Surface Defects from Fractional Branes

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis[1]

Journal

1. *“Surface Defects from Fractional Branes – I"*

   S.K. Ashok, M. Billò, M. Frau, A. Lerda, S. Mahato

   JHEP 07 (2020) 051,arXiv:2005.02050 [hep-th]

2. *“Surface Defects from Fractional Branes – II"*

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List of other Publications, Not included in the thesis

Journal

1. *“Weights, Recursion relations and Projective triangulations for positive geometry of scalar theories"*

   Renjan Rajan John, Ryota Kojima, Sujoy Mahato

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2. *“On positive geometries of quartic interactions: Stokes polytopes,lower forms on associahedra and world-sheet forms"*

   P. Aneesh, P. Banerjee, M. Jagadale, R. R. John, A. Laddha and S. Mahato

   JHEP 04 (2020) 149, arXiv:1911.06008 [hep-th].

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[1]As it is standard in the High Energy Physics Theory (hep-th) community the names of the authors on any paper appear in their alphabetical order.

Conferences and workshops attended

1. *“Hamilton School on Mathematical Physics"* held Online during August 24 -August 28, 2020.

2. *“National String Meet"* held at IISER, Bhopal during December 22 - December 27, 2019.

3. *“Laces 2019"* organized by GGI, Florence from November 25 to December 13, 2019.

4. *“International School on Amplitudes and Cosmology, Holography and Positive Geometries"* organized by INFN, Lecce from May 27 to June 1, 2019.

5. *“IX Workshop on Geometric Correspondences of Gauge Theories"* organized by SISSA, Trieste held during June 17- June 21, 2019.

6. *“Indian String Meet"* held at IISER, Thiruvananthapuram from December 16 to December 21, 2018.

7. *“SERC Main School in THEP"* held at IISER, Pune from November 25 to December 15, 2018.

8. *“National String Meet"* held at NISER, Bhubaneswar from December 5 to December 10, 2017.

Presentation and poster

1. Conference talk on *““Weights, Recursion relations and Projective triangulations for positive geometry of scalar theories" "* in Chennai String Meet held online by IMSc from December 14 to December 18, 2020.

2. Poster presentation on *“Surface Defects from Fractional Branes"* in YITP Workshop on Strings and Fields held Online from November 16 to November 20, 2020.

3. Gong show talk and Poster presentation on *“Surface Defects from Fractional Branes"* in Hamilton School on Mathematical Physics held Online from August 24 to August 28, 2020.

4. Conference talk on *“Surface Defects from String Theory "* in Chennai String Meet held at IMSc from November 21 to November 23, 2019.

মায়ের জন্য...

আজ তুমি নিশ্চয়ই আমার জন্য খুশি
হয়েছ, যেখানেই আছো খুব
ভালো থেকো মা ।।

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A journey of a thousand miles begin with a single step...

- Lao Tzu

Another small step ahead! And well it was a real journey, a journey through so many ups and downs still a really fulfilling one in the end.

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Summary

Surface defects in gauge theories are non-local operators supported on co-dimension two sub manifold. Defects in quantum field theory can provide us with important non-perturbative information eg. different phase structures present in the system. They were first studied by Gukov-Witten in the context of N=4 super Yang-Mills theories. The Gukov-Witten surface defects are characterized by sets of discrete and continuous parameters. In this thesis, we geometrically engineer Gukov-Witten surface defects in maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory with gauge group U(N) within the setup of perturbative Type IIB string theory. In particular, we refine the proposal of Kanno and Tachikawa and realize the defect by a configuration of fractional $D3$ branes on an orbifold background that preserves two dimensional Poincaré invariance.

On this particular orbifold target space in which the D3 world volume is extended partially along the orbifold, we consider closed string fields that act as a background. Moreover, the relevant closed string states are the twisted sector ones that are special to the orbifold space. Due to the presence of the fractional D3 branes which introduce a boundary on the worldsheet, the left and right moving sectors of the closed string fields are identified under some reflection rules. In addition, we consider the open string vertex operators that are invariant under the action of the orbifold group. After providing the necessary details about twisted closed string sectors, Reflection rules, and open string spectra, we calculate open/closed disk correlation functions on the worldsheet involving one massless closed string field from a twisted sector and one massless open string field. By giving a constant background vacuum expectation value to the twisted field, we interpret non-vanishing correlators as sources for the open string field. By Fourier transform to position space, we obtain a space-time profile for the open string field that matches exactly with the expected singular profiles of the four dimensional fields of the gauge theory in the presence of the Gukov-Witten defect. The background values of the twisted closed string

fields are identified with the continuous parameters that define the defect in the gauge theory description. We provide an important check of our proposal by verifying that this identification is consistent with the expected S-duality properties of these continuous parameters.

In the first part, we consider the simplest possible surface defect with two discrete parameters realized via fractional D3 branes on a background with $\mathbb{Z}_2$ orbifolding. In the later part, we generalize the above construction to the most generic type of Gukov-Witten defects with M discrete parameters realized via fractional D3 branes on a $\mathbb{Z}_M$ orbifold.

Contents

Synopsis

0.1 Introduction

The study of defects in QFT has become an active area of research in recent years and their study has provided useful information about the vacuum structure of the gauge theories. They are an important probe of the non-perturbative effects and dualities in gauge theories. One-dimensional or line defects such as Wilson and 't Hooft loops in gauge theories are the best studied in the literature. In this thesis, we focus on higher dimensional defects which are supported on co-dimension two submanifolds, called surface defects. These were first introduced by Gukov and Witten (GW) [1,2] in the context of four dimensional maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory with $U(N)$ gauge group. The defect was defined as a "monodromy defect" by specifying the singular behavior of the four dimensional fields in the gauge theory as one approaches the location of the defect. With these boundary conditions specified for the path integral, correlation functions in the presence of such defects provide us with valuable non-perturbative information about the bulk gauge theory.

In this thesis, we geometrically engineer a Gukov-Witten surface defect within Type II B string theory by considering a configuration of fractional branes on an orbifold background. In addition, we provide a physical interpretation of the defining parameters of the defect in terms of background values for closed string scalars in the twisted sectors of the closed string orbifold.

0.2 Background

0.2.1 Type II B string theory

The massless spectra of *Type II B* string theory in 10d flat spacetime having $SO(1,9)$ symmetry, contains the N = (2, 0) gravity multiplet. In addition, there are two matter multiplets containing the 2-form field $B_{\mu\nu}$ and Dilaton in NS/NS sector and $C_{\mu\nu}$ in R/R sector in addition to other fields. In the Type II B theory, the same GSO projection is taken in both left and right sectors and the spectrum is chiral. The closed string background we will be interested in is given by the following orbifold:

$$\mathbb{C}^3 \times (\mathbb{C}^2/\mathbb{Z}_M) \,. \tag{1}$$

If we do a $\mathbb{Z}_M$ orbifolding along 4 directions, the massless spectra of Type II B theory, in addition to the untwisted sector mentioned above, will contain $(M-1)$ twisted sectors [3].

0.2.2 D-branes

D-branes are explicit realizations of RR charged BPS states in superstring theory. From an open string theory perspective, D-branes are hypersurfaces on which the end-points of the open string lies. In closed string theory, they can be described using boundary state formalism.

Fractional branes are particular to orbifold backgrounds and correspond to D-branes that are stuck at the fixed loci of the orbifolded space. They transform under the irreducible representations of the orbifold group. Regular branes that can freely move along the orbifolded representations are simply linear combinations of these fractional branes.

In this thesis, we realize GW type surface defects within the perturbative string theory. The configuration that we shall consider was first introduced in a work by Kanno-Tachikawa (KT) [4]. Their goal was to derive the instanton partition functions in the presence of surface defects by introducing stacks of fractional D3 and D(-1) branes in the orbifold background. We shall consider just the fractional D3 branes and consider the following brane set-up:

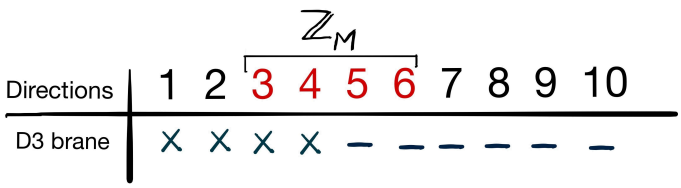

Figure 1: The orbifold setup.

0.2.3 Surface operators

In this section, we review basic facts about surface operators in $\mathcal{N} = 4$ SYM theory with $U(N)$ gauge group, following [1]. One approach to define these operators is by specifying a particular monodromy behaviour of the massless fields of the four dimensional theory near the defect plane. If $z = re^{i\theta}$ is the coordinate for the transverse plane to the defect D, then as one approaches $r \to 0$, the gauge field behaves in the following manner:

$$A \sim \underline{\alpha}\, d\theta = \mathrm{diag}\Big(\underbrace{\alpha_0, \ldots, \alpha_0}_{n_0}, \underbrace{\alpha_1, \ldots, \alpha_1}_{n_1}, \ldots \underbrace{\alpha_{M-1}, \ldots, \alpha_{M-1}}_{n_{M-1}} \Big) d\theta \,, \tag{2}$$

where the n_I set of integers satisfy $\sum_{I=0}^{M-1} n_I = N$.

In the path integral, one needs to integrate over all gauge field configurations with the above prescribed condition. The holonomy of the gauge field is given by $P\ \exp(-\int_l A)$. So, the monodromy of the gauge connection A around a circle of constant r is given by $\exp(-2\pi\underline{\alpha})$. At the location of the defect the gauge group is broken to its Levi subgroup:

$$L = U(n_0) \times U(n_2) \times ... \times U(n_{M-1}) \tag{3}$$

Also, the scalars in the theory behave in the following manner as $z \to 0$:

$$\Phi \sim \frac{\underline{\beta} + i\underline{\gamma}}{2z} \tag{4}$$

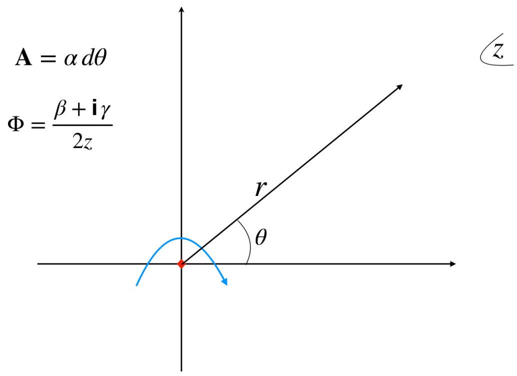

Figure 2: The monodromy behaviour of the fields.

In the path integral, one is also allowed to add a 2d topological term:

$$exp\left(2\pi i \sum_{I=1}^{M} \eta_I \int_D Tr\, F_{U(n_I)}\right) \tag{5}$$

Since $d(d\theta) = 2\pi\delta_D$, the field strength calculated using this gauge field A is singular: $F = 2\pi\underline{\alpha}\delta_D$. A surface operator is characterized by 4M continuous parameters $(\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\eta})$ and M discrete parameters $n_I, I = 0, 1, ..., (M-1)$.

Lastly, a remarkable feature of $\mathcal{N} = 4$ SYM is that it is invariant under the non-perturbative duality group $SL(2, \mathbb{Z})$. This is a strong weak coupling duality as far as the gauge coupling is concerned; in particular an element $\Lambda = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in SL(2, \mathbb{Z})$ induces the transformation

$$\tau \longrightarrow \frac{m\tau + n}{p\tau + q}\,, \tag{6}$$

where τ is the complexified gauge coupling constant $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$.

As shown in [1] the duality group also acts naturally on the continuous parameters of the surface defect as follows:

$$
\begin{aligned}
(\alpha_I, \eta_I) &\to (\alpha_I, \eta_I)\Lambda^{-1} = (q\alpha_I - p\eta_I, -n\alpha_I + m\eta_I), \\
(\beta_I, \gamma_I) &\to |p\tau + q|(\beta_I, \gamma_I) \, .
\end{aligned} \tag{7}
$$

0.3 Surface defects from fractional Branes

0.3.1 Summary of main idea

Our primary goal is to derive the field profiles (2) and (4) as well as the topological term (5) that characterize a Gukov-Witten defect from a world-sheet analysis. In our set-up, the $4d$ gauge theory lives on a system of D3-branes in Type II B string theory placed in a $\mathbb{Z}_M$ orbifold space as shown in the figure 1. We divide the $10d$ target space into five complex directions in the following way:

$$
\mathbb{C}_{(1)} \times \frac{\mathbb{C}_{(2)} \times \mathbb{C}_{(3)}}{\mathbb{Z}_M} \times \mathbb{C}_{(4)} \times \mathbb{C}_{(5)} \tag{8}
$$

The world volume of the D3 brane is along $\mathbb{C}^{(1)} \times \mathbb{C}^{(2)}$ whereas the $\mathbb{Z}_M$ orbifold is along $\mathbb{C}^{(2)} \times \mathbb{C}^{(3)}$. The defect is extended along $\mathbb{C}^{(1)}$ i.e. its located at the fixed point of orbifold action.

The integer partition of N, which determines the unbroken Levi subgroup corresponds to the choice of the N-dimensional representation of $\mathbb{Z}_M$ on the Chan-Paton indices of the D3-branes; in other words, n_I is the number of the fractional branes transforming in the I-th irreducible representation of $\mathbb{Z}_M$. The missing link in the KT description was how the orbifold realization encodes the continuous parameters of the surface defect. In this thesis work, we fill this gap by showing that they correspond to background values for massless closed string fields belonging to the twisted sectors of the Type II B theory on the orbifold.

Schematically, the mechanism goes as follows. We give a constant background value to the closed string scalars in the twisted sector and they act as a classical source for the open string fields. In the presence of a closed string background certain open string fields Φ_{open} attached to a fractional D3-brane of type I acquire a non-zero one point function, i.e. a tadpole. If we denote by $\mathcal{V}_{open}$ the open string vertex operator associated to Φ_{open} and by $\mathcal{V}_{closed}$ the closed string vertex operator corresponding to field Φ_{closed}, the tadpole $\langle \mathcal{V}_{open} \rangle_{closed,I}$ arises from an open/closed string correlator evaluated on a disk which contains an insertion of $\mathcal{V}_{closed}$ in the interior and of the vertex operator $\mathcal{V}_{open}$ on the boundary that lies on a D3-brane of type I:

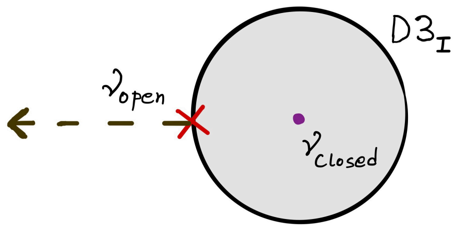

Figure 3: An example of an open/closed string amplitude on a D3-brane with one closed string vertex operator insertion in the interior and one open string vertex operator insertion at the boundary.

The disk diagram presented above acts as a classical source for Φ_{open} and acquires a non-trivial profile in the transverse plane to the defect plane. The explicit expression of this profile near the defect is obtained by attaching a propagator to the source and taking the Fourier transform.

To calculate the open/closed correlators we will need the vertex operator of closed string scalars in twisted sectors, the boundary states for the closed string fields which will provide the reflection rules for the left and right moving sectors, and the open string spectrum.

0.3.2 The $\mathbb{Z}_2$ orbifold

Closed string spectrum

We will be considering the $\mathbb{Z}_2$ orbifold setup first. In this case, the closed string Hilbert space in addition to the usual untwisted sector, contains one twisted sector, associated with the non-trivial conjugacy class of $\mathbb{Z}_2$ [5].

The vertex operator at the massless level in the NS/NS twisted sector is given by combining the left moving and right moving vertex operators. For the left moving one the explicit form of the vertex operator is given by:

$$\mathcal{V}^{\alpha}(z) = \varDelta(z)\, S^{\alpha}(z)\, \mathrm{e}^{-\phi(z)}\, \mathrm{e}^{\mathrm{i}\,\bar{\kappa}\cdot Z(z)+\mathrm{i}\,\kappa\cdot\bar{Z}(z)}\ . \tag{9}$$

which is a conformal field of weight 1 if $\kappa\cdot\bar{\kappa} = \frac{1}{2}k^2 = 0$. The conformal twist operator $\varDelta(z)$ and spin field $S^{\alpha}(z)$ are of conformal weight $\frac{1}{4}$ each. The vertex operator $:\mathrm{e}^{-\phi(z)}:$ is to describe physical vertex operators in the standard (-1)-superghost picture of the NS sector. We have introduced the complexified momentum $\kappa_i = \frac{k_i+ik_{i+1}}{\sqrt{2}}$.

Similarly one can construct the right moving vertex operator $\widetilde{\mathcal{V}}^{\alpha}(\bar{z})$.

$$\widetilde{\mathcal{V}}^{\alpha}(\bar{z}) = \widetilde{\varDelta}(\bar{z})\, \widetilde{S}^{\alpha}(\bar{z})\, \mathrm{e}^{-\widetilde{\phi}(\bar{z})}\, \mathrm{e}^{\mathrm{i}\,\bar{\kappa}\cdot\widetilde{Z}(\bar{z})+\mathrm{i}\,\kappa\cdot\widetilde{\bar{Z}}(\bar{z})}\ , \tag{10}$$

The complete NS/NS vertex operators are constructed by combining $b_{\alpha\beta}\mathcal{V}^{\alpha}(z)\widetilde{\mathcal{V}}^{\beta}(\bar{z})$.

The four independent components can be decomposed into a real scalar b and a triplet b_c (with $c = 1, 2, 3$). They correspond to the following vertex operators:

$$\begin{aligned} b \quad &\longleftrightarrow \quad \mathcal{V}_b(z,\bar{z}) = \mathrm{i}\,\epsilon_{\alpha\beta}\,\mathcal{V}^{\alpha}(z)\,\widetilde{\mathcal{V}}^{\beta}(\bar{z})\ , \\ b_c \quad &\longleftrightarrow \quad \mathcal{V}_{b_c}(z,\bar{z}) = (\epsilon\,\tau_c)_{\alpha\beta}\,\mathcal{V}^{\alpha}(z)\,\widetilde{\mathcal{V}}^{\beta}(\bar{z})\ , \end{aligned} \tag{11}$$

where τ_c are the usual Pauli matrices. Due to the fact that D3 brane is partially extended along the orbifold, the triplet state b_c further breaks down to a scalar b' and a doublet of complex conjugate fields $b_{\pm}$.

Similarly one can construct the R/R massless vertex operators. We write down the vertex operators of two massless scalars c and c' below which we will need later:

$$c \quad\longleftrightarrow\quad \mathcal{V}_c(z,\bar{z}) = C_{A\dot{B}}\,\mathcal{V}^A(z)\,\widetilde{\mathcal{V}}^{\dot{B}}(\bar{z})\;, \tag{12a}$$

$$c' \quad\longleftrightarrow\quad \mathcal{V}_{c'}(z,\bar{z}) = (C\,\varGamma_{12})_{A\dot{B}}\,\mathcal{V}^A(z)\,\widetilde{\mathcal{V}}^{\dot{B}}(\bar{z})\;. \tag{12b}$$

where $\varGamma_{MN} = \frac{1}{2}[\varGamma_M, \varGamma_N]$, with $\varGamma_M$ being the Dirac matrices of $SO(6)$.

Boundary states and Reflection rules

In the $\mathbb{Z}_2$ orbifold case there are two types of fractional D-branes that correspond to the two irreducible representations of the orbifold group. The fractional D3-branes can be schematically represented in the boundary state formalism as follows [6–8]:

$$|\mathrm{D3}; I\rangle = \mathcal{N}\,|\mathrm{U}\rangle + \mathcal{N}'\,|\mathrm{T}; I\rangle \quad \text{with} \quad |\mathrm{T}; I\rangle = (-1)^I\,|\mathrm{T}\rangle\;. \tag{13}$$

$I = 0, 1$ labelling two types of fractional branes. $|\mathrm{U}\rangle$ and $|\mathrm{T}\rangle$ are the untwisted and twisted Ishibashi states.

The boundary state $|\mathrm{D3}; I\rangle$ introduces a boundary on the closed string world-sheet along which the left and right moving modes are identified. Using explicit expressions of the boundary states, one can derive that the right moving parts of the twisted closed string vertex operators are reflected on a boundary of type I with the following rules

$$\widetilde{\mathcal{V}}^{\alpha}(\bar{z}) \;\longrightarrow\; (-1)^I(\gamma_4\gamma_3)^{\alpha}{}_{\beta}\,\mathcal{V}^{\beta}(\bar{z})\;, \tag{14a}$$

$$\widetilde{\mathcal{V}}^{\dot{A}}(\bar{z}) \;\longrightarrow\; (-1)^I(\varGamma_1\varGamma_2)^{\dot{A}}{}_{\dot{B}}\,\mathcal{V}^{\dot{B}}(\bar{z})\;. \tag{14b}$$

γ_m and Γ_M being the Dirac matrices of $SO(4)$ and $SO(6)$ respectively.

0.3.3 Open string spectrum

In the case of D3 branes completely transverse to the orbifold, the open string vertex operators for gauge field A_μ in (0)-picture number is given by:

$$(\mathrm{i}\,\partial X^\mu + k\cdot\psi\,\psi^\mu)\,\mathrm{e}^{\mathrm{i}\,k\cdot X}\ . \tag{15}$$

But in our case where the D3-brane is partially extended along the $\mathbb{Z}_2$ orbifold, we have to take appropriate linear combinations of $e^{ik.X}$ factor which transforms covariantly under the orbifold action.

For eg. the vertex operator for the gauge field component A_1 is given by,

$$A_1 \;\longrightarrow\; \mathcal{V}_{A_1} = \Big[(\mathrm{i}\,\partial Z^1 + \kappa_\parallel\cdot\Psi_\parallel\,\Psi^1)\cos(\kappa_\perp\cdot Z_\perp) + \mathrm{i}\,\kappa_\perp\cdot\Psi_\perp\,\Psi^1\,\sin(\kappa_\perp\cdot Z_\perp)\Big]\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel}\ . \tag{16}$$

where we define the following combinations

$$z_i = \frac{x_{2i-1} + \mathrm{i}\,x_{2i}}{\sqrt{2}} \qquad \text{and} \qquad \bar{z}_i = \frac{x_{2i-1} - \mathrm{i}\,x_{2i}}{\sqrt{2}} \tag{17}$$

$$\kappa_\perp\cdot Z_\perp = \kappa_2\,\overline{Z}^2 + \overline{\kappa}_2\,Z^2\ , \qquad \kappa_\perp\cdot Z_\perp = \kappa_2\,\overline{Z}^2 + \overline{\kappa}_2\,Z^2\ . \tag{18}$$

Also $\kappa_\parallel\cdot\Psi_\parallel$ and $\kappa_\perp\cdot\Psi_\perp$ are defined in similar way. In addition to the gauge field, there are three complex scalars as well in the massless open string spectra denoted as $\Phi,\ \Phi_r\ \{r = 4, 5\}$.

Open/Closed correlators

The couplings of the massless open string fields of a fractional D3-brane of type I with a closed string field is given by,

$$\langle \mathcal{V}_{\text{open}} \rangle_{b;I} = \int \frac{dz\, d\bar{z}\, dx}{dV_{\text{proj}}} \langle \mathcal{V}_{\text{closed}}(z,\bar{z})\, \mathcal{V}_{\text{open}}(x) \rangle_I \tag{19}$$

where $\mathcal{V}_{\text{closed}}, \mathcal{V}_{\text{open}}$ stands for any of the closed and open string vertex operators respectively and dV_{proj} is the invariant projective volume.

The final result of the coupling of the four dimensional fields with the closed string scalars are given as follows:

$$\langle \mathcal{V}_{A_2} \rangle_{b;I} = (-1)^{I+1}\, b\, \kappa_2\, \delta^{(2)}(\kappa_{\|}) \tag{20}$$

$$\langle \mathcal{V}_{\Phi} \rangle_{b_+;I} = (-1)^{I+1}\, \mathrm{i}\, b_+\, \bar{\kappa}_2\, \delta^{(2)}(\kappa_{\|}) \tag{21}$$

$$\langle \mathcal{V}_{A_1} \rangle_{c;I} = (-1)^{I+1}\, 2\mathrm{i}\, c\, \kappa_1\, \delta^{(2)}(\kappa_{\|})\ . \tag{22}$$

Field profiles

The profile of gauge field A_2 in configuration space induced by the NS/NS twisted scalar b is obtained by Fourier transforming the coupling we have just obtained:

$$\begin{aligned} A_2 &= \int \frac{d^2\kappa_{\|}\, d^2\kappa_{\perp}}{(2\pi)^2}\, \mathrm{i}\, \sin(\kappa_{\perp}\cdot z_{\perp})\, \mathrm{e}^{\mathrm{i}\,\kappa_{\|}\cdot z_{\|}}\, \frac{\langle \mathcal{V}_{A_2} \rangle_{b;0}}{2(|\kappa_{\|}|^2 + |\kappa_{\perp}|^2)} \\ &= -\mathrm{i}\, b \int \frac{d^2\kappa_{\perp}}{(2\pi)^2}\, \sin(\kappa_{\perp}\cdot z_{\perp})\, \frac{\kappa_2}{2|\kappa_{\perp}|^2} \end{aligned} \tag{23}$$

After doing the integral, the final expression for the gauge profile is given by

$$A_2 = -\frac{\mathrm{i}\, b}{4\pi \bar{z}_2}\ . \tag{24}$$

The component $\bar{A}_2$ of the gauge field also has a non-trivial profile which is given by the complex conjugate of (24). Using these, the gauge field on a fractional D3-brane of type 0 in the $\mathbb{Z}_2$ orbifold acquires the following profile

$$\mathbf{A} = A \cdot dx = A_2\, d\bar{z}_2 + \bar{A}_2\, dz_2 = -\frac{\mathrm{i}\, b}{4\pi}\left(\frac{d\bar{z}_2}{\bar{z}_2} - \frac{dz_2}{z_2}\right) = -\frac{b}{2\pi}\, d\theta \tag{25}$$

where θ is the polar angle in the $\mathbb{C}_{(2)}$ plane which is perpendicular direction to the defect. A similar calculation for the scalar leads to a simple pole for Φ.

It is quite straightforward to generalize these findings to the case of a system made of n_0 fractional D3-branes of type 0 and n_1 fractional D3-branes of type 1, which describes a gauge theory with group $\mathrm{U}(n_0 + n_1)$ broken to the Levi group $\mathrm{U}(n_0)\times\mathrm{U}(n_1)$. The profiles of the gauge field and scalar are given by

$$\mathbf{A} = -\frac{b}{2\pi}\begin{pmatrix} \mathbb{I}_{n_0} & 0 \\ 0 & -\mathbb{I}_{n_1} \end{pmatrix} d\theta\,, \tag{26a}$$

$$\mathbf{\Phi} = \frac{b_+}{4\pi}\begin{pmatrix} \mathbb{I}_{n_0} & 0 \\ 0 & -\mathbb{I}_{n_1} \end{pmatrix} \frac{1}{z_2}\,. \tag{26b}$$

This is precisely the expected profile for a monodromy defect of GW type. The continuous parameters of the surface defect are related to the background values of the fields in the NS/NS twisted sector as follows

$$\boxed{\alpha_I = (-1)^{I+1}\,\frac{b}{2\pi}\,, \quad \beta_I = (-1)^I\,\frac{\mathrm{Re}(b_+)}{2\pi}\,, \quad \gamma_I = (-1)^I\,\frac{\mathrm{Im}(b_+)}{2\pi}} \tag{27}$$

Similarly, one can show that there is a coupling between the longitudinal component of the gauge field A_1 and the twisted scalar c in the R/R sector. While this does not lead to a profile, we show that this can be interpreted as an effective topological term localized on the defect D and identified with the "η- parameter" in the following way:

$$\boxed{\eta_I = (-1)^I \frac{c}{2\pi}} \tag{28}$$

S-duality

From a geometric view point, the twisted scalars b and c arise by wrapping the NS/NS and R/R 2-form fields $B_{(2)}$ and $C_{(2)}$ of Type II B string theory around the exceptional 2-cycle ω_2 at the orbifold fixed point, namely

$$b = \int_{\omega_2} B_{(2)}\ , \qquad c = \int_{\omega_2} C_{(2)}\ . \tag{29}$$

It leads to an identification of α_I with $\int_{\omega_2} B_{(2)}$ and η_I with $\int_{\omega_2} C_{(2)}$. Similarly the $b_{\pm}$ field is identified with the string-frame metric $G_{\mu\nu}$. So, (β_I, γ_I) transform in the similar way as $G_{\mu\nu}$. Since, the S-duality properties of the parent Type II B theory are inherited within $B_{(2)}, C_{(2)}$ and $G_{\mu\nu}$, it leads to natural transformation properties for all the continuous parameters $\alpha_I, \beta_I, \gamma_I, \eta_I$ which matches exactly with the expected transformation properties (7) prescribed for them in GW-prescription.

0.4 Surface defects from $\mathbb{Z}_M$ orbifolds

We will now generalize the above construction to $\mathbb{Z}_M$ orbifold which captures the generic surface defect of type $[n_0, n_1, ..., n_{M-1}]$.

0.4.1 Closed string spectrum

In this case, the closed string Hilbert space in addition to the usual untwisted sector, contains $(M-1)$ twisted sector, associated with the non-trivial conjugacy class of $\mathbb{Z}_M$

labeled by $a = 1, 2, ..., M - 1$.

When M is even, there is a state with $\nu_a \equiv \frac{a}{M}$ equal to $\frac{1}{2}$ in which case the details are same as the $\mathbb{Z}_2$ case. So, from now onwards without loss of generality, we will consider M to be odd. The salient feature of this case is that the vacuum states are tachyonic and the massless states are the first excited states.

eg. in the case of $\nu_a < \frac{1}{2}$, in NS-sector we have the following vertex operators at the massless level [9, 10]

$$\begin{aligned} \mathcal{V}_a^1(z) &= \sigma_a(z) : \Psi^3(w)\, s_a(z) : \mathrm{e}^{-\phi(z)} \, , \\ \mathcal{V}_a^2(z) &= \sigma_a(z) : \overline{\Psi}^2(w)\, s_a(z) : \mathrm{e}^{-\phi(z)} \, . \end{aligned} \tag{30}$$

The bosonic twist field $\sigma_a(z)$ is a conformal field of weight $\nu_a(1 - \nu_a)$ while the fermionic twist field $s_a(w)$ is a conformal field of weight ν_a^2.

The massless closed string excitations in the twisted NS/NS sectors are obtained by combining the left- and right-moving massless states. In the sectors with twist parameter $\nu_a < \frac{1}{2}$, they are then described by the following vertex operators at zero momentum

$$b^{(a)}_{\alpha\beta}\, \mathcal{V}^\alpha_a(z)\, \widetilde{\mathcal{V}}^\beta_a(\bar{z}) \tag{31}$$

where $b^{(a)}_{\alpha\beta}$ are four constant complex fields.

Similarly, in the sectors with twist parameter $(1 - \nu_a) > \frac{1}{2}$, the massless closed string excitations are described by the vertex operators at zero momentum

$$b^{(M-a)}_{\alpha\beta}\, \mathcal{V}^\alpha_{M-a}(z)\, \widetilde{\mathcal{V}}^\beta_{M-a}(\bar{z}) \tag{32}$$

where again $b^{(M-a)}_{\alpha\beta}$ are four constant complex fields.

In the R/R sector, we only consider non-vanishing background values for the scalars $C^{(a)}$ and $C^{(M-a)}$, since they are the only ones that turn out to be relevant for the description of the continuous parameters of surface defects. Thus, the closed string vertex operators we consider are

$$C^{(a)}\, C_{A\dot{B}}\, \mathcal{V}^{A}_{a}(z)\, \widetilde{\mathcal{V}}^{\dot{B}}_{a}(\bar{z}) \quad \text{and} \quad C^{(M-a)}\, C_{\dot{A}B}\, \mathcal{V}^{\dot{A}}_{M-a}(z)\, \widetilde{\mathcal{V}}^{B}_{M-a}(\bar{z})\ . \tag{33}$$

Boundary states and Reflection rules

The fractional D3-branes can be schematically represented in the boundary state formalism as follows [6–8]:

$$|\mathrm{D3}; I\rangle = \mathcal{N}\, |\mathrm{U}\rangle + \mathcal{N}'\, |\mathrm{T}; I\rangle \quad \text{with} \quad |\mathrm{T}; I\rangle = (-1)^{I}\, |\mathrm{T}\rangle\ . \tag{34}$$

$I = 0, 1, ..., M-1$ corresponding to M irreducible representation of $\mathbb{Z}_M$. $|\mathrm{U}\rangle$ and $|\mathrm{T}; I\rangle$ are the untwisted and twisted components of the boundary states.

As in the $\mathbb{Z}_2$ case due to presence of the D-branes, the right and left moving vertex operators are related. Schematically introducing the following reflection rule for right moving NS/NS vertex operators:

$$\widetilde{\mathcal{V}}^{\beta}_{a}(\bar{z}) \longrightarrow (R_{I,a})^{\beta}{}_{\gamma}\, \mathcal{V}^{\gamma}_{M-a}(\bar{z})\ , \tag{35}$$

The reflection matrix $R_{I,a}$ is found to be

$$R_{I,a} = \mathrm{i}\ \sin\left(\frac{\pi a}{M}\right) \omega^{-I\,a}\, \tau_3 \tag{36}$$

with $a = 1, \ldots, M-1$ and ω is the M^{th} root of unity.

0.4.2 Open string spectrum

To construct the massless open string vertex operators, we need to construct linear combinations of $e^{iK.X}$ factors that transform covariantly under the orbifold group action g.

$$\mathcal{E}_I = \frac{1}{M}\sum_{J=0}^{M-1}\omega^{-IJ}\, g^J\Big[\mathrm{e}^{\mathrm{i}\,\kappa_\perp\cdot Z_\perp}\Big] = \frac{1}{M}\sum_{J=0}^{M-1}\omega^{-IJ}\,\mathrm{e}^{\mathrm{i}\,(\omega^{-J}\kappa_2\,\overline{Z}^2+\omega^J\overline{\kappa}_2\,Z^2)}\,. \tag{37}$$

One can easily check that $g[\mathcal{E}_I] = \omega^I\,\mathcal{E}_I$. A typical vertex operator eg. for the gauge field A_1 is given by

$$\mathcal{V}_{A_1} = \Big[(\mathrm{i}\,\partial Z^1 + \kappa_\| \cdot \Psi_\|\,\Psi^1)\,\mathcal{E}_0 + \kappa_2\,\overline{\Psi}^2\,\Psi^1\,\mathcal{E}_1 + \overline{\kappa}_2\,\Psi^2\,\Psi^1\,\mathcal{E}_{M-1}\Big]\,\mathrm{e}^{\mathrm{i}\,\kappa_\|\cdot Z_\|}\,. \tag{38}$$

Similarly, expressions can be written for the vertex operators for gauge field component A_2 and other three scalars as well as each of their complex conjugate fields.

0.4.3 Open/Closed correlators and field theory profiles

Having both the closed string and open string field vertex operators, we can calculate the various open/closed correlators using (4.54). The complete expression for the open string fields emitted by a fractional D3-brane of type I in the presence of background values for the scalars of the NS/NS twisted sectors is given by summing over all components of $b^{(a)}_{\alpha\beta}$ and over all twisted sectors:

$$\langle\mathcal{V}_{\text{open}}\rangle_I = \sum_{a=1}^{M-1}\sum_{\alpha,\beta=1}^{2}\langle\mathcal{V}_{\text{open}}\rangle_{b^{(a)}_{\alpha\beta};I}\,. \tag{39}$$

We define the singlet combination to be $b^{(a)}_{\mathrm{s}} = \frac{i}{2}(b^{(a)}_{12} - b^{(a)}_{21})$ and also define $b^{(a)}_{+} = b^{(a)}_{22}$ and $b^{(a)}_{-} = -b^{(a)}_{11}$. Then the non-zero coupling of the transverse gauge field is given by:

$$\langle\mathcal{V}_{A_2}\rangle_I = -\kappa_2\,b_I\,\delta^{(2)}(\kappa_\|)\,, \tag{40}$$

with the following combination defined as b_I:

$$b_I = \sum_{a=1}^{\frac{M-1}{2}} \sin \pi\nu_a \left[\omega^{-Ia}\, b_{\rm s}^{(a)} + \omega^{Ia}\, b_{\rm s}^{(M-a)}\right] = \sum_{a=1}^{M-1} \sin\Big(\frac{\pi a}{M}\Big)\, \omega^{-Ia}\, b_{\rm s}^{(a)} \,, \tag{41}$$

After taking the fourier transform with the appropriate insertion of propagator, we have the result:

$$A_{2;I} = -\frac{\mathrm{i}\, b_I}{4\pi \bar{z}_2} \,, \tag{42}$$

Combining this result with the one for the complex conjugate component $\overline{A}_2$, we find that the gauge field on the I-th fractional D3-brane has the following profile:

$$\mathbf{A}_I = A \cdot dx = A_{2;I}\, d\bar{z}_2 + \overline{A}_{2;I}\, dz_2 = -\frac{\mathrm{i}\, b_I}{4\pi}\Big(\frac{d\bar{z}_2}{\bar{z}_2} - \frac{dz_2}{z_2}\Big) = -\frac{b_I}{2\pi}\, d\theta \,, \tag{43}$$

The only other open string field that has a non-vanishing profile in the twisted NS/NS background we have chosen is the complex scalar Φ. The analogous calculation takes the following form:

$$\Phi_I = \frac{b_I^+}{4\pi z_2} \,, \tag{44}$$

with

$$b_I^\pm = \sum_{a=1}^{\frac{M-1}{2}} \sin \pi\nu_a \left[\omega^{-Ia}\, b_\pm^{(a)} + \omega^{Ia}\, b_\pm^{(M-a)}\right] = \sum_{a=1}^{M-1} \sin\Big(\frac{\pi a}{M}\Big)\, \omega^{-Ia}\, b_\pm^{(a)} \,. \tag{45}$$

These results relate $(\alpha_I, \beta_I, \gamma_I)$ that conventionally parametrize the singular profiles near the defect to the background values of the NS/NS twisted scalars as follows:

$$\boxed{\alpha_I = -\frac{b_I}{2\pi} \,, \qquad \beta_I = \frac{\mathrm{Re}(b_I^+)}{2\pi} \,, \qquad \gamma_I = \frac{\mathrm{Im}(b_I^+)}{2\pi}} \tag{46}$$

The coupling of the gauge field component with R/R scalars leads to the following identification of the η_I parameter:

$$\boxed{\eta_I = \frac{c_I}{2\pi}} \tag{47}$$

where c_I is the following combination:

$$c_I = \sum_{a=1}^{\frac{M-1}{2}} \sin \pi\nu_a \left[\omega^{-Ia} C^{(a)} + \omega^{Ia} C^{(M-a)}\right] = \sum_{a=1}^{M-1} \sin\left(\frac{\pi a}{M}\right) \omega^{-Ia} C^{(a)} . \tag{48}$$

The continuous parameters α_I and η_I are identified with the the NS/NS 2-form $B_{(2)}$ and the R/R 2-form $C_{(2)}$ of Type II B supergravity around the exceptional cycles ω_I of the blown-up ALE space.

$$\alpha_I = -\frac{1}{2\pi} \int_{\omega_I} B_{(2)} , \qquad \eta_I = \frac{1}{2\pi} \int_{\omega_I} C_{(2)} . \tag{49}$$

Using the S-duality action on the 2-forms, with simple manipulations one can show that this identification implies that α_I and η_I indeed transform in the expected way.

Similarly, the $b_I^{\pm}$ parameters can be identified with the (string frame) metric moduli corresponding to the complex structure of the blown-up exceptional cycle ω_I. As such they inherit the S-duality transformation properties from the (string frame) metric, which are precisely the ones expected for the parameters β_I and γ_I of the GW defects.

0.5 Conclusion

In this thesis, we have studied a microscopic realization of Gukov-Witten surface defects within the realm of Type II B string theory. The setup we considered is a $10d$ target space of Type II B theory with $\mathbb{Z}_M$ orbifolding along 2 of the complex directions. The world-volume of the D3 brane on which the gauge theory lives is extended partially along the orbifolded directions. It was already clear from the work of Kanno-Tachikawa that the discrete data of the surface defect are encoded within the number of fractional branes of each type. In our work [11, 12], we have shown that the continuous data of the surface defect is encoded as the background value of different twisted closed string scalars.

We provided support for our claim by matching the expected S-duality properties of the continuous parameters from the viewpoint of Type II B theory.

Plan of the thesis

In this thesis, we study the geometric engineering of surface defects in 4d $U(N)$ SYM within Type II B string theory via fractional branes.

- Chapter 1 provides a general introduction to surface defects and summarizes the motivation of our work.
- Chapter 2 provides necessary background material regarding Type II B string theory, orbifold backgrounds, and D-branes.
- Chapter 3 discusses the realization of the simplest Gukov-Witten (GW) defects of type $[n_0, n_1]$ within Type II B string theory via branes on a $\mathbb{Z}_2$ orbifold.
- Chapter 4 generalizes the above construction to generic GW defect of type $[n_0, ..., n_{M-1}]$ within Type II B string theory via branes on a $\mathbb{Z}_M$ orbifold background.
- Chapter 5 will conclude with a discussion of the results.

List of Figures

Chapter 1

Motivation and Introduction

The study of defects in QFT has become an active area of research in recent years and their study has provided useful information about the vacuum structure of the gauge theories. They are an important probe of the non-perturbative effects and dualities in gauge theories. One-dimensional or line defects such as Wilson and 't Hooft loops in gauge theories are the best studied in the literature. In this thesis, we focus on higher dimensional defects which are supported on co-dimension two sub manifolds, called surface defects. These were first introduced by Gukov and Witten (GW) [1, 2] in the context of topologically twisted four dimensional maximally supersymmetric N = 4 Yang-Mills theory with U(N) gauge group. The defect was defined as a "monodromy defect" by specifying the singular behavior of the four dimensional fields in the gauge theory as one approaches the location of the defect. With these boundary conditions specified for the path integral, correlation functions in the presence of such defects provide us with valuable non-perturbative information about the bulk gauge theory.

