Generalized BMS Symmetry and Double Soft Theorems

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis[1]

Journal

1. "Double soft graviton theorems and Bondi-Metzner-Sachs symmetries"

   ***Anupam A H**, Arpan Kundu, Krishnendu Ray*;

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2. "Generalized BMS Algebra at Time-like Infinity"

   ***Anupam A H**, Aniket Khairnar, Arpan Kundu*;

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1. "Generalized coherent states in QCD from asymptotic symmetries"

   ***Anupam A H**, Athira P V*;

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[1]As it is standard in the High Energy Physics Theory (hep-th) community the names of the authors on any paper appear in their alphabetical order.

Conferences and workshops attended

1. “Chennai Strings Meeting 2020” at Institute of Mathematical Sciences”, Chennai from 14-18 December 2020..

2. “Indian Strings Meeting” at “Indian Institute of Science Education and Research(IISER)”, Bhopal from 22-27 December 2019.

3. “Chennai Strings Meeting 2019” at “Institute of Mathematical Sciences”, Chennai from 21-23 November 2019.

4. “25th Saalburg Summer School” at Heigenbrucken, Germany from 2-13 September 2019.

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6. “Indian Strings Meeting” at “Indian Institute of Science Education and Research(IISER)”, Trivandrum from 16-21 December 2018.

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8. “Black Holes: From Classical to Quantum Gravity” at “IIT Gandhinagar”, Gandhinagar from 15-19 December 2017.

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Contents

Synopsis

Introduction

The central theme in this thesis revolves around understanding certain aspects of relationship between asymptotic symmetries and soft theorems in gravity.

Asymptotic symmetries are gauge transformations that do not die down at the infinity. Since these transformations lead to actual physical symmetries,there are conserved charges associated with it. In gravity, the asymptotic symmetry corresponds to those that preserve the asymptotic structure of spacetime. This corresponds to the infinite dimensional extension of the famous Bondi, Metzner and Sachs (BMS) group. There are two known extensions, one being the extended BMS group [1] and the other being the Generalized BMS (GBMS) group [2] [3]. Soft theorems on the other hand are factorization theorems which constrains the scattering amplitude when one or more of the external gravitons becomes soft. The deep underlying connection between these two independent areas were unknown till Strominger et al showed that Weinberg's soft graviton theorem (leading soft graviton theorem) is equivalent to Ward identities corresponding to BMS symmetry of the quantum gravity S-matrix [4]. Later it was shown by Campiglia and Laddha [2] that these statements can be extended to subleading level.

In this thesis we are interested in the aspects of the generalized BMS symmetries and its connection with the Double soft theorems [5]. Double soft theorems are factorization statements which constrain the scattering amplitude when two of the external gravitons become soft. Such theorems are very interesting in the sense that they contain the information of the structure of the (unbroken) symmetry generators.

In particular we try to find a interpretation of a special class of Double Soft theorems called consecutive double soft theorems (CDST) as a consequence of Ward Identities corresponding to generalized BMS symmetry. We also find that algebra of generalized BMS vector fields close at time-like infinity, which will serve as a precursor for understanding the relationship of GBMS charge algebra and consecutive double soft theorems when the external states are massive.

Background

In this section we quickly review the equivalence of the single soft graviton theorems[2] for massless particles (both leading and sub–leading) from asymptotic symmetries at null infinity [2, 4]. This will serve as the background for our main work in this thesis. In the process, we also define the notations that we use later.

According to present understanding, the asymptotic symmetry group of gravity, acting on the asymptotic phase space of gravity is the "Generalized BMS" group — it is a semidirect product of supertranslations and Diff(S^2). They can be thought of as a local generalization of translations and the Lorentz group respectively. While the original BMS group [6, 7] is a semidirect product of supertranslations and $SL(2,\mathbb{C})$, in the generalized BMS group the $SL(2,\mathbb{C})$ symmetry is further extended to Diff(S^2). Each of the supertranslations and Diff(S^2) symmetry gives rise to conserved asymptotic charges, namely, the supertranslation charge (Q_f) and Diff(S^2) charge (Q_V) respectively. These charges are determined completely by the asymptotic "free data" and are parametrized by an arbitrary function $f(z,\bar{z})$ and an arbitrary vector field $V^A(z,\bar{z})$, respectively, where $(z,\bar{z})$ denotes coordinates on 2-sphere.

[2]In this thesis we are analyzing the soft theorems only at tree level S matrix.

To define a symmetry of a gravitational scattering problem at the quantum level, these charges are elevated to a symmetry of the quantum gravity $\mathcal{S}$–matrix. Corresponding to each such symmetry one gets a Ward identity. The authors in [2] showed that each such Ward identity is equivalent to single soft graviton theorem at the leading and subleading level. In the rest of this section, we discuss briefly how the single soft graviton theorems are equivalent to Ward identities of generalized BMS charges.

The leading single soft graviton theorem follows from the Ward identity of the supertranslation charge Q_f [4], which physically corresponds to the conservation of energy at each direction on the conformal sphere at null infinity. The supertranslation charge Q_f is given by [4]

$$Q_f = \int du\, d^2z\, f\, \gamma_{z\bar{z}}\, N_{zz}\, N^{zz} + 2\int du\, d^2z\, f\, \partial_u\Big(\partial_z U_{\bar{z}} + \partial_{\bar{z}} U_z\Big). \tag{1}$$

Here, $U_z = -\frac{1}{2}D^z C_{zz}$, and $N_{zz} = \partial_u C_{zz}$ is the Bondi news tensor, where C_{zz} is the radiative "free data". The derivative D^z is the covariant derivative *w.r.t.* the 2–sphere metric.

The supertranslation charge Q_f is characterized by the arbitrary function $f(z, \bar{z})$, where (z, $\bar{z}$) are coordinates on the conformal sphere at null infinity. The first term in 1 is quadratic in C_{zz} is conventionally called the "hard part"(Q_f^{hard}), while the second is linear in C_{zz} — is called the "soft part" (Q_f^{soft}).

The equivalence between the supertranslation Ward identity and the leading single soft graviton theorem, follows from the fact that the asymptotic charge 1 is a symmetry of the quantum gravity $\mathcal{S}$–matrix [4]. As a result, one gets the Ward identity for supertranslation as:

$$\langle \mathrm{out}|\, [Q_f, \mathcal{S}]\, |\mathrm{in}\rangle = 0 \Leftrightarrow \langle \mathrm{out}|\, [Q_f^{\mathrm{soft}}, \mathcal{S}]\, |\mathrm{in}\rangle = -\, \langle \mathrm{out}|\, [Q_f^{\mathrm{hard}}, \mathcal{S}]\, |\mathrm{in}\rangle\,. \tag{2}$$

where in writing the above, the classical charges have been promoted to quantum operators. This quantization is carried out using the asymptotic quantization of C_{zz} [4]. Upon asymptotic quantization of the charges one can see that Q_f^{soft} is responsible for the creation or annihilation of a soft graviton mode, and Q_f^{hard} acts non trivially on the external states through a contribution from the energy momentum tensor of massless particles at null infinity. Additionally one uses Christodoulou and Klainerman condition to relate positive helicity soft graviton mode with negative helicity soft graviton mode. Finally we can write 2 as

$$\lim_{E_p\to 0}\frac{E_p}{2\pi}\int d^2w\, D_{\bar{w}}^2 f(w,\bar{w})\ \langle\text{out}|\, a_+(E_p,w,\bar{w})\mathcal{S}\,|\text{in}\rangle \\ = -\Big[\sum_{\text{out}} E_i\ f(\hat{k}_i) - \sum_{\text{in}} E_i\ f(\hat{k}_i)\Big]\ \ \langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle\,. \tag{3}$$

Structure of the terms in 3 encourages one to ask whether this can be related to Weinberg's soft graviton theorem [8] which is given as,

$$\lim_{E_p\to 0} E_p\,\langle\text{out}|\, a_+(E_p,w,\bar{w})\ \mathcal{S}\,|\text{in}\rangle = \sum_i \frac{(\epsilon^+(w,\bar{w})\cdot k_i)^2}{(p/E_p)\cdot k_i}\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle\,. \tag{4}$$

where the soft graviton has energy E_p and momentum p. Its direction is parametrized by $(w,\bar{w})$ and its polarization vector is denoted by $\epsilon^+(w,\bar{w})$. We adopt the notation:

$$\hat{S}^{(0)}(p;k_i) \equiv \frac{1}{E_{k_i}}\frac{(\epsilon^+(w,\bar{w})\cdot k_i)^2}{(p/E_p)\cdot k_i}. \tag{5}$$

with which, the leading soft factor in the r.h.s. of 4 can be written as:

$$\sum_i \frac{(\epsilon^+(w,\bar{w})\cdot k_i)^2}{(p/E_p)\cdot k_i} \equiv S^{(0)}(p;\{k_i\}) \equiv \sum_i S^{(0)}(p;k_i) \equiv \sum_i E_{k_i}\,\hat{S}^{(0)}(p;k_i). \tag{6}$$

Now one can show the equivalence of the Ward identity 3 with the leading soft theorem 4 by choosing $f(z,\bar{z})$ to a particular function $s(z,\bar{z};w,\bar{w})$ which satisfies

$$D_{\bar{z}}^2 s(z,\bar{z};w,\bar{w}) = 2\pi\delta^2(w-z). \tag{7}$$

The subleading single soft graviton theorem follows from the Ward identity of the Diff(S^2)

charge Q_V [2], which physically corresponds to the conservation of angular momentum at each angle in a gravitational scattering process. This charge is given by:

$$Q_V = \frac{1}{4}\int du\, d^2z\ \sqrt{\gamma}\,\partial_u C^{AB}\left(\mathcal{L}_V C_{AB} - \alpha\, C_{AB} + \alpha u\,\partial_u C_{AB}\right) \\ +\frac{1}{2}\int du\, d^2z\ \sqrt{\gamma}\left(C^{zz}\, D_z^3 V^z +\ C^{\bar{z}\bar{z}}\, D_{\bar{z}}^3 V^{\bar{z}}\right). \tag{8}$$

where $\alpha = \frac{1}{2}(D_z V^z + D_{\bar{z}} V^{\bar{z}})$ and $V^A(z,\bar{z})$ is an arbitrary vector field on the conformal sphere at null infinity. The covariant derivatives are *w.r.t.* the 2–sphere metric. As explained for the supertranslation case, the first term is the "hard part" Q_V^{hard} and the second is the "soft part" Q_V^{soft} of the Diff(S^2) charge.

Proceeding in a manner similar to the case of supertranslation, the Ward identity for Diff(S^2)s can be written as:

$$\langle\text{out}|\,[Q_V,\mathcal{S}]\,|\text{in}\rangle = 0 \Leftrightarrow \langle\text{out}|\,[Q_V^{\text{soft}},\mathcal{S}]\,|\text{in}\rangle = -\,\langle\text{out}|\,[Q_V^{\text{hard}},\mathcal{S}]\,|\text{in}\rangle\,. \tag{9}$$

Now, using the asymptotic quantization of the "free data" and crossing symmetry one can write the Ward identity as

$$-\frac{1}{4\pi}\lim_{E_p\to 0}\,(1+E_p\,\partial_{E_p}) \\ \times\int d^2w\left[V^{\bar{w}}\,\partial_{\bar{w}}^3\,\langle\text{out}|\,a_+(E_p,w,\bar{w})\,\mathcal{S}\,|\text{in}\rangle + V^w\,\partial_w^3\,\langle\text{out}|\,a_-(E_p,w,\bar{w})\,\mathcal{S}\,|\text{in}\rangle\right] \\ = \left[\sum_{\text{out}} J_{V_i}^{h_i} - \sum_{\text{in}} J_{V_i}^{-h_i}\right]\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle\,. \tag{10}$$

where $J_{V_i}^{h_i}$ is a differential operator characterised by the vector field $V^A(z,\bar{z})$ acting on the external states.

Now, the Cachazo–Strominger (CS) subleading soft graviton theorem reads [9]:

$$\lim_{E_p\to 0}(1+E_p\,\partial_{E_p})\,\langle\text{out}|a_+(E_p,w,\bar{w})\,\mathcal{S}\,|\text{in}\rangle$$
$$=\sum_i \frac{\epsilon^+(w,\bar{w})\cdot k_i}{p\cdot k_i}\,\epsilon^+_\mu(w,\bar{w})\,p_\nu\,J_i^{\mu\nu}\,\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle\,. \tag{11}$$

where, $J_i^{\mu\nu}$ is the angular momentum operator acting on the i^{th} hard particle. For further use, we adopt the notation:

$$S^{(1)}(p;k_i)=\frac{\epsilon^+(w,\bar{w})\cdot k_i}{p\cdot k_i}\,\epsilon^+_\mu(w,\bar{w})\,p_\nu\,J_i^{\mu\nu}. \tag{12}$$

Using this, the subleading soft factor in the r.h.s. of 11 can be written as:

$$\sum_i \frac{\epsilon^+(w,\bar{w})\cdot k_i}{p\cdot k_i}\,\epsilon^+_\mu(w,\bar{w})\,p_\nu\,J_i^{\mu\nu}=\sum_i S^{(1)}(p;k_i)=S^{(1)}(p;\{k_i\}). \tag{13}$$

Now, in the Ward identity 10, if one chooses the vector field V^A as:

$$V^A=K^+_{(w,\bar{w})}\equiv\frac{(\bar{z}-\bar{w})^2}{(z-w)}\,\partial_{\bar{z}}. \tag{14}$$

one can show the equivalence between the Ward identity and the subleading soft graviton theorem.

Double Soft theorems and asymptotic symmetries

Having reviewed the relationship between asymptotic symmetries and the single soft graviton theorems, the next natural question is to ask if such a relationship holds between the generalized BMS algebra and double soft graviton theorems. These theorems (and its generalization to the multiple soft graviton case) have been studied previously using various methods including BCFW recursions [10], CHY amplitudes [11–14] and Feynman diagram techniques [15]. In a recent work [16], the authors have studied the symmetry foundations of the double soft theorems of certain classes of theories like the dilaton, DBI, and special Galileon. As has been analyzed in the literature, there are two kinds of double soft graviton theorems depending upon the relative energy scale of the soft gravitons. The simultaneous soft limit is the one where soft limit is taken on both the gravitons at the same rate. It was shown in [15], that simultaneous soft limit yields a universal factorization theorem. However, from the perspective of Ward identities, it is the consecutive soft limits which arise rather naturally. Consecutive double soft graviton theorems (CDST) elucidate the factorization property of scattering amplitudes when the soft limit is taken on one of the gravitons at a faster rate than the other [10]. We now review this factorization property when such soft limits are taken and show that they give rise to three CDSTs. The first one, we refer to as the leading CDST which is the case where the leading soft limit is taken on both the soft gravitons. The remaining two theorems refer to the case where the leading soft limit is taken with respect to one of the gravitons and the subleading soft limit is taken with respect to the other.

We begin with a $(n+2)$ particle scattering amplitude denoted by $\mathcal{A}_{n+2}(q, p, \{k_m\})$ where p , q are the momenta of the two gravitons which will be taken to be soft and $\{k_m\}$ is the set of momenta of the n hard particles. Consider the consecutive limit where the soft limit is first taken on graviton with momentum q, keeping all the other particles momenta unchanged and then a soft limit is taken on the graviton with momentum p. Following this one finally gets the factorization

$$\begin{aligned}
&\mathcal{A}_{n+2}(q, p, \{k_m\}) = \\
&\quad\Big[\frac{1}{E_p E_q}\sum_{i,j} E_{k_i}\, E_{k_j}\, \hat{S}^{(0)}(q;k_i)\, \hat{S}^{(0)}(p;k_j) + \sum_{i,j}\frac{E_{k_i}}{E_q}\, \hat{S}^{(0)}(q;k_i)\, S^{(1)}(p;k_j) \\
&\qquad + \sum_{i}\frac{E_{k_i}}{E_q}\, \hat{S}^{(0)}(q;p)\, \hat{S}^{(0)}(p;k_i) \; + \sum_{i,j} S^{(1)}(q;k_i)\,\frac{E_{k_j}}{E_p}\, \hat{S}^{(0)}(p;k_j) \\
&\qquad + S^{(1)}(q;p)\sum_{i}\frac{E_{k_i}}{E_p}\, \hat{S}^{(0)}(p;k_i)\Big]\mathcal{A}_n(\{k_m\}) + O(E_p) + O(E_q). \qquad (15)
\end{aligned}$$

This expansion contains three types of terms. The first type scales as $1/(E_pE_q)$ (and hence gives rise to a pole in both the soft graviton energies), giving the leading contribution to the factorization. The second and the third type of terms scale as E_q^0/E_p and E_p^0/E_q respectively, both contributing to the subleading order of the factorization. This gives the *leading* CDST as:

$$\lim_{E_p\to 0} E_p \lim_{E_q\to 0} E_q\, \mathcal{A}_{n+2}(q,p,\{k_m\}) = \Big[\, S^{(0)}(q;\{k_i\})\, S^{(0)}(p;\{k_j\})\Big]\, \mathcal{A}_n(\{k_m\}). \tag{16}$$

At the subleading level one gets two types of factorisation which are given by

$$\begin{aligned}&\lim_{E_p\to 0}(1+E_p\partial_{E_p}) \lim_{E_q\to 0} E_q\, \mathcal{A}_{n+2}(q,p,\{k_m\})\\ &\qquad\qquad = \Big[S^{(0)}(q;\{k_i\})\, S^{(1)}(p;\{k_j\}) + \mathcal{N}(q;p;\{k_i\})\Big]\mathcal{A}_n(\{k_m\}).\end{aligned} \tag{17}$$

and

$$\begin{aligned}&\lim_{E_p\to 0} E_p \lim_{E_q\to 0}(1+E_q\,\partial_{E_q})\, \mathcal{A}_{n+2}(q,p,\{k_m\})\\ &\qquad = \Big[S^{(0)}(p;\{k_i\})\, S^{(1)}(q;\{k_j\}) + \mathcal{M}_1(q;p;\{k_i\}) + \mathcal{M}_2(q;p;\{k_i\})\Big]\mathcal{A}_n(\{k_m\}).\end{aligned} \tag{18}$$

where

$$\mathcal{M}_1(q;p;\{k_i\}) = \sum_i S^{(1)}(q;k_i)\Big(E_{k_i}\,\hat{S}^{(0)}(p;k_i)\Big) = \sum_i S^{(1)}(q;k_i)\Big(S^{(0)}(p;k_i)\Big), \tag{19}$$

$$\mathcal{M}_2(q;p;\{k_i\}) = \sum_i \lim_{E_p\to 0} E_p\, S^{(1)}(q;p)\left(\frac{E_{k_i}}{E_p}\,\hat{S}^{(0)}(p;k_i)\right), \tag{20}$$

$$\mathcal{N}(q;p;\{k_i\}) = \hat{S}^{(0)}(q;p)\, S^{(0)}(p;\{k_i\}). \tag{21}$$

We now ask if there are Ward identities in the theory which are equivalent to the double soft graviton theorems at the leading and sub–leading order. In particular, we look for Ward identities that will lead us to the CDSTs. Let us consider the family of Ward identities whose general structure is:

$$\langle \text{out}| [Q_1, [Q_2, \mathcal{S}]] |\text{in}\rangle = 0. \tag{22}$$

where both Q_1 and Q_2 are either both supertranslation charges or Q_1 is a supertranslation charge and Q_2 is a Diff(S^2) charge.

In [5] following a work by Avery and Schwab [17], we present a derivation of this proposed Ward identity. Using the tools that we discussed in the previous section we show that such a proposal leads to the CDSTs. Depending on the choice of charges one gets the leading as well as the subleading consecutive double soft theorems.

We also show that the CDST's are equivalent to Ward identities associated to Generalized BMS in scattering states defined around (non) Fock vacua parametrized by supertranslations and Diff(S^2) charges. As explained in the previous section the asymptotic charges can be written as a sum of soft and hard part. The soft part is responsible for the creation and annihilation of a soft graviton mode. Therefore the action of the soft charge on the usual Fock vacuum gives rise to a infinite degeneracy in the definition of vaccua. Each such vaccua can be parameterized by a supertranslation soft charge or Diff(S^2) soft charge[3]. In [5] we showed that, CSDT's arise naturally if one considers Ward Identity involving one asymptotic charge on scattering states built around such degenerate vaccua. We summarize the main results in [5] as

$$\langle \text{out}| [Q_f, [Q_g, \mathcal{S}]] |\text{in}\rangle = 0 \Leftrightarrow \langle \text{out}, f| [Q_g, \mathcal{S}] |\text{in}\rangle = 0 \Leftrightarrow \text{Leading CDST}, \tag{23}$$

$$\langle \text{out}| [Q_f, [Q_V, \mathcal{S}]] |\text{in}\rangle = 0 \Leftrightarrow \langle \text{out}, f| [Q_V, \mathcal{S}] |\text{in}\rangle = 0 \Leftrightarrow \text{Sub leading CDST 1}, \tag{24}$$

$$\langle \text{out}| [Q_V, [Q_f, \mathcal{S}]] |\text{in}\rangle = 0 \Leftrightarrow \langle \text{out}, V| [Q_f, \mathcal{S}] |\text{in}\rangle = 0 \Leftrightarrow \text{Sub leading CDST 2}. \tag{25}$$

where Q_f, Q_g corresponds to supertranslation charge and Q_V corresponds to Diff(S^2)

[3] the notion of Diff(S^2) vacuum has some conceptual and mathematical subtleties associated to it and we have treated this formally in our work.

charge. $\langle \text{out}, f|$,$\langle \text{out}, V|$ corresponds to external states built from supertranslated and Diff(S^2) vaccua respectively. Leading CDST, subleading CDST 1 and subleading CDST 2 refers to 16, 17 and 18 respectively.

Generalized BMS algebra at time-like infinity

In the previous section, the focus has been on the asymptotic symmetry group at null infinity and its relationship with the soft theorems where the external particles were massless. In soft theorems, the external particles (other than the soft particle), can be massive or massless. To prove the equivalence between asymptotic symmetries and soft theorems when the external states contain massive particles, one needs to include the phase space for massive particles (whose geodesics start(end) at past(future) time-like infinity) as well. Based on the earlier work on the action of BMS group on massive scalar particle phase space [18], this question was addressed in [19].

The study of extended BMS as well as generalized BMS charge algebra at null infinity has been extensively carried out in [1, 20–23]. Motivated by these works, we were interested in understanding the generalized BMS charge algebra from the perspective at time-like infinity and its relationship with double soft theorems when the external states are massive. Our work [24] serves as a precursor to this goal. In this paper, we are interested in understanding the generalized BMS vector field algebra at time-like infinity with the aim of understanding the generalized BMS charge algebra at time-like infinity in future. We show that there is a closure of generalized BMS vector fields under modified version of Lie bracket as proposed by Barnich et.al [1].

We now summarize the key ideas that were relevant for our analysis. The set of coordinates which we used are the hyperbolic coordinates $(\tau, \rho, \hat{x})$, which are defined in terms of Cartesian coordinates $(t, \vec{x})$ in the region $t \geq r \equiv \sqrt{\vec{x} \cdot \vec{x}}$ as:

$$\tau := \sqrt{t^2 - r^2} \;\; ; \;\; \rho := \frac{r}{\sqrt{t^2 - r^2}} \;\; ; \;\; \hat{x} = \vec{x}/r. \tag{26}$$

We consider a space of metrics g_{ab} which has an asymptotic expansion in τ near time-like infinity of the form:

$$ds^2 = \left(-1 + O(1/\tau)\right) d\tau^2 + \tau^2 h_{\alpha\beta}(\tau, \rho, \hat{x}) dx^\alpha dx^\beta. \tag{27}$$

where $h_{\alpha\beta}(\tau, \rho, \hat{x})$ has the following asymptotic expansion (in τ) around time-like infinity

$$h_{\alpha\beta}(\tau, \rho, \hat{x}) = h^{(0)}_{\alpha\beta}(\rho, \hat{x}) + \frac{h^{(1)}_{\alpha\beta}(\rho, \hat{x})}{\tau} + \frac{h^{(2)}_{\alpha\beta}(\rho, \hat{x})}{\tau^2} + \cdots . \tag{28}$$

where $h^{(0)}_{\alpha\beta}(\rho, \hat{x})$ belongs to the class of metrics diffeomorphic to the hyperboloid part of the usual Minkowski metric. This ansatz is mainly motivated from the asymptotic flatness structure taken in [2, 3] where the authors considered asymptotic flat metrics at null infinity in which leading order sphere metric can be any unit $S^{(2)}$ metric. The notion of asymptotic flatness for metric of this form 27 at time-like infinity have been addressed in [25, 26]. The Minkowski metric (which we denote by $\mathring{g}_{ab}$) belongs to the class of metric 27 that has only the leading components (in τ) and the hyperboloid components take a particular form. The line element for $\mathring{g}_{ab}$ is written as:

$$ds^2 = -d\tau^2 + \tau^2 \mathring{h}_{\alpha\beta}(\rho, \hat{x}) dx^\alpha dx^\beta. \tag{29}$$

where

$$\mathring{h}_{\alpha\beta}(\rho, \hat{x}) dx^\alpha dx^\beta \equiv \frac{d\rho^2}{1 + \rho^2} + \rho^2 q_{AB} dx^A dx^B. \tag{30}$$

Here, q_{AB} is the unit metric on 2-sphere. The greek indices $\alpha, \beta, \cdots$ runs over the coordinates on the hyperboloid. We also denote the small Latin indices $a, b, c, \cdots$ to denote the four spacetime indices.

One can reach time-like infinity i^+ in hyperboloid coordinates by taking $\tau \to \infty$ limit (or in the Cartesian coordinates $t \to \infty$, keeping $t \geq r$). Similarly, one can reach the part of null infinity where $u > 0$ in the hyperboloid coordinates by taking the limit $\tau \to \infty$, $\rho \to \infty$, keeping $\frac{\tau}{2\rho}$ = const.

In order to analyze the asymptotic symmetries at time-like infinity i^+, we suitably adapt the de-Donder gauge in the hyperbolic coordinates. In this gauge, the residual (large) diffeomorphisms are generated by supertranslation and Diff(S^2) vector fields that smoothly matches with the corresponding BMS vector fields at null infinity.

In particular, we consider the following gauge conditions to the metric ansatz[4] 27

$$\mathring{\nabla}_b \mathcal{G}^{ab} = 0, \tag{31}$$

$$\mathrm{Tr}(h^{(1)}_{\alpha\beta}(\rho, \hat{x})) = 0. \tag{32}$$

where $\mathcal{G}^{ab} \equiv \sqrt{g} g^{ab}$ and $\mathring{\nabla}_b$ refers to the covariant derivative w.r.t to the reference Minkowski metric ($\mathring{g}_{ab}$) in 29. The trace free condition 32 of $h^{(1)}_{\alpha\beta}(\rho, \hat{x})$ is taken w.r.t to $h^{(0)}_{\alpha\beta}(\rho, \hat{x})$.

The generalized BMS vector fields at time-like infinity are those that generate the group of diffeomorphisms that preserve the form of the metric 27 and the gauge conditions 31-32. To find the structure of such vector fields we start by taking a general ansatz for the vector fields which has an asymptotic expansion (in τ) of the form:

$$\xi(\tau, \rho, \hat{x}) = \Big(\xi^{(0)\tau}(\rho, \hat{x}) + \frac{\xi^{(1)\tau}(\rho, \hat{x})}{\tau} + \cdots\Big)\partial_\tau + \Big(\xi^{(0)\alpha}(\rho, \hat{x}) + \frac{\xi^{(1)\alpha}(\rho, \hat{x})}{\tau} + \cdots\Big)\partial_\alpha. \tag{33}$$

Using the ansatz above and applying the gauge constraints, one gets the following condi-

[4] We are indebted to Miguel Campiglia for suggesting this gauge choice which was a vital input in this work.

tions on the vector fields,

$$\Big(\varDelta - 3\Big)\xi^{(0)\tau}(\rho, \hat{x}) = 0, \tag{34}$$

$$2\, D^{(\alpha}\xi^{(0)\beta)}\partial_\beta\Big(\ln\Big(\sqrt{\frac{h^{(0)}}{\mathring{h}}}\Big)\Big) \; + \; h^{(0)\alpha\beta}\partial_\beta D_\gamma \xi^{(0)\gamma} \; + \; 2\mathring{D}_\beta D^{(\alpha}\xi^{(0)\beta)} = 0, \tag{35}$$

$$D^\alpha \xi^{(0)\beta}\mathring{h}_{\alpha\beta} = 0. \tag{36}$$

The above constraints are a more general case of the constraints obtained by Campiglia and Laddha in [19] where they considered asymptotic symmetries at time-like infinity as residual gauge transformations of de-Donder gauge around the reference Minkowski metric.

In our work [24], we were primarily interested in the algebra of the generalized BMS vector fields w.r.t reference Minkowski metric ($\mathring{g}_{ab}$). The supertranslation and Diff(S^2) vector fields to leading order at time-like infinity are given by

$$\xi_{ST} \; = \; f_{\mathcal{H}}(\rho, \hat{x})\partial_\tau, \tag{37}$$

$$\xi_{SR} \; = \; V^\alpha_{\mathcal{H}}(\rho, \hat{x})\partial_\alpha. \tag{38}$$

The constraints 35,36 at the reference Minkowski metric therefore becomes

$$\Big(\mathring{\varDelta} - 3\Big)f_{\mathcal{H}}(\rho, \hat{x}) = 0, \tag{39}$$

$$\Big(\mathring{\varDelta} - 2\Big)V^\alpha_{\mathcal{H}} = 0, \tag{40}$$

$$\mathring{D}_\alpha V^\alpha_{\mathcal{H}} = 0. \tag{41}$$

where $\mathring{\varDelta}$ refers to Laplacian w.r.t $\mathring{h}_{\alpha\beta}$. A naive attempt to study the vector field algebra will be to compute the ordinary Lie bracket (as for the null infinity case) of the vector fields and check whether the resulting vector field satisfies the constraint 39 (in case for supertranslation), 40 and 41 (in case for Diff(S^2)). However, as is well known in the literature [1,23], the correct definition of Lie bracket in the case of asymptotic symmetries is more intricate. This can be explained as follows.

