SOME Ω -THEOREMS AND RELATED QUESTIONS IN NUMBER THEORY

A thesis submitted to the University of Madras for the degree of Doctor of Philosophy

by

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॥ मत्यमृतं तव पदं मरणोर्मिनाशं
तस्मात्त्वमेव शरणं मम दीनबन्धो ॥

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CERTIFICATE

This is to certify that the Ph.D. thesis submitted by Mr. Sukumar Das Adhikari, to the University of Madras, entitled :Some Ω-theorems and related questions in Number Theory,is a record of bonafide research work done by him under my supervision. The research work presented in this thesis has not been presented in part or full for any other Degree, Diploma, Associateship or other similar title. It is further certified that the thesis represents independent work on the part of the candidate, and collaboration was necessitated by the nature and scope of the problems dealt with.

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PREFACE

This thesis is concerned with the study of certain error terms in Analytic Number Theory. In the general introduction the philosophy of the problem is discussed in detail.

No words of gratitude will be enough for the help and encouragement I received from my teacher and guide Professor R. Balasubramanian. He initiated me into this subject, taught with patience and encouraged at every step.

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S. D. Adhikari.

NOTATION

Here we explain some of the most common symbols and functions used in this thesis, while others will be explained in the body of the text as they appear.

$\mathbb{N}$	The set of natural numbers $1, 2, 3\ldots$
d, k, l, m, n	Elements of $\mathbb{N}$.
p	A prime.
(m, n)	The greatest common divisor of m and n.
$[m, n]$	The least common multiple of m and n.
$[x]$	The greatest integer not exceeding the real number x.
$\{x\}$	Stands for $x - [x]$.
$d \mid n$	Means that d divides n.
$d \nmid n$	Means that d does not divide n.
$\sum_{n\leq x}$	A sum taken over all natural numbers n not exceeding x; the empty sum is defined to be zero.
$\sum_{d\mid n}$	A sum taken over all divisors of n.
$\prod_{a<p\leq b}$	Product over all primes greater than a and not exceeding b.
$\zeta(s)$	Riemann's zeta function, defined for $\mathrm{Re}(s) > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$; otherwise by analytic continuation.
$\delta_k(n)$	Max $\{d \in \mathbb{N} : d \mid n, (d, k) = 1\}$.
$\sigma(k)$	The sum of the divisors of k.
$d(k)$	The number of divisors of k.

$f(x) = O(g(x))$ or $f(x) << g(x)$	Means $\mid f(x) \mid \leq cg(x)$ for $x \geq X$ and some absolute constant $c > 0$. Here $f(x)$ is a complex function of a real variable and $g(x) \geq 0$.
$f(x) = o(g(x))$	$f(x), g(x)$, being as above , this means that for $\epsilon > 0$ however small, there is X such that $\mid f(x) \mid < \epsilon g(x)$ for $x \geq X$.
$f(x) = \Omega(g(x))$	Stands for the negation of $f(x) = o(g(x))$.
$f(x) = \Omega_+(g(x))$	Means that there exists a suitable constant $c > 0$ such that $f(x) > cg(x)$ holds for a sequence $x = x_n$ with $lim_{n \to \infty} x_n = \infty$.
$f(x) = \Omega_-(g(x))$	Means that there exists a suitable constant $c > 0$ such that $f(x) < -cg(x)$ holds for a sequence $x = x_n$ with $lim_{n \to \infty} x_n = \infty$.
$f(x) = \Omega_\pm(g(x))$	Means that both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ hold.
$f(x) \approx g(x)$	Means that both $f(x) = O(g(x))$ and $g(x) = O(f(x))$ hold. ∎

GENERAL INTRODUCTION

Many arithmetical functions, that is, complex valued functions defined on the set of positive integers, fluctuate wildly as the argument increases. Nevertheless, in many interesting cases the functions behave rather nicely in average, that is,the sum over the arguments upto x has nice asymptototic behaviour as x tends to ∞. The next question is to determine the 'order' of the error term.Thus the famous 'circle problem' and 'Dirichlet divisor problem' are problems related to the error terms of the sums $\sum_{n\leq x} r_2(n)$ and $\sum_{n\leq x} d(n)$ respectively. In this thesis,we have proved some Ω -theorems corresponding to the averages of $\delta_k(n)$,$r_4(n)$ and coefficients of some cusp forms.The history of each problem and the results obtained therein are given in the introduction to the corresponding chapter. ∎

CHAPTER-1

On Ω-constants for the error term of $\sum_{n\leq x}\delta_k(n)$.

1.1 Introduction

Let k be a natural number.

We define an arithmetical function δ_k by

$$\delta_k(n) = \max\{d \in I\!N : d \mid n, (d,k) = 1\}.$$

Clearly, $\delta_k(n) = \delta_{\alpha(k)}(n)$ where $\alpha(k)$ is the square-free core of k, that is, $\alpha(k)$ is the product of the distinct primes which appear in the factorisation of k.

Hence, without any loss of generality, we can assume k to be square-free.

From the definition, it is easy to see that δ_k is multiplicative.

We define $S_k(x)$ and the error term $E_k(x)$ by the formula :

$$S_k(x) = \sum_{n\leq x} \delta_k(x) = \frac{kx^2}{2\sigma(k)} + E_k(x) \qquad (1)$$

where $\sigma(k)$ is the sum of the divisors of k.

D. Suryanarayana [21] obtained the estimate:

$$E_k(x) = O(x\log^2 x). \qquad (2)$$

Later, V. S. Joshi and A. M. Vaidya [16] proved

$$E_k(x) = O(x), \tag{3}$$

where the O-constant depends on k. For the special case $k = p$, Joshi and Vaidya proved that

$$\liminf_{x\to\infty} \frac{E_p(x)}{x} = -\frac{p}{p+1} \tag{4}$$

and

$$\limsup_{x\to\infty} \frac{E_p(x)}{x} = \frac{p}{p+1}. \tag{5}$$

Using the method of Erdös and Shapiro [10] of averaging over arithmetic progressions, we [3] have proved that, for any square-free k,

$$\liminf_{x\to\infty} \frac{E_k(x)}{x} \leq -\frac{k}{\sigma(k)} \tag{6}$$

and

$$\limsup_{x\to\infty} \frac{E_k(x)}{x} \geq \frac{k}{\sigma(k)}. \tag{7}$$

Around the same time, J. Herzog and T. Maxsein [14] also proved the results (6) and (7). They applied Tauberian theorem of Hardy- Littlewood and Karamata to get an asymptotic formula for $\sum_{n\leq x} g_k(n)$, where $g_k(n)$ is defined by the relation $\delta_k(n) = \sum_{d|n} g_k(d)\frac{n}{d}$.

Herzog and Maxsein [14] also observed that

$$\limsup_{x\to\infty} \frac{E_k(x)}{x} \leq \frac{d(k)}{2}, \tag{8}$$

where $d(k)$ denotes the number of divisors of k.

For $\liminf_{x\to\infty} \frac{E_k(x)}{x}$ they did not have any corresponding result.

In [2] , We have proved that:

$$\limsup_{x\to\infty} \frac{|E_k(x)|}{x} \leq \frac{1}{2}(1 - \frac{1}{p+1})d(k), \tag{9}$$

where p is the smallest prime dividing k.

From (9), it follows that

$$\limsup_{x\to\infty} \frac{E_k(x)}{x} \le \frac{1}{2}(1 - \frac{1}{p+1})d(k) \tag{10}$$

and

$$\liminf_{x\to\infty} \frac{E_k(x)}{x} \ge -\frac{1}{2}(1 - \frac{1}{p+1})d(k) \tag{11}$$

where p is as above.

Following Joshi and Vaidya [16] we give the proof of (3) in Theorem 1 of section 1.2. In section 1.2,we prove the statements (4) and (5) in Theorem 2 and the statement (9) in theorem 3. In Theorem 4 and Theorem 5 of section 1.3, we present our proofs [2] of the statements (6) and (7) respectively.

1.2 Proof of Theorems 1,2 and 3

Lemma 1

$$\begin{aligned} S_p(x) &= \frac{x^2 p}{2(p+1)} + x(-\{x\} + (1-\frac{1}{p})(\{\frac{x}{p}\} + \frac{1}{p}\{\frac{x}{p^2}\} \\ &\quad + \cdots + \frac{1}{p^\alpha}\{\frac{x}{p^{\alpha+1}}\})) + o(x), \end{aligned}$$

where α is the integer defined by $p^\alpha \le x < p^{\alpha+1}$.

Proof:

$$\begin{aligned} \sum_{n\le x} \delta_p(n) &= \sum_{\substack{n\le x \\ p|x}} \delta_p(n) + \sum_{\substack{n\le x \\ p\nmid n}} \delta_p(n) \\ &= \sum_{r\le \frac{x}{p}} \delta_p(pr) + \sum_{\substack{n\le x \\ p\nmid n}} n \end{aligned}$$

$$= \sum_{r \le \frac{x}{p}} \delta_p(r) + \frac{[x]^2 + [x]}{2} - \frac{p}{2}([\frac{x}{p}]^2 + [\frac{x}{p}]).$$

Hence,

$$S_p(x) - S_p(\frac{x}{p}) = \frac{x^2}{2}(1 - \frac{1}{p}) - x(\{x\} + \{\frac{x}{p}\}) + O(1). \qquad (12)$$

We go on replacing x by $\frac{x}{p}$ successively in (12),till we get

$$S_p(\frac{x}{p^\alpha}) - S_p(\frac{x}{p^{\alpha+1}}) = \frac{x^2}{2p^{2\alpha}}(1 - \frac{1}{p}) - \frac{x}{p^\alpha}(\{\frac{x}{p^\alpha}\} - \{\frac{x}{p^{\alpha+1}}\}) + O(1).$$

Adding all these $(\alpha + 1)$ relations, we have

$$S_p(x) = \frac{x^2}{2}(1 - \frac{1}{p})((1 - \frac{1}{p^2})^{-1} + O(\frac{1}{x^2}))$$

$$+x(-\{x\} + (1 - \frac{1}{p})(\{\tfrac{x}{p}\} + \frac{1}{p}\{\tfrac{x}{p^2}\} + \cdots + \frac{1}{p^\alpha}\{\tfrac{x}{p^{\alpha+1}}\})) + O(\log x)$$

$$= \frac{x^2 p}{2(p+1)} + x(-\{x\} + (1 - \frac{1}{p})(\{\tfrac{x}{p}\} + \frac{1}{p}\{\tfrac{x}{p^2}\} + \cdots + \frac{1}{p^\alpha}\{\tfrac{x}{p^{\alpha+1}}\})) + o(x).$$