From the gauge theory point of view, there are several ways to analyze surface defects. The original approach of Gukov and Witten (GW) in [1, 2] was to treat them as monodromy defects, in which one specifies the singular behavior of the fields of the gauge theory as one approaches the defect. Another possibility is to describe them as flavour defects: we specify the theory on the defect as a $2d$ quiver gauge theory with a U(N) flavour group, which is then identified with the gauge group of the $4d$ theory [13, 14]. In many cases, these two different descriptions lead to the same results [15]. For example, the low-energy effective action on the Coulomb branch of the $4d$ gauge theory computed in the two approaches exactly match. Moreover, by fruitfully combining the two methods various properties of the surface defects as well as many duality relations and non-perturbative effects can be studied [16–24].

There are also several ways to embed the surface defects in string theory and, more generally, to study the defects from a higher-dimensional perspective. In [25, 26] the GW defects were given a holographic representation in terms of bubbling geometries, which are particular solutions of Type II B supergravity that asymptote to $AdS_5 \times S_5$. Since many $4d$ supersymmetric gauge theories can be obtained by compactification from the $6d$ $(2, 0)$ theory defined on the world-volume of an M5 brane [27, 28], surface defects can also be realized by introducing intersecting M5 branes or an M2 brane inside the M5 brane [18]. From this six dimensional perspective, surface defects have been recently studied in detail [29] following earlier work in [28, 30, 31], by exploiting the relation to the Hitchin integrable system, with the aim of obtaining a complete classification of the surface defects in the $6d$ theory.

In this thesis, we geometrically engineer a GW surface defect within Type II B string theory by considering a configuration of fractional branes on an orbifold background. Further, we provide a physical interpretation of the defining parameters of the defect in terms of background values for closed string scalars in the twisted sectors of the closed string orbifold. In particular, we consider Type II B string theory on the following orbifold space

$$\mathbb{C}_{(1)} \times \frac{\mathbb{C}_{(2)} \times \mathbb{C}_{(3)}}{\mathbb{Z}_M} \times \mathbb{C}_{(4)} \times \mathbb{C}_{(5)} \tag{1.1}$$

Note that we will be considering *Euclidean* string theory for the setup and hence all the 10 real directions in (1.1) are space-like.

We turn on constant vacuum expectation values for particular twisted scalar fields in the Neveu-Schwarz/Neveu-Schwarz (NS/NS) and Ramond/Ramond (R/R) sectors. In this background we engineer a $4d$ gauge theory by introducing stacks of fractional D3-branes that extend along the first two complex planes $\mathbb{C}_{(1)}$ and $\mathbb{C}_{(2)}$. In this combined orbifold/D-brane set-up, which we refer to as the Kanno-Tachikawa [KT] configuration [4], we compute the profile in configuration space of the massless open strings by means of open/closed world-sheet correlators, and show that these exactly reproduce the singular profiles that characterize the GW surface defect in the $\mathcal{N} = 4$ gauge theory [1]. In this way, we are therefore able to provide an explicit identification of the continuous parameters of the GW solution with the vacuum expectation values of the twisted scalars.

Historically, it has been fruitful to embed field theory solutions or phenomena in string theory as it is a much larger framework and usually the embedding leads one to uncover new ideas and/or phenomena. The work presented in this thesis is best understood from this perspective: it is a first step in the study of surface defects using perturbative string theoretic methods. The rest of the thesis is organized as follows, in chapter 2 we provide the basic background concepts used in our calculations. In chapter 3 we introduce surface operators as monodromy defects, providing the path integral definition of it as well as introduce the various parameters needed to define it. We also discuss the string theoretic setup and open/closed string disk diagram that we need to calculate. In chapter 4 we give the explicit string theoretic realization of the simplest possible $[n_0, n_1]$ type of surface defects via fractional branes in a $\mathbb{Z}_2$ orbifolded background. In chapter 5 we generalize the above construction of ch. 4 to the more generic $[n_0, n_1, ..., n_{M-1}]$ type of surface defects realized via fractional branes in a $\mathbb{Z}_M$ orbifolded background. In chapter 6 we finish with a summary of the results and their importance.

Chapter 2

Background

In this chapter, we review some of the important background concepts needed to follow this thesis. It is by no means complete and the reader is suggested to refer to the appropriate literature for more details.

2.1 Type IIB string theory

We follow the discussion in [32, 33]. and begin with the $1+1$ dimensional worldsheet action corresponding to superstrings propagating on ten dimensional flat space $\mathbb{R}^{1,9}$:

$$S = \frac{1}{4\pi} \int d^2z \left(\frac{2}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \widetilde{\psi}^\mu \partial \widetilde{\psi}_\mu \right) \tag{2.1}$$

The index $\mu = 0, 1, \ldots 9$ and the parameter α' is given in terms of the string length as $\alpha' = \ell_s^2$ and is the inverse tension of the fundamental string. This action has $N = (1,1)$ supersymmetry on the worldsheet and describes a superconformal theory. The bosonic fields X_μ and their superpartners fermionic fields $\psi^\mu, \widetilde{\psi}^\mu$ have the following OPEs :

$$
\begin{aligned}
X^\mu(z,\bar{z})\,X^\nu(0,0) &\sim -\frac{\alpha'}{2}\eta^{\mu\nu}\,ln\,|z|^2 \\
\psi^\mu(z)\,\psi^\nu(0) &\sim \frac{\eta^{\mu\nu}}{z} \\
\widetilde{\psi}^\mu(\bar{z})\,\widetilde{\psi}^\nu(0) &\sim \frac{\eta^{\mu\nu}}{\bar{z}}
\end{aligned} \tag{2.2}
$$

Ramond and Neveu Schwarz sectors

Now we investigate the (X^μ, ψ^ν) superconformal field theory (SCFT) on a worldsheet with cylindrical topology. With the cylindrical coordinate $w = \sigma^1 + i\sigma^2$, the fermionic part of the action is given by,

$$
\frac{1}{4\pi}\int d^2w\left(\psi^\mu\partial_{\bar{w}}\psi_\mu + \widetilde{\psi}^\mu\partial_w\widetilde{\psi}_\mu\right) \tag{2.3}
$$

This action must be invariant under the periodic identification $w \sim w + 2\pi$. This allows for two possible periodic conditions for the ψ^μ fields.

$$
\begin{aligned}
\text{Ramond (R)} : \psi^\mu(w+2\pi) &= +\psi^\mu(w) \\
\text{Nevue-Schwarz(NS)} : \psi^\mu(w+2\pi) &= -\psi^\mu(w)
\end{aligned} \tag{2.4}
$$

Similar periodicity conditions are defined for $\widetilde{\psi}^\mu$. It can be written compactly as

$$
\begin{aligned}
\psi^\mu(w+2\pi) &= \exp(2\pi i\nu)\psi^\mu(w) \\
\widetilde{\psi}^\mu(\bar{w}+2\pi) &= \exp(-2\pi i\tilde{\nu})\widetilde{\psi}^\mu(w)
\end{aligned} \tag{2.5}
$$

where ν and $\tilde{\nu}$ takes values 0 and $\frac{1}{2}$. [1]

To maintain Poincare invariance, one considers periodic boundary conditions for bosonic fields X^μ. In later sections, we will relax this condition when we consider orbifold backgrounds in which the X-fields have non-trivial identifications as one traverses the world-sheet circle.

There are four different combinations one can make out of the NS and R sectors i.e. NS-NS, NS-R, R-NS, R-R thus counting for four possible kinds of closed superstrings.

The bosonic and fermionic fields can be mode expanded in the following way:

$$\psi^\mu(w) = i^{-1/2} \sum_{r \in \mathbb{Z}+\nu} \psi^\mu_r \, \exp(irw), \quad \widetilde{\psi}^\mu(\bar{w}) = i^{1/2} \sum_{r \in \mathbb{Z}+\nu} \widetilde{\psi}^\mu_r \, \exp(-ir\bar{w}) \tag{2.6}$$

After the coordinate transformation $z = \exp(-iw)$, the expansion becomes:

$$\psi^\mu(z) = \sum_{r \in \mathbb{Z}+\nu} \frac{\psi^\mu_r}{z^{r+1/2}}, \quad \widetilde{\psi}^\mu(\bar{z}) = \sum_{r \in \mathbb{Z}+\tilde{\nu}} \frac{\widetilde{\psi}^\mu_r}{\bar{z}^{r+1/2}} \tag{2.7}$$

For the bosonic part of the theory, we have the following expansions for ∂X and $\overline{\partial} X$:

$$\partial X^\mu(z) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\alpha^\mu_m}{z^{m+1}}, \quad \overline{\partial} X^\mu(\bar{z}) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\widetilde{\alpha}^\mu_m}{\bar{z}^{m+1}} \tag{2.8}$$

It's clear from the expansion that they are holomorphic and anti-holomorphic respectively. Using these one can write down the expansion of X^μ field:

[1] We will see later that for string theories on orbifolded backgrounds can have more general periodicity conditions leading to more possible values of $\nu, \tilde{\nu}$.

$$X^\mu(z,\bar{z}) = x^\mu - i\frac{\alpha'}{2}p^\mu ln|z|^2 + i\left(\frac{\alpha}{2}\right)^{1/2}\sum_{m=-\infty,m\neq 0}^{\infty}\frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m}+\frac{\widetilde{\alpha}_m^\mu}{\bar{z}^m}\right) \tag{2.9}$$

For open string, there are two possible periodicities:

$$\psi^\mu(0,\sigma^2) = \exp(2\pi i\nu)\,\widetilde{\psi}^\mu(0,\sigma^2), \quad \psi^\mu(\pi,\sigma^2) = \exp(2\pi i\nu')\,\widetilde{\psi}^\mu(\pi,\sigma^2) \tag{2.10}$$

By the redifinition $\widetilde{\psi}^\mu \rightarrow \exp(-2\pi i\nu')\,\widetilde{\psi}^\mu$, one can set $\nu' = 0$. Thus there are only two sectors for open string : NS and R.

Now consider the spectrum generated by a single NS or R field which can correspond to an open string or one of the left/right sectors of the closed string. In the NS sector, there is no mode with $r = 0$ and we define the ground state to be the state annihilated by all $r > 0$ modes:

$$\psi_r^\mu\,|0\rangle_{NS} = 0, \quad r > 0 \tag{2.11}$$

The modes with $r < 0$ are the ones that act as the raising operators. Moreover, because of their anticommuting properties, each mode can be excited only once.

For the R sector states, there are states with $r = 0$ mode. If we define the ground state to be the one annihilated by $r > 0$ modes and since $\{\psi_r^\mu, \psi_0^\nu\} = 0$ for $r > 0$, the ψ_0^μ take ground states into ground states. Due to this reason, the R sector ground state is degenerate. If we define $\Gamma^\mu \sim 2^{1/2}\psi_0^\mu$, it satisfies the gamma matrix algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. So, the ground states of the R sector form a representation of the gamma matrix algebra. One can show that to make the total central charge of the worldsheet CFT be zero (which is related to the cancellation of *Weyl Anomaly*) we must take the spacetime dimension D to be 10 for the superstring theory. In $D = 10$ the dimension of the gamma matrix algebra is **32**.

In the R sector, all the states including the ground state have half-integer spacetime spins.

In the NS sector, all the states have integer spins.

There is an operator that anticommutes will all the ψ^{μ} s. The operator is $\exp(\pi i F)$ where F is the world-sheet fermion number. Since ψ^{μ} changes F by one, it's easy to check it anticommutes with ψ^{μ}. The spacetime Lorentz generators are defined as

$$\Sigma^{\mu\lambda} = -\frac{i}{2} \sum_{r \in \mathbb{Z}+\nu} [\psi^{\mu}_{r}, \psi^{\lambda}_{-r}] \tag{2.12}$$

We also define the spin operator $S_a = i^{\delta_{a,0}} \Sigma^{2a,2a+1}$ whose eigenvalue gives the spacetime spin for a given state. With that generators, the fermion number operator is defined as follows:

$$F = \sum_{a=0}^{4} S_a \tag{2.13}$$

Closed string spectra: The spins of the states in the two closed string sectors are additive. So, for closed string, the NS-NS states have integer spins. The two half integers from R-R sectors also add up to give integer spins. Whereas NS-R and R-NS states have half-integer spin.

We will now denote the two sectors of the open strings by (α, F). Where it's defined as $\alpha = (1 - 2\nu)$ is 1 for R sector and 0 for the NS sector.

For the closed string, there are independent periodicities and fermion numbers on both sides, so there are 16 possible combinations:

$$(\alpha, F, \widetilde{\alpha}, \widetilde{F}) \tag{2.14}$$

But six out of these combinations are ruled out by level matching conditions: $L_0 = \widetilde{L}_0$. $L_0 - \alpha' p^2/4$ is half-integer in NS- sector whereas its integer in NS+, R+, and R- sectors.

So, NS- can't pair with any of the three mentioned sectors.

Further constraints appear as the various vertex operators need to be mutually local i.e. they should not have branch cuts in their operator product expansions. There are total 2^{10} such possible combinations but only some subset of it will lead to consistent closed string theories. It is achieved by imposing the following consistency conditions:

1. All pairs of vertex operators in the theory should be mutually local i.e. if $(\alpha_1, F_1, \widetilde{\alpha}_1, \widetilde{F}_1)$ and $(\alpha_2, F_2, \widetilde{\alpha}_2, \widetilde{F}_2)$, the net phase when one operator encircles another is exp $(F_1\alpha_2 - F_2\alpha_1 - \widetilde{F}_1\widetilde{\alpha}_2 + \widetilde{F}_2\widetilde{\alpha}_1)$ and for them to be local one should have

$$F_1\alpha_2 - F_2\alpha_1 - \widetilde{F}_1\widetilde{\alpha}_2 + \widetilde{F}_2\widetilde{\alpha}_1 \in 2\mathbb{Z} \tag{2.15}$$

2. The OPE must close. If both $(\alpha_1, F_1, \widetilde{\alpha}_1, \widetilde{F}_1)$ and $(\alpha_2, F_2, \widetilde{\alpha}_2, \widetilde{F}_2)$ are in the spectrum so does $(\alpha_1 + \alpha_2, F_1 + F_2, \widetilde{\alpha}_1 + \widetilde{\alpha}_2, \widetilde{F}_1 + \widetilde{F}_2)$.

3. To ensure modular invariance of the one-loop amplitude, there must be atleast one of both left moving R sector ($\alpha = 1$) and right moving R sector ($\widetilde{\alpha} = 1$) in the spectrum.

There are four possible superstring theories with at least one R-NS sector IIA, IIB, IIA′, IIB′ which satisfy all the three conditions above. None of these theories contains any tachyon.

Without any R-NS sector, there are two possible string theories 0A and 0B. Both of these two theories contain tachyons in (NS-,NS-) sector and do not contain any space-time fermion.

In this thesis we will be particularly interested in the Type IIB theory. It contains the following sectors:

$$\text{IIB} : (NS+, NS+)\ \ (R+, NS+)\ \ (NS+, R+)\ \ (R+, R+) \tag{2.16}$$

In 10-d, in each sector, (left or right) massless states are classified by their behavior under the SO(8) rotations which keep the momentum invariant. Thus the massless closed string spectrum is given by a product representation of SO(8) corresponding to combining left and right sectors. Alternative to the product rep., one can as well write the massless spectra by the following tensor representations of SO(8):

$$\text{IIB}: \ [0]^2 + [2]^2 + [4]^2_+ + (2) + \mathbf{8'^2} + \mathbf{56^2} \tag{2.17}$$

where (m) and $[m]$ denote the traceless symmetric and antisymmetric rank m tensor in eight dimensions respectively [2]. Note that $\mathbf{8'}, \mathbf{56}$ denotes objects of dimensions 8 and 56 respectively and is not representable in tensor representation. They are obtained by multiplying a vector rep. and spinor rep. of SO(8): $8_v \times 8$.

In type IIB theory one keep the sectors with

$$\exp(\pi i F) = \exp(\pi i \widetilde{F}) = +1 \tag{2.18}$$

This is known as the **Gliozzi-Scherk-Olive (GSO) projection** where the full spectra are projected onto the eigenstates of $\exp(\pi i F)$ and $\exp(\pi i \widetilde{F})$. The GSO projection ensures spacetime supersymmetry and also removes the tachyonic states to make the tachyon free type II B theory [3]. Since the same GSO projection is taken in both left and right sectors, the spectrum of type IIB theory is chiral.

[2] [0] denotes a scalar of SO(8).

[3] GSO projection similarly removes the tachyon from type II A , II A', II B' theories too

2.2 State-operator correspondence and Vertex operators

In quantum field theory, there is on the one hand the space of states of the theory and on the other hand the set of local operators. In conformal field theory, there is a simple and useful isomorphism between these, with the CFT quantized on a circle [34]. Consider a QFT on a semi-infinite cylinder $\mathbb{R} \times S^P$, the metric is given by $ds^2 = -dt^2 + d\Omega_p^2$. By the coordinate transformation $r = \exp(it)$, the cylinder is mapped to a plane $\mathbb{R}^{p+1}$ with the metric $ds^2 = (dr^2 + r^2 d\Omega_p^2)/r^2$ which is simply flat space with a conformal factor.

If one specifies an initial state $|\psi\rangle$ on the cylinder, one can make a conformal transformation that squeezes it to the origin of the plane $\mathbb{R}^{p+1}$. Thus it is a local perturbation at the origin and corresponds to local operator $O_\psi(r = 0)$. So there is a correspondence between local operators and states in CFT :

$$\text{States specified on cylinder} \leftrightarrow \text{local operators at the origin of the plane} \tag{2.19}$$

One can also go the other way around, given a local operator on the plane the initial state on the cylinder can be recovered in the following way:

$$|O\rangle = lim_{x\to 0}\, O(x)|0\rangle \tag{2.20}$$

where $|0>$ denotes the vacuum or ground state.

Here are a few examples that illustrate this [32]:

The unit operator corresponds to the vacuum state: $|1\rangle = |0; 0\rangle$.

The free state with momentum k corresponds to $: \exp(ik.X) :$ operator. Similarly, one can write the following bosonic excited states and their corresponding vertex operators :

$$\alpha^{\mu}_{-m}|1\rangle \;\rightarrow\; i\left(\frac{2}{\alpha'}\right)^{1/2}\frac{1}{(m-1)!}\partial^{m}X^{\mu}(0), \quad m \geq 1$$
$$\widetilde{\alpha}^{\mu}_{-m}|1\rangle \;\rightarrow\; i\left(\frac{2}{\alpha'}\right)^{1/2}\frac{1}{(m-1)!}\bar{\partial}^{m}X^{\mu}(0), \quad m \geq 1 \tag{2.21}$$

And for the fermionic excitations:

e.g NS states

$$\psi^{\mu}_{-r}|1\rangle \;\rightarrow\; \frac{1}{(r-1/2)!}\partial^{r-1/2}\psi^{\mu}(0), \qquad r \geq \frac{1}{2} \tag{2.22}$$

2.3 D-branes

D-branes are extended (solitonic) dynamical objects in string theory. From the perspective of open strings [32, 33, 35], D-branes are the hyperplanes on which open strings ends Fig.2.1. A Dp brane is obtained by imposing dirichlet boundary conditions in $(d-p-1)$ directions. The world volume of a Dp brane is $(p+1)$ dimensional hyperplane and the end point of the open strings can only move along the world volume of the brane. They are new dynamical objects within superstring theory on their own. Further, if we consider $U(N)$ Chan-Paton factors on the end points of the open string, there would be N such hyperplanes for each dirichlet direction on which the open strings can end. Usually, a gauge field theory lives on the world volume of the D-brane.

An alternative way to describe D-branes which will be more suitable for our purpose is given as follows.

The 10d spacetime SUSY algebra is schematically given by,

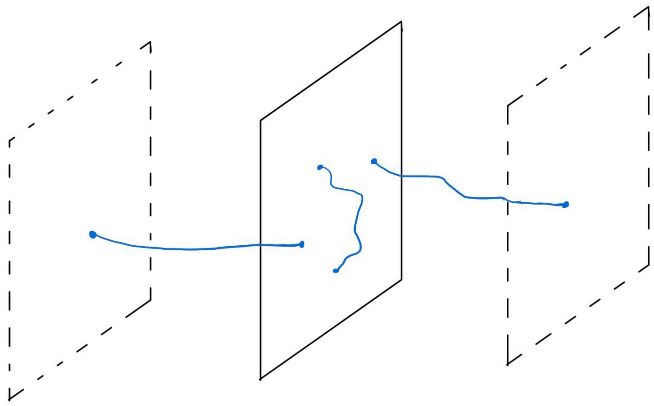

Figure 2.1: Open string end points lying on same D-brane or different D branes.

$$\{Q^{\dot{\mathcal{A}}}, \overline{Q}^{\dot{\mathcal{B}}}\} \sim P_M \Gamma_M^{\dot{\mathcal{A}}\dot{\mathcal{B}}}$$
$$\{\widetilde{Q}^{\dot{\mathcal{A}}}, \widetilde{\overline{Q}^{\dot{\mathcal{B}}}}\} \sim P_M \Gamma_M^{\dot{\mathcal{A}}\dot{\mathcal{B}}} \tag{2.23}$$

All other anti-commutators being zero. $Q^{\dot{\mathcal{A}}}$ and $\widetilde{Q}^{\dot{\mathcal{B}}}$ are the left and right moving supersymmetry charges respectively and similary for the conjugate supercharges $\overline{Q}^{\dot{\mathcal{A}}}, \widetilde{\overline{Q}^{\dot{\mathcal{B}}}}$. $M = 0, 1, ..., 9$, P_M is the momentum and γ^M is the $10d$ Dirac matrices.

The supersymmetry charges are obtained by integrating supersymmetry currents which contains spin field $S^{\dot{\mathcal{A}}}$ transforming as a Weyl tensor of $SO(10)$. The presence of $D3$ brane breaks the $SO(10)$ symmetry of spacetime to $SO(4)\times SO(6)$ and the $\dot{\mathcal{A}}$ index divides into $(S_\alpha S_A, S^{\dot{\alpha}} S^A)$ depending upon positive and negative chiralities respectively. One can show that on the $D3$ branes only the supercharge combination $Q^{\dot{\alpha}A} - \widetilde{Q}^{\dot{\alpha}A}$ and $Q_{\alpha A} + \widetilde{Q}_{\alpha A}$ is preserved on the $D3$ branes whereas the other two combinations $Q^{\dot{\alpha}A} + \widetilde{Q}^{\dot{\alpha}A}$ and $Q_{\alpha A} - \widetilde{Q}_{\alpha A}$ is broken. So, only half of the supersymmetry is preserved ie. 16 supercharges. In fact, one can define D-branes as some BPS states within superstring theory which preserve only half of the supersymmetry. Moreover, these objects are charged under the R-R 2-forms.

From the closed string perspective, the D-branes introduce a boundary in the string worldsheet and are described by a **boundary state** $|B>$ which is a BRST invariant state. They encode the couplings of *all* perturbative closed string fields to the D-brane. They can therefore be expanded in a basis of such closed string fields in which the coefficient encodes the coupling of the brane to the perturbative field. We will discuss more about it in the main text. But here we briefly mention few uses of boundary states to calculate D-brane couplings and interactions [36] : If we denote the closed string propagator by D_a, the matrix element $< B|D_a|B >$ will give rise to an interaction between two D-branes. If we consider a first excited state of momentum k, its coupling with the D-brane is given by the amplitude:

$$A^{\mu\nu} =< 0|a^{\mu}\tilde{a}^{\nu}|B > \tag{2.24}$$

where $a, \tilde{a}$ are the usual left and right moving creation operators respectively. The emission amplitude of a graviton or dilaton can be obtained by contracting $A^{\mu\nu}$ with the appropriate polarization vectors.

$$A^{\text{grav}} \equiv A^{\mu\nu}\epsilon^{\text{grav}}_{\mu\nu}, \qquad A^{\text{dil}} \equiv A^{\mu\nu}\epsilon^{\text{dil}}_{\mu\nu} \tag{2.25}$$

where $\epsilon^{\text{grav}}_{\mu\nu}, \epsilon^{\text{dil}}_{\mu\nu}$ are the polarization vectors corresponding to graviton and dilaton respectively.

In this thesis, we will be interested in what is called fractional D-branes in an orbifold background, which we will briefly review in sec. 2.5. Our goal will be to compute some open/closed correlators on the disk. As a warm up we shall compute some simple disk correlators in the following section.

Calculation of total R-R charge carried by a Dp brane using one-point functions

In the massless spectrum in R-R sectors of type IIB string theory there are antisymmetric form fields $A_{\mu_1\mu_2\ldots\mu_n}$ where n is even for type IIB theory. The vertex operator corresponding to the R-R form field in usual (-1) picture number is given by [36],

$$V^{\text{R-R}}(k,z,\bar{z}) = \mathcal{F}_{\dot{\alpha}\dot{\beta}} \ : \mathcal{V}^{\dot{\alpha}}_{-1/2}(\frac{k}{2},z)\, \widetilde{\mathcal{V}^{\dot{\beta}}_{-1/2}}(\frac{k}{2},\bar{z}) : \tag{2.26}$$

where the holmorphic part is given by

$$\mathcal{V}^{\dot{\alpha}}_{-1/2}(\frac{k}{2},z) = c(z) S^{\dot{\alpha}}(z)\, e^{-\phi(z)/2}\, e^{ik.X(z)} \tag{2.27}$$

$S^{\dot{\alpha}}$ is the spin field, $c(z)$ is the ghost field and $\phi(z)$ is related to superghost fields.

By using the picture changing operator one can write the same R-R vertex operator in (-2) picture number also which we denote as $W^{\text{R-R}}$. One can also write down a boundary state $|B>_R$ for the R-R form field. [4]

To compute the interaction between the R-R field and the Dp-brane, one needs to saturate $|B>_R$ with the corresponding state i.e. one needs to evaluate $<0|\, W^{\text{R-R}}\, |B>_R$. Here we quote the final result [36],

$$\lim_{z\to\infty} <0|\, W^{\text{R-R}}(k,z,\bar{z})\, |B>_R \quad = \pm\sqrt{2} T_p V_{p+1}\, A_{0,\ldots,p} \tag{2.28}$$

Note that p is odd for type IIB theory and the $\pm$ sign refers to brane and anti-brae configuration respectively. V_{p+1} is the world-volume of Dp-brane and T_p is the D-brane tension. From (2.28) one can immediately identify the total R-R charge of the Dp-brane to be

[4] Refer to [36] for the exact expressions of $W^{\text{R-R}}$ and $|B>_R$.

$$\mu_p = \pm \sqrt{2} T_p \tag{2.29}$$

Illustrative computation of 3-pt correlator of fields living on the D3 banes

Let us compute some amplitudes among fields living on D3 branes and their interpretation [37]. It will serve as a warm up for the calculation of disk amplitudes in the later chapters.

In the NS sector, there are a gauge vector A^μ and six scalars ϕ^a at the massless level. They can propagate along the four world-volume directions of the D3 branes. The corresponding vertex operators for these fields are given by,

$$\begin{aligned} V_A^{(-1)}(z) &= A^\mu \, \mathcal{V}_{A^\mu}^{(-1)}(z,p) \\ V_\phi^{(-1)}(z) &= \phi^a \, \mathcal{V}_{\phi_a}^{(-1)}(z,p) \end{aligned} \tag{2.30}$$

where

$$\begin{aligned} \mathcal{V}_{A_\mu}^{(-1)}(z,p) &= \frac{1}{\sqrt{2}} \psi_\mu(z) \, e^{-\phi(z)} \, e^{ip_\nu X^\nu}(z) \\ \mathcal{V}_{\phi^a}^{(-1)}(z,p) &= \frac{1}{\sqrt{2}} \psi_a \, e^{-\phi(z)} \, e^{ip_\nu X^\nu}(z) \end{aligned} \tag{2.31}$$

In the R sector, there are two gauginos at the massless level, $\Lambda^{\alpha A}$ and $\overline{\Lambda}_{\dot{\alpha}A}$. In the $(-1/2)$ picture, the gaugino vertex operators are given by,

$$
\begin{aligned}
V_{\Lambda}^{(-1/2)}(z) &= \Lambda^{\alpha A}(p)\, \mathcal{V}_{\Lambda^{\alpha A}}^{(-1/2)}(z,p) \\
V_{\overline{\Lambda}}^{(-1/2)}(z) &= \overline{\Lambda}_{\dot{\alpha}A}(p)\, \mathcal{V}_{\overline{\Lambda}_{\dot{\alpha}A}}^{(-1/2)}(z,p)
\end{aligned}
\tag{2.32}
$$

where

$$
\begin{aligned}
\mathcal{V}_{\Lambda^{\alpha A}}^{(-1/2)}(z,p) &= S_{\alpha}(z) S_{A}(z)\, e^{-\frac{1}{2}\phi(z)}\, e^{ip_{\nu}X^{\nu}} \\
\mathcal{V}_{\overline{\Lambda}^{\dot{\alpha}A}}^{(-1/2)}(z,p) &= S^{\dot{\alpha}}(z) S^{A}(z)\, e^{-\frac{1}{2}\phi(z)}\, e^{ip_{\nu}X^{\nu}}
\end{aligned}
\tag{2.33}
$$

where $S_{\alpha}, S^{\dot{\alpha}}, S_{A}$ are the spin fields of $SO(4)$ and $SO(6)$ respectively.

Let us consider the amplitude between one gauge boson and two gauginos [5]:

$$
\begin{aligned}
\mathcal{A}_{(\bar{\Lambda}A\Lambda)} &= \langle\langle V_{\bar{\Lambda}}^{(-1/2)}\; V_{A}^{(-1)}\; V_{\Lambda}^{(-1/2)} \rangle\rangle \\
&= C \int \frac{dz_1 dz_2 dz_3}{dV_{123}}\, \langle V_{\bar{\Lambda}}^{(-1/2)}(z_1)\; V_{A}^{(-1)}(z_2)\; V_{\Lambda}^{(-1/2)}(z_3) \rangle
\end{aligned}
\tag{2.34}
$$

where C is a normalization constant and dV_{123} is the projective volume element defined as follows:

$$
dV_{123} = \frac{dz_1 dz_2 dz_3}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}
\tag{2.35}
$$

The correlator breaks into various correlators of free theories and after some simple algebra the final amplitude is given by [37],

[5]We will use use similar calculations to compute disk amplitudes among open/closed string fields in the main text and this calculation will serve as an illustrative example.

$$\mathcal{A}_{\overline{\Lambda}A\Lambda} = C'Tr\left(\overline{\Lambda}_{\dot{\alpha}A}\ \overline{\cancel{A}}^{\dot{\alpha}\beta}\ \Lambda^A_\beta\right)\ \delta^4(p_1+p_2+p_3) \tag{2.36}$$

where C' is a constant, $\overline{\cancel{A}}$ notation means contraction of A_μ with $\overline{\sigma}^\mu$ and we have used the following basic correlators in evaluating the amplitude:

$$\begin{aligned}
\langle\, S^A(z_1)S_B(z_2)\,\rangle &\sim \frac{i\delta^A_B}{(z_1-z_2)^{3/4}} \\
\langle\, S^{\dot{\alpha}}(z_1)\psi_\mu(z_2)S_\beta(z_3)\,\rangle &= \frac{1}{\sqrt{2}}(\overline{\sigma}_\mu)^{\dot{\alpha}}_\beta\ (z_1-z_2)^{-1/2}\ (z_2-z_3)^{-1/2} \\
\langle\, e^{-\frac{\phi}{2}}(z_1)e^{-\phi}(z_2)e^{-\frac{\phi}{2}}(z_3)\,\rangle &\sim (z_1-z_2)^{-1/2}\ (z_2-z_3)^{-1/2}\ (z_3-z_1)^{-1/4} \\
\langle e^{ip_1.X}(z_1)\ e^{ip_2.X}(z_2)\ e^{ip_3.X}(z_3)\rangle &\sim \delta^4(p_1+p_2+p_3)
\end{aligned} \tag{2.37}$$

Where $\overline{\sigma}$ is the Dirac matrices of $SO(4)$. One can calculate the 3-pt and 4-pt amplitude among the various $D3$ string modes in a similar fashion. After taking a Fourier transform, one can show that the 1PI part of them is reproduced by the $\mathcal{N}=4$ SYM action.

2.4 ALE space

In this thesis, we will be interested in studying the superstring theory on an orbifolded background which is a special case of what is called ALE spaces. From a physicist point of view [38], an *Asymptotically Locally Euclidean* or **ALE** manifold is the minimal resolution of singularities of an orbifold $\mathbb{C}^2/\Gamma$ where $\mathbb{C}^2 \equiv \mathbb{R}^4$ is the usual Euclidean 4d flat space and Γ is a finite Kleinian subgroup of $SU(2)$. Since we are dealing with orbifolded theories there is a point group and corresponding to that there are twisted sectors in the closed string spectrum.

The ALE manifolds can be described geometrically as affine complex variety in $\mathbb{C}^3$ i.e. the zero loci of the following polynomial constraints [38]:

$$\{x, y, z\} \in \mathcal{M}_\Gamma\,(t_1, ..., t_r) \;\rightarrow\; \tilde{W}_\Gamma\,(x, y, z; t_1, ..., t_r) = 0 \tag{2.38}$$

that are determined by the algebraic structure of the Kleinian group Γ and that depend on r complex parameters $\{t_i\}$ ($i = 1, ..., r$), r being the number of non-trivial conjugacy classes in Γ. In the limit $t_i \rightarrow 0$ the locus (2.38) reduces to the orbifolded manifold $\mathbb{C}^2/\Gamma$.

In type IIB theory, by viewing the orbifold background as the singular limit of the ALE space for vanishing two-cycles [39], the NS-NS 2-form field $B_{(2)}$ and R-R 2-form field $C_{(2)}$ gives rise to the twisted sector scalars b and c [6] (located at the orbifold fixed point) in the low energy effective action of the six dimensional theory extended in the transverse direction to the orbifold via the following relations [40] :

$$B_{(2)} = b\,\omega_2 \qquad , \qquad C_{(2)} = c\,\omega_2 \tag{2.39}$$

where ω_2 is the 2-form dual to the vanishing 2-cycle of the orbifold. This relation will be

[6] We will describe in details about twisted scalars in sec. 4.1.1, 5.1.1

important when we will study the S-dualty properties of the twisted fields b, c in sec. 4.5, 5.4.

2.5 Fractional Branes

Following the discussion in [41], an open string state is described in the following way :

$$\textbf{CP factor} \times \text{oscillator} \mid \text{momentum eigenstates}\rangle \tag{2.40}$$

Now, we already know that the open strings end on D-branes. So, if there is a single D-brane in flat space, the CP factor is just a number. And for a stack of N D-branes, the CP factor is an $(N \times N)$ matrix. For a D-brane in orbifold space, the CP matrix transforms under different representations of the orbifold group Γ in the following way: let us denote the CP factor by λ, then it transforms as $\lambda \to \lambda' = \mathcal{R}(h)\, \lambda\, \mathcal{R}(h)^{-1}$ where h is any orbifold group element and $\mathcal{R}$ is any specific representation. Since there is various possible representation of a group, there are different kinds of D-branes on orbifolded space.

A D-brane whose CP factors transform under the regular representation[7] of the orbifold group is called *regular D-brane*. But we know that the regular representation is reducible; the regular representation $\mathbf{R}$ is decomposed into its irreducible representations $\mathbf{D_I}$ in the following way:

$$\mathbf{R} = \bigoplus n_I \mathbf{D_I} \tag{2.41}$$

where $n_I;\ I = 0, ..., M-1$ is the dimension of I-th irreducible representation of Γ, and $\sum_{I=0}^{M-1} n_I = |\Gamma|$.

[7]The dimension of regular representation is equal to order $|\Gamma|$ of orbifold group Γ.

This leads one naturally to define another type of D-brane, one whose CP factors transform under the irreducible representations of orbifold group Γ. These branes are called *Fractional D-branes* and there are as many distinct fractional branes as there are irreducible representations of the discrete group. So, a fractional brane of type I is the one whose CP factor transforms under the $\mathbf{D_I}$-th irreducible representation of Γ[8].

When one computes the open string spectrum of a regular brane, one finds massless scalar fields or moduli that correspond to motion along the orbifolded directions. However, the fractional branes do not have any such open string moduli in their spectrum. They are stuck at the orbifold fixed point and are invariant states under the action of the orbifold group.

As already mentioned D-branes in flat space carry charges under the R-R two forms, which can be obtained from the boundary state. The basic idea of constructing the boundary state was first explained by Cardy: we begin with the one-loop open string partition function and then use open/closed duality to rewrite it as a tree level closed string amplitude. The properties of the boundary state can then be read off from this result. By a calculation of the charges carried by the branes labeled by the irreducible representation, it can be shown that these branes carry fractional charge w.r.t the untwisted R-R $(p+1)$ form of a usual Dp-brane. [9]. To be specific if the usual flat space D-brane has charge μ_p under the untwisted RR form, the fractional brane of type I will have the charge $\frac{n_I\,\mu_p}{|\Gamma|}$. Note, for the $\mathbb{Z}_M$ orbifold setup we consider in this thesis, $n_I = 1$. In addition, however, fractional branes are also charged w.r.t. some twisted R-R $(p+1)$ form fields [42].

In Appendix C we explicitly write down the boundary state corresponding to a fractional brane in a $\mathbb{Z}_2$ orbifold, in which the brane is extended partially along the orbifold directions.

[8]For the $\mathbb{Z}_M$ orbifold group considered in this thesis, the dimension of its irreducible representations are always 1 and there are total M types of fractional branes.

[9]Hence the name "fractional brane".

Chapter 3

Surface operators as monodromy defects

Non-local operators or defects in quantum field theories are generally classified by the dimension of their support. In this thesis, we will be mainly interested in four dimensional quantum field theories. So there are four possible kinds of operators [43].

- Codimension 4: These are usual local operators in QFT of zero dimension supported on a point eg. gauge invariant combinations of the fields in any gauge theory.
- Codimension 3: These are non-local operators supported on a one-dimensional line, hence they are called line operators. eg. Wilson and 't Hooft operators.
- Codimension 2: These operators are supported on co-dimension two surfaces. For the four dimensional cases we will be considering, these operators are supported on two dimensional surfaces.[1] These will be the primary objects of study in this thesis and we will be giving more details and defining properties of them as we go along.

[1]The name surface operator is applicable for any non-local operator with co-dimension two support irrespective of the dimension of the total manifold.

- Codimension 1: These operators have support on co-dimesional 3 manifolds. eg. domain walls.

Another way to classify defects is by their construction. An operator is called "*electrical*" if it can be constructed directly from the elementary fields present in the theory eg. local operators or Wilson line. The other type of operators called "*magnetic*" or disorder operators which can not be defined through algebraic combinations of elementary fields but rather by some particular behavior of the some of the elementary fields near these operators eg. 't-Hooft line defects, Gukov-Witten surface defects (which will be the topic of interest in this thesis).

To have an intuitive physical picture in mind, Wilson loops in an $U(1)$ theory can be viewed as world-volume of oppositely moving electron - anti electron bound state [2] Similarly, surface defects can be thought of as world-volume of a Dirac string of dyonic charge which does not obey the *Dirac quantization rule* [43].

3.1 Monodromy defects

We will be mainly interested in studying half-BPS surface defects in $\mathcal{N} = 4$ super Yang-Mills theory. We will define them as monodromy defects: consider a four dimensional manifold M and we modify it along a co-dimensional two sub manifold D in such a way that specific fields in the four dimensional theory develop singularities along surface D. [3]

Let us consider the $\mathcal{N} = 4$ super Yang-Mills theory with gauge group U(N) defined on $\mathbb{R}^4 \simeq \mathbb{C}_{(1)} \times \mathbb{C}_{(2)}$. We will use the complex coordinate z_i on $\mathbb{C}_{(i)}$; we will also use polar coordinates in $\mathbb{C}_{(2)}$ setting $z_2 = r\, \mathrm{e}^{\mathrm{i}\theta}$. It will also be useful at times to denote by $\vec{x}_{||}$ and $\vec{x}_{\perp}$ the real coordinates of these two planes and by $\vec{k}_{||}$ and $\vec{k}_{\perp}$ the corresponding momenta in these directions.

[2] For $U(N)$ theories, one can picture them as quark - anti quark pairs.

[3] This definition is very analogous to the construction of an 't Hooft operator as a co-dimensional three singularity.

A monodromy defect D is a surface defect extended along $\mathbb{C}_{(1)}$ and placed at the origin of $\mathbb{C}_{(2)}$. It is defined by the singular behavior of some of the bosonic fields of the theory, namely the 1-form gauge connection $\mathbf{A}$ and one of the three complex adjoint scalars, which we call $\mathbf{\Phi}$. Near the location of the defect, *i.e.* for $r \to 0$, these fields have the following non-trivial (singular) profile:

$$\mathbf{A} = \begin{pmatrix} \alpha_0\,\mathbb{I}_{n_0} & 0 & \cdots & 0 \\ 0 & \alpha_1\,\mathbb{I}_{n_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{M-1}\,\mathbb{I}_{n_{M-1}} \end{pmatrix} d\theta\,, \tag{3.1}$$

and

$$\mathbf{\Phi} = \begin{pmatrix} (\beta_0 + \mathrm{i}\,\gamma_0)\,\mathbb{I}_{n_0} & 0 & \cdots & 0 \\ 0 & (\beta_1 + \mathrm{i}\,\gamma_1)\,\mathbb{I}_{n_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\beta_{M-1} + \mathrm{i}\,\gamma_{M-1})\,\mathbb{I}_{n_{M-1}} \end{pmatrix} \frac{1}{2z_2}\,. \tag{3.2}$$

Here $\mathbb{I}_{n_I}$ denotes the $(n_I \times n_I)$ identity matrix; α_I, β_I and γ_I are real parameters and the integers n_I are such that

$$\sum_{I=0}^{M-1} n_I = N\,. \tag{3.3}$$

This non-trivial field configuration breaks the $\mathrm{U}(N)$ gauge group to a Levi subgroup

$$\mathrm{U}(n_0) \times \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{M-1})\,. \tag{3.4}$$

If the gauge group is $\mathrm{SU}(N)$ one has to remove the overall $\mathrm{U}(1)$ factor from (3.4) and subtract the trace from (3.1) and (3.2).