Usually, one studies the vector field algebra by considering the commutator of two variations of the vector fields on the metric. An important point to be noted here is the fact that the vector fields themselves are metric dependent. This can be seen from the defining equations for the vector field 39, 40 and 41, which tells us that the vector fields depend upon the hyperboloid metric $\mathring{h}_{\alpha\beta}$ through covariant derivative and Laplacian. Therefore, performing the second variation will affect both the first variation as well as the metric. This can be written as

$$[\delta_{\xi_1(g)}, \delta_{\xi_2(g)}]g_{\mu\nu} = \delta_{\left([\xi_1(g),\xi_2(g)]-\delta^g_{\xi_1}\xi_2(g)+\delta^g_{\xi_2}\xi_1(g)\right)}g_{\mu\nu}.$$

The first term in the above expression is the ordinary Lie bracket. The extra term $\delta^g_{\xi_1}\xi_2(g)$ captures the variation on the vector field $\xi_2(g)$ due to the action of the vector field $\xi_1(g)$ on the metric. Hence to realize the algebra of the vector fields at time-like infinity one needs to take into account such terms. Therefore one defines the modified Lie bracket for realizing the BMS vector fields algebra as

$$[\xi_1, \xi_2]^a_M \equiv [\xi_1, \xi_2]^a - \delta^g_{\xi_1}\xi^a_2 + \delta^g_{\xi_2}\xi^a_1. \tag{42}$$

where $\delta^g_{\xi_1}\xi^a_2$ denotes the change in ξ^a_2 due to the variation in the metric induced by ξ_1. In [24] we find that the vector fields that one gets from the modified Lie bracket obeys the same constraints that the supertranslation and Diff(S^2) vector field does. This can be written as

$$[\xi_{ST1}, \xi_{ST2}]^a_M = 0, \tag{43}$$

$$(\mathring{\Delta} - 3)[\xi_{ST}, \xi_{SR}]_M = 0, \tag{44}$$

$$\mathring{D}_\alpha[\xi_{SR1}, \xi_{SR2}]_M = 0, \tag{45}$$

$$(\mathring{\Delta} - 2)[\xi_{SR1}, \xi_{SR2}]_M = 0. \tag{46}$$

Thus under the modified Lie bracket, the supertranslation vector fields commute, a supertranslation and a Diff(S^2)) vector field gives another supertranslation vector field and two Diff(S^2)) vector field gives another Diff(S^2)) vector field. Thus we show that under the modified Lie bracket the generalized BMS vector fields closes.

Plan of the thesis

In this thesis, we will investigate various aspects of generalized BMS symmetry and relationship with double soft graviton theorems. It will include the following chapters.

- Chapter 1 will provide an introduction to asymptotic symmetries in gravity and soft graviton theorems.
- Chapter 2 reviews the earlier works on asymptotic symmetry groups in gravity, single soft graviton theorems and their equivalence.
- Chapter 3 discusses about the double soft graviton theorem with focus on consecutive double soft graviton theorems at leading and subleading level.
- Chapter 4 will discuss the equivalence of consecutive double soft graviton theorems with generalized BMS symmetry group.
- Chapter 5 will discuss the extension of generalized BMS to time-like infinity and the study of GBMS vector field algebra at time-like infinity.
- Chapter 6 will be the concluding chapter which include the summary of the results and future directions.

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Chapter 1

Introduction

Understanding the symmetries of classical and quantum gravitational scattering has been an active area of research since the 1960s [6, 21, 22, 27–44]. The study of soft theorems in scattering amplitudes also has a similar history [8, 12–14, 38, 45–53]. The relationship between these seemingly independent topics was unknown until the seminal work by Strominger [4]. Since then there were lot of works in exploring the deep connection between the two subjects [2–5, 19, 54–64]. This thesis explores the relationship of the symmetries associated to quantum gravitational scattering amplitudes with a particular class of soft graviton theorems known as the double soft graviton theorems.

Gravitational scattering is studied in asymptotically flat spacetimes. Asymptotic flatness can be understood as the behavior of space-times as one recedes away from isolated gravitational systems. The mathematically rigorous definition for asymptotic flatness can be found in the works by [6,7,27,29–32]. Away from the sources, one expects that such class of metrics asymptotes to the Minkowski metric and therefore one expects the asymptotic symmetries to be the isometries of the Minkowskian space-time, which generates the Poincare group. But in their seminal work [6, 7, 27] Bondi, Van der Berg, Metzner, and Sachs, showed that one gets an infinite dimensional extension of the Poincare group. This extension is known as the BMS (Bondi, Metzner, and Sachs) group. It is a semi-direct product of the Lorentz group and an infinite-dimensional extension of the constant translations known as supertranslations.

There are two known extensions to the BMS group. Motivated from AdS/CFT, Barnich and Troessart proposed one of the extensions which are referred to as the "Extended" BMS group. The extended BMS algebra is implemented by enhancing the six-dimensional global conformal transformations that generate the Lorentz transformations in the original BMS, to infinite-dimensional local conformal transformations. Such local conformal transformations are known as super-rotations. Hence, the extended BMS can be written as a semi-direct product of supertranslations and superrotations. There are several works understanding the many aspects of the "Extended" BMS, especially in connection with the algebra related to "Extended" BMS charges and its relationship with conformal field theories [1, 23, 38].

Campiglia and Laddha proposed a second extension [2, 3] to BMS, which is known as the "generalized" BMS group. Generalized BMS is realized by enhancing the Lorentz group in the BMS to the infinite-dimensional group generated by smooth vector fields on the 2-sphere (which is denoted as Diff(S^2)). Hence, the generalized BMS group can be written as the semi-direct product of supertranslations and Diff(S^2). A lot of aspects of generalized BMS symmetry are currently being explored [3, 20–22, 24, 65]. In this thesis, we will focus on this proposed extension and explore its relationship with soft graviton theorems.

Soft theorems are factorization theorems that are related to the infrared properties of the scattering amplitude. Consider a scattering amplitude of finite energy particles and soft particles [1]. Such an amplitude can be written in terms of *soft factors* and scattering amplitude involving finite energy particles only. The soft factors are an expansion in the soft momenta and depend on the kinematics of the soft particles and finite energy particles. The soft expansion terminates at a particular order in the soft momenta. If one takes

[1]a massless particle of momentum k is considered a soft particle if the ratio $\frac{p_i \cdot k}{p_i \cdot p_j} << 1 \ \forall \ p_i, p_j$ where p_i, p_j denote the momentum of the rest of the particles in the scattering amplitude

appropriate soft limits in the scattering amplitude, one can extract each of the soft factors seperately. The soft factors are not arbitrary and there are certain aspects of the soft factors that are universal in nature and does not depend on a specific Lagrangian under consideration. Hence the factorization is known as the soft theorems. If the soft particles in the external state are photons, then one gets soft photon theorems and similarly, if the soft particles are gravitons, one gets soft graviton theorems.

Weinberg's soft factor [8],which corresponds to the leading term in the soft expansion, is universal in nature and does not depend on a particular theory and is valid up to all orders in the perturbation theory in the scattering amplitude. The sub-leading soft photon theorem was identified by Low [45]. Unlike Weinberg's soft photon factor, the sub-leading soft photon factor is not universal (theory dependent) and is affected by the loop corrections in the scattering amplitude. The sub-leading and sub-subleading soft graviton theorems for tree-level amplitudes were discovered very recently by Cachazo and Strominger [9]. In dimensions greater than four, the sub-leading soft graviton factor is universal, but in four dimensions due to infrared divergences in the scattering amplitude the sub-leading soft factor gets loop corrected [66]. The sub-sub-leading soft graviton factor can be written as a sum of a universal term and a theory dependent term that comes from the non-minimal coupling of the Riemann tensor with finite energy particles. In [15], Sen introduced a new procedure known as "covariantization" technique for deriving the soft photon and graviton theorems for any generic quantum theory of gravity that admits general coordinate invariance and $U(1)$ gauge invariance. Many other recent developments in the area of soft theorems have happened, especially in the understanding of soft theorems in the classical regime and its applications to classical gravitational radiation [67–72].

The deep underlying connection between the two seemingly independent areas of study was unknown until the work by Strominger [4]. In [4], it was shown that the Ward identities corresponding to the supertranslation invariance of the quantum gravity $\mathcal{S}$ matrix are equivalent to Weinberg's soft graviton theorem with massless external particles. This initiated a flurry of activities toward understanding similar relationships in gauge theories and gravity [2, 19, 62, 63, 73–75]. For example, asymptotic symmetries in QED were shown to be equivalent to leading/sub-leading soft photon theorems [56,57,61,75]. Similar relationships were also established in the context of non Abelian gauge theories also [73,76,77].

One of the major implications of this equivalence is that it helps us get new insights to one of the fields by knowing the results of the other. For example, in [59] it was shown that one could derive the Ward identities corresponding to super-rotation invariance of the quantum gravity $\mathcal{S}$ matrix at the tree-level. But there were certain subtleties in proceeding in the opposite direction, i.e if sub-leading soft graviton theorem can be understood as a consequence of super-rotation invariance. This motivated Campiglia and Laddha to propose the generalized BMS symmetry in which such obstacles can be overcome and equivalence can be established [2,3].

Motivated by the connections between asymptotic symmetries and soft theorems, we tried to explore the relationship between generalized BMS group and soft graviton theorems in the context of double soft graviton theorems that are factorization statements in scattering amplitudes involving two soft gravitons. There are two types of double soft graviton theorems in the literature, depending on the energy scale of the soft gravitons. If the soft limit is taken at the same rate, one gets a simultaneous double soft graviton theorem. If the soft limit of one of the gravitons is taken at a faster rate than the other, one gets a consecutive double soft graviton theorem. In [5], we find that generalized BMS symmetries are equivalent to the consecutive double soft graviton thorems. Generalized BMS charges give rise to an infinite set of degenerate vacua which is labeled by the supertranslation and Diff(S^2) charges[2]. We show that the consecutive double soft graviton theorem at the leading and subleading level can be derived from Ward identities corresponding to the generalized BMS charge, but evaluated around the external states built from such de-

[2]there are some subtleties associated with degenerate vacua parametrized by Diff(S^2) charge. We will elaborate on this in section 4

generate vacua. We also show that the consecutive double soft graviton theorems can be derived by evaluating the nested Ward identities constructed from two generalized BMS charges in states built around Fock vacua.

In the second part of the thesis, we explore another aspect of generalized BMS group. In soft theorems the external particle (other than the soft particle) can be massive or massless. Hence, to prove the equivalence between generalized BMS group and soft graviton theorems when the external states are massive, one should understand the role of generalized BMS at time-like infinity. Massive particle geodesics asymptotically reach time-like infinity in a asymptotically flat space-time. The vector fields that generate the generalized BMS symmetry at time-like infinity were already derived by Campiglia and Laddha in [19]. Subsequently, the equivalence between Ward identities associated with generalized BMS symmetry and leading/subleading single soft graviton theorem were external states are massive were established. But the algebra of the vector fields were not investigated. In [24], we explore this aspect. The vector field algebra is conventionally studied by computing the Lie bracket. But at time-like infinity, the vector fields are metric dependent and therefore the ordinary Lie bracket will not suffice to capture the algebra of the vector fields. Motivated by the above, in [24], we show that the super-translations and vector fields that generate the sphere diffeomorphisms close under a modified version of Lie bracket as proposed by Barnich et al. in [1].

This thesis is organized as follows. In chapter 2, we give an brief review of the asymptotic symmetries in gravity with a focus on generalized BMS symmetry. A brief overview of soft graviton theorems and the equivalence between generalized BMS symmetry and soft graviton theorems are presented. In chapter 3, we give the notion of double soft graviton theorems with emphasis on consecutive double soft graviton theorems at the leading and sub-leading level. In chapter 4, we present the equivalence between consecutive double soft graviton theorems and generalized BMS symmetry, which is one of the main results in this thesis. In chapter 5, we study the generalized BMS symmetry at time-like infinity and specifically discuss the algebra of the vector fields that generate generalized BMS symmetry at time-like infinity. In chapter 6, we summarize the main results in this thesis.

Chapter 2

Preliminaries

In this chapter, we review the earlier works on asymptotic symmetries and soft theorems in gravity. This will serve as the background material for our works, which will be explained in detail in sections 3 and 4. In section 2.1, we give an brief review of asymptotic flat spacetimes and corresponding asymptotic symmetries and charges. In section 2.2 a review of soft theorems in gravity is presented and in section 2.3 the equivalence between the two is given following the works of [2, 3].

2.1 Asymptotic flat spacetimes and asymptotic symmetries in gravity

Asymptotic flatness is referred to as the properties of space times as one moves away from localized gravitational sources. It is described in the literature in two regimes, depending on how one moves from isolated gravitational systems. If one moves from the sources in a space-like direction toward the boundary of space-time one has a notion of asymptotic flatness at spatial infinity that is defined as the conformal boundary of space-time in space-like directions. Asymptotic flatness has been explored in detail [33, 95]. In this thesis, we focus on the second regime in which one defines a notion of asymptotic flatness near the null boundaries of space time. This was mainly pursued in the works by Bondi, Sachs, Penrose [6,7,30], where they studied the propagation of gravitational waves from isolated systems. Asymptotic flatness in the null directions is defined around null infinity (denoted by $\mathcal{I}$), which is defined as conformal boundary of space time in null directions [30–32,34]. If one moves away from the sources in the future null direction one reaches future null infinity (denoted by $\mathcal{I}^+$) and if one moves in the past null direction one reaches past null infinity (denoted by $\mathcal{I}^-$). Hence all null geodesics start at $\mathcal{I}^-$ and end at $\mathcal{I}^+$. Future null infinity ($\mathcal{I}^+$) is described in terms of a retarded null coordinate system labeled by $(u, r, z, \bar{z})$. These are related to Cartesian coordinates (t, x^1, x^2, x^3) by

$$u = t - r \quad ; \quad r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \quad ; \quad z = \frac{x^1 + ix^2}{x^3 + r} \quad ; \quad \bar{z} = \frac{x^1 - ix^2}{x^3 + r}. \tag{2.1}$$

One can reach $\mathcal{I}^+$ by taking u = constant and $r \to \infty$ limit. Similarly, the asymptotic flatness in the past null direction is described using an advanced null coordinate system labeled by $(v, r, z, \bar{z})$, which are related to Cartesian coordinates (t, x^1, x^2, x^3) by

$$v = t + r \quad ; \quad r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \quad ; \quad z = \frac{x^1 + ix^2}{x^3 + r} \quad ; \quad \bar{z} = \frac{x^1 - ix^2}{x^3 + r}. \tag{2.2}$$

In [6, 7, 27], the authors showed that any four-dimensional asymptotically flat spacetime in these coordinates takes the following form:

$$ds^2 = (V/r)e^{2\beta}du^2 - 2e^{2\beta}dudr + g_{AB}(dx^A - U^A du)(dx^B - U^B du), \tag{2.3}$$

with the following gauge conditions[1]:

$$g_{rr} = 0 \;\; ; \;\; g_{rA} = 0, \tag{2.4}$$

[1]For our purposes in this thesis, we focus on the description of asymptotic flatness in a coordinate based approach. There also exists a coordinate free description for asymptotic flatness and the corresponding asymptotic symmetries. This is due to the works by Ashtekar et. al [33–37]

$$\det(\, g_{AB}\,) = r^4 \det(\, \gamma_{AB}\,). \tag{2.5}$$

where β, V, U^A, g_{AB} are functions of the coordinates $(u, r, z, \bar{z})$ and A, B indices denotes the coordinates on the sphere expressed in stereographic coordinates $(z, \bar{z})$ and γ_{AB} denotes the metric on unit $2-$sphere given by $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}, \gamma_{zz} = \gamma_{\bar{z}\bar{z}} = 0$.

The asymptotic flatness at null infinity is imposed by assuming appropriate fall-off conditions at large r on the metric components. The fall-offs are chosen such that they are not strong enough to disallow interesting solutions like gravitational waves, but should not be weak enough so that they allow unphysical solutions.

Based on the above criteria, Bondi and Sachs in [6, 7] adopted the following conditions on the metric:

$$\beta \sim O(r^{-2}) \;\; ; \;\; \frac{V}{r} \sim -1 + O(r^{-1}) \;\; ; \;\; U^A \sim O(r^{-2}), \tag{2.6}$$

$$g_{AB} \sim r^2 \gamma_{AB} + O(r). \tag{2.7}$$

where γ_{AB} denotes the metric on unit $2-$sphere given by $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$. Using these fall-offs given in 2.6, 2.7, one can see that at the leading order (in r) of the metric given in 2.3 the metric becomes the Minkowski metric expressed in the $(u, r, z, \bar{z})$ coordinates, i.e 2.3 becomes

$$ds^2 = -du^2 - 2dudr + 2r^2 \gamma_{AB} dx^A dx^B + \cdots \tag{2.8}$$

where $\cdots$ denotes the subleading components in r. With these fall-off conditions, one can now describe the asymptotic symmetry group to such class of metrics. These are described as the group of non-trivial diffeomorphisms (i.e those diffeomorphisms generated by vector fields that survive at null infinity) that preserves the fall-off conditions 2.6 and

2.7. Inorder to find such vector fields, the following conditions are imposed:

$$\mathcal{L}_\xi g_{uu} = O(r^{-1}) \;\; ; \;\; \mathcal{L}_\xi g_{ur} = O(r^{-2}) \;\; ; \;\; \mathcal{L}_\xi g_{uA} = O(1) \;\; ; \;\; \mathcal{L}_\xi g_{AB} = O(r). \tag{2.9}$$

Since the metric asymptotes to Minkowski metric at large r, one might expect that the asymptotic symmetry group will be generated by vector fields that generate the Poincare group. However surprising enough, one gets an infinite number of vector fields that generate the asymptotic symmetry group. These vector fields are known in the literature as Bondi, Metzner and Sachs (BMS) vector fields and the asymptotic symmetry group generated by such vector fields is called the BMS group. The BMS vector fields are given by:

$$\xi_f^{\text{BMS}} = f(z,\bar{z})\partial_u + D_A D^A f \partial_r - \frac{1}{r} D^A f \partial_A + \cdots \tag{2.10}$$

$$\xi_Y^{\text{BMS}} = Y^A(z,\bar{z})\partial_A + u\frac{D_A Y^A}{2}\partial_u - r\frac{D_A Y^A}{2}\partial_r + \cdots \tag{2.11}$$

where $f(z,\bar{z})$ is an arbitrary function on the 2− sphere and Y^A corresponds to those vector fields that satisfies

$$Y^C\partial_C\gamma_{AB} + \gamma_{CB}\partial_A Y^C + \gamma_{AC}\partial_B Y^C = D_C Y^C \gamma_{AB}. \tag{2.12}$$

The vector field ξ_f^{BMS} characterized by the arbitrary function $f(z,\bar{z})$ is called the super-translation vector field. These are an infinite dimensional extension of the vector fields that generate the global translations. The global Poincare translations $x^\mu \to x^\mu + a^\mu$ are realized using the following $f(z,\bar{z})$:

$$f(z,\bar{z}) = \frac{(a^0 - a^3) - (a^1 - ia^2)z - (a^1 - +ia^2)\bar{z} + (a^0 + a^3)z\bar{z}}{1 + z\bar{z}}. \tag{2.13}$$

From 2.12, it can be seen that $Y^z, Y^{\bar{z}}$ satisfies the 2−d conformal Killing equation on the 2−sphere. Hence, the solutions to $Y^z, Y^{\bar{z}}$ can be any holomorphic/anti holomorphic vector fields and hence there are infinitely many solutions of them. The globally well-defined solutions to the 2.12 correspond to those Y^z to be spanned by the generators,$\{1, z, z^2, i, iz, iz^2\}$ and $Y^{\bar{z}}$ to be spanned by the generators,$\{1, \bar{z}, \bar{z}^2, i, i\bar{z}, i\bar{z}^2\}$. These six generators correspond to six generators of the Lorentz transformations. Hence, one can see that ξ_Y are the vector fields that generate the Lorentz transformations.

We now discuss a more relaxed set of fall-off conditions adopted by Campiglia and Laddha [2, 3]. This will be our focus in rest of the thesis. The fall-off conditions that are adopted in [2, 3] for the metric components 2.3 around $\mathcal{I}^+$ are as follows :

$$\beta = \frac{\mathring{\beta}(u,\hat{x})}{r^2} + O(r^{-3}) \quad ; \quad \frac{V}{r} = \mathring{V}(u,\hat{x}) + O(r^{-1}), \tag{2.14}$$

$$U^A = \frac{\mathring{U}(u,\hat{x})}{r^2} + O(r^{-3}), \tag{2.15}$$

$$g_{AB} = r^2 q_{AB}(\hat{x}) + r\left(C_{AB}(u,\hat{x}) + uT_{AB}(\hat{x})\right) + \cdots \tag{2.16}$$

where $\mathring{\beta}, \mathring{V}, \mathring{U}$ are functions of $(u, z, \bar{z})$. Here all the indices are raised w.r.t to the metric q_{AB}. An important thing to note is regarding the metric component q_{AB}. In the previous choice, for defining the notion of asymptotic flatness, q_{AB} was fixed to be proportional to round sphere metric γ_{AB}. In [2, 3], the authors considered more relaxed fall-off conditions, in which q_{AB} can be any sphere metric, which is independent of u and such that volume element of q_{AB} is that of the unit 2-sphere metric γ_{AB}. i.e q_{AB} belong to the class of metrics:

$$\partial_u q_{AB} = 0 \quad ; \quad \sqrt{q} = \sqrt{\gamma}. \tag{2.17}$$

C_{AB} is often referred to as shear tensor in the literature and T_{AB} is a 2D tensor constructed entirely from q_{AB} and vanishes when q_{AB} is chosen to be the round sphere metric γ_{AB}.

Along with the fall-off conditions, one assumes the gauge fixing condition,

$$g_{rr} = 0 \quad ; \quad g_{rA} = 0 \quad ; \quad det(g_{AB}) = r^4 det(q_{AB}) \tag{2.18}$$

One can now solve the Einstein equations using the metric given and find that

$$\mathring{V} = -\frac{1}{2}\mathcal{R} \quad ; \quad \mathring{\beta} = -\frac{1}{32}C^2 \quad ; \quad \mathring{U}^A = -\frac{1}{2}D_B C^{AB}. \tag{2.19}$$

where $\mathcal{R}$ and D_A denote the Ricci scalar curvature and covariant derivative of q_{AB} and $C^2 = C_{AB}C^{AB}$. It can also be shown that all the subleading components in the metric can be expressed in terms of q_{AB} and C_{AB}. q_{AB} is often referred as the kinematical "frame" and the pair (q_{AB}, C_{AB}) are often referred to as the radiative free data as these are the data required to determine all other metric components. The special case in which q_{AB} is the unit round sphere γ_{AB} is known as the "Bondi" frame. From the gauge fixing condition 2.18, it can be easily seen that $q^{AB}C_{AB} = 0$ and hence C_{AB} is trace-free. Therefore, there are two independent degrees of freedom encoded in C_{AB} that indicate that C_{AB} are the two radiative degrees of freedom.

Having defined the fall-off conditions, one can now define the asymptotic symmetry group associated with such class of metrics. The asymptotic symmetries of such asymptotically flat spacetimes are the group of diffeomorphisms that preserves the form of the metric, along with preserving the gauge fixing condition 2.5. Additionally, in [2, 3], it was found that the vector fields that generate these diffeomorphisms are asymptotically divergence free,

$$\lim_{r\to\infty} \nabla_a \xi^a = 0. \tag{2.20}$$

The vector fields that satisy these constraints are of the form[2]:

$$\xi_{f_+} = f_+(\hat{x})\partial_u + \cdots \quad ; \quad \xi_{V_+} = V_+^A(\hat{x})\partial_A + u\alpha\partial_u - r\alpha\partial_r + \cdots \tag{2.21}$$

where $f_+(\hat{x})$ denotes an arbitrary function on the two-sphere, and $V_+^A(\hat{x})$ is an arbitrary vector field on the sphere and $\alpha = D_C V_+^C/2$. The vector fields ξ_{f_+} characterized by $f_+(\hat{x})$

[2]The detailed derivation of the vector fields will be discussed in chapter 5

are called the supertranslation vector fields. On the other hand the vector fields ξ_{V_+} characterized by $V^A_+(\hat{x})$ are called Diff(S^2) vector fields. $\cdots$ indicate the subleading components (in r) of the vector fields. The algebra of the vector fields can be obtained by evaluating the Lie bracket of the vector fields and one finally gets

$$[\xi_{f_1}, \xi_{f_2}] = 0 \;\; ; \;\; [\xi_{V_1}, \xi_{V_2}] = \xi_{[V_1,V_2]} \;\; ; \;\; [\xi_f, \xi_V] = \xi_{L_V f - \alpha f}. \tag{2.22}$$

From 2.22, one can see the Generalized BMS vector fields at null infinity have a semidirect sum algebra structure in which the supertranslation vector fields form an Abelian ideal. We will revisit more of the generalized BMS vector field algebra in chapter 5.

Using the Lie derivative of the generalized BMS vector fields on the metric, the action of the vector fields on the radiative free data can be found to be,

$$\delta_{f_+} q_{AB} = 0 \qquad ; \;\; \delta_{f_+} C_{AB} = f \partial_u C_{AB} - 2(D_A D_B f_+)^{\mathrm{TF}} + f_+ T_{AB}, \tag{2.23}$$

$$\delta_{V_+} q_{AB} = \mathcal{L}_{V_+} q_{AB} - 2\alpha q_{AB} \;\; ; \;\; \delta_{V_+} C_{AB} = \mathcal{L}_{V_+} C_{AB} - \alpha C_{AB} + \alpha u \partial_u C_{AB}. \tag{2.24}$$

where T.F indicates the trace-free component.

Having defined the asymptotic symmetries, one can now construct the corresponding charges. In [3], the authors used covariant phase space formalism to derive the generalized BMS charges. Inorder to construct well defined finite charges, one must define an appropriate radiative phase space that considers the variation in both q_{AB} and C_{AB}, and suitable fall-offs and boundary conditions (in u) for C_{AB}[3]. The radiative phase space Γ^q is given by:

$$\Gamma^q := \Big\{ (q_{AB}, C_{AB}) : q_{AB} C^{AB} = 0 \; ; \; \lim_{u \to \pm\infty} \partial_u C_{AB} = O(1/|u|^{2+\epsilon}) \; ; \; D_z^2 C_{\bar{z}\bar{z}}|_{\mathcal{I}^+_+} = D_{\bar{z}}^2 C_{zz}|_{\mathcal{I}^+_-} \Big\}. \tag{2.25}$$

[3]The fall-offs chosen here are restrictive for a general gravitational scattering. This choice has been made for compatibility with tree-level scattering processes. For a general classic scattering process one should allow for $\partial_u C_{AB}$s to fall as $O(1/u^2)$.

where I^+_+ and I^+_- denote the boundaries of future null infinity as $u \to \pm\infty$ respectively. The generalized BMS charges that correspond to supertranslation symmetry and Diff (S^2) symmetry are defined in a particular subspace of Γ^q in which q_{AB} is the 2-sphere metric γ_{AB}. The charges are found to be [2, 3]:

$$Q^+_{f_+} = \frac{1}{4}\int du\, d^2z\, f_+\ N_{AB}\, N^{AB} + \int du\, d^2z\, f_+\ \lim_{r\to\infty} r^2 T^M_{uu} - \frac{1}{2}\int du\, d^2z\, f_+\, D^A D^B N_{AB}, \tag{2.26}$$

$$\begin{aligned} Q^+_{V_+} = \frac{1}{4}\int du\, d^2z\ N^{AB}\left(\mathcal{L}_{V_+} C_{AB} - \alpha\, C_{AB} + \alpha u\, N_{AB}\right) + \\ \int du\, d^2z \Big(\frac{1}{2} u\alpha \lim_{r\to\infty} r^2 T^M_{uu} + 2V^A_+ \lim_{r\to\infty} T^M_{uA}\Big) + \frac{1}{2}\int du\, d^2z\ \ u\Big(N^{zz}\, D^3_z V^z_+ +\ N^{\bar z\bar z}\, D^3_{\bar z} V^{\bar z}_+\Big). \end{aligned} \tag{2.27}$$

where $N_{AB} \equiv \partial_u C_{AB}$ is the Bondi News tensor and T^M_{uu}, T^M_{uA} correspond to the uu and uA components of the matter stress energy tensor, respectively. We have chosen the convention $8\pi G = 1$. In this thesis, we restrict ourselves to pure gravity and therefore drop the terms involving the matter stress energy terms from here.

One can repeat the similar analysis at past null infinity ($\mathcal{I}^-$) and arrive at a generalized BMS group defined at $\mathcal{I}^-$. We denote the generalized BMS group acting on $\mathcal{I}^+$ and $\mathcal{I}^-$ as $\mathcal{G}^+$ and $\mathcal{G}^-$ respectively. The generalized BMS $\mathcal{G}$ of the asymptotically flat spacetime is defined as the diagonal subgroup of $\mathcal{G}^+ \times \mathcal{G}^-$. The relationship of the generalized BMS group $\mathcal{G}$ with single soft graviton theorems will be discussed in section 2.3.

Motivated by connections from AdS/CFT correspondence, Barnich and Troessart proposed another natural extension to BMS group. This was based on extending the BMS vector field ξ^{BMS}_Y that generated the Lorentz group. In [1, 23], it was found that one can naturally extend the global conformal Killing vector Y^A in ξ^{BMS}_Y to include local conformal Killing vectors, so that the vector fields that generate the Lorentz symmetry is extended to those that generate local conformal transformations. This extension is known as "Extended BMS" group, which is characterized by supertranslation vector fields and "superrotation" vector fields. The superrotation vector fields are the infinite dimensional extension of ξ_Y^{BMS} by allowing Y^A to have local conformal Killing vectors on the sphere. The consequence of such an extension is that one must relax one of the fall-off conditions, namely the g_{AB} component in 2.7, such that the leading order component of g_{AB} is round sphere metric (denoted by γ_{AB}) except at isolated singular points. Since we focus on generalized BMS in this thesis, we are not going into the details of extended BMS here.

2.2 Soft graviton theorems

Soft graviton theorems are factorization statements in scattering amplitudes involving finite energy particles and soft gravitons[4]. It relates a scattering amplitude with finite energy external particles (which can be massive or massless) and soft gravitons with scattering amplitude involving finite energy particles only. From recent works [11, 15, 66, 96], it has now become clear that soft theorems exist for any generic quantum theory of gravity as well as in the classical regime. In classical scattering amplitudes, the soft graviton theorem extracts the low energy spectrum of gravitational radiation emitted during a classical gravitational scattering process and has many interesting applications [67–70]. In this thesis, we focus on understanding the aspects on soft graviton theorems in quantum scattering amplitude in four dimensions and its relationship with asymptotic symmetries.

