Two restriction problems in the representation theory of symmetric groups

By

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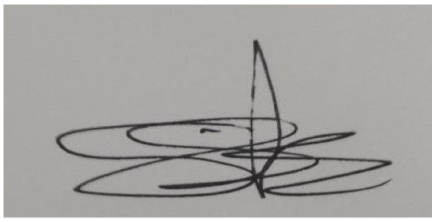

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DECLARATION

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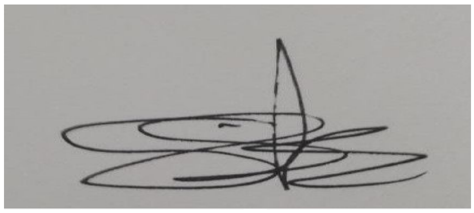

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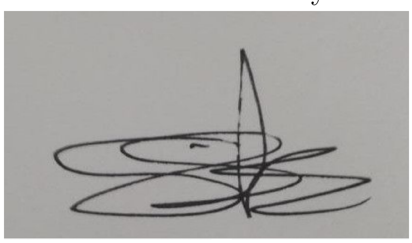

Love to my mother and

to my father rest in peace.

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Contents

Chapter 0

Introduction

Let $\mathrm{Irr}(G)$ denote the set of isomorphism classes of irreducible representations of a group G over a field of characteristic 0. Given a subgroup $H \leq G$ and for all $\phi \in \mathrm{Irr}(G), \psi \in \mathrm{Irr}(H)$ define the restriction coefficient $r_{\phi\psi}$ by

$$\mathrm{Res}^G_H(\phi) = \sum_{\psi \in \mathrm{Irr}(H)} r_{\phi\psi}\psi.$$

In this work we consider two problems on this theme.

In Chapter 1 we consider the restriction of representations from symmetric groups to their Sylow 2-subgroups. This was motivated by the study of odd partitions in Young's lattice by Ayyer, Prasad and Spallone in [4]. Let λ be a partition of n and let f_λ denote the dimension of the irreducible representation of the symmetric group S_n corresponding to λ. Recall that

$$f_\lambda = \frac{n!}{\prod_{x \in \lambda} h(x)},$$

where $h(x)$ is defined to be the hook-length of x, for each cell x in the Young diagram of λ. The dimension of λ is defined to be equal to f_λ. The partition λ is an odd partition if its dimension is odd. It can be shown that when n is a power of 2 then

λ is odd if and only if it is hook shaped.

The Bratteli diagram of irreducible representations of symmetric groups is called the Young's lattice. It is the graph on vertices labelled by integer partitions and edges between each partition and every partition obtained from the former by removing a single cell from it. The subgraph of odd partitions in the Young's lattice was shown in [4] to have the structure of a binary tree, and named the Macdonald tree. Given a finite group G, a prime integer p and a Sylow p-subgroup P of G, the McKay conjecture states that the number of irreducible representations of G with degree coprime to p is equal to the number of irreducible representations of the normaliser of P in G whose degrees are coprime to p. The conjecture is proved for symmetric groups (see Olsson in [32], Macdonald in [24]). Let P_n denote a 2-Sylow subgroup of S_n. It is easy to show that $N_{S_n}(P_n) = P_n$. In [13], Giannelli defines a bijection between $\mathrm{Irr}_{2'}(S_n)$ and $\mathrm{Irr}_{2'}(P_n)$. When n is a power of 2, the image of each odd degree representation V_λ under this bijection is defined to be the unique one-dimensional representation of the Sylow subgroup that occurs in the restriction to P_n. We find a formula for restriction coefficients between symmetric groups and their Sylow 2-subgroups. A combinatorial description of the bijection follows from this.

The theme of restriction continues in the second chapter, where we study the restriction of irreducible representations of the general linear group $GL_n(K)$ over a field K of characteristic 0 to the subgroup of permutation matrices, which is isomorphic to S_n. Let $W_\lambda(K^n)$ denote an irreducible representation of $GL_n(K)$ and V_μ denote an irreducible represenation of S_n. Let $r_{\lambda\mu}$ denote the multiplicity of V_μ in the restriction of $W_\lambda(K^n)$ to S_n. Similarly for partitions λ, μ, ν with $|\lambda| = |\mu| = |\nu|$, the Kronecker coefficient $g(\lambda, \mu, \nu)$ is the multiplicity of V_ν in $V_\lambda \otimes V_\mu$. Combinatorial interpretations for the restriction coefficients and Kronecker coefficients are open problems. They have extensively been studied through the lens of symmetric functions. For instance Littlewood showed in [22] that

$$r_{\lambda\mu} = \langle s_\lambda, s_\mu[H]\rangle,$$

where $H = h_0+h_1+\dots$ for complete symmetric functions h_i, as defined in Appendix 3. The plethysm $s_\mu[H]$ is defined as a substitution of the monomials of H into the variables of s_μ. For a partition μ with $n \geq |\mu|+\mu_1$, let $\mu[n]$ denote the partition with a first row of size $n-|\mu|$ added to μ. Orellana and Zabrocki found an inhomogeneous basis $\tilde{s}_\lambda$ for the ring of symmetric functions Λ, such that

$$s_\lambda = \sum_\mu r_{\lambda\mu[n]}\tilde{s}_\mu,$$

and

$$\tilde{s}_\lambda\tilde{s}_\mu = \sum_\nu \overline{g}(\lambda,\mu,\nu)\tilde{s}_\nu,$$

where the reduced Kronecker coefficients $\overline{g}(\lambda,\mu,\nu)$ denote the stable value of the Kronecker coefficient $g(\lambda[n],\mu[n],\nu[n])$.

An alternative approach to these problems is through the study of character polynomials. Let (V_n) be a sequence of representations of symmetric groups, so that V_n is a representation of S_n, and let $X_i(\sigma)$ be the number of i-cycles in a permutation σ. Let $\mathrm{tr}(\sigma;V_n)$ denote the character of V_n evaluated on a permutation $\sigma \in S_n$. The sequence (V_n) is said to have eventually polynomial character if there exists $p \in K[X_1,X_2,\dots]$ and an integer $N \geq 0$ such that for all $n \geq N$

$$\mathrm{tr}(\sigma;V_n) = p(X_1(\sigma),X_2(\sigma),\dots).$$

The polynomial p is called the character polynomial associated to the sequence (V_n), and we henceforth use $p(\sigma)$ to denote $p(X_1(\sigma),X_2(\sigma),\dots)$.

For a partition λ with at most n parts, let W_λ^n denote the irreducible representation of $GL_n(\mathbb{C})$. The sequence $(\mathrm{Res}_{S_n}^{GL_n(\mathbb{C})}(W_\lambda^n))$ is eventually polynomial. In Chapter 2 we find the character polynomial associated to this sequence, which we denote S_λ. The sequence $(V_{\mu[n]})$ is also eventually polynomial, with character polynomial q_μ. Formulas for q_μ were found by Macdonald in [25] and by Garsia and Goupil in [12]. For an eventually polynomial sequence (V_n) with associated character polynomial p, Garsia and Goupil ([12]) prescribe a method for computing the multiplicity of an irreducible representation $V_{\lambda[n]}$ in V_n by evaluating the inner product

$$\langle V_n, V_{\lambda[n]}\rangle_n = \frac{1}{n!}\sum_{\sigma\in S_n} p(\sigma)q_\mu(\sigma).$$

We find generating functions for the moment $\langle S_\lambda\rangle_n$, and find conditions for the positivity of $r_{\lambda,(n)}$ when λ has a simple shape (i.e. two rows, two columns, hook shapes). We then use the recursive formula for q_μ, by Garsia and Goupil, to compute generating functions for the restriction coefficients and the Kronecker coefficients, and their stable values. Through these generating functions we recover Littlewood's formula, as above, and an expression for reduced Kronecker coefficients that was found by Briand, Orellana and Rosas in [7].

The final chapter is an appendix that provides a quick summary of some results on symmetric functions that are used in the preceding chapters. We refer the reader to [25, Section I.2] for a detailed exposition of these concepts.

Chapter 1

Sylow 2-subgroups of symmetric groups

1.1 Introduction

Let G be a finite group and P be a Sylow p-subgroup of G for some prime p. Let $N_G(P)$ denote the normaliser of P in G, and let $\mathrm{Irr}_{p'}(G)$ (or $\mathrm{Irr}_{p'}(N_G(P))$) denote the set of irreducible representations of G (resp. of $N_G(P)$) whose dimensions are coprime to p. The McKay conjecture states that there is a bijection

$$\mathrm{Irr}_{p'}(G) \leftrightarrow \mathrm{Irr}_{p'}(N_G(P)),$$

The conjecture is proved for the family of symmetric groups (see Olsson in [32], Macdonald in [24]), and for arbitrary groups when $p = 2$ by Malle and Späth in [26]. When $p = 2$ and G is the symmetric group S_n, the Sylow subgroup P is self-normalising. Thus we know that there are as many odd-dimensional representations of S_n as there are one-dimensional representations of a Sylow 2-subgroup of S_n. Let P_n denote a Sylow 2-subgroup of S_n and let $H_k := P_{2^k}$.

Odd-dimensional irreducible representations of symmetric groups were studied by Ayyer, Prasad and Spallone in [4]. In particular, it is known (see [4, Theorem 1]) that the subgraph of the Young graph comprising odd-dimensional representations of S_n is a rooted binary tree that branches at every even level. This tree is called the Macdonald tree. A bijection between odd-dimensional irreducible representations of symmetric groups and one-dimensional irreducible representations of any of its Sylow 2-subgroups was found by Giannelli in [13]. This bijection associates an odd-dimensional irreducible representation of a symmetric group of S_{2^k} to the unique one-dimensional irreducible representation of H_k that occurs in its restriction.

Orellana, Orrison and Rockmore study the structure and representations of iterated wreath products of the cyclic group C_p in [33]. It is known (see [17] for instance) that the kth iterated wreath product of C_p is isomorphic to a Sylow p-subgroup of S_{p^k}. In particular [33] contains a complete description of the conjugacy classes and the irreducible representations of H_k for all $k \geq 0$. We recall this description in Section 1.2. The authors of [33] associate to each irreducible representation (and each conjugacy class of H_k) a labelled binary tree called a 2-*ary tree* (or in general, r-tree for $r \geq 2$). Our description of the conjugacy classes and representations associates them to a different combinatorial object, which we call a *1-2 binary tree.* Although a bijection must exist between these two sets of objects, we do not pursue it here.

In Section 1.3 we describe the branching of representations from P_n to P_{n-1} in terms of 1-2 binary trees, prove that it is multiplicity-free, and obtain the Bratteli diagram for these subgroups. In Section 1.4 we prove that the subgraph of one-dimensional representations in this Bratteli diagram has the structure of a binary tree that branches at every even level. We demonstrate the differences between this subgraph and the Macdonald tree.

A formula for restriction coefficients from S_n to P_n is stated and proved in Section 1.5. This is done first for H_k and then in general. Corollaries of this result include a combinatorial description of the bijection in [13]. The final section contains some generating functions for the sizes of conjugacy classes and the dimensions of irreducible representations.

The main results of this chapter are the following:

- Theorem 1.3.17, which describes the branching rules for a family of Sylow 2-subgroups as combinatorial operations on forests of binary trees[1]. As a result of this we observe a self-similarity in the Bratteli diagram of this family of subgroups.

- Theorem 1.4.11, which rephrases the branching rules for one-dimensional irreducible representations of these subgroups as operations on sequences of binary strings. This allows us to define this subgraph recursively. It has the structure of a binary tree, but is not isomorphic to the Macdonald tree.

- Theorem 1.5.2, which provides a recursive formula for the multiplicity of an irreducible representation of a Sylow 2-subgroup of S_{2^k} in the restriction of a representation of S_{2^k} to this subgroup.

1.2 Preliminaries and Notation

Throughout this paper, n is a positive integer with the binary expansion

$$n = 2^{k_1} + \cdots + 2^{k_s},$$

with $k_1 > \cdots > k_s$.

[1]see Definition 1.2.10 for the definition of a forest of trees.

Definition 1.2.1. *The binary digits of* n*, denoted* $\mathrm{Bin}(n)$ *is the set* $\{k_1, \ldots, k_s\}$.

Recall that Sylow 2-subgroups of S_n are denoted P_n, and when $n = 2^k$ for a nonnegative integer k, the Sylow 2-subgroup is denoted by H_k.

1.2.1 Structure and representation theory of P_n

The structure of Sylow p-subgroups is well studied (see [17]). It is known that:

$$P_n = \prod_{k \in \mathrm{Bin}(n)} H_k.$$

It is also known that $H_k \cong H_{k-1} \wr C_2$, where C_2 is the cyclic group of order 2. Equivalently, H_k is the k-th iterated wreath product of C_2. We refer the reader to [33] for a detailed exposition on iterated wreath products of cyclic groups C_r, and confine ourselves to describing results for the case $r = 2$. An element of H_k is denoted $(\sigma_1, \sigma_2)^\epsilon$, where $\sigma_1, \sigma_2 \in H_{k-1}$ and $\epsilon \in S_2 = \{\pm 1\}$. The identity element of the group is denoted id. Multiplication is defined as follows:

$$(\sigma_1, \sigma_2)^{\epsilon_1} (\tau_1, \tau_2)^{\epsilon_2} = \begin{cases} (\sigma_1 \tau_1, \sigma_2 \tau_2)^{\epsilon_1 \epsilon_2} & \epsilon_1 = 1, \\ (\sigma_1 \tau_2, \sigma_2 \tau_1)^{\epsilon_1 \epsilon_2} & \epsilon_1 = -1. \end{cases}$$

Lemma 4.5 of [33] describes the conjugacy classes for iterated wreath products. We divide the conjugacy classes of H_k into three types:

Definition 1.2.2. *Given an element* σ*, let* $[\sigma]$ *denote its conjugacy class. Then we have:*

- $[\sigma]$ *is of Type I if*

$$[\sigma] = [(\sigma_1, \sigma_1)^1],$$

where $(\sigma_1, \sigma_1)^1 \sim (\sigma_2, \sigma_2)^1$ *iff* $\sigma_1 \sim \sigma_2$ *in* H_{k-1}.

- *$[\sigma]$ is of Type II if*

$$[\sigma] = [(id, \sigma_1)^{-1}],$$

where $(id, \sigma_1)^{-1} \sim (id, \sigma_2)^{-1}$ iff $\sigma_1 \sim \sigma_2$ in H_{k-1}..

- *$[\sigma]$ is of Type III if*

$$[\sigma] = [(\sigma_1, \sigma_2)^1],$$

for elements $\sigma_1, \sigma_2 \in H_{k-1}$ and $[\sigma_1] \neq [\sigma_2]$. We also have $(\sigma_1, \sigma_2)^1 \sim (\sigma_2, \sigma_1)^1$.

Example 1.2.3. *We will enumerate (a representative of each of) the conjugacy classes of H_2. Before this we must know the conjugacy classes of H_1, which in turn requires us to know the conjugacy classes of H_0. H_0 comprises only the identity element id_0. By Definition 1.2.2, H_1 has one conjugacy class of Type I- $id_1 := [(id_0, id_0)^1]$ and one of Type II- $c := [(id_0, id_0)^{-1}]$. There is no conjugacy class of Type III since we cannot find two distinct conjugacy classes in H_0.*

The Type I conjugacy classes of H_2 are $[(id_1, id_1)^1]$ and $[(c, c)^1]$. The Type II conjugacy classes of H_2 are $[(id_1, id_1)^{-1}]$ and $[(id_1, c)^{-1}]$. The only Type III conjugacy class of H_2 is $[(id_1, c)^1]$.

The cardinalities of the above listed classes (denoted $c_k([\sigma])$) and the number of classes of each are listed in Table 1.1. The total number of conjugacy classes of the group H_k is denoted C_k in this table.

Type	Representative	# classes	Size of class(c_k)
I	$[(\sigma, \sigma)^1]$	C_{k-1}	$c_{k-1}([\sigma])^2$
II	$[(id, \sigma)^{-1}]$	C_{k-1}	$\|H_{k-1}\|c_{k-1}([\sigma])$
III	$[(\sigma_1, \sigma_2)^1]$	$\binom{C_{k-1}}{2}$	$2c_{k-1}([\sigma_1])c_{k-1}([\sigma_2])$

Table 1.1: Conjugacy classes of H_k

The enumeration of characters of Sylow 2-subgroups is a particular instance of characters of wreath products; we refer the reader to [10], [18] and [33] for details. All irreducible representatives of H_k are obtained as constituents in the induction of irreducible representations from the normal subgroup $H_{k-1} \times H_{k-1}$ to H_k. The irreducible representations of $H_{k-1} \times H_{k-1}$ are tensor products of two irreducible representations of H_{k-1}.

Let ϕ_1 and ϕ_2 be irreducible representations of H_{k-1}. If ϕ_2 is not isomorphic to ϕ_1, then $\mathrm{Ind}_{H_{k-1} \times H_{k-1}}^{H_k}(\phi_1 \otimes \phi_2)$ is an irreducible representation of H_k. We denote it $\mathrm{Ind}(\phi_1, \phi_2)$. The character values for $\mathrm{Ind}(\phi_1, \phi_2)$ are obtained by [15, Chapter 5, Pg 64]:

$$\mathrm{Ind}(\phi_1, \phi_2)((\sigma_1, \sigma_2)^\epsilon) = \begin{cases} \phi_1(\sigma_1)\phi_2(\sigma_2) + \phi_1(\sigma_2)\phi_2(\sigma_1) & \text{if } \epsilon = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

If ϕ_1 and ϕ_2 are isomorphic, with ϕ the representative of their common isomorphism class, the induced representation $\mathrm{Ind}_{H_{k-1} \times H_{k-1}}^{H_k}(\phi \otimes \phi)$ is the sum of two irreducible representations of H_k. We call these two irreducible representations the extensions of $\phi \otimes \phi$. The restriction of either extension to $H_{k-1} \times H_{k-1}$ is $\phi \otimes \phi$. It remains to find the character values of the two extensions on classes of Type II (see Definition 1.2.2). From [18] we have that the values of the two extensions of $\phi \otimes \phi$ on the class $(id, \sigma)^{-1}$ are $\phi(\sigma)$ and $-\phi(\sigma)$. Thus we denote these extensions $\mathrm{Ext}^+(\phi)$ and $\mathrm{Ext}^-(\phi)$ respectively.

$$\mathrm{Ext}^{\pm}(\phi)((\sigma_1, \sigma_2)^\epsilon) = \begin{cases} \phi(\sigma_1)\phi(\sigma_2) & \text{if } \epsilon = 1, \\ \pm\phi(\sigma_1\sigma_2) & \text{otherwise.} \end{cases} \tag{1.2}$$

Now we define three types of representations, as we did for conjugacy classes in Definition 1.2.2.

Definition 1.2.4. *Given an irreducible representation ϕ of H_k, we have:*

- *ϕ is of Type I if*

$$\phi = \mathrm{Ext}^+(\phi_1),$$

  for an irreducible representation ϕ_1 of H_{k-1}.

- *ϕ is of Type II if*

$$\phi = \mathrm{Ext}^-(\phi_1),$$

  for an irreducible representation ϕ_1 of H_{k-1}.

- *ϕ is of Type III if*

$$\phi = \mathrm{Ind}(\phi_1, \phi_2),$$

  for nonisomorphic irreducible representations ϕ_1 and ϕ_2 of H_{k-1}, and $\mathrm{Ind}(\phi_1, \phi_2) \cong \mathrm{Ind}(\phi_2, \phi_1)$*.*

These results are summarised in Table 1.2. Based on Table 1.2 it may be observed that the character table of H_k can be recursively obtained. The template for doing so is Table 1.5. The recursive process is illustrated for $k = 2$ in Table 1.4.

Table 1.2: Irreducible characters of H_k

Type	Notation	Description	Value on $(\sigma_1, \sigma_2)^1$	Value on $(id, \sigma)^{-1}$
I	$\mathrm{Ext}^+(\phi)$	Positive extension of $\phi \otimes \phi$	$\phi(\sigma_1)\phi(\sigma_2)$	$\phi(\sigma)$
II	$\mathrm{Ext}^-(\phi)$	Negative extension of $\phi \otimes \phi$	$\phi(\sigma_1)\phi(\sigma_2)$	$-\phi(\sigma)$
III	$\mathrm{Ind}(\phi_1, \phi_2)$	Induced from $\phi_1 \otimes \phi_2$	$\phi_1(\sigma_1)\phi_2(\sigma_2)$ $+\phi_1(\sigma_2)\phi_2(\sigma_1)$	0

Example 1.2.5. *We will illustrate the recursive nature of the representation theory of H_k by finding the character table of H_2 by first finding the character table of H_1 from that of H_0. H_0 is a 1×1 matrix with entry 1. Let Id denote the only irreducible representation of H_0. Then the two irreducible representations of H_1 are* $\mathrm{Ext}^{\pm}(Id)$*. Their values may be calculated from Table 1.2:*

Table 1.3: Character table for H_1:

	$C_1 := (id, id)^1$	$C_2 := (id, id)^{-1}$
$\text{Ext}^+(\text{Id})$	1	1
$\text{Ext}^-(\text{Id})$	1	-1

We know from Example 1.2.3 that there are five conjugacy classes of H_2. Therefore there must be five irreducible representations of H_2. The two Type I representations of H_2 are $\text{Ext}^+(\text{Ext}^+(Id))$ and $\text{Ext}^+(\text{Ext}^-(Id))$. The two Type II representations of H_2 are $\text{Ext}^-(\text{Ext}^+(Id))$ and $\text{Ext}^-(\text{Ext}^-(Id))$. The only Type III representation of H_2 is $\text{Ind}(\text{Ext}^+(Id), \text{Ext}^-(Id))$.

Table 1.4: Character table for H_2:

	$(C_1, C_1)^1$	$(C_2, C_2)^1$	$(C_1, C_2)^1$	$(id, C_1)^{-1}$	$(id, C_2)^{-1}$
$\text{Ext}^+(\text{Ext}^+(\text{Id}))$	1	1	1	1	1
$\text{Ext}^+(\text{Ext}^-(\text{Id}))$	1	1	-1	1	-1
$\text{Ext}^-(\text{Ext}^+(\text{Id}))$	1	1	1	-1	-1
$\text{Ext}^-(\text{Ext}^-(\text{Id}))$	1	1	-1	-1	1
$\text{Ind}(\text{Ext}^+(\text{Id}), \text{Ext}^-(\text{Id}))$	2	-2	0	0	0

This outlines a general recursive procedure for the calculation of character tables of H_k, given the character table of H_{k-1}.

Table 1.5: Template for the character table for H_k

	$[(\sigma_1, \sigma_2)^1]$	$[(\sigma, \sigma)^1]$	$[(id, \sigma)^{-1}]$
$\text{Ext}^+(\phi)$	$\phi(\sigma_1)\phi(\sigma_2)$	$\phi(\sigma)\phi(\sigma)$	character table for H_{k-1}
$\text{Ext}^-(\phi)$	$\phi(\sigma_1)\phi(\sigma_2)$	$\phi(\sigma)\phi(\sigma)$	-character table for H_{k-1}
$\text{Ind}(\phi_1, \phi_2)$	$\phi_1(\sigma_1)\phi_2(\sigma_2) + \phi_1(\sigma_2)\phi_2(\sigma_1)$	$2\phi_1(\sigma)\phi_2(\sigma)$	0

Remark 1.2.6. From Table 1.2 we know the dimensions of the representations of each type. Thus we have $\dim(\mathrm{Ext}^{\pm}(\phi)) = \dim(\phi)^2$ and $\dim(\mathrm{Ind}(\phi_1, \phi_2)) = 2\dim(\phi_1)\dim(\phi_2)$.

1.2.2 Binary trees and forests

Binary trees are commonly occurring objects in computer science and mathematics. For a complete introduction to these objects see [20].

A rooted binary tree is a tuple (r, L, R)- a root vertex r, and binary trees L and R, denoted the left and right subtree. They are commonly depicted by connecting the root vertex r to the root vertices of each of the subtrees L and R. The trivial binary tree $(r, \emptyset, \emptyset)$ comprises only the root vertex. Given a vertex y of a binary tree, it is known that there exists a unique path $r = v_0, v_1, \ldots, v_k = y$. The height of the vertex y is k- the number of vertices on this unique path (not counting the root vertex). Each vertex of a binary tree is connected to two possibly trivial subtrees. If both subtrees connected to a vertex are trivial, the vertex is called an external vertex. All vertices that are not external are called internal.

For our purposes the designation of a subtree as either the right or the left is superfluous. Thus we may define binary trees formally as a tuple (r, S) of a root vertex r and a multiset S of at most two binary trees. The trivial tree is defined as the unique tree that has an empty multiset of subtrees S. The height of a vertex is unaffected by this modification in definition. Binary trees where all the external vertices have the same height are called 1-2 binary trees.

Definition 1.2.7. *A 1-2 binary tree of height k is a tuple (r, S) consisting of a root vertex r and multiset S comprising of up to two binary trees, where every external vertex of the tree has height k.*

We refer to 1-2 binary trees as either binary trees or trees when there is no ambiguity in doing so.

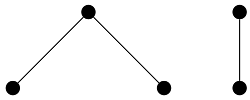


Figure 1.1: 1-2 binary trees of height 1

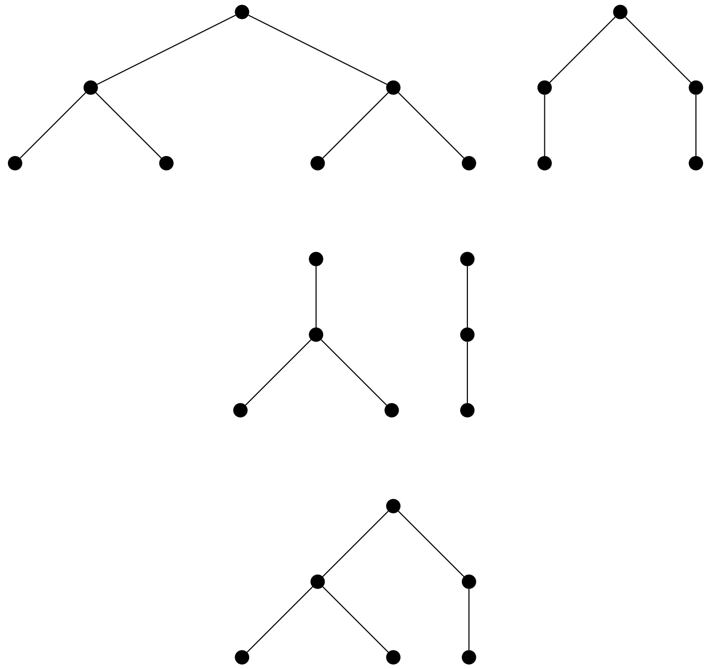


Figure 1.2: 1-2 binary trees of height 2

Example 1.2.8. *The trivial tree is the unique tree of height 0. There are two 1-2 binary trees of height* 1*. These are as in Figure 1.1.*

Example 1.2.9. *There are 5 distinct 1-2 binary trees of height 2. These are as in Figure 1.2.*

Definition 1.2.10. *Given an integer n with $\mathrm{Bin}(n) = \{k_1, \ldots, k_s\}$, a forest of size n is an ordered collection of 1-2 binary trees $(T_1, \ldots, T_s)$, where T_i is a 1-2 binary tree of height k_i for $i = 1, \ldots, s$.*

A forest with a single element is identified with the tree that is its only element.

1.2.3 Representations, classes and trees

We will now show how to associate 1-2 binary trees to irreducible representations and conjugacy classes of H_k. This association was arrived at after noticing that the OEIS entry for the number of representations of H_k (see [42, Sequence A006893]) also counted the number of 1-2 binary trees of height k.

