STUDIES IN
KALUZA - KLEIN APPROACH TO UNIFICATION

THESIS

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By

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DEDICATED TO THE MEMORY OF MY DIDIMA

Late Smt. Subarna Prabha Devi

who meant so much to me.

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CERTIFICATE

This is to certify that the Ph.D. thesis submitted by Mr.Biswajit Chakraborty, to the University of Madras entitled. STUDIES IN KALUZA-KLEIN APPROACH TO UNIFICATION is a record of bondafide research work done by him during 1987-90 under my supervision. The research work presented in this thesis has not formed the basis for the award to the candidate of any degree, diploma, Associateship, Fellowship or other similar title. It is further certified that the thesis represents independent work on the part of the candidate, and collaboration was necessiated by the nature and scope of the problems dealt with.

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PREFACE

This thesis embodies the research work done by me on "Studies in Kaluza-Klein approach to unification " during the years 1987-90 at the Institute of Mathematical Sciences, Madras under the guidance of Dr. R.Parthasarathy with financial assistance from the Institute of Mathematical Sciences. Collaboration with my thesis supervisor Dr. R.Parthasarathy was necessitated by the nature and range of the problems dealt with. Three research papers were published and one more has been accepted for publication. These references have been cited in appropriate places.

This research work is presented in five chapters preceded by an introduction. There are appendices at the end of the chapter 2, 4 and 5 containing some mathematical details used in various places and a list of references at the end of each chapter and the introduction.

Biswajit Chakraborty
18th March '91

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INTRODUCTION

"In the general theory of relativity, in order to characterize world events, the electromagnetic four-potential A_μ has to be introduced separately along with the fundamental metric tensor $g_{\mu\nu}$ of the 4-dimensional world manifold which is regarded as the tensor potential of gravitation. The dualistic nature of gravitation and electricity still remaining here does not actually destroy the ensnaring beauty of the theory but rather affords a new challenge towards its triumph through the entirely unified picture of the world" [1].

In a remarkable paper Th.Kaluza [2] advanced the idea to unify the then known basic forces of Nature, namely, electromagnetism and gravity. According to Kaluza if one starts from pure gravity in 5-dimensions and make a special ansatz for the metric, then the resulting classical equations of motion have both four dimensional gravity and electromagnetism contained in it. The metric contains the electromagnetic vector potential A_μ and so acquires a geometrical nature. In order to reconcile with the observed (3+1) physical dimensions, Kaluza assumed that the fifth dimension is compactified to a circle of very small radius. (Cylinder condition). Klein [3] reformulated Kaluza's 5-dimensional theory on the basis of the action principle and investigated the 5-dimensional wave equation in the hope of obtaining the ordinary quantum theory in 4-dimensions.

The Kaluza-Klein philosophy of unifying gravitation with electromagnetism has been revived in the past few years with great enthusiasm and hope. In its original form, the 5-dimensional world has been taken to be the product of two spaces $M^4 \times S^1$ where M^4 is the usual 4-dimensional Minkowski space and S^1 is a compact space (with no time) of smaller dimensions - a circle of small radius. The U(1) gauge symmetry pertinent to electromagnetism is a consequence of transformations of the extra coordinate (x_5). In this way the extra dimension provides internal symmetry of the elementary particles. To accommodate other known interactions, the internal manifold has to be enlarged, so that the non-abelian gauge fields become part of the metric. This generalization of Dewitt [4] has been elaborated by Rayski [5], Kerner [6], Trautman [7], Cho [8], Cho and Freund [9] Luciani [10], Tanaka [11], along with the coset space approach by Witten [12] and Salam and Strathdee [13]. In Kaluza-Klein point of view, one attributes all interactions other than gravity and the spectrum of elementary particles to the structure of the internal manifold. The process of compactifying the M^{4+n} dimensional manifold to the product $M^4 \times B^n$ is called spontaneous compactification. Following the process of spontaneous compactification one obtains an effective theory in 4-dimensional space time governed by general covariance and a compact internal symmetry group G. The group of isometry of the internal space B^n corresponds to the symmetry group G. For instance B^n could be taken to be the group space G itself. This is not an economical choice though. A more convenient choice is to realize B^n as a coset space G/H where H is the maximal closed subgroup of G. As an example, to accommodate iso-spin symmetry, for which the appropriate group is SU(2) having 3 generators, the group space approach [11] B^n has to be a 3-dimensional SU(2) manifold so that the Kaluza-Klein space(K-K space is 7 dimensional $M^4 \times SU(2)$, while in the coset space approach [12,13] $B^n = SU(2)/U(1) = S^2$and so the K-K space is six dimensional $M^4 \times S^2$. A direct way to obtain the theory in 4-dimensions is to expand all fields in the theory in a complete set of harmonics on G/H. The coefficients in this expansion will be the 4-dimensional fields which carry a sequence of representations of the group G. The action in 4-dimensions is then obtained by integrating out the extra n co-ordinates parametrizing G/H, using the orthonormality of the harmonics. This involved procedure requires a knowledge of the properties of the coset spaces and an understanding of the method of harmonic expansion on coset spaces, see Salam and Strathdee [13] and Viswanathan [14]. To trigger spontaneous compactification, one needs additional fields to be included in the (4+n) dimensional theory. For instance Randjbar - Daemi, Salam and Strathdee [15] demonstrated spontaneous compactification in a 6-dimensional $M^4 \times S^2$ model with a monopole configuration in S^2, Omero and Percacci [16] and Gell-Mann and Zwiebach [17] demonstrated a non-linear sigma field induced compactification which was generalized by Parthasarathy [18] to a general non-linear sigma field induced compactification in which the Kaluza-Klein gauge bosons were made to remain massless at the classical level.

In the original 5-dimensional K-K theory, the U(1) coupling constant (electric charge) e is quantized ; $e = K/R$ where $K^2 = 16\pi G$ and $2\pi R$ is the circumference of the compact 5^{th} dimension. Weinberg [19] (See Mann [20] also) generalized this relation to (4+n) dimensional Kaluza-Klein theory.

The Kaluza-Klein approach provides a geometrical meaning to the gauge fields, as part of the metric in higher dimensional gravity. In fact, a class of co.ordinate transformations gives rise to gauge transformations. In the 5 -dimensional theory, $x^\mu \longrightarrow x'^\mu = x^\mu$ and $x^5 \longrightarrow x'^5 = x^5 + f(x^\mu)$ leads to $A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) - \frac{\partial f}{\partial x^\mu}$, which is the familiar U(1) gauge transformation. In more than 5-dimensions, the class of transformations, $x^\mu \rightarrow x'^\mu = x^\mu$, $y^i \rightarrow y'^i = y^i + \theta^\alpha(x)\, K^i_\alpha(y)$, where $K^i(y)$ are the killing vectors, leads to

$$A^\alpha_\mu(x) \longrightarrow A'^\alpha_\mu(x) = A^\alpha_\mu(x) + f^\alpha_{\beta\gamma}\, \theta^\beta(x)\, A^\gamma_\mu(x) - \partial_\mu \theta^\alpha(x)$$

which is the familiar non-Abelian gauge transformation. These results indicate that although the extra co-ordinates (other than 4) are introduced at a more formal/mathematical level and are finally integrated out in obtaining an effective action in 4-dimensions, they play very important role in determining the interaction (gauge fields) and couplings and hence cannot be classified as unphysical.

Quite apart from attempts to construct a realistic theory of elementary particle interactions, Kaluza-Klein theory has been explored in the context of describing the early universe. This goes under the name 'Kaluza-Klein Cosmology'. In this approach, it is assumed that at the beginning of the universe, the space was higher dimensional (>4). In the radiation dominated universe the space time Geometry is M^{4+n}. Due to some reasons, after the inflation, the physical three spatial dimensions expand while the (internal) extra space B^n contract so that they (B^n) are undetectable at present. This 'dynamical compactification' has been first studied by Chodos and Detweiler [21] using Kasner metric [22] without matter field. They were able to demonstrate the above mentioned asymmetrical expansion. This approach has been used by Freund [23] for a higher dimensional Jordon-Brans-Dicke theory, by Kerner [24] for a 6-dimensional theory with a monopole and a Higgs scalar on S^2 (in $M^4 \times S^2$) and by Kolb [25] for a 6-dimensional monopole induced compactified Kaluza-Klein theory. The other approach to Kaluza-Klein Cosmology is based on thermodynamical arguments. Here the energy momentum tensor is usually taken to correspond to a perfect fluid in all spatial dimensions with an assumption that this is appropriate for a radiation dominated era. Then the dynamical compactification is achieved by conservation of entropy as shared between M^4 and B^n. This approach has been studied by Alvarez and Gavela [26], Abbott, Barr and Ellis [27] Sahdev [28] and Kolb [29].

In spontaneous and dynamical compactification schemes, the size of the extra space B^n have been either assumed to be small or taken to shrink in size. In a remarkable paper Appelquist and

Chodos [30] offered a plausible explanation for this. This is based on the Casimir [31] effect : as a consequence of quantum mechanical vacuum fluctuations of the electromagnetic field, an attractive force exists between two uncharged conducting plates. The presence of the plates separated by a distance a imposes a boundary condition on the field, as a result the wave modes perpendicular to the plane of the plates get restricted to $K_{\perp} = \frac{n\pi}{a}$, n = 0, ± 1,± 2, ... Computing the zero point energy per unit volume of the field, one finds [32] it to be $-\frac{\pi^2}{720}\frac{\hbar c}{a^4}$. If instead of photon, a free massless scalar field ϕ in a space in which one of the dimensions, say the z-direction, is not of infinite extent but a circle of radius a/π is considered, then the appropriate boundary condition $\phi\,(t,x,y,z+2a) = \phi\,(t,x,y,z)$ and the allowed frequencies are $\omega(\vec{k},n) = [k_x^2+k_y^2+n^2\pi^2/a^2]^{1/2}$ The observable finite part of the zero point energy is $-(\pi^2\hbar/720)\,a^3$. Chodos [33] shows that the one loop effective potential for such a theory is again $-(\pi^2\hbar/720)\,a^3$, demonstrating that the Casimir energy can be found either by summing the zero point energy or by evaluating the one loop effective potential, obtained by expanding the action about a suitable background field ϕ_0 and keeping only the quadratic terms in the fluctuations. When applied to 5-dimensional Kaluza-Klein theory with no electromagnetic field in the background metric the one loop effective potential becomes [30]

$$V_{eff} = -\frac{15}{4\pi^2}\,\zeta\,(5)\;/\;(2\pi R)^4$$

where R is the radius of the 5^{th} dimension. The interpretation a la Casimir effect is: there is an attractive force causing the distance $2\pi R$ to shrink. This quantum explanation for the shrinking of the extra dimensions loses its validity when $2\pi R$ becomes lower than planck length similar to the situation when 'a' becomes comparable to the interatomic distance in the original Casimir effect, beyond which the idealized plates must be replaced by assembly of atoms and the treatment becomes involved. This Appelquist-Chodos mechanism has been extended by Rubin and Roth [34] to include finite temperature effects, Rubin and Roth [35] and Tsokos [36] to include massive fermions. Generalization to the background manifold $M^4 \times (S^1)^d$ i.e. 4-dimensional Euclidean space x d-dimensional torus has been made [37]. One result emerging from the inclusion of fermions [35,36] is the stabilization of Casimir energy i.e. The effective potential has a minimum. Candelas and Weinberg [38] presented a detailed study of the quantum one loop effects in Kaluza-Klein theory including matter fields in the form of scalars, spinors and vector fields. In their approach, one starts from Einstein equations in (4+n) dimensions with the energy momentum tensor arising from the quantum fluctuations of matter fields. Candelas and Weinberg [38] find the effect of a single matter field is extremely small for both scalar and Dirac fields and need large number of them to have an appropriate value for the gauge coupling constant. Candelas and Weinberg [38] find that for 4+n = 7,11,15,19 and 23 the theory with a flat background metric (classical solution) is stable and with suitably large number of scalars and fermions, the theory with 4+n = 9,13,17,21,27,31 and 35 are stable. Kato and Saito [39] introduced additional Abelian gauge fields to examine the above scenario. Nevertheless it is important to examine the quantum effects in a specific model with spontaneous compactification. An attempt in this direction was made by Castaldini [40] who evaluated the quantum one-loop potential for the monopole induced compactification of $M^4 \times S^2$ six dimensional theory due to Ranjdbar-Daemi, Salam and Strathdee [15]; however they restricted the physical manifold to M^3.

The Kaluza-Klein philosophy provides a geometrical origin to the gauge fields as part of metric in a higher dimensional gravity. It is remarkable that the gauge transformations (both Abelian and non-Abelian) are nothing but a class of coordinate transformations in an enlarged gravity theory (base point preserving automorphisms). This initiated many [41] to formulate Kaluza-Klein theory from fibre bundle point of view. Dolan and Duff [42] have shown if one keeps $m \neq 0$ modes in the Fourier expansion of the fields then one finds a Kac-Moody extension of Poincare algebra. Weinberg [43] proposed a quasi-Riemannian structure for the Kaluza-Klein space which was studied in further detail by Viswanathan and Wong [44].

While the Kaluza-Klein methodology is very attractive as far as the description of gauge fields and possible unification with gravity (neglecting any propagation on B^n as this requires planck energies) are concerned, it poses serious difficulties once fermions are introduced. Compactification of the extra dimensions triggers a scale, planck mass, rendering fermions to acquire planck mass(Lichenorowicz theorem [45]). The (physical) observed fermions are very light and so to obtain chiral fermions in Kaluza-Klein theory after spontaneous compactification is exceedingly difficult. However in some specific models it is possible to obtain chiral fermions. For example in $M^4 \times S^2$, monopole induced compactification scheme, the gauge invariant coupling of fermions with the S^2 monopole field allows us to realize chiral fermions. One may well consider introducing parallelizable torsion [46] in B^n to obtain chiral fermions, since torsion in B^n will not affect the equivalence principle in M^4.

In view of this many attempts have been made to construct a phenomenologically viable unification scheme. While it may be feasible to construct a unified picture of electroweak theory with gravitation with $M^4 \times B^3$ where $B^3 = S^2 \times S^1$ representing SU(2)/U(1) and U(1) groups relevant for electroweak theory of Salam [47] and Weinberg [48], the introduction of quarks poses a more serious problem. The strong interaction is now accepted to be described by $SU(3)_c$ symmetry and so following Witten [12] the appropriate manifold is $M^4 \times SU(3)_c \times SU(2) \times U(1)$ which can be realized in the coset space approach as $M^4 \times CP^2 \times S^2 \times S^1$ where $CP^2 \simeq SU(3)_c / U(2)$ and $S^2 = SU(2)/U(1)$ and $S^1 \simeq U(1)$. The manifold CP^2 does not admit spin structure [50] and so we cannot introduce quarks! However it is possible to circumvent this problem by appealing to 'generalized spin structure' [51] according to which, such manifolds as CP^2 can admit spin structure provided the spinor is coupled to a topologically non-trivial 1-form. In fact CP^2 naturally admits an U(1) instanton which can be exploited to define spinors consistently on CP^2.

The revival of the Kaluza-Klein theory with added features essentially occupies a major part in supergravity and super string theories which are higher dimensional theories. The technique of Kaluza-Klein spontaneous compactification could be useful for dimensional reduction of higher dimensional theories. The incorporation of fermions and possible realization of chiral fermions provide deep insight into the geometrical structure of the theory.

In this thesis, four important aspects of the Kaluza-Klein theory are studied. They are

(i) Quantum effects in Kaluza-Klein theory with spontaneous compactification induced by a non-linear σ-model.

(ii) Asymmetric expansion of space in a model in which the extra space has a non-linear σ-model field.

(iii) Spontaneous compactification of $M^4 \times CP^2$ induced by instantons in CP^2, and fermions.

and

(iv) Generalized spin structure on CP^2.

Our results are published [51, 52, 53, 54] in various journals. We summarize our results below.

(I) We have evaluated the 1-loop potential for the scalar fields in the form of a non-linear σ-model coupled to gravity in

Kaluza-Klein theory proposed by Gell-Mann and Zwiebach [17]. For the product manifold $M^4 \times S^3$ it is shown that the classical solutions are stable against quantum fluctuations.

(I.1) The quantum fluctuations of the scalar fields with respect to the classical solutions are vectors in the scalar manifold and are treated as divergenceless and gradient of a scalar quantity.

(I.2) The 1-loop effective potential for $M^4 \times S^3$ is found to be

$$V^{1\text{-loop}} = \frac{1}{(2\pi)^6 r^4} \left\{ \frac{135}{4\pi^2} \zeta(7) + 6\zeta(5) + \frac{1}{2}\pi^2 \zeta(3) \right\}$$

where r is the radius of S^3 and ζ is the Riemann ζ function. Parametrizing the 1-loop potential as C_n / r^4, we find $C_n = 2.54 \times 10^{-4}$ and the total potential $V = V_{cl} + V_q$ becomes $\alpha_n r^n + \frac{C_n}{r^4}$ where $\alpha_n = n\alpha/\lambda^2 a^2 > 0$. (see chapter 2 for details). Varying with respect to r, the only parameter, the minimum occurs when $r = r_{min} = \left[\frac{4C_n \lambda^2 a^2}{n^2 \alpha}\right]^{\frac{1}{4+n}}$. So to have a stable minimum $C_n > 0$. Thus the system with classical solutions is stable against quantum fluctuations.(i.3) By demanding spontaneous compactification with energy momentum tensor coming from the one-loop effective potential, we obtain an algebraic constraint on the size r of the internal manifold as $r^2 = \frac{8\pi G_0 (n+4) C_n}{n(n-1)}$, indicating that C_n has to be positive as $r^2 > 0$. Our 1-loop calculations give $C_n > 0$ and so is consistent with this analysis as well.(See chapter 2 for details.)

(II) A Kaluza-Klein theory based on the sigma model induced compactification scheme is shown to admit dynamical solution with the usual space expanding and the extra space contracting.

(II.1) The action for (4+n) dimensional gravity coupled to a non-linear σ-model is

$$S = \int d^{4+n} Z \sqrt{G} \left\{ -\frac{1}{2} \hat{R} + \frac{1}{\lambda^2} G^{MN} m_{ij} (\phi) \partial_M \phi^i \partial_N \phi^j \right\}$$

where $Z^M = (x^\mu, y^i)$ with x^μ as the usual space-time coordinates and y^i the extra space, G_{MN} (4+n) dimensional metric, $\hat{R} = \hat{R}_{MN} G^{MN}$, $m_{ij}(\phi)$ the scalar manifold metric and ϕ^i are the n-scalars.The classical field equations for this action are examined with respect to generalised Friedmann - Robertson metric

$$dS^2 = dt^2 - R_3^2 (t) g_{mn} dx^m dx^n - R_I^2 (t) g_{ij} dy^i dy^i .$$

The scale factors are given respectively by $R_3(t)$ and $R_I(t)$. We assume $r(t+t_0)^\alpha$ and $R(t+t_0)^\beta$ respectively, where t_0 is the time at which all space dimensions have comparable size. It is found then

$\alpha = \dfrac{3+ \sqrt{6n+3n^2}}{3(3+n)}$ and $\beta = \dfrac{n - \sqrt{6n + 3n^2}}{n(3+n)}$ showing the usual dimensions expand while the extra dimensions contract.

(II.2) The classical field equations further give a new constraint $2/\lambda^2 a^2 = K_I$ where K_I is the constant curvature of the compact extra space and $g_{ij} = - a^2 m_{ij}$.

(II.3) The classical field equations in the monopole induced compactification scheme do not admit such solutions. This is found to be due to the *presence* of cosmological constant term in this model.

(III) We propose an instanton induced compactification scheme for $M^4 \times CP^2$ Kaluza-Klein theory. The instanton configuration is in CP^2 which triggers spontaneous compactification. Much in the spirit mentioned above, we show that it is possible to obtain chiral fermions.

(IV). The manifold CP^2 as such does not admit spinors. An explanation based on the concept of parallel transport of vectors (spinors) is given to demonstrate this result.

(IV.1) The manifold CP^2 naturally admits an U(1) instanton. This is then coupled to fermions and using the concept of parallel transport, we show that CP^2 can admit generalized spin structure.

The thesis is organised as below.

Chapter.I: Provides a brief review of 5-dimensional Kaluza-Klein theory and its generalization to (4+n) dimensions. Here we demonstrate how gauge transformations both Abelian and non-Abelian can be obtained from a class of co-ordinate transformations. Also the result that the Dirac operator in the internal space is the essentially the mass operator will be derived. Notation followed subsequently is established.

Chapter.II: Provides an explicit spontaneous compactification scheme proposed by Gell-Mann and Zwiebach [17] and the action is expanded in the background field method. Assuming a flat background the eigenvalues and degeneracies of the fluctuation operator for the scalar field fluctuations are obtained and the regularized 1-loop potential is calculated. Stability is discussed.

Chapter.III: Contains an analysis of the model in Chapter.II with a view to obtain the asymmetric expansion in Kaluza-Klein theory.

Chapter.IV: Provides a spontaneous compactification scheme induced by instantons for $M^4 \times CP^2$ Kaluza-Klein theory. Chiral fermions are discussed.

Chapter.V: Contains the study of CP^2 space for generalised spin structure using 'parallel transport' concept and coupling fermions to instantons.

The contents of Chapters II, III and IV have been published respectively in

(1) Quantum effects in Kaluza-Klein theory in the spontaneous compactification induced by a non-linear sigma model.

(B. Chakraborty and R. Parthasarathy)

Classical and Quantum Gravity 6 (1989) 1455.

(2) Dynamical compactification in Kaluza-Klein Cosmology

(B. Chakraborty and R. Parthasarathy)

Physics Letters A 142 (1989) 75.

(3) Instanton induced compactification and chiral fermions

(B. Chakraborty and R. Parthasarathy)

Classical and Quantum Gravity 7 (1990) 1217.

(4) On generalized spin structure on CP^2 manifold

(B. Chakraborty and R. Parthasarathy)

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CHAPTER I

REVIEW OF FIVE AND HIGHER DIMENSIONAL KALUZA-KLEIN THEORY

Section 1

Here we are going to review briefly the five dimensional original Kaluza-Klein theory[1] where it incorporates four dimensional Maxwell's electromagnetism, along with four - dimensional general relativity. Later in section 2 we shall discuss about its higher dimensional generalization[2] to include non-abelian gauge theories.

Consider pure gravity in a five (4+1) dimensional space time, where the additional dimension is taken to be a compact circle S^1 of very small size to render it inaccessible to present day accelerators. Let $Z^M = \left\{ x^\mu, y \right\}$ refer to the co-ordinates of this five dimensional manifold. The most general co-ordinate transformation is

$$Z^M \longrightarrow Z'^M = Z'^M(Z^N) \tag{1.1}$$

But Kaluza's ansatz is to consider the restricted co-ordinate transformation

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = x'^\mu(x^\nu) \\ y &\longrightarrow y' = y + f(x^\mu) \end{aligned} \tag{1.2}$$

With this ansatz certain consequences are immediate. If p^M is a five dimensional contravariant vector, then the components p^μ transform as a four-dimensional contravariant vector. Similarly the components $p^{\mu\nu}$ of a contravariant second rank tensor P^{MN}

transform like a 2nd rank tensor in four-dimension. P_5 and P_{55} behaves as scalars in four dimension. On the other hand p^5 and p_μ do not have well defined transformation property in four-dimensions. But one can define a covariant counterpart π_μ of p^μ in 4-dimensions as

$$\pi_\mu = p_\mu - \frac{G_{\mu 5}}{G_{55}} \; p_5 \tag{1.3}$$

with G_{MN} is the metric for the five dimensional space-time where it can be shown that π_μ transforms like a covariant vector in four dimensions.

Defining four - dimensional metric tensor $g_{\mu\nu}$ as the one which gives π_μ from p^μ as

$$\pi_\mu = g_{\mu\nu} \; p^\nu \tag{1.4}$$

we get

$$g_{\mu\nu} = G_{\mu\nu} - \frac{G_{\mu 5} \; G_{\nu 5}}{G_{55}} \quad , \; g^{\mu\nu} = G^{\mu\nu} \tag{1.5}$$

Now let us write Lorentz force equation in four dimension in a general covariant form , which is

$$m_0 \left[\frac{d}{d\tau} \left(g_{\mu\nu} \frac{dx^\nu}{d\tau} \right) - \frac{1}{2} \frac{\partial g_{\sigma\rho}}{\partial x^\mu} \; \frac{dx^\sigma}{d\tau} \; \frac{dx^\rho}{d\tau} \right] = q \; F_{\mu\nu} \; \frac{dx^\nu}{d\tau} \tag{1.6}$$

where m_0 is the mass and q is the charge of the particle and τ is the proper time.Now one can ask whether this equation can be obtained by considering the five - dimensional geodesic equation

$$\frac{d^2 Z^M}{d\tau^2} + \Gamma^M_{PQ} \frac{dZ^P}{d\tau} \frac{dZ^Q}{d\tau} = 0$$

or equivalently

$$\frac{d}{d\tau} \left(G_{MN} \frac{dZ^N}{d\tau} \right) = \frac{1}{2} \frac{\partial G_{PQ}}{d Z^M} \frac{dZ^P}{d\tau} \frac{dZ^Q}{d\tau} \qquad (1.7)$$

Following Kaluza, let us assume that the coefficients G_{MN} be independent of y and G_{55} a constant. Then for $M = 5$, equation (1.7) gives a constraint

$$G_{5M} \frac{dZ^M}{d\tau} = a \quad (\text{ a constant})$$

Using this constraint, the $M = \mu$ part of the equation (1.7) becomes

$$\frac{d}{d\tau} \left(g_{\mu\nu} \frac{dx^\nu}{d\tau} \right) - \frac{1}{2} \frac{\partial g_{\sigma\rho}}{\partial x^\mu} \frac{dx^\sigma}{d\tau} \frac{dx^\rho}{d\tau} = \frac{a}{G_{55}} \left(\frac{\partial G_{\nu 5}}{\partial x^\mu} - \frac{\partial G_{\mu 5}}{\partial x^\nu} \right) \frac{dx^\nu}{d\tau} \qquad (1.8)$$

This equation (1.8) when compared to (1.6) suggests that the metric components $G_{\mu 5}$ can be identified with the electromagnetic vector potential.

$$\frac{G_{\mu 5}}{G_{55}} = A_\mu \quad ; \quad a = \frac{q}{m_0} \tag{1.9}$$

with this identification, the 4-dimensional part of the 5-dimensional geodesic equation yields the Lorentz force equation. This can be taken to be the first hint that a 5-dimensional theory with Kaluza's ansatz (1.2) unifies electromagnetism with gravity in the sense that both the 4-dimensional gravitation and electromagnetic fields are components of the metric in five dimensions. The constant 'a' in (1.9) depends upon the specific particle under consideration.