Consider a scattering amplitude involving $(n+1)$ external particles denoted by $\mathcal{A}(p, \{k_m\})$ where $\{k_m\}$ denotes the set of n finite energy particles and p denotes the momentum of a positive helicity soft graviton. In four dimensions, there are two types of soft theorems depending on whether one considers the scattering amplitude at tree-level or at loop-level. The soft graviton theorem at tree-level states that the tree-level scattering amplitude

[4] Recall the definition of a soft particle in footnote 1

$\mathcal{A}_{\text{tree}}(p, \{k_m\})$ can be written as:

$$\mathcal{A}_{\text{tree}}(p; \{k_m\}) = \left(\frac{1}{E_p} S^{(0)}(p; k_m) + S^{(1)}(p; \{k_m\}) + \cdots\right) \mathcal{A}_{\text{tree}}(\{k_m\}). \tag{2.28}$$

where

$$S^{(0)}(p; \{k_m\}) = \sum_{i=\text{out}} \frac{(\epsilon^+(p) \cdot k_i)^2}{(p/E_p) \cdot k_i} - \sum_{i=\text{in}} \frac{(\epsilon^+(p) \cdot k_i)^2}{(p/E_p) \cdot k_i}, \tag{2.29}$$

$$S^{(1)}(p; \{k_m\}) = \sum_{i=\text{out}} \frac{\epsilon^+(p) \cdot k_i}{p \cdot k_i} \epsilon^+_\mu(p)\, p_\nu\, J_i^{\mu\nu} - \sum_{i=\text{in}} \frac{\epsilon^+(p) \cdot k_i}{p \cdot k_i} \epsilon^+_\mu(p)\, p_\nu\, J_i^{\mu\nu} \tag{2.30}$$

In the above expression $\epsilon^+(p)$ denotes the positive helicity polarization of the soft graviton with momentum p, and $J_i^{\mu\nu}$ denotes the angular momentum operator (both orbital angular momentum and spin angular momentum) of the external finite energy particles. "out" and "in" in the summation of the above expression denote outgoing/incoming states, respectively.

As explained in the introduction, the terms inside the bracket are known as the soft factors. The leading soft factor $S^{(0)}(p; \{k_m\})$ known as the Weinberg's soft graviton factor has a pole in the soft momenta [8]. It is a function depending on the external momenta of the finite energy particles and the direction of the soft graviton. Similarly, the sub-leading soft factor also known as Cachazo-Strominger (CS) soft factor correspond to $O(E_p^0)$ in the soft momenta. Unlike the leading soft factor, the CS soft factor is a differential operator that depends upon the angular momentum of the external particle. From the expression 2.28, if one takes the appropriate soft limits, one can extract each soft factor separately. This can be written as,

$$\lim_{E_p \to 0} E_p \mathcal{A}^{\text{tree}}(p; \{k_m\}) = S^{(0)}(p; \{k_m\}) \mathcal{A}^{\text{tree}}(\{k_m\}), \tag{2.31}$$

$$\lim_{E_p \to 0} (1 + E_p \partial_{E_p}) \mathcal{A}^{\text{tree}}(p; \{k_m\}) = S^{(1)}(p; \{k_m\}) \mathcal{A}^{\text{tree}}(\{k_m\}). \tag{2.32}$$

In this thesis, we wil focus on the above factorization theorems 2.31,2.32. Howevever, it is important to note that the soft expansion 2.28 was valid only for tree-level scattering amplitudes in four dimensions. If one considers a scattering amplitude beyond tree-level, one gets a different soft expansion. This can be written as:

$$\mathcal{A}(p;\{k_m\}) = \left(\frac{1}{E_p}S^{(0)}(p;k_m) + \ln(E_p)S^{(\ln)}(p;k_m) + \cdots\right)\mathcal{A}(\{k_m\}). \tag{2.33}$$

Even though the leading order contribution is Weinberg's soft factor, unlike the tree-level soft expansion, the second term $S^{(\ln)}(p;k_m)$ in the soft factors involve a logarithmic divergence in the soft momenta. $S^{(\ln)}(p;k_m)$ was found recently by Sahoo and Sen in [66]. These terms arise due to infrared divergences arising from loop integrals in a quantum scattering amplitude in four dimensions and hence will not appear if one considers the scattering amplitude at tree-level. The Sahoo-Sen soft factor is found to be one loop exact and universal. One can find the explicit expression for the logarithmic soft factor in [66].

2.3 Equivalence between soft graviton theorems and asymptotic symmetries in gravity

In this section, we give a quick review of the deep connection between the two seemingly independent areas of study discussed in section 2.3 and 2.2. This relationship was initially explored in the seminal work by Strominger et.al in [4], where it was discovered that the supertranslation invariance of the quantum gravity $\mathcal{S}$ matrix is equivalent to Weinberg's soft graviton theorem. Later, it was shown by Campiglia and Laddha that if considers the Diff(S^2) subgroup of generalized BMS as the symmetry of the quantum gravity $\mathcal{S}$ matrix, the equivalence can be extended to subleading soft graviton theorems at the tree level. In the rest of this section we will elaborate on this.

As we have discussed in 2.1, there are two asymptotic symmetry groups that act on the radiative free data at $\mathcal{I}^+$ and $\mathcal{I}^-$ respectively. Inorder to define a gravitational scattering problem that takes incoming scattering data defined at fields at $\mathcal{I}^-$ to outgoing scattering data defined at $\mathcal{I}^+$, one must define a common asymptotic symmetry group that acts throughout null infinity. Motivated by this fact following [4], in [2,3] Campiglia and Laddha proposed that the diagonal subgroup of generalized BMS $\mathcal{G}$[5] to be the symmetry of the quantum gravity $\mathcal{S}$ matrix. The diagonal subgroup is identified using the conditions on $\Big(f_+(\hat{x}) \,,\, V^A_+(\hat{x}) \Big)$ and $\Big(f_-(\hat{x}) \,,\, V^A_-(\hat{x}) \Big)$ appearing in the null generators of $\mathcal{G}^+$ and $\mathcal{G}^-$ respectively. The identification is as follows,

$$f_+(\hat{x}) = f_-(-\hat{x}) \;\; ; \;\; V^A_+(\hat{x}) = V^A_-(-\hat{x}). \tag{2.34}$$

From here on, we drop the labels $(+,-)$ on generalized BMS vector fields and the charges and parametrize them by (f, V^A) only. Based on this proposal that generalized BMS being a symmetry of quantum gravity $\mathcal{S}$ matrix, one can write the following Ward identity,

$$\langle \text{out} | \, [Q, \mathcal{S}] \, | \text{in} \rangle = 0. \tag{2.35}$$

where $\langle \text{out} | \,, | \text{in} \rangle$ refers to outgoing/incoming scattering states and Q denote the generalized BMS charges. We will see in the later sections, that the Ward identity of supertranslation charge is equivalent to leading soft graviton theorem and Ward identity corresponding to Diff(S^2) charge is equivalent to subleading soft graviton theorem at tree-level.

[5]the definition of $\mathcal{G}$ was defined in 2.1 as the diagonal subgroup of $\mathcal{G}^+ \times \mathcal{G}^-$

2.3.1 Leading single soft graviton theorem and supertranslation symmetry

The leading single soft graviton theorem follows from the Ward identity of the supertranslation charge Q_f [4]. The supertranslation charge Q_f is given by 2.26[6] [4]

$$Q_f = \frac{1}{4}\int du\, d^2z\, f\ N_{AB}\, N^{AB} - \frac{1}{2}\int du\, d^2z\, f\, D^A D^B N_{AB}. \tag{2.36}$$

where N_{AB} is the Bondi News tensor defined by $N_{AB} \equiv \partial_u C_{AB}$. It is important to note that, the supertranslation charge Q_f is characterized by the arbitrary function $f(z,\bar{z})$, where $(z,\ \bar{z})$ are coordinates on the conformal sphere at null infinity. Notice that, the first term in 2.36 is quadratic in the Bondi-News N_{AB} while the second is linear in N_{AB} — these are conventionally referred to as the "hard part" (Q_f^{hard}) and the "soft part" (Q_f^{soft}) of the supertranslation charge respectively.

We begin by writing the Ward identity for supertranslation symmetry of generalized BMS as 2.35:

$$\langle \text{out}|\, [Q_f, \mathcal{S}]\, |\text{in}\rangle = 0 \Leftrightarrow \langle \text{out}|\, [Q_f^{\text{soft}}, \mathcal{S}]\, |\text{in}\rangle = -\langle \text{out}|\, [Q_f^{\text{hard}}, \mathcal{S}]\, |\text{in}\rangle\,. \tag{2.37}$$

In writing the above, the classical charges have been promoted to quantum operators. This quantization is evaluated using the asymptotic quantization of C_{zz} and $C_{\bar{z},\bar{z}}$ [4], which expresses them in terms of the graviton creation and annihilation operators as:

$$C_{zz}(u,z,\bar{z}) = \frac{-i}{2\pi^2(1+z\bar{z})^2}\int_0^\infty a_+(\omega,z,\bar{z})e^{-i\omega u} - a_-^\dagger(\omega,z,\bar{z})e^{i\omega u}, \tag{2.38}$$

$$C_{\bar{z}\bar{z}}(u,z,\bar{z}) = \frac{i}{2\pi^2(1+z\bar{z})^2}\int_0^\infty a_+^\dagger(\omega,z,\bar{z})e^{i\omega u} - a_-(\omega,z,\bar{z})e^{-i\omega u}. \tag{2.39}$$

where $a_+(\omega,z,\bar{z})/a_+^\dagger(\omega,z,\bar{z})$ denotes positive helicity annihilation/creation graviton oper-

[6] In this thesis we restrict to pure gravity and hence the terms involving the matter stress tensor will not appear in the supertranslation charge.

ator with energy ω and direction parametrized by $(z, \bar{z})$. Similarly $a_-(\omega, z, \bar{z})/a_-^\dagger(\omega, z, \bar{z})$ denotes negative helicity annihilation/creation graviton operator. Using 2.38 and 2.39 one can now express the hard and soft charge in terms of the graviton creation and annihilation operators. Subsequently one can evaluate the action of the hard and soft charges on the "in" and "out" states.

Using 2.38,2.39 and the boundary conditions given in 2.25 ($D_z^2 C_{\bar{z}\bar{z}}|_{\mathcal{I}^+_+} = D_{\bar{z}}^2 C_{zz}|_{\mathcal{I}^+_-}$), the soft charge can be written as:

$$\begin{aligned} Q_f^{\text{soft}} &= \lim_{E_p \to 0} \frac{E_p}{4\pi} \int d^2w \, D_w^2 f(w, \bar{w}) \Big(a_-(E_p, w, \bar{w}) \;+\; a_+^\dagger(E_p, w, \bar{w}) \Big), \\ &= \lim_{E_p \to 0} \frac{E_p}{4\pi} \int d^2w \, D_{\bar{w}}^2 f(w, \bar{w}) \Big(a_+(E_p, w, \bar{w}) + a_-^\dagger(E_p, w, \bar{w}) \Big). \end{aligned} \tag{2.40}$$

Here, E_p is the energy of the soft graviton and $(w, \bar{w})$ characterizes its direction on the conformal sphere. From 2.40 one can see that the soft charge Q_f^{soft} is responsible for the creation and annihilation of soft gravitons.

The hard charge can also be evaluated in a similar procedure, finally giving the action on "in" and "out" states as:

$$Q_f^{\text{hard}} \, |\text{in}\rangle = \sum_{i=\text{in}} E_i \, f(\hat{k}_i) \, |\text{in}\rangle \,, \tag{2.41}$$

$$\langle\text{out}| \, Q_f^{\text{hard}} = \sum_{i=\text{out}} E_i \, f(\hat{k}_i) \, \langle\text{out}| \,. \tag{2.42}$$

Here, the sum $\sum\limits_{i=\text{in}}$ and $\sum\limits_{i=\text{out}}$ is over the hard particles[7] in the "in" and "out" states respectively, with energy $E_i = |\vec{k}_i|$ and the unit spatial vector $\hat{k}_i = \vec{k}_i / E_i$ characterizing the direction of i^{th} particle.

Using 2.40 and 2.41 in 2.37, one obtains a factorization of the form:

$$\lim_{E_p \to 0} \frac{E_p}{4\pi} \int d^2w \, D_{\bar{w}}^2 f(w, \bar{w}) \Bigg(\langle\text{out}| \, a_+(E_p, w, \bar{w}) \mathcal{S} \, |\text{in}\rangle + \langle\text{out}| \, \mathcal{S} a_-^\dagger(E_p, w, \bar{w}) \, |\text{in}\rangle \Bigg),$$

[7] Particles with finite energy are known as hard particles

$$= -\Big[\sum_{\text{out}} E_i \, f(\hat{k}_i) - \sum_{\text{in}} E_i \, f(\hat{k}_i) \Big] \langle \text{out} | \, \mathcal{S} \, | \text{in} \rangle \, . \quad (2.43)$$

Now one can use the crossing symmetry to write the above expression as:

$$\lim_{E_p \to 0} \frac{E_p}{2\pi} \int d^2 w \; D^2_{\bar{w}} f(w, \bar{w}) \; \langle \text{out} | \, a_+(E_p, w, \bar{w}) \mathcal{S} \, | \text{in} \rangle$$
$$= -\Big[\sum_{\text{out}} E_i \, f(\hat{k}_i) - \sum_{\text{in}} E_i \, f(\hat{k}_i) \Big] \langle \text{out} | \, \mathcal{S} \, | \text{in} \rangle \, . \quad (2.44)$$

The structure of the terms in 2.44 encourages one to ask whether this can be related to Weinberg's soft graviton theorem [8] which is related to leading term in the soft expansion given in 2.28 . This reads,

$$\lim_{E_p \to 0} E_p \, \langle \text{out} | \, a_+(E_p, w, \bar{w}) \, \mathcal{S} \, | \text{in} \rangle = \left(\sum_{i = \text{out}} \frac{(\epsilon^+(w, \bar{w}) \cdot k_i)^2}{(p/E_p) \cdot k_i} - \sum_{i = \text{in}} \frac{(\epsilon^+(w, \bar{w}) \cdot k_i)^2}{(p/E_p) \cdot k_i} \right) \langle \text{out} | \, \mathcal{S} \, | \text{in} \rangle \, . \quad (2.45)$$

where the soft graviton has energy E_p and momentum p. Its direction is parametrized by $(w, \bar{w})$ and its polarization is given by $\epsilon^+(w, \bar{w}) = 1/\sqrt{2}(\bar{w}, 1, -i, -\bar{w})$. We adopt the notation:

$$\hat{S}^{(0)}(p; k_i) \equiv \frac{1}{E_{k_i}} \frac{(\epsilon^+(w, \bar{w}) \cdot k_i)^2}{(p/E_p) \cdot k_i} . \quad (2.46)$$

with which, the leading soft factor in the r.h.s. of 2.45 can be written as:

$$S^{(0)}(p; \{k_i\}) \equiv \sum_{i = \text{out}} \frac{(\epsilon^+(w, \bar{w}) \cdot k_i)^2}{(p/E_p) \cdot k_i} - \sum_{i = \text{in}} \frac{(\epsilon^+(w, \bar{w}) \cdot k_i)^2}{(p/E_p) \cdot k_i} , \quad (2.47)$$
$$\equiv \sum_{i = \text{out}} S^{(0)}(p; k_i) - \sum_{i = \text{in}} S^{(0)}(p; k_i) , \quad (2.48)$$
$$\equiv \sum_{i = \text{out}} E_{k_i} \, \hat{S}^{(0)}(p; k_i) - \sum_{i = \text{in}} E_{k_i} \, \hat{S}^{(0)}(p; k_i) . \quad (2.49)$$

It is important to notice that the contribution to the soft factor $S^{(0)}(p; \{k_i\})$ from the i^{th} hard particle with momentum k_i and energy E_{k_i}, namely, $S^{(0)}(p; k_i)$, depends on the energy of the hard particle. But, $\hat{S}^{(0)}(p; k_i)$ does not depend on E_{k_i} — as written in 2.47, the energy dependence has been separated out.

Now, consider a hard particle of momentum k parametrized by $(E, z, \bar{z})$ where E denotes the energy of the particle and $(z, \bar{z})$ denotes its direction. The four momentum of such a particle can be written as:

$$k^{\mu} = E\Big(1, \frac{z+\bar{z}}{1+z\bar{z}}, \frac{-i(z-\bar{z})}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}}\Big) \tag{2.50}$$

Now if one chooses

$$f(z, \bar{z}) = s(z, \bar{z}; w, \bar{w}) \equiv \frac{1+w\bar{w}}{1+z\bar{z}} \cdot \frac{\bar{w}-\bar{z}}{w-z}. \tag{2.51}$$

in 2.44, then the rhs of the soft theorem 2.45 and the Ward identity 2.44 match, since,

$$\frac{(\epsilon^{+}(w, \bar{w}) \cdot k)^2}{(p/E_p) \cdot k} = -E_k\ s(z, \bar{z}; w, \bar{w}) \tag{2.52}$$

Further, the l.h.s. of the soft theorem 2.45 and the Ward identity 2.44 match because of the identity,

$$D_{\bar{z}}^2 s(z, \bar{z}; w, \bar{w}) = 2\pi\delta^2(w-z). \tag{2.53}$$

It is also possible to go from the soft theorem 2.45 to the Ward identity 2.44 by acting $(2\pi)^{-1} \int d^2w\ f(w, \bar{w}) D_{\bar{w}}^2$ on both sides of 2.45. In this case, the r.h.s. matches because of the identity:

$$D_{\bar{w}}^2 s(z, \bar{z}; w, \bar{w}) = 2\pi\delta^2(w-z). \tag{2.54}$$

Hence, the equivalence of Weinberg's soft theorem and supertranslation Ward identity is established. It should also be noted that Weinberg's soft theorem for the negative helicity graviton is not an independent soft theorem and can be obtained through a similar derivation.

We would like to conclude this section by a convention that we will follow in the rest of this thesis. In the rest of this thesis, we will always be concerned with the action of the soft operator at the level of the scattering amplitudes. Therefore, using the notion of crossing symmetry, one can always relate the outgoing postive/negative helicity soft graviton with incoming negative/postive helicity soft graviton. Therefore, one can drop the creation operator terms in the soft charge 2.40 for convenience in calculations, i.e we can write

$$\begin{aligned} Q_f^{\text{soft}} &= \lim_{E_p \to 0} \frac{E_p}{2\pi} \int d^2w \, D_w^2 f(w,\bar{w}) \left(a_-(E_p,w,\bar{w}) \right), \\ &= \lim_{E_p \to 0} \frac{E_p}{2\pi} \int d^2w \, D_{\bar{w}}^2 f(w,\bar{w}) \left(a_+(E_p,w,\bar{w}) \right). \end{aligned} \tag{2.55}$$

2.3.2 Subleading single soft graviton theorem and Diff(S^2) symmetry

The subleading single soft graviton theorem follows from the Ward identity of the Diff(S^2) charge Q_V [2, 3]. This charge is given by 2.27[8]:

$$\begin{aligned} Q_V = \frac{1}{4} \int du \, d^2z \;\; N^{AB} \left(\mathcal{L}_V C_{AB} - \alpha \, C_{AB} + \alpha u \, N_{AB} \right) \\ + \frac{1}{2} \int du \, d^2z \;\; u \Big(N^{zz} \, D_z^3 V^z + \; N^{\bar{z}\bar{z}} \, D_{\bar{z}}^3 V^{\bar{z}} \Big). \end{aligned} \tag{2.56}$$

where $\alpha = \frac{1}{2}(D_z V^z + D_{\bar{z}} V^{\bar{z}})$ and $V^A(z,\bar{z})$ is an arbitrary vector field on the conformal sphere at null infinity and the covariant derivatives are *w.r.t.* the 2–sphere metric. As before, the first term (which is quadratic in N_{AB}) is the "hard part" Q_V^{hard} and the second is known as the "soft part" Q_V^{soft}

Proceeding in a manner similar to the case of supertranslation, the Ward identity for Diff(S^2) charge can be written as:

$$\langle \text{out} | \, [Q_V, \mathcal{S}] \, | \text{in} \rangle = 0 \Leftrightarrow \langle \text{out} | \, [Q_V^{\text{soft}}, \mathcal{S}] \, | \text{in} \rangle = - \langle \text{out} | \, [Q_V^{\text{hard}}, \mathcal{S}] \, | \text{in} \rangle \, . \tag{2.57}$$

[8]In this thesis we restrict ourselves to pure gravity and hence the terms involving the matter stress tensor will not appear in the Diff(S^2) charge.

Now, using the asymptotic quantization of the "free data", one can write the Diff(S^2) soft charge as :

$$Q_V^{\text{soft}} = \frac{1}{4\pi i} \lim_{E_p \to 0} (1 + E_p\, \partial_{E_p}) \\ \times \int d^2 w \left[V^{\bar{w}} \partial_{\bar{w}}^3 a_+(E_p, w, \bar{w}) + V^w\, \partial_w^3 a_-(E_p, w, \bar{w}) \right]. \tag{2.58}$$

Hence, $Q_V^{\text{soft}} \left|\text{in}\right\rangle = 0$ [9]. The action of the hard Diff(S^2) charge gives:

$$\left\langle \text{out} \right| Q_V^{\text{hard}} = i \sum_{\text{out}} J_{V_i}^{h_i} \left\langle \text{out} \right| , \tag{2.59}$$

$$Q_V^{\text{hard}} \left|\text{in}\right\rangle = i \sum_{\text{in}} J_{V_i}^{-h_i} \left|\text{in}\right\rangle . \tag{2.60}$$

Again, the sum $\sum\limits_{i=\text{in}}$ and $\sum\limits_{i=\text{out}}$ is over all the hard particles in the "in" and "out" states respectively, with the i^{th} particle having energy $E_i = |\vec{k}_i|$ and direction characterized by the vector $\hat{k}_i = \vec{k}_i / E_i$. $J_V^{h_i}$ is a differential operator and the detailed expression of $J_V^{h_i}$ is given as [2] .

$$J_V^{h_i} = V^z \partial_z + V^{\bar{z}} \partial_{\bar{z}} - \frac{1}{2}(D_z V^z + D_{\bar{z}} V^{\bar{z}}) \omega_i \partial_{\omega_i} + \frac{h_i}{2}(\partial_z V^z - \partial_{\bar{z}} V^{\bar{z}}). \tag{2.61}$$

where ω_i, h_i denote the energy and helicity of the "i"-th external particle. One can therefore write the Ward identity for Diff(S^2) charge as 2.57 as:

$$-\frac{1}{4\pi} \lim_{E_p \to 0} (1 + E_p\, \partial_{E_p}) \\ \times \int d^2 w \left[V^{\bar{w}}\, \partial_{\bar{w}}^3 \left\langle \text{out} \right| a_+(E_p, w, \bar{w})\, \mathcal{S} \left|\text{in}\right\rangle + V^w\, \partial_w^3 \left\langle \text{out} \right| a_-(E_p, w, \bar{w})\, \mathcal{S} \left|\text{in}\right\rangle \right]$$

[9]Note that in 2.58 there should be terms corresponding to the creation operators $(a_+^\dagger(E_p, w, \bar{w}), a_-^\dagger(E_p, w, \bar{w}))$ in the expression. We have excluded such terms here for the convenience for calculations. As we have done for the supertranslation Ward identity at the level of the scattering amplitudes using crossing symmetry one can relate the incoming positive/negative helicity soft graviton insertion to outgoing negative/positive helicity soft graviton insertion in the $\mathcal{S}$ matrix when one finally evaluates the Diff(S^2) Ward identity. Therefore, at the level of evaluating the Ward identity both conventions give the same result.

$$= \left[\sum_{\text{out}} J^{h_i}_{V_i} - \sum_{\text{in}} J^{-h_i}_{V_i} \right] \langle \text{out} | \, \mathcal{S} \, | \text{in} \rangle \, . \tag{2.62}$$

Now, the Cachazo–Strominger (CS) subleading soft theorem 2.32 reads [9]:

$$\lim_{E_p \to 0} (1 + E_p \, \partial_{E_p}) \langle \text{out} | \, a_+(E_p, w, \bar{w}) \, \mathcal{S} \, | \text{in} \rangle \tag{2.63}$$
$$= \left(\sum_{i=\text{out}} \frac{\epsilon^+(w, \bar{w}) \cdot k_i}{p \cdot k_i} \, \epsilon^+_\mu(w, \bar{w}) \, p_\nu \, J^{\mu\nu}_i - \sum_{i=\text{in}} \frac{\epsilon^+(w, \bar{w}) \cdot k_i}{p \cdot k_i} \, \epsilon^+_\mu(w, \bar{w}) \, p_\nu \, J^{\mu\nu}_i \right) \langle \text{out} | \, \mathcal{S} \, | \text{in} \rangle \, .$$

where, $J^{\mu\nu}_i$ is the angular momentum operator acting on the i^{th} hard particle. For further use, we adopt the notation:

$$S^{(1)}(p; k_i) = \frac{\epsilon^+(w, \bar{w}) \cdot k_i}{p \cdot k_i} \, \epsilon^+_\mu(w, \bar{w}) \, p_\nu \, J^{\mu\nu}_i . \tag{2.64}$$

Using this, the subleading soft factor in the r.h.s. of 2.63 can be written as:

$$S^{(1)}(p; \{k_i\}) = \left(\sum_{i=\text{out}} \frac{\epsilon^+(w, \bar{w}) \cdot k_i}{p \cdot k_i} \, \epsilon^+_\mu(w, \bar{w}) \, p_\nu \, J^{\mu\nu}_i - \sum_{i=\text{out}} \frac{\epsilon^+(w, \bar{w}) \cdot k_i}{p \cdot k_i} \, \epsilon^+_\mu(w, \bar{w}) \, p_\nu \, J^{\mu\nu}_i \right),$$
$$= \left(\sum_{i=\text{out}} S^{(1)}(p; k_i) - \sum_{i=\text{in}} S^{(1)}(p; k_i) \right). \tag{2.65}$$

Now, in the Ward identity 2.62, if one chooses the vector field V^A as:

$$V^A = K^+_{(w,\bar{w})} \equiv \frac{(\bar{z} - \bar{w})^2}{(z - w)} \, \partial_{\bar{z}} . \tag{2.66}$$

the r.h.s. of the soft theorem 2.63 and the Ward identity 2.62 match since:

$$\frac{\epsilon^+(w, \bar{w}) \cdot k_i}{p \cdot k_i} \, \epsilon^+_\mu(w, \bar{w}) \, p_\nu \, J^{\mu\nu}_i = J^i_{K^+_{(w,\bar{w})}} . \tag{2.67}$$

The l.h.s. of the soft theorem 2.63 and the Ward identity 2.62 also match due to the identity:

$$\partial^3_{\bar{z}} \frac{(\bar{z}-\bar{w})^2}{(z-w)} = 4\pi\delta^2(w-z). \tag{2.68}$$

To go from the CS soft theorem 2.63 to the Diff(S^2) Ward identity 2.62 one acts the operator $-(4\pi)^{-1}\int d^2w\ V^{\bar{w}}\partial^3_{\bar{w}}$ on both sides of 2.63. Then, using the linearity of J_V in vector field V,

$$-(4\pi)^{-1}\int d^2w\ V^{\bar{w}}\ \partial^3_{\bar{w}} J^i_{K^+_{(w,\bar{w})}} = -(4\pi)^{-1} J_W. \tag{2.69}$$

and the identity,

$$\partial^3_{\bar{w}} \frac{(\bar{z}-\bar{w})^2}{(z-w)} = -4\pi\delta^2(w-z), \tag{2.70}$$

one recovers Ward identity 2.62 with the vector field $V^{\bar{w}}\partial_{\bar{w}}$. The vector field W in above expression is given by:

$$W = \int V^{\bar{w}}\ \partial^3_{\bar{w}} K^+_{(w,\bar{w})}. \tag{2.71}$$

Here, unlike the Ward identity for the leading case 2.44, it is important to note that the Ward identity for the subleading case 2.62, contains both negative and positive helicity soft graviton amplitudes. To get a factorization that involves only one of the soft gravitons, one of the components of vector field V^A is chosen to be zero, depending upon which soft graviton helicity we want in the soft theorem.

Chapter 3

Double soft graviton theorems

Having reviewed the relationship between asymptotic symmetries and the single soft theorem, the next natural question is to ask if such a relationship holds between the generalised BMS symmetry and double soft graviton theorems. These theorems (and its generalization to the multiple soft graviton case) have been studied previously using various methods including BCFW recursions [10], CHY amplitudes [11–14] and Feynman diagram techniques [15]. In a recent work [16], the authors have studied the symmetry foundations of the double soft theorems of certain classes of theories like the dilaton, DBI, and special Galileon.

As has been analyzed in the literature, there are two kinds of double soft graviton theorems depending upon the relative energy scale of the soft gravitons. The simultaneous soft limit is the one where soft limit is taken on both the gravitons at the same rate. However, as we argue in Appendix A, from the perspective of Ward identities, it is the consecutive soft limits which arise rather naturally. Consecutive double soft graviton theorems (CDST) elucidate the factorization property of scattering amplitudes when the soft limit is taken on one of the gravitons at a faster rate than the other [10]. We now review this factorization property when such soft limits are taken and show that they give rise to three CDSTs. The first one, we refer to as the leading CDST which is the case where the leading soft limit is taken on both the soft gravitons. The remaining two theorems refer to the case where the leading soft limit is taken with respect to one of the gravitons and the subleading soft limit is taken with respect to the other.

3.1 Consecutive double soft graviton theorems (CDST)

We begin with a (n+2) particle scattering amplitude at tree-level denoted by $\mathcal{A}_{n+2}(q, p, \{k_m\})$ where p , q are the momenta of the two gravitons which will be taken to be soft and $\{k_m\}$ is the set of momenta of the n hard particles. Consider the consecutive limit where the soft limit is first taken on graviton with momentum q, keeping all the other particles momenta unchanged and then a soft limit is taken on the graviton with momentum p.

