Around non-vanishing, linear independence and transcendence of L values at rational and integer points

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

Journal

1. "On Dedekind zeta values at 1/2", Neelam Kandhil, *Int. J. Number Theory, 18 (2022), 1289-1299.*

2. "Around a question of Baker", Neelam Kandhil and Purusottam Rath, to appear in *Proc. of Subbarao Symposium to be published by the Fields institute.*

3. "On an extension of a question of Baker", Sanoli Gun and Neelam Kandhil, appeared online in *Int. J. Number Theory, 2022.*

4. "On linear independence of Dirichlet L values", Sanoli Gun, Neelam Kandhil and Patrice Philippon, *J. Number Theory, 244 (2023), 63–83.*

Neelam Kandhil

Neelam

To My Parents And Teachers

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belief and support that always gives me a boost to go through all the struggles.

CONTENTS

SUMMARY

The focal point of this thesis is to study the arithmetic nature of Dirichlet L values at positive integers. While non-vanishing of these values was established in 1837, it took about 130 years to settle the nature of these numbers. Thanks to Dirichlet and Baker, Dirichlet L values corresponding to non-trivial Dirichlet characters at 1 are transcendental. It is an open question of Baker whether the Dirichlet L values at 1 corresponding to non-trivial Dirichlet characters with fixed modulus are linearly independent over the rational numbers. The best-known result is due to Baker, Birch and Wirsing, which affirms this when the modulus of the associated Dirichlet character is co-prime to its Euler's phi value. In this thesis, we discuss an extension of this result to any arbitrary family of moduli. The interplay between the resulting ambient number fields brings new technical issues and complications hitherto absent in the context of a fixed modulus. For k greater than 1, the study of linear independence of Dirichlet L values at k, corresponding to non-trivial Dirichlet characters depends critically on the parity of k and the Dirichlet characters. This has been investigated by a number of authors for Dirichlet characters of a fixed modulus and having the same parity as k. We extend this investigation to families of Dirichlet characters modulo distinct pairwise co-prime natural numbers. The product of these Dirichlet L values gives the Dedekind zeta values associated with abelian number fields.

The main results that make up this thesis are as follows.

- There are at most finitely many abelian number fields for which the derivative of their associated Dedekind zeta value at 1/2 is zero, whenever their associated Dedekind zeta value at 1/2 is non-zero. All such number fields (if exist) have degree less than 46369. This result refines a result of Murty and Tanabe, both qualitatively and quantitatively. We also extend our investigation to Galois as well as arbitrary number fields, borrowing tools from algebraic as well as transcendental number theory.

- Let t and q be co-prime integers such that tq is co-prime to $(t-1)(q-1)$. Then the set of all Dirichlet L values at 1 corresponding to non-trivial Dirichlet characters modulo t and q, is linearly independent over the rational numbers. In the process, we also extend a result of Okada about linear independence of the cotangent values over the rational numbers. We also prove that the set of all Dirichlet L values at 1 corresponding to non-trivial even Dirichlet characters is linearly independent over the algebraic numbers, conditional on co-primality of their moduli. This extends a result of Murty-Murty.

- Let k be a positive integer greater than 1, and t and q be co-prime integers such that tq is co-prime to $(t-1)(q-1)$. Then the set of all Dirichlet L values at k, corresponding to non-trivial Dirichlet characters modulo t and q and having the same parity as k, is linearly independent over the rational numbers. We connect the spaces generated by such L values with the spaces generated by Hurwitz zeta values. Here, we also compute the lower bounds of dimensions of finite sum of generalized Chowla-Milnor spaces over linearly disjoint number fields.

NOTATIONS

Symbol	**Description**
$\emptyset$	The empty set.
$\mathbb{N}$	The set of all natural numbers.
$\mathbb{Z}$	The ring of rational integers.
$\mathbb{Q}$	The field of rational numbers.
$\mathbb{C}$	The field of complex numbers.
$\Re(s)$	The real part of the complex number s.
$\mathbf{K}$	A number field.
$\mathscr{O}_{\mathbf{K}}$	The ring of integers associated to $\mathbf{K}$.
$Gal(\mathbf{K}/\mathbb{Q})$	Galois group of $\mathbf{K}$ over $\mathbb{Q}$.
$d_{\mathbf{K}}$	Discriminant of $\mathbf{K}$ over $\mathbb{Q}$.
r_1	Number of real embeddings of $\mathbf{K}$.
r_2	Number of non-conjugate complex embeddings of $\mathbf{K}$.
$n = r_1 + 2r_2$	Degree of $\mathbf{K}$ over $\mathbb{Q}$.
Γ	Gamma function.
γ	Euler's constant $\approx 0.577\cdots$
ζ_n	An n-th primitive root of unity in $\mathbb{C}$.
$\mathbb{Q}(\zeta_n)$	The n-th cyclotomic field.

$\varphi(n)$	The Euler-totient function.
$\mu(n)$	The Möbius function.
$\mathfrak{N}(\mathfrak{a})$	The absolute norm of an ideal $\mathfrak{a}$.
$Li(x)$	The logarithmic integral from 2 to x.
$\Lambda(n)$	The von Mangoldt function.
p	prime number.
ζ_n	primitive nth root of unity.
$\hat{G}$	character group of G.

CHAPTER 1

INTRODUCTION

This chapter is devoted to describe various lower bounds for discriminants of number fields relevant to our work. The primary goal is to give a complete proof of an elegant result of Ram Murty which shall be applied in due course.

We first begin with a brief account of the Conductor-Discriminant formula for abelian number fields.

1.1 Discriminant

Definition 1.1.1. *(Discriminant of a basis) Let* $\mathbf{K}$ *be any finite field extension of* $\mathbb{Q}$ *and let* $W = (w_1, \dots w_n)$ *be a basis of* $\mathbf{K}$ *over* $\mathbb{Q}$*. We define the discriminant of the basis* W *in the following manner:*

$$D_{\mathbf{K}}(W) = det((\sigma_i w_j)_{i,j})^2$$

where σ_i ranges over the distinct embeddings of $\mathbf{K}$ *into* $\mathbb{C}$ *(i.e. injective ring homomorphisms from* $\mathbf{K} \to \mathbb{C}$*).*

Definition 1.1.2. *(Discriminant of an algebraic number field) Let* $\mathbf{K}$ *be an algebraic number field (i.e. a finite extension of* $\mathbb{Q}$ *). Also, let* $W = (w_1, \ldots w_n)$ *be an integral basis of* $\mathbf{K}$*, i.e.,* W *is a* $\mathbb{Z}$*-basis for the ring of integers* $\mathcal{O}_{\mathbf{K}}$*. Then the discriminant of* $\mathbf{K}$ *is the following integer*

$$d_{\mathbf{K}} = det((\sigma_i w_j)_{i,j})^2$$

where σ_i ranges over the distinct embeddings of $\mathbf{K}$ *into* $\mathbb{C}$*.*

We recall that $\mathcal{O}_{\mathbf{K}}$ is a free $\mathbb{Z}$-module of rank n (degree of $\mathbf{K}$ over $\mathbb{Q}$) and thus such an integral basis of $\mathcal{O}_{\mathbf{K}}$ exists. Let us show that the discriminant of a number field is well defined, that is, if $V = (v_1, \ldots v_n)$ is another integral basis of $\mathcal{O}_{\mathbf{K}}$, then

$$D_{\mathbf{K}}(W) = D_{\mathbf{K}}(V).$$

We have $v_k = \sum_{j=1}^{n} a_{k,j} w_j$ for some $a_{k,j} \in \mathbb{Z}$ and $1 \le k, j \le n$. So, we obtain

$$\begin{aligned} D_{\mathbf{K}}(V) &= det((\sigma_i v_k)_{i,k})^2 \\ &= det((\sum_{j=1}^{n} a_{k,j} \sigma_i w_j)_{i,k})^2 \\ &= det((a_{k,j})_{k,j})^2 \; det((\sigma_k w_j)_{k,j})^2 \\ &= det(T)^2 \; D_{\mathbf{K}}(W), \end{aligned}$$

where T is the transition matrix $(a_{k,j})_{k,j}$. By reversing the roles of bases, we see that inverse of T has entries also in $\mathbb{Z}$. It implies that $det(T)$ is a unit in $\mathbb{Z}$, so $det(T) = \pm 1$. Hence, $D_{\mathbf{K}}(W) = D_{\mathbf{K}}(V)$ as desired. In other words, $d_{\mathbf{K}}$ is independent of the choice of the integral basis W of $\mathbf{K}$ over $\mathbb{Q}$.

Example 1.1.3. *Let* $\mathbf{K}$ *be a quadratic number field. Then there exists a non-zero square free integer* d *such that* $\mathbf{K} = \mathbb{Q}(\sqrt{d})$*. One can show that*

$$d_{\mathbf{K}} = \begin{cases} d & \text{if } d \equiv 1 \bmod 4; \\ 4d & \text{if } d \equiv 2,3 \bmod 4. \end{cases}$$

Another useful example is the following.

Example 1.1.4. *For a cyclotomic field* $\mathbf{K} = \mathbb{Q}(\zeta_n)$*, its discriminant is given by*

$$d_{\mathbf{K}} = (-1)^{\varphi(n)/2} \frac{n^{\varphi(n)}}{\prod_{p|n} p^{\varphi(n)/(p-1)}},$$

where φ *is Euler's totient function.*

Example 1.1.5. *Let us now consider a more involved number field. For an odd prime number* p*, let* $\mathbf{K} = \mathbb{Q}(p^{1/p})$*. Its ring of integers* $\mathcal{O}_{\mathbf{K}}$ *is the ring* $\mathbb{Z}[p^{1/p}]$*. A proof of this can be seen in Proposition 7.3 [47, pg. 110].*

To find the discriminant of $\mathbf{K}$*, we recall a recipe to find discriminant of basis which is of special type. Let* α *be a root of the irreducible polynomial* $f(x) \in \mathbb{Q}[x]$ *and let* $\mathbf{L} = \mathbb{Q}(\alpha)$*. Then the discriminant of the basis* $\{1, \alpha, \cdots, \alpha^{n-1}\}$ *is given by*

$$D_{\mathbf{L}}(1, \alpha, ..., \alpha^{n-1}) = (-1)^{\frac{n(n-1)}{2}} N_{\mathbf{L}/\mathbb{Q}}(f'(\alpha))$$

where $f'(x)$ *is the derivative of* $f(x)$ *(see [63, pg 41]).*

We give a quick proof of it. Denote by $\alpha_1, \alpha_2, \cdots, \alpha_n$ *the roots of* $f(x)$*. Then*

$$\begin{aligned}
D_{\mathbf{L}}(1,\alpha,\alpha^2,...,\alpha^{n-1}) &= det((\sigma_i\alpha^j)_{i,j})^2 \\
&= det(\alpha_i^j)^2 \\
&= (\prod_{i<j}(\alpha_i-\alpha_j))^2 \\
&= (-1)^{\frac{n(n-1)}{2}}\prod_{i\neq j}(\alpha_i-\alpha_j) \\
&= (-1)^{\frac{n(n-1)}{2}}\prod_{i}\prod_{j\neq i}(\alpha_i-\alpha_j) \\
&= (-1)^{\frac{n(n-1)}{2}}\prod_{i} f'(\alpha_i) \\
&= (-1)^{\frac{n(n-1)}{2}} N_{\mathbf{L}/\mathbb{Q}}(f'(\alpha)).
\end{aligned}$$

We want to find out the discriminant of the basis $1, p^{1/p}, \cdots, p^{(p-1)/p}$*. Since* $\mathbb{Z}[p^{1/p}]$ *is the ring of integers of* $\mathbf{K} = \mathbb{Q}(p^{1/p})$ *and* $p^{1/p}$ *is the root of the irreducible polynomial* $x^p - p$*, we have*

$$d_{\mathbf{K}} = (-1)^{\frac{p(p-1)}{2}} N_{\mathbf{K}/\mathbb{Q}}(p^{(2p-1)/p}) = (-1)^{\frac{p(p-1)}{2}} p^{2p-1}.$$

Definition 1.1.6. *Let* $\mathbf{K}$ *and* $\mathbf{F}$ *be algebraic extensions of a field* $\mathbf{L}$*. The fields* $\mathbf{K},\mathbf{F}$ *are said to be linearly disjoint over* $\mathbf{L}$ *if every finite subset of* $\mathbf{K}$ *that is* $\mathbf{L}$ *linearly independent is also* $\mathbf{F}$ *linearly independent. See §1.4 for more details.*

Very often, a number field is expressible as compositum of two number fields each of which is easier to understand. The following result allows us to compute discriminant of compositums of special type.

Proposition 1.1.7. *[42, pg 68] Let* $\mathbf{K},\mathbf{L}$ *be two number fields. Assume that their discriminants are relatively prime and that the fields are linearly disjoint and therefore* *deg*($\mathbf{KL}/\mathbb{Q}$) = *deg*($\mathbf{K}/\mathbb{Q}$) *deg*($\mathbf{L}/\mathbb{Q}$). *Then*

$$\mathcal{O}_{\mathbf{KL}} = \mathcal{O}_{\mathbf{K}}\mathcal{O}_{\mathbf{L}}$$

and

$$d_{\mathbf{KL}} = (d_{\mathbf{K}})^{deg(\mathbf{L}/\mathbb{Q})}\,(d_{\mathbf{L}})^{deg(\mathbf{K}/\mathbb{Q})}.$$

Example 1.1.8. *Let* $\mathbf{K} = \mathbb{Q}(2^{1/3}, \zeta_3)$. *We have* $\mathcal{O}_{\mathbb{Q}(\zeta_3)} = \mathbb{Z}[\zeta_3]$ *and* $\mathcal{O}_{\mathbb{Q}(2^{1/3})} = \mathbb{Z}[2^{1/3}]$ *(see Proposition 7.3 in [47, pg. 110] for more details). So* $\mathcal{O}_{\mathbf{K}}$ *the ring of integers of* $\mathbf{K}$ *is* $\mathbb{Z}[2^{1/3}, \zeta_3]$ *by Proposition 1.1.7 stated above. By 1.1.4 and [63, pg 41],* $d_{\mathbf{K}} = 108^2\,(-3)^3 = -314928$. *Note that* $Gal(\mathbf{K}/\mathbb{Q})$ *is* S_3*, permutation group of order 6 and* $\mathbb{Q}(2^{1/3}, \zeta_3)$ *is an example of a non abelian Galois extension.*

We now state the following classical lower bound for discriminant of a number field $\mathbf{K}$ in terms of its degree.

Theorem 1.1.9. *(Minkowski's Bound) Let* $\mathbf{K}$ *be a number field of degree n. Then*

$$|d_{\mathbf{K}}|^{1/2} \quad \geq \quad \frac{n^n}{n!}\left(\frac{\pi}{4}\right)^{n/2}.$$

An immediate but important corollary of this theorem is the following.

Theorem 1.1.10. *[63, §4.3, Thm 1] For any number field* $\mathbf{K} \neq \mathbb{Q}$ *,* $|d_{\mathbf{K}}| > 1$.

In general, Minkowski's Bound is not optimal. For instance, if $\mathbf{K}$ is a quadratic field, $|d_{\mathbf{K}}| \geq 2.46$ by Minkowski's Bound. But as we vary over all quadratic fields $\mathbf{K}$, the absolute discriminant $|d_{\mathbf{K}}|$ evidently goes to infinity.

Now let us take a cubic field $\mathbf{K} = \mathbb{Q}(2^{1/3})$ whose ring of integers is $\mathbb{Z}[2^{1/3}]$. From [63, pg 41], $|d_{\mathbf{K}}| = 108$ but Minkowski's bound provides $|d_{\mathbf{K}}| > 17.44\cdots$. Thus $|d_{\mathbf{K}}|$ of $\mathbb{Q}(2^{1/3})$ is much larger than lower bound of $|d_{\mathbf{K}}|$ which one derives by applying Theorem 1.1.9.

Of course, these are not the most illuminating examples as the degree is bounded. Now let us consider a family where the degrees are unbounded. For a prime p, let $\mathbf{K} = \mathbb{Q}(\zeta_p)$. Its discriminant $|d_{\mathbf{K}}| = p^{p-2}$ as mentioned earlier (see example 1.1.4). By Minkowski's Bound,

$$|d_{\mathbf{K}}|^{1/2} \geq \frac{(p-1)^{p-1}}{(p-1)!}\left(\frac{\pi}{4}\right)^{(p-1)/2}$$

where the bound is much smaller as compared to the actual value of $|d_{\mathbf{K}}|$.

As observed by Hermite, there are only finitely many number fields with bounded degree and discriminant. But bounded discriminant ensures bounded degree and therefore one has the following fundamental theorem.

Theorem 1.1.11. *[63, ch 4] There exist only finitely many number fields with bounded discriminant.*

Now we slowly proceed towards the conductor-discriminant formula. We begin with some group theoretic results.

1.2 Character group

For a finite abelian group, let $\hat{G}$ be the group of all homomorphisms from G to $\mathbb{C}^\times$. We shall refer this as the character group or the dual group of G. Since G is finite, any such homomorphism actually maps into the unit circle S^1.

Lemma 1.2.1. *Let G be a finite abelian group. Then* $G \cong \hat{G}$.

Proof. By Fundamental theorem of abelian groups,

$$G \cong \bigoplus_{m \in I} \mathbb{Z}/m\mathbb{Z},$$

where I is a finite indexed set of prime powers. So, any element of

$$\widehat{\bigoplus_{m\in I}\mathbb{Z}/m\mathbb{Z}}$$

looks like the product of elements of the groups $\widehat{\mathbb{Z}/m\mathbb{Z}}, m \in I$. Hence,

$$\hat{G} \cong \bigoplus_{m\in I}\widehat{\mathbb{Z}/m\mathbb{Z}}.$$

So, it is enough to show that $\widehat{\mathbb{Z}/m\mathbb{Z}} \cong \mathbb{Z}/m\mathbb{Z}$ for prime power m. We define a map

$$f : \mathbb{Z}/m\mathbb{Z} \to \widehat{\mathbb{Z}/m\mathbb{Z}}$$

by $f(i+m\mathbb{Z}) := \chi_i$ where $\chi_i(x+m\mathbb{Z}) := \zeta_m^{ix}$ for $0 \le i, x \le m-1$ and $\chi_i \in \widehat{\mathbb{Z}/m\mathbb{Z}}$. Clearly, $Ker(f) = 0$. Since, $\widehat{\mathbb{Z}/m\mathbb{Z}}$ is nothing but the collection of all $\chi_i's$ defined above. Hence, f is an isomorphism. ■

Let us now study a bit more carefully the character group of quotients.

Lemma 1.2.2. *Let G be a finite abelian group and H be a subgroup of G. We define* $H^\perp = \{\chi \mid H \subset ker\chi\}$. *Then*

$$H^\perp \cong \widehat{G/H}.$$

Proof. For a subgroup H of G, the natural surjection $G \to G/H$ induces a map

$$\widehat{G/H} \to \hat{G}$$

which is evidently injective. The image is indeed the group $H^\perp$ and hence

$$H^\perp \cong \widehat{G/H}.$$

■

On the other hand, the dual group $\hat{H}$ of H is linked to the group $H^\perp$.

Proposition 1.2.3. *Let G be a finite abelian group and H be a subgroup of G. Then*

$$\hat{H} \cong \hat{G}/H^{\perp}.$$

Proof. Let $f : \hat{G} \to \hat{H}$ defined by $f(\chi) := \chi$, be a homomorphism. Clearly, its kernel is $H^{\perp}$. So, $\hat{G}/H^{\perp}$ injects inside $\hat{H}$. Both $\hat{G}/H^{\perp}$ and $\hat{H}$ have the same cardinality by previous lemma. So, the proposition follows. ■

We note that for any subgroup H of G, the groups $H^{\perp}$ and $H^{\perp\perp}$ are subgroups of $\hat{G}$ and $\hat{\hat{G}}$ respectively and consequently

$$|H||H^{\perp}| = |G|, \ \ |H| = |H^{\perp\perp}|.$$

We note that for any finite abelian group G, G is isomorphic to its dual $\hat{G}$ and consequently G is isomorphic to its double dual $\hat{\hat{G}}$. However, there exists a direct natural isomorphism between these two groups which we highlight below.

Lemma 1.2.4. *Let G be a finite abelian group. The map $f : G \to \hat{\hat{G}}$ defined by $f(g) := \pi_g$ where $\pi_g(\chi) := \chi(g)$ for $\chi \in \hat{G}$ is an isomorphism between G and $\hat{\hat{G}}$.*

Proof. Let $f : G \to \hat{\hat{G}}$ defined by $f(g) := \pi_g$ where $\pi_g(\chi) := \chi(g)$ for $\chi \in \hat{G}$.
We have

$$H := ker(f) = \left\{g \in G | \chi(g) = 1 \ \forall \chi \in \hat{G}\right\}.$$

Clearly $H^{\perp} = \hat{G}$ and hence $|H| = 1$. On the other hand, any injective map from G to $\hat{\hat{G}}$ is surjective as they have the same finite order. Therefore, f is an isomorphism. ■

Thus the double dual is explicitly given by $\hat{\hat{G}} = \{\pi_g | g \in G\}$.

Proposition 1.2.5. *Let G be a finite abelian group and H be a subgroup of G. We define*

$$(H^{\perp})^{\perp} := \{\pi_g \in \hat{\hat{G}} : \pi_g(\chi) := \chi(g) = 1, \forall \chi \in H^{\perp}\}.$$

Then

$$H \cong (H^{\perp})^{\perp}.$$

Proof. Let $f : H \to (H^{\perp})^{\perp}$ defined by, $f(h) := \pi_h$ be a homomorphism. Clearly, f is 1-1. We have the following natural isomorphisms

$$(H^{\perp})^{\perp} \cong \{g \in G | \chi(g) = 1, \forall \chi \in H^{\perp}\} \cong \ char(\hat{G}/H^{\perp}).$$

This implies that $|(H^{\perp})^{\perp}| = \frac{|G|}{|H^{\perp}|}$. By lemma proved above, we have $|H| = \frac{|G|}{|H^{\perp}|}$. So, $|H| = |(H^{\perp})^{\perp}|$ implying the surjectivity of f. ∎

Let us now specialise our discussion to abelian groups which are Galois groups. For an abelian number field $\mathbf{K}$, let G denote the Galois group Gal($\mathbf{K}/\mathbb{Q}$). We shall call the dual group $\hat{G}$ to be the character group of the field $\mathbf{K}$. When $\mathbf{K} = \mathbb{Q}(\zeta_n)$, then clearly the character group of $\mathbf{K}$ can be identified with the set of all Dirichlet characters mod n. By Galois theory, the sub extensions $\mathbb{Q} \subset \mathbf{L} \subset \mathbf{K}$ are in bijection with subgroups of G. The association $H \leftrightarrow H^{\perp}$ gives a bijection between subgroups of G and $\hat{G}$. The inverse of this bijection is given as follows: for any $X \subset \hat{G}$, the corresponding subgroup H of G is simply given by

$$H = \bigcap_{\chi \in X} Ker(\chi).$$

Thus, we have a bijection between the subgroups of the character group of $\mathbf{K}$ and its subextensions over $\mathbb{Q}$. The following theorem gives the precise description of this bijection.

Proposition 1.2.6. *Let X be the character group of the field $\mathbb{Q}(\zeta_n)$, i.e., the dual of the group $Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$. Then there is a bijection*

$$\{Y \mid Y subgroup\ of\ X\} \longleftrightarrow \{\mathbf{L} \mid \mathbb{Q} \subset \mathbf{L} \subset \mathbb{Q}(\zeta_n)\}$$

given by the correspondences

$$Y \longrightarrow \quad \text{fixed field of } \bigcap_{\chi \in Y} Ker(\chi) \ \text{ and}$$

$$Gal(\mathbb{Q}(\zeta_n)/\mathbf{L})^\perp \longleftarrow \mathbf{L},$$

which are inverse to each other.

Furthermore, let X_i corresponds to $\mathbf{K}_i$. Then under this correspondence,

(1)

$$X_1 \subset X_2 \iff \mathbf{K}_1 \subset \mathbf{K}_2.$$

(2) $\langle X_1, X_2 \rangle$ corresponds to the compositum $\mathbf{K}_1\mathbf{K}_2$.

Proof. Denote the fixed field of G by $fix(G)$.

Let $f : S_1 \longrightarrow S_2$ and $g : S_2 \longrightarrow S_1$ be two maps between given sets. If $fog = gof = id$, then they are bijective maps and inverse to each other. Using Fundamental Theorem of Galois Theory, we have

$$\begin{aligned} Y^\perp &= \cap_{\chi \in Y} Ker(\chi) \\ &= \{g \in Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q}) | \chi(g) = 1 \forall \chi \in Y\} \\ &= Gal(\mathbb{Q}(\zeta_n)/\mathbf{L}), \end{aligned}$$

where $\mathbf{L}$ is the fixed field of $\bigcap_{\chi \in Y} Ker(\chi)$. By previous proposition, $(Y^\perp)^\perp = Y$. So, $Gal(\mathbb{Q}(\zeta_n)/\mathbf{L})^\perp = Y$. Conversely, we start with $\mathbf{L}$ subfield of $Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})$.

We have

$$fix(Gal(\mathbb{Q}(\zeta_n)/\mathbf{L})^\perp)^\perp = fix(Gal(\mathbb{Q}(\zeta_n)/\mathbf{L})) = \mathbf{L}.$$

(1) We have

$$
\begin{aligned}
& X_1 \subset X_2 \\
\iff & \bigcap_{\chi \in X_2} Ker(\chi) \subset \bigcap_{\chi \in X_1} Ker(\chi) \\
\iff & fix(\bigcap_{\chi \in X_1} Ker(\chi)) \subset fix(\bigcap_{\chi \in X_2} Ker(\chi)) \\
\iff & \mathbf{K}_1 \subset \mathbf{K}_2.
\end{aligned}
$$

(2) We have the following obvious equality.

$$
\bigcap_{\chi \in \langle X_1, X_2 \rangle} Ker(\chi) = (\bigcap_{\chi \in X_1} Ker(\chi)) \bigcap (\bigcap_{\chi \in X_2} Ker(\chi))
$$

$$
\begin{aligned}
fix(\bigcap_{\chi \in \langle X_1, X_2 \rangle} Ker(\chi)) &= fix((\bigcap_{\chi \in X_1} Ker(\chi)) \bigcap (\bigcap_{\chi \in X_2} Ker(\chi))) \\
&= \mathbf{K}_1 \mathbf{K}_2.
\end{aligned}
$$

The last equality follows from Fundamental Theorem of Galois Theory.

■

Remark 1.2.7. *As indicated before, the character group of* $\mathbb{Q}(\zeta_n)$ *is isomorphic to the group of all Dirichlet characters modulo n. So if L is a subfield of* $\mathbb{Q}(\zeta_n)$*, its character group Y is identified with a subset of the set of all Dirichlet characters modulo n which determines L uniquely.*

Example 1.2.8. *Let us determine the character group of the maximal real subfield* $\mathbf{L} = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$ *of* $\mathbf{K} = \mathbb{Q}(\zeta_n)$*. Note that the Galois group of the extension* $\mathbf{K}/\mathbf{L}$ *is given by complex conjugation and thus identified with the group* $\{\pm 1\}$*. So the character group of L consists of all Dirichlet characters* χ *mod n such that* $\chi(-1) = 1$*. In other words, the character group of L is precisely the set of all even Dirichlet characters mod n.*

Conversely, let us start with some character groups and determine the associated fields.

Example 1.2.9. *Let p be an odd prime number and $\mathbf{K} = \mathbb{Q}(\zeta_p)$. The character group X of $\mathbf{K}$ has a unique element of order two. This quadratic character*

$$\chi : (\mathbb{Z}/p\mathbb{Z})^* \longrightarrow \mathbb{C}^*$$

is given by $\chi(a) := \left(\frac{a}{p}\right)$, where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol.