Theorem 1.2.11. *The number of 1-2 binary trees of height k, the number of irreducible representations of H_k and the number of conjugacy classes of H_k all satisfy the following recurrence relation*

$$\begin{aligned} a_k &= 2a_{k-1} + \binom{a_{k-1}}{2}, \\ a_0 &= 1. \end{aligned} \tag{1.3}$$

Proof. Let a_k be the number of 1-2 binary trees of height k. There is a unique tree of height 0 (the trivial tree) so $a_0 = 1$. A 1-2 binary tree of height k comprises either a single subtree of height $k-1$ attached to the root or two subtrees of height $k-1$ attached to the root. There are a_{k-1} of the former. There are a_{k-1} trees in the latter category whose subtrees are identical, and $\binom{a_{k-1}}{2}$ trees in the latter category whose subtrees are distinct.

Let a_k be the number of irreducible representations of H_k. There are two irreducible representations of H_k associated to each irreducible representation ϕ of H_{k-1}- namely $\mathrm{Ext}^+\phi$ and $\mathrm{Ext}^-\phi$. This makes $2a_{k-1}$ representations so far. There is a single representation of H_k associated to a choice of two nonisomorphic representations ϕ_1, ϕ_2 of H_{k-1}- namely $\mathrm{Ind}\phi_1, \phi_2$. These make up the remaining $\binom{a_{k-1}}{2}$.

Let a_k be the number of conjugacy classes of H_k. Given a conjugacy class $[\sigma]$ of H_{k-1} we can form the conjugacy classes $[(\sigma, \sigma)^1]$ and $[(id, \sigma)^{-1}]$. Given two distinct conjugacy classes $[\sigma_1], [\sigma_2]$ of H_{k-1}, we can form the conjugacy class $[(\sigma_1, \sigma_2)^1]$. $\square$

This observation leads us to define three types of binary trees, in line with Definitions 1.2.2 and 1.2.4.

Definition 1.2.12. *Given a 1-2 binary tree T of height k, we say:*

- *T is of Type I if*

  $$T = (r, \{T_1, T_1\}),$$

  for a 1-2 binary tree T_1 of height $k-1$.

- *T is of Type II if*

  $$T = (r, \{T_1\}),$$

  for a 1-2 binary tree T_1 of height $k-1$.

- *T is of Type III if*

  $$T = (r, \{T_1, T_2\}),$$

  for distinct 1-2 binary trees T_1 and T_2 of height $k-1$.

This division into three types facilitates an understanding of the bijections between representations of H_k or conjugacy classes of H_k on the one hand and binary trees of height k on the other.

Definition 1.2.13. *Define a family of functions θ_{2^k} for nonnegative integers k between the set of irreducible representations of H_k and the set of 1-2 binary trees of height k as under:*

$$\theta_{2^k}(\Gamma) = \begin{cases} (r, \{\theta_{2^{k-1}}(\phi), \theta_{2^{k-1}}(\phi)\}) & \Gamma = \mathrm{Ext}^+(\phi \otimes \phi), \\ (r, \{\theta_{2^{k-1}}(\phi)\}) & \Gamma = \mathrm{Ext}^-(\phi \otimes \phi), \\ (r, \{\theta_{2^{k-1}}(\phi_1), \theta_{2^{k-1}}(\phi_2)\}) & \Gamma = \mathrm{Ind}(\phi_1, \phi_2), \end{cases} \tag{1.4}$$

for $k \geq 1$, and $\theta_0(\phi)$ is defined to be the trivial tree. The dimension of a binary *tree T is denoted* $\dim(T)$ *and is defined to be the dimension of its corresponding irreducible representation.*

Theorem 1.2.14. *For every $k \geq 0$ the function θ_{2^k} is a bijection from the set of irreducible representations of H_k to the set of 1-2 binary trees of height k.*

Proof. We prove this by induction on k. For $k = 0$ the trivial representation of H_0 is mapped to the trivial tree.

For $k \geq 1$, we assume $\theta_{2^{k-1}}$ is a bijection. We will define the inverse of θ_{2^k}:

$$\begin{aligned}
\theta_{2^k}^{-1}((r, \{T, T\})) &= \mathrm{Ext}^+(\theta_{2^{k-1}}^{-1}(T)),\\
\theta_{2^k}^{-1}((r, \{T\})) &= \mathrm{Ext}^-(\theta_{2^{k-1}}^{-1}(T)),\\
\theta_{2^k}^{-1}((r, \{T_1, T_2\})) &= \mathrm{Ind}(\theta_{2^{k-1}}^{-1}(T_1), \theta_{2^{k-1}}^{-1}(T_2)).
\end{aligned}$$

The inverse function is well-defined since $\theta_{2^{k-1}}$ is a bijection. It is easy to verify that this is the inverse to θ_{2^k}. $\square$

Definition 1.2.15. *Choosing class representatives as in Table 1.1, we define a bijection Θ_{2^k} between representatives of conjugacy classes of H_k and 1-2 binary trees of height k as under:*

$$\Theta_{2^k}(\sigma) = \begin{cases} (r, \{\Theta_{2^{k-1}}(\sigma), \Theta_{2^{k-1}}(\sigma)\}) & \sigma = (\sigma, \sigma)^1,\\ (r, \{\Theta_{2^{k-1}}(\sigma)\}) & \sigma = (id, \sigma)^{-1},\\ (r, \{\Theta_{2^{k-1}}(\sigma_1), \Theta_{2^{k-1}}(\sigma_2)\}) & \sigma = (\sigma_1, \sigma_2)^1, \end{cases} \tag{1.5}$$

for $k \geq 1$, and $\Theta_0(id_1)$ defined to be the trivial tree. The order of a binary tree T is denoted $\rho(T)$ and is defined to be the size of its corresponding conjugacy class.

Theorem 1.2.16. *For every $k \geq 0$ the function Θ_{2^k} is a bijection from the set of conjugacy classes of H_k to the set of 1-2 binary trees of height k.*

Proof. We prove this by induction on k. For $k = 0$ the only permutation $id_1 = [1]$ in H_0 is mapped to the trivial tree.

For $k \geq 1$, we assume $\Theta_{2^{k-1}}$ is a bijection. We will define the inverse of Θ_{2^k}:

$$\Theta_{2^k}^{-1}(r, \{T, T\}) = (\Theta_{2^{k-1}}^{-1}(T), \Theta_{2^{k-1}}^{-1}(T))^1,$$

$$\Theta_{2^k}^{-1}(r, \{T\}) = (id, \Theta_{2^{k-1}}^{-1}(T))^{-1},$$

$$\Theta_{2^k}^{-1}(r, \{T_1, T_2\}) = (\Theta_{2^{k-1}}^{-1}(T_1), \Theta_{2^{k-1}}^{-1}(T_2))^1.$$

The inverse function is well-defined since $\Theta_{2^{k-1}}$ is a bijection. It is easy to verify that this is the inverse to Θ_{2^k}. $\square$

Example 1.2.17. *Consider the assignment of the left and right tree in Figure 1.1 to the* $\mathrm{Ext}^+(Id)$ *and* $\mathrm{Ext}^-(Id)$ *of* H_1 *respectively. In Figure 1.3 we list the images of* θ_2*, in the same order as these trees occur in Figure 1.2. The first two trees in Figure are of Type I and correspond to* $\mathrm{Ext}^+(\mathrm{Ext}^+(Id)), \mathrm{Ext}^+(\mathrm{Ext}^-(Id))$ *respectively. The next two are of Type II and correspond to* $\mathrm{Ext}^-(\mathrm{Ext}^+(Id)), \mathrm{Ext}^-(\mathrm{Ext}^-(Id))$ *respectively; the final tree in the figure is of Type III and corresponds to the representation* $\mathrm{Ind}(\mathrm{Ext}^+(Id), \mathrm{Ext}^-(Id))$.

These two families of bijections extend to the case where n is an arbitrary integer as below:

Definition 1.2.18. *With* θ_{2^k} *as in Definition 1.2.13 and* Θ_{2^k} *as in Definition 1.2.15 we define* θ_n *and* Θ_n *as follows:*

$$\theta_n = \theta_{2^{k_1}} \times \cdots \times \theta_{2^{k_s}}$$

$$\Theta_n = \Theta_{2^{k_1}} \times \cdots \times \Theta_{2^{k_s}}.$$

The dimension *of a forest and the* order *of a forest are defined respectively to be the product of the dimensions and the product of the orders of the trees in the forest.*

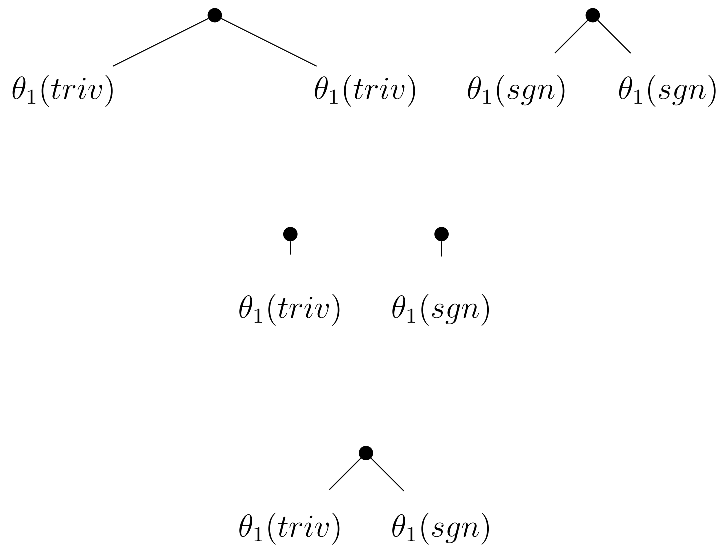


Figure 1.3: The images of θ_2 listed in the same order as Figure 1.2.

Theorem 1.2.19. *The functions θ_n and Θ_n are bijections.*

Proof. This is clear from Theorems 1.2.14 and 1.2.16 since $P_n = H_{k_1} \times \cdots \times H_{k_s}$. $\square$

1.2.4 Cores and quotients of partitions

In this section we recall the definition of the p-core and the p-quotient of a partition for a positive integer p. We refer the reader to [16] for a complete exposition of this topic. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ with $\lambda_i \geq 0$ for $i = 1, \ldots, l$ define

$$\beta(\lambda) = (\lambda_l, \lambda_{l-1} + 1, \ldots, \lambda_1 + l - 1),$$

which we call the beta sequence of λ. A beta sequence is associated to a unique partition but the converse is untrue. A partition may be augmented by adding zeroes to give a longer beta sequence. However for a fixed length, l, that is greater than the length of the partition, this association is bijective.

Example 1.2.20. *Let* $\lambda = (4,3,3,2)$ *(see figure 1.4). Then for* $l = 4$ *we have*

$$\beta(\lambda) = (2+0, 3+1, 3+2, 4+3).$$

So $\beta(\lambda) = (2,4,5,7)$*. However this association is not bijective. For instance for* $l = 6$ *we first add two* 0 *parts to* λ *to give a sequence of length* $l = 6$*:* $\lambda = (4,3,3,2,0,0)$*. Then*

$$\beta(\lambda) = (0+0, 0+1, 2+2, 3+3, 3+4, 4+5).$$

So $\beta(\lambda) = (0,1,4,6,7,9)$*. However for a fixed length we can clearly see that the association is bijective.*

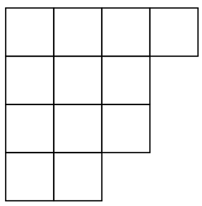


Figure 1.4: The Young diagram for $\lambda = (4,3,3,2)$ with $\beta(\lambda) = (2,4,5,7)$ for $l = 4$.

We define an equivalence on partitions by their beta sequences.

Definition 1.2.21. *Consider two partitions* λ *and* μ *with the same number of parts. If the partitions have an unequal number of parts we add parts of length* 0 *to one of them until they have the same number of parts. Then we define* $\mu \overset{p}{\sim} \lambda$ *if the beta sequences of both have the same number of entries congruent to* $i \mod p$ *for every* $i = 0, \ldots, p-1$*.*

Theorem 1.2.22. *The relation* $\overset{p}{\sim}$ *is an equivalence relation.*

Example 1.2.23. *Consider* $\lambda = (4,3,3,2)$ *as in Figure 1.4 and* $\mu = (4,2,2,2)$ *as in Figure 1.5. We have seen that* $\beta(\lambda) = (2,4,5,7)$*, and we can calculate* $\beta(\mu) = (2,3,4,7)$*. Let* $p = 2$*: the beta sequences of both partitions have two parts congruent to* $0 \mod 2$*, and two parts congruent to* $1 \mod 2$*. Thus* $\lambda \overset{p}{\sim} \mu$*.*

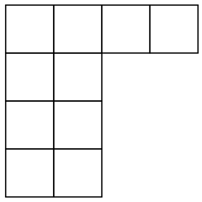


Figure 1.5: The Young diagram for $\mu = (4, 2, 2, 2)$ with $\beta(\mu) = (2, 3, 4, 7)$.

Definition 1.2.24. *For $i = 0, \ldots, p-1$, let a_i denote the number of parts of $\beta(\lambda)$ that are in the ith residue class $\mod p$, and let $A_i = \{i, p+i, \ldots, (a_i - 1)p + i\}$. Then the elements of $\cup_i A_i$ arranged in ascending order form the beta sequence of a partition which is called the p-core of λ and denoted $core_p(\lambda)$.*

Example 1.2.25. *For $\lambda = (4, 3, 3, 2)$ and $p = 2$ we have $a_0 = 2, a_1 = 2$. So $A_0 = \{0, 0+2\}$ and $A_1 = \{1, 2+1\}$. The partition given by the beta sequence $(0, 1, 2, 3)$ is $\emptyset$. Thus $core_2(\lambda) = \emptyset$.*

The partition $\mu = (4, 2, 2, 2)$ has beta sequence $\beta(\mu) = (2, 3, 4, 7)$. We again have $a_0 = 2, a_1 = 2$. So $A_0 = \{0, 0+2\}$ and $A_1 = \{1, 2+1\}$. Thus $core_2(\mu) = \emptyset$.

Observe that for any partition λ, $core_p(\lambda)$ is in the same equivalence class as λ since it has the same number of elements in each congruence class as λ.

Definition 1.2.26. *For $i = 0, \ldots, p-1$ define $\beta(\lambda)_i$ to be the subsequence of $\beta(\lambda)$ consisting of all entries that are in the ith residue class $\mod p$. Let ν_i denote the partition corresponding to the beta sequence whose jth entry is given by*

$$\beta(\nu_i)(j) = \frac{\beta(\lambda)_i(j) - i}{p}.$$

The tuple $\nu_0, \ldots, \nu_{p-1}$ is called the p-quotient of λ, denoted $quo_p(\lambda)$.

Example 1.2.27. *For λ with $\beta(\lambda) = (2, 4, 5, 7)$, decompose it into the disjoint subsequences $\beta(\lambda)_0 = (2, 4)$ and $\beta(\lambda)_1 = (5, 7)$. Corresponding to the sequence $(2, 4)$ we obtain the beta sequence $(\frac{2-0}{2}, \frac{4-0}{2})$ and corresponding to the sequence $(5, 7)$ we obtain the beta sequence $(\frac{5-1}{2}, \frac{7-1}{2})$.*

Let ν_0 be the partition corresponding to the first beta sequence; i.e. $\beta(\nu_0) = (1, 2)$. Then $\nu_0 = (1, 1)$. Let ν_0 be the partition corresponding to the first beta sequence; i.e. $\beta(\nu_1) = (2, 3)$. Then $\nu_1 = (2, 2)$.

Thus the $quo_2(\lambda) = ((1, 1), (2, 2))$.

For μ with $\beta(\lambda) = (2, 3, 4, 7)$, decompose it into the sequences $\beta(\lambda)_0 = (2, 4)$ and $\beta(\lambda)_1 = (3, 7)$.

Thus $\beta(\nu_0) = (1, 2)$ and $\beta(\nu_1) = (1, 3)$. Thus $\nu_0 = (1, 1)$ and $\nu_1 = (2, 1)$.

Thus $quo_2(\mu) = ((1, 1), (2, 1))$.

For any integer p, the pair $(\mathrm{core}_p(\lambda), \mathrm{quo}_p(\lambda))$ uniquely identify the partition λ, and satisfy the relation $|\lambda| = |\mathrm{core}_p(\lambda)| + p \sum_{i=0}^{p-1} |\nu_i|$.

Definition 1.2.28. *Let b be the permutation obtained from $\beta(\lambda)$ by replacing the entries of $\beta(\lambda)_0$ with $1, 3, \ldots$ and the entries of $\beta(\lambda)_1$ with $2, 4, \ldots$. The sign of λ is denoted $sgn(\lambda)$ and is defined to be the sign of the permutation b.*

Example 1.2.29. *For λ with $\beta(\lambda) = (2, 4, 5, 7)$ we replace the entries $2, 4$ with $1, 3$ respectively and the entries $5, 7$ with $2, 4$ respectively to obtain the permutation $b = 1324$. The sign of the permutation b is -1. Thus $sgn(\lambda) = -1$.*

For μ with $\beta(\mu) = (2, 3, 4, 7)$ we replace the entries $2, 4$ by $1, 3$ respectively, and the entries $3, 7$ with $2, 4$ respectively to obtain the permutation $b = 1234$. The sign of the permutation b is 1. Thus $sgn(\mu) = 1$.

For a partition λ of n, let χ_λ denote the character of the representation of S_n corresponding to λ, and for a partition ρ of n, let $\chi_\lambda(\rho)$ be the character value of this representation evaluated at a permutation with cycle type ρ. For a partition ρ with every part of ρ divisible by a fixed integer p, Littlewood expresses the value $\chi_\lambda(\rho)$ in terms of the p-quotient and the p-core of λ ([23, Chapter VIII, Page 143]).

We state the result here for $p = 2$ to suit its application in the proof of Theorem 1.5.2.

Theorem 1.2.30 (Littlewood's formula (for $p = 2$),[23])**.** *Given an even integer n and partitions $\lambda, \rho \vdash n$ with $\rho = 2^{a_1}4^{a_2}6^{a_3}\ldots$ in the exponential notation with all $a_i \geq 0$, let $\sqrt{\rho} = 1^{a_1}2^{a_2}3^{a_3}\cdots$. Then*

$$\chi_\lambda(\rho) = \begin{cases} 0 & \mathit{core}_2(\lambda) \neq \emptyset \\ \operatorname{sgn}(\lambda) \sum_{\nu \vdash \frac{n}{2}} c^{\nu}_{\nu_0,\nu_1} \chi_\nu(\sqrt{\rho}) & \mathit{otherwise}, \end{cases}$$

where $c^{\nu}_{\nu_0,\nu_1}$ is the Littlewood-Richardson coefficient (see Definition 3.0.5).

Example 1.2.31. *In this example we will show how to reconstruct a partition from its 2-core and 2-quotient, and demonstrate Theorem 1.2.30.*

Let λ be such that $\mathit{core}_2(\lambda) = \emptyset$ and $\mathit{quo}_2(\lambda) = ((2,0),(1,1))$.

Denote $\nu_0 = (2,0)$ and $\nu_1 = (1,1)$. Then $\beta(\nu_0) = (0,3)$ and $\beta(\nu_1) = (1,2)$.

We work backwards to obtain $\beta(\lambda)_0 = \beta(\nu_0) + 0 \times 2$ and $\beta(\lambda)_1 = \beta(\nu_1) \times 2 + 1$ (where by adding 1 or 0 to ν_1 or ν_0 we mean adding this integer to each part of the partition). So we have

$$\beta(\lambda)_0 = (0,6),$$
$$\beta(\lambda)_1 = (3,5).$$

Thus $\beta(\lambda) = \beta(\lambda)_0 \cup \beta(\lambda)_1 = (0,3,5,6)$, and thus $\lambda = (3,3,2)$. The sign of λ is -1.

From the Pieri rules (see [43] or Equation 3.4) we know that:

$$c^{\nu}_{\nu_0,\nu_1} = \begin{cases} 1 & \nu = (2,1^2) \text{ or } (3,1) \\ 0 & \text{otherwise} \end{cases}$$

Consider $\rho = (8)$. Then $\sqrt{\rho} = (4)$. By Theorem 1.2.30

$$\chi_{(3,3,2)}\left((8)\right) = -\chi_{(2,1^2)}\left((4)\right) - \chi_{(3,1)}\left((4)\right).$$

1.3 The Bratteli diagram

Given the family of subgroups $\{P_n\}_{n\geq 0}$ with $P_0 \subset \cdots P_{n-1} \subset P_n \subset \cdots$, the Bratteli diagram of this family, denoted $\mathbb{P}$, is the graded poset whose elements at the nth level are indexed by the irreducible representations of P_n, for all $n \geq 0$. An edge exists between the vertex corresponding to the representation γ of P_{n-1} and the vertex corresponding to the representation Γ of P_n if γ is a constituent of $\mathrm{Res}^{P_n}_{P_{n-1}}(\Gamma)$.

Observe that

$$P_{2^k-1} = H_{k-1} \times P_{2^{k-1}-1} \leqslant H_{k-1} \times H_{k-1}.$$

For example, $\mathrm{Bin}(3) = \{1, 0\}$ and thus $P_3 = H_1 \times H_0$. Through the chain of inclusions

$$H_1 \times H_0 \leqslant H_1 \times H_1 \leqslant H_2$$

we see that $P_3 \leqslant H_1 \times H_1$. Similarly $P_7 = H_2 \times H_1 \times H_0$. Through the above chain of inclusions we see that $P_7 \leqslant H_2 \times H_2$.

Thus we have:

$$\mathrm{Res}^{H_k}_{P_{2^k-1}}(\Gamma) = \left(\mathrm{Id} \times \mathrm{Res}^{H_{k-1}}_{P_{2^{k-1}-1}}\right) \circ \mathrm{Res}^{H_k}_{H_{k-1}\times H_{k-1}}(\Gamma).$$

We denote $\mathrm{Res}^{H_k}_{P_{2^k-1}}(\Gamma)$ by r_{2^k}, and note that:

$$r_{2^k} = (\mathrm{Id} \times r_{2^{k-1}}) \circ \mathrm{Res}^{H_k}_{H_{k-1}\times H_{k-1}}. \tag{1.6}$$

For instance if Γ is an irreducible representation of H_k of Type III (see Table 1.2) and $\mathrm{Res}^{H_k}_{H_{k-1}\times H_{k-1}}(\Gamma) = \phi_1 \otimes \phi_2 + \phi_2 \otimes \phi_1$, then:

$$r_{2^k}(\Gamma) = \phi_1 \otimes r_{2^{k-1}}(\phi_2) + \phi_2 \otimes r_{2^{k-1}}(\phi_1).$$

Following as before the convention $\mathrm{Bin}(n) = \{k_1, \dots, k_s\}$ with $k_1 > \dots > k_s$, an irreducible representation of P_n is of the form $\phi_1 \otimes \cdots \phi_s$, where ϕ_i is an irreducible representation of H_{k_i} for $i = 1, \dots, s$. We extend Equation (1.6) to the general case by restricting the last component, ϕ_s. Let r_n denote the restriction $\mathrm{Res}^{P_n}_{P_{n-1}}$. Then:

$$r_n = \mathrm{Id} \times \cdots \mathrm{Id} \times r_{2^{k_s}}. \tag{1.7}$$

Example 1.3.1. *Let* $\phi = \mathrm{Ind}(\mathrm{Ext}^+(\mathrm{Id}), \mathrm{Ext}^-(\mathrm{Id}))$ *be the irreducible representation of* H_2 *(i.e.* P_4*) defined in Table 1.4. Then*

$$\begin{aligned}
r_4(\phi) &= (\mathrm{Id} \times r_2) \circ \mathrm{Res}^{H_k}_{H_{k-1}\times H_{k-1}}(\phi) \\
&= (\mathrm{Id} \times r_2)\left[\mathrm{Ext}^+(\mathrm{Id}) \otimes \mathrm{Ext}^-(\mathrm{Id}) + \mathrm{Ext}^-(\mathrm{Id}) \otimes \mathrm{Ext}^+(\mathrm{Id})\right] \\
&= \mathrm{Ext}^+(\mathrm{Id}) \otimes r_2\left(\mathrm{Ext}^-(\mathrm{Id})\right) + \mathrm{Ext}^-(\mathrm{Id}) \otimes r_2\left(\mathrm{Ext}^+(\mathrm{Id})\right) \\
&= \mathrm{Ext}^+(\mathrm{Id}) \otimes \mathrm{Id} + \mathrm{Ext}^-(\mathrm{Id}) \otimes \mathrm{Id},
\end{aligned}$$

since $r_2(\mathrm{Ext}^+(\mathrm{Id})) = r_2(\mathrm{Ext}^-(\mathrm{Id})) = \mathrm{Id}$.

Now consider $\Phi = \mathrm{Ext}^+(\phi) \otimes \phi$. *This is an irreducible representation of* P_{12}.

The binary digits of 12 *are* $\{3, 2\}$*. Thus* $r_{12} = \mathrm{Id} \times r_4$*.*

$$\begin{aligned} r_{12}(\Phi) &= (\mathrm{Id} \otimes r_4)(\Phi) \\ &= \mathrm{Ext}^+(\phi) \otimes r_4(\phi) \\ &= \mathrm{Ext}^+(\phi) \otimes \mathrm{Ext}^+(\mathrm{Id}) \otimes \mathrm{Id} + \mathrm{Ext}^+(\phi) \otimes \mathrm{Ext}^-(\mathrm{Id}) \otimes \mathrm{Id}. \end{aligned}$$

Definition 1.3.2. *Let* Γ *be an irreducible representation of* P_n*, for some* $n \geq 1$*. Then the down-set of* Γ*, denoted* Γ^-*, is defined to be the multiset of representations of* P_{n-1} *with each representation* γ *occurring in* Γ^- *as many times as* γ *occurs in* $r_n(\Gamma)$*. The up-set of* Γ*, denoted denoted* Γ^+*, is defined to be the multiset of representations of* P_{n+1} *such that* Γ *occurs in their down-set, each repeated as many times as* Γ *occurs in its down-set.*

Example 1.3.3. *In Example 1.3.1, the down-set of* ϕ *is*

$$\phi^- = \{\mathrm{Ext}^+(\mathrm{Id}) \otimes \mathrm{Id}, \mathrm{Ext}^-(\mathrm{Id}) \otimes \mathrm{Id}\}.$$

The up-set of $\varphi = \mathrm{Ext}^+(\mathrm{Id}) \otimes \mathrm{Id}$ *is:*

$$\varphi^+ = \{\mathrm{Ext}^+(\mathrm{Ext}^+(\mathrm{Id})), \mathrm{Ext}^-(\mathrm{Ext}^+(\mathrm{Id})), \phi\}.$$

One can check that φ *occurs in the restriction of each of these representations with multiplicity one, and does not occur in the restriction of* $\mathrm{Ext}^\pm(\mathrm{Ext}^-(\mathrm{Id}))$*.*

We make the analogous definition of the down-set and up-set of a forest of binary trees:

Definition 1.3.4. *Given a forest* F *of size* n*, let* Γ *be the unique irreducible representation* P_n *such that* $F = \theta_n(\Gamma)$ *as in Definition 1.2.13. Define the down-set* F^-

and the up-set F^+ as the multisets

$$F^- = \{\theta_{n-1}(\gamma) | \gamma \in \Gamma^-\},$$
$$F^+ = \{\theta_{n+1}(\gamma) | \gamma \in \Gamma^+\}.$$

The following result follows from the definition of r_n in Equation (1.7).

Proposition 1.3.5. *Let $F = (T_1, \ldots, T_s)$ (remember that T_s is the shortest tree). Then*

$$F^- = \{(T_1, \ldots, T_{s-1}, t_s) \mid t_s \in T_s^-\}.$$

We define an operation on 1-2 binary trees to retrieve the downset τ^- from the tree τ:

Definition 1.3.6. *For a tree τ let $\mathrm{Res}(\tau)$ be defined as the multiset*

$$\mathrm{Res}(\tau) = \begin{cases} T \times \mathrm{Res}(T) & \tau = (r, \{T, T\}), \\ T \times \mathrm{Res}(T) & \tau = (r, \{T\}), \\ T_1 \times \mathrm{Res}(T_2) \cup T_2 \times \mathrm{Res}(T_1) & \tau = (r, \{T_1, T_2\}), \\ \{\emptyset\} & \tau = (r, \emptyset), \end{cases}$$

where $T \times \mathrm{Res}(\tilde{T})$ is the multiset $\{T \times x \mid x \in \mathrm{Res}(\tilde{T})\}$.

