q^{1/9}(\mathbf{1} + 6q^{1/5} + 8q^{2/5} - 6q^{3/5} + 18q^{4/5} + 12q + 74q^{6/5} + 77q^{7/5} + \cdots) \end{pmatrix}$$

$$\mathbf{g}^{5,5}(\tau) = \begin{pmatrix} q^{-5/4}(q + 2q^{6/5} + 6q^{7/5} + 8q^{8/5} + 14q^{9/5} - 16q^2 + 40q^{11/5} + 64q^{12/5} + \cdots) \\ q^{-35/44}(5q - 4q^{6/5} - 12q^{7/5} - 16q^{8/5} - 28q^{9/5} + 73q^2 - 74q^{11/5} + \cdots) \\ q^{-19/44}(-21q + 6q^{6/5} + 18q^{7/5} + 24q^{8/5} + 42q^{9/5} - 194q^2 + 112q^{11/5} + \cdots) \\ q^{-7/44}(-2q^{1/5} - 6q^{2/5} - 8q^{3/5} - 14q^{4/5} + 24q - 40q^{6/5} - 64q^{7/5} + \cdots) \\ q^{1/44}(\mathbf{-1} + 4q^{1/5} + 12q^{2/5} + 16q^{3/5} + 28q^{4/5} - 34q + 72q^{6/5} + \cdots) \\ q^{5/44}(\mathbf{3} - 2q^{1/5} - 6q^{2/5} - 8q^{3/5} - 14q^{4/5} + 73q - 44q^{6/5} - 76q^{7/5} + \cdots) \end{pmatrix}$$

The leading root that appears in $g_4^{5,5}$ is the $\gamma^{(5)}$ image of $q^{1/5}$ and thus appears with the same multiplicity as $q^{1/5}$. The leading term in the first row of all the vvmfs $\mathbf{g}^{5,m}$ for $m = 1, \ldots, 5$ is associated with the simple real root $m\beta$, All appear with multiplicity +1 indicating the fermionic nature of the root. These terms are consistent with adding the following term in Eq. (5.33).

$$\frac{1}{1 - e^{-\beta}} = 1 + \sum_{m=1}^{\infty} e^{-m\beta} \ . \tag{5.35}$$

This is the first term that cannot be a Borcherds correction due to the real nature of the root. However, it would be a Borcherds correction if β were a fermionic null root. The first five terms in the above expansion appear in our character expansions with the correct multiplicity. On the product side given by Eq. (4.18), we can see that the root β appears with multiplicity -1 with the roots $m\beta$ for $m = 2\ldots$ not appearing to the extent that we have checked. This is also consistent with the claim in Eq. (5.35).

For $N = 2, 3, 4$, it expected that $\mathcal{B}_N^{CHL}(A^{(N)})$ is a BKM Lie superalgebra and a suitably enlarged set $\mathcal{I}$ should do the job. As far as we know, an explicit proof is not available in the literature. For $N = 5$, we expect a new set of real roots might appear at $m = 10$. In particular, it is known that following two real roots of norm 2 could appear as they are present in the product side.

$$\tilde{\alpha}_1 = \begin{pmatrix} 4 & 9 \\ 9 & 20 \end{pmatrix} \quad , \quad \tilde{\alpha}_2 = \begin{pmatrix} 6 & 11 \\ 11 & 20 \end{pmatrix} \quad .$$

These are associated with the $\widehat{sl(2)}$ characters $\chi_{40,18}$ and $\chi_{40,22}$. They should appear as the leading coefficient in the $\widehat{sl(2)}$ character decomposition of $\Psi_{0,10}^{(5)}(\mathbf{Z})$ given below. The relevant term in the second row is given in bold face and is vanishing.

$$
\mathbf{g}^{5,10}(\tau) = \begin{pmatrix}
q^{-5/2}(q^2 + 2q^{11/5} + 5q^{12/5} + 12q^{13/5} + 27q^{14/5} + 114q^3 + \cdots) \\
q^{-85/42}(\mathbf{0\, q^2} + 8q^{11/5} + 27q^{12/5} + 20q^{13/5} + 17q^{14/5} - 603q^3 + \cdots) \\
q^{-67/42}(35q^2 - 66q^{11/5} - 207q^{12/5} - 228q^{13/5} - 345q^{14/5} + \cdots) \\
q^{-17/14}(2q^{7/5} - 8q^{8/5} - 26q^{9/5} - 326q^2 + 104q^{11/5} + 461q^{12/5} + \cdots) \\
q^{-37/42}(5q - 16q^{6/5} - 54q^{7/5} - 40q^{8/5} - 34q^{9/5} + 1056q^2 + \cdots) \\
q^{-25/42}(-35q + 66q^{6/5} + 207q^{7/5} + 228q^{8/5} + 345q^{9/5} + \cdots) \\
q^{-5/14}(-q^{2/5} + 4q^{3/5} + 13q^{4/5} + 164q - 80q^{6/5} - 318q^{7/5} + \cdots) \\
q^{-1/6}(2q^{1/5} + 9q^{2/5} - 4q^{3/5} - 25q^{4/5} - 397q + 102q^{6/5} + \cdots) \\
q^{-1/42}(8q^{1/5} - 27q^{2/5} - 20q^{3/5} - 17q^{4/5} + 603q - 352q^{6/5} + \cdots) \\
q^{1/14}(-3 + 14q^{1/5} + 45q^{2/5} + 44q^{3/5} + 59q^{4/5} - 812q + \cdots) \\
q^{5/42}(5 - 16q^{1/5} - 54q^{2/5} - 40q^{3/5} - 34q^{4/5} + 1056q + \cdots)
\end{pmatrix}
$$

The leading term in row 1 has weight 10β and multiplicity one. This is consistent with the expansion of the term involving β conjectured in Eq. (5.35). The multiplicities are given by the $\mathcal{B}_5^{CHL}(\widehat{sl(2)})$ character expansion. The coefficient of $\widetilde{\chi}_{40,20}$ is

$$
\begin{aligned}
f_1^{5,10}(\tau) &= T_5(\tau)(q^2 + 2q^{11/5} + 5q^{12/5} + 12q^{13/5} + 27q^{14/5} + 114q^3 + O(q^{16/5}) \,, \\
&= q^2 + 2q^{13/5} + 3q^{14/5} + 63q^3 + O(q^{16/5}) \,.
\end{aligned}
$$

The other potential real roots associated with $q^{11/5}$ and $q^{12/5}$ do not appear.

$N = 6$

The coefficients of the Fourier series $T_6(\tau)$ give the multiplicity of the imaginary simple roots δ_6' and δ_6''. The coefficient of q^y gives the multiplicity of the roots $y\,\delta_6'$ and $y\,\delta_6''$. One has

$$
\begin{aligned}
T_6(\tau) &= \prod_{j=1}^{\infty} \left(1 - q^{j/2}\right)\left(1 - q^{j/3}\right)\left(1 - q^{j/6}\right)\left(1 - q^{j}\right)^{-2} \\
&= 1 - \mathbf{1}\, q^{1/6} - \mathbf{2}\, q^{1/3} + \mathbf{0}\, q^{1/2} + \mathbf{1}\, q^{2/3} + \mathbf{4}\, q^{5/6} + \mathbf{1}\, q + O(q^{7/6}) \,.
\end{aligned}
$$

These appear as the leading coefficient in the bottom row of each vvmf $\mathbf{g}^{6,m}$ for $m = 1, \ldots, 6$.

$$\mathbf{g}^{6,1}(\tau) = \begin{pmatrix} q^{-1/4}(q^{1/6} + 2q^{1/3} + \cdots) \\ q^{1/12}(-\mathbf{1} - 3q^{1/6} - 6q^{2/6} + \cdots) \end{pmatrix}$$

In the first row, it is $q^{1/6}\tilde{\chi}_{4,2}$ whose weight vector is $(\frac{\delta}{6} - \alpha_2 + \delta_6') = \frac{\delta}{6} + \alpha_3^{(1)}$ which has norm $2/3$. This again is a real fermionic root.

$$\mathbf{g}^{6,2}(\tau) = \begin{pmatrix} q^{-1/2}(q^{1/3} + q^{1/2} + \cdots) \\ q^{-1/10}(q^{1/6} - 2q^{1/2} + \cdots) \\ q^{1/10}(-\mathbf{2} + -3q^{1/6} - 5q^{1/3} + q^{1/2} \cdots) \end{pmatrix}$$

$$\mathbf{g}^{6,3}(\tau) = \begin{pmatrix} q^{-3/4}(q^{1/2} + q^(2/3) + \cdots) \\ q^{-9/28}(q^{3/6} - 3q^{4/6} + \cdots) \\ q^{-1/28}(-q^{1/6} - 5q^{2/6} - 14q^{3/6} - 12q^{4/6} + \cdots) \\ q^{3/28}(3q^{1/6} + 15q^{2/6} + 30q^{3/6} + 44q^{4/6} + \cdots) \end{pmatrix}$$

$$\mathbf{g}^{6,4}(\tau) = \begin{pmatrix} q^{-1}(q^{4/6} + \cdots) \\ q^{-5/9}(2q^{4/6} + \cdots) \\ q^{-2/9}(q^{2/6} - 3q^{3/6} + \cdots) \\ (-3q^{1/6} - 8q^{2/6} - 2(q^{3/6}) + \cdots) \\ q^{1/9}(\mathbf{1} + 9q^{1/6} + 19q^{2/6} + 21q^{3/6} + \cdots) \end{pmatrix}$$

$$\mathbf{g}^{6,5}(\tau) = \begin{pmatrix} q^{-5/4}(q^{5/6} + +q^{6/6} + \cdots) \\ q^{-35/44}(q^{5/6} + 5q^{6/6} + \cdots) \\ q^{-19/44}(2q^{3/6} + 6q^{4/6} + q^{5/6} + \cdots) \\ q^{-7/44}(-4q^{2/6} - 18q^{3/6} - 42q^{4/6} - 39q^{5/6} + \cdots) \\ q^{1/44}(-3q^{1/6} + 5q^{2/6} + 37q^{3/6} + 87q^{4/6} + 78q^{5/6} + \cdots) \\ q^{5/44}(\mathbf{4} + 9q^{1/6} + 5q^{2/6} - 35q^{3/6} - 93q^{4/6} - 66q^{5/6} + \cdots) \end{pmatrix}$$

$$\mathbf{g}^{6,6}(\tau) = \begin{pmatrix} q^{-3/2}(q^{6/6} + q^{7/6} + 3q^{8/6} + \cdots) \\ q^{-27/26}(q^{7/6} + 15q^{8/6} + \cdots) \\ q^{-17/26}(4q^{5/6} + 13q^{6/6} - 8q^{7/6} + \cdots) \\ q^{-9/26}(4q^{3/6} - 4q^{4/6} - 40q^{5/6} - 106q^{6/6} - 57q^{7/6} + \cdots) \\ q^{-3/26}(-q^{1/6} - 7q^{2/6} - 21q^{3/6} - q^{4/6} + 122q^{5/6} + 303q^{6/6} + \cdots) \\ q^{1/26}(-1 + 3q^{1/6} + 25q^{2/6} + 59q^{3/6} + 35q^{4/6} - 182q^{5/6} - 481q^{6/6} + \cdots) \\ q^{3/26}(\mathbf{1} - 4q^{1/6} - 40q^{2/6} - 100q^{3/6} - 72q^{4/6} + 180q^{5/6} + \cdots) \end{pmatrix}$$

The leading root that appears in $g_5^{6,6}$ is the $\gamma^{(6)}$ image of $q^{1/6}$ and thus appears with the same multiplicity as $q^{1/6}$. The leading term in the first row of all the vvmfs $\mathbf{g}^{6,m}$ for $m = 1, \ldots, 6$ is associated with the simple real root $m\beta$, All appear with multiplicity $+1$ indicating the fermionic nature of the root. These terms again fall in line with the addition of the term (5.35) in Eq. (5.33).