Since the defect D is supported on a $2d$ surface, in the definition of the path-integral one is allowed to turn on a $2d$ θ-term, whose coefficient we denote η_I for each factor in the unbroken Levi subgroup. This means that in the path-integral we include the following phase factor:

$$\exp\left(\mathrm{i}\sum_{I=0}^{M-1}\eta_I\int_D \mathrm{Tr}_{\,\mathrm{U}(n_I)}F_I\right)\,. \tag{3.5}$$

The reason for calling the gauge and scalar field profile (3.1), (3.2) respectively to be singular is the presence of $\frac{1}{z}$ pole at the origin $z = 0$ [4]. The introdcution of the surface defect breaks the poincare symmtery along $\mathbb{C}_{(2)}$ plane and also breaks half of the supersymmetry of the $\mathcal{N} = 4$ sYM. Interestingly the specific form of these profiles (3.1), (3.2) are scale invariant [25]. As a result, the Gukov-Witten defects are $2d$ superconformal defects inside the $4d$ $\mathcal{N} = 4$ sYM theory. Since $d(d\theta) = 2\pi\delta_D$, the field strength calculated using this gauge field A is given by: $\boldsymbol{F = 2\pi\underline{\alpha}\delta_D}$ where δ_D is a 2-form delta function supported on D. The holonomy of the singular gauge field A around the singularity is given by

$$P\exp(i\oint A) = \exp(2\pi i\underline{\alpha}) \tag{3.6}$$

Altogether, we can say that a monodromy defect is characterized by the discrete parameters n_I, which constitute a partition of N, and by the four sets of real continuous parameters $\{\alpha_I,\beta_I,\gamma_I,\eta_I\}$, with $I = 0,\ldots,M-1$. Whereas the discrete parameters are analogous to the choice of representation R of the gauge groups for line operators, the continuous parameters are novel to the surface operators. Surface operators that do not depend on continuous parameters do exist and are called **rigid** surface operators.

One of the remarkable features of the $\mathcal{N} = 4$ Yang-Mills theory is its invariance under the action of the non-perturbative duality group $\mathrm{SL}(2,\mathbb{Z})$. It turns out that this duality also acts naturally on the parameters of the surface operator as shown in [1]. In particular, an

[4]Note that $d\theta = \frac{i}{2}\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right)$ and gauge field $\mathbf{A}$ has similar pole singularity as $\boldsymbol{\Phi}$.

element $\Lambda = \left(\begin{smallmatrix} m & n \\ p & q \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ induces the transformation

$$\begin{aligned} (\alpha_I, \eta_I) &\longrightarrow (\alpha_I, \eta_I)\,\Lambda^{-1} = (q\,\alpha_I - p\,\eta_I, -n\,\alpha_I + m\,\eta_I)\,, \\ (\beta_I, \gamma_I) &\longrightarrow |p\,\tau + q|\,(\beta_I, \gamma_I) \end{aligned} \tag{3.7}$$

where τ is the complexified gauge coupling constant.

3.1.1 Origin of GW defect

Gukov-Witten(GW) surface defects were first introduced in the seminal papers [1, 2]. There they studied surface defects in a topological theory, the GL-twisted $\mathcal{N} = 4$ super Yang-Mills theory, which was introduced by Kapustin and Witten in [44]. The topological twist leads to non-canonical spins for the fields and the relevant fields turn out to be the gauge field A and a one-form scalar field ϕ. The Kapustin-Witten equations were derived by imposing the supersymmetry projections consistent with the topological twist:

$$\begin{aligned} F - \phi \wedge \phi &= 0 \\ d_A \phi &= 0 \\ d_A \star \phi &= 0 \end{aligned} \tag{3.8}$$

where the covariant derivative is defined as $d_A \equiv d + A$ and $\star$ is *Hodge star* operator.

Gukov and Witten reduced these equations further to two dimensions, appropriate to the existence of a defect, and found the most general solution to the above set of equations having a singular behavior around the defect to be given by

$$\begin{aligned} A &= \alpha d\theta \\ \phi &= \beta \frac{dr}{r} - \gamma d\theta \end{aligned} \tag{3.9}$$

where α, β, γ are constant elements of the lie algebra of the gauge group and (r, θ) are the polar coordinate in the two-plane transverse to the defect (the defect being at the origin of the coordinate system).

The monodromy of the gauge field around the defect is given by $P\exp(-\oint A)$. Note that the gauge and scalar field used in Gukov-Witten original papers [1, 2] are anti-hermitian in nature whereas in this thesis we used hermitian fields (3.1), (3.2) [5] [6]. That is why the holonomy expression in this thesis (3.6) differs by a factor of i from the above expression used in [1, 2].

3.1.2 Path integral description of surface defects

To understand how surface defects are treated within the path integral formulation, let us suppose we want to calculate an n-point correlator within the four-dimensional gauge theory but in presence of a Gukov-Witten defect i.e., we want to compute $\langle \boldsymbol{O_1(z_1)\, O_2(z_2)\, ... O_(n)(z_n)\, S} \rangle$ where the GW surface operator is denoted by S. Then, such a correlator is given the following path integral:

$$\int_{\mathrm{GW}} [\mathcal{D}A][\mathcal{D}\phi][\mathcal{D}\lambda]\; e^{(-S_{\mathrm{SYM}}+i\,\eta \int_D F)}\;\; O_1(z_1)\, O_2(z_2)\, ... O_n(z_n) \tag{3.10}$$

where λ is the gaugino field, S_{SYM} is the euclidean sYM action and the 2d theta-term has been added by hand in the total action. The important point to note is, by $\int_{\mathrm{GW}}$ we mean carry out the path integral only over the singular field configurations given by (3.1),(3.2). One such example of the above type of correlator is the instanton partition function in the presence of a Gukov-Witten defect [4, 16].

Although in this thesis we will not be exploring the operator properties of the surface

[5] This matches with the conventions of [25].

[6] Also in GW original papers, they studied the GL-twisted $\mathcal{N} = 4$ sYM theory and the scalar field is a one-form. But for the usual $\mathcal{N} = 4$ sYM theory we are interested in this thesis, the scalar field is complex scalar valued.

defects but will primarily focus on reproducing the classical space-time profiles (3.1),(3.2) from a string theoretic setup. We discuss this setup in full detail in the next section.

3.2 Monodromy defects from fractional branes

In this thesis, we shall study the simplest case of GW defects in the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory with gauge group U(N) or SU(N). Our primary goal is to realize these surface defects in perturbative string theory and to recover the singular profiles of the fields in the gauge theory. We do so by calculating perturbative open/closed string amplitudes in Type II B string theory on an orbifold background. Following a proposal of Kanno and Tachikawa (KT) [4], we engineer the $4d$ $\mathcal{N} = 4$ Yang-Mills theory using fractional D3-branes with two world-volume directions along the orbifold, leaving unbroken the Poincaré symmetry in the other two world-volume directions. This is quite different from the more familiar configuration in which the fractional D3-branes are completely transverse to the orbifold [3]. In fact, in this latter case, the resulting gauge theory has Poincaré symmetry in four dimensions but a reduced amount of supersymmetry since only a fraction of the sixteen supercharges of the orbifold background is preserved on the world-volume.

This orbifold set-up has already been studied in earlier works on the subject [4, 16, 20, 45, 46] where also fractional D(–1)-branes have been introduced on top of the fractional D3-branes to derive the so-called ramified instanton partition function in the presence of a surface operator, extending the equivariant localization methods of [47]. In this thesis, instead, we consider only stacks of fractional D3-branes and focus on the gauge theory defined on their world-volume, which has largely remained unexplored. In particular we compute correlators involving both the massless fields of the gauge theory and the massless twisted scalars in the NS/NS and R/R sectors of the closed string background, and show that these correlators precisely encode the singular profiles that define a GW defect. The continuous parameters that appear in these profiles and that are part of the defining data of a surface operator are related to the vacuum expectation values of the twisted scalars. In this way, we clarify in detail how the KT set-up realizes surface defects in the gauge theory.

3.2.1 Setup

Our primary goal is to derive the field profiles (3.1) and (3.2) as well as the topological term (3.5) that characterize a monodromy GW defect from a world-sheet analysis of its orbifold realization proposed in [4].

In this set-up, the gauge theory lives on a system of D3-branes in Type II B string theory placed in a $\mathbb{Z}_M$ orbifold space. The orbifold group acts on two complex planes $\mathbb{C}_{(2)} \times \mathbb{C}_{(3)}$, the first of which is transverse to the defect inside the world-volume of the D3-branes, while the second is transverse to the D3. In this realization, therefore, the defect D is located at the fixed point of the orbifold action. The integer partition of N, (see (3.3)), which determines the unbroken Levi subgroup (3.4) corresponds to the choice of the N-dimensional representation of $\mathbb{Z}_M$ on the Chan-Paton indices of the D3-branes; in other words, n_I is the number of the fractional branes transforming in the I-th irreducible representation of $\mathbb{Z}_M$. We shall refer to these fractional branes as D3-branes of type I.

What is missing in the KT description is how the orbifold realization encodes the continuous parameters of the monodromy defect. Our goal is to fill this gap by showing that they correspond to background values for fields belonging to the twisted sectors of the closed string theory on the orbifold. In particular, the twisted background fields in the NS/NS sector, which here we collectively denote as b, account for the parameters α_I, β_I and γ_I which appear in (3.1) and (3.2), while the twisted scalar of the R/R sector, which we denote c, accounts for the parameters η_I in the topological term (3.5).

Schematically, the mechanism goes as follows. In the presence of a closed string background certain open string fields Φ_{open} attached to a fractional D3-brane of type I acquire a non-zero one-point function, *i.e.* a tadpole. If we denote by $\mathcal{V}_{\text{open}}$ the open string vertex operator associated to Φ_{open} and by $\mathcal{V}_b$ the closed string vertex operator corresponding to b, the tadpole $\langle \mathcal{V}_{\text{open}} \rangle_{b;I}$ arises from an open/closed string correlator evaluated on a disk which contains an insertion of $b\,\mathcal{V}_b$ in the interior and of the vertex operator $\mathcal{V}_{\text{open}}$ on the boundary that lies on a D3-brane of type I:

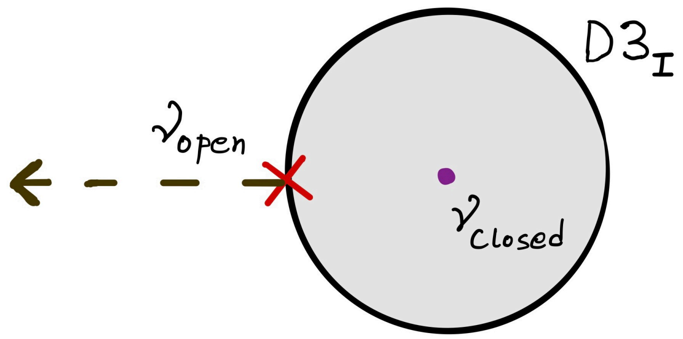

Figure 3.1: An example of a mixed open/closed string amplitude on a D3-brane of type I.

Note that the open string vertex carries momentum along the D3-brane world-volume. While its longitudinal components $\vec{k}_{\parallel}$ along the defect are set to zero by momentum conservation, its transverse components $\vec{k}_{\perp}$ need not be set to zero, as we have pictorially indicated in the diagram. Indeed, the twisted fields, which are localized at the orbifold fixed point, break translation invariance along the orbifold directions and thus $\vec{k}_{\perp}$ does not need to be conserved. Therefore, the disk diagram represented above acts as a classical source for Φ_{open}, which acquires a non-trivial profile in the plane transverse to the defect. The explicit expression of this profile near the defect is obtained by attaching a propagator to the source and taking the Fourier transform ($\mathcal{FT}$), namely

$$\Phi_{\text{open}}(\vec{x}_{\perp}) = \mathcal{FT}\Big[\, \frac{1}{\vec{k}_{\perp}^2} \langle \mathcal{V}_{\text{open}} \rangle_{b;I}(\vec{k}_{\perp}) \Big] \,. \tag{3.11}$$

The fields Φ_{open} which get a tadpole from this mechanism arise from open strings with both ends on the same D3-brane, so they have diagonal Chan-Paton factors. In the following, we will show in detail that the only non-zero tadpoles are those of the diagonal entries of the transverse components[7] $\mathbf{A}_2$ and $\bar{\mathbf{A}}_2$ of the gauge connection 1-form and the diagonal entries of the complex scalar $\mathbf{\Phi}$. These are precisely the fields which have a non-trivial profile in a monodromy defect of GW type. Moreover, we will show that the functional dependence on the transverse coordinates acquired by these fields through (3.11) coincides with that of (3.1) and (3.2), thereby identifying the parameters α_I, β_I and γ_I with some of the background fields of the twisted NS/NS sector.

The mechanism that encodes the non-trivial profile of the surface defect in a perturbative disk diagram is reminiscent of the way in which disks with mixed D3/D(−1) boundary conditions account for the classical profile of the instanton solutions [37, 48]. In that case, however, the defect is point-like and the classical profile of the fields depends on all world-volume coordinates; moreover, the role of parameters that appear in the profile is played by the D3/D(−1) open string moduli, instead of the closed string moduli, as in the present situation.

The twisted fields in the R/R sector also couple to the open string excitations through disk diagrams analogous to the one in (3.1). It turns out that the only non-zero diagrams of this type involve the diagonal entries of the longitudinal components $\mathbf{A}_1$ and $\bar{\mathbf{A}}_1$ of the gauge connection and do not depend on the transverse momentum $\vec{k}_\perp$. Thus, these diagrams are not tadpoles and do not lead to the emission of open string fields with a non-trivial profile; instead, they account for some terms of the defect effective action, and in particular, correspond to the θ-terms of (3.5). This implies that the parameters η_I arise from the twisted R/R background fields.

The description of the monodromy defects that we propose is analogous to the holographic description of defects given by [25, 26] in terms of bubbling geometries. Also in that case one gives a bulk description of the defect that accounts for all of its parameters in terms of a closed string background. The orbifold description that we will discuss in

[7] We use the complex notation $\mathbf{A} = \sum_{i=1}^{2}\left(\mathbf{A}_i d\bar{z}_i + \bar{\mathbf{A}}_i dz_i\right)$ to facilitate the comparison with (3.1).

the following is however quite different since it makes use of perturbative string theory and world-sheet conformal field theory tools.

The perturbative string theory realization of a non-trivial sector of the gauge theory that we have described bears many analogies with the explicit derivation of the gauge instanton profiles from D3/D-instanton systems [49–51] via the emission of open strings from disk diagrams with mixed boundary conditions [37]. The role of the instanton moduli is played in the construction of the surface defect by the insertion of the twisted closed string at zero momentum.

Chapter 4

Surface operators using $\mathbb{Z}_2$ orbifold

In this section, we will consider the particular case $M = 2$, corresponding to simple defects. This case allows us to illustrate all the ingredients and mechanisms involved in our proposal, while at the same time avoiding some of the more technical issues related to the general $\mathbb{Z}_M$ orbifold.

4.1 Closed strings in the $\mathbb{Z}_2$ orbifold

We consider Type II B string theory propagating in a $10d$ target space given by the orbifold

$$\mathbb{C}_{(1)} \times \frac{\mathbb{C}_{(2)} \times \mathbb{C}_{(3)}}{\mathbb{Z}_2} \times \mathbb{C}_{(4)} \times \mathbb{C}_{(5)} \,. \tag{4.1}$$

The i-th complex plane $\mathbb{C}_{(i)}$ is parametrized by the complex coordinates z_i and $\bar{z}_i$ defined as

$$z_i = \frac{x_{2i-1} + \mathrm{i}\, x_{2i}}{\sqrt{2}} \qquad \text{and} \qquad \bar{z}_i = \frac{x_{2i-1} - \mathrm{i}\, x_{2i}}{\sqrt{2}} \tag{4.2}$$

in terms of the ten real coordinates x_μ, and the non-trivial element of the $\mathbb{Z}_2$ orbifold group acts as follows

$$(z_2, z_3) \longrightarrow (-z_2, -z_3) \qquad \text{and} \qquad (\bar{z}_2, \bar{z}_3) \longrightarrow (-\bar{z}_2, -\bar{z}_3)\,. \tag{4.3}$$

This breaks the $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_+ \otimes \mathrm{SU}(2)_-$ isometry of the space $\mathbb{C}_{(2)} \times \mathbb{C}_{(3)}$ down to $\mathrm{SU}(2)_+$.

To describe closed strings in the orbifold (1.1) we use a complex notation analogous to the one in (4.2). We denote the left-moving bosonic string coordinates as $Z^i(z)$ and $\bar{Z}^i(z)$, and the right-moving ones as $\widetilde{Z}^i(\bar{z})$ and $\widetilde{\bar{Z}}^i(\bar{z})$. Here, z and $\bar{z}$ are the complex coordinates that parametrize the world-sheet of the closed strings. In a similar way, we introduce the complex world-sheet fermionic coordinates $\Psi^i(z)$ and $\overline{\Psi}^i(z)$, and their right-moving counterparts $\widetilde{\Psi}^i(\bar{z})$ and $\widetilde{\overline{\Psi}}^i(\bar{z})$. In all of our string computations we will use the convention that $2\pi\alpha' = 1$.

For the $\mathbb{Z}_2$ orbifold under consideration, the Hilbert space of the closed string, in addition to the usual untwisted sector, possesses one twisted sector, associated to the non-trivial conjugacy class of $\mathbb{Z}_2$. In the following, we are going to briefly review[1] the main properties of this twisted sector which will play a crucial role in our analysis.

4.1.1 Twisted closed string sectors

In the twisted sector the left-moving bosonic string coordinates Z^2 and Z^3 has the following anti-periodicity:

$$Z^2(\mathrm{e}^{2\pi\mathrm{i}}\, z) = -\,Z^2(z) \qquad \text{and} \qquad Z^3(\mathrm{e}^{2\pi\mathrm{i}}\, z) = -\,Z^3(z)\,. \tag{4.4}$$

[1] See for instance [3, 38, 52] for more detailed accounts of various properties of the CFT on a $\mathbb{C}^2/\Gamma$ orbifold space.

Of course, the same happens for the complex conjugate coordinates $\overline{Z}^2(z)$ and $\overline{Z}^3(z)$. The vacuum for these twisted bosonic fields is created by the operator

$$\Delta(z) = \sigma^2(z)\,\sigma^3(z)\;, \tag{4.5}$$

where $\sigma^2(z)$ and $\sigma^3(z)$ are the twist fields [5] in the complex directions 2 and 3. Each of these twist fields is a conformal operator of weight 1/8 so that $\Delta(z)$ has weight 1/4.

A completely analogous construction can be made in the right-moving sector, where one has

$$\widetilde{Z}^2(\mathrm{e}^{2\pi\mathrm{i}}\,\bar{z}) = -\,\widetilde{Z}^2(\bar{z}) \qquad \text{and} \qquad \widetilde{Z}^3(\mathrm{e}^{2\pi\mathrm{i}}\,\bar{z}) = -\,\widetilde{Z}^3(\bar{z})\;, \tag{4.6}$$

and similarly for their complex conjugates. Correspondingly, one defines the right-moving twist field $\widetilde{\Delta}(\bar{z})$ of dimension 1/4.

As far as the fermionic coordinates are concerned, one has

$$\Psi^2(\mathrm{e}^{2\pi\mathrm{i}}\,z) = \mp\,\Psi^2(z) \qquad \text{and} \qquad \Psi^3(\mathrm{e}^{2\pi\mathrm{i}}\,z) = \mp\,\Psi^3(z)\;, \tag{4.7}$$

where the upper signs refer to the NS sector and the lower ones to the R sector. The complex conjugate coordinates $\overline{\Psi}^2(z)$ and $\overline{\Psi}^3(z)$ have similar monodromy properties. In the right-moving sector, the fermionic fields are such that

$$\widetilde{\Psi}^2(\mathrm{e}^{2\pi\mathrm{i}}\bar{z}) = \mp\,\widetilde{\Psi}^2(\bar{z}) \qquad \text{and} \qquad \widetilde{\Psi}^3(\mathrm{e}^{2\pi\mathrm{i}}\bar{z}) = \mp\,\widetilde{\Psi}^3(\bar{z})\;, \tag{4.8}$$

with similar expressions for the complex conjugate coordinates $\widetilde{\overline{\Psi}}^2(\bar{z})$ and $\widetilde{\overline{\Psi}}^3(\bar{z})$.

As a consequence of these monodromy properties, in the expansion of the various fields the moding is shifted by $1/2$ with respect to their untwisted values. In particular, the bosonic fields Z^2, Z^3, $\overline{Z}^2$ and $\overline{Z}^3$ have half-integer modes, while the fermions Ψ^2, Ψ^3, $\overline{\Psi}^2$

and $\overline{\Psi}^3$ are integer moded in the NS sector, and half-integer moded in the R sector. The same is true, of course, for their right-moving counterparts.

Massless states in the NS/NS sector

Since in the NS sector the fermionic coordinates along directions 2 and 3 are periodic and possess zero-modes, the vacuum of the world-sheet theory of the fields Ψ^2 and Ψ^3 is degenerate and carries a representation of the $4d$ Clifford algebra formed by their zero-modes. With respect to the SO(4) isometry of $\mathbb{C}_{(2)} \times \mathbb{C}_{(3)}$, these zero-modes build a $4d$ Dirac spinor, which decomposes into its chiral and anti-chiral parts: $(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$. Given our choice for the embedding of the $\mathbb{Z}_2$ action into SO(4), the anti-chiral part $(\mathbf{1},\mathbf{2})$ is not invariant under the orbifold and is projected out. Therefore, we just remain with the chiral spinor $(\mathbf{2},\mathbf{1})$, whose two components are labeled by an index α. From the world-sheet point of view, this chiral spinor is created by a $4d$ chiral spin field [9, 10]

$$S^{\alpha}(z) \tag{4.9}$$

which is a conformal field of weight 1/4.

Due to the twisted boundary conditions (4.4), the bosonic coordinates Z^2 and Z^3 along the orbifold do not possess zero-modes. The momentum can only be defined in the directions Z^1, Z^4 and Z^5 that have the standard behavior. We find it convenient to use a complex notation for the momentum analogous to the one in (4.2), and thus we define

$$\kappa_i = \frac{k_{2i-1} + \mathrm{i}\, k_{2i}}{\sqrt{2}} \qquad \text{and} \qquad \bar{\kappa}_i = \frac{k_{2i-1} - \mathrm{i}\, k_{2i}}{\sqrt{2}} \tag{4.10}$$

where k is the momentum in real notation. Then, in the twisted sector, the usual plane-wave factor $:\mathrm{e}^{\mathrm{i}\,k\cdot X}:$ that appears in the vertex operators describing string exictations is written as follows

$$:\mathrm{e}^{\mathrm{i}\,\bar{\kappa}\cdot Z(z) + \mathrm{i}\,\kappa\cdot\bar{Z}(z)}: \tag{4.11}$$

where only κ_1, κ_4 and κ_5 (and their complex conjugates) are defined. The operator (4.11) is a conformal field of weight $\kappa \cdot \bar{\kappa} = \frac{1}{2}k^2$.

Finally, to describe physical vertex operators in the standard (-1)-superghost picture of the NS sector, one introduces the vertex operator

$$:\mathrm{e}^{-\phi(z)}: \tag{4.12}$$

where $\phi(z)$ is the field appearing in the bosonization formulas of the superghost system [9]. The operator (4.12) is a conformal field of weight 1/2.

We have now all ingredients to construct a vertex operator that describes a physical left-moving excitation at the massless level in the NS twisted sector. This is obtained by taking the product of the twist field (4.5), the spin field (4.9), the plane-wave factor (4.11) and the superghost term (4.12). In this way, we obtain [2]

$$\mathcal{V}^{\alpha}(z) = \varDelta(z)\, S^{\alpha}(z)\, \mathrm{e}^{-\phi(z)}\, \mathrm{e}^{\mathrm{i}\,\bar{\kappa}\cdot Z(z) + \mathrm{i}\,\kappa\cdot\bar{Z}(z)}\,, \tag{4.13}$$

which is a conformal field of weight 1 if $\kappa \cdot \bar{\kappa} = \frac{1}{2}k^2 = 0$. In the following, we will consider the closed strings as providing a constant background for the gauge theory, and thus in these vertex operators we will set the momentum to zero. We also observe that the vertices (4.13) are preserved by the GSO projection of the NS sector. Indeed, the sum of the spinor weights minus the superghost charge is an even integer.

Exploiting the conformal properties of the various factors, it is easy to check that [3]

$$\langle \mathcal{V}^{\alpha}(z)\, \mathcal{V}^{\beta}(z')\rangle = \frac{(\epsilon^{-1})^{\alpha\beta}}{(z-z')^2}\,, \tag{4.14}$$

[2] For simplicity, from now on in all vertex operators we will suppress the : : notation, but the normal ordering will be always present.

[3] Here and in the following, we understand the δ-function enforcing momentum conservation.

where

$$\epsilon = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \tag{4.15}$$

is the chiral part of the charge conjugation matrix $\widehat{C}$ in four dimensions (see Appendix A.0.1 for details and our conventions).

The same construction goes through in the right-moving sector, where one finds the vertex operators

$$\widetilde{\mathcal{V}}^{\alpha}(\bar{z}) = \widetilde{\Delta}(\bar{z})\, \widetilde{S}^{\alpha}(\bar{z})\, \mathrm{e}^{-\widetilde{\phi}(\bar{z})}\, \mathrm{e}^{\mathrm{i}\, \bar{\kappa}\cdot \widetilde{Z}(\bar{z}) + \mathrm{i}\, \kappa\cdot \widetilde{\bar{Z}}(\bar{z})} \tag{4.16}$$

which have the same form of the two-point function as in (4.14) but with anti-holomorphic coordinate dependence.

Overall, the massless spectrum in the twisted NS/NS sector contains four states described by the vertices $\mathcal{V}^{\alpha}(z)\, \widetilde{\mathcal{V}}^{\beta}(\bar{z})$ in the $(-1,-1)$-superghost picture. The four independent components can be decomposed into a real scalar b and a triplet b_c (with $c = 1,2,3$), transforming, respectively, in the $(\mathbf{1},\mathbf{1})$ and $(\mathbf{3},\mathbf{1})$ representations of SO(4). They correspond to the following vertex operators:

$$\begin{aligned} b \quad &\longleftrightarrow \quad \mathcal{V}_b(z,\bar{z}) = \mathrm{i}\, \epsilon_{\alpha\beta}\, \mathcal{V}^{\alpha}(z)\, \widetilde{\mathcal{V}}^{\beta}(\bar{z})\ , \\ b_c \quad &\longleftrightarrow \quad \mathcal{V}_{b_c}(z,\bar{z}) = (\epsilon\, \tau_c)_{\alpha\beta}\, \mathcal{V}^{\alpha}(z)\, \widetilde{\mathcal{V}}^{\beta}(\bar{z})\ , \end{aligned} \tag{4.17}$$

where τ_c are the usual Pauli matrices.

Massless states in the R/R sector

In the twisted R sector, the bosonic coordinates in the complex directions 2 and 3 have, of course, the same monodromy properties as in the NS sector, whereas the corresponding fermionic coordinates Ψ^2, Ψ^3 and their complex conjugates are anti-periodic. This means that in those directions the world-sheet vacuum is non-degenerate. On the contrary, the fermionic fields Ψ^1, Ψ^4 and Ψ^5 and their complex conjugates are periodic as usual in the

R sector and possess zero-modes. Therefore the world-sheet vacuum in this twisted sector is degenerate and carries a representation of the $6d$ Clifford algebra generated by the zero modes of the periodic fermions. These form a Dirac spinor of SO(6) which decomposes into a chiral part, transforming in the **4** of SO(6), plus an anti-chiral part transforming in the $\bar{\mathbf{4}}$.

From the world-sheet point of view, the chiral spinor is created by a $6d$ chiral spin field [9, 10]

$$S^A(z)\,, \tag{4.18}$$

with A taking four values. Likewise, the anti-chiral spinor corresponds to the $6d$ anti-chiral spin field

$$S^{\dot{A}}(z)\,, \tag{4.19}$$

where also the dotted index $\dot{A}$ takes four values. Both S^A and $S^{\dot{A}}$ are conformal fields of weight 3/8.

In the R sector there are two standard choices for the superghost picture: the $(-\frac{1}{2})$-picture and the $(-\frac{3}{2})$-picture, created respectively by the operators

$$\mathrm{e}^{-\frac{1}{2}\phi(z)} \qquad \text{and} \qquad \mathrm{e}^{-\frac{3}{2}\phi(z)}\,, \tag{4.20}$$

which are both conformal fields of weight 3/8.

The GSO projection selects the combinations $S^A(z)\,\mathrm{e}^{-\frac{1}{2}\phi(z)}$ and $S^{\dot{A}}(z)\,\mathrm{e}^{-\frac{3}{2}\phi(z)}$, for which the sum of the spinor weights minus the superghost charge is an even integer. Then, using these ingredients we can build the following vertex operators

$$\mathcal{V}^A(z) = \varDelta(z)\, S^A(z)\, \mathrm{e}^{-\frac{1}{2}\phi(z)}\, \mathrm{e}^{\mathrm{i}\,\bar{\kappa}\cdot Z(z)+\mathrm{i}\,\kappa\cdot\bar{Z}(z)}\,, \tag{4.21a}$$

$$\mathcal{V}^{\dot{A}}(z) = \varDelta(z)\, S^{\dot{A}}(z)\, \mathrm{e}^{-\frac{3}{2}\phi(z)}\, \mathrm{e}^{\mathrm{i}\,\bar{\kappa}\cdot Z(z)+\mathrm{i}\,\kappa\cdot\bar{Z}(z)}\,, \tag{4.21b}$$

which are conformal fields of dimension 1 if $\kappa\cdot\bar{\kappa} = \frac{1}{2}k^2 = 0$. From the conformal properties of the various components, it is easy to check that

$$\langle \mathcal{V}^{A}(z)\,\mathcal{V}^{\dot{B}}(z')\rangle = \frac{(C^{-1})^{A\dot{B}}}{(z-z')^{2}}\,, \tag{4.22}$$

where C is the charge conjugation matrix of the spinor representations of SO(6) (see Appendix A.0.2).

The same construction goes on in the right-moving sector, where one finds the vertex operators

$$\widetilde{\mathcal{V}}^{A}(\bar{z}) = \widetilde{\varDelta}(\bar{z})\,\widetilde{S}^{A}(\bar{z})\,\mathrm{e}^{-\frac{1}{2}\widetilde{\phi}(\bar{z})}\,\mathrm{e}^{\mathrm{i}\,\bar{\kappa}\cdot\widetilde{Z}(\bar{z})+\mathrm{i}\,\kappa\cdot\widetilde{\bar{Z}}(\bar{z})}\,, \tag{4.23a}$$

$$\widetilde{\mathcal{V}}^{\dot{A}}(\bar{z}) = \widetilde{\varDelta}(\bar{z})\,\widetilde{S}^{\dot{A}}(\bar{z})\,\mathrm{e}^{-\frac{3}{2}\widetilde{\phi}(\bar{z})}\,\mathrm{e}^{\mathrm{i}\,\bar{\kappa}\cdot\widetilde{Z}(\bar{z})+\mathrm{i}\,\kappa\cdot\widetilde{\bar{Z}}(\bar{z})}\,, \tag{4.23b}$$

which have the same two-point function as in (4.22).

Using the vertex operators (4.21) and (4.23) we can study the massless spectrum of the twisted R/R sector. In the asymmetric $(-\frac{1}{2},-\frac{3}{2})$-superghost picture the vertex operators $\mathcal{V}^{A}(z)\,\widetilde{\mathcal{V}}^{\dot{B}}(\bar{z})$ describe R/R potentials [4] which have sixteen independent components. These can be decomposed into a scalar c and a 2-index anti-symmetric tensor c_{MN} of SO(6) that correspond to the vertex operators

$$\begin{aligned} c \quad &\longleftrightarrow \quad \mathcal{V}_{c}(z,\bar{z}) = C_{A\dot{B}}\,\mathcal{V}^{A}(z)\,\widetilde{\mathcal{V}}^{\dot{B}}(\bar{z})\,, \\ c_{MN} \quad &\longleftrightarrow \quad \mathcal{V}_{c_{MN}}(z,\bar{z}) = (C\,\varGamma_{MN})_{A\dot{B}}\,\mathcal{V}^{A}(z)\,\widetilde{\mathcal{V}}^{\dot{B}}(\bar{z})\,, \end{aligned} \tag{4.24}$$

where $\varGamma_{MN} = \frac{1}{2}[\varGamma_{M},\varGamma_{N}]$, with $\varGamma_{M}$ being the Dirac matrices of SO(6) (see Appendix A.0.2).

[4]As shown in [53] the full BRST invariant vertex operators describing the R/R potentials in the asymmetric supeghost picture are actually a sum of infinite terms with multiple insertions of superghost zero-modes. Here we only consider the first one of these terms, since all the others decouple from the physical amplitudes we will consider and thus can be neglected for our present purposes.

4.2 Fractional D3-branes in the $\mathbb{Z}_2$ orbifold

We engineer the $4d$ gauge theory supporting the surface defect employing fractional D3-branes in the $\mathbb{Z}_2$ orbifold background (1.1). Differently from the case usually considered in the literature [3] in which the fractional D3-branes are entirely transverse to the orbifold, we take fractional D3-branes whose world-volume extends partially along the orbifold. In particular, using the notation introduced in the previous section, we consider D3-branes that extends along the complex directions 1 and 2, and are transverse to the complex directions 3, 4 and 5. Thus, the $\mathbb{Z}_2$ orbifold acts on one complex longitudinal and one complex transverse direction. This fact has two important consequences: firstly, on the D3-brane world-volume, one finds the same content of massless fields as in $\mathcal{N} = 4$ super Yang-Mills theory; secondly, since the orbifold acts only on one of the two complex directions of the world-volume, a $2d$ surface defect is naturally introduced in the gauge theory. Our goal is to show that this defect is precisely a GW monodromy defect.

To do so we first clarify the properties of the fractional D3-branes in the $\mathbb{Z}_2$ orbifold from the closed string point of view, using the boundary state formalism [5], and then from the open string point of view by analyzing the world-volume massless fields.

4.2.1 Boundary states

In a $\mathbb{Z}_2$ orbifold there are two types of fractional D-branes that correspond to the two irreducible representations of the orbifold group. We label these two types of D-branes by an index $I = 0, 1$. The D-branes with $I = 0$ carry the trivial representation in which the $\mathbb{Z}_2$ element g is represented by $+1$, while the D-branes with $I = 1$ carry the other representation in which g is represented by -1. The two types of fractional branes therefore only differ by a sign in front of the twisted sectors. With this in mind, the fractional D3-branes

[5] For a review on the boundary state formalism, see for example [6, 7].

can be represented in the boundary state formalism in the following schematic way [6–8]:

$$|\mathrm{D3}; I\rangle = \mathcal{N}\,|\mathrm{U}\rangle + \mathcal{N}'\,|\mathrm{T}; I\rangle \quad \text{with} \quad |\mathrm{T}; I\rangle = (-1)^I\,|\mathrm{T}\rangle\,. \tag{4.25}$$

Here $\mathcal{N}$ and $\mathcal{N}'$ are dimensionful normalization factors related to the brane tension, and $|\mathrm{U}\rangle$ and $|\mathrm{T}\rangle$ are the untwisted and twisted Ishibashi states that enforce the identification between the left and right moving modes in the untwisted and twisted sectors, respectively. For our purposes, we do not need to write the explicit expressions of these quantities which can be obtained by factorizing the 1-loop open-string partition function in the closed string channel, and thus we refer to the original literature and in particular to [8], where also the case of D3-branes partially extending along the orbifold has been considered. We do give a very brief review of the boundary state description for fractional branes in Appendix C. However, for clarity, we recall the essential information that will be needed in the following, namely that both $|\mathrm{U}\rangle$ and $|\mathrm{T}\rangle$ have a component in the NS/NS sector and a component in the R/R sector and that, after GSO projection, the twisted part of the boundary state is

$$|\mathrm{T}\rangle = |\mathrm{T}\rangle_{\mathrm{NS}} + |\mathrm{T}\rangle_{\mathrm{R}} \tag{4.26}$$

with

$$|\mathrm{T}\rangle_{\mathrm{NS}} = (\widehat{C}\gamma_3\gamma_4)_{\alpha\beta}\,|\alpha\rangle\,|\widetilde{\beta}\rangle + \cdots\,, \tag{4.27a}$$

$$|\mathrm{T}\rangle_{\mathrm{R}} = (C\Gamma_1\Gamma_2)_{A\dot{B}}\,|A\rangle\,|\widetilde{\dot{B}}\rangle + \cdots\,. \tag{4.27b}$$

Here the kets represent the ground states created by acting on the untwisted vacuum with the vertex operators (4.13), (4.16) of the NS/NS twisted sector and with the vertex operators (4.21a) and (4.23b) of the R/R twisted sector, namely

$$|\alpha\rangle = \lim_{z\to 0}\mathcal{V}^{\alpha}(z)\,|0\rangle\,, \quad |\widetilde{\beta}\rangle = \lim_{\bar{z}\to 0}\widetilde{\mathcal{V}}^{\beta}(\bar{z})\,|\widetilde{0}\rangle\,, \tag{4.28a}$$

$$|A\rangle = \lim_{z\to 0}\mathcal{V}^{A}(z)\,|0\rangle\,, \quad |\widetilde{\dot{B}}\rangle = \lim_{\bar{z}\to 0}\widetilde{\mathcal{V}}^{\dot{B}}(\bar{z})\,|\widetilde{0}\rangle\,. \tag{4.28b}$$

In (4.27) the ellipses stand for terms involving higher excited states which will not play any role in our analysis. We remark that the coefficient $(\widehat{C}\gamma_3\gamma_4)_{\alpha\beta}$ in the NS/NS component (4.27a) is the appropriate one for our D3-branes since in the NS/NS twisted sector the ground states are spinors of the $4d$ space spanned by the real coordinates x_3, x_4, x_5 and x_6, of which only the directions x_3 and x_4 are longitudinal to the D3-brane world-volume. Therefore the product of the SO(4) γ-matrices $\gamma_3\gamma_4$ must appear in the prefactor. Notice that the GSO projection only selects the chiral block of the matrix $\widehat{C}\gamma_3\gamma_4$, as it is indicated by the undotted indices. Likewise, in the R/R component (4.27b) the coefficient $(C\Gamma_1\Gamma_2)_{A\dot{B}}$ is due to the fact that in R/R twisted sector the ground states are spinors in the $6d$ space in which the real coordinates x_1 and x_2 belong to the D3-brane world-volume while the real coordinates x_7, x_8, x_9 and x_{10} are transverse. This explains why the product of the SO(6) Γ-matrices $\Gamma_1\Gamma_2$ appears in the prefactor. Again the GSO projection selects only the chiral/anti-chiral block of the matrix $C\Gamma_1\Gamma_2$, as indicated by the pair of undotted/dotted indices.

Reflection rules

When the world-sheet of the closed string has a boundary, there are non-trivial 2-point functions between the left and right moving parts. We are interested in computing these 2-point functions for the massless fields of the twisted sectors when the boundary is created by a fractional D3-brane of type I in the $\mathbb{Z}_2$ orbifold discussed in Section 4.2.

In the boundary state formalism (see for instance the reviews [6, 7]) the boundary created by a D-brane is the unit circle, *i.e.* the set of points corresponding to the world-sheet time $\tau = 0$ where the boundary state is inserted. The points inside the unit circle define the disk $\mathbb{D}$. When we insert a closed string inside $\mathbb{D}$, the left and right moving modes are reflected at the boundary, and a non-vanishing correlator between them arises. For example, considering the twisted NS/NS sector and in particular the massless states described by the vertex operators (4.17) in the presence of a fractional D3-brane of type I, we have

$$\begin{aligned} \langle \mathcal{V}^\alpha(w)\,\widetilde{\mathcal{V}}^\beta(\bar{w})\rangle_I &= \langle T; I|\,\mathcal{V}^\alpha(w)\,\widetilde{\mathcal{V}}^\beta(\bar{w})\,|0\rangle\widetilde{|0\rangle} \\ &= (-1)^I\ {}_{\mathrm{NS}}\langle T|\,\mathcal{V}^\alpha(w)\,\widetilde{\mathcal{V}}^\beta(\bar{w})\,|0\rangle\widetilde{|0\rangle} \end{aligned} \tag{4.29}$$

for w and $\bar{w} \in \mathbb{D}$. Here we have used the boundary state to represent the fractional D3-brane of type I (see (4.25)) and taken into account that only the NS component of its twisted part is relevant for the calculation. As in the main text, $|0\rangle$ and $\widetilde{|0\rangle}$ denote the left and right vacua.