Example 1.3.7. *Consider the 1-2 binary tree of height 2 in Figure 1.6. In this example we compute the down-set of this tree. We know from Example 1.2.17 that*

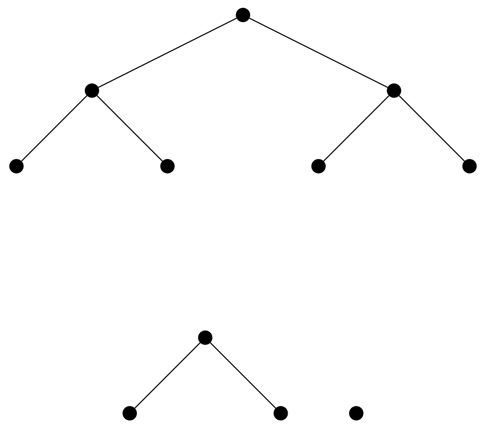


Figure 1.6: A tree and its down-set

this tree corresponds to the representation $\mathrm{Ext}^+(\mathrm{Ext}^+(\mathrm{Id}))$ *of* H_2*.*

$$\begin{aligned} r_4(\mathrm{Ext}^+(\mathrm{Ext}^+(\mathrm{Id}))) &= (\mathrm{Id} \times r_2) \circ \mathrm{Res}^{H_k}_{H_{k-1}\times H_{k-1}} \left(\mathrm{Ext}^+(\mathrm{Ext}^+(\mathrm{Id}))\right) \\ &= (\mathrm{Id} \times r_2) \left[\mathrm{Ext}^+(\mathrm{Id}) \otimes \mathrm{Ext}^+(\mathrm{Id})\right] \\ &= \mathrm{Ext}^+(\mathrm{Id}) \otimes r_2 \left(\mathrm{Ext}^+(\mathrm{Id})\right) \\ &= \mathrm{Ext}^+(\mathrm{Id}) \otimes \mathrm{Id} \end{aligned}$$

Thus $\mathrm{Ext}^+(\mathrm{Ext}^+(\mathrm{Id}))^- = \{\mathrm{Ext}^+(\mathrm{Id}) \otimes \mathrm{Id}\}$*. By Definition 1.2.18 we see that the image of the only element in the down-set is the forest in Figure 1.6*

Example 1.3.8. *In this example we will compute* $\mathrm{Res}(\tau)$ *for* τ *as in Figure 1.6. We have* $\tau = (r, \{T, T\})$ *for* $T = (r, \{\cdot, \cdot\})$*, where* $\cdot = (r, \emptyset)$ *denotes the trivial tree.*

By Definition 1.3.6 we have $\mathrm{Res}(T) = \cdot \times \emptyset$*, which may be identified with* $\cdot$*. So we say* $\mathrm{Res}(T) = \cdot$*. Then*

$$\mathrm{Res}(\tau) = \{(T, \cdot)\}$$

We obtain the same result as the down-set of the tree from the last example.

Remark 1.3.9. For the trees $\tau = (r, \{T, T\})$ and $\tau = (r, \{T\})$, $\mathrm{Res}(\tau)$ is the multiset formed by adding T to the front of every forest in $\mathrm{Res}(T)$. If $\tau = (r, \{T_1, T_2\})$, and T_1 and T_2 are distinct subtrees, the multiset $\mathrm{Res}(\tau)$ is a union of two multisets, one where T_1 was added to the front of every forest of $\mathrm{Res}(T_2)$, and one where T_2 was added to the front of every forest of $\mathrm{Res}(T_1)$. The two terms of this union are disjoint since the largest trees in the forests of each (T_1 and T_2) are distinct.

Proposition 1.3.10. *Let T be a 1-2 binary tree, and T^- be its downset. Then $T^- = \mathrm{Res}(T)$.*

Proof. The proof follows by induction on the height of the tree. When $k = 0$, let $\mathrm{Id} = (r, \emptyset)$ denote the trivial representation of H_0. Then the result is a matter of definition, so we begin with $k = 1$. There are two trees of height 1, namely $T_1 = (r, \{\mathrm{Id}, \mathrm{Id}\})$ and $T_2 = (r, \{\mathrm{Id}\})$. These correspond to the two representations in Table 1.3. As can be seen, the restriction of both of these representations to the diagonal subgroup $D \cong H_0$ in $H_0 \times H_0$ is the trivial representation. The result of $\mathrm{Res}(T_1)$ and $\mathrm{Res}(T_2)$ is Id.

Now assume the result is true for trees of height less than k. Let Γ be the representation of H_k corresponding to T. If T(and thus Γ) are of type III(see Table 1.2), and let $\Gamma = Ind(\phi_1, \phi_2)$, then by Equation (1.2.1) and Equation (1.6):

$$\mathrm{Res}^{H_k}_{P_{2^k-1}}(\Gamma) = \sum_{i,j \in 1,2, i \neq j} \phi_i \otimes \mathrm{Res}^{H_{k-1}}_{P_{2^{k-1}-1}}(\phi_j).$$

Since the result holds for $k-1$, and by Definition 1.2.13, T_i is the tree corresponding to ϕ_i for $i = 1, 2$, the down-set T^- is $\mathrm{Res}(T)$. A similar computation for trees of Type I and Type II completes the proof. $\square$

From Definition 1.3.6 and Proposition 1.3.10 we have:

Corollary 1.3.11. *Given a tree τ of height k:*

$$\tau^- = \begin{cases} T \times T^- & \tau = (r, \{T, T\}), \\ T \times T^- & \tau = (r, \{T\}), \\ T_1 \times T_2^- \cup T_2 \times T_1^- & \tau = (r, \{T_1, T_2\}), \\ \{\emptyset\} & \tau = (r, \emptyset). \end{cases}$$

Corollary 1.3.12. *Given a forest F of size $2^k - 1$ for $k \geq 0$, let $F(1)$ denote the largest tree in the forest and $\overline{F}$ denote the tuple F with the tree $F(1)$ removed:*

$$F^+ = \begin{cases} \{(r, \{F(1), T\}) | T \in \overline{F}^+\} & F(1) \notin \overline{F}^+, \\ \{(r, \{F(1), T\}) | T \in \overline{F}^+\} \cup \{(r, \{F(1)\})\} & F(1) \in \overline{F}^+. \end{cases}$$

Where for the empty tuple we define $\emptyset^+$ to be the trivial tree $\{\bullet\}$.

Example 1.3.13. *Let F denote the forest of size 3 in Figure 1.6. The largest tree in the forest is $F(1) = (r, \{\bullet, \bullet\})$ where $\bullet$ denotes the trivial tree. The forest with this tree removed is $\overline{F} = (\bullet)$.*

The largest tree in $\overline{F}$ is the trivial tree and $\overline{F}$ with its largest tree removed is the empty tuple. The up-set of the empty tuple by definition is the trivial tree $\bullet$. Since the largest tree of $\overline{F}$ occurs in the up-set of the forest with this tree removed, we have

$$\overline{F}^+ = \{F(1), T\},$$

where $T = (r, \{\bullet\})$.

Since $F(1)$ occurs in $\overline{F}^+$, the up-set of F according to Corollary 1.3.12 is

$$F^+ = \{(r, \{F(1), F(1)\}), (r, \{F(1), T\}), (r, \{F(1)\})\}.$$

Corollary 1.3.14. *Given a tree T of height k for any integer $k \geq 1$, let set(T^-) denote the set of distinct elements in T^-. Then $T^- = $ set(T^-).*

Proof. The proof proceeds by induction. The result is easily verified for $k = 1$. If it holds for all integers less than k, consider the down-set of a tree T of height k:

If T is of Type I or Type II, by Corollary 1.3.11, the down-set of T may be identified with the down-set of its only distinct subtree (by deleting the largest tree from each forest). Then $T^- = \text{set}(T^-)$ by the induction hypothesis.

If T is of Type III, and let T_1 and T_2 denote its distinct subtrees. Again by 1.3.11 we know that T^- is the union of two sets (multisets that are known to be multiplicity free by the induction hypothesis). One of these sets consists of forests where the largest tree is T_1, while the other consists of forests where the largest tree is T_2. This union is disjoint since these trees are distinct. □

The operator Res can be extended to forests of arbitrary size, following the cue of Equation (1.7). With $F = (T_1, \ldots, T_n)$ a forest of size n, we have:

Definition 1.3.15. *Define an operator* Res *from forests to multisets of forests as under:*

$$\text{Res}(F) = (T_1, \ldots, T_{s-1}) \times \text{Res}(T_s).$$

The following proposition is easy to observe from this definition.

Proposition 1.3.16. *Let F be a forest of binary trees, and F^- be its down-set. Then $F^- = \text{Res}(F)$.*

We may combine these results into a combinatorial branching rule on forests of binary trees. Recall that a tree T of height k is identified with the forest $F = (T)$ of size 2^k.

Theorem 1.3.17. *Given a forest $F = (F_1, \ldots, F_s)$ of size n:*

1. *Define $\underline{F}$ to be the forest F without the element F_s. The down-set of F is given by:*

$$F^- = \underline{F} \times F_s^-$$

2. *Let d denote the smallest nonnegative integer that does not occur in* $\mathrm{Bin}(n)$*. Partition F as the tuple $F_1 \times F_2$, where F_1 is the tuple of trees in F with height greater than d, and F_2 is the tuple of trees of height less than d. Then the up-set of F is given by:*

$$F^+ = F_1 \times F_2^+.$$

Thus the branching at each level replicates the branching at the 2^kth level, for some nonnegative integer k.

Example 1.3.18. *Consider the forest of size* 10 *whose trees are in Figure 1.7. The largest tree, $F(1)$, is of height* 3*, and may be observed to correspond to the representation* $\mathrm{Ext}^-(\mathrm{Ext}^+(\mathrm{Ext}^+(\mathrm{Id})))$ *of H_3. The smallest tree,F_s, is of size* 1*, corresponding to the representation* $\mathrm{Ext}^+(\mathrm{Id})$ *of H_1.*

We calculate the down-set first. The forest F without its smallest tree is $(F(1))$ *and $F_s^- = \{\cdot\}$ ($\cdot$ is the trivial tree). Thus*

$$F^- = \{(F(1), \cdot)\}.$$

Now we find the up-set. $\mathrm{Bin}(10) = \{3, 1\}$*. The smallest nonnegative integer that does not occur in* $\mathrm{Bin}(10)$ *is* 2*. We partition F into sub-tuples F_1, comprising trees of height more than* 2*, and F_2 comprising trees of height less than* 2*. So we have*

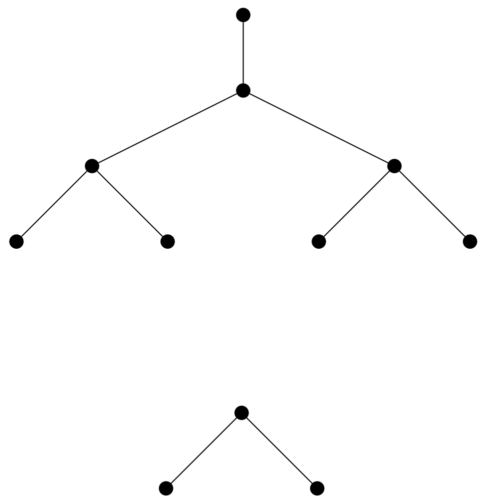


Figure 1.7: A forest of size 10

$F_1 = (F(1))$ *and* $F_2 = (F_s)$, *and*

$$F^+ = F(1) \times F_2^+.$$

We must calculate F_2^+. F_2 *is a forest of size* 2, *and* $\mathrm{Bin}(2) = \{1\}$. *The smallest nonnegative integer that does not occur in* $\mathrm{Bin}(2)$ *is* 0. *Again partitioning* F_2 *into* $F_{21} = (F_s)$, *comprising trees of height greater than* 0, *and* $F_{22} = \emptyset$, *comprising trees of height less than* 0, *we have*

$$F_2^+ = F_{21} \times F_{22}^+,$$

where by 1.3.12 we know that $\emptyset^+ = \{\cdot\}$.

Thus $F_2^+ = \{(F_s, \cdot)\}$ *and*

$$F^+ = \{(F(1), F_s, \cdot)\}.$$

Proposition 1.3.19. *The branching in* $\mathbb{P}$ *is multiplicity free.*

Proof. We may reduce the proof to the branching at every 2^kth level, and we know by Corollary 1.3.14 that this is multiplicity free. From Theorem 1.3.17, we have this in general. □

Another consequence of Theorem 1.3.17 is the self-similarity of $\mathbb{P}$:

Lemma 1.3.20. *Given a forest $F = (F_1, \ldots, F_s)$ of size $2^k + m$ $(0 \leq m < 2^k)$, let $\overline{F} = (F_2, \ldots, F_s)$. For $m < n \leq 2^k$, let $\mathbb{P}_F^n$ denote the subgraph of $\mathbb{P}$ comprising F and all forests of size at least $2^k + m$ and strictly less than $2^k + n$ that are comparable to F. Then:*

$$\mathbb{P}_F^{2^k} = \{F_1\} \times \mathbb{P}_{\overline{F}}^{2^{k-1}}.$$

We end the section with Figure 1.8 which shows a portion of the Bratteli diagram $\mathbb{P}$. The levels are numbered from bottom to top. The lowest node corresponds to the empty set, the node above it to the only representation of P_1, and so on.

Edges in black indicate branching from the $2^k - 1$th to the 2^kth level. Pick a vertex at the 4th level- for instance the rightmost vertex- and consider the subposet consisting of all vertices between the 4th and 7th level that are comparable to it. This subposet is isomorphic to the Bratteli diagram from levels 0 to 3. Similarly between the levels 8 and 15 the subposet attached to any vertex at level 8 is isomoprhic to the Bratteli diagram between 0 and 7. This is the self-similarity alluded to in Lemma 1.3.20.

More specifically, pick a vertex T at level 2^k, and a forest F of size between 0 and $2^k - 1$. The suposet of $\mathbb{P}$ attached to T upto level $2^{k+1} - 1$ is isomorphic to the portion of the Bratteli diagram between 0 and $2^k - 1$, and the vertex on this subposet that is isomorphic to F is labelled $T \times F$.

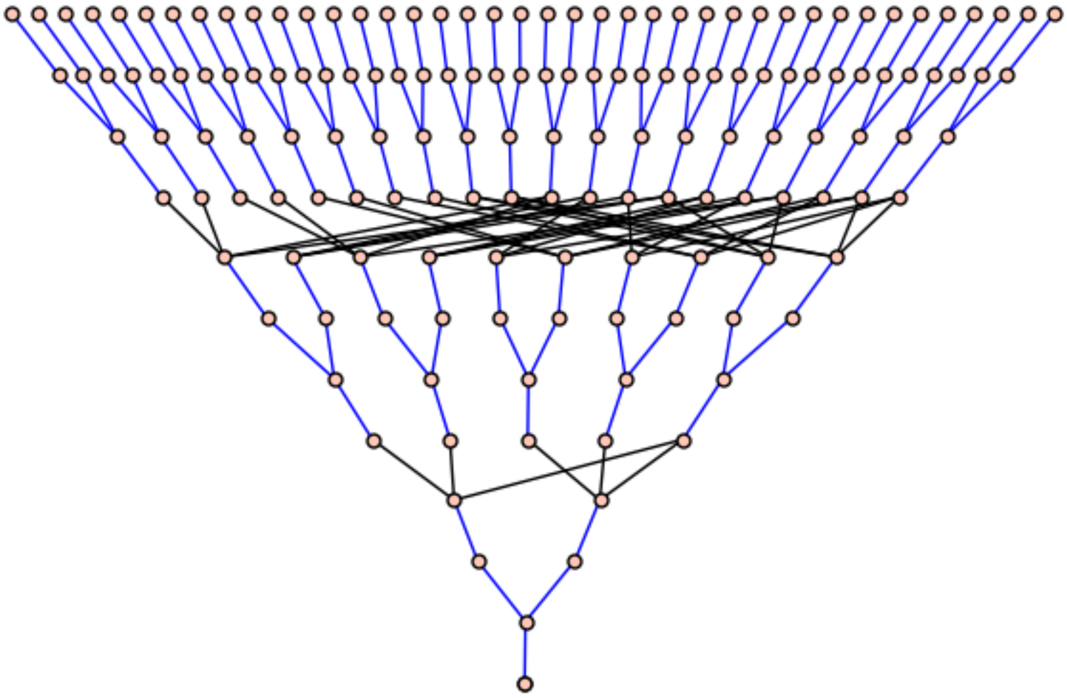


Figure 1.8: Branching of irreducible representations for $n \leq 11$.

1.4 The one-dimensional representations of P_n

We now turn to the subposet of one-dimensional representations of $\mathbb{P}$. Theorem 1 of [4] states that the subgraph of odd partitions in Young's lattice is a binary tree that branches at every even level. We see that the subposet of one-dimensional representations of the family $\{P_n\}$ also has the structure of a binary tree (see Figure 1.9). We show that these graphs are nonisomorphic by describing the structure of the subgraph of one-dimensional representations of $\mathbb{P}$, which we contrast with the description of the Macdonald tree in [4].

By Remark 1.2.6 we conclude that an irreducible representation ϕ of H_k is one-dimensional if $\phi = \mathrm{Ext}^{\pm}(\phi_1)$ for an irreducible one-dimensional representation ϕ_1 of H_{k-1}.

Definition 1.4.1. *Define recursively a binary encoding of one-dimensional trees,*

β_{2^k} acting on one-dimensional trees of height k as below:

$$\beta_{2^k}(\tau) = \begin{cases} 0\beta_{2^{k-1}}(T) & \tau = (r, \{T, T\}), \\ 1\beta_{2^{k-1}}(T) & \tau = (r, \{T\}). \end{cases}$$

and $\beta_1(\cdot) = \emptyset$ for the trivial tree $\cdot$.

Theorem 1.4.2. *The map β_{2^k} is a bijection between one-dimensional irreducible representations of H_k and binary strings of length k.*

For instance if for the tree T, $\beta_{2^{k-1}}(T) = b_1 b_2 \dots b_s$, then $\beta_{2^k}((r, \{T, T\})) = 0b_1 b_2 \dots b_s$ and $\beta_{2^k}((r, \{T\})) = 1b_1 b_2 \dots b_s$.

Example 1.4.3. *There are two one-dimensional representations of H_1, shown in Table 1.3. They are $\mathrm{Ext}^+(\mathrm{Id})$ and $\mathrm{Ext}^-(\mathrm{Id})$. They correspond to the bits 0 and 1 respectively.*

There are four one-dimensional representations of H_2, shown in Table 1.4. They are $\mathrm{Ext}^+(\mathrm{Ext}^+(\mathrm{Id})), \mathrm{Ext}^+(\mathrm{Ext}^-(\mathrm{Id})), \mathrm{Ext}^-(\mathrm{Ext}^+(\mathrm{Id})), \mathrm{Ext}^-(\mathrm{Ext}^-(\mathrm{Id}))$. They correspond to the strings $00, 01, 10, 11$ respectively.

Thus we have an encoding of one-dimensional binary trees as binary strings. The family of maps β_{2^k} may be extended to β_n, acting on every tree in a forest of size n. Thus, with $\mathrm{Bin}(n) = \{k_1, \dots, k_s\}$:

$$\beta_n = \beta_{k_1} \times \cdots \beta_{k_s}. \tag{1.8}$$

Definition 1.4.4. *A sequence of strings of size n is an ordered collection of binary strings $(b_1, \dots, b_s)$ where the length of the string b_i is k_i for $i = 1, \dots, s$.*

We now define an operation Res on binary strings, that is analogous to the operation of the same name defined on binary trees in Definition 1.3.6:

Definition 1.4.5. *Given a binary string b of length k, let $\overline{b}$ be the binary string of length $k-1$ obtained by removing the leading bit of b. Then*

$$\begin{aligned}\mathrm{Res}(b) &= \overline{b} \times \mathrm{Res}(\overline{b}),\\ \mathrm{Res}(0) &= \{\emptyset\},\\ \mathrm{Res}(1) &= \{\emptyset\}.\end{aligned}$$

Remark 1.4.6. Observe that $\mathrm{Res}(b) = \{(\overline{b}, \overline{\overline{b}}, \dots)\}$. For instance $\mathrm{Res}(010) = \{(10, 0, \emptyset)\}$.

Lemma 1.4.7. *If T is a one-dimensional tree of height k:*

$$\mathrm{Res}(\beta_{2^k}(T)) = \beta_{2^k-1}(\mathrm{Res}(T)).$$

Proof. This is a straightforward proof by induction. For $k = 1$, the lemma is true by definition.

Assume it is true for all trees of height less than k. The one-dimensional tree T is either $(r, \{T_1, T_1\})$ or $(r, \{T_1\})$ for some one-dimensional tree T_1. Recall from Definition 1.3.6 that $\mathrm{Res}(T) = T_1 \times \mathrm{Res}(T_1)$. The binary string $\beta_{2^k}(T)$ is either $0\beta_{2^{k-1}}(T_1)$ or $1\beta_{2^{k-1}}(T_1)$. Then

$$\begin{aligned}\beta_{2^k-1}(\mathrm{Res}(T)) &= \beta_{2^{k-1}}(T_1) \times \beta_{2^{k-1}-1}(\mathrm{Res}(T_1))\\ &= \beta_{2^{k-1}}(T_1) \times \mathrm{Res}(\beta_{2^{k-1}-1}(T_1))\\ &= \mathrm{Res}(\beta_{2^k}(T)).\end{aligned}$$

□

This verifies that the operation Res defined on binary strings returns the down-set of the corresponding one-dimensional binary tree. We may extend this operation to act on sequences of binary strings in a manner analogous to Equation 1.3.15. Given a sequence of strings $S = (b_1, \dots, b_s)$ of size n:

$$\text{Res}(S) = (b_1, \dots, b_{s-1}) \times \text{Res}(b_s). \tag{1.9}$$

Corollary 1.4.8. *If F is a one-dimensional forest of size n:*

$$\text{Res}(\beta_n(F)) = \beta_n(\text{Res}(F)).$$

The result of Corollary 1.4.8 is that we may identify the subposet of one-dimensional representations of $\mathbb{P}$ with a poset generated by sequences of binary strings with Res providing the partial order. We denote by $\mathbb{B}$ the set of all sequences of strings of all positive integers.

Theorem 1.4.9. *The subgraph of one-dimensional irreducible representations in The Bratteli diagram of $\{P_n\}_{0 \leq n \leq s}$ is isomorphic to $(\mathbb{B}, \text{Res})$.*

Proof. From Equation (1.8) there is a bijection between one-dimensional representations of P_n and sequences of binary strings of size n. The down-set of a one-dimensional forest is a singleton set. From Corollary 1.4.8 we see that the operation Res acting on sequences of strings corresponding to a forest F returns the binary encoding under Equation (1.8) of the unique element in F^-. $\square$

Definition 1.4.10. *Given a binary string S, let F denote the forest it corresponds to. Then we define the down-set S^- and the up-set S^+ to be F^- and F^+ respectively.*

Note that S^- is a singleton set. The following theorem is the analogue of Theorem 1.3.17.

Theorem 1.4.11. *Given an integer n and a sequence of strings S of size n corresponding to a forest F, define $S(1)$ to be the longest string in S, and define $\overline{S}$ to* *be the sequence S without $S(1)$. Similarly define S_{min} to be the smallest string in S and $\underline{S}$ to be the sequence S without S_{min}.*

1. *The down-set of S is given by:*

$$S^- = \underline{S} \times S^-_{min}$$

2. *Partition S as the tuple $S_1 \times S_2$, where S_1 is the tuple of strings in S with more than d bits, and S_2 is the tuple of strings with less than d bits.*

   The up-set of S is given by:

$$S^+ = \begin{cases} \{S_1 \times 0S_2(1), S_1 \times 1S_2(1)\} & S_2(1) \in \overline{S_2}^+, \\ \emptyset & \textit{otherwise} \end{cases}$$

Remark 1.4.12. $(\mathbb{B}, \mathrm{Res})$ (hereafter referred to as $\mathbb{B}$ when there is no ambiguity) is a binary tree that branches at every even level. Let $\mathbb{B}_k$ denote the first $2^k - 1$ levels of $\mathbb{B}$. The following procedure constructs $\mathbb{B}_k$ recursively:

1. For each binary string b of length $k-1$, let $v_b = (b, \overline{b}, \overline{\overline{b}}, \cdots, \emptyset)$.
2. To each vertex v_b of $\mathbb{B}_{k-1}$, attach two copies of $\mathbb{B}_{k-1}$, and denote them the left and right subtree of v_b.
3. Change the label of each vertex v of the left subtree by appending the string $0b$ to the sequence. Similarly append $1b$ to the string labelling each vertex on the right subtree.

Figure 1.9 uses this method to build the structure $\mathbb{B}_3$ from $\mathbb{B}_2$. The two one-dimensional vertices at level 3 are $(\mathrm{Ext}^+(\mathrm{Id}), \cdot)$ and $(\mathrm{Ext}^+(\mathrm{Id}), \cdot)$.

To obtain $\mathbb{B}_3$, first we attach two branches to each of these vertices. Label the vertices attached to $(\mathrm{Ext}^+(\mathrm{Id}), \cdot)$ $\mathrm{Ext}^+(\mathrm{Ext}^+(\mathrm{Id}))$ and $\mathrm{Ext}^-(\mathrm{Ext}^+(\mathrm{Id}))$. Label the vertices attached to $(\mathrm{Ext}^-(\mathrm{Id}), \cdot)$ $\mathrm{Ext}^+(\mathrm{Ext}^-(\mathrm{Id}))$ and $\mathrm{Ext}^-(\mathrm{Ext}^-(\mathrm{Id}))$. Paste a copy of $\mathbb{B}_2$ on each of these newly created vertices.

To obtain the new labels on these pasted copies, append the existing labels for $\mathbb{B}_2$ with the label of the vertex to which the copy is pasted. For instance, the vertex labeled $(\mathrm{Ext}^-(\mathrm{Id}), \cdot)$ on the copy of $\mathbb{B}_2$ attached to the vertex $\mathrm{Ext}^-(\mathrm{Ext}^-(\mathrm{Id}))$ will now be relabelled $(\mathrm{Ext}^-(\mathrm{Ext}^-(\mathrm{Id})), \mathrm{Ext}^-(\mathrm{Id}), \cdot)$.

The vertices that propagate at the 7th level are labelled $A - D$ in Figure 1.9.

A recursive construction of the Macdonald tree can be found in [4]. In particular the Macdonald tree has only two infinite rays. The subgraph $\mathbb{B}$ by contrast has an infinite number of infinite rays, since each binary string b can be extended by attaching $\epsilon = 0, 1$ to the left of b, and between the vertices ϵb and b, there is a unique path in $\mathbb{B}$.

Corollary 1.4.13. *The Macdonald tree is not isomorphic to* $\mathbb{B}$.

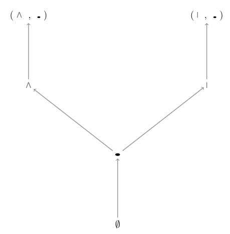
Figure 1.10 is the Macdonald tree and Figure 1.11 is the subgraph of one-dimensional representations of P_n. The vertices of both are labelled $0, 1, 2, \ldots$ from bottom to top. We observe several crucial differences:

1. There are only two infinite rays in the Macdonald tree but there are an infinite number of infinite rays in $\mathbb{B}$.