The metric G_{MN} can then be specifically written (putting $G_{55} = 1$) using equations (1.5) and (1.9)

$$G_{MN} = \left(\begin{array}{c|c} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ \hline A_\nu & 1 \end{array} \right) \tag{1.10}$$

So considering pure gravity action in five dimensions

$$S = - \int d^4 x \, dy \sqrt{G} \; \hat{R} \tag{1.11}$$

and integrating over the y co-ordinate we get the four dimensional effective action

$$S = - \int d^4 x \sqrt{g} \; R - \frac{1}{4} \int d^4 x \sqrt{g} \; F_{\mu\nu} F^{\mu\nu} \tag{1.12}$$

where the first and second term correspond to 4-dimensional gravity and 4-dimensional electromagnetism respectively. From the Einstein's equation that follow from the above action (1.11) is

$$\hat{R}^{MN} - \frac{1}{2} G^{MN} \hat{R} = 0 \tag{1.13}$$

The M = μ, B = ν components of this equation is

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \frac{1}{2} \left\{ \frac{1}{4} g^{\mu\nu} F^2 - F^{\mu\rho} F_{\rho}{}^{\nu} \right\} = 0 \tag{1.14}$$

If the matter fields are present, they are taken into account by putting the corresponding energy-momentum tensor $T^{\mu\nu}$ on the right hand side of the Einstein's equations. While the left hand side describes geometry, the right hand side describes matter. In contrast to this, the pure gravity in 5 - dimensions in Kaluza-Klein yields the left hand side which naturally contains matter fields. Transferring to the right hand side we have the standard equation for gravity and electromagnetism.

Now consider a sub class of co-ordinate transformations in 5-dimensions

$$x^{\mu} \longrightarrow x'^{\mu} = x^{\mu} \quad \text{and} \quad y \longrightarrow y' = y + f(x^{\mu}) \tag{1.15}$$

Using the transformation law of metric G_{MN} under the above co-ordinate transformation (taking G_{55} = 1 again), we find

$$G_{\mu 5} \longrightarrow G'_{\mu 5} = G_{\mu 5} - \partial_\mu f(x^\nu) \qquad (1.16)$$

using (1.9) it follows that the vector potential A_μ transform as

$$A_\mu \longrightarrow A'_\mu = A_\mu - \partial_\mu f(x^\nu) \qquad (1.17)$$

which is the familiar gauge transformation for electromagnetic vector potential. If fermions are introduced in the action (1.11) then one obtains the usual minimal coupling with A_μ.

It is to be noted that the momentum conjugate to the fifth co-ordinate gives the charge of the particle and the vanishing of the divergence of ($\hat{R}^M - \frac{1}{2} G^M_N \hat{R}$) for M = 5 gives the conservation of charge. Here we note that the metric as it stands in (1.10) have all the components of G_{MN} independent of y. This corresponds to the n = 0 terms of the fourier series expansion of the various metric components as they must be periodic in the co-ordinates ($y = r\theta$, r being the radius of the S^1)

$$g_{\mu\nu}(x,\theta) = \sum_{n=-\infty}^{+\infty} g^{(n)}_{\mu\nu}(x)\, e^{in\theta}$$

$$A_\mu(x,\theta) = \sum_{n=-\infty}^{+\infty} A^{(n)}_\mu(x)\, e^{in\theta} \qquad (1.18)$$

with

$$g^{(n)*}_{\mu\nu} = g^{(-n)}_{\mu\nu} \qquad \text{etc.}$$

The vacuum being characterized by the ground state metric G_{MN} in (1.10), is given by

$$< G_{MN} > = \left(\begin{array}{c|c} \eta_{\mu\nu} & 0 \\ \hline 0 & 1 \end{array} \right) \qquad (1.19)$$

i.e. $\quad < g_{\mu\nu} > = \eta_{\mu\nu} , \quad < A_{\mu} > = 0$

which implies that the symmetry of the vacuum is four dimensional Poincare group $\otimes$ R. Note that the gauge group is R rather than U(1) because this truncated n = 0 theory has lost all memory of the periodicity in θ [3].

At last we are going to relate the charge to the size given by the radius 'r' of S^1. For this we consider a complex scalar field ϕ in five dimension being described by the action[4]

$$S = \int d^4x \, dy \sqrt{G} \left[(\partial_M \phi) \, (\partial_N \phi)^{\dagger} \, G^{MN} \right] \qquad (1.20)$$

The assumption of compactification means that ϕ can be expanded in a Fourier series in y (i.e.θ) like (1.18)

$$\phi = \sum_n \phi^{(n)} \, (x^{\mu}) \quad \exp \, (\, in\theta \,) \qquad (1.21)$$

Inserting this (1.21) into the action (1.20) gives on integration over y

$$S = \sum_n \int d^4x \sqrt{g} \left(\left| \left(\partial_\mu + \frac{in}{r} A_\mu \right) \phi^{(n)}(x^\mu) \right|^2 - \frac{4\pi^2 n^2}{r^2} \left| \phi^{(n)} \right|^2 \right) \tag{1.22}$$

showing that the charge is quantized in discrete units (1/r) where e is given by (n/r). If we put the known values of the electronic charge as e we indeed get r to be of the order of Planck length.

Section II

Here we discuss briefly the generalization of this five dimensional theory to still higher dimension in order to accommodate non-abelian gauge theory. This generalization is due to B. deWitt, Salam and Strathdee, Cho, Cho and Freund and several others[2]..

Because of dimensional economy one generally takes the extra space to be a coset space $B^n = G/H$. To this end we start by giving a brief outline of the mathematical formalisms of coset space as discussed by Salam and Strathdee[2].

Let $Z^M = (x^\mu, y^i)$ be the co-ordinates of the entire (4+n) dimensional space-time manifold with x^μ, y^i parametrizing the base four dimensional Minkowski space M^4 and $B^n = G/H$ respectively. Thus y^i corresponds to the cosets of G with respect to H. Let there be a chosen representative element $L_y \in G$ from each coset. Multiplication from the left by an arbitrary element $g \in G$ will generally carry L_y into another coset, one for which the representative element is $L_{y'}$.

$$g\, L_y = L_{y'}\, h \qquad \text{with } h \in H \tag{1.23}$$

Both y'^i and h are determined by this equation as function of y^i and g.

To define a covariant basis consider the left invariant 1-form

$$e(y) = L_y^{-1}\; d\, L_y \tag{1.24}$$

This object belongs to the infinitesimal Lie algebra $\not{G}$ of G and therefore can be expressed as a linear combination of the generators $Q_{\hat{a}}$, satisfying $[Q_{\hat{a}}, Q_{\hat{b}}] = f_{\hat{a}\hat{b}}{}^{\hat{c}}\, Q_{\hat{c}}$ as

$$e(y) = e^{\hat{a}}(y)\, Q_{\hat{a}} = dy^i\, e_i^{\hat{a}}(y)\, Q_{\hat{a}} \tag{1.25}$$

The generators $Q_{\hat{a}}$ falls into two categories : The set $Q_{\bar{a}}$ which generates the subgroup H and the remainder set Q_a a = 1, ..n associated with the cosets G/H. Correspondingly one writes

$$e(y) = e^a(y)\, Q_a + e^{\bar{a}}\, Q_{\bar{a}} \tag{1.26}$$

Of special interest are the 1-form valued coefficients $e^a(y) = e_i^a\, dy^i$ provide the orthonormal basis on the cotangent space. And e_i^a are the n-beins. The adjoint representation D(g) of the group

G is defined

$$g^{-1} Q_{\hat{a}} g = D_{\hat{a}}{}^{\hat{b}}(g) \; Q_{\hat{b}} \tag{1.27}$$

We shall assume that the algebra G is fully reducible i.e.

$$D_{a}{}^{\bar{b}}(h) = D_{\bar{a}}{}^{b}(h) = 0$$

Under an infinitesimal left translation by $g \equiv (1 + \delta g^{\hat{a}} Q_{\hat{a}})$ we have from (1.23)

$$L_y \longrightarrow L_{y'} = L_{y+\delta y} = g \, L_y \, h^{-1} \tag{1.28}$$

Writing $h = 1 + \delta h^{\bar{a}} Q_{\bar{a}}$, we get

$$\delta y^i = \delta g^{\hat{a}} \; K_{\hat{a}}{}^i (y) \tag{1.29}$$

where

$$K_{\hat{a}}{}^i = D_{\hat{a}}{}^b (L_y) \, e_b{}^i (y) \tag{1.30}$$

defines the components of the killing vector $K_{\hat{a}} = K_{\hat{a}}^i \partial_i$ and $e_b^i(y)$ being the inverse of $e_i^a(y)$. It is easy to show that the killing vectors also satisfy G Lie algebra[5]

$$[K_{\hat{a}}, K_{\hat{b}}] = - f_{\hat{a}\hat{b}}{}^{\hat{c}} \, K_{\hat{c}} \tag{1.30a}$$

Now coming back to this higher dimensional theory, we make the ansatz for (4+n) bein $E_M^A(x,y)$ as

$$E_M{}^A(X,Y) = \begin{pmatrix} E_\mu{}^\alpha(X) & -A_\mu{}^a(X)\, D_a{}^c(L_y) \\ 0 & e_i{}^a(y) \end{pmatrix} \qquad (1.31)$$

from which the (4+n) dimensional metric G_{MN} is calculated as $G_{MN} = E_M{}^A E_N{}^B \eta_{AB}$

$$\left(\begin{array}{c|c} g_{\mu\nu}(X) + A_\mu{}^a(X)\, A_\nu{}^b(X)\, K_a{}^m(y)\, K_b{}^n(y)\, g_{mn}(y) & A_\mu^a(X)\, K_{am}(y) \\ \hline A_\nu{}^a(X)\, K_{an}(y) & g_{mn}(y) \end{array}\right)$$

(1.32)

As in the five dimensional case this metric corresponds to the lowest order term of the series obtained by harmonic expansion on G/H.In the low energy sector only the leading term will be relevant. This ansatz (1.32) is compatible with a subgroup of (4+n) dimensional symmetries: the 4-dimensional general co-ordinate transformation

$$x^\mu \longrightarrow x'^\mu = x'^\mu(x^\nu)$$

and x dependent left translation

$$y^i \longrightarrow y'^\mu(x,y) \qquad (1.33)$$

with associated frame rotation $D_a{}^b(h)$. This in turn implies the following transformation Law for $E_\mu{}^a$

$$E_\mu{}^a\,(x,y) \longrightarrow E'^{\,a}_\mu\,(x',y') = \left(\frac{\partial x^\nu}{\partial x'^\mu}\, E_\nu{}^b\,(x,y) + \frac{\partial y^i}{\partial x'^\mu}\, e^b_i\,(y) \right) D_b{}^a(h^{-1})$$

In particular, from the ansatz (1.31) we get

$$- A'^{\,\hat{a}}_\mu(x')D_{\hat{a}}{}^b(L_y) = \left(- \frac{\partial x^\nu}{\partial x'^\mu}\, A_\nu{}^{\hat{b}}(x)\; D_{\hat{b}}{}^c(L_y) + \frac{\partial y^i}{\partial x'^\mu}\, e\,{}_i^c(y) \right) D^b_c\,(h^{-1})$$

From this it can be shown that

$$A'_\mu\,(x') = \frac{\partial x^\nu}{\partial x'^\mu} \left(gA_\nu(x)\, g^{-1} - g\,\partial_\nu\, g^{-1} \right) \qquad (1.34)$$

where $A_\mu = A^{\hat{a}}_\mu\, Q_{\hat{a}}$ and $g \in G$. This is precisely the rule of transformation for a Yang - Mills potential under gauge transformation. Thus we get gauge transformation in 4-dimensional space as a manifestation of co-ordinate transformations in the extra space B^n. In view of the relation (1.30a) this identifies the four dimensional gauge group with the group of isometry of the coset space B^n.

Assuming torsion to be zero, we get the spin connection one form ω as

$$0 = dE^A + \omega^A{}_B \wedge E^B \qquad (1.35)$$

From where the various components of the curvature two form can be computed using

$$\hat{R}^{A}{}_{B} = d\omega^{A}{}_{B} + \omega^{A}{}_{C} \wedge \omega^{C}{}_{B} \tag{1.35a}$$

Finally computing the scalar curvature $\hat{R}$ for the entire (4+n) space one gets

$$\hat{R} = R + \frac{1}{4} F_{\mu\nu}{}^{\hat{a}} F^{\mu\nu\,\hat{b}} \; D^{c}_{\hat{a}}(L_y)\, D_{\hat{b}}{}^{c}(L_y) + R_n \tag{1.36}$$

Here R is the usual 4-dimensional curvature scalar and R_n the constant curvature of G/H. On integrating over the internal co-ordinates y^i one gets usual four dimensional effective action consisting of 4 dimensional gravity with 4-dimensional Newton's constant $G_4 = G_{4+n}/\Omega_n$ (G_{4+n} being the (4+n) dimensional Newton's constant) along with 4-dimensional Yang - Mills field described by the gauge group G. We use the orthonormality relation

$$\frac{1}{\Omega_n} \int d^n y \det e(y)\, D_{\hat{a}}{}^{c}(L_y)\, D_{\hat{b}}{}^{c}(L_y) = k\, \delta_{\hat{a}\hat{b}} \tag{1.37}$$

where $k = \dfrac{\dim (G/H)}{\dim G}$ and Ω_n is the volume of B^n and $\det(e(y)) = \det(e^{a}{}_{i})$. We note that R_n is effective cosmological constant in four dimension. This term was absent in five dimensional case as curvature of S^1 is zero. Now we shall show that R_n or rather Ricci curvature tensor must be non zero in order to realize Yang-Mills fields with non-abelian gauge group G.

As we know the killing vectors $K^{\hat{a}}_{i}$ must satisfy by definition

$$\nabla_i K^{\hat{a}}_j + \nabla_j K^{\hat{a}}_i = 0 \qquad (1.38)$$

Multiplying (1.38) by g^{ij}, one finds that the covariant divergence of the killing fields must vanish

$$\nabla^j K^{\hat{a}}_j = 0 \qquad (1.39)$$

Again applying ∇_k on (1.38) and then multiplying by g^{ik}, we get using (1.39)

$$\nabla_i \nabla^i k^{\hat{a}}_j + g^{ik} \left[\nabla_k, \nabla_j \right] K^{\hat{a}}_i = 0 \qquad (1.40)$$

using

$$\left[\nabla_k , \nabla_j \right] K^{\hat{a}}_i = R^n_{\ ijk} K^{\hat{a}}_n \qquad (1.41)$$

one gets

$$\nabla_i \nabla^i K^{\hat{a}}_j + R^n_{\ j} K^{\hat{a}}_n = 0 \qquad (1.42)$$

Multiplying (1.42) by $g^{mj} K^{\hat{a}}_m$ and integrating over B^n, we get

$$\int d^n y \; e(y) \; g^{mj} g^{ik} \left(\nabla_i k_m^{\ \hat{a}} \right) \left(\nabla_k k_j^{\ \hat{a}} \right) = \int d^n y \; e(y) \; k^{j\hat{a}} R^n_{\ j} k^{\hat{a}}_{\ n} \qquad (1.43)$$

Now from this equation (1.43) it follows that if the Ricci curvature $R^n{}_j$ vanishes, then we must have $\nabla_i k^{\hat{a}}_m = 0$, which in turn implies that the group must be abelian (see equation (1.30)).

$$\left[K_{\hat{a}} , K_{\hat{b}} \right] = K^i_{\hat{a}} \nabla_i K_{\hat{b}} - K^j_{\hat{b}} \nabla_i K_{\hat{a}} = 0 \qquad (1.44)$$

Thus we see that to realize non-abelian symmetries we must have non-Ricci flat extra space B^n. To balance this curvature in B^n one has to put additional matter term on the right hand side of the Einstein equation.

the relation between charge and the size of circle S^1 in the five dimensional case was generalized by Weinberg[4,6] for this non-abelian case also where the relation between the gauge coupling to the appropriate root mean square circumference of the extra space is established.

The ground state geometry of the whole (4+n) dimensional space time is taken to be $M^4 \times B^n$ with the corresponding vacuum expectation value of the metric G_{MN} being

$$\langle G_{MN} \rangle = \left(\begin{array}{c|c} \eta_{\mu\nu} & 0 \\ \hline 0 & g_{mn}(y) \end{array} \right) \text{ i.e. } \langle g_{\mu\nu} \rangle = \eta_{\mu\nu} , \langle A_\mu \rangle = 0 \qquad (1.45)$$

To balance the curvature of B^n in the ground state the vacuum expectation value of the energy momentum tensor T_{MN} also must be non vanishing.

We have seen that the gauge group G in the effective four dimensional theory is given by the isometry group of B^n. However this statement requires modification if the matter fields Φ with non zero vacuum expectation value transform non trivially under G that is

$$\mathcal{L}_K \langle\Phi\rangle \neq 0 \tag{1.46}$$

in which case the unbroken gauge group corresponding to the massless gauge bosons is given by some sub group of G. For example in the Gell-Mann and Zwiebach [7] model we consider (see Chapter 2) the σ - model fields $\langle\phi^i\rangle \neq 0$ break all the symmetries leaving no massless gauge bosons at all in the d = 4 theory.

Though Kaluza-Klein methodology is very attractive as far as the gauge fields are concerned it poses serious difficulties in getting chiral fermions (See Witten [2]).

To understand the basic idea of getting massless chiral fermions let us consider a massless spin 1/2 particle in (4+n) dimensional theory satisfying Dirac equation

$$D\,\Psi = 0 \tag{1.47}$$

i.e.,

$$\Gamma^A E_A{}^M \nabla_M \Psi = 0 \tag{1.48}$$

where $D = \Gamma^A E_A{}^M \nabla_M$ is the (4+n) dimensional Dirac operator. Γ^A are the generalised Dirac matrices satisfying (4+n) dimensional Clifford algebra $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$ and $\nabla_M = \partial_M + \omega_M$,with ω being SO(1,3+n) Lie algebra valued spin connection ($=1/8\ \omega_{M[A,B]}\Sigma^{AB}$).

The wave function Ψ is treated as a world scalar and a SO(1,3+n) spinor having $2^{2+[n/2]}$ ($[n/2]=n/2$ for n even and $(n-1)/2$ for n odd) components. This Dirac operator can be written as a sum of the Dirac operators corresponding to the 4 and n dimensional spaces respectively enabling us to rewrite the equation (1.47) as

$$D^{(4)}\Psi + D^{(n)}\Psi = 0 \qquad (1.49)$$

The expression (1.49) immediately shows us that the eigen value of the Dirac operator in the n dimensional internal space $D^{(n)}$ will be observed in practice as the4-dimensional mass. Thus to get chiral fermions we must get hold of the zero modes of $D^{(n)} = \Gamma^a e_a^{\ i} \nabla_i$. But it can be shown that the square of $D^{(n)}$ is $(-\nabla_a \nabla^a + 1/4\ R^{(n)})$. Since $(-\nabla_a \nabla^a)$ is a positive operator (eigenvalues on compact B^n space are positive) and $R^{(n)}>0$ for a compact B^n we can not have zero eigenvalues for $D^{(n)}$ and hence no chiral fermions . This is the famous Lichnerowicz theorem[8]. To circumvent this difficulty one generally puts topologically non trivial gauge fields, like monopoles in S^2, which besides triggering spontaneous compactification, will couple to fermions and the mass operator will become $\Gamma^a e_a^{\ i} (\nabla_i + A_i)$ whose square may admit zero eigenvalues. Alternatively one may put torsion on B^n. Wu and Zee.Destri,Orzalazi and Rossi[9] have analysed the zero modes on compact group manifolds with parallelizable torsion and have found many such modes. However, they find that these solutions are parity invariant thus belonging to the real representations of G.

By considering B^n to be S^4,Neville[10] demonstrated that S^4 has zero modes if the gravitational field is endowed with torsion where the spin connection corresponds to Belavin-Polyakov-Schwarz-Tyupkin (BPST) SU(2) instanton. Later Tchrakian[11] generalized it for S^{4p}.

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CHAPTER II

QUANTUM EFFECTS IN KALUZA-KLEIN THEORY WITH SPONTANEOUS COMPACTIFICATION INDUCED BY NON-LINEAR SIGMA MODEL[1]

In this Chapter, we study the quantum effects in Kaluza-Klein theory with spontaneous compactification induced by a non-linear sigma model proposed by Gell-Mann and Zwiebach [1]. This study is of two fold interest. It can be used to trigger a dynamical compactification in which the energy momentum tensor arises from the quantum fluctuations of the various matter fields or it can be used to examine the stability of a given classical solution (in which the energy momentum tensor is due to some topologically non-trivial classical field configuration) against the quantum fluctuations. In higher dimensional theories, the extra dimensions are usually assumed to form a compact space of size of the order of Planck length, undetectable at present. This assumption thus far has no natural explanation as to its origin. Appelquist and Chodos[2] showed that the quantum fluctuations in the original 5-dimensional Kaluza-Klein theory may be responsible for the smallness of the 5th dimension - Casimir effect mentioned in the Introduction. However parametrising the quantum one loop potential, when all the fields are massless in (4+n) dimensions, by C/r^4 where r is the size of the manifold of extra dimensions, C

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[1] B. Chakraborty and R. Parthasarathy, Class. and Quantum Grav.6 (1989) 1455.

is found to be negative in [2]. While this provides an attractive force, similar to the attractive force between the parallel plates in the original Casimir effect, responsible for shrinking the extra space to Planck length, poses problems of stability of the theory against quantum fluctuations with respect to the classical background field assumed. In fact inclusion of matter fields [3] render the theory stable. The stability argument need to be understood in the following manner : the effective potential when varied with respect to the size of the extra space r has to have a non-trivial minimum and this happens when $C > 0$. In fact Candelas and Weinberg[4] have evaluated the quantum 1-loop potential due to scalars and fermions assuming flat background metric for gravitation. Using the positivity argument for C, they conclude that a Kaluza-Klein theory with $4+n = 7,11,15,19$ and 23, with their ansatz for classical fields, is stable against quantum fluctuations. With suitably large number of scalars and fermions, Candelas and Weinberg[4] obtain the result that for $4+n = 9,13,17,21,27,31$ and 33 the Kaluza-Klein theory could be made stable. Thus it appears that the matter fields play a dominant role in governing the stability of the theory. Following [4], Kato and Saito [5] evaluated the contribution to the 1-loop effective potential due to Abelian gauge fields in $(4+n)$ dimensions. Although the introduction of gauge fields is against the very spirit of Kaluza-Klein theory in which they arise as part of metric, their [5] results indicate that $4+n = 7$ to 17, C is positive and hence the theory is stable. It is to be noted that the quantum effects evaluated in references [4] and [5] are for arbitrary number of extra matter fields, arbitrary number of scalars and fermions in [4] and Abelian gauge fields in [5]. Thus it is important to examine the quantum effects in *specific models with spontaneous compactification*. Specific models such as monopole induced compactification of Randjbar-Daemi, Salam and Strathdee [6] and sigma-model induced compactification of Gell-Mann and Zwiebach [1] are proposed for possible unification of gravity with gauge fields and so their stability against quantum fluctuations should be examined. Secondly, in these models one has to consider the fluctuations of various fields with respect to the solution of classical Einstein equations for $M^4 \times B^n$ and these fluctuations are of different nature from those studies in [4] and [5]. An attempt in this direction was made by Castaldini [7] who examined the monopole induced compactification model [6]. To avoid conformal anomalies arising in even dimensions [4], Castaldini [7] retained the magnetic monopole classical solution on S^2 but considered the dimensions of the usual space time to be 3,5,7 and 9 which is in contradiction with the reality of the physical world. As the monopole induced compactification [6] is for $M^4 \times S^2$ with monopole configuration in S^2, Castalidini [7] is forced to consider $M^3 \times S^2$, $M^5 \times S^2$, $M^7 \times S^2$ and $M^9 \times S^2$. On the other hand the non-linear Sigma model induced compactification [1] is defined on $M^4 \times S^n$ and so we need not choose odd dimensions for the physical spacetime. To our knowledge this study has not been made in the literature.

Although Gell-Mann and Zwiebach [8] themselves discuss the fluctuations they did not evaluate the 1-loop potential.

§ 1 Sigma model induced compactification :-

In this scheme the spontaneous compactification is triggered by a scalar sector in the form of a non-linear sigma model. Dimensional reduction induced by non-linear sigma model has been studied earlier by Omero and Percacci [9].

This model is based on Einstein gravity in (D+n) dimensions (D can be eventually taken to be 4) coupled to a non-linear σ-model with scalars ϕ^i, $i = D+1, \ldots D+n$, as coordinates of an n-dimensional compact scalar manifold, of metric $m_{ij}(\phi)$. The action is

$$S = \int d^{D+n} Z \sqrt{G} \left\{ -\frac{1}{2} \hat{R} + \frac{1}{\lambda^2} G^{MN} m_{ij}(\phi) \partial_M \phi^i \partial_N \phi^j \right\} \quad (2.1)$$

where $Z^M = (X^\mu, Y^i)$, $\mu = 1,2,\ldots D$; $i = D+1, \ldots D+n$

X^μ the co-ordinates of (non-compact) D-dimensional space

Y^i the co-ordinates of the compact n dimensional extra space

G^{MN} the metric in (D+n) dimensions of signature (+ - ...-)

$\hat{R}$ is $\hat{R}^{MN} G_{MN}$, the scalar curvature in (D+n) dimensions and λ a constant.