Using the single soft factorization, the scattering amplitude $\mathcal{A}_{n+2}(q, p, \{k_m\})$ can be written as:

$$\begin{aligned}\mathcal{A}_{n+2}(q, p, \{k_m\}) = \Bigg[\sum_i \frac{E_{k_i}}{E_q}\, \hat{S}^{(0)}(q; k_i) + \frac{E_p}{E_q}\, \hat{S}^{(0)}(q; p) \\ + \sum_i S^{(1)}(q; k_i) + S^{(1)}(q; p)\Bigg] \mathcal{A}_{n+1}(p, \{k_m\}) + \mathcal{O}(E_q). \qquad (3.1)\end{aligned}$$

where $\mathcal{A}_{n+1}(p, \{k_m\})$ is the $n + 1$ particle scattering amplitude. It is important to recall the notations used here, which we explained in section 2.3 (2.47, 2.64). As mentioned, $S^{(1)}(q; k_i)$ is the contribution to the subleading soft factor with soft momentum q with k_i being the i^{th} hard particle. Similarly $\hat{S}^{(0)}(q; k_i)$ denotes the contribution to the subleading soft factor with soft momentum q with k_i being the i^{th} hard particle, with energy dependences *w.r.t.* both the soft and hard particles seperated out. $\hat{S}^{(0)}(q; p)$ and $S^{(1)}(q; p)$ denote similar contributions to the soft factor where the graviton with momentum p is treated as hard *w.r.t.* the graviton with momentum q.

Now, the amplitude $\mathcal{A}_{n+1}(p,\{k_m\})$ further factorizes as:

$$\mathcal{A}_{n+1}(p,\{k_m\}) = \left[\sum_i \frac{E_{k_i}}{E_p}\,\hat{S}^{(0)}(p;k_i) + \sum_i S^{(1)}(p;k_i)\right]\mathcal{A}_n(\{k_m\}) + O(E_p). \tag{3.2}$$

Note that, according to our notation, $S^{(1)}(p;k_i)$ is the contribution to the subleading soft factor with soft momentum p and k_i is the i^{th} hard particle. Again, $\hat{S}^{(0)}(p;k_i)$ denotes the contribution to the subleading soft factor with soft momentum p and k_i the i^{th} hard particle, with energy dependences *w.r.t.* both the soft and the hard particles seperated out.

Substituting 3.2 in 3.1, we get the factorization of the $(n+2)$ particle amplitude containing two soft gravitons in terms of the amplitude of the n hard particles (up to subleading order in energy of the individual soft particles).

$$\begin{aligned}
\mathcal{A}_{n+2}(q,p,\{k_m\}) = \\
\Bigg[\frac{1}{E_pE_q}\sum_{i,j} E_{k_i}\,E_{k_j}\,\hat{S}^{(0)}(q;k_i)\,\hat{S}^{(0)}(p;k_j) + \sum_{i,j}\frac{E_{k_i}}{E_q}\,\hat{S}^{(0)}(q;k_i)\,S^{(1)}(p;k_j) \\
+ \sum_i \frac{E_{k_i}}{E_q}\,\hat{S}^{(0)}(q;p)\,\hat{S}^{(0)}(p;k_i) \; + \sum_{i,j} S^{(1)}(q;k_i)\,\frac{E_{k_j}}{E_p}\,\hat{S}^{(0)}(p;k_j) \\
+ S^{(1)}(q;p)\sum_i \frac{E_{k_i}}{E_p}\,\hat{S}^{(0)}(p;k_i)\Bigg]\mathcal{A}_n(\{k_m\}) + O(E_p) + O(E_q).
\end{aligned} \tag{3.3}$$

This expansion contains three types of terms. The first type scales as $1/(E_pE_q)$ (and hence gives rise to a pole in both the soft graviton energies), giving the leading contribution to the factorization. The second and the third type of terms scale as E_q^0/E_p and E_p^0/E_q respectively, both contributing to the subleading order of the factorization.

The leading order contribution, described above, is:

$$\left[\frac{1}{E_pE_q}\sum_{i,j} E_{k_i}\,E_{k_j}\,\hat{S}^{(0)}(q;k_i)\,\hat{S}^{(0)}(p;k_j)\right]\mathcal{A}_n(\{k_m\}). \tag{3.4}$$

This gives the *leading* CDST as:

$$\lim_{E_p \to 0} E_p \lim_{E_q \to 0} E_q \, \mathcal{A}_{n+2}(q, p, \{k_m\}) = \Big[\, S^{(0)}(q; \{k_i\}) \, S^{(0)}(p; \{k_j\}) \Big] \, \mathcal{A}_n(\{k_m\}). \tag{3.5}$$

As is evident, the leading double soft factor is just the product of the individual leading soft factors. One obtains this same theorem in the case of the simultaneous double soft limit as well [10, 12, 13, 15]. In section 4.1, we show that this soft theorem matches with the result derived from the Ward identity of two supertranslation charges 4.10.

Let us now consider the subleading soft limit. At this order of factorization we have four terms:

$$\begin{aligned}
&\Big[\sum_{i,j} \frac{E_{k_i}}{E_q} \, \hat{S}^{(0)}(q; k_i) \, S^{(1)}(p; k_j) + \sum_i \frac{E_{k_i}}{E_q} \, \hat{S}^{(0)}(q; p) \, \hat{S}^{(0)}(p; k_i) \\
&\quad + \sum_i S^{(1)}(q; k_i) \sum_j \frac{E_{k_j}}{E_p} \, \hat{S}^{(0)}(p; k_j) + S^{(1)}(q; p) \sum_i \frac{E_{k_i}}{E_p} \, \hat{S}^{(0)}(p; k_i) \Big] \mathcal{A}_n(\{k_m\}).
\end{aligned} \tag{3.6}$$

Notice that the first two terms in (3.6) scale with soft graviton energies as E_p^0/E_q and the second two terms scale as E_q^0/E_p.

From the first two terms of (3.6), one gets a subleading CDST.

$$\begin{aligned}
&\lim_{E_p \to 0} (1 + E_p \partial_{E_p}) \lim_{E_q \to 0} E_q \, \mathcal{A}_{n+2}(q, p, \{k_m\}) \\
&\qquad = \Big[S^{(0)}(q; \{k_i\}) \, S^{(1)}(p; \{k_j\}) + \mathcal{N}(q; p; \{k_i\}) \Big] \mathcal{A}_n(\{k_m\}).
\end{aligned} \tag{3.7}$$

Here, the first term is the product of single soft factors 2.47, 2.65, appearing in the leading and subleading single soft theorems respectively. The second term in the r.h.s of 3.7 contains a single sum over the set of hard particles as opposed to the first term which is the product of single soft factors and contains two sums over the set of hard particles. Such terms are referred to as “contact terms" in the literature. One can evaluate this contact term as:

$$\mathcal{N}(q;p;\{k_i\}) = \hat{S}^{(0)}(q;p)\, S^{(0)}(p;\{k_i\}) = \sum_i \frac{(\epsilon_q \cdot \tilde{p})^2}{\tilde{q}\cdot\tilde{p}} \cdot \frac{(\epsilon_p \cdot k_i)^2}{\tilde{p}\cdot k_i}. \tag{3.8}$$

where $\tilde{p} = p/E_p = (1,\hat{p})$ and similarly, $\tilde{q} = q/E_q = (1,\hat{q})$. ϵ_p and ϵ_q refer to the polarisations of soft gravitons with momentum p and q respectively. This is the well known consecutive double soft graviton theorem [10] .

A different consecutive limit.

We now take a different limit in eq.(3.6) and show how it leads to a distinct factorization theorem. From the last two terms in (3.6) one gets:

$$\begin{aligned}
&\lim_{E_p\to 0} E_p \lim_{E_q\to 0}(1 + E_q\, \partial_{E_q})\, \mathcal{A}_{n+2}(q,p,\{k_m\}) \qquad (3.9)\\
&= \Big[\sum_i S^{(1)}(q;k_i) \sum_j E_{k_j}\, \hat{S}^{(0)}(p;k_j) + \lim_{E_p\to 0} E_p\, S^{(1)}(q;p) \sum_i \frac{E_{k_i}}{E_p}\, \hat{S}^{(0)}(p;k_i)\Big] \mathcal{A}_n(\{k_m\}).
\end{aligned}$$

Now, $S^{(1)}(q;k_i)$ contains the angular momentum operator of the i^{th} hard particle, and thus acts on $E_{k_j}\hat{S}^{(0)}(p;k_j)$, as well as the n particle amplitude $\mathcal{A}_n(\{k_m\})$. However, $S^{(1)}(q;p)$ does not depend on the set of hard particles labelled by momentum $\{k_m\}$. Hence $S^{(1)}(q;p)$ acts only on the soft factor, and one can finally write the subleading CDST as:

$$\begin{aligned}
&\lim_{E_p\to 0} E_p \lim_{E_q\to 0}(1 + E_q\, \partial_{E_q})\, \mathcal{A}_{n+2}(q,p,\{k_m\})\\
&\qquad = \Big[S^{(0)}(p;\{k_i\})\, S^{(1)}(q;\{k_j\}) + \mathcal{M}_1(q;p;\{k_i\}) + \mathcal{M}_2(q;p;\{k_i\})\Big] \mathcal{A}_n(\{k_m\}). \qquad (3.10)
\end{aligned}$$

Similar to the other subleading CDST 3.7, the first term in the r.h.s. of 3.10 is product of single soft factors. However, the important difference is that the role of the soft gravitons with momentum p and q is interchanged in the first term of 3.10 and the first term of 3.7. Here, $\mathcal{M}_1(q;p;\{k_i\})$ and $\mathcal{M}_2(q;p;\{k_i\})$ are contact terms which can be expressed as follows:

$$\mathcal{M}_1(q;p;\{k_i\}) = \sum_i S^{(1)}(q;k_i)\Big(E_{k_i}\,\hat{S}^{(0)}(p;k_i)\Big) = \sum_i S^{(1)}(q;k_i)\Big(S^{(0)}(p;k_i)\Big),$$
$$= \sum_i \Big[-\frac{(\epsilon_q\cdot k_i)^2(\epsilon_p\cdot k_i)^2(p\cdot q)}{(q\cdot k_i)(p\cdot k_i)^2} + \frac{(\epsilon_q\cdot k_i)(\epsilon_q\cdot p)(\epsilon_p\cdot k_i)^2}{(p\cdot k_i)^2}$$
$$+ 2\frac{(\epsilon_q\cdot k_i)^2(\epsilon_p\cdot k_i)(\epsilon_p\cdot q)}{(p\cdot k_i)(q\cdot k_i)} - 2\frac{(\epsilon_q\cdot k_i)(\epsilon_p\cdot \epsilon_q)(\epsilon_p\cdot k_i)}{(p\cdot k_i)}\Big]. \tag{3.11}$$

and,

$$\mathcal{M}_2(q;p;\{k_i\}) = \sum_i \lim_{E_p\to 0} E_p\, S^{(1)}(q;p)\left(\frac{E_{k_i}}{E_p}\,\hat{S}^{(0)}(p;k_i)\right)$$
$$= \sum_i \Big[\frac{(\epsilon_q\cdot\tilde{p})(\epsilon_q\cdot k_i)(\epsilon_p\cdot k_i)^2}{(\tilde{p}\cdot k_i)^2} - \frac{(\epsilon_q\cdot\tilde{p})^2(\epsilon_p\cdot k_i)^2(q\cdot k_i)}{(\tilde{p}\cdot k_i)^2(\tilde{p}.q)}$$
$$-2\,\frac{(\epsilon_q\cdot\tilde{p})(\epsilon_q\cdot k_i)(\epsilon_p\cdot q)(\epsilon_p\cdot k_i)}{(\tilde{p}.q)(\tilde{p}.k_i)} + 2\,\frac{(\epsilon_q\cdot\tilde{p})(\epsilon_q\cdot\epsilon_p)(\epsilon_p\cdot k_i)(q.k_i)}{(\tilde{p}.q)(\tilde{p}.k_i)}\Big]. \tag{3.12}$$

Again, $\tilde{p} = p/E_p = (1,\hat{p})$ and ϵ_p and ϵ_q refer to the polarisation of soft gravitons with momentum p and q respectively.

In [10], the authors have considered similar consecutive limits for the double soft graviton and gluon amplitudes. There, they have imposed a gauge condition $\epsilon_p\cdot q = 0$ and $\epsilon_q\cdot p = 0$. However, our analysis proceeded without imposing any particular gauge condition. With the specific gauge condition used in [10], a few of the terms like $\hat{S}^{(0)}(q;p)$ and $S^{(1)}(q;p)$ drop out from the CDST result that we have obtained at the subleading level and we recover their result. This serves as a consistency check for our calculation.

One can also verify the consistency of both the consecutive limits with the general result which was given in [15]. That is, both the CDST 3.7 and 3.10 are special cases of the double soft limit in [15]. The CDST 3.7 can be recovered by imposing the condition $E_p \gg E_q$ on the result of [15] and taking the leading limit in E_q and subleading limit in E_p. Similarly, the CDST 3.10 can be obtained by imposing the the same $E_p \gg E_q$ condition, but taking the leading limit in E_p and subleading limit in E_q.

Chapter 4

Equivalence between double soft graviton theorem and generalized BMS symmetry

Having reviewed the relationship between Ward identities associated to the asymptotic symmetries and single soft graviton theorems, we now ask if there are Ward identities in the theory which are equivalent to the double soft graviton theorems at the leading and sub–leading order. In particular, we look for Ward identities that will lead us to the consecutive double soft theorems (CDST). Let us consider the family of Ward identities whose general structure is:

$$\langle \text{out}|\, [Q_1, [Q_2, \mathcal{S}]]\, |\text{in}\rangle = 0. \tag{4.1}$$

where both Q_1 and Q_2 are either both supertranslation charges or Q_1 is a supertranslation charge and Q_2 is a Diff(S^2) charge.[1]

Following [17], we present a derivation of this proposed Ward identity in Appendix A.

[1]The alternate case where Q_1 is Diff(S^2) charge and Q_2 is supertranslation charge is riddled with conceptual subtleties which remain unresolved — we return to this in Appendix B.

In the following sections, we show that such a proposal leads to the consecutive double soft theorems discussed in section 3.1. Depending on the choice of charges one gets the leading as well as the subleading consecutive double soft theorems.

4.1 Leading CDST and asymptotic symmetries

4.1.1 Ward identity from asymptotic symmetries

Following the discussion in the previous section, we explore the factorization arising from two supertranslation charges, Q_f and Q_g characterized by arbitrary functions $f(z,\bar{z})$ and $g(z,\bar{z})$, on the conformal sphere. We start with:

$$\langle \text{out}|\, [Q_f, [Q_g, \mathcal{S}]]\, |\text{in}\rangle = 0. \tag{4.2}$$

Proceeding in a manner similar to the single soft case in section 2.3, we can write Q_f and Q_g as sum of hard and soft charges as:

$$Q_f = Q_f^{\text{hard}} + Q_f^{\text{soft}} \quad , \quad Q_g = Q_g^{\text{hard}} + Q_g^{\text{soft}}. \tag{4.3}$$

Thus, the Ward identity 4.2 becomes:

$$\begin{aligned}\langle \text{out}|\, [Q_f^{\text{hard}}, [Q_g^{\text{hard}}, \mathcal{S}]]\, |\text{in}\rangle + \langle \text{out}|\, [Q_f^{\text{hard}}, [Q_g^{\text{soft}}, \mathcal{S}]]\, |\text{in}\rangle \\ + \langle \text{out}|\, [Q_f^{\text{soft}}, [Q_g^{\text{hard}}, \mathcal{S}]]\, |\text{in}\rangle + \langle \text{out}|\, [Q_f^{\text{soft}}, [Q_g^{\text{soft}}, \mathcal{S}]]\, |\text{in}\rangle = 0.\end{aligned} \tag{4.4}$$

Now using the Ward identity of supertranslation, namely $[Q_g^{\text{soft}}, S] = -[Q_g^{\text{hard}}, S]$, the first and the second terms cancel each other. One may be tempted to cancel the third and fourth terms, on similar lines. However, we contend that this isn't quite correct as the action of Q_f^{soft} maps ordinary the Fock vaccuum to a supertranslated vaccuum state parametrised by

f [55,90,91]. As a result, we are really looking at the following Ward identity.

$$\langle \text{out}, f|\,[Q_g, S]\,|\text{in}\rangle = \;0. \tag{4.5}$$

where $|\text{out}, f\rangle$ is a finite energy state defined with respect to the super–translated vacuum. The "in" state is defined w.r.t standard Fock Vacuum because of our prescription $Q_f^{\text{soft}}|\text{in}\rangle \;=\; 0$. We can re–write the above identity as:

$$\langle \text{out}|\,[Q_f^{\text{soft}}, [Q_g^{\text{soft}}, \mathcal{S}]]\,|\text{in}\rangle = -\,\langle \text{out}|\,[Q_f^{\text{soft}}, [Q_g^{\text{hard}}, \mathcal{S}]]\,|\text{in}\rangle\,. \tag{4.6}$$

Now using the Jacobi identity among Q_f^{soft}, Q_g^{hard} and $\mathcal{S}$, the commutation relation $[Q_f^{\text{soft}}, Q_g^{\text{hard}}] = 0$, and the single soft Ward identity, we can finally write

$$\langle \text{out}|\,[Q_f^{\text{soft}}, [Q_g^{\text{soft}}, \mathcal{S}]]\,|\text{in}\rangle = \langle \text{out}|\,[Q_g^{\text{hard}}, [Q_f^{\text{hard}}, \mathcal{S}]]\,|\text{in}\rangle\,. \tag{4.7}$$

. Using the (known) action of charges on external states in 4.7 we finally arrive at the Ward identity:

$$\begin{aligned}
&\lim_{E_p\to 0}\frac{E_p}{2\pi}\lim_{E_q\to 0}\frac{E_q}{2\pi}\int d^2w_1\, d^2w_2\, D^2_{\bar{w}_1} f(w_1,\bar{w}_1)\, D^2_{\bar{w}_2} g(w_2,\bar{w}_2) \\
&\qquad\qquad\qquad\qquad \times \langle \text{out}|\, a_+(E_p, w_1, \bar{w}_1)\, a_+(E_q, w_2, \bar{w}_2)\, \mathcal{S}\,|\text{in}\rangle \\
&= \left[\sum_{\text{out}} f(\hat{k}_i)E_i - \sum_{\text{in}} f(\hat{k}_i)E_i\right]\left[\sum_{\text{out}} g(\hat{k}_j)E_j - \sum_{\text{in}} g(\hat{k}_j)E_j\right]\langle \text{out}|\,\mathcal{S}\,|\text{in}\rangle\,.
\end{aligned} \tag{4.8}$$

The factorization above is just the product of two factors of the type obtained from the Ward identity for supertranslation 2.44. It is natural therefore to expect that the soft theorem we obtain from 4.8 will also be the product of two leading single soft factors. In the next section, we show that this is indeed true.

4.1.2 From Ward identity to soft theorem

From the factorization obtained in 4.8 from the Ward identity with two supertranslation charges, we try to understand what soft theorem follows from it. Motivated from the single soft case, we make the choices for arbitrary function f and g on the conformal sphere as:

$$f(w_1, \bar{w_1}) = s(w_1, \bar{w_1}; w_p, \bar{w_p}) \quad , \quad g(w_2, \bar{w_2}) = s(w_2, \bar{w_2}; w_q, \bar{w_q}). \tag{4.9}$$

where the definition of the functions $s(w_1, \bar{w_1}; w_p, \bar{w_p})$ and $s(w_2, \bar{w_2}; w_q, \bar{w_q})$ can be read from 2.51. Substituting these choices in 4.8, we finally get:

$$\begin{aligned}\lim_{E_p \to 0} E_p \lim_{E_q \to 0} E_q \; \langle \text{out}| \, a_+(E_p, w_p, \bar{w_p}) \, a_+(E_q, w_q, \bar{w_q}) \, \mathcal{S} \, |\text{in}\rangle \\ = \left[S^{(0)}(q; \{k_i\}) \, S^{(0)}(p; \{k_j\}) \right] \langle \text{out}| \, \mathcal{S} \, |\text{in}\rangle \, .\end{aligned} \tag{4.10}$$

This is the same as the leading double soft theorem 3.5 for the case of two positive helicity soft gravitons with momenta p and q, localized at $(w_p, \bar{w_p})$ and $(w_q, \bar{w_q})$ respectively, on the conformal sphere. Although we have chosen both the soft graviton helicities to be positive in the above, one can do a similar analysis for both the helicities being negative or one positive and one negative, and a similar result holds. This provides the equivalence of the leading CDST and the Ward identity 4.2.

We have thus shown that the leading order double soft graviton theorem is equivalent to the supertranslation Ward identity when this identity is evaluated in a Hilbert space built out of a super–translated vacuum that containing a single soft graviton.

4.2 Subleading CDST and asymptotic aymmetries

4.2.1 Ward identity from asymptotic symmetries

As motivated in section 4, and derived in Appendix A, we now analyze the Ward identity corresponding to one supertranslation charge (characterized by arbitrary function f) and one Diff(S^2) charge (characterized by vector field V^A):

$$\langle \text{out}|\, [Q_f, [Q_V, \mathcal{S}]]\, |\text{in}\rangle = 0. \tag{4.11}$$

We begin by writing the charges as sum of hard and soft charges:

$$\begin{aligned}\langle \text{out}|\, [Q_f^{\text{hard}}, [Q_V^{\text{hard}}, \mathcal{S}]]\, |\text{in}\rangle + \langle \text{out}|\, [Q_f^{\text{hard}}, [Q_V^{\text{soft}}, \mathcal{S}]]\, |\text{in}\rangle \\ + \langle \text{out}|[Q_f^{\text{soft}}, [Q_V^{\text{hard}}, \mathcal{S}]]\, |\text{in}\rangle + \langle \text{out}|\, [Q_f^{\text{soft}}, [Q_V^{\text{soft}}, \mathcal{S}]]\, |\text{in}\rangle = 0.\end{aligned} \tag{4.12}$$

Now, using the Ward identity for Diff(S^2), namely $[Q_V^{\text{soft}}, \mathcal{S}] = -[Q_V^{\text{hard}}, \mathcal{S}]$, the first and the second term of 4.12 cancel each other. Again, one may be tempted to cancel the third and the fourth term of 4.12 instead, using the same Diff(S^2) Ward identity. However if we do not cancel them, we are led to

$$\langle \text{out}|\, Q_f^{\text{soft}}\, [Q_V, S\,]\, |\text{in}\rangle = 0, \tag{4.13}$$

$$\langle \text{out}, f|\, [Q_V, S\,]\, |\text{in}\rangle = 0. \tag{4.14}$$

Whence not cancelling the third and forth terms in (4.12) is tantamount to considering Diff(S^2) Ward identity in scattering states which are excitations around supertranslated vacuua. As we show below, it is precisely the Ward identity
$\langle \text{out}, f|\, [Q_V, \mathcal{S}]\, |\text{in}\rangle = 0$ that leads to a specific double soft graviton theorem.

Hence the above identity 4.12 reduces to,

$$\begin{aligned}
&\langle \text{out}|\,[Q_f^{\text{soft}},[Q_V^{\text{soft}},\mathcal{S}]]\,|\text{in}\rangle \\
&\qquad\qquad = -\,\langle \text{out}|\,[Q_f^{\text{soft}},[Q_V^{\text{hard}},\mathcal{S}]]\,|\text{in}\rangle\,, \\
&\qquad\qquad = -\,\langle \text{out}|\,Q_f^{\text{soft}}\;Q_V^{\text{hard}}\;\mathcal{S}\,|\text{in}\rangle + \langle \text{out}|\,Q_f^{\text{soft}}\;\mathcal{S}\;Q_V^{\text{hard}}\,|\text{in}\rangle\,.
\end{aligned} \tag{4.15}$$

Using the known action of the soft and hard charges, first term in the r.h.s. of 4.15 can be written as:

$$\begin{aligned}
&\langle \text{out}|\,Q_f^{\text{soft}}\;Q_V^{\text{hard}}\;\mathcal{S}\,|\text{in}\rangle \\
&\qquad = \frac{1}{2\pi}\lim_{E_p\to 0}\int d^2w_1\;D_{\bar{w}_1}^2 f\;E_p\,\langle \text{out}|\,a_+(E_p\hat{x})\;Q_V^{\text{hard}}\;\mathcal{S}\,|\text{in}\rangle\,, \\
&\qquad = \frac{i}{2\pi}\lim_{E_p\to 0}\int d^2w_1\;D_{\bar{w}_1}^2 f\;E_p\Big(\sum_{\text{out}} J_V^{h_i} + J_V^{+}\Big)\langle \text{out}|\,a_+(E_p\hat{x})\;\mathcal{S}\,|\text{in}\rangle\,.
\end{aligned} \tag{4.16}$$

where $\hat{x}$ denotes the direction of the soft graviton parametrized by $(w_1, \bar{w}_1)$ on the conformal sphere. J_V^{+} represents the action of Q_V^{hard} on the soft graviton with energy E_p.

Similarly, the second term in 4.15 can be evaluated to:

$$\langle \text{out}|\,Q_f^{\text{soft}}\mathcal{S}Q_V^{\text{hard}}\,|\text{in}\rangle = \frac{i}{2\pi}\lim_{E_p\to 0}\int d^2w_1\;D_{\bar{w}_1}^2 f\,\Big(\sum_{\text{in}} J_V^{-h_i}\Big)E_p\,\langle \text{out}|\,a_+(E_p\hat{x})\;\mathcal{S}\,|\text{in}\rangle\,. \tag{4.17}$$

Hence, the Ward identity 4.15 simplifies to:

$$\begin{aligned}
&\langle \text{out}|\,Q_f^{\text{soft}}Q_V^{\text{soft}}\mathcal{S}\,|\text{in}\rangle = \\
&\quad -\frac{i}{2\pi}\lim_{E_p\to 0}\int d^2w_1 D_{\bar{w}_1}^2 f\Big(\sum_{\text{out}} J_V^{h_i} - \sum_{\text{in}} J_V^{-h_i}\Big)\Big[E_p\,\langle \text{out}|\,a_+(E_p\hat{x})\mathcal{S}\,|\text{in}\rangle\,\Big] \\
&\qquad\qquad -\frac{i}{2\pi}\lim_{E_p\to 0}\int d^2w_1\;D_{\bar{w}_1}^2 f\;E_p\Big(J_V^{+}\Big)\Big[\,\langle \text{out}|\,a_+(E_p\hat{x})\;\mathcal{S}\,|\text{in}\rangle\,\Big].
\end{aligned} \tag{4.18}$$

Note that, the l.h.s. of 4.18 can be written as:[2]

$$\lim_{E_p\to 0} \frac{1}{2\pi} E_p \lim_{E_q\to 0} \frac{1}{4\pi i}(1 + E_q\partial_{E_q})\times \int d^2w_1\, d^2w_2\, D^2_{\bar{w}_1} f\, \partial^3_{\bar{w}_2} V^{\bar{w}_2} \langle\text{out}|\, a_+(E_p\hat{x})\, a_+(E_q\hat{y})\, \mathcal{S}\,|\text{in}\rangle\,. \tag{4.19}$$

It is important to note that the soft limits taken in the above equation do not follow any particular order in the energies of the soft gravitons. However as we show in the next section, the right hand side of the Ward identity is equivalent to the right hand side of one of the CDSTs .

From Ward identity to soft theorem

Having derived the Ward identity 4.18, we now ask whether it can be interpreted as a soft theorem. Motivated by the single soft graviton case, we make the following choices for function f and vector field V:

$$\begin{aligned} f(w_1, \bar{w}_1) &= s(w_1, \bar{w}_1; w_p, \bar{w}_p), \\ V^{\bar{w}_2} &= K^{+}_{(w_q, \bar{w}_q)}. \end{aligned} \tag{4.20}$$

where $s(w_1, \bar{w}_1; w_p, \bar{w}_p)$ and $K^{+}_{(w_q,\bar{w}_q)}$ follow the definitions in section 2.3. Using this, 4.19 becomes:

$$\lim_{E_p\to 0} E_p \lim_{E_q\to 0}(1 + E_q\, \partial_{E_q})\, \langle\text{out}|\, a_+(E_p\hat{x})\, a_+(E_q\hat{y})\, \mathcal{S}\,|\text{in}\rangle\,. \tag{4.21}$$

where the unit vectors $\hat{x}$ and $\hat{y}$ denote the coordinates $(w_p, \bar{w}_p)$ and $(w_q, \bar{w}_q)$ on the conformal sphere.

[2] More precise definition of l.h.s. is given in Appendix B.

Further, for the r.h.s. of 4.18, we have:

$$\lim_{E_p\to 0}\sum_i S^{(1)}(q;k_i)\Big[E_p\ \langle\text{out}|a_+(E_p\hat{x})\,\mathcal{S}\,|\text{in}\rangle\Big]$$
$$+\lim_{E_p\to 0} E_p\, S^{(1)}(q;p)\Big[\langle\text{out}|\,a_+(E_p\hat{x})\,\mathcal{S}\,|\text{in}\rangle\Big]. \tag{4.22}$$

In the above expression, notice that in both the subleading factors $S^{(1)}(q;k_i)$ and $S^{(1)}(q;p)$, the soft graviton with momentum q is localized at $\hat{y}$ on the conformal sphere. However, the first one contains an angular momentum operator acting on the i^{th} hard particle and the latter contains an angular momentum operator acting on the soft graviton with momentum p.