On the other hand, conformal invariance implies that the disk 2-point function of $\mathcal{V}^\alpha$ and $\widetilde{\mathcal{V}}^\beta$, which are conformal fields of weight 1, has the following form

$$\langle \mathcal{V}^\alpha(w)\,\widetilde{\mathcal{V}}^\beta(\bar{w})\rangle_I = \frac{M_I^{\alpha\beta}}{(1-w\bar{w})^2} \tag{4.30}$$

where $M_I^{\alpha\beta}$ is a constant to be determined. Combining (4.29) and (4.30), we easily see that

$$M_I^{\alpha\beta} = \lim_{w\to 0}\,\lim_{\bar{w}\to 0}\,\langle \mathcal{V}^\alpha(w)\,\widetilde{\mathcal{V}}^\beta(\bar{w})\rangle_I = (-1)^I\ {}_{\mathrm{NS}}\langle T|\alpha\rangle\widetilde{|\beta\rangle} \tag{4.31}$$

where $|\alpha\rangle$ and $\widetilde{|\beta\rangle}$ are the left and right ground states defined in (4.28a). Thus, the disk 2-point function (4.30) becomes

$$\langle \mathcal{V}^\alpha(w)\,\widetilde{\mathcal{V}}^\beta(\bar{w})\rangle_I = (-1)^I\,\frac{{}_{\mathrm{NS}}\langle T|\alpha\rangle\widetilde{|\beta\rangle}}{(1-w\bar{w})^2}\,. \tag{4.32}$$

Let us now map this result to the complex plane by means of the Cayley map

$$w = \frac{z-\mathrm{i}}{z+\mathrm{i}}\,, \qquad \bar{w} = \frac{\bar{z}+\mathrm{i}}{\bar{z}-\mathrm{i}}\,. \tag{4.33}$$

Notice that w is mapped to the upper half-complex plane and $\bar{w}$ to the lower half. Then, we have

$$\langle \mathcal{V}^{\alpha}(z)\, \widetilde{\mathcal{V}}^{\beta}(\bar{z})\rangle_I = \langle \mathcal{V}^{\alpha}(w)\, \widetilde{\mathcal{V}}^{\beta}(\bar{w})\rangle_I \, \frac{dw}{dz}\, \frac{d\bar{w}}{d\bar{z}} = (-1)^{I+1}\, \frac{{}_{\mathrm{NS}}\langle T|\alpha\rangle|\widetilde{\beta}\rangle}{(z-\bar{z})^2}\ . \tag{4.34}$$

Comparing with (4.14) and using the so-called doubling trick, we are led to introduce the following reflection rule

$$\widetilde{\mathcal{V}}^{\beta}(\bar{z}) \longrightarrow (R_I)^{\beta}{}_{\gamma}\, \mathcal{V}^{\gamma}(\bar{z}) \tag{4.35}$$

with

$$(R_I)^{\beta}{}_{\gamma} = (-1)^I\, \widehat{C}_{\gamma\alpha}\, {}_{\mathrm{NS}}\langle T|\alpha\rangle|\widetilde{\beta}\rangle = (-1)^{I+1}\, \varepsilon_{\gamma\alpha}\, {}_{\mathrm{NS}}\langle T|\alpha\rangle|\widetilde{\beta}\rangle \tag{4.36}$$

where in the second step we have used the fact that the chiral part of the charge conjugation matrix is ϵ (see A.5)). Using the expression (4.27a) for the twisted boundary state in the NS/NS sector, it is easy to show that

$${}_{\mathrm{NS}}\langle T|\alpha\rangle|\widetilde{\beta}\rangle = (\gamma_4\gamma_3\widehat{C}^{-1})^{\beta\alpha}\ . \tag{4.37}$$

Inserting this into (4.36), we find

$$(R_I)^{\beta}{}_{\gamma} = (-1)^I (\gamma_4\gamma_3)^{\beta}{}_{\gamma} \tag{4.38}$$

in agreement with (4.44a) of the main text.

The reflection matrix for the R sector can be obtained in the same way. Indeed, in the presence of a D3-brane of type I the left and right-moving vertex operators $\mathcal{V}^A$ and $\widetilde{\mathcal{V}}^{\dot{B}}$ have the following 2-point function

$$\langle \mathcal{V}^{A}(z)\, \widetilde{\mathcal{V}}^{\dot{B}}(\bar{z})\rangle_I = (-1)^{I+1}\, \frac{{}_{\mathrm{R}}\langle T|A\rangle|\widetilde{\dot{B}}\rangle}{(z-\bar{z})^2}\ . \tag{4.39}$$

Comparing with (4.22), we are led to introduce the reflection rule

$$\widetilde{\mathcal{V}}^{\dot{B}}(\bar{z}) \longrightarrow (R_I)^{\dot{B}}{}_{\dot{C}}\, \mathcal{V}^{\dot{C}}(\bar{z}) \tag{4.40}$$

such that

$$(R_I)^{\dot{B}}{}_{\dot{C}} = (-1)^{I+1}\, C_{\dot{C}A}\, {}_{\mathrm{R}}\langle T|A\rangle|\widetilde{\dot{B}}\rangle\,. \tag{4.41}$$

From the expression (4.27b) for the twisted boundary state in the R/R sector, one can show that

$$_{\mathrm{R}}\langle T|A\rangle|\widetilde{\dot{B}}\rangle = (\Gamma_2\Gamma_1 C^{-1})^{\dot{B}A}\,. \tag{4.42}$$

Inserting this into (4.41), we therefore find

$$(R_I)^{\dot{B}}{}_{\dot{C}} = (-1)^I (\Gamma_1\Gamma_2)^{\dot{B}}{}_{\dot{C}} \tag{4.43}$$

in agreement with (4.44b) of the main text.

The boundary state $|D3; I\rangle$ introduces a boundary on the closed string world-sheet along which the left and right moving modes are identified. From (4.27) one can derive that the right moving parts of the twisted closed string vertex operators are reflected on a boundary of type I with the following rules

$$\widetilde{\mathcal{V}}^{\alpha}(\bar{z}) \;\longrightarrow\; (-1)^I (\gamma_4\gamma_3)^{\alpha}{}_{\beta}\, \mathcal{V}^{\beta}(\bar{z})\,, \tag{4.44a}$$

$$\widetilde{\mathcal{V}}^{\dot{A}}(\bar{z}) \;\longrightarrow\; (-1)^I (\Gamma_1\Gamma_2)^{\dot{A}}{}_{\dot{B}}\, \mathcal{V}^{\dot{B}}(\bar{z})\,. \tag{4.44b}$$

These reflection rules will be important in computing closed string amplitudes involving twisted fields in the presence of the fractional D3-branes.

4.2.2 The open string spectrum

We now analyze the spectrum of the massless excitations defined on the world-volume of the fractional D3-branes. For definiteness, we take a fractional D3-brane of type 0, but of course completely similar considerations apply to a D3-brane of type 1. Since the world-volume extends in the first two complex directions and the orbifold acts on the second one, it is convenient, as remarked at the start of Section 3.1, to distinguish the directions that are along and transverse to the orbifold. We will label the longitudinal variables (momentum, coordinates, and so on) by a subscript ||, which involves the components along the first complex direction. We will similarly use the subscript $\perp$ to label the components along the second complex direction. The reason for these labels is that the first complex direction is longitudinal to the surface defect that the D3-branes realize, while the second direction is transverse to it. In particular, using the complex notation introduced in Section 4.1, we define the combinations

$$\begin{aligned} \kappa_{\|}\cdot Z_{\|} &= \kappa_1\,\overline{Z}^1 + \bar{\kappa}_1\,Z^1\ , \\ \kappa_{\perp}\cdot Z_{\perp} &= \kappa_2\,\overline{Z}^2 + \bar{\kappa}_2\,Z^2\ , \end{aligned} \tag{4.45}$$

and

$$\begin{aligned} \kappa_{\|}\cdot \Psi_{\|} &= \kappa_1\,\overline{\Psi}^1 + \bar{\kappa}_1\,\Psi^1\ , \\ \kappa_{\perp}\cdot \Psi_{\perp} &= \kappa_2\,\overline{\Psi}^2 + \bar{\kappa}_2\,\Psi^2\ . \end{aligned} \tag{4.46}$$

Notice that under the $\mathbb{Z}_2$ orbifold parity, $\kappa_{\|}\cdot Z_{\|}$ and $\kappa_{\|}\cdot\Psi_{\|}$ are even, while $\kappa_{\perp}\cdot Z_{\perp}$ and $\kappa_{\perp}\cdot\Psi_{\perp}$ are odd.

Let us consider the bosonic NS sector. In the familiar case when the D3-branes are completely transverse to the orbifold, the gauge vector field A_μ is typically represented in the (0)-superghost picture by the standard vertex operator

$$(\mathrm{i}\,\partial X^\mu + k\cdot\psi\,\psi^\mu)\,\mathrm{e}^{\mathrm{i}\,k\cdot X}\ . \tag{4.47}$$

In our case things are different. First of all, in the plane wave factor $\mathrm{e}^{\mathrm{i}\,k\cdot X}$ we have to distinguish the parallel and perpendicular parts which behave differently under the orbifold, and thus we are naturally led to consider the following structures

$$
\begin{aligned}
&\cos(\kappa_\perp\cdot Z_\perp)\,\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel}\ ,\\
\mathrm{i}\ &\sin(\kappa_\perp\cdot Z_\perp)\,\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel}\ ,
\end{aligned}
\tag{4.48}
$$

which are respectively even and odd under $\mathbb{Z}_2$. Therefore, they can be combined with other even and odd structures to make invariant vertex operators selected by the orbifold projection. Similarly, also the $k\cdot\psi$ combination appearing in (4.47) has to be split into a parallel and a perpendicular component.

Applying these considerations, it is not difficult to realize that the gauge field A_1 along the parallel directions is described by the following vertex operator in the (0)-superghost picture [6]

$$
A_1 \ \longrightarrow\ \mathcal{V}_{A_1} = \Big[(\mathrm{i}\,\partial Z^1 + \kappa_\parallel\cdot\Psi_\parallel\,\Psi^1)\cos(\kappa_\perp\cdot Z_\perp) + \mathrm{i}\,\kappa_\perp\cdot\Psi_\perp\,\Psi^1\ \sin(\kappa_\perp\cdot Z_\perp)\Big]\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel}\ . \tag{4.49}
$$

Each term in this expression is invariant under $\mathbb{Z}_2$. For instance, the terms $\mathrm{i}\,\partial Z^1$ or $\kappa_\parallel{\cdot}\Psi_\parallel\,\Psi_1$, which are $\mathbb{Z}_2$-even, are multiplied with the cosine combination $\cos(\kappa_\perp Z_\perp)$ which is also even, so that the product is invariant under the orbifold action. Similarly, the odd term $\kappa_\perp\cdot\Psi_\perp\,\Psi^1$ is multiplied by the sine combination $\sin(\kappa_\perp\cdot Z_\perp)$, which is also odd, to make an even expression under $\mathbb{Z}_2$. Furthermore, it is easy to check that $\mathcal{V}_{A_1}$ is a conformal field of weight 1 if $\kappa\cdot\bar\kappa = \frac{1}{2}k^2 = 0$. The vertex operator for the complex conjugate component $\overline{A}_1$ of the gauge field is simply obtained by replacing ∂Z^1 with $\partial\overline{Z}^1$ and Ψ^1 with $\overline{\Psi}^1$ in the above expression.

The gauge field A_2 along the second complex direction of the D3-brane world-volume is

[6] We consider the (0)-superghost picture since it is the relevant one for the applications discussed in Section 4.3, but of course our analysis can be done also in any other superghost picture of the NS sector.

instead described by the following vertex operator

$$A_2 \;\longrightarrow\; \mathcal{V}_{A_2} = \Big[(\mathrm{i}\,\partial Z^2 + \kappa_{\parallel}\cdot\Psi_{\parallel}\,\Psi^2)\,\mathrm{i}\,\sin(\kappa_{\perp}\cdot Z_{\perp}) + \kappa_{\perp}\cdot\Psi_{\perp}\,\Psi^2\,\cos(\kappa_{\perp}\cdot Z_{\perp})\Big]\mathrm{e}^{\mathrm{i}\,\kappa_{\parallel}\cdot Z_{\parallel}}\,. \quad (4.50)$$

Notice that the position of the cosine and sine combinations is different with respect to (5.95), but this is precisely what is needed to obtain an invariant vertex in this case. Again this vertex is a conformal field of weight 1 if the field is massless. The operator describing the complex conjugate component $\bar{A}_2$ is obtained by replacing ∂Z^2 with $\partial\overline{Z}^2$ and Ψ^2 with $\overline{\Psi}^2$ in (5.96).

Let us now consider the massless scalar fields. Without the orbifold, on the D3-brane world-volume there are three complex scalars that together with the gauge vector provide the bosonic content of the $\mathcal{N} = 4$ vector multiplet. When the orbifold acts entirely in the transverse directions, only one of these scalars remains in the invariant spectrum, thus reducing the supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 2$. In our case, instead, when the orbifold acts partially along the world-volume, all three complex scalars remain. Denoting them by Φ and Φ_r with $r = 4, 5$, they are described by the following three $\mathbb{Z}_2$-invariant vertices

$$\Phi \;\longrightarrow\; \mathcal{V}_{\Phi} = \Big[(\mathrm{i}\,\partial Z^3 + \kappa_{\parallel}\cdot\Psi_{\parallel}\,\Psi^3)\,\mathrm{i}\,\sin(\kappa_{\perp}\cdot Z_{\perp}) + \kappa_{\perp}\cdot\Psi_{\perp}\,\Psi^3\,\cos(\kappa_{\perp}\cdot Z_{\perp})\Big]\mathrm{e}^{\mathrm{i}\,\kappa_{\parallel}\cdot Z_{\parallel}}\,. \quad (4.51)$$

and

$$\Phi_r \;\longrightarrow\; \mathcal{V}_{\Phi_r} = \Big[(\mathrm{i}\,\partial Z^r + \kappa_{\parallel}\cdot\Psi_{\parallel}\,\Psi^r)\cos(\kappa_{\perp}\cdot Z_{\perp}) + \mathrm{i}\,\kappa_{\perp}\cdot\Psi_{\perp}\,\Psi^r\,\sin(\kappa_{\perp}\cdot Z_{\perp})\Big]\mathrm{e}^{\mathrm{i}\,\kappa_{\parallel}\cdot Z_{\parallel}}\,. \quad (4.52)$$

Since the scalars are massless, these vertices are conformal operators of weight 1.

A similar analysis can be repeated also for the fermionic R sector, where one can find sixteen massless fermions that are the supersymmetric partners of the bosonic fields listed above.

In conclusion, we see that when the fractional D3-branes extend partially along the orbifold, the latter does not project the open string spectrum by removing some excitations, as it does when the fractional D3-branes are totally transverse, but instead it reorganizes the fields in such a way that they behave differently along the $\|$ and $\perp$ subspaces into which the $4d$ world-volume of the D3-branes is divided. Another piece of evidence for the defect interpretation is the 1-loop open string partition function [8], which receives contributions both from modes that propagate in all four dimensions of the world-volume and also from modes that propagate only in the $\|$ subspace. This is precisely what one expects for a surface defect in the $4d$ gauge theory, extended along the $\|$ subspace. Moreover, if we consider a system made of n_0 fractional D3-branes of type 0 and n_1 fractional D3-branes of type 1, we engineer a $4d$ theory with gauge group $\mathrm{U}(n_0 + n_1)$ broken to the Levi group $\mathrm{U}(n_0)\times\mathrm{U}(n_1)$ at the orbifold fixed plane. In the case of special unitary groups, the overall U(1) factor has to be removed.

4.3 Open/closed correlators

Our next step is to show that there are non-vanishing interactions between the twisted closed string sectors discussed in Section 4.1 and the open string fields introduced in the previous section. In particular, we will show that there are non-vanishing amplitudes corresponding to the diagram represented in (3.1). The reason why such open/closed amplitudes exist is that a D-brane inserts a boundary in the closed string world-sheet along which the left- and right-moving modes are identified. Thus, the two components of the closed string vertex operators effectively behave as two open string vertices which can have a non-vanishing interaction with a third open string vertex operator describing an excitation of the gauge theory on the brane word-volume. In the following, we are going to systematically compute these open/closed string amplitudes, starting from the twisted NS/NS sector.

4.3.1 Correlators with NS/NS twisted fields

As we discussed in the Section 5.1.3, in the twisted NS/NS sector the fermionic fields in the $4d$ space where the $\mathbb{Z}_2$ orbifold acts have zero modes that build a spinor representation of SO(4). A fractional D3-brane that partially extends along the orbifold breaks this SO(4) into SO(2)×SO(2). In this breaking, the singlet b remains, while the triplet $b_c \in (\mathbf{3}, \mathbf{1})$ decomposes into a scalar b' and a doublet $b_\pm$ of complex conjugate fields. The vertex operators corresponding to these four fields can be read from (4.17), which we rewrite here for convenience

$$b \quad \longleftrightarrow \quad \mathcal{V}_b(z,\bar{z}) = \mathrm{i}\, \epsilon_{\alpha\beta}\, \mathcal{V}^\alpha(z)\, \widetilde{V}^\beta(\bar{z})\,, \tag{4.53a}$$

$$b' \quad \longleftrightarrow \quad \mathcal{V}_{b'}(z,\bar{z}) = (\epsilon\tau_3)_{\alpha\beta}\, \mathcal{V}^\alpha(z)\, \widetilde{V}^\beta(\bar{z})\,, \tag{4.53b}$$

$$b_\pm \quad \longleftrightarrow \quad \mathcal{V}_{b_\pm}(z,\bar{z}) = (\epsilon\tau_\pm)_{\alpha\beta}\, \mathcal{V}^\alpha(z)\, \widetilde{V}^\beta(\bar{z}) \tag{4.53c}$$

where $\tau_\pm = (\tau_1 \pm \mathrm{i}\,\tau_2)/2$. Since we are going to regard the closed string fields as a background for the open string excitations, in all vertices (4.53) we set the momentum to zero.

Correlators with b

We begin by evaluating the couplings of the massless open string fields of a fractional D3-brane of type I with the scalar b. These are given by

$$\langle \mathcal{V}_{\text{open}} \rangle_{b;I} = b \int \frac{dz\, d\bar{z}\, dx}{dV_{\text{proj}}}\, \langle \mathcal{V}_b(z,\bar{z})\, \mathcal{V}_{\text{open}}(x) \rangle_I \tag{4.54}$$

where $\mathcal{V}_{\text{open}}$ stands for any of the vertex operators described in Section 5.2.2 and

$$dV_{\text{proj}} = \frac{dz\, d\bar{z}\, dx}{(z-\bar{z})(\bar{z}-x)(x-z)} \tag{4.55}$$

is the projective invariant volume element. In (4.54) the integrals are performed on the string word-sheet. In particular z and $\bar{z}$, where the close string vertex operator is inserted, are points in the upper and lower half complex plane, respectively, while x is a point on the real axis from which the open string is emitted. The integrand of (4.54) is

$$\begin{aligned}\big\langle \mathcal{V}_b(z,\bar{z})\,\mathcal{V}_{\text{open}}(x)\big\rangle_I &= \mathrm{i}\,\epsilon_{\alpha\beta}\,\big\langle \mathcal{V}^{\alpha}(z)\,\widetilde{\mathcal{V}}^{\beta}(\bar{z})\,\mathcal{V}_{\text{open}}(x)\big\rangle_I \\ &= (-1)^I\,\mathrm{i}\,\epsilon_{\alpha\beta}(\gamma_4\gamma_3)^{\beta}{}_{\gamma}\,\big\langle \mathcal{V}^{\alpha}(z)\,\mathcal{V}^{\gamma}(\bar{z})\,\mathcal{V}_{\text{open}}(x)\big\rangle\end{aligned} \tag{4.56}$$

where the second line follows from the reflection rules (4.44a). Our task is therefore to compute the three-point functions $\langle \mathcal{V}^{\alpha}(z)\,\mathcal{V}^{\gamma}(\bar{z})\,\mathcal{V}_{\text{open}}(x)\rangle$ for the various open string fields.

Let us start with the components of the gauge field that are longitudinal to the defect. These are described by the vertex operator (5.95). Factorizing the resulting amplitude in a product of correlation functions for the independent conformal fields, we find

$$\begin{aligned}\langle \mathcal{V}^{\alpha}(z)\,\mathcal{V}^{\gamma}(\bar{z})\,\mathcal{V}_{A_1}(x)\rangle &= \langle \mathrm{e}^{-\phi(z)}\,\mathrm{e}^{-\phi(\bar{z})}\rangle \\ &\times\Big[\mathrm{i}\,\langle \partial Z_1(x)\mathrm{e}^{\mathrm{i}\,\kappa_{\parallel}\cdot Z_{\parallel}(x)}\rangle\langle \varDelta(z)\varDelta(\bar{z})\cos(k_{\perp}\cdot Z_{\perp})(x)\rangle\langle S^{\alpha}(z)S^{\gamma}(\bar{z})\rangle \\ &+\langle \mathrm{e}^{\mathrm{i}\,\kappa_{\parallel}\cdot Z_{\parallel}(x)}\rangle\langle \varDelta(z)\varDelta(\bar{z})\cos(k_{\perp}\cdot Z_{\perp})(x)\rangle\langle S^{\alpha}(z)S^{\gamma}(\bar{z})\rangle\langle \kappa_{\parallel}\cdot\varPsi_{\parallel}(x)\varPsi_1(x)\rangle \\ &+\mathrm{i}\,\langle \mathrm{e}^{\mathrm{i}\,\kappa_{\parallel}\cdot Z_{\parallel}(x)}\rangle\langle \varDelta(z)\varDelta(\bar{z})\,\sin(k_{\perp}\cdot Z_{\perp})(x)\rangle\langle S^{\alpha}(z)S^{\gamma}(\bar{z})\,\kappa_{\perp}\cdot\varPsi_{\perp}(x)\rangle\langle \varPsi_1(x)\rangle\Big]\,.\end{aligned} \tag{4.57}$$

It is not difficult to realize that in each of the three lines in square brackets, there is always one factor that vanishes due to normal ordering. For example, in the first line it is the term containing $\mathrm{i}\,\partial Z_1$ that vanishes, while in the second and third line it is the last factor involving the fermionic field $\varPsi_1$ that gives zero. Therefore,

$$\langle \mathcal{V}^{\alpha}(z)\,\mathcal{V}^{\gamma}(\bar{z})\,\mathcal{V}_{A_1}(x)\rangle = 0\,, \tag{4.58}$$

so that

$$\langle \mathcal{V}_{A_1} \rangle_{b;I} = 0\,. \tag{4.59}$$

Let us now consider the components of the gauge field that are transverse to the defect. Using the corresponding vertex operator (5.96), we obtain

$$\begin{aligned}\langle \mathcal{V}^{\alpha}(z)\,\mathcal{V}^{\gamma}(\bar{z})\,\mathcal{V}_{A_2}(x)\rangle = \langle \mathrm{e}^{-\phi(z)}\,\mathrm{e}^{-\phi(\bar{z})}\rangle\langle \mathrm{e}^{\mathrm{i}\,\kappa_{\parallel}\cdot Z_{\parallel}(x)}\rangle \\ \times\Big[-\langle \varDelta(z)\varDelta(\bar{z})\,\partial Z_2(x)\,\sin(k_{\perp}\cdot Z_{\perp})(x)\rangle\langle S^{\alpha}(z)S^{\gamma}(\bar{z})\rangle \\ +\mathrm{i}\,\langle \varDelta(z)\varDelta(\bar{z})\,\sin(k_{\perp}\cdot Z_{\perp})(x)\rangle\langle \kappa_{\parallel}\cdot \varPsi_{\parallel}(x)\rangle\langle S^{\alpha}(z)S^{\gamma}(\bar{z})\varPsi_2(x)\rangle \\ +\langle \varDelta(z)\varDelta(\bar{z})\cos(k_{\perp}\cdot Z_{\perp})(x)\rangle\langle S^{\alpha}(z)S^{\gamma}(\bar{z})\,\kappa_{\perp}\cdot\varPsi_{\perp}(x)\varPsi_2(x)\rangle\Big]\,.\end{aligned} \tag{4.60}$$

As before, in the first and second lines inside the square brackets there are vanishing factors; instead, the third line is not zero and we remain with

$$\begin{aligned}\langle \mathcal{V}^{\alpha}(z)\,\mathcal{V}^{\gamma}(\bar{z})\,\mathcal{V}_{A_2}(x)\rangle = \langle \mathrm{e}^{-\phi(z)}\,\mathrm{e}^{-\phi(\bar{z})}\rangle\langle \mathrm{e}^{\mathrm{i}\,\kappa_{\parallel}\cdot Z_{\parallel}(x)}\rangle\langle \varDelta(z)\varDelta(\bar{z})\cos(k_{\perp}\cdot Z_{\perp})(x)\rangle \\ \times\langle S^{\alpha}(z)S^{\gamma}(\bar{z})\,\kappa_{\perp}\cdot\varPsi_{\perp}(x)\varPsi_2(x)\rangle\,.\end{aligned} \tag{4.61}$$

Each correlator in this expression can be easily evaluated using standard conformal field theory methods; in particular we have

$$\langle \mathrm{e}^{-\phi(z)}\,\mathrm{e}^{-\phi(\bar{z})}\rangle = \frac{1}{z-\bar{z}}\,, \tag{4.62a}$$

$$\langle \mathrm{e}^{\mathrm{i}\,\kappa_{\parallel}\cdot Z_{\parallel}(x)}\rangle = \delta^{(2)}(\kappa_{\parallel})\,, \tag{4.62b}$$

$$\langle \varDelta(z)\varDelta(\bar{z})\cos(k_{\perp}\cdot Z_{\perp})(x)\rangle = \frac{1}{(z-\bar{z})^{\frac{1}{2}}}\,, \tag{4.62c}$$

$$\langle S^{\alpha}(z)S^{\gamma}(\bar{z})\,\psi_m(x)\psi_n(x)\rangle = \frac{1}{2}\,\frac{(\gamma_n\gamma_m\widehat{C}^{-1})^{\alpha\gamma}}{(z-\bar{z})^{-\frac{1}{2}}(z-x)(\bar{z}-x)}\,. \tag{4.62d}$$

The last correlator implies that

$$\begin{aligned}\big\langle S^{\alpha}(z)S^{\gamma}(\bar z)\,\kappa_\perp\!\cdot\!\Psi_\perp(x)\Psi_2(x)\big\rangle &= \mathrm{i}\,\kappa_2\,\big\langle S^{\alpha}(z)S^{\gamma}(\bar z)\,\psi_3(x)\psi_4(x)\big\rangle \\ &= \mathrm{i}\,\frac{\kappa_2}{2}\,\frac{(\gamma_4\gamma_3\widehat{C}^{-1})^{\alpha\gamma}}{(z-\bar z)^{-\frac12}(z-x)(\bar z-x)}\;.\end{aligned}\tag{4.63}$$

Putting everything together, we obtain

$$\big\langle \mathcal{V}^{\alpha}(z)\,\mathcal{V}^{\gamma}(\bar z)\,\mathcal{V}_{A_2}(x)\big\rangle = \mathrm{i}\,\frac{\kappa_2}{2}\,\frac{(\gamma_4\gamma_3\widehat{C}^{-1})^{\alpha\gamma}}{(z-\bar z)(z-x)(\bar z-x)}\,\delta^{(2)}(\kappa_\parallel)\;.\tag{4.64}$$

Inserting this result in (4.56) and performing the corresponding γ-matrix algebra, in the end we find

$$\big\langle \mathcal{V}_{A_2}\big\rangle_{b;I} = (-1)^{I+1}\,b\,\kappa_2\,\delta^{(2)}(\kappa_\parallel)\;.\tag{4.65}$$

As is clear from this expression, the momentum conservation occurs only in the longitudinal directions, whereas the transverse momenta κ_2 and $\bar\kappa_2$ can be arbitrary. This fact implies that (4.65) can be interpreted as a tadpole-like source for the gauge field A_2 which acquires a non-trivial profile in the transverse space. We will explicitly compute this profile in the following section.

The calculation of the couplings of b with the complex scalar Φ gauge theory proceeds along the same lines. One finds that the only non-vanishing contribution to the correlation function is given by

$$\begin{aligned}\big\langle \mathcal{V}^{\alpha}(z)\,\mathcal{V}^{\gamma}(\bar z)\,\mathcal{V}_{\Phi}(x)\big\rangle &= \big\langle \mathrm{e}^{-\phi(z)}\,\mathrm{e}^{-\phi(\bar z)}\big\rangle\big\langle \mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel(x)}\big\rangle\big\langle \varDelta(z)\varDelta(\bar z)\cos(k_\perp\!\cdot\! Z_\perp)(x)\big\rangle \\ &\quad\times\big\langle S^{\alpha}(z)S^{\gamma}(\bar z)\,\kappa_\perp\!\cdot\!\Psi_\perp(x)\Psi_3(x)\big\rangle\;.\end{aligned}\tag{4.66}$$

The last factor is easily computed using (4.62d) with the result

$$\begin{aligned}\big\langle S^{\alpha}(z)S^{\gamma}(\bar z)\,\kappa_\perp\!\cdot\!\Psi_\perp(x)\Psi_3(x)\big\rangle &= \frac{\kappa_2}{4}\,\frac{((\gamma_5+\mathrm{i}\,\gamma_6)(\gamma_3-\mathrm{i}\,\gamma_4)\widehat{C}^{-1})^{\alpha\gamma}}{(z-\bar z)^{-\frac12}(z-x)(\bar z-x)} \\ &\quad+\frac{\bar\kappa_2}{4}\,\frac{((\gamma_5+\mathrm{i}\,\gamma_6)(\gamma_3+\mathrm{i}\,\gamma_4)\widehat{C}^{-1})^{\alpha\gamma}}{(z-\bar z)^{-\frac12}(z-x)(\bar z-x)}\;.\end{aligned}\tag{4.67}$$

This implies that

$$\langle \mathcal{V}^{\alpha}(z)\,\mathcal{V}^{\gamma}(\bar{z})\,\mathcal{V}_{\Phi}(x)\rangle = \frac{\left[\left(\frac{\kappa_2}{4}\,(\gamma_5+\mathrm{i}\,\gamma_6)(\gamma_3-\mathrm{i}\,\gamma_4)+\frac{\bar{\kappa}_2}{4}\,(\gamma_5+\mathrm{i}\,\gamma_6)(\gamma_3+\mathrm{i}\,\gamma_4)\right)\widehat{C}^{-1}\right]^{\alpha\gamma}}{(z-\bar{z})(z-x)(\bar{z}-x)}\,\delta^{(2)}(\kappa_{\parallel})\,. \tag{4.68}$$

When we plug this expression into (4.56) and perform the resulting γ-matrix algebra we get zero, so that

$$\langle \mathcal{V}_{\Phi}\rangle_{b;I} = 0 \tag{4.69}$$

Finally, considering the scalars Φ_r, we find that

$$\langle \mathcal{V}^{\alpha}(z)\,\mathcal{V}^{\gamma}(\bar{z})\,\mathcal{V}_{\Phi_r}(x)\rangle = 0 \tag{4.70}$$

since, like for A_1, the resulting correlator always contains a vanishing factor. Therefore,

$$\langle \mathcal{V}_{\Phi_r}\rangle_{b;I} = 0\,. \tag{4.71}$$

Correlators with b'

Let us now consider the couplings with the twisted scalar b' whose vertex operator (4.53b) has the polarization $\epsilon\tau_3$. The vanishing of the correlators (4.58) and (4.70) shows that $\mathbf{A}_1$ and $\mathbf{\Phi}_r$ do not couple to any NS/NS twisted field, including b'. Also the non-vanishing correlators (4.64) and (4.68) give a zero result for b' due to the γ-matrix algebra. Therefore the field b' does not couple to any of the massless open string fields of the gauge theory:

$$\langle \mathcal{V}_{A_1}\rangle_{b';I} = \langle \mathcal{V}_{A_2}\rangle_{b';I} = \langle \mathcal{V}_{\Phi}\rangle_{b';I} = \langle \mathcal{V}_{\Phi_r}\rangle_{b';I} = 0\,. \tag{4.72}$$

Correlators with $b_{\pm}$

The couplings of the doublet $b_{\pm}$ with the open string fields can be computed along the same lines. We simply have to use the correlators (4.58), (4.64), (4.68) and (4.70) and the polarizations ($\epsilon\tau_\pm$) corresponding to $b_\pm$. Proceeding in this way we find

$$\langle \mathcal{V}_{A_1} \rangle_{b_\pm;I} = \langle \mathcal{V}_{A_2} \rangle_{b_\pm;I} = \langle \mathcal{V}_{\Phi} \rangle_{b_-;I} = \langle \mathcal{V}_{\Phi_r} \rangle_{b_\pm;I} = 0 \, . \tag{4.73}$$

The vanishing of the coupling of A_2 with $b_\pm$ and of the coupling of Φ with b_- is again due to the structure of the resulting combinations of γ-matrices which have a vanishing trace. On the other hand, the terms proportional to $\bar{\kappa}_2$ in (4.68) yield a non-zero result when contracted with the polarization of b_+, leading to

$$\langle \mathcal{V}_{\Phi} \rangle_{b_+;I} = (-1)^{I+1} \, \mathrm{i} \, b_+ \, \bar{\kappa}_2 \, \delta^{(2)}(\kappa_\|) \, . \tag{4.74}$$

4.3.2 Correlators with R/R twisted fields

For the twisted fields of the R/R sector discussed in Section 4.1.1, the twisted R/R sector the fermionic fields possess zero modes in the six dimensions that are orthogonal to the $\mathbb{Z}_2$ orbifold. They realize spinor representations of SO(6), but when a fractional D3-brane is inserted, this group is broken to SO(2)×SO(4). We are interested in giving a constant background value to some scalars that remain after this breaking. The scalar c obviously remains, while the anti-symmetric tensor $c_{MN} \in \mathbf{15}$ decomposes in various representations of the unbroken subgroup. In particular, we will consider only the component c_{12} which is a scalar of SO(2)×SO(4) that we denote c'. The vertex operators corresponding to c and c' are given in (4.24) which we rewrite here for convenience:

$$c \quad \longleftrightarrow \quad \mathcal{V}_c(z,\bar{z}) = C_{A\dot{B}} \, \mathcal{V}^A(z) \, \widetilde{\mathcal{V}}^{\dot{B}}(\bar{z}) \, , \tag{4.75a}$$

$$c' \quad \longleftrightarrow \quad \mathcal{V}_{c'}(z,\bar{z}) = (C\,\Gamma_{12})_{A\dot{B}} \, \mathcal{V}^A(z) \, \widetilde{\mathcal{V}}^{\dot{B}}(\bar{z}) \, . \tag{4.75b}$$

Again we take these vertices at zero momentum since we want to regard the closed string fields as a constant background.

Correlators with c

The mixed correlators between the R/R twisted scalar c and the open string massless fields of a D3-brane of type I are given by

$$\langle \mathcal{V}_{\text{open}} \rangle_{c;I} = c \int \frac{dz\, d\bar{z}\, dx}{dV_{\text{proj}}} \, \langle \mathcal{V}_c(z,\bar{z})\, \mathcal{V}_{\text{open}}(x) \rangle_I \tag{4.76}$$

with

$$\begin{aligned} \langle \mathcal{V}_c(z,\bar{z})\, \mathcal{V}_{\text{open}}(x) \rangle_I &= C_{A\dot{B}} \, \langle \mathcal{V}^A(z)\, \widetilde{\mathcal{V}}^{\dot{B}}(\bar{z})\, \mathcal{V}_{\text{open}}(x) \rangle_I \\ &= (-1)^I \, C_{A\dot{B}} (\Gamma_1 \Gamma_2)^{\dot{B}}{}_{\dot{C}} \, \langle \mathcal{V}^A(z)\, \mathcal{V}^{\dot{C}}(\bar{z})\, \mathcal{V}_{\text{open}}(x) \rangle \, . \end{aligned} \tag{4.77}$$

where the last step follows from the reflection rules (4.44b).

The first coupling we consider is the one with the gauge field A_1. Using the vertex operator (5.95) we find that there is only a single structure contributing to the amplitude, namely

$$\begin{aligned} \langle \mathcal{V}^A(z)\, \mathcal{V}^{\dot{C}}(\bar{z})\, \mathcal{V}_{A_1}(x) \rangle = \langle \mathrm{e}^{-\frac{1}{2}\phi(z)}\, \mathrm{e}^{-\frac{3}{2}\phi(\bar{z})} \rangle \langle \mathrm{e}^{\mathrm{i}\, \kappa_{\parallel} \cdot Z_{\parallel}(x)} \rangle \langle \Delta(z) \Delta(\bar{z}) \cos(k_{\perp} \cdot Z_{\perp})(x) \rangle \\ \times \langle S^A(z) S^{\dot{C}}(\bar{z})\, \kappa_{\parallel} \cdot \Psi_{\parallel}(x) \Psi_1(x) \rangle \, . \end{aligned} \tag{4.78}$$

The second and third factors are given in (4.62b) and (4.62c), while the other factors are obtained from the standard conformal field theory results, namely

$$\langle \mathrm{e}^{-\frac{1}{2}\phi(z)}\, \mathrm{e}^{-\frac{3}{2}\phi(\bar{z})} \rangle = \frac{1}{(z-\bar{z})^{\frac{3}{4}}} \, , \tag{4.79a}$$

$$\langle S^A(z) S^{\dot{C}}(\bar{z})\, \psi_M(x) \psi_N(x) \rangle = \frac{1}{2} \, \frac{(\Gamma_M \Gamma_N C^{-1})^{A\dot{C}}}{(z-\bar{z})^{-\frac{1}{4}} (z-x)(\bar{z}-x)} \, . \tag{4.79b}$$

The last correlator implies that

$$\begin{aligned} \langle S^A(z) S^{\dot{C}}(\bar{z})\, \kappa_{\parallel} \cdot \Psi_{\parallel}(x) \Psi_1(x) \rangle &= \mathrm{i}\, \kappa_1 \, \langle S^A(z) S^{\dot{C}}(\bar{z})\, \psi_1(x) \psi_2(x) \rangle \\ &= \mathrm{i}\, \frac{\kappa_1}{2} \, \frac{(\Gamma_1 \Gamma_2 C^{-1})^{A\dot{C}}}{(z-\bar{z})^{-\frac{1}{4}} (z-x)(\bar{z}-x)} \end{aligned} \tag{4.80}$$

so that from (4.78) we get

$$\langle \mathcal{V}^{A}(z)\,\mathcal{V}^{\dot{C}}(\bar{z})\,\mathcal{V}_{A_1}(x)\rangle = \mathrm{i}\,\frac{\kappa_1}{2}\,\frac{(\Gamma_1\Gamma_2 C^{-1})^{A\dot{C}}}{(z-\bar{z})(z-x)(\bar{z}-x)}\,\delta^{(2)}(\kappa_{\parallel})\ . \tag{4.81}$$

Plugging this expression into (4.76) and performing the algebra on the Γ-matrices in the end we obtain

$$\langle \mathcal{V}_{A_1}\rangle_{c;I} = (-1)^{I+1}\,2\,\mathrm{i}\,c\,\kappa_1\,\delta^{(2)}(\kappa_{\parallel})\ . \tag{4.82}$$

There are no other non-trivial couplings of c since for A_2, Φ and Φ_r the three-point functions vanish at the level of conformal field theory correlators, namely

$$\langle \mathcal{V}^{A}(z)\,\mathcal{V}^{\dot{C}}(\bar{z})\,\mathcal{V}_{A_2}(x)\rangle = \langle \mathcal{V}^{A}(z)\,\mathcal{V}^{\dot{C}}(\bar{z})\,\mathcal{V}_{\Phi}(x)\rangle = \langle \mathcal{V}^{A}(z)\,\mathcal{V}^{\dot{C}}(\bar{z})\,\mathcal{V}_{\Phi_r}(x)\rangle = 0\ . \tag{4.83}$$

Obviously this implies that

$$\langle \mathcal{V}_{A_2}\rangle_{c;I} = \langle \mathcal{V}_{\Phi}\rangle_{c;I} = \langle \mathcal{V}_{\Phi_r}\rangle_{c;I} = 0\ . \tag{4.84}$$

Correlators with c'

In this case we can be extremely brief since the scalar c' does not couple to any of the massless bosonic open string fields. Indeed we have

$$\langle \mathcal{V}_{A_1}\rangle_{c';I} = \langle \mathcal{V}_{A_2}\rangle_{c';I} = \langle \mathcal{V}_{\Phi}\rangle_{c';I} = \langle \mathcal{V}_{\Phi_r}\rangle_{c';I} = 0\ . \tag{4.85}$$

The last three equalities clearly follow from (4.83), while the vanishing of the coupling of A_1 is due to the fact that the Γ-matrices in the numerator of (4.81) give a zero result when they are contracted with the polarization $(C\,\Gamma_{12})_{A\dot{B}}$. Thus, like b', the scalar c' will also not play any role in our further analysis.

4.4 Continuous parameters of surface operators from world-sheet correlators

In this section, we provide an interpretation of the non-vanishing couplings between the closed string massless fields of the twisted sectors and the massless open string fields on the fractional D3-branes.

Field Profiles

The twisted scalar b of the NS/NS sector produces a tadpole-like source for the gauge field A_2 given in (4.65), which depends on the orthogonal momentum to the surface defect. This source, which is localized at the orbifold fixed point where b is defined, gives rise to a non-trivial profile for A_2 in the transverse directions: This profile is obtained by computing the Fourier transform of the tadpole after including the massless propagator

$$\frac{1}{2(|\kappa_\parallel|^2 + |\kappa_\perp|^2)} = \frac{1}{k_1^2 + k_2^2 + k_3^2 + k_4^2} \ . \tag{4.86}$$

This procedure is the strict analog of what has been discussed in [36] for the profile of the gravitational fields emitted by a Dp-brane and in [37] for the instanton profile of the gauge fields of a D3-brane in the presence of D-instantons.