The classical equations of motion are obtained by the variation of the action (2.1) with respect to the metric G^{MN} and ϕ^i. They are respectively,

$$\hat{R}_{MN}=\frac{2}{\lambda^2}\, m_{ij}(\phi)\ \partial_M\phi^i\partial_N\phi^j \tag{2.2}$$

$$\sqrt{G}\ \left(\frac{\partial}{\partial\phi^p}\, m_{ij}(\phi)\right)\ G^{MN}\ \partial_M\phi^i\partial_N\phi^j \ =\ 2\ \partial_M\left[\sqrt{G}\ \ m_{pj}(\phi)\ \partial^M\phi^j\right] \tag{2.3}$$

Gell-Mann and Zwiebach [1] made the ansatz,

$$\phi^i(X,Y)\ =\ \phi^i(Y)\ =\ Y^i, \tag{2.4}$$

i.e. the scalar fields ϕ^i are assumed independent of x and further identified with the coordinates y^i of the manifold B^n. In fact this ansatz for $B^n = S^2$ corresponds to monopole configuration in S^2 [10]. With this ansatz, (2.2) readily gives,

$$R_{\mu\nu}\ =\ 0$$

$$R_{ij}\ =\ \frac{2}{\lambda^2}\ m_{ij}(Y) \tag{2.5}$$

where the background metric (in the ground state).

$$G^{MN}\ =\ \begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & g^{ij} \end{pmatrix} \tag{2.6}$$

is used. The result $R_{\mu\nu}= 0$ (2.5) allows us to take M^4 to be Minkowskian flat. In B^n, one obtains a very interesting relation (2.5). Thus the problem of scalar induced compactification is reduced to finding a metric g_{ij} for B^n whose Ricci curvature equals to the scalar manifold metric $m_{ij}(Y)$. For positively curved compact B^n, if $g_{ij} = -a^2 m_{ij}$, then a solution exists if $\lambda^2 = \frac{2}{\bar{\alpha}^2}$ where $R_{ij}(m) = \bar{\alpha}^2 m_{ij}$. Here the extra dimensions roll up to form a manifold of the same shape as that of the scalar manifold. With the solution (2.4) and (2.6) the other classical field equation (2.3) is satisfied identically. Eqn.(2.3) gives no new information. The essential features of this scheme are (i) the usual four dimensional space M^4 is flat ($R_{\mu\nu} = 0$) and so can be taken to be Minkowskian flat. (ii) there is no need to introduce a cosmological constant (to be compared with the monopole induced compactification [6] where a cosmological constant has to be introduced) (iii) the scalar fields are defined on the *extra space only* and (iv) the m_{ij} metric must be Einsteinian. Harmonic expansion can be made for S^n for example. However when the background metric G_{MN} in (2.6) is replaced by the Kaluza-Klein metric with gauge fields, namely

$$G^{MN} = \begin{bmatrix} g^{\mu\nu} & -A^{\mu\hat{a}}(X)\, K^i_{\hat{a}}(Y) \\ -A^{\nu\hat{a}}(X)\, K^i_{\hat{a}}(Y) & g^{ij} + A^{\hat{a}}_{\mu}(X)\, A^{\mu\hat{b}}(X)\, K^i_{\hat{a}}(Y)\, K^j_{\hat{b}}(Y) \end{bmatrix} \qquad (2.7)$$

where $A^{\mu\hat{a}}(x)$ are the Kaluza-Klein gauge fields belonging to the

Lie algebra of $G(B^n = G/H)$ and $K^i_{\hat{a}}(Y)$ are the Killing vectors, and is substituted in the action (2.1), the first term gives gravity in 4-dimensions and a Yang-Mills Kinetic energy term for $A^{\hat{a}}_{\mu}(x)$ after integrating over y while the second term gives rise to a mass term for $A^{\hat{a}}_{\mu}(x)$, namely

$$- \frac{1}{2a^2\lambda^2} \int d^4X \sqrt{g}\; A^{\hat{a}}_{\mu}(X)\, A^{\mu\hat{b}}(X) \int d^nY \sqrt{g_{ij}}\; K^i_{\hat{a}}(Y)\, K_{i\hat{b}}(Y).$$

The Y-integration gives δ_{ab} and a constant so that the mass term has a coefficient ~ $1/a^2\lambda^2$. Since a is the size of B^n which is assumed to be small ~ Planck length, the Kaluza-Klein gauge bosons acquire Planck mass. To understand precisely about what goes wrong it is worthwhile here to examine the symmetry properties of the action (2.1). It is invariant under the general co-ordinate transformation, which for infinitesimal transformations are

$$Z^M \longrightarrow Z'^M = Z^M + \varepsilon^M(Z^N)$$

$$\delta G_{MN} = G'_{MN}(Z) - G_{MN}(Z) = -(\varepsilon_{M;N} + \varepsilon_{N;M}) \qquad (2.8)$$

$$\delta\phi^i = -\varepsilon^M \partial_M \phi^i$$

It is also invariant under "internal symmetry" transformation

$$\delta G_{MN} = 0\ ,\ \delta\phi^i = \eta_{\hat{a}} K^i_{\hat{a}}(\phi^k) \qquad (2.9)$$

where $K^i_{\hat{a}}(\phi^k)$ are the Killing vectors in the scalar manifold with $\hat{a}$ being the group index and $\eta_{\hat{a}}$'s are constants.

Now in order to have mass less gauge bosons the isometries must be invariances of the full background configuration, that is not only the gravity but also the scalars. As we have seen in chapter 1 that the gauge transformations correspond to X^{μ} dependent co-ordinate transformations in the extra dimensions along the Killing vectors $y^{i} \longrightarrow y'^{i} = y^{i} + \alpha_{\hat{a}}(X^{\mu})\, K^{i}_{\hat{a}}(y^{k})$. For such transformations the metric tensor is obviously invariant but the scalars are not and yield (equation (2.4)) $\delta\phi^{i} = -\alpha_{\hat{a}}(X^{\mu})\, K_{\hat{a}}(y^{k})$. One can use the internal symmetry of the σ - model given in (2.9) to cancel this provided that we choose $\alpha_{\hat{a}} = \eta_{\hat{a}}$, a constant. But if $\alpha_{\hat{a}}$ depends on x^{α} we cannot have an invariant scalar field background. We therefore have a background invariant under global G transformations only and this is the symmetry of the dimensionally reduced theory. As we do not have local G invariance in d = 4 theory we have no mass less gauge bosons left. This difficulty can be circumvented in a compactification scheme using a general non - linear σ-model [10].

In this chapter we examine the stability of the theory described by (2.1) with the classical solutions (2.4) and (2.6) with $g_{\mu\nu}$ in M^{4} as a flat metric against quantum fluctuations. Therefore the afore mentioned issue of Planck mass for Kaluza-Klein gauge bosons is not relevant here.

§.2. Fluctuation analysis :-

The action (2.1) contains two kinds of fields namely, G_{MN} and

ϕ^i. Let $\bar{G}_{MN}$ and Φ^i denote the solutions to the classical equations of motion, which are

$$\bar{G}_{MN} = \begin{pmatrix} \bar{g}_{\mu\nu} & 0 \\ 0 & \bar{g}_{ij} \end{pmatrix} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & -a^2\, m_{ij}(Y) \end{pmatrix} \tag{2.10}$$

and

$$\Phi^i = Y^i \tag{2.11}$$

The fluctuations around the classical solutions are defined by

$$G_{MN} = \bar{G}_{MN} + h_{MN} \tag{2.12}$$

and

$$\phi^i = Y^i + z^i(X,Y) \tag{2.13}$$

where h_{MN} and $z^i(X,Y)$ are the fluctuations. It is to be noted that while the classical solution $\Phi^i = y^i$ not depending upon X^μ, the fluctuation $z^i(X,Y)$ may have x-dependence as well. We use the *background field expansion* and treat the fluctuations to be small when compared to their classical counterparts so that terms of order more than quadratic in h_{MN} and z^iin the expanded action are neglected. This standard procedure essentially linearises the field equations for the fluctuations. [In the functional integral formalism, this allows Gaussian integration over the quantum fields]. Substituting (2.12) and (2.13) in (2.1)and using the expansion for $\hat{R}$, G^{MN} and $\sqrt{G}$, we obtain

$$S = S_o + S_{hh} + S_{zz} + S_{hz} \tag{2.14}$$

where

S_o is the classical action involving $\overline{G}^{MN}$ and Φ^i

S_{hh} is the part quadratic in h_{MN}

S_{zz} is the part quadratic in z^i

S_{hz} is the part quadratic in the combined fluctuations i.e terms involving $h_{MN}\ z^i$

The detailed expressions of S_{hh}, S_{zz} and S_{hz} are given below

$$S_{hh} = \int d^{4+n}Z \ \sqrt{\overline{G}} \ \Big[\frac{1}{8} h_{MN;P}\, h^{MN;P} - \frac{1}{8} h_{;M} h^{;M} - \frac{1}{4} h^{MN}{}_{;N}\, h_{MP}{}^{;P} + \frac{1}{4} h_{MN}{}^{;N} h^{;M} + \frac{1}{4} \overline{R}_{MN} h^{MP}\, h^{N}{}_{P} - \frac{1}{4}\ \overline{R}^{MNPQ} h_{MP}\, h_{NQ} \Big] \tag{2.15}$$

$$S_{zz} = \int d^{4+n}Z \ \sqrt{\overline{G}}\ \lambda^{-2} \Big[\overline{g}^{\mu\nu}\, m_{ij} \left(D_\mu z^i \right) \left(D_\nu z^j \right) + \overline{g}^{pq} \left\{ m_{ij} \left(\tilde{D}_p\, z^i \right) \left(\tilde{D}_q\, z^j \right) - \tilde{R}_{ipjq}\, z^i z^j \right\} \Big] \tag{2.16}$$

$$S_{hz} = \frac{2}{\lambda^2} \int d^{4+n}Z \ \sqrt{\overline{G}} \ \Big[\overline{g}^{pq}\, z_p (D_\mu h^\mu_q - \frac{1}{2} D_q h) - h^{pq}\, \tilde{D}_p z_q \Big] \tag{2,17}$$

where the covariant derivatives D_μ and $\tilde{D}_P$ are taken with background metric $\bar{G}_{MN}$ and with the scalar field manifold metric m_{pq} respectively. $\tilde{R}_{ipjq}$ is the Riemann tensor for the scalar manifold. Finally $z_i = m_{ij}\, z^j$.

§.3 1-loop operator :-

We examine the fluctuations in the matter field ϕ^i. This is in the spirit of Candelas and Weinberg [4]. Then the expression (2.12) involves S_{zz} only, besides S_o. [we derive the expression for S_{zz} given in (2.16) in the appendix I to this chapter]. By using the classical equation $\bar{g}_{ij} = -a^2 m_{ij}$, it follows

$$\tilde{R}^i{}_{pjq} = \bar{R}^i{}_{pjq} . \qquad (2.18)$$

The extra space B^n is taken to be S^n. This choice had been made by Gell-Mann and Zwiebach [8]. The line element is then given by

$$ds^2 = r^2\, k_{pq}\, dy^p\, dy^q \qquad (2.19)$$

where r is the radius of S^n and so the metric

$$\bar{g}_{pq} = r^2 k_{pq}. \qquad (2.20)$$

using $\bar{g}_{pq} = -a^2 m_{pq}$. S_{zz} becomes after an integration by parts

$$S_{zz} = -\frac{1}{\lambda^2}\int d^{4+n}Z\ \sqrt{\bar{G}}\ z_i\Big[\ \Box + r^{-2}\left\{\nabla_s^2 + (n-1)\right\}\Big]\ z^i \quad (2.21)$$

where

$$\Box = \bar{g}^{\mu\nu} D_\mu D_\nu, \text{ the d'Alembertian in } M^4 \text{space}$$

and $$\nabla^2_s = k^{pq} D_p D_q, \text{ the Laplacian in } S^n \text{ modulo } r^2.$$

The operator in the square bracket in (2.21) is the desired one-loop operator for the quantum fluctuations of ϕ^i in the theory. A few comments are in order. First of all the integral in (2.21) is quadratic in z^i and so in the generating function $Z = \int [dz^i] e^{-S_{zz}}$ the functional integral becomes Gaussian. This is the result of keeping terms up to quadratic in z^i in the expansion. The terms linear in z^i drop out by virtue of equations of motion. Secondly although $\Box$ in (2.21) is in M^4, it still acts on z^i since $z^i = z^i(X,Y)$ will have in general some x-dependence.

§.4 Eigenvalues and Degeneracy of the 1-loop operator :-

As remarked above the one-loop operator acts on $z^i(X,Y)$ which are components of the contravariant vector z on the *scalar manifold*. By the classical equation $\bar{g}_{ij} = -a^2 m_{ij}$($\bar{g}_{ij} = r^2 k_{ij}$ is the metric on the scalar manifold and both these are same except for a scaling factor) we understand that both the scalar manifold and S^n have same topology. With the product manifold structure $M^4 \times S^n$ in mind, the eigenvalues of $\Box = \bar{g}^{\mu\nu} D_\mu D_\nu = \eta^{\mu\nu} \partial_\mu \partial_\nu$ (M^4 is Minkowskian flat) are simply $-k_\mu k^\mu$, (Chodos and Myers [11]).On the other hand, to find the eigenvalues of ∇^2_s which is the Laplacian on S^n we have to do harmonic analysis on S^n. This is carried out in the *Appendix* II.

As z^{i}'s are contravariant vectors on S^n we decompose it into a divergenceless part and a gradient of a scalar part. We need eigenvalues and degeneracies of ∇_s^2 for both the cases which are distinctively different. Let us denote the eigenvalues of ∇_s^2 by $\lambda(n,\ell)$ and degeneracy by $d(n,\ell)$ where the indices (n,ℓ) are due to harmonic expansion in S^n(see appendix II).

<u>Case.1:</u> When ∇_s^2 acts on z^i considered as divergenceless

$$\lambda_D\,(n,\ell) = \ell(\ell+n-1) - 1 \tag{2.22}$$

$$d_D(n,\ell) = (n+1)\left\{\binom{n+\ell}{\ell} + \binom{n+\ell-2}{\ell-2}\,\theta\,(\ell-2)\right\} - \binom{n+\ell+1}{\ell+1} + \binom{n+\ell-3}{\ell-3}\,\theta\,(\ell-3) \tag{2.23}$$

<u>Case.2:</u> When ∇_s^2 acts on z^i considered as gradient of a scalar

$$\lambda_G\,(n,\ell) = \ell^2 + (\ell-1)\,(n-1) \tag{2.24}$$

$$d_G(n,\ell) = \binom{n+\ell}{\ell} - \binom{n+\ell-2}{\ell-2}\,\theta\,(\ell-2) \tag{2.25}$$

Where, the suffixes D and G stand for divergenceless and gradient of a scalar respectively, $\ell = 1,2,\ \ldots\ \infty$ (n same as in S^n).

<u>§ 5. Evaluation of 1-loop potential :-</u>

The 1-loop effective action S_1 for the model is given by

$$e^{iS_t} = \int [dz^i]\, e^{i(S_0+S_{zz})} \qquad (2.26)$$

where S_{zz} is given in (2.21). With the eigenvalues and degeneracies known from (2.22 to 2.25) the functional integral in (2.26) can be evaluated. The 1-loop potential is just due to S_{zz}. Since the degeneracy $d(n,\ell)$ is different for cases 1 and 2 (2.23 and 2.25) we have to sum the contributions separately.

At this stage let us recall that the total effective potential V_t is defined through the total effective action S_t as $S_t = -\int d^4x V_t$. This $V_t = V_{cl} + V^{1-loop}$ is again the sum of two terms V_{cl} - the classical potential coming from S_0 and V^{1-loop} the 1-loop quantum correction coming from S_{zz}. Now using the general result that

$$\int [d\phi]\, e^{-\int \phi(x)A\phi(x)dx} \approx (\det A)^{-1/2}, \text{ one gets}$$

$$V^{1-loop} = -\frac{i}{2}\sum_a \int \frac{d^4k}{(2\pi)^4} \sum_{\ell=1}^{\infty} \ell n \left\{ -k^2 + r^{-2}(\lambda_a(n,\ell) + n-1) \right\}^{d_a(n,\ell)}$$

(2.27)

where the summation over a stands for the sum over the contributions from Case 1 and Case 2, and $\sum_{\ell=1}^{\infty}$ and $\int \frac{d^4k}{(2\pi)^4}$ stand for trace. Eqn.(2.27) gives the 1-loop potential for the specific model (2.1) with specific classical solutions (2.5 and 2.6) and isdifferent from the results obtained in [4,5,7]. In particular, it is different from the contribution of scalars considered by

Candelas and Weinberg [4] in the sense that here z^i are taken as components of contravariant vector in S^n. In the study of 1-loop potential due to gauge fields (vector fields) Kato and Saito [5] find that the contribution from the 'gradient of a scalar' part for vectors is cancelled by the Faddeev - Popov ghost term arising from the gauge fixing and so effectively, the divergenceless part alone contributes to the effective potential. A similar situation is encountered by Castaldini [7]. In our case there are no gauge fixing term and hence the contributions from both the cases need to be taken into account.

We now evaluate (2.27) for some particular cases of interest.

Case.1: The extra space as S^2

The motivation here is to compare our results for the 1-loop potential with that of Castaldini [7] who has evaluated the 1-loop potential for monopole induced compactification scheme of Randjbar - Daemi, Salam and Strathdee [6], in which the coset space is S^2. At first it may appear strange to compare these as we have a non-linear σ - model induced compactification. Nevertheless the coset space is same S^2 and for this the two schemes may be related by appealing to a general non-linear σ - model [10].

For S^2, we have from (2.22) to (2.25)

$$\lambda_D(2,\ell) = \lambda_G(2,\ell) = \ell^2 + \ell - 1, \tag{2.28}$$

and

$$d_D\,(2,\ell) = d_G\,(2,\ell) = 2\ell + 1 \tag{2.29}$$

Consequently the sum over 'a' in (2.27) is just a multiplication by 2. The integral over k^2 in (2.27) is obviously divergent and we use dimensional regularization,

$$\int \frac{d^D k}{(2\pi)^D}\,\ell n\,(-k^2+a^2) = -i\,\frac{\Gamma\,(-D/2)}{(4\pi)^{D/2}}\,a^{D/2} \tag{2.30}$$

Employing this, we find

$$V^{1-loop}_{n=2} = -\mu^{-(D-4)}\,\frac{\Gamma\,(-\frac{1}{2}D)}{(4\pi)^{D/2}\,r^D}\,\sum_{\ell=1}^{\infty}\,(2\ell+1)\,\{\,\ell(\ell+1)\,\}^{D/2}\Big|_{D=4} \tag{2.31}$$

where an arbitrary constant μ of mass dimension has been introduced. The summation over ℓ can be performed for D - odd and so we restrict to D = 3. Parameterising the 1-loop potential as C/r^3, $C \sim -9.6 \times 10^{-4}$. The negativeness of the constant C signals the instability of the classical solution against fluctuations. This will be discussed in the next section.

Case.2: The extra space as S^3

The usual space time is taken to be M^4 which is realistic. So the Kaluza-Klein space is $M^4 \times S^3$.

Fluctuations as divergenceless vectors :-

The eigenvalues and degeneracies in this case are given by Eqs. (2.22) and (2.23) for n = 3. They are

$$\lambda_D\ (3,\ell) = \ell\ (\ell + 2) - 1, \tag{2.32}$$

and

$$d_D\ (3,\ell) = 2\ell\ (\ell + 2). \tag{2.33}$$

Then Eqn.(2.27) becomes

$$V^{1\text{-loop}}_{n=3} = -\frac{i}{2}\int d^4k\ (2\pi)^{-4} \sum_{\ell=1}^{\infty} 2\ell(\ell+2)\ \ell n\left\{ -k^2 + r^{-2}(\ell(\ell+2)+1)\right\} \tag{2.34}$$

The k^2 - integration is done using (2.30) to obtain

$$V^{1\text{-loop}}_{n=3} = -\mu^{-(D-4)} \frac{\Gamma(-D/2)}{(4\pi)^{D/2} r^D} \sum_{\ell=1}^{\infty} \ell(\ell+2)(\ell+1)^D \Big|_{D=4} \tag{2.35}$$

Writing $\qquad \ell(\ell+2) = (\ell+1)^2 - 1,$

we get

$$\sum_{\ell=1}^{\infty} \ell(\ell+2)(\ell+1)^D = \sum_{\ell=1}^{\infty}\left\{(\ell+1)^{D+2} - (\ell+1)^D\right\}$$

and the sum can be formally done to give

$$\sum_{\ell=1}^{\infty} (\ell+1)^{D+2} = \sum_{m=2}^{\infty} m^{D+2} = \sum_{m=0}^{\infty} m^{D+2} - 1 = \zeta\ (-D-2) - 1$$

and

$$\sum_{1}^{\infty} (\ell+1)^D = \sum_{m=2}^{\infty} (m)^D = \sum_{m=0}^{\infty} m^D - 1 = \zeta\ (-D) - 1.$$

Finally substituting in (2.35) yields

$$V^{1-loop}_{n=3} = - \mu^{-(D-4)} \frac{\Gamma\ (-D/2)}{(4\pi)^{D/2}\ r^{D}} \left\{ \zeta\ (-D-2) - \zeta\ (-D) \right\} \Bigg|_{D=4} , \tag{2.36}$$

where ζ is Riemann ζ - function. We make use of the following formula for the ζ - function [12],

$$\zeta\ (Z)\ \Gamma(\frac{Z}{2}) = \pi^{Z-1/2}\ \zeta\ (1-Z)\ \Gamma\ (\frac{1}{2}\ (1-Z)\ ), \tag{2.37}$$

to express $\zeta\ (-D-2)$ and $\zeta\ (-D)$ in terms of $\zeta\ (D+3)$ and $\zeta\ (D+1)$ respectively. i.e,

$$\zeta\ (-D)\ \Gamma\ (\ -\ D/2) = \pi^{-D-1/2}\ \zeta\ (D+1)\ \Gamma\ (\frac{1}{2}\ (D+1)\ ), \tag{2.38}$$

$$\zeta\ (-D-2)\ \Gamma\ (\ -\ \frac{D}{2} - 1) = \pi^{-D-2-1/2}\ \zeta\ (D+3)\ \Gamma\ (\frac{1}{2}\ (D+3)\ ).$$

Using $Z\ \Gamma(Z) = \Gamma(Z+1)$, we have

$$\Gamma\ (\ -\ \frac{D}{2} - 1) = \frac{\Gamma\ (-D/2)}{(-\ \frac{D}{2} - 1)}$$

and so

$$\zeta(-D-2)\ \Gamma(-D/2) = (\ -\ \frac{D}{2} - 1)\ \pi^{-D-5/2}\ \zeta(D+3)\ \Gamma\ (\frac{1}{2}\ (D+3)) \tag{2.39}$$

The crucial point is, the divergent part coming from $\zeta(-D/2)$ when $D = 4$ is removed i.e. the *product* $\Gamma(-D/2) \sum_{\ell=1}^{\infty} \ell(\ell+2)\ (\ell+1)^{D}$ allows us to write down the finite result. Taking the limit $D \longrightarrow 4$ is possible now and the result is, using (2.38) and (2.39),

$$V^{1-loop}_{n=3} = \frac{3}{(2\pi)^6\ r^4} \left[\frac{15}{2\pi^2}\ \zeta(7) + \zeta(5) \right] . \tag{2.40}$$

<u>Fluctuations as gradient of a scalar :-</u>

The eigenvalues and degeneracy of the 1-loop operator when acting upon z^i considered as gradient of a scalar for n = 3, can be obtained from (2.24) and (2.25). For ∇^2_s , we have

$$\lambda_G\ (3,\ell) = \ell^2 + 2(\ell-1), \tag{2.41}$$

and

$$d_G\ (3,\ell) = (\ell+1)^2. \tag{2.42}$$

Then eqn.(2.27) gives

$$V^{1-loop}_{n=3} = - \frac{i}{2} \int d^4k\ (2\pi)^{-4} \sum_{\ell=1}^{\infty} (\ell+1)^2\ \ell n \left\{ -k^2+r^{-2}(\ell^2+2\ell) \right\} \tag{2.43}$$

The k^2- integration is done using the dimensional regularization result (2.30) to give

$$V^{1-loop}_{n=3} = - \frac{1}{2}\ \mu^{-(D-4)}\ \frac{\Gamma\ (-D/2)}{(4\pi)^{D/2}\ r^D} \sum_{\ell=1}^{\infty} (\ell+1)^2\ (\ell^2+2\ell)^{D/2} \Bigg|_{D=4} \tag{2.44}$$

The sum over ℓ is performed by writing $\ell^2+ 2\ell=(\ell+1)^2- 1$ and setting D = 4. We observe that we have to regularize

$$\Gamma(-2) \sum_{\ell=1}^{\infty} \left\{ (\ell+1)^6 + (\ell+1)^2 - 2(\ell+1)^4 \right\}.$$

Formally this is equal to

$$\Gamma(-2)\left\{\zeta(-6)+\zeta(-2)-2\zeta(-4)\right\}.$$

Using (2.37), and the procedure illustrated in the divergenceless case we get

$$V^{1-loop}_{n=3} = \frac{1}{2}\frac{1}{(2\pi)^6 r^4}\left\{\frac{45}{2\pi^2}\zeta(7)+6\,\zeta(5)+\pi^2\zeta(3)\right\} \qquad (2.45)$$

Combining the two cases for the fluctuations, the final result for the 1-loop potential for the model (2.1) with the product manifold as $M^4 \times S^3$ is

$$V^{1-loop} = \frac{1}{(2\pi)^6 r^4}\left\{\frac{135}{4\pi^2}\zeta(7)+6\,\zeta(5)+\frac{1}{2}\pi^2\zeta(3)\right\} \qquad (2.46)$$

We summarize this section by giving the conclusion based upon (2.46) and (2.31). The 1-loop potential for the extra space taken as S^2 and the usual space time to be odd dimensional can be obtained from (2.31). The choice of odd dimensions for the usual space time is unrealistic. Further, in this case, the 1-loop potential becomes (with ordinary space taken as M^3, M^5, M^7, M^9) C_3/r^3 C_5/r^5, C_7/r^7, C_9/r^9 with $C_3 = -9.6 \times 10^{-4}$, $C_5 = -2.3 \times 10^{-5}$, $C_7 = -1.2 \times 10^{-7}$ and $C_9 = -0.5 \times 10^{-9}$ respectively. The effect decreases with increase of dimensions for M^D. More crucial is that in all the cases C is negative. Such a model is not stable against the fluctuations (next section).