One of the main results of this chapter is the occurrence of some exotic roots which do not fit in the original Borcherds' prescription and considering the examples we have at our disposal we were able to conjecture how one should modify the sum-side of the denominator formulae if such roots are present. Besides through the course of this chapter we came across a class of vector valued modular forms while doing $\widehat{sl(2)}$-character decomposition of some Jacobi forms of our interest. We were also able to determine the closed forms of some of those Jacobi forms which we did not know of beforehand.

As a surprise we learn that one of the Umbral Jacobi form ($\psi_{0,4}$) is just a linear combination of a bunch of $\widehat{sl(2)}$-characters with constant coefficients.

Chapter 6

Vector valued modular forms(vvmfs)

In this chapter we explore, for the two families, modular properties of $\mathcal{G}(\tau)$ (5.3) and try to follow the work of Terry Gannon [22] in order to obtain closed formulae for them.

For the family related to Umbral moonshine we have $t = N$ for $\{\Delta^{(N,N)} | N = 1, 2, 3, 4\}$ and $t = 6$ for Δ_0. We study only five cases associated to $m = 1$ i,e. $\Psi^{(1,1)}_{0,1}$, $\Psi^{(2,2)}_{0,2}$, $\Psi^{(3,3)}_{0,3}$, $\Psi^{(4,4)}_{0,4}$ and $\Psi^{\text{Umb}}_{0,6}$. These are the Jacobi forms under full $SL(2,\mathbb{Z})$.

6.1 Vvmfs in the first family

The decomposition of Umbral Jacobi forms in terms of $\widehat{sl(2)}$ characters has given us vvmfs that are weight zero and have multiplier determined by the matrices S and T. We have obtained the first few terms in their Fourier expansions by direct computation. In this section, we will determine them to all orders. For $N \neq 6$, we show that they are solutions to a Matrix Differential Equation (MDE) thereby obtaining explicit analytical formulae for the vvmfs. For $N = 6$, we use a different method to obtain a similar result.

Let $\mathcal{M}^!_w(\rho)$ denote the space of weakly holomorphic vvmf with multiplier ρ of weight w

and rank d. Further, let

$$S = \rho\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad T = \rho\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad , \quad U = \rho\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = ST^{-1} \, .$$

For $j = 0, 1$, let a_j denote the multiplicity of the eigenvalue $(-1)^j$ of S and for $j = 0, 1, 2$ let b_j denote the multiplicity of the eigenvalue $\exp(2\pi i j/3)$ of U. Further, let us assume that T is diagonal

$$T = \exp(2\pi i\Lambda) \quad , \quad \text{where } \Lambda = \text{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_d) \, . \tag{6.1}$$

The exponents λ_i are only defined modulo one. In some situations, the exponents can be fixed. Let $\mathbf{g}(\tau) \in \mathcal{M}^!_w(\rho)$ have the following Laurent series

$$\mathbf{g}(\tau) = q^\Lambda \sum_{n\in\mathbb{Z}} \mathbf{a}_n \, q^n \, . \tag{6.2}$$

Define the principal part map, $\mathcal{P}_\Lambda : \mathcal{M}^!_w(\rho) \to \mathbb{C}^d[q^{-1}]$ as follows:

$$\mathcal{P}_\Lambda(\mathbf{g}) = \sum_{n\leq 0} \mathbf{a}_n \, q^n \, . \tag{6.3}$$

Proposition 6.1.1 (Gannon [22])**.** *Let (ρ, w) be admissible (see A.9) and T diagonal. Then, there exists a choice of exponents Λ for which the principal part map $\mathcal{P}_\Lambda : \mathcal{M}^!_w(\rho) \to \mathbb{C}^d[q^{-1}]$ is a vector space isomorphism.*

Using an index theorem argument, Gannon shows that a necessary but not sufficient condition for the bijectivity described above is

$$\sum_{k=1}^{d} \lambda_k = c(\rho, w) \, , \tag{6.4}$$

where $c(\rho, w) := \frac{wd}{12} - \frac{a_1}{2} - \frac{b_1+2b_2}{3}$. In all our examples, we made choices that satisfied the above condition and for $N \leq 4$ found choices such that the bijection holds.

Theorem 6.1.2 (Theorem 3.3(b) of Gannon [22])**.** *Let (ρ, w) be admissible with rank d, T diagonal and Λ be bijective. Further, let*

$$\Xi(\tau) := \big(\mathbf{g}_1(\tau), \mathbf{g}_2(\tau), \ldots, \mathbf{g}_d(\tau)\big) = q^{\Lambda}\,\left(\mathbf{1}_d + \chi\, q + O(q^2)\right)$$

denote the $d \times d$ matrix whose columns are a basis for $\mathcal{M}^!_w(\rho)$. Then, $\Xi(\tau)$, solves the Matrix Differential Equation (MDE) of the form:

$$\nabla_{1,w}\,\Xi(\tau) = \Xi(\tau)\big((J(\tau) - 984)\,\Lambda_w + \chi_w + [\Lambda_w, \chi_w]\big) \quad , \tag{6.5}$$

where $\Lambda_w = \Lambda - \frac{w}{12}\mathbf{1}_d$ and $\chi_w = \chi + 2w\mathbf{1}_d$.

In all our examples, one column of $\Xi(\tau)$ is obtained from the vvmf that we obtain in Sec. 5.1.1 from the $\widehat{sl(2)}$ decomposition of the Umbral Jacobi forms. We use the Fourier coefficents of the known vvmf to determine the MDE.

6.1.1 Identifying the MDE for vvmfs of interest

The data entering the MDE of Gannon are the following:

1. The pair (ρ, w),
2. an invertible set of exponents Λ, and
3. the $d \times d$ matrix χ defined by

$$\Xi(\tau) = q^{\Lambda}\,\left(\mathbf{1}_d + \chi\, q + O(q^2)\right) . \tag{6.6}$$

For all our examples, the weight $w = 0$ and the multiplier ρ is known. The unknowns are an invertible Λ and χ. Instead we know a solution to the MDE to any order that we desire. We assume that our solution corresponds to one column of $\Xi(\tau)$ – this leads to a choice of Λ and determines one column of χ. We thus have $(d^2 - d)$ unknowns that we determine by using higher orders of the known solution. This method works for $N = 1, 2, 3$ but not for $N = 6$ in part due to the higher dimensionality of the problem. There are no vvmfs for $N = 4$ as the coefficients are constants and there is nothing more to do.

$N = 1$

This is one of rank two and thus there are only two unknown constants to fix. We choose $\Lambda = \text{Diag}(-1/4, -11/12)$ and obtain

$$\chi = \begin{pmatrix} -84 & 32076 \\ 9 & 88 \end{pmatrix} . \tag{6.7}$$

This first column agrees with the $O(q)$ term in $g_1(\tau)$ and $g_2(\tau)$.

The expression for the full vvmf can be expressed in terms of the hypergeometric function [73]

$$\Xi(\tau) = \begin{pmatrix} f(-1/4, 7/6; z) & 32076 f(3/4, 7/6; z) \\ 9 f(1/12, 7/6; z) & f(-11/12, 7/6; z) \end{pmatrix} , \tag{6.8}$$

where $z(\tau) = J(\tau)/1728$ and

$$f(a, c; z) = (1728z)^{-a} \;{}_2F_1(a, a + 2/3; 2a + c; z^{-1}) .$$

The first column of $\Xi(\tau)$ is our solution. Thus, we can obtain the q-series for the vvmf associated with $A^{(1)}$ to arbitrary order.

$N = 2$

We have a rank three vvmf with the exponents $\lambda_1 = 1/2 \mod \mathbb{Z}$, $\lambda_2 = -1/10 \mod \mathbb{Z}$ and $\lambda_3 = 1/10 \mod \mathbb{Z}$, We find that the following exponents lead to an invertible Λ.

$$\lambda_1 = -1/2 \, , \; \lambda_2 = -1/10 \, , \; \lambda_3 = -9/10 \, .$$

The eigenvalues of S are $(1, 1, -1)$ and thus $a_0 = 2$, $a_1 = 1$. The eigenvalues of U are $1, \exp(2\pi i/3), \exp(4\pi i/3)$ and thus $b_0 = b_1 = b_2 = 1$. From this, we see that $\sum_i \lambda_i =$

$-3/2 = c(\rho, 0)$ where $c(\rho, 0) = -a_1/2 - (b_1 + 2b_2)/3$.

The complete solution is given by the data

$$\chi = \begin{pmatrix} 222 & -10 & 4590 \\ -1275 & 3 & 42483 \\ 25 & 3 & 27 \end{pmatrix} , \tag{6.9}$$

and

$$\Xi(\tau) = \begin{pmatrix} G(\lambda_1, \lambda_2, \lambda_3; z(\tau)) & G(\lambda_1+1, \lambda_2-1, \lambda_3; z(\tau)) & G(\lambda_1+1, \lambda_2, \lambda_3-1; z(\tau)) \\ G(\lambda_2+1, \lambda_1-1, \lambda_3; z(\tau)) & G(\lambda_2, \lambda_1, \lambda_3; z(\tau)) & G(\lambda_2+1, \lambda_1, \lambda_3-1; z(\tau)) \\ G(\lambda_3+1, \lambda_1-1, \lambda_2; z(\tau)) & G(\lambda_3+1, \lambda_1, \lambda_2-1; z(\tau)) & G(\lambda_3, \lambda_1, \lambda_2; z(\tau)) \end{pmatrix}, \tag{6.10}$$

where

$$G(a, b, c; z) := (1728z)^{-a} \, {}_3F_2\big(a, a+1/3, a+2/3; a-b, a-c; z^{-1}\big) .$$

The second column of $\Xi(\tau)$ is our vvmf and is expressed in terms of generalized hypergeometric functions. This is true only for ranks ≤ 3.