Theorem 1

$$S_k(x) = \frac{kx^2}{2\sigma(k)} + O(x).$$

Proof: We proceed by induction. We assume that the result is true if k has r distinct prime-factors and then prove it for $m = kp_{r+1}$, where p_{r+1} is a prime not dividing k.

$$\begin{aligned} S_m(x) &= \sum_{n \le x} \delta_m(n) \\ &= \sum_{\substack{n \le x \\ p_{r+1} | n}} \delta_m(n) + \sum_{\substack{n \le x \\ p_{r+1} \nmid n}} \delta_m(n) \\ &= S_m(\frac{x}{p_{r+1}}) + \sum_{\substack{n \le x \\ p_{r+1} \nmid n}} \delta_k(n) \\ &= S_m(\frac{x}{p_{r+1}}) + S_k(x) - p_{r+1} S_k(\frac{x}{p_{r+1}}). \end{aligned}$$

Hence,

$$\begin{aligned} S_m(x) - S_m(\frac{x}{p_{r+1}}) &= S_k(x) - p_{r+1}S_k(\frac{x}{p_{r+1}}) \\ &= \frac{x^2k}{2\sigma(k)} - \frac{x^2k}{2p_{r+1}\sigma(k)} + O(x) \\ &= \frac{x^2k}{2\sigma(k)}(1 - \frac{1}{p_{r+1}}) + O(x). \end{aligned} \tag{13}$$

Now,replacing x by $\frac{x}{p_{r+1}}$ etc. and adding we have, as in the proof of Lemma 1,

$$\begin{aligned} S_m(x) &= \frac{x^2k}{2\sigma(k)}(\frac{p_{r+1}}{p_{r+1}+1}) + O(x) \\ &= \frac{x^2k}{2\sigma(m)} + O(x). \end{aligned}$$

This completes the proof of Theorem 1.

We observe that

$$E_k(x) - E_k([x]) = \frac{k}{2\sigma(k)}(-2[x]\{x\} - \{x\}^2),$$

which shows that $E_k(x)$ decreases continuously in any open interval $(m, m+1)$. Dividing by x,

$$\frac{E_k(x)}{x} = \frac{E_k([x])}{x} - \frac{k}{\sigma(k)}\{x\} + o(1). \tag{14}$$

Theorem 2 *For a prime p, we have*

$$\limsup_{x\to\infty} \frac{E_p(x)}{x} = \frac{p}{p+1}$$

and

$$\liminf_{x\to\infty} \frac{E_p(x)}{x} = -\frac{p}{p+1}.$$

Proof: From (14) it is clear that

$$\limsup_{x\to\infty} E_k(x) = \limsup_{n\to\infty} E_k(n).$$

So, we can assume that x runs on integers to prove the first equality.

Then

$$\{x\} = 0$$

and

$$\begin{aligned}
&(1-\frac{1}{p})\left(\{\tfrac{x}{p}\} + \frac{1}{p}\{\tfrac{x}{p^2}\} + \cdots + \frac{1}{p^\alpha}\{\tfrac{x}{p^{\alpha+1}}\}\right) \\
&\le (1-\frac{1}{p})\left(\frac{p-1}{p} + \frac{1}{p}(\frac{p^2-1}{p^2}) + \cdots + \frac{1}{p^\alpha}(\frac{p^{\alpha+1}-1}{p^{\alpha+1}})\right) \\
&\le (1-\frac{1}{p})\left(1 + \frac{1}{p^2}(\frac{p^2}{p^2-1})\right) + O(\frac{1}{x})
\end{aligned}$$

that is,

$$(1-\frac{1}{p})(\{\tfrac{x}{p}\} + \frac{1}{p}\{\tfrac{x}{p^2}\} + \cdots + \frac{1}{p^\alpha}\{\tfrac{x}{p^{\alpha+1}}\}) \le \frac{p}{p+1} + O(\frac{1}{x}). \qquad (15)$$

Also making x run over integers $-1(\bmod p^{\alpha+1})$, equality in (15) is obtained. Hence from Lemma 1,

$$\limsup_{x\to\infty} \frac{E_p(x)}{x} = \frac{p}{p+1}.$$

By similar reasoning, making x run over numbers θ (mod $p^{\alpha+1}$) where $0 < \theta < 1$ and observing that one can choose θ very close to 1, from Lemma 1 and equation(14) we have

$$\begin{aligned}
\liminf_{x\to\infty} \frac{E_p(x)}{x} &= -1 + (1-\frac{1}{p})(\frac{1}{p} + \frac{1}{p^3} + \cdots) \\
&= -1 + (1-\frac{1}{p}).\frac{1}{p}.\frac{p^2}{p^2-1} \\
&= -\frac{p}{p+1}.
\end{aligned}$$

Theorem 3

$$\limsup_{x\to\infty} \frac{| E_k(x) |}{x} \le \frac{1}{2}(1 - \frac{1}{p+1})d(k).$$

Proof: We arrange the primes dividing k in increasing order and proceed by induction. If k is a prime p, then $d(k)$=2 and the result is true by Theorem 2.

Let

$$\begin{aligned} k &= p_1 p_2 \cdots p_r \\ &= m p_r \text{ say,} \end{aligned}$$

where $p = p_1 < p_2 < \cdots < p_r$; p_i's are primes.

Now, as in the proof of Theorem 1,

$$\begin{aligned} S_k(x) - S_k(\frac{x}{p_r}) &= S_m(x) - p_r S_m(\frac{x}{p_r}) \\ &= \frac{x^2 m}{2\sigma(m)}(1 - \frac{1}{p_r}) + E_m(x) - p_r E_m(\frac{x}{p_r}). \end{aligned} \tag{16}$$

Let

$$p_r{}^{\alpha} \le x < p_r{}^{\alpha+1}.$$

We go on replacing x by $\frac{x}{p_r}$ successively in (16),till we get

$$S_k(\frac{x}{p_r{}^{\alpha}}) - S_k(\frac{x}{p_r{}^{\alpha}}) = \frac{x^2 m}{2\sigma(m)}(1 - \frac{1}{p_r})\frac{1}{p_r{}^{2\alpha}} + E_m(\frac{x}{p_r{}^{\alpha}}) - p_r E_m(\frac{x}{p_r{}^{\alpha+1}}).$$

Summing up,

$$S_k(x) = \frac{x^2 m}{2\sigma(m)}(\frac{p_r - 1}{p_r})(1 - \frac{1}{p_r^2})^{-1} + E_k(x) = \frac{x^2 k}{2\sigma(k)} + E_k(x),$$

where

$$E_k(x) = E_m(x) - (p_{r-1})(E_m(\frac{x}{p_r}) + E_m(\frac{x}{p_r{}^2}) + \cdots + E_m(\frac{x}{p_r{}^{\alpha+1}})) - E_m(\frac{x}{p_r{}^{\alpha+1}}) + o(x).$$

Hence

$$| E_k(x) | \leq \frac{x}{2}(1-\frac{1}{p+1})d(m)(1+(p_r-1)(\frac{1}{p_r}+\frac{1}{p_r^2}+\cdots+\frac{1}{p_r{}^{\alpha+1}}))+o(x)$$

by the induction hypothesis.

Therefore,

$$\begin{aligned}\frac{| E_k(x) |}{x} &\leq \frac{1}{2}(1-\frac{1}{p+1})d(m)(1+\frac{(p_r-1)}{p_r}(1-\frac{1}{p_r})^{-1})+o(1)\\ &= \frac{1}{2}(1-\frac{1}{p+1})d(m)2+o(1)\\ &= \frac{1}{2}(1-\frac{1}{p+1})d(k)+o(1)\end{aligned}$$

and hence the theorem.

1.3 Proof of Theorems 4 and 5

Throughout this section, k will be a fixed square-free integer and we shall write $\delta(n)$ and $E(x)$ instead of $\delta_k(n)$ and $E_k(x)$ respectively.

Let

$$H(x)=\sum_{n\leq x}\frac{\delta(n)}{n}-\frac{xk}{\sigma(k)}. \tag{17}$$

Writing $s=\sigma+it$, for $\sigma>2$,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\delta(n)}{n^s} &= \prod_p(1+\frac{\delta(p)}{p^s}+\frac{\delta(p^2)}{p^{2s}}+\cdots)\\ &= \prod_{p|k}(1-\frac{1}{p^s})^{-1}\prod_{p\nmid k}(1-\frac{1}{p^{s-1}})^{-1}\end{aligned}$$

(Since, $\delta(p^m)=1$, if $p \mid k$

$=p^m$, otherwise)

$$= \prod_{p|k}((1-\frac{1}{p^s})^{-1}(1-\frac{1}{p^{s-1}}))\zeta(s-1)$$

$$= (\sum_{n=1}^{\infty}\frac{g(n)}{n^s})\zeta(s-1), \text{ say.} \quad (18)$$

We have:

$$\begin{aligned} g(p^m) &= 1-p \quad \text{if } p \mid k, \\ &= 0 \quad \text{otherwise.} \end{aligned} \quad (19)$$

and

$$\delta(n) = \sum_{ab=n} ag(b) \quad (20)$$

We observe that

$$| g(n) | \leq \prod_{i=1}^{r}(p_i - 1) \quad (21)$$

where $k = p_1 \cdots p_r$ is the factorisation of k into primes, and

$$\sum_{n\leq x} | g(n) | = O((\log x)^r) = o(x). \quad (22)$$

We also observe that

$$\sum_{n=1}^{\infty}\frac{g(n)}{n^2} = \prod_{p|k}(\frac{p}{p+1}) = \frac{k}{\sigma(k)}, \quad (23)$$

$$\sum_{n=1}^{\infty}\frac{g(n)}{n} = 0, \quad (24)$$

$$\sum_{n=1}^{\infty}\frac{| g(n) |}{n} = O(1) \quad (25)$$

and

$$\left| \sum_{n>x}\frac{g(n)}{n^2} \right| \leq \left(\sum_{m=1}^{\infty} \sum_{2^{m-1}x\leq n\leq 2^m x} \frac{|g(n)|}{n^2} \right) \leq \sum_{m=1}^{\infty}\frac{\epsilon(2^m - 2^{m-1})x}{2^{2m-2}x^2},$$

ϵ being a small positive number which can be taken smaller and smaller tending to 0 as $x \to \infty$, which implies

$$\sum_{n>x} \frac{g(n)}{n^2} = o\left(\frac{1}{x}\right). \tag{26}$$

Lemma 2 *We have*

$$H(x) = -\sum_{b \le x} \frac{g(b)}{b}\{\frac{x}{b}\} + o\,(1) = o\,(x).$$

Proof: From (17) and (20),we have

$$\begin{aligned} H(x) &= \sum_{ab \le x} \frac{ag(b)}{ab} - \frac{xk}{\sigma(k)} \\ &= \sum_{b \le x} \frac{g(b)}{b}[\frac{x}{b}] - \frac{xk}{\sigma(k)} \\ &= x\sum_{b \le x} \frac{g(b)}{b^2} - \sum_{b \le x} \frac{g(b)}{b}\{\frac{x}{b}\} - \frac{xk}{\sigma(k)} \\ &= -\sum_{b \le x} \frac{g(b)}{b}\{\frac{x}{b}\} + o(1) \\ &\qquad \text{(by(23)and(26))} \\ &= O(\log x) \quad (\text{ by}(21)) \\ &= o(x). \end{aligned}$$

Lemma 3 *We have :* $E(x) = xH(x) + o(x)$.