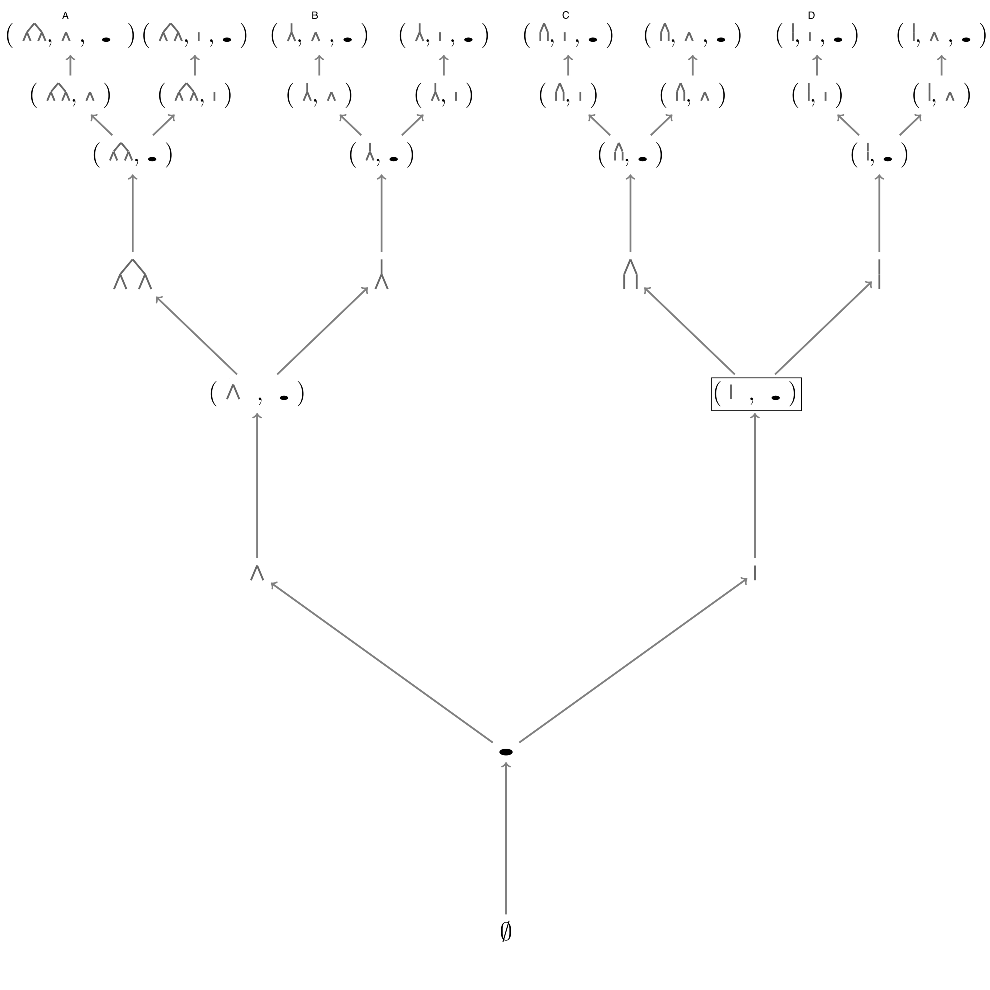
2. Two of the vertices at the 4th level in Figure 1.10 terminate at the 7th level. However all the vertices at the 4th level propagate in Figure 1.11.

1.5 Restriction coefficients

Given a partition λ of n, and a forest F of size n, let χ_λ and χ_F denote the irreducible representations of S_n and P_n corresponding to λ and F respectively. The restric-



$\mathbb{B}_2$



$\mathbb{B}_3$

Figure 1.9: $\mathbb{B}_3$ is built recursively by attaching two copies of $\mathbb{B}_2$ to appropriate nodes on the maximal level of $\mathbb{B}_2$. Nodes on the highest level of $\mathbb{B}_3$ that further propagate are labelled A-D.

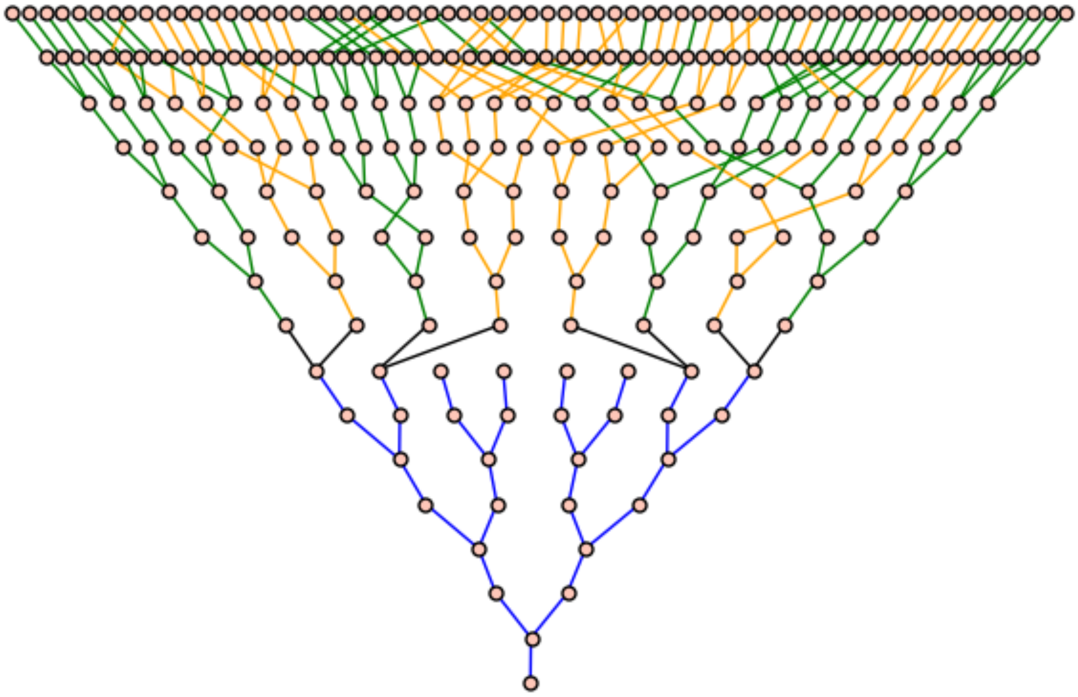


Figure 1.10: The Macdonald tree for levels $n \leq 15$

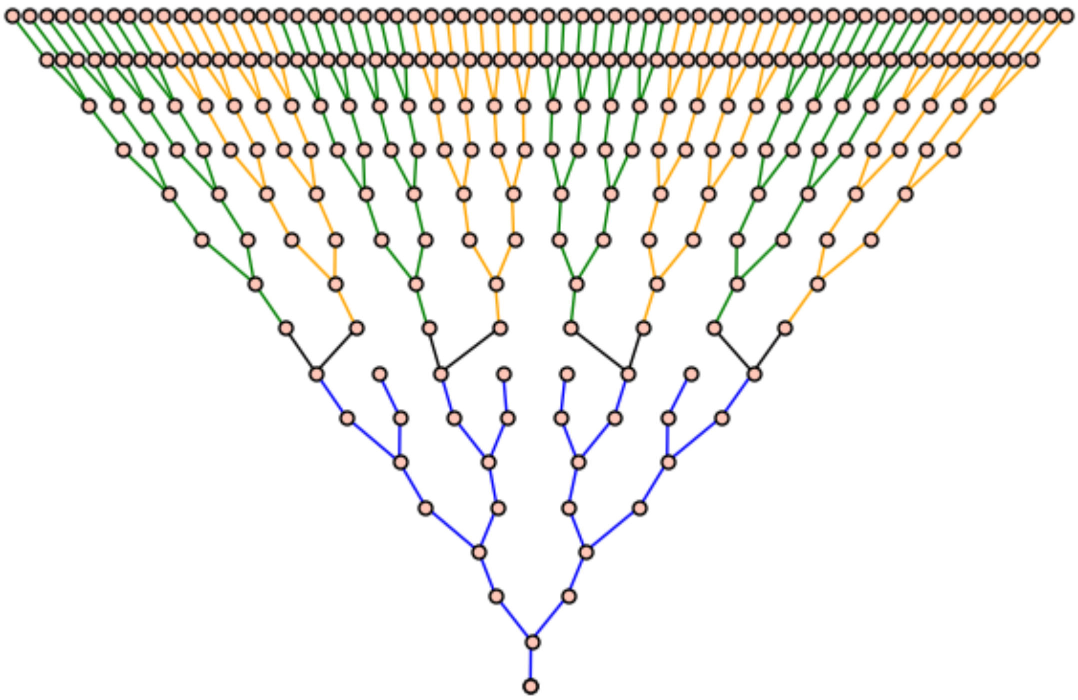


Figure 1.11: The subgraph of one-dimensional representations of P_n for levels $n \leq 15$.

tion coefficient $r_{\lambda F}$ is defined to be the multiplicity of χ_F in the restriction of χ_λ to P_n. In this section we obtain a formula for these coefficients. A refinement of this formula is obtained for odd degree representations of S_{2^k} for each $k \geq 0$, which correspond to hook partitions. Some interesting observations follow thus, including an explicit recursive construction of the bijection between odd partitions of n and one-dimensional representations of a Sylow subgroup of S_n obtained in [13]. Throughout this section, binary trees of Type I, II and III are denoted by $(r, \{T_1, T_1\})$, $(r, \{T_1\})$ and $(r, \{T_1, T_2\})$ respectively. For trees of Type I and Type II, define $\mathrm{ext}(T) = 1$ if T is of Type I and $\mathrm{ext}(T) = -1$ if T is of Type II. The hook partition $(a+1, 1^b)$ is represented in Frobenius notation as $(a|b)$. Let $\mathrm{vert}(a|b) = \left\lceil \frac{(b+1)}{2} \right\rceil$.

Definition 1.5.1. *Given two class functions χ_1 and χ_2 of a group G*

$$\langle \chi_1, \chi_2 \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_1(g^{-1}) \chi_2(g).$$

Note that $\langle \chi_\lambda, \chi_F \rangle_{P_n}$ is the multiplicity of χ_F in the restriction of χ_λ to P_n.

Theorem 1.5.2. *Given a partition λ of 2^k with $quo_2(\lambda) = (\nu_1, \nu_2)$, and a tree T of height k, let $r_{\lambda T} = \langle \chi_\lambda, \chi_T \rangle_{H_k}$. Then we have:*

$$r_{\lambda T} = \begin{cases} A(T_1, \lambda) + \frac{1}{2} \sum_{\mu \vdash 2^{k-1}} c^\lambda_{\mu,\mu} r^2_{\mu T_1} + \frac{ext(T) sgn(\lambda)}{2} C(T_1, \lambda) & T \text{ Type I or II,} \\ B(T_1, T_2, \lambda) + \sum_{\mu \vdash 2^{k-1}} c^\lambda_{\mu,\mu} r_{\mu T_1} r_{\mu T_2} & T \text{ Type III.} \end{cases}$$

The quantities A, B and C are defined as under

$$A(T_1, \lambda) = \sum_{\mu_1, \mu_2} c^\lambda_{\mu_1, \mu_2} r_{\mu_1 T_1} r_{\mu_2 T_1}, \tag{1.10}$$

where the sum is over two-element subsets of partitions $\{\mu_1, \mu_2\}$ of 2^{k-1}, and

$$B(T_1, T_2, \lambda) = \sum_{\mu_1, \mu_2} c^\lambda_{\mu_1, \mu_2} (r_{\mu_1 T_1} r_{\mu_2 T_2} + r_{\mu_1 T_2} r_{\mu_2 T_1}), \tag{1.11}$$

where the sum is over the same range as (1.10), *and*

$$C(T_1, \lambda) = \begin{cases} \sum_{\nu \vdash 2^{k-1}} c^{\nu}_{\nu_1,\nu_2} r_{\nu T_1} & core_2(\lambda) = \emptyset \\ 0 & otherwise. \end{cases} \tag{1.12}$$

Proof. We write out the expression for $r_{\lambda T}$ as in Definition 1.5.1

$$r_{\lambda T} = \frac{1}{|H_k|} \sum_{g \in H_k} \chi_\lambda(g)\chi_T(g),$$

noting that $\chi_\lambda(g^{-1}) = \chi_\lambda(g)$.

The set of elements of H_k may be split into two sets: $\{(\sigma_1, \sigma_2)^1 | \sigma_1, \sigma_2 \in H_{k-1}\}$ and $\{(\sigma_1, \sigma_2)^{-1} | \sigma_1, \sigma_2 \in H_{k-1}\}$. Splitting the sum thus

$$r_{\lambda T} = \frac{1}{|H_k|} \sum_{g=(\sigma_1,\sigma_2)^1} \chi_\lambda(g)\chi_T(g) + \frac{1}{|H_k|} \sum_{g=(\sigma_1,\sigma_2)^{-1}} \chi_\lambda(g)\chi_T(g). \tag{1.13}$$

The set $\{(\sigma_1, \sigma_2)^1 | \sigma_1, \sigma_2 \in H_{k-1}\}$ is the set of all elements of $H_{k-1} \times H_{k-1}$. Since $H_{k-1} \times H_{k-1} \subset S_{2^{k-1}} \times S_{2^{k-1}}$, we have

$$\chi_\lambda((\sigma_1, \sigma_2)^1) = \sum_{\mu,\nu \vdash 2^{k-1}} c^{\lambda}_{\mu,\nu} \chi_\mu(\sigma_1)\chi_\nu(\sigma_2).$$

We substitute this into the expression the first sum in Equation (1.13)

$$\frac{1}{|H_k|} \sum_{g=(\sigma_1,\sigma_2)^1} \chi_\lambda(g)\chi_T(g) =$$
$$\frac{1}{|H_k|} \sum_{\sigma_1,\sigma_2 \in H_{k-1}} \sum_{\mu,\nu \vdash 2^{k-1}} c^{\lambda}_{\mu,\nu} \chi_\mu(\sigma_1)\chi_\nu(\sigma_2)\chi_T((\sigma_1, \sigma_2)^1).$$

Substituting the value of the character corresponding to T, and $|H_k| = 2|H_{k-1}|^2$ into this equation, we have:

For T of Type I or Type II:

$$\frac{1}{|H_k|}\sum_{g=(\sigma_1,\sigma_2)^1}\chi_\lambda(g)\chi_T(g)=\frac{1}{2}\sum_{\mu,\nu\vdash 2^k}c^\lambda_{\mu,\nu}r_{\mu T_1}r_{\nu T_1}.$$

For T of Type III:

$$\frac{1}{|H_k|}\sum_{g=(\sigma_1,\sigma_2)^1}\chi_\lambda(g)\chi_T(g)=\frac{1}{2}\sum_{\mu,\nu\vdash 2^k}c^\lambda_{\mu,\nu}\left(r_{\mu T_1}r_{\nu T_2}+r_{\mu T_2}r_{\nu T_1}\right).$$

It is known that $c^\lambda_{\mu,\nu}=c^\lambda_{\nu,\mu}$. Accounting for this, we modify the summation as under:

(1.14)

$$\frac{1}{|H_k|}\sum_{g=(\sigma_1,\sigma_2)^1}\chi_\lambda(g)\chi_T(g)=\begin{cases}A(T_1,\lambda)+\frac{1}{2}\sum_{\mu\vdash 2^{k-1}}c^\lambda_{\mu,\mu}r^2_{\mu T_1} & T\text{ Type I /Type II,}\\ B(T_1,T_2,\lambda)+\sum_{\mu\vdash 2^{k-1}}c^\lambda_{\mu,\mu}r_{\mu T_1}r_{\mu T_2} & T\text{ Type III,}\end{cases}$$

where $A(T_1,\lambda)$ and $B(T_1,T_2,\lambda)$ are defined as in Equations (1.16) and (1.17) respectively.

The second sum in Equation (1.13) may be simplified as under:

$$\frac{1}{|H_k|}\sum_{g=(\sigma_1,\sigma_2)^{-1}}\chi_\lambda(g)\chi_T(g)=\frac{1}{|H_k|}\sum_{g=(\mathrm{Id},\sigma)^{-1}}|[g]|\chi_\lambda(g)\chi_T(g),$$

where the sum is now over representatives σ of distinct conjugacy classes of H_{k-1}.

Since $|(id,\sigma)^{-1}| = |H_{k-1}||[\sigma]|$:

$$\begin{aligned}\frac{1}{|H_k|}\sum_{g=(\sigma_1,\sigma_2)^{-1}}\chi_\lambda(g)\chi_T(g) &= \frac{1}{|H_k|}\sum_{g=(id,\sigma)^{-1}}|H_{k-1}||[\sigma]|\chi_\lambda(g)\chi_T(g)\\ &= \frac{1}{2|H_{k-1}|}\sum_{g=(id,\sigma)^{-1}}|[\sigma]|\chi_\lambda(g)\chi_T(g)\\ &= \frac{1}{2}\sum_{\sigma\in H_{k-1}}\frac{\chi_\lambda((id,\sigma)^{-1})\chi_T((id,\sigma)^{-1})}{|H_{k-1}|},\end{aligned}$$

where the summation in the final equation is over all elements $\sigma \in H_{k-1}$. Let ν denote the cycle type of an element $(id,\sigma)^{-1}$ for $\sigma \in H_{k-1}$. Then it is easy to prove that if the cycle type of σ is $(1^{k_0}2^{k_1}4^{k_2}\dots)$, then the cycle type of $(id,\sigma)^{-1}$ is $(2^{k_0}4^{k_1}8^{k_2}\dots)$. By Theorem 1.2.30

$$\chi_\lambda((id,\sigma)^{-1}) = \begin{cases} 0 & \mathrm{core}_2(\lambda) \neq \emptyset, \\ \mathrm{sgn}(\lambda)\sum_{\nu\vdash\frac{|\lambda|}{2}} c^{\nu}_{\nu_1,\nu_2}\chi_\nu(\sigma), & \end{cases}$$

Thus:

$$\frac{1}{|H_k|}\sum_{g=(\sigma_1,\sigma_2)^{-1}}\chi_\lambda(g)\chi_T(g) = \begin{cases} \frac{\mathrm{ext}(T)\mathrm{sgn}(\lambda)}{2}C(T_1,\lambda) & T \text{ Type I or II}, \\ 0 & T \text{ of Type III}. \end{cases} \tag{1.15}$$

Adding Equations (1.14) and (1.15) proves the theorem. □

When the partition λ is a hook-partition of 2^k, we can use the results in Section 1.2.4 to obtain the following:

Corollary 1.5.3. *Given a hook-partition $\lambda = (a|b)$ of 2^k and a tree T of height k,* *we have:*

$$r_{\lambda T} = \begin{cases} A(T_1,\lambda) + \frac{1}{2} r^2_{\frac{\lambda}{2}T_1} + \frac{ext(T)(-1)^{vert(\lambda)}}{2} r_{\frac{\lambda}{2}T_1} & T \text{ Type I or II,} \\ B(T_1,T_2,\lambda) + r_{\frac{\lambda}{2}T_1} r_{\frac{\lambda}{2}T_2} & T \text{ Type III.} \end{cases}$$

where $\frac{\lambda}{2} = (\frac{a}{2}|\frac{b-1}{2})$ *if* a *is even and* $(\frac{a-1}{2}|\frac{b}{2})$ *otherwise, and the quantities* A *and* B *are defined as under*

$$A(T_1,\lambda) = \sum_{(a_1|b_1)} r_{(a_1|b_1)T_1}\left(r_{(a-a_1|b-b_1-1)T_1} + r_{(a-a_1-1|b-b_1)T_1}\right), \tag{1.16}$$

where the sum is over hook partitions $(a_1|b_1)$ *of* 2^{k-1} *with* $\frac{\lambda}{2} \neq (a_1|b_1)$ *and* $a \geq a_1 > \frac{a}{2}$*, and*

$$\begin{aligned} B(T_1,T_2,\lambda) = \sum_{(a_1|b_1)} & r_{(a_1|b_1)T_1}\left(r_{(a-a_1|b-b_1-1)T_2} + r_{(a-a_1-1|b-b_1)T_2}\right) + \\ & r_{(a_1|b_1)T_2}\left(r_{(a-a_1|b-b_1-1)T_1} + r_{(a-a_1-1|b-b_1)T_1}\right), \end{aligned} \tag{1.17}$$

where the sum is over the same range as (1.16)*.*

Proof. This follows from Theorem 1.5.2 by noting that $c^{\lambda}_{\mu,\nu}$ is 1 if $\{\mu,\nu\} = \{(a_1|b_1),(a-a_1|b-b_1-1)\}$ or $\{\mu,\nu\} = \{(a_1|b_1),(a-a_1-1|b-b_1)\}$ and 0 otherwise. □

The bijection between odd partitions and one-dimensional representations of Sylow 2-subgroups introduced in [13] maps a hook partition λ of 2^k to the unique one-dimensional representation of H_k occurring with odd multiplicity, which we denote by $\eta(\lambda)$. We think of $\eta(\lambda)$ as a binary string of length k under the encoding described in Definition 1.4.1.

Corollary 1.5.4. *Given a hook partition* $\lambda = (a|b)$ *of size* 2^k*,* $k \geq 0$*, the representation* $\eta(\lambda)$ *is the unique one-dimensional representation occurring in* $R(\lambda)$ *and it* *occurs with multiplicity one. Further:*

$$\eta(\lambda) = \begin{cases} 0\eta(\frac{\lambda}{2}) & a \text{ odd }, b \equiv 0 \mod 4, \\ 1\eta(\frac{\lambda}{2}) & a \text{ odd }, b \equiv 2 \mod 4, \\ 1\eta(\frac{\lambda}{2}) & a \text{ even }, b \equiv 1 \mod 4, \\ 0\eta(\frac{\lambda}{2}) & a \text{ even }, b \equiv 3 \mod 4. \end{cases}$$

Proof. We shall prove this inductively, and skip the case $k = 1$ since it is an easy computation. If the result is true for all integers less than k, let χ_T be the unique one-dimensional representation occurring in λ. We will determine the tree T recursively. By the induction hypothesis,

$$r_{\mu T_1} r_{\nu T_1} = 0$$

for hook partitions $\mu \neq \nu$. Thus:

$$r_{\lambda T} = \frac{r^2_{\frac{\lambda}{2} T_1}}{2} + \frac{\mathrm{ext}(T)(-1)^{\mathrm{vert}(\lambda)}}{2} r_{\frac{\lambda}{2} T_1},$$

where $r_{\frac{\lambda}{2} T_1} = \delta_{T_1, \eta(\frac{\lambda}{2})}$ by the induction hypothesis. Thus $T = (r, \{\eta(\frac{\lambda}{2}), \eta(\frac{\lambda}{2})\})$ if $\mathrm{vert}(\lambda)$ is even and $T = (r, \{\eta(\frac{\lambda}{2})\})$ when $\mathrm{vert}(\lambda)$ is odd. The number of vertical dominoes in λ is even when a is odd and $b \equiv 0 \mod 4$ or when a is even and $b \equiv 3 \mod 4$, and is odd otherwise. $\square$

For an integer n with $\mathrm{Bin}(n) = \{k_1, \ldots, k_s\}$ and an odd partition λ of n, [4, Lemma 1] states that λ has a unique hook h_1 of size 2^{k_1}, and λ/h_1 is an odd partition. We apply this recursively to obtain a decomposition of λ into the tuple $(h_1, \ldots, h_s)$ of hook partitions h_i of size k_i, for $i = 1, \ldots, s$. We call $(h_1, \ldots, h_s)$ the hook decomposition of λ.

Definition 1.5.5. *The restriction set of a partition μ of size n is:*

$$R(\mu) = \{F | \langle \chi_\mu, \chi_F \rangle_{P_n} > 0\},$$

where χ_F is the irreducible character corresponding to a forest F of size n.

Proposition 1.5.6. *For an odd partition λ of n with hook decomposition $(h_1, \ldots, h_s)$, we have:*

$$R(h_1) \times \cdots \times R(h_s) \subset R(\lambda)$$

A cell c of a partition λ is called a border cell if there does not exist a cell to its southeast in the Ferrers diagram of λ. The hook corresponding to c is the set of all cells to its right in the same row or below it in the same column. The leftmost cell in a hook is called the hand, while the lowest cell in the hook is called its leg. The hand and the leg are both border cells, and the set of border cells between the two (including the hand and the leg) form the rim-hook corresponding to the hook of c. The following lemma was communicated to the author by Steven Spallone; we are grateful for sharing his proof, which we reproduce here.

Lemma 1.5.7. *Let λ be a partition and $h = (a|b)$ be a hook in λ with corresponding rim-hook r. Then*

$$c^{\lambda}_{\lambda/r,h} = 1.$$

Proof. We prove this by exhibiting a unique semistandard skew tableau of shape r and content h whose reverse reading word is a lattice permutation. A cell is called an inner cell if there is an element to its right. The cell in the rim-hook that is leftmost and topmost is called the arm of the hook.

The process for labeling the cells is thus:

- Fill a 1 in the arm.

- Fill a 1 in each inner cell.
- Label the remaining cells $\{2, \ldots, b\}$ from top to bottom.

The reverse reading word (entries of tableau read right to left and top to bottom) of this tableau is of the type

$$1 \ldots 12 \cdots s_1 1 \ldots 1(s_1 + 1) \ldots, s_2 1 \ldots 1 \ldots.$$

This is clearly a lattice permutation (the number of is is more than the number of $i+1$s in any initial segment, for all positive integers i) and is unique since the 1s can occupy no other position. $\square$

Proof of Proposition 1.5.6. Since $P_n \subset S_{2^{k_1}} \times \cdots \times S_{2^{k_s}}$, we have

$$\mathrm{Res}_{P_n}^{S_n} \chi_\lambda = \sum_{\mu_i \vdash 2^{k_i} \text{ for } 1,\ldots,s} c^\lambda_{\mu_1,\mu_2,\ldots,\mu_s} \prod_i \mathrm{Res}_{H_{k_i}}^{S_{2^{k_i}}} \chi_{\mu_i},$$

where $c^\lambda_{\mu_1,\mu_2,\ldots,\mu_s}$ is as defined in Definition 3.0.6. Thus given a forest $F = (T_1, \ldots, T_s)$ of size n,

$$\langle \chi_\lambda, \chi_F \rangle_{P_n} = \sum_{\mu_i \vdash 2^{k_i} \text{ for } 1,\ldots,s} c^\lambda_{\mu_1,\mu_2,\ldots,\mu_s} \prod_i r_{\mu_i T_i}.$$

By repeated applications of Lemma 1.5.7 we know that $c^\lambda_{h_1,h_2,\ldots,h_s} = 1$. Thus for a forest $F = (T_1, \ldots, T_s)$ where $T_i \in R(h_i)$, we see that $\langle \chi_\lambda, \chi_F \rangle_{P_n} > 0$. $\square$

1.6 Some generating functions

In this section we find generating functions for conjugacy classes collected by class size and irreducible representations collected by dimension.These generating functions are easily derived from results in [33] on the sizes of conjugacy classes and dimensions of irreducible representations of H_k, which

Proposition 1.6.1. *Let a_{km} denote the number of irreducible representations of H_k of dimension 2^m. Let $g_k(t) = \sum_{m \geq 0} a_{km} t^m$. Then $g_k(t)$ satisfies the recurrence relation:*

$$g_k(t) = \frac{t}{2}[g_{k-1}(t)^2 - g_{k-1}(t^2)] + 2g_{k-1}(t^2). \tag{1.18}$$

Proof. Note that by Definition 1.2.13 we have $g_k(t) = \sum_T t^{log_2(\dim(T))}$, where T runs over all 1-2 binary trees of height k. Let $T^{\pm}$ denote the trees $(r, \{T, T\})$ and $(r, \{T\})$ respectively. From Remark 1.2.6 we know that each of their dimensions satisfies the relation:

$$t^{log_2(\dim(T^{\pm})} = t^{2log_2(\dim(T))}.$$

Thus each of the set of Type I and Type II trees are enumerated by $g_{k-1}(t^2)$ in the generating function.