Now, using the leading single soft theorem, the first term in 4.22 can be written as:

$$\sum_i S^{(1)}(q;k_i)\Big[\sum_j E_{k_j}\hat{S}^{(0)}(p;k_j)\,\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle\Big]. \tag{4.23}$$

For the second term in 4.22, we use the expansion of the $(n+1)$ particle amplitude 3.2 and we get a factorization of the form:

$$\langle\text{out}|\,a_+(E_p\hat{x})\,\mathcal{S}\,|\text{in}\rangle = \Big[\sum_i \frac{E_{k_i}}{E_p}\,\hat{S}^{(0)}(p;k_i) + \sum_i S^{(1)}(p;k_i)\Big]\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle + \mathcal{O}(E_p). \tag{4.24}$$

The second term of 4.24 is at a higher order in soft graviton energy, and so does not contribute to 4.22. Thus, 4.22 finally becomes:

$$\sum_i S^{(1)}(q;k_i)\Big[\sum_j E_{k_j}\hat{S}^{(0)}(p;k_j)\,\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle\Big]$$
$$+\lim_{E_p\to 0}\ E_p\, S^{(1)}(q;p)\Big[\sum_j \frac{E_{k_j}}{E_p}\,\hat{S}^{(0)}(p;k_j)\Big]\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle\,. \tag{4.25}$$

Lastly, since $S^{(1)}(q;k_i)$ is a linear differential operator and $S^{(1)}(q;p)$ acts only on the soft coordinates, we can further simplify 4.25 as:

$$\Big[\sum_{i,j} E_{k_i}\,\hat{S}^{(0)}(p;k_i)\,S^{(1)}(q;k_j) + \sum_i S^{(1)}(q;k_i)\Big(E_{k_i}\,\hat{S}^{(0)}(p;k_i)\Big) \\ + \lim_{E_p\to 0}\,E_p\,S^{(1)}(q;p)\Big(\sum_j \frac{E_{k_j}}{E_p}\,\hat{S}^{(0)}(p;k_j)\Big)\Big]\langle\mathrm{out}|\,\mathcal{S}\,|\mathrm{in}\rangle\,. \quad (4.26)$$

Finally, putting this all together, we get a subleading double soft theorem:

$$\lim_{E_p\to 0} E_p \lim_{E_q\to 0}(1+E_q\,\partial_{E_q})\,\mathcal{A}_{n+2}(q,p,\{k_m\}) \\ = \Big[S^{(0)}(p;\{k_i\})\,S^{(1)}(q;\{k_j\}) + \mathcal{M}_1(q;p;\{k_i\}) + \mathcal{M}_2(q;p;\{k_i\})\Big]\mathcal{A}_n(\{k_m\}) \qquad (4.27)$$

where, $\mathcal{M}_1(q;p;\{k_i\})$ and $\mathcal{M}_2(q;p;\{k_i\})$ are the same contact terms obtained in subleading CDST 3.10, whose expressions can be read off from 3.11, 3.12 respectively. This is the same subleading consecutive double soft theorm 3.10, that we studied in the section 3.1. Note however that, in 4.19 there is no particular ordering in the limits of the soft graviton energy obtained from the successive action of the soft charges. Hence, the l.h.s. of the double soft theorem 4.27 contains independent limits as opposed to 3.10, where the limits have definite ordering. Although we believe this point needs to be better understood, what we have shown here is that the Ward identity of Diff(S^2) charges in a supertranslated vacuum leads to a particular CDST. It is also important to emphasise that there is a definite time ordering in $[Q_f,[Q_V,\mathcal{S}]] = 0$. This is clear from the derivation of the Ward identity $\langle\mathrm{out}|\,[Q_f,[Q_V,\mathcal{S}]]\,|\mathrm{in}\rangle = 0$, which is presented in Appendix A.

4.2.2 Relating the standard CDST to a Ward identity

As we saw above, the Ward identity $[Q_f,[Q_V,\mathcal{S}]] = 0$, gave rise to a double soft theorem whose r.h.s. matched with the consecutive soft theorem, where we considered the subleading limit of the graviton which was taken soft first. This is in contrast to the more standard consecutive soft limit where we consider the leading soft limit of the graviton which is taken soft first and subleading soft limit of the graviton which is taken soft second. We will argue how this CDST could potentially arise out of the Ward identity:

$$\langle\text{out}|\,[Q_V,[Q_f,\mathcal{S}]]\,|\text{in}\rangle = 0. \tag{4.28}$$

Expressing the charges in 4.28 as the sum of hard and soft charges, we get:

$$\begin{aligned}&\langle\text{out}|\,[Q_V^{\text{hard}},[Q_f^{\text{hard}},\mathcal{S}]]\,|\text{in}\rangle + \langle\text{out}|\,[Q_V^{\text{soft}},[Q_f^{\text{hard}},\mathcal{S}]]\,|\text{in}\rangle\\ &\qquad + \langle\text{out}|\,[Q_V^{\text{hard}},[Q_f^{\text{soft}},\mathcal{S}]]\,|\text{in}\rangle + \langle\text{out}|\,[Q_V^{\text{soft}},[Q_f^{\text{soft}},\mathcal{S}]]\,|\text{in}\rangle = 0.\end{aligned} \tag{4.29}$$

Using the Ward identity for supertranslation, namely $[Q_f^{\text{soft}},\mathcal{S}] = -[Q_f^{\text{hard}},\mathcal{S}]$, the first and the third terms cancel each other. Once again, this leads us to the following supertranslation Ward identity evaluated in states defined with respect to “super–rotated vacuum".

$$\langle\text{out}|\,Q_V^{\text{soft}}\,[Q_f,\mathcal{S}]\,|\text{in}\rangle = 0, \tag{4.30}$$

$$\langle\text{out}, V|\,[Q_f,\mathcal{S}]\,|\text{in}\rangle = 0. \tag{4.31}$$

where by $|\text{out}, V\rangle$ we mean a finite energy scattering state defined with respect to a vacuum which contains a subleading soft graviton mode.[3] However, as we explain in appendix B, unlike the action of Q_f^{soft}, the action of Q_V^{soft} is not well understood thus far. Consequently, the proposed Ward identity remains rather formal at this point. We will still proceed further and show that this proposed Ward identity, if well defined is equivalent to the standard CDST. We can rewrite the Ward identity as

$$\begin{aligned}&\langle\text{out}|\,Q_V^{\text{soft}}\,Q_f^{\text{soft}}\,\mathcal{S}\,|\text{in}\rangle\\ &\qquad = -\,\langle\text{out}|\,[Q_V^{\text{soft}},[Q_f^{\text{hard}},\mathcal{S}]]\,|\text{in}\rangle\,,\\ &\qquad = \langle\text{out}|\,Q_V^{\text{soft}}\mathcal{S}Q_f^{\text{hard}} - Q_f^{\text{hard}}Q_V^{\text{soft}}\mathcal{S}\,|\text{in}\rangle + \langle\text{out}|\,[Q_f^{\text{hard}},Q_V^{\text{soft}}]\mathcal{S}\,|\text{in}\rangle\,.\end{aligned} \tag{4.32}$$

[3]It was shown in [3] how Q_V^{soft} maps the vacuum to a different vacuum.

We evaluate the two terms in the r.h.s. of 4.32 one by one. The first term can be written as:

$$\begin{aligned}
&\langle \text{out}|\, Q_V^{\text{soft}}\, \mathcal{S}\, Q_f^{\text{hard}} - Q_f^{\text{hard}}\, Q_V^{\text{soft}}\, \mathcal{S}\, |\text{in}\rangle\,, \\
&\qquad = -\langle \text{out}|\, [Q_f^{\text{hard}}, Q_V^{\text{soft}}\, \mathcal{S}]\, |\text{in}\rangle = -\langle \text{out}|\, [Q_f^{\text{hard}}, [Q_V^{\text{soft}}, \mathcal{S}]]\, |\text{in}\rangle\,, \\
&\qquad = \langle \text{out}|\, [Q_f^{\text{hard}}, [Q_V^{\text{hard}}, \mathcal{S}]]\, |\text{in}\rangle\,.
\end{aligned} \tag{4.33}$$

Then, using the action of Q_f^{hard} and Q_V^{hard} on the external states, we can write the r.h.s. of 4.33 as:

$$\begin{aligned}
&\langle \text{out}|\, [Q_f^{\text{hard}}, [Q_V^{\text{hard}}, \mathcal{S}]]\, |\text{in}\rangle = \\
&\qquad i\left[\sum_{\text{out}} f(\hat{k}_i)E_i - \sum_{\text{in}} f(\hat{k}_i)E_i\right]\left[\sum_{\text{out}} J_{V_i}^{h_i} - \sum_{\text{in}} J_{V_i}^{-h_i}\right] \langle \text{out}|\, \mathcal{S}\, |\text{in}\rangle\,.
\end{aligned} \tag{4.34}$$

To evaluate the second term in 4.32, note that for a single particle state $|k\rangle$,

$$\begin{aligned}
&\langle k|\, [Q_f^{\text{hard}}, Q_V^{\text{soft}}] \\
&\qquad = -\frac{1}{4\pi i} \lim_{E_p \to 0} (1 + E_p\, \partial_{E_p}) \int d^2 w_2\, \partial_{\bar{w}_2}^3 V^{\bar{w}_2}\, E_p\, f(w_2, \bar{w}_2)\, \langle k|\, a_+(E_p, w_2, \bar{w}_2) \\
&\qquad = -\frac{1}{4\pi i} \lim_{E_p \to 0} \int d^2 w_2\, \partial_{\bar{w}_2}^3 V^{\bar{w}_2}\, E_p\, f(w_2, \bar{w}_2)\, \langle k|\, a_+(E_p, w_2, \bar{w}_2)
\end{aligned} \tag{4.35}$$

Where, in going from the first line to the second, we have used the fact that $a_+(E_p, w_2, \bar{w}_2) \sim \frac{1}{E_p}$. Therefore,

$$-\frac{1}{4\pi i} \lim_{E_p \to 0} E_p\, \partial_{E_p} \int d^2 w_2\, \partial_{\bar{w}_2}^3 V^{\bar{w}_2}\, E_p\, f(w_2, \bar{w}_2)\, \langle k|\, a_+(E_p, w_2, \bar{w}_2) = 0. \tag{4.36}$$

Using the above expression 4.35, we can evaluate the second term of (4.32) as:

$$\begin{aligned}
\langle \text{out}|\, [Q_f^{\text{hard}}, Q_V^{\text{soft}}]\, \mathcal{S}\, |\text{in}\rangle = -\frac{1}{4\pi i} \lim_{E_p \to 0} \int d^2 w_2\, \partial_{\bar{w}_2}^3 V^{\bar{w}_2}\, E_p \\
\times f(w_2, \bar{w}_2)\, \langle \text{out}|\, a_+(E_p, w_2, \bar{w}_2)\, \mathcal{S}\, |\text{in}\rangle\,.
\end{aligned} \tag{4.37}$$

Lastly, using the single soft graviton theorem (with energy E_p), 4.37 simplifies to:

$$\langle \text{out}|\,[Q_f^{\text{hard}}, Q_V^{\text{soft}}]\,\mathcal{S}\,|\text{in}\rangle$$
$$= -\frac{1}{4\pi i}\sum_i \int d^2w_2\,\partial^3_{\bar{w}_2}V^{\bar{w}_2}\,f(w_2,\bar{w}_2)\,E_{k_i}\,\hat{S}^{(0)}(p;k_i)\,\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle\,. \tag{4.38}$$

Finally, substituting (4.34) and (4.38) in (4.32), we arrive at the Ward identity:

$$\begin{aligned}&\langle\text{out}|Q_V^{\text{soft}}\;Q_f^{\text{soft}}\;\mathcal{S}\,|\text{in}\rangle\\ &= i\Bigg[\sum_{\text{out}} f(\hat{k}_i)E_i - \sum_{\text{in}} f(\hat{k}_i)E_i\Bigg]\Bigg[\sum_{\text{out}} J^{h_i}_{V_i} - \sum_{\text{in}} J^{-h_i}_{V_i}\Bigg]\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle\\ &\quad -\frac{1}{4\pi i}\sum_{\text{hard}}\int d^2w_2\,\partial^3_{\bar{w}_2}V^{\bar{w}_2}\,f(w_2,\bar{w}_2)\,E_{k_i}\,S^{(0)}(w_2,\bar{w}_2;k_i)\,\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle\,.\end{aligned} \tag{4.39}$$

where the l.h.s. can be expressed as:

$$\begin{aligned}&\frac{1}{4\pi i}\lim_{E_p\to 0}\,(1+E_p\,\partial_{E_p})\,\frac{1}{2\pi}\lim_{E_q\to 0}\,E_q\int d^2w_1\,d^2w_2\,D^2_{\bar{w}_1}f(w_1,\bar{w}_1)\,\partial^3_{\bar{w}_2}V^{\bar{w}_2}\\ &\qquad\qquad\times\langle\text{out}|a_+(E_q,w_1,\bar{w}_1)\,a_+(E_p,w_2,\bar{w}_2)\,\mathcal{S}\,|\text{in}\rangle\,.\end{aligned} \tag{4.40}$$

In order to proceed from the Ward identity 4.39 to a soft theorem we make the following choices for f and V:

$$f(w_1,\bar{w}_1) = s(w_1,\bar{w}_1;w_q,\bar{w}_q)\;,\quad V^{\bar{w}_2} = K^+_{(w_p,\bar{w}_p)}. \tag{4.41}$$

Substituting these in 4.39, we formally get the subleading CDST for positive helicity gravitons as:

$$\begin{aligned}&\lim_{E_p\to 0}(1+E_p\,\partial_{E_p})\lim_{E_q\to 0}E_q\,\langle\text{out}|\,a_+(E_q\hat{y})\,a_+(E_p\hat{x})\,\mathcal{S}\,|\text{in}\rangle\\ &= \Big[S^{(0)}(q;\{k_i\})\,S^{(1)}(p;\{k_j\}) + \hat{S}^{(0)}(q;p)\,S^{(0)}(p;\{k_i\})\Big]\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle\,.\end{aligned} \tag{4.42}$$

Again, $\hat{x}$ and $\hat{y}$ denote the points $(w_p, \bar{w_p})$, $(w_q, \bar{w_q})$ on the conformal sphere. This is the same consecutive double soft theorem 3.7 discussed in section 3.1.

However, as discussed in Appendix B, there are some important subtleties in the definition of soft operators, especially the soft Diff(S^2) charge Q_V^{soft}. Due to this, in the evaluation of the Ward identity $\langle \text{out} | [Q_V, [Q_f, \mathcal{S}]] | \text{in} \rangle = 0$, the steps which involve the operation of charge Q_V^{soft} first on the "out" state before the other charge are not mathematically rigorous. However, we present this calculation here, in the hope that this might give some hint to the structure of a more mathematically sound proof of this soft theorem as well as a more rigorous understanding of the operation of the soft Diff(S^2) charge.

Chapter 5

Generalized BMS algebra at time-like infinity

In this chapter, we will be interested in understanding the aspects of the asymptotic symmetries in gravity at time-like infinity. In most of the earlier literature, the study of asymptotic symmetries were focused, at null infinity and spatial infinity. However, to describe asymptotic symmetries throughout space-time one should understand the asymptotic symmetries at time-like infinity (denoted by i^0) too. Time-like infinity is described as the conformal boundary of the space-time in the time-like direction. All the massive geodesics start at past time-like infinity and end at future time-like infinity. We will elaborate on this further in this section.

One of the main motivations for our study on generalized BMS symmetry at time-like infinity is from the soft graviton theorems. In soft graviton theorems the external particles (other than the soft particle) can be massive or massless. To prove the equivalence between asymptotic symmetries and soft theorems when the external states contain the massive particles, one should include the phase space for massive particles as well. Based on the earlier work on the action of BMS group on massive scalar particle phase space [18], this question was addressed in [19]. To describe time-like infinity in [19], the authors considered constant time Euclidean-AdS hypersurface foliations of the Minkowski space. In the limit when the time coordinate in their coordinate system tends to infinity, one reaches near time-like infinity. The boundary of such hypersurfaces resides on the null infinity, and therefore one can express the vector fields preserving large time fall off behavior of Minkowski space, using generalized BMS vector fields, by use of bulk-boundary Green's functions of standard AdS/CFT dictionary. In this way, one has a natural action of generalized BMS vector fields near time-like infinity, which are intrinsically defined from the perspective of null infinity. Using this, the authors derived the generalized BMS charges at time-like infinity, which have a natural action on the phase space of the massive particles and the equivalence between soft graviton theorems and generalized BMS symmetries were established.

Although the vector fields that generate the generalized BMS algebra at time-like infinity were defined in the literature, the algebra has not been investigated. In this chapter, we are interested in this aspect. The rest of the chapter is organized as follows. Section 5.1.1 deals with the algebra of generalized BMS vector fields at null infinity. In section 5.1.2, we discuss the asymptotic flatness at time-like infinity and associated generalized BMS vector fields at time-like infinity. We also discuss the constraints on the vector fields and what we mean by supertranslation and the Diff(S^2) vector fields from the perspective of time-like infinity. The need for modified Lie bracket for realizing the vector field algebra is also summarized. In section 5.2, we show the algebra between generalized BMS vector fields at time-like infinity and prove that there is a closure of the vector fields.

5.1 Generalized BMS vector fields

5.1.1 Generalized BMS vector fields at null infinity

In the previous chapters, the generalized BMS vector fields were derived by considering the vector fields that preserve the asymptotic flatness and the Bondi gauge conditions. In this chapter, we adopt an alternate way to derive the generalized BMS vector fields, as this method is more suited for describing generalized BMS vector fields at time-like infinity. This was adopted by Campiglia and Laddha in [62, 63] following the work by Avery and Schwab in [97].

We start by reviewing the generalized BMS vector fields and their algebra at null infinity . We discuss the case for future null infinity ($\mathcal{I}^+$) following [62, 63], but similar analysis can be done for past null infinity.

Recall that the coordinates that are well adapted for describing future null infinity are (u, r, x^A), where $u = t - r$ is the retarded time, r is the radial coordinate, and x^A denote the direction along the unit sphere S^2. One can reach future null infinity by taking $u = \text{const}$ and $r \to \infty$ limit. The flat Minkowski metric in these coordinates is given by the line element

$$ds^2 = -du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B \tag{5.1}$$

where γ_{AB} is the unit S^2 metric. The generalized BMS vector fields can be described as follows. These are vector fields (denoted by ξ^a) that survive at null infinity and generate residual gauge transformations in the de-Donder gauge (w.r.t to Minkowski metric). Such vector fields obey the wave equation. Additionally, they satisfy the asymptotic divergence free condition as given in [2]. These two conditions can be written as

$$\Box\xi^a = 0, \tag{5.2}$$

$$\lim_{r\to\infty} \nabla_a \xi^a = 0. \tag{5.3}$$

where $\Box, \nabla$ refers to the flat space Laplacian and flat space covariant derivative, respectively. To understand the structure of the vector fields that satisfy these conditions, one starts with the following ansatz:

$$\xi^a\partial_a = \left(\xi^{(0)u}(u, x^B) + O(r^\epsilon)\right)\partial_u + \left(r\xi^{(1)r}(u, x^B) + O(r^0)\right)\partial_r \\ + \left(\xi^{(0)A}(u, x^B) + r^{-1}\xi^{(-1)A}(u, x^B) + O(r^{-1-\epsilon})\right)\partial_A. \tag{5.4}$$

One can find the vector field components by substituting the above ansatz in 5.2 and solving them perturbatively in r. The details of the computation can be found in [62]. Finally, one gets the generalized BMS vector field as:

$$\xi = (f + u\alpha)\partial_u - r\alpha\partial_r + V^A\partial_A + \cdots \tag{5.5}$$

Here, $f = f(\hat{q})$ is a free scalar function and $V^A = V^A(\hat{q})$ is a free vector field, that depends on the sphere coordinates $\hat{q}$. Also, $\alpha = \frac{1}{2}D_A V^A$, where D_A is the covariant derivative compatible with q_{AB}. The vector fields characterized by the function $f(\hat{q})$ (i.e by setting $V^A = 0$ in 5.5) are called the supertranslation vector fields. Similarly, the vector fields characterized by V^A (by setting $f = 0$ in 5.5) are called Diff(S^2) vector fields. The subleading components in 1/r expansion are also characterized by $f(\hat{q})$ and $V^A(\hat{q})$. The supertranslation and Diff(S^2) vector fields at future null infinity can therefore be written as:

$$\xi_f = f\partial_u, \tag{5.6}$$

$$\xi_V = u\alpha\partial_u - r\alpha\partial_r + V^A\partial_A. \tag{5.7}$$

One can study the algebra of the vector fields by computing the commutator of two variations of the metric w.r.t to the vector fields.

$$\begin{aligned}[\delta_{\xi_1}, \delta_{\xi_2}]g_{\mu\nu} &= \delta_{\xi_1}\delta_{\xi_2}g_{\mu\nu} - \delta_{\xi_2}\delta_{\xi_1}g_{\mu\nu}, \\ &= \delta_{\xi_1}\mathcal{L}_{\xi_2}g_{\mu\nu} - \delta_{\xi_2}\mathcal{L}_{\xi_1}g_{\mu\nu}, \\ &= \mathcal{L}_{\xi_1}\mathcal{L}_{\xi_2}g_{\mu\nu} - \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_1}g_{\mu\nu}, \\ &= \delta_{[\xi_1,\xi_2]}g_{\mu\nu}. \end{aligned} \tag{5.8}$$

where $[\xi_1, \xi_2]$ denotes the Lie bracket of the vector fields which is defined as,

$$[\xi_1, \xi_2]^a = \xi_1^b \partial_b \xi_2^a - \xi_2^b \partial_b \xi_1^a. \tag{5.9}$$

Therefore, the generalized vector field algebra at null infinity is found to be,

$$[\xi_{f_1}, \xi_{f_2}] = 0 \;\; ; \;\; [\xi_V, \xi_f] = \xi_{\widetilde{f}} \;\; ; \;\; [\xi_{V_1}, \xi_{V_2}] = \xi_{\widetilde{V}}. \tag{5.10}$$

Here, ξ_{f_1} and ξ_{f_2} are two supertranslation vector fields characterized by two functions on the sphere namely, f_1 and f_2. Similarly, ξ_{V_1} and ξ_{V_2} are two Diff(S^2) vector fields characterized by two vector fields on the sphere namely V_1 and V_2. Here, $\xi_{\widetilde{f}}$ is another supertranslation vector field characterized by $\widetilde{f} = \mathcal{L}_V f - \alpha f$. Also, $\xi_{\widetilde{V}}$ is another Diff(S^2) vector field characterized by $\widetilde{V}^A = V_1^B \partial_B V_2^A - V_2^B \partial_B V_1^A$. Clearly, supertranslation forms an abelian ideal of the generalized BMS group.

5.1.2 Generalized BMS vector fields at time-like infinity

Having discussed the algebra of vector fields at null infinity, our main goal in this chapter will be to investigate the algebra at time-like infinity. Following [19, 65], we summarize the key ideas relevant for our analysis. The set of coordinates, that we shall use are the hyperbolic coordinates $(\tau, \rho, \hat{x})$, which are defined in terms of Cartesian coordinates $(t, \vec{x})$

in the region $t \geq r \equiv \sqrt{\vec{x} \cdot \vec{x}}$ as:

$$\tau := \sqrt{t^2 - r^2} \;\; ; \;\; \rho := \frac{r}{\sqrt{t^2 - r^2}} \;\; ; \;\; \hat{x} = \vec{x}/r. \tag{5.11}$$

We consider a space of metrics g_{ab} which has an asymptotic expansion in τ near time-like infinity of the form:

$$ds^2 = (-1 + O(1/\tau))\, d\tau^2 + \tau^2 h_{\alpha\beta}(\tau, \rho, \hat{x}) dx^\alpha dx^\beta. \tag{5.12}$$

where $h_{\alpha\beta}(\tau, \rho, \hat{x})$ has the following asymptotic expansion (in τ) around time-like infinity

$$h_{\alpha\beta}(\tau, \rho, \hat{x}) = h^{(0)}_{\alpha\beta}(\rho, \hat{x}) + \frac{h^{(1)}_{\alpha\beta}(\rho, \hat{x})}{\tau} + \frac{h^{(2)}_{\alpha\beta}(\rho, \hat{x})}{\tau^2} + \cdots . \tag{5.13}$$

where $h^{(0)}_{\alpha\beta}(\rho, \hat{x})$ belongs to the class of metrics diffeomorphic to the hyperboloid part of the Minkowski metric, which we will describe shortly. The asymptotic flatness for metric of this form 5.12 at time-like infinity has been addressed in [25, 26]. The Minkowski metric (which we denote by $\mathring{g}_{ab}$) belongs to the class of metric 5.12 that has only the leading components (in τ) and the hyperboloid components take a particular form. The line element for $\mathring{g}_{ab}$ is written as:

$$ds^2 = -d\tau^2 + \tau^2 \mathring{h}_{\alpha\beta}(\rho, \hat{x}) dx^\alpha dx^\beta, \tag{5.14}$$

where

$$\mathring{h}_{\alpha\beta}(\rho, \hat{x}) dx^\alpha dx^\beta \equiv \frac{d\rho^2}{1 + \rho^2} + \rho^2 \gamma_{AB} dx^A dx^B. \tag{5.15}$$

Here, γ_{AB} is the unit metric on 2-sphere. The greek indices $\alpha, \beta, \cdots$ runs over the coordinates on the hyperboloid and the capital Latin indices $A, B, C, \cdots$ runs over the coordinates of the 2-sphere. Here after, we denote the small Latin indices $a, b, c, \cdots$ to denote the four spacetime indices. The Riemann tensor for the above mentioned hyperboloid metric ($\mathring{h}_{\alpha\beta}$) can be written as

$$\mathring{R}_{\alpha\beta\gamma\delta} = \mathring{h}_{\alpha\delta}\mathring{h}_{\beta\gamma} - \mathring{h}_{\alpha\gamma}\mathring{h}_{\beta\delta} \quad ; \quad \mathring{R}^{\alpha}_{\beta\rho\gamma} = \delta^{\alpha}_{\gamma}\mathring{h}_{\beta\rho} - \delta^{\alpha}_{\rho}\mathring{h}_{\beta\gamma}. \tag{5.16}$$

One can reach time-like infinity i^+ in hyperboloid coordinates by taking $\tau \to \infty$ limit (or in the Cartesian coordinates $t \to \infty$, keeping $t \geq r$). Similarly, one can reach the part of null infinity where $u > 0$ in the hyperboloid coordinates by taking the limit $\tau \to \infty$, $\rho \to \infty$, keeping $\frac{\tau}{2\rho} = \text{const}$.

To analyze the asymptotic symmetries at time-like infinity i^+, we suitably adapt the de-Donder gauge in the hyperbolic coordinates. In this gauge, the residual (large) diffeomorphisms are precisely generated by supertranslation and Diff(S^2) vector fields that smoothly matches with the corresponding BMS vector fields at null infinity.

We consider the following gauge conditions to the metric ansatz 5.12.

$$\mathring{\nabla}_b \mathcal{G}^{ab} = 0, \tag{5.17}$$

$$\mathrm{Tr}(h^{(1)}_{\alpha\beta}(\rho, \hat{x})) = 0. \tag{5.18}$$

where $\mathcal{G}^{ab} \equiv \sqrt{g} g^{ab}$ and $\mathring{\nabla}_b$ refers to the covariant derivative w.r.t to the reference Minkowski metric ($\mathring{g}_{ab}$) in 5.14. One can see that the gauge condition 5.17 reduces to the de-Donder gauge condition when one uses the linearized metric around the Minkowski metric, i.e $g_{ab} \to \mathring{g}_{ab} + h_{ab}$[1]. It is also important to note that, the trace free condition 5.18 of $h^{(1)}_{\alpha\beta}(\rho, \hat{x})$ is taken w.r.t to $h^{(0)}_{\alpha\beta}(\rho, \hat{x})$.

The generalized BMS vector fields at time-like infinity are those that generate the group of diffeomorphisms that preserve the form of the metric 5.12 and the gauge conditions 5.17-5.18. To find the structure of such vector fields, we start by taking a general ansatz

[1]Not to be confused h_{ab} here with the hyperboloid metric defined earlier 5.15. Here h_{ab} refers to a small perturbation around the Minkowski metric

for vector fields which have an asymptotic expansion (in τ) of the form:

$$\xi(\tau, \rho, \hat{x}) = \Big(\xi^{(0)\tau}(\rho, \hat{x}) + \frac{\xi^{(1)\tau}(\rho, \hat{x})}{\tau} + \cdots\Big)\partial_\tau + \Big(\xi^{(0)\alpha}(\rho, \hat{x}) + \frac{\xi^{(1)\alpha}(\rho, \hat{x})}{\tau} + \cdots\Big)\partial_\alpha. \tag{5.19}$$

From the form of the metric ansatz given in 5.12, we note that the metric component $g_{\tau\alpha}$ is absent. This imposes the following condition on the vector field:

$$\mathcal{L}_\xi g_{\tau\alpha} = 0 \quad \Longleftrightarrow \quad \xi^{(1)\alpha}(\rho, \hat{x}) = D^\alpha \xi^{(0)\tau}(\rho, \hat{x}). \tag{5.20}$$

Here D^α refers to the covariant derivative w.r.t $h^{(0)}_{\alpha\beta}(\rho, \hat{x})$. Similarly the trace free condition 5.18 leads to the following constraint.

$$h^{(0)\alpha\beta}\mathcal{L}_\xi g_{\alpha\beta} = 0 \ \text{ at } O(\tau^0) \quad \Longleftrightarrow \Big(\Delta - 3\Big)\xi^{(0)\tau}(\rho, \hat{x}) = 0. \tag{5.21}$$

Here Δ refers to the Laplacian w.r.t $h^{(0)}_{\alpha\beta}(\rho, \hat{x})$. The remaining gauge condition 5.17 can also be written as

$$g^{ab}\partial_b\Big(\ln\Big(\sqrt{\frac{h}{\mathring{h}}}\Big)\Big) + \mathring{\nabla}_b g^{ab} = 0. \tag{5.22}$$

The above expression puts the following contraints on the vector fields (details are given in the Appendix-D):

$$2\, D^{(\alpha}\xi^{(0)\beta)}\partial_\beta\Big(\ln\Big(\sqrt{\frac{h^{(0)}}{\mathring{h}}}\Big)\Big) \ + \ h^{(0)\alpha\beta}\partial_\beta D_\gamma \xi^{(0)\gamma} \ + \ 2\mathring{D}_\beta D^{(\alpha}\xi^{(0)\beta)} = 0, \tag{5.23}$$

$$D^\alpha \xi^{(0)\beta}\mathring{h}_{\alpha\beta} = 0. \tag{5.24}$$

In the above expression $\mathring{D}_\beta$ refers to the covariant derivative w.r.t reference hyperboloid metric $\mathring{h}_{\alpha\beta}$. As one can see from the constraints 5.21, 5.23 and 5.24, the vector field components (to the leading order in τ) depend on the hyperboloid metric $h^{(0)}_{\alpha\beta}$ as well as the reference hyperboloid metric $\mathring{h}_{\alpha\beta}$. The dependence on $\mathring{h}_{\alpha\beta}$ arises due to the gauge condition 5.17 that we have chosen in which divergence is taken w.r.t to the reference metric $\mathring{g}_{ab}$.