Proof: From (1) and (17), we have

$$E(x) = \sum_{b \le x} g(b) \sum_{a \le \frac{x}{b}} a - \frac{1}{2}\frac{x^2 k}{\sigma(k)}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{b\le x} g(b)([\frac{x}{b}]^2 + [\frac{x}{b}]) - \frac{1}{2}\frac{x^2 k}{\sigma(k)} \\
&= \frac{1}{2}\sum_{b\le x} g(b)(\frac{x^2}{b^2} - 2\frac{x}{b}\{\frac{x}{b}\} + \{\frac{x}{b}\}^2) \\
&\qquad + \frac{1}{2}\sum_{b\le x} g(b)(\frac{x}{b} - \{\frac{x}{b}\}) - \frac{1}{2}\frac{x^2 k}{\sigma(k)} \\
&= -x\sum_{b\le x}\frac{g(b)}{b}\{\frac{x}{b}\} + o(x)
\end{aligned}
$$

(by (22), (23) and (24)).

Now the lemma follows from Lemma 2.

Lemma 4 *If x is an integer, then*

$$
\sum_{n\le x} H(n) = (x+1)H(x) - E(x) + \frac{kx}{2\sigma(k)} = \frac{kx}{2\sigma(k)} + o(x).
$$

Proof: From (17),

$$
\begin{aligned}
\sum_{n\le x} H(n) &= \sum_{n\le x}(\sum_{m\le n}\frac{\delta(m)}{m} - \frac{nk}{\sigma(k)}) \\
&= \sum_{n\le x}(x+1-n)\frac{\delta(n)}{n} - \frac{k}{\sigma(k)}\frac{x(x+1)}{2} \\
&= (x+1)\sum_{n\le x}\frac{\delta(n)}{n} - \sum_{n\le x}\delta(n) - \frac{k}{2\sigma(k)}x(x+1) \\
&= (x+1)\left(H(x) + \frac{xk}{\sigma(k)}\right) - \left(E(x) + \frac{kx^2}{2\sigma(k)}\right) - \frac{k}{2\sigma(k)}x(x+1) \\
&= (x+1)H(x) - E(x) + \frac{kx}{2\sigma(k)} \\
&= H(x) + \frac{kx}{2\sigma(k)} + o(x) \text{ (by Lemma 3)}
\end{aligned}
$$

$$= \frac{kx}{2\sigma(k)} + o(x) \text{ (by Lemma 2).}$$

Lemma 5 *Let* $A = 2P^n$, $n \geq 2$ *where*

$$\begin{aligned} P &= 2 \quad if \quad 2 \mid k, \\ &= \quad an \ odd \ prime \ which \ divides \ k, if \quad 2 \mid k. \end{aligned}$$

If β *is an integer with* $0 < \beta < A$, *then*

$$\sum_{\substack{m \leq z \\ m \equiv \beta(A)}} \frac{\delta(m)}{m} = \frac{z}{A} \frac{k}{\sigma(k)} (\frac{1+P}{P}) \sum_{\substack{t \mid (A,\beta) \\ P^n \mid t}} \frac{g(t)}{t} + \frac{z}{A} O(\frac{1}{P^n}) + O(\log z).$$

Proof:

$$\begin{aligned} \sum_{\substack{m \leq z \\ m \equiv \beta(A)}} \frac{\delta(m)}{m} &= \sum_{\substack{m \leq z \\ m \equiv \beta(A)}} \sum_{d \mid m} \frac{g(d)}{d} \\ &= \sum_{d \leq z} \frac{g(d)}{d} \sum_{\substack{m \leq z \\ m \equiv \beta(A) \\ m \equiv 0(d)}} 1 \\ &= \sum_{\substack{d \leq z \\ (d,A) \mid \beta}} \frac{g(d)}{d} (\frac{z}{[d,A]} + O(1)) \end{aligned}$$

(Since the congruences $m \equiv \beta(A)$ and $m \equiv 0(d)$ are simultaniously solvable iff $(d, A) \mid \beta$,where (d, A) is the g.c.d. of d and A,and in that case, there is a unique solution modulo [d,A] , the l.c.m. of d and A.)

Therefore,

$$\sum_{\substack{m \leq z \\ m \equiv \beta(A)}} \frac{\delta(m)}{m}$$

$$= \frac{z}{A} \sum_{\substack{d \le z \\ (d,A)|\beta}} \frac{g(d)}{d^2}(d, A) + O(\sum_{d \le z} \frac{g(d)}{d})$$

$$= \frac{z}{A} \sum_{t|(A,\beta} t \sum_{\substack{d \le z \\ (d,A)=t}} \frac{g(d)}{d^2} + O(\log z)$$

$$= \frac{z}{A} \sum_{t|(A,\beta)} \frac{1}{t} \sum_{\substack{m \le \frac{z}{t} \\ (m,\frac{A}{t})=1}} \frac{g(mt)}{m^2} + O(\log z)$$

$$= \frac{z}{A} \sum_{\substack{t|(A,\beta) \\ P^n \nmid t}} \frac{g(t)}{t} \sum_{\substack{m \le \frac{z}{t} \\ (m,P)=1}} \frac{g(m)}{m^2} + O(\frac{1}{P^n}) \frac{z}{a} + O(\log z)$$

(Since the first term corresponds to $(m,t) = 1$)

$$= \frac{z}{A} \sum_{\substack{t|(A,\beta) \\ P^n \nmid t}} \frac{g(t)}{t} \left(\sum_{\substack{m=1 \\ (m,P)=1}}^{\infty} \frac{g(m)}{m^2} - \sum_{\substack{m \ge \frac{z}{t} \\ (m,P)=1}} \frac{g(m)}{m^2} \right) + O(\frac{1}{P^n}) \frac{z}{a} + O(\log z)$$

$$= \frac{z}{A} \frac{k}{\sigma(k)} (\frac{1+P}{P}) \sum_{\substack{t|(A,\beta) \\ P^n \nmid t}} \frac{g(t)}{t} + O(\frac{1}{P^n}) \frac{z}{a} + O(\log z)$$

(Since, by putting $X = \frac{z}{t}$,we have

$$\sum_{\substack{m \le \frac{z}{t} \\ (m,P)=1}} \frac{g(m)}{m^2} = \sum_{\substack{m=1 \\ (m,P)=1}}^{\infty} \frac{g(m)}{m^2} - \sum_{\substack{m \ge X \\ (m,P)=1}} \frac{g(m)}{m^2}$$

$$= (1 - \frac{1}{P^2})(1 - \frac{1}{P})^{-1} (\sum_{n=1}^{\infty} \frac{g(n)}{n^2}) + O(\sum_{n=1}^{\infty} \sum_{X2^{n-1} < m < x2^n} \frac{|g(m)|}{m^2})$$

$$= \frac{k}{\sigma(k)} (\frac{1+P}{P}) + O(\sum_n \frac{(\log X2^n)^r}{2^{2n} X^2})$$

(by (18), (22) and (23), observing that $P \mid k$)

and

$$\sum_n \frac{(\log X2^n)^r}{2^{2n}X^2} = O(\sum_{n=1}^{\infty} \frac{(\log X + n\log 2)^r}{2^{2n}X^2})$$
$$= O(\sum_{n>\log X} \frac{n^r}{2^{2n}X^2} + \sum_{n\le \log X} \frac{\log X)^r}{2^{2n}X^2})$$
$$= O(\frac{(\log X)^r}{X^2}) \quad).$$

Lemma 6 *If, A is as in Lemma 5, then, for* $0 < B < A$ *and integral* x*, we have:*

$$\sum_{l\le x} H(Al - B)$$

$$= \frac{xk}{\sigma(k)} \left(B - \frac{A}{2} + (1+\frac{1}{P}) \sum_{\substack{t\|(A,B) \\ P^n \nmid t}} \frac{g(t)}{t} - (1+\frac{1}{P})\frac{1}{A}\sum_{a=0}^{A-1}(B-a) \sum_{\substack{t\|(A,a-B) \\ P^n \nmid t}} \frac{g(t)}{t} \right)$$
$$+O(\frac{1}{Pn})\ \frac{x}{A}\sum_{a=0}^{A-1}(B-a) + o(x).$$

Proof:

$$\sum_{l\le x} H(Al - B)$$

$$= \sum_{l\le x}\left(\sum_{m\le Al-B} \frac{\delta(m)}{m} - \frac{(Al-B)k}{\sigma(k)}\right)$$
$$= \sum_{m\le Ax-B} \frac{\delta(m)}{m}(x - [\frac{m+B}{A}]) + \sum_{\substack{m\le Ax-B \\ m\equiv -B(A)}} \frac{\delta(m)}{m} - \sum_{l\le x}(Al-B)\frac{k}{\sigma(k)}. \tag{27}$$