Let us determine the associated quadratic number field $\mathbf{L}$. Since p is the only prime which ramifies in $\mathbb{Q}(\zeta_p)$, we have only two possibilities, namely $\mathbb{Q}(\sqrt{p})$ or $\mathbb{Q}(\sqrt{-p})$. The above character is even if and only if $p \equiv 1 \pmod 4$ and hence the associated quadratic field $\mathbf{L}$ is real if and only if $p \equiv 1 (mod 4)$. Hence the quadratic field associated to the Legendre character is $\mathbb{Q}(\sqrt{p})$ if $p \equiv 1 (mod 4)$. On the other hand, $p \equiv 3 (mod 4)$, then the corresponding field of given character is complex and hence equal to $\mathbb{Q}(\sqrt{-p})$.

1.3 Conductor

Definition 1.3.1. *The conductor f_χ of a Dirichlet character $\chi : (\mathbb{Z}/n\mathbb{Z})^* \longrightarrow \mathbb{C}$ is the smallest positive integer f such that χ factors through $(\mathbb{Z}/f\mathbb{Z})^*$.*

We now come to the final piece needed to define the conductor of an abelian extension $\mathbf{K}$ of $\mathbb{Q}$. The celebrated Kronecker-Weber theorem asserts that any such $\mathbf{K}$ is contained in a cyclotomic extension $\mathbb{Q}(\zeta_n)$. Evidently, such a cyclotomic extension is not unique as $\mathbb{Q}(\zeta_n) \subset \mathbb{Q}(\zeta_m)$ if n divides m.

Definition 1.3.2. *The conductor of an abelian extension $\mathbf{K}$ is the smallest positive integer f such that $\mathbf{K} \subset \mathbb{Q}(\zeta_f)$.*

Example 1.3.3. *Since 2 and 3 ramify in $\mathbb{Q}(\sqrt{3})$ [63, ch 5, §5.4], they ramify in all extensions of $\mathbb{Q}(\sqrt{3})$ [42, ch 1]. By Prop. 2.3 [72], the possible cyclotomic extensions are $\mathbb{Q}(\zeta_6)$, $\mathbb{Q}(\zeta_{12})$ etc. We have $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\zeta_6) = \mathbb{Q}(\sqrt{-3})$ which does not contain $\mathbb{Q}(\sqrt{3})$. Also, $\mathbb{Q}(\zeta_{12}) = \mathbb{Q}(\zeta_3)\mathbb{Q}(\zeta_4)$ which clearly contains $\mathbb{Q}(\sqrt{3})$. So the conductor of $\mathbb{Q}(\sqrt{3})$ is* 12. *Note that* 12 *is also its discriminant.*

We are now all set to state the Conductor-Discriminant formula. For any natural number δ, we set

$$m(\delta) := [\mathbf{K} \cap \mathbb{Q}(\zeta_\delta) : \mathbb{Q}],$$

where ζ_δ denotes a primitive δth root of unity. Let f be the conductor of $\mathbf{K}$.

Theorem 1.3.4. *[69](Conductor-Discriminant Formula) Let $\mathbf{K}/\mathbb{Q}$ be any finite abelian extension. Then*

$$|d_{\mathbf{K}}| = \prod_{\chi \in X} f_\chi$$

where X denotes the group of Dirichlet characters associated to $\mathbf{K}$, that is, the character group of $\mathbf{K}$.

Example 1.3.5. *Let χ be an even character mod 12 such that $\chi(5) = \chi(7) = -1$. We now determine its conductor by appealing to above. Its corresponding field is $\mathbb{Q}(\zeta_{12} + \zeta_{12}^{-1}) = \mathbb{Q}(\sqrt{3})$. The discriminant of $\mathbb{Q}(\sqrt{3})$ is 12. So, conductor of χ is 12 by the Conductor-Discriminant Formula.*

Remark 1.3.6. *The character group of the field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ is precisely the group of all even characters mod p. Other than the trivial character, all the other even characters have conductor p. Thus the Conductor-Discriminant Formula implies that discriminant of $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ is $p^{(p-3)/2}$.*

Now we come to the main point of the chapter. As indicated before, the lower bound given by Minkowski is not always optimal. In this connection, Ram Murty proved an elegant result which improves the Minkowski bound for abelian extensions. We shall give a complete proof of this pretty result describing all the intermediate steps. This result of Ram Murty will be pivotal to one of our works. We begin with the classical Mobius-Inversion formula.

Theorem 1.3.7. *(Möbius-Inversion formula) Let g and f be arithmetic functions then*

$$g(n) = \sum_{d|n} f(d) \quad \text{for every integer } n \geq 1,$$

if and only if

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d) \quad \text{for every integer } n \geq 1$$

where μ is the Möbius function.

Definition 1.3.8. *The Von Mangoldt function, denoted by $\Lambda(n)$, is defined as*

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and integer } m \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.3.9. *Let $\mathbf{K}/\mathbb{Q}$ be an abelian extension. Then we have*

$$\log|d_{\mathbf{K}}| = m(f)\ \log f - \sum_{\delta|f} m\left(\frac{f}{\delta}\right)\ \Lambda(\delta).$$

Proof. Let $s(e)$ be the number of characters of $Gal(\mathbf{K}/\mathbb{Q})$ which have conductor e. It is easy to see that for any g,

$$\sum_{e|g} s(e) = m(g) =: [\mathbf{K} \cap \mathbb{Q}(\zeta_g) : \mathbb{Q}] \tag{1.3.1}$$

since the left hand side is nothing but the total number of Dirichlet characters of Galois group of $\mathbf{K} \cap \mathbb{Q}(\zeta_g)$ over $\mathbb{Q}$.

Using Möbius-Inversion formula on (1.3.1), we have

$$s(e) = \sum_{\delta | e} \mu\left(\frac{e}{\delta}\right) m(\delta),$$

where μ denotes the Möbius function.

For an abelian extension $\mathbf{K}$, Theorem 1.3.4 can be rewritten as

$$|d_{\mathbf{K}}| = \prod_{e|f} e^{s(e)},$$

where $s(e)$ is the number of Dirichlet characters of $Gal(\mathbf{K}/\mathbb{Q})$ which have conductor e.

On the other hand, by taking log on both sides of above identity we have

$$\begin{aligned} \log|d_{\mathbf{K}}| &= \sum_{e|f} s(e)\log(e), \\ &= \sum_{e|f} \log(e) \sum_{\delta|e} \mu\left(\frac{e}{\delta}\right) m(\delta), \qquad (1.3.2) \\ &= \sum_{\delta|f} m(\delta) \sum_{e|f} \mu\left(\frac{e}{\delta}\right) \log(e). \end{aligned}$$

Since $\sum_{t|v} \mu(t)\log(t) = -\Lambda(v)$, we have

$$\begin{aligned} \sum_{e|f} \mu\left(\frac{e}{\delta}\right) \log(e) &= \sum_{t|(f/\delta)} \mu(t) \log(\delta t) \\ &= \log(\delta) \sum_{t|(f/\delta)} \mu(t) + \sum_{t|(f/\delta)} \mu(t)\log(t) \qquad (1.3.3) \\ &= \log(\delta) \sum_{t|(f/\delta)} \mu(t) - \Lambda\left(\frac{f}{\delta}\right). \end{aligned}$$

We know that

$$\sum_{d|n} \mu(n) = \begin{cases} 1 & \text{if } n = 1; \\ 0 & \text{else }. \end{cases}$$

Therefore, we obtain using (1.3.3) and (1.3.2)

$$\log|d_{\mathbf{K}}| = m(f)\log(f) - \sum_{\delta|f} m\left(\frac{f}{\delta}\right) \Lambda(\delta).$$

■

Corollary 1.3.10. *For any abelian extension* $\mathbf{K}/\mathbb{Q}$ *of degree* n*, discriminant* $d_{\mathbf{K}}$*, and conductor* f*, we have*

$$\frac{1}{2}\log f \leq \frac{\log|d_{\mathbf{K}}|}{n} \leq \log f.$$

Proof. As $m\left(\frac{f}{\delta}\right) = [\mathbf{K} \cap \mathbb{Q}(\zeta_{\frac{f}{\delta}}) : \mathbb{Q}]$, so $m\left(\frac{f}{\delta}\right)$ divides n.
Also, $m(f) = n$ since f is the conductor of the extension $\mathbf{K}/\mathbb{Q}$.
Hence for $\delta \neq n$, we obtain

$$m\left(\frac{f}{\delta}\right) \leq \frac{m(f)}{2} = \frac{n}{2}.$$

So by Lemma 1.3.9, we have

$$\begin{aligned}
\log|d_{\mathbf{K}}| &= m(f)\log(f) - \sum_{\delta|f} m\left(\frac{f}{\delta}\right)\Lambda(\delta) \\
&\geq m(f)\log(f) - \frac{m(f)}{2}\sum_{\delta|f}\Lambda(\delta) \\
&\geq \frac{m(f)}{2}\log f,
\end{aligned}$$

as $\log(n) = \sum_{d|n}\Lambda(d)$.
So, we have the first inequality of our corollary

$$\frac{1}{2}\log f \leq \frac{\log|d_{\mathbf{K}}|}{n}.$$

From (1.3.2), we obtain

$$\log|d_{\mathbf{K}}| = \sum_{e|f} s(e)\log(e) \leq \log(f)\sum_{e|f} s(e) = n\log(f).$$

■

In 1984, Ram Murty [49, Cor 2] proved the following elegant result on lower bounds of discriminants of abelian number fields in terms of their degrees. More precisely,

Theorem 1.3.11. *Let* $\mathbf{K}$ *be an abelian extension of* $\mathbb{Q}$ *of degree* n*. Then,*

$$\log|d_{\mathbf{K}}| \geq \frac{n\log n}{2}.$$

Proof. As n divides $\phi(f)$ and $\phi(f) < f$, so f can not be less than n.

Hence, $\log f \geq \log n$. From Corollary 1.3.10, we have our desired result. ∎

Remark 1.3.12. *The discriminant of abelian extension* $\mathbb{Q}(\zeta_p)$ *is* p^{p-2}*. So*

$$\frac{|d_{\mathbf{K}}|}{n^{n/2}} > \frac{p^{p-2}}{p^{(p-1)/2}} = p^{\frac{p-3}{2}}$$

which goes to ∞ *as* $p \to \infty$*. It indicates that the bound of* $|d_{\mathbf{K}}|$ *in above theorem is not optimal.*

Example 1.3.13. *Theorem 1.3.11 can hold for certain number fields which are not even Galois. For instance, let us consider the field* $\mathbf{K} = \mathbb{Q}(p^{1/p})$*, where* p *is an odd prime number. We see that this extension is not normal as no primitive* p*-th root of unity is in* $\mathbf{K}$*. In example 1.1.5, we computed its discriminant* $|d_{\mathbf{K}}| = p^{(2p-1)}$*. So*

$$\frac{|d_{\mathbf{K}}|}{n^{n/2}} = \frac{p^{(2p-1)}}{p^{p/2}} \to \infty,$$

as $p \to \infty$*.*

Now let us consider a non-abelian Galois extension of $\mathbb{Q}$*, namely* $\mathbf{K} = \mathbb{Q}(p^{1/p}, \zeta_p)$*, where* p *is a prime number. This extension is Galois with Galois group isomorphic to the following group of matrices*

$$\left\{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right),\ a, b \in \mathbf{F}_p, a \neq 0\right\}.$$

This can be seen by sending $\sigma_{a,b}$ *to* $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ *where* $\sigma_{a,b}(p^{1/p}) := p^{1/p}\zeta_p^b$*,* $\sigma_{a,b}(\zeta_p) := \zeta_p^b$*. Thus, it is not an abelian extension. Its ring of integers is given by* $\mathcal{O}_{\mathbf{K}} = \mathbb{Z}[p^{1/p}, \zeta_p]$*. The* *discriminant of* $\mathbf{K}$ *is given by* $d_{\mathbf{K}} = (p^{2p-1})^{(p-1)} \cdot (p^{(p-2)})^p = p^{3p^2-5p+1}$ *by Proposition 1.1.7. Therefore,*

$$\frac{|d_{\mathbf{K}}|}{n^{n/2}} = \frac{p^{(3p^2-5p+1)}}{(p(p-1))^{(p(p-1))/2}} \to \infty,$$

as $p \to \infty$*.*

1.4 Linearly disjoint fields

Definition 1.4.1. *Let* $\mathbf{K}$ *and* $\mathbf{F}$ *be algebraic extensions of a field* $\mathbf{L}$*. The fields* $\mathbf{K}, \mathbf{F}$ *are said to be linearly disjoint over* $\mathbf{L}$ *if every finite subset of* $\mathbf{K}$ *that is* $\mathbf{L}$ *linearly independent is also* $\mathbf{F}$ *linearly independent.*

A family of examples of linearly disjoint fields is the following. Take any two co-prime integers, then the cyclotomic fields generated by their primitive roots of unity are linearly disjoint over the field of rational numbers. Well, it is not a miracle. The following theorem provides us the proof of their linearly disjointness.

Theorem 1.4.2. *[14, Ch 5, Thm 5.5] Let* $\mathbf{K}$ *and* $\mathbf{F}$ *be algebraic extensions of a field* $\mathbf{L}$*. Also let at least one of* $\mathbf{K}, \mathbf{F}$ *is separable and one (possibly the same) is normal. Then* $\mathbf{K}$ *and* $\mathbf{F}$ *are linearly disjoint over* $\mathbf{L}$ *if and only if* $\mathbf{K} \cap \mathbf{F} = \mathbf{L}$*.*

If we remove the normality condition from hypothesis, does the above theorem hold? The answer of this question is false. We provide a famous counterexample for this. Take $\mathbf{L} = \mathbb{Q}$, $\mathbf{K} = \mathbb{Q}(\omega\alpha)$, $\mathbf{F} = \mathbb{Q}(\alpha)$, where $\alpha = 2^{1/3}$ and ω is a primitive third root of unity. Since degree of $\mathbf{K} \cap \mathbf{F}$ is either 1 or 3, it can not be 3 as $\mathbf{K}$ is not real but $\mathbf{F}$ is a real field. So $\mathbf{K} \cap \mathbf{F} = \mathbf{L}$. Also, $1, \omega\alpha, \omega^2\alpha^2$ the elements of $\mathbf{K}$, are linearly independent over $\mathbb{Q}$ but are linearly dependent over $\mathbf{F}$ as

$$1 + \frac{1}{\alpha}(\omega\alpha) + \frac{1}{\alpha^2}(\omega^2\alpha^2) = 0.$$

Let us also record the following equivalent criterion for linearly disjoint fields.

Proposition 1.4.3. *[14, Ch 5, Prop 5.2] Let* $\mathbf{L} \subset \mathbf{K}$ *and* $\mathbf{L} \subset \mathbf{E} \subset \mathbf{F}$ *be algebraic extensions of a field* $\mathbf{L}$*. Then* $\mathbf{K}$ *and* $\mathbf{F}$ *are linearly disjoint over* $\mathbf{L}$ *if and only if* $\mathbf{K}$ *and* $\mathbf{E}$ *are linearly disjoint over* $\mathbf{L}$ *and* $\mathbf{KE}$ *and* $\mathbf{F}$ *are linearly disjoint over* $\mathbf{E}$*.*

1.5 Organisation of the thesis

The next chapter is devoted to Hurwitz zeta function. The third chapter will deal with some preliminaries from transcendental number theory required for our work.

In fourth chapter, we will survey the generalisations and extensions of a question of Baker ([3]) regarding linear independence of Dirichlet characters.

In fifth chapter, we will talk about the extension of his question and will investigate it to the families of Dirichlet characters modulo distinct natural numbers. The sixth chapter extends the results of Okada and Murty-Murty ([50]). Finally, in chapter seven, we will study the relation between the non-vanishing of Dedekind zeta function and its derivative at 1/2.

CHAPTER 2

HURWITZ ZETA FUNCTION

2.1 Definition

Hurwitz zeta function is defined as

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \ ,$$

where x is a real number with $0 < x \leqslant 1$ and s is a complex number with $\Re(s) > 1$. This series is absolutely and uniformly convergent in the domain $\Re(s) > 1+\delta$, for every $\delta \geq 0$. It therefore represents an analytic function in the half- plane $\Re(s) > 1$. Further, it has the analytic continuation to whole complex plane except $s = 1$ where it has a simple pole of residue 1. The Hurwitz zeta function is named after Adolf Hurwitz who introduced it in 1882. The Riemann zeta function is $\zeta(s,1)$. Now we state the functional equation of $\zeta(s,x)$ when x is rational.

Theorem 2.1.1. *[1] If h and k are integers, $1 \le h \le k$, then for all $s \in \mathbb{C}$, we have*

$$\zeta(1-s, \frac{h}{k}) = \frac{2\Gamma(s)}{(2\pi k)^s} \sum_{r=1}^{k} \cos\left(\frac{\pi s}{2} - \frac{2\pi r h}{k}\right) \zeta(s, \frac{r}{k}).$$

2.2 Zeroes

We have Euler product for Riemann zeta function, i.e,

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}},$$

where p runs through prime numbers. It implies that Riemann zeta function has no zeroes for $\Re(s) > 1$. The gamma function $\Gamma(s)$ is nowhere 0 and has simple poles at non-positive integers. The functional equation therefore shows that the only zeroes of $\zeta(s)$ in the domain $\Re(s) < 0$ are at $s = -2n,\ n \in \mathbb{N}$. These are called the trivial zeroes of $\zeta(s)$ and the non-trivial zeroes lie in critical strip $0 \le \Re(s) \le 1$.

Conjecture 2.2.1. *(Riemann Hypothesis) All non-trivial zeroes of $\zeta(s)$ lie on the line $\Re(s) = \frac{1}{2}$.*

However, if $0 < x < 1$ and $x \neq 1/2$, then there are zeros of Hurwitz's zeta function in the strip $1 < \Re(s) < 1+\epsilon$ for any positive real number ϵ. This was proved by Davenport and Heilbronn for rational or transcendental irrational x and by Cassels for algebraic irrational x. Therefore, there is no Euler product for Hurwitz zeta function. For more information, please see [9, 16].

2.3 Specific Values

The values of $\zeta(s,x)$ at $s = 0, -1, -2, \cdots$ are related to the Bernoulli polynomials:

$$\zeta(-n, a) = -\frac{B_{n+1}(a)}{n+1}.$$

Now put $s = 2n,\ h = k = 1$ in the functional equation of Hurwitz zeta function which is Theorem 2.1.1 to get

$$\zeta(1-2n) = 2(2\pi)^{-2n}\Gamma(2n)\cos(\pi n)\zeta(2n).$$

It implies that

$$-\frac{B_{2n}}{2n} = 2(2\pi)^{-2n}(2n-1)!(-1)^n\zeta(2n).$$

Hence for any positive even integer $2n$, we obtain

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!}B_{2n},$$

where B_{2n} is the $2n$th Bernoulli number. This implies that $\zeta(2n)/\pi^{2n}$ is a rational number.

If we put $s = 2n+1$ in the functional equation then we see that right hand side vanishes and we get no information about $\zeta(2n+1)$. As yet no simple formula analogous to above derived formula is known for $\zeta(2n+1)$. Similarly, this technique does not work for other values of Hurwitz Zeta Function.

Also, we have (see Ch. 22 ,[53])

$$\lim_{s\to 1}\left(\zeta(s,a) - \frac{1}{s-1}\right) = \frac{-\Gamma'(a)}{\Gamma(a)} = -\psi(a),$$

where ψ is the digamma function, i.e, logarithmic derivative of gamma function.

2.4 Relation to periodic Dirichlet series

For periodic arithmetic functions f with period $q > 1$ and consider the associated L function

$$L(s,f) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

for $s \in \mathbb{C}$ with $\Re(s) > 1$. It is easy to see that when $k > 1$, we have

$$L(k,f) = q^{-k} \sum_{a=1}^{q} f(a)\zeta(k,a/q) \tag{2.4.1}$$

and hence in particular

$$\zeta(k)\prod_{p|q}(1-p^{-k}) = L(k,\chi_0) = q^{-k} \sum_{\substack{1\le a<q \\ (a,q)=1}} \zeta(k,a/q). \tag{2.4.2}$$

The Dirichlet series $L(s,f)$ converges absolutely for $\Re(s) > 1$ and has meromorphic continuation to whole complex plane except possibly at $s = 1$ where it has a simple pole with residue $q^{-1}\sum_{a=1}^{q} f(a)$. So, when $\sum_{a=1}^{q} f(a) = 0$, we have

$$L(1,f) = \sum_{n=1}^{\infty} \frac{f(n)}{n}.$$

Now we record the functional equation of $L(s,f)$.

Theorem 2.4.1. *[44] Let f be an arithmetic function with period q and $\hat{f}$ be its Fourier transform. Define $\hat{f}^{-}(n) := \hat{f}(-n)$ for $n \in \mathbb{Z}$. For $s \in \mathbb{C}$, we have the following expression*

$$L(s,f) = \frac{1}{2\pi i}\left(\frac{2\pi}{q}\right)^{s} \Gamma(1-s)\left[L(1-s,\hat{f}^{-})e^{i\pi s/2} - L(1-s,\hat{f})e^{-i\pi s/2}\right].$$

Theorem 2.4.2. *[54] Let f be a periodic function with period q. If k and f have the same parity, and $k > 1$, then*

$$2L(k,f) = (-1)^{k-1}\frac{(2\pi i)^k}{k!} \sum_{a=1}^{q} \hat{f}(a)B_k(a/q).$$

Thus, if f takes algebraic values, then it is an algebraic multiple of π^k. If in addition $L(k,f)$ is non-zero, then it is transcendental.

Proof. Define the Fourier transform of f by

$$\hat{f}(n) = q^{-1}\sum_{a=1}^{q} f(a)e^{2\pi i a n/q}.$$

By orthogonality, this implies that

$$f(n) = \sum_{a=1}^{q} \hat{f}(a)e^{-2\pi i a n/q}.$$

Thus, we may write

$$L(k,f) = \sum_{n=1}^{\infty} n^{-k}\sum_{a=1}^{q} \hat{f}(a)e^{-2\pi i a n/q} = \sum_{a=1}^{q} \hat{f}(a)Li_k(e^{-2\pi i a/q}),$$

where polylogarithm function $Li_k(z)$ is defined by $\sum_{n=1}^{\infty}\frac{z^n}{n^k}$. It implies that

$$2L(k,f) = \sum_{a=1}^{q} \hat{f}(a)\left(Li_k(e^{-2\pi i a/q}) + (-1)^k Li_k(e^{2\pi i a/q})\right) \tag{2.4.3}$$

It is well known that for $0 < x < 1$, the kth Bernoulli polynomial, $B_k(x)$ has the Fourier series expression

$$B_k(x) = \frac{-k!}{(2\pi i)^k}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{n^k}.$$

This means that

$$B_k(x) = \frac{-k!}{(2\pi i)^k}\left(Li_k(e^{2\pi i x}) + (-1)^k Li_k(e^{-2\pi i x})\right).$$

Hence plugging the above expreesion in (2.4.3), we get the desired result. The other parts of the theorem are obvious now.

■

Theorem 2.4.3. *Let χ be a Dirichlet character mod q and let $k > 1$. If k and χ have the same parity, then*

$$L(k,\chi) = \frac{(-q)^{k-1}(2\pi i)^k}{2k!}\tau(\chi)\sum_{b=1}^{q} \overline{\chi}(b)B_k(b/q).$$

Proof. For a co-prime to q, we consider the following function of period q

$$\delta_a(n) = \begin{cases} 1 & \text{n = a} \\ (-1)^k & \text{n = q - a} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} 2L(k,\delta_a) &= (-1)^{k-1}\frac{(2\pi i)^k}{k!}\sum_{b=1}^{q}\hat{\delta_a}(b)B_k(b/q) \\ &= (-1)^{k-1}\frac{(2\pi i)^k}{q.k!}\sum_{b=1}^{q}(\zeta_q^{ab}+(-1)^k\zeta_q^{-ab})B_k(b/q). \end{aligned} \tag{2.4.4}$$

On the other hand,

$$\begin{aligned} L(k,\delta_a) &= q^{-k}\sum_{b=1}^{q}\delta_a(b)\zeta(k,b/q) \\ &= q^{-k}[\zeta(k,a/q)+(-1)^k\zeta(k,1-a/q]. \end{aligned} \tag{2.4.5}$$

From (2.4.4) and (2.4.5), we have

$$q^{-k}[\zeta(k,a/q)+(-1)^k\zeta(k,1-a/q] = (-1)^{k-1}\frac{(2\pi i)^k}{2q.k!}\sum_{b=1}^{q}(\zeta_q^{ab}+(-1)^k\zeta_q^{-ab})B_k(b/q). \tag{2.4.6}$$

For any character χ mod q, the associated Gauss sum $\tau(\chi)$ is given by

$$\tau(\chi) = \sum_{a=1}^{q}\chi(a)\zeta_q^a.$$

Multiplying both sides of above equation by $\overline{\chi}(b)$, we have

$$\begin{aligned} \tau(\chi)\overline{\chi}(b) &= \sum_{a=1}^{q}\chi(a)\overline{\chi}(b)\zeta_q^a \\ &= \sum_{a=1}^{q}\chi(a)\zeta_q^{ab} \\ &= \sum_{a=1}^{q/2}\chi(a)(\zeta_q^{ab}+(-1)^k\zeta_q^{-ab}). \end{aligned} \tag{2.4.7}$$

Multiplying both sides of above equation by $B_k(b/q)$ and sum over $b = 1$ to q to get

$$\begin{aligned}\tau(\chi)\sum_{b=1}^{q}\overline{\chi}(b)B_k(b/q) &= \sum_{a=1}^{q/2}\chi(a)\sum_{b=1}^{q}(\zeta_q^{ab}+(-1)^k\zeta_q^{-ab})B_k(b/q)\\ &= \frac{2k!}{(-q)^{k-1}(2\pi i)^k}\sum_{a=1}^{q/2}\chi(a)[\zeta(k,a/q)+(-1)^k\zeta(k,1-a/q)] \qquad (2.4.8)\\ &= \frac{2k!}{(-q)^{k-1}(2\pi i)^k}L(k,\chi),\end{aligned}$$

where $\chi(1) = (-1)^k$. Therefore, if k and χ are of same parity then

$$L(k,\chi) = \frac{(-q)^{k-1}(2\pi i)^k}{2k!}\tau(\chi)\sum_{b=1}^{q}\overline{\chi}(b)B_k(b/q).$$

■

CHAPTER 3

TRANSCENDENTAL PRE-REQUISITES

Definition 3.0.1. *A transcendental number is a number that is not the root of a non-zero polynomial with rational coefficients. The best known transcendental numbers are e and π.*

3.1 Lindemann-Weierstrass theorem

In 1882, Lindemann published the first complete proof of the transcendence of π. We record here the following application of Lindemann-Weierstrass theorem [57].

Lemma 3.1.1. *[57, Cor 1.3] If α is an algebraic number different from 0 and 1, then $\log\alpha$ is a transcendental number where $\log$ denotes any branch of logarithmic function.*

3.2 Gelfond-Schneider Theorem

Lindemann-Weierstrass theorem does not give us any information about the nature of e^{π}. It is Gelfond–Schneider theorem which establishes the transcendence of this number. More precisely,

Theorem 3.2.1. *[21, 57] If α and β are non-zero algebraic numbers with $\beta \neq 1$ and $\log\alpha/\log\beta \notin \mathbb{Q}$, then $\log\alpha/\log\beta$ is transcendental.*

3.3 Baker's Theorem

Basically, the previous theorem says that if α_1, α_2 are non-zero algebraic numbers such that $\log\alpha_1, \log\alpha_2$ are linearly independent over the rationals, then $\log\alpha_1, \log\alpha_2$ are linearly independent over the field of algebraic numbers. But does the same thing hold if we talk about more than two numbers? To answer this question, we state the Baker's seminal theorem on linear forms in logarithms of algebraic numbers.

Theorem 3.3.1. *[3, Thm 2.1] If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are non-zero algebraic numbers such that $\log\alpha_1, \log\alpha_2, \cdots, \log\alpha_n$ are linearly independent over the rationals, then $1, \log\alpha_1, \log\alpha_2, \cdots, \log\alpha_n$ are linearly independent over the field of algebraic numbers.*

Now, we also record a useful corollary of Theorem 3.3.1.

