On the Hardy type potentials

By

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution/University.

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Dedicated to my family, teachers, and friends.

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Contents

Summary

In this thesis, we study the weighted Hardy, Hardy-Rellich, and the weighted logarithmic Sobolev inequalities. More precisely, for an open set Ω in $\mathbb{R}^N$, we look for $g \in L^1_{loc}(\Omega)$ for which one of the following inequalities hold:

(i) **Weighted Hardy inequality:**

$$\int_\Omega |g||u|^p \, dx \leq \mathrm{C} \int_\Omega |\nabla u|^p \, dx, \ \forall u \in \mathrm{C}^1_c(\Omega)\,, \text{where } p \in (1, N).$$

(ii) **Weighted Hardy-Rellich inequality:**

$$\int_\Omega |g||u|^2 \, dx \leq \mathrm{C} \int_\Omega |\Delta u|^2 \, dx, \ \forall u \in \mathrm{C}^2_c(\Omega)\,.$$

(iii) **Weighted logarithmic-Sobolev inequality:**

$$\int_{\mathbb{R}^N} |g||u|^p \log |u|^p \, dx \leq \gamma \log \left(\mathrm{C}(g,\gamma) \int_{\mathbb{R}^N} |\nabla u|^p \, dx \right), \ \forall u \in \mathrm{C}^1_c(\mathbb{R}^N)$$

with $\displaystyle\int_{\mathbb{R}^N} |u|^p \, dx = 1$, where $p \in (1, N)$.

For $g = \frac{1}{|x|^p}, \frac{1}{|x|^4}, 1$, inequalities $(i), (ii)$, and (iii) corresponds to the classical Hardy, Hardy-Rellich, and the logarithmic Sobolev inequality respectively. We study each of these inequalities separately.

■ **Weighted Hardy inequality.** We say $g \in L^1_{loc}(\Omega)$ is a Hardy potential if g satisfies the above weighted Hardy inequality and we define $\mathcal{H}_p(\Omega) = \Big\{ g \in L^1_{loc}(\Omega) : g \text{ is a Hardy potential} \Big\}$. Using p-capacity, we define the following Banach function

space norm on $\mathcal{H}_p(\Omega)$:

$$\|g\|_{\mathcal{H}_p} = \sup \left\{ \frac{\int_F |g| \ dx}{\mathrm{Cap}_p(F, \Omega)} : F \subset\subset \Omega; |F| \neq 0 \right\}.$$

The Mazya's p-capacity condition help us to identify $\mathcal{H}_p(\Omega) = \{g \in L^1_{loc}(\Omega) : \|g\|_{\mathcal{H}_p} < \infty\}$. Further, we characterise the set of all g in $\mathcal{H}_p(\Omega)$ for which the map $G_p(u) = \displaystyle\int_\Omega |g||u|^p$ is compact on the Beppo-Levi space $\mathcal{D}^{1,p}_0(\Omega)$. We use a variation of the concentration compactness lemma to give a sufficient condition on $g \in \mathcal{H}_p(\Omega)$ so that the best constant in the above inequality is attained in $\mathcal{D}^{1,p}_0(\Omega)$.

■ **Weighted Hardy-Rellich inequality.** We say $g \in L^1_{loc}(\Omega)$ is a Hardy-Rellich potential if g satisfies the weighted Hardy-Rellich inequality and we define $\mathcal{HR}(\Omega) = \left\{ g \in L^1_{loc}(\Omega) : g \text{ is a Hardy-Rellich potential} \right\}$. Using the Muckenhoupt necessary and sufficient conditions for the one-dimensional weighted Hardy inequalities and a point-wise inequality for the symmetrization, we obtain certain Lorentz spaces in $\mathcal{HR}(\Omega)$. Furthermore, using the fundamental theorem of calculus, we obtain certain weighted Lebesgue space in $\mathcal{HR}(\Omega)$. Indeed, we show that these two classes are not comparable.

■ **Weighted logarithmic-Sobolev inequality.** For $g \in L^1_{loc}(\mathbb{R}^N)$ and $q \geq p$, using the p-capacity, we define

$$\|g\|_{\mathcal{H}_{p,q}} = \sup \left\{ \frac{\int_F |g| \ dx}{[\mathrm{Cap}_p(F, \Omega)]^{\frac{q}{p}}} : F \subset\subset \Omega; |F| \neq 0 \right\},$$

and define the Banach function space $\mathcal{H}_{p,q}(\Omega) = \left\{ g \in L^1_{loc}(\Omega) : \|g\|_{\mathcal{H}_{p,q}} < \infty \right\}$. For each $q \in (p, \frac{Np}{N-p}]$, we show that the weighted logarithmic Sobolev inequality holds for $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$. For $\gamma > \frac{r}{r-p}$, we also find a class of g for which the best constant $\mathrm{C}(g, \gamma)$ in the above inequality is attained in $\mathcal{D}^{1,p}_0(\mathbb{R}^N)$. For $\gamma > \frac{q}{q-p}$, we also find a class of g for which the best constant $\mathrm{C}(g, \gamma)$ in the weighted logarithmic Sobolev inequality is attained in $\mathcal{D}^{1,p}_0(\mathbb{R}^N)$.

Notations

- $\mathrm{C}_{\mathrm{c}}^{\mathrm{k}}(\Omega)$ = the space of k differentiable, compactly supported, real-valued functions on an open set Ω.

- $\mathrm{C}_{\mathrm{b}}^{\mathrm{k}}(\Omega)$ = the space of k differentiable, bounded, real-valued functions on an open set Ω.

- $L^1_{loc}(\Omega)$ = the space of real-valued, locally integrable functions on an open set Ω.

- $|E|$ = the Lebesgue measure of the set E.

- $\mathcal{M}(\Omega)$ = the space of extended real valued measurable function on an open set Ω.

- $\mathbb{M}(\Omega)$ = the space of all bounded signed measures on an open set Ω.

- $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N}\right)$ for $u \in \mathrm{C}_{\mathrm{c}}^{1}(\Omega)$.

- $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$ for $u \in \mathrm{C}_{\mathrm{c}}^{2}(\Omega)$.

- $|\nabla^2 u|^2 = \sum_{i,j=1}^{N} \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2$ for $u \in \mathrm{C}_{\mathrm{c}}^{2}(\Omega)$.

- $\mathcal{D}_0^{1,p}(\Omega)$ = the completion of $\mathrm{C}_{\mathrm{c}}^{1}(\Omega)$ with respect to the norm $\|u\|_{\mathcal{D}_0^{1,p}} := \left[\int_\Omega |\nabla u|^p \, dx\right]^{\frac{1}{p}}$.

- $\mathcal{D}_0^{2,p}(\Omega)$ = the completion of $\mathrm{C}_{\mathrm{c}}^{2}(\Omega)$ with respect to the norm $\|u\|_{\mathcal{D}_0^{2,p}} := \left[\int_\Omega |\nabla^2 u|^p \, dx\right]^{\frac{1}{p}}$.

- $\mathbb{S}^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$, which is the unit sphere in $\mathbb{R}^N$.

- $B_r(x)$ = the r-radius ball centred at x.

- ω_N = the Lebesgue measure of the unit ball in $\mathbb{R}^N$.

Chapter 1

Introduction

This thesis is devoted to study the weighted Hardy, the weighted Hardy-Rellich, and the weighted logarithmic Sobolev inequality. More precisely, for an open set Ω in $\mathbb{R}^N$, we look for $g \in L^1_{loc}(\Omega)$ for which one of the following inequalities hold:

(i) **Weighted Hardy inequality:**

$$\int_\Omega |g||u|^p \; dx \leq \mathrm{C} \int_\Omega |\nabla u|^p \; dx, \;\; \forall u \in \mathrm{C}^1_c(\Omega)\,, \text{where } p \in (1, N)\,,$$

(ii) **Weighted Hardy-Rellich inequality:**

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(iii) **Weighted logarithmic-Sobolev inequality:**

$$\int_{\mathbb{R}^N} |g||u|^p \log |u|^p \; dx \leq \gamma \log \left(\mathrm{C}(g, \gamma) \int_{\mathbb{R}^N} |\nabla u|^p \; dx \right), \;\; \forall u \in \mathrm{C}^1_c(\mathbb{R}^N)$$

with $\displaystyle\int_{\mathbb{R}^N} |g||u|^p \; dx = 1, \text{where } p \in (1, N).$

Further, we discuss the sufficient conditions on the weight functions that ensure the best constants in these inequalities are achieved.

1.1 The optimal space of Hardy potentials

For $p \in (1, N)$ and an open set Ω in $\mathbb{R}^N$, recall the classical Hardy inequality due to G. H. Hardy [43]:

$$\int_\Omega \frac{|u|^p}{|x|^p}\, dx \le \left(\frac{p}{N-p}\right)^p \int_\Omega |\nabla u|^p\, dx,\ \ u \in \mathrm{C}_c^1(\Omega)\,. \tag{1.1.1}$$

For a more detailed discussion on this inequality, we refer to [46]. Inequality (1.1.1) has been extended and generalised in several directions. One of which is the improved Hardy inequality, and it concerns with replacing the Hardy potential $\frac{1}{|x|^p}$ with $\frac{1}{|x|^p}$ + lower-order radial weights, see [19, 2, 38] and the references therein. On the other hand, many authors have shown interest in producing more general weight functions in (1.1.1) in place of $\frac{1}{|x|^p}$ i.e., $g \in L^1_{loc}(\Omega)$ for which the following weighted Hardy inequality holds:

$$\int_\Omega |g||u|^p\, dx \le C \int_\Omega |\nabla u|^p\, dx, \quad \forall\, u \in \mathrm{C}_c^1(\Omega), \tag{1.1.2}$$

for some $C > 0$. Let

$$\mathcal{H}_p(\Omega) = \left\{ g \in L^1_{loc}(\Omega) : g \text{ satisfies (1.1.2)} \right\}.$$

We call a function $g \in \mathcal{H}_p(\Omega)$ as **Hardy potential**. If Ω is bounded in one direction, then the Poincaré inequality implies that $L^\infty(\Omega) \subseteq \mathcal{H}_p(\Omega)$. Next we see how the various embeddings of the Beppo-Levi space $\mathcal{D}_0^{1,p}(\Omega)$ (the completion of $\mathrm{C}_c^1(\Omega)$ with respect to the norm $\|u\|_{\mathcal{D}_0^{1,p}} := \left[\int_\Omega |\nabla u|^p\, dx\right]^{\frac{1}{p}}$) provide other classes of function spaces in $\mathcal{H}_p(\Omega)$. The Sobolev embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ ($p^* := \frac{Np}{N-p}$) ensures that $L^{\frac{N}{p}}(\Omega) \subseteq \mathcal{H}_p(\Omega)$. Further, one may use finer embeddings of $\mathcal{D}_0^{1,p}(\Omega)$ to produce larger space in $\mathcal{H}_p(\Omega)$. For instance, using the Lorentz-Sobolev embedding $\mathcal{D}_0^{1,2}(\Omega) \hookrightarrow L^{2^*,2}(\Omega)$, Visciglia [71, Theorem 1.1] show that $L^{\frac{N}{2},\infty}(\Omega) \subseteq \mathcal{H}_2(\Omega)$.

Moreover, for $p \in (1, N)$, one can use the embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{p^*,p}(\Omega)$ and follow Visciglia's arguments to show that $L^{\frac{N}{p},\infty}(\Omega) \subseteq \mathcal{H}_p(\Omega)$. Indeed, it is known that $L^{\frac{N}{p},\infty}(\Omega)$ does not exhaust $\mathcal{H}_p(\Omega)$. For instance, for $\Omega = \overline{B_1(0)}^c$, there are Hardy potentials in certain weighted Lebesgue space that do not belong to $L^{\frac{N}{p},\infty}(\Omega)$, where $\overline{B_1(0)}^c$ denotes the exterior of the closed unit ball centered at the origin, see [10, Theorem 1.1].

In [57], Maz′ya has given a necessary and sufficient condition for the Hardy potentials using the notion of p-capacity. For $F \subset\subset \Omega$, the p-capacity of F relative to Ω is defined as

$$\mathrm{Cap}_p(F, \Omega) = \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in \mathcal{N}_p(F) \right\},$$

where $\mathcal{N}_p(F) = \{u \in \mathcal{D}_0^{1,p}(\Omega) : u \geq 1 \text{ in a neighbourhood of } F\}$. Notice that, for $g \in \mathcal{H}_p(\Omega)$ and $w \in \mathcal{N}_p(F)$, we have

$$\int_F |g| \, dx \leq \int_\Omega |g||w|^p \, dx \leq C \int_\Omega |\nabla w|^p \, dx.$$

By taking the infimum over $\mathcal{N}_p(F)$ and as F is arbitrary, we get a necessary condition:

$$\sup_{F \subset\subset \Omega} \frac{\int_F |g| \, dx}{\mathrm{Cap}_p(F, \Omega)} \leq C.$$

Maz'ya proved that the above condition is also sufficient for g to be in $\mathcal{H}_p(\Omega)$ [57, Section 2.3.2, page 111]. Motivated by this, for $g \in L^1_{loc}(\Omega)$, we define,

$$\|g\|_{\mathcal{H}_p} = \sup \left\{ \frac{\int_F |g| \, dx}{\mathrm{Cap}_p(F, \Omega)} : F \subset\subset \Omega; |F| \neq 0 \right\}.$$

Therefore, $\mathcal{H}_p(\Omega)$ can be identified as

$$\mathcal{H}_p(\Omega) = \left\{ g \in L^1_{loc}(\Omega) : \|g\|_{\mathcal{H}_p} < \infty \right\}.$$

In fact, $\|.\|_{\mathcal{H}_p}$ is a Banach function norm, and $\mathcal{H}_p(\Omega)$ is the Banach function space with respect to this norm (see Section 2.2.1 for the precise definition of Banach function space).

For $g \in \mathcal{H}_p(\Omega)$, let $\mathcal{B}_g$ be the best constant in (1.1.2) i.e., $\mathcal{B}_g$ is the least possible constant so that (1.1.2) holds. Therefore, for $g \in \mathcal{H}_p(\Omega)$, we have

$$\mathcal{B}_g^{-1} = \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in \mathcal{D}_0^{1,p}(\Omega), \int_\Omega |g||u|^p \, dx = 1 \right\}. \tag{1.1.3}$$

It is clear that if the above minimisation problem admits a solution $w \in \mathcal{D}_0^{1,p}(\Omega)$, then the equality holds in (1.1.2) with $C = \mathcal{B}_g$ and $u = w$. In this case, we say $\mathcal{B}_g$ is attained at w. Now we are interested in identifying the weight functions $g \in \mathcal{H}_p(\Omega)$ for which $\mathcal{B}_g$ is attained in $\mathcal{D}_0^{1,p}(\Omega)$. It is worth mentioning that this problem has been extensively studied in the context of finding the first (least) positive eigenvalue of the following problem:

$$-\Delta_p u = \lambda |g||u|^{p-2}u \ \text{ in } \Omega; \quad u \in \mathcal{D}_0^{1,p}(\Omega). \tag{1.1.4}$$

This weighted non-linear eigenvalue problem arises in the mathematical modelling of the population distribution of certain species, where the weight function g represents the distribution of resources on the domain. It is natural to expect that g is not uniformly distributed throughout the domain. This motivates us to find $g \in L^1_{loc}(\Omega)$ such that (1.1.4) admits a positive solution. Using variational methods, it is not difficult to see that (1.1.4) admits a positive solution if (1.1.3) has a minimizer in $\mathcal{D}_0^{1,p}(\Omega)$ or equivalently the best constant $\mathcal{B}_g$ in (1.1.2) is attained in $\mathcal{D}_0^{1,p}(\Omega)$.

One of the simplest conditions that guarantees a minimizer for (1.1.3) is the compactness of the map

$$G_p(u) = \int_\Omega |g||u|^p \, dx \ \text{ on } \mathcal{D}_0^{1,p}(\Omega)$$

(i.e., for $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$, $G_p(u_n) \to G_p(u)$ as $n \to \infty$). Many authors have given various sufficient conditions for the compactness of the map G_p. For example, for $p = 2$ and Ω bounded, the compactness of G_p is proved for $g \in L^r(\Omega)$ with $r > \frac{N}{2}$ in [55] and $r = \frac{N}{2}$ in [4], and for $p \in (1, \infty)$ and general domain Ω, $g \in L^{\frac{N}{p},d}(\Omega)$ with $d < \infty$ [71], and $g \in \mathcal{F}_{\frac{N}{p}}(\Omega) := \overline{\mathrm{C}_c^\infty(\Omega)}$ in $L^{\frac{N}{p},\infty}(\Omega)$ [6]. Motivated by this, we consider

$$\mathcal{FH}_p(\Omega) := \overline{\mathrm{C_c}(\Omega)} \text{ in } \mathcal{H}_p(\Omega)\,.$$

The following theorem extends and unifies all the existing sufficient conditions for the compactness of G_p. Moreover, we prove that $\mathcal{FH}_p(\Omega)$ is the optimal space for the compactness of G_p.

Theorem 1.1.1. *[8, Theorem 1] G_p is compact on $\mathcal{D}_0^{1,p}(\Omega)$ if and only if $g \in \mathcal{FH}_p(\Omega)$.*

Next, we give a characterisation of $\mathcal{FH}_p(\Omega)$ by using the notion of the absolutely continuous norm in $\mathcal{H}_p(\Omega)$.

Definition 1.1.2 (Absolute continuous norm)**.** *Let $(X(\Omega), \|.\|_X)$ be a Banach function space. We say $g \in X(\Omega)$ has absolutely continuous norm in $X(\Omega)$, if for any sequence of measurable subsets (A_n) of Ω with χ_{A_n} converges to 0 a.e. in Ω, we have $\|g\chi_{A_n}\|_X$ converges to 0.*

Now we have the following result.

Theorem 1.1.3. *[8, Theorem 2] $g \in \mathcal{FH}_p(\Omega)$ if and only if g has absolute continuous norm in $\mathcal{H}_p(\Omega)$.*

As we mentioned before, the compactness of G_p ensures that the best constant $\mathcal{B}_g$ in (1.1.2) is attained in $\mathcal{D}_0^{1,p}(\Omega)$. Indeed, the best constant $\mathcal{B}_g$ may attain in $\mathcal{D}_0^{1,p}(\Omega)$ without G_p being compact. Such cases are treated by Tertikas [69] for $p = 2$, $\Omega = \mathbb{R}^N$, and Smets [67] for $p \in (1, N)$ and general Ω. In [69, 67], authors have considered the following concentration function of g

$$\begin{aligned}
\mathcal{S}_g(x) &= \lim_{r\to 0} \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in \mathcal{D}_0^{1,p}(\Omega \cap B_r(x)), \int_\Omega |g||u|^p \, dx = 1 \right\}, \\
\mathcal{S}_g(\infty) &= \lim_{R\to\infty} \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in \mathcal{D}_0^{1,p}(\Omega \cap B_R^c), \int_\Omega |g||u|^p \, dx = 1 \right\},
\end{aligned}$$

and defined the singular set of g as $\Sigma_g' := \left\{ x \in \overline{\Omega} : \mathcal{S}_g(x) < \infty \right\}$. Under the assumption that $\overline{\Sigma_g'}$ is countable ((H) of [69] and (H1) of [67]), they have provided a sufficient condition on the concentration function $\mathcal{S}_g$ so that $\mathcal{B}_g$ is attained in $\mathcal{D}_0^{1,p}(\Omega)$. In this thesis, we introduce a new concentration function using the norm on $\mathcal{H}_p(\Omega)$.

Definition 1.1.4 (Concentration function $\mathcal{C}_g$)**.** *For $g \in \mathcal{H}_p(\Omega)$, we define the concentration function of g as*

$$\mathcal{C}_g(x) = \lim_{r\to 0} \|g\chi_{B_r(x)}\|_{\mathcal{H}_p}, \ \forall x \in \overline{\Omega} \ ; \qquad \mathcal{C}_g(\infty) = \lim_{R\to\infty} \|g\chi_{B_R(0)^c}\|_{\mathcal{H}_p}.$$

Observe that, our concentration function $\mathcal{C}_g$ captures the local behaviour of g in terms of the norm in $\mathcal{H}_p(\Omega)$. Next we give another characterisation for the compactness of the map G_p using the concentration function $\mathcal{C}_g$.

Theorem 1.1.5. *[8, Theorem 3] G_p is compact on $\mathcal{D}_0^{1,p}(\Omega)$ if and only if $\mathcal{C}_g \equiv 0$.*

We define the singular set of $\mathcal{C}_g$ as

$$\sum\nolimits_g := \left\{ x \in \overline{\Omega} : \mathcal{C}_g(x) > 0 \right\}.$$

Indeed, $\sum_g$ coincides with the singular set considered by Tertikas and Smets i.e.,

$\sum_g = \sum_g^{'}$ (see (3.1.12)). Here, we provide a sufficient condition for the existence of minimiser for (1.1.3) under the assumption that the closure of the singular set is of Lebesgue measure zero (relaxing the countability assumption on $\overline{\Sigma_g}$ of [67, 69]).

Theorem 1.1.6. *[8, Theorem 4] Let $g \in \mathcal{H}_p(\Omega)$ be such that $\left|\overline{\sum_g}\right| = 0$ and*

$$C_H \mathcal{C}_g(x) < \mathcal{B}_g, \forall x \in \overline{\Omega} \cup \{\infty\},$$

where $\mathcal{B}_g$ is the best constant in (1.1.2) and $C_H = p^p(p-1)^{1-p}$. Then $\mathcal{B}_g$ is attained in $\mathcal{D}_0^{1,p}(\Omega)$.

Remark 1.1.7. (i) We provide cylindrical Hardy potentials g for which $|\sum_g| = 0$, but $\sum_g$ is not countable (see Remark 3.1.22). Such cylindrical weights were considered by Badiale and Tarantello in [12] (for $N = 3$), Mancini et. al in [54] (for $N \geq 3$) to study certain semi-linear PDE involving Sobolev critical exponent. In astrophysics, such critical exponent problems with cylindrical weights often arises in the dynamics of galaxies [17, 24].

(ii) For a cylindrical Hardy potential $g \in \mathcal{H}_p(\Omega)$ with $|\overline{\sum_g}| = 0$, one can consider its perturbation $\tilde{g} := g + \phi$ by a suitable $\phi \in \mathrm{C}_c^\infty(\Omega)$ and apply the above theorem to ensure $\mathcal{B}_{\tilde{g}}$ is attained in $\mathcal{D}_0^{1,p}(\Omega)$ (see Remark 3.1.22 for a precise example). It is worth noticing that $|\overline{\sum_{\tilde{g}}}| = 0$ but not countable. Indeed, the results of [69, 67] are not applicable for such Hardy potentials.

1.2 The Hardy-Rellich and the Hardy-Hessian potentials

For $N \geq 5$ and an open subset Ω of $\mathbb{R}^N$, Rellich [65, Section 7, Chapter 2, page 90-101] has proved the following second order generalization of (1.1.1):

$$\int_{\Omega} \frac{|u|^2}{|x|^4}\, dx \leq \left[\frac{16}{N^2(N-4)^2}\right] \int_{\Omega} |\Delta u|^2\, dx,\ \forall\, u \in \mathrm{C}_c^2(\Omega)\,. \tag{1.2.1}$$

This inequality is known as Hardy-Rellich inequality. The above inequality is later extended for any $p \in (1, \frac{N}{2})$ as follows [28]:

$$\int_{\Omega} \frac{|u|^p}{|x|^{2p}}\, dx \leq \left[\frac{p^2}{N(p-1)(N-2p)}\right]^p \int_{\Omega} |\Delta u|^p\, dx,\ \forall\, u \in \mathrm{C}_c^2(\Omega). \tag{1.2.2}$$

In this thesis, we look for $g \in L^1_{loc}(\Omega)$ for which the following weighted Hardy-Rellich inequality holds:

$$\int_{\Omega} |g||u|^p\, dx \leq C \int_{\Omega} |\Delta u|^p\, dx\,, \forall\, u \in \mathrm{C}_c^2(\Omega), \tag{1.2.3}$$

for some $C > 0$. Let

$$\mathcal{HR}_p(\Omega) = \left\{ g \in L^1_{loc}(\Omega) : g \text{ satisfies } (1.2.3) \right\}.$$

We call the functions in $\mathcal{HR}_p(\Omega)$ as **Hardy-Rellich potentials.** Unlike the Hardy potentials, the Maz′ya type characterisation is not available for Hardy-Rellich potentials in general domain. However, we are able to provide certain weighted Lebesgue spaces and Lorentz spaces in $\mathcal{HR}_p(\Omega)$.

Using the Muckenhoupt necessary and sufficient conditions for the one dimensional weighted Hardy inequalities [60, Theorem 1 and Theorem 2] and a pointwise inequality for the symmetrization obtained in [23], we prove the following theorem.

Theorem 1.2.1. *Let Ω be an open set in $\mathbb{R}^N$ with $N > \max\{2p, 2p'\}$.*

(i) (A sufficient condition) $L^{\frac{N}{2p},\infty}(\Omega) \subseteq \mathcal{HR}_p(\Omega)$.

(ii) (A necessary condition) Let Ω be a ball centered at the origin or entire $\mathbb{R}^N$ and g be radial, radially decreasing. Then $g \in \mathcal{HR}_p(\Omega)$, only if g belongs to $L^{\frac{N}{2p},\infty}(\Omega)$.

For $p = 2$, the above theorem is proved in our article [9, Theorem 1.2].

Next we consider the particular case: $p = 2$ and $N = 4$. For a measurable function g, we denote its one-dimensional decreasing rearrangement by g^* and we define $g^{**}(t) = \frac{1}{t}\int_0^t g^*(s)ds$. For a bounded domain Ω, we consider the following space introduced in [7]

$$\mathcal{M}\log L(\Omega) := \left\{ g \text{ measurable } : \sup_{0<t<|\Omega|} t \log\left(\frac{|\Omega|}{t}\right) g^{**}(t) < \infty \right\}.$$

Now, for $p = 2$ and $N = 4$, we have the following results:

Theorem 1.2.2. *[9, Theorem 1.4]. Let Ω be a bounded domain in $\mathbb{R}^4$. Then*

(i) $\mathcal{M}\log L(\Omega) \subseteq \mathcal{HR}_2(\Omega)$.

(ii) Let $\Omega = B_R(0)$ with $R \in (0, \infty)$. Let $g \in \mathcal{HR}_2(\Omega)$ be radial, radially decreasing. Then g must belong to $\mathcal{M}\log L(\Omega)$.

As a consequence of Theorem 1.2.1 and Theorem 1.2.2, we could give a simple proof for the Lorentz-Sobolev embedding $\mathcal{D}_0^{2,2}(\Omega) \hookrightarrow L^{2^{**},2}(\Omega)$ ($N \geq 5$, Ω is a general domain, [59]) and the Hansson embedding $\mathcal{D}_0^{2,2}(\Omega) \hookrightarrow L^{\infty,2}(\log L)^{-1}(\Omega)$ ($N = 4$, Ω is a bounded domain, [42]) respectively, where $\mathcal{D}_0^{2,2}(\Omega)$ is the completion of $\mathrm{C}_c^2(\Omega)$ with respect to the norm $\|u\|_{\mathcal{D}_0^{2,2}} := \left[\int_\Omega |\nabla^2 u|^2 \, dx\right]^{\frac{1}{2}}$, where $\nabla^2 u$ is the Hessian matrix of u and $|\nabla^2 u| = \left(\sum_{i,j=1}^N \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2\right)^{\frac{1}{2}}$.

Next we consider the weighted Hardy-Hessian inequality, namely, we are interested to identify $g \in L^1_{loc}(\Omega)$ so that the following **weighted Hardy-Hessian inequality** holds

$$\int_\Omega |g||u|^p \, dx \leq C \int_\Omega |\nabla^2 u|^p \, dx \, , \ \forall u \in \mathrm{C}^2_c(\Omega) \, . \tag{1.2.4}$$

Let

$$\mathbb{H}_p(\Omega) = \left\{ g \in L^1_{loc}(\Omega) : g \text{ satisfies } (1.2.4) \right\}.$$

We call the functions in $\mathbb{H}_p(\Omega)$ as **Hardy-Hessian potentials.** Notice that the right hand side of (1.2.4) involves the Hessian and that of (1.2.3) involves the Laplacian, and in general $\int_\Omega |\Delta u|^p \, dx \leq \int_\Omega |\nabla^2 u|^p \, dx$. Therefore,

$$\mathcal{HR}_p(\Omega) \subseteq \mathbb{H}_p(\Omega).$$

Using the integration by parts, we also have $\int_\Omega |\nabla^2 u|^2 \, dx \leq \mathrm{C} \int_\Omega |\Delta u|^2 \, dx$, $\forall u \in \mathrm{C}^2_c(\Omega)$. Therefore, for $p = 2$, the Hardy-Rellich potentials are same as the Hardy-Hessian potentials i.e.,

$$\mathbb{H}_2(\Omega) = \mathcal{HR}_2(\Omega).$$

In this thesis, we identify certain weighted Lebesgue spaces in $\mathbb{H}_p(\Omega)$. For an open set Ω in $\mathbb{R}^N$ and a radial, non-negative function w on $\mathbb{R}^N$, we define

$$L^1_{rad}(\mathbb{R}^N, w) = \left\{ g \in L^1(\mathbb{R}^N, w) : g \text{ is radial} \right\},$$
$$L^1_{rad}(\Omega, w) = \left\{ \, g|_\Omega : g \in L^1_{rad}(\mathbb{R}^N, w) \right\}.$$

Also, for an open subset S of $\mathbb{S}^{N-1}$ and $a, b \in [0, \infty]$ with $b > a$, we consider the sectorial open set

$$\Omega_{a,b,S} = \left\{x \in \mathbb{R}^N : a < |x| < b, \frac{x}{|x|} \in S\right\}. \tag{1.2.5}$$

We have the following theorem. For $p = 2$, the following theorem appears in our work [9, Theorem 1.3, Theorem 1.5].