$N = 3$

This is a rank 4 case and hence we do not anticipate that the solution can be expressed in terms of generalized hypergeometric functions. We choose the following exponents:

$$\lambda_1 = -3/4 \quad , \quad \lambda_2 = -25/28 \quad , \quad \lambda_1 = -1/28 \quad , \quad \lambda_4 = -9/28 \quad .$$

The multiplicity of eigenvalues of S and U are

$$a_0 = a_1 = 2 \quad , \quad b_0 = 2 \quad , \quad b_1 = b_2 = 1 \; .$$

The choice of exponents satisfies the condition $\sum_i \lambda_i = c(\rho, 0) = -2$. We obtain

$$\chi = \begin{pmatrix} -150 & 550 & -2 & 36 \\ 49 & 25 & 1 & 15 \\ -10829 & 37400 & 2 & -104 \\ 2499 & 10625 & -1 & -117 \end{pmatrix} \tag{6.11}$$

The solution to the matrix DE is the following

$$q^{\Lambda} \begin{pmatrix} 1-150q-39249q^2-1624394q^3 & 550q+248490q^2+15046550q^3 & -2q-2q^2-4q^3 & 36q+918q^2+9284q^3 \\ 49q+20874q^2+1007244q^3 & 1+25q+27625q^2+1978625q^3 & q+q^2+3q^3 & 15q+576q^2+6183q^3 \\ -10829q-614754q^2-14799078q^3 & 37400q+3220140q^2+106417025q^3 & 1+2q+3q^2+5q^3 & -104q-1107q^2-8181q^3 \\ 2499q+217854q^2+6319628q^3 & 10625q+1485800q^2+60356369q^3 & -q-q^3 & 1-117q-1647q^2-13461q^3 \end{pmatrix} + O\left(q^4\right)$$

where the third column is the vvmf of interest. We have checked that column three of the above matrix agrees with expressions for $(g_1, \ldots, g_4)$ to $O(q^{16})$. Thus, even though we do not have simple expression in terms of hypergeometric functions as before, we have identified the MDE that the vvmf satisfies. We can easily solve the recursion relation to obtain the q-series to fairly high orders.

$N = 6$

We choose the exponents as follows:

$$(\lambda_1, \ldots, \lambda_7) = \left(-\frac{1}{2}, -\frac{1}{26}, -\frac{17}{26}, -\frac{9}{26}, -\frac{3}{26}, -\frac{25}{26}, -\frac{23}{26}\right) \tag{6.12}$$

with $\sum\limits_i \lambda_i = -\frac{7}{2}$. The multiplicity of eigenvalues of S and U are

$$a_0 = 4 \quad , \quad a_1 = 3 \quad , \quad b_0 = 3 \quad , \quad b_1 = b_2 = 2 \, .$$

Hence $c_{\rho,0} = -7/2 = \sum_i \lambda_i$. However, we have not been able to determine whether the choice of exponents is bijective. The problem is the large number of constants that need to be determined using the data from the known vvmf. Using the action of $\nabla_{i,w}$ for $i = 1, 2, 3$, we can generate three linear combinations of the solutions. This leaves us with

21 unknown constants and this space is too large for us to solve on a computer. Hence we chose an alternate method to get an all orders formula for the vvmf that we discuss next.

6.1.2 Determining an explicit formula for the $N = 6$ vvmf

We observe that the theta expansion of the Umbral Jacobi form takes a very simple form after dividing out by a factor of $\eta(\tau)$.

$$\begin{aligned}\psi_{0,6}(\tau,z) &= \frac{1}{\eta(\tau)}(\theta_{24,2}(\tau,z)+\theta_{24,26}(\tau,z)+\theta_{24,22}(\tau,z)+\theta_{24,46}(\tau,z) \\ &\qquad -\theta_{24,10}(\tau,z)-\theta_{24,34}(\tau,z)-\theta_{24,14}(\tau,z)-\theta_{24,38}(\tau,z)) \qquad (6.13)\\ &= \frac{1}{\eta(\tau)}(\mathcal{M}_{24,2}(\tau,z)+\mathcal{M}_{24,22}(\tau,z)-\mathcal{M}_{24,10}(\tau,z)-\mathcal{M}_{24,14}(\tau,z)\ , \qquad (6.14)\end{aligned}$$

where $\mathcal{M}_{k,m}(\tau,z) = \theta_{k,m}(\tau,z) + \theta_{k,-m}(\tau,z)$. Kac and Peterson [74, see section 5.5] express the characters of $\widehat{sl(2)}$ in terms of theta functions that appear above. The transformation matrix is given by Hecke modular forms. Explicitly, one has

$$\chi_{k,\lambda}(\tau,z) = \sum_{\substack{0\le n<2m \\ n\equiv\lambda \bmod 2}} \frac{C^{(k)}_{\lambda,n}(\tau)}{\eta(\tau)^3}\,\theta_{k,n}(\tau,z)\ , \qquad (6.15)$$

where $C^{(k)}_{\lambda,n}$ is defined in terms of Hecke indefinite modular forms as follows:

$$C^{(k)}_{\lambda,n}(\tau) = \sum_{\substack{(x,y)\in\mathbb{R}^2 \\ -|x|<y\le|x| \\ (x,y)\text{ or }\left(\frac{1}{2}-x,\frac{1}{2}+y\right)\in\left(\frac{\lambda+1}{2(k+2)},\frac{n}{2k}\right)+\mathbb{Z}^2}} \text{sign}(x)\, q^{(k+2)x^2-ky^2}\ . \qquad (6.16)$$

We need to express the theta functions in terms of $\widehat{sl(2)}$ characters. This is given by

$$\mathcal{M}_{k,n}(\tau,z) = \sum_{\substack{0\le\lambda<2k \\ \lambda\equiv n \bmod 2}} \mathcal{D}^{(k)}_{n,\lambda}(\tau)\,\chi_{k,\lambda}(\tau,z)\ , \qquad (6.17)$$

where

$$\mathcal{D}^{(k)}_{n,\lambda}(\tau) = \sum_{\substack{m\in\mathbb{Z} \\ m\equiv\pm n \bmod 2k}} (-1)^{\frac{\lambda+m}{2}}\, q^{\frac{k(k+2)}{8}\left(\frac{m}{k}+\frac{\lambda+1}{k+2}\right)^2}\ . \qquad (6.18)$$

Substituting Eq. (6.17) in Eq. (6.14), we obtain

$$\begin{aligned}\psi_{0,6}(\tau,z) &= \sum_{\lambda\equiv 0 \bmod 2} \frac{(\mathcal{D}^{(24)}_{2,\lambda}(\tau)+\mathcal{D}^{(24)}_{22,\lambda}(\tau)-\mathcal{D}^{(24)}_{10,\lambda}(\tau)-\mathcal{D}^{(24)}_{14,\lambda}(\tau))}{\eta(\tau)}\chi_{24,\lambda}(\tau,z)\\ &= \sum_{\lambda \text{ even}}\left(\sum_{\substack{m\in\mathbb{Z}\\ m\equiv\pm 10 \bmod 24}} - \sum_{\substack{m\in\mathbb{Z}\\ m\equiv\pm 2 \bmod 24}}\right)\frac{(-1)^{\lambda/2}q^{78\left(\frac{m}{24}+\frac{\lambda+1}{26}\right)^2}}{\eta(\tau)}\chi_{24,\lambda}(\tau,z) \qquad (6.19)\end{aligned}$$

Comparing the above expression with Eq. (5.17), we obtain explicit formulae for $(g_1(\tau),\ldots,g_7(\tau))$ that agree with the expressions to the order that we have determined them. We thus have obtained explicit formulae for the vvmf associated with $N=6$ even though we have not determined the MDE satisfied by the vvmf.

6.1.3 Interpreting the vvmfs

We have seen that the vvmf that we denote by $\mathbf{g}(\tau)$ captures the contribution of simple roots. Combining this result with the invariance of the Siegel modular forms under the action of the dihedral group, we obtain formulae that extend our results. The Fourier-Jacobi expansion of the Siegel modular form is compatible with the action of the subgroup $\langle w_2,\widehat{\delta}\rangle$ as these are realised as elements of the Jacobi group which preserve the cusp at $\tau'=i\infty$. The generator $\gamma^{(N)}$ does not belong to the Jacobi group. Including its action on the Umbral Jacobi form its decomposition into $\widehat{sl(2)}$ characters enables us to organise the result in terms of orbits of the extended Weyl group.

The Siegel modular form $\Delta_{k(N)}(\mathbf{Z})$ can be written as a sum of terms of the kind that follow from our $\widehat{sl(2)}$ decomposition of the Umbral Jacobi form.

1. The real roots α_0 and α_3 are accounted from the expansion of

$$\sum_{w\in W}\det(w)\,w(e^{-\varrho^{(N)}})$$

2. Let us denote the set of imaginary roots with zero norm $(\alpha_1+\alpha_2)$, $(\alpha_0+\alpha_1)$, $(\alpha_2+\alpha_3)$ and their $\gamma^{(N)}$ translates by S_0, These appear in the expansion of a Borcherds

correction of the form

$$\sum_{w\in W}\det(w)\ \sum_{a\in S_0}\sum_{n=1}^{\infty}\sigma_N(n)\ w(e^{-\varrho^{(N)}+na}) \tag{6.20}$$

where $\sigma_N(n)$ is defined as follows:

$$\prod_{m=1}^{\infty}(1-q^m)^{3(4-N)/N} = 1+\sum_{n=1}^{\infty}\sigma_N(n)\ q^N\ . \tag{6.21}$$

It is easy to see that $\sigma_4(n) = 0$ for all $n > 1$ and $\sigma_6(n) = p(n)$ where $p(n)$ is the number of partitions of the positive integer n.

3. For $N \leq 4$, all other terms correspond to imaginary simple roots and provide Borcherds correction terms. To see how to do this, consider a term of the form

$$g(\tau)\,\chi_\Lambda(\tau,z) = \sum_{m=0}^{\infty} b(m)\,q^m\ \chi_\Lambda(\tau,z)\ .$$

The m-th term in the above sum is associated with the $\widehat{sl(2)}$ weight vector $(\Lambda+m\delta)$ with multiplicity $b(m)$. A W-covariant expression that accounts for these roots is

$$\sum_{w\in W}\det(w)\ w\left(\sum_{m=1}^{\infty}b(m)\ e^{-\varrho-\Lambda+m\delta}\right)\ . \tag{6.22}$$

4. The case of $N = 6$ needs special attention. First, we get new simple real roots that we denoted by $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$. The Siegel modular form is not invariant under Weyl reflections generated by these roots. Further $\langle\rho^{(N)},\widetilde{\alpha}_i\rangle = +1$ and not equal to -1. The multiplicity on the product side is -1 and hence they are fermionic roots. It appears that the term that we obtain arises as follows:

$$\sum_{w\in W}\det(w)\ w\left(\frac{e^{-\varrho^{(N)}}}{(1-e^{-\widetilde{\alpha}_i})}\right) = \sum_{w\in W}\det(w)\ w\left(e^{-\varrho^{(N)}}(1+e^{-\widetilde{\alpha}_i}+\cdots)\right) \tag{6.23}$$

The above formula is conjectural as we have not checked if the pieces indicated by the ellipsis do appear. We also see that further imaginary roots involving the tilde roots also appear. They are of the form $(\widetilde{\alpha}_i + \alpha_j)$ for $i, j = 1, 2$. These do not appear in the set of positive roots that we obtain from the product formula. This is also true for the weights associated with labels 5 and 7. There is a cancellation of the form $1 - 1 = 0$. We can see a similar cancellation in the WKB denominator formula for $\mathcal{B}_6(\widehat{sl(2)})$. The root $\delta = (\alpha_1 + \alpha_2)$ does not appear on the product side. This is because this root appears as a non-simple bosonic imaginary root as well as a fermionic imaginary simple root (with the same weight). This suggests that the positive roots given by the product formula is incomplete and we need to take into account cancellations that occur. Our decomposition in terms of $\widehat{sl(2)}$ characters is able to account for this.