One new feature in this orbifold case is that for functions f_+ and f_- which are, respectively, even and odd under $\mathbb{Z}_2$, the Fourier transform is given by

$$\begin{aligned} \mathcal{FT}[f_+](z) &= \int \frac{d^2\kappa_\parallel\, d^2\kappa_\perp}{(2\pi)^2}\, \cos(\kappa_\perp\!\cdot\! z_\perp)\, \mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot z_\parallel}\, f_+(\kappa)\ , \\ \mathcal{FT}[f_-](z) &= \int \frac{d^2\kappa_\parallel\, d^2\kappa_\perp}{(2\pi)^2}\, \mathrm{i}\,\sin(\kappa_\perp\!\cdot\! z_\perp)\, \mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot z_\parallel}\, f_-(\kappa)\ . \end{aligned} \tag{4.87}$$

Let us consider for simplicity a fractional D3-brane type 0. Applying the above procedure, the profile of its gauge field A_2 in configuration space induced by the NS/NS twisted scalar b is

$$\begin{aligned} A_2 &= \int \frac{d^2\kappa_\| \, d^2\kappa_\perp}{(2\pi)^2} \, \mathrm{i} \, \sin(\kappa_\perp \cdot z_\perp) \, \mathrm{e}^{\mathrm{i}\,\kappa_\| \cdot z_\|} \, \frac{\langle \mathcal{V}_{A_2} \rangle_{b;0}}{2(|\kappa_\||^2 + |\kappa_\perp|^2)} \\ &= -\mathrm{i}\, b \int \frac{d^2\kappa_\perp}{(2\pi)^2} \, \sin(\kappa_\perp \cdot z_\perp) \, \frac{\kappa_2}{2|\kappa_\perp|^2} \end{aligned} \tag{4.88}$$

where in the second line we have used (4.65) with $I = 0$ and taken into account the δ-function enforcing momentum conservation in the parallel directions to perform the integral over $\kappa_\|$. This shows that, as anticipated, the propagation of the source is only in the transverse directions. With a simple calculation we can see that

$$\int \frac{d^2\kappa_\perp}{(2\pi)^2} \, \sin(\kappa_\perp \cdot z_\perp) \, \frac{\kappa_2}{2|\kappa_\perp|^2} = \frac{1}{4\pi \bar{z}_2} \, , \tag{4.89}$$

so that

$$A_2 = -\frac{\mathrm{i}\, b}{4\pi \bar{z}_2} \, . \tag{4.90}$$

The component $\bar{A}_2$ of the gauge field also has a non-trivial profile which is given by the complex conjugate of (5.131).

As we have seen in the previous section, there are no other tadpole-like sources for A_2, so that (5.131) is the full result. One might think that the R/R scalar c can act as a source for the longitudinal component A_1 of the gauge field in view of (4.82). However, if one takes into account the δ-function that enforces momentum conservation along the first complex direction, one easily realizes that this actually vanishes. Therefore, the vector field is only sourced by the NS/NS twisted scalar b which yields (5.131) and its complex conjugate.

In conclusion, the gauge field on a fractional D3-brane of type 0 in the $\mathbb{Z}_2$ orbifold acquires the following profile

$$\mathbf{A} = A \cdot dx = A_2 \, d\bar{z}_2 + \bar{A}_2 \, dz_2 = -\frac{\mathrm{i}\, b}{4\pi} \left(\frac{d\bar{z}_2}{\bar{z}_2} - \frac{dz_2}{z_2} \right) = -\frac{b}{2\pi} \, d\theta \tag{4.91}$$

where θ denote the polar angle in the $\mathbb{C}_{(2)}$ plane . If we take a fractional D3-brane of type

1, we obtain the same profile but with an overall minus sign due to the different sign in twisted component of the boundary state and in the reflection rules (see (4.44a)).

Let us now consider the scalar field Φ, the only other open string field that has a non-vanishing tadpole produced by b_+. Applying the same procedure discussed above and using (4.74), for a fractional D3-brane of type 0 we obtain

$$\begin{aligned}\Phi &= \int \frac{d^2\kappa_\parallel\, d^2\kappa_\perp}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot z_\parallel}\, \mathrm{i}\, \sin(\kappa_\perp\cdot z_\perp)\, \frac{\langle \mathcal{V}_\Phi\rangle_{b_+;0}}{2(|\kappa_\parallel|^2+|\kappa_\perp|^2)} \\ &= b_+ \int \frac{d^2\kappa_\perp}{(2\pi)^2}\, \sin(\kappa_\perp\cdot z_\perp)\, \frac{\bar\kappa_2}{2|\kappa_\perp|^2} = \frac{b_+}{4\pi z_2}\,.\end{aligned} \tag{4.92}$$

Of course, for a fractional D3-brane of type 1 we get the same result with an overall minus sign.

It is quite straightforward to generalize these findings to the case of a system made of n_0 fractional D3-branes of type 0 and n_1 fractional D3-branes of type 1, which describes a gauge theory with group $\mathrm{U}(n_0+n_1)$ broken to the Levi group $\mathrm{U}(n_0)\times\mathrm{U}(n_1)$. In fact, we simply obtain

$$\mathbf{A} = -\frac{b}{2\pi}\begin{pmatrix}\mathbb{I}_{n_0} & 0\\ 0 & -\mathbb{I}_{n_1}\end{pmatrix} d\theta\,, \tag{4.93a}$$

$$\mathbf{\Phi} = \frac{b_+}{4\pi}\begin{pmatrix}\mathbb{I}_{n_0} & 0\\ 0 & -\mathbb{I}_{n_1}\end{pmatrix} \frac{1}{z_2}\,. \tag{4.93b}$$

This is precisely the expected profile for a monodromy defect of GW type. Comparing with (3.1) and (3.2) we see that the continuous parameters of the surface defect are related to the background fields of the NS/NS twisted sector as follows

$$\alpha_I = (-1)^{I+1}\,\frac{b}{2\pi}\,,\quad \beta_I = (-1)^I\,\frac{\mathrm{Re}(b_+)}{2\pi}\,,\quad \gamma_I = (-1)^I\,\frac{\mathrm{Im}(b_+)}{2\pi}\,. \tag{4.94}$$

Notice that in our realization we have $\sum_I \alpha_I = \sum_I \beta_I = \sum_I \gamma_I = 0$. This is not a limitation since a generic GW solution can always be brought to this form by adding a U(1) term proportional to the identity without changing the Levi subgroup $\mathrm{U}(n_0)\times\mathrm{U}(n_1)$.

To obtain the profile in the case of special unitary groups we have to remove the overall U(1) factor. This is simply done as follows

$$\mathbf{A} \;\longmapsto\; \mathbf{A} - \frac{1}{n_0+n_1}\,(\mathrm{Tr}\,\mathbf{A})\,\mathbb{I}_{n_0+n_1} \;=\; -\frac{b}{2\pi}\begin{pmatrix} \frac{n_1}{n_0+n_1}\,\mathbb{I}_{n_0} & 0 \\ 0 & -\frac{n_0}{n_0+n_1}\,\mathbb{I}_{n_1}\end{pmatrix} d\theta\,, \tag{4.95a}$$

$$\mathbf{\Phi} \;\longmapsto\; \mathbf{\Phi} - \frac{1}{n_0+n_1}\,(\mathrm{Tr}\,\mathbf{\Phi})\,\mathbb{I}_{n_0+n_1} \;=\; \frac{b_+}{4\pi}\begin{pmatrix} \frac{n_1}{n_0+n_1}\,\mathbb{I}_{n_0} & 0 \\ 0 & -\frac{n_0}{n_0+n_1}\,\mathbb{I}_{n_1}\end{pmatrix} \frac{1}{z^2}\,. \tag{4.95b}$$

Let us now comment on the meaning of the result (4.82), which indicates a coupling between the longitudinal component of the gauge field A_1 and the twisted scalar c in the R/R sector. This cannot be interpreted as a source for the gauge field A_1 because it is not proportional to the transverse momentum but to the longitudinal one, which is set to zero by the momentum conserving δ-function. However, a different and interesting interpretation is possible. If we multiply the amplitude (4.82) and its complex conjugate by the corresponding polarizations of the gauge field, namely $\bar{A}_1$ and A_1, the resulting sum can be interpreted as an effective interaction term involving the gauge field strength in the longitudinal directions. Such a term can be non-zero even in the presence of the momentum conserving δ-function provided the field strength is kept fixed. To make this explicit, let us consider a D3-brane of type 0 and use (4.82) for $I=0$. Then we have

$$\bar{A}_1\,\langle\mathcal{V}_{A_1}\rangle_{c;0} + A_1\,\langle\mathcal{V}_{\bar{A}_1}\rangle_{c;0} = -2\,\mathrm{i}\,c\,(\kappa_1\,\bar{A}_1 - \bar{\kappa}_1\,A_1)\,\delta^{(2)}(k_\parallel) = 2\,\mathrm{i}\,c\,\widetilde{F}_0\,\delta^{(2)}(\kappa_\parallel) \tag{4.96}$$

where $\widetilde{F}_0 = \bar{\kappa}_1\,A_1 - \kappa_1\,\bar{A}_1$ is the (momentum space) field strength in the $2d$ space where the surface defect is extended. The Fourier transform of (4.96), computed according to

(4.87), is

$$ \mathrm{i}\, c \int d^2k_{\parallel}\, \widetilde{F}_0\, \delta^{(2)}(\kappa_{\parallel}) \times 2\, \delta^{(2)}(z_{\perp}) \;=\; \frac{\mathrm{i}\, c}{2\pi} \int d^2x_{\parallel}\, F_0 \times 2\, \delta^{(2)}(z_{\perp}) \tag{4.97} $$

where F_0 is the field strength in configuration space. If we assume that this $2d$ space instead of being simply $\mathbb{C}$ is a manifold D where the gauge field strength has a non-vanishing first Chern class, then (4.97) can be interpreted as an effective interaction term localized [7] on D, meaning that in the path-integral of the underlying (abelian) gauge theory one has the following phase factor

$$ \exp\left(\frac{\mathrm{i}\, c}{2\pi} \int_D F_0\right) . \tag{4.98} $$

If we extend this argument to a system made of n_0 fractional D3-branes of type 0 and n_1 fractional D3-branes of type 1, the phase factor becomes

$$ \exp\left(\mathrm{i} \sum_I (-1)^I \frac{c}{2\pi} \int_D \mathrm{Tr}_{\,\mathrm{U}(n_I)} F_I\right) \tag{4.99} $$

which has exactly the same form of the one of the GW monodromy defect given in (3.5) with

$$ \eta_I = (-1)^I \frac{c}{2\pi} . \tag{4.100} $$

In the case of special unitary groups, we have to remove the overall U(1) factor and this leads to

$$ \exp\left(\frac{\mathrm{i}\, c}{2\pi} \frac{n_1}{n_0 + n_1} \int_D \mathrm{Tr}_{\,\mathrm{U}(n_0)} F_0 - \frac{\mathrm{i}\, c}{2\pi} \frac{n_0}{n_0 + n_1} \int_D \mathrm{Tr}_{\,\mathrm{U}(n_1)} F_1\right) . \tag{4.101} $$

[7] Notice that the term that localizes on the defect placed at the origin is $2\,\delta^{(2)}(z_{\perp})$, where the factor of 2 compensates the fact that the orbifold halves the volume of the transverse space.

4.5 S-duality properties

We have shown that a system of n_0 fractional D3-branes of type 0 and n_1 fractional D3-branes of type 1 that partially extend along a $\mathbb{Z}_2$ orbifold, supports a gauge theory with a surface defect of the GW type, whose discrete data (n_0, n_1) are encoded in the representation of the orbifold group assigned to the fractional D3-branes and whose continuous data are encoded in the expectation values of the closed string fields in the orbifold twisted sectors according to

$$\{\alpha_I, \beta_I, \gamma_I, \eta_I\} = \left\{(-1)^{I+1}\,\frac{b}{2\pi}, (-1)^I\,\frac{\mathrm{Re}(b_+)}{2\pi}, (-1)^I\,\frac{\mathrm{Im}(b_+)}{2\pi}, (-1)^I\,\frac{c}{2\pi}\right\}. \tag{4.102}$$

In the case of special unitary gauge groups, the parameters with $I = 0$ must be multiplied by $\frac{n_1}{n_0+n_1}$ and those with $I = 1$ by $\frac{n_0}{n_0+n_1}$ in order to enforce the decoupling of the overall U(1) factor.

This explicit realization of the continuous parameters of the surface defect in terms of closed string fields allows us to also discuss how they behave under duality transformations. To do so we first recall that, from a geometric point of view, the twisted scalars b and c arise by wrapping the NS/NS and R/R 2-form fields $B_{(2)}$ and $C_{(2)}$ of Type II B string theory around the exceptional 2-cycle ω_2 at the orbifold fixed point [40, 52, 54], namely [8]

$$b = \int_{\omega_2} B_{(2)}\,, \qquad c = \int_{\omega_2} C_{(2)}\,. \tag{4.103}$$

Using this fact, we can then rewrite the parameters α_I and η_I given in (4.102) in the following suggestive way

$$\alpha_I = \frac{(-1)^{I+1}}{2\pi}\int_{\omega_2} B_{(2)}\,, \qquad \eta_I = \frac{(-1)^I}{2\pi}\int_{\omega_2} C_{(2)}\,. \tag{4.104}$$

[8]Note that the symbols b and c are used for closed string twisted scalars, not to be confused with conventional notation for (b, c) ghost fields.

These formulas, including the relative minus sign, are reminiscent of those obtained in [25, 26] where a holographic representation of the GW surface defects has been proposed in terms of bubbling geometries, which are particular solutions of Type II B supergravity with an $AdS_5 \times S_5$ asymptotic limit. Our explicit realization in terms of perturbative string theory, however, is very different, although the identification of the parameters α_I and η_I with the holonomies of the two 2-forms of Type II B is similar.

The exceptional 2-cycle ω_2 has a vanishing size in the orbifold limit but when the orbifold singularity is resolved in a smooth space, it acquires a finite size. The other three fields of the twisted NS/NS sector, b' and $b_\pm$, correspond precisely to the blow-ups of the orbifold fixed point [38, 54]. In particular, b' is the Kähler modulus while $b_\pm$ is the complex structure moduli of the blown-up 2-cycle. Hence they are directly related to the string-frame metric $G_{\mu\nu}$ of the Type II B string theory.

This geometric interpretation fixes the duality transformations of the twisted fields since they are inherited from those of the parent Type II B fields from which they descend. It is well-known [9] that under a duality transformation $\Lambda = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ the two 2-forms rotate among themselves according to

$$\begin{pmatrix} C_{(2)} \\ B_{(2)} \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_{(2)} \\ B_{(2)} \end{pmatrix} \tag{4.105}$$

while the string-frame metric $G_{\mu\nu}$ transforms as

$$G_{\mu\nu} \longrightarrow |c\,\tau + d|\, G_{\mu\nu} \tag{4.106}$$

where τ is the axio-dilaton field. Therefore, under a duality b and c rotate as in (4.105) and $b_\pm$ transform as the metric in (4.106). From this and the identification (4.102), it follows

[9] See, for instance, [55].

with straightforward manipulations that the surface operator parameters transform as

$$
\begin{aligned}
(\alpha_I, \eta_I) &\longrightarrow (d\,\alpha_I - c\,\eta_I, -b\,\alpha_I + a\,\eta_I)\,, \\
(\beta_I, \gamma_I) &\longrightarrow |c\,\tau + d|\,(\beta_I, \gamma_I)
\end{aligned}
\tag{4.107}
$$

Comparing with (3.7), we see that this is precisely the expected behavior of the parameters of the GW defect as originally shown in [1]. This agreement is an important check of our proposal for the realization of surface operators using perturbative string theory.

Chapter 5

Surface operators using $\mathbb{Z}_M$ orbifold

In the previous section, we already worked out this identification for the simple surface defects that correspond to the $\mathbb{Z}_2$ orbifold. In this section, we extend our analysis to the $\mathbb{Z}_M$ orbifolds for $M > 2$ which can describe the most generic half-BPS surface defect corresponding to the breaking of the $U(N)$ gauge group to the Levi subgroup $\mathrm{U}(n_0)\times\ldots\times\mathrm{U}(n_{M-1})$ with $\sum_I n_I = N$.

While the basic conceptual issues in realizing such a surface defect using fractional D3-branes remain the same for all M, the main difference with respect to ch.4 lies in the treatment of the closed string background. For $M = 2$ the massless fields of the NS/NS and R/R twisted sectors correspond to degenerate ground states and their vertex operators are realized using spin fields [9]. This is no longer the case for $M > 2$ and, in fact, the massless fields of the NS/NS sector arise from excited states created by the oscillators of the fermionic string coordinates. Furthermore, pairs of twisted sectors are related by complex conjugation and this turns out to play an important role in the identification of the closed string background with the real parameters in the GW profiles.

In the $\mathbb{Z}_M$ orbifold, there are $(M-1)$ twisted sectors. One could treat all of them at once using the bosonization formalism [9, 10], but in order to keep track of all the relative phases it would be necessary to introduce the so-called cocycle factors. Since dealing with these cocycle factors is quite involved, and since the relative phases are crucial to obtain the correct identification of the continuous parameters of the GW surface defects, we adopt an explicit fermionic approach and use the bosonization formalism only where no phase ambiguities arise. The advantage of this method is that the relative phases among the contributions from different sectors are easily tracked and fixed by the fermionic statistics. Moreover, in this fermionic approach we can describe the fractional D3-branes using boundary states (for a review see for example [6, 7]). Even though the KT brane configuration has not been explicitly considered so far from the boundary state point of view, we can exploit many of the results that already exist in the literature [3, 8, 42, 52] and generalize them to the present case, in which the fractional D3-branes partially extend along the orbifold. The price we have to pay for using this fermionic approach is that we have to distinguish between the twisted sectors and treat separately those whose twist parameter is smaller or bigger than $\frac{1}{2}$.

The open string sector, instead, is similar to that of the $M = 2$ case. We recall that for the KT configuration, the fractional D3-branes have the same field content as the regular D3-branes, since in this case the orbifold does not project away any of the open string excitations, unlike the case when the branes are entirely transverse to the orbifolded space. Indeed, on the world-volume of the fractional D3-branes we find a gauge vector and three complex massless scalars plus their fermionic partners. However, the corresponding vertex operators are linear combinations that behave covariantly under the action of the orbifold. When $M > 2$, these combinations are slightly more involved than for $M = 2$ and are written in terms of generalized plane-waves.

Once the vertex operators for the massless open and closed string states are derived, the discussion proceeds along the same lines as in the $M = 2$ case, but with the important technical differences and peculiarities that we have just mentioned.

Our analysis provides an explicit realization of the monodromy defects of the $\mathcal{N} = 4$ super Yang-Mills theory using perturbative string theory methods. As we discuss in the concluding chapter 6 we believe that this stringy realization may prove to be useful in further investigations of surface defects and their properties, and it may even offer an alternative approach to the study of extended objects in ordinary field theory through their embedding into string theory.

5.1 Twisted closed strings in the $\mathbb{Z}_M$ orbifold

We consider Type II B string theory on the orbifold (1.1). The i-th complex plane $\mathbb{C}_{(i)}$ is parametrized by

$$z_i = \frac{x_{2i-1} + \mathrm{i}\, x_{2i}}{\sqrt{2}} \qquad \text{and} \qquad \bar{z}_i = \frac{x_{2i-1} - \mathrm{i}\, x_{2i}}{\sqrt{2}} \tag{5.1}$$

where x_μ are the ten real coordinates of space-time. The orbifold group $\mathbb{Z}_M$ is generated by an element g such that $g^M = 1$, with the following action on z_2 and z_3:

$$g \;:\; (z_2\,,\, z_3) \longrightarrow (\omega\, z_2\,,\, \omega^{-1} z_3) \tag{5.2}$$

where

$$\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{M}}\ . \tag{5.3}$$

The action of g on $\bar{z}_2$ and $\bar{z}_3$ follows from complex conjugation. This breaks the $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ isometry group of $\mathbb{C}_{(2)} \times \mathbb{C}_{(3)}$ to $\mathrm{SU}(2)_+$.

To describe the closed strings propagating on this orbifold, we use the complex notation and denote the bosonic string coordinates by $\left\{Z^i(z), \overline{Z}^i(z)\right\}$ for the left-movers and $\{\widetilde{Z}^i(\bar{z}), \widetilde{\overline{Z}}^i(\bar{z})\}$ for the right-movers, with z and $\bar{z}$ parametrizing the closed string world-sheet. Similarly, we denote the fermionic string coordinates by $\{\Psi^i(z), \bar{\Psi}^i(z)\}$ for the left-movers and $\{\widetilde{\Psi}^i(\bar{z}), \widetilde{\bar{\Psi}}^i(\bar{z})\}$ for the right-movers.

5.1.1 Twisted sectors

In the $\mathbb{Z}_M$ orbifold theory, there are $(M-1)$ twisted sectors labeled by the index $\widehat{a} = 1,\ldots,M-1$. If M is odd, we can divide the twisted sectors in two sets, each one containing $\frac{M-1}{2}$ elements. The sectors of the first set are labeled by $\widehat{a} = a = 1,\cdots,\frac{M-1}{2}$ and are characterized by a twist parameter

$$\nu_a = \frac{a}{M} < \frac{1}{2}\ . \tag{5.4}$$

The sectors of the second set have, instead, a twist parameter

$$1-\nu_a = \frac{M-a}{M} > \frac{1}{2} \tag{5.5}$$

and are labeled by $\widehat{a} = (M-a)$. If M is even, in addition, there is an extra sector with twist parameter $\frac{1}{2}$, which has to be treated separately. For most of the discussion, we will assume that M is odd and briefly comment on the special case with twist $\frac{1}{2}$, occurring when M is even, only at the very end, since this case has already been discussed in detail in the previous chapter [4].

In the sectors with label a and twist parameter as in (5.4), the left-movers of the bosonic and fermionic string coordinates satisfy the following monodromy properties on the world-sheet:

$$Z^2(\mathrm{e}^{2\pi\mathrm{i}}\,z) = \mathrm{e}^{2\pi\mathrm{i}\nu_a}\,Z^2(z)\,, \qquad Z^3(\mathrm{e}^{2\pi\mathrm{i}}\,z) = \mathrm{e}^{-2\pi\mathrm{i}\nu_a}\,Z^3(z)\,, \tag{5.6a}$$

$$\Psi^2(\mathrm{e}^{2\pi\mathrm{i}}\,z) = \pm\,\mathrm{e}^{2\pi\mathrm{i}\nu_a}\,\Psi^2(z)\,, \qquad \Psi^3(\mathrm{e}^{2\pi\mathrm{i}}\,z) = \pm\,\mathrm{e}^{-2\pi\mathrm{i}\nu_a}\,\Psi^3(z)\,, \tag{5.6b}$$

where the $+(-)$ sign refers to the NS (R) sector. The analogous relations for $\overline{Z}^i$ and $\overline{\Psi}^i$ can be obtained by complex conjugation. On the other hand, the right-movers satisfy the monodromy properties:

$$\widetilde{Z}^2(\mathrm{e}^{2\pi\mathrm{i}}\,\bar{z}) = \mathrm{e}^{-2\pi\mathrm{i}\nu_a}\,\widetilde{Z}^2(\bar{z})\,, \qquad \widetilde{Z}^3(\mathrm{e}^{2\pi\mathrm{i}}\,\bar{z}) = \mathrm{e}^{2\pi\mathrm{i}\nu_a}\,\widetilde{Z}^3(\bar{z})\,, \tag{5.7a}$$

$$\widetilde{\Psi}^2(\mathrm{e}^{2\pi\mathrm{i}}\,\bar{z}) = \pm\,\mathrm{e}^{-2\pi\mathrm{i}\nu_a}\,\widetilde{\Psi}^2(\bar{z})\,, \qquad \widetilde{\Psi}^3(\mathrm{e}^{2\pi\mathrm{i}}\,\bar{z}) = \pm\,\mathrm{e}^{2\pi\mathrm{i}\nu_a}\,\widetilde{\Psi}^3(\bar{z})\,. \tag{5.7b}$$

Again, the relations for $\overline{\widetilde{Z}}^i$ and $\overline{\widetilde{\Psi}}^i$ are obtained by complex conjugation.

For the sectors with label $(M-a)$ and twist parameter as in (5.5), similar monodromy relations hold for the world-sheet fields but with ν_a everywhere replaced by $(1-\nu_a)$.

5.1.2 Twisted NS sectors

We now turn to a discussion of the spectrum of massless string states in the various twisted sectors, focusing mainly on the fermionic fields in the complex directions 2 and 3. In the fermionic formalism, when the NS boundary conditions are imposed, we have to treat separately the sectors with twist parameter smaller than $\frac{1}{2}$ and those with twist parameter bigger than $\frac{1}{2}$.

Sectors with twist parameter $\nu_a < \frac{1}{2}$

In this case the monodromy properties (5.6b) and their complex conjugate lead to the following mode expansions for the left-moving fermionic fields (see for example [56] and references therein):

$$\begin{aligned}
\Psi^2(z) &= \sum_{r=1/2}^{\infty}\left(\overline{\Psi}^2_{r-\nu_a}\,z^{-r+\nu_a-\frac{1}{2}} + \Psi^2_{-r-\nu_a}\,z^{r+\nu_a-\frac{1}{2}}\right),\\
\overline{\Psi}^2(z) &= \sum_{r=1/2}^{\infty}\left(\Psi^2_{r+\nu_a}\,z^{-r-\nu_a-\frac{1}{2}} + \overline{\Psi}^2_{-r+\nu_a}\,z^{r-\nu_a-\frac{1}{2}}\right),
\end{aligned} \tag{5.8}$$

and

$$\Psi^3(z) = \sum_{r=1/2}^{\infty} \left(\overline{\Psi}^3_{r+\nu_a} \, z^{-r-\nu_a-\frac{1}{2}} + \Psi^3_{-r+\nu_a} \, z^{r-\nu_a-\frac{1}{2}} \right) ,$$
$$\overline{\Psi}^3(z) = \sum_{r=1/2}^{\infty} \left(\Psi^3_{r-\nu_a} \, z^{-r+\nu_a-\frac{1}{2}} + \overline{\Psi}^3_{-r-\nu_a} \, z^{r+\nu_a-\frac{1}{2}} \right) . \tag{5.9}$$

The oscillators $\Psi^2_{-r-\nu_a}$, $\overline{\Psi}^2_{-r+\nu_a}$, $\Psi^3_{-r+\nu_a}$ and $\overline{\Psi}^3_{-r-\nu_a}$ are creation modes acting on the twisted vacuum of the a-th sector which we denote by $|\Omega_a\rangle$. Such a state is defined by

$$|\Omega_a\rangle = \lim_{z\to 0} \sigma_a(z) \, s_a(z) \, |0\rangle \tag{5.10}$$

where $|0\rangle$ is the Fock vacuum and $\sigma_a(z)$ and $s_a(z)$ are, respectively, the bosonic and fermionic twist fields [5]. More precisely, these twist fields take the form

$$\sigma_a(z) = \sigma^2_{\nu_a}(z) \, \sigma^3_{1-\nu_a}(z) \quad \text{and} \quad s_a(z) = s^2_{\nu_a}(z) \, s^3_{-\nu_a}(z) \, , \tag{5.11}$$

where the superscripts refer to the complex directions where the twist takes place, and the subscripts indicate the twist parameters. The bosonic twist field $\sigma_a(z)$ is a conformal field of weight $\nu_a(1-\nu_a)$ while the fermionic twist field $s_a(w)$ is a conformal field of weight ν_a^2. Therefore, the total conformal weight of the operator associated to the twisted ground state is ν_a. This means that $|\Omega_a\rangle$ is massive with a mass m given by

$$m^2 = \nu_a - \frac{1}{2} < 0 \, . \tag{5.12}$$

This tachyonic state is removed by the GSO projection.

The first set of physical states one finds in the GSO projected spectrum are those obtained by acting with one fermionic creation mode with index $r = \frac{1}{2}$ on the twisted vacuum. In particular, the oscillators $\Psi^3_{-\frac{1}{2}+\nu_a}$ and $\overline{\Psi}^2_{-\frac{1}{2}+\nu_a}$ increase the energy by $(\frac{1}{2}-\nu_a)$ and thus, when acting on the twisted vacuum, they create two massless states [1]. The vertex operators

[1]The oscillators $\Psi^2_{-\frac{1}{2}-\nu_a}$ and $\overline{\Psi}^3_{-\frac{1}{2}-\nu_a}$, instead, carry an energy $(\frac{1}{2}+\nu_a)$ and, upon acting on the twisted vacuum, they create massive states with $m^2 = 2\nu_a$.

corresponding to these massless excitations, in the (-1)-superghost picture and at zero momentum [2], are:

$$
\begin{aligned}
\mathcal{V}^1_a(z) &= \sigma_a(z) :\Psi^3(w)\, s_a(z) \colon \mathrm{e}^{-\phi(z)} \,, \\
\mathcal{V}^2_a(z) &= \sigma_a(z) :\overline{\Psi}^2(w)\, s_a(z) \colon \mathrm{e}^{-\phi(z)} \,.
\end{aligned}
\tag{5.13}
$$

Here $\phi(z)$ is the bosonic field appearing in the bosonization formulas of the superghosts [9] and, as usual, the symbol : : denotes the normal ordering. The vertex operators (5.13) are conformal fields of weight 1 and we collectively denote them as $\mathcal{V}^\alpha_a(z)$ with $\alpha = 1, 2$. As explained in Appendix A.0.1, they form a doublet transforming as a spinor of $\mathrm{SU}(2)_+$.

In the right-moving part, the monodromy properties (5.7b) lead to the following mode expansions for the fermionic fields

$$
\begin{aligned}
\widetilde{\Psi}^2(\bar z) &= \sum_{r=1/2}^{\infty} \left(\widetilde{\overline{\Psi}}^2_{r+\nu_a}\, \bar z^{-r-\nu_a-\frac12} + \widetilde{\Psi}^2_{-r+\nu_a}\, \bar z^{r-\nu_a-\frac12} \right), \\
\widetilde{\overline{\Psi}}^2(\bar z) &= \sum_{r=1/2}^{\infty} \left(\widetilde{\Psi}^2_{r-\nu_a}\, \bar z^{-r+\nu_a-\frac12} + \widetilde{\overline{\Psi}}^2_{-r-\nu_a}\, \bar z^{r+\nu_a-\frac12} \right),
\end{aligned}
\tag{5.14}
$$

and

$$
\begin{aligned}
\widetilde{\Psi}^3(\bar z) &= \sum_{r=1/2}^{\infty} \left(\widetilde{\overline{\Psi}}^3_{r-\nu_a}\, \bar z^{-r+\nu_a-\frac12} + \widetilde{\Psi}^3_{-r-\nu_a}\, \bar z^{r+\nu_a-\frac12} \right), \\
\widetilde{\overline{\Psi}}^3(\bar z) &= \sum_{r=1/2}^{\infty} \left(\widetilde{\Psi}^3_{r+\nu_a}\, \bar z^{-r-\nu_a-\frac12} + \widetilde{\overline{\Psi}}^3_{-r+\nu_a}\, \bar z^{r-\nu_a-\frac12} \right).
\end{aligned}
\tag{5.15}
$$

The oscillators $\widetilde{\Psi}^2_{-r+\nu_a}$, $\widetilde{\overline{\Psi}}^2_{-r-\nu_a}$, $\widetilde{\Psi}^3_{-r-\nu_a}$ and $\widetilde{\overline{\Psi}}^3_{-r+\nu_a}$ are creation modes acting on the twisted vacuum of the right sector which we denote by $|\widetilde{\Omega}_a\rangle$. This is defined by

$$
|\widetilde{\Omega}_a\rangle = \lim_{\bar z \to 0} \widetilde{\sigma}_a(\bar z)\, \widetilde{s}_a(\bar z)\, |\widetilde{0}\rangle
\tag{5.16}
$$

[2] The reason to write the vertex operators at zero momentum is because, as in chapter 4, ultimately we will be interested in describing a constant twisted closed string background to account for the continuous parameters of the GW surface defects.

where $|\widetilde{0}\rangle$ is the Fock vacuum of this sector and

$$\widetilde{\sigma}_a(\bar{z}) = \widetilde{\sigma}^2_{1-\nu_a}(\bar{z})\,\widetilde{\sigma}^3_{\nu_a}(\bar{z}) \quad \text{and} \quad \widetilde{s}_a(\bar{z}) \;=\; \widetilde{s}^2_{-\nu_a}(\bar{z})\,\widetilde{s}^3_{\nu_a}(\bar{z})\,. \tag{5.17}$$

The bosonic twist field $\widetilde{\sigma}_a(\bar{z})$ is a conformal field of weight $(1-\nu_a)\nu_a$ while the fermionic twist field $\widetilde{s}_a(\bar{z})$ is a conformal field of weight ν_a^2, so that the total conformal weight of the operator associated to $|\widetilde{\Omega}_a\rangle$ is ν_a. The right-moving ground state is then tachyonic with a mass given by (5.12) and it is removed by the GSO projection.

The first set of physical states in the GSO projected spectrum are those created by a fermionic creation mode with index $r = \frac{1}{2}$. In particular those generated by the oscillators $\widetilde{\Psi}^2_{-\frac{1}{2}+\nu_a}$ and $\widetilde{\overline{\Psi}}^3_{-\frac{1}{2}+\nu_a}$ are massless since the energy carried by these modes exactly cancels that of the vacuum. Therefore, the vertex operators at zero momentum associated to these right-moving massless excitations in the (-1)-superghost picture are:

$$\begin{aligned}\widetilde{\mathcal{V}}^1_a(\bar{z}) &= -\widetilde{\sigma}_a(\bar{z}) :\widetilde{\Psi}^2(\bar{z})\,\widetilde{s}_a(\bar{z}):\mathrm{e}^{-\widetilde{\phi}(\bar{z})}\,,\\ \widetilde{\mathcal{V}}^2_a(\bar{z}) &= \quad\widetilde{\sigma}_a(\bar{z}) :\widetilde{\overline{\Psi}}^3(\bar{z})\,\widetilde{s}_a(\bar{z}):\mathrm{e}^{-\widetilde{\phi}(\bar{z})}\,.\end{aligned} \tag{5.18}$$

These are conformal fields of weight 1 and we collectively denote them as $\widetilde{\mathcal{V}}^\beta_a(\bar{z})$ with $\beta = 1, 2$. We point out that the $-$ sign in the first line above is introduced because in this way the two operators form a doublet transforming in the spinor representation of $\mathrm{SU}(2)_+$, as explained in Appendix A.0.1.

Sectors with twist parameter $(1-\nu_a) > \frac{1}{2}$

Apart from a few subtleties, the conclusions obtained in the previous subsection for the twisted sectors with $\nu_a < \frac{1}{2}$, are valid also in the twisted sectors with $(1-\nu_a) > \frac{1}{2}$ provided one exchanges the role of the complex directions 2 and 3, and uses the sector label $(M-a)$. Thus, we can rather brief in our presentation.

In the left-moving part, the fermionic creation modes are the oscillators $\Psi^2_{-r+\nu_a}$, $\overline{\Psi}^2_{-r-\nu_a}$, $\Psi^3_{-r-\nu_a}$ and $\overline{\Psi}^3_{-r+\nu_a}$ where r is a positive half-integer. They act on the twisted vacuum $|\Omega_{M-a}\rangle$ which is defined by

$$|\Omega_{M-a}\rangle = \lim_{z\to 0} \sigma_{M-a}(z)\, s_{M-a}(z)\,|0\rangle \tag{5.19}$$

with

$$\sigma_{M-a}(z) = \sigma^2_{1-\nu_a}(z)\,\sigma^3_{\nu_a}(z)\ , \quad \text{and} \quad s_{M-a}(z) = s^2_{-\nu_a}(z)\, s^3_{\nu_a}(z)\ . \tag{5.20}$$

The ground state $|\Omega_{M-a}\rangle$ is tachyonic with a mass given by (5.12) and is removed by the GSO projection. At the first excited level, instead, we find two massless states created by the oscillators $\Psi^2_{-\frac{1}{2}+\nu_a}$ and $\overline{\Psi}^3_{-\frac{1}{2}+\nu_a}$, which correspond to the following vertex operators at zero momentum:

$$\begin{aligned} \mathcal{V}^1_{M-a}(z) &= -\sigma_{M-a}(z) :\Psi^2(z)\, s_{M-a}(z):\, \mathrm{e}^{-\phi(z)}\ , \\ \mathcal{V}^2_{M-a}(z) &= \sigma_{M-a}(z) :\overline{\Psi}^3(z)\, s_{M-a}(z):\, \mathrm{e}^{-\phi(z)}\ . \end{aligned} \tag{5.21}$$

These are conformal fields of weight 1 which we collectively denote as $\mathcal{V}^\alpha_{M-a}(z)$ with $\alpha = 1, 2$. Again the $-$ sign in the first line is inserted so that these two operators transform as a doublet in the spinor representation of $\mathrm{SU}(2)_+$ (see Appendix A.0.1).

Finally, in the right-moving part the oscillators $\widetilde{\Psi}^2_{-r-\nu_a}$, $\widetilde{\overline{\Psi}}^2_{-r+\nu_a}$, $\widetilde{\Psi}^3_{-r+\nu_a}$ and $\widetilde{\overline{\Psi}}^3_{-r-\nu_a}$ where r is a positive half-integer, are creation modes. They act on the twisted vacuum defined by

$$|\widetilde{\Omega}_{M-a}\rangle = \lim_{\bar{z}\to 0} \widetilde{\sigma}_{M-a}(\bar{z})\,\widetilde{s}_{M-a}(\bar{z})\,|0\rangle \tag{5.22}$$

where

$$\widetilde{\sigma}_{M-a}(\bar{z}) = \widetilde{\sigma}^2_{\nu_a}(\bar{z})\,\widetilde{\sigma}^3_{1-\nu_a}(\bar{z})\ , \quad \text{and} \quad \widetilde{s}_{M-a}(\bar{z}) = \widetilde{s}^2_{\nu_a}(\bar{z})\,\widetilde{s}^3_{-\nu_a}(\bar{z})\ . \tag{5.23}$$

As before this vacuum state is tachyonic and removed by the GSO projection. On the other hand, the states created by $\widetilde{\overline{\Psi}}^2_{-\frac{1}{2}+\nu_a}$ and $\widetilde{\Psi}^3_{-\frac{1}{2}+\nu_a}$ are massless and selected by the

GSO projection. They correspond to the following vertex operators at zero momentum:

$$\begin{aligned}\widetilde{\mathcal{V}}^1_{M-a}(\bar{z}) &= \widetilde{\sigma}_{M-a}(\bar{z}) : \widetilde{\Psi}^3(\bar{z})\, \widetilde{s}_{M-a}(\bar{z}) : \mathrm{e}^{-\widetilde{\phi}(\bar{z})} \,, \\ \widetilde{\mathcal{V}}^2_{M-a}(\bar{z}) &= \widetilde{\sigma}_{M-a}(\bar{z}) : \widetilde{\overline{\Psi}}^2(\bar{z})\, \widetilde{s}_{M-a}(\bar{z}) : \mathrm{e}^{-\widetilde{\phi}(\bar{z})} \,,\end{aligned} \tag{5.24}$$

which are conformal fields of weight 1. We collectively denote these vertex operators as $\widetilde{\mathcal{V}}^\beta_{M-a}(\bar{z})$ with $\beta = 1, 2$, since they transform as a doublet of $\mathrm{SU}(2)_+$ (see Appendix A.0.1).

We summarize our results on the massless vertex operators of the twisted NS sectors in Table 5.1 below.

Vertex operator	State
$\mathcal{V}^1_a(z) = \sigma_a(z)\ :\Psi^3(z)\, s_a(z): \mathrm{e}^{-\phi(z)}$	$\Psi^3_{-\frac{1}{2}+\nu_a}\, \lvert\Omega_a\rangle_{(-1)}$
$\mathcal{V}^2_a(z) = \sigma_a(z)\ :\overline{\Psi}^2(z)\, s_a(z): \mathrm{e}^{-\phi(z)}$	$\overline{\Psi}^2_{-\frac{1}{2}+\nu_a}\, \lvert\Omega_a\rangle_{(-1)}$
$\widetilde{\mathcal{V}}^1_a(\bar{z}) = -\widetilde{\sigma}_a(\bar{z})\ :\widetilde{\Psi}^2(\bar{z})\, \widetilde{s}_a(\bar{z}): \mathrm{e}^{-\widetilde{\phi}(\bar{z})}$	$-\widetilde{\Psi}^2_{-\frac{1}{2}+\nu_a}\, \lvert\widetilde{\Omega}_a\rangle_{(-1)}$
$\widetilde{\mathcal{V}}^2_a(\bar{z}) = \widetilde{\sigma}_a(\bar{z})\ :\widetilde{\overline{\Psi}}^3(\bar{z})\, \widetilde{s}_a(\bar{z}) : \mathrm{e}^{-\widetilde{\phi}(\bar{z})}$	$\widetilde{\overline{\Psi}}^3_{-\frac{1}{2}+\nu_a}\, \lvert\widetilde{\Omega}_a\rangle_{(-1)}$
$\mathcal{V}^1_{M-a}(z) = -\sigma_{M-a}(z)\ :\Psi^2(z)\, s_{M-a}(z): \mathrm{e}^{-\phi(z)}$	$-\Psi^2_{-\frac{1}{2}+\nu_a}\, \lvert\Omega_{M-a}\rangle_{(-1)}$
$\mathcal{V}^2_{M-a}(z) = \sigma_{M-a}(z)\ :\overline{\Psi}^3(z)\, s_{M-a}(z): \mathrm{e}^{-\phi(z)}$	$\overline{\Psi}^3_{-\frac{1}{2}+\nu_a}\, \lvert\Omega_{M-a}\rangle_{(-1)}$
$\widetilde{\mathcal{V}}^1_{M-a}(\bar{z}) = \widetilde{\sigma}_{M-a}(\bar{z})\ :\widetilde{\Psi}^3(\bar{z})\, \widetilde{s}_{M-a}(\bar{z}): \mathrm{e}^{-\widetilde{\phi}(\bar{z})}$	$\widetilde{\Psi}^3_{-\frac{1}{2}+\nu_a}\, \lvert\widetilde{\Omega}_{M-a}\rangle_{(-1)}$
$\widetilde{\mathcal{V}}^2_{M-a}(\bar{z}) = \widetilde{\sigma}_{M-a}(\bar{z})\ :\widetilde{\overline{\Psi}}^2(\bar{z})\, \widetilde{s}_{M-a}(\bar{z}): \mathrm{e}^{-\widetilde{\phi}(\bar{z})}$	$\widetilde{\overline{\Psi}}^2_{-\frac{1}{2}+\nu_a}\, \lvert\widetilde{\Omega}_{M-a}\rangle_{(-1)}$

Table 5.1: The vertex operators and the corresponding states in the left- and right-moving parts of the various twisted NS sectors. Here the label a takes values in the range $[1, \frac{M-1}{2}]$, and in the last column the subscript (-1) on the kets identifies the superghost picture.