In the case when the coset space is taken to be S^3, the usual space time can be taken as M^4 and the Kaluza-Klein manifold is then $M^4 \times S^3$. The effective 1-loop potential is given by (2.46).

Parametrising it as C/r^4, $C = 2.54 \times 10^{-4}$, *positive.* So the system with classical solutions is stable against the fluctuations. The one loop potential due to fluctuations in the background gravity (with no matter fields) in 5-dimensional Kaluza-Klein theory has been obtained by Appelquist and Chodos [2] as $-15\ (2\pi)^{-6}\zeta(5)/r^4$ where r is the radius of S^1.This may be taken to be 1-loop potential due to fluctuations of the background gravity assuming the absence of fluctuations of the metric for S^3, as they involve large energy. When added to (2.46) it still remains positive. In the next section, we analyse the stability against quantum fluctuations and obtain new results for the algebraic constraint relating r with λ, a , the parameters of the model and for Λ cosmological constant as well.

§.7. Stability Analysis

As is evident from the action (2.1) we have considered, we have taken the Newton's gravitational constant G in (4+n) dimensions to be (1/8π) so that Einstein's equation in (4+n) dimension is

$$\hat{R}_{MN} - \frac{1}{2} G_{MN} \hat{R} = T_{MN} \tag{2.47}$$

With the ground state ansatz $\phi^i = y^i$, we have from (2.5)

$$R_{\mu\nu} = 0$$

$$R_{pq} = \frac{2}{\lambda^2} m_{pq}$$

From these it follows that $R = G^{MN} R_{MN} = \dfrac{-2n}{\lambda^2 a^2}$, as $g_{pq} = -a^2 m_{pq}$. Thus the various components of T_{MN} can be read off easily as

$$T_{\mu\nu} = \frac{n}{\lambda^2 a^2} g_{\mu\nu}$$

$$T_{pq} = \left(\frac{2-n}{\lambda^2} \right) m_{pq} \tag{2.48}$$

In particular we note that $T_{oo} = \frac{n}{\lambda^2 a^2}$ as the energy density in (4+n) dimensional space-time. This is the classical contribution to the energy density. To get the 4-dimensional energy density V_{cl} we just multiply T_{oo} by the volume 'Ω_n' of the extra space taken to be S^n. Thus

$$V_{cl} = \frac{n\Omega_n}{\lambda^2 a^2} \tag{2.49}$$

One can get it from the classical matter action also, as can be seen easily from

$$S_{matter} = \int d^{4+n} z \sqrt{G} \frac{G^{MN}}{\lambda^2} m_{ij} \left(\partial_M \phi^i \right) \left(\partial_N \phi^i \right)$$

$$\equiv - \int d^4 x \, V_{cl} \tag{2.50}$$

This way of identifying effective potential is same as that of Candelas and Weinberg.

Now with the ansatz $\phi^i = y^i$ and using $g_{pq} = -a^2 m_{pq}$ and the fact that M^4 is Ricci flat the corresponding metric can be taken as

$g_{\mu\nu} = \eta_{\mu\nu}$ and hence $\sqrt{G}$ can be written as $\sqrt{g}$ where $g = |\det g_{mn}|$ giving

$$S_{matter} = - \frac{n\Omega_n}{\lambda^2 a^2} \int d^4x \qquad (2.51)$$

giving $V_{cl} = \frac{n\Omega_n}{\lambda^2 a^2}$ agreeing with (2.49).

We have already calculated the one-loop effective potential $V^{1\text{-}loop}$ (henceforth written simply as V) coming from the quantum fluctuations of the matter fields. This when added to the classical contribution will give the total effective potential V_t

$$V_t = V_{cl} + V \qquad (2.52)$$

and is the integrand of the total effective action

$$S_t = - \int d^4x \; V_t = - \int d^4x \; (V_{cl} + V) \qquad (2.53)$$

Since the volume of the n-sphere S^n is $\Omega_n = \alpha \; r^n$ for some constant α we have

$$V_t = \alpha_n \; r^n + \frac{C_n}{r^4}$$

with

$$\alpha_n = \frac{n\alpha}{\lambda^2 a^2} \qquad (2.54)$$

Since for $M^4 \times S^n$ with odd n we have seen that $C_n > 0$ and since α_n is already positive, we see that V_t has a non-trivial minimum thus indicating stability. Naively varying V_t with respect to r, one gets

$$r_{min} = \left(\frac{4C_n \lambda^2 a^2}{n^2 \alpha} \right)^{1/(n+4)} \qquad (2.55)$$

Thus for stability we must have $C_n > 0$, which we have for $M^4 \times S^3$, but not for $M^{3,5,7,9} \times S^2$.

Here we note that V_{cl} term (coming from matter term only)was entirely absent in the Candelas and Weinberg [4] case as they did not have spontaneous compactification at the classical level. They had compactification at the quantum level only, where the energy momentum tensor comes from the one loop effective potential obtained by integrating out the scalar fields. Thus it was a compactification of dynamical nature. On the other hand we have spontaneous compactification at the classical level itself. Now integrating out the fluctuations of the fields around the classical background solutions we get an additional contribution of quantum 1-loop effective potential over and above the classical one. Now demanding that the energy-momentum tensor comes from the total effective potential V_t and implement the conditions of spontaneous compactification even at this quantum level. we get a certain algebraic constraint on the size of the internal manifold taken to be S^n in our case different from equation (2.55). This is explained below.

We start with Einsteins equation in (4+n) dimensions (2.47)

$$R_{MN} - \frac{1}{2} G_{MN}(R+\Lambda) = T_{MN} \qquad (2.56)$$

Here the energy-momentum tensor T_{MN} arises from the total matter action S_t (i.e. both classical and quantum) of equation (2.53). We have included a cosmological constant Λ into the Einstein's equation, because we do not know a priori whether with this modified energy-momentum tensor we can have compactification without cosmological term or not. However we note that its absence was a virtue in the classical theory.

Thus we have the energy-momentum tensor T^{MN} given by

$$\frac{1}{2} T^{MN} = - \frac{\delta S_t}{\delta G_{MN}} \tag{2.56a}$$

We seek a vacuum solution with Poincare invariance in four dimensions. That is the metric is taken to have the components

$$G_{\mu\nu} = \eta_{\mu\nu} \quad ; \quad G_{\mu n} = 0 \quad ; \quad G_{mn} = g_{mn}(y) \tag{2.57}$$

which corresponds to our classical case (2.6).

The Ricci tensor and the curvature scalar appearing in Einstein's equations are then

$$R_{\mu\nu} = 0 = R_{\mu n} \quad ; \quad R_{mn} = \tilde{R}_{mn}(y) \quad ; \quad R = \tilde{R}(y) \tag{2.58}$$

where $\tilde{R}_{mn}(y)$ and $\tilde{R}$ are the n-dimensional Ricci tensor and the curvature scalar respectively. Poincare invariance gives the energy-momentum tensor the structure

$$T_{\mu\nu} = B(y)\, \eta_{\mu\nu} \quad ; \quad T_{\mu n} = 0 \quad ; \quad T_{mn} = T_{mn}(y) \tag{2.59}$$

Then the field equations reduce to

$$-\frac{1}{2}\left(\tilde{R}(y)+\Lambda\right) = B(y)$$

$$\tilde{R}_{mn}(y)-\frac{1}{2}g_{mn}(y)\left[\tilde{R}(y)+\Lambda\right] = T_{mn} \qquad (2.60)$$

In calculating, $T_{mn}(y)$, using (2.56), we must evaluate a variational derivative of S_t in (2.53) with respect to g_{mn} for which we can fix $g_{\mu n}$ and $g_{\mu\nu}$ at their classical values. The matter effective action is

$$S_t = -\int d^4x\, V_t\left[g_{mn}(x,y)\right]$$

$$= -\int d^4x\left\{V_{cl}\left[g_{mn}(x,y)\right]+V\left[g_{mn}(x,y)\right]\right\} \qquad (2.61)$$

Then using equations (2.56) and (2.61) we get

$$\frac{1}{2}T^{mn} = -\left.\frac{\delta S_t}{\delta g_{mn}(x,y)}\right|_{g_{mn}(x',y)=g_{mn}(y)}$$

$$= \frac{\delta}{\delta g_{mn}(x,y)}\left.\int d^4x'\, V_t\left[g_{mn}(x',y)\right]\right|_{g_{mn}(x',y)\,=\,g_{mn}(y)}$$

which leads to

$$\frac{1}{2}T^{mn} = \frac{\delta V_t[g_{mn}(y)]}{\delta g_{mn}(y)} \qquad (2.62)$$

The extra space we are considering is S^n which is a homogeneous coset space SO(n+1)/SO(n), for which

$$R_{mn} = - \frac{(n-1)}{r^2} g_{mn} \qquad (2.63)$$

Also the energy-momentum tensor here takes the form

$$T_{mn} = A g_{mn} \qquad (2.64)$$

as all tensors S_{mn} formed by the variational derivatives of scalar functionals F[g] with respect to the metric has all the symmetries of the metric

$$\frac{1}{2} S^{mn}(y) \equiv \frac{\delta F[g]}{\delta g_{mn}(y)} \qquad (2.65)$$

For T_{mn}, F is V_t and for $(R^{mn} - \frac{1}{2}g^{mn} R)$ it is $\int d^n y \sqrt{g} R(y)$. Thus T_{mn} can be taken proportional to the metric. Also the homogeneity of the space S^n implies that the constant of proportionality B(y) in $T_{\mu\nu} = B(y)\, \eta_{\mu\nu}$ is actually independent of y. In fact this can be seen from equations (2.60) and (2.63) for instance. Equation (2.63)gives

$$R(y) = - \frac{n(n-1)}{r^2}$$

and so from the equation (2.60) it follows that

$$B = - \frac{1}{2} \left[- \frac{n(n-1)}{r^2} + \Lambda \right]$$

essentially making B independent of y.

Now coming to the calculation of A, we go back to the relation (2.62).

$$\frac{1}{2} \quad T^{mn} = \frac{\delta V_t[g_{mn}(y)]}{\delta g_{mn}(y)}$$

Multiplying both sides by $g_{mn}\sqrt{g(y)}$ and using (2.64), we get

$$\frac{1}{2} \, nA \sqrt{g(y)} = \sqrt{g(y)} \quad g_{mn}(y) \quad \frac{\delta V_t[g_{mn}(y)]}{\delta g_{mn}(y)}$$

Integrating both sides with respect to y, one gets

$$\frac{1}{2} \, nA \quad \Omega_n = \int d^n y \sqrt{g(y)} \quad g_{mn}(y) \quad \frac{\delta V_t[g_{mn}(y)]}{\delta g_{mn}(y)} \tag{2.66}$$

To evaluate the right hand side, we consider a function $f(x_1,\ldots,x_n)$ of several variables x_i. Let us denote this by $f(x_j)$. Then

$$\frac{df(X_j)}{d(\ln X_i)} = \frac{df(X_j)}{dX_k} \quad \frac{dX_k}{d(\ln X_i)} = \frac{df}{dX_k} \; X_i \; \delta_{ik} = X_i \frac{df}{dX_i}$$

(2.66a)

In this expression in the right hand side of (2.66), g_{mn} plays the role of X and Y plays the role of the index i, the only difference being is that the index i is discrete and is summed over here, whereas the index y in (2.66) is continuous and is integrated over with respect to the invariant measure $\sqrt{g(y)}d^n y$.

Thus we have from (2.66).

$$\frac{1}{2}\, nA\, \Omega_n = \frac{\delta V_t[g_{mn}]}{\delta(\ln g_{mn})} \qquad (2.67)$$

In this case the change δg_{mn} in g_{mn} is coming from the overall scale change, as the shape of S^n is not changed even under fluctuation. We can write $g_{mn} = r^2 k_{mn}$ where k_{mn} is the corresponding metric for a unit S^n. So we can instead parametrize $V_t[g]$ by the radius r and write $V_t[r]$. Also since

$$\delta(\ln g_{mn}) = \delta(\ln(r^2 k_{mn})) = \delta(\ln r^2),$$

as k_{mn} is a constant metric in the functionalspace, we can rewrite (2.67) as

$$\frac{1}{2}\, n\, \Omega_n A = \frac{dV_t(r)}{d(\ln r^2)} = r^2 \frac{dV_t(r)}{dr^2} \qquad (2.68)$$

To calculate B in equation (2.60), we note B is the energy density (classical + 1 loop quantum correction) in (4+n), whereas V_t is the corresponding energy density in 4-dimensions, we must have $V_t = B\Omega_n$, as we have done earlier.

Now the 4-dimensional gravitational constant G_o is obtained by dividing the (4+n) dimensional gravitational constant ($1/8\pi$) by the volume Ω_n of S^n ,i.e.,

$$G_o = \frac{1}{8\pi\Omega_n}. \qquad (2.69)$$

From the field equations (2.60), we get using (2.63) and

$$-\frac{1}{2}\left(-\frac{n(n-1)}{r^2}+\Lambda\right) = 8\pi G_o V_t$$

$$-\frac{(n-1)}{r^2}-\frac{1}{2}\left(-\frac{n(n-1)}{r^2}+\Lambda\right) = A \qquad (2.70)$$

But from equation (2.68) and (2.69) it follows that

$$A = \frac{8\pi G_o}{n} r \frac{dV_t(r)}{dr}. \qquad (2.71)$$

Thus the equations (2.70) can be written as

$$-\frac{1}{2}\left(-\frac{n(n-1)}{r^2}+\Lambda\right) = 8\pi G_o V_t,$$

$$-\frac{(n-1)}{r^2}-\frac{1}{2}\left(-\frac{n(n-1)}{r^2}+\Lambda\right) = \frac{8\pi G_o}{n} r \frac{dV_t(r)}{dr}. \qquad (2.72)$$

Substituting the first of these equation (2.72) into the second, we get

$$-\frac{(n-1)}{r^2}+8\pi G_o V_t(r) = \frac{8\pi G_o}{n} r \frac{dV_t(r)}{dr}. \qquad (2.73)$$

From the equation (2.54) we had $V_t = \alpha_n r^n + \frac{C_n}{r^4}$, using this into the above equation, we get

$$r^2 = \frac{8\pi G_o (4+n) C_n}{n(n-1)}. \qquad (2.74)$$

Thus we see that the size of the extra space given by the radius r in equation (2.74) makes sense if and only if $C_n > 0$. This relation is the desired algebraic constraint on the size of the extra space in the theory. Incidentally we note that this relation is the same as that of Candelas and Weinberg[4].

Now let us consider the cosmological constant Λ. The first of the equations (2.72) gives a value of Λ, according to which

$$\Lambda = \frac{n(n-1)}{r^2} - 16\pi G_o V_t. \tag{2.75}$$

Using (2.54) for V_t and substituting for α_n as $\frac{n\alpha}{\lambda^2 a^2}$ and using $\frac{1}{8\pi} = G = G_o \Omega_n = G_o \alpha r^n$, we obtain

$$\Lambda = \frac{1}{r^2}\left[n(n-1) - \frac{16\pi G_o C_n}{r^2}\right] - \frac{2n}{\lambda^2 a^2}. \tag{2.76}$$

Now doing a fine tuning on Λ by employing the algebraic constraint (2.74) for r^2, we finally get

$$\Lambda = \frac{n^2(n-1)^2 (n+2)}{8\pi G_o (n+4)^2 C_n} - \frac{2n}{\lambda^2 a^2}.$$

The first term is the same as that of Candelas and Weinberg and the second term is the new term in our case is the classical term. Incidentally we note that the parameter 'a' has been a free parameter so far. We can again fine tune it to a value given by

$$a^2 = \frac{16\pi G_o (n+4)^2 C_n}{\lambda^2 n(n-1)^2 (n+2)} \tag{2.77}$$

so that the cosmological constant 'Λ' vanishes even at this quantum level.

Now we shall discuss about the stability of the system containing both gravity and scalars. In this case it is enough to consider total action under the constraint of Poincare invariance (as we have done earlier) which allows it to be written as

$$S_{total} = - \int d^4x \, V_{eff},$$

where

$$V_{eff} = \frac{1}{2} \int d^n y \sqrt{g(y)} \left[R(y) + \Lambda \right] + V_t[g]. \qquad (2.78)$$

As is well known a potential V_{eff} constructed in this way is equal to the minimum energy of any state in which the expectation value of the fields has the value indicated by the argument of the potential in our case $g_{mn}(y)$.

Therefore provided that there are no negative energy perturbations, a stable solution is one associated with a minimum of V_{eff}.

Since during perturbations the shape of S^n is not supposed to change V_{eff} can be parameterized by r only and thus can be rewritten as (using $\Omega_n = 1/8\pi G_o$).

$$V_{eff}(r) = \frac{\Omega_n}{2} \left[- \frac{n(n-1)}{r^2} + \Lambda \right] + V_t(r)$$

$$= \frac{1}{16\pi G_o} \left[- \frac{n(n-1)}{r^2} + \Lambda \right] + V_t(r) \qquad (2.79)$$

We can also easily see that

$$r \frac{dV_{eff}}{dr} = \frac{n}{16\pi G_o}\left[- \frac{n(n-1)}{r^2} + \Lambda \right] + \frac{n(n-1)}{8\pi G_o r^2} + r \frac{dV_t(r)}{dr}. \qquad (2.80)$$

Using the field equation (2.72) and the equation (2.80) we see that V_{eff} is stationary in r. The other field equation (2.79) just says that Λ takes a value that makes V_{eff} vanish at its stationary point. Since we know that the (4+n) dimensional gravitational constants G is r independent not G_o -the 4 dimensional one, we can rewrite V_{eff} using (2.79) and $\Omega_n = \alpha r^n$

$$V_{eff}(r) = \frac{1}{2}\alpha r^n \left[- \frac{n(n-1)}{r^2} + \Lambda \right] + \alpha_n r^n + \frac{C_n}{r^4}. \qquad (2.81)$$

From this we see that $V_{eff}(r) \longrightarrow \infty$ as $r \longrightarrow 0$ and $r \longrightarrow \infty$ provided that $\left(\frac{1}{2}\alpha\Lambda + \alpha_n\right) > 0$ and C_n is positive in our case showing that it must have a minimum at some finite r. In fact if we choose 'a' as given by (2.77), we can have Λ=0 and α_n is always positive and thus the condition $\left(\frac{1}{2}\alpha\Lambda + \alpha_n\right) > 0$ is always satisfied. We have already found one stationary point and it must therefore correspond to the minimum of the whole system of matter and gravitation whereas the stationary point given in (2.55) corresponds to matter only.

§.8. Summary

In this chapter we have analysed a specific model for spontaneous compactification - the one proposed by Gell-Mann and Zwiebach [1] in which the compactification is triggered by scalars in the form of non-linear σ-model coupled to gravity in $(4+n)$ dimensions. The classical equations of motion require, for M^4 to be flat, the scalars to live on the extra space only. Upon identifying the scalars ϕ^i to the co-ordinates Y^i of B^n themselves, the scalar manifold is found to have the same topology as B^n. We examine the quantum effects in such a model using the background field method. Here in the action (2.1), the scalars are expanded around their classical configuration. Keeping terms up to quadratic in quantum fluctuations (the linear terms drop out by virtue of the classical equations of motion), the effective 1-loop potential is evaluated, taking into account the fact that the fluctuations of the scalars, z^i could be taken as either divergenceless or gradient of a scalar. This result shows that the 1-loop potential has the form C/r^4 where r is the radius of $B^n = S^n$, with the constant C being positive. We have examined the stability of the system in § 7. Two new results are obtained. First of all, a new algebraic constraint relating r, the size of S^n, and C is derived. It is, for D = 4 (2.74)

$$r^2 = \frac{8\pi G_0 (n+4)C}{n(n-1)}$$

showing that the size of extra space, r above makes sense iff $C > 0$. Indeed we have C positive. The stability analysis further gives a fine tuned value for the cosmological constant Λ as

$$\Lambda = \frac{n(n-1)(n+2)}{r^2 (n+4)} - \frac{2n}{\lambda^2 a^2}$$

The parameter 'a' can be chosen (2.77) so that Λ vanishes even at the quantum level. The vanishing of Λ is *supported* by observations. Thus we see that the compactification at the classical level can be maintained at the quantum level also (with the energy-momentum tensor coming from the effective action, which includes the classical part) by putting a cosmological constant — which can be made to vanish by proper choice of 'a' the free parameter of the theory.

APPENDIX I

Here we derive the expression for S_{zz} appearing in (2.16). The matter action is given by

$$S = \frac{1}{\lambda^2} \int d^{D+n} Z \sqrt{G} \; G^{AB} \; m_{ij}(\varphi) \, (\partial_A \varphi^i) \, (\partial_B \varphi^j) \qquad (2.83)$$

Fluctuating the scalars ϕ^i around their classical background (2.11), $\phi^i = y^i + z^i(x,y)$, we get (we do not fluctuate the metric)

$$S = \frac{1}{\lambda^2} \int d^{D+n} Z \sqrt{G} \; G^{AB} \, m_{ij} \, (y^k + z^k) \, \partial_A(y^i + z^i) \, \partial_B(y^j + z^j) \qquad (2.84)$$

Writing

$$\partial_A(y^i + z^i) = (\delta^i_A + z^i_{,A})$$

and

$$m_{ij} \, (y^k + z^k) = m_{ij}(y^k) + z^k \, m_{ij,} + \frac{1}{2!} z^k z^\ell \, m_{ij,kl},$$

using Taylor's expansion and collecting terms up to quadratic in (2.84) fluctuation z^i, we get, noting that term linear in z vanishes by equation of motion,

$$S = S_o + S_{zz}$$

where

$$S_{zz} = \frac{1}{\lambda^2} \int d^{D+n} Z \sqrt{G}\ G^{AB} \Big[m_{ij}\, z^i_{,A}\, z^j_{,B} + z^k\, m_{ij,k} \left(\delta^i_A\, z^j_{,B} + \delta^i_B\, z^j_{,A} \right) + \frac{1}{2} z^k z^l\, m_{ij,kl}\, \delta^i_A\, \delta^j_B \Big]. \qquad (2.85)$$

Here G^{AB} corresponds to the classical solution given in equation (2.8) which can be used to further simplify this as

$$S_{zz} = \frac{1}{\lambda^2} \int d^{D+n}Z \sqrt{G} \Big[g^{\mu\nu}\, m_{ij}\, z^i_{,\mu}\, z^j_{,\nu} + g^{pq}\, m_{ij}\, z^i_{,p}\, z^j_{,q} + 2\, g^{ip}\, m_{ij,k}\, z^k\, z^j_{,p} + \frac{1}{2} g^{ij}\, m_{ij,kl}\, z^k\, z^l \Big]. \qquad (2.86)$$

Now we note that the fluctuations z^i are scalars with respect to the ordinary space-time manifold but are treated as the components of a contravariant vector with respect to the scalar manifold. Thus in the first term within the parenthesis of (2.86) we can replace the ordinary derivatives $(z^i_{,\mu})$ by the covariant derivatives $(D_\mu z^i)$ where D_μ is the covariant derivative operator calculated with respect to the metric $g_{\mu\nu}$, whereas in the second term within the parenthesis of equation (2.86), we can replace

$z^i_{,p}$ by $\left(\tilde{D}_p z^i - \tilde{\Gamma}^i_{pq} z^q\right)$ where D and $\tilde{\Gamma}$ are calculated with respect to m_{pq} metric. With these substitutions we get

$$S_{zz} = \frac{1}{\lambda^2} \int d^{D+n} Z \sqrt{G} \left[g^{\mu\nu} m_{ij} (D_\mu z^i)(D_\nu z^j) + g^{pq} m_{ij} (\tilde{D}_p z^i)(\tilde{D}_q z^j) \right.$$

$$+ 2 g^{ip} m_{ij,k} z^k z^j_{,p} + \frac{1}{2} g^{ij} m_{ij,kl} z^k z^l - 2 g^{pq} m_{ij} \tilde{D}_p z^i \tilde{\Gamma}^j_{qr} z^r +$$

$$\left. g^{pq} m_{ij} \tilde{\Gamma}^i_{pr} \tilde{\Gamma}^j_{qs} z^r z^s \right]. \qquad (2.87)$$

We write $\tilde{D}_p z^i = z^i_{,p} + \tilde{\Gamma}^i_{ps} z^s$

and

$$\tilde{\Gamma}^i_{pq} = \frac{1}{2} m^{ir} \left(m_{rp,q} + m_{rq,p} - m_{pq,r} \right)$$

in the third and fifth term in the integrand of (2.87), getting after some simplification

$$- 2 g^{pq} m_{ij} (\tilde{D}_p z^i) \tilde{\Gamma}^j_{qr} z^r + 2 g^{ip} m_{ij,k} z^k z^j_{,p}$$

$$= 2 g^{pq} m_{jq} \tilde{\Gamma}^j_{ir} z^r z^i_{,p} - 2 g^{pq} m_{ij} \tilde{\Gamma}^i_{ps} \tilde{\Gamma}^j_{qr} z^r z^s \qquad (2.88)$$