6.2 Vvmfs in the second family

In the last section of previous chapter, we obtained vector-valued modular forms of the congruence group $\Gamma^0(N)$. We would like to obtain closed formulae for the Fourier coefficients of these modular forms. In [1], this was done by showing that the vvmfs satisfied a modular differential equation. However, those examples involved modular forms of the full modular group, $PSL(2,\mathbb{Z})$. So we construct vector-valued modular forms for the whole group following a two-step procedure[1] First, we convert the Jacobi forms of $\Gamma^0(N)$ into Jacobi forms of the full modular group. We obtain vector-valued Jacobi forms in this fashion. Next, we carry out the character decomposition of the these vector-valued Jacobi forms and obtain vector-valued modular forms of the whole modular group. The price we pay is that the rank of the vector-valued modular forms increases by the index of the subgroup in $PSL(2,\mathbb{Z})$.

6.2.1 Vector-Valued Jacobi Forms

The Jacobi forms $\Psi^{(N)}_{0,m}(\tau, z)$ belong to $J_{0,m}(\Gamma^0(N))$. The Jacobi forms, obtained by the action of S, $\psi^{(N)[1,g]}_{0,m}(\tau, z)|S \in J_{0,m}(\Gamma_0(N))$. For prime $N = 2, 3, 5$, there are two cusps of

[1]We learned this method from the work of Borcherds who obtains modular forms for the full modular group in this fashion [75]. This procedure is called lifting by Bajpai in [76].

width 1 and N respectively. We restrict our discussion to only these three cases. We form a rank $(N+1)$ vector-valued Jacobi Form (vvJF) of the full modular group, $PSL(2,\mathbb{Z})$. Let $\psi \equiv \Psi^{(N)}_{0,m}(\tau,z)$ and define

$$\widetilde{\mathcal{V}}(\psi) = \begin{pmatrix} \psi(\tau,z)|S \\ \psi(\tau,z) \\ \psi(\tau,z)|T \\ \vdots \\ \psi(\tau,z)|T^{N-1} \end{pmatrix} .$$

The first entry is the contribution from the cusp at infinity and the other N are the contribution from the cusp at zero. Note that $T^N = 1$ at the cusp at zero. The vvJF, $\widetilde{\mathcal{V}}$, is reducible with T having an off-diagonal action. We first make a change of basis so that T is diagonal. Consider the Jacobi forms (with $\omega_N = \exp(2\pi i/N)$)

$$\widetilde{\psi}_i(\tau,z) = \frac{1}{N}\sum_{j=0}^{N-1} \omega_N^{ij}\, \psi(\tau,z)|T^j\ , \quad i = 0,1,\ldots,(N-1) \mod N \tag{6.24}$$

Now T has a diagonal action i.e.,

$$\widetilde{\psi}_i(\tau,z)|T = (\omega_N)^i\, \widetilde{\psi}_i \quad \text{and} \quad \psi(\tau,z)|ST = \psi(\tau,z)|S\ .$$

The rank $(N+1)$ vvJF $\widetilde{\mathcal{V}}$ is reducible and decomposes into a Jacobi form for the full modular group and another one that is a rank N vvJF. The rank one Jacobi Form is given by the combination

$$\mathcal{A}^{(N)}(\tau,z) := \psi(\tau,z)|S\ + N\widetilde{\psi}_0(\tau,z)\ . \tag{6.25}$$

and the irreducible rank N vvJF is given by

$$\mathcal{V}^{(N)}(\psi) := \begin{pmatrix} \psi(\tau,z)|S - \widetilde{\psi}_0(\tau,z) \\ \widetilde{\psi}_1(\tau,z) \\ \vdots \\ \widetilde{\psi}_{N-1}(\tau,z) \end{pmatrix} . \tag{6.26}$$

The T matrix of the vvJF is

$$T_V = \mathrm{diag}(1, \omega_N, \ldots, (\omega_N)^{N-1})$$

and the S-matrix can be obtained from the following formulae.

$$\big(\psi(\tau,z)|S - \widetilde{\psi}_0(\tau,z)\big)\,|S = -\frac{1}{N}\big(\psi(\tau,z)|S - \widetilde{\psi}_0(\tau,z)\big) + \frac{N+1}{N}\sum_{j=1}^{N-1}\widetilde{\psi}_j(\tau,z)$$

$$\psi(\tau+j,z)|S = \psi(\tau-j',z) \text{ where } j \neq 0 \text{ and } jj' = 1 \bmod N\,.$$

For fixed N, the S-matrix is independent of the index of the Jacobi form, $\Psi_{0,m}(\tau,z)$ We thus give the S-matrices for the three cases of interest.

$$S_V^{N=2} = \frac{1}{2}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \quad , \quad S_V^{N=3} = \frac{1}{3}\begin{pmatrix} -1 & 4 & 4 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix} \tag{6.27}$$

$$S_V^{N=5} = \frac{1}{5}\begin{pmatrix} -1 & 6 & 6 & 6 & 6 \\ 1 & \frac{1}{2}\left(3-\sqrt{5}\right) & -1-\sqrt{5} & -1+\sqrt{5} & \frac{1}{2}\left(3+\sqrt{5}\right) \\ 1 & -1-\sqrt{5} & \frac{1}{2}\left(3+\sqrt{5}\right) & \frac{1}{2}\left(3-\sqrt{5}\right) & -1+\sqrt{5} \\ 1 & \sqrt{5}-1 & \frac{1}{2}\left(3-\sqrt{5}\right) & \frac{1}{2}\left(3+\sqrt{5}\right) & -1-\sqrt{5} \\ 1 & \frac{1}{2}\left(3+\sqrt{5}\right) & -1+\sqrt{5} & -1-\sqrt{5} & \frac{1}{2}\left(3-\sqrt{5}\right) \end{pmatrix} \tag{6.28}$$

6.2.2 Vector-valued modular forms

The procedure of the previous sub-section can be applied to all the Jacobi forms, $\Psi^{(N)}_{0,m}(\tau,z)$. In the process we obtain one weight zero modular form that we denote by $\mathcal{A}^{(N)}_m$ and a vvmf of weight zero and rank N that we denote by $\mathcal{V}^{(N)}_m$ in obvious notation.

One can decompose the rank m Jacobi form $\mathcal{V}^{(N)}_m$ in terms of $\widehat{sl(2)}$ characters, $\chi_{4m,2\ell}$ for $\ell = 0,\ldots,2m$ to obtain a rank $(m+1)N$ vector-valued modular form for the full modular group, $PSL(2,\mathbb{Z})$. Since the rank grows fast, we will first study the $N = 2$ case where we get vvmfs of rank 4 and rank 6. We are able to completely characterize the rank 4 example. The decomposition is as follows: (with $x = (N-1)(m+1)$)

$$\mathcal{V}^{(N)}_m = \begin{pmatrix} g_1\,\chi_{4m,2m} + g_2\,(\chi_{4m,2m-2} + \chi_{4m,2m+2}) + \cdots + g_{m+1}\,(\chi_{4m,0} + \chi_{4m,4m}) \\ \vdots \\ g_{x+1}\chi_{4m,2m} + g_{x+2}\,(\chi_{4m,2m-2} + \chi_{4m,2m+2}) + \cdots + g_{N(m+1)}(\chi_{4m,0} + \chi_{4m,4m}) \end{pmatrix},$$

which leads to the vvmf $\mathcal{G} = (g_1, g_2, \ldots, g_{N(m+1)})^T$. The S and T matrices are, however, easy to write out. Let $S^{(m)}_\chi$ and $T^{(m)}_\chi$ denote the matrices obtained from scalar Jacobi forms of index m as was considered in paper I [1]. Then, the S-matrix for the vvmf obtained from $\mathcal{V}^{(N)}_m$ is given by

$$T = T^{(N)}_V \otimes T^{(m)}_\chi \quad \text{and} \quad S = S^{(N)}_V \otimes S^{(m)}_\chi \tag{6.29}$$

In this fashion, we obtain the data needed to determine the modular differential equation of Gannon [22].

An example

Consider $\mathcal{V}_1^{(2)}$ which leads to a rank 4 example. We obtain the following T and S matrices.

$$T = \text{diag}\left(e^{-\frac{i\pi}{2}}, e^{\frac{i\pi}{6}}, e^{\frac{i\pi}{2}}, e^{-\frac{i5\pi}{6}}\right) \quad , \quad S = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & -2 & -3 & 6 \\ -1 & -1 & 3 & 3 \\ -1 & 2 & -1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{6.30}$$

The first few terms in the Fourier expansion of the vvmf are given below.

$$\begin{pmatrix} q^{-1/4}\left(1 + 36q + 375q^2 + 2162q^3 + 10017q^4 + 38550q^5 + 132446q^6 + 413478q^7 + \cdots\right) \\ q^{-11/12}\left(-3q - 93q^2 - 681q^3 - 3723q^4 - 15879q^5 - 58974q^6 - 195186q^7 + \cdots\right) \\ q^{-3/4}\left(-8q - 128q^2 - 936q^3 - 4784q^4 - 19968q^5 - 72432q^6 - 236392q^7 + \cdots\right) \\ q^{-5/12}\left(24q + 264q^2 + 1656q^3 + 7848q^4 + 31104q^5 + 108552q^6 + 343992q^7 + \cdots\right) \end{pmatrix}$$

Equipped with this data, we can determine the matrix differential equation of Gannon [22] to which $\mathcal{G}$ is one of the independent solutions. The data that we need for a rank d situation are the following:

1. an invertible set of exponents Λ, and
2. a $d \times d$ matrix χ defined by

$$\Xi(\tau) := (\mathcal{G}_1(\tau), \mathcal{G}_2(\tau), \ldots, \mathcal{G}_d(\tau)) = q^{\Lambda}\left(\mathbf{1}_d + \chi\, q + O(q^2)\right) \tag{6.31}$$

For our rank four example, we obtain

$$\Lambda = \left(-\frac{1}{4}, -\frac{11}{12}, -\frac{3}{4}, -\frac{5}{12}\right) \text{ and} \tag{6.32}$$

$$\chi = \begin{pmatrix} -8400 & 1296 & 36 & -15876 \\ 72 & 24 & -3 & -32 \\ -102 & 54 & -8 & 432 \\ 1125 & 106 & 24 & 2800 \end{pmatrix} . \tag{6.33}$$

leading to the four solutions (column 3 is our solution)

$$q^\Lambda = \begin{pmatrix} -8400q - 651744q^2 - 17978112q^3 & 1296q + 28512q^2 + 311040q^3 & 1 + 36q + 375q^2 + 2162q^3 & -15876q - 2094498q^2 - 84825468q^3 \\ 72q + 43056q^2 + 2127528q^3 & 24q + 2064q^2 + 33336q^3 & -3q - 93q^2 - 681q^3 & 1 - 32q - 50161q^2 - 3921788q^3 \\ 1 - 102q - 30051q^2 - 1240398q^3 & 54q + 2268q^2 + 33372q^3 & -8q - 128q^2 - 936q^3 & 432q + 228096q^2 + 14648688q^3 \\ 1125q + 115650q^2 + 3602097q^3 & 1 + 106q + 3047q^2 + 35814q^3 & 24q + 264q^2 + 1656q^3 & 2800q + 518224q^2 + 24040112q^3 \end{pmatrix} + O(q^4)$$

Other examples

We are unable to determine the modular differential equations in the other cases. The next lowest rank is six and we need to numerically determine twelve unknown constants. Our attempts to numerically determine the modular differential equation failed. Until rank four, it is easy to determine the modular differential equation making use of an observation of Gannon in [22] which enables us to generate three linearly independent solutions given a solution. This puts rank five within reach of numerical computation.