Theorem 1.2.3. *Let Ω be an open subset in $\mathbb{R}^N$ with $N > 2p$. For $2 \le N \le 2p$, assume that $\Omega = \Omega_{a,b,S}$ with $a > 0$ and S is an open subset of $\mathbb{S}^{N-1}$. Let $g \in L^1_{loc}(\Omega)$ be such that*

$$g \in X_{rad}(\Omega) := \begin{cases} L^1_{rad}(\Omega, |x|^{2p-N}), & N > 2p \\ L^1_{rad}(\Omega, |x|^{p}), & p < N \le 2p; b = \infty \\ L^1_{rad}(\Omega, |x|^{p}[\log(\frac{|x|}{a})]^{p-1}), & N = p; b = \infty \\ L^1_{rad}(\Omega), & 2 \le N \le 2p; b < \infty. \end{cases}$$

Then $g \in \mathbb{H}_p(\Omega)$.

Notice that, for $p = 2$ and $N \ge 5$, Theorem 1.2.1 implies $L^{\frac{N}{4},\infty}(\Omega) \subseteq \mathcal{HR}_2(\Omega)$, and Theorem 1.2.3 implies $L^1_{rad}(\Omega, |x|^{4-N}) \subseteq \mathcal{HR}_2(\Omega)$. Indeed, we show that these two spaces are not contained in one another.

1.3 The logarithmic-Sobolev potentials

For $N \ge 3$ and $p \in (1, N)$, recall the Sobolev inequality

$$\left[\int_{\mathbb{R}^N} |u|^{p^*}\, dx\right]^{\frac{1}{p^*}} \le \mathrm{c} \left[\int_{\mathbb{R}^N} |\nabla u|^p\, dx\right]^{\frac{1}{p}}, \forall\, u \in \mathcal{D}^{1,p}_0(\mathbb{R}^N),$$

i.e., $\mathcal{D}^{1,p}_0(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$. Since $p^* \to p$ as $N \to \infty$, the gain in the integrability of u disappears as $N \to \infty$. Thus, it is natural to look for an inequality that is dimension independent and plays the role of Sobolev inequality. One such inequality is the Gross's *logarithmic Sobolev inequality* [41]:

$$\int_{\mathbb{R}^N} |u|^2 \log |u|^2 \ d\mu \leq 2 \int_{\mathbb{R}^N} |\nabla u|^2 \ d\mu, \ \forall u \in \mathrm{C}_c^1(\mathbb{R}^N)\,, \tag{1.3.1}$$

where μ is a probability measure given by $d\mu(x) = (2\pi)^{-\frac{N}{2}} e^{-\frac{|x|^2}{2}} \ dx$ and $\displaystyle\int_{\mathbb{R}^N} |u|^2 d\mu = 1$. This shows that the Sobolev space $\mathrm{H}_0^1(\mathbb{R}^N, d\mu)$ is embedded into the Orlicz space $L^2(LogL)(\mathbb{R}^N, d\mu)$ and the gain in the integrability of u does not depend on N. An analogue of (1.3.1) for the Lebesgue measure is obtained in [72] (for $p = 2$) and in [30] (for general p), namely,

$$\int_{\mathbb{R}^N} |u|^p \log |u|^p \ dx \leq \frac{N}{p} \log \left(\mathrm{C} \int_{\mathbb{R}^N} |\nabla u|^p \ dx \right), \ \forall u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N) \tag{1.3.2}$$

with $\displaystyle\int_{\mathbb{R}^N} |u|^p \ dx = 1$, for some $\mathrm{C} > 0$. Unlike (1.3.1), the integrability of u (with respect to Lebesgue measure) follows from (1.3.2) is not dimension independent. This form of logarithmic Sobolev inequality arises in the study of heat-diffusion semigroup, see [72].

We are interested to identify a general class of weight functions $g \in L^1_{loc}(\mathbb{R}^N)$ such that the following *weighted logarithmic Sobolev inequality*:

$$\int_{\mathbb{R}^N} |g||u|^p \log |u|^p \ dx \leq \gamma \log \left(C_\gamma \int_{\mathbb{R}^N} |\nabla u|^p \ dx \right), \quad \forall u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N) \tag{1.3.3}$$

with $\int_{\mathbb{R}^N} |g||u|^p = 1$ holds for some $\gamma, C_\gamma > 0$. We define the space $\mathcal{H}_{p,q}(\mathbb{R}^N)$ consisting of all $g \in L^1_{loc}(\mathbb{R}^N)$ such that the following weighted Hardy-Sobolev inequality holds:

$$\left[\int_{\mathbb{R}^N} |g||u|^q \ dx \right]^{\frac{1}{q}} \leq \mathrm{C} \left[\int_{\mathbb{R}^N} |\nabla u|^p \ dx \right]^{\frac{1}{p}}, \quad \forall u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)\,. \tag{1.3.4}$$

We call a function $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$ as a (p,q)-**Hardy potential**. Now, analogous to the norm $\|.\|_{\mathcal{H}_p}$ on $\mathcal{H}_p(\Omega)$, for $1 < p \leq q \leq p^*$, we define

$$\|g\|_{\mathcal{H}_{p,q}} = \sup_{F \subset\subset \mathbb{R}^N} \left\{ \frac{\displaystyle\int_F |g| \; dx}{[\mathrm{Cap}_p(F)]^{\frac{q}{p}}} \right\}.$$

By Maz′ya's result [58, Theorem 8.5], it follows that

$$\mathcal{H}_{p,q}(\mathbb{R}^N) = \left\{ g \in L^1_{loc}(\mathbb{R}^N) : \|g\|_{\mathcal{H}_{p,q}} < \infty \right\}.$$

$\mathcal{H}_{p,q}(\mathbb{R}^N)$ is a Banach function space. For $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$, we prove the following result.

Theorem 1.3.1. *[27, Theorem 1.1] Let $N \geq 3$, $p \in (1, N)$ and $q \in (p, p^*]$. If $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} |g||u|^p \log |u|^p \; dx \leq \frac{q}{q-p} \log \left(C_H \|g\|^{\frac{p}{q}}_{\mathcal{H}_{p,q}} \int_{\mathbb{R}^N} |\nabla u|^p \; dx \right), \tag{1.3.5}$$

for all $u \in \mathcal{D}^{1,p}_0(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} |g||u|^p \; dx = 1$, where $C_H = p^p(p-1)^{(1-p)}$.

Notice that, for $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$, inequality (1.3.3) holds for $\gamma \geq \frac{q}{q-p}$. Let $\mathrm{C}_B(g, \gamma)$ be the best constant in (1.3.3). Then,

$$\frac{1}{\mathrm{C}_B(g,\gamma)} = \inf \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^p \; dx}{e^{\frac{1}{\gamma}\left(\int_{\mathbb{R}^N} |g||u|^p \log |u|^p \; dx\right)}} : u \in \mathcal{D}^{1,p}_0(\mathbb{R}^N), \int_{\mathbb{R}^N} |g||u|^p \; dx = 1 \right\}.$$

It is clear that $\mathrm{C}_B(g,\gamma) \leq \mathrm{C}_H \|g\|^{\frac{p}{q}}_{\mathcal{H}_{p,q}}$ for $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$ with $\gamma \geq \frac{q}{q-p}$. Next we would like to find $g \in \mathcal{H}_p(\Omega)$ and values of γ for which $\mathrm{C}_B(g,\gamma)$ is attained in $\mathcal{D}^{1,p}_0(\mathbb{R}^N)$. In this context, we define the closed sub-space

$$\mathcal{F}\mathcal{H}_{p,q}(\mathbb{R}^N) = \overline{\mathrm{C_c}(\mathbb{R}^N)} \text{ in } \mathcal{H}_{p,q}(\mathbb{R}^N).$$

Now we have the following result:

Theorem 1.3.2. *[27, Theorem 1.2] Let $N \geq 3$, $p \in (1, N)$ and $q \in (p, p^*]$. If $g \in \mathcal{H}_{p,q}(\mathbb{R}^N) \cap \mathcal{FH}_{p,p}(\mathbb{R}^N)$ and $\gamma > \frac{q}{q-p}$, then $\mathrm{C}_B(g, \gamma)$ is attained in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$.*

1.4 The logarithmic-Hardy potentials

In this thesis, we discuss another inequality called *logarithmic Hardy inequality.* In [31] authors obtained the following logarithmic Hardy inequality:

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \log\left(|x|^{N-2}|u|^2\right)\, dx \leq \frac{N}{2} \log\left(C \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right), \tag{1.4.1}$$

for all $u \in \mathrm{C}_c^1(\mathbb{R}^N)$ with $\displaystyle\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx = 1$. Instead of $|x|$, we consider distance from a general closed set E in $\mathbb{R}^N$ and generalise this inequality. In this regard, we recall the notion of Assouad dimension of a set.

Definition 1.4.1 (Assouad dimension)**.** For a subset E of $\mathbb{R}^N$, $\eta(E, r)$ denotes the minimal number of open balls of radius r with centers in E that are needed to cover the set E. Let

$$\Lambda = \left\{ \lambda \geq 0 : \exists\, C_\lambda > 0 \text{ so that } \eta(E \cap B_R(x), r) \leq C_\lambda \left(\frac{r}{R}\right)^{-\lambda}, \right.$$
$$\left. \forall x \in E, 0 < r < R < \mathrm{diam}(E) \right\}.$$

The $\inf \Lambda$ is called the Assouad dimension of E and it is denoted by $\dim_A(E)$.

Now we state our result.

Theorem 1.4.2. *[27, Theorem 1.3] Let $N \geq 3$, $p \in (1, N)$ and E be a closed set in $\mathbb{R}^N$ with $\dim_{\mathrm{A}}(E) = d < N$. Then, for $a \in (-\frac{(N-d)(p-1)}{p}, \frac{(N-p)(N-d)}{Np})$, there exists*

$C > 0$ *such that*

$$\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}} \log\left(\delta_E^{N-p-pa}|u|^p\right) \, dx \le \frac{N}{p} \log\left(C \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{\delta_E^{pa}} \, dx \right), \tag{1.4.2}$$

for all $u \in \mathrm{C}^1_\mathrm{c}(\mathbb{R}^\mathrm{N})$ *with* $\displaystyle\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}} \, dx = 1.$

We also obtained a second order analogue of (1.4.2) as in the following theorem.

Theorem 1.4.3. *[27, Theorem 1.4] Let* $N \ge 3$, $p \in (1, \frac{N}{2})$ *and* E *be a closed set in* $\mathbb{R}^N$ *with* $\dim_A(E) = d < \frac{N(N-2p)}{(N-p)}$. *Then, for each* $a \in (1 - \frac{(N-d)(p-1)}{p}, \frac{(N-p)(N-d)}{Np})$ *there exists* $C > 0$ *such that*

$$\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}} \log\left(\delta_E^{N-p-pa}|u|^p\right) \, dx \le \frac{N}{p} \log\left(C \int_{\mathbb{R}^N} \frac{|\nabla^2 u|^p}{\delta_E^{(a-1)p}} \, dx \right) \tag{1.4.3}$$

for all $u \in \mathrm{C}^2_\mathrm{c}(\mathbb{R}^\mathrm{N})$ *with* $\displaystyle\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}} \, dx = 1$, *where* $\nabla^2 u$ *denotes the Hessian matrix of* u *and* $\displaystyle|\nabla^2 u|^2 = \sum_{i,j=1}^N \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2$.

Chapter 2

Preliminaries

2.1 Symmetrization

For a domain Ω in $\mathbb{R}^N$, let $\mathcal{M}(\Omega)$ be the set of all extended real valued Lebesgue measurable functions that are finite a.e. in Ω. For $f \in \mathcal{M}(\Omega)$ and for $s > 0$, we define the *one dimensional decreasing rearrangement* f^* of f as below:

$$f^*(t) := \begin{cases} \text{ess } \sup f, & t = 0 \\ \inf \big\{ s > 0 \, : \, \big| \{ x : |f(x)| > s \} \big| \leq t \big\}, \ t > 0 \, , \end{cases}$$

where $|A|$ denotes the Lebesgue measure of a set $A \subseteq \mathbb{R}^N$. Here we have used the convention that $\inf \phi = \infty$. The map $f \mapsto f^*$ is not sub-additive i.e., $(f+g)^* \nleq f^* + g^*$. However, we define a sub-additive function using f^* as below:

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(\tau) d\tau, \quad t > 0.$$

The sub-additivity of f^{**} with respect to f helps us to define norms in certain function spaces. We refer to [34, 45] for more details on symmetrization.

In the next proposition, we enlist some properties of f^*, see [45] or [34] for proof.

Proposition 2.1.1. *For $f, g \in \mathcal{M}(\Omega)$, the following statements are true.*

(i) *f^* is non-negative and decreasing,*

(ii) *f^* is right continuous,*

(iii) *$|f| \le |g|$ implies $f^* \le g^*$,*

(iv) *$(cf)^* = |c| f^*$, $c \in \mathbb{R}$,*

(v) *f and f^* are equimeasurable i.e., for all $s > 0$*

$$\left|\{x \in \Omega : |f(x)| > s\}\right| = \left|\{t \in [0, |\Omega|) : f^*(t) > s\}\right|,$$

(vi) *for $p \in [1, \infty]$ and $f \in L^p(\Omega)$, $f^* \in L^p((0, |\Omega|))$, and $\|f\|_{L^p(\Omega)} = \|f^*\|_{L^p((0,|\Omega|))}$.*

The *Schwarz symmetrization* of f is defined by

$$f^\star(x) = f^*(\omega_N |x|^N), \quad \forall\, x \in \Omega^\star,$$

where ω_N is the measure of the unit ball in $\mathbb{R}^N$ and $\Omega^\star$ is the open ball centered at the origin with same measure as Ω. Next, we state two important inequalities concerning the Schwarz symmetrization.

Proposition 2.1.2. *Let $N \ge 2$.*

(i) *Hardy-Littlewood inequality [34, Theorem 3.2.10]: Let f and g be nonnegative measurable functions on Ω. Then*

$$\int_\Omega f(x) g(x)\, dx \le \int_{\Omega^\star} f^\star(x) g^\star(x)\ dx = \int_0^{|\Omega|} f^*(t) g^*(t)\ dt.$$

(ii) *Pólya-Szegö inequality [63]: Let $1 \le p < \infty$. Then*

$$\int_{\Omega^\star} |\nabla \phi^\star(x)|^p\ dx \le \int_\Omega |\nabla \phi(x)|^p\ dx\,,\ \forall \phi \in \mathcal{D}_0^{1,p}(\Omega)\,.$$

The Pólya-Szegö type inequality does not hold for the second-order derivatives. In general, the Schwarz symmetrization of a $\mathcal{D}_0^{2,p}(\mathbb{R}^N)$ function does not admit the second-order weak derivatives; even if they do, the second-order derivatives may not satisfy the Pólya-Szegö type inequality, see [59, 22] for more discussion on this. Next, we state an inequality (1.14 of [23]) that plays the role of Pólya-Szegö inequality for the second-order derivatives. This inequality is obtained using the rearrangement inequality for the convolution due to O'Neil [61].

Lemma 2.1.3 (A point-wise rearrangement inequality). *For $u \in \mathrm{C}_c^\infty(\mathbb{R}^\mathrm{N})$ with $N \geq 3$, let u^* be the decreasing rearrangement of u. Then the following inequality holds:*

(2.1.1)
$$u^*(s) \leq \frac{1}{2(N-2)\omega_N^{\frac{2}{N}}} \left(s^{-1+\frac{2}{N}} \int_0^s |\Delta u|^*(t)\ dt + \int_s^\infty |\Delta u|^*(t) t^{-1+\frac{2}{N}}\ dt \right), \ \forall\, s > 0.$$

2.2 The Banach function spaces

Definition 2.2.1 (Banach function space). *A normed linear space $(X(\Omega), \|.\|_X)$ of functions in $\mathcal{M}(\Omega)$ is called a Banach function space if the following conditions are satisfied:*

1. $\|f\|_X = \|\ |f|\ \|_X$, *for all* $f \in X(\Omega)$,
2. *if (f_n) is a non-negative sequence of function in $X(\Omega)$, increases to f, then $\|f_n\|_X$ increases to $\|f\|_X$.*

The norm $\|.\|_X$ is called a Banach function space norm on $X(\Omega)$ [73, Section 30, Chapter 6]. Indeed, the Banach function spaces are complete [73, Theorem 2, Section 30, Chapter 6]. Corresponding to a Banach function space $(X(\Omega), \|.\|_X)$, we also have a notion of 'associated Banach function space'.

Definition 2.2.2 (Associate space)**.** *Let* $(X(\Omega), \|.\|_X)$ *be a Banach function space. For* $u \in \mathcal{M}(\Omega)$*, define*

$$\|u\|_{X'} = \sup\left\{\int_\Omega |fu| : f \in X(\Omega),\ \|f\|_X \le 1\right\}.$$

Then the associate space $X(\Omega)'$ *of* $X(\Omega)$ *is given by*

$$X(\Omega)' = \big\{u \in \mathcal{M}(\Omega) : \|u\|_{X'} < \infty\big\}.$$

Indeed, $X(\Omega)'$ is also a Banach function space with respect to the norm $\|.\|_{X'}$. We refer to [73, 16] for further readings on Banach function spaces.

The Lebesgue spaces, Lorentz spaces, and Lorentz-Zygmund spaces are classical examples of Banach function space. Next, we discuss these spaces in detail.

2.2.1 The Lorentz spaces

The Lorentz spaces are two parameter family of function spaces introduced by Lorentz in [52] that refine the classical Lebesgue spaces. Let Ω be an open set in $\mathbb{R}^N$ and $f \in \mathcal{M}(\Omega)$. For $(p,q) \in (0,\infty) \times (0,\infty]$ we consider the following quantity:

(2.2.1)

$$|f|_{L^{p,q}} := \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q((0,|\Omega|))} = \begin{cases} \left(\displaystyle\int_0^\infty \left(t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\right)^q \mathrm{d}t\right)^{\frac{1}{q}}, & 0 < q < \infty; \\ \displaystyle\sup_{t>0} t^{\frac{1}{p}} f^*(t), & q = \infty. \end{cases}$$

The Lorentz space $L^{p,q}(\Omega)$ is defined as

$$L^{p,q}(\Omega) := \big\{f \in \mathcal{M}(\Omega) : |f|_{L^{p,q}} < \infty\big\},$$

where $|f|_{L^{p,q}}$ is a complete quasi norm on $L^{p,q}(\Omega)$. For $(p,q) \in (1,\infty] \times (0,\infty]$, let

$$\|f\|_{L^{p,q}} = \|t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t)\|_{L^q((0,|\Omega|))}.$$

Then $\|f\|_{L^{p,q}}$ is a norm on $L^{p,q}(\Omega)$ and it is equivalent to $|f|_{L^{p,q}}$ [34, Lemma 3.4.6]. Note that $L^{p,p}(\Omega) = L^p(\Omega)$ for $p \in [1,\infty)$. For a detailed study on the Lorentz spaces, we refer to [1, 34].

In the following proposition we list some properties of the Lorentz spaces.

Proposition 2.2.3. *Let $p, q, \tilde{p}, \tilde{q} \in [1,\infty]$.*

(i) *For $\alpha > 0$, $\||f|^\alpha\|_{L^{\frac{p}{\alpha},\frac{q}{\alpha}}} = \|f\|^\alpha_{L^{p,q}}$.*

(ii) *Generalized Hölder inequality: Let $f \in L^{p_1,q_1}(\Omega)$ and $g \in L^{p_2,q_2}(\Omega)$, where $(p_i, q_i) \in (1,\infty) \times [1,\infty]$ for $i = 1,2$. If (p,q) be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, then*

$$\|fg\|_{L^{p,q}} \leq C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}},$$

where $C = C(p) > 0$ is a constant such that $C = 1$, if $p = 1$ and $C = p'$, if $p > 1$.

(iii) *If $q \leq \tilde{q}$, then $L^{p,q}(\Omega) \hookrightarrow L^{p,\tilde{q}}(\Omega)$, i.e., there exists a constant $C > 0$ such that*

$$\|f\|_{L^{p,\tilde{q}}} \leq C\|f\|_{L^{p,q}}, \quad \forall\, f \in L^{p,q}(\Omega). \tag{2.2.2}$$

(iv) *If $\tilde{p} < p$, then $L^{p,q}(\Omega) \hookrightarrow L^{\tilde{p},\tilde{q}}_{loc}(\Omega)$.*

(v) *Let $1 < p < \infty$ and $1 \leq q \leq \infty$ (or $p = q = 1$). Then the dual space of $L^{p,q}(\Omega)$ is, up to equivalence of norms, the Lorentz space $L^{p',q'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.*

Proof. Proof of (i) directly follows using the definition of the Lorentz space. Proof of (ii) follows using [44, Theorem 4.5]. For the proof of (iii) and (iv), see [34, Proposition 3.4.3 and Proposition 3.4.4] and a proof of (v) can be found in [16, Corollary 4.8, page 221]. $\square$

Next proposition identifies the associate space of Lorentz spaces, see [16, Theorem 4.7, page 220].

Proposition 2.2.4. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$ (or $p = q = 1$ or $p = q = \infty$). Then the associate space of $L^{p,q}(\Omega)$ is, up to equivalence of norms, the Lorentz space $L^{p',q'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.*

Notice that, for $1 < p < \infty$ and $1 \leq q \leq \infty$ (or $p = q = 1$), the associate space of $L^{p,q}(\Omega)$ is same as the dual space (by Proposition 2.2.3-(v) and Proposition 2.2.4). This is not just a mere coincidence. In general, we have the following result [16, Theorem 4.1, Chapter 1, page 20].

Proposition 2.2.5. *Let $(X(\Omega), \|.\|_X)$ be a Banach function space which is an ordered ideal of $\mathcal{M}(\Omega)$ i.e., if $|f| \leq |g|$ a.e. in Ω and $g \in X(\Omega)$, then $f \in X(\Omega)$. Further, assume that $X(\Omega)$ contains all the simple functions. Then $X(\Omega)^* = X(\Omega)'$ if and only if every function in $X(\Omega)$ has absolute continuous norm.*

2.2.2 The Lorentz-Zygmund spaces

The Lorentz-Zygmund spaces refine the Lorentz spaces. For more information on Lorentz-Zygmund spaces, we refer to [14]. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $l_1(t) = \log\left(\frac{e|\Omega|}{t}\right)$. Given a function $f \in \mathcal{M}(\Omega)$ and for $(p, q, \alpha) \in (0, \infty] \times (0, \infty] \times \mathbb{R}$, we consider the following quantity:

$$|f|_{L^{p,q}(\log L)^\alpha} := \|t^{\frac{1}{p}-\frac{1}{q}} l_1(t)^\alpha f^*(t)\|_{L^q((0,|\Omega|))}$$
$$= \begin{cases} \left(\int_0^{|\Omega|} \left(t^{\frac{1}{p}-\frac{1}{q}} l_1(t)^\alpha f^*(t) \right)^q \, \mathrm{d}t \right)^{\frac{1}{q}}, & 0 < q < \infty; \\ \sup\limits_{0<t<|\Omega|} t^{\frac{1}{p}} l_1(t)^\alpha f^*(t), & q = \infty. \end{cases}$$

Then the Lorentz-Zygmund space $L^{p,q}(\log L)^\alpha(\Omega)$ is defined as

$$L^{p,q}(\log L)^\alpha(\Omega) := \left\{ f \in \mathcal{M}(\Omega) : |f|_{L^{p,q}(\log L)^\alpha} < \infty \right\},$$

where $|f|_{L^{p,q}(\log L)^\alpha}$ is the quasi norm on $L^{L^{p,q}(\log L)^\alpha}(\Omega)$. Observe that, if $\alpha = 0$, then $L^{p,q}(\log L)^\alpha(\Omega)$ coincide with the Lorentz space $L^{p,q}(\Omega)$. For $(p,q,\alpha) \in (1,\infty) \times [1,\infty] \times \mathbb{R}$,

$$\|f\|_{L^{p,q}(\log L)^\alpha} := \|t^{\frac{1}{p}-\frac{1}{q}} l_1(t)^\alpha f^{**}(t)\|_{L^q((0,|\Omega|))} \tag{2.2.3}$$

is a norm in $L^{p,q}(\log L)^\alpha(\Omega)$ equivalent to $|f|_{L^{p,q}(\log L)^\alpha}$ [14, Corollary 8.2]. The next proposition provides the equivalence of the quasinorm $|u|_{L^{\infty,2}(\log L)^{-1}}$ and the norm $\|u\|_{L^{\infty,2}(\log L)^{-1}}$. We adapt the proof of Theorem 6.4 of [15] to our case.

Proposition 2.2.6. *Let Ω be a bounded subset of $\mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ be a measurable function. Then there exist a constant $C > 0$ such that*

$$\int_0^{|\Omega|} \left(\frac{u^{**}(t)}{\log(\frac{e|\Omega|}{t})} \right)^2 \frac{dt}{t} \le C \int_0^{|\Omega|} \left(\frac{u^*(t)}{\log(\frac{e|\Omega|}{t})} \right)^2 \frac{dt}{t}.$$

Proof. Choose $0 < \delta < 1$ and write $u^*(s) = [s^\delta u^*(s)][s^{1-\delta}]s^{-1}$. Using the Hölder's inequality we obtain,

$$\left| \int_0^t u^*(s) ds \right|^p \le C_1 t^{q-q\delta} \left(\int_0^t [s^\delta u^*(s)]^q \frac{ds}{s} \right). \tag{2.2.4}$$

Multiplying by $\left(\dfrac{1}{t^{q+1}(\log(\frac{e|\Omega|}{t}))^q}\right)$ and integrating over $(0, |\Omega|)$ we get

$$\begin{aligned}
\int_0^{|\Omega|}\left|\frac{u^{**}(t)}{\log(\frac{e|\Omega|}{t})}\right|^q \frac{dt}{t} &\leq C_1\int_0^{|\Omega|}\frac{1}{t^{q\delta}|\log(\frac{e|\Omega|}{t})|^q}\left(\int_0^t [s^\delta u^*(s)]^q\frac{ds}{s}\right)\frac{dt}{t}\\
&\leq C_1\int_0^{|\Omega|}[s^\delta u^*(s)]^q\left(\int_s^{|\Omega|}\frac{1}{t^{q\delta}|\log(\frac{e|\Omega|}{t})|^q}\frac{dt}{t}\right)\frac{ds}{s}\\
&\leq C_1\int_0^{|\Omega|}\frac{[s^\delta u^*(s)]^q}{s^{(q-1)\delta}|\log(\frac{e|\Omega|}{s})|^q}\left(\int_s^{|\Omega|}\frac{dt}{t^{1+\delta}}\right)\frac{ds}{s}. \qquad (2.2.5)
\end{aligned}$$

The last two inequalities of (2.2.5) follows from Fubini's theorem and monotonic decreasing property of $\dfrac{1}{t^{(q-1)\delta}|\log(\frac{e|\Omega|}{t})|^q}$ respectively. Further, we estimate the right hand side of (2.2.5) as below,

(2.2.6)

$$\int_0^{|\Omega|}\frac{[s^\delta u^*(s)]^q}{s^{(q-1)\delta}|\log(\frac{e|\Omega|}{s})|^q}\left(\int_s^{|\Omega|}\frac{dt}{t^{1+\delta}}\right)\frac{ds}{s} \leq C_2\int_0^{|\Omega|}\frac{[s^\delta u^*(s)]^q}{s^{(q-1)\delta}|\log(\frac{e|\Omega|}{s})|^q}\left(\frac{1}{s^\delta}\right)\frac{ds}{s}.$$

Hence by combining (2.2.5) and (2.2.6) we have the following inequality as required

$$\int_0^{|\Omega|}\left|\frac{u^{**}(t)}{\log(\frac{e|\Omega|}{t})}\right|^q\frac{dt}{t} \leq C\int_0^{|\Omega|}\left|\frac{u^{*}(t)}{\log(\frac{e|\Omega|}{t})}\right|^q\frac{dt}{t}.$$

□

In the following proposition we discuss some important properties of the Lorentz-Zygmund spaces.

Proposition 2.2.7. *Let $p, q, \tilde{q} \in [1,\infty]$ and $\alpha, \beta \in (-\infty, \infty)$.*

(i) Let $p, q \in (1,\infty], \alpha \in \mathbb{R}$, and $\gamma > 0$. Then there exists $C > 0$ such that $\||f|^\gamma\|_{L^{\frac{p}{\gamma},\frac{q}{\gamma}}(\log L)^{\alpha\gamma}} \leq C\|f\|^\gamma_{L^{p,q}(\log L)^\alpha}$, $\forall\, f \in L^{p,q}(\log L)^\alpha(\Omega)$.