Let us define

$$T = \int d^{D+n} Z \sqrt{G}\, g^{pq} m_{jq} \tilde{\Gamma}^j_{ir} z^r z^i_{,p} \qquad (2.89)$$

which can be integrated by parts to give

$$T = -\int d^{D+n} Z \left(\sqrt{G}\, g^{pq}\, m_{jq}\, \tilde{\Gamma}^{j}_{ir}\, z^{r}\right)_{,p} z^{i}$$

$$= -\int d^{D+n} Z \left[\left(\sqrt{G}\, g^{pq}\, m_{jq}\right)_{,p} \tilde{\Gamma}^{j}_{ir}\, z^{r} + \sqrt{G}\, g^{pq}\, m_{jq}\, (\partial_{p}\, \tilde{\Gamma}^{j}_{ir})z^{r}\right] z^{i}$$

$$- \int d^{D+n} Z \sqrt{G}\, g^{pq}\, m_{jq}\, \tilde{\Gamma}^{j}_{ir}\, z^{r}_{,p}\, z^{i} \qquad (2.90)$$

We note that the last term in (2.90) in T itself. Bringing it to the left one can solve for T getting

$$T = -\frac{1}{2}\int d^{D+n} Z \left[\left(\sqrt{G}\, g^{pq}\, m_{jq}\right)_{,p} \tilde{\Gamma}^{j}_{ir}\, z^{r} + \sqrt{G}\, g^{pq} m_{jq} (\partial_{p}\, \tilde{\Gamma}^{j}_{ir})z^{r}\right] z^{i}$$

$$(2.91)$$

But the factor $(\sqrt{G}\, g^{pq}\, m_{jq})_{,p}$ appearing in the first term of the equation (2.91) appears in the right hand side of the equation of motion (2.3) and hence with ansatz (2.4) can simply be written as

$$\frac{\sqrt{G}}{2}\, g^{pq}\, \frac{\partial m_{pq}}{\partial y^{j}} ,$$

thus getting

$$T = -\frac{1}{2}\int d^{D+n} Z \sqrt{G} \left[\frac{1}{2}\, g^{pq}\, m_{pq,j}\, \tilde{\Gamma}^{j}_{ir}\, z^{r} + g^{pq}\, m_{jq} (\partial_{p}\, \tilde{\Gamma}^{j}_{ir})z^{r}\right] z^{i}$$

$$(2.92)$$

So we can rewrite S_{zz} using (2.87), (2.88), (2.92) as

$$S_{zz} = \frac{1}{\lambda^2} \int d^{D+n} Z \sqrt{G} \Big[g^{\mu\nu} m_{ij} (D_\mu z^i)(D_\nu z^j) + g^{pq} m_{ij} (\tilde{D}_p z^i)(\tilde{D}_q z^j)$$

$$- \frac{1}{2} g^{pq} m_{pq,j} \tilde{\Gamma}^j_{sr} z^r z^s - g^{pq} m_{jq} (\partial_p \tilde{\Gamma}^j_{rs}) z^r z^s - 2 g^{pq} m_{ij} \tilde{\Gamma}^i_{ps} \tilde{\Gamma}^j_{qr} z^r z^s$$

$$+ \frac{1}{2} g^{ij} m_{ij,kl} z^k z^l + g^{pq} m_{ij} \tilde{\Gamma}^i_{pr} \tilde{\Gamma}^j_{qs} z^r z^s \Big]. \qquad (2.93)$$

Barring the first two terms within the parenthesis in (2.93) the rest of the terms can be written in a compact form as

$$\frac{1}{\lambda^2} \int d^{D+n} Z \sqrt{G} g^{pq} \Big[\frac{1}{2} m_{pq,rs} - m_{ij} \tilde{\Gamma}^i_{pr} \tilde{\Gamma}^j_{qs} - \frac{1}{2} m_{pq,j} \tilde{\Gamma}^j_{sr} - m_{jq}$$

$$(\partial_p \tilde{\Gamma}^j_{rs}) \Big] z^r z^s . \qquad (2.94)$$

Now using the fact that the metric is covariantly constant ($m_{pq;j}=0$), we get

$$m_{pq,j} = \tilde{\Gamma}^\ell_{pj} m_{\ell q} + \tilde{\Gamma}^\ell_{qj} m_{\ell p} . \qquad (2.95)$$

Using this one gets after some manipulation

$$m_{pq,rs} = (m_{pq,r}),s$$

$$= m_{\ell q} (\partial_s \tilde{\Gamma}^\ell_{pr}) + m_{\ell p} (\partial_s \tilde{\Gamma}^\ell_{qr}) + m_{aq} \tilde{\Gamma}^\ell_{pr} \tilde{\Gamma}^a_{st}$$

$$+ m_{a\ell} \tilde{\Gamma}^\ell_{pr} \tilde{\Gamma}^a_{sq} + m_{ap} \tilde{\Gamma}^\ell_{qr} \tilde{\Gamma}^a_{s\ell} + m_{a\ell} \tilde{\Gamma}^\ell_{qr} \tilde{\Gamma}^a_{sp}$$

$$(2.96)$$

Substituting (2.96) in (2.94) the expression in the parenthesis of (2.94) becomes

$$- g^{pq} \tilde{R}_{prqs} z^r z^s \tag{2.97}$$

Thus we finally get from (2.93) using (2.94) and (2.97)

$$S_{zz} = \frac{1}{2} \int d^{D+n} Z \sqrt{G} \Big[g^{\mu\nu} m_{ij} (D_\mu z^i)(D_\nu z^j) + g^{pq}\Big(m_{ij} \tilde{D}_p z^i \tilde{D}_q z^j - \tilde{R}_{ipjq} z^i z^j \Big) \Big],$$

which is the same expression appearing in (2.16).

APPENDIX - II

Harmonic analysis on S^n

Since the one-loop operator containing ∇_s^2 juxtaposed between vectors z^i in S^n in (2.21), we have to find the eigenvalues and degeneracy of this operator acting in vectors. As any vector in S^n can be decomposed into a gradient of a scalar and a divergenceless vector, we have to consider both of these cases separately.

Though in our cases ∇_s^2 act on vectors, but the knowledge of its eigenvalues and degeneracy for the scalar case will be useful for the determination of the vector case as well. Hence we start with scalars. Here we closely follow Chodos and Myers [11].

(i) Scalars

To obtain the eigenvalues λ of the equation

$$\nabla^2 \Phi = \lambda \Phi \tag{2.98}$$

on S^n, let us consider the set of all homogeneous harmonic polynomials $\hat{\Phi}$ of degree ℓ in $(n+1)$ dimensional Euclidean space $\mathbb{R}^{n+1}$ where the S^n can be thought of being embedded. Any homogeneous polynomial of degree 'ℓ' in $(n+1)$ variables may be written as

$$\hat{\Phi}(x_1, x_2, \ldots, x_{n+1}) = C_{a_1 \ldots a_\ell} x_{a_1} \ldots x_{a_\ell} \tag{2.99}$$

where $C_{a_1 \ldots a_\ell}$ is a constant symmetric tensor with ℓ-indices each of which takes $(n+1)$ values. The number of independent components of C is evidently equal to the number of terms in the expansion $(X_1+X_2+\ldots+X_{n+1})^\ell$. It can be easily seen by induction that the number of terms in the above expansion and hence the number of independent components of the tensor C is $\binom{n+\ell}{\ell}$.

Now the condition that Φ be harmonic in $\mathbb{R}^{n+1}$ is $\nabla_E^2 \hat{\Phi} = 0$ or equivalently,

$$\sum \delta_{a_1 a_2} C_{a_1 \ldots a_\ell} = 0 \tag{2.100}$$

which represents $\binom{n+\ell-2}{\ell-2}$ independent conditions for $\ell \geq 2$.

The eigenvalues of ∇^2 can be obtained by going to polar co-ordinates

$$ds^2 = dx_1^2 + dx_2^2 + \ldots + dx_{n+1}^2 = dr^2 + r^2 \ell_{ij} \, d\theta^i \, d\theta^j \tag{2.101}$$

where ℓ_{ij} is metric on the unit S^n.

Thus $$\nabla_E^2 \hat{\Phi} = \frac{1}{r^n} \frac{\partial}{\partial r} \left(r^n \frac{\partial \hat{\Phi}}{\partial r} \right) + \frac{1}{r^2} \nabla^2 \hat{\Phi} = 0. \tag{2.102}$$

Since $\hat{\Phi}$ is homogeneous function of degree ℓ we must have

$$\hat{\Phi} = r^{\ell} \Phi (\theta). \tag{2.103}$$

Substituting this in (2.102), we get

$$\nabla^2 \hat{\Phi} (\theta) = - \ell(\ell + n-1) \Phi (\theta). \tag{2.104}$$

Thus the eigenvalue $\lambda_s = -\ell(\ell+n-1)$ and the degeneracy $d_s(n,\ell)$ is given by

$$d_s(n,\ell) = \binom{n + \ell}{\ell} - \binom{n + \ell - 2}{\ell - 2} \theta (\ell - 2). \tag{2.105}$$

Let us now come to the case of vectors.

(ii) Vectors

Proceeding as above, we have the equation

$$\nabla^2_E \hat{V}_{\hat{a}} = 0, \tag{2.106}$$

with $\hat{V}_{\hat{a}} = C_{\hat{a}\,;\, a_1 \ldots a_l} \chi^{a_1} \ldots . \chi^{a_l}$. The total number of such entities is clearly

$$(n+1) \left[\binom{n + \ell}{\ell} - \binom{n + \ell - 2}{\ell - 2} \right]. \tag{2.107}$$

When we go to the polar co-ordinates, $\hat{V}_{\hat{a}}$ will decompose into a scalar which is the r-th component $\hat{V}_r = \hat{\rho}$ and a vector $\hat{V}_i$ on S^n. $\hat{V}_i$ in turn can be decomposed into two parts, one being the gradient of some scalar $\hat{\sigma}$ and the other a divergenceless vector $\hat{W}_i$

$$\hat{V}_i = \hat{W}_i + \nabla_i \hat{\sigma}, \tag{2.108}$$

with

$$\ell^{ij} \nabla_i \hat{W}_j = 0. \tag{2.109}$$

Before we go for the calculation of $\nabla^2 W_i$, let us write the metric corresponding to the line element (2.101) as

$$g = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & r^2 \ell_{ij} \end{array} \right). \tag{2.110}$$

Here μ,ν takes values from 1 to (n+1) and i,j takes values from 1 to n. It is easy to calculate the various components of affine connection

$$\Gamma^r_{rr} = \Gamma^r_{ir} = \Gamma^i_{rr} = 0,$$

$$\Gamma^r_{ij} = - r \ell_{ij} \; ; \; \Gamma^i_{rj} = \frac{1}{r} \delta^i_j \; ; \; \Gamma^i_{jk} = \frac{1}{2} \ell^{ip} \left(\ell_{pj,k} + \ell_{pk,j} - \ell_{jk,p} \right) \tag{2.111}$$

Then one can easily see that

$$\nabla^2_E V_\lambda = g^{\mu\nu} D_\mu D_\nu V_\lambda$$

$$= g^{\mu\nu} \frac{\partial^2 V_\lambda}{\partial x^\mu \partial x^\nu} - g^{\mu\nu} \frac{\partial \Gamma^\alpha_{\mu\lambda}}{\partial x^\nu} V_\alpha + g^{\mu\nu} \Gamma^\rho_{\nu\lambda} \Gamma^\alpha_{\rho\mu} V_\alpha$$

$$+ g^{\mu\nu} \Gamma^\rho_{\mu\nu} \Gamma^\alpha_{\lambda\rho} V_\alpha - 2 g^{\mu\nu} \Gamma^\alpha_{\mu\lambda} V_{\alpha,\nu} - g^{\mu\nu} \Gamma^\rho_{\mu\nu} V_{\lambda,\rho}$$

(2.112)

Putting $\lambda = i$, we calculate the above expression (2.112) term by term to get after some simplification using (2.111)

1st term $$g^{\mu\nu} V_{i,\mu\nu} = \frac{\partial^2 V_i}{\partial r^2} + g^{ik} \frac{\partial^2 V_i}{\partial x^i \partial x^k} \tag{2.113}$$

2nd term $$- g^{\mu\nu} \frac{\partial \Gamma^{\alpha}_{\mu i}}{\partial x^{\nu}} V_{\alpha}$$

$$= \frac{V_i}{r^2} + \frac{1}{r} \ell^{mn} \frac{\partial \ell_{mi}}{\partial x^n} V_r - g^{mn} \frac{\partial \Gamma^{j}_{mi}}{\partial x^n} V_j \tag{2.114}$$

3rd term $$g^{\mu\nu} \Gamma^{\rho}_{\nu i} \Gamma^{\alpha}_{\rho\mu} V_{\alpha}$$

$$= - \frac{1}{r} \Gamma^{j}_{ji} V_r + g^{mn} \Gamma^{j}_{mi} \Gamma^{p}_{jn} V_p \tag{2.115}$$

4th term $$g^{\mu\nu} \Gamma^{\rho}_{\mu\nu} \Gamma^{\alpha}_{i\rho} V_{\alpha} = - \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\beta}} \left(\sqrt{g} \; g^{\rho\beta} \right) \Gamma^{\alpha}_{i\rho} V_{\alpha}$$

$$= - \frac{n}{r^2} V_i + \frac{1}{r\sqrt{\ell}} \frac{\partial}{\partial x^m} \left(\sqrt{\ell} \, \ell^{mj} \right) \ell_{ij} V_r$$

$$- \frac{1}{r^2\sqrt{\ell}} \frac{\partial}{\partial x^m} \left(\sqrt{\ell} \, \ell^{mj} \right) \Gamma^{p}_{ij} V_p \tag{2.116}$$

5th term $$-2g^{\mu\nu} \Gamma^{\alpha}_{\mu i} V_{\alpha,\nu}$$

$$= - \frac{2}{r} \frac{\partial V_i}{\partial r} + \frac{2}{r} \frac{\partial V_r}{\partial x^i} - \frac{2}{r^2} \ell^{pq} \Gamma^{m}_{pi} V_{m,q} \tag{2.117}$$

6th term $\quad - g^{\mu\nu} \Gamma^{\rho}_{\mu\nu} V_{i,p} = \frac{n}{r} \frac{\partial V_i}{\partial r} + \frac{1}{\sqrt{\ell}} \frac{\partial}{\partial x^j} \left(\sqrt{\ell}\, g^{jm} \right) V_{i,m}$

(2.118)

Adding all the terms from (2.113) to (2.118), we get

$$
\begin{aligned}
\nabla^2 V_i = {} & \frac{\partial^2 V_i}{\partial r^2} + \frac{V_i}{r^2} - \frac{n}{r^2} V_i + \frac{(n-2)}{r} \frac{\partial V_i}{\partial r} + \frac{1}{r^2} \ell^{pq} \frac{\partial^2 V_i}{\partial x^p \partial x^q} \\
& + \frac{1}{r} \ell^{mn} \frac{\partial \ell_{mi}}{\partial x^n} V_r - \frac{1}{r^2} \ell^{mn} \frac{\partial \Gamma^j_{mi}}{\partial x^n} V_j - \frac{1}{r\sqrt{\ell}} \frac{\partial \sqrt{\ell}}{\partial x^i} V_r \\
& + \frac{1}{r^2} \ell^{mn} \Gamma^j_{mi} \Gamma^p_{jn} V_p + \frac{1}{r\sqrt{\ell}} \frac{\partial}{\partial x^m} \left(\sqrt{\ell}\, \ell^{mj} \right) \ell_{ij} V_r \\
& - \frac{1}{r^2\sqrt{\ell}} \frac{\partial}{\partial x^m} \left(\sqrt{\ell}\, \ell^{mj} \right) \Gamma^p_{ij} V_p + \frac{2}{r} \frac{\partial V_r}{\partial x^i} - \\
& \frac{2}{r^2} \ell^{pq} \Gamma^m_{pi} V_{m,q} + \frac{1}{r^2\sqrt{\ell}} \frac{\partial}{\partial x^j} \left(\sqrt{\ell}\, \ell^{jm} \right) V_{i,m}. \qquad (2.119)
\end{aligned}
$$

Now writing $V_r = \rho$ and $V_i = W_i + \frac{\partial \sigma}{\partial x^i}$ and substituting in (2.119) and after a long but straight-forward calculation one finds that W_i satisfies

$$\frac{\partial^2 W_i}{\partial r^2} + \frac{(n-2)}{r} \frac{\partial W_i}{\partial r} - \frac{(n-1)}{r^2} W_i + \frac{1}{r^2} \nabla^2 W_i = 0. \qquad (2.120)$$

Now putting $\lambda = r$ in (2.112), gives

$$0 = \nabla_E^2 \quad V_r \quad = \quad g^{\mu\nu} \frac{\partial^2 \rho}{\partial x^\mu \, \partial x^\nu} - g^{\mu\nu} \frac{\partial \Gamma^\alpha_{\mu r}}{\partial x^\nu} + g^{\mu\nu} \, \Gamma^\beta_{\nu r} \, \Gamma^\alpha_{\beta\mu} + g^{\mu\nu} \, \Gamma^\beta_{\mu\nu} \times$$

$$\Gamma^\alpha_{\beta r} \, V_\alpha - 2g^{\mu\nu} \, \Gamma^\alpha_{\mu r} \, V_{\alpha,\nu} - g^{\mu\nu} \, \Gamma^\alpha_{\mu\nu} \, \frac{\partial \rho}{\partial x^\alpha}. \qquad (2.121)$$

Proceeding as before we calculate term by term to get the

First term

$$g^{\mu\nu} \frac{\partial^2 \rho}{\partial x^\mu \, \partial x^\nu} = \frac{\partial^2 \rho}{\partial r^2} + \frac{1}{r^2} \, \ell^{ij} \, \frac{\partial^2 \rho}{\partial x^i \, \partial x^j} \qquad (2.122)$$

Second term vanishes and the third term becomes

$$g^{\mu\nu} \, \Gamma^\beta_{\nu r} \, \Gamma^\alpha_{\beta\mu} \, V_\alpha = - \frac{n}{r^2} \, \rho - \frac{1}{r^2 \sqrt{\ell}} \, \frac{\partial}{\partial x^n} \left(\sqrt{\ell} \, \ell^{mn} \right) V_m \qquad (2.123)$$

The fourth term becomes

$$g^{\mu\nu} \, \Gamma^\beta_{\nu\mu} \, \Gamma^\alpha_{\beta r} \, V_\alpha = - \frac{1}{r^2} \, \frac{1}{\sqrt{\ell}} \, \frac{\partial}{\partial x^i} \left(\sqrt{\ell} \, \ell^{ij} \right) V_j \qquad (2.124)$$

The fifth term gives

$$-2g^{\mu\nu} \, \Gamma^\alpha_{\mu r} \, V_{\alpha,\nu} = - \frac{2}{r^3} \, \ell^{ij} \, \frac{\partial V_i}{\partial x^j} \qquad (2.125)$$

while the sixth term gives

$$-g^{\mu\nu} \, \Gamma^\alpha_{\mu\nu} \, \frac{\partial \rho}{\partial x^\alpha} \, \frac{1}{\sqrt{g}} \, \frac{\partial}{\partial x^\beta} \left(\sqrt{g} \, g^{\alpha\beta} \right) \frac{\partial \rho}{\partial x^\alpha} = \frac{n}{r} \, \frac{\partial \rho}{\partial r} +$$

$$\frac{1}{r^2 \sqrt{\ell}} \, \frac{\partial}{\partial x^i} \left(\sqrt{\ell} \, \ell^{ij} \right) \frac{\partial \rho}{\partial x^j} \qquad (1.126)$$

Adding all the terms from (2.122) to (2.126) and making some simplifications yields

$$\frac{\partial^2\rho}{\partial r^2} + \frac{n}{r}\frac{\partial\rho}{\partial r} - \frac{n}{r^2}\rho + \frac{1}{r^2}\nabla^2\rho - \frac{2}{r^3}\ell^{ij}\frac{\partial(W_i+\partial_i\sigma)}{\partial x^j} - \frac{2}{r^3}\frac{1}{\sqrt{\ell}}$$

$$\frac{\partial}{\partial x^j}\left(\sqrt{\ell}\,\ell^{ij}\right)(W_i + \partial_i\sigma) = 0 \qquad (2.127)$$

Using $\ell^{ij}\ \nabla_i W_j = 0$ this further reduces to a pair of coupled equations for ρ and σ^i

$$\frac{\partial^2\rho}{\partial r^2} + \frac{n}{r}\frac{\partial\rho}{\partial r} - \frac{n}{r^2}\rho + \frac{1}{r^2}\nabla^2\rho - \frac{2}{r^3}\nabla^2\sigma = 0 \qquad (2.128)$$

$$\frac{\partial^2\sigma}{\partial r^2} + \left(\frac{n-2}{r}\right)\frac{\partial\sigma}{\partial r} + \frac{1}{r^2}\nabla^2\sigma + \frac{2}{r}\rho = 0 \qquad (2.129)$$

Now we note that $\hat{V}_{\hat{a}}$ is homogeneous of degree ℓ : $\hat{V}_{\hat{a}}\ \alpha\ r^{\ell}$ then $\rho = r^{\ell}\tilde{\rho}(\theta)$, $\sigma = r^{\ell+1}\tilde{\sigma}(\theta)$ and $\hat{W}_i = r^{\ell+1}\tilde{W}(\theta)$ (the extra factor of r comes from the transformation to polar co-ordinates).

Putting this into the equation (2.120) we can easily get the eigenvalues as

$$\nabla^2 W_i = -\left[\ell(\ell + n-1) - 1\right] W_i \qquad (2.130)$$

In order to compute the degeneracy,however,we must also know what is happening to ρ and σ.For that we again substitute $\rho = r^{\ell}\tilde{\rho}(\theta)$, $\sigma = r^{\ell+1}\tilde{\sigma}(\theta)$ and $W_i = r^{\ell+1}\tilde{W}(\theta)$ into the equation (2.128) and

(2.129),getting after some simplification

$$(\ell-1)(\ell+n)\tilde{\rho} + \nabla^2\tilde{\rho} = 2\,\nabla^2\tilde{\sigma} \qquad (2.131)$$

$$(\ell+1)\,(n+\ell-2)\,\tilde{\sigma} + \nabla^2\,\tilde{\sigma} = -2\,\tilde{\rho} \qquad (2.132)$$

At this stage to simplify matters let us write

$$a = (\ell-1)\,(\ell+n)$$

and

$$b = (\ell+1)\,(\ell+n-2) \qquad (2.133)$$

With that we can rewrite the pair of equations (2.131) and (2.132) after substituting the value of $\nabla^2\,\tilde{\sigma}$ from (2.132) into the right hand side of (2.131) giving an eigenvalue equation in matrix form as

$$\nabla^2 \begin{pmatrix} \tilde{\rho} \\ \tilde{\sigma} \end{pmatrix} = \begin{pmatrix} -(a+4) & -2b \\ -2 & -b \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{\sigma} \end{pmatrix} \qquad (2.134)$$

The eigenvalues λ of this matrix can be easily determined to be

$$\lambda = \frac{1}{2}\left[-(a+b+4) \pm \sqrt{(a-b)^2 + 8(a+b) + 16}\,\right] \qquad (2.135)$$

Substituting the values of a and b from (2.133) one gets the eigenvalues

$$\lambda = -(\ell-1)\,(\ell+n-2) \quad \text{or} \quad -(\ell+1)\,(\ell+n) \qquad (2.136)$$

i.e., $\lambda = -L(L+n-1)$ with $L = (\ell-1)$ and $(\ell+1)$ respectively.

To get the number of W's which we call $d_v(n,\ell)$, we must subtract the number of these scalar eigenfunctions from the total number of $\hat{V}_{\hat{a}}$'s:

$$d_v(n,\ell) = (n+1)\, d_s(n,\ell) - d_s(n,\ell+1) - d_s(n,\ell-1)$$

$$= (n+1)\left[\binom{n+1}{\ell} - \binom{n+\ell-2}{\ell-2}\theta(\ell-2)\right] - \binom{n+\ell+1}{\ell+1} + \binom{n+\ell-3}{\ell-3}\theta(\ell-3)$$

This is the same expression which appeared in (2.23). We note that $d_v(n,0) = 0$.

Now coming to the gradient case where the vector X_i on S^n can be written as $X_i = \nabla_i\phi$, we can see using the rules for commuting covariant derivatives that

$$\nabla^2(\nabla_i\phi) = \nabla_i(\nabla^2\phi) + (n-1)\,\nabla_i\phi \qquad (2.137)$$

The eigenvalues of the Laplacian ∇^2 can beeasily read out from (2.137) to be

$$-\ell(\ell+n-1) + (n-1) = -\left[\ell^2+(\ell-1)(n-1)\right] \quad \ell=1,2,.. \qquad (2.138)$$

With degeneracy $d_s(n,\ell)$. We omit $\ell = 0$ case as the corresponding scalar ϕ satisfies $\nabla^2\phi = 0$ and any such function on S^n also satisfies $\nabla_i\phi = 0$ and hence $X_i = 0$ in this case.

As the signature of the metric used in this Chapter is $(+,-,\ldots,-)$, the metric k_{pq} defined in (2.19) is the negative of ℓ_{pq} defined in (2.101) i.e., $k_{pq} = -\ell_{pq}$. Thus,

$$\nabla_s^2 = k^{pq}D_pD_q = -\ell^{pq}\,D_pD_q = -\nabla^2$$

and hence the eigen values of ∇_s^2 will carry an additional negative sign (see equations (2.22) and (2.24).