Lemma 3.3.2. *[3, Thm 2.2] If $\alpha_1, \alpha_2, \cdots, \alpha_n \in \overline{\mathbb{Q}} \setminus \{0\}$ and $\beta_1, \beta_2, \cdots, \beta_n \in \overline{\mathbb{Q}}$, then*

$$\sum_{j=1}^{n} \beta_j \log\alpha_j$$

is either zero or transcendental. The latter case arises if $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly independent over $\mathbb{Q}$ and $\beta_1, \beta_2, \cdots, \beta_n$ are not all zero.

Now we state and prove the following important application of the above theorems which will be used multiple times in next chapters.

Lemma 3.3.3. *[53, p. 154, Lemma 25.4] Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be positive algebraic numbers. If $c_0, c_1, \cdots, c_n$ are algebraic numbers with $c_0 \neq 0$, then*

$$c_0\pi + \sum_{j=1}^{n} c_j \log \alpha_j$$

is a transcendental number.

Proof. Let S be such that $\{\log \alpha_j : j \in S\}$ be a maximal $\mathbb{Q}$-linearly independent subset of

$$\log \alpha_1, \log \alpha_2, \cdots, \log \alpha_n.$$

We write $\pi = -i \log(-1)$. We can re-write our linear form as

$$-ic_0 \log(-1) + \sum_{j \in S} d_j \log \alpha_j,$$

for algebraic numbers d_j. By Theorem 3.3.2, this is either zero or transcendental. The former case cannot arise if we show that $\log(-1), \log \alpha_j, j \in S$ are linearly independent over $\mathbb{Q}$. But this is indeed the case since

$$b_0 \log(-1) + \sum_{j \in S} b_j \log \alpha_j = 0$$

for integers $b_0, b_j, j \in S$ implies that

$$\prod_{j \in S} \alpha_j^{2b_j} = 1,$$

which in turn implies $b_j = 0$ for all $j \in S$ since α_j, for $j \in S$ are multiplicatively independent. Consequently, $b_0 = 0$. ∎

CHAPTER 4

DIRICHLET CHARACTERS

4.1 Introduction

The central theme of this chapter is the Dirichlet characters. These are one dimensional Galois representations of Cyclotomic extensions. More concretely, for an integer $n > 1$, a Dirichlet character χ is simply a homomorphism from the group $(\mathbb{Z}/n\mathbb{Z})^{\times}$ of co-prime residue classes mod n to the multiplicative group $\mathbb{C}^{\times}$. By assigning the value zero at the other classes mod n, we can extend χ to a function from $\mathbb{Z}$ to $\mathbb{C}$ which is completely multiplicative and periodic with period n.

Each such character χ gives rise to a Dirichlet series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where the series is absolutely convergent in the region $\Re(s) > 1$. Furthermore, since χ

is completely multiplicative, one has the Euler product representation

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \chi(p)p^{-s}\right)^{-1}$$

which is a consequence of prime factorisation of integers. When χ is a non-trivial character, we know that $L(s,\chi)$ extends to an entire function and we have

$$L(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

and it is these complex numbers which are the centre of our focus. From now on, the Dirichlet characters are assumed to be non-trivial, unless stated otherwise.

A celebrated result of Dirichlet asserts that $L(1,\chi)$ is non-zero. This work of Dirichlet laid the foundation of Analytic Number theory. This also ushered in the application of Character theory into the realms of Number theory, a theme which has now spectacularly morphed into the enigmatic interplay between Harmonic Analysis and Arithmetic of Galois representations. Furthermore, these special values have deep arithmetic . For instance, for any quadratic number field K, one has a quadratic Dirichlet character χ associated to K and $L(1,\chi)$ subsumes deep arithmetic data like the class number and Regulator of K.

Let us now come to the focal point of this note. Since the $L(1,\chi)$'s are non-zero, it is natural to ask about the algebraic nature of these complex numbers. While the non-vanishing was established in 1837, it took about 130 years more to settle the nature of these numbers. The seminal work of Baker in mid 1960's established that these numbers are transcendental.

For a positive integer $q \geq 3$, each of the $\varphi(q)-1$ non-trivial Dirichlet characters give rise to seemingly unrelated transcendental numbers. But these numbers do not quite live in complete isolation from each other. Let $K = \mathbb{Q}(\zeta_q)$ be the q-th Cyclotomic field and $\zeta_K(s)$ be its Dedekind zeta function. The product of these $L(1,\chi)$ values gives the residue of $\zeta_K(s)$ at $s=1$. So it is natural to wonder about any relation, linear or algebraic, existing between these mysterious numbers.

In 1973, Baker, Birch and Wirsing [4] in a beautiful work proved that for a prime p, the numbers $L(1,\chi)$ where χ runs over the non-trivial characters mod p, are linearly independent over $\mathbb{Q}$. Baker, in his book Transcendental number Theory ([3], p.48), stated that it would be of interest to know if this is true for an arbitrary modulus q. This remains unanswered till now and is the raison d'etre of our note.

We attempt to give a broad account of the history and the state-of-the-art of this question of Baker. We highlight the mathematical tools and techniques that enter into this circle of questions. We also describe a number of generalisations and extensions of this question, namely extensions to Number fields, to class group L-functions as well as specialisations at larger integers. There are some essential ingredients common to each of the above themes, viz, Galois action on linear forms of certain periods, Transcendence theory of linear forms in logarithms, non-vanishing of L-values from Arithmetic and finally Dedekind-Frobenius determinants from Linear algebra. We shall try to illustrate the commonality in the above circle of questions as well as the issues intrinsic to each of these separate themes.

Let us end this section by noting that the complex numbers $L(1,\chi)$'s are examples of *Periods*. A period as defined by Kontsevich and Zagier [38] is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients. Clearly, all algebraic numbers are periods. An important class of periods is supplied by the special values of the Riemann zeta function. For example, we have for $k \geq 2$

$$\zeta(k) = \int_{1>t_1>\cdots>t_k>0} \frac{dt_1}{t_1} \cdots \frac{dt_{k-1}}{t_{k-1}} \frac{dt_k}{1-t_k},$$

as is easily verified by direct integration. Also π is a period, it is expected that e is not a period.

The set of periods is countable and a $\bar{\mathbb{Q}}$-algebra of infinite dimension. Since logarithms of algebraic numbers are periods, we shall see that each $L(1,\chi)$ is indeed a period. We recommend the original delightful article of Kontsevich and Zagier [38] as well as the account of Waldschmidt [70] for further details. We also heartily recommend the comprehensive book by Huber and Müller-Stach [32].

4.2 A question of Chowla and the remarkable result of Baker, Birch and Wirsing

In a lecture at the Stony Brook conference on number theory in 1969, Sarvadaman Chowla posed the question whether there exists a non-zero rational-valued arithmetic function f, periodic with prime period p such that $\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0$.

In 1973, Baker, Birch and Wirsing [4] answered this question in the following theorem:

Theorem 4.2.1. *If f is a non-zero function defined on the integers with algebraic values and period q such that $f(n) = 0$ whenever $1 < (n,q) < q$ and the q-th cyclotomic polynomial is irreducible over the field $F = \mathbb{Q}(f(1),\ldots,f(q))$, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

In particular, if f is rational valued, the second condition holds trivially. If q is prime, then the first condition is vacuous. Thus, the theorem resolves Chowla's question.

The above theorem of Baker, Birch and Wirsing is remarkable as it brought out a new aspect of Transcendence theory hitherto undiscovered. More often than not, non-vanishing is a major obstacle in Transcendence and typically non-vanishing follows from Arithmetic considerations. For instance, transcendence of $L(1,\chi)$ follows only when its non-vanishing is ensured. But the ideas in the proof of Baker-Birch-Wirsing indicated that the tables can be turned and transcendence can ensure non-vanishing. This perspective has been exploited quite fruitfully in recent times. For instance, Kumar Murty and Ram Murty [50] have used Transcendence theory to prove the non-vanishing of $L(1,\chi)(\chi \neq 1)$ for even characters, an illustrative instance of Transcendence theory returning the favours to Arithmetic. Recall that a character χ is even or odd according as $\chi(-1)$ is 1 or -1.

Let us now highlight the main ingredients in the proof of Baker-Birch-Wirsing Theorem.

1. The series $\sum_{n=1}^{\infty} \frac{f(n)}{n}$ converges if and only if $\sum_{a=1}^{q} f(a) = 0$. Once this is ensured, one derives that

$$\mathrm{L}(1,f) = \sum_{n=1}^{\infty} \frac{f(n)}{n} = \frac{-1}{q} \sum_{(a,q)=1} f(a)(\psi(a/q)+\gamma).$$

Here ψ is the digamma function which shows up since it is the constant term of the Hurwitz zeta function around $s=1$. We will need to come back to the Hurwitz zeta function in the last section. But the digamma function regrettably is a rather difficult function to handle, for instance, $-\psi(1)$ is the enigmatic Euler's constant γ.

2. Since f is periodic, we can Fourier analyse and go to the dual set up. Let

$$\hat{f}(n) = \frac{1}{q}\sum_{m=1}^{q} f(m)e^{-2\pi imn/q}$$

be the Fourier transform of f. By inverse Fourier transform and some functional manipulation, one is led to the following new identity

$$\mathrm{L}(1,f) = -\sum_{a=1}^{q-1}\widehat{f}(a)\log(1-\zeta_q^a).$$

While we no longer have the digamma function, the caveat now is the added complexity of the sequence of Fourier coefficients $\{\hat{f}(n)\}$ as they need not lie in the field generated by the original sequence $\{f(n)\}$. But the redeeming feature is that the digamma function is replaced by logarithms of algebraic numbers, setting the tone for the entry of Baker's seminal theory. Without further ado, let us state Baker's theorem (see [3], for instance) which is a pivotal ingredient in our context.

Theorem 4.2.2. *(Baker) If $\alpha_1,\dots,\alpha_m$ are non-zero algebraic numbers such that $\log\alpha_1,\dots,\log\alpha_m$ are linearly independent over $\mathbb{Q}$, then $1,\log\alpha_1,\dots,\log\alpha_m$ are linearly independent over $\overline{\mathbb{Q}}$.*

3. Baker's theorem allows one to show that vanishing of $L(1,f)$ ensures the vanishing of the "conjugate" L-values $L(1,f^\sigma)$ for σ in the Galois group G of the extension $F(\zeta_q)/F$. Let us be more concrete. We note that G is isomorphic to the group $(\mathbb{Z}/q\mathbb{Z})^\times$. For $(h,q)=1$, let $\sigma_h \in G$ be such that $\sigma_h(\zeta_q) = \zeta_q^h$. Define $f^{\sigma_h} = f_h(n) := f(nh^{-1})$ for $(h,q)=1$. Then, Baker, Birch and Wirsing beautifully exploited Baker's theorem to show that the vanishing of $L(1,f)$ results in

$$\mathrm{L}(1,f_h) = \sum_{n=1}^{\infty}\frac{f_h(n)}{n} = 0$$

for all $(h,q)=1$.

4. We then reverse the Fourier process and come back to the original set up

$$L(1, f_h) = \frac{-1}{q} \sum_{(a,q)=1} f_h(a)(\psi(a/q) + \gamma) = 0$$

leading to the following system of identities (for each h co-prime to q)

$$\sum_{(a,q)=1} f(a)(\psi(ah/q) + \gamma) = 0.$$

5. We now notice that the matrix $A := \big(\psi(ah/q) + \gamma\big)_{(ah,q)=1}$ associated to the system of identities obtained in the previous step is a Dedekind-Frobenius matrix on the group $H = (\mathbb{Z}/q\mathbb{Z})^\times$ and its determinant (up to a sign) is given by

$$\prod_{\chi \in \widehat{H}} \left(\sum_{h \in H} \chi(h)(\psi(h/q) + \gamma) \right).$$

If we show that the matrix A is invertible, then f vanishes everywhere and we are done.

6. This is where "Arithmetic" enters the picture. The non-vanishing of determinant of A is ensured by the non-vanishing of $L(1, \chi)$ for non-trivial χ while the monotonicity of ψ takes care of the trivial character. This completes the proof of the theorem.

We mention in passing that in [27], a generalization of the above theorem has been derived.

Remark 4.2.3. *In 1966, Lang [41] proved a multi-dimensional generalisation of the classical theorem of Schneider, motivated by a question of Cartier about the analogue of transcendence of* $e^\alpha, \alpha \in \bar{\mathbb{Q}}^\times$ *for arbitrary group varieties. Lang did answer this question in affirmative. In retrospect, as evinced by a later work of Bertrand and Masser, we see that Baker's Theorem in the form stated above could have been proved by Lang in 1966 building on the Galois conjugate idea in Baker-Birch-Wirsing*

Theorem and his generalisation of Schneider's theorem. But perhaps it is fortuitous that Lang did not prove this theorem. It is because the approach of Baker is different who not only proved the above theorem, but also obtained lower bounds for linear forms in logarithms of algebraic numbers. These lower bounds are of seminal importance in the study of diophantine equations and form a subject of its own. For instance, these lower bounds allowed Baker to classify all imaginary quadratic fields with class number one, a venerable theme in Number theory set in motion by Gauss.

As indicated above, Bertrand and Masser [7] gave a new proof of Baker's theorem by Galois action on linear forms. They also exploited these ideas to prove an elliptic analog of Baker's theorem. Let us state this result. For a Weierstrass $\wp$-function with algebraic invariants g_2 and g_3 and field of endomorphisms k, the set $\mathscr{L} = \{\alpha \in \mathbb{C} \,:\, \wp(\alpha) \in \overline{\mathbb{Q}} \cup \{\infty\}\}$ is the two-dimensional analogue of logarithms of algebraic numbers. Bertrand and Masser proved that if $u_1, \cdots, u_n \in \mathscr{L}$ are linearly independent over $\mathbb{Q}$, then $1, u_1, \cdots, u_n$ are linearly independent over $\overline{\mathbb{Q}}$.

4.3 Settling for prime modulus and over $\overline{\mathbb{Q}}$

Let us begin by noting that the above theorem of Baker, Birch and Wirsing settles the question of Baker for prime modulus. It is because the values taken by the Dirichlet characters mod p lie in the field $\mathbb{Q}(\zeta_{(p-1)})$ which is linearly disjoint with the p-th cyclotomic field. So all the hypotheses of Baker-Birch-Wirsing theorem are satisfied and hence for an odd prime p, the numbers $L(1,\chi)$ where χ runs over the non-trivial characters mod p, are indeed linearly independent over $\mathbb{Q}$. In a recent work [23], the above result has been extended to any arbitrary family of moduli.

Let us now consider the possible $\bar{\mathbb{Q}}$-linear relations between these values of $L(1,\chi)$ as χ ranges over all non-trivial Dirichlet characters mod q with $q > 2$. It is remarkable that over $\bar{\mathbb{Q}}$, we have a complete answer for all moduli q. This follows from a natural extension of the works of Ram and Kumar Murty [50]. Let us give a summary of their work. One of the crucial results in their work is the following:

Theorem 4.3.1. *For any integer $q > 2$, the numbers $L(1,\chi)$ as χ ranges over non-trivial even characters mod q are linearly independent over $\bar{\mathbb{Q}}$.*

Remark 4.3.2. *This in particular furnishes a new proof of non vanishing of $L(1,\chi)$ for even non-trivial characters by transcendental means. The possibility of such an approach could not have been envisaged, but for the work of Baker-Birch-Wirsing. Furthermore this result shows that the dimension of space generated by $L(1,\chi)$ for even characters remains the same over any number field. As we shall see a little later, this is a luxury which is not at all afforded to us for $L(k,\chi)$ with $k > 1$.*

Let us briefly indicate the main points in the proof of this result. The new ingredient in the proof of the above theorem is the properties of a set of real multiplicatively independent units in the cyclotomic field discovered by K. Ramachandra (see Theorem 8.3 on page 147 of [72] as well as [61]). These marvellous units allowed Ram and Kumar Murty to work with new expressions of $L(1,\chi)$ for even characters in terms the logarithms of positive real numbers. Thereafter, they appeal to Baker's theorem and its variants leading to a system of equations involving characters of finite groups. Finally, they prove a variant of Artin's theorem on linear independence of irreducible characters which establishes the desired linear independence over $\bar{\mathbb{Q}}$. Let us state this elegant group theoretic result for the sake of completion. Let G be a finite group.

Suppose that $\sum_{\chi\neq 1}\chi(g)u_{\chi}=0$ for all $g\neq 1$ and all irreducible characters $\chi\neq 1$ of G. Then $u_{\chi}=0$ for all $\chi\neq 1$.

So the space generated by the even characters is now well and truly done. Let us now consider the odd $L(1,\chi)$ values. We note that for any odd Dirichlet character χ, $L(1,\chi)$ is a non-zero algebraic multiple of π. This follows from the expressions we indicated in the previous section for $L(1,f)$ after applying Fourier transform of f. Therefore, the space generated by the odd $L(1,\chi)$ values is one dimensional over $\bar{\mathbb{Q}}$.

But do these subspaces intersect? Here we come to the following pretty application of Baker's theory which has been proved in Chapter 3 and has a large number of applications in various different set ups.

Lemma 4.3.3. *Let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be positive algebraic numbers. If $c_0,c_1,\ldots,c_n$ are algebraic numbers with $c_0\neq 0$, then*

$$c_0\pi+\sum_{j=1}^{n}c_j\log\alpha_j$$

is a transcendental number and hence non-zero.

The above lemma leads to the following result since for an even character $\chi\neq 1$, the number $L(1,\chi)$ is a linear form in logarithms of positive real algebraic numbers.

Theorem 4.3.4. *The $\overline{\mathbb{Q}}$-space generated by the values $L(1,\chi)$ with χ non-trivial even character is linearly disjoint from the space generated by the values $L(1,\chi)$ with χ odd character.*

Consequently, one has the following satisfying result.

Theorem 4.3.5. *For any integer $q>2$, the $\bar{\mathbb{Q}}$-vector space generated by the values $L(1,\chi)$ as χ ranges over the non-trivial Dirichlet characters (mod q) has dimension $\varphi(q)/2$.*

Let us end this section with specifying what exactly is the issue which hinders us from settling Baker's question for an arbitrary modulus. As we noted in the proof of Baker-Birch-Wirsing theorem, the central part of the proof was to show that the vanishing of $L(1,f)$ ensues the vanishing of the Galois conjugates $L(1,f^\sigma)$. A careful look in the proof of this part reveals that it was important that the Galois elements σ kept the coefficients $\{f(n)\}$ unchanged. Let us give some more indication why and when this comes up. In the course of the proof, Baker's theory eventually leads to a family of expressions (indexed by b) of the form

$$\sum_{a=1}^{q-1} \widehat{f}(a)\mathrm{r}_{ab} = 0,$$

where the numbers r_{ab} lie in the field of definition F of f. Then for any automorphism $\sigma \in \mathrm{G}al(F(\zeta_q)/F)$, we need to act σ on each of these identities. The condition of irreducibility of the q-th cyclotomic polynomial over F ensures that

$$\sum_{a=1}^{q-1} \sigma(\widehat{f}(a))\mathrm{r}_{ab} = 0,$$

which then leads to

$$\sum_{a=1}^{q-1} \sigma(\widehat{f}(a))\log(1-\zeta_q^a) = 0$$

which is what we desire. In general, if the automorphisms do act nontrivially on the sequence $\{f(n)\}$, the issue becomes involved, some instances of which we shall see in the next section.

On the other hand, the space generated by the even $L(1,\chi)$ values present no problem, thanks to the result of Ram and Kumar Murty described in this section. So it is only the linear independence of odd $L(1,\chi)$ values which remain unresolved for an arbitrary modulus. But since these are algebraic multiples of π, perhaps transcendence theory has no more role to play. This parity conundrum will show up again a bit later when we work with $L(k,\chi)$ with $k>1$.

4.4 Extension to Number fields

In this section, we consider the extension of Baker's question to arbitrary number fields. Since the question is open for arbitrary modulus over $\mathbb{Q}$, it is prudent to consider the number field extension only for prime modulus for now. This constitutes the ethos of a recent work [8].

Let K be a number field and p be an odd prime. Let us consider the K-vector space in $\mathbb{C}$ generated by the $L(1,\chi)$ values for non-trivial characters χ modulo p. Let $d(K,p)$ denote its dimension. In view of the discussions in the previous section, we have the following bounds:

$$\frac{p-1}{2} \leq d(K,p) \leq p-2.$$

When $K = \mathbb{Q}$, the upper bound is attained. On the other hand, when $K = \mathbb{Q}(\zeta_p, \zeta_{p-1})$, the lower bound is attained. Therefore, one can ask the following question: Which numbers in the interval

$$\left(\frac{p-1}{2} \ , \ p-2\right)$$

can be equal to $d(K,p)$ as K runs over all number fields? This is not known. From now onwards all primes are at least 7.

The next question is to ask whether for any prime $p > 5$, there is a number field such that

$$\frac{p-1}{2} < d(K,p) < p-2.$$

This question is answered in the affirmative in [8]. The initial strategy in [8] is to look for primes of specific type which may be more amenable to work with. The family of primes which seem more tractable in this context are the Sophie Germain Primes. A prime p is called a *Sophie Germain prime* if $2p+1$ is also a prime. It is a folklore conjecture that there are infinitely many Sophie Germain primes. Following theorem is proved in [8].

Theorem 4.4.1. *Let $p > 5$ be an odd prime. Then there exists a number field K such that*

$$\frac{p-1}{2} < d(K,p) < p-2.$$

This theorem is proved in two steps. In the first step, one considers primes that are not Sophie Germain where one investigates arithmetic of number fields K with $\mathbb{Q}(\zeta_{p-1}) \subset K \subset \mathbb{Q}(\zeta_{p-1}, \zeta_p)$. In the second step, one proves the result for Sophie Germain primes by working with fields K such that $\mathbb{Q}(\zeta_p) \subset K \subset \mathbb{Q}(\zeta_{p-1}, \zeta_p)$.

Consequently for any $p > 7$, at least one number in the interval

$$\left(\frac{p-1}{2} \quad , \quad p-2\right)$$

is realised as the dimension of space of $L(1,\chi)$ values over some number field. Let $b(p)$ count the numbers in the above interval that can be realised as this. More precisely, for a prime $p > 5$, let

$$b(p) := \left| \left\{ n \;\middle|\; \frac{p-1}{2} < n < p-2 \quad and \quad d(K,p) = n \quad for\ some\ number\ field\ K \right\} \right|.$$

The above theorem implies that $b(p) > 0$ for every prime $p > 5$ and hence we have,

$$1 \leq b(p) \leq \frac{p-3}{2}.$$

Then in [8], the following is proved.

Theorem 4.4.2. *The sequence $\{b(p)\}$ satisfies*

$$\limsup_{p\to\infty} \ b(p) = \infty.$$

Thus the sequence $\{b(p)\}$ is unbounded. One can ask about its growth. In [8], the following Omega result is established.

Theorem 4.4.3. *There exists a constant $c > 0$ such that*

$$b(p) > exp\left(\frac{c\log p}{\log\log p}\right)$$

for infinitely many primes p.

For the proof of these results, one has to work with families of number fields for which the methods and approaches of the earlier sections no longer work and therefore we shall not dwell further. We shall just indicate one of the ingredients, a folklore result of Linnik ([45], [46]) which constitutes a celebrated theme in Analytic number theory with far-reaching implications.

Theorem 4.4.4. *Let a, n be two positive integers with $(a,n) = 1, n \geq 2$. Let $p(a,n)$ denote the least prime p such that $p \equiv a \pmod{n}$. There exists absolute positive constants C and L such that*

$$p(a,n) < Cn^L.$$

The constant L is known as *"Linnik's constant"*. It is conjectured that $p(a,n) < n^2$. The best known value for L is due to Xylouris [73] who proves that L can be taken to be 5.18. On the other hand, Lamzouri, Li and Soundararajan [66] have shown, under Generalised Riemann Hypothesis, that $p(a,n) \leq \varphi(n)^2 \log^2 n$ for all $n > 3$.

4.5 Analogous question for class group L-functions

In this section, we consider the analog of the question of Baker for class group L-functions. This has been carried out by Ram and Kumar Murty [51].

Let K be a number field. Let $\mathcal{O}_K$ be its ring of integers and $\mathcal{H}_K$ be its ideal class group. It is this finite group on which our functions will act.

Let f be a complex-valued function of the ideal class group $\mathcal{H}_K$ of K. For such an f, we consider the Dirichlet series for $\Re(s) > 1$

$$L(s,f) := \sum_{\mathfrak{a}} \frac{f(\mathfrak{a})}{\mathbb{N}(\mathfrak{a})^s},$$

where the summation is over non-zero ideals $\mathfrak{a}$ of $\mathcal{O}_K$. If $f \equiv 1$, $L(s,f)$ is simply the Dedekind zeta function of K. We hope to study the numbers $L(1,f)$ as and when they exist. A necessary and sufficient condition of existence is given by the following (see [53] as well as [42]):

Theorem 4.5.1. *$L(s,f)$ extends analytically for all $s \in \mathbb{C}$ except possibly at $s = 1$ where it may have a simple pole with residue a non-zero multiple of*

$$\rho_f := \sum_{\mathfrak{a} \in \mathcal{H}_K} f(\mathfrak{a}).$$

Consequently, the series $\sum_{\mathfrak{a}} \frac{f(\mathfrak{a})}{\mathbb{N}(\mathfrak{a})}$ converges and equal to $L(1,f)$ if and only if $\rho_f = 0$.

We want to investigate the values $L(1,f)$ when K is imaginary quadratic and when f takes algebraic values, in particular the values $L(1,\chi)$ when χ runs over ideal class characters. We note that complex conjugation acts on the group of ideal class characters and $L(1,\chi) = L(1,\overline{\chi})$ for any ideal class character χ. Let $\mathcal{H}_K^*$ denote a set of orbit representatives under this action. Here is a pretty result proved in [51].

Theorem 4.5.2. *Let K be an imaginary quadratic field and $\mathcal{H}_K$ its ideal class group. The values $L(1,\chi)$ as χ ranges over the non-trivial characters of $\mathcal{H}_K^*$ and π are linearly independent over $\overline{\mathbb{Q}}$.*

We add that unlike the case of Dirichlet characters, the above does not prove the transcendence of $L(1,\chi)$. However it does prove that at most one of the values $L(1,\chi)$, as χ ranges over the non-trivial characters of $\mathscr{H}_K^*$, can be algebraic.

Recall that Baker's question for primes is a consequence of the Baker-Birch-Wirsing Theorem. Here is the analogue of Baker-Birch-Wirsing of which the above theorem is an immediate consequence.

Theorem 4.5.3. *Let K be an imaginary quadratic field and $f : \mathscr{H}_K \to \bar{\mathbb{Q}}$ be not identically zero. Suppose that $\rho_f = 0$. Then, $L(1,f) \neq 0$ unless $f(\mathfrak{C}) + f(\mathfrak{C}^{-1}) = 0$ for every ideal class $\mathfrak{C} \in \mathscr{H}_K$.*

So we need to prove the above theorem. Let us indicate the salient features in the proof of the above, indicating the commonality with that of Baker-Birch-Wirsing as well as the new ingredients intrinsic to this set up. One needs to use Kronecker's limit formula as discussed in the works of Siegel [64], Ramachandra [60] and Lang [43]. We shall need Baker's theorem from Transcendence theory as well as Chebotarev density theorem from Algebraic number theory. Finally, we shall need a few results from Theory of Complex Multiplication. However, we can only take a cursory glance into this delightful realm and shall enthusiastically direct the interested reader to the original work [51] (and to [15] and [43]).

We begin with the celebrated discriminant functions $\Delta(z)$:

$$\Delta(z) = (2\pi)^{12}\, q \prod_{n=1}^{\infty} (1-q^n)^{24} = (2\pi)^{12} \eta(z)^{24}, \qquad q = e^{2\pi i z}$$

where η^{24} is the ubiquitous Ramanujan cusp form. As before, let K be an imaginary quadratic field and $\mathfrak{b}$ be an ideal of $\mathscr{O}_K$. If $[\beta_1, \beta_2]$ is an integral basis (i.e. a basis as a

$\mathbb{Z}$-module) of $\mathfrak{b}$ with $\Im(\beta_2/\beta_1) > 0$, we define

$$g(\mathfrak{b}) = (2\pi)^{-12}(\mathbb{N}(\mathfrak{b}))^6|\Delta(\beta_1,\beta_2)|,$$

where

$$\Delta(\omega_1,\omega_2) := \omega_1^{-12}\Delta\left(\frac{\omega_2}{\omega_1}\right).$$

One can verify that $g(\mathfrak{b})$ is well-defined and furthermore depends only on the ideal class $[\mathfrak{b}]$ belonging to in the ideal class group (see [60] and [43]). Let us now describe the main steps in the proof of the above theorem.