(ii) If either $q \leq \tilde{q}$ and $\alpha \geq \beta$ or, $q > \tilde{q}$ and $\alpha + \frac{1}{q} > \beta + \frac{1}{\tilde{q}}$, then $L^{p,q}(\log L)^\alpha(\Omega) \hookrightarrow$

$L^{p,\tilde{q}}(\log L)^{\beta}(\Omega)$, i.e., there exists $C > 0$ such that

$$\|f\|_{L^{p,\tilde{q}}(\log L)^{\beta}} \leq C\|f\|_{L^{p,q}(\log L)^{\alpha}}, \quad \forall\, f \in L^{p,q}(\log L)^{\alpha}(\Omega).$$

Proof. (i) This assertion immediately follows from the definition of the Lorentz-Zygmund spaces.

(ii) Proof follows using [14, Theorem 9.3]. □

2.3 p-capacity and Maz′ya's condition

Definition 2.3.1 (p-capacity)**.** *For $F \subset\subset \Omega$, the p-capacity of F relative to Ω is defined as*

$$\mathrm{Cap}_p(F, \Omega) = \inf\left\{ \int_{\Omega} |\nabla u|^p \, dx : u \in \mathcal{N}_p(F) \right\},$$

where $\mathcal{N}_p(F) = \left\{ u \in \mathcal{D}_0^{1,p}(\Omega) : u \geq 1 \text{ in a neighbourhood of } F \right\}$.

If $\Omega = \mathbb{R}^N$, we write $\mathrm{Cap}_p(F, \mathbb{R}^N) = \mathrm{Cap}_p(F)$. Here we enlist some properties of capacity that will be used in the subsequent chapters.

Proposition 2.3.2. *(a) If $\Omega_1 \subseteq \Omega_2$ are open in $\mathbb{R}^N$, then $\mathrm{Cap}_p(., \Omega_2) \leq \mathrm{Cap}_p(., \Omega_1)$.*

(b) Cap_p is an outer measure on $\mathbb{R}^N$.

(c) For $\lambda > 0$ and $F \subset\subset \mathbb{R}^N$, $\mathrm{Cap}_p(\lambda F) = \lambda^{N-p}\mathrm{Cap}_p(F)$.

(d) For $F \subset\subset \mathbb{R}^N$, $\exists C > 0$ depending on p, N such that $|F| \leq C\mathrm{Cap}_p(F)^{\frac{N}{N-p}}$.

(e) For $N > p$, $\mathrm{Cap}_p(B_1) = N\omega_N \left(\frac{N-p}{p-1}\right)^{p-1}$, where B_1 is the unit ball in $\mathbb{R}^N$.

(f) $\mathrm{Cap}_p(L(F)) = \mathrm{Cap}_p(F)$, for any affine isometry $L : \mathbb{R}^N \mapsto \mathbb{R}^N$.

Proof. (a) Follows easily from the definition of capacity.

(b) See Theorem 4.14 of [35](page 174).

$(c), (d), (f)$ See Theorem 4.15 of [35](page 175).

(e) Section 2.2.4 of [57] (page 106). $\square$

Remark 2.3.3. If a set A is measurable with respect to Cap_p then $\text{Cap}_p(A)$ must be 0 or ∞ [35, Theorem 4.14, Page 174].

Next, we prove an interesting property of capacity, which allows us to localize the norm on $\mathcal{H}_p(\Omega)$.

Lemma 2.3.4. *There exists $C_1, C_2 > 0$ such that for each $x \in \overline{\Omega}$ and $F \subset\subset \Omega$,*

(i) $\text{Cap}_p(F \cap B_r(x), \Omega \cap B_{2r}(x)) \leq C_1 \text{Cap}_p(F \cap B_r(x), \Omega), \ \forall r > 0.$

(ii) $\text{Cap}_p(F \cap B_{2R}^c, \Omega \cap \overline{B_R}^c) \leq C_2 \text{Cap}_p(F \cap B_{2R}^c, \Omega), \ \forall R > 0.$

Proof. (i) Let $\Phi \in \mathrm{C}_c^\infty(\mathbb{R}^N)$ be such that $0 \leq \Phi \leq 1$, $\Phi = 1$ on $\overline{B_1(0)}$ and $Supp(\Phi) \subseteq B_2(0)$. Take $\Phi_r(z) = \Phi(\frac{z-x}{r})$. Let $\epsilon > 0$ be given. Then for $F \subset\subset \Omega$, $\exists u \in \mathcal{N}_p(F \cap B_r(x))$ such that $\int_\Omega |\nabla u|^p \, dx < \text{Cap}_p(F \cap B_r(x), \Omega) + \epsilon$. If we set $w_r(z) = \Phi_r(z) u(z)$, then it is easy to see that $w_r \in \mathcal{D}_0^{1,p}(\Omega \cap B_{2r}(x))$ and $w_r \geq 1$ on $F \cap B_r(x)$. Further, we have the following estimate:

$$
\begin{aligned}
\int_\Omega |\nabla w_r|^p \, dx &\leq C \left[\int_\Omega |\Phi_r|^p |\nabla u|^p \, dx + \int_\Omega |u|^p |\nabla \Phi_r|^p \, dx \right] \\
&\leq C \left[\int_\Omega |\nabla u|^p \, dx + \left(\int_\Omega |u|^{p*} \, dx \right)^{p/p^*} \left(\int_\Omega |\nabla \Phi_r|^N \, dx \right)^{p/N} \right].
\end{aligned}
$$

By noticing $\int_\Omega |\nabla \Phi_r|^N \, dx \leq \int_{\mathbb{R}^N} |\nabla \Phi|^N \, dx$ and then using the Sobolev embedding, we obtain

$$
\int_\Omega |\nabla w_r|^p \, dx \leq C_1 \int_\Omega |\nabla u|^p \, dx,
$$

where C_1 is a constant independent of F, r and ϵ. Therefore,

$$
\text{Cap}_p(F \cap B_r(x), \Omega \cap B_{2r}(x)) \leq C_1 \text{Cap}_p(F \cap B_r(x), \Omega) + C_1 \epsilon.
$$

Now as $\epsilon > 0$ is arbitrary we obtain the desired result.

(ii) For $\Phi \in C_b^\infty(\mathbb{R}^N)$ with $0 \le \Phi \le 1$, $\Phi = 0$ on $\overline{B_1}(0)$ and $\Phi = 1$ on $B_2(0)^c$, we take $\Phi_R(z) = \Phi(\frac{z}{R})$. The rest of the proof is similar to the proof of (i). □

Let us recall that the space of (p,q)-Hardy potentials $\mathcal{H}_{p,q}(\Omega)$ consists of all $g \in L^1_{loc}(\Omega)$ such that

$$\left[\int_{\mathbb{R}^N} |g||u|^q \, dx\right]^{\frac{1}{q}} \le \mathrm{C} \left[\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right]^{\frac{1}{p}}, \quad \forall u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)\,.$$

If $g \in \mathcal{H}_{p,q}(\Omega)$, it can be easily verified that

$$\sup_{F \subset\subset \Omega} \frac{\int_F |g| \, dx}{[\mathrm{Cap}_p(F,\Omega)]^{\frac{q}{p}}} \le C.$$

Furthermore, Maz'ya proved that the above condition is also sufficient for g to be in $\mathcal{H}_{p,q}(\Omega)$ [58, Theorem 8.5]. Motivated by this, for $g \in L^1_{loc}(\Omega)$, we define

$$\|g\|_{\mathcal{H}_{p,q}} = \sup \left\{ \frac{\int_F |g| \, dx}{[\mathrm{Cap}_p(F,\Omega)]^{\frac{q}{p}}} : F \subset\subset \Omega; |F| \neq 0 \right\}.$$

Now we state the Mazya's condition as follows.

Theorem 2.3.5. *[58, Theorem 8.5. Maz'ya's condition.] Let* $1 < p \le q < \infty$. *Then* $g \in \mathcal{H}_{p,q}(\Omega)$ *if and only if*

$$\left[\int_{\Omega} |g||u|^q \, dx\right]^{\frac{p}{q}} \le C_H \|g\|^{\frac{p}{q}}_{\mathcal{H}_{p,q}(\Omega)} \left[\int_{\Omega} |\nabla u|^p \, dx\right], \quad \forall u \in \mathcal{D}_0^{1,p}(\Omega)\,, \tag{2.3.1}$$

where $C_H = p^p(p-1)^{(1-p)}$.

The above theorem immediately identifies the space of (p,q)-Hardy potentials as

$$\mathcal{H}_{p,q}(\Omega) = \left\{ g \in L^1_{loc}(\Omega) : \|g\|_{\mathcal{H}_{p,q}} < \infty \right\}.$$

In fact, $\mathcal{H}_{p,q}(\Omega)$ is a Banach function space with respect to the norm $\|.\|_{\mathcal{H}_{p,q}}$. It is worth mentioning that $\mathcal{H}_{p,p}(\Omega) = \mathcal{H}_p(\Omega)$.

Consider the associate space $\mathcal{H}_p(\Omega)'$ of $\mathcal{H}_p(\Omega)$ as defined below:

$$\|u\|_{\mathcal{H}_p'} = \sup\left\{\int_\Omega |fu|\,dx : f \in X,\ \|f\|_{\mathcal{H}_p} \le 1\right\},$$
$$\mathcal{H}_p(\Omega)' = \left\{u \in \mathcal{M}(\Omega)\ :\ \|u\|_{\mathcal{H}_p'} < \infty\right\}.$$

Now, we define

$$\mathcal{E}_p(\Omega) := \left\{u \in \mathcal{M}(\Omega) : |u|^p \in \mathcal{H}_p(\Omega)'\right\}$$

equipped with the norm

$$\|u\|_{\mathcal{E}_p} := \left(\||u|^p\|_{\mathcal{H}_p(\Omega)'}\right)^{\frac{1}{p}}.$$

Clearly $v \in \mathcal{E}_p(\Omega)$ if and only if $|v|^p \in \mathcal{H}_p(\Omega)'$. Later we show that $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow \mathcal{E}_p(\Omega)$ which is finer than the Lorentz-Sobolev embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{p^*,p}(\Omega)$.

2.4 Assouad dimension

In this section, we recall the notion of the Assouad dimension of a set.

Definition 2.4.1 (Assouad dimension)**.** Let (X, d) be a metric space. For a subset E of X, $\eta(E, r)$ denotes the minimal number of open balls of radius r with centers in E that are needed to cover the set E. Let

$$\Lambda = \Bigg\{\lambda \ge 0 : \exists\ C_\lambda > 0 \text{ so that } \eta(E \cap B_R(x), r) \le C_\lambda \left(\frac{r}{R}\right)^{-\lambda},$$
$$\forall x \in E, 0 < r < R < \operatorname{diam}(E)\Bigg\}.$$

The Assouad dimension is denoted by $\dim_A(E)$ and is defined by $\dim_A(E) = \inf \Lambda$.

In the case when $\mathrm{diam}(E) = 0$, we remove the restriction $R < \mathrm{diam}(E)$ from above definition. By this convention one can see that, if $E = \{x_0\}$ for some $x_0 \in X$ then $\dim_A(E) = 0$. We refer to [53] for a historical background of the Assouad dimension and its basic properties. More recent results on this can be found in [39]. Here, we enlist some of its basic properties in the following proposition; for proof, see [53].

Proposition 2.4.2. Let (X, d) be a metric space. Then the following statements are true:

(i) X has finite Assouad dimension if and only if it is a doubling space, i.e. there exists a finite constant $C > 0$ such that every ball of radius r can be covered by no more than C balls of radius $\frac{r}{2}$.

(ii) If $Y \subset X$, then $\dim_A(Y) \leq \dim_A(X)$. Equality holds if $\overline{Y} = X$.

(iii) $\dim_H(X) \leq \dim_A(X)$, where $\dim_H$ denotes the Hausdroff dimension of X.

(iv) Let $X = \mathbb{R}^N$ with usual metric and $E \subset \mathbb{R}^N$. Then $\dim_A(E) < N$ if and only if E is porous in $\mathbb{R}^N$ i.e. there is a constant $\alpha \in (0, 1)$ such that for every $x \in E$ and all $0 < r < \mathrm{diam}(E)$ there exists a point $y \in \mathbb{R}^N$ such that $B_{\alpha r}(y) \subset B_r(x) \setminus E$.

Remark 2.4.3. Let $X = \mathbb{R}^N$ with usual metric. Notice that, for $x \in \partial B_1$ and $r \in (0, 2)$ we can find $y \in B_r(x)$ such that $B_{\frac{r}{4}}(y) \subseteq B_r(x) \setminus \partial B_1$. Hence, by the definition of porousity, it follows that the boundary of a unit ball is porous in $\mathbb{R}^N$. Hence, $\dim_A(\partial B_1) < N$. Similarly, it can be seen that $\mathbb{R}^{N-1} \times \{0\}$ is porous in $\mathbb{R}^N$. Hence, $\dim_A(\mathbb{R}^{N-1} \times \{0\}) < N$.

2.5 Some important results

In this section, we recall two important results: (i) Muckenhoupt condition, (ii) Brézis-Lieb lemma, which will be used extensively in the subsequent chapters.

2.5.1 Muckenhoupt condition

For $p \in (1, \infty)$, we denote its Hölder conjugate by p' which is defined by $\frac{1}{p} + \frac{1}{p'} = 1$. In this sub-section, we recall the Muckenhoupt necessary and sufficient conditions [60, Theorem 1 and Theorem 2] for the one-dimensional weighted Hardy inequalities.

Lemma 2.5.1. *Let u, v be nonnegative measurable functions such that $v > 0$. Then for any $a \in (0, \infty]$,*

(i) the inequality

$$\int_0^a \left| \int_0^s f(t)\ dt \right|^p u(s)\ ds \leq C \int_0^a |f(s)|^p v(s)\ ds, \tag{2.5.1}$$

holds for all measurable function f on $(0, a)$ if and only if

$$A_1 := \sup_{0<t<a} \left(\int_t^a u(s)\ ds \right)^{\frac{1}{p}} \left(\int_0^t v(s)^{1-p'}\ ds \right)^{\frac{1}{p'}} < \infty. \tag{2.5.2}$$

(ii) the dual inequality

$$\int_0^a \left| \int_s^a f(t)\ dt \right|^p u(s)\ ds \leq C \int_0^a |f(s)|^p v(s)\ ds, \tag{2.5.3}$$

holds for all measurable function f on $(0, a)$ if and only if

$$A_2 := \sup_{0<t<a} \left(\int_0^t u(s)\ ds \right)^{\frac{1}{p}} \left(\int_t^a v(s)^{1-p'}\ ds \right)^{\frac{1}{p'}} < \infty. \tag{2.5.4}$$

Remark 2.5.2. Let B_1, B_2 be the best constants in (2.5.1) and (2.5.3) respectively. Then,

$$A_i \leq B_i \leq (p)^{\frac{1}{p}} (p')^{\frac{1}{p'}} A_i \, , \; i = 1, 2 \, .$$

2.5.2 Brézis-Lieb lemma

Let $J : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function with $J(0) = 0$ such that, for every $\epsilon > 0$ there exist two continuous, non-negative functions $\phi_\epsilon, \psi_\epsilon$ satisfying

$$|J(a+b) - J(a)| \leq \epsilon \phi_\epsilon(a) + \psi_\epsilon(b) \, , \; \forall a, b \in \mathbb{R} \, . \tag{2.5.5}$$

Now we state a lemma proved by Brézis and Lieb in [18].

Lemma 2.5.3. *Let* $J : \mathbb{R} \mapsto \mathbb{R}$ *satisfies* (2.5.5) *and* $f_n = f + g_n$ *be a sequence of measurable functions on* Ω *to* $\mathbb{R}$ *such that*

(i) $g_n \to 0$ *a.e.,*

(ii) $J(f) \in L^1(\Omega)$,

(iii) $\displaystyle\int_\Omega \phi_\epsilon(g_n(x)) \, d\mu(x) \leq C < \infty$, *for some* $C > 0$ *independent of* n, ϵ,

(iv) $\displaystyle\int_\Omega \psi_\epsilon(f(x)) \, d\mu(x) < \infty$, *for all* $\epsilon > 0$.

Then

$$\lim_{n \to \infty} \int_\Omega |J(f + g_n) - J(g_n) - J(f)| \, d\mu = 0 \, .$$

We require the following inequality (see [51], page 22) that played an important role in the proof of Brésiz-Lieb lemma: for $a, b \in \mathbb{C}$,

$$\big| |a+b|^p - |a|^p \big| \leq \epsilon |a|^p + C(\epsilon, p) |b|^p \tag{2.5.6}$$

valid for each $\epsilon > 0$ and $0 < p < \infty$.

Remark 2.5.4. If J is convex on $\mathbb{R}$, then J satisfies (2.5.5). In particular, if $J(t) = |t|^p$; $p \in (1, \infty)$, then (2.5.5) is valid with $\phi_\epsilon(t) = |t|^p$ and $\psi_\epsilon(t) = C_\epsilon |t|^p$ for sufficiently large C_ϵ, see [18].

Using the above remark, we have the following special case of Lemma 2.5.3.

Lemma 2.5.5. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and (f_n) be a sequence of real - valued measurable functions which are uniformly bounded in $L^p(\Omega, \mu)$ for some $0 < p < \infty$. Moreover, if (f_n) converges to f a.e., then*

$$\lim_{n\to\infty} \big| \|f_n\|_{(p,\mu)} - \|f_n - f\|_{(p,\mu)} \big| = \|f\|_{(p,\mu)}.$$

Example 2.5.6. *Let $J(t) = t^p \log t$, for $t \geq 0$. Then J is continuous and $J(0) = 0$. Further, for $a, b \geq 0$, using mean value theorem we obtain*

$$\begin{aligned} |J(a+b) - J(a)| &\leq (a+b)\left[(a+b)^{p-1} + p(a+b)^{p-1}\log(a+b)\right] \\ &\leq \begin{cases} (a+b)^p\,, & \text{if } a+b \leq 1\,, \\ (p+1)(a+b)^{p^*}\,, & \text{if } a+b \geq 1\,. \end{cases} \end{aligned}$$

Thus, it follows from Remark 2.5.4 that J satisfies (2.5.5).

Chapter 3

The compactness and the concentration compactness via p-capacity

In this chapter, we study the optimal space of Hardy potentials in detail. Here we prove Theorem 1.1.1, Theorem 1.1.3, Theorem 1.1.5 and Theorem 1.1.6. Maz′ya's p-capacity condition helps us to define the Banach function space norm $\|.\|_{\mathcal{H}_p}$ on the space of Hardy potentials $\mathcal{H}_p(\Omega)$. We identify $\mathcal{F}\mathcal{H}_p(\Omega)$ as the optimal space of Hardy potentials for which the map $G_p(u) = \displaystyle\int_\Omega |g||u|^p \, dx$ is compact on $\mathcal{D}_0^{1,p}(\Omega)$. Further, using the notion of absolute continuous norm on $\mathcal{H}_p(\Omega)$, we characterize the space $\mathcal{F}\mathcal{H}_p(\Omega)$. We derive a variation of the concentration compactness lemma to give a sufficient condition on $g \in \mathcal{H}_p(\Omega)$ so that the best constant in the above inequality is attained in $\mathcal{D}_0^{1,p}(\Omega)$. Also, we establish an embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow \mathcal{E}_p(\Omega)$ which is finer that the Lorentz-Sobolev embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{p^*,p}(\Omega)$.

As we have pointed out in the introduction, various inequalities and embeddings of $\mathcal{D}_0^{1,p}(\Omega)$ helps us to provide many classes of function spaces in $\mathcal{H}_p(\Omega)$. We list them again here:

- $L^{\infty}(\Omega) \subseteq \mathcal{H}_p(\Omega)$ if Ω is bounded in one direction (using Poincaré inequality),
- $L^{\frac{N}{p}}(\Omega) \subseteq \mathcal{H}_p(\Omega)$ [4] (using the Sobolev embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{\frac{Np}{N-p}}(\Omega)$),
- $L^{(\frac{N}{p},\infty)}(\Omega) \subseteq \mathcal{H}_p(\Omega)$ [71] (using the Lorentz-Sobolev embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{(p^*,p)}(\Omega)$).

Further, if $\Omega = B_1^c$ (the exterior of the closed unit ball centered at the origin) then examples of Hardy potentials outside the $L^{(\frac{N}{p},\infty)}(\Omega)$ are provided in [10]. For an open set Ω in $\mathbb{R}^N$ and a radial, non-negative function w on $\mathbb{R}^N$, let us recall that

$$\begin{aligned} L^1_{rad}(\mathbb{R}^N, w) &= \left\{ g \in L^1(\mathbb{R}^N, w) : g \text{ is radial} \right\}, \\ L^1_{rad}(\Omega, w) &= \left\{ \left. g \right|_{\Omega} : g \in L^1_{rad}(\mathbb{R}^N, w) \right\}. \end{aligned}$$

In [10], authors have shown that $L^1_{rad}(B_1^c, |x|^{p-N}) \subseteq \mathcal{H}_p(B_1^c)$ and the next example shows that $L^1_{rad}(B_1^c, |x|^{p-N})$ and $L^{(\frac{N}{p},\infty)}(B_1^c)$ are not contained in one another.

Example 3.0.1. (i) For $p = 2$, $N \geq 3$ and $\beta \in (\frac{2}{N}, 1)$ consider the following function

$$g(x) = \begin{cases} (|x|-1)^{-\beta}, & 1 < |x| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

It can be verified that $g \in L^1_{rad}(B_1^c, |x|^{2-N})$ and $g \notin L^{(\frac{N}{2},\infty)}(B_1^c)$ (Example 3.8 of [9]).

(ii) Let $g_2(x) = \frac{1}{|x|^2}$, $x \in B_1^c$ with $N \geq 3$. By Example 4.1.4, $g_2 \in L^{\frac{N}{2},\infty}(B_1^c)$. On the other hand

$$\int_{B_1^c} g_2(x)|x|^{2-N} \, dx = N\omega_N \int_1^{\infty} \tilde{g}(r) r \, dr \geq N\omega_N \int_1^{\infty} r^{-2} \times r \, dr = \infty.$$

Thus $g_2 \notin L^1_{rad}(B_1^c, |x|^{2-N})$.

3.0.1 Some embeddings

In this subsection, we provide several embedding theorems concerning the spaces $\mathcal{H}_p(\Omega)$ and $\mathcal{D}_0^{1,p}(\Omega)$. The next proposition can be obtained from Lorentz-Sobolev embedding and the generalised Hölder's inequality. However, we give a direct proof using the norm in $\mathcal{H}_p(\Omega)$.

Proposition 3.0.2. *Let $p \in (1, N)$ and an open subset Ω in $\mathbb{R}^N$. Then $L^{\frac{N}{p},\infty}(\Omega)$ is continuously embedded in $\mathcal{H}_p(\Omega)$.*

Proof. Observe that, $\mathrm{Cap}_p(F^\star) \le \mathrm{Cap}_p(F^\star, \Omega^\star) \le \mathrm{Cap}_p(F, \Omega)$. The first inequality comes from (a)-th property of Proposition 2.3.2 and the latter one follows from Polya-Szego inequality. $\mathrm{Cap}_p(F^\star) = N\omega_N(\frac{N-p}{p-1})^{p-1}R^{N-p}$, where R is the radius of $F^\star$ (by (e)-th property of Proposition 2.3.2). Now, for a relatively compact set F,

$$\frac{\int_F |g|(x)dx}{\mathrm{Cap}_p(F,\Omega)} \le \frac{\int_{F^\star} g^\star(x)dx}{\mathrm{Cap}_p(F^\star,\mathbb{R}^N)} = \frac{\int_0^{|F|} g^*(t)dt}{N\omega_N(\frac{N-p}{p-1})^{p-1}R^{N-p}} = \frac{R^p g^{**}(\omega_N R^N)}{N(\frac{N-p}{p-1})^{p-1}}.$$

By setting $\omega_N R^N = t$ we get,

$$\frac{\int_F |g|(x)dx}{\mathrm{Cap}_p(F,\Omega)} \le C(N,p)\|g\|_{L^{\frac{N}{p},\infty}}.$$

Now take the supremum over $F \subset\subset \Omega$ to obtain,

$$\|g\|_{\mathcal{H}_p} \le C(N,p)\|g\|_{L^{\frac{N}{p},\infty}} \text{ with } C(N,p) = \frac{1}{N(\omega_N)^{\frac{p}{N}}(\frac{N-p}{p-1})^{p-1}}.$$

□

As we mentioned before, our proof of Proposition 3.0.2 does not use the Lorentz-Sobolev embedding of $\mathcal{D}_0^{1,p}(\Omega)$. In fact, using the above proposition, we give an alternate proof for the Lorentz-Sobolev embedding of $\mathcal{D}_0^{1,p}(\Omega)$. The idea is similar to that of Corollary 3.6 of [9].

Theorem 3.0.3. *Let $p \in (1, N)$ and an open subset Ω in $\mathbb{R}^N$. Then $\mathcal{D}_0^{1,p}(\Omega)$ is continuously embedded in $L^{p^*,p}(\Omega)$.*

Proof. Without loss of generality we may assume $\Omega = \mathbb{R}^N$ (for a general domain Ω, the result will follow by considering the zero extension to $\mathbb{R}^N$). Let $g \in \mathcal{H}_p(\Omega)$ be such that $g^\star \in \mathcal{H}_p(\Omega)$. Then, using Lemma 2.3.5 we have,

$$\int_{\mathbb{R}^N} g^\star |u^\star|^p dx \leq C_H \|g^\star\|_{\mathcal{H}_p} \int_{\mathbb{R}^N} |\nabla u^\star|^p \, dx \leq C_H \|g^\star\|_{\mathcal{H}_p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx, \, \forall u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N).$$

In particular, for $g(x) = \frac{1}{\omega_N^{\frac{p}{N}} |x|^p}$, $g^*(s) = \frac{1}{s^{\frac{p}{N}}}$ and one can compute $\|g^\star\|_{\mathcal{H}_p} = \frac{(p-1)^{p-1}}{N(N-p)^{p-1}}$. Now $\int_{\mathbb{R}^N} g^\star |u^\star|^p dx = \int_0^\infty g^*(s) |u^*(s)|^p ds$. Thus from the above inequality we obtain,

$$\int_0^\infty \frac{|u^*(s)|^p}{s^{\frac{p}{N}}} \, ds \leq C(N,p) \int_{\mathbb{R}^N} |\nabla u|^p \, dx, \, \forall \, u \in \mathcal{D}_0^{1,p}(\Omega).$$

The left hand side of the above inequality is $|u|^p_{L^{p^*,p}}$, a quasi-norm equivalent to the norm $\|u\|^p_{L^{p^*,p}}$ in $L^{p^*,p}(\Omega)$. This completes the proof. □

Corollary 3.0.4. *Let $p \in (1, N)$ and an open subset Ω in $\mathbb{R}^N$. Then $\mathcal{D}_0^{1,p}(\Omega)$ is compactly embedded in $L^p_{loc}(\Omega)$.*

Proof. Clearly $\mathcal{D}_0^{1,p}(\Omega)$ is continuously embedded into $W^{1,p}_{loc}(\Omega)$. Since $W^{1,p}_{loc}(\Omega)$ is compactly embedded in $L^p_{loc}(\Omega)$, we have the required embedding. □

Notice that we used just one Hardy potential $\frac{1}{|x|^p}$ to obtain the Lorentz-Sobolev embedding in Theorem 3.0.3. Instead, if we consider the entire $\mathcal{H}_p(\Omega)$, then we get an embedding finer than the above one.

Theorem 3.0.5. *Let $1 < p < N$ and Ω be open in $\mathbb{R}^N$. Then*

(a) $\mathcal{D}_0^{1,p}(\Omega)$ is continuously embedded into $\mathcal{E}_p(\Omega)$,

(b) $\mathcal{E}_p(\Omega)$ *is a proper subspace of* $L^{p^*,p}(\Omega)$.

Proof. (a) For $g \in \mathcal{H}_p(\Omega)$, by Theorem 2.3.5,

$$\int_\Omega |g||u|^p \, dx \le C_H \|g\|_{\mathcal{H}_p} \int_\Omega |\nabla u|^p \, dx, \forall u \in \mathcal{D}_0^{1,p}(\Omega).$$

Now taking the supremum over the unit ball in $\mathcal{H}_p(\Omega)$ we obtain,

$$\|u\|_{\mathcal{E}_p} \le C_H^{\frac{1}{p}} \|u\|_{\mathcal{D}_0^{1,p}}, \forall u \in \mathcal{D}_0^{1,p}(\Omega).$$

(b) Clearly $v \in \mathcal{E}_p(\Omega)$ if and only if $|v|^p \in \mathcal{H}_p(\Omega)'$. Further, $L^{\frac{N}{p},\infty}(\Omega) \subsetneq \mathcal{H}_p(\Omega)$ and hence $\mathcal{H}_p(\Omega)' \subsetneq L^{\frac{p^*}{p},1}(\Omega)$ (by Proposition 2.2.4). Therefore, there exists $w \in L^{\frac{p^*}{p},1}(\Omega)$ such that $w \notin \mathcal{H}_p(\Omega)'$. Hence, $v := |w|^{\frac{1}{p}} \in L^{p^*,p}(\Omega)$ such that $v \notin \mathcal{E}_p(\Omega)$. This shows that $\mathcal{E}_p(\Omega) \subsetneq L^{p^*,p}(\Omega)$. □

Remark 3.0.6. Let $p \in (1, N)$ and $\Omega = B_d \setminus \overline{B_c}$ with $0 \le c < d \le \infty$. Then the weighted Lebesgue space $L^1_{rad}(\Omega, |x|^{p-N})$ is continuously embedded in $\mathcal{H}_p(\Omega)$. To see this, we use [10, Lemma 2.1] to obtain

$$\int_F |g| \, dx \le \int_\Omega |g||u|^p dx \le C_H \|g\|_{L^1_{rad}(\Omega,|x|^{p-N})} \int_\Omega |\nabla u|^p dx, \ \forall u \in \mathcal{N}_p(F),$$

where C depends only on N, p. Taking the infimum over $\mathcal{N}_p(F)$ and then the supremum over F we obtain $\|g\|_{\mathcal{H}_p} \le C_H \|g\|_{L^1_{rad}(\Omega,|x|^{p-N})}$.