In [19], Campiglia and Laddha derived the generalized BMS vector fields at time-like infinity as residual gauge transformations (that survive at time-like infinity) of de-Donder gauge around the fixed Minkowski background $\mathring{g}_{ab}$. The conditions that we obtained for the vector fields are more general in the sense that these are the constraints for the vector fields that preserve the form of the metric ansatz[2] together with the gauge conditions. Inorder to make connection with [19], we consider the above constraints 5.21, 5.23 and 5.24 evaluated at $h^{(0)}_{\alpha\beta} = \mathring{h}_{\alpha\beta}$. Therefore, substituting $h^{(0)}_{\alpha\beta} = \mathring{h}_{\alpha\beta}$ in 5.21, 5.23 and 5.24 we get,

$$\left(\mathring{\Delta} - 3\right)\xi^{(0)\tau}(\rho, \hat{x}) = 0, \tag{5.25}$$

$$\left(\mathring{\Delta} - 2\right)\xi^{(0)\alpha}(\rho, \hat{x}) = 0, \tag{5.26}$$

$$\mathring{D}_{\alpha}\xi^{(0)\alpha}(\rho, \hat{x}) = 0. \tag{5.27}$$

where $\mathring{\Delta}$ refers to the Laplacian w.r.t $\mathring{h}_{\alpha\beta}$. These are the same conditions that the authors arrive in [19] for the vector fields at time-like infinity. The following boundary conditions are also imposed to make a connection with the generalized BMS vector fields at null-infinity.

$$\lim_{\rho\to\infty} \rho^{-1}\xi^{(0)\tau}(\rho, \hat{x}) = f(\hat{x}), \tag{5.28}$$

$$\lim_{\rho\to\infty} \xi^{(0)A}(\rho, \hat{x}) = V^{A}(\hat{x}). \tag{5.29}$$

From the above equations, the leading component of these vector fields can be written in terms of the functions characterizing supertranslation and Diff(S^2) vector field at null

[2]The fixed Minkowski metric is one of the metric that satisfies the ansatz.

infinity

$$\xi^{(0)\tau}(\rho,\hat{x}) = \int_{S^2} d^2\hat{q}\; G_{ST}(\rho,\hat{x};\hat{q}) f(\hat{q}) \equiv f_{\mathcal{H}}(\rho,\hat{x}), \tag{5.30}$$

$$\xi^{(0)\alpha}(\rho,\hat{x}) = \int_{S^2} d^2\hat{q}\; G^{\alpha}_{A}(\rho,\hat{x};\hat{q}) V^{A}(\hat{q}) \equiv V^{\alpha}_{\mathcal{H}}(\rho,\hat{x}). \tag{5.31}$$

The Green's functions in turn follows the following constraints:

$$(\mathring{\Delta} - 3)G_{ST} = 0 \quad ; \quad \lim_{\rho\to\infty} \rho^{-1} G_{ST}(\rho,\hat{x};\hat{q}) = \delta^{(2)}(\hat{x},\hat{q}), \tag{5.32}$$

$$(\mathring{\Delta} - 2)G^{\alpha}_{A} = 0 \quad ; \quad \mathring{D}_{\alpha} G^{\alpha}_{A} = 0 \;\; ; \;\; \lim_{\rho\to\infty} G^{A}_{B}(\rho,\hat{x};\hat{q}) = \delta^{A}_{B}\, \delta^{(2)}(\hat{x},\hat{q}). \tag{5.33}$$

For detailed expressions of the Green's functions and further discussions, one can refer to [65].

In this work, we are primarily interested in the algebra of the generalized BMS vector fields w.r.t reference Minkowski metric ($\mathring{g}_{ab}$). Therefore, the supertranslation and Diff(S^2) vector fields to leading order at time-like infinity are given by

$$\xi_{ST} = f_{\mathcal{H}}(\rho,\hat{x})\partial_{\tau}, \tag{5.34}$$

$$\xi_{SR} = V^{\alpha}_{\mathcal{H}}(\rho,\hat{x})\partial_{\alpha}. \tag{5.35}$$

One can verify that variation w.r.t. the supertranslation vector field does not alter the leading order (in τ) structure of 5.12 (and hence 5.14) but the variation under Diff(S^2) vector field does. This can be seen from evaluating the Lie derivative of the metric w.r.t supertranslation/Diff(S^2) vector field.

$$\mathcal{L}_{\xi_{ST}} \mathring{g}_{\tau\tau} = 0 \quad ; \quad \mathcal{L}_{\xi_{ST}} \mathring{g}_{\alpha\beta} = O(\tau), \tag{5.36}$$

$$\mathcal{L}_{\xi_{SR}} \mathring{g}_{\tau\tau} = 0 \quad ; \quad \mathcal{L}_{\xi_{SR}} \mathring{g}_{\alpha\beta} = O(\tau^2). \tag{5.37}$$

One can clearly see that the Diff(S^2) vector field changes the hyperboloid components of the metric at order τ^2. The relevance of the above mentioned point will become clear in further sections where we verify the algebra of the vector fields.

Our main interest in this chapter is to understand whether the supertranslation and Diff(S^2) vector fields defined above form a closed algebra at time-like infinity. A naive attempt to study these algebra will be to compute the ordinary Lie bracket (as we have done for the null infinity case) of the vector fields and check whether the resulting vector field satisfies the constraint 5.25 (in case for supertranslation), 5.26 and 5.27 (in case for Diff(S^2)). However, as is well known in the literature [1, 23], the correct definition of Lie bracket in the case of asymptotic symmetries is more intricate. This can be explained as follows.

The vector field algebra is studied by considering the commutator of two variations of the vector fields on the metric. An important point to be noted here is the fact that the vector fields themselves are metric dependent[3]. This can be seen from the defining equations for the vector field 5.25, 5.26 and 5.27, which tells us that the vector fields depend upon the hyperboloid metric $\mathring{h}_{\alpha\beta}$ through covariant derivative and Laplacian. Therefore, performing the second variation will affect both the first variation as well as the metric. This can be seen as

$$\begin{aligned}
[\delta_{\xi_1(g)}, \delta_{\xi_2(g)}]g_{\mu\nu} &= \delta_{\xi_1(g)}\delta_{\xi_2(g)}g_{\mu\nu} - \delta_{\xi_2(g)}\delta_{\xi_1(g)}g_{\mu\nu}, \\
&= \delta_{\xi_1(g)}\mathcal{L}_{\xi_2(g)}g_{\mu\nu} - \delta_{\xi_2(g)}\mathcal{L}_{\xi_1(g)}g_{\mu\nu}, \\
&= \mathcal{L}_{\xi_1(g)}\mathcal{L}_{\xi_2(g)}g_{\mu\nu} - \mathcal{L}_{\delta^g_{\xi_1}\xi_2(g)}g_{\mu\nu} - \mathcal{L}_{\xi_2(g)}\mathcal{L}_{\xi_1(g)}g_{\mu\nu} + \mathcal{L}_{\delta^g_{\xi_2}\xi_1(g)}g_{\mu\nu}, \\
&= \delta_{[\xi_1(g),\xi_2(g)]}g_{\mu\nu} \;-\; (\delta_{\delta^g_{\xi_1}\xi_2(g)} - \delta_{\delta^g_{\xi_2}\xi_1(g)})g_{\mu\nu}, \\
&= \delta_{\left([\xi_1(g),\xi_2(g)]-\delta^g_{\xi_1}\xi_2(g)+\delta^g_{\xi_2}\xi_1(g)\right)}g_{\mu\nu}. \qquad (5.38)
\end{aligned}$$

As one can see, this is different from 5.8. The first term in the above expression is the

[3]This was not the case at null infinity, where the generalized BMS vector fields were metric independent.

ordinary Lie bracket, which is the same as that we encountered in the null infinity case. The extra term $\delta^g_{\xi_1}\xi_2(g)$ captures the variation on the vector field $\xi_2(g)$ due to the action of the vector field $\xi_1(g)$ on the metric. Hence, to realize the algebra of the vector fields at time-like infinity, one should take into account such terms. One defines the modified Lie bracket for realizing the BMS vector field algebra as

$$[\xi_1, \xi_2]^a_M \equiv [\xi_1, \xi_2]^a - \delta^g_{\xi_1}\xi^a_2 + \delta^g_{\xi_2}\xi^a_1. \tag{5.39}$$

where $\delta^g_{\xi_1}\xi^a_2$ denotes the change in ξ^a_2 due to the variation in the metric induced by ξ_1. The exact computation of these terms will be shown in the next section.

We end this section by emphasizing the difference between the two sets of constraints we have derived for the vector fields. The first set of constraints (equations 5.21, 5.23 and 5.24) are defining equations for vector fields that preserve the gauge conditions and metric ansatz 5.12. The second set of constraints (equations 5.25, 5.26 and 5.27) are the conditions on the vector fields when, one chooses a particular metric from the metric ansatz, i.e, the reference Minkowski metric 5.14.

5.2 Generalised BMS vector field algebra at time-like infinity

In this section, we show the closure of the generalized BMS vector fields at time-like infinity using the modified Lie-bracket. We first consider the algebra between two super-translations and then, in the next sub-section, we look at the algebra between a supertranslation and Diff(S^2) vector field. Finally, we would be considering the algebra between two Diff(S^2) vector fields. In each case, we find a similar result like one gets of the algebra for generalized BMS vector fields at null infinity.

5.2.1 Algebra between two supertranslations

We start with the case of two supertranslations. Consider two supertranslation vector fields:

$$\xi_{ST1} = f_{\mathcal{H}}(\rho, \hat{x})\partial_\tau, \tag{5.40}$$

$$\xi_{ST2} = g_{\mathcal{H}}(\rho, \hat{x})\partial_\tau. \tag{5.41}$$

where, $f_{\mathcal{H}}$ and $g_{\mathcal{H}}$ is defined as follows:

$$f_{\mathcal{H}}(\rho, \hat{x}) = \int d^2\hat{q}_1\, G_{ST1}(\rho, \hat{x}; \hat{q}_1) f(\hat{q}_1), \tag{5.42}$$

$$g_{\mathcal{H}}(\rho, \hat{x}) = \int d^2\hat{q}_2\, G_{ST2}(\rho, \hat{x}; \hat{q}_2) g(\hat{q}_2). \tag{5.43}$$

Here, G_{ST1} and G_{ST2} is the same Green's function satisfying the constraints 5.32. To compute the algebra of two supertranslation vectors, we evaluate the modified Lie bracket as defined in 5.39. We expect an algebra similar to the case of null infinity, where the supertranslation vector fields commute.

The modified Lie bracket is written as:

$$[\xi_{ST1}, \xi_{ST2}]^a_M = [\xi_{ST1}, \xi_{ST2}]^a - \delta^g_{\xi_{ST1}}\xi^a_{ST2} + \delta^g_{\xi_{ST2}}\xi^a_{ST1}. \tag{5.44}$$

As we have explained in the previous section, supertranslation vector fields do not change the Minkowski metric at the leading order in τ. This can be seen from 5.36. Hence, the terms $\delta^g_{\xi_{ST1}}\xi^a_{ST2}$ and $\delta^g_{\xi_{ST2}}\xi^a_{ST1}$ do not contribute at time-like infinity. Consequently, the above expression of modified Lie bracket reduces to the ordinary Lie bracket, namely:

$$[\xi_{ST1}, \xi_{ST2}]^a_M = [\xi_{ST1}, \xi_{ST2}]^a. \tag{5.45}$$

Now, using the expressions of the vector fields 5.40 and 5.41, it is then easy to see that ordinary Lie bracket also vanishes. Hence we finally get

$$[\xi_{ST1}, \xi_{ST2}]^a_M = 0. \tag{5.46}$$

This matches with the case of null infinity. We see that supertranslations form an Abelian ideal.

5.2.2 Algebra between a supertranslation and a Diff(S^2) vector field

We now consider the modified Lie bracket between a supertranslation and a Diff(S^2) vector field. i.e:

$$\xi_{ST} = f_{\mathcal{H}}(\rho, \hat{x})\partial_\tau, \tag{5.47}$$

$$\xi_{SR} = V^\alpha_{\mathcal{H}}(\rho, \hat{x})\partial_\alpha. \tag{5.48}$$

where, $f_{\mathcal{H}}$ and $V^\alpha_{\mathcal{H}}$ are already defined in 5.30 and 5.31, and they satisfy:

$$\mathring{\Delta} f_{\mathcal{H}}(\rho, \hat{x}) = 3 f_{\mathcal{H}}(\rho, \hat{x}) \quad ; \quad \mathring{D}_\alpha V^\alpha_{\mathcal{H}}(\rho, \hat{x}) = 0 \quad ; \quad \mathring{\Delta} V^\alpha_{\mathcal{H}}(\rho, \hat{x}) = 2 V^\alpha_{\mathcal{H}}(\rho, \hat{x}). \tag{5.49}$$

From the equations above it is clear that $f_{\mathcal{H}}$ and $V^\alpha_{\mathcal{H}}$ depend upon the metric $\mathring{h}_{\alpha\beta}$ (through covariant derivative $\mathring{D}_\alpha$ and Laplacian $\mathring{\Delta}$).

Using 5.39 the modified Lie bracket of supertranslation and Diff(S^2) vector field can be written as:

$$[\xi_{ST}, \xi_{SR}]^a_M = [\xi_{ST}, \xi_{SR}]^a - \delta^g_{\xi_{ST}} \xi^a_{SR} + \delta^g_{\xi_{SR}} \xi^a_{ST}. \tag{5.50}$$

As explained in the beginning of this section the Diff(S^2) vector field depends upon $\mathring{h}_{\alpha\beta}$, and $\delta^g_{\xi_{ST}} \xi^a_{SR}$ represents the variation in ξ^a_{SR} due to the change in the metric induced by the supertranslation vector field ξ_{ST}. But we already saw in the previous section that the supertranslation does not alter the Minkowski metric to the leading order 5.36 and hence, does not alter $\mathring{h}_{\alpha\beta}$. Therefore, the term $\delta^{g}_{\xi_{ST}}\xi^{a}_{SR}$ in the above expression vanishes and the modified Lie bracket becomes

$$[\xi_{ST}, \xi_{SR}]^{a}_{M} = [\xi_{ST}, \xi_{SR}]^{a} + \delta^{g}_{\xi_{SR}}\xi^{a}_{ST}. \tag{5.51}$$

From the definitions of the vector fields given in 5.47, 5.48, it is clear that, only the τ component contributes to the above expression of modified Lie bracket. For the null infinity case, the algebra of one supertranslation and one Diff(S^2) vector field gives another supertranslation. Hence, it is natural to expect that a similar algebra holds at time-like infinity. Namely, the modified Lie bracket 5.51 gives us another supertranslation. In order to verify this, we check whether the conditions on a supertranslation vector field hold for the modified Lie bracket, *i.e.*

$$(\mathring{\Delta} - 3)[\xi_{ST}, \xi_{SR}]^{\tau}_{M} \stackrel{?}{=} 0. \tag{5.52}$$

Or, equivalently,

$$(\mathring{\Delta} - 3)[\xi_{ST}, \xi_{SR}]^{\tau} + (\mathring{\Delta} - 3)\delta^{g}_{\xi_{SR}}\xi^{\tau}_{ST} \stackrel{?}{=} 0. \tag{5.53}$$

In the rest of this section, we show that this is indeed true. We start with the contribution from the ordinary Lie bracket term.

$$(\mathring{\Delta} - 3)[\xi_{ST}, \xi_{SR}]^{\tau} = -(\mathring{\Delta} - 3)\Big[V^{\alpha}_{\mathcal{H}}\mathring{D}_{\alpha}f_{\mathcal{H}}\Big]. \tag{5.54}$$

Using the properties of $V^{\alpha}_{\mathcal{H}}$ and $f_{\mathcal{H}}$ given in 5.49, the r.h.s of the above expression finally becomes (Details of the calculation are given in Appendix-E.1.1):

$$(\mathring{\Delta} - 3)[\xi_{ST}, \xi_{SR}]^{\tau} = -2\mathring{D}^{\beta}V^{\alpha}_{\mathcal{H}}\mathring{D}_{\beta}\mathring{D}_{\alpha}f_{\mathcal{H}}. \tag{5.55}$$

We now proceed to evaluate the second term in 5.53. As we have mentioned in the previous section, the Diff(S^2) vector field changes the Minkowski metric at the leading order. It can be easily seen that, under the Lie derivative action of the Diff(S^2) vector field, the hyperboloid components of the reference Minkowski metric is shifted, i.e

$$\mathcal{L}_{\xi_{SR}}\mathring{g}_{\alpha\beta} = \tau^2(\mathring{D}_\alpha\xi_{SR\beta} + \mathring{D}_\beta\xi_{SR\alpha}). \tag{5.56}$$

where $\mathring{D}$ refers to the covariant derivative w.r.t. to reference hyperboloid metric $\mathring{h}_{\alpha\beta}$. Thereby, the gauge condition on the supertranslation vector fields shift to

$$(\Delta - 3)\xi^\tau_{ST} = 0, \tag{5.57}$$

where, Δ refers to the Laplacian w.r.t. to shifted hyperboloid $h^{(0)}_{\alpha\beta} = \mathring{h}_{\alpha\beta} + \mathcal{L}_{\xi_{SR}}\mathring{h}_{\alpha\beta}$. This indicates that the change in the vector field ξ_{ST} due to the change in the metric induced by ξ_{SR} is reflected in the variation of the Laplacian induced by ξ_{SR}. Therefore, the second term in 5.53 can be evaluated as:

$$\begin{aligned}(\mathring{\Delta} - 3)\delta^g_{\xi_{SR}}\xi^\tau_{ST} &= \delta^g_{\xi_{SR}}\Big((\mathring{\Delta} - 3)\xi^\tau_{ST}\Big) - \delta^g_{\xi_{SR}}\Big(\mathring{\Delta} - 3\Big)\xi^\tau_{ST}, \\ &= -\delta^g_{\xi_{SR}}\Big(\mathring{\Delta} - 3\Big)\xi^\tau_{ST}.\end{aligned} \tag{5.58}$$

In going from first line to the second in the above expression we have used the fact $(\mathring{\Delta} - 3)\xi^\tau_{ST} = 0$. One can evaluate r.h.s of 5.58 to (Details of this calculation are given in Appendix-E.1.2):

$$(\mathring{\Delta} - 3)\delta^g_{\xi_{SR}}\xi^\tau_{ST} = 2\mathring{D}^\beta V^\alpha_{\mathcal{H}}\mathring{D}_\beta\mathring{D}_\alpha f_{\mathcal{H}}. \tag{5.59}$$

Therefore, summing 5.55 and 5.59 we finally get:

$$(\mathring{\Delta} - 3)[\xi_{ST}, \xi_{SR}]^\tau + (\mathring{\Delta} - 3)\delta^g_{\xi_{SR}}\xi^\tau_{ST} = 0. \tag{5.60}$$

This shows that the modified Lie bracket of a supertranslation and a Diff(S^2) vector field is indeed another supertranslation.

5.2.3 Algebra between two Diff(S^2) vector fields

We now proceed to compute the algebra of two Diff(S^2) vector fields at i^+. The Diff(S^2) vector fields at i^+ are

$$\xi_{SR1} = V^{\alpha}_{\mathcal{H}}(\rho, \hat{x})\partial_{\alpha}, \tag{5.61}$$

$$\xi_{SR2} = W^{\alpha}_{\mathcal{H}}(\rho, \hat{x})\partial_{\alpha}, \tag{5.62}$$

where, $V^{\alpha}_{\mathcal{H}}$ and $W^{\alpha}_{\mathcal{H}}$ are defined as in 5.31. Therefore, we can write:

$$V^{\alpha}_{\mathcal{H}} = \int d^2\hat{q}_1 \; G^{\alpha}_{A}(\rho, \hat{x}; \hat{q}_1)V^{A}(\hat{q}_1), \tag{5.63}$$

$$W^{\alpha}_{\mathcal{H}} = \int d^2\hat{q}_2 \; G^{\alpha}_{B}(\rho, \hat{x}; \hat{q}_2)W^{B}(\hat{q}_2), \tag{5.64}$$

where $V^A(\hat{q}_1)$, $W^B(\hat{q}_2)$ are two vector fields on the 2−sphere at $\mathcal{I}^+$. The vector fields $V^{\alpha}_{\mathcal{H}}$, $W^{\alpha}_{\mathcal{H}}$ follow the constraints 5.33.

$$\mathring{D}_{\alpha}V^{\alpha}_{\mathcal{H}}(\rho, \hat{x}) = 0 \;\; ; \;\; \mathring{\Delta}V^{\alpha}_{\mathcal{H}}(\rho, \hat{x}) = 2V^{\alpha}_{\mathcal{H}}(\rho, \hat{x}) \tag{5.65}$$

$$\mathring{D}_{\alpha}W^{\alpha}_{\mathcal{H}}(\rho, \hat{x}) = 0 \;\; ; \;\; \mathring{\Delta}W^{\alpha}_{\mathcal{H}}(\rho, \hat{x}) = 2W^{\alpha}_{\mathcal{H}}(\rho, \hat{x}) \tag{5.66}$$

In order to understand the algebra between two Diff(S^2) vector fields, we evaluate the modified Lie bracket *i.e.*

$$[\xi_{SR1}, \xi_{SR2}]^{a}_{M} = [\xi_{SR1}, \xi_{SR2}]^{a} - \delta^{g}_{\xi_{SR1}}\xi^{a}_{SR2} + \delta^{g}_{\xi_{SR2}}\xi^{a}_{SR1}. \tag{5.67}$$

It is easy to see, from the form of the vector fields ξ_{SR1}, ξ_{SR2} given in 5.61, 5.62 that the τ component of the modified Lie bracket vanishes and only the hyperboloid component exists. Therefore, we need to evaluate

$$[\xi_{SR1}, \xi_{SR2}]^\alpha_M = [\xi_{SR1}, \xi_{SR2}]^\alpha - \delta^g_{\xi_{SR1}} \xi^\alpha_{SR2} + \delta^g_{\xi_{SR2}} \xi^\alpha_{SR1}, \tag{5.68}$$

where, α runs over the hyperboloid components only. At null infinity we have already seen that, the Lie bracket of two Diff(S^2) vector fields is another Diff(S^2) vector field. We expect similar result to hold at time-like infinity. Therefore, we want to check whether the vector field that one gets from the modified Lie bracket obeys the constraints

$$\mathring{D}_\alpha [\xi_{SR1}, \xi_{SR2}]^\alpha_M \stackrel{?}{=} 0, \tag{5.69}$$

$$(\mathring{\Delta} - 2)[\xi_{SR1}, \xi_{SR2}]^\alpha_M \stackrel{?}{=} 0. \tag{5.70}$$

Here, written explicitly in terms of expression of modified Lie bracket the above expressions are equivalent to

$$\mathring{D}_\alpha [\xi_{SR1}, \xi_{SR2}]^\alpha - \mathring{D}_\alpha \delta^g_{\xi_{SR1}} \xi^\alpha_{SR2} + \mathring{D}_\alpha \delta^g_{\xi_{SR2}} \xi^\alpha_{SR1} \stackrel{?}{=} 0, \tag{5.71}$$

$$\left(\mathring{\Delta} - 2\right)[\xi_{SR1}, \xi_{SR2}]^\alpha - \left(\mathring{\Delta} - 2\right)\delta^g_{\xi_{SR1}} \xi^\alpha_{SR2} + \left(\mathring{\Delta} - 2\right)\delta^g_{\xi_{SR2}} \xi^\alpha_{SR1} \stackrel{?}{=} 0. \tag{5.72}$$

We start with the verification of 5.71. The first term in the l.h.s of 5.71 vanishes. This can be shown as

$$\begin{aligned}
\mathring{D}_\alpha [\xi_{SR1}, \xi_{SR2}]^\alpha &= V^\beta_{\mathcal{H}} \mathring{D}_\alpha \mathring{D}_\beta W^\alpha_{\mathcal{H}} - W^\beta_{\mathcal{H}} \mathring{D}_\alpha \mathring{D}_\beta V^\alpha_{\mathcal{H}}, \\
&= \mathring{R}^\alpha_{\rho\alpha\beta} V^\beta_{\mathcal{H}} W^\rho_{\mathcal{H}} - \mathring{R}^\alpha_{\rho\alpha\beta} W^\beta_{\mathcal{H}} V^\rho_{\mathcal{H}}, \\
&= -2\mathring{h}_{\rho\beta}\left(V^\beta_{\mathcal{H}} W^\rho_{\mathcal{H}} - W^\beta_{\mathcal{H}} V^\rho_{\mathcal{H}}\right) \\
&= 0.
\end{aligned} \tag{5.73}$$

In going from the first line to the second we used the divergence free condition of the Diff(S^2) vector fields. We now proceed to evaluate the contribution from the modification terms (the last two terms in 5.68) in the modified Lie bracket. In the earlier section 5.2.2, we showed that the change in the supertranslation vector field due to the change in the metric induced by the Diff(S^2) vector field was reflected in the variation of the Laplacian in 5.25. But the situation is more intricate for the case of Diff(S^2) vector field. The gauge conditions 5.26 and 5.27 for one of the Diff(S^2) vector field (say ξ_{SR1}) now shift to 5.23 and 5.24 respectively where $h^{(0)}_{\alpha\beta}(\rho,\hat{x})$ will be now defined by $h^{(0)}_{\alpha\beta} = \mathring{h}_{\alpha\beta} + \mathcal{L}_{\xi_{SR2}}\mathring{h}_{\alpha\beta}$,where ξ_{SR2} is another Diff(S^2) vector field. Keeping this in mind, in order to evaluate the last two terms in the l.h.s of 5.71, we use the residual gauge condition 5.24, which is one of the defining condition for the Diff(S^2) vector field for an arbitrary $h^{(0)}_{\alpha\beta}(\rho,\hat{x})$. We vary this gauge condition w.r.t. another Diff(S^2) vector field and finally evaluate the expression at $h^{(0)}_{\alpha\beta}(\rho,\hat{x}) = \mathring{h}_{\alpha\beta}(\rho,\hat{x})$. We demonstrate this in detail further in this section.

We start with the gauge condition 5.24 for an arbitrary $h^{(0)}_{\alpha\beta}$

$$D^\alpha \xi^{(0)\beta}\mathring{h}_{\alpha\beta} = 0. \tag{5.74}$$

Under variation w.r.t. to the Diff(S^2) vector field ξ_{SR1}, the above condition becomes

$$\delta^g_{\xi_{SR1}}\Big(D^\alpha \xi^\beta \mathring{h}_{\alpha\beta}\Big) = \delta^g_{\xi_{SR1}}\Big(D^\alpha\Big)\xi^\beta\mathring{h}_{\alpha\beta} + D^\alpha\delta^g_{\xi_{SR1}}\xi^\beta\mathring{h}_{\alpha\beta} = 0. \tag{5.75}$$

It is important to note that, the variation is not taken on the reference metric $\mathring{h}_{\alpha\beta}$. Hence, the above expression can be written as

$$D^\alpha\delta^g_{\xi_{SR1}}\xi^\beta\mathring{h}_{\alpha\beta} = -\delta^g_{\xi_{SR1}}\Big(D^\alpha\Big)\xi^\beta\mathring{h}_{\alpha\beta}. \tag{5.76}$$

In order to compute $\mathring{D}_\alpha\delta^g_{\xi_{SR1}}\xi^\alpha_{SR2}$ in 5.71, we evaluate the above expression at $h^{(0)}_{\alpha\beta} = \mathring{h}_{\alpha\beta}$, then

$$\mathring{D}^\alpha\delta^g_{\xi_{SR1}}\xi^\beta_{SR2}\mathring{h}_{\alpha\beta} = -\delta^g_{\xi_{SR1}}\Big(\mathring{D}^\alpha\Big)\xi^\beta_{SR2}\mathring{h}_{\alpha\beta}. \tag{5.77}$$

The r.h.s of the above expression can be evaluated as

$$\begin{aligned}
\delta^{g}_{\xi_{SR1}}\Big(\mathring{D}^{\alpha}\Big)\xi^{\beta}_{SR2}\mathring{h}_{\alpha\beta} &= \delta^{g}_{\xi_{SR1}}\Big(\mathring{h}^{\alpha\gamma}\mathring{D}_{\gamma}\Big)\xi^{\beta}_{SR2}\mathring{h}_{\alpha\beta},\\
&= \Big(\delta^{g}_{\xi_{SR1}}\mathring{h}^{\alpha\gamma}\Big)\mathring{D}_{\gamma}\xi^{\beta}_{SR2}\mathring{h}_{\alpha\beta} + \mathring{h}^{\alpha\gamma}\Big(\delta^{g}_{\xi_{SR1}}\mathring{D}_{\gamma}\Big)\xi^{\beta}_{SR2}\mathring{h}_{\alpha\beta},\\
&= \Big(\mathring{D}^{\alpha}V^{\gamma}_{\mathcal{H}} + \mathring{D}^{\gamma}V^{\alpha}_{\mathcal{H}}\Big)\mathring{D}_{\gamma}W^{\beta}_{\mathcal{H}}\mathring{h}_{\alpha\beta} + \Big(\delta^{g}_{\xi_{SR1}}\mathring{\Gamma}^{\gamma}_{\gamma\rho}\Big)W^{\rho}_{\mathcal{H}}\mathring{h}_{\alpha\beta},\\
&= \Big(\mathring{D}^{\alpha}V^{\gamma}_{\mathcal{H}} + \mathring{D}^{\gamma}V^{\alpha}_{\mathcal{H}}\Big)\mathring{D}_{\gamma}W^{\beta}_{\mathcal{H}}\mathring{h}_{\alpha\beta} + 0,\\
&= \Big(\mathring{D}^{\alpha}V^{\gamma}_{\mathcal{H}} + \mathring{D}^{\gamma}V^{\alpha}_{\mathcal{H}}\Big)\mathring{D}_{\gamma}W^{\beta}_{\mathcal{H}}\mathring{h}_{\alpha\beta}.
\end{aligned} \tag{5.78}$$

In evaluating the above expression, we have used the fact that $\delta^{g}_{\xi_{SR1}}\mathring{\Gamma}^{\gamma}_{\gamma\rho} = 0$. This can be easily seen from C.5. Therefore, 5.77 becomes

$$\mathring{D}_{\alpha}\delta^{g}_{\xi_{SR1}}\xi^{\alpha}_{SR2} = -\mathring{D}_{\gamma}W_{\alpha}\Big(\mathring{D}^{\alpha}V^{\gamma} + \mathring{D}^{\gamma}V^{\alpha}\Big). \tag{5.79}$$

Similarly, the last term in 5.71 *i.e.* $\mathring{D}_{\alpha}\delta^{g}_{\xi_{SR2}}\xi^{\alpha}_{SR1}$ can be evaluated as

$$\mathring{D}_{\alpha}\delta^{g}_{\xi_{SR2}}\xi^{\alpha}_{SR1} = -\Big(\mathring{D}^{\alpha}W^{\gamma} + \mathring{D}^{\gamma}W^{\alpha}\Big)\mathring{D}_{\gamma}V_{\alpha}. \tag{5.80}$$

Therefore,

$$\mathring{D}_{\alpha}\delta^{g}_{\xi_{SR1}}\xi^{\alpha}_{SR2} - \mathring{D}_{\alpha}\delta^{g}_{\xi_{SR2}}\xi^{\alpha}_{SR1} = 0. \tag{5.81}$$

Hence, the divergence of the modified terms sums to zero, thereby verifying one of the conditions for a Diff(S^2) vector field. *i.e.*

$$\mathring{D}_{\alpha}[\xi_{SR1}, \xi_{SR2}]^{\alpha}_{M} = 0. \tag{5.82}$$

Now, one needs to verify 5.72. We start by evaluating the first term in 5.72.

$$\Big(\mathring{\Delta} - 2\Big)[\xi_{SR1}, \xi_{SR2}]^{\alpha} = 2\Big(\mathring{D}^{\lambda}V^{\beta}\mathring{D}_{\lambda}\mathring{D}_{\beta}W^{\alpha} - \mathring{D}^{\lambda}W^{\beta}\mathring{D}_{\lambda}\mathring{D}_{\beta}V^{\alpha}\Big)$$

$$+\left(V^{\beta}\mathring{\varDelta}\,\mathring{D}_{\beta}W^{\alpha}-W^{\beta}\mathring{\varDelta}\,\mathring{D}_{\beta}V^{\alpha}\right). \tag{5.83}$$

The last two terms can be simplified more using 5.26 and 5.27 and using the identity

$$\begin{aligned}[\mathring{\varDelta},\mathring{D}_{\alpha}]T^{a_1..a_n} &= 2\left(\delta^{a_1}_{\alpha}\mathring{D}_{\rho}T^{\rho a_2..a_n}+..\delta^{a_n}_{\alpha}\mathring{D}_{\rho}T^{a_1a_2..\rho}\right)\\ &\quad -2\left(\mathring{D}^{a_1}T^{a_2..a_n}_{\alpha}+...\mathring{D}^{a_n}T^{a_1..a_{n-1}}_{\alpha}+\mathring{D}_{\alpha}T^{a_1...a_n}\right).\end{aligned} \tag{5.84}$$

The above expression can be derived using the Riemann tensor of the hyperboloid metric 5.16 and $T^{a_1..a_n}$ is a arbitrary tensor on the hyperboloid. Therefore, 5.83 finally evaluates to

$$\begin{aligned}\left(\mathring{\varDelta}-2\right)[\xi_{SR1},\xi_{SR2}]^{\alpha} &= 2\left(\mathring{D}^{\lambda}V^{\beta}\mathring{D}_{\lambda}\mathring{D}_{\beta}W^{\alpha}-\mathring{D}^{\lambda}W^{\beta}\mathring{D}_{\lambda}\mathring{D}_{\beta}V^{\alpha}\right)\\ &\quad -2\left(V^{\beta}\mathring{D}^{\alpha}W_{\beta}-W^{\beta}\mathring{D}^{\alpha}V_{\beta}\right).\end{aligned} \tag{5.85}$$

To evaluate the last two terms in 5.71, we proceed similarly as we have done earlier for the verification of divergence free condition. We use residual gauge condition 5.23 to evaluate the last two terms. The details of the calculation is given in the Appendix E.2.1 . Finally, we get

$$\begin{aligned}&\left(\mathring{\varDelta}-2\right)\delta^{g}_{\xi_{SR1}}\xi^{\alpha}_{SR2}-\left(\mathring{\varDelta}-2\right)\delta^{g}_{\xi_{SR2}}\xi^{\alpha}_{SR1}\\ &\quad =2\left(\mathring{D}^{\beta}V^{\gamma}\mathring{D}_{\beta}\mathring{D}_{\gamma}W^{\alpha}-\mathring{D}^{\beta}W^{\gamma}\mathring{D}_{\beta}\mathring{D}_{\gamma}V^{\alpha}\right)\\ &\quad\quad -2\left(V_{\gamma}\mathring{D}^{\alpha}W^{\gamma}-W_{\gamma}\mathring{D}^{\alpha}V^{\gamma}\right).\end{aligned} \tag{5.86}$$

Therefore, substituting 5.85 and 5.86 in 5.72, we get

$$\left(\mathring{\varDelta}-2\right)[\xi_{SR1},\xi_{SR2}]^{\alpha}-\left(\mathring{\varDelta}-2\right)\delta^{g}_{\xi_{SR1}}\xi^{\alpha}_{SR2}+\left(\mathring{\varDelta}-2\right)\delta^{g}_{\xi_{SR2}}\xi^{\alpha}_{SR1}=0. \tag{5.87}$$

Hence, we have a closure of Diff(S^2) vector field at time-like infinity similar to the case of null infinity. In all the three cases, the desired relations are satisfied and hence we show that the BMS vector field algebra closes under the modified Lie bracket.