1. In the first step, we need to get an expression for $L(1,f)$. For this we need Kronecker's limit formula. For an ideal class $\mathfrak{C}$, by Kronecker's limit formula we have

$$\zeta(s,\mathfrak{C}) = \sum_{\mathfrak{a}\in\mathfrak{C}} \frac{1}{\mathbb{N}(\mathfrak{a})^s} = \frac{2\pi}{w\sqrt{|d_K|}}\left(\frac{1}{s-1} + 2\gamma - \log|d_K| - \frac{1}{6}\log|g(\mathfrak{C}^{-1})|\right) + O(s-1),$$

as $s \to 1^+$. Here d_K is the discriminant of K and w is the number of roots of unity in $\mathcal{O}_K$

2. In the above expression apart from γ, one has the mysterious number $|g(\mathfrak{C}^{-1})|$. But by CM theory, we know that if $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are ideal classes, then $g(\mathfrak{C}_1)/g(\mathfrak{C}_2)$ is an algebraic number lying in the Hilbert class field of K.

3. Kronecker's limit formula gives rise to the following expression for $L(1,f)$

$$\frac{L(1,f)}{\pi} = \frac{-1}{3w\sqrt{|d_K|}} \sum_{\mathfrak{C}\in\mathcal{H}_K} f(\mathfrak{C})\log|g(\mathfrak{C}^{-1})|.$$

Since $g(\mathfrak{C}^{-1})/g(\mathfrak{C}_0)$ is algebraic for the identity class $\mathfrak{C}_0$ and $\rho_f = 0$, we rewrite the above as

$$\frac{L(1,f)}{\pi} = \frac{-1}{3w\sqrt{|d_K|}} \sum_{\mathfrak{C}\in\mathcal{H}_K} f(\mathfrak{C})\log|g(\mathfrak{C}^{-1})/g(\mathfrak{C}_0)|,$$

and hence $L(1,f)/\pi$ is a linear form in logarithms of algebraic numbers.

4. Now we come to the Galois conjugation step. In particular, one shows that $L(1,f)=0$ implies that $L(1,f^{\sigma})=0$ for any Galois automorphism σ of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$. This is using Baker's theory similar to the approach adopted in the Baker-Birch-Wirsing Theorem.

5. In the next step, one uses the above lemma to reduce it to the case when f is actually rational valued. More precisely, one proves the following: Let M be the number field of degree r generated by the values of f. Then for any basis $\beta_1,\ldots,\beta_r$ of M over $\mathbb{Q}$ and $f(\mathfrak{C})=\sum_{i=1}^{r} f_i(\mathfrak{C})\beta_i$ with $f_i(\mathfrak{C})\in\mathbb{Q}$, we have $L(1,f)=0$ if and only if $L(1,f_i)=0$ for $i=1,\ldots,r$. Thus we may assume without loss of generality that our function f is actually rational valued.

6. In the penultimate step, it is shown that if f is a rational-valued function and $L(1,f)=0$, then $f(\mathfrak{C})+f(\mathfrak{C}^{-1})=0$ for every ideal class $\mathfrak{C}$. For this one needs to appeal to the Chebotarev density theorem.

7. Finally as before, one views these equations as a matrix equation $DV=0$ where V is the transpose of the row vector $(f(\mathfrak{C})+f(\mathfrak{C}^{-1}))_{\mathfrak{C}\in H_K}$ and D is the "Dedekind-Frobenius" matrix whose (i,j)-th entry is given by $\log g(\mathfrak{C}_i^{-1}\mathfrak{C}_j)/g(\mathfrak{C}_i^{-1})$ with $\mathfrak{C}_i,\mathfrak{C}_j$ running over the elements of the ideal class group. The non-vanishing of this determinant is a consequence of non-vanishing of each $L(1,\chi),\chi\neq 1$ and consequently $f(\mathfrak{C})+f(\mathfrak{C}^{-1})=0$ for all $\mathfrak{C}\neq\mathfrak{C}_0$. However since the sum $\sum f(\mathfrak{C})=0$, we have $f(\mathfrak{C}_0)=0$ as well, completing the proof of Theorem 4.5.3.

4.6 Linear Independence of L(k,χ) values with $k>1$

In this final section, let us consider the analogous question for $L(k,\chi)$, when $k>1$. Here we can also include the principal character in our list.

As we saw earlier, the parity of χ plays a crucial role in the context of $L(1,\chi)$, a phenomenon which continues for $k > 1$. For values in the domain of absolute convergence, it is worthwhile to introduce Hurwitz zeta values as these form a natural generating set for the study of special values of Dirichlet series associated to periodic arithmetic functions.

For a real number x with $0 < x \leq 1$ and $s \in \mathbb{C}$ with $\Re(s) > 1$, the Hurwitz zeta function is defined by

$$\zeta(s,x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}.$$

This can be analytically extended to the entire complex plane except at $s = 1$ where it has a simple pole with residue one. Note that $\zeta(s,1) = \zeta(s)$. For any Dirichlet character mod q, running over arithmetic progressions mod q, one immediately deduces that

$$L(k,\chi) = q^{-k} \sum_{a=1}^{q} \chi(a)\zeta(k,a/q).$$

For $q > 1$, let K_q be the $\varphi(q)$-th cyclotomic field. Suppose that k and χ have the same parity, that is $\chi(-1) = (-1)^k$. Then the above identity yields that $L(k,\chi)$ in this case is a K_q-linear combination of elements of the following set

$$X := \{\lambda_a \;:\; 1 \leq a \leq q/2, (a,q) = 1\} \;\; where \;\; \lambda_a := \zeta(k,a/q) + (-1)^k \zeta(k,1-a/q).$$

Now differentiating the series expansion of $\pi \cot \pi z$ for $z \notin \mathbb{Z}$, one has

$$\lambda_a = \zeta(k,a/q) + (-1)^k \zeta(k,1-a/q) = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z)|_{(z=a/q)}.$$

On the other hand for $z \notin \mathbb{Z}$, we have

$$\frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z) = \pi^k \sum_{\substack{r,s \geq 0 \\ r+2s=k}} \beta_{r,s}^{(k)} \cot^r \pi z \; (1 + \cot^2 \pi z)^s,$$

where $\beta_{r,s}^{(k)} \in \mathbb{Z}$. Since $i \cot \frac{\pi a}{q} \in \mathbb{Q}(\zeta_q)$, we see that

$$\zeta(k,a/q) + (-1)^k \zeta(k,1-a/q) = (i\pi)^k \alpha_{a,q},$$

where $\alpha_{a,q} \in \mathbb{Q}(\zeta_q)$. Thus when k and χ have the same parity, we deduce that $L(k,\chi)$ is an algebraic multiple of π^k reminiscent of the fact that $L(1,\chi)$ is an algebraic multiple of π. This also generalises Euler's classical result that $\zeta(2n)$ is a rational multiple of π^{2n}.

The following theorem now allows us to settle the dimension of the *same-parity* space for prime modulus, namely that its dimension over $\mathbb{Q}$ is $\frac{p-1}{2}$. Let $\cot^{(k-1)}(z_0)$ denote the $(k-1)$-th derivative $\frac{d^{k-1}}{dz^{k-1}}(\cot z)|_{z=z_0}$.

Theorem 4.6.1. *Let $k > 1$ and $q > 2$ be positive integers and K be a number field such that $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then the set of real numbers*

$$\cot^{(k-1)}(\pi a/q), \qquad 1 \leq a \leq q/2, (a,q) = 1$$

is linearly independent over K.

The above result does seem to be in the spirit of the Baker-Birch-Wirsing theorem and was proved by Okada [59]. But as noted by Girstmair [22], it is a much simpler result and we indicate his proof which is short and elegant. The first point to note is that the numbers

$$i^k \frac{d^{k-1}}{dz^{k-1}}(\cot z)|_{(z=\pi a/q)}, \qquad 1 \leq a \leq q/2, \ \ (a,q) = 1$$

are Galois conjugates. Now consider any non-trivial $\mathbb{Q}$-linear combination of the form

$$\sum_a r_a \cot^{(k-1)}(\pi a/q) = \sum_a r_a \left((-1)^{(k-1)}(k-1)! \frac{q^k}{\pi^k} \sum_{\substack{n \in \mathbb{Z} \\ n \equiv a \bmod (q)}} n^{-k} \right).$$

The Galois conjugacy observation allows us to assume that the first coefficient r_1 has the largest modulus. Thus, for $k \geq 2$, modulus of the sum in the right hand side of the above identity is at least

$$\frac{(k-1)! q^k}{\pi^k} |r_1| \left(1 - \sum_{n=2}^{\infty} n^{-k} \right) > 0$$

and we are done. Note that when $k = 1$, the matter is indeed more delicate as it is linked to the non-vanishing of $L(1,\chi)$.

What about the case when k and χ have opposite parity? Recall for $s = 1$, the $L(1,\chi)$ values for even characters χ are linearly independent over $\bar{\mathbb{Q}}$ for all moduli. However for larger integers, the situation is rather bleak with almost no information. Let us indicate the issue here. When $s = 1$, the Fourier transform approach allowed us to express $L(1,\chi)$ as a linear form in logarithms of algebraic numbers and thereafter Baker's theory took over. An analogous course of action for larger k leads us to linear forms in polylogarithms.

For an integer $k \geqslant 2$ and complex numbers $z \in \mathbb{C}$ with $|z| \leqslant 1$, the polylogarithm function $Li_k(z)$ is defined as

$$Li_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

Then we can deduce the following:

$$\mathrm{L}(k,\chi) = \sum_{a=1}^{p} \widehat{\chi}(a) Li_k(\zeta_p^a).$$

However we do not have an analogue of Baker's theorem for polylogarithms and hence have no information when k and χ have different parity. We also do not know whether the space generated by the even and odd $L(k,\chi)$ values intersect trivially. Presumably, we need deeper results in transcendence to make any further progress. Finally, there is a conjecture of P. Chowla and S. Chowla [13] on non-vanishing of certain $L(k,f)$ which was later generalised by Milnor [48]. These are deep conjectures which are linked to the irrationality of numbers of the form $\zeta(2n+1)/\pi^{2n+1}$ as well as to a folklore conjecture of Zagier on linear independence of Multiple zeta values (see [25], [10] for more details, generalisations and partial results in this direction).

CHAPTER 5

EXTENSION OF A QUESTION OF BAKER

5.1 Introduction

For an integer $q > 1$ and a Dirichlet character χ with period q, consider the Dirichlet L function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re(s) > 1.$$

When χ is non-trivial, we know that $L(s,\chi)$ extends to an entire function, $L(1,\chi)$ is non-zero and is equal to $\sum_{n=1}^{\infty} \frac{\chi(n)}{n}$. For q as before, consider the set

$$X_q = \{L(1,\chi) \mid \chi \bmod q, \chi \neq \chi_0\},$$

where χ_0 is the trivial Dirichlet character with period q. In [3, p. 48], Baker asked whether the numbers in X_q are linearly independent over $\mathbb{Q}$. In 1973, Baker, Birch and Wirsing [4] in an elegant work proved that the numbers in the set $\{L(1,\chi) \in X_q \mid (q,\varphi(q)) = 1\}$ are linearly independent over $\mathbb{Q}$ (see [37] for an exposition on this topic). In this context, we prove the following theorem.

Theorem 5.1.1. *Let $q_j > 2$ for $1 \le j \le \ell$ be pairwise co-prime natural numbers such that the number $q_1 \cdots q_\ell$ is co-prime to $\varphi(q_1)\cdots\varphi(q_\ell)$. Then the numbers in the set $X_{q_1} \cup \cdots \cup X_{q_\ell}$ are linearly independent over $\mathbb{Q}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$.*

More generally, we derive the following theorem.

Theorem 5.1.2. *Let $q_j > 2$ for $1 \le j \le \ell$ be pairwise co-prime natural numbers. Also let $\mathbf{K}$ be a number field with $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)}) \cap \mathbb{Q}(\zeta_{q_1\cdots q_\ell}) = \mathbb{Q}$, where ζ_q denotes a primitive qth root of unity. Then the numbers in the set $X_{q_1} \cup \cdots \cup X_{q_\ell}$ are linearly independent over $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$.*

Note that $X_q = X_{q,e} \cup X_{q,o}$, where

$$\begin{aligned} X_{q,e} &= \{L(1,\chi) \mid \chi \bmod q,\ \chi(-1) = 1, \chi \neq \chi_0\} \\ \text{and} \quad X_{q,o} &= \{L(1,\chi) \mid \chi \bmod q,\ \chi(-1) = -1\}. \end{aligned}$$

In 2011, Murty-Murty refined Baker-Birch-Wirsing result to show that

Theorem 5.1.3. *(Murty-Murty [50]) Let $q > 2$ be a natural number. Then the numbers in the set $X_{q,e}$ are linearly independent over $\overline{\mathbb{Q}}$.*

In this chapter, we prove the following theorem.

Theorem 5.1.4. *Let $q_j > 2$ for $1 \le j \le \ell$ be pairwise co-prime natural numbers. Then the numbers in the set $X_{q_1,e} \cup \cdots \cup X_{q_\ell,e}$ are linearly independent over $\overline{\mathbb{Q}}$.*

In 1981, Okada [59] (see also [54, 71]) proved that

Theorem 5.1.5. *(Okada [59]) Let $q > 2$ be a natural number and $\mathbf{K}$ be a number field with the property that $\mathbf{K}(\zeta_{\varphi(q)}) \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then the numbers in the set $X_{q,o}$ are linearly independent over $\mathbf{K}(\zeta_{\varphi(q)})$.*

Here we prove the following theorem.

Theorem 5.1.6. *For $1 \le j \le \ell$, let $q_j > 2$ be pairwise co-prime natural numbers. If $\mathbf{K}$ is a number field such that $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)}) \cap \mathbb{Q}(\zeta_{q_1\cdots q_\ell}) = \mathbb{Q}$, then the numbers in the set $X_{q_1,o} \cup \cdots \cup X_{q_\ell,o}$ are linearly independent over $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$.*

Remark 5.1.7. *In Theorem 5.1.2 and Theorem 5.1.6, $\varphi(q_1)\cdots\varphi(q_\ell)$ in $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$ and $q_1 \cdots q_\ell$ in the $\mathbb{Q}(\zeta_{q_1\cdots q_\ell})$ can be replaced by their respective least common multiples.*

As consequences to Theorem 5.1.2, Theorem 5.1.4 and Theorem 5.1.6, we derive the following corollaries. Before we state the corollaries, let us introduce the notion of Dirichlet type functions as defined by Murty and Saradha (see [54]).

Definition 5.1.8. *An arithmetical function f with period $q > 1$ with values in $\overline{\mathbb{Q}}$ is called Dirichlet type if $f(a) = 0$ whenever $(a,q) \neq 1$.*

Definition 5.1.9. *A periodic function f with period $q > 1$ is called an Erdösian function if $f(a) = \pm 1$ for all $1 \le a < q$ and $f(q) = 0$.*

For an arithmetical function f with period q, consider the series $L(s,f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ for $\Re(s) > 1$. This series has a meromorphic continuation to $\mathbb{C}$ with a possible simple pole at $s = 1$ of residue $q^{-1}\sum_{a=1}^{q} f(a)$ (see [53, Ch 22] for further details). From now onwards, we assume that $\sum_{a=1}^{q} f(a) = 0$.

For a natural number $q > 2$ and a number field $\mathbf{K}$, let $Y_q(\mathbf{K})$ be $\mathbf{K}$ linearly independent set of Dirichlet type functions of period q. Define

$$\begin{aligned} X_q(\mathbf{K}) &= \{L(1,f) \mid f \in Y_q(\mathbf{K})\} \\ \text{and} \quad X_{q,e}(\mathbf{K}) &= \{L(1,f) \mid f \in Y_q(\mathbf{K}),\ f(-a) = f(a) \text{ for } 1 \le a < q\}. \end{aligned}$$

In this set-up, we have the following corollaries.

Corollary 5.1.10. *For $1 \le j \le \ell$, let $q_j > 2$ be pairwise co-prime natural numbers. Then the numbers in the set $X_{q_1,e}(\overline{\mathbb{Q}}) \cup \cdots \cup X_{q_\ell,e}(\overline{\mathbb{Q}})$ are $\overline{\mathbb{Q}}$ linearly independent.*

Corollary 5.1.11. *For any odd prime p, choose an Erdösian function f_p with period p which is not an odd function. Then the numbers in the set $\{L(1,f_p) \mid p \text{ odd}\}$ are linearly independent over $\overline{\mathbb{Q}}$.*

Corollary 5.1.12. *Let $q_j > 2, 1 \le j \le \ell$ be pairwise co-prime natural numbers. Also let $f_j \in Y_{q_j}(\mathbf{K})$ with values in a number field $\mathbf{K}$. If $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)}) \cap \mathbb{Q}(\zeta_{q_1\cdots q_\ell}) = \mathbb{Q}$, then the elements in $X_{q_1}(\mathbf{K}) \cup \cdots \cup X_{q_\ell}(\mathbf{K})$ are $\mathbf{K}$ linearly independent. In particular, choose Erdösian functions f_{p_i} with odd prime periods p_i, then the numbers $L(1,f_{p_i})$ for $1 \le i \le \ell$ are linearly independent over a number field $\mathbf{K}$ which satisfies the condition $\mathbf{K}(\zeta_{\varphi(p_1\cdots p_\ell)}) \cap \mathbb{Q}(\zeta_{p_1\cdots p_\ell}) = \mathbb{Q}$.*

Remark 5.1.13. *Consider the sets*

$$A = \left\{ \text{pairs } \left(\frac{p-1}{2},\ p\right) \;\middle|\; \text{both } \frac{p-1}{2} \text{ and } p \text{ are primes} \right\} \quad \text{and} \quad B = \left\{ p \;\middle|\; \left(\frac{p-1}{2},\ p\right) \in A \right\}.$$

Any prime pair in the set A is called a Sophie-Germain prime pair. Dickson's conjecture (see preliminaries for precise statement) implies the existence of infinitely many Sophie-Germain prime pairs (see [17]). Let

$$C = \left\{ p_i \;\middle|\; i \ge 1,\ p_i \in B,\ p_{i+1} > 2p_i + 1 \right\}.$$

Since by Dickson's conjecture A is an infinite set, so is B and hence C is an infinite set. Choose Erdösian functions f_p for $p \in C$. Then the numbers in the set $\{L(1, f_p) \mid p \in C\}$ are linearly independent over any Galois number field $\mathbf{K}$ whose discriminant $d_{\mathbf{K}}$ is co-prime to $\{p\varphi(p) \mid p \in C\}$. Note that $\mathbf{K}(\zeta_{\varphi(q)}) \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$ if and only if $\mathbf{K} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$ and $\mathbf{K}(\zeta_{\varphi(q)}) \cap \mathbf{K}(\zeta_q) = \mathbf{K}$ (see Proposition 1.4.3 in preliminaries), where $q > 2$ is a natural number. Further, the property $\mathbf{K}(\zeta_{\varphi(q)}) \cap \mathbf{K}(\zeta_q) = \mathbf{K}$ is not necessarily true for $(q, \varphi(q)) = 1$. But when $\mathbf{K}$ is Galois number field whose discriminant $d_{\mathbf{K}}$ is co-prime to $q\varphi(q)$, where $(q, \varphi(q)) = 1$, then $\mathbf{K}(\zeta_{\varphi(q)}) \cap \mathbf{K}(\zeta_q) = \mathbf{K}$ (see Theorem 1.8 in [25]).

5.2 Preliminaries

In this section, we state the results which will play an important role in proving our main theorems. We start with the following non-vanishing result of Baker, Birch and Wirsing [4] (see also chapter 23 of [53]).

Theorem 5.2.1. *(Baker, Birch and Wirsing). Let f be a non-zero algebraic valued periodic function with period q. Also let $f(n) = 0$ whenever $1 < (n, q) < q$ and the q-th cyclotomic polynomial $\Phi_q(X)$ be irreducible over $\mathbb{Q}(f(1), \cdots, f(q))$, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

Chowla [12] proved that if p is an odd prime, then the numbers

$$\cot(2\pi a/p), \quad 1 \leq a \leq (p-1)/2$$

are linearly independent over the field of rational numbers. This result was reproved by various authors (see, for instance, [30, 34]). In 1981, Okada [59] (see also Wang

[71]) extended Chowla's theorem to natural number $q > 2$ which are not necessarily primes. In the same theorem he also considered derivatives of higher orders of $\cot x$. Both Okada and Wang made use of the fact that $L(k,\chi) \neq 0$ though their proofs were different. More precisely, Okada [59] proved the following theorem.

Theorem 5.2.2. *Let k and q be positive integers with $k > 0$ and $q > 2$. Let T be a set of $\varphi(q)/2$ representatives mod q such that the union $T \cup (-T)$ is a complete set of co-prime residues modulo q. Then the set of real numbers*

$$\frac{d^{k-1}}{dz^{k-1}}(\cot \pi z)|_{(z=a/q)}, \quad a \in T$$

is linearly independent over $\mathbb{Q}$.

Five years later, Girstmair [22] gave a much simpler proof of this result of Okada using Galois Theory in the case when order of the derivative of $\cot x$ is at least 1. In 2009, Murty and Saradha [54] extended the work of Okada to show the following theorem.

Theorem 5.2.3. *Let k and q be positive integers with $k > 0$ and $q > 2$. Let T be a set of $\varphi(q)/2$ representatives mod q such that the union $T \cup (-T)$ is a complete set of co-prime residues modulo q. Let $\mathbf{K}$ be an algebraic number field over which the qth cyclotomic polynomial is irreducible. Then the set of real numbers*

$$\frac{d^{k-1}}{dz^{k-1}}(\cot \pi z)|_{(z=a/q)}, \quad a \in T$$

is linearly independent over $\mathbf{K}$.

See the recent work of Hamahata [29] for a multi-dimensional generalization of Theorem 6.2.1. We deduce another generalization of Theorem 6.2.1 required for our work.

For an integer $q > 4$, Ramachandra [61] discovered a set of multiplicatively independent units in the cyclotomic field $\mathbb{Q}(\zeta_q)$, where ζ_q is a primitive qth root of unity. For $1 < a < q/2$ and $(a,q) = 1$, define

$$\xi_a = \zeta_q^{d_a} \eta_a \quad \in \mathbb{Q}(\zeta_q + \zeta_q^{-1}),$$

where

$$d_a = \frac{1}{2}(1-a) \sum_{\substack{d|q,\ d\neq q \\ (d,\frac{q}{d})=1}} d, \qquad \eta_a = \prod_{\substack{d|q,\ d\neq q \\ (d,\frac{q}{d})=1}} \frac{1-\zeta_q^{ad}}{1-\zeta_q^{d}}.$$

It is easy to see that ξ_a is a unit in $\mathbb{Q}(\zeta_q + \zeta_q^{-1})$ for $1 < a < q/2$ and $(a,q) = 1$. Ramachandra proved the following important theorem about these units.

Theorem 5.2.4. *[61, 72] The set of real units $\{\xi_a \mid 1 < a < q/2, (a,q) = 1\}$ is multiplicatively independent.*

These units are now known as Ramachandra units. Using these units, one can express $L(1,\chi)$ when χ is an even non-trivial character with period q as follows.

Lemma 5.2.5. *[72, p. 149] For a natural number $q > 4$, let χ be an even non-trivial character with period q. Then we have*

$$L(1,\chi) = \delta_\chi \sum_{\substack{1<a<q/2 \\ (a,q)=1}} \overline{\chi}(a) \log \xi_a,$$

where δ_χ is a non-zero algebraic number. Further, $L(1,\chi)$ can also be written as algebraic linear combination of logarithms of positive algebraic numbers.

Using Lemma 5.2.5, Ram Murty and Kumar Murty [50] proved the following theorem.

Theorem 5.2.6. *[50, Thm 8] Let $q > 2$ be a natural number and f be a non-zero Dirichlet type function with period q. Write $f = f_e + f_o$, where f_e is an even function* *and f_o is an odd function. Let $\mathbf{K}$ be the field generated by the values of f_o over $\mathbb{Q}$. If $\mathbf{K} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, then $L(1, f) \neq 0$.*

We end this section by recalling a group theoretic pre-requisite [50] as well as a conjecture of Dickson [18].

Lemma 5.2.7. *Let G be a finite group. Suppose that for all $g \in G$, $g \neq 1$, we have*

$$\sum_{\substack{\chi \neq 1, \\ \chi \text{ irreducible}}} \mu_\chi \chi(g) = 0, \quad \mu_\chi \in \mathbb{C},$$

where the summation varies over all non-trivial irreducible characters of G. Then $\mu_\chi = 0$ for all $\chi \neq 1$.

Conjecture 5.2.8 (Dickson's conjecture)**.** *Let s be a positive integer and $F_1, F_2, \ldots, F_s$ be s linear polynomials with integral coefficients and positive leading coefficient such that their product has no fixed prime divisor [1]. Then there exist infinitely many positive integers t such that $F_1(t), F_2(t), \ldots, F_s(t)$ are all primes.*

5.3 Proofs of the Main Theorems

For $1 \leq j \leq \ell$ and $q_j > 2$, consider the sets

$$S_j = \{1 < a_j < q_j/2 \mid (a_j, q_j) = 1\} \quad \text{and} \quad T_j = \{1 \leq a_j < q_j/2 \mid (a_j, q_j) = 1\}.$$

Throughout this section, we shall be using these notations.

[1] We say that the prime number p is a *fixed prime divisor* of a polynomial G if we have: $\forall t \in \mathbb{Z} : p | G(t)$.

5.3.1 Proof of Theorem 5.1.4

We first show that the set of Ramachandra units

$$\bigcup_{1\le j\le \ell} \{\xi_{a_j} \mid a_j \in S_j\}$$

is multiplicatively independent. For $\ell = 1$, it follows from the work of Ramachandra (see Theorem 5.2.4). Now suppose that

$$\bigcup_{1\le j< \ell} \{\xi_{a_j} \mid a_j \in S_j\}$$

is multiplicatively independent. If there exist $\alpha_{a_j} \in \mathbb{Z}$ for $a_j \in S_j$, $1 \le j \le \ell$ such that

$$\prod_{1\le j\le \ell} \prod_{a_j \in S_j} \xi_{a_j}^{\alpha_{a_j}} = 1,$$

then

$$\prod_{1\le j< \ell} \prod_{a_j \in S_j} \xi_{a_j}^{\alpha_{a_j}} = \prod_{a_\ell \in S_\ell} \xi_{a_\ell}^{-\alpha_{a_\ell}}. \tag{5.3.1}$$

Note that

$$\prod_{1\le j< \ell} \prod_{a_j \in S_j} \xi_{a_j}^{\alpha_{a_j}} = \prod_{a_\ell \in S_\ell} \xi_{a_\ell}^{-\alpha_{a_\ell}} \in \mathbb{Q}(\zeta_{q_1\cdots q_{\ell-1}}) \cap \mathbb{Q}(\zeta_{q_\ell}) = \mathbb{Q}. \tag{5.3.2}$$

Let us call this rational number β. If $\mathbf{F} = \mathbb{Q}(\zeta_{q_1\cdots q_\ell})$ and $N_{\mathbf{F}/\mathbb{Q}}(\alpha)$ denotes the norm of α of $\mathbf{F}$ over $\mathbb{Q}$, then taking $N_{\mathbf{F}/\mathbb{Q}}$ of the quantities on both sides of (5.3.1), we get that $\beta^{\varphi(q_1\cdots q_\ell)} = 1$ as $\varphi(q_1\cdots q_\ell)$ is even. This implies that $\beta = \pm 1$. Thus

$$\prod_{1\le j< \ell} \prod_{a_j \in S_j} \xi_{a_j}^{2\alpha_{a_j}} = \prod_{a_\ell \in S_\ell} \xi_{a_\ell}^{-2\alpha_{a_\ell}} = 1.$$

Applying induction hypothesis, we obtain $\alpha_{a_j} = 0$ for all $a_j \in S_j, 1 \le j \le \ell$. This implies that the set of real numbers $\bigcup_{1\le j\le \ell}\{\xi_{a_j} \mid a_j \in S_j\}$ is multiplicatively independent.