Two-point functions in the twisted NS sectors

Given the explicit form of the vertex operators that we have derived, it is rather straightforward to compute their two-point functions. As a first step, we observe that there are no non-vanishing correlators between left (or right) operators of the same twisted sector, due to the presence of the bosonic twist fields; in fact for any complex direction j one has [5]

$$\langle \sigma^j_{\nu_a}(z_1)\, \sigma^j_{\nu_b}(z_2) \rangle = \frac{\delta_{\nu_b, 1-\nu_a}}{(z_1 - z_2)^{\nu_a(1-\nu_a)}} \ , \tag{5.25}$$

and similarly in the right sector. This implies that only the correlator $\langle \sigma_a(w_1)\, \sigma_{M-a}(w_2) \rangle$ is non vanishing. Therefore, only the two-point functions between vertex operators in sectors a and $(M - a)$ are non-zero. Another important point to consider is that these vertex operators inherit the fermionic statistics from the fermionic fields that are present in their definitions.

Let us then compute the two-point function between $\mathcal{V}^1_a$ and $\mathcal{V}^2_{M-a}$. Using (5.25) and the basic conformal field theory correlators

$$\begin{aligned} \langle :\Psi^3(z_1)\, s_a(z_1):\, :\overline{\Psi}^3(z_2)\, s_{M-a}(z_2): \rangle &= \frac{1}{(z_1 - z_2)^{1-2\nu_a(1-\nu_a)}} \ , \\ \langle \mathrm{e}^{-\phi(z_1)}\, \mathrm{e}^{-\phi(z_2)} \rangle &= \frac{1}{z_1 - z_2} \ , \end{aligned} \tag{5.26}$$

we obtain

$$\langle \mathcal{V}^1_a(z_1)\, \mathcal{V}^2_{M-a}(z_2) \rangle = \frac{1}{(z_1 - z_2)^2} \ . \tag{5.27}$$

In a similar way, using

$$\langle :\overline{\Psi}^2(z_1)\, s_a(z_1):\, :\Psi^2(z_2)\, s_{M-a}(z_2): \rangle = \frac{1}{(z_1 - z_2)^{1-2\nu_a(1-\nu_a)}} \ , \tag{5.28}$$

and taking into account the explicit negative sign in $\mathcal{V}^1_{M-a}$, we get

$$\langle \mathcal{V}^2_a(z_1)\, \mathcal{V}^1_{M-a}(z_2) \rangle = \frac{-1}{(z_1 - z_2)^2} \ . \tag{5.29}$$

Furthermore, the two-point functions between $\mathcal{V}^1_a$ and $\mathcal{V}^1_{M-a}$ and between $\mathcal{V}^2_a$ and $\mathcal{V}^2_{M-a}$ vanish since their fermionic charges do not match. Thus, altogether, we have

$$\langle \mathcal{V}^\alpha_a(z_1)\, \mathcal{V}^\beta_{M-a}(z_2) \rangle = \frac{(\epsilon^{-1})^{\alpha\beta}}{(z_1 - z_2)^2} \tag{5.30}$$

where we have defined

$$\epsilon = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} . \tag{5.31}$$

By taking into account the fermionic statistics of the vertex operators and the anti-symmetry of ϵ, we also find

$$\langle \mathcal{V}^{\alpha}_{M-a}(z_1)\, \mathcal{V}^{\beta}_{a}(z_2)\rangle = \frac{(\epsilon^{-1})^{\alpha\beta}}{(z_1 - z_2)^2} . \tag{5.32}$$

Notice that (5.30) and (5.32) may be unified in a single formula by promoting the index a to the complete index $\widehat{a}$. This shows that despite the differences in the structure of the states and vertex operators in the fermionic formalism, all twisted sectors are actually treated on equal footing.

Similarly, in the right-moving sector, we obtain

$$\langle \widetilde{\mathcal{V}}^{\alpha}_{M-a}(\bar{z}_1)\, \widetilde{\mathcal{V}}^{\beta}_{a}(\bar{z}_2)\rangle = \langle \widetilde{\mathcal{V}}^{\alpha}_{a}(\bar{z}_1)\, \widetilde{\mathcal{V}}^{\beta}_{M-a}(\bar{z}_2)\rangle = \frac{(\epsilon^{-1})^{\alpha\beta}}{(\bar{z}_1 - \bar{z}_2)^2} . \tag{5.33}$$

From these two-point functions it is possible to infer the conjugate vertex operators as follows:

$$\begin{aligned} (\mathcal{V}_{M-a}(z))^{\dagger}_{\alpha} &= \mathcal{V}^{\beta}_{a}(z)\, \epsilon_{\beta\alpha} , \quad (\mathcal{V}_{a}(z))^{\dagger}_{\alpha} = \mathcal{V}^{\beta}_{M-a}(z)\, \epsilon_{\beta\alpha} , \\ (\widetilde{\mathcal{V}}_{a}(z))^{\dagger}_{\alpha} &= \widetilde{V}^{\beta}_{M-a}(z)\, \epsilon_{\beta\alpha} , \quad (\widetilde{\mathcal{V}}_{M-a}(z))^{\dagger}_{\alpha} = \widetilde{V}^{\beta}_{a}(z)\, \epsilon_{\beta\alpha} . \end{aligned} \tag{5.34}$$

5.1.3 The massless NS/NS vertex operators

The massless closed string excitations in the twisted NS/NS sectors are obtained by combining the left- and right-moving massless states that we have obtained in the previous subsection. In the sectors with twist parameter $\nu_a < \frac{1}{2}$, they are then described by the following vertex operators at zero momentum

$$b^{(a)}_{\alpha\beta}\, \mathcal{V}^{\alpha}_{a}(z)\, \widetilde{\mathcal{V}}^{\beta}_{a}(\bar{z}) \tag{5.35}$$

where $b^{(a)}_{\alpha\beta}$ are four constant complex fields.

Similarly, in the sectors with twist parameter $(1-\nu_a) > \frac{1}{2}$, the massless closed string excitations are described by the vertex operators at zero momentum

$$b^{(M-a)}_{\alpha\beta}\,\mathcal{V}^{\alpha}_{M-a}(z)\,\widetilde{\mathcal{V}}^{\beta}_{M-a}(\bar{z}) \tag{5.36}$$

where again $b^{(M-a)}_{\alpha\beta}$ are four constant complex fields.

The constants $b^{(a)}$ and $b^{(M-a)}$ can be considered as a background in which the string theory on the orbifold is defined. Given the structure of the vertex operators, there are non-trivial relations among them. In particular, using (5.34) one finds that

$$\left(b^{(a)}_{\alpha\beta}\,\mathcal{V}^{\alpha}_{a}(z)\,\widetilde{\mathcal{V}}^{\beta}_{a}(\bar{z})\right)^{\dagger} = b^{(M-a)}_{\alpha\beta}\,\mathcal{V}^{\alpha}_{M-a}(z)\,\widetilde{\mathcal{V}}^{\beta}_{M-a}(\bar{z}) \tag{5.37}$$

where

$$b^{(M-a)}_{11} = -b^{(a)\,\star}_{22}\ , \quad b^{(M-a)}_{12} = b^{(a)\,\star}_{21}\ , \quad b^{(M-a)}_{21} = b^{(a)\,\star}_{12}\ , \quad b^{(M-a)}_{22} = -b^{(a)\,\star}_{11}\ , \tag{5.38}$$

or, equivalently in matrix notation,

$$b^{(M-a)} = \epsilon\, b^{(a)\,\star}\,\epsilon\ . \tag{5.39}$$

These relations, which also appear in [3], show that if one turns on background values for the closed string fields in the twisted sector a, one also turns on background values for the fields in the twisted sector $(M-a)$ and *viceversa*, in such a way that the total background configuration is real.

5.1.4 Twisted R sectors

The $\mathbb{Z}_M$ orbifold (1.1) breaks the isometry of the ten-dimensional space as follows:

$$\mathrm{SO}(10) \;\longrightarrow\; \mathrm{SO}(6) \times \mathrm{SO}(2) \times \mathrm{SO}(2) \,, \tag{5.40}$$

where SO(6) acts on the first, fourth and fifth complex directions, which are not affected by the orbifold action. Correspondingly, the untwisted vacuum of the R sector which carries the 32-dimensional spinor representation of SO(10) decomposes into eight massless spinors of SO(6). Four of these are chiral and four anti-chiral. We denote the four chiral vacuum states by

$$\left|A, \pm\frac{1}{2}, \pm\frac{1}{2}\right\rangle \tag{5.41}$$

where $A \in \mathbf{4}$ labels the four different components of the chiral spinor representation of SO(6) and the four pairs of $\pm\frac{1}{2}$ denote the spinor weights along the second and third complex directions where the orbifold acts. Similarly, the four anti-chiral vacuum states are denoted by

$$\left|\dot{A}, \pm\frac{1}{2}, \pm\frac{1}{2}\right\rangle \tag{5.42}$$

where $\dot{A} \in \bar{\mathbf{4}}$ spans the four-dimensional anti-chiral spinor representation of SO(6).

In the twisted R sectors, not all such chiral and anti-chiral states remain massless. Indeed, the fermionic twist fields change the spinor weights in the orbifolded directions, so that conformal dimensions and the GSO parities of the corresponding vertex operators are modified. In the following we present a brief description of the spectrum in the various twisted R sectors, focusing on the massless excitations.

Sectors with twist parameter $\nu_a < \frac{1}{2}$

In these sectors the left-moving bosonic and fermionic twist fields σ_a and s_a are given in (5.11). When we act with s_a on the states (5.41) and (5.42), the charges in the directions 2 and 3 become

$$\varepsilon_2 = \pm\frac{1}{2} + \nu_a \quad \text{and} \quad \varepsilon_3 = \pm\frac{1}{2} - \nu_a \tag{5.43}$$

depending on their initial values. Because of this, not all choices of signs lead to massless configurations. In fact, the mass vanishes only if

$$\varepsilon_2^2 = \varepsilon_3^2 = \left(\frac{1}{2} - \nu_a\right)^2 . \tag{5.44}$$

Combining this with (5.43), we see that the only solution is

$$\varepsilon_2 = -\varepsilon_3 = -\frac{1}{2} + \nu_a , \tag{5.45}$$

so that, instead of $s_a(z)$, we can consider the effective fermionic twist

$$r_a(z) = s^2_{\nu_a-\frac{1}{2}}(z)\, s^3_{-\nu_a+\frac{1}{2}}(z) \tag{5.46}$$

which is a conformal field of weight $(\frac{1}{4} - \nu_a(1-\nu_a))$.

In the R sector, there are two fundamental superghost pictures that one considers: the $(-\frac{1}{2})$- and the $(-\frac{3}{2})$-pictures [9]. Enforcing the GSO projection, in the $(-\frac{1}{2})$-picture one selects the chiral spinor of SO(6), while in the $(-\frac{3}{2})$-picture one selects the anti-chiral one. In this way, in fact, the sum of the spinor weights minus the superghost-charge is always an even integer. Thus we are led to introduce the following two vertex operators at zero momentum

$$\mathcal{V}_a^A(z) = \sigma_a(z)\, r_a(z)\, S^A(z) \mathrm{e}^{-\frac{1}{2}\phi(z)} , \tag{5.47a}$$

$$\mathcal{V}_a^{\dot{A}}(z) = \sigma_a(z)\, r_a(z)\, S^{\dot{A}}(z) \mathrm{e}^{-\frac{3}{2}\phi(z)} , \tag{5.47b}$$

where S^A and $S^{\dot{A}}$ are, respectively, the chiral and anti-chiral spin-fields of SO(6) [9, 10]. Both vertex operators are conformal fields of weight 1 and define the following massless twisted vacuum states:

$$\begin{aligned} |A_a\rangle_{(-\frac{1}{2})} &= \lim_{z\to 0} \mathcal{V}_a^{A}(z)\, |0\rangle , \\ |\dot{A}_a\rangle_{(-\frac{3}{2})} &= \lim_{z\to 0} \mathcal{V}_a^{\dot{A}}(z)\, |0\rangle . \end{aligned} \tag{5.48}$$

As far as the right-moving part is concerned, the bosonic and fermionic twist fields are given in (5.17). Therefore, we can repeat the previous analysis by simply replacing everywhere ν_a with $(1-\nu_a)$. In this way we find the following two physical vertex operators of weight 1:

$$\widetilde{\mathcal{V}}_a^{A}(\bar{z}) = \widetilde{\sigma}_a(\bar{z})\widetilde{r}_a(\bar{z})\, \widetilde{S}^{A}(\bar{z})\, \mathrm{e}^{-\frac{1}{2}\widetilde{\phi}(\bar{z})} , \tag{5.49a}$$

$$\widetilde{\mathcal{V}}_a^{\dot{A}}(\bar{z}) = \widetilde{\sigma}_a(\bar{z})\widetilde{r}_a(\bar{z})\, \widetilde{S}^{\dot{A}}(\bar{z})\, \mathrm{e}^{-\frac{3}{2}\widetilde{\phi}(\bar{z})} , \tag{5.49b}$$

where the effective fermionic twist is given by

$$\widetilde{r}_a(\bar{z}) = \widetilde{s}^{2}_{-\nu_a+\frac{1}{2}}(\bar{z})\, \widetilde{s}^{3}_{\nu_a-\frac{1}{2}}(\bar{z}) . \tag{5.50}$$

The massless states corresponding to these vertex operators are

$$\begin{aligned} |\widetilde{A}_a\rangle_{(-\frac{1}{2})} &= \lim_{\bar{z}\to 0} \widetilde{\mathcal{V}}_a^{A}(\bar{z})\, |0\rangle , \\ |\widetilde{\dot{A}}_a\rangle_{(-\frac{3}{2})} &= \lim_{\bar{z}\to 0} \widetilde{\mathcal{V}}_a^{\dot{A}}(\bar{z})\, |0\rangle . \end{aligned} \tag{5.51}$$

Sectors with twist parameter $(1-\nu_a) > \frac{1}{2}$

These sectors can be described in the same manner as before by simply exchanging the roles of the complex directions 2 and 3, and using $(M-a)$ as twist label. Thus, we merely present the physical GSO projected massless vertex operators at zero momentum. In the left-moving part they are

$$\mathcal{V}^{A}_{M-a}(z) = \sigma_{M-a}(z)\, r_{M-a}(z)\, S^{A}(z)\, \mathrm{e}^{-\frac{1}{2}\phi(z)}\ , \tag{5.52a}$$

$$\mathcal{V}^{\dot{A}}_{M-a}(z) = \sigma_{M-a}(z)\, r_{M-a}(z)\, S^{\dot{A}}(z)\, \mathrm{e}^{-\frac{3}{2}\phi(z)}\ , \tag{5.52b}$$

with

$$r_{M-a}(z) = s^{2}_{-\nu_a+\frac{1}{2}}(z)\, s^{3}_{\nu_a-\frac{1}{2}}(z)\ . \tag{5.53}$$

In the right-moving part, instead, they are

$$\widetilde{\mathcal{V}}^{A}_{M-a}(\bar{z}) = \widetilde{\sigma}_{M-a}(\bar{z})\widetilde{r}_{M-a}(\bar{z})\, \widetilde{S}^{A}(\bar{z})\, \mathrm{e}^{-\frac{1}{2}\widetilde{\phi}(\bar{z})}\ , \tag{5.54a}$$

$$\widetilde{\mathcal{V}}^{\dot{A}}_{M-a}(\bar{z}) = \widetilde{\sigma}_{M-a}(\bar{z})\widetilde{r}_{M-a}(\bar{z})\, \widetilde{S}^{\dot{A}}(\bar{z})\, \mathrm{e}^{-\frac{3}{2}\widetilde{\phi}(\bar{z})}\ . \tag{5.54b}$$

where

$$\widetilde{r}_{M-a}(\bar{z}) = \widetilde{s}^{2}_{\nu_a-\frac{1}{2}}(\bar{z})\, \widetilde{s}^{3}_{-\nu_a+\frac{1}{2}}(\bar{z})\ . \tag{5.55}$$

When acting on the Fock vacuum these vertex operators create the twisted ground states which have the same expressions as in (5.48) and (5.51) with the obvious changes in notation.

We summarize our findings in Table 5.2 below.

Two-point functions in the twisted R sectors

As we have seen in the twisted NS sectors, the only non-vanishing two-point functions necessarily involve the left-moving (or right-moving) vertex operators in complementary sectors a and $(M-a)$, because of the two-point functions (5.25). Of course, the same is true in the twisted R sectors. Furthermore, in order to soak up the background charge in the superghost sector, only the overlaps between states in the $(-\frac{1}{2})$- and $(-\frac{3}{2})$-pictures, or *viceversa*, are non-zero. Taking this into account and using standard results from confor-

Vertex operator	State
$\mathcal{V}_a^A(z) = \sigma_a(z)\, r_a(z)\, S^A(z)\mathrm{e}^{-\frac{1}{2}\phi(z)}$	$\lvert A_a\rangle_{(-\frac{1}{2})}$
$\mathcal{V}_a^{\dot A}(z) = \sigma_a(z)\, r_a(z)\, S^{\dot A}(z)\mathrm{e}^{-\frac{3}{2}\phi(z)}$	$\lvert \dot A_a\rangle_{(-\frac{3}{2})}$
$\widetilde{\mathcal{V}}_a^A(\bar z) = \widetilde\sigma_a(\bar z)\widetilde r_a(\bar z)\,\widetilde S^A(\bar z)\,\mathrm{e}^{-\frac{1}{2}\widetilde\phi(\bar z)}$	$\lvert \widetilde A_a\rangle_{(-\frac{1}{2})}$
$\widetilde{\mathcal{V}}_a^{\dot A}(\bar z) = \widetilde\sigma_a(\bar z)\widetilde r_a(\bar z)\,\widetilde S^{\dot A}(\bar z)\,\mathrm{e}^{-\frac{3}{2}\widetilde\phi(\bar z)}$	$\lvert \widetilde{\dot A}_a\rangle_{(-\frac{3}{2})}$
$\mathcal{V}_{M-a}^A(z) = \sigma_{M-a}(z)\, r_{M-a}(z)\, S^A(z)\,\mathrm{e}^{-\frac{1}{2}\phi(z)}$	$\lvert A_{M-a}\rangle_{(-\frac{1}{2})}$
$\mathcal{V}_{M-a}^{\dot A}(z) = \sigma_{M-a}(z)\, r_{M-a}(z)\, S^{\dot A}(z)\,\mathrm{e}^{-\frac{3}{2}\phi(z)}$	$\lvert \dot A_{M-a}\rangle_{(-\frac{3}{2})}$
$\widetilde{\mathcal{V}}_{M-a}^A(\bar z) = \widetilde\sigma_{M-a}(\bar z)\widetilde r_{M-a}(\bar z)\,\widetilde S^A(\bar z)\,\mathrm{e}^{-\frac{1}{2}\widetilde\phi(\bar z)}$	$\lvert \widetilde A_{M-a}\rangle_{(-\frac{1}{2})}$
$\widetilde{\mathcal{V}}_{M-a}^{\dot A}(\bar z) = \widetilde\sigma_{M-a}(\bar z)\widetilde r_{M-a}(\bar z)\,\widetilde S^{\dot A}(\bar z)\,\mathrm{e}^{-\frac{3}{2}\widetilde\phi(\bar z)}$	$\lvert \widetilde{\dot A}_{M-a}\rangle_{(-\frac{3}{2})}$

Table 5.2: The vertex operators and the corresponding states in the left- and right-moving parts of the twisted R sectors.

mal field theory, we find

$$\begin{aligned}
\langle \mathcal{V}_a^A(z_1)\, \mathcal{V}_{M-a}^{\dot B}(z_2)\rangle = \langle \mathcal{V}_{M-a}^A(z_1)\, \mathcal{V}_a^{\dot B}(z_2)\rangle = \frac{(C^{-1})^{A\dot B}}{(z_1 - z_2)^2}\ , \\
\langle \mathcal{V}_a^{\dot A}(z_1)\, \mathcal{V}_{M-a}^{B}(z_2)\rangle = \langle \mathcal{V}_{M-a}^{\dot A}(z_1)\, \mathcal{V}_a^{B}(z_2)\rangle = \frac{(C^{-1})^{\dot A B}}{(z_1 - z_2)^2}\ ,
\end{aligned} \tag{5.56}$$

where C is the charge conjugation matrix of SO(6) (see Appendix A.0.2). Of course, analogous correlators hold for the right-moving vertex operators.

From the first line of (5.56), we read the following conjugation rules

$$\begin{aligned}
(\mathcal{V}_{M-a}(z))^{\dagger}_{\dot B} = \mathcal{V}_a^A(z)\, C_{A\dot B}\ , \\
(\mathcal{V}_{a}(z))^{\dagger}_{\dot B} = \mathcal{V}_{M-a}^A(z)\, C_{A\dot B}\ ,
\end{aligned} \tag{5.57}$$

while from the second line we obtain the same relations with dotted and undotted indices exchanged. The same formulas apply also for the right-moving vertices.

5.1.5 The massless R/R vertex operators

The massless closed string excitations in the twisted R/R sectors are obtained by combining left and right movers. We shall work with the asymmetric superghost pictures $(-\frac{1}{2}, -\frac{3}{2})$ or $(-\frac{3}{2}, -\frac{1}{2})$, so that the corresponding closed string fields are R/R potentials. In the twisted sector labeled by a we choose the $(-\frac{1}{2}, -\frac{3}{2})$-picture and write the following massless vertex operators at zero momentum: [3].

$$C^{(a)}_{A\dot{B}}\, \mathcal{V}^{A}_{a}(z)\, \widetilde{\mathcal{V}}^{\dot{B}}_{a}(\bar{z}) \tag{5.58}$$

where $C^{(a)}_{A\dot{B}}$ are sixteen constant complex fields. These constants can be considered as a background in which the orbifold closed string theory is defined.

In the twisted sector labeled by $(M - a)$ we choose, instead, the other asymmetric superghost picture, namely the $(-\frac{3}{2}, -\frac{1}{2})$-picture, and consider the following massless vertex operators

$$C^{(M-a)}_{\dot{A}B}\, \mathcal{V}^{\dot{A}}_{M-a}(z)\, \widetilde{\mathcal{V}}^{B}_{M-a}(\bar{z}) \tag{5.59}$$

where $C^{(M-a)}_{A\dot{B}}$ are other sixteen constant complex fields contributing to the background in which the closed string propagates.

Notice that in writing the vertex operators (5.58) and (5.59) for the twisted R/R potentials, we have correlated the choice of picture numbers with the twisted sector. Of course, we could have made different choices, but they would lead to the same results. In fact, it is well-known that in a BRST invariant framework like ours, the way in which the superghost pictures are distributed is completely arbitrary, provided one satisfies the global constraints due to the presence of a background charge, and that the physical results do

[3] In [53] it is shown that the complete BRST invariant vertex operators in the asymmetric superghost pictures are an infinite sum of terms characterized by the number of superghost zero modes. For our purposes, however, only the first (and simplest) terms in these sums is relevant since all the others decouple and thus can be discarded.

not depend on this choice. However, our picture assignment is particularly convenient because it immediately implies that the R/R potentials in the a-th twisted sector are naturally related to those in the sector $(M-a)$ by complex conjugation, exactly as it happens in the twisted NS/NS sectors. Indeed, we have

$$\Big(C^{(a)}_{A\dot{B}}\,\mathcal{V}^{A}_{a}(z)\,\widetilde{\mathcal{V}}^{\dot{B}}_{a}(\bar{z})\Big)^{\dagger}=C^{(M-a)}_{\dot{A}B}\,\mathcal{V}^{\dot{A}}_{M-a}(z)\,\widetilde{\mathcal{V}}^{B}_{M-a}(\bar{z})\;, \tag{5.60}$$

where, in matrix notation,

$$C^{(M-a)}=C\,C^{(a)\star}\,C\;, \tag{5.61}$$

which is the strict analogue of (5.39) holding in the NS/NS sectors. We therefore see that by turning on a R/R background potential value in the twisted sector a, one also turns on a background R/R potential in the twisted sector $(M-a)$ and *viceversa*, in such a way that the total configuration is real.

5.2 Fractional D3-branes in the $\mathbb{Z}_M$ orbifold

We now turn to discuss the open strings in the $\mathbb{Z}_M$ orbifold with the aim of analyzing surface defects in $4d$ gauge theories engineered on stacks of (fractional) D3-branes. As is well-known, a D-brane introduces a boundary on the string world-sheet where non-trivial relations between the left and the right movers of the closed strings take place. We will investigate these relations using the boundary state formalism (for a review, see for example [6, 7]) and then will analyze the massless open string spectrum on the brane world-volume. Since our ultimate goal is to recover a string theory description of the surface defects in a $4d$ gauge theory, we place the (fractional) D3-branes in such a way that they are partially extended along the orbifold as originally proposed in [4]. More precisely, we take the D3-brane world-volume to be $\mathbb{C}_{(1)}\times\mathbb{C}_{(2)}$ in such a way that the orbifold action breaks the $4d$ Poincaré symmetry leaving unbroken the one in the first complex direction along which the surface defect is extended.

5.2.1 Boundary states and reflection rules

In the $\mathbb{Z}_M$ orbifold there are M different types of fractional D-branes, labeled by an index $I = 0, 1, \ldots, M-1$, corresponding to the M irreducible representations of $\mathbb{Z}_M$. A fractional D3-brane of type I can be described by a boundary state which contains an untwisted component $|U\rangle$, which is the same for all types of branes, and a twisted component $|T; I\rangle$, which depends on the type of brane considered:

$$|\mathrm{D3}; I\rangle = \mathcal{N}\,|U\rangle + \mathcal{N}'\,|T; I\rangle \tag{5.62}$$

where $\mathcal{N}$ and $\mathcal{N}'$ are appropriate normalization factors related to the brane tensions (whose explicit expression is not relevant for our purposes). This schematic structure holds of course both in the NS/NS and R/R sectors, which we now discuss in turn, focusing on the fermionic twisted components.

NS/NS sector

The twisted component of the boundary state for a fractional D3-brane of type I is a sum of $(M-1)$ terms which refer to the $(M-1)$ twisted sectors of the closed strings on the orbifold and whose coefficients have to be chosen in a specific way in order to have a consistent description of the D-brane. By this, we mean that the cylinder amplitude between two such boundary states, once translated into the open string channel, must correctly reproduce the $\mathbb{Z}_M$-invariant one-loop annulus amplitude. In [42] a thorough analysis of this issue was carried out in general, using the Cardy condition for the construction of consistent boundary states in rational conformal field theories [57]. Borrowing these results and adapting them to our case, we can write the twisted component of the boundary state for a D3-brane of type I in the NS/NS sector and its conjugate as follows:

$$
\begin{aligned}
|T;I\rangle_{\rm NS} &= \sum_{\widehat{a}=1}^{M-1} \sin\Big(\frac{\pi\widehat{a}}{M}\Big)\,\omega^{I\widehat{a}}\,|\widehat{a}\,\rangle\!\rangle_{\rm NS}\ ,\\
{}_{\rm NS}\langle T;I| &= \sum_{\widehat{a}=1}^{M-1} \sin\Big(\frac{\pi\widehat{a}}{M}\Big)\,\omega^{-I\widehat{a}}\;{}_{\rm NS}\langle\!\langle\widehat{a}\,|\ .
\end{aligned}
\tag{5.63}
$$

Here, the sum runs over all twisted sectors, ω is the M-th root of unity as in (5.3) and $|\widehat{a}\,\rangle\!\rangle_{\rm NS}$ is the GSO projected Ishibashi state for the twisted sector $\widehat{a}$. These Ishibashi states enforce the appropriate gluing conditions between the left-moving and right-moving modes. For our purposes, it is not necessary to write the complete expression of these Ishibashi states, but it is enough to write the terms which may have a non-zero overlap with the massless states of the closed string twisted sectors discussed in Section 5.1.

Let us suppose again that M is odd. If $\widehat{a} = a \in [1, \frac{M-1}{2}]$, we have

$$
|a\rangle\!\rangle_{\rm NS} = \Big(\mathrm{i}\,\overline{\Psi}^{2}_{-\frac{1}{2}+\nu_a}\,\widetilde{\Psi}^{2}_{-\frac{1}{2}+\nu_a} - \mathrm{i}\,\Psi^{3}_{-\frac{1}{2}+\nu_a}\,\widetilde{\overline{\Psi}}^{3}_{-\frac{1}{2}+\nu_a}\Big)|\Omega_a\rangle_{(-1)}\ |\widetilde{\Omega}_a\rangle_{(-1)} + \cdots \tag{5.64}
$$

where the ellipses stand for terms involving a higher number of oscillators or massive fermionic modes. The relative minus sign in the brackets of (5.64) is due to the fact that the complex direction 3 is transverse to the D3-brane while the complex direction 2 is longitudinal. If $\widehat{a} = (M-a)$, instead, we have

$$
|M-a\rangle\!\rangle_{\rm NS} = \Big(\mathrm{i}\,\Psi^{2}_{-\frac{1}{2}+\nu_a}\,\widetilde{\overline{\Psi}}^{2}_{-\frac{1}{2}+\nu_a} - \mathrm{i}\,\overline{\Psi}^{3}_{-\frac{1}{2}+\nu_a}\,\widetilde{\Psi}^{3}_{-\frac{1}{2}+\nu_a}\Big)|\Omega_{M-a}\rangle_{(-1)}\ |\widetilde{\Omega}_{M-a}\rangle_{(-1)} + \cdots\ . \tag{5.65}
$$

The corresponding Ishibashi bra states are

$$
\begin{aligned}
{}_{\rm NS}\langle\!\langle a| &= {}_{(-1)}\langle\widetilde{\Omega}_a|\ {}_{(-1)}\langle\Omega_a|\Big(-\mathrm{i}\,\widetilde{\Psi}^{2}_{\frac{1}{2}-\nu_a}\,\overline{\Psi}^{2}_{\frac{1}{2}-\nu_a} + \mathrm{i}\,\widetilde{\overline{\Psi}}^{3}_{\frac{1}{2}-\nu_a}\,\Psi^{3}_{\frac{1}{2}-\nu_a}\Big) + \cdots\\
{}_{\rm NS}\langle\!\langle M-a| &= {}_{(-1)}\langle\widetilde{\Omega}_{M-a}|\ {}_{(-1)}\langle\Omega_{M-a}|\Big(-\mathrm{i}\,\widetilde{\overline{\Psi}}^{2}_{\frac{1}{2}-\nu_a}\,\Psi^{2}_{\frac{1}{2}-\nu_a} + \mathrm{i}\,\widetilde{\Psi}^{3}_{\frac{1}{2}-\nu_a}\,\overline{\Psi}^{3}_{\frac{1}{2}-\nu_a}\Big) + \cdots
\end{aligned}
\tag{5.66}
$$

where the conjugate vacuum states are normalized in such a way that

$$ {}_{(-1)}\langle \Omega_a | \Omega_a \rangle_{(-1)} = 1 \quad \text{and} \quad {}_{(-1)}\langle \Omega_{M-a} | \Omega_{M-a} \rangle_{(-1)} = 1 \,, \tag{5.67} $$

and similarly for the right-moving sectors.

When the fractional D3-branes are present, the left and right moving parts of a twisted closed string have non-trivial correlation functions since the closed string world-sheet has a boundary. In the boundary state formalism, this boundary is the unit circle on which the Ishibashi states enforce an identification between the left and the right movers of the closed strings. In particular for the massless vertex operators of the twisted NS sector with label a and twist parameter ν_a described in Section 5.1.2, given any two points w and $\bar{w}$ inside the unit disk $\mathbb{D}$ corresponding to a D3-brane of type I, we have

$$ \langle \mathcal{V}_a^\alpha(w)\, \widetilde{\mathcal{V}}_a^\beta(\bar{w}) \rangle_I \;\equiv\; {}_{\mathrm{NS}}\langle T; I |\, \mathcal{V}_a^\alpha(w)\, \widetilde{\mathcal{V}}_a^\beta(\bar{w}) |0\rangle|\widetilde{0}\rangle = \frac{M_{I,a}^{\alpha\beta}}{(1 - w\bar{w})^2} \,, \tag{5.68} $$

where the last step is a consequence of the conformal invariance which fixes the form of the two-point function of conformal fields of weight 1 on $\mathbb{D}$. The constant in the numerator can be obtained from the overlap between the twisted boundary state and the states created by the vertex operators $\mathcal{V}_a^\alpha$ and $\widetilde{\mathcal{V}}_a^\beta$. For example, fixing $\alpha = 1$ and $\beta = 2$ and referring to the explicit expressions in Table 5.1, we have

$$ \begin{aligned} M_{I,a}^{12} &= \lim_{w \to 0} \lim_{\bar{w} \to 0} {}_{\mathrm{NS}}\langle T; I |\, \mathcal{V}_a^1(w)\, \widetilde{\mathcal{V}}_a^2(\bar{w}) |0\rangle|\widetilde{0}\rangle \\ &= {}_{\mathrm{NS}}\langle T; I |\, \Psi^3_{-\frac{1}{2}+\nu_a}\, \widetilde{\overline{\Psi}}{}^3_{-\frac{1}{2}+\nu_a} |\Omega_a\rangle_{(-1)} |\widetilde{\Omega}_a\rangle_{(-1)} \\ &= \sin(\pi\nu_a)\, \omega^{-Ia}\, {}_{\mathrm{NS}}\langle\!\langle a |\, \Psi^3_{-\frac{1}{2}+\nu_a}\, \widetilde{\overline{\Psi}}{}^3_{-\frac{1}{2}+\nu_a} |\Omega_a\rangle_{(-1)} |\widetilde{\Omega}_a\rangle_{(-1)} \\ &= \mathrm{i}\, \sin(\pi\nu_a)\, \omega^{-Ia} \,. \end{aligned} \tag{5.69} $$

Proceeding in a similar way, we find that $M_{I,a}^{21}$ is identical to (5.69), while $M_{I,a}^{11} = M_{I,a}^{22} = 0$, since in these cases the fermionic oscillators are unbalanced. We can thus summarize this result by rewriting (5.68) as

$$\langle \mathcal{V}_a^\alpha(w)\, \widetilde{\mathcal{V}}_a^\beta(\bar{w})\rangle_I = \frac{\mathrm{i}\, \sin(\pi\nu_a)\, \omega^{-Ia}\, (\tau_1)^{\alpha\beta}}{(1-w\bar{w})^2} \tag{5.70}$$

where τ_1 is the first Pauli matrix.

We now map this disk two-point function onto the complex plane by using the Cayley transformation

$$w = \frac{z-\mathrm{i}}{z+\mathrm{i}}\,, \tag{5.71}$$

obtaining

$$\langle \mathcal{V}_a^\alpha(z)\, \widetilde{\mathcal{V}}_a^\beta(\bar{z})\rangle_I = \langle \mathcal{V}_a^\alpha(w)\, \widetilde{\mathcal{V}}_a^\beta(\bar{w})\rangle_I \,\frac{dw}{dz}\,\frac{d\bar{w}}{d\bar{z}} = \frac{-\mathrm{i}\, \sin(\pi\nu_a)\, \omega^{-Ia}\, (\tau^1)^{\alpha\beta}}{(z-\bar{z})^2}\,. \tag{5.72}$$

Thus, using the doubling trick, we are led to introduce the following reflection rule for right moving vertex operators:

$$\widetilde{\mathcal{V}}_a^\beta(\bar{z}) \longrightarrow (R_{I,a})^\beta{}_\gamma\, \mathcal{V}_{M-a}^\gamma(\bar{z})\,, \tag{5.73}$$

so that

$$\langle \mathcal{V}_a^\alpha(z)\, \widetilde{\mathcal{V}}_a^\beta(\bar{z})\rangle_I \longrightarrow (R_{I,a})^\beta{}_\gamma\, \langle \mathcal{V}_a^\alpha(z)\, \mathcal{V}_{M-a}^\gamma(\bar{z})\rangle = (R_{I,a})^\beta{}_\gamma\, \frac{(\epsilon^{-1})^{\alpha\gamma}}{(z-\bar{z})^2} \tag{5.74}$$

where, in the last step, we used (5.30). Comparing with (5.72) we find that the reflection matrix $R_{I,a}$ is given by

$$R_{I,a} = \mathrm{i}\, \sin(\pi\nu_a)\, \omega^{-Ia}\, \tau_3 \tag{5.75}$$

where τ_3 is the third Pauli matrix. Repeating the same calculations in the twisted sector labeled by $(M-a)$, we get

$$R_{I,M-a} = \mathrm{i}\, \sin(\pi\nu_a)\, \omega^{Ia}\, \tau_3\,. \tag{5.76}$$

Notice that even though the oscillator structure of the boundary states in the sectors a and

$(M-a)$ is different, in the end the reflection matrices (5.75) and (5.76) have the same form and can be simultaneously written as

$$R_{I,\widehat{a}} = \mathrm{i}\,\sin\Big(\frac{\pi\widehat{a}}{M}\Big)\,\omega^{-I\widehat{a}}\,\tau_3 \tag{5.77}$$

with $\widehat{a} = 1, \ldots, M-1$.

R/R sector

The above analysis can be easily extended to the R/R sector where, in analogy with (5.63), the twisted components of the boundary state are given by

$$\begin{aligned} |T;I\rangle\!\rangle_{\mathrm{R}} &= \sum_{\widehat{a}=1}^{M-1} \sin\Big(\frac{\pi\widehat{a}}{M}\Big)\,\omega^{I\widehat{a}}\,|\widehat{a}\,\rangle\!\rangle_{\mathrm{R}}\,, \\ {}_{\mathrm{R}}\langle\!\langle T;I| &= \sum_{\widehat{a}=1}^{M-1} \sin\Big(\frac{\pi\widehat{a}}{M}\Big)\,\omega^{-I\widehat{a}}\,{}_{\mathrm{R}}\langle\!\langle\widehat{a}\,|\,. \end{aligned} \tag{5.78}$$

In writing the expressions for the GSO-projected Ishibashi states $|\widehat{a}\,\rangle\!\rangle_{\mathrm{R}}$ and their conjugates, we adopt the same picture assignments discussed in Section 5.1.4: the $(-\frac{1}{2},-\frac{3}{2})$-picture for the twisted sectors labeled by $\widehat{a} = a \in [1, \frac{M-1}{2}]$, and the $(-\frac{3}{2},-\frac{1}{2})$-picture for the sectors with $\widehat{a} = (M-a)$. Apart from this, the structure of these states is similar to that of the twisted boundary states for D3-branes in the $\mathbb{Z}_2$ orbifold obtained in [8] from the factorization of the one-loop open string partition function, and already used in pevious chapter [4] of this thesis. In particular, for $\widehat{a} = a$ we have

$$|a\rangle\!\rangle_{\mathrm{R}} = (C\varGamma_1\varGamma_2)_{A\dot{B}}\,|A_a\rangle_{(-\frac{1}{2})}|\widetilde{\dot{B}_a}\rangle_{(-\frac{3}{2})} + \ldots \tag{5.79}$$

where the ellipses stand for contributions from massive fermionic modes, the vacuum states have been defined in (5.48) and (5.51), and $\varGamma_1$ and $\varGamma_2$ are the SO(6) Dirac matrices along the first two real longitudinal directions of the D3-branes. Likewise, when $\widehat{a}$ =

$(M-a)$ we have

$$|M-a\rangle\!\rangle_{\rm R} = (C\Gamma_1\Gamma_2)_{\dot{A}B}\,|\dot{A}_{M-a}\rangle_{(-\frac{3}{2})}|\widetilde{B}_{M-a}\rangle_{(-\frac{1}{2})} + \ldots\ . \tag{5.80}$$

The corresponding Ishibashi conjugate states are

$$\begin{aligned} {}_{\rm R}\langle\!\langle a| &= {}_{(-\frac{3}{2})}\langle\widetilde{\dot{A}}_a|\ {}_{(-\frac{1}{2})}\langle B_a|\,(\Gamma_2\Gamma_1C^{-1})^{\dot{A}B} + \ldots \\ {}_{\rm R}\langle\!\langle M-a| &= {}_{(-\frac{1}{2})}\langle\widetilde{A}_{M-a}|\ {}_{(-\frac{3}{2})}\langle\dot{B}_{M-a}|\,(\Gamma_2\Gamma_1C^{-1})^{A\dot{B}} + \ldots \end{aligned} \tag{5.81}$$

where the bra vacuum states are defined such that

$${}_{(-\frac{1}{2})}\langle B_a|A_a\rangle_{(-\frac{1}{2})} = \delta^A_B \quad \text{and} \quad {}_{(-\frac{3}{2})}\langle\widetilde{\dot{B}}_{M-a}|\widetilde{\dot{A}}_{M-a}\rangle_{(-\frac{3}{2})} = \delta^{\dot{A}}_{\dot{B}} \tag{5.82}$$

with analogous relations for the right-moving vacua [4].