6.2.3 The Jacobi Forms $\mathcal{A}_m^{(N)}$

The $\mathcal{A}_m^{(N)}$ are Jacobi forms for the full modular group. One can expand these as follows:

$$\mathcal{A}_m^{(N)}(\tau, z) = \sum_{j=0}^{m} h_{2j}(\tau)\, A(\tau, z)^{m-j} B(\tau, z)^j \,, \tag{6.34}$$

where $h_{2j}(\tau)$ ($j = 0, 1, \ldots, m$) are modular forms of weight $2j$. Since the ring of modular forms of $PSL(2, \mathbb{Z})$ is generated by polynomials in $E_4(\tau)$ and $E_6(\tau)$, we can characterize $\mathcal{A}_m^{(N)}(\tau, z)$ by a few constants. $h_2(\tau) = 0$ since there is no weight two modular form for the full modular group.

In this fashion, we can show that

$$\mathcal{A}_1^{(N)}(\tau,z) = A(\tau,z) = \psi_{0,1}(\tau,z) \text{ for } N = 2,3,5\ ,$$
$$\mathcal{A}_2^{(2)}(\tau,z) = -\psi_{0,2}\tau,z)\ .$$

In these two cases we obtain Umbral Jacobi forms defined in Eq. (A.23). That is not true in general. For instance,

$$\mathcal{A}_2^{(3)} = 0\ ,$$
$$\mathcal{A}_3^{(3)} = \frac{1}{216}A(\tau,z)^3 + \frac{5}{72}E_4(\tau)A(\tau,z)B(\tau,z)^2 - \frac{2}{27}E_6(\tau)B(\tau,z)^3 \neq \psi_{0,3}(\tau,z)\ .$$

$\mathcal{A}_3^{(3)}$ is however a linear combination of two solutions of the matrix differential equation satisfied by the umbral Jacobi form [1]. We are not presenting the Jacobi forms that appear for $N = 5$.

This chapter is dedicated to find the closed formulae for the vvmfs appeared in the previous chapter. We were able to achieve that, by following the work Terry Gannon, as long as the vvmf is modular under full $SL(2,\mathbb{Z})$ and the rank of the vvmf is less then four.
Using the fact that $\psi_{0,6}$ can be written as linear combination of some classical theta functions upto a factor of $1/\eta(\tau)$ which by the way is an astonishing findings in and of itself came up in this chapter, we were able to figure out how to find the closed formulae for a special case of rank seven vvmf associated to Umbral Jacobi form $\psi_{0,6}$, using a prescription due to Kac and Peterson.
Besides in a number of examples the application of Gannon's work helped us to determine all the generators of space of vvmfs of a given rank.

Chapter 7

Conclusion

The main result of this thesis is a preliminary study of the WKB superdenominator formulae associated with BKM Lie superalgebras using a $\widehat{sl(2)}$ subalgebra (and its Borcherds extension). In the first paper [1], we have restricted our study to include the first two additional simple real roots (and corresponding imaginary simple roots) that appear in the first Fourier-Jacobi coefficients of the Siegel modular forms. This leads to an interesting connection with vector-valued modular forms associated with some Umbral Jacobi forms. In all cases, we obtained relatively simple formulae for the Fourier coefficients of the vvmfs. These Fourier coefficients correspond to the multiplicities of simple roots, both imaginary and real, of the BKM Lie superalgebras.

The next step would be to carry out a similar decomposition for all Fourier-Jacobi coefficients. The connection with umbral moonshine gives a second formula for the Siegel modular forms. Extending heuristic arguments given in [17] (see also [77]) for Mathieu moonshine to Umbral moonshine, one has

$$\Delta_{k(N)}(\mathbf{Z}) := s^{1/2}\ \phi_{k(N),1/2}(\tau,z)\ \exp\left[-\sum_{m=1}^{\infty} s^{mN}\ \psi_{0,N}\ |V_m\ (\tau,z)\right]\ , \tag{7.1}$$

where

$$\psi_{0,N}\,|V_m\,(\tau,z) = \frac{1}{m}\sum_{ad=m}\sum_{b=0}^{d-1}\psi_{0,N}\left(\tfrac{a\tau+b}{d},az\right)\ . \tag{7.2}$$

The same formula also appears in [30, see Eq. (2.7)]. This formula is very useful in obtaining explicit formulae for higher Fourier-Jacobi coefficients of the $\Delta_{k(N)}(\mathbf{Z})$. For instance, the second coefficient is given by

$$\psi_{0,2N}(\tau,z) = \frac{1}{2}\left(\psi_{0,N}(\tau,z)\right)^2 - \sum_{ad=2}\sum_{b=0}^{1}\psi_{0,N}\left(\tfrac{a\tau+b}{d},az\right)\ . \tag{7.3}$$

The above formula has a nice interpretation. Let $\mathcal{V}_N$ denote a $\widehat{sl(2)}$ module such that (H is the Cartan subalgebra of $\widehat{sl(2)}$)

$$\mathcal{V}_N = \oplus_{\mu\in H} V_\mu\ ,$$

and the Umbral Jacobi form is equal to supercharacter of $\mathcal{V}_N$ i.e.,

$$s^N\,\psi_{0,N}(\tau,z) = \text{Sch}(\mathcal{V}_N) := \sum_{\lambda\in H}\left(\text{dim}V_{0\,\mu} - \text{dim}V_{1\,\mu}\right)\,e^{-\mu}\ .$$

where $V_{0\,\mu}$ (resp. $V_{1\,\mu}$) is the bosonic (resp. fermionic) subspace of $\mathcal{V}_N$ of weight μ. Then, $\psi_{0,2N}(\tau,z)$ is obtained as the supertrace over direct sum of the $\widehat{sl(2)}$ modules: $\Lambda^2\mathcal{V}_N$ and $\mathcal{V}_N^{[2]}$. The latter module $\mathcal{V}_N^{[2]}$ is obtained via the following *scaling procedure* [78]. The Lie subalgebra $\widehat{sl(2)}^{[2]} = sl(2)\otimes\mathbb{C}[t^2,t^{-2}]\oplus\mathbb{C}\,\hat{k}\oplus\mathbb{C}\,d$ of $\widehat{sl(2)}$ is in fact isomorphic to $\widehat{sl(2)}$.[1] The $\widehat{sl(2)}$-module $\mathcal{V}_N$ is $\mathbb{Z}_+$-graded, with the highest weight state being of grade zero and each application of $X\otimes t^{-m}$ increasing the grade by m. The subspace of $\mathcal{V}_N$ comprising its graded pieces of even grade is a module for the subalgebra $\widehat{sl(2)}^{[2]} \cong \widehat{sl(2)}$. This module is denoted $\mathcal{V}_N^{[2]}$. It is easy to see that

$$\text{Sch}\left(\mathcal{V}_N^{[2]}\right) = s^{2N}\sum_{b=0}^{1}\psi_{0,N}\left(\tfrac{\tau+b}{2},z\right)\ .$$

[1]via the isomorphism $X\otimes t^{2m}\mapsto X\otimes t^m$ for all $X\in sl(2), m\in\mathbb{Z}$ and $\hat{k}\mapsto\hat{k}/2,\ d\mapsto 2d$ (*cf.* §**??**).

Formulae such as these will enable us to write explicit formulae using the $\widehat{sl(2)}$ decomposition obtained in the first paper [1]. This should, in principle, enable us to rewrite the sum side of the WKB denominator formula first in terms of $\widehat{sl(2)}$ representations and then in sums where the covariance under the full Weyl group is manifest.

In [30], Gritsenko and Nikulin point out that the $\Delta_{k(N)}(\mathbf{Z})$ for $N = 1, 2, 3, 4$ are three-dimensional generalizations of the Dedekind eta function. Rankin [79] showed that the weight-twelve modular form $\Psi = \eta(\tau)^{24}$ of Γ_1 satisfies the following nonlinear ODE: (see Zagier [80] for a derivation)

$$13\Psi_1^4 + 10\Psi^2\Psi_1\Psi_3 - 24\Psi\Psi_1^2\Psi_2 + 3\Psi^2\Psi_2^2 - 2\Psi^3\Psi_4 = 0\,, \tag{7.4}$$

where $\Psi_p \equiv \frac{d^p\Psi}{d\tau^p}$. Defining

$$y = \frac{1}{2}\frac{d}{d\tau}\log\Psi = \frac{1}{2}\frac{\Psi_1}{\Psi}\,, \tag{7.5}$$

Rankin's ODE becomes the Chazy equation:

$$y''' - 2yy'' + 3(y')^2 = 0\,. \tag{7.6}$$

This nonlinear equation satisfies the Painlevé property and connections with integrable systems (see [81, 82] and references therein). We have found MDE's for vvmf's associated with the Umbral Jacobi forms. Do all these combine to give a nice three-dimensional modular ODE for the logarithm of the Siegel modular forms? In this context, it is known that the logarithmic derivatives of the genus two theta constants satisfy a system of equations. [83, 84]. These methods might help one obtain similar nonlinear modular differential equations for the Siegel modular forms.

In the second paper [2], we have begun a study of the decomposition of the Siegel modular forms $\Delta_{k(N)}^{(N)}(\mathbf{Z})$ as denominator formulae for a Lie algebra under two sub-algebras of a Lie algebra, $\mathcal{B}_N^{CHL}(A^{(N)})$, that we wish to understand. There is a natural product formula that provides the product side of the denominator formula – this provides a description of the positive roots with their multiplicities. The character decomposition that we study is a probe on the sum side of the denominator formula. The work is preliminary as we focused on the first N terms that appear. The $N = 5$ case provides the first example of something new. It is the simple real root that we called β with $e^{-\beta} \sim q^{1/5}rs$. Roots of type $m\beta$ appear consistent with the expansion of

$$\frac{1}{1 - e^{-\beta}} = 1 + \sum_{m=1}^{\infty} e^{-m\beta} .$$

The terms for $m = 1, 2, 3, 4, 5, 10$ that appear in our study agree with the above formula. A preliminary study shows that similar root with $e^{-\beta} \sim q^{1/6}rs$ appears for the $N = 6$ CHL orbifold. We have checked that it again fits the above formula – we have verified that the first six terms do appear with the correct multiplicity. While the evidence for this is compelling, an all-orders proof is lacking. What is the Lie algebraic interpretation of this kind of 'correction' term? There is a conflict between the following two properties of β.