Given two distinct trees T_1 and T_2 of height $k-1$ let $T = (r, \{T_1, T_2\})$. Then from Remark 1.2.6

$$t^{log_2(\dim(T)} = t^{log_2(\dim(T_1)\dim(T_2))+1}.$$

The factor $\frac{g_{k-1}(t)^2 - g_{k-1}(t^2)}{2}$ is the sum $\sum_T t^{log_2(\dim(T_1)\dim(T_2))}$ over all trees T of Type III. This must then be multiplied by t to enumerate such trees by their dimension. $\square$

Remark 1.6.2. In particular $g_k(2)$ is the sum of the dimensions of all representations, which is equal to the number of involutions of H_k. $g_k(4)$ is the sum of the squares of dimensions of representations, thus $g_k(4) = 2^{2^k-1}$. Substituting into the equation, we have

$$g_k(2) = g_{k-1}(2)^2 + g_{k-1}(4).$$

The involutions of H_k are enumerated by the elements $(\sigma_1, \sigma_2)^1$ for involutions σ_1

and σ_2 of H_{k-1}, and by the elements $(\sigma, \sigma^{-1})^{-1}$ where σ is an element of H_{k-1}.

Proposition 1.6.3. *Let b_{km} be the number of conjugacy classes of H_k of size 2^m. Define the generating function $f_k(t) := \sum_{m \geq 0} b_{km} t^m$. The ordinary generating function $f_k(t)$ satisfies the recurrence relation:*

$$f_k(t) = \frac{t}{2}[f_{k-1}(t)^2 - f_{k-1}(t^2)] + f_{k-1}(t^2) + t^{2^{k-1}-1} f_{k-1}(t). \tag{1.19}$$

Proof. By Definition 1.2.15 we have $f_k(t) = \sum_T t^{log_2(\rho(T))}$. The sizes of conjugacy classes may be found in Table 1.1.

Let $T^{\pm}$ be as defined in the last proof. Since $\rho(T^+) = \rho(T)^2$, the monomials in the sum $f_k(t)$ corresponding to such trees are enumerated by the term $f_{k-1}(t^2)$. Since $\rho(T^-) = 2^{2^{k-1}-1}\rho(T)$, the monomials corresponding to such trees are enumerated by $t^{2^{k-1}-1} f_{k-1}(t)$.

Given two distinct trees T_1 and T_2 of height $k-1$, let $T = (r, \{T_1, T_2\})$. In this case we have $\rho(T) = 2\rho(T_1)\rho(T_2)$, and as in the proof above, $\frac{t}{2} g_{k-1}(t)^2 - g_{k-1}(t^2)$ is an enumeration of the monomials corresponding to such trees. □

The following corollary generalises the above generating functions.

Corollary 1.6.4. *Let α_{nm} denote the number of forests of size n and dimension m and let β_{nm} denote the number of forests of size n and order m. Define $G(v,t) = \sum_{n,m} \alpha_{nm} v^n t^m$ and $F(v,t) = \sum_{n,m} \beta_{nm} v^n t^m$. Then we have*

$$G(v,t) = \prod_{i \geq 0} (1 + g_i(t) v^{2^i})$$

$$F(v,t) = \prod_{i \geq 0} (1 + f_i(t) v^{2^i})$$

Chapter 2

Character polynomials and restriction from GL_n to S_n

2.1 Introduction

Let K be a field of characteristic 0. Let $P = K[X_1, X_2, \dots]$, a ring of polynomials in infinitely many variables. Regard P as a graded algebra where the variable X_i has degree i.

Definition 2.1.1. *For each $n \geq 1$, let V_n be a representation of the symmetric group S_n. The collection $\{V_n\}_{n=1}^{\infty}$ is said to have* eventually polynomial character*, if there exists $q \in P$ and a positive integer N such that, for each $n \geq N$ and each $w \in S_n$,*

$$\mathrm{trace}(w; V_n) = q(X_1(w), X_2(w), \dots),$$

where $X_i(w)$ is the number of i-cycles in w. The collection $\{V_n\}$ is said to have polynomial character *if $N = 1$. The polynomial q is called the* character polynomial *of $\{V_n\}$.*

Character polynomials have been used to study characters of families of representations of symmetric groups that occur naturally in combinatorics, topology and other areas. A survey of their history can be found in the article of Garsia and Goupil [12]. More recently, Church, Ellenberg and Farb [9] developed the theory of FI-modules. They showed that each finitely generated FI-module gives rise to a family of representations with eventually polynomial character.

Any polynomial $q \in P$ gives rise to a class function on S_n for every positive integer n. The value of this function at $w \in S_n$ is obtained by substituting for X_i the number of i-cycles in w. For each n, we define the moment of q as the average value of the associated class function on S_n. The ring P has a basis indexed by integer partitions, which we call the binomial basis (Definition 2.2.2). We give an explicit formula for the moment of a binomial basis element (Theorem 2.2.3). This formula can be used to compute inner products of class functions coming from character polynomials. It implies that such an inner product achieves a constant value for large n (Corollary 2.2.4). This is a character-theoretic analogue of [9, Theorem 1.13], which establishes representation stability for finitely generated FI-modules.

For each partition λ, let W_λ denote the Weyl functor (see [1, Definition II.1.3]) associated to λ. Let Par denote the set of partitions, $\mathrm{Par}(d)$ the set of partitions of d and $\mathrm{Par}(n, d)$ the set of all partitions of d with at most n parts. Then $W_\lambda(K^n)$, as λ runs over $\mathrm{Par}(n, d)$, are the irreducible polynomial representations of the general linear group $GL_n(K)$ of degree d. For a partition $\mu = (\mu_1, \dots, \mu_m)$ of size $|\mu|$ and an integer $n \geq \mu_1 + |\mu|$, let $\mu[n]$ denote the padded partition $(n - |\mu|, \mu_1, \mu_2, \dots, \mu_m)$. Let $V_{\mu[n]}$ denote the Specht module of S_n corresponding to $\mu[n]$. In Section 2.2.3 we show that the family $\{\mathrm{Res}^{GL_n(K)}_{S_n} W_\lambda(K^n)\}$ has polynomial character and determine its character polynomial S_λ (Theorem 2.2.8) by applying the Jacobi–Trudi identities to the character polynomials of symmetric and exterior powers of K^n (Corollary 2.2.7). The character polynomials of symmetric and exterior tensor powers of K^n have generating functions with Eulerian factorization (Theorem 2.2.6).

Consider the decomposition of the restriction of $W_\lambda(K^n)$ to S_n into Specht modules:

$$\mathrm{Res}_{S_n}^{GL_n(K)} W_\lambda(K^n) = \bigoplus_{\mu} V_{\mu[n]}^{\oplus r_{\lambda\mu[n]}},$$

where the sum is over partitions μ such that $n - |\mu| \geq \mu_1$. It is well-known that the coefficients $r_{\lambda\mu[n]}$, called the *restriction coefficients* are eventually constant for large n. This result is attributed by Assaf and Speyer in [3] to [22, Theorem XI] of D. E. Littlewood. Let $\overline{r}_{\lambda\mu}$ be their eventually constant value, which is called the *stable restriction coefficient*. Finding a combinatorial interpretation of $\overline{r}_{\lambda\mu}$ is known as the *restriction problem*. Notwithstanding several interesting recent developments [3, 14, 35, 34], a solution to the restriction problem remains elusive. Let q_μ be the character polynomial of the collection of Specht modules $\{V_{\mu[n]}\}$. We resolve a recursive formula for q_μ from [12] in Theorem 2.2.21. Multiplying S_λ by the character polynomial q_μ of Specht modules $\{V_{\mu[n]}\}$ (which was computed by Macdonald [25, Example I.7.14(b)] and Garsia–Goupil [12]), and then taking moments (Theorem 2.2.3) gives an algorithm to compute the stable restriction coefficients (Theorem 2.4.1). Theorem 2.2.21 enables us to find multivariate generating functions for restriction coefficients and their stable limits in Section 2.3.2. We provide generating functions in λ for the dimension of the space of S_n-invariant vectors in $W_\lambda(K^n)$ (Corollary 2.3.4) and characterize partitions with two rows, two columns and hook partitions which have non-zero S_n-invariant vectors.

Assaf and Speyer [3] and independently, Orellana and Zabrocki [35] introduced *Specht symmetric functions* to study the restriction problem. In Section 2.2.6 we explain the relationship between these two approaches.

For $\lambda, \mu \in \mathrm{Par}$ and appropriately large n, consider the decomposition of the product $V_{\lambda[n]} \otimes V_{\mu[n]}$ as an S_n module (by the diagonal action):

$$V_{\lambda[n]} \otimes V_{\mu[n]} = \bigoplus_{\nu} V_{\nu[n]}^{\oplus g(\lambda[n],\mu[n],\nu[n])},$$

where the sum is over partitions ν such that $n-|\nu| \geq \nu_1$. The coefficients $g(\lambda[n], \mu[n], \nu[n])$ are called the *Kronecker coefficients*. They were shown by Murnaghan in [28],[30] to stabilise to a value called the *reduced Kronecker coefficients*, which we denote by $\overline{g}(\lambda, \mu, \nu)$. Abundant results exist for their asymptotic behaviour ([27],[41]), bounds ([36],[38]), computational complexity ([37]), generating functions ([11], [2]), etc. The problem of finding a combinatorial interpretation for these coefficients remains open. Some of the advances in this direction are [5], [40], [11]. The expression for q_μ in 2.2.21 gives us generating functions for the Kronecker coefficients and their stable limits (Theorem 2.4.2).

The main new results in this thesis are the following:

- We find the character polynomials for $\mathrm{Sym}^d(K^n)$ and $\wedge^d(K^n)$ in Theorem 2.2.6. We use this to find the character polynomials for $\mathrm{Res}_{S_n}^{GL_n(K)} W_\lambda(K^n)$ in Theorem 2.2.11.

  In Corollary 2.3.4 we find the generating function for the multiplicity of the trivial representation in $\mathrm{Res}_{S_n}^{GL_n(K)} W_\lambda(K^n)$. In Theorem 2.3.15 we find positivity condition for this multiplicity when λ has either two rows or two columns or is a hook shape. These results may be found in [31], coauthored with A. Prasad, D. Paul and S. Srivastava.

- Theorem 2.2.21 gives an expression for the character polynomial q_μ of Specht modules $\{V_{\mu[n]}\}$. This is the main new result of the chapter that does not occur in [31].

  Theorem 2.2.21 is used in Theorem 2.3.6 to find a generating function for the multiplicity of $V_{\mu[n]}$ in $\mathrm{Res}_{S_n}^{GL_n(K)} W_\lambda(K^n)$. This generating function is used in

Theorem 2.3.10 to prove Littlewood's identity

$$r_{\lambda,\mu[n]} = \langle s_\lambda, s_\nu[1 + h_1 + \dots]\rangle.$$

The expression for q_μ also gives a generating function for the Kronecker coefficients in Theorem 2.4.2. This generating function is used in Lemma 2.4.4 to find an expression for reduced Kronecker coefficients in terms of Kronecker coefficients and Littlewood-Richardson coefficients, which occurs as Lemma 2.1 of [7].

We must note that the generating functions in Theorem 2.3.6 and Theorem 2.4.2 are known, and may be derived using symmetric functions; this thesis is our attempt to recover them using character polynomials.

2.2 Character Polynomials and their Moments

2.2.1 Moments and Stability

Definition 2.2.1 (Moment). *The moment of $q \in P$ at n is defined as:*

$$\langle q\rangle_n = \frac{1}{n!}\sum_{w\in S_n} q(X_1(w), X_2(w), \dots).$$

We shall express integer partitions in *exponential notation*: given a partition α with largest part r, we write:

$$\alpha = 1^{a_1}2^{a_2}\cdots r^{a_r},$$

where a_i is the number of parts of α of size i for each $1 \le i \le r$. Thus α is a partition of the integer $|\alpha| := a_1 + 2a_2 + \dots + ra_r$. For every integer partition $\alpha = 1^{a_1}\cdots r^{a_r}$

define $\binom{X}{\alpha} \in P$ by:

$$\binom{X}{\alpha} = \binom{X_1}{a_1}\binom{X_2}{a_2}\cdots\binom{X_r}{a_r}.$$

Definition 2.2.2 (Binomial basis)**.** *The basis of P consisting of elements*

$$\left\{\binom{X}{\alpha} \;\middle|\; \alpha \textit{ is an integer partition}\right\}$$

is called the binomial basis *of P.*

For an integer partition $\alpha = 1^{a_1}2^{a_2}\cdots r^{a_r}$, define $z_\alpha = \prod_{i=1}^{r} i^{a_i}a_i!$. This is the order of the centralizer in S_n of a permutation with cycle-type α.

Theorem 2.2.3. *For every integer partition $\alpha = 1^{a_1}2^{a_2}\cdots$, we have:*

$$\left\langle\binom{X}{\alpha}\right\rangle_n = \begin{cases} 0 & \textit{if } n < |\alpha|, \\ 1/z_\alpha & \textit{otherwise.} \end{cases}$$

Proof. We have:

$$\sum_{n\geq 0}\left\langle\binom{X}{\alpha}\right\rangle_n v^n = \sum_{n\geq 0}\frac{1}{n!}\sum_{w\in S_n}\prod_{i\geq 1}\binom{X_i(w)}{a_i}v^{iX_i(w)}.$$

Replace the sum $w \in S_n$ by a sum over conjugacy classes in S_n. If $\beta = 1^{b_1}2^{b_2}\cdots$ is a partition of n, then the number of elements in S_n with cycle type β is $\frac{n!}{\prod_i i^{b_i}b_i!}$. We get:

$$\begin{aligned}\sum_{n\geq 0}\left\langle\binom{X}{\alpha}\right\rangle_n v^n &= \sum_{n\geq 0} v^n \sum_{\beta\vdash n}\frac{1}{z_\beta}\binom{\beta}{\alpha} \\ &= \prod_{i\geq 1}\sum_{b_i\geq a_i}\frac{v^{ib_i}}{i^{b_i}b_i!}\binom{b_i}{a_i} \\ &= \prod_{i\geq 1}\sum_{b_i\geq a_i}\frac{v^{ib_i}}{i^{b_i}a_i!(b_i-a_i)!} \\ &= \prod_{i\geq 1}\sum_{b_i\geq a_i}\frac{v^{ia_i}}{i^{a_i}a_i!}\frac{v^{i(b_i-a_i)}}{i^{b_i-a_i}(b_i-a_i)!}.\end{aligned}$$

Setting $c_i = b_i - a_i$ gives:

$$\begin{aligned}\sum_{n\geq 0}\left\langle\binom{X}{\alpha}\right\rangle_n v^n &= \frac{v^{|\alpha|}}{z_\alpha}\prod_{i\geq 1}\sum_{c_i\geq 0}\frac{v^{ic_i}}{i^{c_i}c_i!}\\ &= \frac{v^{|\alpha|}}{z_\alpha}\sum_{n\geq 0}v^n\sum_{\gamma\vdash n}\frac{1}{z_\gamma}.\end{aligned}$$

Since $\sum_{\gamma\vdash n} 1/z_\gamma = 1$ for every n, we get:

$$\sum_{n\geq 0}\left\langle\binom{X}{\alpha}\right\rangle_n v^n = \frac{v^{|\alpha|}}{z_\alpha}\frac{1}{1-v}, \tag{2.1}$$

from which Theorem 2.2.3 follows. □

For two representations V and W of S_n, let:

$$\langle V, W\rangle_n = \dim \operatorname{Hom}_{S_n}(V, W),$$

which is the same as the Schur inner product of their characters:

$$\langle V, W\rangle_n = \frac{1}{n!}\sum_{w\in S_n}\operatorname{trace}(w; V)\operatorname{trace}(w, W).$$

Corollary 2.2.4. *For any $q \in P$ of degree d, $\langle q\rangle_n = \langle q\rangle_d$ for all $n \geq d$. In particular, if $\{V_n\}$ and $\{W_n\}$ are families of representations with polynomial characters of degree d_1 and d_2, then $\langle V_n, W_n\rangle_n$ stabilizes for $n \geq d_1 + d_2$.*

Proof. This follows from the fact that the polynomials $\binom{X}{\alpha}$, as α runs over the set of integer partitions, form a basis of P. □

Definition 2.2.5 (Stable moment)**.** *For a polynomial $q \in P$ we define the stable* *moment $\langle q \rangle$ of q to be the eventually constant value of $\langle q \rangle_n$:*

$$\langle q \rangle = \lim_{n\to\infty} \langle q \rangle_n .$$

Let $V_n = V_{\lambda[n]}$, the Specht module of S_n corresponding to the padded partition $\lambda[n]$. It is well-known that $\{V_n\}$ is a family of representations with eventually polynomial character [12, Prop. I.1]. In other words, for every partition λ, there exists a polynomial $q_\lambda \in P$ such that

$$\chi_{\lambda[n]}(w) = q_\lambda(X_1(w), X_2(w), \dots) \text{ for } n \geq |\lambda| + \lambda_1, \tag{2.2}$$

where $\chi_{\lambda[n]}$ denotes the character of the Specht module $V_{\lambda[n]}$. Given three partitions λ, μ, and ν of the same integer k, recall that $g(\lambda[n], \mu[n], \nu[n])$ denotes the multiplicity of $V_{\lambda[n]}$ in $V_{\mu[n]} \otimes V_{\nu(n)}$. Then

$$g(\lambda[n], \mu[n], \nu[n]) = \langle q_\lambda q_\mu q_\nu \rangle_n .$$

By Corollary 2.2.4, $g(\lambda[n], \mu[n], \nu[n])$ is eventually constant, recovering a well-known theorem of Murnaghan (see [22]). Church, Ellenberg, and Farb [9, Section 3.4] point out that this result can also be obtained by showing that the families $V_{\mu[n]} \otimes V_{\nu[n]}$ and $V_{\lambda[n]}$ come from finitely generated FI-modules.

2.2.2 Symmetric and Alternating Tensors

Let Sym^d and $\wedge^d$ denote the symmetric and alternating tensor functors respectively. Then for every $n \geq 0$, $\mathrm{Sym}^d(K^n)$ and $\wedge^d(K^n)$ can be regarded as representations of S_n. In this section, we will prove that they have polynomial character by computing their character polynomials.

Given $w \in S_n$ for any n and $q \in P$, let $q(w) = q(X_1(w), X_2(w), \dots)$.

Theorem 2.2.6. *Let $\{H_d\}_{d=0}^{\infty}$ be the sequence of polynomials in P defined by:*

$$\sum_{d=0}^{\infty} H_d t^d = \prod_{i=1}^{\infty} (1 - t^i)^{-X_i}, \tag{2.3}$$

an identity in the formal power series ring $P[[t]]$. Then for every $n \geq 1$ and every $w \in S_n$,

$$H_d(w) = \operatorname{trace}(w; \operatorname{Sym}^d(K^n)).$$

Let $\{E_d\}_{d=0}^{\infty}$ be the sequence of polynomials in P defined by:

$$\sum_{d=0}^{\infty} E_d t^d = \prod_{i=1}^{\infty} (1 - (-t)^i)^{X_i}. \tag{2.4}$$

Then for every $n \geq 1$ and every $w \in S_n$,

$$E_d(w) = \operatorname{trace}(w; \wedge^d(K^n)).$$

Proof. $\operatorname{Sym}^d K^n$ has an obvious basis indexed by multisets of size d with elements drawn from $[n]$. The character of $\operatorname{Sym}^d K^n$ at $w \in S_n$ is the number of such multisets that are fixed by w. In a multiset fixed by w, all the elements in any cycle of w appear with the same multiplicity. Therefore, a multiset fixed by w may be regarded as a multiset of cycles of w with total weight d, where the weight of each i-cycle is i. Hence the number of multisets of size d with elements drawn from the i-cycles of w is the coefficient of t^d in $(1 - t^i)^{-X_i(w)}$. It follows that the total number of multisets of size d with elements drawn from all cycles of w is the coefficient of t^d in $\prod_{i \geq 1}(1-t^i)^{-X_i(w)}$ as claimed. A similar argument works for $\wedge^d K^n$, taking d-element subsets of $[n]$ instead of d-element multisets with elements drawn from $[n]$. Also, each i-cycle contributes $(-1)^{i+1}$ $\square$

Expanding out the products in (2.3) and (2.4) gives:

Corollary 2.2.7. *For every positive integer d, we have:*

$$H_d = \sum_{\alpha \vdash d} \prod_{i=1}^{d} \left(\!\!\binom{X_i}{a_i}\!\!\right), \tag{2.5}$$

$$E_d = \sum_{\alpha \vdash d} (-1)^{a_2+a_4+\cdots} \prod_{i=1}^{d} \binom{X_i}{a_i}. \tag{2.6}$$

Here the partition α has exponential notation $1^{a_1}2^{a_2}\cdots$, and $\left(\!\binom{X_i}{a_i}\!\right)$ denotes the multiset binomial coefficient $\binom{X_i+a_i-1}{a_i}$.

2.2.3 Character Polynomials of Weyl Modules

Applying the Jacobi–Trudi identities (3.3) to the character polynomials of Sym^d and $\wedge^d$ gives character polynomials for Weyl functors. For a partition λ, let λ' denote its conjugate partition.

Theorem 2.2.8. *For every partition λ, the element of P defined by*

$$S_\lambda = \det(H_{\lambda_i+j-i}) = \det(E_{\lambda'_i+j-i}) \tag{2.7}$$

is such that for every positive integer n and every $w \in S_n$,

$$S_\lambda(w) = \mathrm{trace}(w; W_\lambda(K^n)).$$

The polynomials S_λ for partitions λ of integers at most 5 are given in Table 2.1.

Theorem 2.2.9. *Let λ be a partition of a positive integer d. For every partition $\alpha = 1^{a_1}2^{a_2}\cdots$ of d, the coefficient of $\binom{X}{\alpha}$ in the expansion of S_λ in the binomial basis (Definition 2.2.2) is $\chi_\lambda(w_\alpha)$, where w_α is a permutation with cycle type α.*

The theorem will be a consequence of the following lemma:

λ	S_λ
	$n=1$
(1)	X_1
	$n=2$
(2)	$\frac{1}{2}X_1^2+\frac{1}{2}X_1+X_2$
(1^2)	$\frac{1}{2}X_1^2-\frac{1}{2}X_1-X_2$
	$n=3$
(3)	$\frac{1}{6}X_1^3+\frac{1}{2}X_1^2+X_1X_2+\frac{1}{3}X_1+X_3$
$(2,1)$	$\frac{1}{3}X_1^3-\frac{1}{3}X_1-X_3$
(1^3)	$\frac{1}{6}X_1^3-\frac{1}{2}X_1^2-X_1X_2+\frac{1}{3}X_1+X_3$
	$n=4$
(4)	$\frac{1}{24}X_1^4+\frac{1}{4}X_1^3+\frac{1}{2}X_1^2X_2+\frac{11}{24}X_1^2+\frac{1}{2}X_1X_2+\frac{1}{2}X_2^2+X_1X_3+\frac{1}{4}X_1+\frac{1}{2}X_2+X_4$
$(3,1)$	$\frac{1}{8}X_1^4+\frac{1}{4}X_1^3+\frac{1}{2}X_1^2X_2-\frac{1}{8}X_1^2-\frac{1}{2}X_1X_2-\frac{1}{2}X_2^2-\frac{1}{4}X_1-\frac{1}{2}X_2-X_4$
$(2,2)$	$\frac{1}{12}X_1^4-\frac{1}{12}X_1^2+X_1X_2+X_2^2-X_1X_3$
$(2,1^2)$	$\frac{1}{8}X_1^4-\frac{1}{4}X_1^3-\frac{1}{2}X_1^2X_2-\frac{1}{8}X_1^2-\frac{1}{2}X_1X_2-\frac{1}{2}X_2^2+\frac{1}{4}X_1+\frac{1}{2}X_2+X_4$
(1^4)	$\frac{1}{24}X_1^4-\frac{1}{4}X_1^3-\frac{1}{2}X_1^2X_2+\frac{11}{24}X_1^2+\frac{1}{2}X_1X_2+\frac{1}{2}X_2^2+X_1X_3-\frac{1}{4}X_1-\frac{1}{2}X_2-X_4$
	$n=5$
(5)	$\frac{1}{120}X_1^5+\frac{1}{12}X_1^4+\frac{1}{6}X_1^3X_2+\frac{7}{24}X_1^3+\frac{1}{2}X_1^2X_2+\frac{1}{2}X_1X_2^2+\frac{1}{2}X_1^2X_3+\frac{5}{12}X_1^2+\frac{5}{6}X_1X_2+\frac{1}{2}X_1X_3+X_2X_3+X_1X_4+\frac{1}{5}X_1+X_5$
$(4,1)$	$\frac{1}{30}X_1^5+\frac{1}{6}X_1^4+\frac{1}{3}X_1^3X_2+\frac{1}{6}X_1^3+\frac{1}{2}X_1^2X_3-\frac{1}{6}X_1^2-\frac{1}{3}X_1X_2-\frac{1}{2}X_1X_3-X_2X_3-\frac{1}{5}X_1-X_5$
$(3,2)$	$\frac{1}{24}X_1^5+\frac{1}{12}X_1^4+\frac{1}{6}X_1^3X_2-\frac{1}{24}X_1^3+\frac{1}{2}X_1^2X_2+\frac{1}{2}X_1X_2^2-\frac{1}{2}X_1^2X_3-\frac{1}{12}X_1^2-\frac{1}{6}X_1X_2+\frac{1}{2}X_1X_3+X_2X_3-X_1X_4$
$(3,1^2)$	$\frac{1}{20}X_1^5-\frac{1}{4}X_1^3-X_1^2X_2-X_1X_2^2+\frac{1}{5}X_1+X_5$
$(2^2,1)$	$\frac{1}{24}X_1^5-\frac{1}{12}X_1^4-\frac{1}{6}X_1^3X_2-\frac{1}{24}X_1^3+\frac{1}{2}X_1^2X_2+\frac{1}{2}X_1X_2^2-\frac{1}{2}X_1^2X_3+\frac{1}{12}X_1^2+\frac{1}{6}X_1X_2-\frac{1}{2}X_1X_3-X_2X_3+X_1X_4$
$(2,1^3)$	$\frac{1}{30}X_1^5-\frac{1}{6}X_1^4-\frac{1}{3}X_1^3X_2+\frac{1}{6}X_1^3+\frac{1}{2}X_1^2X_3+\frac{1}{6}X_1^2+\frac{1}{3}X_1X_2+\frac{1}{2}X_1X_3+X_2X_3-\frac{1}{5}X_1-X_5$
(1^5)	$\frac{1}{120}X_1^5-\frac{1}{12}X_1^4-\frac{1}{6}X_1^3X_2+\frac{7}{24}X_1^3+\frac{1}{2}X_1^2X_2+\frac{1}{2}X_1X_2^2+\frac{1}{2}X_1^2X_3-\frac{5}{12}X_1^2-\frac{5}{6}X_1X_2-\frac{1}{2}X_1X_3-X_2X_3-X_1X_4+\frac{1}{5}X_1+X_5$

Table 2.1: Character polynomials of Weyl modules

Lemma 2.2.10. *Let λ be a partition of a positive integer d and let $n \geq d$. For every partition α of d, the coefficient of $\binom{X}{\alpha}$ in the expansion of H_λ in the binomial basis is $\sigma_\lambda(w_\alpha)$, where $\sigma_\lambda(w_\alpha)$ is the value of the character σ_λ of the permutation representation of S_n induced from the trivial representation of the Young subgroup $S_{n-d} \times S_{\lambda_1} \times \cdots \times S_{\lambda_l}$ (see [16, Section 2.2] or [39, Section 2.3]) at a permutation with cycle type α.*

Proof. For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of d, let $\lambda_0 = n - d$. Let

$$X_{n,\lambda} = \{(T_0, \ldots, T_l) \mid [n] = T_0 \cup \cdots \cup T_l \text{ is a partition, with } |T_i| = \lambda_i\}.$$

Let $K[X_{n,\lambda}]$ be the permutation representation associated to the action of S_n on $X_{n,\lambda}$. Then σ_λ is the character of $K[X_{n,\lambda}]$, and $\sigma_\lambda(w) = |X_{n,\lambda}^w|$, the cardinality of the number of fixed points of a permutation w in $X_{n,\lambda}$. Take $(T_0, \ldots, T_l) \in X_{n,\lambda}^w$. Then each T_i is formed by taking a union of cycles of w. Suppose that b_{ij} is the number of j cycles of w in T_i. Then the array (b_{ij}) satisfies the constraints:

$$b_{i1} + 2b_{i2} + \cdots = \lambda_i \text{ for each } i, \tag{2.8}$$

$$b_{0j} + b_{1j} + b_{2j} + \cdots = X_j(w) \text{ for each } j. \tag{2.9}$$

Let $B(\lambda; w)$ denote the set of such arrays. We have:

$$\operatorname{trace}(w, K[X_{n,\lambda}]) = \sum_{(b_{ij}) \in B(\lambda;w)} \prod_j \binom{X_j(w)}{b_{0j}\ b_{1j}\ \ldots}. \tag{2.10}$$

The multinomial coefficient $\binom{X_j(w)}{b_{0j}\ b_{1j}\ \ldots}$ is a polynomial in $X_j = X_j(w)$ with top degree term $\prod_{i\geq 1} X_j^{b_{ij}}/b_{ij}!$. Therefore, $\operatorname{trace}(w, K[X_{n,\lambda}])$ is given by a polynomial with top degree terms:

$$\sum_{b_{i1}+2b_{i2}+\cdots=\lambda_i} \prod_{i\geq 1} \prod_{j \geq 1} \frac{X_j^{b_{ij}}}{b_{ij}!}. \tag{2.11}$$

On the other hand, by (2.5),

$$H_\lambda = \sum_{b_{i1}+2b_{i2}+\cdots=\lambda_i} \prod_{i\geq 1}\prod_{j\geq 1} \left(\!\!\binom{X_j}{b_{ij}}\!\!\right). \tag{2.12}$$

The terms of homogeneous degree d in this product come from the top degree terms in each factor. Hence the top degree terms in H_λ coincide with the top degree terms in the expression on the right hand side of (2.10) and the lemma easily follows. □

Proof of Theorem 2.2.9. For partitions λ and μ of d, let $K_{\mu\lambda}$ denote the number of semi-standard Young tableaux of shape μ and weight λ. Then $K = (K_{\mu\lambda})$ is a unitriangular integer matrix with rows and columns indexed by partitions of d. We have:

$$H_\lambda = \sum_\mu K_{\mu\lambda} S_\mu, \tag{2.13}$$

$$\sigma_\lambda = \sum_\mu K_{\mu\lambda} \chi_\mu. \tag{2.14}$$

Let $K^{-1}_{\mu\lambda}$ be the entries of the inverse matrix K^{-1}. Then

$$\begin{aligned}
S_\lambda &= \sum_\mu K^{-1}_{\mu\lambda} H_\mu && \text{by (2.13)}\\
&\equiv \sum_\mu K^{-1}_{\mu\lambda} \sum_\alpha \sigma_\mu(\alpha)\binom{X}{\alpha} && \text{ignoring lower deg. terms (Lemma 2.2.10)}\\
&= \sum_{\alpha\vdash d}\sum_\mu K^{-1}_{\mu\lambda}\sigma_\mu(\alpha)\binom{X}{\alpha} \\
&= \sum_\alpha \chi_\lambda(\alpha)\binom{X}{\alpha} && \text{by (2.14),}
\end{aligned}$$

thereby completing the proof of Theorem 2.2.9. □

Theorem 2.2.11. *For every partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, S_λ is the coefficient of $t_1^{\lambda_1}\cdots t_l^{\lambda_l}$ in*

$$\Delta(t_1, \ldots, t_l) \prod_{r=1}^{l}\prod_{i\geq 1}(1-t_r^i)^{-X_i},$$

where $\Delta(t_1, \ldots, t_l) = \prod_{i>j}(1 - \frac{t_i}{t_j})$.