We can now repeat the same steps followed in the NS sector to prove that the boundary state enforces an identification between left-moving and right-moving vertex operators in the twisted R sector a according to

$$\widetilde{\mathcal{V}}^{\dot{B}}_a(\bar{z}) \longrightarrow (R_{I,a})^{\dot{B}}{}_{\dot{C}}\,\mathcal{V}^{\dot{C}}_{M-a}(\bar{z})\ , \tag{5.83}$$

where the reflection matrix is the anti-chiral/anti-chiral block of

$$R_{I,a} = \sin(\pi\nu_a)\,\omega^{-Ia}\,\Gamma_1\Gamma_2\ . \tag{5.84}$$

Similarly, in the twisted R sector labeled by $(M-a)$ the reflection matrix is the chiral/chiral block of

$$R_{I,M-a} = \sin(\pi\nu_a)\,\omega^{Ia}\,\Gamma_1\Gamma_2\ . \tag{5.85}$$

[4]We remark that in (5.82) the superghost charges of the bra and ket states exactly soak up the background charge anomaly. For example the superghost charge of ${}_{(-\frac{1}{2})}\langle B_a|$ is $-\frac{3}{2}$, and that of $|A_a\rangle_{(-\frac{1}{2})}$ is $-\frac{1}{2}$.

We can combine the last two formulas into

$$R_{I,\widehat{a}} = \sin\Big(\frac{\pi\widehat{a}}{M}\Big)\,\omega^{-I\widehat{a}}\,\Gamma_1\Gamma_2 \tag{5.86}$$

with the understanding that one has to take the lower-right and upper-left blocks for $\widehat{a} = a$ and $\widehat{a} = (M - a)$, respectively, as a consequence of the picture assignments.

5.2.2 Massless open string spectrum

We now analyze the spectrum of massless open strings that live on a configuration made of stacks of n_I fractional D3-branes of type I for $I = 0, \ldots, M-1$, that engineer a theory with gauge group $\mathrm{U}(n_0) \times \ldots \times \mathrm{U}(n_{M-1})$. We will restrict ourselves to listing the fields in the adjoint representation of $\mathrm{U}(n_I)$ as these will be the only fields that are sourced by the background values given to the twisted closed string scalars. We tailor our notations and conventions to be as close as possible to those in chapter 4.

In the familiar case of D3-branes in flat space, in the (0)-superghost picture the bosonic massless open string states are represented by vertex operators of the form [5]

$$(\mathrm{i}\,\partial Z^i + \kappa\cdot\Psi\,\Psi^i)\,\mathrm{e}^{\mathrm{i}\,\kappa\cdot Z}\,. \tag{5.87}$$

where

$$\kappa_i = \frac{k_{2i-1} + \mathrm{i}\,k_{2i}}{\sqrt{2}} \quad \text{and} \quad \overline{\kappa}_i = \frac{k_{2i-1} + \mathrm{i}\,k_{2i}}{\sqrt{2}}\,, \tag{5.88}$$

with k_μ being the real momentum along the direction x^μ of the D3-brane world-volume. In addition we denote the complex direction 1 by the symbol $\|$ and the complex direction 2 by the symbol $\perp$, since these directions are, respectively, longitudinal and perpendicular to the surface defect realized by the D3-brane configuration on the orbifold. We also

[5] Here and in the following we always assume the operators to be normal ordered, unless this causes ambiguities.

introduce the following convenient notation

$$\begin{aligned}\kappa_{\|}\cdot Z_{\|} &= \kappa_1\,\overline{Z}^1 + \bar{\kappa}_1\,Z^1\ , \quad \kappa_{\perp}\cdot Z_{\perp} = \kappa_2\,\overline{Z}^2 + \bar{\kappa}_2\,Z^2\ ,\\ \kappa_{\|}\cdot \Psi_{\|} &= \kappa_1\,\overline{\Psi}^1 + \bar{\kappa}_1\,\Psi^1\ , \quad \kappa_{\perp}\cdot \Psi_{\perp} = \kappa_2\,\overline{\Psi}^2 + \bar{\kappa}_2\,\Psi^2\ ,\end{aligned}\tag{5.89}$$

so that

$$\kappa\cdot Z = \kappa_{\|}\cdot Z_{\|} + \kappa_{\perp}\cdot Z_{\perp}\tag{5.90}$$

and similarly for $\kappa\cdot\Psi$. Clearly, the parallel terms $\kappa_{\|}\cdot Z_{\|}$ and $\kappa_{\|}\cdot\Psi_{\|}$ are invariant under the orbifold group $\mathbb{Z}_M$, but the perpendicular terms are not, since

$$g:\ \begin{cases}\kappa_{\perp}\cdot Z_{\perp} \ \longrightarrow\ g[\kappa_{\perp}\cdot Z_{\perp}] = \omega^{-1}\kappa_2\,\overline{Z}^2 + \omega\,\bar{\kappa}_2\,Z^2\ ,\\[2ex] \kappa_{\perp}\cdot \Psi_{\perp} \ \longrightarrow\ g[\kappa_{\perp}\cdot \Psi_{\perp}] = \omega^{-1}\kappa_2\,\overline{\Psi}^2 + \omega\,\bar{\kappa}_2\,\Psi^2\ .\end{cases}\tag{5.91}$$

This in particular implies that in order to write the open string vertex operators for the fractional D3-branes one cannot use the plane waves $\mathrm{e}^{\mathrm{i}\,\kappa_{\perp}\cdot Z_{\perp}}$ but instead decomposes these into functions that transform in the irreducible representations of $\mathbb{Z}_M$. These functions, which we denote by $\mathcal{E}_I$ with $I = 0,\ldots,M-1$, are simply obtained by summing the plane waves $\mathrm{e}^{\mathrm{i}\,\kappa_{\perp}\cdot Z_{\perp}}$ over the orbits of the group with coefficients chosen such that the combination transforms covariantly under the group action. So we are led to define:

$$\mathcal{E}_I = \frac{1}{M}\sum_{J=0}^{M-1}\omega^{-IJ}\,g^J\Big[\mathrm{e}^{\mathrm{i}\,\kappa_{\perp}\cdot Z_{\perp}}\Big] = \frac{1}{M}\sum_{J=0}^{M-1}\omega^{-IJ}\,\mathrm{e}^{\mathrm{i}\,(\omega^{-J}\kappa_2\,\overline{Z}^2+\omega^J\bar{\kappa}_2\,Z^2)}\ .\tag{5.92}$$

One can easily check that

$$g[\mathcal{E}_I] = \frac{1}{M}\sum_{J=0}^{M-1}\omega^{-IJ}\,\mathrm{e}^{\mathrm{i}\,(\omega^{-J-1}\kappa_2\,\overline{Z}^2+\omega^{J+1}\bar{\kappa}_2\,Z^2)} = \omega^I\,\mathcal{E}_I\ ,\tag{5.93}$$

which shows that $\mathcal{E}_I$ transforms in the I-th irreducible representation of $\mathbb{Z}_M$. For $M = 2$ and $\omega = -1$, the functions $\mathcal{E}_I$ are simply

$$\mathcal{E}_0 = \cos(\kappa_{\perp}\cdot Z_{\perp}) \quad\text{and}\quad \mathcal{E}_1 = \mathrm{i}\,\sin(\kappa_{\perp}\cdot Z_{\perp})\ ,\tag{5.94}$$

which are exactly the two combinations used in the case of the $\mathbb{Z}_2$ orbifold in previous chapter.

In a similar way, we have to break up the operators multiplying the plane wave in (5.87) into various pieces with definite charge I under the orbifold action and form invariant combinations with $\mathcal{E}_{M-I}$. In the orbifold theory, only such combinations represent vertex operators describing physical fields on the world-volume of the fractional D3-brane.

Applying these considerations, we see that the gauge field A_1 along the parallel directions is described by the following vertex operator in the (0)-superghost picture:

$$\mathcal{V}_{A_1} = \Big[(\mathrm{i}\,\partial Z^1 + \kappa_\parallel\cdot\Psi_\parallel\,\Psi^1)\,\mathcal{E}_0 + \kappa_2\,\overline{\Psi}^2\,\Psi^1\,\mathcal{E}_1 + \overline{\kappa}_2\,\Psi^2\,\Psi^1\,\mathcal{E}_{M-1}\Big]\,\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel}\,. \tag{5.95}$$

Each term in square brackets is invariant under $\mathbb{Z}_M$. For instance, the terms $\partial\overline{Z}^1$ and $\kappa_\parallel\cdot\Psi_\parallel\,\overline{\Psi}^1$, which are $\mathbb{Z}_M$ invariant, are multiplied with the invariant function $\mathcal{E}_0$. Similarly the term $\kappa_2\,\overline{\Psi}^2\,\Psi^1$, which gets a factor ω^{-1} under the orbifold action, is multiplied by $\mathcal{E}_1$ to make a $\mathbb{Z}_M$-invariant combination. Likewise, it is easy to see that the third term in (5.95) is also $\mathbb{Z}_M$ invariant. The vertex operator for the complex conjugate field component $\overline{A}_1$ is obtained by simply replacing ∂Z^1 and Ψ^1 with $\partial\overline{Z}^1$ and $\overline{\Psi}^1$.

In a similar way we can write the vertex operators for the gauge field A_2 in the directions transverse to the surface defect, which is

$$\mathcal{V}_{A_2} = \Big[(\mathrm{i}\,\partial Z^2 + \kappa_\parallel\cdot\Psi_\parallel\,\Psi^2)\,\mathcal{E}_{M-1} + \kappa_2\,\overline{\Psi}^2\,\Psi^2\,\mathcal{E}_0\Big]\,\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel}\,. \tag{5.96}$$

The vertex operator for $\overline{A}_2$ can be obtained from the above expression by replacing ∂Z^2 and Ψ^2 with $\partial\overline{Z}^2$ and $\overline{\Psi}^2$, and $\mathcal{E}_{M-1}$ with $\mathcal{E}_1$.

Finally, let us consider the scalar fields. On the fractional D3-brane world-volume there are three complex scalars that together with the gauge vector provide the bosonic content of the $\mathcal{N} = 4$ vector multiplet. When the orbifold acts partially along the world-volume as in our case, all three complex scalars remain in the spectrum. Denoting them by Φ and Φ_r with $r = 4, 5$, they and their complex conjugates are described by the following $\mathbb{Z}_M$-invariant vertices:

$$\begin{aligned}
\mathcal{V}_{\Phi} &= \Big[(\mathrm{i}\,\partial Z^3 + \kappa_{\|}\cdot\Psi_{\|}\,\Psi^3)\,\mathcal{E}_1 + \kappa_2\,\overline{\Psi}^2\,\Psi^3\,\mathcal{E}_2 + \overline{\kappa}_2\,\Psi^2\,\Psi^3\,\mathcal{E}_0\Big]\,\mathrm{e}^{\mathrm{i}\,\kappa_{\|}\cdot Z_{\|}}\,,\\
\mathcal{V}_{\overline{\Phi}} &= \Big[(\mathrm{i}\,\partial\overline{Z}^3 + \kappa_{\|}\cdot\Psi_{\|}\,\overline{\Psi}^3)\,\mathcal{E}_{M-1} + \kappa_2\,\overline{\Psi}^2\,\overline{\Psi}^3\,\mathcal{E}_0 + \overline{\kappa}_2\,\Psi^2\,\overline{\Psi}^3\,\mathcal{E}_{M-2}\Big]\,\mathrm{e}^{\mathrm{i}\,\kappa_{\|}\cdot Z_{\|}}\,,
\end{aligned} \tag{5.97}$$

and

$$\mathcal{V}_{\Phi_r} = \Big[(\mathrm{i}\,\partial Z^r + \kappa_{\|}\cdot\Psi_{\|}\,\Psi^r)\,\mathcal{E}_0 + \kappa_2\,\overline{\Psi}^2\,\Psi^r\,\mathcal{E}_1 + \overline{\kappa}_2\,\Psi^2\,\Psi^r\,\mathcal{E}_{M-1}\Big]\,\mathrm{e}^{\mathrm{i}\,\kappa_{\|}\cdot Z_{\|}}\,, \tag{5.98}$$

with $\mathcal{V}_{\overline{\Phi}_r}$ obtained by simply replacing Ψ^r with $\overline{\Psi}^r$.

All these vertex operators have conformal dimension 1 provided the corresponding fields are massless, *i.e.* if $\kappa\cdot\overline{\kappa} = \frac{1}{2}k^2 = 0$.

5.3 Open/closed correlators

In this section we study the mixed amplitudes between the twisted closed string fields discussed in Section 5.1 and the massless open string fields introduced in the previous section by calculating open/closed disk correlators (see [58] for a review of scattering of strings off D-branes). An example of such a mixed amplitude is shown in Figure 3.1, in which the closed string field is the NS/NS scalar $b^{\widehat{a}}_{\alpha\beta}$ in the twisted sector $\widehat{a}$.

The open/closed string amplitudes we consider correspond to disk diagrams with a closed string vertex insertion at the center and an open string vertex inserted on the boundary. These diagrams are generically non-vanishing due to the D3-brane boundary conditions that enforce an identification between the left and right movers of the closed strings.

We now explain how to compute these mixed amplitudes starting from the NS/NS twisted fields.

5.3.1 Correlators with NS/NS twisted fields

Let us consider the scalar $b^{(a)}_{\alpha\beta}$ in the NS/NS twisted sector $\widehat{a}$. Its coupling with a massless open string excitation on a D3-brane of type I described by the vertex operator $\mathcal{V}_{\text{open}}$ is given by the following expression:

$$\langle \mathcal{V}_{\text{open}} \rangle_{b^{(a)}_{\alpha\beta};I} = b^{(a)}_{\alpha\beta} \int \frac{dz\, d\bar{z}\, dx}{dV_{\text{proj}}} \langle \mathcal{V}^{\alpha}_{\widehat{a}}(z) \widetilde{\mathcal{V}}^{\beta}_{\widehat{a}}(\bar{z})\, \mathcal{V}_{\text{open}}(x) \rangle_I \,, \tag{5.99}$$

where

$$dV_{\text{proj}} = \frac{dz\, d\bar{z}\, dx}{(z-\bar{z})(\bar{z}-x)(x-z)} \tag{5.100}$$

is the projective invariant volume element and the integrals are performed on the string world-sheet. In particular the closed string insertion points z and $\bar{z}$, are in the upper and lower half complex plane, respectively, while the open string insertion point x is on the real axis.

Since we are interested in the couplings with constant background fields $b^{(a)}_{\alpha\beta}$, the left and right vertex operators in (5.99) are at zero momentum. The open string vertex, instead, has a non-vanishing momentum. Since the fractional brane is located at the orbifold fixed point $z_2 = 0$, translation invariance is broken in the complex direction 2. Therefore, the components κ_2 and $\bar{\kappa}_2$ of the open string momentum are arbitrary, while the components κ_1 and $\bar{\kappa}_1$ are set to zero by momentum conservation in the parallel directions and the final amplitude will be proportional to $\delta^{(2)}(\kappa_{\|})$.

Using the reflection rule (5.77), the integrand of (5.99) can be rewritten as

$$\langle \mathcal{V}^{\alpha}_{\widehat{a}}(z) \widetilde{\mathcal{V}}^{\beta}_{\widehat{a}}(\bar{z})\, \mathcal{V}_{\text{open}}(x) \rangle_I = \mathrm{i}\, \sin\left(\frac{\pi \widehat{a}}{M}\right) \omega^{-I\widehat{a}}\, (\tau_3)^{\beta}{}_{\gamma}\, \langle \mathcal{V}^{\alpha}_{\widehat{a}}(z)\, \mathcal{V}^{\gamma}_{M-\widehat{a}}(\bar{z})\, \mathcal{V}_{\text{open}}(x) \rangle \,. \tag{5.101}$$

Thus, the calculation is reduced to the evaluation of a three-point function of vertex operators of conformal weight 1. The functional dependence on the word-sheet variables is fixed by conformal invariance and exactly cancels that of the projective invariant volume (5.100) so that in the end the result will be a constant that depends on the detailed structure of the vertex operators.

There are, however, some features that can be described in generality, and are independent of the specific components of $b^{(\widehat{a})}_{\alpha\beta}$ and of the particular open string vertices that are considered. When we write the three-point functions in (5.101) as products of correlators for each of the independent conformal fields, we easily recognize that the superghost contribution is always given by

$$\big\langle \mathrm{e}^{-\phi(z)}\, \mathrm{e}^{-\phi(\bar{z})} \big\rangle = \frac{1}{z-\bar{z}} \;. \tag{5.102}$$

It is perhaps less obvious but it turns out that also the contribution arising from the bosonic string coordinates is the same for all amplitudes. Indeed, the only non-vanishing correlator involving the bosonic coordinates along the parallel direction is

$$\big\langle \mathrm{e}^{\mathrm{i}\,\kappa_{\parallel}\cdot Z_{\parallel}}(x) \big\rangle = \delta^{(2)}(\kappa_{\parallel}) \;, \tag{5.103}$$

which enforces the anticipated momentum conservation for $\kappa_{\parallel}$, while the terms containing ∂Z^1 or $\partial \overline{Z}^1$ always vanish inside the correlators and thus they can be ignored. As far as the perpendicular direction is concerned, we have to take into account the presence of the bosonic twist fields and the fact that the plane waves appear in the combinations $\mathcal{E}_I$ defined in (5.92). Thus, one typically has to evaluate a correlator of the form

$$\big\langle \sigma_{\widehat{a}}(z)\, \sigma_{M-\widehat{a}}(\bar{z})\, \mathcal{E}_I(x) \big\rangle = \frac{1}{M} \sum_{J=0}^{M-1} \omega^{-IJ} \, \big\langle \sigma_{\widehat{a}}(z)\, \sigma_{M-\widehat{a}}(\bar{z})\, \mathrm{e}^{\mathrm{i}\left(\omega^{-J}\kappa_2\,\overline{Z}^2(x)+\omega^{J}\bar{\kappa}_2\,Z^2(x)\right)} \big\rangle \;. \tag{5.104}$$

For any value of J, the correlator in the sum is equal simply to $\langle \sigma_{\widehat{a}}(z)\, \sigma_{M-\widehat{a}}(\bar{z}) \rangle$, so that

$$\big\langle \sigma_{\widehat{a}}(z)\, \sigma_{M-\widehat{a}}(\bar{z})\, \mathcal{E}_I(x) \big\rangle = \frac{1}{M}\Big(\sum_{J=0}^{M-1} \omega^{-IJ} \Big) \big\langle \sigma_{\widehat{a}}(z)\, \sigma_{M-\widehat{a}}(\bar{z}) \big\rangle = \delta_{I,0}\, \big\langle \sigma_{\widehat{a}}(z)\, \sigma_{M-\widehat{a}}(\bar{z}) \big\rangle \;. \tag{5.105}$$

This means that in the open string vertex operators we can just focus on the terms proportional to $\mathcal{E}_0$ and disregard the other terms, as they will not contribute. Furthermore, we can also neglect the terms involving ∂Z^2 or $\partial \overline{Z}^2$, since they always give a vanishing contribution inside the correlators. With this in mind, we can proceed to the explicit evaluation of the mixed amplitudes with the twisted NS/NS scalars.

Explicit computations

We start by considering the correlator (5.101) with $\widehat{a} = a \in [1, \frac{M-1}{2}]$ and $\alpha = 1$ and $\beta = 2$, corresponding to the twisted field $b_{12}^{(a)}$. Applying the above considerations, one realizes that this scalar does not couple to any open string field except A_2 and $\overline{A}_2$. Indeed, the terms of the vertex operators of A_1, Φ, Φ_r and their conjugates which contain $\mathcal{E}_0$ always contain other structures with unbalanced bosonic or fermionic fields, which therefore vanish inside the correlator. Let us then consider the coupling with A_2. In this case, inserting the explicit expressions of the vertex operators in (5.101), we have

$$\big\langle \mathcal{V}_a^1(z)\widetilde{\mathcal{V}}_a^2(\bar{z})\,\mathcal{V}_{A_2}(x)\big\rangle_I = -\mathrm{i}\,\sin\pi\nu_a\,\omega^{-Ia}\,\big\langle \mathcal{V}_a^1(z)\,\mathcal{V}_{M-a}^2(\bar{z})\,\mathcal{V}_{A_2}(x)\big\rangle \tag{5.106}$$

with

$$\begin{aligned}\big\langle \mathcal{V}_a^1(z)\,\mathcal{V}_{M-a}^2(\bar{z})\,\mathcal{V}_{A_2}(x)\big\rangle = \kappa_2\,\big\langle \mathrm{e}^{-\phi(z)}\,\mathrm{e}^{-\phi(\bar{z})}\big\rangle\,\big\langle \mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot Z_\parallel(x)}\big\rangle\,\big\langle \sigma_a(z)\,\sigma_{M-a}(\bar{z})\big\rangle \\ \times\,\big\langle :\Psi^3(z)s_a(z):\,:\overline{\Psi}^3(\bar{z})s_{M-a}(\bar{z}):\,:\overline{\Psi}^2(x)\Psi^2(x):\big\rangle\,.\end{aligned} \tag{5.107}$$

The fermionic correlator in the second line above can be evaluated by factorizing it in the two independent directions 2 and 3 and using the bosonization method [10]. In this way we have

$$\begin{aligned}\big\langle :\Psi^3(z)s_a(z):\,:\overline{\Psi}^3(\bar{z})s_{M-a}(\bar{z}):\,:\overline{\Psi}^2(x)\Psi^2(x):\big\rangle = \big\langle s^2_{\nu_a}(z)\,s^2_{-\nu_a}(\bar{z})\,:\overline{\Psi}^2(x)\Psi^2(x):\big\rangle \\ \times\big\langle :\Psi^3(z)s^3_{-\nu_a}(z):\,:\overline{\Psi}^3(\bar{z})s^3_{\nu_a}(\bar{z}):\big\rangle\end{aligned} \tag{5.108}$$

where [6]

$$\begin{aligned}\langle s^2_{\nu_a}(z)\, s^2_{-\nu_a}(\bar z) :\overline{\Psi}^2(x)\Psi^2(x):\rangle &= \langle\, \mathrm{e}^{\mathrm{i}\,\nu_a\,\phi_2}(z)\,\mathrm{e}^{-\mathrm{i}\,\nu_a\,\phi_2}(\bar z)\,(-\mathrm{i}\,\partial\phi_2(x))\,\rangle \\ &= \frac{-\nu_a}{(z-\bar z)^{\nu_a^2-1}(z-x)(\bar z - x)}\ ,\end{aligned} \tag{5.109}$$

and

$$\langle :\Psi^3(z)s^3_{-\nu_a}(z): :\overline{\Psi}^3(\bar z)s^3_{\nu_a}(\bar z):\rangle = \langle \mathrm{e}^{\mathrm{i}\,(1-\nu_a)\phi_3}(z)\,\mathrm{e}^{-\mathrm{i}\,(1-\nu_a)\phi_3}(\bar z)\rangle = \frac{1}{(z-\bar z)^{(1-\nu_a)^2}}\ . \tag{5.110}$$

Combining everything together in (5.107), we obtain

$$\langle \mathcal{V}^1_a(z)\,\mathcal{V}^2_{M-a}(\bar z)\,\mathcal{V}_{A_2}(x)\rangle = \frac{\kappa_2\,\nu_a}{(z-\bar z)(\bar z - x)(x-z)}\,\delta^{(2)}(\kappa_\parallel)\ . \tag{5.111}$$

Finally, inserting this into (5.106) and (5.99), we find that the coupling of $b^{(a)}_{12}$ with A_2 is

$$\langle \mathcal{V}_{A_2}\rangle_{b^{(a)}_{12};I} \;=\; -\mathrm{i}\, b^{(a)}_{12}\,\kappa_2\,\nu_a\,\sin\pi\nu_a\,\omega^{-Ia}\,\delta^{(2)}(\kappa_\parallel)\ . \tag{5.112}$$

The same calculation shows that $b^{(a)}_{12}$ also couples to $\overline{A}_2$ and the result is simply obtained by replacing κ_2 with $-\bar\kappa_2$ in the above expression.

We can similarly repeat the analysis for the other components $b^{(a)}_{\alpha\beta}$. For example, taking $b^{(a)}_{21}$ we find that its only non-vanishing coupling is

$$\langle \mathcal{V}_{A_2}\rangle_{b^{(a)}_{21};I} \;=\; \mathrm{i}\, b^{(a)}_{21}\,\kappa_2\,(1-\nu_a)\,\sin\pi\nu_a\,\omega^{-Ia}\,\delta^{(2)}(\kappa_\parallel)\ , \tag{5.113}$$

with a similar result for $\overline{A}_2$ in which κ_2 is replaced with $-\bar\kappa_2$. The diagonal components $b^{(a)}_{11}$ and $b^{(a)}_{22}$, instead, only couple to the complex scalars Φ and $\overline{\Phi}$ according to

$$\begin{aligned}\langle \mathcal{V}_{\Phi}\rangle_{b^{(a)}_{22};I} &= -\mathrm{i}\, b^{(a)}_{22}\,\bar\kappa_2\,\sin\pi\nu_a\,\omega^{-Ia}\,\delta^{(2)}(\kappa_\parallel)\ , \\ \text{and}\quad \langle \mathcal{V}_{\overline{\Phi}}\rangle_{b^{(a)}_{11};I} &= \mathrm{i}\, b^{(a)}_{11}\,\kappa_2\,\sin\pi\nu_a\,\omega^{-Ia}\,\delta^{(2)}(\kappa_\parallel)\ .\end{aligned} \tag{5.114}$$

[6] Here ϕ_2 and ϕ_3 denote the fields that bosonize the fermionic systems in the complex directions 2 and 3.

It is equally straightforward to compute the open/closed string correlators in the twisted sectors with $\widehat{a} = (M - a)$. In this case, we find again that the off-diagonal components $b_{12}^{(M-a)}$ and $b_{21}^{(M-a)}$ only interact with A_2 and $\overline{A}_2$, and that the couplings with A_2 are

$$\begin{aligned} \langle \mathcal{V}_{A_2} \rangle_{b_{12}^{(M-a)};I} &= -\mathrm{i}\, b_{12}^{(M-a)}\, \kappa_2\, (1 - \nu_a) \sin \pi\nu_a\, \omega^{Ia}\, \delta^{(2)}(\kappa_\parallel) \,, \\ \text{and} \quad \langle \mathcal{V}_{A_2} \rangle_{b_{21}^{(M-a)};I} &= \mathrm{i}\, b_{21}^{(M-a)}\, \kappa_2\, \nu_a\, \sin \pi\nu_a\, \omega^{Ia}\, \delta^{(2)}(\kappa_\parallel) \,, \end{aligned} \tag{5.115}$$

while those with $\overline{A}_2$ follow by replacing κ_2 with $-\overline{\kappa}_2$ in the above expressions. The diagonal components $b_{11}^{(M-a)}$ and $b_{21}^{(M-a)}$ interact instead with $\varPhi$ and $\overline{\varPhi}$ with the following couplings:

$$\begin{aligned} \langle \mathcal{V}_{\varPhi} \rangle_{b_{22}^{(M-a)};I} &= -\mathrm{i}\, b_{22}^{(M-a)}\, \overline{\kappa}_2\, \sin \pi\nu_a\, \omega^{Ia}\, \delta^{(2)}(\kappa_\parallel) \,, \\ \text{and} \quad \langle \mathcal{V}_{\overline{\varPhi}} \rangle_{b_{11}^{(M-a)};I} &= \mathrm{i}\, b_{11}^{(M-a)}\, \kappa_2\, \sin \pi\nu_a\, \omega^{Ia}\, \delta^{(2)}(\kappa_\parallel) \,. \end{aligned} \tag{5.116}$$

As a consistency check of our results, we observe that the formulas (5.115) and (5.116) can be obtained from (5.112), (5.113) and (5.114) by simply replacing everywhere a with $(M - a)$. Thus, despite the fact that the fermionic approach we have used introduces differences in the explicit expressions for the twisted sector vertex operators, in the end, all sectors are treated on an equal footing.

Results

We are finally in a position to write down the complete expression for the open string fields emitted by a fractional D3-brane of type I in the presence of background values for the scalars of the NS/NS twisted sectors. This is given by summing over all components of $b_{\alpha\beta}^{(\widehat{a})}$ and over all twisted sectors:

$$\langle \mathcal{V}_{\mathrm{open}} \rangle_I = \sum_{\widehat{a}=1}^{M-1} \sum_{\alpha,\beta=1}^{2} \langle \mathcal{V}_{\mathrm{open}} \rangle_{b_{\alpha\beta}^{(\widehat{a})};I} \,. \tag{5.117}$$

As we have seen, the components of the gauge field along the parallel direction 1 and the complex scalars $\varPhi_r$ do not couple to any NS/NS twisted field, while we have a non-vanishing source for A_2, Φ and their complex conjugates. For A_2 the above formula gives

$$\begin{aligned}\langle \mathcal{V}_{A_2} \rangle_I = -\mathrm{i}\,\kappa_2 \sum_{a=1}^{\frac{M-1}{2}} \sin \pi\nu_a \Big[\nu_a\, \omega^{-Ia}\, b^{(a)}_{12} - (1-\nu_a)\, \omega^{-Ia}\, b^{(a)}_{21} \\ - \nu_a\; \omega^{Ia}\, b^{(M-a)}_{21} + (1-\nu_a)\, \omega^{Ia}\, b^{(M-a)}_{12}\Big]\, \delta^{(2)}(\kappa_\parallel)\ .\end{aligned} \tag{5.118}$$

Taking into account the relations (5.38), it is easy to realize that the quantity in square brackets is purely imaginary. A similar result holds for $\overline{A}_2$ with κ_2 replaced by $-\overline{\kappa}_2$.

For the complex scalars Φ and $\overline{\Phi}$ we have instead

$$\begin{aligned}\langle \mathcal{V}_{\Phi} \rangle_I &= -\mathrm{i}\,\overline{\kappa}_2 \sum_{a=1}^{\frac{M-1}{2}} \sin \pi\nu_a \Big[\omega^{-Ia}\, b^{(a)}_{22} + \omega^{Ia}\, b^{(M-a)}_{22}\Big]\, \delta^{(2)}(\kappa_\parallel)\ ,\\ \langle \mathcal{V}_{\overline{\Phi}} \rangle_I &= \quad \mathrm{i}\,\kappa_2 \sum_{a=1}^{\frac{M-1}{2}} \sin \pi\nu_a \Big[\omega^{-Ia}\, b^{(a)}_{11} + \omega^{Ia}\, b^{(M-a)}_{11}\Big]\, \delta^{(2)}(\kappa_\parallel)\ .\end{aligned} \tag{5.119}$$

5.3.2 Correlators with R/R twisted fields

We now turn to the calculation of the interactions between the massless open string fields and the twisted R/R potentials. For definiteness, we only consider non-vanishing background values for the scalars $C^{(a)}$ and $C^{(M-a)}$, since they are the only ones that turn out to be relevant for the description of the continuous parameters of surface defects. Thus, the closed string vertex operators we consider are

$$C^{(a)}\, C_{A\dot{B}}\, \mathcal{V}^{A}_{a}(z)\, \widetilde{\mathcal{V}}^{\dot{B}}_{a}(\bar{z}) \quad \text{and} \quad C^{(M-a)}\, C_{\dot{A}B}\, \mathcal{V}^{\dot{A}}_{M-a}(z)\, \widetilde{\mathcal{V}}^{B}_{M-a}(\bar{z})\ . \tag{5.120}$$

By inspecting the fermionic structure of these vertex operators and comparing it with that of the open string vertices, one realizes that only the longitudinal component of the gauge field A_1 and its conjugate $\overline{A}_1$ can have a non-vanishing coupling.

Let us start by considering the interaction between A_1 and $C^{(a)}$. This is given by

$$\langle \mathcal{V}_{A_1} \rangle_{C^{(a)},I} = C^{(a)}\, C_{A\dot{B}} \int \frac{dz\, d\bar{z}\, dx}{dV_{\text{proj}}}\, \langle \mathcal{V}^{A}_{a}(z)\, \widetilde{\mathcal{V}}^{\dot{B}}_{a}(\bar{z})\, \mathcal{V}_{A_1}(x) \rangle_I\,, \tag{5.121}$$

where the projective invariant volume element is defined in (5.100). Using the reflection rules (5.84) for the R/R fields, the integrand of (5.121) becomes

$$\langle \mathcal{V}^{A}_{a}(z)\, \widetilde{\mathcal{V}}^{\dot{B}}_{a}(\bar{z})\, \mathcal{V}_{A_1}(x) \rangle_I = \sin(\pi\nu_a)\, \omega^{-Ia}\, (\Gamma_1\Gamma_2)^{\dot{B}}{}_{\dot{C}}\, \langle \mathcal{V}^{A}_{a}(z)\, \mathcal{V}^{\dot{C}}_{M-a}(\bar{z})\, \mathcal{V}_{A_1}(x) \rangle\,. \tag{5.122}$$

Using the explicit form of the vertex operators given in (5.47a), (5.52b) and (5.95), and taking into account the points discussed at the beginning of this section, the above correlator can be written as follows:

$$\begin{aligned}\langle \mathcal{V}^{A}_{a}(z)\, \mathcal{V}^{\dot{C}}_{M-a}(\bar{z})\, \mathcal{V}_{A_1}(x) \rangle = \kappa_1 \langle \mathrm{e}^{-\frac{1}{2}\phi(z)}\, \mathrm{e}^{-\frac{3}{2}\phi(\bar{z})} \rangle\, \langle \mathrm{e}^{\mathrm{i}\,\kappa_\parallel \cdot Z_\parallel} \rangle\, \langle \sigma_a(z)\, \sigma_{M-a}(\bar{z}) \rangle \\ \times \langle r_a(z)\, r_{M-a}(\bar{z}) \rangle\, \langle S^{A}(z)\, S^{\dot{C}}(\bar{z}) \;:\overline{\Psi}^1 \Psi^1: (x) \rangle\,.\end{aligned} \tag{5.123}$$

Each factor in this expression can be easily computed using standard conformal field theory methods. The new ingredients with respect to the calculations in the NS/NS sectors are the following two-point functions:

$$\begin{aligned}\langle \mathrm{e}^{-\frac{1}{2}\phi(z)}\, \mathrm{e}^{-\frac{3}{2}\phi(\bar{z})} \rangle &= \frac{1}{(z-\bar{z})^{\frac{3}{4}}}\,,\\ \langle r_a(z)\, r_{M-a}(\bar{z}) \rangle &= \frac{1}{(z-\bar{z})^{\frac{1}{2}-2\nu_a(1-\nu_a)}}\,,\\ \text{and}\quad \langle S^{A}(z)\, S^{\dot{C}}(\bar{z}) \;:\overline{\Psi}^1 \Psi^1: (x) \rangle &= \frac{\mathrm{i}}{2}\, \frac{(\Gamma_1\Gamma_2 C^{-1})^{A\dot{C}}}{(z-\bar{z})^{-\frac{1}{4}}(z-x)(\bar{z}-x)}\,.\end{aligned} \tag{5.124}$$

Putting everything together, we have

$$\langle \mathcal{V}^{A}_{a}(z)\, \mathcal{V}^{\dot{C}}_{M-a}(\bar{z})\, \mathcal{V}_{A_1}(x) \rangle = -\frac{\mathrm{i}}{2}\, \frac{(\Gamma_1\Gamma_2 C^{-1})^{A\dot{C}}}{(z-\bar{z})(\bar{z}-x)(x-z)}\, \delta^{(2)}(\kappa_\parallel)\,. \tag{5.125}$$

Inserting this into (5.122) and (5.121), and performing the Γ-matrix algebra, we finally obtain

$$\langle \mathcal{V}_{A_1} \rangle_{C^{(a)},I} = -2\mathrm{i}\,\kappa_1\ \sin\pi\nu_a\,\omega^{-Ia}\,C^{(a)}\,\delta^{(2)}(\kappa_{\|})\ . \tag{5.126}$$

In a very similar way we find

$$\langle \mathcal{V}_{A_1} \rangle_{C^{(M-a)},I} = -2\mathrm{i}\,\kappa_1\ \sin\pi\nu_a\,\omega^{Ia}\,C^{(M-a)}\,\delta^{(2)}(\kappa_{\|})\ . \tag{5.127}$$

Thus, the full amplitude becomes

$$\langle \mathcal{V}_{A_1} \rangle_{I} = -2\mathrm{i}\,\kappa_1 \sum_{a=1}^{\frac{M-1}{2}} \Big[\sin\pi\nu_a(\omega^{-Ia}\,C^{(a)} + \omega^{Ia}\,C^{(M-a)})\Big]\,\delta^{(2)}(\kappa_{\|})\ . \tag{5.128}$$

Taking into account that $C^{(M-a)} = C^{(a)\,\star}$, as it follows from (5.61), we see that the expression inside the square brackets is real.

5.4 Continuous parameters of surface defects

We are now ready to identify the twisted closed string background that leads to a monodromy surface defect in the gauge theory on the world-volume of the fractional D3-branes. It is convenient to decompose the twisted fields of the NS/NS sectors into irreducible representations of the unbroken SU(2)$_+$ symmetry group of the orbifolded space (see the discussion in Section 5.1). In each twisted sector $\widehat{a}$, this can be done by writing

$$b^{\widehat{a}}_{\alpha\beta} = \mathrm{i}\,b^{\widehat{a}}_{\mathrm{s}}\,\epsilon_{\alpha\beta} + b^{\widehat{a}}_{+}\,(\epsilon\tau_+)_{\alpha\beta} + b^{\widehat{a}}_{-}\,(\epsilon\tau_-)_{\alpha\beta} + b^{\widehat{a}}_{3}\,(\epsilon\tau_3)_{\alpha\beta} \tag{5.129}$$

where ϵ is defined in (5.31) and $\tau_\pm = (\tau_1 \pm \mathrm{i}\tau_2)/2$. In the $M = 2$ case studied in chapter 4 it was found that only the singlet component $b^{\widehat{a}}_{\mathrm{s}}$ (which we denoted b in that reference) acted as a source for the gauge field. This can also be seen from (5.118) by setting $\nu_1 = \frac{1}{2}$ and $\omega = -1$ for the only twisted sector that is present when $M = 2$. For the general $M > 2$ case, however, we see that the gauge field couples to both the scalars $b^{\widehat{a}}_{\mathrm{s}}$ and $b^{\widehat{a}}_{3}$. Since we wish to have a uniform description of surface defects for all values of M, in what follows, we will set $b_3^{\widehat{a}} = 0$ and only turn on the background value for $b_\mathrm{s}^{\widehat{a}}$. Furthermore, we also turn on the doublet components $b_\pm^{\widehat{a}}$ which source the scalar fields $\varPhi$ and $\overline{\varPhi}$. This means that, in terms of the initial fields $b_{\alpha\beta}^{\widehat{a}}$, our background reads

$$\begin{aligned} b_{12}^{\widehat{a}} &= -b_{21}^{\widehat{a}} = -\mathrm{i}\, b_\mathrm{s}^{\widehat{a}}\ , \\ b_{22}^{\widehat{a}} &= b_+^{\widehat{a}}\ , \quad b_{11}^{\widehat{a}} = -b_-^{\widehat{a}}\ , \end{aligned} \tag{5.130}$$

with $(b_\mathrm{s}^{\widehat{a}})^* = b_\mathrm{s}^{(M-\widehat{a})}$ and $(b_+^{\widehat{a}})^* = b_-^{(M-\widehat{a})}$ for all twisted sectors, as follows from the relations (5.38).

Inserting these background values in (5.118) and (5.119), we have

$$\langle \mathcal{V}_{A_2} \rangle_I = -\kappa_2\, b_I\, \delta^{(2)}(\kappa_\parallel)\ , \tag{5.131}$$

and

$$\langle \mathcal{V}_{\varPhi} \rangle_I = -\mathrm{i}\, \bar{\kappa}_2\, b_I^+\, \delta^{(2)}(\kappa_\parallel)\ , \quad \langle \mathcal{V}_{\overline{\varPhi}} \rangle_I = -\mathrm{i}\, \kappa_2\, b_I^-\, \delta^{(2)}(\kappa_\parallel)\ , \tag{5.132}$$

where we have defined the combinations

$$b_I = \sum_{a=1}^{\frac{M-1}{2}} \sin \pi\nu_a \left[\omega^{-Ia}\, b_\mathrm{s}^{(a)} + \omega^{Ia}\, b_\mathrm{s}^{(M-a)}\right] = \sum_{\widehat{a}=1}^{M-1} \sin\left(\frac{\pi\widehat{a}}{M}\right) \omega^{-I\widehat{a}}\, b_\mathrm{s}^{\widehat{a}}\ , \tag{5.133}$$

and

$$b_I^\pm = \sum_{a=1}^{\frac{M-1}{2}} \sin \pi\nu_a \left[\omega^{-Ia}\, b_\pm^{(a)} + \omega^{Ia}\, b_\pm^{(M-a)}\right] = \sum_{\widehat{a}=1}^{M-1} \sin\left(\frac{\pi\widehat{a}}{M}\right) \omega^{-I\widehat{a}}\, b_\pm^{\widehat{a}}\ . \tag{5.134}$$

Notice that b_I is real, while $(b_I^+)^* = b_I^-$. It is interesting to note that a similar change of basis for profiles of closed string fields between the fractional branes (labelled by irreducible representations) and the twisted sectors (labelled by conjugacy classes) has been observed previously for fractional branes at orbifolds in [52].