Chapter 6

Conclusion

In the first part of this thesis, we studied the relationship of generalized BMS symmetry with tree-level double soft graviton theorems at leading and subleading level. The action of generalized BMS charges on Fock vaccua gives rise to an infinite set of degenerate vacua, which are parametrized by the supertranslation and Diff(S^2) charges. We show that a particular class of a class of double soft factorization theorems at tree-level can be derived at the leading and subleading level, if one considers the Ward identity of generalized BMS charges evaluated in finite energy states which are built from the degenerate vacua which are parametrized by a single generalized BMS charge. These double soft graviton theorems are identified with the consecutive double soft graviton theorems (CDST), which elucidate the factorization property of scattering amplitude involving two soft gravitons in which energy of one of the soft graviton falls at a faster rate than the other. We also show one can arrive at consecutive double soft theorems at leading and sub-leading levels if one considers nested Ward identities of two generalized BMS charges evaluated in states built from Fock vaccua. Using a method proposed by Avery and Schwab using Noether's second theorem and path integral techniques, we give a derivation for these nested Ward identities in appendix A. The results in the first part of the thesis can be summarized as follows:

$$\langle \text{out}|\,[Q_f,[Q_g,\mathcal{S}]]\,|\text{in}\rangle = 0 \Leftrightarrow \langle \text{out}, f|\,[Q_g,\mathcal{S}]\,|\text{in}\rangle = 0 \Leftrightarrow \text{Leading CDST}, \tag{6.1}$$

$$\langle \text{out}|\,[Q_f,[Q_V,\mathcal{S}]]\,|\text{in}\rangle = 0 \Leftrightarrow \langle \text{out}, f|\,[Q_V,\mathcal{S}]\,|\text{in}\rangle = 0 \Leftrightarrow \text{Sub leading CDST 1}, \tag{6.2}$$

$$\langle \text{out}|\,[Q_V,[Q_f,\mathcal{S}]]\,|\text{in}\rangle = 0 \Leftrightarrow \langle \text{out}, V|\,[Q_f,\mathcal{S}]\,|\text{in}\rangle = 0 \Leftrightarrow \text{Sub leading CDST 2}. \tag{6.3}$$

where Q_f, Q_g corresponds to supertranslation charge and Q_V corresponds to Diff(S^2) charge. $\langle \text{out}, f|$,$\langle \text{out}, V|$ corresponds to external states built from supertranslated and Diff(S^2) vaccua respectively.

In the second part of this thesis, we studied the algebra of generalized BMS vector fields at time-like infinity. Inspired by [25, 26], we defined the notion of asymptotic flatness at time-like infinity. The generalized BMS vector fields at time-like infinity correspond to those vector fields that preserve this asymptotic flatness structure. Similar to the case at null infinity, the generalized BMS vector fields at time-like infinity are characterized by supertranslation and Diff(S^2) vector fields. But unlike null infinity, the generalized BMS vector fields at time-like infinity are metric dependent. The metric dependence appears through the differential equations that the generalized BMS vectors obey 5.21, 5.23, 5.24. In order for the vector fields to give a faithful representation of generalized BMS algebra, one should use the modified Lie bracket proposed by Barnich et.al in [1] instead of ordinary Lie bracket. The algebra is found to be similar to that at null infinity, in which, supertranslation vector fields form an Abelian subgroup. The (modified) Lie bracket between one supertranslation and a Diff(S^2) vector field is found to be another supertranslation and the algebra between two Diff(S^2) vector fields is found to be another Diff(S^2) vector field.

Appendix A

Ward identities from the Avery–Schwab method

In this appendix we derive the asymptotic Ward identity $\langle\text{out}|\,[Q_f,[Q_V,\mathcal{S}]]\,|\text{in}\rangle = 0$, based on a method that was proposed in [17]. The basic idea is to use Noether's second theorem and path integral techniques to derive Ward identities for asymptotic symmetries.

As shown in [17], given a asymptotic symmetry or large gauge transformation with a gauge parameter λ, at the level of correlation functions one obtains the following Ward identity.

$$-i\,\langle 0|\,\delta_\lambda T\Big(\varPhi(x_1)\ldots\varPhi(x_n)\Big)\,|0\rangle = \langle 0|\,T\Big((Q_{\mathcal{I}^+}[\lambda]\;-\;Q_{\mathcal{I}^-}[\lambda])\varPhi(x_1)\ldots\varPhi(x_n)\Big)\,|0\rangle \tag{A.1}$$

Here we use a generic label $\varPhi$ to label the quantum field associated to scattering particles. $Q_{\mathcal{I}^\pm}[\lambda]$ are the asymptotic charges associated to large gauge transformations λ at future and past null infinity respectively.

Before deriving the identity associated to the insertion of two charge operators, we first revisit the supertranslation Ward identity $\langle\text{out}|\,[Q_f,\mathcal{S}]\,|\text{in}\rangle = 0$. Let $\varPhi$ be any massless field that interacts with gravity and $\delta_\lambda = \delta_f$ be the generator of supertranslation on the fields.

We begin by noting that through LSZ reduction we have the following[1]

$$\begin{aligned}\prod_{i=1}^{m} p_i^2 \int d^4x_i\, e^{-ip_i\cdot x_i} \prod_{j=m+1}^{n} p_j^2 \int d^4x_j\, e^{ip_j\cdot x_j} \langle 0|\, \delta_f T\left(\Phi(x_1)\ldots\Phi(x_n)\right)|0\rangle \\ = -i\,\langle p_1,\ldots,p_m|\, Q_f^{\text{hard}+}\, \mathcal{S} - \mathcal{S}\, Q_f^{\text{hard}-}\, |p_{m+1},\ldots,p_n\rangle\end{aligned} \tag{A.2}$$

We can schematically represent this step as,

$$\langle 0|\, \delta_\lambda T\left(\Phi(x_1)\ \ldots\ \Phi(x_n)\right)|0\rangle \underset{\text{LSZ}}{\longrightarrow} \langle p_1,\ldots,p_m|\, [Q_f^{\text{hard}}, \mathcal{S}]\ |p_{m+1},\ldots,p_n\rangle \tag{A.3}$$

where we have used the fact that

$$\delta_f \Phi(p) = -i\,[Q_f, \Phi(p)] \tag{A.4}$$

On the other hand, once again via LSZ and the fact that

$$\begin{aligned} Q_f^{\text{hard}}\,|0\rangle &= 0 \\ Q_f^{\text{soft}}\,|0\rangle &= 0 \\ \langle 0|\, Q_f^{\text{soft}} &\neq 0 \end{aligned} \tag{A.5}$$

we see that

$$\langle 0|\, T\Big((Q_{\mathcal{I}^+}[\lambda] - Q_{\mathcal{I}^-}[\lambda])\Phi(x_1)\ldots\Phi(x_n)\Big)|0\rangle \underset{\text{LSZ}}{\longrightarrow} \langle p_1,\ldots,p_m|\,[Q_f^{\text{soft}}, \mathcal{S}]\,|p_{m+1},\ldots,p_n\rangle \tag{A.6}$$

[1]These arguments are formal because they are tied to the fact that the usual Dyson $\mathcal{S}$–matrix with massless particles is only formally defined. However, as we are only analyzing symmetries of the tree–level $\mathcal{S}$–matrix, we will not worry about the issue of infra–red divergence.

Substituting eqns. (A.3,A.6) in eq.(A.1) we recover the super-translation Ward identity,

$$\langle \text{out}|\, [Q_f, \mathcal{S}]\, |\text{in}\rangle = 0 \tag{A.7}$$

We note that an identical derivation for Ward identity associated to large $U(1)$ gauge transformations was already given in [76].

We will now derive the Ward identities $[Q_f, [Q_V, \mathcal{S}]] = 0$ using this method. That is, we begin with the Ward identity where the Diff(S^2) δ_V is applied after the supertranslation δ_f. The starting point for the derivation is (45) in [17], which in the present context can be written as

$$\begin{aligned} &-\langle 0|\, T\Big((Q_{\mathcal{I}^+}[f] - Q_{\mathcal{I}^-}[f])(Q_{\mathcal{I}^+}[V] - Q_{\mathcal{I}^-}[V])\Phi(x_1)\ldots\Phi(x_n)\Big)\,|0\rangle \\ &\qquad\qquad\qquad\qquad = \langle 0|\, \delta_f\, \delta_V T\Big(\Phi(x_1)\ldots\Phi(x_n)\Big)\,|0\rangle \end{aligned} \tag{A.8}$$

With our prescription that the soft charges annihilate the "in" vacuum, the l.h.s. of (A.8) reduces to

$$\begin{aligned} &-\langle 0|\, T\Big((Q_{\mathcal{I}^+}[f] - Q_{\mathcal{I}^-}[f])(Q_{\mathcal{I}^+}[V] - Q_{\mathcal{I}^-}[V])\Phi(x_1)\ldots\Phi(x_n)\Big)\,|0\rangle \\ &\qquad = -\langle 0|\, Q^{\text{soft}}_{\mathcal{I}^+}[f]\,\Big(Q^{\text{soft}}_{\mathcal{I}^+}[V] + Q^{\text{hard}}_{\mathcal{I}^+}[V]\Big)T(\Phi(x_1)\ldots\Phi(x_n))\,|0\rangle \end{aligned} \tag{A.9}$$

On the other hand, using (A.4), it is easy to see that the r.h.s. of (A.8) is given by

$$\begin{aligned} &\langle 0|\, \delta_f\, \delta_V\, T\,(\Phi(x_1)\ldots\Phi(x_n))\;|0\rangle \\ &\qquad = -\langle 0| \sum_{i,j} T\left(\Phi(x_1)\ldots[Q_f, \Phi(x_i)]\ldots[Q_V, \Phi(x_j)]\ldots\Phi(x_n)\right)|0\rangle \\ &\qquad\qquad\qquad\qquad \xrightarrow[\text{LSZ}]{} -\langle \text{out}|\,[Q^{\text{hard}}_f, [Q^{\text{hard}}_V, \mathcal{S}]]\,|\text{in}\rangle \end{aligned} \tag{A.10}$$

Thus the path integral identity and the LSZ formula lead to (equating the r.h.s. of (A.9) with r.h.s. of (A.10)),

$$\begin{aligned}\langle \text{out}|\, Q_f^{\text{soft}} \quad & Q_V^{\text{soft}}\, \mathcal{S}\, |\text{in}\rangle \\ & = -\langle \text{out}|\, Q_f^{\text{soft}}\, Q_V^{\text{hard}}\, \mathcal{S}\ |\text{in}\rangle \ + \ \langle \text{out}|\, [Q_f^{\text{hard}}, [Q_V^{\text{hard}},\ \mathcal{S}]]\, |\text{in}\rangle\end{aligned} \tag{A.11}$$

A straightforward manipulation shows that above equation is equivalent to

$$\langle \text{out}|\, [Q_f, [Q_V, \mathcal{S}]]\, |\text{in}\rangle = \ 0 \tag{A.12}$$

This is one of the Ward identities used in the main text of the thesis. The remaining identities can be derived similarly.

Appendix B

Subtleties associated to the domain of soft operators

We will now comment on the assumption that was implicitly used in previous section, and which has been used frequently in relating single soft theorems to BMS Ward identities.

From the expressions of the supertranslation and Diff(S^2) soft charges, we can see that these are singular limits of single graviton annihilation operators.

$$\begin{aligned} Q_f^{\text{soft}} &\sim \lim_{E\to 0} E\, a_+(E, w, \bar{w}) \\ Q_V^{\text{soft}} &\sim \lim_{E\to 0}\ (1 + E\partial_E) a_+(E, w, \bar{w}) \end{aligned} \tag{B.1}$$

For simplicity we have just considered the expression of the soft charges for positive helicity graviton creation operators only. In the case of Ward identities associated to the single soft theorems, it has been implicitly assumed that the super-translation soft charge can be defined as (apart from the extra factors),

$$\langle \text{out}| \lim_{E\to 0} E\, a_+(E, w, \bar{w})\, \mathcal{S}\, |\text{in}\rangle = \lim_{E\to 0} E\, \langle \text{out}|\, a_+(E, w, \bar{w})\, \mathcal{S}\, |\text{in}\rangle \tag{B.2}$$

A similar assumption is also made for the Diff(S^2) soft charge Q_V^{soft}.

However, this does not take into account the fact that the supertranslation soft charge shifts the vacuum. This subtlety is now well understood for supertranslations. It was shown in [55, 90, 91, 98] that the action of the supertranslation soft charge maps a standard Fock vaccuum to a supertranslated state which can be thought of as being labelled by a single soft graviton. With this is in mind the precise definition of $\langle\text{out}|Q_f^{\text{soft}}\, Q_V^{\text{soft}}\, \mathcal{S}\,|\text{in}\rangle$ would be

$$\langle\text{out}|\, Q_f^{\text{soft}}\, Q_V^{\text{soft}}\, \mathcal{S}\,|\text{in}\rangle :\approx \int d^2w\, D_{\bar{w}}^3 V^{\bar{w}}\;\; \langle\text{out}, f|\lim_{E\to 0}\,(1 + E\partial_E)\, a_+(E, w, \bar{w})\, \mathcal{S}\,|\text{in}\rangle \qquad \text{(B.3)}$$

where $\langle\text{out}, f|$ is the "out" state defined over the shifted vaccuum parametrized by f, generated by the action of supertranslation charge (Q_f^{soft}) on the Fock vaccuum.

In going from (4.18) to (4.19) we have made the same assumption for defining Q_V^{soft} on the shifted vacuum as has been made in the literature for defining it on the Fock vacuum, namely:

$$\langle\text{out}, f|\lim_{E\to 0}\,(1 + E\partial_E)\, a_+(E, w, \bar{w})\;\; := \;\; \lim_{E\to 0}\,(1 + E\partial_E)\;\langle\text{out}, f|\, a_+(E, w, \bar{w}) \qquad \text{(B.4)}$$

However for reasons which can be traced back to the classical theory, it is still not clear what the precise definition of Q_V^{soft} is. That is, just as a rigorous definition of Q_f^{soft} being defined as an operator which maps the ordinary Fock vacuum to a super–translated state [55, 90], no corresponding definition is available for Q_V^{soft} as yet. Consquently, operator insertions like $\langle\text{out}|Q_V^{\text{soft}}\, Q_f^{\text{soft}}\, \mathcal{S}\,|\text{in}\rangle$ are not mathematically well–defined, and we do not know how to make sense of them.

Appendix C

Variation of Christoffel symbols

In this section, we compute the variation of Christoffel symbol under a Diff(S^2) vector field. We start with:

$$\mathring{\Gamma}^{\alpha}_{\lambda\gamma} = \frac{1}{2}\mathring{h}^{\alpha\eta}\Big(\partial_\gamma \mathring{h}_{\lambda\eta} + \partial_\lambda \mathring{h}_{\eta\gamma} - \partial_\eta \mathring{h}_{\lambda\gamma}\Big) \tag{C.1}$$

Now,

$$\begin{aligned}\delta^{g}_{\xi_{SR}}\Big(\mathring{\Gamma}^{\alpha}_{\lambda\gamma}\Big) = &\frac{1}{2}\delta^{g}_{\xi_{SR}}(\mathring{h}^{\alpha\eta})\Big(\partial_\gamma \mathring{h}_{\lambda\eta} + \partial_\lambda \mathring{h}_{\gamma\eta} - \partial_\eta \mathring{h}_{\lambda\gamma}\Big) \\ &+ \frac{1}{2}\mathring{h}^{\alpha\eta}\Big(\partial_\gamma \delta^{g}_{\xi_{SR}}(\mathring{h}_{\lambda\eta}) + \partial_\lambda \delta^{g}_{\xi_{SR}}(\mathring{h}_{\eta\gamma}) - \partial_\eta \delta^{g}_{\xi_{SR}}(\mathring{h}_{\lambda\gamma})\Big)\end{aligned} \tag{C.2}$$

To evaluate the above expression we compute the variation of metric by taking the Lie derivative w.r.t. the Diff(S^2) vector field. Using this, after some algebraic manipulation we finally get C.2 as:

$$\delta^{g}_{\xi_{SR}}\Big(\mathring{\Gamma}^{\alpha}_{\lambda\gamma}\Big) = \frac{1}{2}(\mathring{D}_\gamma \mathring{D}_\lambda + \mathring{D}_\lambda \mathring{D}_\gamma)V^{\alpha}_{\mathcal{H}} + \frac{1}{2}\mathring{h}^{\alpha\eta}\left(\mathring{R}_{\lambda\zeta\gamma\eta} + \mathring{R}_{\gamma\zeta\lambda\eta}\right)V^{\zeta}_{\mathcal{H}} \tag{C.3}$$

Here, $\mathring{R}_{\lambda\zeta\gamma\eta}$ and $\mathring{R}_{\gamma\zeta\lambda\eta}$ are the Riemann tensor for the hyperboloid metric $\mathring{h}_{\alpha\beta}$. For the hyperboloid metric we can write the Riemann tensor as:

$$\mathring{R}_{\alpha\beta\gamma\delta} = \mathring{h}_{\alpha\delta}\mathring{h}_{\beta\gamma} - \mathring{h}_{\alpha\gamma}\mathring{h}_{\beta\delta} \tag{C.4}$$

Substituting C.4 in C.3 we finally get the variation of Christoffel Symbols as:

$$\delta^{g}_{\xi_{SR}}\left(\mathring{\Gamma}^{\alpha}_{\lambda\gamma}\right) = \frac{1}{2}(\mathring{D}_{\gamma}\mathring{D}_{\lambda} + \mathring{D}_{\lambda}\mathring{D}_{\gamma})V^{\alpha}_{\mathcal{H}} + \frac{1}{2}V_{\mathcal{H}\gamma}\delta^{\alpha}_{\lambda} + \frac{1}{2}V_{\mathcal{H}\lambda}\delta^{\alpha}_{\gamma} - h_{\gamma\lambda}V^{\alpha}_{\mathcal{H}} \tag{C.5}$$

Appendix D

Details of calculation of constraints on the generalized BMS vector fields at time-like infinity

In this section, we give the sketch of the calculation that leads to the constraints on the generalized BMS vector fields 5.23, 5.24. We start with the gauge condition 5.22

$$g^{ab}\partial_b\Big(\ln\Big(\sqrt{\frac{h}{\mathring{h}}}\Big)\Big) + \mathring{\nabla}_b g^{ab} = 0 \tag{D.1}$$

If we consider the τ component of the above expression at leading order in τ we get:

$$\mathring{\nabla}_\alpha g^{\tau\alpha} = 0 \tag{D.2}$$

In evaluating the l.h.s of the above expression one can use the non-zero Christoffel symbols for the Minkowski metric

$$\mathring{\Gamma}^{\tau}_{\alpha\beta} = \tau\mathring{h}_{\alpha\beta} \qquad ; \quad \mathring{\Gamma}^{\alpha}_{\beta\tau} = \tau^{-1}\delta^{\alpha}_{\beta} \tag{D.3}$$

to get

$$(\mathring{h}_{\alpha\beta}h^{(0)\alpha\beta} - 3) = 0 \tag{D.4}$$

Similarly, one can find that at the leading order in τ the hyperboloid components in D.1 evaluates to

$$h^{(0)\alpha\beta}\partial_{\beta}(\ln(\sqrt{\frac{h^{(0)}}{\mathring{h}}})) + \mathring{\nabla}_{\beta}h^{(0)\alpha\beta} = 0 \tag{D.5}$$

The residual gauge transformations that preserves the above gauge conditions namely D.4 and D.5 can be found by varying the metric g_{ab} w.r.t. to the vector field as given by 5.19. In both of the expressions we can see that only the metric component $h^{(0)\alpha\beta}$ is involved. One can easily check that $h^{(0)\alpha\beta}$ will be altered only by the $\xi^{(0)\alpha}$ part of the vector field (the τ component of the vector field only alters the hyperboloid part of the metric at $O(\tau)$). *i.e.* one can see that

$$\mathcal{L}_{\xi}h^{(0)}_{\alpha\beta} = \left(D_{\alpha}\xi^{(0)}_{\beta} + D_{\beta}\xi^{(0)}_{\alpha}\right) \tag{D.6}$$

Hence, substituting $h^{(0)}_{\alpha\beta} \to h^{(0)}_{\alpha\beta} + \left(D_{\alpha}\xi^{(0)}_{\beta} + D_{\beta}\xi^{(0)}_{\alpha}\right)$ in the gauge conditions D.4 and D.5 one finally gets the constraints:

$$2\, D^{(\alpha}\xi^{(0)\beta)}\partial_{\beta}\Big(\ln\Big(\sqrt{\frac{h^{(0)}}{\mathring{h}}}\Big)\Big) \;+\; h^{(0)\alpha\beta}\partial_{\beta}D_{\gamma}\xi^{(0)\gamma} \;+\; 2\mathring{D}_{\beta}D^{(\alpha}\xi^{(0)\beta)} = 0 \tag{D.7}$$

$$D^{\alpha}\xi^{(0)\beta}\mathring{h}_{\alpha\beta} = 0 \tag{D.8}$$

Appendix E

Details of calculation for modified Lie bracket

E.1 Calculation of modified Lie bracket between super-translation and Diff(S^2) vector field

In this section, we provide the details of the calculation for the modified bracket between one supertranslation and one Diff(S^2) vector field at i^+.

E.1.1 Contribution from ordinary Lie bracket

We start with evaluating the expression 5.54. This can be written as

$$
\begin{aligned}
&-(\mathring{\Delta}-3)(V^\alpha_{\mathcal{H}}\mathring{D}_\alpha f_{\mathcal{H}})\\
&\qquad = -\mathring{D}^\beta\mathring{D}_\beta(V^\alpha_{\mathcal{H}}\mathring{D}_\alpha f_{\mathcal{H}}) + 3V^\alpha_{\mathcal{H}}\mathring{D}_\alpha f_{\mathcal{H}}\\
&\qquad = -(\Delta V^\alpha_{\mathcal{H}}\mathring{D}_\alpha f_{\mathcal{H}} + \mathring{D}_\beta V^\alpha_{\mathcal{H}}\mathring{D}^\beta\mathring{D}_\alpha f_{\mathcal{H}} + \mathring{D}^\beta V^\alpha_{\mathcal{H}}\mathring{D}_\beta\mathring{D}_\alpha f_{\mathcal{H}} + V^\alpha_{\mathcal{H}}\mathring{D}^\beta\mathring{D}_\beta\mathring{D}_\alpha f_{\mathcal{H}}) + 3V^\alpha_{\mathcal{H}}\mathring{D}_\alpha f_{\mathcal{H}}\\
&\qquad = -(2\mathring{D}^\beta V^\alpha_{\mathcal{H}}\mathring{D}_\beta\mathring{D}_\alpha f_{\mathcal{H}} + V^\alpha_{\mathcal{H}}\mathring{D}_\beta\mathring{D}^\beta\mathring{D}_\alpha f_{\mathcal{H}}) + V^\alpha_{\mathcal{H}}\mathring{D}_\alpha f_{\mathcal{H}} \qquad \text{(E.1)}
\end{aligned}
$$

We have used $\mathring{\Delta} V^{\alpha}_{\mathcal{H}} = 2V^{\alpha}_{\mathcal{H}}$ in going from third line to the last line. The second term in the above expression E.1 can be further simplified as

$$\begin{aligned} V^{\alpha}_{\mathcal{H}}\mathring{D}_{\beta}\mathring{D}^{\beta}\mathring{D}_{\alpha}f_{\mathcal{H}} &= V^{\alpha}_{\mathcal{H}}\mathring{D}_{\beta}\mathring{D}_{\alpha}\mathring{D}^{\beta}f_{\mathcal{H}} \\ &= V^{\alpha}_{\mathcal{H}}(\mathring{D}_{\alpha}\mathring{D}_{\beta}\mathring{D}^{\beta}f_{\mathcal{H}} + R^{\beta}{}_{\gamma\beta\alpha}\mathring{D}^{\gamma}f_{\mathcal{H}}) \\ &= V^{\alpha}_{\mathcal{H}}\mathring{D}_{\alpha}f_{\mathcal{H}} \end{aligned} \tag{E.2}$$

Here, in going from the second line to the third we have used the Riemann tensor $\mathring{R}^{\beta}{}_{\gamma\beta\alpha} = \mathring{R}_{\gamma\alpha} = -2\mathring{h}_{\gamma\alpha}$ for $EAdS_3$ metric $\mathring{h}_{\alpha\beta}$ and the constraint $\mathring{\Delta} f_{\mathcal{H}} = 3f_{\mathcal{H}}$. Using E.2 in E.1, we finally get

$$-(\Delta - 3)(V^{\alpha}_{\mathcal{H}}\mathring{D}_{\alpha}f_{\mathcal{H}}) = -2\mathring{D}^{\beta}V^{\alpha}_{\mathcal{H}}\mathring{D}_{\beta}\mathring{D}_{\alpha}f_{\mathcal{H}} \tag{E.3}$$

E.1.2 Contribution from modification terms

In this section, we evaluate the details of the calculation to arrive at 5.59. We have:

$$\begin{aligned} \delta^{g}_{\xi_{SR}}\Big(\mathring{\Delta} - 3\Big)\xi^{\tau}_{ST} &= \delta^{g}_{\xi_{SR}}\Big(\mathring{\Delta}\Big)f_{\mathcal{H}} \\ &= \delta^{g}_{\xi_{SR}}(\mathring{h}^{\alpha\beta}\mathring{D}_{\alpha}\mathring{D}_{\beta})f_{\mathcal{H}} \\ &= \delta^{g}_{\xi_{SR}}(\mathring{h}^{\alpha\beta})\mathring{D}_{\alpha}\mathring{D}_{\beta}f_{\mathcal{H}} + \mathring{h}^{\alpha\beta}\delta^{g}_{\xi_{SR}}(\mathring{D}_{\alpha})\mathring{D}_{\beta}f_{\mathcal{H}} + \mathring{h}^{\alpha\beta}\mathring{D}_{\alpha}\delta^{g}_{\xi_{SR}}(\mathring{D}_{\beta})f_{\mathcal{H}} \\ &= \delta^{g}_{\xi_{SR}}(\mathring{h}^{\alpha\beta})\mathring{D}_{\alpha}\mathring{D}_{\beta}f_{\mathcal{H}} - \mathring{h}^{\alpha\beta}\delta^{g}_{\xi_{SR}}(\Gamma^{\gamma}_{\alpha\beta})\mathring{D}_{\gamma}f_{\mathcal{H}} \end{aligned} \tag{E.4}$$