We now apply the above observation to complete the proof of Theorem 5.1.4. Let

$$C_j = \{\chi_j \bmod q_j \mid \chi_j(-1) = 1, \chi_j \neq 1\}$$

be the set of non-trivial even characters with periods q_j for $1 \le j \le \ell$. Suppose that there exist algebraic numbers α_{χ_j} for $\chi_j \in C_j, 1 \le j \le \ell$ such that

$$\sum_{1 \le j \le \ell} \sum_{\chi_j \in C_j} \alpha_{\chi_j} L(1, \chi_j) = 0. \tag{5.3.3}$$

Substituting (see Lemma 5.2.5)

$$L(1, \chi_j) = \delta_{\chi_j} \sum_{a_j \in S_j} \overline{\chi_j}(a_j) \log \xi_{a_j}$$

for $\chi_j \in C_j, 1 \le j \le \ell$ in (5.3.3), we obtain

$$\sum_{1 \le j \le \ell} \sum_{a_j \in S_j} \left(\sum_{\chi_j \in C_j} \alpha_{\chi_j} \delta_{\chi_j} \overline{\chi_j}(a_j) \right) \log \xi_{a_j} = 0.$$

Applying Baker's theorem (Theorem 3.3.1) and our observation about linear independence of Ramachandra units for $q_1, \cdots, q_\ell$, we get

$$\sum_{\chi_j \in C_j} \alpha_{\chi_j} \delta_{\chi_j} \overline{\chi_j}(a_j) = 0,$$

for $a_j \in S_j, 1 \le j \le \ell$. Since $\delta_{\chi_j} \neq 0$ (see Lemma 5.2.5), the even characters of $(\mathbb{Z}/q_j\mathbb{Z})^\times$ can be viewed as characters of the quotient group $(\mathbb{Z}/q_j\mathbb{Z})^\times/\{\pm 1\}$. As these characters are of dimension one and hence irreducible, applying Lemma 5.2.7, we have $\alpha_{\chi_j} = 0$ for $\chi_j \in C_j, 1 \le j \le \ell$. This completes the proof of Theorem 5.1.4.

5.3.2 Proof of Theorem 5.1.6

We first show that the set of real numbers

$$\bigcup_{1 \le j \le \ell} \{\cot(\frac{\pi a_j}{q_j}) \mid a_j \in T_j\}$$

is linearly independent over $\mathbb{Q}$. For $\ell = 1$, it follows from the work of Okada (see Theorem 5.2.2). Suppose that the set of real numbers

$$\bigcup_{1 \le j < \ell} \{\cot(\frac{\pi a_j}{q_j}) \mid a_j \in T_j\}$$

is linearly independent over $\mathbb{Q}$. If there exist rational numbers α_{a_j} for $a_j \in T_j, 1 \le j \le \ell$ such that

$$\sum_{1\le j\le \ell}\sum_{a_j\in T_j} \alpha_{a_j}\cot(\frac{\pi a_j}{q_j}) = 0,$$

then

$$\sum_{1\le j<\ell}\sum_{a_j\in T_j} \alpha_{a_j}\cot(\frac{\pi a_j}{q_j}) = -\sum_{a_\ell\in T_\ell} \alpha_{a_\ell}\cot(\frac{\pi a_\ell}{q_\ell}). \tag{5.3.4}$$

Since

$$-i\cot\frac{\pi a_j}{q_j} = \frac{\zeta_{q_j}^{a_j}+1}{\zeta_{q_j}^{a_j}-1} \in \mathbb{Q}(\zeta_{q_j}),$$

where $i=\sqrt{-1}$, it follows that

$$i\sum_{1\le j<\ell}\sum_{a_j\in T_j} \alpha_{a_j}\cot(\frac{\pi a_j}{q_j}) = -i\sum_{a_\ell\in T_\ell} \alpha_{a_\ell}\cot(\frac{\pi a_\ell}{q_\ell}) \in \mathbb{Q}(\zeta_{q_1\cdots q_{\ell-1}})\cap\mathbb{Q}(\zeta_{q_\ell}) = \mathbb{Q}.$$

Since a purely imaginary number is a rational number if and only if it is 0, we have

$$\sum_{1\le j<\ell}\sum_{a_j\in T_j} \alpha_{a_j}\cot(\frac{\pi a_j}{q_j}) = -\sum_{a_\ell\in T_\ell} \alpha_{a_\ell}\cot(\frac{\pi a_\ell}{q_\ell}) = 0.$$

Applying induction hypothesis, we get that $\alpha_{a_j}=0$ for all $a_j\in T_j, 1\le j\le \ell$. Hence the set of real numbers

$$\bigcup_{1\le j\le\ell}\{\cot(\frac{\pi a_j}{q_j}) \mid a_j\in T_j\}$$

is linearly independent over $\mathbb{Q}$.

We now apply the above observation to complete the proof of Theorem 5.1.6. Let

$$D_j = \{\chi_j \bmod q_j \mid \chi_j(-1) = -1\}$$

be the set of odd characters with periods q_j for $1\le j\le \ell$. Let $\mathbf{K}$ be as in Theorem 5.1.6. Suppose that there exist $\alpha_{\chi_j}\in \mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$ for $\chi_j\in D_j, 1\le j\le\ell$ such that

$$\sum_{1\le j\le\ell}\sum_{\chi_j\in D_j} \alpha_{\chi_j} L(1,\chi_j) = 0. \tag{5.3.5}$$

Substituting (see [37, 59])

$$L(1,\chi_j) = \frac{\pi}{q_j} \sum_{a_j \in T_j} \chi_j(a_j) \cot(\frac{\pi a_j}{q_j}), \tag{5.3.6}$$

for $\chi_j \in D_j, 1 \le j \le \ell$ in (6.4.2), we obtain

$$\sum_{1 \le j \le \ell} \sum_{a_j \in T_j} \left(\sum_{\chi_j \in D_j} \frac{\alpha_{\chi_j}}{q_j} \chi_j(a_j) \right) \cot(\frac{\pi a_j}{q_j}) = 0. \tag{5.3.7}$$

By given hypothesis and Theorem 1.4.2, the number fields $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$ and $\mathbb{Q}(\zeta_{q_1\cdots q_\ell})$ are linearly disjoint over $\mathbb{Q}$. Therefore $\mathbb{Q}$-linearly independent elements $i\cot(\frac{\pi a_j}{q_j})$ in (6.4.3) which belong to $\mathbb{Q}(\zeta_{q_1\cdots q_\ell})$ are also linearly independent over $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$. Since the coefficients of $\cot(\frac{\pi a_j}{q_j})$ in (6.4.3) belong to $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$, we have

$$\sum_{\chi_j \in D_j} \frac{\alpha_{\chi_j}}{q_j} \chi_j(a_j) = 0$$

for $a_j \in T_j, 1 \le j \le \ell$. Since all the characters in the set $D_j, 1 \le j \le \ell$ are of same parity, it follows that

$$\sum_{\chi_j \in D_j} \frac{\alpha_{\chi_j}}{q_j} \chi_j(a_j) = 0$$

for $a_j \in (\mathbb{Z}/q_j\mathbb{Z})^\times, 1 \le j \le \ell$. It then follows from linear independence of characters that $\alpha_{\chi_j} = 0$ for $\chi_j \in D_j, 1 \le j \le \ell$. This completes the proof of Theorem 5.1.6.

5.3.3 Proofs of Theorem 5.1.1 and Theorem 5.1.2

Note that Theorem 5.1.1 follows by considering $\mathbf{K} = \mathbb{Q}$ in Theorem 5.1.2. Hence it is sufficient to prove Theorem 5.1.2. It follows from Lemma 5.2.5 that for an even non-trivial Dirichlet character χ, the number $L(1,\chi)$ is a linear form in logarithms of positive real algebraic numbers. We know from (5.3.6) that for an odd character χ, the number $L(1,\chi)$ is an algebraic multiple of π. Then Lemma 4.2.2 implies that the space generated by $L(1,\chi)$ for non-trivial even χ do not intersect with the space generated by $L(1,\chi)$ for odd χ. Theorem 5.1.2 now follows by applying Theorem 5.1.4 and Theorem 5.1.6.

5.3.4 Proof of Corollary 6.1.10

Let us denote by $Y_{j,e} = \{f \in Y_{q_j}(\overline{\mathbb{Q}}) \mid f(-a) = f(a) \text{ for } 1 \le a < q\}$. Suppose that there exist $\alpha_{f_j} \in \overline{\mathbb{Q}}$ for $f_j \in Y_{j,e}, 1 \le j \le \ell$ such that

$$\sum_{1\le j\le \ell} \sum_{f_j \in Y_{j,e}} \alpha_{f_j} L(1, f_j) \;=\; 0.$$

Then

$$\sum_{1\le j\le \ell} L(1, F_j) \;=\; 0, \tag{5.3.8}$$

where $F_j = \sum_{f_j \in Y_{j,e}} \alpha_{f_j} f_j$. For $1 \le j \le \ell$, $f_j \in Y_{j,e}$ and hence F_j's are even Dirichlet type functions with periods q_j. Therefore we can write F_j as a linear combination of χ_j, where χ_j belong to the set $C_j = \{\chi \bmod q_j \mid \chi(-1) = 1, \chi \ne 1\}$. This implies that

$$L(1, F_j) = \sum_{\chi_j \in C_j} \beta_{\chi_j} L(1, \chi_j),$$

where β_{χ_j} are algebraic numbers. Substituting this expression in (5.3.8), we get

$$\sum_{1\le j\le \ell} \sum_{\chi_j \in C_j} \beta_{\chi_j} L(1, \chi_j) = 0.$$

Applying Theorem 5.1.4, we obtain $\beta_{\chi_j} = 0$ for $\chi_j \in C_j$, $1 \le j \le \ell$. Thus $L(1, F_j) = 0$ for $1 \le j \le \ell$. Using Theorem 5.2.6 (see also [50, Th. 6]), we then have $F_j = 0$ for $1 \le j \le \ell$. Since by hypothesis, the elements of $Y_{j,e}$ are $\overline{\mathbb{Q}}$ linearly independent, we have $\alpha_{f_j} = 0$ for $f_j \in Y_{j,e}$, $1 \le j \le \ell$. This completes the proof of Corollary 6.1.10.

5.3.5 Proof of Corollary 7.1.3

Let $\{f_p \mid p \text{ odd prime}\}$ be as in Corollary 7.1.3. Note that we can write f_p as a sum of an even function and an odd function, i.e., $f_p = f_{p,e} + f_{p,o}$, where

$$f_{p,e}(a) = \frac{f_p(a)+f_p(-a)}{2} \quad \text{and} \quad f_{p,o}(a) = \frac{f_p(a)-f_p(-a)}{2}$$

for $1 \le a \le q$. If the corollary is not true, then there exist a finite subset $\mathscr{P}$ of prime numbers and algebraic numbers α_p (not all zero) for $p \in \mathscr{P}$ such that

$$\sum_{p\in\mathscr{P}} \alpha_p L(1, f_p) = 0. \tag{5.3.9}$$

This implies that

$$\sum_{p\in\mathscr{P}} \alpha_p L(1, f_{p,e}) + \sum_{p\in\mathscr{P}} \alpha_p L(1, f_{p,o}) = 0.$$

Since each $L(1, f_{p,e})$ for $p \in \mathscr{P}$ can be written as algebraic linear combination of $L(1,\chi)$'s for non-trivial even Dirichlet characters χ with period p, it follows from Lemma 5.2.5 that the summation $\sum_{p\in\mathscr{P}} \alpha_p L(1, f_{p,e})$ is an algebraic linear combination of logarithms of positive algebraic numbers. Similarly each $L(1, f_{p,o})$ for $p \in \mathscr{P}$ can be written as algebraic linear combination of $L(1,\chi)$'s for odd Dirichlet characters χ with period p, we see that $\sum_{p\in\mathscr{P}} \alpha_p L(1, f_{p,o})$ is an algebraic multiple of π by identity (5.3.6). Now by applying Lemma 4.2.2, we have

$$\sum_{p\in\mathscr{P}} \alpha_p L(1, f_{p,e}) = -\sum_{p\in\mathscr{P}} \alpha_p L(1, f_{p,o}) = 0.$$

Since $f_{p,e}$ are non-zero even Dirichlet type functions with distinct prime periods $p \in \mathscr{P}$, we have $L(1, f_{p,e})$ are non-zero for $p \in \mathscr{P}$. Now applying Corollary 6.1.10, we have $\alpha_p = 0$ for $p \in \mathscr{P}$, a contradiction to (5.3.9). This completes the proof of Corollary 7.1.3.

5.3.6 Proof of Corollary 5.1.12

Let $\mathbf{K}$ be as in Corollary 5.1.12 and for $1 \le j \le \ell$, $Y_j(\mathbf{K})$ denotes $Y_{q_j}(\mathbf{K})$ for the sake of brevity. As in Corollary 7.1.3, let us write $f_j = f_{j,e} + f_{j,o}$, where

$$f_{j,e}(a) = \frac{f_j(a)+f_j(-a)}{2} \quad \text{and} \quad f_{j,o}(a) = \frac{f_j(a)-f_j(-a)}{2}$$

for $1 \le a \le q$ and $1 \le j \le \ell$. Suppose that there exist $\alpha_{f_j} \in \mathbf{K}$ for $f_j \in Y_j(\mathbf{K}), 1 \le j \le \ell$ such that

$$\sum_{1\le j\le \ell}\sum_{f_j\in Y_j(\mathbf{K})} \alpha_{f_j} L(1,f_j) = 0.$$

This implies that

$$\sum_{1\le j\le \ell}\sum_{f_j\in Y_j(\mathbf{K})} \alpha_{f_j} L(1,f_{j,e}) + \sum_{1\le j\le \ell}\sum_{f_j\in Y_j(\mathbf{K})} \alpha_{f_j} L(1,f_{j,o}) = 0. \tag{5.3.10}$$

Proceeding as in Corollary 7.1.3, we note that the first term in (5.3.10) is an algebraic linear combination of logarithms of positive algebraic numbers by Lemma 5.2.5 and the second term of (5.3.10) is an algebraic multiple of π by identity (5.3.6). Applying Lemma 4.2.2, we have

$$\sum_{1\le j\le \ell}\sum_{f_j\in Y_j(\mathbf{K})} \alpha_{f_j} L(1,f_{j,e}) = 0 \quad \text{and} \quad \sum_{1\le j\le \ell}\sum_{f_j\in Y_j(\mathbf{K})} \alpha_{f_j} L(1,f_{j,o}) = 0.$$

This implies that

$$\sum_{1\le j\le \ell} L(1,F_{j,e}) = 0 \quad \text{and} \quad \sum_{1\le j\le \ell} L(1,F_{j,o}) = 0, \tag{5.3.11}$$

where

$$F_{j,e} = \sum_{f_j\in Y_j(\mathbf{K})} \alpha_{f_j} f_{j,e} \quad \text{and} \quad F_{j,o} = \sum_{f_j\in Y_j(\mathbf{K})} \alpha_{f_j} f_{j,o}. \tag{5.3.12}$$

Since $F_{j,e}$'s are even Dirichlet type functions with distinct periods q_j for $1 \le j \le \ell$, applying Corollary 6.1.10, we have

$$F_{j,e} = 0 \quad \text{for } 1 \le j \le \ell. \tag{5.3.13}$$

Note that $F_{j,o}$'s are odd Dirichlet type functions with periods q_j for $1 \le j \le \ell$ with values in $\mathbf{K}$. Let V_j be the $\mathbf{K}(\zeta_{\varphi(q_j)})$ vector space of functions from $(\mathbb{Z}/q_j\mathbb{Z})^\times$ to $\mathbf{K}(\zeta_{\varphi(q_j)})$. Dirichlet characters with periods q_j are contained in V_j and they form a basis of V_j over $\mathbf{K}(\zeta_{\varphi(q_j)})$. Let D_j be the set of all odd Dirichlet characters with periods q_j. Since $F_{j,o}$ can be written as

$$F_{j,o} = \sum_{\chi_j \in D_j} \beta_{\chi_j} \chi_j$$

where $\beta_{\chi_j} \in \mathbf{K}(\zeta_{\varphi(q_j)})$ for $\chi_j \in D_j, 1 \le j \le \ell$, we have $L(1, F_{j,o}) = \sum_{\chi_j \in D_j} \beta_{\chi_j} L(1, \chi_j)$. Substituting this expression in (5.3.11), we have

$$\sum_{1 \le j \le \ell} \sum_{\chi_j \in D_j} \beta_{\chi_j} L(1, \chi_j) = 0.$$

Since by hypothesis, $\mathbf{K}(\zeta_{\varphi(q_1) \cdots \varphi(q_\ell)}) \cap \mathbb{Q}(\zeta_{q_1 \cdots q_\ell}) = \mathbb{Q}$, applying Theorem 5.1.6, we get $\beta_{\chi_j} = 0$ for $\chi_j \in D_j$, $1 \le j \le \ell$. This implies that

$$L(1, F_{j,o}) = 0$$

for $1 \le j \le \ell$. Theorem 5.2.6 then implies that

$$F_{j,o} = 0 \quad \text{for } 1 \le j \le \ell. \tag{5.3.14}$$

Then for $1 \le j \le \ell$, we have

$$\sum_{f_j \in Y_j(\mathbf{K})} \alpha_{f_j} f_j = \sum_{f_j \in Y_j(\mathbf{K})} \alpha_{f_j} (f_{j,e} + f_{j,o}) = F_{j,e} + F_{j,o} = 0.$$

Since by hypothesis, elements of $Y_j(\mathbf{K})$ are $\mathbf{K}$ linearly independent, we have $\alpha_{f_j} = 0$ for any $f_j \in Y_j(\mathbf{K}), 1 \le j \le \ell$. This completes the proof of Corollary 5.1.12.

CHAPTER 6

LINEAR INDEPENDENCE OF DIRICHLET L VALUES

6.1 Introduction and statements of Theorems

For a Dirichlet character χ modulo $q > 1$ and $s \in \mathbb{C}$, consider the Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

which converges absolutely for $\Re(s) > 1$. Further, it is holomorphic in this region. The study of irrationality of $L(k,\chi)$ for natural numbers $k > 1$ has an intriguing history starting from the work of Euler. He found a closed formula for $L(k,\chi_0)$ when χ_0 is the trivial character modulo $q \geq 1$ and k is even. When k is odd, it follows from the work of Ball and Rivoal [6] (see also [20, 39]) that there are infinitely many irrational numbers as $k > 1$ varies over odd natural numbers. When χ is the non-trivial character modulo 4, similar results were established by Rivoal and Zudilin

[62]. For arbitrary non-trivial character χ modulo q, infinitude of irrationality of $L(k,\chi)$ when $\chi(-1) = (-1)^{k+1}$ follows from the recent work of Fischler [19]. When $\chi(-1) = (-1)^k$, infinitude of irrationality of $L(k,\chi)$ is well known (see [58, Ch. VII, §2]).

In this article, we study linear independence of $L(k,\chi)$ when k is fixed and χ varies over Dirichlet characters modulo pairwise co-prime natural numbers. If we fix $k > 1$ and vary χ modulo a natural number $q > 2$, the question of linear independence of $L(k,\chi)$'s over $\mathbb{Q}$ was first investigated by Okada [59]. As noted by Murty-Saradha [54], this result can be extended over number fields which are disjoint to the qth cyclotomic field. To proceed further, we need to introduce some notations. For a natural number $k > 1$, let us set

$$X_{q,k} = \{L(k,\chi) \mid \chi \bmod q,\ \chi \neq \chi_0\}.$$

We can write $X_{q,k} = X_{q,k,e} \cup X_{q,k,o}$, where

$$\begin{aligned} X_{q,k,e} &= \{L(k,\chi) \mid \chi \bmod q,\ \chi(-1) = 1, \chi \neq \chi_0\} \\ \text{and} \quad X_{q,k,o} &= \{L(k,\chi) \mid \chi \bmod q,\ \chi(-1) = -1\}. \end{aligned} \tag{6.1.1}$$

In this set-up, Okada [59] (see also Murty-Saradha [54]) proved the following theorems.

Theorem 6.1.1. *(Okada [59]) Let $k \geq 1, q > 2$ be natural numbers and $\mathbf{K}$ be a number field with $\mathbf{K}(\zeta_{\varphi(q)}) \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then the numbers in the set $X_{q,2k+1,o}$ are linearly independent over $\mathbf{K}(\zeta_{\varphi(q)})$.*

Theorem 6.1.2. *(Okada [59]) Let $k \geq 1, q > 2$ be natural numbers and $\mathbf{K}$ be a number field with $\mathbf{K}(\zeta_{\varphi(q)}) \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then the numbers in the set $\{\zeta(2k)\} \cup X_{q,2k,e}$ are linearly independent over $\mathbf{K}(\zeta_{\varphi(q)})$.*

From now on, for an integer $q > 2$, we shall denote the maximal real subfield of $\mathbb{Q}(\zeta_q)$ by $\mathbb{Q}(\zeta_q)^+$. In this set-up, we have the following theorems.

Theorem 6.1.3. *For $1 \le j \le \ell$, let $q_j > 2$ be pairwise co-prime natural numbers and $k \ge 1$ be an integer. If $\mathbf{K}$ is a number field such that $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)}) \cap \mathbb{Q}(\zeta_{q_1\cdots q_\ell})^+ = \mathbb{Q}$, then the numbers in the set*

$$X_{q_1,2k+1,o} \cup \cdots \cup X_{q_\ell,2k+1,o}$$

are linearly independent over $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$.

Theorem 6.1.4. *For $1 \le j \le \ell$, let $q_j > 2$ be pairwise co-prime natural numbers and $k \ge 1$ be an integer. If $\mathbf{K}$ is a number field such that $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)}) \cap \mathbb{Q}(\zeta_{q_1\cdots q_\ell})^+ = \mathbb{Q}$, then the numbers in the set*

$$\{\zeta(2k)\} \cup X_{q_1,2k,e} \cup \cdots \cup X_{q_\ell,2k,e}$$

are linearly independent over $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$.

If one replace Dirichlet characters by arbitrary periodic arithmetic functions f with period $q > 1$ and consider the associated L-function

$$L(s,f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

for $s \in \mathbb{C}$ with $\Re(s) > 1$, then non-vanishing of $L(k,f)$ is intricately related to a conjecture of Chowla-Milnor [25]. For describing this conjecture, we need to introduce Hurwitz zeta function

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} ,$$

where x is a real number with $0 < x \leqslant 1$ and s is a complex number with $\Re(s) > 1$. It is easy to see that when $k > 1$, we have

$$L(k,f) = q^{-k} \sum_{a=1}^{q} f(a)\zeta(k,a/q) \tag{6.1.2}$$

and hence in particular, when χ_0 is the trivial character modulo $q \geq 2$,

$$\zeta(k) \prod_{\substack{p|q \\ p \text{ prime}}} (1 - p^{-k}) = L(k, \chi_0) = q^{-k} \sum_{\substack{1 \leq a < q \\ (a,q)=1}} \zeta(k, a/q). \tag{6.1.3}$$

For example, $\zeta(k, 1/2) = (2^k - 1)\zeta(k) \neq 0$, for all $k > 1$.

Remark 6.1.5. *Since $\zeta(s,x)$ extends analytically to the entire complex plane, apart from $s = 1$, where it has a simple pole with residue 1, we have by (6.1.2) that $L(s, f)$, for a periodic function f modulo q, extends meromorphically to the complex plane with a possible simple pole at $s = 1$ with residue $q^{-1} \sum_{a=1}^{q} f(a)$. Thus when $f = \chi$, a non-trivial Dirichlet character modulo q, the number $L(1, \chi)$ makes sense. See the articles [4, 23, 37, 50, 59] for linear independence of such values.*

P. Chowla and S. Chowla [13] were the first to study non-vanishing of $L(2, f)$ for arbitrary periodic functions f and made the following conjecture.

Conjecture 6.1.6. *(Chowla-Chowla) Let p be any prime and f be any rational valued periodic function with period p. Then $L(2, f) \neq 0$ except in the case when*

$$f(1) = f(2) = \cdots = f(p-1) = \frac{f(p)}{1 - p^2}.$$

Milnor [48] reformulated the conjecture of Chowla-Chowla as follows;

Conjecture 6.1.7. *(Milnor). For any integer $k > 1$ and prime p, the real numbers*

$$\zeta\big(k, 1/p\big), \;\; \zeta\big(k, 2/p\big), \cdots, \zeta\big(k, (p-1)/p\big)$$

are all linearly independent over $\mathbb{Q}$.

When q is not necessarily prime, Milnor suggested the following generalization of the Chowla conjecture.

Conjecture 6.1.8. *(Chowla-Milnor) Let $k > 1, q > 2$ be integers. Then the following $\varphi(q)$ real numbers*

$$\zeta(k, a/q) \quad with \;\; (a,q) = 1, \;\; 1 \le a < q$$

are linearly independent over $\mathbb{Q}$.

In relation to the Chowla-Milnor conjecture, we define the following linear spaces (see [25]).

Definition 6.1.9. *For a number field $\mathbf{K} \subset \mathbb{C}$ and integers $k > 1$, $q \ge 1$, the $\mathbf{K}$-vector space*

$$V_{\mathbf{K},k}(q) = \mathbf{K} - span\ of\ \{\zeta(k, a/q) : 1 \le a \le q, (a,q) = 1\}$$

is defined to be the Chowla-Milnor space for $\mathbf{K}$ and q. In particular, $V_{\mathbf{K},k}(1) = \mathbf{K}\zeta(k,1) = \mathbf{K}\zeta(k)$ and $V_{\mathbf{K},k}(2) = \mathbf{K}\zeta(k,1/2) = \mathbf{K}\zeta(k)$.

Conjecture 6.1.8 is equivalent to $\dim_{\mathbb{Q}}(V_{\mathbb{Q},k}(q)) = \varphi(q)$ for $q > 2$. We observe that $\sum_{d|q} V_{\mathbf{K},k}(d) = \sum_{a=1}^{q} \mathbf{K}\zeta(k, a/q)$.

For $q \ge 1$, we can write the space $V_{\mathbf{K},k}(q)$ as $V_{\mathbf{K},k}(q) = V^{+}_{\mathbf{K},k}(q) + V^{-}_{\mathbf{K},k}(q)$, where for $q > 2$

$$\begin{aligned} V^{\pm}_{\mathbf{K},k}(q) &= \sum_{\substack{1 \le a < q/2 \\ (a,q)=1}} \mathbf{K}\Big(\zeta(k, a/q) \pm (-1)^k \zeta(k, 1 - a/q)\Big) \\ \text{and} \qquad V^{\pm}_{\mathbf{K},k}(2) &= \mathbf{K}\zeta(k,1/2)(1 \pm (-1)^k), \qquad V^{\pm}_{\mathbf{K},k}(1) = \mathbf{K}\zeta(k,1)(1 \pm (-1)^k). \end{aligned}$$

For $q = 1,2$, we have $\dim_{\mathbf{K}}(V^{\pm}_{\mathbf{K},k}(1)) = \dim_{\mathbf{K}}(V^{\pm}_{\mathbf{K},k}(2)) = \frac{1}{2}(1 \pm (-1)^k)$.

For $q > 2$, it results from Okada's theorem 6.2.1 that $\dim_{\mathbb{Q}}(V^{+}_{\mathbb{Q},k}(q)) = \varphi(q)/2$.

Since $\dim_{\mathbb{Q}}(V^{-}_{\mathbb{Q},k}(q)) \le \varphi(q)/2$, Conjecture 6.1.8 is equivalent to $V^{+}_{\mathbb{Q},k}(q) \cap V^{-}_{\mathbb{Q},k}(q) = 0$ and $\dim_{\mathbb{Q}}(V^{-}_{\mathbb{Q},k}(q)) = \varphi(q)/2$. In this set-up, we have the following theorem.