Field Profiles

As explained in detail in previous chapter [4], these amplitudes are interpreted as a source for the corresponding open string field (see also Figure 3.1), whose profile in configuration space is obtained by taking the Fourier transform, after attaching the massless propagator along the D3-brane world-volume:

$$\frac{1}{k^2} = \frac{1}{2(|\kappa_{\parallel}|^2 + |\kappa_{\perp}|^2)} \,. \tag{5.135}$$

For example, for the gauge field A_2 we have

$$A_{2;I} = \mathcal{F}\mathcal{T}\Big[\frac{\langle \mathcal{V}_{A_2}\rangle_I}{k^2}\Big] \,. \tag{5.136}$$

In Appendix B we show how to organize the calculation of this Fourier transform in terms of the generalized plane-waves $\mathcal{E}_I$ that transform covariantly with charge I under the orbifold group. Applying these methods to the present case, we see that since the source (5.131) is proportional to κ_2, which has charge (-1), only the term proportional to $\mathcal{E}_1$ remains so that (5.136) becomes

$$\begin{aligned} A_{2;I} &= \int \frac{d^2\kappa_{\parallel} d^2\kappa_{\perp}}{(2\pi)^2} \frac{\langle \mathcal{V}_{A_2}\rangle_I}{2(\kappa_{\parallel}^2 + \kappa_{\perp}^2)}\, \mathrm{e}^{\mathrm{i}\,\kappa_{\parallel}\cdot z_{\parallel}}\, \mathcal{E}_1 \\ &= -b_I\, \frac{1}{M}\sum_{J=0}^{M-1} \omega^{-J} \int \frac{d^2\kappa_{\perp}}{(2\pi)^2} \frac{\kappa_2}{2|\kappa_{\perp}|^2}\, \mathrm{e}^{\mathrm{i}\,(\omega^{-J}\kappa_2\,\bar{z}_2 + \omega^{J}\,\bar{\kappa}_2\, z_2)} \;=\; -\frac{\mathrm{i}\, b_I}{4\pi\bar{z}_2} \,, \end{aligned} \tag{5.137}$$

where the last equality is a consequence of the fact that all M terms in the sum are actually all equal to each other and equal to $\mathrm{i}/(4\pi\bar{z}_2)$.

Combining this result with the one for the complex conjugate component $\overline{A}_2$, we find that the gauge field on the I-th fractional D3-brane has the following profile:

$$\mathbf{A}_I = A\cdot dx = A_{2;I}\, d\bar{z}_2 + \overline{A}_{2;I}\, dz_2 = -\frac{\mathrm{i}\, b_I}{4\pi}\Big(\frac{d\bar{z}_2}{\bar{z}_2} - \frac{dz_2}{z_2}\Big) = -\frac{b_I}{2\pi}\, d\theta \,, \tag{5.138}$$

where θ is the usual polar angle in the $\mathbb{C}_{(2)}$-plane.

The only other open string field that has a non-vanishing profile in the twisted NS/NS background we have chosen is the complex scalar Φ. The analogous calculation takes the following form:

$$\begin{aligned}\Phi_I &= \mathcal{FT}\Big[\frac{\langle\mathcal{V}_\Phi\rangle_I}{k^2}\Big] = \int\frac{d^2\kappa_\parallel d^2\kappa_\perp}{(2\pi)^2}\,\frac{\langle\mathcal{V}_\Phi\rangle_I}{2(\kappa_\parallel^2+\kappa_\perp^2)}\,\mathrm{e}^{\mathrm{i}\,\kappa_\parallel\cdot z_\parallel}\,\mathcal{E}_{M-1}\\ &= -\mathrm{i}\,b_I^+\,\frac{1}{M}\sum_{J=0}^{M-1}\omega^J\int\frac{d^2\kappa_\perp}{(2\pi)^2}\,\frac{\overline{\kappa}_2}{2|\kappa_\perp|^2}\,\mathrm{e}^{\mathrm{i}\,(\omega^J\kappa_2\,\bar{z}_2+\omega^J\,\overline{\kappa}_2\,z_2)} \;=\; \frac{b_I^+}{4\pi z_2}\,.\end{aligned} \tag{5.139}$$

If we now consider a general configuration with n_I fractional D3-branes of type I for all values of I, as in the KT proposal [4], we obtain the following profiles:

$$\mathbf{A} = -\frac{d\theta}{2\pi}\begin{pmatrix} b_0\,\mathbb{I}_{n_0} & 0 & \cdots & 0\\ 0 & b_1\,\mathbb{I}_{n_1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & b_{M-1}\,\mathbb{I}_{n_{M-1}}\end{pmatrix}, \tag{5.140}$$

and

$$\mathbf{\Phi} = \frac{1}{4\pi\,z_2}\begin{pmatrix} b_0^+\,\mathbb{I}_{n_0} & 0 & \cdots & 0\\ 0 & b_1^+\,\mathbb{I}_{n_1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & b_{M-1}^+\end{pmatrix}. \tag{5.141}$$

These are precisely the profiles of a GW surface defect in the $\mathcal{N}=4$ theory corresponding to the breaking of $\mathrm{U}(N)$ group to the Levi subgroup $\mathrm{U}(n_0)\times\ldots\times\mathrm{U}(n_{M-1})$, provided the continuous parameters $(\alpha_I,\beta_I,\gamma_I)$ that conventionally parametrize the singular profiles near the defect are related to the background values of the NS/NS twisted scalars as follows:

$$\alpha_I = -\frac{b_I}{2\pi}\,,\qquad \beta_I = \frac{\mathrm{Re}(b_I^+)}{2\pi}\,,\qquad \gamma_I = \frac{\mathrm{Im}(b_I^+)}{2\pi}\,. \tag{5.142}$$

If the original gauge group is $\mathrm{SU}(N)$, the corresponding field profiles are obtained by removing the overall trace from each of the above expressions.

We now turn to discussing the coupling of the open string fields with the twisted scalars in the R/R sector. As we have seen in Section 5.3.2, we only need to consider the coupling with the longitudinal component A_1 of the gauge field. This is given in (5.128), which we rewrite as

$$\langle \mathcal{V}_{A_1} \rangle_I = -2\mathrm{i}\, \kappa_1 c_I\, \delta^{(2)}(\kappa_\parallel) \tag{5.143}$$

where

$$c_I = \sum_{a=1}^{\frac{M-1}{2}} \sin \pi\nu_a \left[\omega^{-Ia}\, C^{(a)} + \omega^{Ia}\, C^{(M-a)}\right] = \sum_{\widehat{a}=1}^{M-1} \sin\left(\frac{\pi\widehat{a}}{M}\right) \omega^{-I\widehat{a}}\, C^{(\widehat{a})}\,. \tag{5.144}$$

This real quantity is the R/R counterpart of b_I defined in (5.133) for the NS/NS sectors.

At face value, the coupling (5.143) is vanishing because of the δ-function. However, as was explained in the $\mathbb{Z}_2$ in ch. 4, if we multiply this amplitude and its complex conjugate with the corresponding gauge field polarizations, the resulting sum can be interpreted as an interaction term between the R/R scalars and the longitudinal components of the gauge field strength. Indeed,

$$\overline{A}_{1,I}\, \langle \mathcal{V}_{A_1} \rangle_I + A_{1,I}\, \langle \mathcal{V}_{\overline{A}_1} \rangle_I = -2\,\mathrm{i}\, c_I(\overline{\kappa}_1 A_1 - \kappa_1 \overline{A}_1)\, \delta^2(\kappa_\parallel) = 2\,\mathrm{i}\, c_I\, \widetilde{F}_I\, \delta^2(\kappa_\parallel)\,, \tag{5.145}$$

where $\widetilde{F}_I$ is the gauge field strength on the Ith fractional brane (along the defect), in momentum space. Performing the Fourier transform, this expression becomes an effective interaction term localized on the surface defect:

$$\frac{\mathrm{i}\, c_I}{2\pi} \int d^2 z_\parallel\, F_I\,, \tag{5.146}$$

where F_I is the gauge field strength in configuration space, on the Ith fractional brane. If this has a non-trivial first Chern class, then this effective interaction can be understood as the $2d$ topological θ-term that can be included in the path integral definition of the theory with surface defect. When a generic configuration with n_I D3-branes of type I is considered, the following phase factor is therefore introduced in the path integral

$$\exp\Big(\mathrm{i}\sum_{I=0}^{M-1}\frac{c_I}{2\pi}\int d^2z_{\parallel}\,\mathrm{Tr}_{\mathrm{U}(n_I)}F_I\Big)\,, \tag{5.147}$$

leading to the following identification of the η-parameters of the surface defect:

$$\eta_I = \frac{c_I}{2\pi}\,. \tag{5.148}$$

This completes the identification of all the parameters of the generic GW monodromy defect with the background values of the twisted scalars in the $\mathbb{Z}_M$ orbifold. We note that these formulas generalize those in chapter 4 and exactly reduce to them when $M = 2$. We also remark that if we write the parameters b_I, $b_I^{\pm}$ and c_I as sums over all twisted sectors, their relation with the parameters of the surface defects holds also for even M. In this case, in fact, besides the twisted sectors we have described at length in this chapter, there is also a sector with twist $\frac{1}{2}$ whose contribution is exactly the same as in the $M = 2$ case. For this reason, therefore, we see that the restriction we made at the beginning to restrict to odd values of M does not lead to any loss of generality.

We end this section by observing that the identifications (5.142) and (5.148), namely

$$\{\alpha_I,\beta_I,\gamma_I,\eta_I\} = \Big\{-\frac{b_I}{2\pi},\frac{\mathrm{Re}(b_I^+)}{2\pi},\frac{\mathrm{Im}(b_I^+)}{2\pi},\frac{c_I}{2\pi}\Big\}\,, \tag{5.149}$$

are consistent with the behavior of the GW parameters under S-duality, as given in [1].

S-duality properties

In fact, even though our world-sheet analysis has been at the orbifold fixed point, it is possible to blow-up the $\mathbb{Z}_M$-singularity into an ALE space and provide an interpretation to the twisted scalars of the orbifold theory as massless moduli in the low-energy supergravity (see for instance [3, 38]). In such a geometric approach, the combinations b_I and c_I, which are made of the singlets $b_s^{(a)}$ and $C^{(a)}$ from each twisted sector as shown in (5.133) and (5.144), arise by integrating, respectively, the NS/NS 2-form $B_{(2)}$ and the R/R 2-form $C_{(2)}$ of Type II B supergravity around the exceptional cycles ω_I of the blown-up ALE space. Therefore, from (5.149) we read

$$\alpha_I = -\frac{1}{2\pi}\int_{\omega_I} B_{(2)}\,, \qquad \eta_I = \frac{1}{2\pi}\int_{\omega_I} C_{(2)}\,. \tag{5.150}$$

Using the S-duality action on the 2-forms, with simple manipulations as in sec.4.5 one can show that this identification implies that α_I and η_I indeed transform in the expected way.

Similarly, the $b_I^{\pm}$ parameters can be identified with the (string frame) metric moduli corresponding to the complex structure of the blown-up exceptional cycle ω_I. As such they inherit the S-duality transformation properties from the (string frame) metric, which are precisely the ones expected for the parameters β_I and γ_I of the GW defects.

We finally remark that when $M > 2$ also the scalars $b_3^{(a)}$ can couple to the gauge fields, differently from what happens in the $M = 2$ case [ch.4]. To have a uniform description for all M we have therefore chosen to set $b_3^{(a)} = 0$ in each twisted sector. As we have just seen, this choice has allowed us to identify a perturbative closed string realization of the generic GW defects that is fully consistent with S-duality. However, our approach offers the possibility of considering more general backgrounds with also $b_3^{(a)}$ turned on, and it would be interesting to further investigate their meaning and implications for the world-volume theory on the D3-branes and their defects.

Chapter 6

Conclusion

In a quantum field theory, *observables* are the primary objects used to extract and understand the underlying physics of that theory. Scattering amplitudes, n-point correlation functions, etc. are among the most used and well studied observables. In addition, there are various other operators e.g. Wilson loops, t'Hooft line operators, etc. which can be inserted inside the correlation function and can give important non-perturbative information about the theory under consideration. In particular, they can reveal information about different phase structures within the theory and their expectation values can serve as the order parameter. Surface operators or defects are also one such kind of disorder (defect) operator supported on a co-dimensional two surface which can extract non-perturbative information including different existing phase structures.

In [4], it was already suggested by Kanno and Tachikawa (KT) that the surface defects can be realized within a string theory setup using fractional branes on an orbifolded background. They showed in that paper the instanton contributions to the effective theory of surface defects are organized in terms of chain-saw quivers and described as D-instanton corrections to a system of fractional D3-branes with two world-volume directions extended along the orbifold background. From the KT construction, it was clear that the discrete data of the surface defect i.e. the number of elements in the partition of N (total rank of gauge group) is captured by the order M of the orbifold group, and each partition n_I is captured by the number of fractional branes of type I transforming under I-th irreducible representation of orbifold group $\mathbb{Z}_M$. But their construction did not give any hint as to how to realize the continuous parameters that are also part of the defining data of the Gukov-Witten surface defect. In this thesis, we addressed this issue and showed that these continuous data are in fact encoded within this fractional brane setup as the background values of certain twisted closed string scalar fields. Further, we derived the singular profiles of the gauge fields and scalars within this setup via open/closed correlators. This result has now provided us with a microscopic realization of GW surface defects within perturbative Type II B string theory using fractional D3 branes.

We want to point out that the orbifold setup considered in this thesis is quite different from the usual orbifold structures considered in the literature where the orbifolding is done entirely in transverse directions to the D-brane world volume. In our case, two of the world-volume directions are along the orbifold, and in Section 5.2.2, we saw that the orbifold group in addition to acting on the oscillators and Chan-Paton factors, also acts on the momentum factor $e^{i.k\ X}$ [1]. The open string vertex operators are combinations of momentum factors that have specific charges under the action of the orbifold group and the corresponding transformation can be compensated by the transformation of the oscillators and CP factors. This leads to the conclusion that none of the massless open string states is projected out of the open string spectrum. This is in line with our expectation as well since the presence of the defect does not alter the basic spectrum of the $\mathcal{N} = 4$ SYM. It will be an interesting exercise to calculate all the 3-pt and 4-pt couplings among the various fields and reproduce the $\mathcal{N} = 4$ SYM action.

The work presented in this thesis is very similar in spirit to the string theoretic realization of the gauge instantons using D3/D(-1) brane system [37]. In that work, one point emission of the open string field is calculated with disk diagrams having mixed boundary conditions and the derived profile matched that of the classical instanton. In our

[1] To be more specific it acts non-trivially only on the part of the momentum factors along the orbifold.

case, the role of the D-instantons is being played by the twisted closed string scalars at zero momentum. We believe that this string theoretic construction of surface defects using perturbative string theory is interesting because it provides an explicit and calculable framework. Moreover, we hope such a microscopic realization may be useful for various generalizations, applications in deformed theories and possibly letting us explore some novel effects at higher energies which are unseen in usual effective field theory descriptions which operates on much lower energy scales [2].

On the other hand, the construction may turn out to be useful in computing other quantities that characterize the superconformal defect field theory and help to establish connections with alternative approaches to the study of defects. One such alternative approach is the holographic description of surface defects where surface defects are realized using bubbling geometries of Type II B supergravity that asymptote to $AdS_5 \times S^5$ [25, 26]. Our construction of surface defects based on an exactly solvable string background of fractional D3 branes on orbifold space facilitates us with a framework where world-sheet computations are possible and usual string theoretic techniques can be applied. In contrast to the supergravity description, we are on the gauge theory side of holographic correspondence: the D-branes are not dissolved into the geometry, and the open string degrees of freedom that capture the gauge theory fields are explicitly present. A study to explore the interconnection between these two approaches: bubbling geometries and D-brane on orbifolds can shed some more light on the gauge/gravity correspondence.

Throughout the thesis, we have considered GW surface defects in $\mathcal{N} = 4$ U(N) theories as half-BPS objects from the total 10d string theory perspective. The natural step ahead would be to extend our explicit realization of surface defects in theories with lower supersymmetry and/or with other gauge groups. For example, by introducing a mass deformation in two of the directions transverse to the D3-branes [20] we can realize the so-called $\mathcal{N} = 2^*$ theory. To study gauge theories with less number of supersymmetry we can im-

[2]These expectations are based on the applications and constructions of various novel instanton configurations using the D3/D(-1) brane realization of gauge instantons [71–76]

plement another orbifold acting purely in directions transverse to the D3-branes: e.g. in our setup (1.1) employing one orbifold along $\mathbb{C}_{(4)}$ would lead to an $\mathcal{N} = 2$ theory living on the D-brane world -volume, consequent orbifolding along $\mathbb{C}_{(5)}$ would lead to an $\mathcal{N} = 1$ theory and so on. To study defects with orthogonal SO or symplectic Sp gauge groups, one can consider additional orientifold planes. One can also explore if this construction can be generalized to more general surface defects with distinct singularity structures other than simple poles and which can lead to quarter-BPS or $\frac{1}{8}$-BPS defects [59, 60].

As already mentioned line operators such as Wilson lines, t'Hooft operators can help us in distinguishing different phases within the theory. Surface operators also do a similar job but in some cases, surface operators can distinguish between phases that are otherwise indistinguishable using line operators [43]. It will be wonderful if the string theoretic realization of surface defects presented here could be applied to such scenarios and might provide us with an extremely important understanding of the phase dynamics from a string theory perspective.

Appendix A

Dirac matrices

In this appendix, we define in detail our conventions for the Dirac matrices used in the main text.

A.0.1 4d

We consider the 4d Euclidean space spanned by the coordinates x_m with $m \in \{3, 4, 5, 6\}$. These are the real coordinates corresponding to the complex coordinates z_2 and z_3 (see (4.2)) along which the $\mathbb{Z}_M$ orbifold acts.

An explicit realization of the Dirac matrices γ_m satisfying the 4d Euclidean Clifford algebra

$$\{\gamma_m, \gamma_n\} = 2\delta_{mn} , \tag{A.1}$$

is given by

$$\gamma_3 = \begin{pmatrix} 0 & \tau_1 \\ \tau_1 & 0 \end{pmatrix} , \quad \gamma_4 = \begin{pmatrix} 0 & -\tau_2 \\ -\tau_2 & 0 \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} 0 & \tau_3 \\ \tau_3 & 0 \end{pmatrix} , \quad \gamma_6 = \begin{pmatrix} 0 & +\mathrm{i}\,\mathbb{I}_2 \\ -\mathrm{i}\,\mathbb{I}_2 & 0 \end{pmatrix} \tag{A.2}$$

where τ_c are the usual Pauli matrices and $\mathbb{I}_2$ is the 2×2 identity matrix.

The chirality matrix $\widehat{\gamma}$ is given by

$$\widehat{\gamma} = -\gamma_3\gamma_4\gamma_5\gamma_6 = \begin{pmatrix} +\mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} . \tag{A.3}$$

This shows that in this basis a $4d$ Dirac spinor is written as

$$\begin{pmatrix} S^{\alpha} \\ S^{\dot{\alpha}} \end{pmatrix} \tag{A.4}$$

where α and $\dot{\alpha}$ label, respectively, the chiral and anti-chiral components.

Finally, the charge conjugation matrix $\widehat{C}$ is given by

$$\widehat{C} = \begin{pmatrix} +\epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} \tag{A.5}$$

where $\epsilon = -\mathrm{i}\,\tau_2$ (see (5.31)), and is such that

$$\widehat{C}\,\gamma_m\,\widehat{C}^{-1} = (\gamma_m)^{\mathtt{t}} \tag{A.6}$$

where $\mathtt{t}$ denotes the transpose.

Spinors in 4d

Introducing the following matrices

$$\sigma^m = (\tau^1, \tau^2, \tau^3, -i\,\mathbf{1}_2) , \tag{A.7}$$

we can form the combination

$$X_{\alpha\dot\beta} = \frac{1}{\sqrt{2}}\, x_m\, (\sigma^m)_{\alpha\dot\beta} = \begin{pmatrix} \bar z_3 & \bar z_2 \\ z_2 & -z_3 \end{pmatrix}. \tag{A.8}$$

The $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ isometry group acts on X as follows

$$X \quad \longrightarrow \quad U_+\, X\, U_-^\dagger \tag{A.9}$$

where $U_\pm \in \mathrm{SU}(2)_\pm$. Therefore, the two columns of X are two doublets transforming as spinors of $\mathrm{SU}(2)_+$:

$$y_\alpha = \begin{pmatrix} \bar z_3 \\ z_2 \end{pmatrix} \quad \text{and} \quad w_\alpha = \begin{pmatrix} \bar z_2 \\ -z_3 \end{pmatrix}. \tag{A.10}$$

Raising the indices, we have

$$y^\alpha = y_\beta\, (\epsilon^{-1})^{\beta\alpha} = \begin{pmatrix} -z_2 \\ \bar z_3 \end{pmatrix} \quad \text{and} \quad w^\alpha = w_\beta\, (\epsilon^{-1})^{\beta\alpha} = \begin{pmatrix} z_3 \\ \bar z_2 \end{pmatrix} \tag{A.11}$$

where $\epsilon = -\mathrm{i}\,\tau_2$ as in (5.31). Of course the same combinations can be made with the fermionic coordinates leading to the doublets

$$\begin{pmatrix} -\Psi^2 \\ \overline{\Psi}^3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Psi^3 \\ \overline{\Psi}^2 \end{pmatrix}. \tag{A.12}$$

These are precisely the structures that have been used in Section 5.1 to write the massless vertex operators of the twisted NS/NS sectors.

A.0.2 6*d*

We consider the 6*d* Euclidean space spanned by the coordinates x_M with $M \in \{1, 2, 7, 8, 9, 10\}$. These are the real coordinates corresponding to the complex coordinates z_1, z_4 and z_5 (see

(4.2)) that are transverse to the $\mathbb{Z}_M$ orbifold.

An explicit realization of the Dirac matrices Γ_M satisfying the 6d Euclidean Clifford algebra

$$\{\Gamma_M, \Gamma_N\} = 2\delta_{MN} \ , \tag{A.13}$$

is given by

$$\begin{aligned}
\Gamma_1 &= \begin{pmatrix} 0 & 0 & -\mathrm{i}\,\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & -\mathrm{i}\,\mathbb{I}_2 \\ \mathrm{i}\,\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & \mathrm{i}\,\mathbb{I}_2 & 0 & 0 \end{pmatrix} , \qquad
\Gamma_2 = \begin{pmatrix} 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & -\tau_3 \\ \tau_3 & 0 & 0 & 0 \\ 0 & -\tau_3 & 0 & 0 \end{pmatrix} , \\
\Gamma_7 &= \begin{pmatrix} 0 & 0 & -\tau_2 & 0 \\ 0 & 0 & 0 & \tau_2 \\ -\tau_2 & 0 & 0 & 0 \\ 0 & \tau_2 & 0 & 0 \end{pmatrix} , \qquad
\Gamma_8 = \begin{pmatrix} 0 & 0 & \tau_1 & 0 \\ 0 & 0 & 0 & -\tau_1 \\ \tau_1 & 0 & 0 & 0 \\ 0 & -\tau_1 & 0 & 0 \end{pmatrix} , \\
\Gamma_9 &= \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i}\,\mathbb{I}_2 \\ 0 & 0 & \mathrm{i}\,\mathbb{I}_2 & 0 \\ 0 & -\mathrm{i}\,\mathbb{I}_2 & 0 & 0 \\ \mathrm{i}\,\mathbb{I}_2 & 0 & 0 & 0 \end{pmatrix} , \qquad
\Gamma_{10} = \begin{pmatrix} 0 & 0 & 0 & \mathbb{I}_2 \\ 0 & 0 & \mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 & 0 & 0 \\ \mathbb{I}_2 & 0 & 0 & 0 \end{pmatrix} .
\end{aligned} \tag{A.14}$$

The chirality matrix $\widehat{\Gamma}$ is

$$\widehat{\Gamma} = \mathrm{i}\,\Gamma_1\Gamma_2\Gamma_7\Gamma_8\Gamma_9\Gamma_{10} = \begin{pmatrix} \mathbb{I}_2 & 0 & 0 & 0 \\ 0 & \mathbb{I}_2 & 0 & 0 \\ 0 & 0 & -\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & -\mathbb{I}_2 \end{pmatrix} . \tag{A.15}$$

This shows that in this basis a $6d$ Dirac spinor is written as

$$\begin{pmatrix} S^{A} \\ S^{\dot{A}} \end{pmatrix} \tag{A.16}$$

where A and $\dot{A}$ label, respectively, the chiral and anti-chiral components.

The charge conjugation matrix C is

$$C = \begin{pmatrix} 0 & 0 & 0 & +\epsilon \\ 0 & 0 & +\epsilon & 0 \\ 0 & -\epsilon & 0 & 0 \\ -\epsilon & 0 & 0 & 0 \end{pmatrix} \tag{A.17}$$

where, as before, $\epsilon = -\mathrm{i}\,\tau_2$. The above charge conjugation matrix is such that

$$C\,\mathit{\Gamma}_M\,C^{-1} = -(\mathit{\Gamma}_M)^{\mathrm{t}}\ . \tag{A.18}$$

Appendix B

$\mathbb{Z}_M$ in momentum space

Here we briefly comment on how to define the $\mathbb{Z}_M$ orbifold action in momentum space. Let us take the complex plane $C_{(2)}$ with coordinates z_2 and $\bar{z}_2$ on which $\mathbb{Z}_M$ acts as in (5.2), and define the momenta κ_2 and $\overline{\kappa}_2$ as in (5.88). For simplicity, however, we can drop the index 2 since in this appendix this does not cause any ambiguity.

First of all, we observe that the orbifold action on the coordinates can be equivalently read as an inverse action on the momenta. Consider for example the scalar product

$$\kappa\,\bar{z} + \overline{\kappa}\,z\,, \tag{B.1}$$

which, under the action of $\mathbb{Z}_M$ on the coordinates, is mapped to

$$\omega^{-1}\kappa\,\bar{z} + \omega\,\overline{\kappa}\,z\,. \tag{B.2}$$

Clearly, this result can also be interpreted as due to the following action of $\mathbb{Z}_M$ on the momentum variables:

$$\hat{g}\;:\;(\kappa\,,\,\overline{\kappa}) \longrightarrow (\omega^{-1}\,\kappa\,,\,\omega\,\overline{\kappa}) \tag{B.3}$$

with the coordinates held fixed.

Then, let us consider a function in momentum space, $f(\kappa, \bar{\kappa})$, and define its images under the orbifold group according to

$$\Pi_I(\kappa, \bar{\kappa}) = \frac{1}{M} \sum_{J=0}^{M-1} \omega^{-IJ} f(\omega^{-J} \kappa, \omega^J \bar{\kappa}) \tag{B.4}$$

where $I = 0, \ldots, M - 1$, modulo M. Using (B.3), it is immediate to check that

$$\hat{g}[\Pi_I] = \omega^I \Pi_I , \tag{B.5}$$

namely that Π_I transforms in the I-th representation of $\mathbb{Z}_M$. Inverting (B.4), we get

$$f(\kappa, \bar{\kappa}) = \sum_{I=0}^{M-1} \Pi_I(\kappa, \bar{\kappa}) . \tag{B.6}$$

Applying these definitions to the plane wave $\mathrm{e}^{\mathrm{i}(\kappa \bar{z} + \bar{\kappa} z)}$, we get

$$\mathcal{E}_I = \frac{1}{M} \sum_{J=0}^{\infty} \omega^{-IJ} \, \mathrm{e}^{\mathrm{i}(\omega^{-J} \kappa \bar{z} + \omega^J \bar{\kappa} z)} , \tag{B.7}$$

with

$$\hat{g}[\mathcal{E}_I] = \omega^I \mathcal{E}_I . \tag{B.8}$$

These functions $\mathcal{E}_I$ have exactly the same form and properties of the functions introduced in Section 5.2.2 when we described the $\mathbb{Z}_M$-invariant open string states. In terms of them, the plane wave can be written as

$$\mathrm{e}^{\mathrm{i}(\kappa \bar{z} + \bar{\kappa} z)} = \sum_{I=0}^{M-1} \mathcal{E}_I . \tag{B.9}$$

Let us now consider the Fourier transform of f. Using (B.6) and (B.9), we have

$$\mathcal{FT}[f](z) = \int \frac{d^2\kappa}{2\pi} f(\kappa, \bar{\kappa}) \, \mathrm{e}^{\mathrm{i}(\kappa \bar{z} + \bar{\kappa} z)} = \int \frac{d^2\kappa}{2\pi} \sum_{I,J=0}^{M-1} \Pi_I(\kappa, \bar{\kappa}) \, \mathcal{E}_J . \tag{B.10}$$

Since the integration measure is $\mathbb{Z}_M$-invariant, only the invariant products $\Pi_I\,\mathcal{E}_{M-I}$ survive, and thus

$$\mathcal{FT}[f](z) = \int \frac{d^2\kappa}{2\pi} \sum_{I=0}^{M-1} \Pi_I(\kappa,\bar{\kappa})\,\mathcal{E}_{M-I} \;=\; \frac{1}{M}\int \frac{d^2\kappa}{2\pi} \sum_{I=0}^{M-1} \hat{g}^I\Big[f(\kappa,\bar{\kappa})\,\mathrm{e}^{\mathrm{i}\,(\kappa\,\bar{z}+\bar{\kappa}\,z)}\Big]\,. \tag{B.11}$$

This shows that the Fourier transform leads to a well-defined function in the orbifolded theory. In particular, the Fourier transform of a function in the I-th irreducible representation of $\mathbb{Z}_M$ in momentum space is a function in configuration space that transforms in the representation $(M-I)$, and *viceversa*.

Appendix C

Fractional Branes: boundary state perspective

In this appendix, we will discuss how the fractional D-brane can be described from the perspective of boundary states [8] in the simplest case of $\mathbb{Z}_2$ orbifolding. Let us start with the expression of vacuum energy of open strings stretched between two fractional Dp branes in a $\mathbb{C}^3 \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold background:

$$Z = \int_0^\infty \frac{ds}{s} Tr_{\mathrm{NS}-R}\left[\left(\frac{1+(-1)^F}{2}\right)\left(\frac{e+g}{2}\right)e^{-2\pi s\ (L_0-a)}\right] \tag{C.1}$$

with the first term inside the trace performs the GSO projection, e and g are the two elements of $\mathbb{Z}_2$ and $a = 1/2, 0$ for NS and R sector respectively.

When the integral is evaluated with the factor $\frac{e}{2}$ gives the contribution from the untwisted sector and its half the contribution of the open strings stretched between usual Dp branes in flat space. The other $\frac{g}{2}$ factor gives the contribution from the twisted sector. If we denote the untwisted contribution as Z_e and twisted one as Z_g, we can interpret them respectively as the tree level closed string amplitude between two untwisted and two twisted boundary states.

$$
\begin{aligned}
Z_e &= \frac{\alpha'\pi}{2}\int_0^\infty dt\ \ {}^U\langle Dp|\, e^{-\pi t(L_0+\tilde{L}_0-2a)}\, |Dp\rangle^U \\
Z_g &= \frac{\alpha'\pi}{2}\int_0^\infty dt\ \ {}^T\langle Dp|\, e^{-\pi t(L_0+\tilde{L}_0)}\, |Dp\rangle^T
\end{aligned} \tag{C.2}
$$

The untwisted and twisted parts of the boundary state can be expressed as a linear combination of NS and R sector parts :

$$
\begin{aligned}
|Dp\rangle^U &= \frac{T_p}{2\sqrt{2}}\left(|Dp\rangle^U_{\rm NS} + |Dp\rangle^U_{\rm R}\right) \\
|Dp\rangle^T &= -\frac{1}{2^{s/2}}\frac{T_p}{2\sqrt{2}\pi^2\alpha'}\left(|Dp\rangle^T_{\rm NS} + |Dp\rangle^T_{\rm R}\right)
\end{aligned} \tag{C.3}
$$

where T_p and T_r are normalization factors. The number of D brane worldvolume directions along the orbifold is denoted by s.

In this thesis, we will be more interested in the twisted part of boundary states as they are used to calculate the refection rules. [1]

Now, both $|Dp\rangle^T_{\rm NS,R}$ are expressed as the following linear combination of Ishibashi states:

$$
|Dp\rangle^T_{\rm NS,R} = \frac{1}{2}\left(|Dp,+\rangle^T_{\rm NS,R} + |Dp,-\rangle^T_{\rm NS,R}\right) \tag{C.4}
$$

We note down the explicit forms of the Ishibashi states $|Dp,\eta\rangle^T_{\rm NS,R};\eta = \pm$ below [8]. We also give a short note on ishibashi states at the end of this appendix.

In the NS-NS twisted sector,

$$
|Dp,\eta\rangle^T_{\rm NS} = |Dp_X\rangle^T\ |Dp_\psi,\eta\rangle^T_{\rm NS} \tag{C.5}
$$

[1] The explicit forms of untwisted boundary states are given in Ref. [36, 53]

And in the R-R twisted sector,

$$|Dp,\eta\rangle_{\rm R}^{T} = |Dp_X\rangle^{T}\ |Dp_\psi,\eta\rangle_{\rm R}^{T} \tag{C.6}$$

where $\eta = \pm$ and

$$|Dp_X\rangle^{T} = \delta^{(5-r)}\ (\hat{q}^i - y^i)\prod_{n=1}^{\infty} e^{-\frac{1}{n}\alpha_{-n}.S.\tilde{\alpha}_{-n}}\ \prod_{r=1/2}^{\infty} e^{-\frac{1}{r}\alpha_{-r}.S.\tilde{\alpha}_{-r}}\ \prod_{\alpha}^{\prime}|p_\alpha = 0\rangle\prod_{i}^{\prime}|p_i\rangle \tag{C.7}$$

$$|Dp_\psi,\eta\rangle_{\rm NS}^{T} = \prod_{r=1/2}^{\infty} e^{i\eta\psi_{-r}.S.\tilde{\psi}_{-r}}\prod_{n=1}^{\infty} e^{i\eta\psi_{-n}.S.\tilde{\psi}_{-n}}\ |Dp_\psi,\eta\rangle_{\rm NS(0)}^{T} \tag{C.8}$$

$$|Dp_\psi,\eta\rangle_{\rm R}^{T} = \prod_{r=1/2}^{\infty} e^{i\eta\psi_{-r}.S.\tilde{\psi}_{-r}}\prod_{n=1}^{\infty} e^{i\eta\psi_{-n}.S.\tilde{\psi}_{-n}}\ |Dp_\psi,\eta\rangle_{\rm R(0)}^{T} \tag{C.9}$$

with $S = (\delta_{\alpha\beta}, -\delta_{ij}$ and $|Dp_\psi,\eta\rangle_{\rm NS(0)}^{T}$ and $|Dp_\psi,\eta\rangle_{\rm R(0)}^{T}$ are the zero modes in the fermionic sector. The longotudinal indices are denoted by $\alpha,\beta = 1,2,...,(p+1)$ and the transverse indices are denoted by $i,j = (p+2),...,10$. Note the prime in the product over α and i in eq. (C.7) signifies the fact the product is over only those longitudinal and transverse directions which are not along the orbifold, since there is no zero mode along those directions.

Lets now look at the zero mode parts; in the NS-NS sector its given by,

$$|Dp_\psi,\eta\rangle_{\rm NS(0)}^{T} = \left(\hat{C}\gamma_3\gamma_4\frac{1+i\eta\hat{\gamma}}{1+i\eta}\right)_{LM}|L\rangle|\widetilde{M}\rangle \tag{C.10}$$

where γ_m and C are the 6d gamma matrices and charge conjugation matrix of $SO(6)$ (defined in Appendix A), and $\hat{\gamma} = -\gamma_3\gamma_4\gamma_5\gamma_6$. Also $|L\rangle, |\widetilde{M}\rangle$ are spinors of $SO(4)$.

In the R-R sector the zero mode part looks like,

$$|Dp_\psi,\eta\rangle_{\rm R(0)}^{T} = \left(\hat{C}\Gamma_1\Gamma_2\frac{1+i\eta\hat{\Gamma}}{1+i\eta}\right)_{AB}|A\rangle|\widetilde{B}\rangle \tag{C.11}$$

where Γ_M and $\hat{C}$ are the 4d gamma matrices and charge conjugation matrix of $SO(4)$ (defined in Appendix A), and $\hat{\Gamma} = i\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6$. Also $|A\rangle, |\widetilde{B}\rangle$ are spinors of $SO(6)$.

There are two types of fractional branes corresponding to two irreducible representations of $\mathbb{Z}_2$. The untwisted and twisted boundary states given in eq. C.3 are the building block for the fractional branes of both type 0 and 1:

$$
\begin{aligned}
|Dp\rangle_0 &= |Dp\rangle^U + |Dp\rangle^T \\
|Dp\rangle_1 &= |Dp\rangle^U - |Dp\rangle^T
\end{aligned}
\tag{C.12}
$$

This discussion can be straight forwardly generalized to the $\mathbb{Z}_M$ orbifold case in which there will be M different types of fractional branes corresponding to M irreducible representations of the orbifold group. The only difference being in the $\mathbb{Z}_2$ case the Ishibashi states we are interested in are parts of the zero modes, but in the general $\mathbb{Z}_M$ case the relevant Ishibashi states are from the first excited states (5.64),(5.65).

Short note on Ishibashi states [61] :

In a rational CFT made out of a chiral algebra $\mathcal{A}$ and $\bar{\mathcal{A}}$, for $\mathcal{A} = \bar{\mathcal{A}}$, one can associate to each highest weight representation $\mathcal{R}_i$ of $\mathcal{A}$ a unique state (upto a constant) $|\mathcal{B}\rangle_i$ which satisfies the gluing conditions arising out of conformal symmetry and extended symmetry (e.g supersymmetry):

$$(L_n - \bar{L}_{-n})\,|B_i\rangle = 0; \quad \text{conformal symmetry condition}$$

$$(W^i_n - (-1)^{h_i} \bar{W}^i_{-n}) |B_i\rangle = 0; \quad \text{extended symmetry condition} \tag{C.13}$$

where L_n is the usual Virasoro generator and W^i_n is the holomorphic Laurent mode of the extended symmetry generator W^i with conformal weight h_i. The barred objects denote the corresponding anti-holomorphic counterparts.

The states $|\mathcal{B}_i\rangle$ are known as *Ishibashi states* [62, 63]. Since we are considering rational CFTs which contain finite highest weight states, the number of ishibashi states is also finite. The true boundary states build-out of linear combinations of ishibashi states [2]:

$$|B\rangle = \sum_i C^B_i \, |\mathcal{B}\rangle_i \tag{C.14}$$

where C^B_i s are complex coefficients.

The ishibashi states are constructed to satisfy the gluing conditions. But the actual boundary state made out of a specific linear combination of Ishibashi states transforms properly under the modular transformations and satisfies the "Cardy's Condition" [42, 57].

We study the boundary states for free boson CFT as an example to illustrate how boundary states are constructed [61] . The bosonic field can X satisfy the following two possible boundary conditions:

$$\begin{aligned} \partial_\sigma X_{\text{open}}|_{\,\sigma=0,\pi} &= 0 && \text{Neumann b.c} \\ \delta X_{\text{open}}|_{\,\sigma=0,\pi} &= 0 = \partial_\tau X|_{\,\sigma=0,\pi} && \text{Dirichlet b.c} \end{aligned} \tag{C.15}$$

where (τ, σ) are the worldsheet coordinates.

The above conditions are for the open string description of the bosonic field X. To study

[2] In eq. C.4 the boundary states are constructed with such a linear combination of ishibashi states.

the boundary states we need to the closed string sector. Using the open-closed duality, we interchange between $(\tau, \sigma) \to (\sigma, \tau)$. So, the conditions in (C.15) transform into the following:

$$\begin{aligned}
\partial_\tau X_{\text{closed}}|_{\tau=0} \, |B_N\rangle &= 0 && \text{Neumann b.c} \\
\partial_\sigma X_{\text{closed}}|_{\tau=0} \, |B_D\rangle &= 0 && \text{Dirichlet b.c}
\end{aligned} \tag{C.16}$$

where we have introduced the boundary states $|B_N\rangle$ and $|B_D\rangle$ satisfying Neumann and Dirchlet conditions respectively.

In terms of the mode expansion of X (2.9), the above conditions are translated to the following:

$$\begin{aligned}
(\alpha_m + \tilde{\alpha}_{-m}) \, |B_N\rangle &= 0 \\
(\alpha_m - \tilde{\alpha}_{-m}) \, |B_D\rangle &= 0
\end{aligned} \tag{C.17}$$

These are the *gluing conditions* for the free boson case. The solutions to these conditions are given by the following boundary states:

$$\begin{aligned}
|B\rangle_N &= \frac{1}{C_N} \exp\left(- \sum_{k=1}^{\infty} \frac{1}{k} \alpha_{-k} \tilde{\alpha}_{-k} \right) |0\rangle \\
|B\rangle_D &= \frac{1}{C_D} \exp\left(+ \sum_{k=1}^{\infty} \frac{1}{k} \alpha_{-k} \tilde{\alpha}_{-k} \right) |0\rangle
\end{aligned} \tag{C.18}$$

where C_N and C_D are normalization constants. Note the different sign inside the exponential for the two different boundary conditions. It is possible to check explicitly by applying these states with α_m and $\tilde{\alpha}_m$, they do satisfy the gluing conditions (C.17).

These are coherent states and flat space counterparts of the Ishibashi states which are relevant in orbifold space. For comparison, notice the similarity between (C.18) and (C.7)(C.8)(C.9). The difference being whereas (C.18) is the total boundary states, for the orbifold case the Ishibashi states are being the building blocks, and their linear combinations are taken to construct a total boundary state to make them satisfy " Cardy's condition".

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