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CHAPTER III

DYNAMICAL COMPACTIFICATION IN KALUZA-KLEIN COSMOLOGY[1]

Introduction

If the extra dimensions really exist, then can one see classically how in the course of time these extra dimensions shrunk to such a small size, to render it practically inaccessible to present day accelerators ? Thus the question that naturally arises is that whether it is possible to get an asymmetrical solution of Einstein's equation at the classical level itself where ordinary space-time go on expanding and extra dimensions go on contracting with time.

The first attempt in these directions was made by Chodos and Detweiler[1] using Kasner[2] metric for 5 dimensions.

$$ds^2 = dt^2 - \sum_{i=1}^{4} (t/t_o)^{2p_i} (dx^i)^2 \qquad (3.1)$$

where the time 't' can be chosen as a continuous real parameter ($-\infty < t < +\infty$) and the spatial four co-ordinates x^i to be periodic $0 \leq x^i < L$ in a suitable co-ordinate system. They find that this metric solves matter free Einstein's equation $R_{MN} = 0$ provided that

$$\sum_{i=1}^{4} p_i = \sum_{i=1}^{4} p_i^2 = 1 \qquad (3.2)$$

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[1]B. Chakraborty and R. Parthasarathy, Phys. Lett. A142 (1989) 75.

To get an isotropic three-dimensional space, they put $p_1=p_2=p_3$ and solve equation (3.2) to get

$$p_1 = p_2 = p_3 = - p_5 = 1/2 \qquad (3.3)$$

Substituting this back in equation (3.1) the metric takes the form

$$ds^2 = - dt^2 + (t/t_0)\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + (t_0/t)(dx^5)^2 \qquad (3.4)$$

which has the desirable feature of 3-space expanding and the additional one-dimension contracting with time.

Later the same approach was taken by (i) Freund [3] for a higher dimensional Jordan-Brans-Dicke (JBD)theory. He starts with (4+n) dimensional JBD action without matter fields to find Kasner type solutions where the ordinary 3 spatial dimensions go on expanding and the additional n - dimensions go on shrinking in size. In their analysis they assume that both the 3 and n dimensional subspaces are essentially flat i.e., the curvature constants of these spaces vanish, thus in their case the extra space is non-compact. Later (ii) Kerner [4] considered a six dimensional theory with a monopole and a Higgs field on S^2 and (iii) Kolb[5] considered a six dimensional theory with a monopole alone on S^2 towards obtaining asymmetric expansion.

The other approach is based on thermodynamical arguments. Here the energy momentum tensor is invariably taken to correspond to a perfect fluid in all spatial dimensions with an assumption that this is appropriate for a radiation dominated era. This approach has been used by Alvarez and Gavela[6]; Abott, Barr and Ellis[7]; Sahdev and Kolb[8] and others. There have been attempts to use Gauss-Bonnet type of action for higher dimensional gravity as well, to obtain asymmetric expansion and inflation(eg.Shafi and Wetterich[9]).

Compactification with the σ-model

Here in this Chapter we consider the same compactification scheme discussed in Chapter 2, and study the same Gell-Mann Zwiebach model for its possible cosmological implications. We attempt to solve the field equations (2.5)

$$R_{\mu\nu} = 0 \quad \text{and} \quad R_{ij} = 2/\lambda^2 m_{ij}$$

assuming that the line element is given by

$$ds^2 = dt^2 - \left[R_3(t) \right]^2 g_{mn}\, dx^m\, dx^n - \left[R_I(t) \right]^2 g_{ij}\, dy^i\, dy^j \qquad (3.5)$$

which is a generalization of the Friedmann-Robertson Walker metric where $R_3(t)$ and $R_I(t)$ are the scale factors of the three ordinary and extra dimensions respectively. The three dimensional subspace is assumed to be flat and the internal space is taken to be S^n. Thus $R_I(t)$ can be taken as the radius of S^n at time t.

The various components of R_{MN} corresponding to the above line element are given by

$$R_{oo} = - \frac{3\ddot{R}_3(t)}{R_3(t)} - \frac{n\ddot{R}_I(t)}{R_I(t)} \qquad (3.6)$$

$$R_{mn} = \left[\frac{d}{dt}\left(\frac{\dot{R}_3(t)}{R_3(t)}\right) + 3\left(\frac{\dot{R}_3(t)}{R_3(t)}\right)^2 + n\,\frac{\dot{R}_3(t)\,\dot{R}_I(t)}{R_3(t)\,R_I(t)}\right] R_3(t)^2\, g_{mn} \quad (3.7)$$

$$R_{ij} = \left[\frac{K_I}{R_I^2(t)} + \frac{d}{dt}\left(\frac{\dot{R}_I(t)}{R_I(t)}\right) + \left(3\frac{\dot{R}_3(t)}{R_3(t)} + n\frac{\dot{R}_I(t)}{R_I(t)}\right)\frac{\dot{R}_I(t)}{R_I(t)}\right] R_I^2(t) g_{ij} \quad (3.8)$$

where K_I is the constant curvature of the compact extra space, taken to be S^n. For the scale factors $R_3(t)$ and $R_I(t)$we assume

$$R_3(t) = r(t + t_o)^{\alpha} \quad , \quad R_I(t) = R(t + t_o)^{\beta} \quad (3.9)$$

where α and β are constants and $t = t_o$ is the time at which all space dimensions have comparable size. r and R are the actual common sizes of these dimensions. Substituting this ansatz (3.9) into the equations of motion given by (2.5) and using equations (3.6) and (3.7), we obtain

$$3\alpha(\alpha-1) + n\beta(\beta-1) = 0,$$

$$3\alpha + n\beta = 1 \quad (3.10)$$

which yield

$$\alpha = \frac{3 \pm \sqrt{6n + 3n^2}}{3(3+n)} \quad , \quad \beta = \frac{n \mp \sqrt{6n + 3n^2}}{n(3+n)} \quad (3.11)$$

At this stage we would like to have the (physical) three spatial dimensions expanding and the extra n dimensions contracting. So the appropriate solutions is

$$\alpha = \frac{3 + \sqrt{6n + 3n^2}}{3(3+n)} \quad , \quad \beta = \frac{n - \sqrt{6n + 3n^2}}{n(3+n)} \quad (3.12)$$

Incidentally these solutions for α and β have the same form for the generalized Kasner metric of Chodos and Detweiler[1,2]. We are able to obtain solutions to the cosmological model based on a non-linear sigma model interacting with gravity in which the usual space dimensions expand and the extra dimensions contract. However the expansion is not exponential. Now let us consider(3.8). We observe (see eqn.3.5) that with the time dependence of the metric for the extra space coming through the scale factor $R_I(t)$,the extra space in no longer Einsteinian as is clear from the expression for R_{ij} in (3.8) unless the second and third term within the parenthesis vanishes. Further,unless it is Einsteinian we will not have the kind of solution as we have described in the last chapter. But fortunately for us with the ansatz (3.9) and the relations (3.10) we find that the second and third term vanishes identically making the Ricci tensor proportional to the metric

$$R_{ij} = K_I g_{ij} \tag{3.13}$$

and hence Einsteinian. Now instead of taking $G_{ij} = R_I^2(t) g_{ij}$ as proportional to the scalar manifold metric m_{ij},we take $g_{ij} = a^2 m_{ij}$thus m_{ij} like g_{ij} is time independent (Note that there is no negative sign here).

Thus we take

$$R_{ij} = K_I a^2 m_{ij} \tag{3.14}$$

But from (2.5), we also have $R_{ij} = -\frac{2}{\lambda^2} m_{ij}$, thus we must have $K_I = \frac{2}{\lambda^2 a^2}$, as a new constraint on the parameters of the model of

Gell-Mann and Zwiebach[10], when the classical equations are solved for Friedmann-Robertson-Walker geometry. It may appear that this new constraint is a direct consequence of the specific ansatz (3.9) and the relations (3.10). We now show that this is not the case, just by assuming a monotonic decrease for $R_I(t)$ and $\dot{R}_I(t)$ with time. This makes the seconds and the third term within the parenthesis of equation (3.8) vanish at large time t giving us the above constraint involving the constant parameters of the theory. Since this constraint involving constants is true for asymptotically large time, we must have this constraint satisfied at all times. Thus from equation (3.8) and (2.5) we get

$$\frac{d}{dt}\left(\frac{\dot{R}_I(t)}{R_I(t)}\right) + \left(3\frac{\dot{R}_3(t)}{R_3(t)} + n\frac{\dot{R}_I(t)}{R_I(t)}\right)\frac{\dot{R}_I(t)}{R_I(t)} = 0 \qquad (3.15)$$

Then defining the Hubble constants for the ordinary space and that of the extra space respectively

$$H_3(t) = \frac{\dot{R}_3(t)}{R_3(t)} \quad , \quad H_I(t) = \frac{\dot{R}_I(t)}{R_I(t)} \qquad (3.16)$$

and

$$3H_3(t) + nH_I(t) = f(t) \qquad (3.17)$$

equations $R_{mn} = 0$ and (3.15) imply

$$\dot{H}_3 + 3H_3^2 + nH_3H_I = 0$$

$$\dot{H}_I + 3H_3H_I + nH_I^2 = 0 \qquad (3.18)$$

which gives

$$\frac{\dot{H}_3}{H_3} = \frac{\dot{H}_I}{H_I} = -f(t) \tag{3.19}$$

Differentiating and using (3.17) we get

$$\dot{f} = -f^2 \tag{3.20}$$

which on integrating yields $f = \frac{1}{t + t_0}$, where t_0 is a constant of integration. Substituting in (3.19), we get on integration

$$H_3(t) = \frac{\alpha}{(t + t_0)} \quad \text{and} \quad H_I(t) = \frac{\beta}{(t + t_0)} \tag{3.21}$$

where α and β being another pairs of constant of integration, which agree with our ansatz (3.9). A monotonic increase for $R_3(t)$ with t can be obtained with $\alpha > 0$ and a monotonic decrease for $R_I(t)$ with $\beta < 0$. We mention that our this solution resembles very closely to the above mentioned Freund's solution [3].

Thus it is possible to have an asymmetric expansion in Kaluza-Klein cosmology with the Gell-Mann and Zwiebach model[10]. We are unable to obtain such an asymmetric expansion with the monopole induced compactification scheme of Randjbar-Daemi, Salam and Strathdee[11]. This is due to the presence of the cosmological constant term in the model. The radiation dominated era can be easily accommodated in this model by introducing a stress tensor corresponding to a perfect fluid in (4+n) dimensions. As the treatment of this has been carried out by Alvarez and Gavela[6] and Sahdev[8] we will not repeat it here except for the remark that the model presented above along with the stress tensor for a perfect fluid allow us to realise inflation and non-abelian gauge symmetries eventually.

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CHAPTER IV

ON INSTANTON INDUCED SPONTANEOUS COMPACTIFICATION IN $M^4\times CP^2$ AND CHIRAL FERMIONS[1]

It may happen that in some interesting models of spontaneous compactification the internal space B^n does not admit spin structure. In such cases normally one cannot even expect to have any fermions, let alone chiral fermions, as spinors cannot be defined in such spaces. As we have seen in Chapter I that for a compact coset space G/H the Dirac operator does not have any zero modes (Lichnerowicz theorem[1]) but has massive (Planck mass) modes. Thus in these cases one can still define spinors. To get zero modes one generally puts additional gauge fields (with non-trivial topology) coupled to fermions so that, besides triggering compactification, one can have zero modes corresponding to the modified Dirac operator. Alternately one can put torsion in the internal manifold B^n. But if the spin structure is not allowed then no fermions (massless or massive) can be obtained in general. We now briefly outline some of the successful models for spontaneous compactification admitting chiral fermions.

Randjbar-Daemi, Salam and Strathdee[2] considered six dimensional gravity in the product space M^4xS^2 with a monopole configuration on S^2only. The same monopole triggers spontaneous

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[1]B.Chakraborty and R.Parthasarathy,Class.Quant.Grav.7(1990)1217.

compactification and ensures fermion chirality.

Randjbar-Daemi et al[3] provided another example of this mechanism, an eight dimensional gravity based on $M^4 \times S^4$ in which gravity is coupled to a SU(2) Yang-Mills field which takes the form of a one-instanton configuration in S^4. The idea of introducing elementary gauge fields which trigger spontaneous compactification and ensure fermion chirality is perhaps reasonable as they might naturally arise in supergravity theory in still higher dimensions.

Here we attempt a spontaneous compactification of an eight dimensional gravity based on $M^4 \times CP^2$ where gravity is coupled to U(1) instanton on CP^2. Although it is known that CP^2 does not admit spin structure[4] but we shall see that by coupling fermions of definite charges to these instantons we can have generalized spin structure, following Hawking and Pope[5] and thus getting chiral fermions.

Instanton induced compactification $M^4 \times CP^2$

We first give a brief account of the CP^2 instanton(details of the construction of this instanton have been given in the appendix). The set of three complex variables (Z^1, Z^2, Z^3) not all zero, with the identification $(Z^1, Z^2, Z^3) \sim (\lambda Z^1, \lambda Z^2, \lambda Z^3)$ for any non-zero complex number λ is said to define CP^2. By imposing the constraint $|Z^1|^2+|Z^2|^2+|Z^3|^2=1$, we get rid of the scale factor.

One can define

$$A_i(Y) = \frac{i}{2} Z^{\alpha}(Y)^* \overleftrightarrow{\partial_i} Z^{\alpha}(Y) \qquad (\alpha = 1,2,3) \qquad (4.1)$$

with the above constraint $Z^{\alpha^*} Z^{\alpha}=1$. This $A_{\mu}(Y)$ has all the desirable features of a vector potential under U(1) group (Gava, et al[6]). We obtain the usual projective co-ordinates for CP^2 in the chart $Z^3 \neq 0$ by forming the ratios

$$\rho^1 = \frac{Z^1}{Z^3} \quad \text{and} \quad \rho^2 = \frac{Z^2}{Z^3} \quad \text{. with} \quad Z^3 \neq 0 \qquad (4.2)$$

Setting $\rho^1 = (y^1 + iy^2)$ and $\rho^2 = (y^3 + iy^4)$ we can introduce a U(1) covariant derivative

$$D_i Z^{\alpha} = (\partial_i + \frac{i}{2} Z^{\beta^*} \overleftrightarrow{\partial_i} Z^{\beta}) Z^{\alpha} \qquad (4.3)$$

This allows us to form the squared invariant distance

$$ds^2 = \frac{1}{2} dy^i dy^j \left[(D_i Z^{\alpha})^* (D_j Z^{\alpha}) + (D_j Z^{\alpha})^* (D_i Z^{\alpha}) \right] \qquad (4.4)$$

from which the CP^2 metric can be found to be

$$g_{ij} = \frac{1}{1 + y^2} \left(\delta_{ij} - \frac{y_i y_j + \tilde{y}_i \tilde{y}_j}{1+y^2} \right) \qquad (4.5)$$

where $y^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2$ while .

$$\tilde{y}_i = C_{ij}\, y_j \qquad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.6}$$

It has been shown by Gava et al[6] that the gauge field A(y) defined by (4.1) happens to be the same which solves the Einstein-Maxwell equations for the CP^2 space. In terms of the internal space coordinates, we have

$$A_i(y) = \frac{\tilde{y}_i}{(1+y^2)} = \frac{C_{ij}\, y_j}{(1+y^2)} \tag{4.7}$$

This is the CP^2instanton. We now introduce angular co-ordinates (χ ,θ, ϕ ,ψ) as

$$y_1 = \tan\chi \; \mathrm{Cos}\left(\frac{\theta}{2}\right) \; \mathrm{Cos}\; \frac{1}{2}\,(\phi + \psi)$$

$$y_2 = \tan\chi \; \mathrm{Cos}\left(\frac{\theta}{2}\right) \; \mathrm{Sin}\; \frac{1}{2}\,(\phi + \psi)$$

$$y_3 = \tan\chi \; \mathrm{Sin}\left(\frac{\theta}{2}\right) \; \mathrm{Cos}\; \frac{1}{2}\,(\phi - \psi)$$

$$y_4 = \tan\chi \; \mathrm{Sin}\left(\frac{\theta}{2}\right) \; \mathrm{Sin}\; \frac{1}{2}\,(\phi - \psi) \tag{4.8}$$

with the ranges $0 \le \chi < \frac{\pi}{2}$, $0 \le \theta < \pi$, $0 < \phi < 2\pi$, $0 < \psi < 4\pi$.

The $M^4 \times CP^2$ manifold is an eight-dimensional space and for the spontaneous compactification we propose the following action

$$S = - \int d^4x\, d^4y \sqrt{G} \left(\frac{R}{K^2} + \frac{1}{4} F_{MN} F^{MN} + \Lambda \right) \tag{4.9}$$

Here Λ is the cosmological constant in eight dimensions and

$F_{MN} = \partial_M A_N - \partial_N A_M$ where we take A_M to live on the internal space CP^2 only with its instanton configuration given in (4.7). Thus our proposed model for spontaneous compactification of an eight dimensional gravity in $M^4 x CP^2$ is very similar to that of the six-dimensional gravity in $M^4 \times S^2$ by Randjbar-Daemi, Salam and Strathdee[2] in the sense that the role of the monopole configuration in S^2 is played by the U(1) instanton configuration in CP^2. In (4.9) the matter field is due to the CP^2 instanton. The Einstein field equation that follow from (4.9) are

$$R_{MN} = -\frac{K^2}{2}\left(T_{MN} - \frac{1}{6} TG_{MN} + \frac{\Lambda}{3} G_{MN}\right) \tag{4.10}$$

where T_{MN} is the energy-momentum tensor for the CP^2 instanton (non-gravitational) and $T = G^{MN}T_{MN}$. We are interested in those solution of (4.10) which admit a product structure of the form $M^4 \times CP^2$ with M^4 being Minkowskian flat. We follow the procedure of Randjbar-Daemi et al[3] which is that instead of explicitly solving the field equations, we impose the requirement that M^4 should be Minkowskian flat and obtain algebraic constraints which are checked for consistency. Using the orthonormal basis (we use α,β, to denote M^4 and a,b, to denote CP^2) we set Ricci curvature in M^4, $R_{\alpha\beta} = 0$ so that M^4 can be taken as Minkowskian flat. This gives,

$$T_{\alpha\beta} - \frac{1}{6} T\eta_{\alpha\beta} + \frac{\Lambda}{3} \eta_{\alpha\beta} = 0 \tag{4.11}$$

Equation (4.11) can be rewritten as

$$T_{\alpha\beta} = \frac{c}{K^4}\,\eta_{\alpha\beta} \tag{4.12}$$

with

$$\frac{c}{K^4} = \frac{1}{6}\,(T-2\Lambda) \tag{4.13}$$

This structure of $T_{\alpha\beta}$ is expected from the four-dimensional Poincare invariance, with c as a constant. Now CP^2 is an Einsteinian manifold which implies

$$T_{ab} = \frac{c'}{K^4}\;\delta_{ab} \tag{4.14}$$

The algebraic condition for spontaneous compactification is $c' > c$. By explicit calculation starting from (4.11) - (4.14)using

$$T_{MN} = F_{ML}\,F_N{}^L - \frac{1}{4}\;G_{MN}\,F^2$$

and noting that the instanton is defined only in CP^2, we arrive at the following relations

$$\begin{aligned} T &= \frac{4(c + c')}{K^4} = -F^2 \\ \Lambda &= (2c' - c)\,/\,K^4 \\ R_{ab} &= -\frac{(c' - c)}{2K^2}\;\delta_{ab} \\ T_{\alpha\beta} &= -\frac{1}{4}\;\eta_{\alpha\beta}\,F^2 \end{aligned} \tag{4.15}$$

A consistent solution is found to be

$$c' = 0 \qquad\qquad c = -\frac{1}{4} K^4 F^2 \tag{4.16}$$

which shows $c' > c$ and so CP^2 instanton induced compactification with M^4 Minkowskian flat is possible.(This is ensured by the negativity of the Ricci tensor R_{ab}). However the appropriate solution for the instanton in the internal space CP^2 must have constant F^2, which is directly verified below. In terms of the angular co-ordinates introduced in(4.8) the non-vanishing components of the field tensor are found to be

$$\begin{aligned} F_{\chi\phi} &= -\operatorname{Sin}\chi \operatorname{Cos}\chi \\ F_{\theta\psi} &= \frac{1}{2}\operatorname{Sin}^2\chi \operatorname{Sin}\theta \\ F_{\chi\psi} &= -\operatorname{Sin}\chi \operatorname{Cos}\chi \operatorname{Cos}\theta \end{aligned} \tag{4.17}$$

and the various non-vanishing metric components for CP^2 are

$$g_{\chi\chi} = 1 \;,\quad g_{\theta\theta} = \frac{1}{4}\operatorname{Sin}^2\chi \;,\qquad g_{\psi\psi} = \frac{1}{4}\left(1-\operatorname{Sin}^2\chi \operatorname{Cos}^2\theta\right)\operatorname{Sin}^2\chi$$

$$g_{\psi\phi} = \frac{1}{4}\operatorname{Sin}^2\chi \operatorname{Cos}^2\chi \operatorname{Cos}\theta, \qquad g_{\phi\phi} = \frac{1}{4}\operatorname{Sin}^2\chi \operatorname{Cos}^2\chi \tag{4.18}$$

and after a lengthy but straightforward calculation we find $F^2 = 16$, thus ensuring the constancy of F^2. Thus we have demonstrated the existence of ground state solution for eight dimensional gravity with the geometry $M^4 \times CP^2$ where the internal space of CP^2 supports a U(1) instanton. We call this CP^2 induced compactification. One can induce spontaneous compactification by the SU(3) invariant SU(2) ×U(1) non-abelian connection also as has been shown in general by Randjbar-Daemi et al.[8] where they considered the spontaneous compactification of $M^4 \times G/H$ induced by the G-invariant H gauge field.

Generalised spin structure in CP^2

The mathematical conditions under which an orientable simply-connected manifold allows spin structure is the vanishing of the second Stiefel-Whitney class[4]. For CP^2 the second Stiefel-Whitney does not vanish showing that it does not allow spin structure. A comparatively simple explanation of why spin structure cannot be defined on an arbitrary manifold has been given by Hawking and Pope[5]. Accordingly the crucial quantity of interest here is the index of the Dirac operator. Detailed considerations (Eguchi et al[4] , Hawking and Pope[5], Delbourgo and Salam[7]) give the following index theorem for four-dimensional CP^2 space

$$n_R - n_L = -\frac{1}{384\pi^2}\int {}^*R^{ijab}\, R_{ijab}\sqrt{g}\, d^4y \qquad (4.19)$$

where ${}^*R$ is the dual, ${}^*R^{ijab}=\frac{1}{2}\frac{\epsilon^{ijmn}}{\sqrt{g}}R_{mn}{}^{ab}$ (ϵ^{abcd} = +1 (-1)if abcd is an even(odd) permutation of1234 and 0 otherwise) and n_R and n_L are the number of zero modes with right handed ($\psi = \gamma^5\psi$) and left handed $(\psi = -\gamma^5\psi)$ helicity. The CP^2 manifold has its natural Fubini-Study metric using which the above expression can be calculated. From this it turns out that $n_R - n_L = -\frac{1}{8}$, which is not an integer and so one cannot define spinors consistently on CP^2. Nevertheless one can still define a generalized spin structure[5] by consistently coupling fermions to some topologically non-trivial gauge field. We now give the procedure.

Consider a closed 2-surface 'Σ' homeomorphic to S^2 being spanned by a one-parameter family of loops $\lambda(v)$ with $\lambda(0)$ and $\lambda(1)$ being the trivial loops - the point itself (see Chapter 5 for details). Now parallel transporting a scalar field ϕ of charge 'e' coupled to a gauge field A_i around $\lambda(v)$, we get the new field ϕ' given by

$$\phi'(v) = \exp\,(ief(v))\phi \tag{4.20}$$

where

$$f(v) = \int_0^1 A_i\; dy^i \tag{4.21}$$

As f(0) and f(1) corresponds to the trivial loops, we must have

$$\phi'(0) = \phi'(1) \tag{4.22}$$

which gives

$$e[f(0)-f(1)] = 2\pi n \tag{4.23}$$

It can be shown that(for details see chapter 5)

$$f(0) - f(1) = \int_{\Sigma} F \qquad (4.24)$$

where $F = \frac{1}{2} F_{ij} dy^i \wedge dy^j = dA$ is the field 2-form for A. We take this U(1) gauge potential to be the instanton configuration (4.7) for which the non-vanishing components have been given in (4.17). For computational convenience we consider $(\chi, \theta, \alpha, \beta)$, where α and β are linear combinations of ϕ and ψ as $\alpha = \frac{1}{2}(\phi + \psi)$ and $\beta = \frac{1}{2}(\psi - \phi)$, as new co-ordinates with $0 < \alpha,\ \beta < 2\pi$. Then,

$$F_{\chi\alpha} = - \operatorname{Sin} 2\chi \ \operatorname{Cos}^2\left(\frac{\theta}{2} \right)$$

$$F_{\chi\beta} = \operatorname{Sin} 2\chi \ \operatorname{Sin}^2\left(\frac{\theta}{2} \right) \qquad (4.25)$$

$$F_{\theta\alpha} = \frac{1}{2} \operatorname{Sin}^2 \chi \ \operatorname{Sin}\theta = F_{\theta\beta}$$

are the non-vanishing components of F_{ij}. As $\pi_2(CP^2) = \mathbb{Z}$ and CP^1 is the generator i.e. belongs to the first homotopy class, we take Σ to be CP^1 itself. Now two different embeddings of CP^1 in CP^2 can be obtained by setting $\theta = 0$ or $\theta = \pi$. We take $\theta = \pi$. Then the only non-vanishing component of F is $F_{\chi\beta} = \operatorname{Sin} 2\chi$. Then

$$\int_{CP^1} F = \int \operatorname{Sin} 2\chi \ d\chi \wedge d\beta = 2\pi \qquad (4.26)$$

If we consider the integration of F over a surface Σ^n belonging to the n-th homotopy class then the above integral will get an additional factor of n as CP^1 is the generator. (In particular if we consider S^2 defined by $y_1^1 + y_2^2 + y_3^2 = 1$ and $y_4 = 0$ in CP^2, then we verify that the above surface integral vanishes, as expected, as the surface is contractible).