3.1 On the best constant

Recall that, the best constant $\mathcal{B}_g$ in (1.1.2) is given by

$$\frac{1}{\mathcal{B}_g} = \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in \mathcal{D}_0^{1,p}(\Omega), \int_\Omega |g||u|^p \, dx = 1 \right\}. \tag{3.1.1}$$

It is easy to see that the following inequalities hold:

$$\|g\|_{\mathcal{H}_p} \leq \mathcal{B}_g \leq C_H \|g\|_{\mathcal{H}_p}. \tag{3.1.2}$$

In this section, we are interested in finding the Hardy potentials for which $\mathcal{B}_g$ is attained in $\mathcal{D}_0^{1,p}(\Omega)$. Towards this, we recall the map

$$G_p(u) = \int_\Omega |g||u|^p \, dx \text{ on } \mathcal{D}_0^{1,p}(\Omega).$$

The next proposition shows that $\mathcal{B}_g$ is attained if G_p is compact.

Proposition 3.1.1. *Let G_p be compact on $\mathcal{D}_0^{1,p}(\Omega)$. Then $\mathcal{B}_g$ is attained in $\mathcal{D}_0^{1,p}(\Omega)$.*

Proof. It is easy to see that any minimising sequence (u_n) of (3.1.1) is bounded in $\mathcal{D}_0^{1,p}(\Omega)$. Hence $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$ (upto a sub-sequence) and $\int_\Omega |\nabla u|^p \, dx \leq \lim_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx = \mathcal{B}_g$. Now, in addition, since the map G_p is compact on $\mathcal{D}_0^{1,p}(\Omega)$, we have $G_p(u) = \lim_{n\to\infty} G_p(u_n) = 1$. This implies that $\mathcal{B}_g$ is attained at $u \in \mathcal{D}_0^{1,p}(\Omega)$. Thus, by the definition of $\mathcal{B}_g$, we get $\int_\Omega |\nabla u|^p \, dx \geq \mathcal{B}_g$. Hence, $\int_\Omega |\nabla u|^p \, dx = \mathcal{B}_g$ i.e., $\mathcal{B}_g$ is attained in $\mathcal{D}_0^{1,p}(\Omega)$. □

In this chapter, we use the Banach function space structure of $\mathcal{H}_p(\Omega)$ to characterise the set of Hardy potentials for which the map G_p is compact. We also treat the cases when G_p is not compact using a version of g depended concentration compactness lemma.

3.1.1 A g depended concentration compactness lemma

Let $\mathbb{M}(\mathbb{R}^N)$ be the space of all regular, finite, Borel signed-measures on $\mathbb{R}^N$. Then $\mathbb{M}(\mathbb{R}^N)$ is a Banach space with respect to the norm $\|\mu\| = |\mu|(\mathbb{R}^N)$ (total variation of the measure μ). A sequence (μ_n) is said to be weak* convergent to μ in $\mathbb{M}(\mathbb{R}^N)$, if

$$\int_{\mathbb{R}^N} \phi \, d\mu_n \to \int_{\mathbb{R}^N} \phi \, d\mu, \ as\ n \to \infty, \forall\, \phi \in \mathrm{C}_0(\mathbb{R}^N),$$

where $\mathrm{C}_0(\mathbb{R}^N) := \overline{\mathrm{C}_c(\mathbb{R}^N)}$ in $L^\infty(\mathbb{R}^N)$. In this case we denote $\mu_n \overset{*}{\rightharpoonup} \mu$. By the Reisz Representation theorem [3, Theorem 14.14, Chapter 14], $\mathbb{M}(\mathbb{R}^N)$ is the dual of $\mathrm{C}_0(\mathbb{R}^N)$.

The following proposition is a consequence of the Banach-Alaoglu theorem [25, Chapter 5, Section 3] which states that for any normed linear space X, the closed unit ball in X^* is weak* compact.

Proposition 3.1.2. *Let (μ_n) be a bounded sequence in $\mathbb{M}(\mathbb{R}^N)$, then there exists $\mu \in \mathbb{M}(\mathbb{R}^N)$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ up to a subsequence.*

Proof. Recall that, if $X = \mathrm{C}_0(\mathrm{\mathbb{R}^N})$, then by the Reisz Representation theorem [3, Theorem 14.14, Chapter 14] $X^* = \mathbb{M}(\mathbb{R}^N)$. Thus, the proof follows from the Banach-Alaoglu theorem [25, Chapter 5, Section 3]. □

The next proposition follows from the uniqueness part of the Riesz representation theorem.

Proposition 3.1.3. *Let $\mu \in \mathbb{M}(\mathbb{R}^N)$ be a positive measure. Then for an open $V \subseteq \Omega$,*

$$\mu(V) = \sup \left\{ \int_{\mathbb{R}^N} \phi \, d\mu : 0 \leq \phi \leq 1, \phi \in \mathrm{C_c^\infty}(\mathrm{\mathbb{R}^N}) \text{ with } \mathrm{Supp}(\phi) \subseteq \mathrm{V} \right\},$$

and for any Borel set $E \subseteq \mathbb{R}^N$, $\mu(E) := \inf \left\{ \mu(V) : E \subseteq V \text{ and } V \text{is open } \right\}$.

A function in $\mathcal{D}_0^{1,p}(\Omega)$ can be considered as a function in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$ by extending by zero outside Ω. With this convention, for $u_n, u \in \mathcal{D}_0^{1,p}(\Omega)$ and a Borel set E in

$\mathbb{R}^N$, we denote

$$\nu_n(E) = \int_E g|u_n - u|^p \, dx\,, \quad \Gamma_n(E) = \int_E |\nabla(u_n - u)|^p \, dx\,, \quad \widetilde{\Gamma}_n(E) = \int_E |\nabla u_n|^p \, dx.$$

If $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$, then ν_n, μ_n and $\widetilde{\Gamma}_n$ have weak* convergent sub-sequences (Proposition 3.1.2) in $\mathbb{M}(\mathbb{R}^N)$. Without loss of generality assume that

$$\nu_n \overset{*}{\rightharpoonup} \nu\,, \qquad \Gamma_n \overset{*}{\rightharpoonup} \Gamma\,, \qquad \widetilde{\Gamma}_n \overset{*}{\rightharpoonup} \widetilde{\Gamma} \ \text{ in } \mathbb{M}(\mathbb{R}^N).$$

We develop a g-depended concentration compactness lemma using our concentration function $\mathcal{C}_g$ (see for the definition). Our results are analogous to the results of Tertikas [69] and Smets [67].

First, we prove the absolute continuity of ν with respect to Γ.

Lemma 3.1.4. *Let $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$. Then the following statements are true.*

(i) Let $\Phi \in \mathrm{C}_\mathrm{b}^1(\Omega)$ be such that $\nabla\Phi$ has compact support. Then

$$\overline{\lim_{n\to\infty}} \int_\Omega |\nabla((u_n - u)\Phi)|^p \, dx = \overline{\lim_{n\to\infty}} \int_\Omega |\nabla(u_n - u)|^p |\Phi|^p \, dx.$$

(ii) Let $g \in \mathcal{H}_p(\Omega)$ with $g \geq 0$. Then for any Borel set E in $\mathbb{R}^N$,

$$\nu(E) \leq C_H \ \mathcal{C}_g^* \Gamma(E), \ \text{where } \mathcal{C}_g^* = \sup_{x \in \overline{\Omega}} \mathcal{C}_g(x)\ .$$

Proof. (i) Let $\epsilon > 0$ be given. Using (2.5.6),

$$\begin{aligned}\left| \int_\Omega |\nabla((u_n \right. & - \left. u)\Phi)|^p \, dx - \int_\Omega |\nabla(u_n - u)|^p |\Phi|^p \, dx \right| \\ & \leq \ \epsilon \int_\Omega |\nabla(u_n - u)|^p |\Phi|^p \, dx + C(\epsilon, p) \int_\Omega |u_n - u|^p |\nabla\Phi|^p \, dx.\end{aligned}$$

Since $\nabla\Phi$ is compactly supported, the second term in the right-hand side of the above inequality goes to 0 as $n \to \infty$ (by Rellich compactness theorem). Further, as (u_n) is bounded in $\mathcal{D}_0^{1,p}(\Omega)$ and $\epsilon > 0$ is arbitrary, we obtain the desired result.

(ii) As $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$, $u_n \to u$ in $L^p_{loc}(\Omega)$ (by Rellich compactness theorem). For $\Phi \in \mathrm{C}_c^\infty(\mathbb{R}^N)$, $(u_n - u)\Phi \in \mathcal{D}_0^{1,p}(\Omega)$ and thus by Theorem 2.3.5,

$$\begin{aligned}\int_{\mathbb{R}^N} |\Phi|^p \, d\nu_n = \int_\Omega g|(u_n - u)\Phi|^p \, dx &\leq C_H \|g\|_{\mathcal{H}_p} \int_\Omega |\nabla((u_n - u)\Phi)|^p \, dx \\ &= C_H \|g\|_{\mathcal{H}_p} \int_{\mathbb{R}^N} |\nabla((u_n - u)\Phi)|^p \, dx.\end{aligned}$$

Take $n \to \infty$ and use part (i) to obtain

$$\int_{\mathbb{R}^N} |\Phi|^p \, d\nu \leq C_H \|g\|_{\mathcal{H}_p} \int_{\mathbb{R}^N} |\Phi|^p \, d\Gamma. \tag{3.1.3}$$

Now, by Proposition 3.1.3, we get

$$\nu(E) \leq C_H \|g\|_{\mathcal{H}_p} \Gamma(E) \ , \forall E \ Borel \ in \ \mathbb{R}^N. \tag{3.1.4}$$

In particular, $\nu \ll \Gamma$ and hence by Radon-Nikodym theorem,

$$\nu(E) = \int_E \frac{d\nu}{d\Gamma} \, d\Gamma \ , \forall E \ Borel \ in \ \mathbb{R}^N. \tag{3.1.5}$$

Further, by Lebesgue differentiation theorem (page 152-168 of [37]) we have

$$\frac{d\nu}{d\Gamma}(x) = \lim_{r \to 0} \frac{\nu(B_r(x))}{\Gamma(B_r(x))}. \tag{3.1.6}$$

Now replacing g by $g\chi_{B_r(x)}$ and proceeding as before,

$$\nu(B_r(x)) \leq C_H \|g\chi_{B_r(x)}\|_{\mathcal{H}_p} \ \Gamma(B_r(x)).$$

Thus from (3.1.6) we get

$$\frac{d\nu}{d\Gamma}(x) \le C_H \mathcal{C}_g(x) \tag{3.1.7}$$

and hence $\|\frac{d\nu}{d\Gamma}\|_\infty \le C_H \mathcal{C}_g^*$. Now from (3.1.5) we obtain $\nu(E) \le C_H \mathcal{C}_g^* \Gamma(E)$ for all Borel subsets E of $\mathbb{R}^N$. $\square$

Remark 3.1.5. As we have already mentioned in the introduction, Tertikas [69] (for $p = 2$ and $\Omega = \mathbb{R}^N$) and Smets [67] (for $p \in (1, N)$ and $\Omega \subseteq \mathbb{R}^N$) have considered the concentration function $\mathcal{S}_g(.)$ and assumed that the closure of the singular set $\overline{\sum_g'}$ (which is same as $\overline{\sum_g}$) is at most countable (see (H) of [69] and (H1) of [67]). The countability assumption allowed them to describe ν as a countable sum of Dirac measures located on $\sum_g'$, and then they have obtained the absolute continuity of ν with respect to Γ (see Lemma 2.1 of [67] and Lemma 3.1 of [69]). Whereas, we use the Radon-Nikodym theorem and the Lebesgue differentiation theorem to prove the absolute continuity of ν with respect to Γ. It is worth pointing out that we do not need the countability assumption on the closure of the singular set in order to show the absolute continuity of ν with respect to Γ.

The next lemma gives a lower estimate for the measure $\widetilde{\Gamma}$. Similar estimate is obtained in Lemma 2.1 of [67]. We make a weaker assumption, $\overline{\sum_g}$ is of Lebesgue measure 0, than the assumption $\overline{\sum_g}$ is countable.

Lemma 3.1.6. *Let $g \in \mathcal{H}_p(\Omega)$ be such that $g \ge 0$ and $|\overline{\sum_g}| = 0$. If $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$, then*

$$\widetilde{\Gamma} \ge \begin{cases} |\nabla u|^p + \frac{\nu}{C_H \mathcal{C}_g^*}, & \text{if } \mathcal{C}_g^* \ne 0, \\ |\nabla u|^p, & \text{otherwise.} \end{cases}$$

Proof. Our proof splits in to three steps.

Step 1: $\widetilde{\Gamma} \ge |\nabla u|^p$. Let $\phi \in C_c^\infty(\mathbb{R}^N)$ with $0 \le \phi \le 1$, we need to show that

$\int_{\mathbb{R}^N} \phi \, d\widetilde{\Gamma} \geq \int_{\mathbb{R}^N} \phi |\nabla u|^p \, dx$. Notice that,

$$\int_{\mathbb{R}^N} \phi \, d\widetilde{\Gamma} = \lim_{n\to\infty} \int_{\mathbb{R}^N} \phi \, d\widetilde{\Gamma}_n = \lim_{n\to\infty} \int_\Omega \phi |\nabla u_n|^p \, dx = \lim_{n\to\infty} \int_\Omega F(x, \nabla u_n(x)) \, dx,$$

where $F : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}$ is defined as $F(x,z) = \phi(x)|z|^p$. Clearly, F is a Caratheodory function and $F(x,.)$ is convex for almost every x. Hence, by [66, Theorem 2.6, page 28], we have $\lim_{n\to\infty} \int_\Omega \phi |\nabla u_n|^p \, dx \geq \int_\Omega \phi |\nabla u|^p \, dx = \int_{\mathbb{R}^N} \phi |\nabla u|^p \, dx$ and this proves our claim 1.

Step 2: $\widetilde{\Gamma} = \Gamma$, on $\overline{\sum_g}$. Let $E \subset \overline{\sum_g}$ be a Borel set. Thus, for each $m \in \mathbb{N}$, there exists an open subset O_m containing E such that $|O_m| = |O_m \setminus E| < \frac{1}{m}$. Let $\epsilon > 0$ be given. Then, for any $\phi \in \mathrm{C}_c^\infty(O_m)$ with $0 \leq \phi \leq 1$, using (2.5.6) we have

$$\begin{aligned} \left| \int_\Omega \phi \, d\Gamma_n \, dx - \int_\Omega \phi \, d\widetilde{\Gamma}_n \, dx \right| &= \left| \int_\Omega \phi |\nabla(u_n - u)|^p \, dx - \int_\Omega \phi |\nabla u_n|^p \, dx \right| \\ &\leq \epsilon \int_\Omega \phi |\nabla u_n|^p \, dx + \mathrm{C}(\epsilon, p) \int_\Omega \phi |\nabla u|^p dx \\ &\leq \epsilon L + \mathrm{C}(\epsilon, p) \int_{O_m} |\nabla u|^p \, dx, \end{aligned}$$

where $L = \sup_n \left\{ \int_\Omega |\nabla u_n|^p \, dx \right\}$. Letting $n \to \infty$, we obtain $\left| \int_\Omega \phi \, d\Gamma - \int_\Omega \phi \, d\widetilde{\Gamma} \right| \leq \epsilon L + \mathrm{C}(\epsilon, p) \int_{O_m} |\nabla u|^p \, dx$. Therefore,

$$\begin{aligned} \left| \Gamma(O_m) - \widetilde{\Gamma}(O_m) \right| &= \sup \left\{ \left| \int_\Omega \phi \, d\Gamma - \int_\Omega \phi \;\; d\tilde{\Gamma} \right| : \phi \in \mathrm{C}_c^\infty(O_m), 0 \leq \phi \leq 1 \right\} \\ &\leq \epsilon L + \mathrm{C}(\epsilon, p) \int_{O_m} |\nabla u|^p \, dx, \end{aligned}$$

Now as $m \to \infty$, $|O_m| \to 0$ and hence $|\Gamma(E) - \widetilde{\Gamma}(E)| \leq \epsilon L$. Since $\epsilon > 0$ is arbitrary, we conclude $\Gamma(E) = \widetilde{\Gamma}(E)$.

Step 3: $\widetilde{\Gamma} \geq |\nabla u|^p + \frac{\nu}{C_H \mathcal{C}_g^*}$, if $\mathcal{C}_g^* \neq 0$. Let $\mathcal{C}_g^* \neq 0$. Then from Lemma 3.1.4 we have $\Gamma \geq \frac{\nu}{C_H \mathcal{C}_g^*}$. Furthermore, (3.1.7) and (3.1.5) ensures that ν is supported on $\sum_g$.

Hence Step 1 and Step 2 yields the following:

$$\widetilde{\Gamma} \geq \begin{cases} |\nabla u|^p, \\ \frac{\nu}{C_H \mathcal{C}_g^*}. \end{cases} \tag{3.1.8}$$

Since $|\overline{\sum_g}| = 0$, the measure $|\nabla u|^p$ is supported inside $\overline{\sum_g}^c$ and hence from (3.1.8) we easily obtain $\widetilde{\Gamma} \geq |\nabla u|^p + \frac{\nu}{C_H \mathcal{C}_g^*}$. □

Lemma 3.1.7. *Let $g \in \mathcal{H}_p(\Omega)$, $g \geq 0$ and $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$ and $\Phi_R \in \mathrm{C}_\mathrm{b}^1(\mathbb{R}^\mathrm{N})$ with $0 \leq \Phi_R \leq 1$, $\Phi_R = 0$ on $\overline{B_R}$ and $\Phi_R = 1$ on B_{R+1}^c. Then,*

(A) $\displaystyle \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega\cap\overline{B_R}^c} g|u_n|^p \, dx = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \nu_n(\Omega \cap \overline{B_R}^c) = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_\Omega \Phi_R \, d\nu_n,$

(B) $\displaystyle \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega\cap\overline{B_R}^c} |\nabla u_n|^p \, dx = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \Gamma_n(\Omega \cap \overline{B_R}^c) = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_\Omega \Phi_R \, d\Gamma_n.$

Proof. By Brezis-Lieb lemma,

$$\begin{aligned} &\overline{\lim_{n\to\infty}} \left| \nu_n(\Omega \cap \overline{B_R}^c) - \int_{\Omega\cap\overline{B_R}^c} g|u_n|^p \, dx \right| \\ &= \overline{\lim_{n\to\infty}} \left| \int_{\Omega\cap\overline{B_R}^c} g|u_n - u|^p \, dx - \int_{\Omega\cap\overline{B_R}^c} g|u_n|^p \, dx \right| \\ &= \int_{\Omega\cap\overline{B_R}^c} g|u|^p \, dx. \end{aligned}$$

As $g|u|^p \in L^1(\Omega)$, the right-hand side integral goes to 0 as $R \to \infty$. Thus, we get the first equality in (A). For the second equality, it is enough to observe that

$$\int_{\Omega\cap\overline{B_{R+1}}^c} g|u_n - u|^p \, dx \leq \int_\Omega g|u_n - u|^p \Phi_R \, dx \leq \int_{\Omega\cap\overline{B_R}^c} g|u_n - u|^p \, dx.$$

Now by taking $n, R \to \infty$ respectively we get the required equality. Now we proceed to prove (B). For $\epsilon > 0$, there exists $\mathrm{C}(\epsilon, p) > 0$ (by (2.5.6)) such that

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \left| \Gamma_n(\Omega \cap \overline{B_R}^c) - \int_{\Omega\cap\overline{B_R}^c} |\nabla u_n|^p \, dx \right| \\
&= \overline{\lim_{n\to\infty}} \left| \int_{\Omega\cap\overline{B_R}^c} |\nabla (u_n - u)|^p \, dx - \int_{\Omega\cap\overline{B_R}^c} |\nabla u_n|^p \, dx \right| \\
&\leq \epsilon \overline{\lim_{n\to\infty}} \int_{\Omega\cap\overline{B_R}^c} |\nabla u_n|^p \, dx + \mathrm{C}(\epsilon, p) \int_{\Omega\cap\overline{B_R}^c} |\nabla u|^p \, dx \\
&\leq \epsilon L + \mathrm{C}(\epsilon, p) \int_{\Omega\cap\overline{B_R}^c} |\nabla u|^p \, dx \,,
\end{aligned}$$

where $L \geq \int_\Omega |\nabla u_n|^p \, dx$ for all n. Thus, by taking $R \to \infty$ and then $\epsilon \to 0$, we obtain the first equality of (B). The second equality of part (B) follows from the same argument as that of part (A). □

Now we prove a g-depended concentration compactness principle as in [67].

Lemma 3.1.8 (Concentration compactness principle). *Let $g \in \mathcal{H}_p(\Omega)$ with $g \geq 0$. Also assume that $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$. Set*

$$\nu_\infty = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \nu_n(\Omega \cap \overline{B_R}^c) \quad \text{and} \quad \Gamma_\infty = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \Gamma_n(\Omega \cap \overline{B_R}^c).$$

Then

(i) $\nu_\infty \leq C_H \mathcal{C}_g(\infty) \Gamma_\infty$,

(ii) $\overline{\lim}_{n\to\infty} \int_\Omega g|u_n|^p \, dx = \int_\Omega g|u|^p \, dx + \|\nu\| + \nu_\infty$.

(iii) Further, if $|\overline{\sum_g}| = 0$, then we have

$$\overline{\lim}_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx \geq \begin{cases} \displaystyle\int_\Omega |\nabla u|^p \, dx + \frac{\|\nu\|}{C_H \mathcal{C}_g^*} + \Gamma_\infty, & \text{if } \mathcal{C}_g^* \neq 0 \\ \displaystyle\int_\Omega |\nabla u|^p \, dx + \Gamma_\infty, & \text{otherwise.} \end{cases}$$

Proof. (i) For $R>0$, choose $\Phi_R \in \mathrm{C}_b^1(\mathbb{R}^N)$ satisfying $0 \leq \Phi_R \leq 1$, $\Phi_R = 0$ on $\overline{B_R}$ and $\Phi_R = 1$ on B_{R+1}^c. Clearly, $(u_n - u)\Phi_R \in \mathcal{D}_0^{1,p}(\Omega \cap \overline{B_R}^c)$. Since $\|g\chi_{\overline{B_R}^c}\|_{\mathcal{H}_p} < \infty$, by Theorem 2.3.5,

$$\int_{\Omega\cap\overline{B_R}^c} g|(u_n-u)\Phi_R|^p \, dx \leq C_H \, \|g\chi_{\overline{B_R}^c}\|_{\mathcal{H}_p} \int_{\Omega\cap\overline{B_R}^c} |\nabla((u_n-u)\Phi_R)|^p \, dx.$$

By part (i) of Lemma 3.1.8 we have,

$$\overline{\lim_{n\to\infty}} \int_{\Omega\cap\overline{B_R}^c} |\nabla((u_n-u)\Phi_R)|^p \, dx = \overline{\lim_{n\to\infty}} \int_{\Omega\cap\overline{B_R}^c} |\Phi_R|^p \, d\Gamma_n \, .$$

Therefore, letting $n\to\infty$, $R\to\infty$ and using Lemma 3.1.7 successively in the above inequality we obtain $\nu_\infty \leq C_H \mathcal{C}_g(\infty)\, \Gamma_\infty$.

(ii) By choosing Φ_R as above and using Brézis-Lieb lemma together with part (A) of Lemma 3.1.7 we have,

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \int_\Omega g|u_n|^p \, dx \\
&= \overline{\lim_{n\to\infty}} \left[\int_\Omega g|u_n|^p(1-\Phi_R)\, dx + \int_\Omega g|u_n|^p\Phi_R \, dx\right] \\
&= \overline{\lim_{n\to\infty}} \left[\int_\Omega g|u|^p(1-\Phi_R)\, dx + \int_\Omega g|u_n-u|^p(1-\Phi_R)\, dx + \int_\Omega g|u_n|^p\Phi_R \, dx\right] \\
&= \int_\Omega g|u|^p \, dx + \|\nu\| + \nu_\infty.
\end{aligned}$$

(iii) Notice that

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \int_\Omega |\nabla u_n|^p \, dx &= \overline{\lim_{n\to\infty}} \left[\int_\Omega |\nabla u_n|^p(1-\Phi_R)\, dx + \int_\Omega |\nabla u_n|^p \Phi_R \, dx\right] \\
&= \widetilde{\Gamma}(1-\Phi_R) + \overline{\lim_{n\to\infty}} \int_\Omega |\nabla u_n|^p \Phi_R \, dx
\end{aligned}$$

By taking $R \to \infty$ and using part (B) Lemma 3.1.7 we get

$$\overline{\lim_{n\to\infty}} \int_\Omega |\nabla u_n|^p \, dx = \|\widetilde{\Gamma}\| + \Gamma_\infty .$$

Now, using Lemma 3.1.6, we obtain

$$\overline{\lim}_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx \geq \begin{cases} \displaystyle\int_\Omega |\nabla u|^p \, dx + \frac{\|\nu\|}{C_H \mathcal{C}_g^*} + \Gamma_\infty, & \text{if } \mathcal{C}_g^* \neq 0 \\ \displaystyle\int_\Omega |\nabla u|^p \, dx + \Gamma_\infty, & \text{otherwise.} \end{cases}$$

□

3.1.2 The compactness

In this subsection, we discuss the compactness of the map

$$G_p(u) = \int_{\mathbb{R}^N} |g||u|^p \, dx \ \text{ on } \mathcal{D}_0^{1,p}(\Omega) .$$

As we have mentioned in the introduction, many authors proved the compactness of G_p under various assumptions on g. Here we list those results:

- For $p = 2$ and Ω bounded, the compactness of G_p is proved for $g \in L^r(\Omega)$ with $r > \frac{N}{2}$ in [55] and $r = \frac{N}{2}$ in [4],
- For $p \in (1, \infty)$ and for general domain Ω, the compactness of G_p has been proved if $g \in L^{\frac{N}{p},d}(\Omega)$ with $d < \infty$ [71],
- The result has been further extended for $g \in \mathcal{F}_{\frac{N}{p}}(\Omega) := \overline{\mathrm{C}_c^\infty(\Omega)}$ in $L^{\frac{N}{p},\infty}(\Omega)$ [11, for $p = 2$], [6, for $p \in (1, N)$],
- In [10], authors obtained the compactness for $g \in L^1_{rad}(\overline{B}_1^c, |x|^{p-N})$.

Recall that

$$\mathcal{FH}_p(\Omega) = \overline{\mathrm{C}_c(\Omega)} \text{ in } \mathcal{H}_p(\Omega).$$

Proposition 3.1.9. *Let $p \in (1, N)$ and an open subset Ω in $\mathbb{R}^N$. Then the following statements are true.*

(a) $\mathcal{F}_{\frac{N}{p}}(\Omega) \subseteq \mathcal{FH}_p(\Omega)$ *for any open subset Ω in $\mathbb{R}^N$.*

(b) $L^1_{rad}(\Omega, |x|^{p-N}) \subseteq \mathcal{FH}_p(\Omega)$ *for $\Omega = B_d \setminus \overline{B_c}$; $0 \le c < d \le \infty$.*

Proof. (a) From the definition of $\mathcal{F}_{\frac{N}{p}}(\Omega)$ and $\mathcal{FH}_p(\Omega)$, we have $\mathcal{F}_{\frac{N}{p}}(\Omega)$ is the closure of $\mathrm{C}_c^\infty(\Omega)$ in $L^{\frac{N}{p},\infty}(\Omega)$ and $\mathcal{FH}_p(\Omega)$ is closure of $\mathrm{C}_c(\Omega)$ in $\mathcal{H}_p(\Omega)$. Furthermore, since $\|.\|_{\mathcal{H}_p} \le C\|.\|_{L^{\frac{N}{p},\infty}}$, it is immediate that $\mathcal{F}_{\frac{N}{p}}(\Omega)$ is contained in $\mathcal{FH}_p(\Omega)$. This proves (a).