Now,

$$\sum_{m\le Ax-B}\frac{\delta(m)}{m}(x-[\frac{m+B}{A}])$$

$$=\left(x\sum_{m\le Ax-B}\frac{\delta(m)}{m}-\frac{1}{A}\sum_{m\le Ax-B}\delta(m)\right)-\sum_{a=0}^{A-1}\frac{B-a}{A}\sum_{\substack{m\le Ax-B\\ m+B\equiv a(A)}}\frac{\delta(m)}{m}$$

$$=\frac{1}{A}\left((Ax-B+1)\left(\sum_{m\le Ax-B}\frac{\delta(m)}{m}-\frac{(Ax-B)k}{\sigma(k)}\right)\right.$$

$$\left.-\left(\sum_{m\le Ax-B}\delta(m)-\frac{1}{2}(Ax-B)^2\frac{k}{\sigma(k)}\right)+\frac{(Ax-B)k}{2\sigma(k)}\right)$$

$$+\frac{B-1}{A}\sum_{m\le Ax-B}\frac{\delta(m)}{m}+\frac{1}{2A}((Ax-B)+(Ax-B)^2)\frac{k}{\sigma(k)}$$

$$-\sum_{a=0}^{A-1}\frac{B-a}{A}\sum_{\substack{m\le Ax-B\\ m+B\equiv a(A)}}\frac{\delta(m)}{m}$$

$$=\frac{1}{A}\left((Ax-B+1)H(Ax-B)-E(Ax-B)+\frac{(Ax-B)k}{2\sigma(k)}\right)$$

$$+\frac{B-1}{A}\left(\sum_{m\le Ax-B}\frac{\delta(m)}{m}-\frac{(Ax-B)k}{2\sigma(k)}\right)+\frac{k}{2A\sigma(k)}(A^2x^2-B^2-Ax+B)$$

$$-\sum_{a=0}^{A-1}\frac{B-a}{A}\sum_{\substack{m\le Ax-B\\ m+B\equiv a(A)}}\frac{\delta(m)}{m}$$

$$=\frac{1}{A}\sum_{m\le Ax-B}H(m)+\frac{Ax^2k}{2\sigma(k)}-\frac{xk}{2\sigma(k)}-\sum_{a=0}^{A-1}\frac{B-a}{A}\sum_{\substack{m\le Ax-B\\ m+B\equiv a(A)}}\frac{\delta(m)}{m}+o(x)$$

(By Lemma 4 and Lemma 2).

(28)

By Lemma 5,

$$\sum_{a=0}^{A-1}\frac{B-a}{A}\sum_{\substack{m\le Ax-B\\ m+B\equiv a(A)}}\frac{\delta(m)}{m}$$

$$= \sum_{a=0}^{A-1}\frac{B-a}{A}\Big(\frac{Ax-B}{A}\ \frac{k}{\sigma(k)}(1+\frac{1}{P})\sum_{\substack{t|(A,a-B)\\ P^n\ |\ t}}\frac{g(t)}{t}$$

$$+O(\frac{1}{P^n})\ \frac{(Ax-B)}{a}+O(\log x)\ \Big)$$

$$= (1+\frac{1}{P})\frac{xk}{\sigma(k)}\ \frac{1}{A}\sum_{a=0}^{A-1}(B-a)\sum_{\substack{t|(A,a-B)\\ P^n\ \nmid\ t}}\frac{g(t)}{t}$$

$$+O(\frac{1}{P^n})\ \frac{x}{A}\sum_{a=0}^{A-1}(B-a)+O(\log x)$$

(29)

$$\sum_{\substack{m\le Ax-B\\ m\equiv -B(A)}}\frac{\delta(m)}{m}$$

$$= \frac{(Ax-B)k}{A\sigma(k)}(1+\frac{1}{P})\sum_{\substack{t\ |\ (A,B)\\ P^n\ \nmid\ t}}\frac{g(t)}{t}+O(\frac{1}{P^n})\ \frac{Ax-B}{A}+O(\log x).$$

(30)

From (27),(28),(29) and (30), we get the lemma.

Theorem 4 *We have the following inequality :*

$$\limsup_{x\to\infty}\frac{E(x)}{x}\ge\frac{k}{\sigma(k)}.$$

Proof: In Lemma 6 we take $B = P^n + 1$.

Then,

$$B - \frac{A}{2} + (1+\frac{1}{P}) \sum_{\substack{t|(A,B) \\ P^n \nmid t}} \frac{g(t)}{t} - (1+\frac{1}{P})\frac{1}{A}\sum_{a=0}^{A-1}(B-a) \sum_{\substack{t|(A,a-B) \\ P^n \nmid t}} \frac{g(t)}{t}$$
$$= 1 + (1+\frac{1}{P}) - (1+\frac{1}{P})\frac{1}{2P^n}(\sum_{r=1}^{P^n+1} r - \sum_{r=1}^{P^n-2} r) \sum_{\substack{t|(A,r) \\ P^n \nmid t}} \frac{g(t)}{t} \tag{31}$$

(Since, as a varies from 0 to A-1,$B - a$ takes on the values:
$P^n + 1, P^n, \cdots, -(P^n - 2)$).

Now,

$$(1+\frac{1}{P})\frac{1}{2P^n}(\sum_{r=1}^{P^n+1} r - \sum_{r=1}^{P^n-2} r) \sum_{\substack{t|(A,r) \\ P^n \nmid t}} \frac{g(t)}{t}$$
$$= (1+\frac{1}{P})\frac{1}{2P^n}\left((P^n+1)\,1 + P^n \sum_{t|P^{n-1}} \frac{g(t)}{t} + (P^n - 1)\right)$$
$$= (1+\frac{1}{P})\frac{1}{2P^n}(2P^n + P^n \sum_{t|P^{n-1}} \frac{g(t)}{t})$$
$$= (1+\frac{1}{P})\frac{1}{2}\left(2 + (1 + (1-P)(\frac{1}{P} + \frac{1}{P^2} + \cdots + \frac{1}{P^{n-1}}))\right)$$
$$= (1+\frac{1}{P}) + \frac{1}{2}(1+\frac{1}{P})(1+\frac{1}{P})\frac{1}{P^{n-1}}$$
$$= 1 + \frac{1}{P} + O(\frac{1}{P^{n-1}}). \tag{32}$$

Again,

$$\sum_{a=0}^{A-1}(B-a) = B + (B-1) + \cdots + (B - A + 1)$$

$$= (P^n + 1) + P^n + \cdots + (2 - P^n)$$
$$= 3P^n.$$

(33)

Therefore, from Lemma 6 with $B = P^n + 1$, we have by (31),(32) and (33)

$$\sum_{l \leq x}(Al - B) = \frac{xk}{\sigma(k)}(1 + O(\frac{1}{P^{n-1}})) + O(\frac{1}{P^n})x + o(x).$$

This is true for all n, n -arbitrarily large.

Therefore,

$$\limsup_{n\to\infty} H(n) \geq \frac{k}{\sigma(k)}.$$

This alongwith Lemma 3 will imply

$$\limsup_{n\to\infty} \frac{E(n)}{n} \geq \frac{k}{\sigma(k)}.$$

Hence the theorem.

Theorem 5 *We have the following inequality:*

$$\liminf_{x\to\infty} \frac{E(x)}{x} \leq -\frac{k}{\sigma(k)}.$$

Proof: By choosing $B = P^n$ in Lemma 6, it is easy to see that:

$$\liminf_{n\to\infty} H(n) \leq 0.$$

So, by Lemma 3,

$$\liminf_{n\to\infty} \frac{E(n)}{n} \leq 0. \tag{34}$$

Now,

$$E(x) - E([x]) = \frac{k}{2\sigma(k)}(-2[x]\{x\} - \{x\}^2)$$

which shows that $E(x)$ decreases continuously in the interval $(m, m+1)$.

Now,

$$\begin{aligned}\frac{E(x)}{x} &= \frac{E([x])}{[x]} - \frac{k}{\sigma(k)}\frac{[x]}{x}\{x\} - \frac{k}{2x\sigma(k)}\{x\}^2 \\ &= \frac{E([x])}{[x]} - \frac{k}{\sigma(k)}\{x\} + o(1). \qquad (35)\end{aligned}$$

In an open interval $(m, m+1)$, as $x \to m+1, \{x\} \to 1$.

Therefore, (34) and (35) give

$$\liminf_{x\to\infty} \frac{E(x)}{x} \leq -\frac{k}{\sigma(k)}.$$

■

Ch. 2 of Adhikari's thesis contains
1) Hardy-Ramanujan 12, 20-30 (1989)
&
2) Acta Arith, 59 329-338 (1991)

CHAPTER-2

An Ω-result related to $r_4(n)$

2.1 Introduction

Let $r_k(n)$ denote the number of representations of the positive integer n as a sum of k squares.

Let $P_k(x)$ be the error term defined by

$$\sum_{n\leq x} r_k(n) = \frac{(\pi x)^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)} + P_k(x). \tag{36}$$

Szego [22] showed that, if $k = 2,3,4(mod8)$, then

$$P_k(x) = \Omega_-((x\log x)^{\frac{k-1}{4}}) \tag{37}$$

and if $k = 6,7,0(mod8)$, then

$$P_k(x) = \Omega_+((x\log x)^{\frac{k-1}{4}}). \tag{38}$$

For the particular case $k = 2$, the result had been proved by Hardy [12]. The best Ω_+ and Ω_- results to date in this case are due to Corradi and Katai [6] and Hafner [11] respectively.

They are

$$P_2(x) = \Omega_+(x^{\frac{1}{4}} \exp\ (c(\log\ \log x)^{\frac{1}{4}}(\log\ \log\ \log x)^{\frac{-3}{4}})) \tag{39}$$

and

$$P_2(x) = \Omega_-((x\log x)^{\frac{1}{4}}(\log\ \log x)^{\frac{\log 2}{4}} \exp\ (-B(\log\ \log\ \log x)^{\frac{-3}{4}})). \tag{40}$$

For the case $k = 4$, we [4] have proved the following :

$$P_4(x) = \Omega_+(x \log \log x). \tag{41}$$

Our proof of the above was inspired by a paper of Montgomery [18],which deals with the error term $R(x) = \sum_{n \leq x} \phi(n)$, where $\phi(n)$ is the Euler totient function; Montgomery has proved $R(x) = \Omega_{\pm}(x\sqrt{\log \log x})$, thereby improving the earlier result $R(n) = \Omega\pm(x \log \log \log \log x)$ of Erdös and Shapiro [10].