Proof. For every vector $\lambda = (\lambda_1, \ldots, \lambda_l)$ with non-negative integer coefficients, define:

$$S_\lambda = \det(H_{\lambda_i - i + j}).$$

When λ is a partition this coincides with the character polynomial of the Weyl module W_λ. Then, for every partition λ, S_λ is the coefficient of t^λ in $\sum_{\lambda \geq 0} S_\lambda t^\lambda$. Here $\lambda \geq 0$ indicates that the sum is over all vectors in $\mathbf{Z}^l_{\geq 0}$, and $t^\lambda = t_1^{\lambda_1} \cdots t_l^{\lambda_l}$. Now

$$\begin{aligned}
\sum_{\lambda \geq 0} S_\lambda t^\lambda &= \sum_{\lambda \geq 0} \sum_{w \in S_l} \operatorname{sgn}(w) \prod_{r=1}^{l} H_{\lambda_r - r + w(r)} t_r^{\lambda_r} \\
&= \sum_{w \in S_l} \operatorname{sgn}(w) \prod_{r=1}^{l} t_r^{r - w(r)} \sum_{\lambda_r \geq 0} H_{\lambda_r - r + w(r)} t_r^{\lambda_r - r + w(r)} \\
&= \sum_{w \in S_l} \operatorname{sgn}(w) \prod_{r=1}^{l} t_r^{r - w(r)} \sum_{\lambda_r \geq 0} H_{\lambda_r} t_r^{\lambda_r} \\
&= \prod_{i<j} (1 - t_j/t_i) \prod_{r=1}^{l} \prod_{i \geq 1} (1 - t_r^i)^{-X_i}.
\end{aligned}$$

Here we have used the convention that $H_d = 0$ when $d < 0$. □

2.2.4 Duality

Going through the entries of Table 2.1, the reader may have noticed that the coefficients in the expansion of S_λ agree up to sign with those of $S_{\lambda'}$ for every partition λ. For each vector $\mu = (\mu_1, \ldots, \mu_m)$ with non-negative integer entries, let $X^\mu = X_1^{\mu_1} \cdots X_m^{\mu_m}$, and $|\mu| = \mu_1 + \cdots + \mu_m$.

Theorem 2.2.12. *For every partition λ, if $S_\lambda = \sum_\mu a_\mu^\lambda X^\mu$, then*

$$S_{\lambda'} = \sum_\mu (-1)^{|\lambda| - |\mu|} a_\mu^\lambda X^\mu.$$

Proof. Let $\tau_d : P \to P$ denote the linear involution defined by $X^\mu \mapsto (-1)^{d - |\mu|} X^\mu$.

Comparing equations (2.3) and (2.4) shows that:

$$\tau_d(H_d) = E_d.$$

It follows that, if $|\mu| = d$, then

$$\tau_d(H_{\mu_1} \cdots H_{\mu_m}) = E_{\mu_1} \cdots E_{\mu_m}.$$

When λ is a partition of d, then every term in the expansion of the Jacobi–Trudi determinants $\det(H_{\lambda_i+j-i})$ is of the form H_μ or E_μ for integer vector μ with $|\mu| = d$. Therefore $\tau_d(\det(H_{\lambda_i+j-i})) = \det(E_{\lambda_i+j-i})$. By the Jacobi–Trudi identities (2.7),

$$\tau_d(S_\lambda) = \tau_d(\det(H_{\lambda_i+j-i})) = \det(E_{\lambda_i+j-i}) = S_{\lambda'},$$

as claimed. □

2.2.5 Character Polynomials of Specht Modules

For any partition λ, Macdonald [25, Example I.7.14(b)] gave the character polynomials $q_\mu \in P$ of (2.2) as follows:

$$q_\mu = \sum_{\{\nu \mid \mu-\nu \text{ is a vertical strip}\}} (-1)^{|\mu|-|\nu|} \sum_{\alpha \vdash |\nu|} \chi_\nu(\alpha) \binom{X}{\alpha}. \tag{2.15}$$

It immediately follows that the leading coefficients of q_λ in the binomial basis are the same as those of S_λ (see Theorem 2.2.9):

Theorem 2.2.13. *Let λ be a partition of a positive integer d. For every partition α of d, the coefficient of $\binom{X}{\alpha}$ in the expansion of q_λ in the binomial basis of P is $\chi_\lambda(\alpha)$.*

Corollary 2.2.14. *The sets:*

$$\mathbf{S} = \{S_\lambda \mid \lambda \text{ is an integer partition}\},$$
$$\mathbf{q} = \{q_\lambda \mid \lambda \text{ is an integer partition}\}$$

are bases of P.

Proof. Regard P as a graded algebra where the degree of X_i is i for each $i \geq 1$. Let P_d denote the homogeneous elements of degree d in P. The degree d homogeneous parts of $\binom{X}{\alpha}$, as α runs over all partitions of d, form a basis of P_d. Theorem 2.2.9 and Corollary 2.2.14 imply that the degree d homogeneous parts of S_λ and q_λ also form such a basis as λ runs over all partitions of d, since the character table of S_d forms a non-singular matrix. Therefore $\mathbf{S}$ and $\mathbf{q}$ are bases of P. $\square$

The umbral operator acts on every monomial X_i^b as

$$X_i^b \downarrow = \binom{X_i}{b} b!,$$

and on polynomials $p \in K[X_i]$ by extending this action. Garsia and Goupil provide the following recursive umbral formula for q_μ in [12, Theorem 1.1]

$$q_\mu(X_1, X_2, \dots) = \sum_{\alpha \vdash |\mu|} \frac{q_{r(\mu)}(a_1, a_2, \dots)}{z_\alpha} \prod_i (iX_i - 1)^{a_i} \downarrow, \tag{2.16}$$

where $r(\mu)$ is the partition formed from removing the largest row of μ.

Example 2.2.15. *Let $\mu = (2)$. Since $r(\mu) = \emptyset$ and $q_\emptyset = 1$, we have:*

$$\begin{aligned} q_{(2)} &= \frac{1}{z_{(2)}}(2X_2 - 1) \downarrow + \frac{1}{z_{(1,1)}}(X_1 - 1)^2 \downarrow \\ &= \frac{1}{2}(2X_2 \downarrow - 1) + \frac{1}{2}(X_1^2 \downarrow - 2X_1 \downarrow + 1) \\ &= X_2 - X_1 + \binom{X_1}{2}. \end{aligned}$$

We find a non-recursive formula for q_μ that is more suited to calculations of moments.

To state our result we must introduce some simplifying notation.

Definition 2.2.16. *For all positive integers i, define the polynomials*

$$p_i(v_1, v_2, \ldots, v_i) = (1 - v_1)(1 - v_1 v_2) \cdots (1 - v_1 \cdots v_i),$$

and

$$P_i(v_1, v_2, \ldots, v_i) = p_1(v_i) p_2(v_{i-1}, v_i) \cdots p_i(v_1, \ldots, v_i),$$

for all positive integers i.

Example 2.2.17. *The polynomials $p_2(v_1, v_2)$ and $P_2(v_1, v_2)$ are defined as under:*

$$\begin{aligned} p_2(v_1, v_2) &= (1 - v_1)(1 - v_1 v_2) \\ P_2(v_1, v_2) &= (1 - v_1)(1 - v_1 v_2)(1 - v_2), \end{aligned}$$

since $P_1(v_2) = (1 - v_2)$.

Definition 2.2.18. *Let R be a ring and a $f : Par^l \to R$. Let μ be a partition with at most l parts and let $p(v_1, \ldots, v_l) = \sum_{\nu=(c_1,\ldots,c_l)} a_\nu v^\nu$ be a polynomial in $\mathbb{Z}[v_1, v_2, \ldots, v_l]$. Then*

$$\sum_{p,\mu} f(\beta_1, \ldots, \beta_l) := \sum_{\nu \in \mathbb{Z}^l} a_\nu \sum_{\beta_i \vdash \sum_{j \geq i} \mu_j - c_i} f(\beta_1, \beta_2, \ldots, \beta_l),$$

where the sum is over all monomials v^ν in p such that $\sum_{j \geq i} \mu_j - c_i \geq 0$ for each i.

Example 2.2.19. *Consider the function $f : Par^2 \to \mathbb{Z}[X_1, X_2, \ldots] :$*

$$f(\beta_1, \beta_2) = \binom{X}{\beta_1}\binom{\beta_1}{\beta_2}.$$

We will evaluate the summation $\sum_{p,\mu} f(\beta_1, \beta_2)$ when $\mu = (1, 1)$ and $p = P_2(v_1, v_2)$. From definition 2.2.16 we have

$$\begin{aligned} P_2(v_1, v_2) &= (1 - v_1)(1 - v_1 v_2)(1 - v_2) \\ &= 1 - v_1 - v_2 + v_1^2 v_2 + v_1 v_2^2 - v_1^2 v_2^2 \end{aligned}$$

Let $\mu = (1,1)$ and let $f(\beta_1, \beta_2) = \binom{X}{\beta_1}\binom{\beta_1}{\beta_2}$. We consider the inner sum associated to each monomial in turn.

The summation corresponding to $v_1^0 v_2^0$ is over the range $\beta_1 \vdash \mu_1+\mu_2-0$ and $\beta_2 \vdash \mu_2-0$. Thus the inner sum is $\sum_{\beta_1 \vdash 2, \beta_2 \vdash 1} f$.

The monomials $v_1^1 v_2^0$ and $v_1^0 v_2^1$ are associated to the sums $\sum_{\beta_1 \vdash 1, \beta_2 \vdash 1} f$ and $\sum_{\beta_1 \vdash 2, \beta_2 \vdash 0} f$ respectively.

The monomial corresponding to $v_1^2 v_2^1$ corresponds to a summation over the range $\beta_1 \vdash \mu_1 + \mu_2 - 2$ and $\beta_2 \vdash \mu_2 - 1$; thus to $\sum_{\beta_1 \vdash 0, \beta_2 \vdash 0} f$.

The monomials corresponding to $v_1 v_2^2$ and $v_1^2 v_2^2$ are ignored since in both cases we have $\mu_2 - c_2 < 0$.

Thus we have

$$\sum_{P_{2,\lambda}} f(\beta_1, \beta_2) = \sum_{\beta_1 \vdash 2, \beta_2 \vdash 1} f - \sum_{\beta_1 \vdash 1, \beta_2 \vdash 1} f - \sum_{\beta_1 \vdash 2, \beta_2 \vdash 0} f + \sum_{\beta_1 \vdash 0, \beta_2 \vdash 0} f$$

Remark 2.2.20. Let f and μ be as in definition 2.2.18 and let $\operatorname{cum}(\mu) = (\sum_{j=1}^{l} \mu_j, \sum_{j=2}^{l} \mu_j, \ldots, \mu_l)$. Define the generating function

$$F(v_1, \ldots, v_l) = \sum_{\beta_1, \ldots, \beta_l} f(\beta_1, \ldots, \beta_l) v_1^{|\beta_1|} \ldots v_l^{|\beta_l|}.$$

The summation $\sum_{p,\mu} f$ is the coefficient of $v^{\operatorname{cum}(\mu)}$ in $p(v_1, \ldots, v_l) F(v_1, \ldots, v_l)$.

Theorem 2.2.21. *Given a partition μ with at most l parts we have*

$$q_\mu(X_1, X_2, \ldots) = \sum_{P_{l,\mu}} \binom{X}{\beta_1}\binom{\beta_1}{\beta_2} \cdots \binom{\beta_{l-1}}{\beta_l}.$$

The proof of this theorem is split into the following three propositions.

Proposition 2.2.22. *The Specht character polynomial q_μ may be expressed as follows*

$$q_\mu(X_1, X_2, \dots) = \sum_\beta (-1)^{l(\beta)} z_\beta \binom{X}{\beta} \langle Q(\mu, \beta) \rangle_{|\mu|}, \tag{2.17}$$

where $Q(\mu, \beta) = (-1)^{\sum_i X_i} q_{r(\mu)}(X_1, X_2, \dots) \binom{X}{\beta}$.

Proof. Expand Equation (2.16) as under:

$$\begin{aligned} q_\mu &= \sum_{\alpha \vdash |\mu|} \frac{q_{r(\mu)}(\alpha)}{z_\alpha} \prod_i (iX_i - 1)^{a_i} \downarrow \\ &= \sum_{\alpha \vdash |\mu|} \frac{q_{r(\mu)}(\alpha)}{z_\alpha} \prod_i \sum_{b_i \le a_i} \binom{a_i}{b_i} (-1)^{a_i - b_i} i^{b_i} X_i^{b_i} \downarrow \\ &= \sum_{\alpha \vdash |\mu|} \frac{q_{r(\mu)}(\alpha)}{z_\alpha} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (-1)^{l(\alpha) - l(\beta)} z_\beta \binom{X}{\beta}, \end{aligned}$$

where $\beta = 1^{b_1} 2^{b_2} \dots$ and $\beta \le \alpha$ denotes that $b_i \le a_i$ for all i. Since $\binom{\alpha}{\beta} = 0$ for all $\beta > \alpha$, we may drop this condition from the second sum and rearrange the order of summation

$$q_\mu = \sum_\beta (-1)^{l(\beta)} z_\beta \binom{X}{\beta} \sum_{\alpha \vdash |\mu|} \frac{(-1)^{l(\alpha)}}{z_\alpha} q_{r(\mu)}(\alpha) \binom{\alpha}{\beta}.$$

The inner sum in the above expression is the moment $\langle Q(\mu, \beta) \rangle_{|\mu|}$. □

Proposition 2.2.23. *We have the following expression for the class function $Q(\mu, \beta_1)$:*

$$Q(\mu, \beta_1) = \sum_{\beta_2} (-1)^{\sum X_i} \binom{X}{\beta_1} \binom{X}{\beta_2} (-1)^{l(\beta_2)} z_{\beta_2} \langle Q(r(\mu), \beta_2) \rangle_{|r(\mu)|}. \tag{2.18}$$

Proof. Recall that

$$Q(\mu, \beta_1) = (-1)^{\sum_i X_i} q_{r(\mu)}(X_1, X_2, \dots) \binom{X}{\beta_1}.$$

We substitute from Equation (2.17) for $q_{r(\mu)}(X_1, X_2, \dots)$

$$Q(\mu, \beta_1) = (-1)^{\sum_i X_i} \binom{X}{\beta_1} \sum_{\beta_2} (-1)^{l(\beta_2)} z_{\beta_2} \binom{X}{\beta_2} \langle Q(r(\mu), \beta_2) \rangle_{|r(\mu)|}$$

$\square$

Example 2.2.24. *When* $\mu = (\mu_1)$, *then* $r(\mu) = \emptyset$, *and* $q_\emptyset(X) = 1$. *With* $\beta_1 = 1^{b_{11}}2^{b_{12}}\dots$, *we have:*

$$Q(\mu, \beta_1) = (-1)^{\sum X_i}\binom{X}{\beta_1}.$$

Thus:

$$\langle Q(\mu, \beta_1)\rangle_{|\mu_1|} = \sum_{\alpha_1 \vdash |\mu_1|} \frac{(-1)^{l(\alpha_1)}}{z_{\alpha_1}}\binom{\alpha_1}{\beta_1}.$$

This is the coefficient of $v_1^{|\mu_1|-|\beta_1|}$ *in*

$$\prod_i \frac{1}{b_{1i}!}(\frac{\partial}{\partial v_1^i})^{b_{1i}} exp(-\frac{v_1^i}{i}).$$

Since $\prod_i \frac{1}{b_{1i}!}(\frac{\partial}{\partial v_1^i})^{b_{1i}} exp(-\frac{v_1^i}{i}) = \frac{(-1)^{l(\beta_1)}}{z_{\beta_1}}(1 - v_1)$,

$$\langle Q((\mu_1), \beta_1)\rangle_{\mu_1} = \begin{cases} \frac{(-1)^{l(\beta_1)}}{z_{\beta_1}} & |\beta_1| = \mu_1 \\ -\frac{(-1)^{l(\beta_1)}}{z_{\beta_1}} & |\beta_1| = \mu_1 - 1 \\ 0 & otherwise. \end{cases} \tag{2.19}$$

We may rewrite this in terms of the summation introduced in Definition 2.2.18

$$\langle Q((\mu_1), \beta_1)\rangle_{\mu_1} = \sum_{P_1,\mu} \delta_{\beta_1,\overline{\beta_1}} \frac{(-1)^{l(\overline{\beta_1})}}{z_{\overline{\beta_1}}},$$

where the term $\delta_{\beta_1,\overline{\beta_1}}$ *denotes that the moment is zero when* β_1 *is neither a partition of* $|\mu|$ *or* $|\mu| - 1$.

Example 2.2.25. *Let* $\mu = (\mu_1, \mu_2)$ *and* $\beta_i = 1^{b_{i1}}2^{b_{i2}}\dots$ *for* $i = 1, 2$. *From* (2.18) *and* (2.19) *we have:*

$$Q(\mu, \beta_1) = (-1)^{\sum X_i}\binom{X}{\beta_1}\binom{X}{\beta_2}(-1)^{l(\beta_2)} z_{\beta_2} \sum_{P_1,r(\mu)} \frac{(-1)^{l(\overline{\beta_2})}}{z_{\overline{\beta_2}}},$$

where $P_1 = P_1(v_2)$. Note that the moment $\langle Q(\mu, \beta_1)\rangle_{|\mu|}$ is the coefficient of $v_1^{|\mu|-|\beta_1|}v_2^{\mu_2}$ in

$$\prod_i \frac{1}{b_{1i}!}\frac{\partial^{b_{1i}}}{\partial {v_1^i}^{b_{1i}}} exp(-\frac{v_1^i(1+v_2^i)}{i}) - [v_1^{|\mu|-|\beta_1|}v_2^{\mu_2-1}]\prod_i \frac{1}{b_{1i}!}\frac{\partial^{b_{1i}}}{\partial {v_1^i}^{b_{1i}}} exp(-\frac{v_1^i(1+v_2^i)}{i}).$$

Thus it is the coefficient of $v_1^{|\mu|-|\beta_1|}v_2^{\mu_2}$ in

$$(1-v_2)\prod_i \frac{1}{b_{1i}!}\frac{\partial^{b_{1i}}}{\partial {v_1^i}^{b_{1i}}} exp(-\frac{v_1^i(1+v_2^i)}{i}).$$

The expression $\prod_i \frac{1}{b_{1i}!}\frac{\partial^{b_{1i}}}{\partial {v_1^i}^{b_{1i}}} exp(-\frac{v_1^i(1+v_2^i)}{i})$ simplifies to

$$\frac{(-1)^{l(\beta_1)}}{z_{\beta_1}}(1-v_1)(1-v_1v_2)\prod_i (1+v_2^i)^{b_{1i}}.$$

We express $\prod_i (1+v_2^i)^{b_{1i}} = \sum_{\beta_2}\binom{\beta_1}{\beta_2}v_2^{|\beta_2|}$. Thus we have that $\langle Q(\mu, \beta_1)\rangle_{|\mu|}$ is the coefficient of $v_1^{|\mu|-|\beta_1|}v_2^{\mu_2}$ in

$$(1-v_1)(1-v_1v_2)(1-v_2)\frac{(-1)^{l(\beta_1)}}{z_{\beta_1}}\sum_{\beta_2}\binom{\beta_1}{\beta_2}v_2^{|\beta_2|}$$

Thus

$$\langle Q(\mu, \beta_1)\rangle_{|\mu|} = \sum_{P_2,\mu} \delta_{\overline{\beta_1},\beta_1}\frac{(-1)^{l(\overline{\beta_1})}}{z_{\overline{\beta_1}}}\binom{\overline{\beta_1}}{\beta_2}.$$

With these motivating examples we are ready to find an expression for the moment of $Q(\mu, \beta)$ in general.

Proposition 2.2.26. *For a partition $\mu = (\mu_1, \ldots, \mu_l)$ and a partition β_1:*

$$\langle Q(\mu, \beta_1)\rangle_{|\mu|} = \sum_{P_l,\mu} \delta_{\beta_1,\overline{\beta_1}}\frac{(-1)^{l(\overline{\beta_1})}}{z_{\overline{\beta_1}}}\binom{\overline{\beta_1}}{\beta_2}\binom{\beta_2}{\beta_3}\cdots\binom{\beta_{l-1}}{\beta_l}.$$

Proof. We prove this by induction on the length of the partition μ. The assertion holds for partitions with a single part, as can be seen in Example 2.2.24. Assume that the assertion is true of all partitions with less than l parts. We know from Proposition 2.2.23 that

$$Q(\mu,\beta_1)=\sum_{\beta_2}(-1)^{\sum X_i}\binom{X}{\beta_1}\binom{X}{\beta_2}(-1)^{l(\beta_2)}z_{\beta_2}\langle Q(r(\mu),\beta_2)\rangle_{|r(\mu)|}.$$

The induction hypothesis effects a substitution into $\langle Q(r(\mu),\beta_2)\rangle_{|r(\mu)|}$ to yield

$$Q(\mu,\beta_1)=\sum_{\beta_2}(-1)^{\sum X_i}\binom{X}{\beta_1}\binom{X}{\beta_2}(-1)^{l(\beta_2)}z_{\beta_2}\sum_{P_{l-1},r(\mu)}\delta_{\beta_2,\overline{\beta_2}}\frac{(-1)^{l(\overline{\beta_2})}}{z_{\overline{\beta_2}}}\binom{\overline{\beta_2}}{\beta_3}\cdots\binom{\beta_{l-1}}{\beta_l}.$$

Thus

$$\langle Q(\mu,\beta_1)\rangle_{|\mu|}=\sum_{\alpha\vdash|\mu|}\sum_{P_{l-1},r(\mu)}\frac{(-1)^{l(\alpha)}}{z_\alpha}\binom{\alpha}{\beta_1}\binom{\alpha}{\beta_2}\binom{\beta_2}{\beta_3}\cdots\binom{\beta_{l-1}}{\beta_l}.$$

This expression is the coefficient of $v_1^{|\mu|}v_2^{|r(\mu)|}\cdots v_l^{\mu_l}$ in the expression

$$P_{l-1}\prod_i\frac{1}{b_{1i}!}(\frac{\partial}{\partial v_1^i})^{b_{1i}}exp(-\frac{v_1^i(1+v_2^i(1+\cdots v_{l-1}^i(1+v_l^i))\cdots)}{i}),$$

where $P_{l-1}=P_{l-1}(v_2,\ldots,v_l)$. This expression simplifies to

$$P_{l-1}p_l\frac{(-1)^{l(\beta_1)}}{z_{\beta_1}}\prod_i(1+v_2^i(1+\cdots v_{l-1}^i(1+v_l^i))\cdots)^{b_1 i},$$

where $p_l=p_l(v_1,\ldots,v_l)$. Note that $p_l(v_1,\ldots,v_l)P_{l-1}(v_2,\ldots,v_l)=P_l(v_1,\ldots,v_l)$. We expand the product $\prod_i(1+v_2^i(1+\cdots v_{l-1}^i(1+v_l^i))\cdots)$ by repeated applications of the binomial theorem into the form

$$\prod_i(1+v_2^i(1+\cdots v_{l-1}^i(1+v_l^i))\cdots)=\sum_{\beta_2,\ldots,\beta_l}\binom{\beta_1}{\beta_2}\cdots\binom{\beta_{l-1}}{\beta_l}v_2^{|\beta_2|}\cdots v_l^{|\beta_l|}.$$

The coefficient of $v_1^{|\mu|}v_2^{|r(\mu)|}\cdots v_l^{\mu_l}$ in the expression

$$P_l\frac{(-1)^{l(\beta_1)}}{z_{\beta_1}}\sum_{\beta_2,\ldots,\beta_l}\binom{\beta_1}{\beta_2}\cdots\binom{\beta_{l-1}}{\beta_l}v_2^{|\beta_2|}\cdots v_l^{|\beta_l|}$$

is precisely the sum

$$\sum_{P_l,\mu} \delta_{\beta_1,\overline{\beta_1}} \frac{(-1)^{l(\overline{\beta_1})}}{z_{\overline{\beta_1}}} \binom{\overline{\beta_1}}{\beta_2} \binom{\beta_2}{\beta_3} \cdots \binom{\beta_{l-1}}{\beta_l},$$

where the term $\overline{\beta_1}$ is used to indicate that the expression is zero if the size of the partition β_1 lies outside the values permitted by the sum $\sum_{P_l,\mu}$. □

Proof of Theorem 2.2.21. From Proposition 2.2.22

$$q_\mu(X_1, X_2, \dots) = \sum_{\beta_1} (-1)^{l(\beta_1)} z_{\beta_1} \binom{X}{\beta_1} \langle Q(\mu, \beta_1) \rangle_{|\mu|},$$

Substituting this expression for the value of $\langle Q(\mu, \beta_1) \rangle_{|\mu|}$ from Proposition 2.2.26 yields the desired result. □

2.2.6 Relation to Symmetric Functions

Let Λ denote the ring of symmetric functions (see Appendix 3). Macdonald [25, Example I.7.13] constructed an isomorphism $\phi : \Lambda \to P$ taking the Schur function s_λ to the character polynomial q_λ. In this section we study a different isomorphism $\Phi : \Lambda \to P$, due to Orellana and Zabrocki [35], which takes s_λ to S_λ. Under this isomorphism q_λ is the image of $\tilde{s}_\lambda$ the *Specht symmetric functions* of [3, 35].