Theorem 6.1.10. *For $1 \le j \le \ell$, let $q_j \ge 1$ be pairwise co-prime natural numbers and $k > 1$ be an integer. If $\mathbf{K}$ is a number field such that $\mathbf{K} \cap \mathbb{Q}(\zeta_{q_1 \cdots q_\ell})^+ = \mathbb{Q}$, then*

$$\dim_{\mathbf{K}}\left(\sum_{j=1}^{\ell} V_{\mathbf{K},k}^+(q_j)\right) = \sum_{j=1}^{\ell} \dim_{\mathbf{K}}(V_{\mathbf{K},k}^+(q_j)) - (\ell-1)\dim_{\mathbf{K}}(V_{\mathbf{K},k}^+(1)).$$

In particular,

$$\dim_{\mathbf{K}}\left(\sum_{j=1}^{\ell} V_{\mathbf{K},k}^+(q_j)\right) = \begin{cases} \sum_{j=1}^{\ell} \frac{\varphi(q_j)}{2} - \frac{\ell-1}{2}(1+(-1)^k) & \text{if } 2 < q_1, \cdots, q_\ell, \\ \sum_{j=2}^{\ell} \frac{\varphi(q_j)}{2} - \frac{\ell-2}{2}(1+(-1)^k) & \text{if } 2 = q_1 < q_2, \cdots, q_r. \end{cases}$$

When $\mathbf{K} = \mathbb{Q}$, Chowla-Milnor conjecture predicts that the dimension of $V_{\mathbb{Q},k}(q)$ over $\mathbb{Q}$ is equal to $\varphi(q)$. Here we have the following corollary.

Corollary 6.1.11. *For $1 \le j \le \ell$, let $q_j > 2$ be pairwise co-prime natural numbers and $k > 1$ be an integer. If $\mathbf{K}$ is a number field such that $\mathbf{K} \cap \mathbb{Q}(\zeta_{q_1 \cdots q_\ell})^+ = \mathbb{Q}$, then*

$$\frac{1}{2}\sum_{j=1}^{\ell} \varphi(q_j) - \frac{\ell-1}{2}(1+(-1)^k) \;\le\; \dim_{\mathbf{K}}\left(\sum_{j=1}^{\ell} V_{\mathbf{K},k}(q_j)\right) \;\le\; \sum_{j=1}^{\ell} \varphi(q_j) - (\ell-1).$$

Remark 6.1.12. *Let $k, q > 1$ be integers and χ_b, $b \in (\mathbb{Z}/q\mathbb{Z})^\times$, the Dirichlet characters modulo q. For a, b running over $(\mathbb{Z}/q\mathbb{Z})^\times$, we have*

$$q^k L(k, \chi_b) = \sum_{(a,q)=1} \chi_b(a)\zeta(k, a/q). \tag{6.1.4}$$

By the orthogonality relations satisfied by Dirichlet characters, the matrix $\left(\chi_b(a)\right)_{a,b}$ has inverse

$$\frac{1}{\varphi(q)}\left(\chi_b(a^{-1})\right)_{b,a}.$$

Let $\mathbf{K} \subset \mathbb{C}$ be a number field containing the $\varphi(q)$-th roots of unity, the $\mathbf{K}$-vector spaces

$$\sum_{(b,q)=1} \mathbf{K}L(k, \chi_b) \quad \text{and} \quad \sum_{(a,q)=1} \mathbf{K}\zeta(k, a/q) = V_{\mathbf{K},k}(q)$$

are equal. Furthermore, it follows from (6.1.4) *that*

$$\sum_{\chi(-1)=\pm(-1)^k} \mathbf{K}L(k,\chi) = V^{\pm}_{\mathbf{K},k}(q). \tag{6.1.5}$$

Definition 6.1.13. *Let* $\mathbf{K}$ *be a number field,* V *and* W *be two* $\mathbf{K}$*-vector spaces in* $\mathbb{C}$*. We define the product* VW *as the* $\mathbf{K}$*-span of the set of numbers* vw *with* $v \in V$ *and* $w \in W$*.*

Following Hamahata [29], we consider generalized Chowla-Milnor spaces.

Definition 6.1.14. *Let* $k_1, \cdots, k_r > 1$ *and* $q_1, \cdots, q_r \geq 1$ *be integers. Set* $\vec{k} = (k_1, \cdots, k_r)$ *and* $\vec{q} = (q_1, \cdots, q_r)$*. For a number field* $\mathbf{K} \subset \mathbb{C}$*, the generalized Chowla-Milnor space is defined by*

$$V_{\mathbf{K},\vec{k}}(\vec{q}) = \mathbf{K} - \mathit{span\ of}\ \{\zeta(k_1, a_1/q_1)\cdots\zeta(k_r, a_r/q_r)\ :\ 1 \leq a_i \leq q_i, (a_i, q_i) = 1, 1 \leq i \leq r\}.$$

We observe that $V_{\mathbf{K},\vec{k}}(\vec{q}) = \prod_{i=1}^{r} V_{\mathbf{K},k_i}(q_i)$ *and we define* $V^{+}_{\mathbf{K},\vec{k}}(\vec{q}) = \prod_{i=1}^{r} V^{+}_{\mathbf{K},k_i}(q_i)$*.*

In 2020, Hamahata proved the following theorem.

Theorem 6.1.15. *(Hamahata [29]) Let* $q_1, \cdots, q_r$ *be pairwise co-prime integers,* $k_1, \cdots, k_r > 1$ *be positive integers and* $\vec{k}$*,* $\vec{q}$ *be as in Definition 6.1.14. If* $\mathbf{K}$ *is a number field such that* $\mathbf{K} \cap \mathbb{Q}(\zeta_{q_1,\cdots,q_r})^{+} = \mathbb{Q}$*, then*

$$\dim_{\mathbf{K}} V_{\mathbf{K},\vec{k}}(\vec{q})\ \geq\ 2^{-r} \prod_{i=1}^{r} \varphi(q_i).$$

Here we have the following extensions of Hamahata's theorem.

Theorem 6.1.16. *Let* $q_{t,j} \geq 1$ *be inegers for* $1 \leq j \leq \ell$ *and* $1 \leq t \leq r$*. Set* $q_t = \prod_{j=1}^{\ell} q_{t,j}$ *and* $\vec{q}_j = (q_{1,j}, \cdots, q_{r,j})$*. Assume* $q_1, \cdots, q_r$ *are pairwise co-prime integers,* $k_1, \cdots, k_r > 1$ *be positive integers and* $\vec{k}$*,* $\vec{q}$ *be as in Definition 6.1.14. If* $\mathbf{K}$ *is a number field such that*

$\mathbf{K} \cap \mathbb{Q}(\zeta_{q_1,\cdots,q_r})^+ = \mathbb{Q}$, *then*

$$\dim_{\mathbf{K}} \left(\sum_{j=1}^{\ell} V^+_{\mathbf{K},\vec{k}}(\vec{q}_j) \right) = \begin{cases} 2^{-r} \sum_{j=1}^{\ell} \prod_{t=1}^{r} \varphi(q_{t,j}) & \text{when at least one } k_t \text{ is odd,} \\ 2^{-r} \sum_{j=1}^{\ell} \prod_{t=1}^{r} \varphi(q_{t,j}) - \ell + 1 & \text{when all } k_t \text{ are even.} \end{cases}$$

Theorem 6.1.17. *Let $r, \ell_1, \ldots, \ell_r$ and $q_{t,j} \geq 1$ be positive integers for $1 \leq j \leq \ell_t$ and $1 \leq t \leq r$. Set $q_t = \prod_{j=1}^{\ell_t} q_{t,j}$ and $\vec{q}_j = (q_{1,j}, \cdots, q_{r,j})$. Assume $q_1, \cdots, q_r$ are pairwise co-prime integers, $k_1, \cdots, k_r > 1$ be positive integers and $\vec{k}$ be as in Definition 6.1.14. If $\mathbf{K}$ is a number field such that $\mathbf{K} \cap \mathbb{Q}(\zeta_{q_1,\cdots,q_r})^+ = \mathbb{Q}$, then*

$$\dim_{\mathbf{K}} \left(\prod_{t=1}^{r} \sum_{j=1}^{\ell_t} V^+_{\mathbf{K},k_t}(q_{t,j}) \right) = 2^{-r} \prod_{t=1}^{r} \left(\sum_{j=1}^{\ell_t} \varphi(q_{t,j}) - (\ell_t - 1)(1 + (-1)^{k_t}) \right).$$

Remark 6.1.18. *Let $[q_1, \cdots, q_\ell]$ denote the least common multiple of $q_1, \cdots, q_\ell$. The number fields $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$ and $\mathbb{Q}(\zeta_{q_1\cdots q_\ell})^+$ can be replaced by $\mathbf{K}(\zeta_{[\varphi(q_1),\cdots,\varphi(q_\ell)]})$ and $\mathbb{Q}(\zeta_{[q_1,\cdots,q_\ell]})^+$ respectively in Theorems 7.1.1, 7.1.2, 6.1.10, 6.1.16 and Corollary 7.1.3.*

The article is organized as follows: in §2 we list required results needed for our proofs, in §3 we derive main propositions and a corollary which are extensions of Okada's theorem (as well as extensions of Murty-Saradha) and a theorem of Hamahata. Finally in the last section, we complete the proofs of Theorems 7.1.1, 7.1.2, 6.1.10, 6.1.16, 6.1.17 and Corollary 7.1.3.

6.2 Preliminaries

In this section, we fix some notations and state the results which will be used in the proofs of the main theorems. When p is an odd prime number, it is a result of Chowla

[12] that the set of numbers

$$\{\cot(2\pi a/p) \mid 1 \le a \le (p-1)/2\}$$

are linearly independent over $\mathbb{Q}$. This result was reproved by various authors (see for instance [30, 34]). In 1981, Okada [59] (see also Wang [71]) extended Chowla's theorem to natural numbers $q > 2$. In the same article, he also considered higher order derivatives of cotangent function. More precisely, Okada [59] proved the following theorem. In order to state the theorem, let us denote $\frac{d^{k-1}}{dz^{k-1}}(\cot z)|_{z=z_0}$ by $\cot^{(k-1)}(z_0)$.

Theorem 6.2.1. *Let k and q be positive integers with $k > 0$ and $q > 2$. Let T be a set of $\varphi(q)/2$ representatives modulo q such that $T \cup (-T)$ is a complete set of co-prime residues modulo q. Then the set of real numbers $\{\cot^{(k-1)}(\pi a/q) \mid a \in T\}$ is linearly independent over $\mathbb{Q}$.*

Using Galois theory, Girstmair [22] gave an alternate proof for $\mathbb{Q}$ linear independence of derivatives of cotangent function. Murty-Saradha [54] noticed that Okada's result can be extended to any number field $\mathbf{K}$ provided $\mathbf{K} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. We note that the condition $\mathbf{K} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$ in Murty-Saradha's result can be replaced by $\mathbf{K} \cap \mathbb{Q}(\zeta_q)^+ = \mathbb{Q}$. Recently Hamahata [29] derived the following multi-dimensional generalization of the above result.

Theorem 6.2.2. *Let $k_1, \cdots, k_r \ge 1$ be natural numbers and $q_1, \cdots, q_r > 2$ be pairwise co-prime natural numbers. For $1 \le i \le r$, let T_i be a set of $\varphi(q_i)/2$ representatives modulo q_i such that $T_i \cup (-T_i)$ is a complete set of co-prime residues modulo q_i. Set $q = q_1 \cdots q_r$. If $\mathbf{K}$ is a number field with $\mathbf{K} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, then the $\varphi(q)/2^r$ numbers*

$$\prod_{i=1}^{r} \cot^{(k_i-1)}(\pi a_i/q_i), \quad a_i \in T_i, \quad i = 1, \cdots, r,$$

are linearly independent over $\mathbf{K}$.

In the next section, we extend Theorems 6.2.1, 6.2.2 for a finite set of pairwise co-prime natural numbers $q_1, \cdots, q_\ell > 2$.

6.3 Requisite Propositions

We first relate the vector spaces $V^+_{\mathbb{Q},k}(q)$ to $\mathbb{Q}$-vector subspaces of cyclotomic fields, via the cotangent values mentioned in the previous section.

Indeed, since $\pi\cot(\pi z)$ is the logarithmic derivative of $\sin(\pi z) = z\prod_{m=1}^{\infty}(1 - z^2/m^2)$, one checks for $k > 1$ that

$$\pi^k \cot^{(k-1)}(\pi z) = (-1)^{k-1}(k-1)!\left(\frac{1}{z^k} + \sum_{m=1}^{\infty}\left(\frac{1}{(z+m)^k} + \frac{1}{(z-m)^k}\right)\right)$$

and, evaluating at $z = a/q$ with $a/q \notin \mathbb{Z}\cup\{\infty\}$, it follows

$$\zeta(k, a/q) + (-1)^k \zeta(k, 1-a/q) = \frac{\pi^k(-1)^{k-1}}{(k-1)!}\cot^{(k-1)}(\pi a/q). \tag{6.3.1}$$

Note that for $z \notin \mathbb{Z}$, we have (see [37])

$$\cot^{(k-1)}(\pi z) = \sum_{\substack{a,b\geq 0\\ a+2b=k}} \beta^{(k)}_{a,b}\cot^a \pi z\,(1+\cot^2\pi z)^b, \tag{6.3.2}$$

where $\beta^{(k)}_{a,b} \in \mathbb{Z}$ and also

$$-i\cot\frac{\pi a}{q} = \frac{\zeta_q^a + 1}{\zeta_q^a - 1} \in \mathbb{Q}(\zeta_q), \tag{6.3.3}$$

where $i = \sqrt{-1}$. We then observe that

$$\begin{aligned} (i\pi)^{-k}V^+_{\mathbb{Q},k}(q) &= \sum_{\substack{1\leq a\leq q/2\\ (a,q)=1}} \mathbb{Q}(i^k\cot^{(k-1)}(\pi a/q)) \subset \mathbb{Q}(\zeta_q)\cap(i^k\mathbb{R}) \\ \text{and} \qquad & (\zeta_q - \zeta_q^{-1})^k i^k \cot^{(k-1)}(\pi a/q) \in \mathbb{Q}(\zeta_q)^+. \end{aligned}$$

Since the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q}(\zeta_q)\cap(i^k\mathbb{R})$ is $\varphi(q)/2$, and $\dim_{\mathbb{Q}} V^+_{\mathbb{Q},k}(q) = \varphi(q)/2$ by Okada's theorem 6.2.1, we have

$$\mathbb{Q}(\zeta_q)\cap(i^k\mathbb{R}) = (i\pi)^{-k}V^+_{\mathbb{Q},k}(q) \tag{6.3.4}$$

for $q > 2$. It is easy to see that the above relation also holds for $q = 1, 2$.

Lemma 6.3.1. *If $k > 1$, q, q_1, $q_2 \geq 1$ are positive integers and $d = (q_1, q_2)$, then*

$$(i\pi)^{-k} V^+_{\mathbb{Q},k}(q) = \mathbb{Q}(\zeta_q) \cap (i^k \mathbb{R}),\ V^+_{\mathbb{Q},k}(q) \subset V^+_{\mathbb{Q},k}(q_1) \text{ if } q|q_1 \text{ and } \quad V^+_{\mathbb{Q},k}(q_1) \cap V^+_{\mathbb{Q},k}(q_2) = V^+_{\mathbb{Q},k}(d).$$

Proof. Identity (6.3.4) asserts precisely the first statement

$$\mathbb{Q}(\zeta_q) \cap (i^k \mathbb{R}) = (i\pi)^{-k} V^+_{\mathbb{Q},k}(q).$$

If $q|q_1$, we have $\mathbb{Q}(\zeta_q) \subset \mathbb{Q}(\zeta_{q_1})$. We also know from Galois theory that

$$\mathbb{Q}(\zeta_{q_1}) \cap \mathbb{Q}(\zeta_{q_2}) = \mathbb{Q}(\zeta_d). \tag{6.3.5}$$

Then the second and the third results follow by intersecting both sides of the inclusion and Equation (6.3.5) with $i^k \mathbb{R}$ and applying (6.3.4). ■

Proposition 6.3.2. *For an integer $k > 1$ and pairwise co-prime positive integers $q_1, \cdots, q_\ell$, the kernel of the surjective map*

$$\begin{array}{ccc} \oplus_{j=1}^{\ell} V^+_{\mathbb{Q},k}(q_j) & \longrightarrow & \sum_{j=1}^{\ell} V^+_{\mathbb{Q},k}(q_j) \\ (x_1, \cdots, x_\ell) & \longmapsto & x_1 + \cdots + x_\ell \end{array}$$

is 0 if k is odd. When k is even, the kernel of the above map is the $\mathbb{Q}$-vector space $V^+_{\mathbb{Q},k}(1)^{\ell-1}$ of dimension $\ell - 1$ which is parametrised as

$$(z_1, \cdots, z_{\ell-1}) \in \mathbb{Q}^{\ell-1} \longmapsto (z_1\zeta(k,1), \cdots, z_{\ell-1}\zeta(k,1), -(z_1 + \cdots + z_{\ell-1})\zeta(k,1)) \in \oplus_{j=1}^{\ell} V^+_{\mathbb{Q},k}(q_j).$$

Proof. The kernel of the map consists of elements $(x_1, \cdots, x_\ell)$ where $x_j \in V^+_{\mathbb{Q},k}(q_j) \cap \sum_{t \neq j} V^+_{\mathbb{Q},k}(q_t)$ for all $j = 1, \cdots, \ell$. By Lemma 6.3.1, $\sum_{t \neq j} V^+_{\mathbb{Q},k}(q_t) \subset V^+_{\mathbb{Q},k}(\prod_{t \neq j} q_t)$ and, since q_j is co-prime to $\prod_{t \neq j} q_t$, it also implies $x_j \in V^+_{\mathbb{Q},k}(1)$ for $j = 1, \cdots, \ell$. ■

We will need a multivariate extension of this result. To do this, we first prove another lemma.

Lemma 6.3.3. *Let $k_1, \cdots, k_r > 1$ be integers and $q_{1,1}, \cdots, q_{r,1}, q_{1,2}, \cdots, q_{r,2}$ be positive integers which are pairwise co-prime. Set $\vec{k} = (k_1, \cdots, k_r)$ and $\vec{q}_j = (q_{1,j}, \cdots, q_{r,j})$ for $j = 1, 2$, then*

$$V^+_{\mathbb{Q},\vec{k}}(\vec{q}_1) \cap V^+_{\mathbb{Q},\vec{k}}(\vec{q}_2) = V^+_{\mathbb{Q},\vec{k}}(\vec{1}).$$

Proof. Applying Lemma 6.3.1, it is easy to see that

$$V^+_{\mathbb{Q},\vec{k}}(\vec{1}) \subseteq V^+_{\mathbb{Q},\vec{k}}(\vec{q}_1) \cap V^+_{\mathbb{Q},\vec{k}}(\vec{q}_2).$$

Set $q_j = \prod_{t=1}^r q_{t,j}$ and $k = \sum_{t=1}^r k_t$. Again Lemma 6.3.1 shows that $(i\pi)^{-k} V^+_{\mathbb{Q},\vec{k}}(\vec{q}_j) \subset \mathbb{Q}(\zeta_{q_j})$ for $j = 1, 2$ and, since q_1 and q_2 are relatively prime, the intersection is contained in $\mathbb{Q}$. Now for $j = 1$ or 2, if some rational number α belongs to $(i\pi)^{-k} V^+_{\mathbb{Q},\vec{k}}(\vec{q}_j)$, we have for each $t = 1, \cdots, r$, a linear relation over $\mathbb{Q}(\zeta_{\prod_{h\neq t} q_{h,j}})$ between α and the elements of a basis (over $\mathbb{Q}$) of $(i\pi)^{-k_t} V^+_{\mathbb{Q},k_t}(q_{t,j})$. The field $\mathbb{Q}(\zeta_{\prod_{h\neq t} q_{h,j}})$ is linearly disjoint from $\mathbb{Q}(\zeta_{q_{t,j}})$ because $q_{t,j}$ is co-prime to $\prod_{h\neq t} q_{h,j}$. Thus the above linear relation over $\mathbb{Q}(\zeta_{\prod_{h\neq t} q_{h,j}})$ implies a linear relation over $\mathbb{Q}$, which involves α, since the elements of a basis of $(i\pi)^{-k_t} V^+_{\mathbb{Q},k_t}(q_{t,j})$ are linearly independent over $\mathbb{Q}$. Hence $\alpha \in (i\pi)^{-k_t} V^+_{\mathbb{Q},k_t}(q_{t,j})$ for all $t = 1, \cdots, r$ and $j = 1, 2$. Applying Lemma 6.3.1, we see that $\alpha \in (i\pi)^{-k_t} V^+_{\mathbb{Q},k_t}(1)$. Note that $(i\pi)^{-k_t} V^+_{\mathbb{Q},k_t}(1)$ is 0 if k_t is odd and $\mathbb{Q}$ if k_t is even. Therefore, if $\alpha \neq 0$ then all k_t must be even and

$$V^+_{\mathbb{Q},\vec{k}}(\vec{q}_1) \cap V^+_{\mathbb{Q},\vec{k}}(\vec{q}_2) \subseteq \prod_{t=1}^r V^+_{\mathbb{Q},k_t}(1) = \begin{cases} 0 & \text{when at least one } k_t \text{ is odd,} \\ \mathbb{Q} \prod_{t=1}^r \zeta(k_t, 1) & \text{when all } k_t \text{ are even,} \end{cases}$$

which is equal to $V^+_{\mathbb{Q},\vec{k}}(\vec{1})$. ■

Proposition 6.3.4. *For integers $k_1,\cdots,k_r > 1$ and pairwise co-prime positive integers $q_{t,j}$, $t = 1,\cdots,r$, $j = 1,\cdots,\ell$, we define $\vec{k} = (k_1,\cdots,k_r)$ and $\vec{q}_j = (q_{1,j},\cdots,q_{r,j})$. Then the kernel of the surjective map*

$$\begin{array}{ccc} \oplus_{j=1}^{\ell} V^+_{\mathbb{Q},\vec{k}}(\vec{q}_j) & \longrightarrow & \sum_{j=1}^{\ell} V^+_{\mathbb{Q},\vec{k}}(\vec{q}_j) \\ (x_1,\cdots,x_\ell) & \longmapsto & x_1+\cdots+x_\ell \end{array}$$

is 0 if at least one k_t is odd and is the $\mathbb{Q}$-vector space $\left(V^+_{\mathbb{Q},\vec{k}}(\vec{1})\right)^{\ell-1}$ of dimension $\ell-1$ parametrised as

$$\begin{array}{ccc} \mathbb{Q}^{\ell-1} & \longrightarrow & \oplus_{j=1}^{\ell} V^+_{\mathbb{Q},\vec{k}}(\vec{q}_j) \\ (z_1,\cdots,z_{\ell-1}) & \longmapsto & \left(z_1\prod_{t=1}^r \zeta(k_t,1),\cdots,z_{\ell-1}\prod_{t=1}^r \zeta(k_t,1),-(z_1+\cdots+z_{\ell-1})\prod_{t=1}^r \zeta(k_t,1)\right) \end{array}$$

if all $k_1,\cdots,k_r$ are even.

Proof. The kernel of the map consists of elements $(x_1,\cdots,x_\ell)$, where

$$x_j \in V^+_{\mathbb{Q},\vec{k}}(\vec{q}_j) \cap \sum_{h\neq j} V^+_{\mathbb{Q},\vec{k}}(\vec{q}_h)$$

for $j = 1,\cdots,\ell$. By Lemma 6.3.1, we have $\sum_{h\neq j} V^+_{\mathbb{Q},\vec{k}}(\vec{q}_h) \subset V^+_{\mathbb{Q},\vec{k}}(\vec{Q})$, where $\vec{Q} = (Q_1,\cdots,Q_r)$ with $Q_t = \prod_{h\neq j} q_{t,h}$. Since $q_{1,j},\cdots,q_{r,j},Q_1,\cdots,Q_r$ are pairwise co-prime, Lemma 6.3.3 implies that $x_j = 0$ if at least one k_t is odd and $x_j \in \prod_{t=1}^r V^+_{\mathbb{Q},k_t}(1) = \mathbb{Q}\prod_{t=1}^r \zeta(k_t,1)$ if all $k_1,\cdots,k_r$ are even, for $j = 1,\cdots,\ell$. ■

For $1 \le j \le \ell$ and $q_j > 2$, consider the sets

$$S_{q_j} = \{1 < a_j < q_j/2 \ : \ (a_j,q_j) = 1\} \quad \text{and} \quad T_{q_j} = \{1 \le a_j < q_j/2 \ : \ (a_j,q_j) = 1\},$$

$$U_j(k) = \begin{cases} T_{q_j} & \text{when } k \text{ is odd,} \\ T_{q_1} & \text{when } j = 1 \text{ and } k \text{ is even,} \\ S_{q_j} & \text{when } j \neq 1 \text{ and } k \text{ is even.} \end{cases} \tag{6.3.6}$$

We have the following proposition about linear independence of derivatives of cotangent function evaluated at certain rational points. It forms a basis for the space $\sum_{j=1}^{\ell} V^{+}_{\mathbf{K},k}(q_j)$.

Proposition 6.3.5. *For $1 \le j \le \ell$, let $q_j > 2$ be pairwise co-prime natural numbers and $k > 1$ be an integer. If $\mathbf{K}$ is a number field such that $\mathbf{K} \cap \mathbb{Q}(\zeta_{q_1 \cdots q_\ell})^+ = \mathbb{Q}$, then the set of real numbers*

$$\bigcup_{1\le j\le \ell} \left\{ \cot^{(k-1)}\left(\frac{\pi a_j}{q_j}\right) \;:\; a_j \in U_j(k) \right\}$$

is linearly independent over $\mathbf{K}$.

Proof. By (6.3.1), the numbers $\bigcup_{1\le j\le \ell}\left\{\pi^k \cot^{(k-1)}\left(\frac{\pi a_j}{q_j}\right) \;:\; a_j \in T_{q_j}\right\}$ span $\sum_{j=1}^{\ell} V^{+}_{\mathbf{K},k}(q_j)$. When k is even and $q > 2$ is any integer, using (6.1.3) and replacing k by $2k$, we have

$$\zeta(2k,1) \;=\; \alpha \sum_{\substack{1\le a<q/2\\(a,q)=1}} (\zeta(2k,a/q)+\zeta(2k,1-a/q)) \;=\; \alpha \sum_{\substack{1\le a<q/2\\(a,q)=1}} \pi^{2k}\cot^{(2k-1)}\left(\frac{\pi a}{q}\right),$$

where α is some non zero rational number. It implies that, removing one arbitrary element from any $\ell-1$ sets of the form $\left\{\pi^{2k}\cot^{(2k-1)}\left(\frac{\pi a_j}{q_j}\right) \;:\; a_j \in T_{q_j}\right\}$ in the union

$$\bigcup_{1\le j\le \ell} \left\{ \pi^{2k}\cot^{(2k-1)}\left(\frac{\pi a_j}{q_j}\right) \;:\; a_j \in T_{q_j} \right\}$$

does not change the space they span, that is $\sum_{j=1}^{\ell} V^{+}_{\mathbf{K},2k}(q_j)$. Whatever the value of $k > 1$ is, the number of generators left is equal to the dimension of this span. Thus these generators must be linearly independent. ■

We have the following corollary of Proposition 6.3.5 which gives a basis of the $\mathbf{K}$ vector space $\sum_{j=1}^{\ell} V^{+}_{\mathbf{K},\vec{k}}(\vec{q}_j)$.

Corollary 6.3.6. *Let $q_{t,j} > 2, T_{q_{t,j}}$ be defined as before Proposition 6.3.5 and $q_t = \prod_{j=1}^{\ell} q_{t,j}$. Assume $q_1, \cdots, q_r$ are pairwise co-prime and $k_1, \cdots, k_r$ be positive integers. Suppose that*

$$U_j(\vec{k}) = \begin{cases} \prod_{t=1}^{r} T_{q_{t,j}} & \text{when one of the } k_h \text{ is odd,} \\ \prod_{t=1}^{r} T_{q_{t,1}} & \text{when } j=1 \text{ and all the } k_h \text{ are even,} \\ \prod_{t=1}^{r} T_{q_{t,j}} \setminus \{(1,\cdots,1)\} & \text{when } j \neq 1 \text{ and all the } k_h \text{ are even.} \end{cases}$$

If $\mathbf{K}$ is a number field such that $\mathbf{K} \cap \mathbb{Q}(\zeta_{q_1 \cdots q_r})^+ = \mathbb{Q}$, then the set of real numbers

$$\bigcup_{j=1}^{\ell} \left\{ \prod_{t=1}^{r} \cot^{(k_t-1)}\left(\frac{\pi a_t}{q_{t,j}}\right) \;:\; (a_1, \cdots, a_r) \in U_j(\vec{k}) \right\}$$

is linearly independent over $\mathbf{K}$.