Thus for an arbitrary surface $\sum^n$ belonging tothe n-th homotopy class we have

$$\int_{\Sigma^n} F = 2\pi n \tag{4.27}$$

Now taking the surface to be CP^1 itself we get from (4.23) and (4.26) $e = n$. If we consider the parallel transport of spinors ψ with charge e' then we have as has again been shown in Chapter 5 in detail

$$\psi'(0) = -\psi'(1) \tag{4.28}$$

(Compare this with (4.22) for a scalar field) and this in turn implies

$$e'[f(0) - f(1)] = 2\pi(m + \frac{1}{2}) \tag{4.29}$$

Considering the surface to be CP^1itself and using (4.26) we obtain the allowed values of e' as $e' = (m + \frac{1}{2})$. The index theorem now incorporates the contribution from an instanton coupled to a fermion besides the usual one given in (4.19) and the final form is given by[5].

$$n_R - n_L = \frac{e'^2}{16\pi^2}\int {}^*F^{ij}F_{ij}\sqrt{g}\,d^4y - \frac{1}{384\pi^2}\int {}^*R^{ijab}R_{ijab}\sqrt{g}\,d^4y \quad (4.30)$$

For CP^2 instanton the field two form F is self dual $^*F = F$ (we choose $\varepsilon^{\chi\theta\phi\psi} = +1$). One can also readily verify thus $^*F^{ij}\,F_{ij} = F^{ij}F_{ij} = 16$, as we have seen earlier. The volume of CP^2 given by $\int\sqrt{g}d^4y$ can be easily calculated with respect to the metric given in (4.18) to be $\frac{\pi^2}{2}$. The second term in (4.30) is same as (4.19) and is $-\frac{1}{8}$, as have been seen. So with the allowed values of $e' = (m+\frac{1}{2})$, we get from (4.30)

$$n_R - n_L = \frac{1}{2}\,m(m+1) \quad (4.31)$$

which is always an integer, showing that though ordinarily spin structure do not exist on CP^2, one may define a generalized spin structure by coupling the instanton to fermions of definite charges. Thus in our case the internal Dirac-operator has zero modes with chiral asymmetry showing that in d=4 theory we will have chiral fermions.

4. CONCLUSION

We have presented a spontaneous compactification scheme for eight-dimensional gravity based on $M^4 x CP^2$. As the internal space CP^2 can be considered to be the homogeneous space $SU(3)/SU(2) \times U(1)$ the internal symmetry will correspond to SU(3). CP^2 admits topologically non-trivial gauge fields, the U(1) instanton and this triggers spontaneous compactification. The algebraic conditions for this have been derived and verified in section 2. This mechanism of spontaneous compactification by introducing topologically non-trivial gauge fields in the internal space is very similar to the monopole induced compactification scheme for $M^4 x S^2$ or to G invariant H- gauge field induced compactification scheme for $M^4 \times G/H$ for a symmetric G/H [8] - which in this case corresponds SU(3) invariant SU(2) × U(1) gauge field induced compactification for $M^4 \times CP^2$. One of the problems with CP^2 internal space is that it does not admit spin structure. This is briefly explained in section 3 using index theorem. This aspect has been studied previously by Hawking and Pope (1978) without recourse to spontaneous compactification. We have here shown how one can possibly define a spin structure by coupling fermions to the CP^2instanton field. Usually classical backgrounds can be topologically non-trivial and are expected to induce chiral fermion spectra. If the internal space is even dimensional then it can be said that there will be a set of fermionic chiral modes. The number of such modes or more precisely the difference in the number of left and right chiral modes is governed by the index theorem. We have explicitly evaluated the index of the Dirac operator in the internal space CP^2 and found it to be an integer. This is due to the contribution from the instanton. In this way the CP^2instanton here plays two roles ; it triggers the spontaneous compactification with M^4 Minkowskian flat and it makes $n_R - n_L$ an integer so that spinors may be consistently defined on the internal space as well. Once again the analogy with monopole induced compactification of $M^4 \times S^2$ is striking as in S^2, a 2-sphere, the index of the Dirac operator is solely due to the monopole contribution (the first term in (4.30) were now A_μ as a monopole in S^2 is the sole contributor) which turns out to be an integer.

In such higher dimensional theories it has been known (see Chapter 1) that the mass operator for fermions is the Dirac operator in the internal space. This operator will have zero modes as dictated by the index theorem. In contrast to the above mechanism of introducing topologically non-trivial gauge fields in the internal space to obtain chiral fermions, there have been attempts to distort the internal space by introducing torsion. In the Dirac operator for the internal space, instead of the minimal coupling term, the spin connection will now have the torsion contribution as well. In particular Neville[9] has constructed fermions with chiral asymmetry in the presence of torsion on S^4 and this idea has been generalised to S^{4p} by Tchrakian[10] with a suggestion of application to Kaluza-Klein theories. Nevertheless, the action will now contain a torsion Lagrangian as well and under dimensional reduction to four-dimensional theory one obtains the usual Einstein-Hilbert action and additional problems due to the higher derivative nature of the residual Lagrangian. Therefore it is desirable to introduce additional gauge fields which do not have such problems.

Appendix

Here we are going to elaborate the construction of CP^2 instanton. We start by outlining the general methods of construction of instanton as given by Atiyah [11] and then apply it to the CP^2 case.

Let us recall that a vector bundle E over a space M consists of a family of vector spaces E_y parametrized (continuously or differentiably) by $y \in M$. E can be said to be a sub bundle of the trivial bundle $M \times \mathbb{R}^N$ if each E_y is embedded as a vector sub space in $\mathbb{R}^N$, the embedding varying continuously (or differentially) with y. Thus for example if M is embedded as a manifold in $\mathbb{R}^N$ its tangent space (translated to the origin) gets identified with subspaces of $\mathbb{R}^N$. When E is a sub bundle of $M \times \mathbb{R}^N$,any section of E,namely a function f(y) taking its values in E_y can be thought of as a function taking values in $\mathbb{R}^N$. If we now take partial derivatives of f, then this will not in general take values in E_y .Thus projecting this ordinary derivatives into E_y defines the co-variant derivative on E.

$$\nabla f = P\, df \qquad (4.32)$$

where P is the projection operator. For example if E be the tangent bundle over M and if P be the orthogonal projection then we get the ordinary Levi-Civita connection of Riemannian geometry.

Choosing a orthogonal gauge or local frame for the bundle E (here we assume E is given canonical inner product or a metric induced from the bundle $M \times \mathbb{R}^N$) will give rise to linear maps

$\mathbb{R}^n \longrightarrow E_y$ (if $E_y \cong \mathbb{R}^n$) which are in fact isomorphisms preserving orthogonality. Now composing these isomorphisms with the continuous (or differentiable) embedding of E_y in $\mathbb{R}^N$, we can write $U_y : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ and image of $U_y = E_y$,we note that $U_y^\dagger U_y = 1$ and $U_y U_y^\dagger = P_y$ is the projection operator projecting orthogonally elements in $\mathbb{R}^N$ onto E_y.To compute the covariant derivative ∇ in the gauge 'U' we put $f = U\lambda$ where λ is now a function on M taking values in $\mathbb{R}^n$ and find

$$\nabla(U\lambda) = P\, d(U\lambda) = UU^\dagger\, d(U\lambda) = U\left[d\lambda - i(iU^\dagger dU)\lambda\right] \qquad (4.33)$$

Showing that the gauge potential is given by

$$A = iU^\dagger dU \qquad (4.34)$$

The same formalism goes through if the vector bundles are over $\mathbb{C}$ rather than over $\mathbb{R}$. Indeed the manifold we are interested in is CP^2 which is a complex manifold of dimension 2 i.e. real 4-dimensional manifold.

To determine the U(1) instanton connection over CP^2 we consider the tautological line bundle over CP^2. It may be recalled that CP^2 is the space of all complex triplets (Z^1,Z^2,Z^3) not all zero with the identification $(Z^1,Z^2,Z^3) \sim (\lambda Z^1,\lambda Z^2,\lambda Z^3)$ for a non-zero complex number λ. Thus it is the space of all complex lines through the origin in $\mathbb{C}^3$. To find a representative point in this line we can initially get rid of the scale factor by imposing the constraint $|Z^1|^2+|Z^2|^2+|Z^3|^2=1$ which defines S^5. But we still have a U(1) freedom left corresponding to the transformation $Z^i \longrightarrow e^{i\theta}Z^i (i = 1,2,3)$. Further quotienting out this U(1) transformation we get $CP^2 = S^5/S^1$ (Hopf fibration of S^5). Thus S^5 is the principal bundle over CP^2 with gauge group U(1). Alternatively in the neighbourhood $Z^3 \neq 0$ (say) we can give a projective co-ordinate to CP^2 as

$$\left(Z^1,\ Z^2, Z^3\right) \sim \left(\frac{Z^1}{Z^3}\ ,\ \frac{Z^2}{Z^3}\ ,\ 1\right) = \left(\rho_1,\ \rho_2,\ 1\right) \tag{4.35}$$

where $\rho_1 = y_1 + iy_2$ and $\rho_2 = y_3 + iy_4$ are two complex numbers. These $y \equiv (y_1, y_2, y_3, y_4)$ define CP^2 co-ordinates.

At this stage we note that the point $(\rho_1, \rho_2, 1)$ is a point where the complex line through y intersects $Z^3 = 1$ plane and hence does not represent a point on S^5. To get a corresponding point on S^5, we scale it by a factor

$\left(\dfrac{1}{\sqrt{|\rho_1|^2 + |\rho_2|^2 + 1}}\right)$, so that the point

$$\left(\left(\frac{\rho_1}{\sqrt{|\rho_1|^2 + |\rho_2|^2 + 1}}\right), \left(\frac{\rho_2}{\sqrt{|\rho_1|^2 + |\rho_2|^2 + 1}}\right), \left(\frac{1}{\sqrt{|\rho_1|^2 + |\rho_2|^2 + 1}}\right)\right) \tag{4.36}$$

now belongs to S^5. We note that the last entity

$\left(\frac{1}{\sqrt{|\rho_1|^2 + |\rho_2|^2 + 1}} \right)$ in (4.36) is real and positive.

Alternatively one can get (4.36) in the following way. Since $|Z^1|^2+|Z^2|^2+|Z^3|^2 = 1$ defines S^5, all of Z^1, Z^2, Z^3 cannot be parametrized by $y \equiv (y_1, y_2, y_3, y_4)$. But by making use of the residual U(1) symmetry $Z^i \longrightarrow e^{i\theta} Z^i$ in S^5, mentioned earlier, we can choose Z^3 to be real and positive, (which corresponds to choosing a gauge in U(1) and thereby can now be parametrized by ρ_1, ρ_2 i.e. by y as given in (4.36)

$$Z_1 = \left(\frac{\rho_1}{\sqrt{|\rho_1|^2 + |\rho_2|^2 + 1}} \right) ; \; Z_2 = \left(\frac{\rho_2}{\sqrt{|\rho_1|^2 + |\rho_2|^2 + 1}} \right) ;$$

$$Z_3 = \left(\frac{1}{\sqrt{|\rho_1|^2 + |\rho_2|^2 + 1}} \right) \qquad (4.37)$$

We also note that this is a unit vector in $\mathbb{C}^3$ and multiplication by all non-zero complex numbers λ generates the whole complex line through this point in CP^2 and thereby defining a section or gauge in this tautological line bundle over CP^2. This line bundle is <u>tautological</u> because the line parametrized by a point $y \in CP^2$ is just the line in $\mathbb{C}^3$ through the origin represented by the point $y \in CP^2$ when CP^2 is considered as the manifold of complex lines through the origin in $\mathbb{C}^3$.

Now coming back to the calculation of instanton we note that the section we have chosen gives rise to a linear map U_y: $\mathbb{C}^1 \longrightarrow \mathbb{C}^3$, where U_y is given by

$$U_y = \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix} = \begin{pmatrix} \dfrac{y_1 + iy_2}{\sqrt{1 + y_1^2 + y_2^2 + y_3^2 + y_4^2}} \\ \dfrac{y_3 + iy_4}{\sqrt{1 + y_1^2 + y_2^2 + y_3^2 + y_4^2}} \\ \dfrac{1}{\sqrt{1 + y_1^2 + y_2^2 + y_3^2 + y_4^2}} \end{pmatrix}$$

which takes any $\lambda \in \mathbb{C}^1$ to $U_y \lambda \in \mathbb{C}^3$.

At this point one can easily verify that $U_y^{\dagger} U_y = 1$ and projection operator $P_y = U_y U_y^{\dagger}$ indeed satisfies $P_y^2 = P_y$.

Substituting the value of U_y from (4.38) in (4.34) one gets after a straightforward calculation

$$A = iU_y^{\dagger} \, dU_y$$

$$= \frac{y_2 \, dy_1 - y_1 \, dy_2 + y_4 \, dy_3 - y_3 \, dy_4}{(1 + y_1^2 + y_2^2 + y_3^2 + y_4^2)} \qquad (4.39)$$

agreeing with (4.7). This is the U(1) instanton on CP^2. Actually the first deRham Cohomology class $H^1(\mathbb{C}P^2,\mathbb{R})$ of CP^2 vanishes showing that all U(1) connections are actually gauge equivalent. But it will not be true in general and this method givesa general method of construction of instanton having self dual field strength.

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CHAPTER V

ON GENERALIZED SPIN STRUCTURE ON CP^2 MANIFOLD[1]

We have seen in Chapter 4 that CP^2 does not admit spin structure. A topological argument for this runs as: All CP^n manifolds are orientable as their first Stiefel-Whitney class vanishes. However the second Stiefel-Whitney class vanishes for CP^n manifolds with odd n only. To realize spin structure, the necessary and sufficient condition is the vanishing of the second Stiefel-Whitney class [1]. Therefore, CP^2 does not admit spin structure. Nevertheless it has been pointed out by Hawking and Pope [2] that on CP^2, one may be able to define a generalized spin structure by which it is meant that the spinors can be defined on the manifold only when they are minimally coupled to a non-trivial one-form, in the form of a gauge field(see chapter 4).

In this Chapter we use the important and familiar concept of 'parallel transport' to show how CP^2 does not admit spin-structure but admits generalized spin-structure. This is possible as there is a non-trivial U(1) connection in the form of an instanton for CP^2(see the appendix of Chapter 4). In [2], the index of the Dirac operator has been used to demonstrate the above result, along with a subtle mention of the parallel transport, which we elaborate here.

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The result that one can consistently define a generalized spin structure on CP^2 manifold is also important for Kaluza-Klein unification of the basic interactions. In this approach [3], general relativity in an enlarged space-time, say (4+n) dimensional space, with a product structure $M^4 \times B^n$ is considered. M^4 is the usual (3+1) dimensional Minkowski space and B^n is usually taken to be a compact extra space of very small size undetectable at present day available energies. The symmetries of B^n are taken to represent the internal symmetries of elementary particles. The original idea of Kaluza [4] and Klein [5] to unify gravity with electromagnetism is to consider a 5-dimensional world as $M^4 \times S^1$, thereby realising the U(1) symmetry of electromagnetic interactions(see chapter 1). It has been realized [6] that to include weak and strong interactions the B^n space should be identified with the coset space G/H with G = SU(3) x SU(2) x U(1) and H = SU(2) x U(1) x U(1) so that G/H becomes $CP^2 \times S^2 \times S^1$. Setting aside the question of obtaining right quantum numbers for fermions after compactification, one cannot even include fermions at the beginning as CP^2 does not allow spinors. However CP^2 admits instanton configuration in the form of a U(1) gauge field which then can be used to define generalized spin-structure. In the last chapter we [7] have shown that the same non-trivial topological U(1) instanton field in CP^2 can be used to trigger spontaneous compactification of $M^4 \times CP^2$ Kaluza-Klein theory with M^4 as Minkowskian flat space. The present study allows to introduce fermions in $M^4 \times CP^2$ by coupling them with the same instanton field. Here in the following we discuss the spin-structure in $CP^1 \sim S^2$ in section I and that of CP^2 i section II, along with the generalized spin structure.

I. Parallel transport on S^2

Before discussing the spin structure of CP^2, as an illustration we consider CP^1 case. It is well known that CP^1 is homeomorphic to S^2, the two sphere. Consider a vector referred to a orthonormal basis, after being parallel transported around a-closed test loop C (Fig.1) of constant latitude, the transported vector is rotated with respect to the initial vector at P(say) with rotation matrix being an element of SO(2).

The infinitesimal change in the vector under infinitesimal parallel transport is given by

$$\delta V^a = -\omega^a_{\mu b} V^b dx^\mu = -\omega^a{}_b V^b \tag{5.1}$$

where $\omega^a{}_b$ is the spin connection 1-form. Assuming torsion-free case, we have the spin connectionexplicitly as :

$$\omega^1{}_2 = - \text{Cos}\ \theta\ d\varphi \ ; \ \omega^2{}_1 = -\omega^1{}_2 \tag{5.2}$$

and so equation (5.1) upon integration around C yields

$$\begin{pmatrix} V^1 \\ V^2 \end{pmatrix}_{\text{final}} = g_1(\theta) \begin{pmatrix} V^1 \\ V^2 \end{pmatrix}_{\text{initial}}$$

with

$$g_1(\theta) = \begin{pmatrix} \mathrm{Cos}(2\pi \ \mathrm{Cos}\ \theta) & \mathrm{Sin}(2\pi \ \mathrm{Cos}\ \theta) \\ -\mathrm{Sin}(2\pi \ \mathrm{Cos}\ \theta) & \mathrm{Cos}(2\pi \ \mathrm{Cos}\ \theta) \end{pmatrix} \in SO(2) \qquad (5.3)$$

At north pole ($\theta = 0$) the vector just rotates by 2π, at equator the vector undergoes no rotation on being parallel transported along the geodesic (as expected) and at the south pole ($\theta = \pi$) the vector again undergoes a rotation by an angle 2π but in the opposite sense. Similarly when a spinor ψ (2-component) is parallel transported, we have

$$\psi_{\mathrm{FINAL}} = g_{1/2}(\theta)\ \psi_{\mathrm{INITIAL}} \qquad (5.4)$$

where the matrix $g_{1/2}(\theta)$ is given by

$$g_{1/2}(\theta) = \begin{pmatrix} \exp(i\pi\ \mathrm{Cos}\theta) & 0 \\ 0 & \exp(-i\pi\ \mathrm{Cos}\theta) \end{pmatrix} \qquad (5.5)$$

We have used :

$D_\mu \psi \equiv \partial_\mu \psi + \frac{1}{4} W_{\mu ab} \Gamma^{ab} \psi = 0$ for parallel transport with $\Gamma^{ab} = \frac{1}{2} [\ \Gamma^a,\ \Gamma^b\ ]$, $\{\ \Gamma^a,\ \Gamma^b\ \} = 2\delta^{ab}$ and in2-dimensions Γ^as are nothing but the Pauli matrices σ^a. At north pole $\theta = 0$, we have $g_{1/2}(0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and at the south pole $g_{1/2}(\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. A few remarks are in order from this simple observation. At north and south pole the spinor ψ changes sign after completing one revolution. This is a defining property of a spinor. Further $g_{1/2}(0) = g_{1/2}(\pi)$. We have considered circular loops parallel to the equatorial plane forconvenience. The matrices $g_1(\theta)$ and $g_{1/2}(\theta)$ are independent of the orientation of the loop but depends only on the angle θ specifying each loop.

As θ varies from 0 to π we get a set of closed loops on S^2whose planes are parallel to the equatorial plane. Now these loops can be brought into one-to-one correspondence with a 1-parameter family of loops all based on some point on S^2. This is because the rotation matrix is independent of the orientation of the loop so we can conveniently slide them on S^2 so that they originate and terminate at a single point N say. These one-parameter family of loops originating and terminating at north pole will span the whole surface S^2 (Fig.2). The above considerations show that a 2-vector on parallel transport produces elements belonging to SO(2) and since $g_1(0) = g_1(\pi) = 1$, we have closed loop in SO(2). The 2-component spinors also give closed loop in the corresponding spin group. We now present an alternative description by keeping the loops without sliding. Consider the two representative loops of cone angle θ_1 and θ_2 (Fig.3). The loops each lie in a plane intersecting the XY-plane in a line $Y = \lambda$ and passes through the N-pole. Thus the loops can also be parameterized by λ ; $-\infty < \lambda < \infty$. If we now integrate a one-form ω along the loops parameterized by θ_1 and θ_2 , then

$$\int_{\theta_2} \omega - \int_{\theta_1} \omega = \int_{\Omega} d\omega , \qquad (5.6)$$

where Ω is the surface enclosed by θ_1 and θ_2 loops on S^2. Under stereographic projection, the loops θ_1 and θ_2 correspond to the lines $Y = \lambda_1$ and $Y = \lambda_2$ and Ω corresponds to the rectangular strip bounded between $Y = \lambda_1$ and $Y = \lambda_2$. Taking the limit $\theta_1 \longrightarrow 0$ ($\lambda_1 \longrightarrow +\infty$) and $\theta_2 \longrightarrow \pi$ ($\lambda_2 \longrightarrow -\infty$), Ω becomes the entire S^2 projected on the XY plane. Thus the difference between the loop integrals between N and S pole is the surface integral over S^2. So

$$\int_{\lambda \longrightarrow -\infty} \omega \quad - \int_{\lambda \longrightarrow +\infty} \omega \;=\; \int_{S^2} d\omega \tag{5.7}$$

For the parallel transport of vector, we have from (5.1)

$$g_1(\lambda) = \exp\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \int_{\lambda} \cos\theta \; d\,\phi \right] \tag{5.8}$$

and so

$$\begin{aligned} g_1(-\infty)\, g_1^{-1}(\infty) &= \exp\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \int_{S^2} d\cos\theta \wedge d\,\phi \right] \\ &= \exp\left[-4\pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \\ &= \begin{pmatrix} \cos(-4\pi) & \sin(-4\pi) \\ -\sin(-4\pi) & \cos(-4\pi) \end{pmatrix} = 1 \end{aligned} \tag{5.9}$$

which agrees with (5.3). Equivalently using the polar co-ordinates (r,α) for the xy-plane, obtained by stereographic projection and noting that $\omega_{12} = \left(\frac{r^2-1}{r^2+1}\right) d\alpha$, we have

$$g_1(\lambda) = \exp\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \int_{\lambda} -\omega_{12} \right] \tag{5.8a}$$

and so

$$g_1(-\infty)\, g_1^{-1}(\infty) = \exp\left[- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \int_{S^2} d\omega_{12} \right] \tag{5.9a}$$

Since $\int_{S^2} d\omega_{12} = 2\pi \int_0^\infty d\left(\frac{r^2-1}{r^2+1}\right) = 4\pi$, we get back (5.9).

Similarly for the parallel transport of spinors, we have

$$g_{1/2}(-\infty)\quad g_{1/2}^{-1}(\infty) = \exp\left[- \frac{i}{2}\ \sigma^3 \int_{S^2} d\omega_{12} \right]$$

$$= \exp(-2\pi\ i\ \sigma^3)$$

$$= \begin{pmatrix} \exp(-2\pi i) & 0 \\ 0 & \exp(2\pi i) \end{pmatrix} = 1 \tag{5.10}$$

agreeing with (5.5). The above example of S^2 shows that to define spinors consistently on a manifold we must have $g_{1/2}(+\infty) = g_{1/2}(-\infty)$ so that we get a closed loop in the corresponding spin group also providing a double cover [2]. In the next section we examine the CP^2 case. Before proceeding to CP^2 we realize that in the case of S^2, the one parameter family of loops span the same S^2. When we calculate the parallel transport of a vector or spinor on CP^2, the loops will span a 2-surface homeomorphic to S^2, embedded in CP^2. Since $\pi_2(CP^2) = \mathbb{Z}$ and CP^1 is the generator of this group we take CP^1 itself as the 2-surface on which the vectors and spinors will be propagating.

2. Parallel Transport in CP^2

Since CP^2 is defined as the space of all triplets (z^1,z^2,z^3) not all zero with the identification $(z^1,z^2,z^3) \sim (\lambda z^1,\lambda z^2,\lambda z^3)$ with $\lambda \neq 0$. Now in the neighbourhood where $z^3 \neq 0$, we can take $\lambda = \frac{1}{z^3}$. Thus $(z^1, z^2, z^3) \sim (\frac{z^1}{z^3}, \frac{z^2}{z^3}, 1) = (\rho^1, \rho^2, 1)$ where ρ^1 and ρ^2 are any two complex numbers. Thus CP^2 is two-complex and hence four real dimensional manifold. Similarly two more, three in total, charts are necessary to cover the whole manifold corresponding to $z^1 \neq 0$ and $z^2 \neq 0$. Now a canonical embedding of CP^1 inside CP^2 in the chart where $z^3 \neq 0$ can be obtained by setting $z^2 = 0$ and similarly in other charts. Now we can parametrize (ρ^1, ρ^2) of the CP^2 co-ordinates as

$$\begin{aligned} \rho^1 &= y^1 + iy^2 = r \operatorname{Cos} \varphi \, e^{i\alpha} \\ \rho^2 &= y^3 + iy^4 = r \operatorname{Sin} \varphi \, e^{i\beta} \end{aligned} \tag{5.11}$$

with the ranges $0 \le r < \infty$; $0 \le \varphi \le \frac{\pi}{2}$; $0 \le \alpha, \beta \le 2\pi$ where $r = 1$ defines $S^3 \sim SU(2)$ topologically. We get the above mentioned embedding of CP^1 in CP^2 by putting $\varphi = 0$, where $z^3 \neq 0$.