Following [35, Proposition 12], define an algebra homomorphism $\Phi : \Lambda \to P$ by:

$$\Phi : p_k \mapsto \sum_{d|k} d X_d. \tag{2.20}$$

For each $k > 0$, define

$$\Xi_k = 1, e^{2\pi i/k}, e^{4\pi i/k}, \dots e^{2(k-1)\pi i/k},$$

and for an integer partition $\mu = (\mu_1, \dots, \mu_m)$,

$$\Xi_\mu = \Xi_{\mu_1}, \dots, \Xi_{\mu_m}.$$

Let R_n denote the space of K-valued class functions on S_n. For every $n \geq 0$ there is a map $\mathrm{ev}^n_\Lambda : \Lambda \to R_n$ defined by:

$$\mathrm{ev}^n_\Lambda f(w) = f(\Xi_\mu),$$

where μ is the cycle type of w. In other words, the symmetric function is evaluated on $|\mu|$ variables, whose values are given by the list Ξ_μ. With this definition, $\mathrm{ev}^n_\Lambda(s_\lambda)$ is the character of $\mathrm{Res}^{GL_n(K)}_{S_n} W_\lambda(K^n)$.

For each $q \in P$ consider the function $\mathrm{ev}^n_P(q) \in R_n$ given by:

$$\mathrm{ev}^n_P(q)(w) = q(X_1(w), X_2(w), \dots).$$

This defines a ring homomorphism $\mathrm{ev}^n_P : P \to R_n$.

Observe that $\oplus_{n=1}^\infty \mathrm{ev}^n_P : P \to \oplus_{n=1}^\infty R_n$ is injective, for if $\mathrm{ev}^n_P(q) \equiv 0$ for all n, then q vanishes whenever $X_1, X_2, \dots$ take non-negative integer values, and hence q must be identically 0.

Theorem 2.2.27. *The algebra homomorphism Φ is the unique K-linear map $\Lambda \to P$ such that the diagram*

$$\begin{tikzcd} \Lambda \arrow[rr, "\Phi"] \arrow[dr, "\mathrm{ev}^n_\Lambda"'] & & P \arrow[dl, "\mathrm{ev}^n_P"] \\ & R_n & \end{tikzcd} \tag{2.21}$$

commutes for every $n \geq 1$.

Proof. From the definition of ev^n_Λ,

$$\mathrm{ev}^n_\Lambda(p_k)(w) = \sum_d X_d(w) \sum_{j=0}^{d-1} (e^{2\pi i/d})^{jk}.$$

Now observe that

$$\sum_{j=0}^{d-1} (e^{2\pi i/d})^{jk} = \begin{cases} d & \text{if } d|k, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\mathrm{ev}^n_\Lambda(p_k)(w) = \sum_{d|k} dX_d(w) = \mathrm{ev}^n_P(\Phi(p_k)).$$

Since $\oplus_n \mathrm{ev}^n_P : P \to \oplus R_n$ is injective, $\Phi(p_k)$ is completely determined by the commutativity of (2.21). Since the polynomials $\{p_k\}_{k\geq 1}$ generate Λ, Φ is completely determined by its values on p_k. □

Lemma 2.2.28. *The homomorphism $\Phi : \Lambda \to P$ is an isomorphism of rings.*

Proof. The inverse of Φ is obtained using the Möbius inversion formula:

$$X_k \mapsto \frac{1}{d} \sum_{d|k} \mu(k/d) p_d.$$

□

Theorem 2.2.29. *For every partition λ, we have:*

$$\Phi(s_\lambda) = S_\lambda, \tag{2.22}$$

$$\Phi(\tilde{s}_\lambda) = q_\lambda. \tag{2.23}$$

Proof. This follows immediately from Theorem 2.2.27. □

Remark 2.2.30. The second identity (2.23) is [35, Prop. 12].

2.3 Restriction Coefficients

2.3.1 Stable Restriction Coefficients

The restriction coefficients $r_{\lambda\mu[n]}$ determine the decomposition of a Weyl module $W_\lambda(K^n)$ into irreducible representations of S_n:

$$\mathrm{Res}_{S_n}^{GL_n(K)} W_\lambda(K^n) = \bigoplus_\mu V_{\mu[n]}^{\oplus r_{\lambda\mu[n]}}.$$

These coefficients eventually stabilise as a function of n. We call this constant value the stable restriction coefficients, and denote them $\overline{r}_{\lambda\mu}$. They are the multiplicities in the expansion of the elements of the basis $\mathbf{S}$ in terms of the basis $\mathbf{q}$:

$$S_\lambda = \sum_\mu \overline{r}_{\lambda\mu} q_\mu$$

The following result, which is now immediate, is an algorithm for computing the stable restriction coefficients:

Theorem 2.3.1. *For any partitions λ and μ and an integer n such that $n \geq |\mu| + \mu_1$,*

$$r_{\lambda\mu[n]} = \langle S_\lambda q_\mu \rangle_n ,$$

$$\overline{r}_{\lambda\mu} = \langle S_\lambda q_\mu \rangle .$$

The polynomial S_λ can be computed using Theorem 2.2.8, q_μ using Theorem 2.2.21. The matrix of the stable restriction coefficients $\overline{r}_{\lambda\mu}$, as λ and μ run over partitions of

$0 \le n \le 5$ is given by:

$$
\left(\begin{array}{c|c|cc|ccc|ccccc|ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
5 & 7 & 5 & 2 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 7 & 5 & 6 & 2 & 3 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 3 & 4 & 1 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 3 & 0 & 2 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
7 & 12 & 9 & 5 & 5 & 3 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 14 & 13 & 12 & 6 & 9 & 3 & 2 & 3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
4 & 10 & 11 & 8 & 6 & 8 & 2 & 1 & 3 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 3 & 4 & 8 & 1 & 7 & 6 & 0 & 2 & 1 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 3 & 4 & 3 & 2 & 5 & 1 & 0 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 3 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

The blocks demarcate the partitions of each integer n, and within each block, the partitions of n are enumerated in reverse lexicographic order.

2.3.2 Generating functions

In this section we obtain generating functions for the multiplicity of irreducible representations of the symmetric group in several GL_n modules. In this and the following subsections, we shall frequently use the following elementary identities:

$$\text{(A)} \qquad \exp(t/i) = \sum_{b\geq 0} \frac{1}{i^b b!} t^b,$$

$$\text{(B)} \qquad \log \frac{1}{1-t} = \sum_{i=1}^{\infty} t^i/i.$$

We shall use $\alpha = 1^{a_1}2^{a_2}\cdots$, $\beta = 1^{b_1}2^{b_2}\cdots$, $\gamma = 1^{c_1}2^{c_2}\cdots$. We use the notation $\lambda = (\lambda_1,\dots,\lambda_l)$, $\mu = (\mu_1,\dots,\mu_m)$. Also, $t^\lambda = t_1^{\lambda_1}\cdots t_l^{\lambda_l}$ and $u^\mu = u_1^{\mu_1}\cdots u_m^{\mu_m}$. We shall interpret $\lambda \geq 0$ as $\lambda_i \geq 0$ for $i = 1,\dots,l$ and $\mu \geq 0$ as $\mu_i \geq 0$ for $i = 1,\dots,m$.

We use the notation $R \sqsubset [l]$ to signify that R is a multiset with elements drawn from $[l]$. We write t^R for the monomial where t_i is raised to the multiplicity of i in R. Similarly, for any $S \subset [m]$, we write $u^S = \prod_{i\in S} u_i$.

Theorem 2.3.2. *We have:*

$$\sum_{n\geq 0,\lambda\geq 0,\mu\geq 0} \langle H_\lambda E_\mu\rangle_n t^\lambda u^\mu v^n = \prod_{R\sqsubset[l]} \frac{\prod_{S\subset[m],\ |S|\ odd}(1+u^S t^R v)}{\prod_{S\subset[m],\ |S|\ even}(1-u^S t^R v)}.$$

Proof. Using (2.3) and (2.4), we have:

$$\sum_{\lambda\geq 0,\mu\geq 0} H_\lambda E_\mu t^\lambda u^\mu = \prod_{r=1}^{l}\prod_{s=1}^{m}\prod_{i\geq 1}\left(\frac{1-(-u_s)^i}{1-t_r^i}\right)^{X_i}.$$

Now proceeding as in the proof of Theorem 2.2.3,

$$\begin{aligned}
\sum_{n\geq 0,\lambda\geq 0,\mu\geq 0} \langle H_\lambda E_\mu\rangle_n t^\lambda u^\mu v^n &= \prod_{i\geq 1}\sum_{b_i\geq 0}\frac{v^{ib_i}}{i^{b_i}b_i!}\prod_{r=1}^{l}\prod_{s=1}^{m}\left(\frac{1-(-u_s)^i}{1-t_j^i}\right)^{b_i}\\
&\overset{(A)}{=} \prod_{i\geq 1}\exp\left(\frac{v^i}{i}\prod_{r=1}^{l}\prod_{s=1}^{m}\left[\frac{1-(-u_s)^i}{1-t_r^i}\right]\right)\\
&= \exp\left(\sum_{i\geq 1}\sum_{R\sqsubset[l]}\sum_{S\subset[m]}(-1)^{|S|}\frac{(t^R(-1)^{|S|}u^S v)^i}{i}\right)\\
&\overset{(B)}{=} \prod_{R\sqsubset[l]}\prod_{S\subset[m]}\left(1-(-1)^{|S|}t^R u^S v\right)^{(-1)^{|S|+1}},
\end{aligned}$$

which is equivalent to the desired expression. ∎

Corollary 2.3.3. *We have:*

$$\sum_{\lambda\geq 0,\mu\geq 0} \langle H_\lambda E_\mu\rangle\, t^\lambda u^\mu = \prod_{R,S}(1-(-1)^{|S|}u^S t^R)^{(-1)^{|S|+1}},$$

where the product is over $R \sqsubset [l]$, $S \subset [m]$, with at least one of R and S non-empty.

Corollary 2.3.4. *For every partition λ, $r_{\lambda\emptyset[n]}$ is the coefficient of $t^\lambda v^n$ in*

$$\Delta(t_1,\dots,t_l)\prod_{R\sqsubset[l]}(1-t^R v)^{-1}.$$

Proof. From Theorem 2.3.2 we get:

$$\sum_{\lambda\geq 0, n\geq 0} \langle H_\lambda\rangle_n\, t^\lambda v^n = \prod_{R\sqsubset[l]}(1-t^R v)^{-1}. \tag{2.24}$$

Using this, the corollary can be deduced from Theorem 2.2.11 by taking moments. □

Corollary 2.3.5. *For every partition λ, $\overline{r}_{\lambda\emptyset}$ is the coefficient of t^λ in*

$$\Delta(t_1,\dots,t_l)\prod_{R\sqsubset[l]\neq\emptyset}(1-t^R)^{-1}.$$

Theorem 2.3.6. *For every partition λ with at most l parts, and every partition μ with at most m parts, $r_{\lambda\mu[n]}$ is the coefficient of $t^\lambda v^n u^\mu$ in*

$$\Delta(t_1,\dots,t_l)\Delta(u_1,\dots,u_m)E(u_1,\dots,u_m)\prod_{i=0}^{m}\prod_{R\sqsubset[l]}(1-u_i t^R v)^{-1}, \tag{2.25}$$

where $u_0 := 1$ and $E(u_1,\dots,u_m) = (1-u_1)\dots(1-u_m)$.

Proof. First we prove that $\langle H_\lambda q_\mu\rangle_n$ is the coefficient of $t^\lambda v^n u^\mu$ in

$$\Delta(u_1,\dots,u_m)E(u_1,\dots,u_m)\prod_{i=0}^{m}\prod_{R\sqsubset[l]}(1-u_i t^R v)^{-1},$$

Recall the character polynomials of q_μ and H_λ from Theorem 2.2.21 and Equation (2.12) respectively. Thus

$$\langle H_\lambda q_\mu \rangle_n = \sum_{\alpha \vdash n} \frac{1}{z_\alpha} \left[\prod_{j=1}^{l} \sum_{\nu_j \vdash \lambda_j} \binom{\alpha}{\nu_j} \right] \left[\sum_{P_l,\mu} \binom{\alpha}{\beta_1} \cdots \binom{\beta_{l-1}}{\beta_l} \right]$$

With $\nu_j = 1^{c_{j1}} 2^{c_{j2}} \ldots$, and $\beta_i = 1^{b_{i1}} 2^{b_{i2}} \ldots$, we may rewrite this as a product

$$\langle H_\lambda q_\mu \rangle_n = \sum_{\alpha \vdash n} \prod_i \frac{1}{i^{a_i} a_i!} \left[\prod_{j=1}^{l} \sum_{\nu_j \vdash \lambda_j} \binom{a_i}{c_{ji}} \right] \left[\sum_{P_l,\mu} \binom{a_i}{b_{1i}} \cdots \binom{b_{l-1i}}{b_{li}} \right]$$

The expression on the right is the coefficient of $t^\lambda v^n w_1^{|\mu|} w_2^{|r(\mu)|} \ldots w_m^{|\mu_m|}$ in

$$P_m \prod_i \exp\left(\frac{v^i}{i} \frac{1 + w_1^i(1 + w_2^i(1 + \ldots + w_{m-1}^i(1 + w_m^i)}{(1 - t_1^i) \ldots (1 - t_l^i)} \right),$$

where $P_m = P_m(w_1, \ldots, w_m)$. Upon substituting $w_i = \frac{u_i}{u_{i-1}}$ for $i = 2, \ldots, m$ and $w_1 = u_1$ we have $P_m(w_1, \ldots, w_m) = E(u_1, \ldots, u_m)\Delta(u_1, \ldots, u_m)$ and

$$\langle H_\lambda q_\mu \rangle_n = E(u_1, \ldots, u_m)\Delta(u_1, \ldots, u_m) \prod_i \exp\left(\frac{v^i}{i} \frac{1 + u_1^i + \ldots + u_m^i}{(1 - t_1^i) \ldots (1 - t_l^i)} \right).$$

Which simplifies to the expression

$$\langle H_\lambda q_\mu \rangle_n = \Delta(u_1, \ldots, u_m) E(u_1, \ldots, u_m) \prod_{i=0}^{m} \prod_{R \sqsubset [l]} (1 - u_i t^R v)^{-1}.$$

Thus

$$\langle S_\lambda q_\mu \rangle_n = \Delta(t_1, \ldots, t_l)\Delta(u_1, \ldots, u_m) E(u_1, \ldots, u_m) \prod_{i=0}^{m} \prod_{R \sqsubset [l]} (1 - u_i t^R v)^{-1}.$$

$\square$

Corollary 2.3.7. *For every partition λ with at most l parts, and every partition μ with at most m parts, $\overline{r}_{\lambda\mu}$ is the coefficient of $t^\lambda u^\mu$ in*

$$\Delta(t_1,\ldots,t_l)\Delta(u_1,\ldots,u_m)\prod_{i=0}^{m}\prod_{R\sqsubset[l]\neq\emptyset}(1-u_it^R)^{-1},$$

where $u_0 := 1$.

This generating function is symmetric in each of the sets of variables. The next proposition allows us to express it in a basis of the symmetric functions.

Proposition 2.3.8. *Let λ be a partition with l parts. The coefficient of v^λ in a generating function $\Delta(v_1,\ldots,v_l)G(v_1,\ldots,v_l)$ is the coefficient of $s_\lambda(v_1,\ldots,v_l)$ in $G(v_1,\ldots,v_l)$.*

Proof. This is immediate from [39, Theorem 5.4.10], which states that the coefficient of $s_\lambda(v_1,\ldots,v_l)$ in $G(v_1,\ldots,v_l)$ is the coefficient of $V^{\lambda+\delta}$ in $G(v_1,\ldots,v_l)a_\delta(v_1,\ldots,v_l)$, where a_δ is the Vandermonde determinant. Dividing by V^δ, we have that the coefficient of $s_\lambda(v_1,\ldots,v_l)$ in $G(v_1,\ldots,v_l)$ is the coefficient of V^λ in $\Delta(v_1,\ldots,v_l)G(v_1,\ldots,v_l)$. □

Define

$$H(u_1,\ldots,u_m) := \prod_{i=1}^{m}\frac{1}{1-u_i}.$$

Equivalently $H(u_1,\ldots,u_m) = h_0+h_1+\cdots$. Similarly $E(u_1,\ldots,u_m)$ as defined in Theorem 2.3.6.

We replace the product $\prod_{i=0}^{m}\prod_{R\sqsubset[l]}(1-u_it^Rv)^{-1}$ in Equation (2.25) by the plethysm

$$H[v(1+u_1+\ldots+u_m)H(t_1,\ldots,t_l)].$$

Notation 2.3.9. *Let $u_1,u_2,\ldots$ be a possibly infinite set of variables. Then define $U = u_1+u_2+\ldots$, so that $f[U] = f(u_1,u_2,\ldots)$ for any symmetric function f.*

We now use the generating function for $r_{\lambda,\mu[n]}$ to prove Littlewood's identity.

Theorem 2.3.10 (Littlewood, [22])**.** *The restriction coefficient $r_{\lambda\mu[n]}$ is the coefficient of s_λ in the plethysm $s_{\mu[n]}[H]$.*

Proof. Following Proposition 2.3.8 we seek the coefficient of $v^n s_\lambda[T]s_\mu[U]$ in

$$E[U]H[v(1+U)H[T]]. \tag{2.26}$$

We have

$$H[v(1+U)H[T]] = \sum_{n\geq 0} v^n h_n[(1+U)H[T]].$$

The coefficient of v^n is $E[U]h_n[(1+U)H[T]]$. Note that

$$h_n[(1+U)H[T]] = \sum_{k\leq n} h_k[UH[T]]h_{n-k}[H[T]].$$

Further, from the RSK correspondence, we have:

$$h_k[UH[T]] = \sum_{\nu\vdash k} s_\nu[U]s_\nu[H[T]].$$

Equation (2.26) is rewritten as

$$\sum_{k\leq n}\sum_{j\geq 0}(-1)^j e_j[U]h_{n-k}[H[T]]\sum_{\nu\vdash k} s_\nu[U]s_\nu[H[T]].$$

We use the Pieri rules to write the coefficient of $s_\mu[U]$ in this expression as

$$\sum_{\nu:\mu-\nu \text{ is a vert strip of lt j}} (-1)^j s_\nu[H[T]]h_{n-|\nu|}[H[T]].$$

we have

$$\sum_{\nu:\mu-\nu \text{ is a vert strip of lt j}} (-1)^j s_\nu[H[T]]h_{n-|\nu|}[H[T]] = s_{\mu[n]} \qquad [8, \text{Theorem } 1].$$

Thus the restriction coefficient $r_{\lambda\mu[n]}$ is the coefficient of $s_\lambda[T]$ in $s_{\mu[n]}[H[T]]$. $\square$

2.3.3 S_n-invariant Vectors and vector partitions

For a representation V_n of S_n, let $V_n^{S_n}$ denote the subspace of S_n-invariant vectors. If a family $\{V_n\}$ of representations has polynomial character $q \in P$, then

$$\langle q \rangle_n = \dim(V_n^{S_n}) \text{ for all } n \geq 0.$$

Therefore, for any partition λ, $\dim W_\lambda(K^n)^{S_n} = \langle S_\lambda \rangle_n$. In particular, $W_\lambda(K^n)$ has a non-zero S_n-invariant vector if and only if $\langle S_\lambda \rangle_n \neq 0$.

Definition 2.3.11 (Vector Partitions)**.** *Let* $\mathbf{v} \in \mathbf{Z}^l_{\geq 0}$*. A* vector partition *of* $\mathbf{v}$ *is an unordered collection* $\mathbf{v}_1, \dots, \mathbf{v}_n$ *of non-zero vectors in* $\mathbf{Z}^l_{\geq 0}$ *such that*

$$\mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_n.$$

Let $p_n(\mathbf{v})$ *(resp.* $p_{\leq n}(\mathbf{v})$*) denote the number of vector partitions of* $\mathbf{v}$ *with exactly (resp. at most) n parts.*

Theorem 2.3.12. *For every partition* $\lambda = (\lambda_1, \dots, \lambda_l)$,

$$\dim W_\lambda(K^n)^{S_n} = \sum_{w \in S_l} \operatorname{sgn}(w) p_{\leq n}(\lambda_1 - 1 + w(1), \dots, \lambda_l - l + w(l)).$$

Proof. The coefficient of $t^\lambda v^n$ in the right hand side of (2.24) is $p_{\leq n}(\lambda)$. Therefore,

$$\langle H_\lambda \rangle_n = p_{\leq n}(\lambda) \text{ for every } \lambda \in \mathbf{Z}^l_{\geq 0}. \tag{2.27}$$

By the Jacobi–Trudi identity (2.7),

$$S_\lambda = \sum_{w \in S_l} H_{\lambda_1 - 1 + w(1), \dots, \lambda_l - l + w(l)},$$

so by (2.27),

$$\langle S_\lambda \rangle_n = \sum_{w \in S_l} \operatorname{sgn}(w) p_{\leq n}(\lambda_1 - 1 + w(1), \dots, \lambda_l - l + w(l)),$$

as claimed. □

Remark 2.3.13. In general, we do not know of a combinatorial proof of the non-negativity of $\sum_{w \in S_l} \mathrm{sgn}(w) p_{\leq n}(\lambda_1 - 1 + w(1), \ldots, \lambda_l - l + w(l))$, which follows from Theorem 2.3.12. When $l = 2$, this is the main result of Kim and Hahn [19], who refer to it as a conjecture of Landman, Brown and Portier [21].

Theorem 2.3.14. *For every positive integer n and every partition λ with at most n parts,*

$$\dim W_\lambda(K^n)^{S_n} \leq \dim W_\lambda(K^{n+1})^{S_{n+1}}.$$

Proof. We use Littlewood's plethystic formula [22, Theorem XI] for restriction coefficients. Taking $\mu = (n)$ in Littlewood's formula gives

$$\dim W_\lambda(K^n)^{S_n} = (s_\lambda, h_n[H]).$$

Recall [12, Eq. 1.8] that

$$h_{n+1}[H] - h_n[H] = h_{n+1}[H - 1],$$

so that

$$\dim W_\lambda(K^{n+1})^{S_{n+1}} - \dim W_\lambda(K^n)^{S_n} = (s_\lambda, h_{n+1}[H - 1]) \geq 0.$$

The inequality above holds because the plethystic substitution of a Schur-positive symmetric function into another is Schur-positive. □

The problem of characterizing those partitions λ for which $W_\lambda(K^n)$ has a non-zero S_n-invariant vector for large n appears to be quite hard. The following result solves this problem for partition with two rows, two columns, and for hook-partitions.

Theorem 2.3.15. *Let λ be a partition.*

1. *If $\lambda = (\lambda_1, \lambda_2)$, then $\langle S_\lambda \rangle > 0$ unless $\lambda = (1, 1)$.*

2. *If $\lambda' = (\lambda_1, \lambda_2)$, then $\langle S_\lambda \rangle > 0$ if and only if $\lambda_1 = \lambda_2$ (in which case $\langle S_\lambda \rangle = 2$) or $\lambda_1 = \lambda_2 + 1$ (in which case $\langle S_\lambda \rangle = 1$).*

3. *If $\lambda = (a+1, 1^b)$, then $\langle S_\lambda \rangle > 0$ if and only if $a \geq \binom{b+1}{2}$.*

Proof of (1). By Theorem 2.3.12 we need to show that, for every $\lambda_1 \geq \lambda_2 \geq 1$,

$$p_{\leq n}(\lambda_1, \lambda_2) > p_{\leq n}(\lambda_1 + 1, \lambda_2 - 1)$$

for sufficiently large n, unless $\lambda_1 = \lambda_2 = 1$. From the main result of Kim and Hahn [19] (the result on the last line of the first page), it follows that

$$p_n(\lambda_1, \lambda_2) \geq p_n(\lambda_1 + 1, \lambda_2 - 1) \text{ for all } n \geq 1.$$

Therefore, it suffices to prove that $p_n(\lambda_1, \lambda_2) > p_n(\lambda_1 + 1, \lambda_2 - 1)$ for at least one value of n. When $k \geq l \geq 1$ are such that at least one of k and l is even, $p_2(k, l) > p_2(k+1, l-1)$. When both k and l are odd and $(k, l) \neq (1, 1)$, $p_3(k, l) > p_3(k+1, l-1)$. These inequalities will be proved in Lemmas 2.3.17 and 2.3.18 below. □

Lemma 2.3.16. *For all $k, l \geq 0$,*

$$p_2(k,l) = \begin{cases} \frac{(k+1)(l+1)-1}{2} & \text{if both } k \text{ and } l \text{ are even,} \\ \frac{(k+1)(l+1)}{2} - 1 & \text{otherwise,} \end{cases} \tag{2.28}$$

$$p_3(k,l) = \frac{1}{6}(A + 3B + 2C), \tag{2.29}$$

where

$$A = \binom{k+2}{2}\binom{l+2}{2} - 3(k+1)(l+1) + 3,$$

$$B = \begin{cases} (k/2+1)(l/2+1) - 2 & \textit{if } k \textit{ and } l \textit{ are even,} \\ (k+1)(l+2)/4 - 1 & \textit{if } k \textit{ is odd and } l \textit{ is even,} \\ (k+2)(l+1)/4 - 1 & \textit{if } k \textit{ is even and } l \textit{ is odd,} \\ (k+1)(l+1)/4 - 1 & \textit{otherwise,} \end{cases}$$

$$C = \begin{cases} 1 & \textit{if } k \textit{ and } l \textit{ are divisible by } 3, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Consider the set of all ordered triples $((k_1, l_1), (k_2, l_2), (k_3, l_3))$ such that $\sum_i (k_i, l_i) = (k, l)$ and no $(k_i, l_i) = (0, 0)$. The group S_3 acts by permutation on the set of all such triples, and the number of orbits if $p_3(k, l)$. The quantities A, B, and C in Lemma 2.3.16 are the number of such triples that are fixed by permutations in S_3 of cycle types $(1, 1, 1)$, $(2, 1)$, and (3), respectively. The formula for $p_3(k, l)$ then follows from Burnside's lemma. The formula for $p_2(k, l)$ is obtained in a similar fashion. □

Lemma 2.3.17. *For all integers $k \geq l \geq 1$ such that at least one of k and l is even,*

$$p_2(k, l) > p_2(k+1, l-1).$$

Proof. By Lemma 2.3.16, we also have:

$$p_2(k+1, l-1) = \begin{cases} \frac{(k+2)l-1}{2} & \text{if } k \text{ and } l \text{ are odd,} \\ \frac{(k+2)l}{2} - 1 & \text{otherwise.} \end{cases}$$

Thus, if k and l are both even and $k \geq l$, then

$$\begin{aligned} p_2(k, l) - p_2(k+1, l-1) &= \frac{(k+1)(l+1) - 1}{2} - \left(\frac{(k+2)l}{2} - 1\right) \\ &= \frac{k-l}{2} + 1 > 0. \end{aligned}$$