2.2 Some Notations

Let

$$f(n) = \sum_{d|\ n} h(d) \tag{42}$$

where h is a multiplicative function such that $\sum_{d=1}^{\infty} h(d)$ is convergent and $h(d) = O(\frac{1}{d})$. Let

$$M_0(x) = x\sum_{d=1}^{\infty} \frac{h(d)}{d}, \tag{43}$$

$$M_1(x) = \frac{x^2}{2}\sum_{d=1}^{\infty} \frac{h(d)}{d}, \tag{44}$$

$$R_0(x) = \sum_{n \leq x} f(n) - M_0(x) \tag{45}$$

and

$$R_1(x) = \sum_{n \leq x} nf(n) - M_1(x). \tag{46}$$

2.3 Some lemmas

Lemma 7

$$R_0(x) = -x \sum_{d > x} \frac{h(d)}{d} - \sum_{d \le x} h(d)\{\frac{x}{d}\}.$$

proof:

$$\begin{aligned} R_0(x) &= \sum_{n \le x} f(n) - M_0(x) \\ &= \sum_{n \le x} \sum_{d|n} h(d) - M_0(x) \\ &= \sum_{d \le x} h(d)(\frac{x}{d} - \{\frac{x}{d}\}) - M_0(x) \\ &= -x \sum_{d > x} \frac{h(d)}{d} - \sum_{d \le x} h(d)\{\frac{x}{d}\} \end{aligned}$$

(by the definition of $M_0(x)$).

Lemma 8 *If b, r (> 0) are integers such that $(b, r) = 1$ and β is a real number, then*

$$\sum_{n=1}^{r} \{\frac{bn}{r} + \beta\} = \frac{r-1}{2} + \{r\beta\}.$$

.

Proof: Both sides are periodic with period $\frac{1}{r}$.

So,we can assume that $b = 1$ and $0 \le \beta < \frac{1}{r}$.

If $\beta = 0$,then the left hand side is

$$\sum_{n=1}^{r-1} \frac{n}{r} = \frac{r-1}{2}.$$

If $0 < \beta < \frac{1}{r}$,then

$$\begin{aligned}\sum_{n=1}^{r}\{\frac{n}{r}+\beta\} &= \frac{1}{r}\frac{r(r-1)}{2}+r\beta \\ &= \frac{r-1}{2}+\{r\beta\}.\end{aligned}$$

Lemma 9 *With notations as in Lemma 8, for any positive integer N ,*

$$\sum_{n=1}^{N}\{\frac{nb}{r}+\beta\} = \frac{N}{r}\{r\beta\}+\frac{N}{2}(\frac{r-1}{r})+O(r).$$

Proof: $N = Qr + R$ for some $0 \leq R < r$.

$$\begin{aligned}\sum_{n=1}^{N}\{\frac{nb}{r}+\beta\} &= q\{r\beta\}+q\ (\frac{r-1}{2})+\sum_{n=1}^{R}\{\frac{n}{r}+\beta\} \\ &\qquad (\text{ by Lemma 8}) \\ &= \frac{N}{r}\{r\beta\}+\frac{N}{2}(\frac{r-1}{r})+O(r).\end{aligned}$$

Lemma 10

$$\frac{R_1(x)}{x} - R_0(x) = \frac{x}{2}\sum_{d>x}\frac{h(d)}{d}+\frac{1}{2}\sum_{d\leq x}h(d)-\frac{1}{2x}\sum_{d\leq x}h(d)\ d(\{\frac{x}{d}\}-\{\frac{x}{d}\}^2).$$

Proof:

$$\begin{aligned}R_1(x) &= \sum_{n\leq x} nf(n) - M_1(x) \\ &= \sum_{n\leq x} n \sum_{n=d_1d_2} h(d_1) - M_1(x) \\ &= \sum_{d_1d_2\leq x} h(d_1)d_1d_2 - M_1(x) \\ &= \sum_{d_1\leq x} h(d_1)d_1 \sum_{d_2\leq[\frac{x}{d_1}]} d_2 - M_1(x)\end{aligned}$$

$$= \frac{1}{2}\sum_{d_1\le x} h(d_1)d_1(\{\frac{x}{d}\} - \{\frac{x}{d}\})(\{\frac{x}{d}\} + 1 - \{\frac{x}{d}\}) - M_1(x)$$

$$= \frac{1}{2}\sum_{d\le x} h(d)d(\frac{x^2}{d^2} + \frac{x}{d} - 2\{\frac{x}{d}\}\frac{x}{d} - \{\frac{x}{d}\} + \{\frac{x}{d}\}^2) - M_1(x)$$

$$= -\frac{x^2}{2}\sum_{d\ge x}\frac{h(d)}{d} + \frac{x}{2}\sum_{d\le x} h(d)$$

$$-x\sum_{d\le x} h(d)\{\frac{x}{d}\} - \frac{1}{2}\sum_{d\le x} h(d)d\{\frac{x}{d}\} + \frac{1}{2}\sum_{d\le x} h(d)d\{\frac{x}{d}\}^2.$$

But,by Lemma 7,

$$-\sum_{d\le x} h(d)\{\frac{x}{d}\} = R_0(x) + x\sum_{d\ge x}\frac{h(d)}{d}.$$

Hence,

$$\frac{R_1(x)}{x} - R_0(x) = \frac{x}{2}\sum_{d>x}\frac{h(d)}{d} + \frac{1}{2}\sum_{d\le x} h(d) - \frac{1}{2x}\sum_{d\le x} h(d)\ d(\{\frac{x}{d}\} - \{\frac{x}{d}\}^2).$$

This completes the proof.

Now, from Lemma 7,

$$R_0(x) = -\sum_{d=1}^{\infty} h(d)\{\frac{x}{d}\}.$$

We assume that there exists some $G(x)$, which is an increasing function of x, such that $\frac{x}{G(x)}$ is increasing and

$$\sum_{d>y} h(d)\{\frac{x}{d}\} = O(1) \quad \text{for } y \ge \frac{x}{G(x)} \tag{47}$$

so that

$$R_0(x) = -\sum_{d\le y} h(d)\{\frac{x}{d}\} + O(1) \quad \text{for } y \ge \frac{x}{G(x)} \tag{48}$$

Of course, such a $G(x)$ may not exist for an arithmetical function $f(n)$.

Lemma 11 *For integers $q \approx G(N)$ with $G(n)$ as above, $\beta \le q$ and* $y = \frac{(N+1)q}{G(N)}$,

$$\sum_{n=1}^{N} R_0(nq+\beta) = N \sum_{\substack{e \le y \\ p|e \Rightarrow p|q}} h(e)\frac{(e,q)}{e}\left(\frac{1}{2} - \left\{\frac{\beta}{(e,q)}\right\}\right) \sum_{\substack{l \le \frac{y}{e} \\ (l,q)=1}} \frac{h(l)}{l} + O(N).$$

Proof:

$$\sum_{n=1}^{N} R_0(nq+\beta)$$

$$= -\sum_{n=1}^{N}\sum_{d \le y} h(d)\left\{\frac{nq+\beta}{d}\right\} + O(N)$$

(By (48); because, $y = \frac{(N+1)q}{G(N)} > \frac{(N+1)q}{G((N+1)q)}$ (since, $G(x)$ is increasing) $> \frac{nq+\beta}{G(nq+\beta)}$ (since, $\frac{x}{G(x)}$ is increasing))

$$= -\sum_{d \le y} h(d) \sum_{n=1}^{N}\left\{\frac{nq+\beta}{d}\right\} + O(N)$$

$$= -\sum_{d \le y} h(d) \sum_{n=1}^{N}\left\{\frac{q/(d,q)}{d/(d,q)}.n + \frac{\beta}{d}\right\} + O(N)$$

$$= -\sum_{d \le y} h(d)\left(\frac{N}{d}(d,q)\left\{\frac{\beta}{(d,q)}\right\} + \frac{N(d,q)}{d}\left(\frac{\frac{d}{(d,q)}-1}{2}\right) + O\left(\frac{d}{(d,q)}\right)\right) + O(N)$$

(by Lemma 9)

$$= -\sum_{d \le y} h(d)\frac{N}{d}(d,q)\left(\left\{\frac{\beta}{(d,q)}\right\} - \frac{1}{2}\right) + O(N)$$

$$= -N \sum_{\substack{e \le y \\ p|e \Rightarrow p|q}} h(e)\frac{(e,q)}{e}\left(\left\{\frac{\beta}{(e,q)}\right\} - \frac{1}{2}\right) \sum_{\substack{l \le \frac{y}{e} \\ (l,q)=1}} \frac{h(l)}{l} + O(N)$$

(Writing $d = el$,where $p \mid e$ implies $p \mid q$ and $(l,q) = 1$.)

Hence the lemma.

2.4 Proof of Theorem 6

We shall prove the statement (41), in Theorem 6.

In equation (43),we put

$$h(d) = \frac{a(d)}{d}$$

where

$$\begin{aligned} a(d) &= -3, \qquad \text{if } 4 \mid d \\ &= 1, \qquad \text{otherwise.} \end{aligned}$$

Then

$$f(n) = \frac{r_4(n)}{8n}.$$

Therefore,

$$R_0(x) = \sum_{n \leq x} \frac{r_4(n)}{8n} - x \sum_{n=1}^{\infty} \frac{a(n)}{n^2} = \sum_{n \leq x} \frac{r_4(n)}{8n} - \frac{\pi^2}{8} x$$

and

$$\frac{P_4(x)}{8} = R_1(x) = \frac{1}{8} \sum_{n \leq x} r_4(n) - \frac{\pi^2}{16} x^2 \tag{49}$$

Throughout this section, the notations $R_0(x), R_1(x), h(n)$ and $f(n)$ will be used for the present particular case.

We observe that $\mid \sum_{n \leq x} a(n) \mid \leq 3$ for all x and hence $\sum \frac{a(n)}{n^s}$ is convergent in the half plane $\sigma > 0$.

Lemma 12 *We have*

$$\sum_{n \leq x} \frac{a(n)}{n} = 2 \log 2 + O(\frac{1}{x}).$$

Proof:

$$\begin{aligned}\sum_{n\le x}\frac{\alpha(n)}{n} &= -\sum_{4n\le x}\frac{3}{4n}+\sum_{n\le x}\frac{1}{n}-\sum_{4n\le x}\frac{1}{4n}\\ &= -\sum_{n\le x}\frac{1}{n}-\sum_{n\le[\frac{x}{4}]}\frac{1}{n}\\ &= 2\log 2+O(\frac{1}{x}).\end{aligned}$$

Lemma 13

$$\frac{R_1(x)}{x}-R_0(x)=O(1).$$

Proof: By Lemma 10,

$$\begin{aligned}\frac{R_1(x)}{x}-R_0(x) &= \frac{x}{2}\sum_{d>x}\frac{h(d)}{d}+\frac{1}{2}\sum_{d\le x}h(d)-\frac{1}{2x}\sum_{d\le x}h(d)\ d(\{\frac{x}{d}\}-\{\frac{x}{d}\}^2)\\ &= \frac{x}{2}\sum_{d>x}\frac{\alpha(d)}{d^2}+\frac{1}{2}\sum_{d\le x}\frac{\alpha(d)}{d}-\frac{1}{2x}\sum_{d\le x}\alpha(d)\ (\{\frac{x}{d}\}-\{\frac{x}{d}\}^2).\end{aligned}$$

Since $\sum_{d=1}^{\infty}\frac{\alpha(d)}{d}$ is convergent and $|\alpha(d)|\le 3$, the result is clear.