(b) Observe that, it is enough to show $\mathrm{C}_c^\infty(\Omega)$ is dense in $L^1_{rad}(\Omega, |x|^{p-N})$. For this, let $g \in L^1_{rad}(\Omega, |x|^{p-N})$ and $\epsilon > 0$ be arbitrary. Since $g \in L^1_{rad}(\Omega, |x|^{p-N})$, there exists $\widetilde{g} \in L^1((c,d), r^{p-1})$ such that $g(x) = \widetilde{g}(|x|)$. As $\mathrm{C}_c^\infty((c,d))$ is dense in $L^1((c,d), r^{p-1})$, there exists $\phi \in \mathrm{C}_c^\infty((c,d))$ such that $\|\tilde{g} - \phi\|_{L^1((c,d),r^{p-1})} < \epsilon$. Now, for $x \in \Omega$, let $\psi(x) := \phi(|x|)$. By denoting, $h = g - \psi$ we have, $\tilde{h}(r) = \tilde{g}(r) - \phi(r)$. Therefore,

$$\begin{aligned} \|g - \psi\|_{L^1_{rad}(\Omega,|x|^{p-N})} &= \int_c^d |\tilde{h}|(r) r^{p-1} dr \\ &= \|\tilde{g} - \phi\|_{L^1((0,\infty),r^{p-1})} = \|\tilde{g} - \phi\|_{L^1((c,d),r^{p-1})} < \epsilon. \end{aligned}$$

□

Remark 3.1.10. In [6, Lemma 3.5], authors have shown that $\mathcal{F}_{\frac{N}{p}}(\Omega)$ contains the Hardy potentials that have faster decay than $\frac{1}{|x-a|^p}$ at all points $a \in \overline{\Omega}$ and at infinity. Such Hardy potentials arise in the work of Szulkin and Willem [68]. Above proposition assures that they belong to $\mathcal{FH}_p(\Omega)$.

In the following proposition, we approximate $\mathcal{FH}_p(\Omega)$ functions using certain

$L^\infty(\Omega)$ functions. A similar result is obtained for $\mathcal{F}_{\frac{N}{p}}(\Omega)$ in [11, Proposition 3.2].

Proposition 3.1.11. *$g \in \mathcal{FH}_p(\Omega)$ if and only if for every $\epsilon > 0$, $\exists g_\epsilon \in L^\infty(\Omega)$ such that $|Supp(g_\epsilon)| < \infty$ and $\|g - g_\epsilon\|_{\mathcal{H}_p} < \epsilon$.*

Proof. Let $g \in \mathcal{FH}_p(\Omega)$ and $\epsilon > 0$ be given. By definition of $\mathcal{FH}_p(\Omega)$, $\exists g_\epsilon \in \mathrm{C}_c(\Omega)$ such that $\|g - g_\epsilon\|_{\mathcal{H}_p} < \epsilon$. This g_ϵ fulfils our requirements. For the converse part, take a g satisfying the hypothesis. Let $\epsilon > 0$ be arbitrary. Then $\exists g_\epsilon \in L^\infty(\Omega)$ such that $|Supp(g_\epsilon)| < \infty$ and $\|g - g_\epsilon\|_{\mathcal{H}_p} < \frac{\epsilon}{2}$. Thus, $g_\epsilon \in L^{\frac{N}{p}}(\Omega)$ and hence there exists $\phi_\epsilon \in \mathrm{C}_c(\Omega)$ such that $\|g_\epsilon - \phi_\epsilon\|_{L^{\frac{N}{p}}} < \frac{\epsilon}{2C}$, where C is the embedding constant for the embedding $L^{\frac{N}{p}}(\Omega)$ into $\mathcal{H}_p(\Omega)$. Now by triangle inequality, we obtain $\|g - \phi_\epsilon\|_{\mathcal{H}_p} < \epsilon$ as required. $\square$

Next we show that $\mathcal{FH}_p(\Omega)$ is the optimal space for the compactness of G_p, and this will unify and extend all the existing results that guarantee the compactness of G_p. Furthermore, we characterise this space using the Banach function space structure of $\mathcal{H}_p(\Omega)$. First we prove the following lemma which gives a sufficient condition for the compactness of G_p.

Lemma 3.1.12. *Let $g \in \mathcal{H}_p(\Omega)$ and $G_p : \mathcal{D}_0^{1,p}(\Omega) \mapsto \mathbb{R}$ is compact. Then,*

(i) if (A_n) is a sequence of bounded measurable subsets such that χ_{A_n} decreases to 0, then $\|g\chi_{A_n}\|_{\mathcal{H}_p} \to 0$ as $n \to \infty$.

(ii) $\|g\chi_{B_n^c}\|_{\mathcal{H}_p} \to 0$ as $n \to \infty$.

Proof. (i) Let (A_n) be a sequence of bounded measurable subsets such that χ_{A_n} decreases to 0. If $\|g\chi_{A_n}\|_{\mathcal{H}_p} \nrightarrow 0$, then $\exists a > 0$ such that $\|g\chi_{A_n}\|_{\mathcal{H}_p} > a, \forall n$ (by the monotonicity of the norm). Thus, $\exists F_n \subset\subset \Omega$ and $u_n \in \mathcal{N}_p(F_n)$ such that

$$\int_\Omega |\nabla u_n|^p \, dx < \frac{1}{a} \int_{F_n \cap A_n} |g| \, dx \leq \frac{1}{a} \int_{\{|u_n| \geq 1\}} |g||u_n|^p \, dx. \tag{3.1.9}$$

Since A_n's are bounded and χ_{A_n} decreases to 0, it follows that $|A_n| \to 0$, as $n \to \infty$. Further, as $g \in L^1(A_1)$, we also have $\int_{F_n \cap A_n} |g| \; dx \to 0$. Hence from the above inequalities, $u_n \to 0$ in $\mathcal{D}_0^{1,p}(\Omega)$. For $0 < \epsilon < 1$, consider $w_n^\epsilon = \frac{|u_n|^p}{(|u_n| + \epsilon)^{p-1} \|u_n\|_{\mathcal{D}_0^{1,p}}}$. One can check that for each n, $w_n^\epsilon \in \mathcal{D}_0^{1,p}(\Omega)$ and it is bounded uniformly (with respect to n) in $\mathcal{D}_0^{1,p}(\Omega)$. Thus up to a sub sequence, w_n^ϵ converges weakly to w in $\mathcal{D}_0^{1,p}(\Omega)$ as $n \to \infty$. Now using the embedding of $\mathcal{D}_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ we obtain that $\|w_n^\epsilon\|_{L^{\frac{p^*}{p}}} \leq C \frac{\|u_n\|^{p-1}_{\mathcal{D}_0^{1,p}}}{\epsilon^{(p-1)}}$. Thus $\|w_n^\epsilon\|_{L^{\frac{p^*}{p}}} \to 0$ as $n \to \infty$ and hence $w = 0$ i.e. $w_n^\epsilon \rightharpoonup 0$ in $\mathcal{D}_0^{1,p}(\Omega)$ as $n \to \infty$. By the compactness of G_p we infer $\lim_{n \to \infty} \int_\Omega |g||w_n^\epsilon|^p \; dx = 0$. On the other hand, for each $n \in \mathbb{N}$ and $0 < \epsilon < 1$,

$$\begin{aligned} \int_\Omega |g||w_n^\epsilon|^p \; dx &= \int_\Omega \frac{|g||u_n|^{p^2}}{(|u_n| + \epsilon)^{p^2-p} \|u_n\|^p_{\mathcal{D}_0^{1,p}}} \; dx \geq \int_{|u_n| \geq \epsilon} \frac{|g||u_n|^{p^2}}{(2|u_n|)^{p^2-p} \|u_n\|^p_{\mathcal{D}_0^{1,p}}} \; dx \\ &= \frac{1}{2^{p^2-p}} \int_{\{|u_n| \geq \epsilon\}} \frac{|g||u_n|^p}{\|u_n\|^p_{\mathcal{D}_0^{1,p}}} \; dx > \frac{a}{2^{p^2-p}} \end{aligned}$$

which is a contradiction.

(ii) If $\|g\chi_{B_n^c}\|_{\mathcal{H}_p} \nrightarrow 0$, as $n \to \infty$, then there exists $F_n \subset\subset \Omega$ such that

$$a < \frac{\int_{F_n \cap B_n^c} |g| \; dx}{\mathrm{Cap}_p(F_n, \Omega)} \leq \frac{\int_{F_n \cap B_n^c} |g| \; dx}{\mathrm{Cap}_p(F_n \cap B_n^c, \Omega)} \leq \frac{C \int_{F_n \cap B_n^c} |g| \; dx}{\mathrm{Cap}_p(F_n \cap B_n^c, \Omega \cap \overline{B}^c_{\frac{n}{2}})}$$

for some $a > 0$ and $C > 0$. Last inequality follows from the part (ii) of Proposition 2.3.4. Thus, for each n there exists $z_n \in \mathcal{D}_0^{1,p}(\Omega \cap \overline{B}^c_{\frac{n}{2}})$ with $z_n \geq 1$ on $F_n \cap B_n^c$ such that

$$\int_\Omega |\nabla z_n|^p \; dx < \frac{C}{a} \int_{F_n \cap B_n^c} |g| \; dx \leq \frac{C}{a} \int_\Omega |g||z_n|^p \; dx.$$

By taking $w_n = \frac{z_n}{\|z_n\|_{\mathcal{D}_0^{1,p}}}$ and following a same argument as in (i) we contradict the compactness of G_p. $\square$

Next, we prove that $\mathcal{C}_g$ vanishes for all $g \in \mathrm{C_c}(\Omega)$.

Proposition 3.1.13. *Let* $\phi \in \mathrm{C_c}(\Omega)$. *Then* $\mathcal{C}_\phi(x) = 0$, $\forall x \in \overline{\Omega} \cup \{\infty\}$.

Proof. Observe that, for $\phi \in \mathrm{C}_c(\Omega)$,

$$\|\phi\chi_{B_r(x)}\|_{\mathcal{H}_p} = \sup_{F\subset\subset\Omega}\left[\frac{\int_{F\cap B_r(x)}|\phi|\,dx}{\mathrm{Cap}_p(F,\Omega)}\right] \leq \sup_{F\subset\subset\Omega}\left[\frac{\sup(|\phi|)|(F\cap B_r)^\star|}{\mathrm{Cap}_p((F\cap B_r)^\star)}\right].$$

If d is the radius of $(F\cap B_r)^\star$ then

$$\frac{|(F\cap B_r)^\star|}{\mathrm{Cap}_p((F\cap B_r)^\star)} = \frac{\omega_N d^N}{N\omega_N(\frac{N-p}{p-1})^{p-1}d^{(N-p)}} = C(N,p)d^p \leq C(N,p)r^p.$$

Thus, $\mathcal{C}_\phi(x) = \lim\limits_{r\to 0}\|\phi\chi_{B_r(x)}\|_{\mathcal{H}_p} = 0$. Also, one can easily see that $\mathcal{C}_\phi(\infty) = 0$ as ϕ has compact support . In fact, one can see that, if $\phi \in L^\infty(\Omega)$ with compact support then $\mathcal{C}_\phi \equiv 0$. □

Now we are in a position to state the chareterization theorem for the compactness of G_p. The next theorem combines the results stated in **Theorem 1.1.1, Theorem 1.1.3,** and **Theorem 1.1.5** together.

Theorem 3.1.14. *Let $g \in \mathcal{H}_p(\Omega)$. Then the following statements are equivalent:*

(i) $G_p : \mathcal{D}_0^{1,p}(\Omega) \mapsto \mathbb{R}$ *is compact,*

(ii) g *has the absolute continuous norm in* $\mathcal{H}_p(\Omega)$*,*

(iii) $g \in \mathcal{FH}_p(\Omega)$*,*

(iv) $\mathcal{C}_g^* = 0 = \mathcal{C}_g(\infty)$.

Proof. $(i) \implies (ii)$: Let G_p be compact. Take a sequence of measurable subsets (A_n) of Ω such that χ_{A_n} decreases to 0 a.e. in Ω. Part (ii) of Lemma 3.1.12 gives $\|g\chi_{B_n^c}\|_{\mathcal{H}_p} \to 0$, as $n\to\infty$. Choose $\epsilon > 0$ arbitrarily. There exists $N_0 \in \mathbb{N}$, such that $\|g\chi_{B_n^c}\|_{\mathcal{H}_p} \leq \frac{\epsilon}{2}, \forall n \geq N_0$. Now $A_n = (A_n \cap B_{N_0}) \cup (A_n \cap B_{N_0}^c)$, for each n. Thus,

$$\|g\chi_{A_n}\|_{\mathcal{H}_p} \leq \|g\chi_{A_n\cap B_{N_0}}\|_{\mathcal{H}_p} + \|g\chi_{A_n\cap B_{N_0}^c}\|_{\mathcal{H}_p} \leq \|g\chi_{A_n\cap B_{N_0}}\|_{\mathcal{H}_p} + \frac{\epsilon}{2}.$$

By part (i) of Lemma 3.1.12, there exists $N_1(\geq N_0) \in \mathbb{N}$ such that $\|g\chi_{A_n \cap B_{N_0}}\|_{\mathcal{H}_p} \leq \frac{\epsilon}{2}$, $\forall n \geq N_1$ and hence $\|g\chi_{A_n}\|_{\mathcal{H}_p} \leq \epsilon$ for all $n \geq N_1$. Therefore, g has absolutely continuous norm.

$(ii) \implies (iii)$: Let g has absolute continuous norm in $\mathcal{H}_p(\Omega)$. Then, $\|g\chi_{B_m^c}\|_{\mathcal{H}_p}$ converge to 0 as $m \to \infty$. Let $\epsilon > 0$ be arbitrary. We choose $m_\epsilon \in \mathbb{N}$ such that $\|g\chi_{B_m^c}\|_{\mathcal{H}_p} < \epsilon$, $\forall m \geq m_\epsilon$. Now for any $n \in \mathbb{N}$,

$$g = g\chi_{\{|g| \leq n\} \cap B_{m_\epsilon}} + g\chi_{\{|g| > n\} \cap B_{m_\epsilon}} + g\chi_{B_{m_\epsilon}^c} := g_n + h_n.$$

where $g_n = g\chi_{\{|g| \leq n\} \cap B_{m_\epsilon}}$ and $h_n = g\chi_{\{|g| > n\} \cap B_{m_\epsilon}} + g\chi_{B_{m_\epsilon}^c}$. Clearly, $g_n \in L^\infty(\Omega)$ and $|Supp(g_n)| < \infty$. Furthermore,

$$\|h_n\|_{\mathcal{H}_p} \leq \|g\chi_{\{|g| > n\} \cap B_{m_\epsilon}}\|_{\mathcal{H}_p} + \|g\chi_{B_{m_\epsilon}^c}\|_{\mathcal{H}_p} < \|g\chi_{\{|g| > n\} \cap B_{m_\epsilon}}\|_{\mathcal{H}_p} + \epsilon\,.$$

Now, $g \in L^1_{loc}(\Omega)$ ensures that $\chi_{\{|g| > n\} \cap B_{m_\epsilon}} \to 0$ as $n \to \infty$. As g has absolutely continuous norm, $\|g\chi_{\{|g| > n\} \cap B_{m_\epsilon}}\|_{\mathcal{H}_p} < \epsilon$ for large n. Therefore, $\|h_n\|_{\mathcal{H}_p} < 2\epsilon$ for large n. Hence, Proposition 3.1.11 concludes that $g \in \mathcal{FH}_p(\Omega)$.

$(iii) \implies (iv)$: Let $g \in \mathcal{FH}_p(\Omega)$ and $\epsilon > 0$ be arbitrary. Then there exists $g_\epsilon \in \mathrm{C}_c(\Omega)$ such that $\|g - g_\epsilon\|_{\mathcal{H}_p} < \epsilon$. Thus, Proposition 3.1.13 infers that $\mathcal{C}_{g_\epsilon}$ vanishes. Now as $g = g_\epsilon + (g - g_\epsilon)$, it follows that $\mathcal{C}_g(x) \leq \mathcal{C}_{g_\epsilon}(x) + \mathcal{C}_{g-g_\epsilon}(x) \leq \|g - g_\epsilon\|_{\mathcal{H}_p} < \epsilon$ and hence $\mathcal{C}_g^* = 0$. By a similar argument one can show $\mathcal{C}_g(\infty) = 0$.

$(iv) \implies (i)$: Assume that $\mathcal{C}_g^* = 0 = \mathcal{C}_g(\infty)$. Let (u_n) be a bounded sequence in $\mathcal{D}^{1,p}_0(\Omega)$. Then by Lemma 3.1.8, up to a sub-sequence we have,

$$\begin{aligned}
\nu_\infty &\leq C_H \mathcal{C}_g(\infty)\Gamma_\infty, \\
\|\nu\| &\leq C_H \mathcal{C}_g^* \|\Gamma\|, \\
\lim_{n\to\infty} \int_\Omega |g||u_n|^p \, dx &= \int_\Omega |g||u|^p \, dx + \|\nu\| + \nu_\infty.
\end{aligned}$$

As $\mathcal{C}_g^* = 0 = \mathcal{C}_g(\infty)$ we immediately conclude that $\lim_{n\to\infty} \int_\Omega |g||u_n|^p \, dx = \int_\Omega |g||u|^p \, dx$ and hence $G_p : \mathcal{D}_0^{1,p}(\Omega) \mapsto \mathbb{R}$ is compact.

$\square$

Remark 3.1.15. Let $N > p$ and $g(x) = \frac{1}{|x|^p}$ in $\mathbb{R}^N$. Then for any $r > 0$, using Proposition 2.3.2 we get

$$\frac{\int_{B_r(0)} \frac{dx}{|x|^p}}{\mathrm{Cap}_p(B_r(0))} = \frac{(p-1)^{p-1}}{(N-p)^p}.$$

Thus $\mathcal{C}_g(0) = \frac{(p-1)^{p-1}}{(N-p)^p}$ and hence $g \notin \mathcal{FH}_p(\mathbb{R}^N)$.

Remark 3.1.16. Let $X = (X(\Omega), \|.\|_X)$ be a Banach function space and $f \in X$. Then f is said to have continuous norm in X, if for each $x \in \Omega$, $\|f\chi_{B_r(x)}\|_X$ converges to 0, as $r \to 0$. Observe that by Theorem 3.1.14, the set of all functions having continuous norm and the set of all function having absolute continuous norm are one and the same on $\mathcal{H}_p(\Omega)$. However, in [47], authors constructed a Banach function space where these two sets are different.

3.1.3 A concentration compactness criteria

Recall that, for $g \in \mathcal{H}_p(\Omega)$, the best constant $\mathcal{B}_g$ in (1.1.2) is given by

$$\frac{1}{\mathcal{B}_g} = \inf_{u \in G_p^{-1}\{1\}} \int_\Omega |\nabla u|^p \, dx. \tag{3.1.10}$$

In this subsection, first we prove Theorem 1.1.6. Then we give several ways to produce Hardy potentials for which $\mathcal{B}_g$ is attained in $\mathcal{D}_0^{1,p}(\Omega)$ but G_p is not compact.

Proof of Theorem 1.1.6. Let $(u_n) \in G_p^{-1}\{1\}$ minimizes $\int_\Omega |\nabla u|^p \, dx$ over $G_p^{-1}\{1\}$. Then up to a sub-sequence we can assume that $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$ and $u_n \to u$ a.e. in Ω. Further, $|\nabla u_n - \nabla u|^p \rightharpoonup \Gamma$, $|g||u_n - u|^p \rightharpoonup \nu$ in $\mathbb{M}(\Omega)$. Since $u_n \in G_p^{-1}\{1\}$, using Lemma 3.1.8 we have

$$1 = \int_\Omega |g||u|^p \, dx + \|\nu\| + \nu_\infty.$$

Suppose $\|\Gamma\|$ or Γ_∞ is nonzero. Then using Hardy-Sobolev inequality, part (i) of Lemma 3.1.8 and Lemma 3.1.4, we obtain the following estimate:

$$\begin{aligned}\int_\Omega |g||u|^p \, dx + \nu_\infty + \|\nu\| &\leq \mathcal{B}_g \int_\Omega |\nabla u|^p \, dx + C_H \left(\mathcal{C}_g^* \|\Gamma\| + \mathcal{C}_g(\infty)\Gamma_\infty \right) \\ &< \mathcal{B}_g \left(\int_\Omega |\nabla u|^p \, dx + \|\Gamma\| + \Gamma_\infty \right) \\ &\leq \mathcal{B}_g \times \overline{\lim}_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx.\end{aligned}$$

A contradiction, as $\overline{\lim}_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx = \dfrac{1}{\mathcal{B}_g}$. Thus $\|\Gamma\| = 0 = \Gamma_\infty$. Consequently, $\|\nu\| = 0 = \nu_\infty$ (by part (i) of Lemma 3.1.8 and Lemma 3.1.4) and $\int_\Omega |g||u|^p \, dx = 1$. Therefore, $\mathcal{B}_g$ is attained at u. □

Remark 3.1.17. For $g(x) = \frac{1}{|x|^p}$ in $\mathbb{R}^N$, it is well known that $\mathcal{B}_g$ is not attained in $\mathcal{D}_0^{1,p}(\Omega)$. Further, $\mathcal{C}_g(0) = \frac{(p-1)^{p-1}}{(N-p)^p}$ and hence $C_H\mathcal{C}_g^* = \mathcal{B}_g$.

Corollary 3.1.18. *Let $g \in \mathcal{H}_p(\Omega)$ and $|\overline{\sum_g}| = 0$. If*

$$C_H[dist(g, \mathcal{F}\mathcal{H}_p(\Omega))] < \|g\|_{\mathcal{H}_p},$$

then $\mathcal{B}_g$ is attained in $\mathcal{D}_0^{1,p}(\Omega)$.

Proof. For $g, h \in L^1_{loc}(\Omega)$ and $F \subset\subset \Omega$,

$$\frac{\int_F |g|\chi_{B_r(x)} \, dx}{\mathrm{Cap}_p(F,\Omega)} \leq \frac{\int_F |g-h|\chi_{B_r(x)} \, dx}{\mathrm{Cap}_p(F,\Omega)} + \frac{\int_F |h|\chi_{B_r(x)} \, dx}{\mathrm{Cap}_p(F,\Omega)}.$$

By taking the supremum over all such F and r tends to 0 respectively, we obtain

$\mathcal{C}_g(x) \leq \mathcal{C}_{g-h}(x) + \mathcal{C}_h(x)$ and hence

$$\mathcal{C}_g^* \leq \mathcal{C}_{g-h}^* + \mathcal{C}_h^*. \tag{3.1.11}$$

Now as $C_H[dist(g, \mathcal{FH}_p(\Omega))] < \|g\|_{\mathcal{H}_p}$, $\exists \phi \in \mathcal{FH}_p(\Omega)$ such that $C_H\|g - \phi\|_{\mathcal{H}_p} < \|g\|_{\mathcal{H}_p}$. Thus by (3.1.11), $C_H\mathcal{C}_g^* \leq C_H\mathcal{C}_{g-\phi}^* \leq C_H\|g - \phi\|_{\mathcal{H}_p} < \|g\|_{\mathcal{H}_p} \leq \mathcal{B}_g$ and similarly $C_H\mathcal{C}_g(\infty) < \mathcal{B}_g$. Now the result follows from Theorem 1.1.6. □

Theorem 1.1.6 helps us to produce Hardy potentials for which the map G_p is not compact, however, $\mathcal{B}_g$ is attained. The following theorem is an analogue of Theorem 1.3 of [69]:

Theorem 3.1.19. *Let $h \in \mathcal{H}_p(\Omega)$ and $\left|\overline{\sum_h}\right| = 0$. Then for any non-zero, non-negative $\phi \in \mathcal{FH}_p(\Omega)$, there exists $\epsilon_0 > 0$ such that $\mathcal{B}_g$ is attained in $\mathcal{D}_0^{1,p}(\Omega)$ for $g = h + \epsilon\phi$, for all $\epsilon > \epsilon_0$.*

Proof. Let $h \in \mathcal{H}_p(\Omega)$ be such that $|\overline{\sum_h}| = 0$. Take a non-zero, non-negative $\phi \in \mathcal{FH}_p(\Omega)$ and $\epsilon_0 = \frac{(2C_H-1)\|h\|_{\mathcal{H}_p}}{\|\phi\|_{\mathcal{H}_p}}$, then for $\epsilon > \epsilon_0$, let $g = h + \epsilon\phi$. Clearly, $|\overline{\sum_g}| = 0$ and

$$C_H\mathcal{C}_g^* = C_H\mathcal{C}_{h+\epsilon\phi}^* = C_H\mathcal{C}_h^* \leq C_H\|h\|_{\mathcal{H}_p} < \frac{\|h\|_{\mathcal{H}_p} + \epsilon\|\phi\|_{\mathcal{H}_p}}{2} \leq \|g\|_{\mathcal{H}_p} \leq \mathcal{B}_g.$$

Similarly, we can show $C_H\mathcal{C}_g(\infty) < \|g\|_{\mathcal{H}_p} \leq \mathcal{B}_g$. Therefore, by Theorem 1.1.6, $\mathcal{B}_g$ is attained. □

Remark 3.1.20. Recall the definition of $\mathcal{S}_g(x)$, $x \in \overline{\Omega} \cup \{\infty\}$. In [67], author also considered the following quantities :

$$\begin{aligned}
\mathcal{S}_g^* &:= \sup_{x \in \overline{\Omega}} \mathcal{S}_g(x), \\
\mathcal{S}_g &:= \inf\left\{\int_\Omega |\nabla u|^p \, dx : u \in \mathcal{D}_0^{1,p}(\Omega), \ \int_\Omega |g||u|^p \, dx = 1\right\}.
\end{aligned}$$

Since $\mathcal{S}_g(.)$ captures the best constant in the Hardy inequality locally at the points of Ω and at the infinity, by (3.1.2), we have

$$\|g\|_{\mathcal{H}_p} \leq \frac{1}{\mathcal{S}_g} \leq C_H\|g\|_{\mathcal{H}_p}, \quad \mathcal{C}_g^* \leq \frac{1}{\mathcal{S}_g^*} \leq C_H \mathcal{C}_g^*, \tag{3.1.12}$$

$$\mathcal{C}_g(\infty) \leq \frac{1}{\mathcal{S}_g(\infty)} \leq C_H \mathcal{C}_g(\infty). \tag{3.1.13}$$

Therefore, if $C_H\mathcal{C}_g^* < \|g\|_{\mathcal{H}_p}$ and $C_H\mathcal{C}_g(\infty) < \|g\|_{\mathcal{H}_p}$ then $\mathcal{S}_g < \mathcal{S}_g^*$ and $\mathcal{S}_g < \mathcal{S}_g(\infty)$. Thus, if in addition $\overline{\sum_g}$ $(= \overline{\sum_g'})$ is countable, then Theorem 1.1.6 follows from [67, Theorem 3.1]. Therefore, our sufficient condition is slightly weaker than that of [67]. This is mainly because of the gap in the Hardy inequality given in (2.3.5) (see (3.1.2)). However, on the other hand, our sufficient condition assumes $|\overline{\sum_g}| = 0$ instead of its countability.

Example 3.1.21. For $2 \leq k < N$ and for $x \in \mathbb{R}^N$, we write $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. Now consider $g(x) = g(y, z) = \frac{1}{|y|^p}$ in $\mathbb{R}^k \times \mathbb{R}^{N-k}$. By Theorem 2.1 of [12], $g \in \mathcal{H}_p(\mathbb{R}^N)$. Next we show that $\sum_g = \{0\} \times \mathbb{R}^{N-k}$. For any $(0, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and $r > 0$, using the translation invariance of both the integral and the Cap_p, we have

$$\frac{\int_{B_r(0,z)} g(x)\,dx}{\mathrm{Cap}_p(B_r(0,z))} = \frac{\int_{B_r(0,0)} \frac{1}{|y|^p}\,dx}{\mathrm{Cap}_p(B_r(0,0))} \geq \frac{\int_{B_r(0,0)} \frac{1}{|x|^p}\,dx}{\mathrm{Cap}_p(B_r(0,0))}.$$

Now by taking $r \to 0$ we have $\mathcal{C}_g(0, z) \geq \mathcal{C}_{\frac{1}{|x|^p}}((0, 0)) > 0$ and hence $\sum_g \supseteq \{0\} \times \mathbb{R}^{N-k}$. Next for $x_0 = (y_0, z_0) \notin \{0\} \times \mathbb{R}^{N-k}$, let $0 < r < |y_0.|$ Then by Proposition 2.3.2 we obtain

$$\frac{\int_{B_r(x_0)} \frac{1}{|y|^p}\,dx}{\mathrm{Cap}_p(B_r(x_0))} \leq \frac{\frac{1}{(|y_0|-r)^p}\int_{B_r(x_0)}\,dx}{\mathrm{Cap}_p(B_r(x_0))} = \left(\frac{p-1}{N-p}\right)^{p-1}\left(\frac{r^p}{N(|y_0|-r)^p}\right).$$

Now by taking $r \to 0$, we obtain $\mathcal{C}_g(x_0) = 0$. Hence, $\sum_g = \{0\} \times \mathbb{R}^{N-k}$.

Remark 3.1.22. We consider $g(x) = \frac{1}{|y|^p}$, for $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ $(2 \leq k < N)$. In Example 3.1.21, we have seen that $g \in \mathcal{H}_p(\Omega)$ with $\sum_g$ is uncountable and

$|\overline{\sum_g}| = 0$. Now choose any $\phi \in \mathcal{FH}_p(\Omega)$ and consider $\tilde{g} := g + \epsilon\phi$. Then, there exists $\epsilon_0 > 0$ such that $\mathcal{B}_{\tilde{g}}$ is attained if $\epsilon > \epsilon_0$ (by Theorem 3.1.19). Further, in Example 3.1.21, we have seen that $\sum_{\tilde{g}}$ is also uncountable and $|\overline{\sum_{\tilde{g}}}| = 0$. Thus, $\tilde{g}$ lies outside the class of functions considered in [67, 69].