1. The root β has positive norm which suggests that it is a real root and should generate a rank one $osp(1, 2)$ Lie superalgebra.
2. It appears on the sum side like a Borcherds correction term for an isotropic root. It should generate a rank one $sl(1, 2)$ Lie superalgebra.

A resolution of this conflict will go a long way in understanding the Lie superalgebra that we seek.

We also need to work out the cases of $N = 4, 6$. The eventual goal is the following: (i) Rewriting the sum term in terms of orbits of the extended Weyl group, (ii) Verifying that the orbits are indeed Borcherds extensions for $N \leq 4$, (iii) For the $N = 5, 6$ examples, we need to have a good description of *all* terms that don't fit into a Borcherds extension.

The additive lift for the modular forms $\Delta^{(N)}_{k(N)}(\mathbf{Z})$ was studied in [14]. This was done, for a case by case basis, by working out the S-transform of the Hecke operator appearing in an additive lift of Cléry and Gritsenko. It would be interesting to carry it out for *all* cases and obtain a closed formula for the sum side. This might enable us to prove that the examples for $N \leq 4$ are indeed Borcherds extensions of $\mathfrak{g}(A^{(N)})$.

Our approach to arriving at modular differential equations was blighted by the large ranks that appeared when we constructed vvmfs for the full modular group. The ranks grew as $N(m+1)$ – the factor of N coming in this process. Is there a way to write modular differential equations for the congruence subgroup? The work of Bajpai might be a way to proceed [76]. Gottesman has studied rank 2 examples of $\Gamma_0(2)$ in his work [85].

Appendix A

Modular background

A.1 Basic Group theory

Let $\mathbb{H}_1$ denote the upper half-plane and $\mathbb{H}_1^* = \mathbb{H}_1 \cup \mathbb{Q} \cup \{\infty\}$ denote the extended upper half-plane. The group $\Gamma^{(1)} := SL(2,\mathbb{Z})$ acts on $\mathbb{H}_1$ as follows:

$$\gamma \cdot \tau := \frac{a\tau + b}{c\tau + d}\,, \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma^{(1)} \text{ and } \tau \in \mathbb{H}_1\,. \tag{A.1}$$

Let $\mathbb{H}_2$ denote the upper half-space with coordinates $\mathbf{Z} = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$. The group $Sp(4,\mathbb{Q})$ is the set of 4×4 matrices, M, written in terms of four 2×2 matrices A, B, C, D with entries in $\mathbb{Q}$ as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

satisfying $AB^T = BA^T$, $CD^T = DC^T$ and $AD^T - BC^T = I$. This group acts naturally on the Siegel upper half space, $\mathbb{H}_2$, as

$$\mathbf{Z} = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \longmapsto M \cdot \mathbf{Z} \equiv (A\mathbf{Z} + B)(C\mathbf{Z} + D)^{-1}\,. \tag{A.2}$$

The paramodular group at paramodular level t that we denote by Γ_t is defined as follows (we follow [71] for all definitions) (for $t \in \mathbb{Z}_{>0}$):

$$\Gamma_t = \left\{ \begin{pmatrix} * & *t & * & * \\ * & * & * & *t^{-1} \\ * & *t & * & * \\ *t & *t & *t & * \end{pmatrix} \in Sp(4, \mathbb{Q}),\ \text{all } * \in \mathbb{Z} \right\} . \tag{A.3}$$

When $t = 1$, then $\Gamma_1 = Sp(4, \mathbb{Z}) \equiv \Gamma^{(2)}$ is the usual symplectic group.

Let $\Gamma_t^+ = \Gamma_t \cup \Gamma_t V_t$ a normal double extension of Γ_t in $Sp(4, \mathbb{R})$ with

$$V_t = \frac{1}{\sqrt{t}} \begin{pmatrix} 0 & t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & t & 0 \end{pmatrix} , \tag{A.4}$$

with $\det(CZ + D) = -1$. This acts on $\mathbb{H}_2$ as

$$(\tau, z, \tau') \longrightarrow (t\tau', z, \tau/t) . \tag{A.5}$$

The group Γ_t^+ is generated by V_t and its parabolic subgroup

$$\Gamma_t^\infty = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & 1 & * & *t^{-1} \\ * & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_t,\ \text{all } * \in \mathbb{Z} \right\} . \tag{A.6}$$

The Jacobi group is defined by

$$\Gamma^J = (\Gamma_t^\infty \cap Sp(4, \mathbb{Z}))/\pm \mathbf{1}_4 \simeq \Gamma^{(1)} \ltimes H(\mathbb{Z}) . \tag{A.7}$$

The embedding of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ in Γ_t is given by

$$\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \equiv \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{A.8}$$

The above matrix acts on $\mathbb{H}_2$ as

$$(\tau, z, \tau') \longrightarrow \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, \tau' - \frac{cz^2}{c\tau + d} \right) , \tag{A.9}$$

with $\det(C\mathbf{Z} + D) = (c\tau + d)$. The Heisenberg group, $H(\mathbb{Z})$, is generated by $Sp(4, \mathbb{Z})$ matrices of the form

$$[\lambda, \mu, \kappa] \equiv \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{with } \lambda, \mu, \kappa \in \mathbb{Z} \tag{A.10}$$

The above matrix acts on $\mathbb{H}_2$ as

$$(\tau, z, \sigma) \longrightarrow \left(\tau,\ z + \lambda\tau + \mu,\ \tau' + \lambda^2\tau + 2\lambda z + \lambda\mu + \kappa\right) , \tag{A.11}$$

with $\det(C\mathbf{Z} + D) = 1$. It is easy to see that Γ^J preserves the one-dimensional cusp at $\text{Im}(\tau') = \infty$.

A.2 Modular Forms

Let $\mathbb{H} = (\tau \mid \text{Im}(\tau) > 0)$ denote the upper half plane.

Definition A.2.1. *A modular form, of weight k and character χ, is a function $f : \mathbb{H} \to \mathbb{C}$ such that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z})$, one has*

$$f|_k\gamma(\tau) = \chi(\gamma)\, f(\tau) , \tag{A.12}$$

where

$$f|_k\gamma(\tau) := (c\tau + d)^{-k}\, f(\gamma \cdot \tau) ,$$

and $\gamma \cdot \tau = \frac{a\tau+b}{c\tau+d}$.

The level N sub-group $\Gamma_0(N) \subseteq PSL(2,\mathbb{Z})$ comprises those γ with $c = 0 \bmod N$. Similarly, the subgroup $\Gamma^0(N)$ is defined by requiring $b = 0 \bmod N$.

The group $SL(2,\mathbb{Z})$ is generated by two generators that are conventionally called the T and S. One has

$$T : \tau \to \tau + 1 \quad , \quad S : \tau \to -\frac{1}{\tau} .$$

Let $f(\tau)$ be a modular form of $PSL(2,\mathbb{Z})$ with weight k. Then, $f(N\tau)$ is a modular form of $\Gamma_0(N)$ and $f(\tau/N)$ is a modular form of $\Gamma^0(N)$ with weight k [86]. Let j be such that $(j,N) = 1$. Then we have the following two identities that are very useful.

$$\begin{aligned} f\left(\tau/N\right)|_k S &= N^k f\left(N\tau\right) \\ f\left(\tfrac{\tau+j}{N}\right)|_k S &= f\left(\tfrac{\tau-j'}{N}\right) \end{aligned} \tag{A.13}$$

with $jj' = 1 \mod N$. The second line follows from the observation that [87]

$$S \cdot \frac{\tau + j}{N} = \frac{j\tau - 1}{N\tau} = G \cdot \left(\frac{\tau - j'}{N}\right) ,$$

where $G = \begin{pmatrix} j & (jj'-1)/N \\ N & j' \end{pmatrix} \in \Gamma_0(N)$.

Examples

A very nice and practical introduction to modular forms is by Zagier [80]. We define the modular forms that appear in our work.

1. The Dedekind eta function $\eta(\tau)$ defined by (with $q = \exp(2\pi i\tau)$)

$$\eta(\tau) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m) , \tag{A.14}$$

is a modular form of weight half and character given by a twenty-fourth root of unity.

2. The modular J function defined below is a weakly holomorphic modular form of weight zero.

$$J(\tau) := \frac{E_4(\tau)^3}{\eta(\tau)^{24}} = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

The J function bijectively maps $\Gamma^{(1)}\backslash\mathbb{H}_1$ to the complex sphere. At special points, $J(\exp(2\pi i/3) = 0$ and $J(i) = 1728$. Define $z(\tau) = J(\tau)/1728$. Thus, $z(\exp(2\pi i/3) = 0$ and $z(i) = 1$.

3. **The Eisenstein series:** Let

$$\begin{aligned} E_2(\tau) &= 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)\, q^n\ , \\ E_4(\tau) &= 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)\, q^n\ , \\ E_6(\tau) &= 1 - 504\sum_{n=1}^{\infty} \sigma_5(n)\, q^n\ . \end{aligned} \tag{A.15}$$

$E_4(\tau)$ and $E_6(\tau)$ are holomorphic modular forms of $PSL(2,\mathbb{Z})$ with weights 4 and 6 respectively. They generate the ring of holomorphic modular forms of $PSL(2,\mathbb{Z})$. Any holomorphic modular form of $PSL(2,\mathbb{Z})$ can be expressed a polynomial of these two modular forms. $E_2(\tau)$ is not modular but

$$E_2^*(\tau) = E_2(\tau) - \frac{2}{\text{Im}(\tau)}\ ,$$

is a non-holomorphic modular form of weight 2.

4. The sub-group $\Gamma_0(N)$ (for $N > 1$) has a holomorphic modular form of weight 2 given by

$$E_2^{(N)}(\tau) := \frac{1}{N-1}\left(NE_2^*(N\tau) - E_2^*(\tau)\right) = \frac{1}{N-1}\left(NE_2(N\tau) - E_2(\tau)\right)\ , \tag{A.16}$$

where we observe that the non-holomorphic pieces cancel away in writing the definition in the second form. It is easy to show that

$$E_2^{(N)}|S(\tau) = -\frac{1}{N}E_2^{(N)}(\tau/N)\ .$$

5. Let $\rho = 1^{a_1}2^{a_2}\cdots N^{a_N}$ be a cycle shape, for a conjugacy class of M_{24}, with $\sum_j ja_j = 24$. Then, the product

$$\eta_\rho(\tau) := \prod_{j=1}^{N} \eta(j\tau)^{a_j}\ ,$$

is a modular form $\Gamma_0(N)$ with character given by an N-th root of unity [88] (also see [89] for a slightly different version).

A.3 Ring of Generators for $\Gamma_0(N)$

Let $M(\Gamma_0(N))$ denote the ring of holomorphic modular forms of $\Gamma_0(N)$. We list the generators of this ring for the cases of interest (obtained from [90]).