Lemma 14

$$R_0(x)=-\sum_{d\le y}\frac{\alpha(d)}{d}(\{\frac{x}{d}\})+O(1),$$

uniformly for $x\ge 2,\ y\ge\sqrt{x}$.

Proof: From Lemma 7,here

$$R_0(x)=-x\sum_{d>x}\frac{\alpha(d)}{d^2}-\sum_{d\le x}\frac{\alpha(d)}{d}\{\frac{x}{d}\}.$$

Now, $\sum_{d>x}\frac{\alpha(d)}{d^2}=O(\frac{1}{x})$.

Hence,we have only to show that:

$$\sum_{y\le d\le x}\frac{\alpha(d)}{d}\{\frac{x}{d}\}=O(1),$$

for $\sqrt{x} \leq y \leq x$.

We choose K such that $1 \leq K \leq \frac{x}{y}$.
In $\frac{x}{K} < d \leq \frac{x}{K-1}$, $\{\frac{x}{d}\}$ is monotone.
Hence

$$\sum_{\frac{x}{K} \leq d \leq \frac{x}{K-1}} \frac{\alpha(d)}{d}\{\frac{x}{d}\} = O(\frac{K}{x}) \qquad \text{(by Lemma 12)}$$

Summing up, over K such that

$$1 \leq K \leq \frac{x}{y},$$

$$\begin{aligned}\sum_{y \leq d \leq x} \frac{\alpha(d)}{d}\{\frac{x}{d}\} &= O(\frac{1}{x} \sum_{1 \leq K \leq \frac{x}{y}} K) \\ &= O(\frac{1}{x}\frac{x^2}{y^2}) \\ &= O(1) \qquad \text{(Since } y \geq \sqrt{x})\end{aligned}$$

Theorem 6

$$P_4(x) = \Omega_+(x \log \log x)$$

Proof: From Lemmas (11) and (14),

$$\sum_{n=1}^{N} R_0(nq+\beta) = N \sum_{\substack{e \leq y \\ p|e \Rightarrow p|q}} \frac{\alpha(e)}{e}\frac{(e,q)}{e}(\frac{1}{2} - \{\frac{\beta}{(e,q)}\}) \sum_{\substack{l \leq \frac{y}{e} \\ (l,q)=1}} \frac{\alpha(l)}{l^2} + O(N) \tag{50}$$

for $q \approx \sqrt{N}$, $\beta \leq q$ and $y = \frac{(N+1)q}{\sqrt{N}} = O(N)$.
Since,

$$\sum_{l \geq \frac{y}{e}} \frac{|\alpha(l)|}{l^2} = O(\sum_{l \geq \frac{y}{e}} \frac{1}{l^2}) = O(\frac{e}{y}),$$

we have,

$$N\sum_{e\leq y}\frac{|\alpha(e)|}{e}\frac{(e,q)}{e}\left(\frac{1}{2}-\left\{\frac{\beta}{(e,q)}\right\}\right)\sum_{l\geq\frac{y}{e}}\frac{|\alpha(l)|}{l^2}$$

$$\ll N\sum_{e\leq y}\frac{|\alpha(e)|}{e}\frac{(e,q)}{e}\left|\frac{e}{y}\right|$$

$$\ll \frac{N}{y}\sum_{e\leq y}\frac{(e,q)}{e}$$

$$= O(N).$$

Hence from (47),

$$\sum_{n=1}^{N} R_0(nq+\beta)$$

$$= N\left(\prod_{p\nmid q}\left(1+\frac{\alpha(p)}{p^2}+\frac{\alpha(p)}{p^4}+\cdots\right)\right)\sum_{\substack{e\leq y\\ p|e\Rightarrow p|q}}\frac{\alpha(e)}{e^2}(e,q)\left(\frac{1}{2}-\left\{\frac{\beta}{(e,q)}\right\}\right)+O(N) \tag{51}$$

Now,we assume that

$$q=\prod_{p\leq z}p,\quad \beta = \prod_{2<p\leq z}p = \frac{q}{2},\quad \text{where } z = [\frac{1}{2}\log N].$$

We observe that $q\approx\sqrt{N}$.

If $2\mid e$, then $(\frac{1}{2}-\{\frac{\beta}{(e,q)}\})=0$

and if $2 \mid e$, then $\alpha(e)=1$ and $(\frac{1}{2}-\{\frac{\beta}{(e,q)}\})=\frac{1}{2}$.

Therefore,

$$\sum_{\substack{e\leq y\\ p|e\Rightarrow p|q}}\frac{\alpha(e)}{e}\frac{(e,q)}{e}\left(\frac{1}{2}-\left\{\frac{\beta}{(e,q)}\right\}\right) = \sum_{\substack{e\leq y\\ p|e\Rightarrow p|q\\ 2\nmid e}}\frac{1}{e^2}(e,q).\frac{1}{2}$$

$$\geq \frac{1}{2}\sum_{\substack{e|q\\ 2\nmid e}}\frac{1}{e}$$

Therefore, from (48), as $N \to \infty$,

$$\sum_{n=1}^{N} R_0(nq+\beta) \geq N\frac{1}{2}\sum_{\substack{e|q \\ 2 \nmid e}} \frac{1}{e} + O(N)$$

where,

$$\sum_{\substack{e|q \\ 2 \nmid e}} \frac{1}{e} \approx \log z \approx \log \log N.$$

Therefore,

$$R_0(x) = \Omega_+(\log \log x)$$

and hence from Lemma 13,

$$R_1(x) = \Omega_+(x \log \log x)$$

Since, $P_4(x) = 8R_1(x)$ (see (46)),
this completes the proof of the theorem. ∎

CHAPTER-3

Ω - results for sums of Fourier coefficients of cusp forms.

3.1 Introduction

Let f be a normalized Hecke eigen form of weight k for the full modular group which is a cusp form and let $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ be its Fourier expansion at the cusp $i\infty$.

Hardy [13] and Rankin [20] showed

$$a(n) = \Omega(n^{\frac{k-1}{2}})$$

and

$$\limsup_{n\to\infty} \frac{|\, a(n)\, |}{n^{\frac{(k-1)}{2}}} = +\infty$$

respectively.

R.Balasubramanian and M.Ram Murty [5] proved:

$$a(n) = \Omega(n^{\frac{k-1}{2}} \exp\, (c(\log n)^{\frac{1}{k}-\epsilon})).$$

Later, for an arbitrary cusp form, which is not necessarily an eigen function, Ram murty [19] proved:

$$a(n) = \Omega(n^{\frac{k-1}{2}} \exp\, (\frac{c\log n}{\log\ \log n})),$$

which is the best possible in view of Deligne's result.

In the same paper [19], Ram Murty had conjectured that, if $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ is an arbitrary cusp form of weight k for the full modular group and $a(n) \in I\!R$, then

$$\sum_{p\leq x} a(p)p^{-\frac{k-1}{2}} = \Omega_{\pm}(\frac{x^{\frac{1}{2}} \log \log \log x}{\log x})$$

and has proved that for a normalized eigenform $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ of weight k for the full modular group, this is true, provided

$$L_f(s) = \prod_p (1 - \frac{e^{i\theta(p)}}{p^s})^{-1}(1 - \frac{e^{-i\theta(p)}}{p^s})^{-1}$$

has no real zero in $\frac{1}{2} \leq s \leq 1$.
Here $\theta(p)$ is given by

$$a(p) = 2p^{\frac{k-1}{2}} \cos(\theta(p)).$$

From the work of Deligne [7], we know that $\theta(p)$ is real, which gives $| a(p) | \leq 2p^{\frac{k-1}{2}}$.

We [1] have proved the following:

Theorem 7 *If $F(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$ $(c(n) \in I\!R)$ is a cusp form of integral weight k for $\Gamma_0(N)$ (for some integer $N \geq 1$) with real character, $F(z)$ is an eigen function of the Hecke operators and it does not vanish on $\{iy \mid 0 < y < \infty\}$, then*

$$\sum_{p\leq x} c(p)p^{-\frac{k-1}{2}} \log p = \Omega_{\pm}(x^{\frac{1}{2}} \log \log \log x) \tag{52}$$

When applied to the classical result:

$$\psi(x) - x = \Omega_{\pm}(x^{\frac{1}{2}} \log \log \log x), \tag{53}$$

where $\psi(x) = \sum_{p^m \leq x} \log p$, our proof provides a simpler proof for (53).

Now we give some examples of functions satisfying the conditions for (52) above.

Dummit, Kisilevsky and Mckay [9] have characterized the products of η - function whose Fourier coefficients are multiplicative.
They are:

$$\prod_{i=1}^{t} \eta(n_i z),$$

with

$$\sum_{i=1}^{t} n_i = 24,$$

where the corresponding partitions of 24 are given by :

$$(24), (8^3), (23, 1), (22, 2), (21, 3), (20, 4), (18, 6), (16, 8),$$
$$(12^2), (15, 5, 3, 1), (14, 7, 2, 1), (12, 6, 4, 2), (11^2, 1^2), (10^2, 2^2),$$
$$(9^2, 3^2), (8^2, 4^2), (6^4), (8^2, 4, 2, 1^2), (7^3, 1^3), (6^3, 2^3),$$
$$(4^6), (6^2, 3^2, 2^2, 1^2), (5^4, 1^4), (4^4, 2^4), (3^6, 1^6),$$
$$(3^8), (4^4, 2^2, 1^4), (2^{12}), (2^8, 1^8), (1^{24}).$$

If, $(n_1, \cdots, n_t)$ is one of the above partitions, then $\phi(z) = \prod_{i=1}^{t} \eta(n_i z)$ is a cusp from of weight k for $\Gamma_0(N)$ with real character, where $k = \frac{t}{2}$ and $N = (\min . \eta_i)\ (\max . \eta_i)$.