We have used the fact that variation of the partial derivative term in the covariant derivative does not contribute since this does not depend on the metric. The first term in E.4 can be evaluated by taking the Lie derivative on the hyperboloid metric $\mathring{h}^{\alpha\beta}$ w.r.t ξ_{SR}. To evaluate the second term in E.4, we need the variation of the Christoffel symbols w.r.t the Diff(S^2)

vector field. This is computed in Appendix-C and using this we finally evaluate E.4 as

$$
\begin{aligned}
&\delta^g_{\xi_{SR}}\Big(\mathring{\Delta}-3\Big)\xi^\tau_{ST}\\
&\qquad= -\mathring{D}^\alpha V^\beta_{\mathcal{H}}(\mathring{D}_\alpha\mathring{D}_\beta+\mathring{D}_\beta\mathring{D}_\alpha)f_{\mathcal{H}} - \frac{1}{2}\mathring{h}^{\alpha\beta}(\mathring{D}_\alpha\mathring{D}_\beta+\mathring{D}_\beta\mathring{D}_\alpha)V_{\mathcal{H}\gamma}\mathring{D}_\gamma f_{\mathcal{H}}\\
&\qquad\qquad - \frac{1}{2}\mathring{h}^{\alpha\beta}(V_{\mathcal{H}\beta}\delta^\gamma_\alpha + V_{\mathcal{H}\alpha}\delta^\gamma_\beta)\mathring{D}_\gamma f_{\mathcal{H}} + \mathring{h}^{\alpha\beta}\mathring{h}_{\alpha\beta}V^\gamma_{\mathcal{H}}\mathring{D}_\gamma f_{\mathcal{H}}\\
&\qquad= -\mathring{D}^\alpha V^\beta_{\mathcal{H}}(\mathring{D}_\alpha\mathring{D}_\beta+\mathring{D}_\beta\mathring{D}_\alpha)f_{\mathcal{H}} - \frac{1}{2}(\mathring{D}_\alpha\mathring{D}^\alpha+\mathring{D}_\alpha\mathring{D}^\alpha)V^\gamma_{\mathcal{H}}\mathring{D}_\gamma f_{\mathcal{H}}\\
&\qquad\qquad - \frac{1}{2}\mathring{D}_\gamma f_{\mathcal{H}}\Big(V_{\mathcal{H}\beta}\mathring{h}^{\gamma\beta} + V_{\mathcal{H}\alpha}\mathring{h}^{\gamma\alpha}\Big) + 3V^\gamma_{\mathcal{H}}\mathring{D}_\gamma f_{\mathcal{H}}\\
&\qquad= -\mathring{D}^\alpha V^\beta_{\mathcal{H}}(\mathring{D}_\alpha\mathring{D}_\beta+\mathring{D}_\beta\mathring{D}_\alpha)f_{\mathcal{H}} - \mathring{\Delta}V^\gamma_{\mathcal{H}}\mathring{D}_\gamma f_{\mathcal{H}} - \mathring{D}_\gamma f_{\mathcal{H}}V^\gamma_{\mathcal{H}} + 3V^\gamma_{\mathcal{H}}\mathring{D}_\gamma f_{\mathcal{H}}\\
&\qquad= -\mathring{D}^\alpha V^\beta_{\mathcal{H}}(\mathring{D}_\alpha\mathring{D}_\beta+\mathring{D}_\beta\mathring{D}_\alpha)f_{\mathcal{H}}\\
&\qquad= -2\mathring{D}^\alpha V^\beta_{\mathcal{H}}\mathring{D}_\alpha\mathring{D}_\beta f_{\mathcal{H}}
\end{aligned}
\tag{E.5}
$$

Hence, we can finally write

$$
\delta^g_{\xi_{SR}}\Big(\mathring{\Delta}-3\Big)\xi^\tau_{ST} = -2\mathring{D}^\alpha V^\beta_{\mathcal{H}}\mathring{D}_\alpha\mathring{D}_\beta f_{\mathcal{H}} \tag{E.6}
$$

E.2 Details of calculation for modified Lie bracket of two Diff(S^2) vector fields

E.2.1 Contribution from the modification terms

In this section, we give the details of the computation of last two terms in 5.72, i.e we evaluate the expression

$$
\Big(\mathring{\Delta}-2\Big)\delta^g_{\xi_{SR1}}\xi^\alpha_{SR2} - \Big(\mathring{\Delta}-2\Big)\delta^g_{\xi_{SR2}}\xi^\alpha_{SR1} \tag{E.7}
$$

In order to evaluate the above, we start with the variation w.r.t to one of the Diff(S^2) vector field on the gauge condition 5.24

$$2\,D^{(\alpha}\xi^{(0)\beta)}\partial_\beta\Big(\ln\Big(\sqrt{\frac{h^{(0)}}{\mathring{h}}}\Big)\Big) \;+\; h^{(0)\alpha\beta}\partial_\beta D_\gamma\xi^{(0)\gamma} \;+\; 2\mathring{D}_\beta D^{(\alpha}\xi^{(0)\beta)} = 0 \tag{E.8}$$

Under variation w.r.t ξ_{SR1} the first term in the above expression becomes

$$\delta^g_{\xi_{SR1}}\Big(2\,D^{(\alpha}\xi^{(0)\beta)}\partial_\beta\Big(\ln\Big(\sqrt{\frac{h^{(0)}}{\mathring{h}}}\Big)\Big)\Big) = \delta^g_{\xi_{SR1}}\Big(2\,D^{(\alpha}\xi^{\beta)}\Big)\partial_\beta\Big(\ln\Big(\sqrt{\frac{h^{(0)}}{\mathring{h}}}\Big)\Big)\Big) \\ + 2\,D^{(\alpha}\xi^{\beta)}\partial_\beta\Big(D_\gamma\xi^\gamma_{SR1}\Big) \tag{E.9}$$

The r.h.s of the above expression vanishes when one considers the variation of E.8 on the Diff(S^2) vector field ξ_{SR2}. This corresponds to evaluating the above expression at $h^{(0)}_{\alpha\beta} = \mathring{h}_{\alpha\beta}$ and $\xi = \xi_{SR2}$.

Consider the variation of the second term in E.8 w.r.t ξ_{SR1}.

$$\delta^g_{\xi_{SR1}}\Big(h^{(0)\alpha\beta}\partial_\beta D_\gamma\xi^\gamma\Big) = \delta^g_{\xi_{SR1}}\Big(h^{(0)\alpha\beta}\Big)\partial_\beta D_\gamma\xi^\gamma + h^{(0)\alpha\beta}\partial_\beta\Big(\delta^g_{\xi_{SR1}}\Big(D_\gamma\xi^\gamma\Big)\Big) \tag{E.10}$$

As we have done previously, the first term in the r.h.s of the above expression will vanish when we finally substitute $h^{(0)}_{\alpha\beta} = \mathring{h}_{\alpha\beta}$, due the divergence free condition of ξ^γ_{SR2}. The second term in E.10 can be evaluated when $h^{(0)}_{\alpha\beta} = \mathring{h}_{\alpha\beta}$ and $\xi = \xi_{SR2}$ as

$$\begin{aligned} h^{(0)\alpha\beta}\partial_\beta\Big(\delta^g_{\xi_{SR1}}\Big(D_\gamma\xi^\gamma_{SR2}\Big)\Big)\Big|_{h^{(0)}_{\alpha\beta}=\mathring{h}_{\alpha\beta}} &= \mathring{h}^{\alpha\beta}\partial_\beta\Big(\delta^g_{\xi_{SR1}}(D_\gamma)\xi^\gamma_{SR2}\Big)\Big|_{h^{(0)}_{\alpha\beta}=\mathring{h}_{\alpha\beta}} + \mathring{h}^{\alpha\beta}\partial_\beta\Big(D_\gamma\delta^g_{\xi_{SR1}}(\xi^\gamma_{SR2})\Big)\Big|_{h^{(0)}_{\alpha\beta}=\mathring{h}_{\alpha\beta}} \\ &= \mathring{h}^{\alpha\beta}\partial_\beta\Big(\delta^g_{\xi_{SR1}}(\mathring{D}_\gamma)\xi^\gamma_{SR2}\Big) + \mathring{h}^{\alpha\beta}\partial_\beta\Big(\mathring{D}_\gamma\delta^g_{\xi_{SR1}}(\xi^\gamma_{SR2})\Big) \\ &= \mathring{h}^{\alpha\beta}\partial_\beta\Big(\delta^g_{\xi_{SR1}}(\mathring{\Gamma}^\gamma_{\gamma\rho})\xi^\rho_{SR2}\Big) + \mathring{h}^{\alpha\beta}\partial_\beta\Big(\mathring{D}_\gamma\delta^g_{\xi_{SR1}}(\xi^\gamma_{SR2})\Big) \\ &= 0 + \mathring{h}^{\alpha\beta}\partial_\beta\Big(\mathring{D}_\gamma\delta^g_{\xi_{SR1}}(\xi^\gamma_{SR2})\Big) \qquad \text{(E.11)} \\ &= \mathring{h}^{\alpha\beta}\partial_\beta\Big(\Big(\mathring{D}^\rho V^\gamma_{\mathcal{H}} + \mathring{D}^\gamma V^\rho_{\mathcal{H}}\Big)\mathring{D}_\gamma W_{\mathcal{H}\rho}\Big) \qquad \text{(E.12)} \end{aligned}$$

Here, in going from E.11 to E.12 we have used 5.77 and 5.78.

At this point, it will be useful to remember the expression E.7. There is a term $\Big(\mathring{\varDelta}-2\Big)\delta^g_{\xi_{SR2}}\xi^\alpha_{SR1}$ which also needs to be evaluated. This corresponds to doing the same analysis as we have done till now but interchanging $V^\alpha_{\mathcal{H}}$ with $W^\alpha_{\mathcal{H}}$. This will help us in eliminating many terms which will not appear in the final expression. Therefore, contribution of E.12 corresponding to doing this procedure is equal to

$$h^{(0)\alpha\beta}\partial_\beta\Big(\delta^g_{\xi_{SR2}}\Big(D_\gamma\xi^\gamma_{SR1}\Big)\Big)\Big|_{h^{(0)}_{\alpha\beta}=\mathring{h}_{\alpha\beta}} = \mathring{h}^{\alpha\beta}\partial_\beta\Big(\Big(\mathring{D}^\rho W^\gamma_{\mathcal{H}} + \mathring{D}^\gamma W^\rho_{\mathcal{H}}\Big)\mathring{D}_\gamma V_{\mathcal{H}\rho}\Big). \tag{E.13}$$

Therefore, we get

$$h^{(0)\alpha\beta}\partial_\beta\Big(\delta^g_{\xi_{SR1}}\Big(D_\gamma\xi^\gamma_{SR2}\Big)\Big)\Big|_{h^{(0)}_{\alpha\beta}=\mathring{h}_{\alpha\beta}} - h^{(0)\alpha\beta}\partial_\beta\Big(\delta^g_{\xi_{SR2}}\Big(D_\gamma\xi^\gamma_{SR1}\Big)\Big)\Big|_{h^{(0)}_{\alpha\beta}=\mathring{h}_{\alpha\beta}} = 0. \tag{E.14}$$

Hence, the second term in E.8 will not contribute.

We are now left with the variation of the third term in E.8

$$\delta^g_{\xi_{SR1}}\Big(2\mathring{D}_\beta D^{(\alpha}\xi^{\beta)}_{SR2}\Big) = 2\mathring{D}_\beta\Big(D^{(\alpha}\delta^g_{\xi_{SR1}}\xi^{\beta)}_{SR2} + \delta^g_{\xi_{SR1}}(D^{(\alpha})\xi^{\beta)}_{SR2}\Big) \tag{E.15}$$

The first term in the above expression evaluated at $h^{(0)}_{\alpha\beta} = \mathring{h}_{\alpha\beta}$ and $\xi = \xi_{SR2}$ can be written as

$$\begin{aligned}
2\Big(\mathring{D}_\beta D^{(\alpha}\delta^g_{\xi_{SR1}}\xi^{\beta)}_{SR2}\Big)\Big|_{h^{(0)}_{\alpha\beta}=\mathring{h}_{\alpha\beta}} &= \mathring{D}_\beta\mathring{D}^\alpha\delta^g_{\xi_{SR1}}\xi^\beta_{SR2} + \mathring{D}_\beta\mathring{D}^\beta\delta^g_{\xi_{SR1}}\xi^\alpha_{SR2} &\text{(E.16)}\\
&= \mathring{h}^{\alpha\gamma}\mathring{D}_\gamma\mathring{D}_\beta\delta^g_{\xi_{SR1}}\xi^\beta_{SR2} + (\mathring{\varDelta}-2)\delta^g_{\xi_{SR1}}\xi^\alpha_{SR2} &\text{(E.17)}\\
&= (\mathring{\varDelta}-2)\delta^g_{\xi_{SR1}}W^\alpha_{\mathcal{H}} &\text{(E.18)}
\end{aligned}$$

The second term in E.15 evaluated at $h^{(0)}_{\alpha\beta} = \mathring{h}_{\alpha\beta}$ and $\xi = \xi_{SR2}$ can be written as

$$2\mathring{D}_\beta\Big(\delta^g_{\xi_{SR1}}(D^{(\alpha})\xi^{\beta)}_{SR2}\Big)\Big|_{h^{(0)}_{\alpha\beta}=\mathring{h}_{\alpha\beta}} = \mathring{D}_\beta\Big(\delta^g_{\xi_{SR1}}(D^\alpha)\xi^\beta_{SR2}\Big)\Big|_{h^{(0)}_{\alpha\beta}=\mathring{h}_{\alpha\beta}} + \mathring{D}_\beta\Big(\delta^g_{\xi_{SR1}}(D^\beta)\xi^\alpha_{SR2}\Big)\Big|_{h^{(0)}_{\alpha\beta}=\mathring{h}_{\alpha\beta}} \tag{E.19}$$

$$= \mathring{D}_\beta\Big(\delta^g_{\xi_{SR1}}(\mathring{D}^\alpha)\xi^\beta_{SR2}\Big) + \mathring{D}_\beta\Big(\delta^g_{\xi_{SR1}}(\mathring{D}^\beta)\xi^\alpha_{SR2}\Big)$$
$$= \mathring{D}_\beta\Big((\delta^g_{\xi_{SR1}}\mathring{h}^{\alpha\gamma})\mathring{D}_\gamma\xi^\beta_{SR2}\Big) + \mathring{D}_\beta\Big((\delta^g_{\xi_{SR1}}\mathring{h}^{\gamma\beta})\mathring{D}_\gamma\xi^\alpha_{SR2}\Big)$$
$$+ \mathring{D}_\beta\Big(\mathring{h}^{\alpha\gamma}(\delta^g_{\xi_{SR1}}\mathring{D}_\gamma)\xi^\beta_{SR2}\Big) + \mathring{D}_\beta\Big(\mathring{h}^{\beta\gamma}(\delta^g_{\xi_{SR1}}\mathring{D}_\gamma)\xi^\alpha_{SR2}\Big) \quad \text{(E.20)}$$

The first two terms in the above expression can be computed using $\delta^g_{\xi_{SR1}}\mathring{h}^{\alpha\gamma} = -\Big(\mathring{D}^\alpha V^\gamma_{\mathcal{H}} + \mathring{D}^\gamma V^\alpha_{\mathcal{H}}\Big)$ to get

$$\mathring{D}_\beta\Big((\delta^g_{\xi_{SR1}}\mathring{h}^{\alpha\gamma})\mathring{D}_\gamma\xi^\beta_{SR2}\Big) + \mathring{D}_\beta\Big((\delta^g_{\xi_{SR1}}\mathring{h}^{\gamma\beta})\mathring{D}_\gamma\xi^\alpha_{SR2}\Big)$$
$$= -\Big[\mathring{D}_\beta(\mathring{D}^\alpha V^\gamma_{\mathcal{H}} + \mathring{D}^\gamma V^\alpha_{\mathcal{H}})\mathring{D}_\gamma W^\beta_{\mathcal{H}} + \mathring{D}_\beta(\mathring{D}^\beta V^\gamma_{\mathcal{H}} + \mathring{D}^\gamma V^\beta_{\mathcal{H}})\mathring{D}_\gamma W^\alpha_{\mathcal{H}}$$
$$+ (\mathring{D}^\alpha V^\gamma_{\mathcal{H}} + \mathring{D}^\gamma V^\alpha_{\mathcal{H}})\mathring{D}_\beta\mathring{D}_\gamma W^\beta_{\mathcal{H}} + (\mathring{D}^\beta V^\gamma_{\mathcal{H}} + \mathring{D}^\gamma V^\beta_{\mathcal{H}})\mathring{D}_\beta\mathring{D}_\gamma W^\alpha_{\mathcal{H}}\Big] \quad \text{(E.21)}$$

The second term in the above expression can be shown to vanish using 5.65 and 5.16. The third term can be further simplified using 5.65 and 5.16 to

$$(\mathring{D}^\alpha V^\gamma_{\mathcal{H}} + \mathring{D}^\gamma V^\alpha_{\mathcal{H}})\mathring{D}_\beta\mathring{D}_\gamma W^\beta_{\mathcal{H}} = (\mathring{D}^\alpha V^\gamma_{\mathcal{H}} + \mathring{D}^\gamma V^\alpha_{\mathcal{H}})\mathring{R}^\beta_{\rho\beta\gamma}W^\rho_{\mathcal{H}} = -2W_{\mathcal{H}\gamma}(\mathring{D}^\alpha V^\gamma_{\mathcal{H}} + \mathring{D}^\gamma V^\alpha_{\mathcal{H}}) \quad \text{(E.22)}$$

Therefore, E.21 can be written as

$$\mathring{D}_\beta\Big((\delta^g_{\xi_{SR1}}\mathring{h}^{\alpha\gamma})\mathring{D}_\gamma\xi^\beta_{SR2}\Big) + \mathring{D}_\beta\Big((\delta^g_{\xi_{SR1}}\mathring{h}^{\gamma\beta})\mathring{D}_\gamma\xi^\alpha_{SR2}\Big)$$
$$= -\Big[(\mathring{D}^\beta V^\gamma_{\mathcal{H}} + \mathring{D}^\gamma V^\beta_{\mathcal{H}})\mathring{D}_\beta\mathring{D}_\gamma W^\alpha_{\mathcal{H}} + \mathring{D}_\beta(\mathring{D}^\alpha V^\gamma_{\mathcal{H}} + \mathring{D}^\gamma V^\alpha_{\mathcal{H}})\mathring{D}_\gamma W^\beta_{\mathcal{H}}$$
$$- 2W_{\mathcal{H}\gamma}(\mathring{D}^\alpha V^\gamma_{\mathcal{H}} + \mathring{D}^\gamma V^\alpha_{\mathcal{H}})\Big] \quad \text{(E.23)}$$

Now, as we have done previously, the terms in E.23 that will contribute to E.8 can be found by interchanging V^α with W^α and ignoring the terms that are same. Finally, the terms that contribute to E.8 in the above expression can be found to be

$$\Big(\mathring{D}_\beta\Big((\delta^g_{\xi_{SR1}}\mathring{h}^{\alpha\gamma})\mathring{D}_\gamma\xi^\beta_{SR2}\Big)+\mathring{D}_\beta\Big((\delta^g_{\xi_{SR1}}\mathring{h}^{\gamma\beta})\mathring{D}_\gamma\xi^\alpha_{SR2}\Big)\Big)_{\text{non-vanishing}}$$
$$=-\mathring{D}_\beta\mathring{D}^\alpha V^\gamma_{\mathcal{H}}\mathring{D}_\gamma W^\beta_{\mathcal{H}}+2W_{\mathcal{H}\gamma}(\mathring{D}^\alpha V^\gamma_{\mathcal{H}}+\mathring{D}^\gamma V^\alpha_{\mathcal{H}})-\mathring{D}_\beta\mathring{D}_\gamma W^\alpha_{\mathcal{H}}\mathring{D}^\beta V^\gamma_{\mathcal{H}} \tag{E.24}$$

where, “non-vanishing" denotes the terms that contribute to E.8. Now, let us simplify the last two terms in E.20.

$$\mathring{D}_\beta\Big(\mathring{h}^{\alpha\gamma}(\delta^g_{\xi_{SR1}}\mathring{D}_\gamma)\xi^\beta_{SR2}\Big)+\mathring{D}_\beta\Big(\mathring{h}^{\beta\gamma}(\delta^g_{\xi_{SR1}}\mathring{D}_\gamma)\xi^\alpha_{SR2}\Big)$$
$$=\mathring{D}_\beta\Big(\mathring{h}^{\alpha\gamma}(\delta^g_{\xi_{SR1}}\mathring{\Gamma}^\beta_{\gamma\rho})\xi^\rho_{SR2}\Big)+\mathring{D}_\beta\Big(\mathring{h}^{\beta\gamma}(\delta^g_{\xi_{SR1}}\mathring{\Gamma}^\alpha_{\gamma\rho})\xi^\rho_{SR2}\Big) \tag{E.25}$$

The variation of Christoffel symbol under Diff(S^2) vector field C.5 is

$$\delta^g_{\xi_{SR1}}\mathring{\Gamma}^\alpha_{\beta\gamma}=\frac{1}{2}\Big(\mathring{D}_\beta\mathring{D}_\gamma+\mathring{D}_\gamma\mathring{D}_\beta\Big)V^\alpha_{\mathcal{H}}+\frac{1}{2}V_{\mathcal{H}\beta}\delta^\alpha_\gamma+\frac{1}{2}V_{\mathcal{H}\gamma}\delta^\alpha_\beta-\mathring{h}_{\beta\gamma}V^\alpha_{\mathcal{H}}. \tag{E.26}$$

Let us denote the first term in the above expression involving two covariant derivatives as “DD" term, the terms containing delta function as “δ” term and the last term as “h” term. We can show that, δ term and h term does not contribute to E.25.

The δ piece contribution of E.26 in E.25 can be evaluated as

$$2\mathring{D}_\beta(\mathring{h}^{\alpha\beta}V_{\mathcal{H}\rho}W^\rho_{\mathcal{H}})+\mathring{D}_\beta(W^\alpha_{\mathcal{H}}V^\beta+W^{\mathcal{H}\beta}V^\alpha_{\mathcal{H}}), \tag{E.27}$$

which will not contribute because of the similar contribution when we interchange $V^\alpha_{\mathcal{H}}$ with $W^\alpha_{\mathcal{H}}$ when evaluating E.8.

The h piece contribution of E.26 in E.25 can be evaluated similarly as

$$\mathring{D}_\beta(V^\beta_{\mathcal{H}}W^\alpha_{\mathcal{H}}+V^\alpha_{\mathcal{H}}W^\beta_{\mathcal{H}}), \tag{E.28}$$

which will also not contribute when we interchange $V^\alpha_{\mathcal{H}}$ with $W^\alpha_{\mathcal{H}}$. Therefore, we are left with only the contribution of the "DD" piece which can be written as

$$\begin{aligned}
&\mathring{D}_\beta\Big(\mathring{h}^{\alpha\gamma}(\delta^g_{\xi_{SR1}}\mathring{D}_\gamma)\xi^\beta_{SR2}\Big)+\mathring{D}_\beta\Big(\mathring{h}^{\beta\gamma}(\delta^g_{\xi_{SR1}}\mathring{D}_\gamma)\xi^\alpha_{SR2}\Big)\\
&\qquad=\frac{1}{2}\mathring{D}_\beta\Big(\mathring{h}^{\alpha\gamma}W^\rho(\mathring{D}_\gamma\mathring{D}_\rho V^\beta_{\mathcal{H}}+\mathring{D}_\rho\mathring{D}_\gamma V^\beta_{\mathcal{H}})\Big)+\frac{1}{2}\mathring{D}_\beta\Big(\mathring{h}^{\beta\gamma}W^\rho_{\mathcal{H}}(\mathring{D}_\gamma\mathring{D}_\rho V^\alpha_{\mathcal{H}}+\mathring{D}_\rho\mathring{D}_\gamma V^\alpha_{\mathcal{H}})\Big)
\end{aligned} \tag{E.29}$$

Now, using 5.16 the above expression can be written as

$$\begin{aligned}
&\mathring{D}_\beta\Big(\mathring{h}^{\alpha\gamma}(\delta^g_{\xi_{SR1}}\mathring{D}_\gamma)\xi^\beta_{SR2}\Big)+\mathring{D}_\beta\Big(\mathring{h}^{\beta\gamma}(\delta^g_{\xi_{SR1}}\mathring{D}_\gamma)\xi^\alpha_{SR2}\Big)\\
&\qquad=\frac{1}{2}\mathring{D}_\beta\Big(\mathring{h}^{\alpha\gamma}W^\rho_{\mathcal{H}}(2\mathring{D}_\gamma\mathring{D}_\rho V^\beta_{\mathcal{H}}+\mathring{R}^\beta_{\sigma\rho\gamma}V^\sigma_{\mathcal{H}})\Big)+\frac{1}{2}\mathring{D}_\beta\Big(\mathring{h}^{\beta\gamma}W^\rho_{\mathcal{H}}(2\mathring{D}_\gamma\mathring{D}_\rho V^\alpha_{\mathcal{H}}+\mathring{R}^\alpha_{\sigma\rho\gamma}V^\sigma_{\mathcal{H}})\Big)\\
&\qquad=\mathring{D}_\beta\Big(\mathring{h}^{\alpha\gamma}W^\rho_{\mathcal{H}}\mathring{D}_\gamma\mathring{D}_\rho V^\beta_{\mathcal{H}}\Big)+\mathring{D}_\beta\Big(\mathring{h}^{\beta\gamma}W^\rho_{\mathcal{H}}\mathring{D}_\gamma\mathring{D}_\rho V^\alpha_{\mathcal{H}}\Big)\\
&\qquad\qquad+\frac{1}{2}\mathring{D}_\beta\Big(2\mathring{h}^{\alpha\beta}W_{\mathcal{H}\rho}V^\rho_{\mathcal{H}}\Big)-\frac{1}{2}\mathring{D}_\beta\Big(V^\alpha_{\mathcal{H}}W^\beta_{\mathcal{H}}+W^\alpha_{\mathcal{H}}V^\beta_{\mathcal{H}}\Big)
\end{aligned} \tag{E.30}$$

Only the first term contributes in the above expression when $V^\alpha_{\mathcal{H}}$ interchanged with $W^\alpha_{\mathcal{H}}$. Therefore, the contribution of E.30 becomes

$$\begin{aligned}
&\Big(\mathring{D}_\beta\Big(\mathring{h}^{\alpha\gamma}(\delta^g_{\xi_{SR1}}\mathring{D}_\gamma)\xi^\beta_{SR2}\Big)+\mathring{D}_\beta\Big(\mathring{h}^{\beta\gamma}(\delta^g_{\xi_{SR1}}\mathring{D}_\gamma)\xi^\alpha_{SR2}\Big)\Big)_{\text{non-vanishing}}\\
&=\mathring{h}^{\alpha\gamma}\mathring{D}_\beta\Big(W^\rho_{\mathcal{H}}\mathring{D}_\gamma\mathring{D}_\rho V^\beta_{\mathcal{H}}\Big)+\mathring{h}^{\beta\gamma}\mathring{D}_\beta\Big(W^\rho_{\mathcal{H}}\mathring{D}_\gamma\mathring{D}_\rho V^\alpha_{\mathcal{H}}\Big)\\
&=\mathring{h}^{\alpha\gamma}\mathring{D}_\beta W^\rho_{\mathcal{H}}\mathring{D}_\gamma\mathring{D}_\rho V^\beta_{\mathcal{H}}+\mathring{h}^{\beta\gamma}\mathring{D}_\beta W^\rho_{\mathcal{H}}\mathring{D}_\gamma\mathring{D}_\rho V^\alpha_{\mathcal{H}}+\mathring{h}^{\alpha\gamma}W^\rho_{\mathcal{H}}[\mathring{D}_\beta,\mathring{D}_\gamma\mathring{D}_\rho]V^\beta_{\mathcal{H}}+\mathring{h}^{\beta\gamma}W^\rho_{\mathcal{H}}[\mathring{D}_\beta,\mathring{D}_\gamma\mathring{D}_\rho]V^\alpha_{\mathcal{H}}\\
&\qquad+\mathring{h}^{\beta\gamma}W^\rho_{\mathcal{H}}\mathring{D}_\rho\mathring{D}_\beta\mathring{D}_\gamma V^\alpha_{\mathcal{H}}\\
&=\mathring{D}_\beta W^\rho_{\mathcal{H}}\mathring{D}^\alpha\mathring{D}_\rho V^\beta_{\mathcal{H}}+\mathring{D}^\gamma W^\rho_{\mathcal{H}}\mathring{D}_\gamma\mathring{D}_\rho V^\alpha-5W^\rho_{\mathcal{H}}\mathring{D}^\alpha V_{\mathcal{H}\rho}-2W^\rho_{\mathcal{H}}\mathring{D}_\rho V^\alpha_{\mathcal{H}}
\end{aligned} \tag{E.31}$$

Finally, adding up E.24 and E.31 and interchanging $V^\alpha_{\mathcal{H}}$ with $W^\alpha_{\mathcal{H}}$, E.8 evaluates to

$$\begin{aligned}
&\Big(\mathring{\varDelta}-2\Big)\delta^g_{\xi_{SR1}}W^\alpha_{\mathcal{H}}-\Big(\mathring{\varDelta}-2\Big)\delta^g_{\xi_{SR2}}V^\alpha_{\mathcal{H}}\\
&\qquad=2\Big(\mathring{D}^\beta V^\gamma_{\mathcal{H}}\mathring{D}_\beta\mathring{D}_\gamma W^\alpha_{\mathcal{H}}-\mathring{D}^\beta W^\gamma_{\mathcal{H}}\mathring{D}_\beta\mathring{D}_\gamma V^\alpha_{\mathcal{H}}\Big)\\
&\qquad\qquad-2\Big(V_{\mathcal{H}\gamma}\mathring{D}^\alpha W^\gamma_{\mathcal{H}}-W_{\mathcal{H}\gamma}\mathring{D}^\alpha V^\gamma_{\mathcal{H}}\Big).
\end{aligned} \tag{E.32}$$

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