1. $PSL(2,\mathbb{Z})$ has two generators: $E_4(\tau)$ and $E_6(\tau)$.
2. $M(\Gamma_0(2))$ has two generators: $E_2^{(2)}(\tau)$ and $E_4(2\tau)$.
3. $M(\Gamma_0(3))$ has three generators: $E_2^{(3)}(\tau)$, $E_4(3\tau)$ and $E_6(3\tau)$.
4. $M(\Gamma_0(5))$ has three generators: $E_2^{(5)}(\tau)$, $E_4(5\tau)$ and $\eta_{1^45^4} = \eta(\tau)^4\eta(5\tau)^4$.

A.4 Jacobi forms

In the limit $\tau' \to i\infty$ or $s = \exp(2\pi i\tau') \to 0$, a Siegel modular form $\Phi_k(\mathbf{Z})$ has the following Fourier-Jacobi expansion:

$$\Phi_k(\mathbf{Z}) = \sum_{m=0}^{\infty} s^m\ \phi_{k,m}(\tau, z)\ .$$

The Jacobi group Γ_J is the sub-group of $Sp(4,\mathbb{Z})$ that preserves the condition $s = 0$. The transformation of the Fourier-Jacobi coefficients, $\phi_{k,m}(\tau, z)$, under the Jacobi group is a natural definition of a Jacobi form. It is generated by two sub-groups, one is the modular group $PSL(2,\mathbb{Z})$ embedded suitably in $Sp(4,\mathbb{Z})$ and the other is the Heisenberg group defined below.

The embedding of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2,\mathbb{Z})$ in $Sp(4,\mathbb{Z})$ is given by

$$\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \equiv \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{A.17}$$

The above matrix acts on $\mathbb{H}_2$ as

$$(\tau, z, \sigma) \longrightarrow \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, \sigma - \frac{cz^2}{c\tau + d} \right) , \tag{A.18}$$

with $\det(C\mathbf{Z} + D) = (c\tau + d)$. The Heisenberg group, $H(\mathbb{Z})$, is generated by $Sp(2,\mathbb{Z})$ matrices of the form

$$[\lambda, \mu, \kappa] \equiv \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{with } \lambda, \mu, \kappa \in \mathbb{Z} \tag{A.19}$$

The above matrix acts on $\mathbb{H}_2$ as

$$(\tau, z, \sigma) \longrightarrow \left(\tau,\ z + \lambda\tau + \mu,\ \sigma + \lambda^2\tau + 2\lambda z + \lambda\mu + \kappa \right) , \tag{A.20}$$

with $\det(C\mathbf{Z} + D) = 1$.

Definition A.4.1. *A Jacobi form of weight k and index m is a map* $\phi : \mathbb{H}\times\mathbb{Z} \to \mathbb{C}$ *satisfying*

$$\Phi|_k M(\mathbf{Z}) = \Phi(\mathbf{Z}) .$$

where $\Phi(\mathbf{Z}) := s^m \phi_{k,m}(\tau, z)$.

The power of s cancels the phases that appear for the Heisenberg group in the usual definition.

Examples

The genus-one theta functions are defined by

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau, z) = \sum_{l\in\mathbb{Z}} q^{\frac{1}{2}(l+\frac{a}{2})^2}\, r^{(l+\frac{a}{2})}\, e^{i\pi l b} \,, \tag{A.21}$$

where $a,\ b \in (0,1) \bmod 2$. We define $\theta_1(\tau,z) \equiv i\,\theta\begin{bmatrix}1\\1\end{bmatrix}(\tau,z)$, $\theta_2(\tau,z) \equiv \theta\begin{bmatrix}1\\0\end{bmatrix}(z_1,z)$, $\theta_3(\tau,z) \equiv \theta\begin{bmatrix}0\\0\end{bmatrix}(\tau,z)$ and $\theta_4(\tau,z) \equiv \theta\begin{bmatrix}0\\1\end{bmatrix}(\tau,z)$.

The following two index 1 Jacobi forms (with weights 0 and −2 respectively) are important.

$$\begin{aligned} A_{0,1}(\tau,z) &= 4\left[\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + \frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right] = (r^{-1} + 10 + r) + O(q)\,, \\ B_{-2,1}(\tau,z) &= \eta(\tau)^{-6}\theta_1(\tau,z)^2 = (r^{-1} - 2 + r) + O(q)\,. \end{aligned} \tag{A.22}$$

We usually drop writing the weight and index of these two basic Jacobi forms. All weak Jacobi forms are given by polynomials in these two Jacobi forms with coefficients given by modular forms of appropriate weight [91, see Prop. 6.1].

Let $f_i = \theta_i(\tau,z)/\theta_i(\tau,0)$ for $i \in \{2,3,4\}$. The Umbral Jacobi forms at lambency ℓ are weak Jacobi forms of weight zero and index $(\ell - 1)$ [92]. We list the three that are relevant for us.

$$
\begin{aligned}
\psi_{0,1}(\tau,z) &= 4(f_2^2+f_3^2+f_4^2) = \left(\tfrac{1}{r}+10+r\right)+\cdots,\\
\psi_{0,2}(\tau,z) &= 2(f_2^2f_3^2+f_3^2f_4^2+f_4^2f_2^2) = \left(\tfrac{1}{r}+4+r\right)+\cdots,\\
\psi_{0,3}(\tau,z) &= 4f_2^2f_3^2f_4^2 = \left(\tfrac{1}{r}+2+r\right)+\cdots, \qquad\qquad (A.23)\\
\psi_{0,4}(\tau,z) &= \frac{1}{4}\left(\psi_{0,1}(\tau,z)\psi_{0,3}(\tau,z)-(\psi_{0,2}(\tau,z))^2\right) = \left(\tfrac{1}{r}+1+r\right)+\cdots,\\
\psi_{0,6}(\tau,z) &= \psi_{0,2}(\tau,z)\psi_{0,4}(\tau,z)-(\psi_{0,3}(\tau,z))^2 = \left(\tfrac{1}{r}+r\right)+\cdots,
\end{aligned}
$$

A.5 Twisted-Twining Elliptic Genera of $K3$

Let g denote a finite symplectic automorphism of $K3$ of order N. We denote one half of the elliptic genus of $K3$ twisted by g^r and twined by g^s by $\psi_{0,1}^{[g^s,g^r]}(\tau,z)$

$$
\psi_{0,1}^{[g^s,g^r]}(\tau,z) = \frac{N}{2}\,F_{(N)}^{(r,s)}(\tau,z)\,, \qquad (A.24)
$$

where $F_{(N)}^{(r,s)}(\tau,z)$ are defined in [6] for prime $N=2,3,5$ as follows:

$$
\begin{aligned}
F_{(N)}^{(0,0)}(\tau,z) &= \frac{2}{N}A(\tau,z)\\
F_{(N)}^{(0,s)}(\tau,z) &= \frac{2}{N(N+1)}\left[A(\tau,z)+NB(\tau,z)E_2^{(N)}(\tau)\right] \quad \text{for } 1\le s\le(N-1)\\
F_{(N)}^{(r,rl)}(\tau,z) &= \frac{2}{N(N+1)}\left[A(\tau,z)-B(\tau,z)E_2^{(N)}\left(\tfrac{\tau+l}{N}\right)\right]\\
&\qquad\qquad \text{for } 1\le r\le(N-1),\ 0\le l\le(N-1),
\end{aligned}
$$

A.6 Second Quantized Twisted-Twining Elliptic Genera of $K3$ [14]

In CFT's on a torus on considers the following traces

$$g \underset{h}{\square} = \mathrm{Tr}_{\mathcal{H}_h}(g\ \cdots)\ , \tag{A.25}$$

where g and h are (commuting) symmetries of the CFT and $\mathcal{H}_h$ is a module h twisted sector. We will also denote the same object by $[g, h]$ in a more compact notation. Let g and h denote, for simplicity, discrete symplectic automorphisms of $K3$ that are commute with each other. For instance, consider the elliptic genus in a twisted sector $\mathcal{K}_h$ of an orbifold of the K3 CFT by the element h twined by the element g.

$$\psi_{0,1}^{[g,h]}(\tau, z) := \mathrm{Tr}_{\mathcal{K}_h}(g\, q^{L_0-1} r^{J_L}(-1)^F)\ , \tag{A.26}$$

Following [17](see also [18, 93–96]), define the following twisted Hecke operator

$$\psi_{0,1}^{[g,h]}\ |V_m\ (\tau, z) \equiv \frac{1}{m} \sum_{ad=m} \sum_{b=0}^{d-1} \psi_{0,1}^{[g^a h^{-b}, h^d]} \left(\tfrac{a\tau+b}{d}, az\right)\ . \tag{A.27}$$

The second-quantized elliptic h-twisted genus twined by the element g is defined to be

$$\mathcal{E}^{[g,h]}(\mathbf{Z}) := \exp\left[-\sum_{m=1}^{\infty} s^m\, \psi_{0,1}^{[g,h]}\ |V_m\ (\tau, z)\right] \tag{A.28}$$

The Siegel modular form is obtained by

$$\Phi_k^{[g,h]}(\mathbf{Z}) := s\, \phi_{k,1}^{[g,h]}(\tau, z)\, \mathcal{E}^{[g,h]}(\mathbf{Z})\ , \tag{A.29}$$

where

$$\phi_{k,1}^{[g,h]}(\tau, z) = \frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} \times \eta^{[g,h]}(\tau)\ .$$

Specialising to the case when both when we consider a twisting by g^y and twining by g^x where g is an element of order N, we obtain the following product formula starting from Eq. (A.27).

$$\Phi_k^{[g^x,g^y]}(\mathbf{Z}) = s\phi_{k,1}^{[g^x,g^y]} \times \prod_{m=1}^{\infty}\prod_{\alpha=0}^{N-1}\prod_{n\in\mathbb{Z}-\frac{\alpha y}{N}}\prod_{\ell\in\mathbb{Z}}(1-e^{\frac{2\pi i\alpha x}{N}}q^n r^\ell s^m)^{c^{[\alpha,my]}(nmN,\ell)}\ . \tag{A.30}$$

A.7 Classical Theta functions

$$\theta_{k,\lambda}(\tau,z) = \sum_{m\in\mathbb{Z}} q^{k(m+\frac{\lambda}{2k})^2} r^{k(m+\frac{\lambda}{2k})}\ .\quad \lambda\in\mathbb{Z}/2k\mathbb{Z} \tag{A.31}$$

This is a vector-valued Jacobi form of weight half and index $k/4$. Dividing by $\eta(\tau)$ makes the weight to zero.

$$\alpha_{k,\lambda}(\tau,z) := \frac{\theta_{k,\lambda}(\tau,z)}{\eta(\tau)} \tag{A.32}$$

Under the T and S modular transformations, the $\alpha_{k,\lambda}$ transform as follows:

$$\begin{aligned}\alpha_{k,\lambda}(\tau+1,z) &= e^{2\pi i\left(\frac{\lambda^2}{4k}-\frac{1}{24}\right)}\ \alpha_{k,\lambda}(\tau,z)\\ \alpha_{k,\lambda}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) &= e^{2\pi i\frac{kz^2}{4\tau}}\sum_{\mu=0}^{2k-1}\frac{e^{2\pi i\left(-\frac{\lambda\mu}{2k}\right)}}{\sqrt{2k}}\alpha_{k,\mu}(\tau,z)\end{aligned} \tag{A.33}$$