For weight ≥ 2, these functions are eigen functions of the Hecke operators. Also, since $\eta^{24}(z) = \Delta(z)$ does not vanish on the upper half plane,

$\phi(z)$ does not vanish there. In section 3.2, we prove some lemmas and in section 3.3, we give the proof of Theorem 7.

3.2 Some lemmas

Let $S_k(N,\chi)$ denote the space of the cusp forms of weight k for $\Gamma_0(N)$ with a real character χ.

Then, the map :

$$f \longmapsto f \mid \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}_k := N^{\frac{k}{2}}(Nz)^{-k} f(-\frac{1}{Nz})$$

is an isomorphism of the vectorspace $S_k(N,\chi)$.

Defining,

$$f^+ = \frac{1}{2}(f + i^k f \mid \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}_k)$$

and

$$f^- = \frac{1}{2}(f - i^k f \mid \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}_k),$$

we see that $f = f^+ + f^-$, where,

$$f \mid \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}_k = i^{-k} f^+$$

and

$$f \mid \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}_k = -i^{-k} f^- \tag{54}$$

Let $F(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi i n z}$ be as is in (52) .

Then,

$$\mid c(p) \mid \leq 2\, p^{\frac{k-1}{2}}$$

for primes p not dividing N.(Deligne [7] for $k \geq 2$, Deligne and Serre [8] for $k = 1$).

Let,

$$F^+(z) = \sum_{n=1}^{\infty} c_1(n) e^{2\pi i n z},$$

$$F^-(z) = \sum_{n=1}^{\infty} c_2(n) e^{2\pi i n z}$$

be the Fourier expansions of F^+ and F^- respectively.

Let

$$L(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

$$L_1(s) = \sum_{n=1}^{\infty} \frac{c_1(n)}{n^s}$$

and

$$L_2(s) = \sum_{n=1}^{\infty} \frac{c_2(n)}{n^s} \quad (\text{ Re } s > \frac{k}{2} + \frac{1}{2})$$

be the Dirichlet series corresponding to $F(z)$, $F^+(z)$, and $F^-(z)$ respectively.

Using (54), by standard methods (see e.g. Koblitz [17], page -140) one gets functional equations for $L_1(s)$ and $L_2(s)$, and hence the following lemma:

Lemma 15 *If*

$$V(s) = (\frac{\sqrt{N}}{2\pi})^s \Gamma(s)[L_1(s) + L_2(s)]$$

and

$$V^*(s) = (\frac{\sqrt{N}}{2\pi})^s \Gamma(s)[L_1(s) - L_2(s)],$$

then we have $V(s) = V^*(k-s)$.

Also, $(L_1(s) + L_2(s))$ and $(L_1(s) - L_2(s))$ have analytic continuation to the whole complex plane as entire functions.

Now, we write :

$$\tilde{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

and

$$\tilde{L}_j(s) = \sum_{n=1}^{\infty} \frac{a_j(n)}{n^s}$$

where, $a(n) = c(n)n^{-\frac{k-1}{2}}$ and $a_j(n) = c_j(n)n^{-\frac{k-1}{2}}$, $j = 1, 2$.

Therefore, writing

$$\bigwedge(s) = (\frac{\sqrt{N}}{2\pi})^{s+\frac{k}{2}-\frac{1}{2}}\Gamma(s + \frac{k}{2} - \frac{1}{2})[\tilde{L}_1(s) + \tilde{L}_2(s)],$$

and

$$\bigwedge^*(s) = (\frac{\sqrt{N}}{2\pi})^{s+\frac{k}{2}-\frac{1}{2}}\Gamma(s + \frac{k}{2} - \frac{1}{2})[\tilde{L}_1(s) - \tilde{L}_2(s)],$$

from Lemma 15 we have :

$$\bigwedge(s) = \bigwedge^*(1-s). \tag{55}$$

Now,

$$\begin{aligned} L(s) &= \prod_p (1 - c(p)p^{-s} + \chi(p)p^{k-1-2s})^{-1} \\ &= \prod_p (1 - \beta_p p^{-s})^{-1}(1 - \chi(p)\overline{\beta}_p p^{-s})^{-1}, \text{ say,} \end{aligned}$$

where, $\overline{\beta}_p$ is the complex conjugate of β_p .

Therefore, we have,

$$\begin{aligned}\widetilde{L}(s) &= L(s+\frac{k}{2}-\frac{1}{2}) \\ &= \prod_p (1-\gamma_p p^{-s})^{-1}(1-\chi(p)\overline{\gamma}_p p^{-s})^{-1}\end{aligned} \tag{56}$$

where,

$\gamma_p = \beta_p p^{-\frac{k}{2}+\frac{1}{2}}$, $\gamma_p + \chi(p)\overline{\gamma}_p = a(p)$ and $|\gamma_p| = |\overline{\gamma}_p| = 1$.

From (56), we have

$$\begin{aligned}-\frac{\widetilde{L}'}{\widetilde{L}}(s) &= \sum_p [(\gamma_p p^{-s}\log p + \gamma_p^2 p^{-2s}\log p + \cdots) \\ &\qquad (\Psi(p)\overline{\gamma}_p p^{-s}\log p + \Psi^2(p)\overline{\gamma}_p^2 p^{-2s}\log p + \cdots)] \\ &= \sum_{n=1}^{\infty} Y(n)n^{-s},\end{aligned}$$

say.

(57)

Here,

$$\begin{aligned}Y(n) &= (\gamma_p^m + \chi^m(p)\overline{\gamma}_p^m)\log p, \\ &\qquad \text{if } n = p^m \ \ (m > 1), \text{ for some prime } p \\ &= 0, \ \text{otherwise.}\end{aligned}$$

We note that

$$Y(p) = (\gamma_p + \chi(p)\overline{\gamma}_p)\log p = a(p)\log p.$$

Now, the following lemma follows by standard methods (see e.g. Ingham [15] pp 68-70).

Lemma 16 *For* $T > 0$, *let* $N(T)$ *denote the number of zeros of* $\tilde{L}(s)$ *in the rectangle* $0 \leq \sigma \leq 1, \ 0 \leq t \leq T$.

Then, as $T \to \infty$,

$$N(T) = T \log T + (\log(\frac{\sqrt{N}}{2\pi}) - 1)T + O(\log T).$$

Following are easy consequences of Lemma 16.

Corollary 16.1: *If* h *is a fixed positive number, then*

$$N(T+h) - N(T) = O(\log T) \tag{58}$$

Corollary 16.2: *If* $\rho = \beta + \gamma i, \ 0 \leq \beta \leq 1$ *are zeros of* $\tilde{L}(s)$ *in the critical strip, then*

$$\sum_{0 \leq \gamma \leq T} \frac{1}{\gamma} = O(\log^2 T)$$

and

$$\sum_{\gamma > T} \frac{1}{\gamma^2} = O(\frac{\log T}{T})$$

(59)

Definitions:

$$\Psi_1(x) = \sum_{n \leq x} Y(n) \tag{60}$$

and

$$\Psi_0(x) = \frac{\Psi_1(x+0) + \Psi_1(x-0)}{2}. \tag{61}$$

Remark 3.2.1 $\Psi_0(x)$ differs from $\Psi_1(x)$ only when x is prime-power p^m, the difference then being $\frac{1}{2}(\gamma_p^m + \chi^m(p)\overline{\gamma}_p^m) \log p$.

Now, by standard methods (see Ingham [15], Theorems 26-29) one gets the explicit formula:

$$\Psi_0(x) = -\sum_\rho \frac{x^\rho}{\rho} - \frac{\tilde{L}'}{\tilde{L}}(0) - 2\log(1 - \frac{1}{\sqrt{x}}) \tag{62}$$

We have $F(iy) \neq 0$ for all $y > 0$. Hence, the equation

$$(-2\pi i)^{-s}\Gamma(s)L(s) = \int_0^{i\infty} F(z)z^{s-1}dz$$

implies that $L(s) \neq 0$ for $s > 0$ and hence the following lemma:

Lemma 17 *$\tilde{L}(s)$ has no zeros on the part of the real axis given by $s > \frac{-k+1}{2}$.*

Lemma 18 *If θ denotes the upper bound of the real parts of the complex zeros of $\tilde{L}(s)$, then $\Psi_1(x) = \Omega_\pm(x^{\theta-\delta})$ for any fixed positive number δ.*

By Abel's identity,

$$-\frac{\tilde{L}'}{\tilde{L}}(s) = s\int_1^\infty \frac{\Psi_1(x)}{x^{s+1}}dx, \quad (s > 1)$$

Writing,

$$c(x) = \frac{\Psi_1(x) - x^\alpha}{x},$$

for some $0 < \alpha < \theta$,

$$\begin{aligned} \int_1^\infty \frac{c(x)}{x^s}dx &= -(\frac{1}{s})\frac{\tilde{L}'}{\tilde{L}}(s) - \frac{1}{s-\alpha} \quad (s > 1) \\ &= f(s), \text{ say.} \end{aligned} \tag{63}$$

If σ_0 is the abscissa of convergence of the Dirichlet integral in (63), then $\sigma_0 \geq \theta$. Also, $f(s)$ has no singularities on the stretch $s > \alpha$, since $\tilde{L}(s)$ is regular and has no zeros on the positive real axis by Lemma 17.

Since, $\sigma_0 \geq \theta > \alpha$, $s = \sigma_0$ is not a singularity of $f(s)$.

Therefore, by Landau's theorem, we can not have either $c(x) \geq 0$ or $c(x) \leq 0$ for sufficiently large x, which proves the lemma.

Remark 3.2.2 If $\theta > \frac{1}{2}$, Lemma 18 would imply

$$\Psi_1(x) = \Omega_{\pm}(x^{\frac{1}{2}}) \tag{64}$$

If $\theta = \frac{1}{2}$, writing $c(x) = \frac{\Psi_1(x) - cx^{\frac{1}{2}}}{x}$ and $f(s) = -(\frac{1}{s})\frac{\tilde{L}'}{\tilde{L}}(s) + \frac{c}{s-\frac{1}{2}}$ for some $c > 0$, we have

$$\int_1^{\infty} \frac{c(x)}{x^s} = f(s), \quad (\sigma > 1) \tag{65}$$

If $f(s)$ has no singularities on the real axis to the right of $\frac{1}{2}$, Landau's theorem would imply that the abscissa of convergence of the integral in (63) is $\sigma = \frac{1}{2}$ and hence (63) is valid for $\sigma > \frac{1}{2}$.