Proof. We will use induction on r to complete the proof. When $r = 1$ and $k_1 = 1$, the result follows from the work of the first and second author [23]. If $r = 1$ and $k_1 > 1$, result follows from Proposition 6.3.5. Now assume that the corollary is true for any natural number strictly less than r. Set $q = q_1 \cdots q_r$. Suppose that there exist rational numbers $\alpha_{j,a_1,\cdots,a_r}$ such that

$$\sum_{j=1}^{\ell} \sum_{a_1} \cdots \sum_{a_r} \alpha_{j,a_1,\cdots,a_r} \prod_{t=1}^{r} i^{k_t} \cot^{(k_t-1)}\left(\frac{\pi a_t}{q_{t,j}}\right) = 0,$$

where $(a_1, \cdots, a_r)$ runs over the elements of $U_j(\vec{k})$. This implies that

$$\sum_{j=1}^{\ell} \sum_{a_r} \left(\sum_{a_1} \cdots \sum_{a_{r-1}} \alpha_{j,a_1,\cdots,a_r} \prod_{t=1}^{r-1} i^{k_t} \cot^{(k_t-1)}\left(\frac{\pi a_t}{q_{t,j}}\right)\right) i^{k_r} \cot^{(k_r-1)}\left(\frac{\pi a_r}{q_{r,j}}\right) = 0.$$

Since $\mathbb{Q}(\zeta_{q_1 \cdots q_{r-1}}) \cap \mathbb{Q}(\zeta_{q_r}) = \mathbb{Q}$, Proposition 6.3.5 implies

$$\sum_{a_1} \cdots \sum_{a_{r-1}} \alpha_{j,a_1,\cdots,a_r} \prod_{t=1}^{r-1} i^{k_t} \cot^{(k_t-1)}\left(\frac{\pi a_t}{q_{t,j}}\right) = 0,$$

for $1 \le j \le \ell$ and $(a_1, \cdots, a_r) \in U_j(\vec{k})$. By induction hypothesis, $\alpha_{j,a_1,\cdots,a_r} = 0$ for all $1 \le j \le \ell$ and $(a_1, \cdots, a_r) \in U_j(\vec{k})$. Therefore, the set of real numbers

$$\bigcup_{j=1}^{\ell} \left\{ \prod_{t=1}^{r} i^{k_t} \cot^{(k_t-1)}\left(\frac{\pi a_t}{q_{t,j}}\right) \,:\, (a_1, \cdots, a_r) \in U_j(\vec{k}) \right\}$$

is linearly independent over $\mathbb{Q}$. It implies that

$$\bigcup_{j=1}^{\ell} \left\{ \prod_{t=1}^{r} i^{k_t} (\zeta_q - \zeta_q^{-1})^{k_t} \cot^{(k_t-1)}\left(\frac{\pi a_t}{q_{t,j}}\right) \,:\, (a_1, \cdots, a_r) \in U_j(\vec{k}) \right\}$$

is linearly independent over $\mathbb{Q}$. As in Proposition 6.3.5, the numbers inside the products belong to $\mathbb{Q}(\zeta_q)^+$ and by given hypothesis, $\mathbf{K}$ and $\mathbb{Q}(\zeta_q)^+$ are linearly disjoint. Hence the numbers in the union are linearly independent over $\mathbf{K}$ as well. It implies that

$$\bigcup_{j=1}^{\ell} \left\{ \prod_{t=1}^{r} \cot^{(k_t-1)}\left(\frac{\pi a_t}{q_{t,j}}\right) \,:\, (a_1, \cdots, a_r) \in U_j(\vec{k}) \right\}$$

is linearly independent over $\mathbf{K}$. ■

Remark 6.3.7. *Let χ be a character modulo q. If χ has the same parity as k (*i.e.* $\chi(-1) = (-1)^k$), then*

$$\frac{\pi^k (-1)^{k-1}}{q^k (k-1)!} \sum_{\substack{1 \le a < q/2, \\ (a,q)=1}} \chi(a) \cot^{(k-1)}(\pi a/q) = q^{-k} \sum_{(a,q)=1} \chi(a) \zeta(k, a/q) = L(k, \chi).$$

*If χ and k have different parity (*i.e. $\chi(-1) = (-1)^{k-1}$*), then*

$$\sum_{\substack{1 \le a < q, \\ (a,q)=1}} \chi(a) \cot^{(k-1)}(\pi a/q) = 0$$

since $\cot^{(k-1)}(-\pi a/q) = (-1)^k \cot^{(k-1)}(\pi a/q)$.

6.4 Proofs of the Main Theorems

6.4.1 Proof of Theorem 6.1.10

Since, by hypothesis, the fields $\mathbf{K}$ and $\mathbb{Q}(\zeta_{q_1\cdots q_\ell})^+$ are linearly disjoint, it suffices to determine the dimension of the $\mathbb{Q}$-vector space $\sum_{j=1}^{\ell}(i\pi)^{-k}V_{\mathbb{Q},k}^+(q_j)$. But, this is the dimension of $\oplus_{j=1}^{\ell}(i\pi)^{-k}V_{\mathbb{Q},k}^+(q_j)$ minus the dimension of the kernel of the map shown in Proposition 6.3.2. This gives the first equality, the remaining ones coming from

$$\dim_{\mathbb{Q}}(V_{\mathbb{Q},k}^+(q_j)) = \varphi(q_j)/2 \quad \text{and} \quad \dim_{\mathbb{Q}}(V_{\mathbb{Q},k}^+(1)) = \dim_{\mathbb{Q}}(V_{\mathbb{Q},k}^+(2)) = \frac{1}{2}(1+(-1)^k).$$

□

6.4.2 Proof of Corollary 7.1.3

The lower bounds follow directly from Theorem 6.1.10. Recall that for $k>1$ and $q>2$, we have

$$\zeta(k)\prod_{\substack{p\mid q\\ p \text{ prime}}}(1-p^{-k}) = \frac{1}{q^k}\sum_{\substack{1\le a<q\\ (a,q)=1}}\zeta(k,a/q) \tag{6.4.1}$$

and

$$V_{\mathbf{K},k}(q) = \mathbf{K}-\text{span of } \{\zeta(k,a/q) : 1\le a<q, (a,q)=1\}.$$

The number of generators of $\sum_{j=1}^{\ell}V_{\mathbf{K},k}(q_j)$ is $\sum_{j=1}^{\ell}\varphi(q_j)$. But, by (6.4.1), there are at least $\ell-1$ independent linear relations between them. Hence

$$\dim_{\mathbf{K}}\left(\sum_{j=1}^{\ell}V_{\mathbf{K},k}(q_j)\right) \le \sum_{j=1}^{\ell}\varphi(q_j)-(\ell-1).$$

□

6.4.3 Proof of Theorem 7.1.1

Let $\mathbf{F} = \mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$. By (6.1.5), the dimension of the $\mathbf{F}$ vector space generated by the elements of $X_{q_1,2k+1,o} \cup \cdots \cup X_{q_\ell,2k+1,o}$ is equal to the dimension of $\sum_{j=1}^{\ell} V^{+}_{\mathbf{F},2k+1}(q_j)$. Since, by hypothesis, $\mathbf{F} \cap \mathbb{Q}(\zeta_{q_1\cdots q_\ell})^+ = \mathbb{Q}$, Theorem 6.1.10 shows that this dimension is equal to the number of elements of $X_{q_1,2k+1,o} \cup \cdots \cup X_{q_\ell,2k+1,o}$, which must therefore be linearly independent over $\mathbf{F}$.

Alternatively we can argue as follows. For $1 \le j \le \ell$, let $D_j = \{\chi_j \bmod q_j \mid \chi_j(-1) = -1\}$ be the set of odd characters modulo q_j. Suppose that there exist $\alpha_{\chi_j} \in \mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$ for $\chi_j \in D_j, 1 \le j \le \ell$ such that

$$\sum_{1 \le j \le \ell} \sum_{\chi_j \in D_j} \alpha_{\chi_j} L(2k+1, \chi_j) = 0. \tag{6.4.2}$$

Substituting (see [37])

$$L(2k+1, \chi_j) = \frac{\pi^{2k+1}}{(2k)!\ q_j^{2k+1}} \sum_{a_j \in T_{q_j}} \chi_j(a_j) \cot^{(2k)}(\frac{\pi a_j}{q_j}),$$

for $\chi_j \in D_j, 1 \le j \le \ell$ in (6.4.2), we obtain

$$\sum_{1 \le j \le \ell} \sum_{a_j \in T_{q_j}} \frac{1}{q_j^{2k+1}} \left(\sum_{\chi_j \in D_j} \alpha_{\chi_j} \chi_j(a_j) \right) \cot^{(2k)}(\frac{\pi a_j}{q_j}) = 0. \tag{6.4.3}$$

Here T_{q_j} is as defined in (6.3). By given hypothesis, we have $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)}) \cap \mathbb{Q}(\zeta_{q_1\cdots q_\ell})^+ = \mathbb{Q}$. It then follows from Proposition 1.4.2 that $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$ and $\mathbb{Q}(\zeta_{q_1\cdots q_\ell})^+$ are linearly disjoint over $\mathbb{Q}$. Note that the coefficients of $\cot^{(2k)}(\frac{\pi a_j}{q_j})$'s in (6.4.3) belong to $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$ and hence Proposition 6.3.5 implies that

$$\sum_{\chi_j \in D_j} \alpha_{\chi_j} \chi_j(a_j) \;=\; 0$$

for $a_j \in T_{q_j}, 1 \le j \le \ell$. Since all the characters in the set $D_j, 1 \le j \le \ell$ are of same parity, it follows that

$$\sum_{\chi_j \in D_j} \alpha_{\chi_j} \chi_j(a_j) = 0$$

for $a_j \in (\mathbb{Z}/q_j\mathbb{Z})^\times, 1 \le j \le \ell$. It then follows from linear independence of characters that $\alpha_{\chi_j} = 0$ for $\chi_j \in D_j, 1 \le j \le \ell$. This completes the proof of Theorem 7.1.1. $\square$

6.4.4 Proof of Theorem 7.1.2

Let $\mathbf{F} = \mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$. By (6.1.5), the dimension of the $\mathbf{F}$ vector space generated by the elements of $\{\zeta(2k)\} \cup X_{q_1,2k,e} \cup \cdots \cup X_{q_\ell,2k,e}$ is equal to the dimension of $\sum_{j=1}^{\ell} V^+_{\mathbf{F},2k}(q_j)$. Since, by hypothesis, $\mathbf{F} \cap \mathbb{Q}(\zeta_{q_1\cdots q_\ell})^+ = \mathbb{Q}$, it follows from Theorem 6.1.10 that the dimension of $\sum_{j=1}^{\ell} V^+_{\mathbf{F},2k}(q_j)$ is equal to the number of elements of $\{\zeta(2k)\} \cup X_{q_1,2k,e} \cup \cdots \cup X_{q_\ell,2k,e}$, which must therefore be linearly independent over $\mathbf{F}$.

Alternatively we can argue as follows. Let $C_1 = \{\chi_1 \bmod q_1 \mid \chi_1(-1) = 1\}$ and for $1 < j \le \ell$, let $C_j = \{\chi_j \bmod q_j \mid \chi_j(-1) = 1, \chi_j \ne 1\}$ be the set of non-trivial even characters modulo q_j. Suppose that there exist $\alpha_{\chi_j} \in \mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$ for $\chi_j \in C_j, 1 \le j \le \ell$ such that

$$\sum_{1\le j\le \ell} \sum_{\chi_j \in C_j} \alpha_{\chi_j} L(2k, \chi_j) = 0. \tag{6.4.4}$$

Substituting (see [37])

$$L(2k, \chi_j) = \frac{-\pi^{2k}}{(2k-1)!\ q_j^{2k}} \sum_{a_j \in T_{q_j}} \chi_j(a_j) \cot^{(2k-1)}\left(\frac{\pi a_j}{q_j}\right), \tag{6.4.5}$$

for $\chi_j \in C_j, 1 \le j \le \ell$ in (6.4.4), we obtain

$$\sum_{1\le j\le \ell} \sum_{a_j \in T_{q_j}} \frac{1}{q_j^{2k}} \left(\sum_{\chi_j \in C_j} \alpha_{\chi_j} \chi_j(a_j) \right) \cot^{(2k-1)}\left(\frac{\pi a_j}{q_j}\right) = 0, \tag{6.4.6}$$

where T_{q_j} is as in (6.3). For $a_j \in T_{q_j}, 1 \le j \le \ell$, let us denote

$$A_j(a_j) = \frac{1}{q_j^{2k}} \sum_{\chi_j \in C_j} \alpha_{\chi_j} \chi_j(a_j).$$

From (6.4.6), we have

$$\sum_{1\le j\le \ell} \sum_{a_j \in T_{q_j}} A_j(a_j) \cot^{(2k-1)}\left(\frac{\pi a_j}{q_j}\right) \ = \ 0. \tag{6.4.7}$$

This implies that

$$\sum_{a_1\in T_{q_1}} A_1(a_1)\cot^{(2k-1)}(\frac{\pi a_1}{q_1}) + \sum_{1<j\le \ell} A_j(1) \sum_{a_j\in T_{q_j}} \cot^{(2k-1)}(\frac{\pi a_j}{q_j})$$
$$+ \sum_{1<j\le \ell} \sum_{a_j\in S_{q_j}} (A_j(a_j)-A_j(1))\cot^{(2k-1)}(\frac{\pi a_j}{q_j}) = 0,$$

where $S_{q_j} = \{1 < a_j < q_j/2 \mid (a_j, q_j) = 1\}$. As in the previous section, for $1 < j \le \ell$, recalling

$$\sum_{a_j\in T_{q_j}} \cot^{(2k-1)}(\frac{\pi a_j}{q_j}) = \frac{tr_j}{tr_1} \sum_{a_1\in T_{q_1}} \cot^{(2k-1)}(\frac{\pi a_1}{q_1}),$$

we have

$$\sum_{a_1\in T_{q_1}} (A_1(a_1)+\sum_{1<j\le \ell} \frac{A_j(1)tr_j}{tr_1})\cot^{(2k-1)}(\frac{\pi a_1}{q_1}) + \sum_{1<j\le \ell} \sum_{a_j\in S_{q_j}} \big(A_j(a_j)-A_j(1)\big)\cot^{(2k-1)}(\frac{\pi a_j}{q_j}) = 0. \tag{6.4.8}$$

As in Theorem 7.1.1, given hypothesis implies that the fields $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$ and $\mathbb{Q}(\zeta_{q_1\cdots q_\ell})^+$ are linearly disjoint. Since the coefficients of $\cot^{(2k-1)}(\frac{\pi a_j}{q_j})$'s in (6.4.8) belong to $\mathbf{K}(\zeta_{\varphi(q_1)\cdots\varphi(q_\ell)})$, using Proposition 6.3.5, we obtain

$$A_1(a_1) + \sum_{1<j\le \ell} \frac{A_j(1)tr_j}{tr_1} = 0 \qquad \text{and} \qquad A_j(a_j) = A_j(1), \tag{6.4.9}$$

where $a_1 \in T_{q_1}$ and $a_j \in S_{q_j}$ for $1 < j \le \ell$. The second equality implies that

$$\frac{1}{q_j^{2k}} \sum_{\chi_j\in C_j} \alpha_{\chi_j}\chi_j(a_j) \;=\; A_j(1)\chi_0(a_j),$$

where χ_0 is the trivial character modulo q_j and $a_j \in T_{q_j}$. Since all the characters in the set $C_j, 1 \le j \le \ell$ are even, it follows that

$$\frac{1}{q_j^{2k}} \sum_{\chi_j\in C_j} \alpha_{\chi_j}\chi_j(a_j) \;=\; A_j(1)\chi_0(a_j),$$

where $a_j \in (\mathbb{Z}/q_j\mathbb{Z})^\times, 1 < j \le \ell$. Linear independence of characters implies that $A_j(1) = 0$ and $\alpha_{\chi_j} = 0$ for $\chi_j \in C_j, 1 < j \le \ell$. Replacing $A_j(1) = 0$ in the first equality of (6.4.9), we get $A_1(a_1) = 0$ for $a_1 \in T_1$. Arguing as above, we get that $\alpha_{\chi_1} = 0$ for $\chi_1 \in C_1$. This completes the proof of Theorem 7.1.2.

6.4.5 Proof of Theorem 6.1.16

Set $k=\sum_{t=1}^{r}k_t$. As in the proof of Theorem 6.1.10, by hypothesis, the fields $\mathbf{K}$ and $\mathbb{Q}(\zeta_{q_1\cdots q_\ell})^+$ are linearly disjoint, thus it suffices to determine the dimension of the $\mathbb{Q}$-vector space $\sum_{j=1}^{\ell}(i\pi)^{-k}V^{+}_{\mathbb{Q},\vec{k}}(\vec{q}_j)$. But, this is the dimension of $\oplus_{j=1}^{\ell}(i\pi)^{-k}V^{+}_{\mathbb{Q},\vec{k}}(\vec{q}_j)$ minus the dimension of the kernel of the map shown in Proposition 6.3.4. Since

$$\dim_{\mathbb{Q}}\left(\oplus_{j=1}^{\ell}(i\pi)^{-k}V^{+}_{\mathbb{Q},\vec{k}}(\vec{q}_j)\right)=2^{-r}\sum_{j=1}^{\ell}\prod_{t=1}^{r}\varphi(q_{t,j}),$$

the result follows from the same Proposition 6.3.4. □

6.4.6 Proof of Theorem 6.1.17

The map

$$\begin{array}{ccc}\otimes_{t=1}^{r}\sum_{j=1}^{\ell_t}(i\pi)^{-k_t}V^{+}_{\mathbb{Q},k_t}(q_{t,j}) & \longrightarrow & \prod_{t=1}^{r}\sum_{j=1}^{\ell_t}(i\pi)^{-k_t}V^{+}_{\mathbb{Q},k_t}(q_{t,j})\\ x_1\otimes\cdots\otimes x_r & \longmapsto & x_1\cdots x_r\end{array}$$

is a bijection. Indeed, a nonzero element in the kernel gives a non trivial relation, over the field $\mathbb{Q}(\zeta_{\prod_{t\neq r}q_t})$, between the elements of a $\mathbb{Q}$-basis of $\sum_{j=1}^{\ell_t}(i\pi)^{-k_r}V^{+}_{\mathbb{Q},k_r}(q_{r,j})$. But this contradicts the linear disjointness of the fields $\mathbb{Q}(\zeta_{\prod_{t\neq r}q_t})$ and $\mathbb{Q}(\zeta_{q_r})$. Thus, the dimension of the $\mathbb{Q}$ vector space on the right is the product $\prod_{t=1}^{r}\dim_{\mathbb{Q}}\left(\sum_{j=1}^{\ell_t}(i\pi)^{-k_t}V^{+}_{\mathbb{Q},k_t}(q_{t,j})\right)$ which is equal to $2^{-r}\prod_{t=1}^{r}\left(\sum_{j=1}^{\ell_t}\varphi(q_{t,j})-(\ell_t-1)(1+(-1)^{k_t})\right)$ by Theorem 6.1.10. □

6.5 Appendix: alternative proof of Proposition 6.3.5.

6.5.1 The case when k is odd

Proof. Set $q = q_1 \cdots q_\ell$. Thus, applying (6.3.2) and (6.3.3), we have

$$\begin{aligned}
& -i^{2k+1}(\zeta_q - \zeta_q^{-1})\cot^{(2k)}(\pi a_j/q_j) \\
= \ & (\zeta_q - \zeta_q^{-1}) \sum_{\substack{a,b\ge 0 \\ a+2b=2k+1}} \beta_{a,b}^{(2k+1)}(-i)^a(-1)^b(\cot\frac{\pi a_j}{q_j})^a \ (1+(\cot\frac{\pi a_j}{q_j})^2)^b \\
= \ & (\zeta_q - \zeta_q^{-1}) \sum_{\substack{a,b\ge 0 \\ a+2b=2k+1}} (-1)^b \beta_{a,b}^{(2k+1)} \left(-i\cot\frac{\pi a_j}{q_j}\right)^a \left(1-(-i\cot\frac{\pi a_j}{q_j})^2\right)^b \\
= \ & (\zeta_q - \zeta_q^{-1}) \sum_{\substack{a,b\ge 0 \\ a+2b=2k+1}} (-1)^b \beta_{a,b}^{(2k+1)} \left(\frac{\zeta_{q_j}^{a_j}+1}{\zeta_{q_j}^{a_j}-1}\right)^a \left(1-(\frac{\zeta_{q_j}^{a_j}+1}{\zeta_{q_j}^{a_j}-1})^2\right)^b .
\end{aligned}$$

This implies that $i(\zeta_q - \zeta_q^{-1})\cot^{(2k)}(\frac{\pi a_j}{q_j}) \in \mathbb{Q}(\zeta_q)^+$ for $a_j \in T_{q_j}$, $1 \le j \le \ell$. The proposition then follows for $\ell = 1$ along the lines of the proof of Okada and Murty-Saradha. Suppose that the proposition is true for any natural number $1 \le n < \ell$. We want to show that

$$\bigcup_{1\le j\le \ell} \{\cot^{(2k)}(\frac{\pi a_j}{q_j}) \ : \ a_j \in T_{q_j}\} \tag{6.5.1}$$

is linearly independent over $\mathbf{K}$. By the given hypothesis, we have $\mathbf{K} \cap \mathbb{Q}(\zeta_{q_1\cdots q_\ell})^+ = \mathbb{Q}$. Appealing to Theorem 1.4.2, it is now sufficient to show that the numbers in the set (6.5.1) are $\mathbb{Q}$ linearly independent. There exist rational numbers α_{a_j} for $a_j \in T_{q_j}, 1 \le j \le \ell$ such that

$$\sum_{1\le j\le \ell} \sum_{a_j \in T_{q_j}} \alpha_{a_j} \cot^{(2k)}(\frac{\pi a_j}{q_j}) \ = \ 0.$$

This implies that

$$\sum_{1\le j<\ell}\sum_{a_j\in T_{q_j}} \alpha_{a_j}\cot^{(2k)}(\frac{\pi a_j}{q_j}) \;=\; -\sum_{a_\ell\in T_{q_\ell}} \alpha_{a_\ell}\cot^{(2k)}(\frac{\pi a_\ell}{q_\ell}).$$

Alternative to the identity (6.3.2), one can write

$$\cot^{(h-1)}(z) = i^h\left(2X\frac{d}{dX}\right)^{h-1}\left(\frac{X+1}{X-1}\right)|_{X=e^{2iz}}$$

for all h since $\cot(z) = i\frac{e^{2iz}+1}{e^{2iz}-1}$. Evaluating the above expression at $z = \pi a_j/q_j$, one gets $i^h\cot^{(h-1)}(\pi a_j/q_j)\in\mathbb{Q}(\zeta_{q_j})\cap i^h\mathbb{R}$. Since $h = 2k+1$ is an odd integer, it then follows that

$$i\sum_{1\le j<\ell}\sum_{a_j\in T_{q_j}} \alpha_{a_j}\cot^{(2k)}(\frac{\pi a_j}{q_j}) \;=\; -i\sum_{a_\ell\in T_{q_\ell}} \alpha_{a_\ell}\cot^{(2k)}(\frac{\pi a_\ell}{q_\ell}) \;\in\; \mathbb{Q}(\zeta_{q_1\cdots q_{\ell-1}})\cap\mathbb{Q}(\zeta_{q_\ell}) \;=\; \mathbb{Q}.$$

Since a purely imaginary number is a rational number if and only if it is 0, we have

$$\sum_{1\le j<\ell}\sum_{a_j\in T_{q_j}} \alpha_{a_j}\cot^{(2k)}(\frac{\pi a_j}{q_j}) \;=\; -\sum_{a_\ell\in T_{q_\ell}} \alpha_{a_\ell}\cot^{(2k)}(\frac{\pi a_\ell}{q_\ell}) \;=\; 0.$$

Applying induction hypothesis, we get that $\alpha_{a_j} = 0$ for all $a_j\in T_{q_j}, 1\le j\le\ell$. This completes the proof of the proposition. ■

6.5.2 The case when k is even

Proof. Applying (6.3.2), we see that $\cot^{(2k-1)}(\frac{\pi a_j}{q_j})\in\mathbb{Q}(\zeta_{q_j})^+$ for $a_j\in T_{q_j}$, $1\le j\le\ell$. The proposition then follows for $\ell=1$ from the works of Okada and Murty-Saradha. Suppose that the proposition is true for any $1\le n<\ell$. We want to show that the numbers

$$\{\cot^{(2k-1)}(\frac{\pi a_1}{q_1}) \;:\; a_1\in T_{q_1}\}\bigcup_{1<j\le\ell}\{\cot^{(2k-1)}(\frac{\pi a_j}{q_j}) \;:\; a_j\in S_{q_j}\} \tag{6.5.2}$$

are linearly independent over $\mathbf{K}$. Since $\mathbf{K}\cap\mathbb{Q}(\zeta_{q_1\cdots q_\ell})^+ = \mathbb{Q}$, applying Theorem 1.4.2, we see that it is sufficient to prove that the numbers in the set (6.5.2) are $\mathbb{Q}$ linearly independent. There exist rational numbers α_{a_j} for $a_1 \in T_{q_1}, a_j \in S_{q_j}, 1 < j \le \ell$ such that

$$\sum_{a_1 \in T_{q_1}} \alpha_{a_1} \cot^{(2k-1)}(\frac{\pi a_1}{q_1}) \; + \sum_{1<j\le \ell} \sum_{a_j \in S_{q_j}} \alpha_{a_j} \cot^{(2k-1)}(\frac{\pi a_j}{q_j}) \; = \; 0.$$

Then

$$\begin{aligned} &\sum_{a_1 \in T_{q_1}} \alpha_{a_1} \cot^{(2k-1)}(\frac{\pi a_1}{q_1}) \; + \sum_{1<j<\ell} \sum_{a_j \in S_{q_j}} \alpha_{a_j} \cot^{(2k-1)}(\frac{\pi a_j}{q_j}) \\ = &\sum_{a_\ell \in S_{q_\ell}} \alpha_{a_\ell} \cot^{(2k-1)}(\frac{\pi a_\ell}{q_\ell}) \; \in \; \mathbb{Q}(\zeta_{q_1 \cdots q_{\ell-1}}) \cap \mathbb{Q}(\zeta_{q_\ell}) \; = \; \mathbb{Q}. \end{aligned}$$

Let us call this rational number β. If $\beta \neq 0$, then without loss of generality, we may assume that $\beta = -1$. So

$$\sum_{a_\ell \in S_{q_\ell}} \alpha_{a_\ell} \cot^{(2k-1)}(\frac{\pi a_\ell}{q_\ell}) \; = \; 1. \tag{6.5.3}$$

Let us denote $\mathbb{Q}(\zeta_j)$ by $\mathbf{F}_j$. Let $\mathrm{Tr}_{\mathbf{F}_j/\mathbb{Q}}(\alpha)$ denotes the trace of α over $\mathbb{Q}$ for $\alpha \in \mathbf{F}_j$. For any $a_j \in T_j$, using identities (6.3.2) and (6.3.3), we get

$$\begin{aligned} tr_j \; &= \; \mathrm{Tr}_{\mathbf{F}_j/\mathbb{Q}}\left(\cot^{(2k-1)}(\frac{\pi a_j}{q_j})\right) \\ &= \sum_{\substack{1\le a_j \le q_j \\ (a_j,q_j)=1}} \cot^{(2k-1)}(\frac{\pi a_j}{q_j}) \; = \; 2 \sum_{a_j \in T_{q_j}} \cot^{(2k-1)}(\frac{\pi a_j}{q_j}). \end{aligned} \tag{6.5.4}$$

It now follows from (6.5.3) and (6.5.4) that

$$\sum_{a_\ell \in S_{q_\ell}} \alpha_{a_\ell} \cot^{(2k-1)}(\frac{\pi a_\ell}{q_\ell}) = \frac{2}{tr_\ell} \left(\sum_{a_\ell \in T_{q_\ell}} \cot^{(2k-1)}(\frac{\pi a_\ell}{q_\ell}) \right).$$

Hence we have

$$\frac{-2}{tr_\ell} \cot^{(2k-1)}(\frac{\pi a_1}{q_1}) + \sum_{a_\ell \in S_{q_\ell}} (\alpha_{a_\ell} - \frac{2}{tr_\ell}) \cot^{(2k-1)}(\frac{\pi a_\ell}{q_\ell}) = 0,$$

a contradiction to Theorem 6.2.1. This implies that $\beta = 0$. Applying induction hypothesis, we then obtain $\alpha_{a_j} = 0$ for $a_1 \in T_{q_1}$, $a_j \in S_{q_j}, 1 < j \le \ell$. ∎

CHAPTER 7

Dedekind zeta values at 1/2

7.1 Introduction

For a number field $\mathbf{K}$, let $\zeta_{\mathbf{K}}$ be the Dedekind zeta function associated to $\mathbf{K}$. The non-vanishing of $\zeta_{\mathbf{K}}(s)$ at $s = 1/2$ is a deep arithmetic question. Armitage [2] gave examples of number fields $\mathbf{K}$ for which $\zeta_{\mathbf{K}}(1/2) = 0$. On the other hand, it is believed that $\zeta_{\mathbf{K}}(1/2) \neq 0$ when $\mathbf{K}$ is an S_n - number field, that is, a number field of degree n whose normal closure has Galois group S_n over $\mathbb{Q}$. Furthermore, very little is known about the transcendental nature of the non-zero values of $\zeta_{\mathbf{K}}(1/2)$. For instance, one has

$$\zeta(1/2) = \frac{1}{1-\sqrt{2}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \approx -1.46035450880\cdots,$$

where ζ is the Riemann zeta function.