Below we define the normalized $\widehat{sl(2)}$ characters which have nice modular properties.

$$\chi_{k,\lambda}(\tau,z) = \frac{\theta_{k+2,\lambda+1}(\tau,z)-\theta_{k+2,-\lambda-1}(\tau,z)}{\theta_{2,1}(\tau,z)-\theta_{2,-1}(\tau,z)};\ \text{for } k,\lambda\in\mathbb{Z}_{\geq 0}\ ,\lambda\leq k\ . \tag{A.34}$$

Under the T and S modular transformations, one has

$$\begin{aligned}\chi_{k,\lambda}(\tau+1,z) &= e^{2\pi i\left[\frac{(\lambda+1)^2}{4(k+2)}-\frac{1}{8}\right]}\chi_{k,\lambda}(\tau,z) \\ \chi_{k,\lambda}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) &= e^{2\pi i k\frac{z^2}{4\tau}}\left(\frac{2}{k+2}\right)^{\frac{1}{2}}\sum_{\mu=0}^{k}\sin\left[\tfrac{\pi(\lambda+1)(\mu+1)}{k+2}\right]\chi_{k,\mu}(\tau,z)\end{aligned} \tag{A.35}$$

A.8 Siegel Modular Forms

Definition A.8.1. *A Siegel modular form of weight k and character v with respect to* Γ_t *is a holomorphic function* $F : \mathbb{H}_2 \to \mathbb{C}$ *satisfying*

$$F(M\cdot\mathbf{Z}) = v(M)\ \det(C\mathbf{Z}+D)^k\ F(\mathbf{Z})\,, \tag{A.36}$$

for all $\mathbf{Z}\in\mathbb{H}_2$ *and* $M\in\Gamma_t$*.*

The Fourier expansion of a Siegel modular form (with trivial character) with respect to the variable τ' (also called the Fourier-Jacobi expansion)

$$F(\mathbf{Z}) = \sum_{m=0}^{\infty}\phi_m(\tau,z)\ s^{tm}\ , \tag{A.37}$$

where $s=\exp(2\pi i\tau')$. For each m, $\phi_m(\tau,z)$ is a Jacobi form of weight k and index mt. This can be understood by observing that the cusp at $\tau'=i\infty$ is preserved by the subgroup Γ^J and studying their transformation under this subgroup. We refer to the first non-vanishing term in the above Fourier expansion as the *zeroth Fourier-Jacobi coefficient* of the Siegel modular form.

The character of Siegel modular forms are determined in part by their transformation under the Jacobi group Γ^J. Consider the Jacobi form of weight -1 and index $\frac{1}{2}$:

$$\frac{\vartheta_1(\tau,z)}{\eta(\tau)^3}\ .$$

This has trivial character under modular transformations and the following character

$$v_H([\lambda,\mu,\kappa]) = (-1)^{\lambda+\mu+\lambda\mu+\kappa} \,. \tag{A.38}$$

Multiplying the above Jacobi form by modular form $f(\tau)$ of Γ_1 with character χ leads to another Jacobi form of index half with character $(\chi \times v_H)$. This data can be obtained from the zeroth Fourier-Jacobi coefficient of the Siegel modular form. We need to determine the character under the involution V_t ($q \leftrightarrow s^t$) and $[0, 0, \kappa/t]$ (for $t > 1$).

A.9 Vector Valued Modular Forms

Definition A.9.1. *An admissible multiplier systems* (ρ, w) *consists of* $w \in \mathbb{C}$ *called the weigth and map* $\rho : \Gamma^{(1)} \to GL(d, \mathbb{C})$ *called the multiplier, for some positive integer* d*, called the rank, such that the following holds:*

(i) the associated automorphy factor (with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^{(1)}$*)*

$$\widetilde{\rho}_w(\gamma,\tau) := \rho(\gamma)(c\tau + d)^w$$

satisfies, for all $\gamma_1, \gamma_2 \in \Gamma^{(1)}$*,*

$$\widetilde{\rho}_w(\gamma_1\gamma_2,\tau) = \widetilde{\rho}_w(\gamma_1, \gamma_2\cdot\tau)\, \widetilde{\rho}_w(\gamma_2,\tau) \,, \tag{A.39}$$

(ii) $\rho(\mathbf{1}_2) = e^{-\pi i w}\rho(-\mathbf{1}_2) = \mathbf{1}_d$*, where* $\mathbf{1}_k$ *is the* $k \times k$ *identity matrix.*

Definition A.9.2. *Let* (ρ, w) *be an admissible multiplier system of rank* d*. A vector valued modular form (vvmf)* $\mathbf{g}(\tau) = (g_1, g_2, \ldots, g_d)^T$ *(of weight* w*, multiplier* ρ *and rankd* d*) is a map* $\mathbb{H}_1 \to \mathbb{C}^d$ *provided*

$$\mathbf{g}(\gamma\cdot\tau) = \widetilde{\rho}_w(\gamma,\tau)\, \mathbf{g}(\tau) \,, \tag{A.40}$$

for all $\gamma \in \Gamma^{(1)}$*,* $\tau \in \mathbb{H}_1$ *and each component* $g_i(\tau)$ *is meromorphic in* $\mathbb{H}_1^*$*.*

Let $\mathcal{M}^!_w(\rho)$ denote the space of weakly holomorphic vvmf i.e., those which are holomorphic in $\mathbb{H}_1$.

A.10 Modular Differential Operators

Let f be a modular form of weight w and D_w denote the modular derivative i.e.,

$$D_w f(\tau) := \left(\frac{1}{2\pi i}\frac{d}{d\tau} - \frac{w}{12}\, E_2(\tau)\right) f(\tau)\,. \tag{A.41}$$

This maps a modular form of weight w to a modular form of weight $(w+2)$. The following differential operators do not change weight [22].

$$\nabla_{1,w} = \frac{E_4(\tau)E_6(\tau)}{\eta(\tau)^{24}}\, D_w\,, \quad \nabla_{2,w} = \frac{E_4(\tau)^2}{\eta(\tau)^{24}}\, D_w^2\,, \quad \nabla_{3,w} = \frac{E_6(\tau)}{\eta(\tau)^{24}}\, D_w^3\,. \tag{A.42}$$

Appendix B

Supercharacter formula for BKM Lie superalgebras

B.1 The superdenominator identity

Let $\mathfrak{g}$ be a BKM Lie superalgebra. The Weyl-Kac-Borcherds superdenominator identity of $\mathfrak{g}$ has the form $\mathcal{S} = \mathcal{P}$ (sum equals product). We describe this in greater detail here, closely following [15].

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the decomposition of $\mathfrak{g}$ into bosonic (even) and fermionic (odd) subspaces. For $p = 0, 1$, let L_p^+ denote the set of positive roots of bosonic ($p = 0$) or fermionic ($p = 1$) type, and let $m_p(\alpha) = \dim(\mathfrak{g}_p)_\alpha$ denote the multiplicity of the root α in the subspace of appropriate parity.

The product side is given by:

$$\mathcal{P} = \frac{\prod_{\alpha \in L_0^+}(1 - e^{-\alpha})^{m_0(\alpha)}}{\prod_{\alpha \in L_1^+}(1 - e^{-\alpha})^{m_1(\alpha)}} \tag{B.1}$$

To describe the sum side, let $\alpha_i\,(i \in I)$ denote the simple roots of $\mathfrak{g}$. Let $I^{re} = \{i \in I : \langle\alpha_i, \alpha_i\rangle > 0\}$ and $I^{im} = I \backslash I^{re}$ be the subsets of real and imaginary simple roots. The

Weyl group W of $\mathfrak{g}$ (when $\mathfrak{g}$ is infinite-dimensional) is the group generated by the simple reflections w_{α_i} for $i \in I^{re}$. We can also decompose $I = I_0 \cup I_1$ into the disjoint union of simple roots of bosonic and fermionic types. Consider the set $\mathcal{T}$ of all elements μ in the root lattice of $\mathfrak{g}$ which can be expressed as a finite sum $\mu = \sum_{i \in I} k_i \alpha_i$ satisfying the following conditions:

1. k_i is a non-negative integer for all i.
2. $k_i = 0$ for $i \in I^{re}$.
3. If k_i and k_j are nonzero for some $i \neq j$, then $\langle \alpha_i, \alpha_j \rangle = 0$.
4. $k_i = 1$, unless α_i is an isotropic fermionic simple root, i.e., $i \in I_1$ with $\langle \alpha_i, \alpha_i \rangle = 0$.

Given $\mu = \sum_{i \in I} k_i \alpha_i \in \mathcal{T}$, let $k_0(\mu) = \sum_{i \in I_0} k_i$ and $\epsilon_0(\mu) = (-1)^{k_0(\mu)}$. We define the *Borcherds correction*

$$T = \sum_{\mu \in \mathcal{T}} \epsilon_0(\mu) e^{-\mu}$$

Finally, the sum side $\mathcal{S}$ is given by:

$$\mathcal{S} = e^{\rho} \sum_{w \in W} \det(w)\, w\left(e^{-\rho}\, T\right) \tag{B.2}$$

The superdenominator identity is the equality of (B.2) and (B.1).

An example: Consider a situation where has m distinct bosonic simple roots of weight $(\delta, 2\delta, 3\delta, \ldots)$ with $\langle \delta, \delta \rangle = 0$. The Borcherds correction factor due to these imaginary simple roots takes the form

$$T = \prod_{k=1}^{\infty} (1 - e^{-k\delta})^m .$$

A negative value for m corresponds to isotropic fermionic simple roots. We will encounter such Borcherds extensions of $\widehat{sl(2)}$ i.e., $\widehat{sl(2)}$ with the addition of the imaginary simple roots of the form discussed above.

B.2 The supercharacter formula

More generally, one has the Weyl-Kac-Borcherds formula for the supercharacter of an irreducible integrable highest weight module $L(\Lambda)$ of $\mathfrak{g}$. Here Λ is a dominant integral weight of $\mathfrak{g}$, i.e., (Λ, α_i) is a non-negative integer (resp. real number) for $i \in I^{re}$ (resp. $i \in I^{im}$). We define a subset $\mathcal{T}_\Lambda$ of $\mathcal{T}$ by imposing the following extra condition in addition to (1)-(4) above:

5. $k_i = 0$ if $\langle \Lambda, \alpha_i \rangle < 0$.

Analogous to the above, define $\widetilde{T}_\Lambda = \sum_{\mu \in \mathcal{T}_\Lambda} \epsilon'(\mu) e^{-\mu}$ and

$$\mathcal{S}_\Lambda = e^\rho \sum_{w \in W} \det(w)\, w \left(e^{-\rho-\Lambda}\, \widetilde{T}_\Lambda \right).$$

The WKB supercharacter formula states that the supercharacter χ_Λ of $L(\Lambda)$ is given by the quotient

$$\mathrm{Sch}(L(\lambda)) := \chi_\Lambda = \frac{\mathcal{S}_\Lambda}{\mathcal{P}} \quad . \tag{B.3}$$

Since $L(\Lambda)$ is the one-dimensional trivial representation when $\Lambda = 0$, this reduces to the superdenominator identity in that case [15].

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