If possible, let $c(x) \geq 0$ for all $x \geq X(> 1)$.

Then for $\sigma > \frac{1}{2}$,

$$\begin{aligned} |f(\sigma + ti)| &\leq \int_1^X \frac{|c(x)|}{x^\sigma} dx + \int_X^\infty \frac{c(x)}{x^\sigma} dx \\ &= \int_1^X \frac{|c(x)| - c(x)}{x^\sigma} dx + f(\sigma) \\ &\leq \int_1^X \frac{|c(x)|}{x^{\frac{1}{2}}} dx + f(\sigma) \\ &= K + f(\sigma) \end{aligned}$$

where K is independent of σ and t.

If $\frac{1}{2}+\gamma_1 i$ is the zero with least positive γ, let $t = \gamma_1$ and then multiplying both sides by $\sigma - \frac{1}{2}$ and making $\sigma \to \frac{1}{2} + 0$, we get from above

$$\frac{m_1}{|\frac{1}{2} + \gamma_1 i|} \leq c,$$

where m_1 is the order of multiplicity of the zero $\frac{1}{2} + \gamma_1 i$. But we could have chosen $0 < c < \frac{m_1}{|\frac{1}{2}+\gamma_1 i|}$ and that shows that the supposition $c(x) \geq 0$ for $x \geq X$ leads to a contradiction.

So $c(x) < 0$ for arbitrary large x. Similarly one can show that $c(x) > 0$ for arbitrary large x, i.e.,(61) holds in the case $\theta = \frac{1}{2}$, as well.

3.3 Proof of theorem 7

We multiply the explicit formula (62) by $x^{-\frac{3}{2}}$, make the change of variable $x = e^u$ and integrate the resulting expression in u from $w - \eta$ to $w + \eta$ (where w, η are parameters to be chosen).
This gives

$$\int_{w-\eta}^{w+\eta} e^{-\frac{u}{2}}(\Psi_0(e^u) + \frac{\tilde{L}'}{\tilde{L}}(0) + 2\log(1 - e^{-\frac{u}{2}})) = -\sum_\rho \int_{w-\eta}^{w+\eta} \frac{e^{u(\rho-\frac{1}{2})}}{\rho} du \tag{66}$$

We put

$$G(u) = e^{-\frac{u}{2}}(\Psi_0(e^u) + \frac{\tilde{L}'}{\tilde{L}}(0) + 2\log(1 - e^{-\frac{u}{2}})).$$

Clearly,

$$G(u) = \Omega_\pm(\log\ \log u) \quad \Leftrightarrow \quad \Psi_0(x) = \Omega_\pm(x^{\frac{1}{2}}\ \log\ \log\ \log x).$$

If Riemann hypothesis is false for $\tilde{L}$, i.e. ,$\theta > \frac{1}{2}$, then from Lemma 18 (see remark 3.2.1) a result stronger than

$$\Psi_0(x) = \Omega_\pm(x^{\frac{1}{2}}\ \log\ \log\ \log x) \tag{67}$$

is true.

So, we can assume Riemann hypothesis.

Putting $\rho = \frac{1}{2} + i\gamma$ in (63) and integrating

$$\frac{1}{2\eta}\int_{w-\eta}^{w+\eta} G(u)du = -\sum_{\rho} \frac{\sin \gamma\eta}{\gamma\eta} \frac{e^{i\gamma w}}{\rho}.$$

Let $\frac{1}{2} + \gamma_1 i$ be the first zero of $\tilde{L}(s)$ on the line $\frac{1}{2}$ and let $T >$ max $\{e^2, \gamma_1\}$.

Then,

$$\sum_{\rho} \frac{\sin \gamma\eta}{\gamma\eta} \frac{e^{i\gamma w}}{\rho} = \sum_{|\gamma| \leq T} \frac{\sin \gamma\eta}{\gamma\eta} \frac{e^{i\gamma w}}{\rho} + \sum_{|\gamma| \geq T} \frac{\sin \gamma\eta}{\gamma\eta} \frac{e^{i\gamma w}}{\rho}$$

Now,

$$\begin{aligned}
\sum_{|\gamma| \leq T} \frac{\sin \gamma\eta}{\gamma\eta} \frac{e^{i\gamma w}}{\rho} &= \sum_{|\gamma| \leq T} \frac{\sin \gamma\eta}{\gamma\eta} \frac{e^{i\gamma w}}{i\gamma} + O(\sum_{|\gamma| \leq T} \gamma^{-2}) \\
&= \sum_{|\gamma| \leq T} \frac{\sin \gamma\eta}{\gamma\eta} \frac{\cos \gamma w}{i\gamma} + \sum_{|\gamma| \leq T} \frac{\sin \gamma\eta}{\gamma\eta} \frac{\sin \gamma w}{\gamma} + O(\sum_{|\gamma| \leq T} \gamma^{-2}) \\
&= \sum_{|\gamma| \leq T} \frac{\sin \gamma\eta}{\gamma\eta} \frac{\sin \gamma w}{\gamma} + O(1).
\end{aligned}$$

On the otherhand, by (59)

$$\sum_{|\gamma| \geq T} \frac{\sin \gamma\eta}{\gamma\eta} \frac{e^{i\gamma w}}{\rho} = O(\sum_{|\gamma| \geq T} (\eta\gamma^2)^{-1}) = O(\frac{\log T}{\eta T})$$

Therefore,

$$\frac{1}{2\eta}\int_{w-\eta}^{w+\eta} G(u)du = -2S(w) + O(1) + O(\frac{\log T}{\eta T}),$$

where

$$S(w) = \sum_{0<\gamma \leq T} \frac{\sin \gamma\eta}{\gamma\eta} \frac{\sin \gamma w}{\gamma}.$$

Now, we utilize the theorem of Dirichlet (see Titchmarsh, Theory of functions) :

Given $\theta_1, \cdots, \theta_N$, -$N$ real numbers, $q > 1, \tau > 0$, the interval $[\tau, \tau q^N]$ contains a $U \in \mathbf{Z}$, such that $\| U\theta_i \| < \frac{1}{q}$. $1 \leq i \leq N$.

Now, applying this to $\theta_j = \frac{\gamma_j}{2\pi}, 1 \leq j \leq N(T)$, for $\tau = q^{N(T)}$ (q will be chosen later) we obtain :

There is $U \in \mathbf{Z}$, $q^{N(T)} \leq U \leq q^{2N(T)}$, such that

$$\| \frac{U\gamma_j}{2} \| < \frac{1}{q} \tag{68}$$

Therefore, for all real v,

$$| \pm S(U \pm v) - S(v) | \leq \frac{2\pi}{q} \sum_{\gamma < T} \frac{1}{\gamma}$$

by the mean value theorem and (66).

Therefore,by (56),

$$| \pm S(U \pm v) - S(v) | = O(\frac{\log^2 T}{q}).$$

Let $0 < \eta < \frac{1}{2}$.

Setting, $w = U \pm 2\eta$, $w = 2\eta$ in (65) and subtracting the corresponding expressions, we have by the above results:

$$\frac{1}{2\eta} \int_{w-\eta}^{w+\eta} [\pm G(U \pm 2\eta + y) - G(2\eta + y)] dy = O(\frac{\log^2 T}{q}) + O(1) + O(\frac{\log T}{\eta T})$$

Choosing, $q = \log^2 T; \eta = \frac{\log T}{T}$,

$$\frac{1}{2\eta} \int_{w-\eta}^{w+\eta} [\pm G(U \pm 2\eta + y) - G(2\eta + y)] dy = O(1) \tag{69}$$

Since, $y \in [-\eta, \eta]$, we have, $2\eta + y = (2 + \theta)\eta$ where,$| \theta | \leq 1$.

As $\eta \to 0$,

$$G(2\eta + y) = 2 \log(1 - e^{-\eta - \frac{y}{2}}) + O(1)$$

$$= 2\log(\eta + \frac{y}{2}) + O(1)$$
$$= 2\log\eta + O(1)$$

Therefore,

$$\frac{1}{2\eta}\int_{w-\eta}^{w+\eta} G(2\eta + y)dy \le 2\log\eta + O(1).$$

Hence, from (67),

$$\frac{1}{2\eta}\int_{w-\eta}^{w+\eta} \pm G(U \pm 2\eta + y)dy \le 2\log\eta + O(1).$$

Now,

$$\frac{1}{2\eta}\int_{w-\eta}^{w+\eta} G(U + 2\eta + y)dy \le 2\log\eta + O(1)$$

implies that there exists $u \in [U+\eta, U+3\eta]$, such that $G(u) \le 2\log\eta + O(1)$.

But,

$$\begin{aligned} \log\log U &= \log N(T) + O(\log\log q) \\ &= \log T + O(\log\log q) + O(\log\log T) \end{aligned}$$

(by lemma 16)

Hence,

$$\liminf_{u\to\infty} \frac{G(u)}{\log\log u} \le -2,$$

that is , $G(u) = \Omega_{-}(\log\log u)$.

A similar analysis with $-G(u)$ yields:

$$\limsup_{u\to\infty} \frac{G(u)}{\log\log u} \ge 2.$$

Hence,(64) is true.

By Remark 3.2.1,

$$\Psi_1(x) = \Omega_{\pm}(x^{\frac{1}{2}} \log\ \log\ \log x).$$

Since,

$$\Psi_1(x) = \sum_{p\leq x} Y(p) + \sum_{p^2\leq x} Y(p^2) + \cdots + \sum_{p^m\leq x} Y(p^m), \text{ where } m = [\frac{\log x}{\log 2}],$$

$$\sum_{p^2\leq x} Y(p^2) \leq 2 \sum_{p\leq\sqrt{x}} \log p = O(\sqrt{x})$$

and

$$\begin{aligned}\sum_{p^3\leq x} Y(p^3) + \cdots + \sum_{p^m\leq x} Y(p^m) &\leq \left[\frac{\log x}{\log 2}\right] \quad (2 \sum_{p\leq x^{\frac{1}{3}}} \log p) \\ &= O(x^{\frac{1}{3}} \log x)\end{aligned}$$

with m as above, we get,

$$\sum_{p\leq x} Y(p) \leq \sum_{p\leq x} a(p) \log p = \Omega_{\pm}(x^{\frac{1}{2}} \log \log \log x)$$

which proves Theorem 7. ■

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