Now let us consider the parallel transport of a vector V^a $(a = 0,1,2,3)$ and a spinor ψ in CP^2 around a closed loop lying in this CP^1. Loops of this type will span the whole of CP^1. Now as before the infinitesimal change of the vector is given by

$$\delta V^a = -\omega^a{}_b V^b \tag{5.12}$$

where $\omega^a{}_b$ are the connection one form referred totheorthonormal basis defined by the vielbein 1-forms E^α on CP^2. From the Fubini study metric on CP^2 [1] we have

$$ds^2 = \frac{dr^2 + r^2 e_z^2}{(1+r^2)^2} + \frac{r^2 (e_x^2 + e_y^2)}{(1+r^2)} \tag{5.13}$$

where e_x , e_y and e_z are the left invariant one-forms defined as

$$g^{-1} dg = i(e_x \sigma_1 + e_y \sigma_2 + e_z \sigma_3) \tag{5.14}$$

where g is an arbitrary element of SU(2) and parametrized as

$$g = \begin{pmatrix} \text{Cos } \varphi \, e^{i\alpha} & - \text{Sin } \varphi \, e^{-i\beta} \\ \text{Sin } \varphi \, e^{i\beta} & \text{Cos } \varphi \, e^{-i\alpha} \end{pmatrix} \tag{5.15}$$

the explicit expressions of e_x, e_y and e_z are given by

$$\begin{aligned} e_x &= -\text{Sin}\varphi \, \text{Cos}\varphi \, \text{Cos}(\alpha+\beta) \, d\alpha + \text{Sin}\varphi \, \text{Cos}\varphi \, \text{Cos}(\alpha+\beta) \, d\beta + \text{Sin}(\alpha+\beta) \, d\varphi \\ e_y &= -\text{Sin}\varphi \, \text{Cos}\varphi \, \text{Sin}(\alpha+\beta) \, d\alpha + \text{Sin}\varphi \, \text{Cos}\varphi \, \text{Sin}(\alpha+\beta) \, d\beta - \text{Cos}(\alpha+\beta) \, d\varphi \\ e_z &= \text{Cos}^2 \varphi \, d\alpha + \text{Sin}^2\varphi \, d\beta \end{aligned} \tag{5.16}$$

with $0 \leq \alpha, \ \beta \leq 2\pi$, $0\leq \varphi \leq \frac{\pi}{2}$, we have the vielbeins E^α $(\alpha=0,1,2,3)$ defined as$ds^2 = E^\alpha E^\beta \delta_{\alpha\beta}$ are given by

$$E^0 = \frac{dr}{(1+r^2)} \ ; E^1 = \frac{r \, e_x}{\sqrt{(1+r^2)}} \ ; E^2 = \frac{r \, e_y}{\sqrt{(1+r^2)}} \ ; \ E^3 = \frac{r \, e_z}{1+r^2} \tag{5.17}$$

From the cartan's equation $dE^a + \omega^a{}_b \wedge E^b = 0$, we obtain the connection one-forms on CP^2 as

$$\omega^0{}_1 = -\frac{e_x}{\sqrt{1+r^2}} \quad ; \quad \omega^0{}_2 = -\frac{e_y}{\sqrt{1+r^2}} \quad ; \quad \omega^0{}_3 = \left(\frac{r^2-1}{r^2+1}\right) e_z$$

$$\omega^2{}_3 = \frac{e_x}{\sqrt{1+r^2}} \quad ; \quad \omega^3{}_1 = \frac{e_y}{\sqrt{1+r^2}} \quad ; \quad \omega^1{}_2 = \left(\frac{1+2r^2}{1+r^2}\right) e_z \qquad (5.18)$$

As the loops lie on CP^1, we get by setting $\varphi = 0$, $e_z = d\alpha$, $e_x = 0$ and $e_y = 0$. Thus the only non vanishing spin connections are

$$\omega^0{}_3 = \left(\frac{r^2-1}{r^2+1}\right) d\alpha \quad \text{and} \quad \omega^1{}_2 = \left(\frac{1+2r^2}{1+r^2}\right) d\alpha \qquad (5.19)$$

The infinitesimal change of the vector is given by

$$\begin{pmatrix} dV^1 \\ dV^2 \\ dV^3 \\ dV^0 \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & -\omega_{12} & 0 & 0 \\ \omega_{12} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\omega_{30} \\ 0 & 0 & \omega_{30} & 0 \end{array}\right) \begin{pmatrix} V^1 \\ V^2 \\ V^3 \\ V^0 \end{pmatrix} \qquad (5.20)$$

parallel transporting along λ, we get the matrix element for this, as we have done before,

$$g_1(\lambda) = \exp\left[\left(-\int_\lambda \omega_{12} \qquad -\int_\lambda \omega_{30}\right) \left(\begin{array}{cc|cc} 0 & +1 & \multicolumn{2}{c}{0} \\ -1 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & +1 \\ & & -1 & 0 \end{array}\right)\right] \qquad (5.21)$$

Thus it follows that

$$g_1(-\infty)\; g_1^{-1}\; (+\infty) = \exp\left[-\int_{CP^1} d\omega_{12} \qquad -\int_{CP^1} d\omega_{30}\right] \left(\begin{array}{cc|cc} 0 & +1 & & \\ -1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & +1 \\ \multicolumn{2}{c|}{0} & -1 & 0 \end{array} \right) \tag{5.22}$$

But it is trivial to see that

$$\int_{CP^1} d\omega_{12} = \frac{1}{2} \int d\omega_{03} = \frac{1}{2} \times 4\pi = 2\pi \tag{5.23}$$

i.e.,

$$g_1(-\infty)\; g_1^{-1}(+\infty) = \exp\left[(-2\pi\; 4\pi) \left(\begin{array}{cc|cc} 0 & +1 & & \\ -1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & +1 \\ \multicolumn{2}{c|}{0} & -1 & 0 \end{array} \right) \right] = 1 \tag{5.24}$$

thus getting a closed loop in the SO(4) space as expected. The loop can be written more precisely as

$$\left(\begin{array}{cc|cc} \cos(-\theta) & \sin(-\theta) & & \\ -\sin(-\theta) & \cos(-\theta) & \multicolumn{2}{c}{0} \\ \hline & & \cos(2\theta) & \sin(2\theta) \\ \multicolumn{2}{c|}{0} & -\sin(2\theta) & \cos(2\theta) \end{array} \right) \qquad 0 \leq \theta < 2\pi \tag{5.25}$$

Though closed, this loop is non-trivial i.e., can't be continuously shrunk to a point. (We prove this in the appendix). And hence the lift of the loop in SO(4) to the corresponding double cover Spin(4) has to be open, as follows from the theory of the covering spaces. We verify this below by parallel transporting spinors instead of vectors over CP^1. But for the spin structure to exist we must have closed loop even in Spin(4). Thus showing that spin structure do not exist in CP^2. At this stage one can make the following general statement that for any differentiable manifold M^n ($n \geq 3$), if we parallel transport a vector along a family of closed loops spanning any 2-surface, belonging to $\pi_2(M)$ and get a non-trivial loop in SO(n) then M^n does not allow spin structure as $\pi_1(SO(n)) = \mathbb{Z}_2$ for all $n \geq 3$ and lift of the non-trivial loops in SO(n) to Spin(n) has to be open. Now coming to spinors in CP^2, we note that the Dirac matrices in this SO(4) case are

$$\Gamma^o = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \Gamma^i = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix} \tag{5.26}$$

where they satisfy the anticommutation relation

$$\left\{ \Gamma^a, \Gamma^b \right\} = 2 \, \delta^{ab} \, 1$$

Define

$$\Gamma^{ab} \equiv \frac{1}{2} \left[\Gamma^a, \Gamma^b \right] \tag{5.27}$$

The infinitesimal change in the spinor ψ when parallel transported is given by

$$d\psi = - \frac{1}{4} \, \omega_{ab} \, \Gamma^{ab} \, \psi \tag{5.28}$$

Now

$$\frac{1}{4} \, \omega_{ab} \, \Gamma^{ab} = \frac{1}{2} \, (\, \omega_{03} \, \Gamma^{03} + \omega_{12} \, \Gamma^{12} \,)$$

From (5.27)

$$\Gamma^{03} = \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix} \quad \text{and} \quad \Gamma^{12} = i\sigma^3 \, 1. \tag{5.29}$$

We note that $[\Gamma^{03}, \Gamma^{12}] = 0$.

Integrating along a loop 'λ' as before

$$g_{1/2}(\lambda) = \exp\left[-\frac{1}{2} \left\{ \int_{\lambda} \omega_{03}\, \Gamma^{03} + \omega_{12}\, \Gamma^{12} \right\} \right]. \qquad (5.30)$$

Thus

$$g_{1/2}(-\infty)\, g^{-1}_{1/2}(+\infty) = \exp\left[-\frac{1}{2} \int_{CP^1} \left(d\omega_{03}\, \Gamma^{03} + d\omega_{12}\, \Gamma^{12} \right) \right]$$

$$= \exp\left[-\pi \left(\Gamma^{12} + 2\, \Gamma^{03} \right) \right] = -1. \qquad (5.31)$$

The fact that $g_{1/2}(-\infty) \neq g_{1/2}(+\infty)$ shows that we do not have a closed loop in the spin group, implying that there is no spin structure on CP^2.

Now we introduce the instanton U(1) 1-form 'A' admitted by CP^2 and couple it to the spinor ψ of charge e'. Then under infinitesimal parallel transformation along any loop will result in the infinitesimal change in the spinor wave function ψ given by

$$d\psi = \left\{ -\frac{1}{4} W_{ab}\, \Gamma^{ab} + ie' A \right\} \psi \qquad (5.32)$$

where the 1-form A is given in Chapter 4 (see equations (4.7),(4.39))

$$A = \frac{y_2\, dy_1 - y_1\, dy_2 + y_4\, dy_3 - y_3\, dy_4}{(1 + y_1^2 + y_2^2 + y_3^2 + y_4^2)}. \qquad (5.33)$$

Integrating over a loop λ as before we get the effective matrix element

$$g^{eff}_{1/2}(\lambda) = \exp\left[-\frac{1}{4}\int_{\lambda} \omega_{ab}\,\Gamma^{ab} + ie' \int_{\lambda} A \right] \qquad (5.34)$$

implying that

$$g^{eff}_{1/2}(-\infty)\left(g^{eff}_{1/2}(+\infty)\right)^{-1} = \left(g_{1/2}(-\infty)\; g^{-1}_{1/2}(+\infty)\right)\left(\exp(ie' \int_{CP^1} dA\right)$$

$$= -\exp\left(ie' \int_{CP^1} F\right) \qquad (5.35)$$

using equation (5.31). Here F=dA is the field two-form. So with $e' = (m+\frac{1}{2})$ and $\int_{CP^1} F = 2\pi$ (see chapter 4 equation 4.26), we have

$$g^{eff}_{1/2}(-\infty) = g^{eff}_{1/2}(+\infty) \qquad (5.36)$$

Showing that we can have a generalized spin structure on CP^2 where fermions of definite charges are coupled to the instanton field configuration.

Appendix

Here we shall prove that the SO(4) element obtained by parallel transporting a four-vector in CP^2, around CP^1 given in (5.25) is a non-trivial loop i.e. this loop cannot be continuously shrunk to a point.

As we know that $\pi_1(SO(4)) = \mathbb{Z}_2$ the group containing two elements (0,1) with the binary operation being the addition modulo

2. Thus all the loops in SO(4) space can be brought into two equivalent classes, where in one class all the loop can be shrunk to a point and corresponds to the element 0 of the fundamental group and in the other class none of the loops are contractible and corresponds to the element 1 of the group.

In order to prove that the SO(4) loop given in (5.25) is really non contractible, let us consider a more general loop given by the one parameter family of matrices A(m,n) (characterized by arbitrary integers m and n) is

$$A(m,n) = \left(\begin{array}{cc|cc} \cos(m\theta) & -\sin(m\theta) & \multicolumn{2}{c}{0} \\ \sin(m\theta) & \cos(m\theta) & & \\ \hline & & \cos(n\theta) & -\sin(n\theta) \\ \multicolumn{2}{c|}{0} & \sin(n\theta) & \cos(n\theta) \end{array}\right) \qquad 0 \le \theta < 2\pi \tag{5.37}$$

and try to find out in which class this matrix element belongs to.

Before proceeding further let us recall that for a principal bundle P over a base B with structure group H, one can write the following <u>exact</u> sequence.

$$.. \longrightarrow \Pi_n(H) \longrightarrow \Pi_n(P) \longrightarrow \Pi_n(B) \longrightarrow \Pi_{n-1}(H) \longrightarrow \Pi_{n-1}(P) \longrightarrow \Pi_{n-1}(B) \longrightarrow .. \longrightarrow \Pi_o(H) \tag{5.38}$$

with $\Pi_o(H)$ being the number of connected components in H, where the kernel of any one map in the sequence is the image of the preceding map. More precisely considering the map $\Pi_n(B) \longrightarrow \Pi_{n-1}(H)$, we take the element zero $\in \Pi_{n-1}(H)$ and its pre-image $\mathcal{J} \subset \Pi_n(B)$. Now the exact sequence means that the image of $\Pi_n(P)$ under this map is precisely the pre-image $\mathcal{J}$ - called the kernel.

In particular this holds for a homogeneous coset space $M = G/H$ which is a H Principal-bundle over M. The exact sequence for this case then becomes

$$\cdots \longrightarrow \Pi_n(H) \longrightarrow \Pi_n(G) \longrightarrow \Pi_n(M) \longrightarrow \Pi_{n-1}(H) \longrightarrow \cdots \longrightarrow \Pi_o(H) \tag{5.39}$$

Now in our case we can see that the matrix A(m,n) consists of two SO(2)'s in the form of SO(2) x SO(2) embedded in SO(4). So we consider the coset space M = SO(4)/ SO(2)xSO(2). The justification that this is well defined manifold follows from the fact that SO(2) x SO(2) is a closed subgroup of SO(4). This manifold is a well known oriented Grassmanian manifold which is the space of oriented 2 - planes in $\mathbb{R}^4$ through origin,which is known to be simply connected.Now writing the exact sequence for M = SO(4)/SO(2)xSO(2)

$$\cdots \longrightarrow \Pi_2(M) \longrightarrow \Pi_1\left(SO(2) \times SO(2)\right) \longrightarrow \Pi_1(SO(4)) \longrightarrow \Pi_1(M) \longrightarrow \Pi_o(M) \tag{5.40}$$

As we have seen earlier $\Pi_1(M) = 0$ and hence the whole of $\Pi_1(SO(4))$

$= \mathbb{Z}_2$ is the Kernel and the mapping

$$\Pi_1\left(SO(2) \times SO(2)\right) = \Pi_1\left(SO(2)\right) \times \Pi_1\left(SO(2)\right) = \mathbb{Z} \times \mathbb{Z}$$
$$\longrightarrow \Pi_1\left(SO(4)\right) = \mathbb{Z}_2$$

must be onto.

The group of homomorphism Hom $(\mathbb{Z} \times \mathbb{Z}, \mathbb{Z}_2)$ has four elements given by (i) $(m,n) \longrightarrow 0$ (ii) $(m,n) \longrightarrow [m]$ (iii) $(m,n) \longrightarrow [n]$ (iv) $(m,n) \longrightarrow [m+n]$ where the square bracket means "modulo 2". We reject the first one as the map in our case should be "onto" as we have seen, the second and third are rejected on the basis of symmetry. Thus we are left with the fourth one, which shows that the loop defined by $A(m,n)$ will be contractible if $(m+n)$ is even and non-contractible if $(m+n)$ is odd.

Our loop given in (5.25) was of $(-1,2)$ type and hence non-contractible.

Here we would like to give a justification for the remark we have made earlier in this chapter that if all homotopy class of maps $f: S^2 \longrightarrow M$ give rise to a non-trivial loop in $SO(n)$ given by parallel transporting along the one parameter family of loops $\sigma_v(t)$ spanning S^2 (here t is the parameter along the loop and v is the parameter parameterizing the family of loops, earlier parameterized by θ or λ (see fig 3.) with ranges $0 \le t, v \le 1$) discussed earlier, then the manifold M does not admit a spin structure.

To prove this we recall a general result from Obstruction

Theory [8] which says that the second Stiefel-Whitney Class completely determines the primary obstruction class - which is the obstruction to constructing (n-1) linearly independent sections of the tangent bundle over the 2-Skeleton of M. But as M is orientable, if the tangent bundle admits (n-1) linearly independent sections, it is trivial. This can also be recast as :

M admits a spin structure iff the pulled back tangent bundle $f^* T_M$ is trivial over S^2 for all smooth maps $f:S^2 \longrightarrow M$. ...(*)

Here we would like to mention that parallel transport along loops $\sigma_v(t)$ can be thought of as horizontal lifts $\tilde{\sigma}_v(t)$ in the principal SO(n) bundle $f^* P$ where P is the principal SO(n) bundle associated to the tangent bundle T_M of M (M is taken to be oriented, so the structure group of T_M can be reduced to SO(n)). Also the loop in SO(n) obtained by parallel transporting, as above, is just the loop formed by the end points $\tilde{\sigma}_v$ (1) $0 \leq v \leq 1$ of the horizontal lifts in the fibre above the point N- which can be identified with SO(n). (N is the fixed point where all the loops are originating and terminating at.)

Now suppose that M admits a spin structure then by (*) we can assume that $f^* P$ is trivial over S^2. That is ,f^*P is homeomorphic to the product $S^2 \times SO(n)$. Notice that $\tilde{\sigma}_v(t)$ $0 \leq v \leq 1$ as t tends to zero gives a homotopy of $\tilde{\sigma}_v(1)$ $0 \leq v \leq 1$ to the constant loop based at identity in SO(n). This is a homotopy in the total space of the principal bundle f^* P . But this is not serious. Writing the long exact homotopy sequence for the principal bundle

$f^* P$ over S^2,

$$\Pi_2(S^2) \longrightarrow \Pi_1(SO(n)) \longrightarrow \Pi_1(f^*P) \longrightarrow \Pi_1(S^2) \longrightarrow \ldots$$

and noting that $\Pi_1(S^2) = 0$ and $\Pi_1(f^*P) \simeq \Pi_1(S^2 \times SO(n)) \simeq \Pi_1(SO(n))$, we see,

$$\Pi_1(SO(n)) \longrightarrow \Pi_1(f^* P) \text{ is an isomorphism.}$$

This means that any loop in SO(n), embedded as a fibre in $f^* P$, which can be shrunk in $f^* P$ can be shrunk in SO(n).

The point here is that horizontal lifts of $\sigma_v(t)$ $0 < t < 1$ and $0 < v < 1$ actually give rise to a section of the bundle $f^* P$ over $S^2 - N$. To extend this section to N also one has to know whether the loop $\tilde{\sigma}_v(1)$ in the fibre above N can be shrunk in the fibre above N or not. So, by (*) this knowledge determines whether M admits spin structure or not. The obstruction to existence of spin structure on the manifold M is given by the homotopy class of this loop in the fibre above N which is identified with SO(n) (notice that this loop depends on the homotopy class of maps $f: S^2 \longrightarrow M$).

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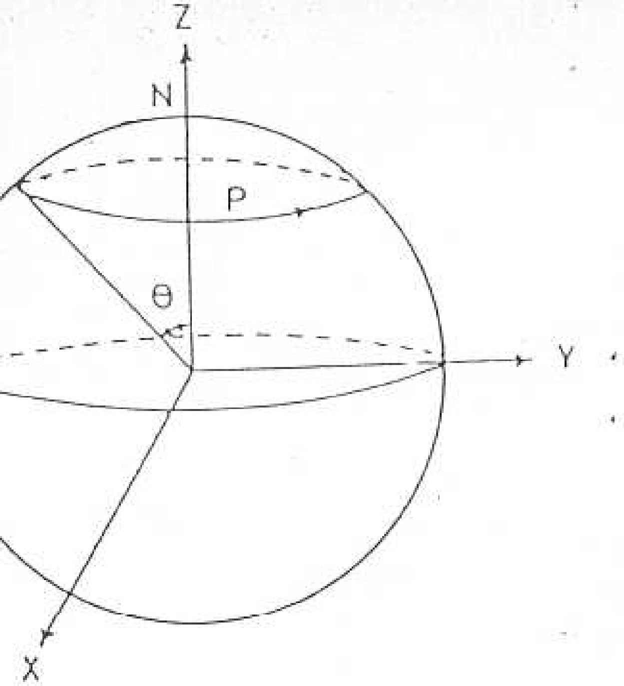
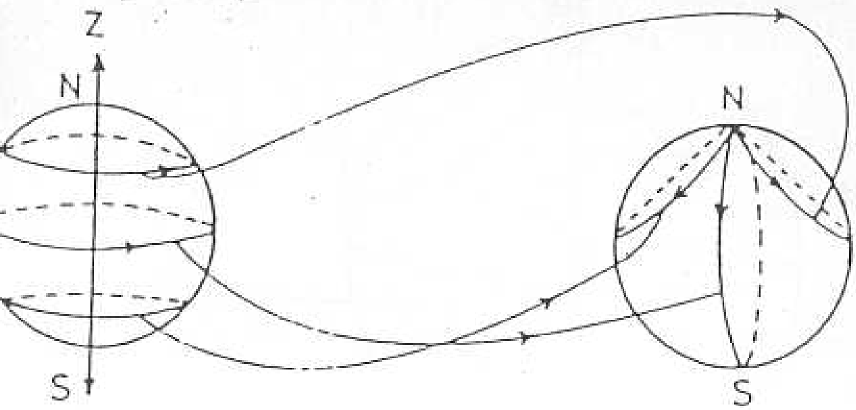
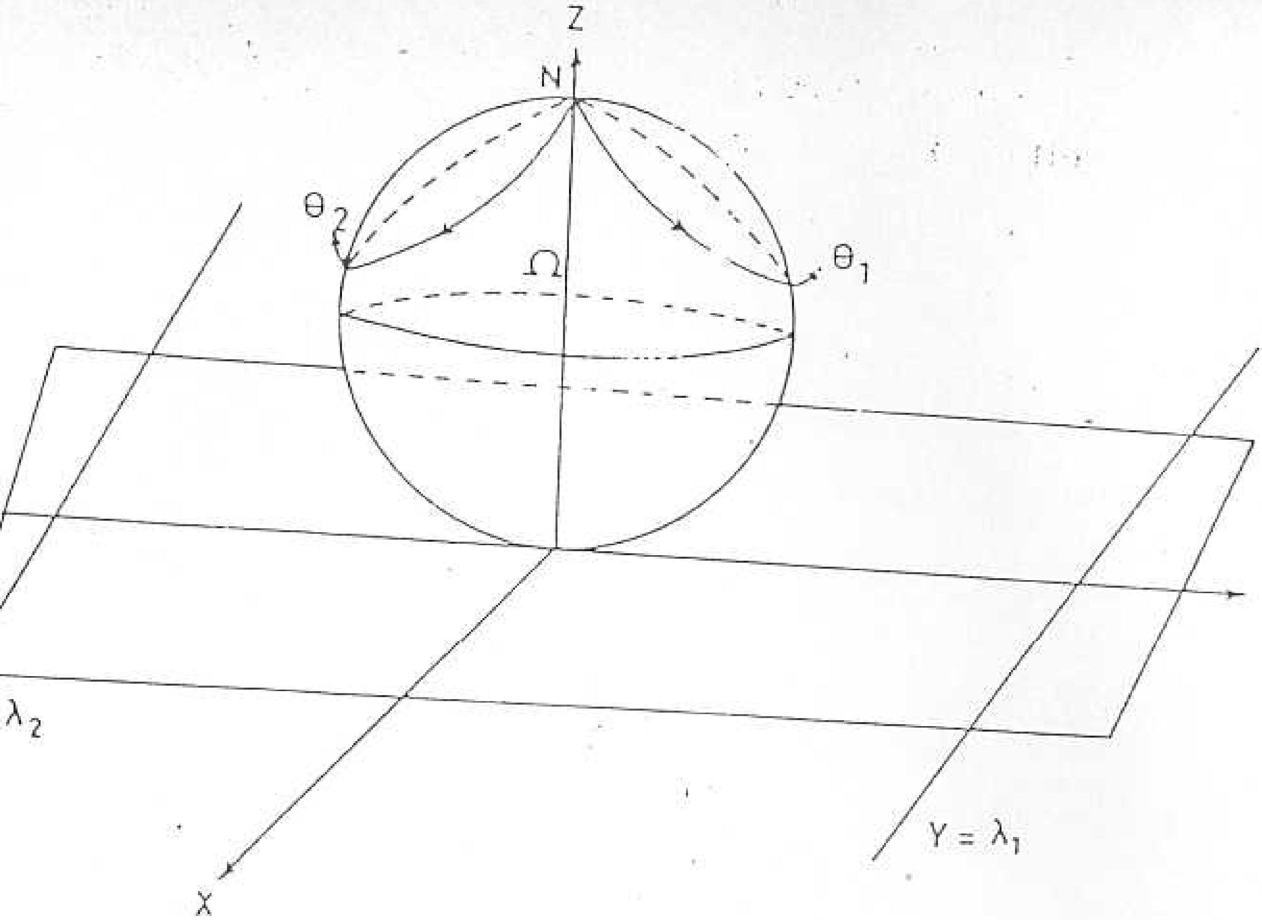


ure 1. A 2-vector and a 2-spinor are parallel transported around a closed loop of constant
ude, parametrized by θ ($0 \leq \theta \leq \pi$).



gure 2. The one-parameter family of loops, each starting and terminating at a common
oint N, spanning the whole of S^2 can be brought, by sliding, to one-to-one correspondence
th the family of loops of constant latitude.



re 3. Under stereographic projection into the XY plane the loops starting and terminat-
at N give a line and thus can be alternatively parametrized by λ $(-\infty < \lambda < +\infty)$.