Chapter 4

The Hardy-Rellich and Hardy-Hessian potentials

In this chapter, we study the Hardy-Rellich potentials and the Hardy-Hessian potentials in detail. Here we prove Theorem 1.2.1, Theorem 1.2.2, and Theorem 1.2.3.

Let us recall

- the space of Hardy-Rellich potentials

$$\mathcal{HR}_p(\Omega) = \left\{ g \in L^1_{loc}(\Omega) : g \text{ satisfies } (1.2.3) \right\},$$

- the space of Hardy-Hessian potentials

$$\mathbb{H}_p(\Omega) = \left\{ g \in L^1_{loc}(\Omega) : g \text{ satisfies } (1.2.4) \right\}.$$

Also, we have observed that $\mathcal{HR}_p(\Omega) \subseteq \mathbb{H}_p(\Omega)$, and for $p = 2$, they are the same i.e., $\mathcal{HR}_2(\Omega) = \mathbb{H}_2(\Omega)$. The Poincaré inequality ensures that $L^\infty(\Omega) \subseteq \mathcal{HR}_p(\Omega)$ if Ω is bounded in one direction. Whereas, (1.2.2) assures that $\frac{1}{|x|^{2p}} \in \mathcal{HR}_p(\Omega)$ even if Ω contains the origin. Unlike the Hardy potentials, Maz′ya type characterisation

(Theorem 2.3.5) for Hardy-Rellich potentials and Hardy-Hessian potentials are not known. Although, Maz′ya has given a necessary and sufficient condition (using higher-order capacity) on g so that (1.2.3) holds for all non-negative $u \in \mathrm{C}_c^2(\Omega)$; however, this can not be extended for sign-changing functions, see [57, Section 8.2.1, page 363]. In this chapter, we provide several function spaces that lie in $\mathcal{HR}_p(\Omega)$ or $\mathbb{H}_p(\Omega)$.

4.1 Lorentz spaces in $\mathbb{H}_p(\Omega)$ and $\mathcal{HR}_p(\Omega)$

In this section, we prove Theorem 1.2.1. Our aim is to identify certain Lorentz spaces in $\mathcal{HR}_p(\Omega)$ and $\mathbb{H}_p(\Omega)$. One may use various embeddings of $\mathcal{D}_0^{2,p}(\Omega)$ (the completion of $\mathrm{C}_c^2(\Omega)$ with respect to the norm $\|u\|_{\mathcal{D}_0^{2,p}} := [\int_\Omega |\nabla^2 u|^p]^{\frac{1}{p}}$), to show certain Lebesgue and Lorentz space are contained in $\mathbb{H}_p(\Omega)$. For instance, using the Lorentz-Sobolev embedding $\mathcal{D}_0^{2,p}(\Omega) \hookrightarrow L^{p^{**},p}(\Omega)$, one can deduce the following:

$$
\begin{aligned}
\int_\Omega |g||u|^p \, dx &\leq \|g\|_{L^{\frac{N}{2p},\infty}} \||u|^p\|_{L^{\frac{p^{**}}{p},1}} = \|g\|_{L^{\frac{N}{p},\infty}} \|u\|^p_{L^{p^{**},p}} \\
&\leq C\|g\|_{L^{\frac{N}{p},\infty}} \|u\|^p_{\mathcal{D}_0^{2,p}} = C\|g\|_{L^{\frac{N}{p},\infty}} \int_\Omega |\nabla^2 u|^p \, dx \,. \qquad (4.1.1)
\end{aligned}
$$

This inequality clearly shows that $L^{\frac{N}{p},\infty}(\Omega) \subseteq \mathbb{H}_p(\Omega)$. Now, for any $p \in (1, \frac{N}{2})$, we prove $L^{\frac{N}{p},\infty}(\Omega) \subseteq \mathcal{HR}_p(\Omega)$ (which is Theorem 1.2.1-(i)) by using the Muckenhoupt necessary and sufficient conditions (Lemma 2.5.1) for the one dimensional weighted Hardy inequalities and a pointwise inequality for the symmetrization (Lemma 2.1.3) obtained in [23].

First, we prove the following lemma which is an immediate consequence of the Muckenhoupt conditions (Lemma 2.5.1).

Lemma 4.1.1. *For $N > \max\{2p, 2p'\}$, let Ω be an open set in $\mathbb{R}^N$ and $g \in L^{\frac{N}{2p},\infty}(\Omega)$. Then, there exists a constant $C = C(N) > 0$ such that the following two inequalities* *hold:*

$$\int_0^{|\Omega|} g^*(s) s^{-p+\frac{2p}{N}} \left(\int_0^s |f(t)|\ dt \right)^p\ ds \le C \|g\|_{L^{\frac{N}{2p},\infty}} \int_0^{|\Omega|} |f(s)|^p\ ds, \tag{4.1.2}$$

$$\int_0^{|\Omega|} g^*(s) \left(\int_s^{|\Omega|} |f(t)| t^{-1+\frac{2}{N}}\ dt \right)^p\ ds \le C \|g\|_{L^{\frac{N}{2p},\infty}} \int_0^{|\Omega|} |f(s)|^p\ ds, \tag{4.1.3}$$

for any measurable function f on $(0, |\Omega|)$.

Proof. For proving (4.1.2), we set $a = |\Omega|, u(s) = g^*(s) s^{-p+\frac{2p}{N}}$ and $v(s) = 1$ in (2.5.1). Thus $\int_0^t v(s)^{1-p'}\ ds = \int_0^t\ ds = t$. Further, since $N > 2p'$, we can get

$$\begin{aligned} \int_t^a u(s)\ ds = \int_t^{|\Omega|} g^*(s) s^{-p+\frac{2p}{N}}\ ds &\le g^*(t) \int_t^{|\Omega|} s^{-p+\frac{2p}{N}}\ ds \\ &\le \frac{N}{p(N-2)-N} t^{\frac{2p}{N}-p+1} g^*(t)\,. \end{aligned}$$

Therefore,

$$A_1 = \sup_{0<t<a} \left(\int_t^a u(s)\ ds \right)^{\frac{1}{p}} \left(\int_0^t v(s)^{1-p'}\ ds \right)^{\frac{1}{p'}} \le C \|g\|^p_{L^{\frac{N}{2p},\infty}} < \infty$$

and hence (4.1.2) follows from part (i) of Lemma 2.5.1.

To prove (4.1.3) we set $a = |\Omega|, u(s) = g^*(s)$ and $v(s) = s^{p-\frac{2p}{N}}$ in (2.5.3). Now $\int_0^t u(s)\ ds = \int_0^t g^*(s)\ ds = t g^{**}(t)$ and since $N > 2p$, we get

$$\int_t^a v(s)^{1-p'}\ ds = \int_t^{|\Omega|} s^{(p-\frac{2p}{N})(1-p')}\ ds \le \frac{N}{p'(N-2)-N} t^{(p-\frac{2p}{N})(1-p')+1}\,.$$

Therefore,

$$A_2 = \sup_{0<t<a} \left(\int_0^t u(s)\ ds \right)^{\frac{1}{p}} \left(\int_t^a v(s)^{1-p'}\ ds \right)^{\frac{1}{p'}} \le C \|g\|^p_{L^{\frac{N}{2p},\infty}} < \infty.$$

Hence (4.1.3) follows from part (ii) of Lemma 2.5.1. □

Now we prove Theorem 1.2.1.

Proof of Theorem 1.2.1. (i) Let $u \in C_c^2(\Omega)$. Then by the Hardy-Littlewood inequality (Proposition 2.1.2-(i)) we have

$$\int_{\Omega} |g(x)||u(x)|^p \, dx \leq \int_0^{|\Omega|} g^*(s)u^*(s)^p \, ds. \tag{4.1.4}$$

Furthermore, Lemma 2.1.3 gives

$$\begin{aligned}\int_0^{|\Omega|} g^*(s)u^*(s)^p \, ds \leq 2^{p-1} \int_0^{|\Omega|} g^*(s)s^{-p+\frac{2p}{N}} \left(\int_0^s |\Delta u|^*(t)\mathrm{d}t\right)^p \, ds \\ + 2^{p-1} \int_0^{|\Omega|} g^*(s) \left(\int_s^{\infty} |\Delta u|^*(t)t^{-1+\frac{2}{N}} \, dt\right)^p \, ds.\end{aligned} \tag{4.1.5}$$

Since $g \in L^{\frac{N}{2p},\infty}(\Omega)$, using Lemma 4.1.1 we can estimate each term in the right hand side of the inequality by

$$C\|g\|_{L^{\frac{N}{2p},\infty}} \int_0^{|\Omega|} (|\Delta u|^*(t))^p \, dt.$$

Noting that $\||\Delta u|^*\|_{L^p((0,|\Omega|))} = \|\Delta u\|_{L^p(\Omega)}$, (4.1.4) and (4.1.5) yields

$$\int_{\Omega} |g(x)||u(x)|^p \, dx \leq C\|g\|_{L^{\frac{N}{2p},\infty}} \int_{\Omega} |\Delta u|^p \, dx, \ \forall u \in C_c^2(\Omega). \tag{4.1.6}$$

Hence $g \in \mathcal{HR}_p(\Omega)$.

(ii) Let $R \in (0, \infty]$ and let $\Omega = B(0; R) \subset \mathbb{R}^N$. Let $g : \Omega \to [0, \infty)$ be a radial and radially decreasing function in $\mathcal{HR}_p(\Omega)$. We will show that $g \in L^{\frac{N}{2p},\infty}(\Omega)$. For each $r \in (0, R)$, consider the following function:

$$u_r(x) = \begin{cases} (r - |x|)^2 & |x| \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

By differentiating twice, we get

$$\Delta u_r(x) = \begin{cases} 2N - (2N-2)\frac{r}{|x|} & |x| < r, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} \int_\Omega |\Delta u_r|^p \, dx = \int_{B_r} |\Delta u_r|^p \, dx &= \int_{B_r} \left[2N - (2N-2)\frac{r}{|x|}\right]^p \, dx \\ &\leq 2^{p-1}\left[2^p N^p \omega_N r^N + (2N-2)^p r^p \int_{B_r} \frac{1}{|x|^p} \, dx\right] \\ &\leq C_1 \left[r^N + r^p \int_0^r s^{N-p-1} \, ds\right] \leq C_2 r^N, \end{aligned} \tag{4.1.7}$$

where C_1, C_2 are constants that depends only on N. Thus for each $r \in (0,R)$, $u_r \in \mathcal{D}_0^{2,p}(\Omega)$. Furthermore, since $g \in \mathcal{HR}_p(\Omega)$ and $\mathrm{C}_c^2(\Omega)$ is dense in $\mathcal{D}_0^{2,p}(\Omega)$, we have

$$\int_\Omega |g(x)||u_r(x)|^p \, dx \leq C \int_\Omega |\Delta u_r|^p \, dx, \ \forall r \in (0,R). \tag{4.1.8}$$

Since g is radial and radially decreasing, the left hand side of the above inequality can be estimated as below:

$$\begin{aligned} \int_\Omega |g(x)||u_r(x)|^p \, dx \geq \int_{B_{\frac{r}{2}}} |g(|x|)||u_r(x)|^p \, dx &\geq \left(r - \frac{r}{2}\right)^{2p} \int_{B_{\frac{r}{2}}} |g(|x|)| \, dx \\ &= \left(\frac{r}{2}\right)^{2p} \int_{B_{\frac{r}{2}}} g^\star(x) \, dx = \left(\frac{r}{2}\right)^{2p} \int_0^{\omega_N (\frac{r}{2})^N} g^*(s) \, ds. \end{aligned} \tag{4.1.9}$$

From (4.1.7), (4.1.8) and (4.1.9), we obtain

$$\left(\frac{r}{2}\right)^{2p} \int_0^{\omega_N (\frac{r}{2})^N} g^*(s) \, ds \leq C C_2 r^N.$$

Now by setting $\omega_N(\frac{r}{2})^N = t$ and since $0 < r < R$ is arbitrary, we conclude that

$$\sup_{t \in (0, \frac{|\Omega|}{2^N})} t^{\frac{2p}{N}} g^{**}(t) \leq C_3.$$

As $t^{\frac{2p}{N}} g^{**}(t)$ is bounded on $(\frac{|\Omega|}{2^N}, |\Omega|)$, g must belong to $L^{\frac{N}{2p},\infty}(\Omega)$. □

Remark 4.1.2. Recall that $\mathbb{H}_2(\Omega) = \mathcal{HR}_2(\Omega)$. Thus, by taking $p = 2$ and $N \geq 5$ in Theorem 1.2.1, we have the following:

(*i*) (A sufficient condition) $L^{\frac{N}{4},\infty}(\Omega) \subseteq \mathcal{HR}_2(\Omega)$.

(*ii*) (A necessary condition) Let Ω be a ball centered at the origin or entire $\mathbb{R}^N$ and g be radial, radially decreasing. Then $g \in \mathcal{HR}_2(\Omega)$, only if g belongs to $L^{\frac{N}{4},\infty}(\Omega)$.

Remark 4.1.3. Let $p = 2$ and $N \geq 5$. Let C_R be the best constant in (1.2.3). Then from (2.1.1), Remark 2.5.2 and Lemma 4.1.1 one can deduce that

$$C_R \leq \frac{N}{(N-4)(N-2)^2 \omega_N^{\frac{4}{N}}} \|g\|_{L^{\frac{N}{4},\infty}}.$$

Example 4.1.4. Let $p = 2$ and $N \geq 5$. For $\alpha \in (0, N)$ and $R \in (0, \infty]$ let $g(x) = \dfrac{1}{|x|^\alpha}, x \in B_R(0)$. It is easy to calculate

$$g^*(t) = \begin{cases} \left(\frac{\omega_N}{t}\right)^{\frac{\alpha}{N}} & 0 < t < \omega_N R^N, \\ 0 & t \geq \omega_N R^N. \end{cases} \qquad g^{**}(t) = \begin{cases} \frac{N}{N-\alpha}\left(\frac{\omega_N}{t}\right)^{\frac{\alpha}{N}} & 0 < t < \omega_N R^N, \\ 0 & t \geq \omega_N R^N. \end{cases}$$

Therefore,

$$g \in L^{\frac{N}{4},\infty}(B_R(0)) \text{ with } \begin{cases} R < \infty \text{ if and only if } & \alpha \leq 4 \\ R = \infty \text{ if and only if } & \alpha = 4. \end{cases}$$

Remark 4.1.5. A similar computation as in Example 4.1.4 shows that, if $N > 2p$, then $g(x) = \frac{1}{|x|^{2p}}$ belongs to $L^{\frac{N}{2p},\infty}(\mathbb{R}^N)$ and $\|g\|_{L^{\frac{N}{2p},\infty}} = \frac{N\omega_N^{\frac{2p}{N}}}{N-2p}$. This verifies that g is a Hardy-Rellich potential (by Theorem 1.2.2-(i)).

As a consequence of Theorem 1.2.1-(i), we have a simple proof for the Lorentz-Sobolev embedding:

Corollary 4.1.6. *Let $\Omega \subset \mathbb{R}^N$ is an open set and $N > \max\{2p, 2p'\}$. Then we have the following embedding:*

$$\mathcal{D}_0^{2,p}(\Omega) \hookrightarrow L^{p^{**},p}(\Omega), \text{ where } p^{**} = \frac{Np}{N-2p}.$$

Proof. Without loss of generality we may assume $\Omega = \mathbb{R}^N$ (for a general domain Ω, the result will follow by considering the zero extension to $\mathbb{R}^N$). Using the density of $\mathrm{C}_c^2(\mathbb{R}^N)$ in $\mathcal{D}_0^{2,p}(\mathbb{R}^N)$, for each $g \in L^{\frac{N}{2p},\infty}(\mathbb{R}^N)$ we have

$$\int_0^\infty g^*(t)|u^*(t)|^p \, dt \le C\|g\|_{L^{\frac{N}{2p},\infty}} \int_{\mathbb{R}^N} |\Delta u|^p \, dx, \, \forall \, u \in \mathcal{D}_0^{2,p}(\mathbb{R}^N).$$

In particular, if we choose $g(x) = \frac{1}{|x|^{2p}}$, then $g^*(t) = \left(\frac{\omega_N}{t}\right)^{\frac{2p}{N}}$ and $\|g\|_{L^{\frac{N}{2p},\infty}} = \frac{N\omega_N^{\frac{2p}{N}}}{N-2p}$. Substituting this in the above inequality, we get

$$\int_0^\infty t^{-\frac{2p}{N}}|u^*(t)|^p \, dt \le C_1 \int_{\mathbb{R}^N} |\Delta u|^p \, dx, \, \forall \, u \in \mathcal{D}_0^{2,p}(\mathbb{R}^N),$$

where C_1 is a constant that depends only on N. Since $\displaystyle\int_0^\infty t^{-\frac{2p}{N}}|u^*(t)|^p \, dt = |u|_{(p^{**},p)}^p$ is equivalent to $\|u\|_{L^{p^{**},p}}^p$, we obtain the required embedding

$$\|u\|_{L^{p^{**},p}}^p \le C_2 \int_{\mathbb{R}^N} |\Delta u|^p \mathrm{d}x, \;\; \forall u \in \mathcal{D}_0^{2,p}(\mathbb{R}^N).$$

This proves the corollary. ∎

4.2 Weighted Lebesgue spaces in $\mathbb{H}_p(\Omega)$

In this section, we prove Theorem 1.2.3. We use the fundamental theorem of integral calculus to identify certain weighted Lebesgue spaces in $\mathbb{H}_p(\Omega)$.

We commence with the following proposition.

Proposition 4.2.1. *Let $p \in (1, \infty)$. For $u \in \mathrm{C}_\mathrm{c}^2(\mathbb{R}^\mathrm{N})$, the following inequality holds:*

$$\int_0^\infty \int_{\mathbb{S}^{N-1}} r^{N-1} \left| \frac{\partial^2 u}{\partial r^2}(r, w) \right|^p \, dS_w \, dr \leq \int_{\mathbb{R}^N} |\nabla^2 u|^p \, dx \, .$$

Proof. Observe that

$$\frac{\partial u}{\partial \eta} = \nabla u \cdot \eta \text{ and } \frac{\partial^2 u}{\partial \eta^2} = \nabla(\nabla u \cdot \eta) \cdot \eta = \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \eta_i \eta_j .$$

Further, have the following inequality for an $N \times N$ real matrix $A = (a_{ij})$ and $x \in \mathbb{R}^N$:

$$|\langle Ax, x \rangle|^2 \leq |\sum_{i=1}^N \sum_{j=1}^N a_{ij} x_i x_j|^2 \leq \left(\sum_{i=1}^N \sum_{j=1}^N a_{ij}^2 \right) \left(\sum_{i=1}^N x_i^2 \right) \left(\sum_{j=1}^N x_j^2 \right). \tag{4.2.1}$$

Now by writing $x = (r, \omega) \in (0, \infty) \times \mathbb{S}^{N-1}$ for $x \in \mathbb{R}^N \setminus \{0\}$, and using (4.2.1), we obtain

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{S}^{N-1}} \left| \frac{\partial^2 u}{\partial r^2}(r, \omega) \right|^p r^{N-1} \, dS_\omega \, dr &= \int_{\mathbb{R}^N} \left| \frac{\partial^2 u}{\partial r^2} \right|^p \, dx \\
&= \int_{\mathbb{R}^N} \left| \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{x_i}{|x|} \frac{x_j}{|x|} \right|^p \, dx \\
&\leq \int_{\mathbb{R}^N} \left[\sum_{i=1}^N \sum_{j=1}^N \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 \right]^{\frac{p}{2}} \, dx \\
&= \int_{\mathbb{R}^N} |\nabla^2 u|^p \, dx,
\end{aligned}$$

and this concludes our proof. □

For $N > 2p$, in Theorem 1.2.3, we exhibit a weighted Lebesgue space in $\mathbb{H}_p(\Omega)$. A similar result for the Hardy potentials is obtained in [33, Lemma 1.1]. Let us recall that, for an open set Ω in $\mathbb{R}^N$ and a radial, non-negative function w on $\mathbb{R}^N$,

$$L^1_{rad}(\mathbb{R}^N, w) = \left\{ g \in L^1(\mathbb{R}^N, w) : g \text{ is radial} \right\},$$
$$L^1_{rad}(\Omega, w) = \left\{ \, g|_\Omega : g \in L^1_{rad}(\mathbb{R}^N, w) \right\}.$$

Now we prove Theorem 1.2.3 for $N > 2p$.

Proof of Theorem 1.2.3 (*for* $N > 2p$). For $x \in \mathbb{R}^N \setminus \{0\}$, using the polar coordinates, we write $x = (r, \omega) \in (0, \infty) \times \mathbb{S}^{N-1}$. Thus for $u \in \mathrm{C}^2_c(\mathbb{R}^N)$,

$$u(r, \omega) = -\int_r^\infty \frac{\partial u}{\partial t}(t, \omega)\, dt = -r \frac{\partial u}{\partial t}(r, \omega) + \int_r^\infty t \frac{\partial^2 u}{\partial t^2}(t, \omega)\, dt$$
$$= \int_r^\infty (t - r) \frac{\partial^2 u}{\partial t^2}(t, \omega)\, dt. \tag{4.2.2}$$

Hence

$$|u(r, \omega)| \leq \int_r^\infty t \left| \frac{\partial^2 u}{\partial t^2}(t, \omega) \right| \, dt = \int_r^\infty t\, t^{\frac{1-N}{p}} t^{\frac{N-1}{p}} \left| \frac{\partial^2 u}{\partial t^2}(t, \omega) \right| \, dt.$$

Now by Hölder inequality, we get

$$|u(r, \omega)|^p \leq \left(\int_r^\infty t^{p'} t^{(1-N)\frac{p'}{p}} \, dt \right)^{\frac{p}{p'}} \left(\int_r^\infty t^{N-1} \left| \frac{\partial^2 u}{\partial t^2}(t, \omega) \right|^p \, dt \right)$$
$$= \frac{1}{N - 2p} r^{2p-N} \int_r^\infty t^{N-1} \left| \frac{\partial^2 u}{\partial t^2}(t, \omega) \right|^p \, dt. \tag{4.2.3}$$

Since $g \in L^1_{rad}(\Omega, |x|^{2p-N})$, there exists $\widetilde{g} : [0, \infty) \mapsto [0, \infty)$ such that $|g(x)| = \widetilde{g}(|x|)$ and $\int_{\mathbb{R}^N} \widetilde{g}(|x|)|x|^{2p-N} \, dx < \infty$. Multiply both sides of (4.2.3) by $\widetilde{g}(r)$ and integrate over $\mathbb{S}^{N-1}$ to obtain

$$\begin{aligned}\int_{\mathbb{S}^{N-1}} \widetilde{g}(r)|u(r,\omega)|^p \, dS_\omega &\leq \frac{1}{N-2p} r^{2p-N}\widetilde{g}(r) \int_0^\infty \int_{\mathbb{S}^{N-1}} t^{N-1} \left|\frac{\partial^2 u}{\partial t^2}(t,\omega)\right|^p \, dS_\omega \, dt \\ &\leq \frac{1}{N-2p} r^{2p-N}\widetilde{g}(r) \left(\int_{\mathbb{R}^N} |\nabla^2 u|^p \, dx\right), \end{aligned} \tag{4.2.4}$$

where the last inequality follows from Lemma 4.2.1. Finally, multiplying both the sides of (4.2.4) by $N\omega_N r^{N-1}$ and integrating over $(0,\infty)$ with respect to r yields:

$$\int_{\mathbb{R}^N} \widetilde{g}(|x|)|u(x)|^p \, dx \leq \frac{1}{N-2p} \left(\int_{\mathbb{R}^N} \widetilde{g}(|x|)|x|^{2p-N} \, dx\right) \left(\int_{\mathbb{R}^N} |\nabla^2 u|^p \, dx\right).$$

Thus, we have

$$\int_\Omega |g(x)||u(x)|^p \, dx \leq \mathrm{C}(N,p) \left(\int_{\mathbb{R}^N} \widetilde{g}(x)|x|^{2p-N} \, dx\right) \left(\int_\Omega |\nabla^2 u|^p \, dx\right), \ \forall u \in \mathrm{C}_c^2(\Omega).$$

Hence $g \in \mathbb{H}_p(\Omega)$. □

Remark 4.2.2. We would like to remark that, for an open subset Ω in $\mathbb{R}^N$ with $N > 2p$, if $g \in L^1_{loc}(\Omega)$ is such that $|g(x)| \leq w(|x|)$ for some measurable function $w : [0,\infty) \mapsto [0,\infty)$ and $\int_0^\infty r^{2p-1} w(r) \, dr < \infty$, then Theorem 1.2.3 infers that $g \in \mathbb{H}_p(\Omega)$.

Remark 4.2.3. Since $\mathbb{H}_2(\Omega) = \mathcal{HR}_2(\Omega)$, by taking $p = 2$ and $N \geq 5$ in Theorem 1.2.3, we conclude that $L^1_{loc}(\Omega, |x|^{4-N}) \subseteq \mathcal{HR}_2(\Omega)$.

Next, we show that the spaces identified in Theorem 1.2.2 and the spaces identified in Theorem 1.2.3 are not contained in each other.

Example 4.2.4. Let $p = 2$, $\Omega = \mathbb{R}^N$ with $N \geq 5$ and let $\beta \in (\frac{4}{N}, 1)$. Consider

$$g_1(x) = \begin{cases} (|x|-1)^{-\beta}, & 1 < |x| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

We can compute the distribution function α_{g_1} and the one dimensional decreasing rearrangement g_1^* as below:

$$\alpha_{g_1}(s) = \begin{cases} \omega_N 2^N - \omega_N, & 0 \le s < 1 \\ \omega_N \left(s^{-\frac{1}{\beta}} + 1\right)^N - \omega_N, & s \ge 1\,. \end{cases}$$

$$g_1^*(t) = \begin{cases} 0, & t > \omega_N(2^N - 1) \\ \left(\left(\frac{t}{\omega_N} + 1\right)^{\frac{1}{N}} - 1\right)^{-\beta}, & t \le \omega_N(2^N - 1). \end{cases}$$

Hence, for $t \le \omega_N(2^N - 1)$,

$$t^{\frac{4}{N}} g_1^*(t) = t^{\frac{4}{N}} \left(\left(\frac{t}{\omega_N} + 1\right)^{\frac{1}{N}} - 1\right)^{-\beta} \ge t^{\frac{4}{N}} \left(\left(\frac{t}{\omega_N} + 1\right) - 1\right)^{-\beta} = t^{\frac{4}{N}} \left(\frac{t}{\omega_N}\right)^{-\beta}.$$

Since $\beta > \frac{4}{N}$, $\sup_{t\in(0,\infty)} t^{\frac{4}{N}} g_1^*(t) = \infty$ and hence $g \notin L^{\frac{N}{4},\infty}(\mathbb{R}^N)$.

Let $w(r) = (r-1)^{-\beta}\chi_{(1,2)}(r)$. Clearly $g_1(x) \le w(|x|), \forall x \in \mathbb{R}^N$ and since $\beta < 1$,

$$\int_0^\infty w(r) r^3 \, dr = \int_1^2 (r-1)^{-\beta} r^3 \, dr \le 8 \int_0^1 s^{-\beta} \, ds < \infty.$$

Thus g_1 is a Hardy-Rellich potential by Theorem 1.2.3.

Example 4.2.5. Let $p = 2$, $g_2(x) = \frac{1}{|x|^4}$, $x \in \mathbb{R}^N$ with $N \ge 5$. By Example 4.1.4, $g_2 \in L^{\frac{N}{4},\infty}(\mathbb{R}^N)$ and hence $g_2 \in \mathcal{HR}_2(\mathbb{R}^N)$ by Theorem 1.2.1. Let w be a function on $(0,\infty)$ such that $g(x) \le w(|x|)$. Then

$$\int_0^\infty w(r) r^3 \, dr \ge \int_0^\infty r^{-4} \times r^3 \, dr = \infty.$$

Thus g_2 does not satisfy the assumptions of Theorem 1.2.3.

Remark 4.2.6. The above examples shows that the the sufficient conditions given by Theorem 1.2.2 and Theorem 1.2.3 are independent. The question whether these spaces exhaust all the Hardy-Rellich potentials is open.

Remark 4.2.7. There are Hardy-Rellich potentials whose Schwarz symmetrizations are not Hardy-Rellich potentials. For example, the Schwarz symmetrization $g_1^\star$ of g_1 (in Example 4.2.4) does not belong to $L^{\frac{N}{4},\infty}(\Omega^\star)$ and hence by part of (ii) of Theorem 1.2.2, $g_1^\star$ can not be an Hardy-Rellich potentials.