In this connection, one has a classical conjecture of Dedekind which asserts that if $\mathbf{L}/\mathbf{K}$ is an extension of number fields, then $\zeta_{\mathbf{K}}(s)$ divides $\zeta_{\mathbf{L}}(s)$, in other words, the function $\zeta_{\mathbf{L}}(s)/\zeta_{\mathbf{K}}(s)$ is entire. This conjecture is open in general, but holds when $\mathbf{L}/\mathbf{K}$ is Galois, thanks to the works of Aramata and Brauer. Also the celebrated Artin's conjecture for holomorphicity of his L-functions will establish Dedekind's conjecture.

If $\mathbf{L}/\mathbb{Q}$ is a Galois number field with Galois group S_n, then $\mathbf{L}$ contains a quadratic subfield $\mathbf{K}$. Dedekind's conjecture ensures that vanishing of $\zeta_{\mathbf{K}}(1/2)$ will ensure vanishing of $\zeta_{\mathbf{L}}(1/2)$.

In this note, we study various aspects of the derivative $\zeta'_{\mathbf{K}}(s)$ at $s = 1/2$. As discussed above, study of these circle of questions for quadratic fields merits special attention. We note that for quadratic fields $\mathbf{K}$, the non vanishing of $\zeta_{\mathbf{K}}(1/2)$ is equivalent to the non vanishing of $L(1/2,\chi)$ for quadratic character χ. Non vanishing of such $L(1/2,\chi)$ has been conjectured by Chowla. We begin with the following theorem for quadratic fields.

Theorem 7.1.1. *Let* $\mathbf{K}$ *and* $\mathbf{L}$ *be distinct quadratic fields such that* $\zeta_{\mathbf{K}}(1/2)\zeta_{\mathbf{L}}(1/2) \neq 0$. *Then*

$$\frac{\zeta'_{\mathbf{K}}(1/2)}{\zeta_{\mathbf{K}}(1/2)} \neq \frac{\zeta'_{\mathbf{L}}(1/2)}{\zeta_{\mathbf{L}}(1/2)}.$$

The above theorem in particular shows that there is at most one quadratic field $\mathbf{K}$ for which $\zeta_{\mathbf{K}}(1/2) \neq 0$ while the derivative $\zeta'_{\mathbf{K}}(1/2) = 0$. We then prove the following quantitative theorem where the existence of the fictitious exception alluded to above is ruled out for both quadratic and cubic number fields.

Theorem 7.1.2. *Let* $\mathbf{K}$ *be an algebraic number field with degree* ≤ 3. *Then*

$$\zeta_{\mathbf{K}}(1/2) = 0 \iff \zeta'_{\mathbf{K}}(1/2) = 0.$$

As a corollary, we have the following courtesy of the seminal work by K. Soundararajan [65].

Corollary 7.1.3. *For at least* 87.5% *of quadratic number fields* $\mathbf{K}$ *with discriminant* $8d$ *with odd positive square-free integers* d*, one has* $\zeta'_{\mathbf{K}}(1/2) \neq 0$.

Let us very briefly describe the context as well as content of the work of Soundararajan indicated above. The Generalised Riemann Hypothesis (GRH) does not preclude the possibility that $L(1/2,\chi) = 0$ for some primitive Dirichlet character χ. But it is believed that there is no rational linear relation between the ordinates of the non-trivial zeros of the Dirichlet $\mathbf{L}$-functions and consequently, $L(1/2,\chi)$ is expected to be non-zero for any primitive Dirichlet character χ. In particular when χ is a quadratic character, this seems to have been conjectured first by Chowla [12] as indicated earlier. In his outstanding work [65], Soundararajan showed that for at least 87.5% of the odd square-free integers $d \geq 0$, $L(1/2,\chi_{8d}) \neq 0$. Here for a fundamental discriminant (discriminant of some quadratic number field) d, $\chi_d(n) := \left(\frac{d}{n}\right)$ where $\left(\frac{d}{n}\right)$ denotes the Kronecker-Legendre symbol. Results along this direction were obtained earlier in [5], [33] and [35].

We now have the following theorem for higher degree number fields.

Theorem 7.1.4. *Let* $\mathbf{K}$ *be an algebraic number field of degree* $n > 3$ *such that the absolute value of its discriminant* $|d_{\mathbf{K}}| \in \mathbb{R} \setminus [(44.763)^n, (215.333)^n]$. *Then*

$$\zeta_{\mathbf{K}}(1/2) = 0 \iff \zeta'_{\mathbf{K}}(1/2) = 0.$$

We now consider the analogous question for Galois number fields.

Theorem 7.1.5. *Consider the following sets.*

$$X = \{\mathbf{K}\, Galois : \mathbf{K} \subset \mathbb{R},\ \zeta_{\mathbf{K}}(1/2) \neq 0,\ \ \zeta'_{\mathbf{K}}(1/2) = 0\}$$

$$Y = \{\mathbf{K}\, Galois : \mathbf{K} \not\subset \mathbb{R},\ \zeta_{\mathbf{K}}(1/2) \neq 0,\ \ \zeta'_{\mathbf{K}}(1/2) = 0\}.$$

Then at least one of the sets X and Y is empty. Furthermore, there are at most finitely many abelian number fields for which $\zeta'_{\mathbf{K}}(1/2) = 0$ but $\zeta_{\mathbf{K}}(1/2) \neq 0$. All such number fields (if exist) have degree less than 46369.

Remark 7.1.6. *Suppose $\zeta_{\mathbf{K}}(1/2) \neq 0$ and $\zeta'_{\mathbf{K}}(1/2) = 0$, then degree of $\mathbf{K}/\mathbb{Q}$ is precisely*

$$\frac{\log|d_{\mathbf{K}}|}{\pi/2 + \log 8\pi + \gamma}$$

and

$$\frac{\log|d_{\mathbf{K}}|}{\log 8\pi + \gamma}$$

in case of totally real and totally complex Galois number fields respectively, where $|d_{\mathbf{K}}|$ denotes the absolute discriminant of $\mathbf{K}$ and γ is the ubiquitous Euler's constant.

The above theorem refines a result of Ram Murty and Tanabe [56, Cor 3.9]. Investigations similar to ours for Elliptic curves over $\mathbb{Q}$ as well as Modular forms were initiated by Gun, Murty and Rath [25]. Furthermore in [56], it has been proved that there are only finitely many *abelian totally real number fields* $\mathbf{K}$ for which $\zeta_{\mathbf{K}}(1/2) \neq 0$ while the derivative $\zeta'_{\mathbf{K}}(1/2) = 0$. One of our objectives in this note was to further this line of investigation to arbitrary number fields, obtain some quantitative results and finally study transcendental nature of these deeply mysterious numbers. In particular, we use Baker's seminal theorem (see [25], [28], [51] and [52] for some other applications of Baker's theorem). In this context, we have the following theorem.

Theorem 7.1.7. *Let $\mathbf{K}$ and $\mathbf{L}$ be distinct algebraic number fields of degree n and m respectively and $\zeta_{\mathbf{K}}(1/2)\zeta_{\mathbf{L}}(1/2) \neq 0$. If one of the two following conditions*

1.

$$|d_{\mathbf{K}}|^m \neq |d_{\mathbf{L}}|^n;$$

2.

$$m\frac{\zeta'_{\mathbf{K}}(1/2)}{\zeta_{\mathbf{K}}(1/2)} \neq n\frac{\zeta'_{\mathbf{L}}(1/2)}{\zeta_{\mathbf{L}}(1/2)},$$

hold then at least one of the following two numbers

$$\frac{\zeta'_{\mathbf{K}}(1/2)}{\zeta_{\mathbf{K}}(1/2)} \quad and \quad \frac{\zeta'_{\mathbf{L}}(1/2)}{\zeta_{\mathbf{L}}(1/2)}$$

is transcendental.

Now, we obtain the following interesting corollaries from Theorem 7.1.7.

Corollary 7.1.8. *Let n be a positive integer. Then the set*

$$\left\{\frac{\zeta'_{\mathbf{K}}(1/2)}{\zeta_{\mathbf{K}}(1/2)} : \zeta_{\mathbf{K}}(1/2) \neq 0, [\mathbf{K}:\mathbb{Q}] = n\right\}$$

has at most one algebraic number. Furthermore,

$$\frac{\zeta'_{\mathbf{K}}(1/2)}{\zeta_{\mathbf{K}}(1/2)} - \frac{n}{2}(\log 8\pi + \gamma)$$

is a transcendental number.

Two non-zero integers u and v are said to be multiplicatively independent if for integers n and m, $u^n = v^m$ implies $n = m = 0$. In this context, we deduce the following corollary for arbitrary degree number fields.

Corollary 7.1.9. *Let $\mathscr{F}$ be a family of number fields with pairwise multiplicatively independent discriminants. If $\zeta_{\mathbf{K}}(1/2) \neq 0$ for every $\mathbf{K} \in \mathscr{F}$, then the set*

$$\left\{\frac{\zeta'_{\mathbf{K}}(1/2)}{\zeta_{\mathbf{K}}(1/2)} : \mathbf{K} \in \mathscr{F}\right\}$$

has at most one algebraic number.

We refer to [51] and [52] for investigations related to the non-vanishing of derivatives of $\zeta_{\mathbf{K}}(s)$ and $L(s, f)$ at $s = 1$, where f is a periodic arithmetic function.

7.2 Preliminaries

We recall some relevant facts about the Dedekind zeta function $\zeta_{\mathbf{K}}(s)$ associated to a number field $\mathbf{K}$. $\zeta_{\mathbf{K}}(s)$ initially given by the following Dirichlet series

$$\zeta_{\mathbf{K}}(s) = \sum_{\mathscr{I} \neq 0} \frac{1}{N(\mathscr{I})^s}$$

for $\Re(s) > 1$ has a meromorphic continuation to the complex plane with a simple pole at $s = 1$. Furthermore, the function

$$Z_{\mathbf{K}}(s) := \Gamma_{\mathbb{C}}(s)^{r_2} \Gamma_{\mathbb{R}}(s)^{r_1} \zeta_{\mathbf{K}}(s)$$

extends meromorphically to the complex plane with simple poles at $s = 0$ and $s = 1$ and satisfies the functional equation $Z_{\mathbf{K}}(s) = |d_{\mathbf{K}}|^{1/2-s} Z_{\mathbf{K}}(1-s)$. Also $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s), \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. But we shall use the following version of the functional equation which is amenable for our purpose, namely

$$\zeta_{\mathbf{K}}(1-s) = A_{\mathbf{K}}(s)\zeta_{\mathbf{K}}(s)$$

for $s \in \mathbb{C} \setminus \{1\}$ (see [58, p. 467], for instance) with the factor

$$A_{\mathbf{K}}(s) := |d_{\mathbf{K}}|^{s-1/2} \left(\cos \frac{\pi s}{2}\right)^{r_1+r_2} \left(\sin \frac{\pi s}{2}\right)^{r_2} \left(2(2\pi)^{-s}\Gamma(s)\right)^n .$$

Now we quickly recall the discriminant of a quadratic number field $\mathbf{K}$. Let d be a square-free integer, then the discriminant $d_{\mathbf{K}}$ of the field $\mathbf{K} = \mathbb{Q}(\sqrt{d})$ is

$$d_{\mathbf{K}} = \begin{cases} d & \text{if } d \equiv 1 \pmod 4 \\ 4d & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

Now, we state the following deep theorem of Soundararajan which we shall need to prove Corollary 7.1.3.

Theorem 7.2.1. *[65, Thm 1] For at least* 87.5% *of the odd square-free integers* $d \geq 0$*,* $L(1/2, \chi_{8d}) \neq 0$.

7.3 Proofs of the Main Theorems

As indicated in the earlier section, we shall work with the following functional equation of $\zeta_{\mathbf{K}}(s)$ for $s \in \mathbb{C} \setminus \{1\}$ [58, p. 467]

$$\zeta_{\mathbf{K}}(1-s) = A_{\mathbf{K}}(s)\zeta_{\mathbf{K}}(s) \tag{7.3.1}$$

with the factor $A_{\mathbf{K}}(s) := |d_{\mathbf{K}}|^{s-1/2}\left(\cos\frac{\pi s}{2}\right)^{r_1+r_2}\left(\sin\frac{\pi s}{2}\right)^{r_2}(2(2\pi)^{-s}\Gamma(s))^n$.

Let us begin with an easy, but important observation that $A_{\mathbf{K}}(1/2) = 1$.

We now differentiate (7.3.1) w.r.t s and substitute at $s = 1/2$ to obtain

$$\zeta'_{\mathbf{K}}(1/2) = -(1/2)A'_{\mathbf{K}}(1/2)\zeta_{\mathbf{K}}(1/2). \tag{7.3.2}$$

On the other hand, taking the logarithmic derivative of $A_{\mathbf{K}}(s)$ we obtain

$$\frac{A'_{\mathbf{K}}(s)}{A_{\mathbf{K}}(s)} = \log|d_{\mathbf{K}}| - \frac{\pi}{2}(r_1+r_2)\tan\frac{\pi s}{2} + r_2\frac{\pi}{2}\cot\frac{\pi s}{2} - n\log 2\pi + n\frac{\Gamma'(s)}{\Gamma(s)},$$

where log is the natural logarithm.

Since the value of digamma function $\frac{\Gamma'(s)}{\Gamma(s)}$ at $s = 1/2$ is $-\gamma - 2\log 2$ (see [68], p. 427), we have

$$A'_{\mathbf{K}}(1/2) = \log|d_{\mathbf{K}}| - r_1\frac{\pi}{2} - n(\log 8\pi + \gamma). \tag{7.3.3}$$

3.1. Proof of Theorem 7.1.1

Let $\mathbf{K} = \mathbb{Q}(\sqrt{d_1})$ and $\mathbf{L} = \mathbb{Q}(\sqrt{d_2})$, where d_1 and d_2 are distinct square-free integers. Since $\mathbf{K}$ and $\mathbf{L}$ are distinct quadratic fields, we have $d_{\mathbf{K}} \neq d_{\mathbf{L}}$.

Using (7.3.2) and (7.3.3), we obtain

$$-2\left(\frac{\zeta'_{\mathbf{K}}(1/2)}{\zeta_{\mathbf{K}}(1/2)}-\frac{\zeta'_{\mathbf{L}}(1/2)}{\zeta_{\mathbf{L}}(1/2)}\right)=\log\frac{|d_{\mathbf{K}}|}{|d_{\mathbf{L}}|}+\frac{\pi}{2}(r_1^{(\mathbf{L})}-r_1^{(\mathbf{K})}),$$

where $r_1^{(\mathbf{L})}$ and $r_1^{(\mathbf{K})}$ denote the number of real embeddings of $\mathbf{L}$ and $\mathbf{K}$ respectively. It follows from Theorem 3.2.1 that e^{π} is a transcendental number. So if $r_1^{(\mathbf{L})}-r_1^{(\mathbf{K})}\neq 0$, then the right hand side of the above equation is non-zero by Theorem 3.2.1. On the other hand if $r_1^{(\mathbf{L})}-r_1^{(\mathbf{K})}=0$, then the right hand side of the above equation is actually transcendental by Lemma 3.1.1.

Thus,

$$\frac{\zeta'_{\mathbf{K}}(1/2)}{\zeta_{\mathbf{K}}(1/2)}-\frac{\zeta'_{\mathbf{L}}(1/2)}{\zeta_{\mathbf{L}}(1/2)}$$

is non-zero.

We note that our proof along with lemma 4.2.2 gives a stronger assertion, namely the number

$$\frac{\zeta'_{\mathbf{K}}(1/2)}{\zeta_{\mathbf{K}}(1/2)}-\frac{\zeta'_{\mathbf{L}}(1/2)}{\zeta_{\mathbf{L}}(1/2)}$$

is actually transcendental.

3.2. Proof of Theorem 7.1.2

By (7.3.2), it is enough to show that $A'_{\mathbf{K}}(1/2)\neq 0$. We have

$$A'_{\mathbf{K}}(1/2)=0 \iff |d_{\mathbf{K}}|=\exp(r_1\frac{\pi}{2}+n(\log 8\pi+\gamma)). \tag{7.3.4}$$

$A'_{\mathbf{K}}(1/2)$ is evidently non-zero for $\mathbf{K}=\mathbb{Q}$. In fact, we have $\zeta'(1/2)=-3.922\cdots$.

So we have the following two cases.

Case (i). Assume $\mathbf{K}$ is a quadratic field. So r_1 could be either 0 or 2.

At $r_1=0$, we have

$$2003<\exp(r_1\frac{\pi}{2}+2(\log 8\pi+\gamma))<2004.$$

At $r_1 = 2$, we have

$$46368 < \exp(r_1 \frac{\pi}{2} + 2(\log 8\pi + \gamma)) < 46369.$$

Since $d_{\mathbf{K}}$ is always an integer, $A'_{\mathbf{K}}(1/2)$ can never be zero in case of quadratic fields.

Case (ii). Now we consider $\mathbf{K}$ a cubic field. So r_1 is either 1 or 3.

At $r_1 = 1$, we have

$$431471 < \exp(r_1 \frac{\pi}{2} + 3(\log 8\pi + \gamma)) < 431472.$$

At $r_1 = 3$, we have

$$9984558 < \exp(r_1 \frac{\pi}{2} + 3(\log 8\pi + \gamma)) < 9984559.$$

Since $d_{\mathbf{K}}$ is always an integer, $A'_{\mathbf{K}}(1/2)$ can not be zero in case of cubic fields also.

3.3. Proof of Corollary 7.1.3

For a quadratic number field $\mathbf{K}$, we have

$$\zeta_{\mathbf{K}}(s) = \zeta(s) L(s, \chi_{d_{\mathbf{K}}}), \ \Re(s) > 1,$$

where

$$L(s, \chi_{d_{\mathbf{K}}}) := \sum_{n=1}^{\infty} \frac{\chi_{d_{\mathbf{K}}}(n)}{n^s}.$$

We refer the reader to [31, Ch. VII] and [58] for further details. We recall that this generalization of Riemann zeta function also has the analytic continuation to whole complex plane except $s = 1$. By uniqueness of analytic continuation of complex functions, one could get the same identity for $s \in \mathbb{C} \setminus \{1\}$.

Now, we let $\mathbf{K} = \mathbb{Q}(\sqrt{2d})$, where d is a square-free positive odd integer. It is easy to see that discriminant of $\mathbf{K}$ is $8d$ (for instance, see §5.3, [63]). Hence,

$$\zeta_{\mathbf{K}}(1/2) = \zeta(1/2) L(1/2, \chi_{8d}).$$

Using Theorem 7.2.1, we have our desired result.

3.4. Proof of Theorem 7.1.4

We show that in the given interval of $d_{\mathbf{K}}$,

$$A'_{\mathbf{K}}(1/2) = \log|d_{\mathbf{K}}| - r_1\frac{\pi}{2} - n(\log 8\pi + \gamma) \neq 0.$$

Using hypothesis, we have

$$A'_{\mathbf{K}}(1/2) < n(\log(44.763) - \log 8\pi - \gamma) < 0.$$

Similarly,

$$A'_{\mathbf{K}}(1/2) > n(\log(215.333) - \frac{\pi}{2} - \log 8\pi - \gamma) > 0.$$

So our result follows from (7.3.2).

3.5. Proof of Theorem 7.1.5

If possible, let us assume that there exist Galois number fields $\mathbf{K} \in X$ and $\mathbf{L} \in Y$ of degree n and m respectively. From (7.3.2), we have

$$A'_{\mathbf{K}}(1/2) = A'_{\mathbf{L}}(1/2) = 0.$$

From (7.3.3), we have

$$nA'_{\mathbf{K}}(1/2) = \log|d_{\mathbf{K}}|^{1/n} - r_1^{(\mathbf{K})}\frac{\pi}{2n} - \log 8\pi - \gamma \tag{7.3.5}$$

and

$$mA'_{\mathbf{L}}(1/2) = \log|d_{\mathbf{L}}|^{1/m} - r_1^{(\mathbf{L})}\frac{\pi}{2m} - \log 8\pi - \gamma, \tag{7.3.6}$$

where $r_1^{(\mathbf{K})}$ and $r_1^{(\mathbf{L})}$ denote the number of real embeddings of $\mathbf{K}$ and $\mathbf{L}$ respectively. From (7.3.5) and (7.3.6), we obtain

$$\log|d_{\mathbf{K}}|^{1/n} - r_1^{(\mathbf{K})}\frac{\pi}{2n} - \log|d_{\mathbf{L}}|^{1/m} + r_1^{(\mathbf{L})}\frac{\pi}{2m} = 0.$$

Since Galois fields are the normal extensions of their base fields, so there does not exist any complex embedding in real Galois fields. Similarly, there are no real embeddings in non-real Galois fields. Therefore, $r_1^{(\mathbf{K})} = n$ and $r_1^{(\mathbf{L})} = 0$. Hence,

$$\log \frac{|d_{\mathbf{K}}|^{1/n}}{|d_{\mathbf{L}}|^{1/m}} - \frac{\pi}{2} = 0,$$

which is a contradiction as e^{π} is transcendental by Theorem 3.2.1. This completes the first part.

We now proceed with the second part of Theorem 7.1.5. From (7.3.3), we have

$$A'_{\mathbf{K}}(1/2) = \log|d_{\mathbf{K}}| - r_1\frac{\pi}{2} - n(\log 8\pi + \gamma).$$

Using Theorem 1.3.11, we obtain

$$A'_{\mathbf{K}}(1/2) \geq n(\log(n)/2 - \pi/2 - \log 8\pi - \gamma) > 0, \ \forall\, n \geq 46369.$$

This implies that for all $n \geq 46369$, $\zeta'_{\mathbf{K}}(1/2) = 0$ if and only if $\zeta_{\mathbf{K}}(1/2) = 0$.

Now we aim to prove that the set

$$S := \{\mathbf{K} : \zeta'_{\mathbf{K}}(1/2) = 0,\ \zeta_{\mathbf{K}}(1/2) \neq 0,\ \ n < 46369\}$$

has finite cardinality. By (7.3.2), we see that

$$S \subseteq S' := \{\mathbf{K} : A'_{\mathbf{K}}(1/2) = 0,\ \ n < 46369\}\ .$$

So it is enough to show that the set S' has finite cardinality. By (7.3.4), we have

$$A'_{\mathbf{K}}(1/2) = 0 \iff |d_{\mathbf{K}}| = \exp(r_1\frac{\pi}{2} + n(\log 8\pi + \gamma)).$$

Since n and r_1 are bounded in the latter set, the discriminant d_K is also bounded. So S' is a set of number fields with bounded discriminant. Hence, we conclude our result by Theorem 1.1.11.

3.6. Proof of Theorem 7.1.7

From (7.3.2) and (7.3.3), we obtain

$$-2\left(m\frac{\zeta'_{\mathbf{K}}(1/2)}{\zeta_{\mathbf{K}}(1/2)}-n\frac{\zeta'_{\mathbf{L}}(1/2)}{\zeta_{\mathbf{L}}(1/2)}\right)=\log\frac{|d_{\mathbf{K}}|^m}{|d_{\mathbf{L}}|^n}+\frac{\pi}{2}(nr_1^{(\mathbf{L})}-mr_1^{(\mathbf{K})}),$$

where $r_1^{(\mathbf{L})}$ and $r_1^{(\mathbf{K})}$ denote the number of real embeddings of $\mathbf{L}$ and $\mathbf{K}$ respectively. If $nr_1^{(\mathbf{L})}-mr_1^{(\mathbf{K})}\neq 0$, then the right hand side of the above equation is a transcendental number by Lemma 4.2.2. On the other hand, if $nr_1^{(\mathbf{L})}-mr_1^{(\mathbf{K})}=0$, then the right hand side of the above equation is a transcendental number by Lemma 3.1.1. So both real numbers

$$\frac{\zeta'_{\mathbf{K}}(1/2)}{\zeta_{\mathbf{K}}(1/2)} \quad \text{and} \quad \frac{\zeta'_{\mathbf{L}}(1/2)}{\zeta_{\mathbf{L}}(1/2)}$$

can not be algebraic.

3.7. Proof of Corollary 7.1.8 and 7.1.9

If there exist two distinct numbers from the set given in Corollary 7.1.8, then it would be a contradiction to Theorem 7.1.7. So the first statement is a direct consequence of Theorem 7.1.7. Now we prove the second part of Corollary 7.1.8. Combining (7.3.2) and (7.3.3), we obtain

$$\frac{\zeta'_{\mathbf{K}}(1/2)}{\zeta_{\mathbf{K}}(1/2)}-\frac{n}{2}(\log 8\pi+\gamma)=r_1\frac{\pi}{4}-(1/2)\log|d_{\mathbf{K}}|.$$

If $\mathbf{K}=\mathbb{Q}$, then the right hand side of above equation is $\frac{\pi}{4}$. By Theorem 1.1.10, $|d_{\mathbf{K}}|>1$ for all $\mathbf{K}$ different from $\mathbb{Q}$. So

$$\frac{\zeta'_{\mathbf{K}}(1/2)}{\zeta_{\mathbf{K}}(1/2)}-\frac{n}{2}(\log 8\pi+\gamma)$$

is a transcendental number by Lemma 4.2.2 and 3.1.1.

For the proof of Corollary 7.1.9, note that the hypothesis given on discriminants ensures that the first condition of Theorem 7.1.7 is satisfied. Consequently, it follows from Theorem 7.1.7.

7.4 Concluding remarks

We believe that for any number field **K**, one should have

$$\zeta_{\mathbf{K}}(1/2) \neq 0 \implies \zeta'_{\mathbf{K}}(1/2) \neq 0.$$

Such results hold for Elliptic curves over $\mathbb{Q}$ as well as Modular forms [25]. The nature of the functional equation in these set ups are amenable to deduce the above supposition. The classical bounds between degree and discriminant in our context do not seem to be strong enough to prove the above supposition, at least through our approach.

Furthermore, if there does exist a number field **K** such that $\zeta_{\mathbf{K}}(1/2) \neq 0$ while $\zeta'_{\mathbf{K}}(1/2) = 0$, we shall have $\log \pi + \gamma$ being equal to a linear form in logarithm of algebraic numbers, an unlikely possibility from a transcendental perspective since neither $\log \pi$ nor γ is expected to be a Baker period, that is, a $\overline{\mathbb{Q}}$ - linear combination of logarithms of algebraic numbers (see [53] for details on Baker periods).

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