If one of k and l is even and the other is odd, then

$$\begin{aligned} p_2(k,l) - p_2(k+1,l-1) &= \frac{(k+1)(l+1)}{2} - 1 - \left(\frac{(k+2)l}{2} - 1\right) \\ &= \frac{k-l+1}{2} > 0, \end{aligned}$$

thereby completing the proof of Lemma 2.3.17. $\square$

Lemma 2.3.18. *If $k \geq l \geq 1$, both k and l are odd, and $(k,l) \neq (1,1)$, then $p_3(k,l) > p_3(k+1,l-1)$.*

Proof. When k and l are odd, Lemma 2.3.16 gives:

$$\begin{aligned} &12(p_3(k,l) - p_3(k+1,l-1)) \\ &\qquad = \begin{cases} (kl+2l)(k-l) + k(k-3) + 3l + 4 & \text{if } 3|k \text{ and } 3|l, \\ (kl+2l)(k-l) + k(k-3) + 3l & \text{otherwise.} \end{cases} \end{aligned}$$

This is clearly positive for all $k \geq l$ such that $k \geq 3$ and $l \geq 1$. $\square$

Proof of (2). By the second Jacobi–Trudi identity,

$$W_{\lambda'} = E_{\lambda_1} E_{\lambda_2} - E_{\lambda+1} E_{\lambda_2 - 1}. \tag{2.30}$$

Taking $l = 0$ and $m = 2$ Corollary 2.3.3 gives:

$$\sum_{\lambda_1, \lambda_2 \geq 0} \langle E_{\lambda_1} E_{\lambda_2} \rangle \, u_1^{\lambda_1} u_2^{\lambda_2} = \frac{(1+u_1)(1+u_2)}{(1-u_1u_2)}.$$

Therefore the coefficient of $u_1^{\lambda_1} u_2^{\lambda_2}$ is the number $p'(\lambda_1, \lambda_2)$ of ways of writing (λ_1, λ_2) as a sum of vectors of the form $(1,1)$, $(1,0)$ and $(0,1)$, where the vectors $(0,1)$ and $(1,0)$ are used at most once. Clearly

$$p'(\lambda_1, \lambda_2) = \begin{cases} 2 & \text{if } \lambda_1 = \lambda_2 \geq 1, \\ 1 & \text{if } |\lambda_1 - \lambda_2| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By (2.30), for any partition $\lambda = (\lambda_1, \lambda_2)$ with two parts,

$$\langle W_{\lambda'} \rangle = p'(\lambda_1, \lambda_2) - p'(\lambda_1 + 1, \lambda_2 - 1) = \begin{cases} 2 & \text{if } \lambda_1 \geq \lambda_2 \geq 1, \\ 1 & \text{if } \lambda_1 - \lambda_2 = 1, \\ 0 & \text{otherwise,} \end{cases}$$

as claimed. $\square$

Proof of (3). Using Pieri's rule, we have:

$$h_k e_l = s_{(k-1|l)} + s_{(k|l-1)},$$

whence

$$s_{(a|b)} = h_{a+1} e_b - h_{a+2} e_{b-1} + \cdots + (-1)^b h_{a+b+1} e_0.$$

It follows that

$$\left\langle W_{(a|b)} \right\rangle = \sum_{j=0}^{b} (-1)^j \left\langle H_{a+j+1} E_{b-j} \right\rangle. \tag{2.31}$$

Taking $l = m = 1$ in Corollary 2.3.3 gives:

$$\sum_{a,b \geq 0} \langle H_i E_j \rangle \, t^i u^j = \frac{\prod_{k=0}^{\infty} (1 + t^k u)}{\prod_{k=1}^{\infty} (1 - t^k)}. \tag{2.32}$$

The coefficient of $t^i u^j$ in the above expression is the number $\tilde{p}(i, j)$ of ways of writing the vector (i, j) as a sum of vectors of the form $(a, 0)$ where $a > 0$, and $(a, 1)$ where $a \geq 0$, and vectors of the form $(a, 1)$ are used at most once. If $\tilde{p}(i, j) > 0$, then j distinct vectors of the form $(a, 1)$ are used, so that $i \geq \binom{j}{2}$. Therefore

$$\tilde{p}(i, j) = 0 \text{ for all } i < \binom{j}{2}. \tag{2.33}$$

If $a < \binom{b+1}{2}$, then $a - j < \binom{b+j+1}{2}$ for all $j \geq 0$. By (2.33) $\langle H_{a-j} E_{b+j+1} \rangle = 0$ for all $j \geq 0$. This allows us to extend the index of summation in the right hand side of (2.31) without changing the sum:

$$\begin{aligned} \left\langle W_{(a|b)} \right\rangle &= \sum_{j=-a-1}^{b} (-1)^j \langle H_{a+j+1} E_{b-j} \rangle \\ &= \sum_{k+l=a+b+1} \langle H_k E_l \rangle = \left\langle W_{(0|a+b+1)} \right\rangle. \end{aligned}$$

Taking $l = 0$ and $m = 1$ in Corollary 2.3.3 can be used to show that $\langle E_k \rangle = 0$ for all $k > 1$, so $\left\langle W_{(0|a+b+1)} \right\rangle = \langle E_{a+b+2} \rangle = 0$ for all $a, b \geq 0$. Therefore, $\left\langle W_{(a|b)} \right\rangle = 0$ for $a < \binom{b+1}{2}$.

Conversely, suppose $a \geq \binom{b+1}{2}$. By Theorem 2.3.14, it suffices to show that $W_\lambda(K^n)$ contains a non-zero S_n-invariant vector for some positive integer n. We shall show that $W_\lambda(K^{b+1})$ contains a non-zero S_{b+1}-invariant vector. The hook partition $\lambda = (a + 1, 1^b)$ dominates the partition $\mu = (a - \binom{b}{2} + 1, b, b - 1, \ldots, 2, 1)$, which has $b + 1$ distinct parts. Therefore $W_{(a|b)}(K^{a+b+1})$ contains a non-zero vector v with weight μ. For each $w \in S_n$ let $v_w = \rho_{(a|b)}(w)v$. Then v_w lies in the weight space of $w \cdot \mu$. Hence the vectors $\{v_w \mid w \in S_n\}$ are linearly independent, and generate a representation that is isomorphic to the regular representation of S_n. In particular, the trivial representation is contained in $W_{(a|b)}(K^{a+b+1})$. □

2.4 Kronecker coefficients

2.4.1 Reduced Kronecker coefficients

The Kronecker coefficients $g(\lambda, \mu, \nu)$ defined on triples of partitions of the same size, determine the multiplicity of the trivial representation in the tensor product $V_\lambda \otimes V_\mu \otimes V_\nu$.

Equivalently they may be defined by the decomposition

$$V_\lambda \otimes V_\nu = \bigoplus_\mu V_\mu^{\oplus g(\lambda,\mu,\nu)}.$$

The coefficient is symmetric in its arguments. For a suitably large n, the coefficient $g(\lambda[n], \mu[n], \nu[n])$ stabilises as a function of n to a value called the reduced Kronecker coefficients, $\overline{g}(\lambda, \mu, \nu)$. Thus we have:

$$q_\lambda q_\nu = \sum_\mu \overline{g}(\lambda, \mu, \nu) q_\mu$$

The following result is an obvious reformulation of these coefficients in terms of moments of Specht character polynomials:

Theorem 2.4.1. *For any partitions λ, μ and ν and an integer n such that $n \geq max\{|\mu| + \mu_1, |\nu| + \nu_1, |\lambda| + \lambda_1\}$,*

$$g(\lambda[n]\mu[n]\nu[n]) = \langle q_\lambda q_\mu q_\nu \rangle_n ,$$
$$\overline{g}(\lambda, \mu, \nu) = \langle q_\lambda q_\mu q_\nu \rangle .$$

2.4.2 Generating functions

In this section we find generating function for Kronecker-coefficients and their stable limits. We use this generating function to derive some well known results.

Theorem 2.4.2. *Given partitions λ, μ, ν with at most l parts, and an integer n such that $n \geq max\{|\lambda| + \lambda_1, |\mu| + \mu_1, |\nu| + \nu_1\}$, the Kronecker coefficient $g(\lambda[n], \mu[n], \nu[n])$ is the coefficient of $z^n u^\lambda v^\mu w^\nu$ in the rational function*

$$\Delta(U)\Delta(V)\Delta(W)E(U)E(V)E(W) \prod_{i,j,k\geq 0} (1 - u_i v_j w_k z)^{-1},$$

where $u_0 = v_0 = w_0 = 1$.

Proof. From Theorem 2.4.1 we know that $g(\lambda[n]\mu[n]\nu[n]) = \langle q_\lambda q_\mu q_\nu \rangle_n$. This moment is the following sum

$$\sum_{\alpha \vdash n} \frac{1}{z_\alpha} \left(\sum_{P_{l,\lambda}} \binom{\alpha}{\beta_1} \cdots \binom{\beta_{l-1}}{\beta_l} \right) \left(\sum_{P_{l,\mu}} \binom{\alpha}{\gamma_1} \cdots \binom{\gamma_{l-1}}{\gamma_l} \right) \left(\sum_{P_{l,\nu}} \binom{\alpha}{\theta_1} \cdots \binom{\theta_{l-1}}{\theta_l} \right).$$

Let $B_i(U) = 1+u_1^i(1+\ldots+u_{l-1}^i(1+u_l^i)\ldots)$, and define $B_i(V)$ and $B_i(W)$ for the variables $V = v_1, \ldots, v_l$ and $W = w_1, \ldots, w_l$ similarly. Following our familiar procedure we express this sum as the coefficient of $z^n u^{\mathrm{cum}(\lambda)} v^{\mathrm{cum}(\mu)} w^{\mathrm{cum}(\nu)}$ in

$$P_l(U)P_l(V)P_l(W) \exp\left(\frac{z^i}{i} B_i(U)B_i(V)B_i(W) \right),$$

where $\mathrm{cum}(\lambda) = (|\lambda|, |r(\lambda)|, \ldots, |r^l(\lambda)|)$, and defined analogously for μ and ν. Simplifying this expression, and making the substitution $u_1 = u_1$, $u_i = \frac{u_i}{u_{i-1}}$ for $i = 2, \ldots, l$ and analogous substitutions in the other sets of variables, we have the desired expression. $\square$

Corollary 2.4.3. *Given partitions λ, μ, ν with at most l parts, the reduced Kronecker coefficient $\overline{g}(\lambda, \mu, \nu)$ is the coefficient of $u^\lambda v^\mu w^\nu$ in the rational function*

$$\Delta(U)\Delta(V)\Delta(W) \prod_{i,j,k \geq 0, i+j+k \geq 2} (1 - u_i v_j w_k z)^{-1},$$

where $u_0 = v_0 = w_0 = 1$.

Once again we rewrite the generating function as a plethysm of symmetric functions. Notice that

$$\prod_{i,j,k \geq 0, i+j+k \geq 2} (1 - u_i v_j w_k z)^{-1} = H[UV + VW + UW + UVW].$$

This allows us a simple proof of the following identity, which is Lemma 2.1 of [7].

Lemma 2.4.4. *Given partitions λ, μ, ν we have*

$$\overline{g}(\lambda, \mu, \nu) = \sum_{\theta_1, \theta_2, \theta_3, \theta_{12}, \theta_{23}, \theta_{13} \in Par} g(\theta_1, \theta_2, \theta_3) c^\lambda_{\theta_1 \theta_{12} \theta_{13}} c^\mu_{\theta_2 \theta_{12} \theta_{23}} c^\nu_{\theta_3 \theta_{13} \theta_{23}}. \tag{2.34}$$

Proof. By Proposition 2.3.8 and Corollary 2.4.3 we know that $\overline{g}(\lambda, \mu, \nu)$ is the coefficient of $s_\lambda[U]s_\mu[V]s_\nu[W]$ in $H[UV + UW + VW + UVW]$. By definition

$$H[UV + UW + VW + UVW] = H[UV]H[UW]H[VW]H[UVW].$$

By the RSK correspondence ((3.1)) we have

$$H[UV] = \sum_{\lambda \in \mathrm{Par}} s_\lambda[U]s_\lambda[V],$$

and by the generalised Cauchy identity ((3.2))

$$H[UVW] = \sum_{\lambda,\mu,\nu} g(\lambda, \mu, \nu)s_\lambda[U]s_\mu[V]s_\nu[W].$$

Thus $H[UV + UW + VW + UVW]$ may be rewritten as

$$\sum_{\theta_{12}} s_{\theta_{12}}[U]s_{\theta_{12}}[V] \sum_{\theta_{13}} s_{\theta_{13}}[U]s_{\theta_{13}}[W] \sum_{\theta_{23}} s_{\theta_{23}}[V]s_{\theta_{23}}[W]$$
$$\sum_{\theta_1,\theta_2,\theta_3} g(\theta_1, \theta_2, \theta_3)s_{\theta_1}[U]s_{\theta_2}[V]s_{\theta_3}[W].$$

Gathering together terms in the same variable, the coefficient of $s_\lambda[U]$ in this expression is $c^\lambda_{\theta_1\theta_{12}\theta_{13}}$, where the generalised Littlewood-Richardson coefficients are as defined in 3.0.6. Seeking out the coefficients for the sets of variables V and W similarly gives us the desired expression. □

Lemma 2.4.4 (or [7, Lemma 2.1]) is an interesting result with several useful and easily proved consequences. We list a few of them here as exposition.

The first of these results combines a triangle inequality due to Murnaghan ([28],[29])- called Murnaghan's inequalities in [7]- with an observation by Littlewood ([22]) on reduced Kronecker coefficients generalising Littlewood-Richardson coefficients.

Proposition 2.4.5 (Murnaghan, Littlewood)**.** *Let λ, μ, ν be three partitions such that $|\lambda| \geq |\mu| + |\nu|$. Then*

$$\overline{g}(\lambda, \mu, \nu) = \begin{cases} 0 & |\lambda| > |\mu| + |\nu| \\ c^{\lambda}_{\mu,\nu} & |\lambda| = |\mu| + |\nu| \end{cases}$$

Proof. Let $|\lambda| > |\mu| + |\nu|$. Consider the system of equations on the sizes of the partitions in Equation (2.34)

$$\begin{aligned} |\theta_1| + |\theta_{12}| + |\theta_{13}| &= |\lambda| \\ |\theta_2| + |\theta_{12}| + |\theta_{23}| &= |\mu| \\ |\theta_3| + |\theta_{13}| + |\theta_{23}| &= |\nu| \end{aligned} \tag{2.35}$$

Let $j := |\theta_1| = |\theta_2| = |\theta_3|$. The solution to this system is $|\theta_{12}| = \frac{|\lambda|+|\mu|-|\nu|-k}{2}, |\theta_{13}| = \frac{|\lambda|-|\mu|+|\nu|-k}{2}, |\theta_{23}| = \frac{-|\lambda|+|\mu|+|\nu|-k}{2}$. The system has no solution when $|\lambda| > |\mu| + |\nu|$, and thus $\overline{g}(\lambda, \mu, \nu) = 0$.

When $|\lambda| = |\mu| + |\nu|$, a solution to the system exists when $k = 0$, in which case $\theta_{23} = \emptyset$ and $\theta_1 = \theta_2 = \theta_3 = \emptyset$ and $|\theta_{12}| = |\mu|$ and $|\theta_{13}| = |\nu|$. Note that $c^{\mu}_{\theta_{12},\emptyset} = c^{\nu}_{\theta_{13},\emptyset} = 1$ precisely when $\theta_{12} = \mu$ and $\theta_{13} = \nu$. Thus Equation (2.34) reduces to

$$\overline{g}(\lambda, \mu, \nu) = c^{\lambda}_{\mu,\nu}.$$

$\square$

Sections 3,5 and 6 of [40] contain formulas for Kronecker coefficients when two of the partitions are either hook-shapes or have two rows. The reader is referred to [40] for elegant proofs of these identities. Here we use Lemma 2.4.4 to get formulas for the corresponding reduced Kronecker coefficients. The reader is also referred to [6, Corollary 5.1] for the more general result (only one partition is required to either be a hook or have two rows).

Proposition 2.4.6. *For natural numbers $k \geq l$ and $\nu \in Par$, we have:*

1. *If $\lambda = (k)$ and $\mu = (l)$*

$$\overline{g}(\lambda, \mu, \nu) = \sum c^{\nu}_{(x),(y),(y+k-l)},$$

where the sum is over pairs of nonnegative integers x, y such that $x + y \leq l$.

2. *If $\lambda = (k)$ and $\mu = (1^l)$*

$$\overline{g}(\lambda, \mu, \nu) = \sum_{t \leq l} c^{\nu}_{(1^t),(k-t),(1^{l-t})} + \sum_{t \leq l-1} c^{\nu}_{(1^t),(k-1-t),(1^{l-1-t})}.$$

3. *If $\lambda = (l)$ and $\mu = (1^k)$*

$$\overline{g}(\lambda, \mu, \nu) = \sum_{t \leq l} c^{\nu}_{(1^t),(1^{k-t}),(l-t)} + \sum_{t \leq l-1} c^{\nu}_{(1^t),(1^{k-1-t}),(l-1-t)}.$$

4. *If $\lambda = (1^k)$ and $\mu = (1^l)$*

$$\overline{g}(\lambda, \mu, \nu) = \sum c^{\nu}_{(x),(1^y),(1^{y+k-l})},$$

where the sum is over pairs of nonnegative integers x, y such that $x + y \leq l$.

Proof. These are easy consequences of Equation (2.35). When $\lambda = (k), \mu = (l)$, we have that all partitions are single row shapes. Let $\theta_3 = (x)$ and $\theta_{23} = (y)$. Then we have $l = x + y + |\theta_{12}|$ and $k = x + |\theta_{12}| + |\theta_{13}|$, which together imply that $|\theta_{13}| = y + k - l$.

When $\lambda = (k)$ and $\mu = (1^l)$ we must have that $\theta_{12} = \emptyset$ or (1), since it must simultaneously be a single row and a single column. We also require that $\theta_1 = (s)$ and $\theta_2 = \theta_3 = (1^s)$ for a nonnegative integer s, since the Kronecker coefficient is nonzero ($g((s), (1^s), (1^s)) = 1$) only for such a combination. Evaluating the system of equations with these added constraint yields the desired expression.

The omitted case follows similarly. $\square$

Chapter 3

Appendix: Symmetric functions and plethysms

This appendix introduces basic facts and notation for symmetric functions. The reader is referred to [25, Section I.2] for a complete description. The symmetric group S_n acts on the space of polynomials $\mathbb{Z}[x_1, \ldots, x_n]$ by permuting the variables $x_1, \ldots, x_n$. A polynomial $p \in \mathbb{Z}[x_1, \ldots, x_n]$ is symmetric if it is invariant under the action of S_n. This subspace is denoted Λ_n. Λ_n is a ring graded by the degree of the polynomial

$$\Lambda_n = \oplus \Lambda_n^k,$$

where Λ_n^k consists of homogeneous symmetric polynomials of degree k, along with 0. For $m \geq n$, consider the homomorphism

$$\mathbb{Z}[x_1, \ldots, x_m] \to \mathbb{Z}[x_1, \ldots, x_n],$$

defined by setting $x_{n+1} = \ldots = x_m = 0$. Let $\rho_n : \Lambda_{m,n} \to \Lambda_n$ denote the restriction of this homomorphism to Λ_m. The restriction of $\rho_{m,n}$ to each graded component we have the surjective homomorphisms

$$\rho_{m,n}^k : \Lambda_m^k \to \Lambda_n^k$$

for all $k \geq 0$ and $m \geq n$. The map is bijective when $m \geq n \geq k$. Define the space Λ^k to be the inverse limit of the $\mathbb{Z}$-modules Λ_n^k relative to the homomorphisms $\rho_{m,n}^k$. The ring of symmetric functions in countably many variables is then defined as

$$\Lambda = \oplus_{k \geq 0} \Lambda^k.$$

For a partition with at most n parts, define

$$m_\lambda(x_1, \dots, x_n) = \sum_\alpha x^\alpha,$$

where the sum is over distinct rearrangements of λ. This is clearly a symmetric function, and the set $\{m_\lambda(x_1, \dots, x_n) | \lambda \vdash k, l(\lambda) \leq n\}$ is a $\mathbb{Z}$ basis of Λ_n^k. The projection map $\rho_n^k : \Lambda^k \to \Lambda_n^k$ is an isomorphism when $n \geq k$. Henceforth we use the plethystic notation $f[X]$ to denote an element $f(x_1, x_2, \dots) \in \Lambda$.

Definition 3.0.1. *The monomial symmetric functions $m_\lambda \in \Lambda$ corresponding to a partition λ is defined such that*

$$\rho_n^k(m_\lambda[X]) = m_\lambda(x_1, \dots, x_n),$$

for all $n \geq k$.

The ring Λ is the free $\mathbb{Z}$-module generated by the monomial symmetric functions; Λ^k is generated by the set $\{m_\lambda | \lambda \vdash k\}$.

Definition 3.0.2. *The elementary symmetric functions are defined for every nonnegative integer k such that*

$$\rho_n^k(e_k[X]) = \sum_{i_1 < \ldots < i_k \leq n} x_{i_1} \dots x_{i_n},$$

for all $n \geq k$. They are defined thus for every integer vector λ by $e_\lambda[X] = e_{\lambda_1}[X] \dots e_{\lambda_{l(\lambda)}}[X]$.

The set $\{e_\lambda | \lambda \in \mathrm{Par}\}$ is a $\mathbb{Z}$-basis of Λ. Define the generating function $E[X] = \sum_j (-1)^j e_j[X]$. Note that

$$E[X] = \prod_i (1 - x_i).$$

Definition 3.0.3. *The complete homogeneous symmetric functions are defined for every nonnegative integer k such that*

$$\rho_n^k(h_k[X]) = \sum_{i_1 \leq \ldots \leq i_k \leq n} x_{i_1} \ldots x_{i_n},$$

for all $n \geq k$. They are defined thus for every integer vector λ by $h_\lambda[X] = h_{\lambda_1}[X] \ldots h_{\lambda_{l(\lambda)}}[X]$.

The set $\{h_\lambda | \lambda \in \text{Par}\}$ is a $\mathbb{Z}$-basis of Λ. Define the generating function $H[X] = \sum_j h_j[X]$. Note that

$$H[X] = \prod_i \frac{1}{(1 - x_i)}.$$

Definition 3.0.4. *Define the Schur function $s_\lambda \in \Lambda_n$ as*

$$s_\lambda(x_1, \ldots, x_n) = \frac{det(x_i^{\lambda_j + n - j})}{det(x_i^{n-j})},$$

and define $s_\lambda \in \Lambda$ such that

$$\rho_n^k(s_\lambda[X]) = s_\lambda(x_1, \ldots, x_n).$$

The Schur functions form an orthonormal $\mathbb{Z}$-basis of Λ. They may be defined combinatorially as the generating function of semistandard Young tableaux by their content. The set of semistandard tableaux of shape λ is denoted SSYT(λ). Given a tableau $T \in SSYT(\lambda)$, its content is an integer vector $(a_1, a_2, \ldots)$ where a_i is the number of occurrences of i in T. The monomial corresponding to T, denoted x^T, is defined to be $x_1^{a_1} x_2^{a_2} \ldots$. Then we have

$$s_\lambda[X] = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T.$$

Schur functions are interesting for the wealth of combinatorial identities that surround them. The product of the the Schur functions is expanded into a Schur basis using the Littlewood-Richardson coefficients

$$s_\mu[X] s_\nu[X] = \sum_{\lambda \vdash |\mu| + |\nu|} c_{\mu,\nu}^\lambda s_\lambda[X].$$

Definition 3.0.5. *The Littlewood-Richardson coefficient* $c^{\lambda}_{\mu,\nu}$ *is the multiplicity of the Specht module* V_λ *of* $S_{|\lambda|}$ *in* $\mathrm{Ind}^{S_{|\lambda|}}_{S_{|\mu|}\times S_{|\nu|}} V_\mu \otimes V_\nu$.

The coefficient $c^{\lambda}_{\mu,\nu}$ has several equivalent combinatorial descriptions; it is equal to the number of semistandard tableaux of skew shape λ/μ and content ν whose reverse-reading word is a lattice permutation. The reader is directed to [43] for details. We define the generalised Littlewood-Richardson coefficients for use through this chapter.

Definition 3.0.6. *For partitions* $\mu_1, \mu_2, \ldots, \mu_r, \lambda$ *such that* $|\lambda| = \sum_{i=1}^{r} |\mu_i|$, *define the generalised Littlewood-Richardson coefficient* $c^{\lambda}_{\mu_1,\ldots,\mu_r}$ *to be the multiplicity of the Specht module* V_λ *in* $\mathrm{Ind}^{S_{|\lambda|}}_{S_{|\mu_1|}\times\ldots\times S_{|\mu_r|}} V_{\mu_1} \otimes \ldots \otimes V_{\mu_r}$.

Equivalently they may be defined such that $s_{\mu_1}[X]\ldots s_{\mu_r}[X] = \sum_{\lambda\vdash\sum_i |\mu_i|} c^{\lambda}_{\mu_1,\ldots,\mu_r} s_\lambda[X]$.

A related family of coefficients is also obtained by defining the Kronecker product on Schur functions. For three partitions μ, ν of the same size

$$s_\mu[X] * s_\nu[X] = \sum_{\lambda\vdash|\mu|} g(\lambda, \mu, \nu) s_\lambda[X]$$

These coefficients are called the Kronecker coefficients. They also occur in the coproduct formula

$$s_\lambda[XY] = \sum_{\mu,\nu} g(\lambda, \mu, \nu) s_\mu[X] s_\nu[Y],$$

where for variable sets $X = x_1, x_2, \ldots$ and $Y = y_1, y_2, \ldots$ the variable set XY comprises $x_i y_j$ for $i, j \geq 1$.

The RSK correspondence between integer matrices and pairs of semistandard tableau of the same shape sets up the equivalent correspondence between the complete symmetric functions and Schur functions

$$h_k[XY] = \sum_{\lambda\vdash k} s_\lambda[X] s_\lambda[Y]. \tag{3.1}$$

In the same spirit is the generalised Cauchy identity, marking another occurrence of Kronecker coefficients

$$h_k[XYZ] = \sum_{\lambda,\mu,\nu\vdash k} g(\lambda,\mu,\nu)s_\lambda[X]s_\mu[Y]s_\nu[Z]. \tag{3.2}$$

The Jacobi-Trudi identity and its dual ([25, Section I.3] express the Schur functions in the basis of the complete symmetric functions and the elementary symmetric functions respectively.

$$\begin{aligned} s_\lambda[X] &= \det(h_{\lambda_i-i+j}[X]), \\ s_{\lambda'}[X] &= \det(e_{\lambda_i-i+j}[X]), \end{aligned} \tag{3.3}$$

where λ' denotes the conjugate of the partition λ. A subset of cells in a Young diagram is called a horizontal strip if no two of them occupy the same column; it is called a vertical strip if no two cells occupy the same row. Then we have the Pieri identities

$$\begin{aligned} s_\lambda[X]h_r[X] &= \sum_{\nu:\nu/\lambda \text{ is horizontal strip}} s_\nu[X], \\ s_\lambda[X]e_r[X] &= \sum_{\nu:\nu/\lambda \text{ is vertical strip}} s_\nu[X] \end{aligned} \tag{3.4}$$

Definition 3.0.7. *Define the power sum functions $p_k(X)$ for positive integers k such that*

$$\rho_n^k(p_k[X]) = \sum_{i\le n} x_i^k,$$

for all $n \geq k$. They are defined thus for every integer vector λ by $p_\lambda = p_{\lambda_1}\dots p_{\lambda_{l(\lambda)}}$.

The power sum functions are a $\mathbb{Q}$-basis of Λ. The Schur functions are expanded in this basis as

$$s_\lambda[X] = \sum_{\nu\vdash|\lambda|} \frac{\chi_\lambda^\nu}{z_\nu} p_\nu[X],$$

where χ_λ^ν is the value at the irreducible character χ_λ on the conjugacy class of cycle-type ν.

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