Next we prove Theorem 1.2.3 for $2 \leq N \leq 2p$. In this case, we assume $\Omega = \Omega_{a,b,S}$, where S is an open subset of $\mathbb{S}^{N-1}$ and $a, b \in (0,\infty]$ with $b > a$, and

$$\Omega_{a,b,S} = \left\{ x \in \mathbb{R}^N : a < |x| < b, \frac{x}{|x|} \in S \text{ if } x \neq 0 \right\}.$$

Proof of Theorem 1.2.3 (*for* $2 \leq N \leq 2p$). Let $\Omega = \Omega_{a,b,S}$ with $a > 0$. As before, for $x \in \Omega$, we write $x = (r, \omega) \in (a, b) \times S$. For $u \in \mathrm{C}_c^2(\Omega)$, we use the fundamental theorem of calculus to get

$$u(r,\omega) = \int_a^r \frac{\partial u}{\partial t}(t,\omega)\, dt.$$

As in the proof of Theorem 1.2.3 for $N > 2p$, we deduce

$$u(r,\omega) = \int_a^r (r-t)\frac{\partial^2 u}{\partial t^2}(t,\omega)\, dt = \int_a^r (r-t)t^{-\frac{N-1}{p}} t^{\frac{N-1}{p}} \frac{\partial^2 u}{\partial t^2}(t,\omega)\, dt.$$

Now Hölder inequality yields

$$|u(r,\omega)|^p \leq r^p \left(\int_a^r t^{\frac{1-N}{p-1}}\, dt \right)^{p-1} \left(\int_a^r t^{N-1} \left| \frac{\partial^2 u}{\partial t^2}(t,\omega) \right|^p dt \right).$$

Since $g \in X_{rad}(\Omega)$, there exists $\widetilde{g} : [0,\infty) \mapsto [0,\infty)$ such that $|g(x)| = \widetilde{g}(|x|)$. Multiply the above inequality by $N\omega_N r^{N-1}\widetilde{g}(r)$ and integrate over $S \times (a, b)$ and use Lemma 4.2.1 to obtain

$$\begin{aligned}
&\int_\Omega \widetilde{g}(|x|)|u(x)|^p\, dx \\
&\leq \left(\int_a^b \left[\int_a^r t^{\frac{1-N}{p-1}}\, dt \right]^{p-1} N\omega_N r^{N+p-1}\widetilde{g}(r)\, dr \right) \left(\int_S \int_a^b t^{N-1} \left| \frac{\partial^2 u}{\partial t^2}(t,\omega) \right|^p dt\, dS_\omega \right).
\end{aligned}$$

Let $A = \left(\int_a^b r^{N+p-1} \left[\int_a^r t^{\frac{1-N}{p-1}} \, dt \right]^{p-1} \widetilde{g}(r) \, dr \right)$. Then the above inequality yields

$$\int_\Omega \widetilde{g}(|x|)|u(x)|^p \, dx \leq N\omega_N A \left(\int_\Omega |\nabla^2 u|^2 \, dx \right). \tag{4.2.5}$$

Notice that

$$A = \begin{cases} \int_\Omega \widetilde{g}(|x|)|x|^p \, dx, & p < N \leq 2p; b = \infty \\ \int_\Omega \widetilde{g}(|x|)|x|^p [\log(\frac{|x|}{a})]^{p-1} \, dx, & N = p; b = \infty \\ \int_\Omega \widetilde{g}(|x|) \, dx, & 2 \leq N \leq 2p; b < \infty. \end{cases} \tag{4.2.6}$$

Now the assumptions on g together with (4.2.5) and (4.2.6) infers that $g \in \mathbb{H}_p(\Omega)$.

$\square$

Remark 4.2.8. Let $p = 2$ and $\Omega = \Omega_{a,b,S}$ (with $a > 0$) be a sectorial open set in $\mathbb{R}^N$ with $2 \leq N \leq 4$. Let $g \in L^1_{loc}(\Omega)$ be such that

$$g \in \begin{cases} L^1_{rad}(\Omega, |x|^2), & N = 3, 4; b = \infty \\ L^1_{rad}(\Omega, |x|^2 \log(\frac{|x|}{a})), & N = 2; b = \infty \\ L^1_{rad}(\Omega), & 2 \leq N \leq 4; b < \infty. \end{cases}$$

Then Theorem 1.2.3 infers that $g \in \mathcal{HR}_2(\Omega)$ (as $\mathcal{HR}_2(\Omega) = \mathbb{H}_2(\Omega)$).

4.3 The critical case ($N = 4$, $p = 2$)

In this section, we consider the particular case: $p = 2$ and $N = 4$ and prove Theorem 1.2.2. In this case, we assume that the domain Ω is bounded. Recall that, for a bounded domain Ω, we define

$$\mathcal{M} \log L(\Omega) = \left\{ g \text{ measurable } : \sup_{0<t<|\Omega|} t \log\left(\frac{|\Omega|}{t}\right) g^{**}(t) < \infty \right\}.$$

$\mathcal{M}\log L(\Omega)$ is a rearrangement invariant Banach function space with the norm

$$\|g\|_{\mathcal{M}\log L} = \sup_{0<t<|\Omega|} t \log\left(\frac{|\Omega|}{t}\right) g^{**}(t),$$

First we show that, as a vector space $\mathcal{M}\log L(\Omega)$ is nothing but the Lorentz-Zygmund sapce $L^{1,\infty}(\log L)^2(\Omega)$.

Proposition 4.3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set. Then $L^{1,\infty}(\log L)^2(\Omega) = \mathcal{M}\log L(\Omega)$.*

Proof. First we prove that $\|f\|_{\mathcal{M}\log L} \leq |f|_{L^{1,\infty}(\log L)^2}$. For $f \in \mathcal{M}(\Omega)$ and $t \in (0, |\Omega|)$, we have

$$\begin{aligned} tf^{**}(t) &= \int_0^t f^*(s)\, ds = \int_0^t f^*(s)s\left[\log\left(\frac{|\Omega|}{s}\right)\right]^2 \frac{1}{s\left[\log\left(\frac{|\Omega|}{s}\right)\right]^2}\, ds \\ &\leq \sup_{0<s\leq t} f^*(s)\, s\, \left[\log\left(\frac{|\Omega|}{s}\right)\right]^2 \int_0^t \frac{1}{s\left[\log\left(\frac{|\Omega|}{s}\right)\right]^2}\, ds \\ &\leq \frac{|f|_{L^{1,\infty}(\log L)^2}}{\log\left(\frac{|\Omega|}{t}\right)}. \end{aligned}$$

This yields $\|f\|_{\mathcal{M}\log L} \leq |f|_{L^{1,\infty}(\log L)^2}$ and hence

$$L^{1,\infty}(\log L)^2(\Omega) \subseteq \mathcal{M}\log L(\Omega).$$

If the above inclusion is strict, then $\exists f \in \mathcal{M}\log L(\Omega) \setminus L^{1,\infty}(\log L)^2(\Omega)$, i.e.,

$$\sup_{0<t<|\Omega|} f^{**}(t)\, t\left[\log\left(\frac{|\Omega|}{t}\right)\right] < \infty; \quad \sup_{0<t<|\Omega|} f^*(t)\, t\, \left[\log\left(\frac{e|\Omega|}{t}\right)\right]^2 = \infty.$$

Now consider the function

$$g(t) = f^*(t)\, t\, \left[\log\left(\frac{e|\Omega|}{t}\right)\right]^2, 0 < t < |\Omega|.$$

Claim: $\lim_{t\to 0} g(t) = \infty$.

If the claim is not true, then $\exists t_0 > 0$ such that $\sup_{t\geq t_0} g(t) = \infty$. Since $t\,\left[\log\left(\frac{e|\Omega|}{t}\right)\right]^2$ is bounded, we must have $f^*(t) = \infty$ for $t \leq t_0$. A contradiction as $f \in \mathcal{M}\log L(\Omega)$ hence claim must be true.

Now by the claim, there exists a decreasing sequence (t_n) in $(0, |\Omega|)$ such that (t_n) converging to 0 and $g(t) > n$, for $t \in (0, t_n)$. Consequently,

$$t_n f^{**}(t_n) = \int_0^{t_n} \frac{g(t)}{t\left[\log\left(\frac{e|\Omega|}{t}\right)\right]^2} dt \geq n \int_0^{t_n} \frac{1}{t\left[\log\left(\frac{e|\Omega|}{t}\right)\right]^2} dt \geq \frac{n}{\log\left(\frac{e|\Omega|}{t_n}\right)}.$$

Therefore,

$$\lim_{n\to\infty} t_n f^{**}(t_n)\log\left(\frac{|\Omega|}{t_n}\right) \geq \lim_{n\to\infty} n\frac{\log\left(\frac{|\Omega|}{t_n}\right)}{\log\left(\frac{e|\Omega|}{t_n}\right)} = \infty.$$

A contradiction as $f \in \mathcal{M}\log L(\Omega)$. Hence $L^{1,\infty}(\log L)^2(\Omega) = \mathcal{M}\log L(\Omega)$. $\square$

Remark 4.3.2. From the above proposition, one can observe that the quasi-norm $|f|_{L^{1,\infty}(\log L)^2}$ and the norm $\|f\|_{\mathcal{M}\log L}$ defines the same vector space, however, they are not equivalent. To see this, let $\Omega = B(0; R) \subset \mathbb{R}^N$ and for each $n \in \mathbb{N}$, consider the function $\{f_n\}$ on Ω defined as

$$f_n(x) = \begin{cases} \dfrac{1}{|x|^N[\log((\frac{R}{|x|})^N e)]^{n+2}}, & x \in B(0, Re^{-(\frac{n+1}{N})}) \\ 0, & \text{otherwise.} \end{cases}$$

Let $T = |B(0; Re^{-(\frac{n+1}{N})})|$. Then, we have

$$f_n^*(t) = \begin{cases} \dfrac{\omega_N}{t[\log(\frac{e|\Omega|}{t})]^{n+2}}, & t \in (0, T) \\ 0, & t \in [T, |\Omega|), \end{cases}$$

and

$$f_n^{**}(t) = \begin{cases} \dfrac{\omega_N}{(n+1)t[\log(\frac{e|\Omega|}{t})]^{n+1}}, & t \in (0,T] \\ (\frac{T}{t}) f_n^{**}(T), & t \in (T, |\Omega|). \end{cases}$$

Therefore,

$$|f_n|_{(1,\infty,2)} = \sup_{0<t<|\Omega|} t \left[\log\left(\frac{e|\Omega|}{t}\right)\right]^2 f_n^*(t) = \omega_N \sup_{0<t<T} \frac{1}{\left[\log\left(\frac{e|\Omega|}{t}\right)\right]^n}.$$

Now, notice that $t\left[\log\left(\frac{|\Omega|}{t}\right)\right] f_n^{**}(t) \leq T\left[\log\left(\frac{|\Omega|}{T}\right)\right] f_n^{**}(T)$ for $t \geq T$. Thus,

$$\|f_n\|_{\mathcal{M}\log L} = \sup_{0<t<|\Omega|} t \left[\log\left(\frac{|\Omega|}{t}\right)\right] f_n^{**}(t) \leq \frac{\omega_N}{n+1} \sup_{0<t<T} \frac{1}{\left[\log\left(\frac{e|\Omega|}{t}\right)\right]^n}.$$

Hence $(n+1)\|f_n\|_{\mathcal{M}\log L} \leq |f_n|_{(1,\infty,2)}$.

Now we prove an analogue of Lemma 4.1.1 for $N = 4$.

Lemma 4.3.3. *Let Ω be a bounded open set in $\mathbb{R}^4$ and $g \in \mathcal{M}\log L(\Omega)$. Then, there exists a constant $C = C(N) > 0$ such that the following two inequalities hold:*

$$\int_0^{|\Omega|} g^*(s)s^{-1}\left(\int_0^s f(t)\ dt\right)^2 ds \leq C\|g\|_{\mathcal{M}\log L}\int_0^{|\Omega|} f(s)^2\ ds, \tag{4.3.1}$$

$$\int_0^{|\Omega|} g^*(s)\left(\int_s^{|\Omega|} f(t)t^{-\frac{1}{2}}\ dt\right)^2 ds \leq C\|g\|_{\mathcal{M}\log L}\int_0^{|\Omega|} f(s)^2\ ds, \tag{4.3.2}$$

for any measurable function f on $(0, |\Omega|)$.

Proof. For proving (4.3.1), we set $a = |\Omega|, u(s) = g^*(s)s^{-1}$ and $v(s) = 1$ in (2.5.1).

Thus $\int_0^t v(s)^{-1}\, ds = \int_0^t \, ds = t$ and

$$\int_t^a u(s)\, ds = \int_t^{|\Omega|} g^*(s)s^{-1}\, ds \leq g^*(t)\int_t^{|\Omega|} s^{-1}\, ds = \log(\frac{|\Omega|}{t})g^*(t).$$

Therefore,

$$A_1 = \sup_{0<t<a}\left(\int_t^a u(s)\, ds\right)\left(\int_0^t v(s)^{-1}\, ds\right) \leq C\|g\|_{\mathcal{M}\log L} < \infty$$

and hence (4.3.1) follows from part (i) of Lemma 2.5.1.

To prove (4.3.2) we set $a = |\Omega|, u(s) = g^*(s)$ and $v(s) = s$ in (2.5.3). Now $\int_0^t u(s)\, ds = \int_0^t g^*(s)\, ds = tg^{**}(t)$ and

$$\int_t^a v(s)^{-1}\, ds = \int_t^{|\Omega|} s^{-1}\, ds = \log(\frac{|\Omega|}{t})$$

Therefore,

$$A_2 = \sup_{0<t<a}\left(\int_0^t u(s)\, ds\right)\left(\int_t^a v(s)^{-1}\, ds\right) \leq C\|g\|_{\mathcal{M}\log L} < \infty.$$

Hence (4.1.3) follows from part (ii) of Lemma 2.5.1. □

Now we prove Theorem 1.2.2 (for $N = 4$). An analogue of this theorem for Hardy potentials is proved in [7].

Proof of Theorem 1.2.2. (i) Our proof follows in the same line as in the proof of Theorem 1.2.1. Let $u \in C_c^\infty(\Omega)$. Then, by the Hardy-Littlewood inequality (Proposition 2.1.2-(i)) we have

$$\int_\Omega |g(x)||u(x)|^2\, dx \leq \int_0^{|\Omega|} g^*(s)u^*(s)^2\, ds. \tag{4.3.3}$$

Further, using (2.1.3) we have

$$\begin{aligned}\int_0^{|\Omega|} g^*(s)u^*(s)^2\,ds &\le 2\int_0^{|\Omega|} g^*(s)s^{-1}\left(\int_0^s |\Delta u|^*(t)\,dt\right)^2 ds\\ &\quad + 2\int_0^{|\Omega|} g^*(s)\left(\int_s^\infty |\Delta u|^*(t)t^{\frac{1}{2}}\,dt\right)^2 ds\\ &\le C\|g\|_{\mathcal{M}\log L}\int_0^{|\Omega|}\left(|\Delta u|^*(t)\right)^2\,dt, \end{aligned} \tag{4.3.4}$$

where the last inequality follows from Lemma 4.3.3 (as $g \in \mathcal{M}\log L(\Omega)$.) From (4.3.3) and (4.3.4), we get

$$\int_\Omega |g(x)||u(x)|^2\,dx \le \int_0^{|\Omega|} g^*(s)u^*(s)^2\,ds \le C\|g\|_{\mathcal{M}\log L}\int_\Omega |\Delta u|^2\,dx\,,\ \forall u \in \mathrm{C}_c^\infty(\Omega)\,.$$

Hence, $g \in \mathcal{HR}_2(\Omega)$.

(ii) Let $R \in (0,\infty)$ and let $\Omega = B_R(0) \subset \mathbb{R}^4$. Let g be a radial and radially decreasing Hardy-Rellich potential on Ω. To show $g \in \mathcal{M}\log L(\Omega)$, for each $r \in (0,R)$, we consider the following test function:

$$u_r(x) = \begin{cases} \frac{1}{e^2}\left(\log(\frac{R}{r})\right)^2, & |x| \le r\\ \left(\log(\frac{R}{|x|})\right)^2\Phi_r(x), & r < |x| < R\end{cases}$$

where $\Phi_r(x) = \exp\left(-\frac{2\log(\frac{R}{|x|})}{\log(\frac{R}{r})}\right)$. In our computations we use the notation $D_i \equiv \frac{\partial}{\partial x_i}$ and $D_{ii} \equiv \frac{\partial^2}{\partial x_i^2}$. For $r \le |x| \le R$, noting that $D_i\Phi_r(x) = \frac{2x_i}{|x|^2\log(\frac{R}{r})}\Phi_r(x)$ and $D_i\log(\frac{R}{|x|}) = -\frac{x_i}{|x|^2}$, we compute the derivatives of u_r as below:

$$D_iu_r(x) = 2\Phi_r(x)\frac{x_i}{|x|^2}\log\left(\frac{R}{|x|}\right)\left[\frac{\log(\frac{R}{|x|})}{\log(\frac{R}{r})} - 1\right].$$

Furthermore,

$$
\begin{aligned}
&D^2_{ii}u_r(x) = \\
&\Phi_r(x)\left\{\frac{4x_i^2}{|x|^4}\frac{\log(\frac{R}{|x|})}{\log(\frac{R}{r})} + 2\left(-\frac{2x_i^2}{|x|^4}+\frac{1}{|x|^2}\right)\log\left(\frac{R}{|x|}\right) - \frac{2x_i^2}{|x|^4}\right\}\left[\frac{\log(\frac{R}{|x|})}{\log(\frac{R}{r})} - 1\right] \\
&- 2\Phi_r(x)\frac{x_i^2}{|x|^4}\frac{\log(\frac{R}{|x|})}{\log(\frac{R}{r})}.
\end{aligned}
$$

Thus for $r \leq |x| \leq R$,

$$
\begin{aligned}
\Delta u_r = \Phi_r(x)&\left\{\frac{4\log(\frac{R}{|x|})}{|x|^2\log(\frac{R}{r})} + \frac{4}{|x|^2}\log\left(\frac{R}{|x|}\right) - \frac{2}{|x|^2}\right\}\left[\frac{\log(\frac{R}{|x|})}{\log(\frac{R}{r})} - 1\right] \\
&- 2\Phi_r(x)\frac{1}{|x|^2}\frac{\log(\frac{R}{|x|})}{\log(\frac{R}{r})}.
\end{aligned}
$$

Observe that $\Phi_r(x) \leq 1$, $\log(\frac{R}{|x|}) \leq \log(\frac{R}{r})$ and $1 \leq \log(\frac{R}{r})$ for $r \leq \frac{R}{e}$. Hence

$$
|\Delta u_r(x)| \leq \frac{16}{|x|^2}\log\left(\frac{R}{|x|}\right) + \frac{4}{|x|^2} + \frac{2}{|x|^2}\log\left(\frac{R}{|x|}\right) \leq \frac{18}{|x|^2}\log\left(\frac{R}{|x|}\right) + \frac{4}{|x|^2}.
$$

Thus for $r < \frac{R}{e}$, we have

$$
\begin{aligned}
\int_\Omega |\Delta u_r(x)|^2\,dx &\leq C_1\int_{\Omega\setminus B(0,r)}\left[\frac{1}{|x|^4}\log\left(\frac{R}{|x|}\right)^2 + \frac{1}{|x|^4}\right]\,dx \\
&\leq C_1\left\{\left[\log\left(\frac{R}{r}\right)\right]^3 + \log\left(\frac{R}{r}\right)\right\} \leq C_1\left[\log\left(\frac{R}{r}\right)\right]^3, \qquad (4.3.5)
\end{aligned}
$$

where C_1 is a positive constant independent of r. Notice that u_r is a C^1 function such that u_r and ∇u_r vanish when $|x| = R$, hence $u_r \in \mathrm{H}^2_0(\Omega)$. Further, as g is radial, radially decreasing, we easily obtain the following estimate:

(4.3.6)

$$
\int_\Omega |g(x)|u_r^2(x)\,dx \geq \int_{B(0,r)}|g(x)|u_r(x)^2\,dx = \left[\frac{1}{e^2}\log\left(\frac{R}{r}\right)\right]^4\int_0^{\omega_4 r^4} g^*(s)\,ds
$$

The assumption that g is a Hardy-Rellich potential together with (4.3.5) and (4.3.6) yields

$$\log\left(\frac{R}{r}\right)\int_0^{\omega_4 r^4} g^*(s)\,ds \le C, \quad \forall\, r \in (0, \frac{R}{e}).$$

By taking $t = \omega_4 r^4$, we get

$$\frac{1}{4}\log\left(\frac{|\Omega|}{t}\right)\int_0^{t} g^*(s)\,ds \le C, \quad \forall\, t \in \left(0, \frac{|\Omega|}{e^4}\right).$$

Since $tg^{**}(t)\log(\frac{|\Omega|}{t})$ is bounded on $\frac{|\Omega|}{e^4} \le t \le |\Omega|$, from the above inequality we conclude that

$$\sup_{t\in(0,|\Omega|)} tg^{**}(t)\log(\frac{|\Omega|}{t}) < \infty.$$

Hence $g \in \mathcal{M}\log L(\Omega)$. $\square$

As a corollary of our previous theorem, we give a simple, alternate proof for the embedding of $\mathrm{H}_0^2(\Omega)$ into the Lorentz-Zygmund space $L^{\infty,2}(\log L)^{-1}(\Omega)$ obtained independently by Brezis and Wainger [20], and Hansson [42] (one can also see [26] for an alternate proof).

Corollary 4.3.4. *Let $\Omega \subset \mathbb{R}^4$ is an open bounded set. Then we have the following embedding:*

$$\mathrm{H}_0^2(\Omega) \hookrightarrow L^{\infty,2}(\log L)^{-1}(\Omega).$$

Proof. First, assume that Ω is a ball of radius R and centred at the origin i.e., $\Omega = B_R(0)$. Let $X(\Omega) = \mathcal{M}\log L(\Omega)$. For each $g \in X(\Omega)$, (4.3.4) gives,

$$\int_0^{|\Omega|} g^*(t)(u^*(t))^2\,dt \le C\|g\|_X \int_\Omega |\Delta u|^2\,dx,\ \forall u \in \mathrm{H}_0^2(\Omega). \tag{4.3.7}$$

Let

$$g_1(x) = \begin{cases} \left[|x|^2 \log((\frac{R}{|x|})^4 e)\right]^{-2}, & x \in B_{Re^{-\frac{1}{4}}}(0) := \widetilde{\Omega} \\ \frac{1}{4|x|^4}, & x \in \Omega \setminus \widetilde{\Omega}. \end{cases}$$

A straight forward calculation gives that

$$g_1^*(t) = \begin{cases} \dfrac{\omega_4}{t(\log \frac{e|\Omega|}{t})^2}, & t \in (0, |\widetilde{\Omega}|) \\ \frac{\omega_4}{4t}, & t \in [|\widetilde{\Omega}|, |\Omega|). \end{cases}$$

Thus, $g_1 \in L^{1,\infty}(\log L)^2(\Omega)$ and hence, $g_1 \in \mathcal{M} \log L(\Omega)$ (by Proposition 4.3.1). Now, by using (4.3.7), we have

$$\begin{aligned} \int_0^{|\Omega|} \frac{(u^*(t))^2}{t\left[\log\left(\frac{e|\Omega|}{t}\right)\right]^2}\, dt &\leq \int_0^{|\widetilde{\Omega}|} \frac{(u^*(t))^2}{t\left[\log\left(\frac{e|\Omega|}{t}\right)\right]^2}\, dt + \int_{|\widetilde{\Omega}|}^{|\Omega|} \frac{(u^*(t))^2}{t}\, dt \\ &= \frac{1}{\omega_4} \int_0^{|\widetilde{\Omega}|} g_1^*(t)(u^*(t))^2\, dt + \frac{4}{\omega_4} \int_{|\widetilde{\Omega}|}^{|\Omega|} g_1^*(t)(u^*(t))^2\, dt \\ &\leq \frac{4}{\omega_4} \int_0^{|\Omega|} g_1^*(t)(u^*(t))^2\, dt \\ &\leq C_1 \int_\Omega |\Delta u|^2\, dx,\ \forall u \in \mathrm{H}_0^2(\Omega). \end{aligned}$$

The left hand side of the above inequality is equivalent to $\|u\|^2_{L^{\infty,\,2}(\log L)^{-1}}$ (by Proposition 2.2.6). Therefore,

$$\|u\|^2_{L^{\infty,\,2}(\log L)^{-1}} \leq C_2 \int_\Omega |\Delta u|^2\, dx, \forall u \in \mathrm{H}_0^2(\Omega).$$

Now for a general bounded set Ω, there exists $R > 0$ such that $\Omega \subset B_R(0)$. In this case, we obtain the required embedding by considering the above inequality for the zero extension to $B_R(0)$. □

Chapter 5

The logarithmic-Sobolev and the logarithmic-Hardy potentials

This chapter is devoted to the study of the weighted logarithmic-Sobolev and the weighted logarithmic-Hardy inequalities. In this chapter, we give a proof of Theorem 1.3.1, Theorem 1.3.2, and Theorem 1.4.2.

5.1 A weighted logarithmic Sobolev inequality

In this section, we look for a general class of weight functions $g \in L^1_{loc}(\mathbb{R}^N)$ so that the following *weighted logarithmic Sobolev inequality*:

$$(5.1.1) \qquad \int_{\mathbb{R}^N} |g||u|^p \log |u|^p \ dx \leq \gamma \log \left(C_\gamma \int_{\mathbb{R}^N} |\nabla u|^p \ dx \right), \forall u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$$

with $\int_{\mathbb{R}^N} |g||u|^p \ dx = 1$ holds for some $\gamma, C_\gamma > 0$. Indeed, the above inequality gives the logarithmic Sobolev type inequalities involving the measure $|g|dx$ which is neither the Lebesgue measure nor a probability measure. There are weighted logarithmic Sobolev inequalities where the weights are coupled with the gradient term in the right hand side of (1.3.1), see [62, 64]. However, to the best of our knowledge the study of weighted logarithmic Sobolev inequality of the form (5.1.1) has not been explored yet. Here we identify a general function space for g so that the weighted logarithmic Sobolev inequality (5.1.1) holds.

Let us recall that, for $1 < p \leq q \leq p^*$,

$$\begin{aligned} \|g\|_{\mathcal{H}_{p,q}} &= \sup_{F \subset\subset \mathbb{R}^N} \left\{ \frac{\int_F |g| \, dx}{[\mathrm{Cap}_p(F)]^{\frac{q}{p}}} \right\}, \\ \mathcal{H}_{p,q}(\mathbb{R}^N) &= \left\{ g \in L^1_{loc}(\mathbb{R}^N) : \|g\|_{\mathcal{H}_{p,q}} < \infty \right\}. \end{aligned}$$

In fact, $\mathcal{H}_{p,q}(\mathbb{R}^N)$ is a Banach function space equipped with the norm $\|.\|_{\mathcal{H}_{p,q}}$. Theorem 1.3.1 assures that (5.1.1) holds for $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$. Now we prove this theorem.

Proof of Theorem 1.3.1. Let $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$ for some $q \in (p, p^*]$. For $r \in [p, q)$, take $k = p\frac{q-r}{q-p}$. Now, using the Hölder's inequality we estimate the following integral:

$$\begin{aligned} \int_{\mathbb{R}^N} |g||u|^r \, dx &= \int_{\mathbb{R}^N} |g|^{\frac{k}{p}} |u|^k |g|^{\frac{p-k}{p}} |u|^{r-k} \, dx \\ &\leq \left[\int_{\mathbb{R}^N} |g||u|^p \, dx \right]^{\frac{k}{p}} \left[\int_{\mathbb{R}^N} |g||u|^{\frac{p(r-k)}{p-k}} \, dx \right]^{\frac{p-k}{p}} \\ &= \left[\int_{\mathbb{R}^N} |g||u|^p \, dx \right]^{\frac{q-r}{q-p}} \left[\int_{\mathbb{R}^N} |g||u|^q \, dx \right]^{\frac{r-p}{q-p}}, \quad \forall u \in \mathcal{D}^{1,p}_0(\mathbb{R}^N). \end{aligned}$$

For small $t > 0$, take $r = p + t$ in the above inequality to obtain

$$\int_{\mathbb{R}^N} |g||u|^{p+t} \, dx \leq \left[\int_{\mathbb{R}^N} |g|u|^p \, dx \right]^{\frac{q-(p+t)}{q-p}} \left[\int_{\mathbb{R}^N} |g||u|^q \, dx \right]^{\frac{(p+t)-p}{q-p}}.$$

Notice that, for $t = 0$ equality holds. Thus

$$\int_{\mathbb{R}^N} \frac{1}{t} \left[|g||u|^{p+t} - |g||u|^p \right] \, dx \leq \frac{1}{t} \left[\mathrm{A}_1^{\frac{q-(p+t)}{q-p}} \mathrm{B}_1^{\frac{(p+t)-p}{q-p}} - \mathrm{A}_1^{\frac{q-p}{q-p}} \mathrm{B}_1^{\frac{p-p}{q-p}} \right], \tag{5.1.2}$$

where $A_1 = \int_{\mathbb{R}^N} |g||u|^p \, dx$ and $B_1 = \int_{\mathbb{R}^N} |g||u|^q \, dx$. Furthermore,

$$\begin{aligned} \lim_{t\to 0} \frac{1}{t}\left[A_1^{\frac{q-(p+t)}{q-p}} B_1^{\frac{(p+t)-p}{q-p}} - A_1^{\frac{q-p}{q-p}} B_1^{\frac{p-p}{q-p}} \right] &= \left(\frac{1}{q-p}\right) A_1 \log\left(\frac{B_1}{A_1}\right), \\ \lim_{t\to 0} |g| \left[\frac{|u|^{p+t} - |u|^p}{t}\right] &= \left(\frac{1}{p}\right) |g||u|^p \log\left(|u|^p\right). \end{aligned}$$

By taking limit $t \to 0$ in (5.1.2) and using Fatou's lemma we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |g||u|^p \log\left(|u|^p\right) \, dx &\leq \left(\frac{p}{q-p}\right) A_1 \log\left(\frac{B_1}{A_1}\right) \\ &= \frac{q}{q-p} A_1 \log\left(\frac{B_1^{\frac{p}{q}}}{A_1}\right) + A_1 \log A_1 . \end{aligned} \tag{5.1.3}$$

This gives,

(5.1.4)

$$\int_{\mathbb{R}^N} |g||u|^p \log\left(\frac{|u|^p}{\int_{\mathbb{R}^N} |g||u|^p}\right) dx \leq \frac{q}{q-p}\left(\int_{\mathbb{R}^N} |g||u|^p \, dx\right) \log\left(\frac{\left[\int_{\mathbb{R}^N} |g||u|^q \, dx\right]^{\frac{p}{q}}}{\int_{\mathbb{R}^N} |g||u|^p \, dx}\right).$$

Since $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$, it follows from Theorem 2.3.5 that

$$\left[\int_{\mathbb{R}^N} |g||u|^q \, dx\right]^{\frac{p}{q}} \leq C_H \|g\|_{\mathcal{H}_{p,q}}^{\frac{p}{q}} \int_{\mathbb{R}^N} |\nabla u|^p \, dx . \tag{5.1.5}$$

From (5.1.4) and (5.1.5) we get

$$\int_{\mathbb{R}^N} |g||u|^p \log(|u|^p) \, dx \leq \frac{q}{q-p} \log\left(C_H \|g\|_{\mathcal{H}_{p,q}}^{\frac{p}{q}} \int_{\mathbb{R}^N} |\nabla u|^p \, dx\right), \tag{5.1.6}$$

for all $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} |g||u|^p \, dx = 1$. This proves the Theorem 1.3.1. □

Now we provide some examples of classical function spaces in $\mathcal{H}_{p,q}(\mathbb{R}^N)$.

Proposition 5.1.1. *Let $p \in (1, N)$ and $q \in [p, p^*]$. Then* *(i)* $L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N) \subseteq \mathcal{H}_{p,q}(\mathbb{R}^N)$,

(ii) $L^{\frac{p^*}{p^*-q},\infty}(\mathbb{R}^N) \subseteq \mathcal{H}_{p,q}(\mathbb{R}^N)$.

Proof. (i) Let $g \in L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)$ for some $q \in (p, p^*]$. Notice that, for $F \subset\subset \mathbb{R}^N$,

$$\frac{\int_F |g|}{[\mathrm{Cap}_p(F)]^{\frac{q}{p}}} \leq \|g\|_{L^{\frac{p^*}{p^*-q}}} \left[\frac{|F|^{\frac{q}{p^*}}}{[\mathrm{Cap}_p(F)]^{\frac{q}{p}}}\right].$$

Since $|F| \leq C[\mathrm{Cap}_p(F)]^{\frac{N}{N-p}}$ (Theorem 4.15, [36]), it follows that

$$\|g\|_{\mathcal{H}_{p,q}} = \sup_{F \subset\subset \mathbb{R}^N} \frac{\int_F |g|}{[\mathrm{Cap}_p(F)]^{\frac{q}{p}}} \leq C\|g\|_{L^{\frac{p^*}{p^*-q}}} < \infty .$$

Thus $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$. Hence, $L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N) \subseteq \mathcal{H}_{p,q}(\mathbb{R}^N)$.

(ii) Let $g \in L^{\frac{p^*}{p^*-q},\infty}(\mathbb{R}^N)$. Then, using part (ii) of Proposition 2.2.3 we obtain

$$\int_{\mathbb{R}^N} |g||u|^q \leq \|g\|_{L^{\frac{p^*}{p^*-q},\infty}} \||u|^q\|_{L^{\frac{p^*}{q},1}} \ = \ \|g\|_{L^{\frac{p^*}{p^*-q},\infty}} \|u\|^q_{L^{p^*,q}}, \quad \forall u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N) .$$

The Lorentz-Sobolev embedding and part (iii) of Proposition 2.2.3 ensure that $\mathcal{D}_0^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*,p}(\mathbb{R}^N) \hookrightarrow L^{p^*,q}(\mathbb{R}^N)$. Thus, the above inequality gives

$$\int_{\mathbb{R}^N} |g||u|^q \, dx \leq \mathrm{C}\|g\|_{L^{\frac{p^*}{p^*-q},\infty}} \left[\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right]^{\frac{q}{p}}, \quad \forall u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N) .$$

Hence, $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$ (by Theorem 2.3.5). □

Example 5.1.2. (A) For $q \in [p, p^*]$, consider the function

$$g_1(x) = \frac{1}{|x|^{N-\frac{q}{p}(N-p)}} \text{ in } \mathbb{R}^N .$$

It can be verified that $g_1 \in L^{\frac{p^*}{p^*-q},\infty}(\mathbb{R}^N)$. Hence, $g_1 \in \mathcal{H}_{p,q}(\mathbb{R}^N)$ (by Proposition 5.1.1-(ii)). Clearly $g_1 \notin L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)$.

(B) For $q \in \left((\frac{N-2}{N-p})p, p^* \right]$, let $g_2(x) = \frac{1}{|y|^{N\left(\frac{p^*-q}{p^*}\right)}}$ for $x = (y, z) \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$. By [12, Theorem 2.1] we have

$$\left[\int_{\mathbb{R}^N} |g_2||u|^q \, dx \right]^{\frac{1}{q}} \leq \mathrm{C} \left[\int_{\mathbb{R}^N} |\nabla u|^p \, dx \right]^{\frac{1}{p}}, \quad \forall u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N).$$

Hence, $g_2 \in \mathcal{H}_{p,q}(\mathbb{R}^N)$ (by Theorem 2.3.5).

Remark 5.1.3. $L^{\frac{p^*}{p^*-q},\infty}(\mathbb{R}^N) \subsetneq \mathcal{H}_{p,q}(\mathbb{R}^N)$ for $q \in [p, p^*]$. By Proposition 5.1.1-(ii) we have $L^{\frac{p^*}{p^*-q},\infty}(\mathbb{R}^N) \subseteq \mathcal{H}_{p,q}(\mathbb{R}^N)$. Now, consider the function g_2 in the above example. We show that $g_2 \notin L^{\frac{p^*}{p^*-q},\infty}(\mathbb{R}^N)$. On the contrary, if $g_2 \in L^{\frac{p^*}{p^*-q},\infty}(\mathbb{R}^N)$, then part (iii) of Proposition 2.2.3 implies $g_2 \in L_{loc}^{\frac{p^*\delta}{p^*-q}}(\mathbb{R}^N)$, $\forall \delta \in [\frac{p^*-q}{p^*}, 1)$. In that case, choosing $\delta > \max\{\frac{2}{N}, \frac{p^*-q}{p^*}\}$ we obtain

$$\int_{[-1,1]^N} g_2^{\frac{p^*\delta}{p^*-q}}(z) \, dz = \int_{[-1,1]^N} \frac{1}{|x|^{N\delta}} \, dz = 2^{N-2} \int_{[-1,1]^2} \frac{1}{|x|^{N\delta}} \, dx = \infty.$$

This is a contradiction. Hence, $L^{\frac{p^*}{p^*-q},\infty}(\mathbb{R}^N) \subsetneq \mathcal{H}_{p,q}(\mathbb{R}^N)$.

Notice that (5.1.1) holds for any $\gamma \geq \frac{q}{q-p}$. Let $\mathrm{C}_B(g, \gamma)$ be the best constant in (5.1.1). Then,

$$\frac{1}{\mathrm{C}_B(g,\gamma)} = \inf \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{e^{\frac{1}{\gamma}\left(\int_{\mathbb{R}^N} |g||u|^p \log |u|^p \, dx\right)}} : u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N), \int_{\mathbb{R}^N} |g||u|^p \, dx = 1 \right\}.$$

It is clear that $\mathrm{C}_B(g,\gamma) \leq \mathrm{C}_H \|g\|_{\mathcal{H}_{p,q}}^{\frac{p}{q}}$ for $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$ and $\gamma \geq \frac{q}{q-p}$. It is natural to look for a class of weights g and values of γ for which $\mathrm{C}_B(g,\gamma)$ is attained in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$. In this context, we consider the following closed sub-space

$$\mathcal{F}\mathcal{H}_{p,q}(\mathbb{R}^N) = \overline{\mathrm{C}_c(\mathbb{R}^N)} \text{ in } \mathcal{H}_{p,q}(\mathbb{R}^N).$$

Theorem 1.3.2 ensure that, for $N \geq 3$, $p \in (1, N)$ and $q \in (p, p^*]$, if $g \in \mathcal{H}_{p,q}(\mathbb{R}^N) \cap \mathcal{F}\mathcal{H}_{p,p}(\mathbb{R}^N)$ and $\gamma > \frac{q}{q-p}$, then the best constant $\mathrm{C}_B(g,\gamma)$ is attained in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$.

In order to prove this theorem, we first recall that the map $G_p(u) = \int_{\mathbb{R}^N} |g||u|^p \, dx$ on $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$ is compact if $g \in \mathcal{FH}_{p,p}(\mathbb{R}^N)$. In fact, we have the following result.

Lemma 5.1.4. *Let $g \in \mathcal{FH}_{p,p}(\mathbb{R}^N)$ and $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$. Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |g||u_n - u|^p \, dx = 0\,.$$

Proof. Let $\epsilon > 0$ be arbitrary. Then there exists $g_\epsilon \in \mathrm{C}_c(\mathbb{R}^N)$ such that $\|g - g_\epsilon\|_{\mathcal{H}_{p,p}} < \epsilon$. Now

$$\int_{\mathbb{R}^N} |g||u_n - u|^p \, dx \leq \int_{\mathbb{R}^N} |g - g_\epsilon||u_n - u|^p \, dx + \int_{\mathbb{R}^N} |g_\epsilon||u_n - u|^p \, dx\,. \tag{5.1.7}$$

Further, using Theorem 3.0.5 we obtain

$$\int_{\mathbb{R}^N} |g - g_\epsilon||u_n - u|^p \, dx \leq \|g - g_\epsilon\|_{\mathcal{H}_{p,p}} \|u_n - u\|_{\mathcal{E}_p} \leq \mathrm{C}\epsilon \left[\int_{\mathbb{R}^N} |\nabla(u_n - u)| \, dx \right]^{\frac{1}{p}}.$$

Since $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$, it follows that $\int_{\mathbb{R}^N} |\nabla(u_n - u)|^p \, dx$ is uniformly bounded and $\int_{\mathbb{R}^N} |g_\epsilon||u_n - u|^p \, dx \to 0$ as $n \to \infty$. Thus, (5.1.7) gives

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |g||u_n - u|^p \, dx \leq \mathrm{C}_1\epsilon\,.$$

This completes our proof, since $\epsilon > 0$ is arbitrary. □

Proof of Theorem 1.3.2. Let $g \in \mathcal{H}_{p,q}(\mathbb{R}^N) \cap \mathcal{FH}_{p,p}(\mathbb{R}^N)$ for some $q \in (p, p^*]$ and $\gamma > \frac{q}{q-p}$. Let $u_n \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$ be a minimising sequence of $\frac{1}{\mathrm{C}_B(g,\gamma)}$ i.e.

$$\frac{1}{\mathrm{C}_B(g,\gamma)} = \lim_{n\to\infty} \left[\frac{\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx}{e^{\frac{1}{\gamma}\left(\int_{\mathbb{R}^N} |g||u_n|^p \log|u_n|^p \, dx\right)}} \right]. \tag{5.1.8}$$

with $\int_{\mathbb{R}^N} |g||u_n|^p \, dx = 1$. We claim that u_n is bounded in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$. By construction of u_n we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx \leq \frac{(1+\frac{1}{n})}{\mathrm{C}_B(g,\gamma)} \left[e^{\frac{1}{\gamma}\left(\int_{\mathbb{R}^N} |g||u_n|^p \log |u_n|^p \, dx\right)} \right] . \tag{5.1.9}$$

It follows from (5.1.6) that

$$\int_{\mathbb{R}^N} |g||u_n|^p \log |u_n|^p \, dx \leq \frac{q}{q-p} \log \left(C \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx \right) .$$

As a consequence, (5.1.9) gives

$$\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx \leq \frac{2C_1}{\mathrm{C}_B(g)} \left[\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx \right]^{\frac{q}{\gamma(q-p)}} .$$

Since $\frac{q}{\gamma(q-p)} < 1$, u_n is bounded in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$. Hence, $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$ upto a sub-sequence. Certainly, we have $\int_{\mathbb{R}^N} |\nabla u|^p \, dx \leq \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx$. Further, Lemma 2.5.3 infers that

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^N} |g||u_n|^p \log |u_n|^p \, dx \\ &= \int_{\mathbb{R}^N} |g||u|^p \log |u|^p \, dx + \lim_{n\to\infty} \int_{\mathbb{R}^N} |g||u_n - u|^p \log |u_n - u|^p \, dx \, . \end{aligned}$$

Using this equality in (5.1.8) we obtain

$$\frac{1}{\mathrm{C}_B(g,\gamma)} \geq \left[\frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{e^{\frac{1}{\gamma}\left(\int_{\mathbb{R}^N} |g||u|^p \log |u|^p \, dx\right)}} \right] \left[\frac{1}{e^{\frac{1}{\gamma}\left(\lim_{n\to\infty} \int_{\mathbb{R}^N} |g||u_n-u|^p \log |u_n-u|^p \, dx\right)}} \right] . \tag{5.1.10}$$

Lemma 5.1.4 ensures that $\displaystyle\int_{\mathbb{R}^N} |g||u|^p \, dx = 1$ and hence,

$$\frac{1}{\mathrm{C}_B(g,\gamma)} \geq \frac{1}{\mathrm{C}_B(g,\gamma)} \left[\frac{1}{e^{\frac{1}{\gamma}\left(\lim_{n\to\infty} \int_{\mathbb{R}^N} |g||u_n-u|^p \log |u_n-u|^p \, dx\right)}} \right] .$$

This gives

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |g||u_n - u|^p \log |u_n - u|^p \, dx \geq 0 \, .$$

On the other hand, from (5.1.3), it follows that

$$\int_{\mathbb{R}^N} |g||u_n - u|^p \log|u_n - u|^p \, dx \le \left[\frac{p}{q-p}\right] \left[\mathrm{A}_n \log\left(\int_{\mathbb{R}^N} |g||u_n - u|^q \, dx\right) - \mathrm{A}_n \log\left(\mathrm{A}_n\right)\right],$$

where $\mathrm{A}_n = \int_{\mathbb{R}^N} |g||u_n - u|^p \, dx$. Notice that $A_n \to 0$ as $n \to \infty$ (by Lemma 5.1.4). Further, since u_n is bounded in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$, it follows that $\int_{\mathbb{R}^N} |g||u_n - u|^q \, dx$ is uniformly bounded. Thus, the above inequality yields

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |g||u_n - u|^p \log|u_n - u|^p \, dx \le 0\,.$$

Therefore, $\lim_{n\to\infty} \int_{\mathbb{R}^N} |g||u_n - u|^p \log|u_n - u|^p \, dx = 0$. Consequently, (5.1.10) implies

$$\frac{1}{\mathrm{C}_B(g,\gamma)} \ge \left[\frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{e^{\frac{1}{\gamma}\left(\int_{\mathbb{R}^N} |g||u|^p \log|u|^p \, dx\right)}}\right].$$

We also have $\int_{\mathbb{R}^N} |g||u|^p \, dx = 1$. Hence, u is a minimiser of $\frac{1}{\mathrm{C}_B(g,\gamma)}$. □

The forthcoming proposition ensures that the spaces considered in Theorem 1.3.2 contain certain Lebesgue spaces.

Proposition 5.1.5. *Let $p \in (1, N)$ and $q \in [p, p^*]$. Then $L^1(\mathbb{R}^N) \cap L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N) \subseteq \mathcal{H}_{p,q}(\mathbb{R}^N) \cap \mathcal{FH}_{p,p}(\mathbb{R}^N)$.*

Proof. Let $g \in L^1(\mathbb{R}^N) \cap L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)$. Then Proposition 5.1.1-(i) implies $g \in \mathcal{H}_{p,q}(\mathbb{R}^N)$. For $q = p$, it follows from Proposition 3.1.9 that $g \in \mathcal{FH}_{p,p}(\mathbb{R}^N)$. For $q > p$, we will use Theorem 3.1.14 to show that $g \in \mathcal{FH}_{p,p}(\mathbb{R}^N)$. Since $1 < \frac{N}{p} < \frac{p^*}{p^*-q}$, we have $g \in L^{\frac{N}{p}}(\mathbb{R}^N)$ and hence, $g \in \mathcal{H}_{p,p}(\mathbb{R}^N)$. For any $y \in \mathbb{R}^N$, $t > 0$ and $F \subset\subset \mathbb{R}^N$, we have $\mathrm{Cap}_p(F) \ge \mathrm{Cap}_p(F \cap B_t(y))$ due to the monotonicity of Cap_p. Further, by Pólya-Szegö inequality it follows that $\mathrm{Cap}_p(F \cap B_t(y)) \ge \mathrm{Cap}_p((F \cap B_t(y))^\star)$, where

$(F \cap B_t(y))^\star$ represents a ball centered at the origin having the same Lebesgue measure as $F \cap B_t(y)$. Let d be the radius of $(F \cap B_t(y))^\star$. Clearly, $d \leq t$. Now, using the fact that $\mathrm{Cap}_p(B_d(0)) = N\omega_N \left(\frac{N-1}{p-1}\right)^{p-1} d^{N-p}$ [36, Theorem 4.15], we estimate

$$\frac{\int_{F\cap B_t(y)} |g|\, dx}{\mathrm{Cap}_p(F)} \leq \frac{\int_{F\cap B_t(y)} |g|\, dx}{\mathrm{Cap}_p((F \cap B_t(y))^\star)} \leq \frac{\left(\int_{\mathbb{R}^N} |g|^{\frac{p^*}{p^*-q}}\, dx\right)^{\frac{p^*-q}{p^*}} \left(\omega_N d^N\right)^{\frac{q}{p^*}}}{N\omega_N \left(\frac{N-1}{p-1}\right)^{p-1} d^{N-p}}$$
$$\leq \mathrm{C} d^{\left(\frac{q}{p}-1\right)(N-p)} \leq \mathrm{C} t^{\left(\frac{q}{p}-1\right)(N-p)}\,.$$

Thus,

$$\lim_{t\to 0} \left[\sup_{F \subset\subset \mathbb{R}^N} \frac{\int_{F\cap B_t(y)} |g|\, dx}{\mathrm{Cap}_p(F)} \right] = 0\,. \tag{5.1.11}$$

Similarly, one can obtain

$$\frac{\int_{F\cap B_t(0)^c} |g|\, dx}{\mathrm{Cap}_p(F)} \leq \begin{cases} \int_{B_t(0)^c} |g|\, dx\,, & \text{if } \mathrm{Cap}_p(F) \geq 1 \\ \left[\int_{B_t(0)^c} |g|^{\frac{p^*}{p^*-q}}\, dx\right]^{\frac{p^*-q}{p^*}}\,, & \text{if } \mathrm{Cap}_p(F) < 1\,. \end{cases}$$

This infers

$$\lim_{t\to \infty} \left[\sup_{F \subset\subset \mathbb{R}^N} \frac{\int_{F\cap B_t(0)^c} |g|\, dx}{\mathrm{Cap}_p(F)} \right] = 0\,. \tag{5.1.12}$$

Since (5.1.11), (5.1.12) hold and $g \in \mathcal{H}_{p,p}(\mathbb{R}^N)$, it follows from Theorem 3.1.14 that $g \in \mathcal{FH}_{p,p}(\mathbb{R}^N)$. Therefore, $L^1(\mathbb{R}^N) \cap L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N) \subseteq \mathcal{H}_{p,q}(\mathbb{R}^N) \cap \mathcal{FH}_{p,p}(\mathbb{R}^N)$.

□

Example 5.1.6. Let $p > 2$. For $q \in \left((\frac{N-2}{N-p})p, p^*\right]$, consider

$$\tilde{g}_2(x) = \begin{cases} \frac{1}{|y|^{N\left(\frac{p^*-q}{p^*}\right)}}\,, & \text{if } x \in [-1,1]^N \\ 0\,, & \text{otherwise} \end{cases}$$

where $x=(y,z)\in\mathbb{R}^2\times\mathbb{R}^{N-2}$. We show that $\tilde{g}_2\in\mathcal{H}_{p,q}(\mathbb{R}^N)\cap\mathcal{FH}_{p,p}(\mathbb{R}^N)$. By part (B) of Example 5.1.2, $\tilde{g}_2\in\mathcal{H}_{p,q}(\mathbb{R}^N)$. To show $\tilde{g}_2\in\mathcal{FH}_{p,p}(\mathbb{R}^N)$, we will use Theorem 3.1.14. Since support of $\tilde{g}_2$ is bounded, it can be easily seen that

$$\int_{\mathbb{R}^N}|\tilde{g}_2||u|^p\,dx\le \mathrm{C}\int_{\mathbb{R}^N}|\nabla u|^p\,dx,\quad \forall u\in\mathcal{D}_0^{1,p}(\mathbb{R}^N)\,.$$

Hence, $\tilde{g}_2\in\mathcal{H}_{p,p}(\mathbb{R}^N)$ (by Theorem 2.3.5). Now, for any $\xi\in\mathbb{R}^N$, let $Q_t(\xi)$ be the cube of length $2t$ and centered at ξ. Then $B_t(\xi)\subseteq Q_t(\xi)\subseteq B_{\sqrt{N}t}(\xi)$. Now, using similar arguments as in part (ii) of Proposition 5.1.1, for any $\xi\in[-1,1]^N$, $t>0$ and $F\subset\subset\mathbb{R}^N$, we estimate

$$\begin{aligned}\frac{\int_{F\cap Q_t(\xi)}|\tilde{g}_2|\,dx}{\mathrm{Cap}_p(F)}\le\frac{\int_{F\cap Q_t(\xi)}|\tilde{g}_2|\,dx}{\mathrm{Cap}_p((F\cap Q_t(\xi))^\star)}&\le C(N,p)\left[\frac{t^{N-2}\int_{[-1,1]^2}|\tilde{g}_2|\,dy}{d^{N-p}}\right]\\&\le \mathrm{C}\left[\frac{d}{t}\right]^2 d^{p-2}\,,\end{aligned}$$

where d is the radius of $(F\cap Q_t(\xi))^\star$. One can see that $d\le\sqrt{N}t$. Since $p>2$, the above inequality infers

$$\lim_{t\to 0}\left[\sup_{F\subset\subset\mathbb{R}^N}\frac{\int_{F\cap Q_t(\xi)}|\tilde{g}_2|\,dx}{\mathrm{Cap}_p(F)}\right]=0,\quad\forall x\in[-1,1]^N\,. \tag{5.1.13}$$

As $\tilde{g}_2$ vanishes outside $[-1,1]^N$, (5.1.13) holds for all $x\in\mathbb{R}^N$. For the same reason

$$\lim_{t\to\infty}\left[\sup_{F\subset\subset\mathbb{R}^N}\frac{\int_{F\cap Q_t(0)^c}|\tilde{g}_2|\,dx}{\mathrm{Cap}_p(F)}\right]=0\,. \tag{5.1.14}$$

Since (5.1.13), (5.1.14) hold and $\tilde{g}_2\in\mathcal{H}_{p,p}(\mathbb{R}^N)$, it follows from Theorem 3.1.14 that $\tilde{g}_2\in\mathcal{FH}_{p,p}(\mathbb{R}^N)$.

Remark 5.1.7. $L^1(\mathbb{R}^N)\cap L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)\subsetneq\mathcal{H}_{p,q}(\mathbb{R}^N)\cap\mathcal{FH}_{p,p}(\mathbb{R}^N)$. By part (ii) of Proposition 5.1.1, we have $L^1(\mathbb{R}^N)\cap L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)\subseteq\mathcal{H}_{p,q}(\mathbb{R}^N)\cap\mathcal{FH}_{p,p}(\mathbb{R}^N)$. The above example shows that $\tilde{g}_2\in\mathcal{H}_{p,q}(\mathbb{R}^N)\cap\mathcal{FH}_{p,p}(\mathbb{R}^N)$. But, following similar computations as for part (B) in Example 5.1.2 one can easily verify that $\tilde{g}_2\notin L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)$.

5.2 A logarithmic Hardy inequality

We recall the classical Hardy inequality

$$(5.2.1) \qquad \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\, dx \le \left(\frac{p}{N-p}\right)^p \int_{\mathbb{R}^N} |\nabla u|^p\, dx\,, \forall u \in \mathrm{C}^1_c(\mathbb{R}^N)\,.$$

A variant of (5.2.1) involving the distance from the boundary of the domain instead of a distance to a point singularity has been studied extensively, see [5, 29, 70, 50, 40]. If Ω is a convex domain, then the following variant of Hardy inequality [56]

$$(5.2.2) \qquad \int_{\Omega} \frac{|u|^p}{|\delta_{\partial\Omega}(x)|^p}\, dx \le \left(\frac{p}{p-1}\right)^p \int_{\Omega} |\nabla u|^p\, dx\,, \forall u \in \mathrm{C}^1_c(\Omega)\,,$$

holds for $p \in (1, \infty)$, where $\delta_A(x)$ denotes the distance of x from a closed set A. Later, the convexity assumptions are relaxed in [13, 49]. Further, recently in [48, 32] authors have considered the distance from a general closed set E in $\mathbb{R}^N$ instead of $\partial\Omega$. Under certain conditions on the Assouad dimension of E, we have the following global Hardy inequality:

$$(5.2.3) \qquad \int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^p}\, dx \le C \int_{\mathbb{R}^N} |\nabla u|^p\, dx, \quad \forall u \in \mathrm{C}^1_c(\mathbb{R}^N)\,,$$

for $1 < p < \infty$ [32, Remark 6.2]. Notice that, for $E = \{0\}$ and $E = \partial\Omega$, (5.2.3) corresponds to (5.2.1) and (5.2.2) respectively.

In [31], M. del Pino, J. Dolbeault, S. Filippas, and A. Tertikas obtained the following *logarithmic Hardy inequality*:

$$(5.2.4) \qquad \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \log\left(|x|^{N-2}|u|^2\right)\, dx \le \frac{N}{2} \log\left(C \int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)\,,$$

for all $u \in \mathrm{C}^1_c(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\, dx = 1$. Notice that the integrals in (5.2.4) are scale invariant which distinguishes them from logarithmic Sobolev inequalities (1.3.2).

The Caffarelli-Kohn-Nirenberg type inequality [21] played a key role in [31] to achieve (5.2.4). In a recent work [32], authors proved a variant of Caffarelli-Kohn-Nirenberg type inequalities involving δ_E under certain restriction on Assouad dimension of E. Here we state it precisely.

Lemma 5.2.1. *[32, Remark 6.2]. Let $N \geq 3$ and $1 < p \leq q \leq p^*$. Further, for a closed set E in $\mathbb{R}^N$ and $\beta \in \mathbb{R}$ assume*

$$\dim_A(E) < \frac{q}{p}\,(N-p+\beta) \qquad \textit{and} \quad \dim_A(E) < N - \frac{\beta}{p-1}\,.$$

Then, there exists $C > 0$ such that

$$\left[\int_{\mathbb{R}^N} \frac{|u|^q}{\delta_E^{N-\frac{q}{p}(N-p+\beta)}}\,dx\right]^{\frac{1}{q}} \leq C\left[\int_{\mathbb{R}^N} |\nabla u|^p \delta_E^\beta\,dx\right]^{\frac{1}{p}}, u \in \mathrm{C}_\mathrm{c}^1(\mathbb{R}^\mathrm{N})\,.$$

The above lemma facilitates us to achieve a logarithmic Hardy inequality involving δ_E, as stated in Theorem 1.4.2. Now we prove Theorem 1.4.2. The underlying ideas are the same as the proof of Theorem 1.3.1. However, we give the proof for the sake of completeness.

Proof of Theorem 1.4.2. For $r \in [p, p^*)$, take $k = p\frac{p^*-r}{p^*-p}$. Clearly, $k \in (0,p]$ and $k = N - \frac{(N-p)r}{p}$. Using the Hölder's inequality, we estimate the following integral:

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|u|^r}{\delta_E^{N-\frac{r}{p}(N-p-pa)}}\,dx &= \int_{\mathbb{R}^N} \frac{|u|^k}{\delta_E^{k(a+1)}} \frac{|u|^{r-k}}{\delta_E^{N-k(a+1)-\frac{r}{p}(N-p-pa)}}\,dx \\
&\leq \left[\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}}\,dx\right]^{\frac{k}{p}} \left[\int_{\mathbb{R}^N} \frac{|u|^{\frac{p(r-k)}{p-k}}}{\delta_E^{[\frac{p}{p-k}][N-k(a+1)-\frac{r}{p}(N-p-pa)]}}\,dx\right]^{\frac{p-k}{p}} \\
&= \left[\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}}\,dx\right]^{\frac{p^*-r}{p^*-p}} \left[\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{\delta_E^{N-\frac{p^*}{p}[N-p-pa]}}\,dx\right]^{\frac{r-p}{p^*-p}},
\end{aligned}$$

for all $u \in \mathrm{C}_c^1(\mathbb{R}^N)$. For small $t > 0$, we take $r = p + t$ in the above inequality to obtain

$$\int_{\mathbb{R}^N} \frac{|u|^{p+t}}{\delta_E^{N-\frac{p+t}{p}(N-p-pa)}}\ dx \le \left[\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}}\ dx\right]^{\frac{p^*-(p+t)}{p^*-p}} \left[\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{\delta_E^{N-\frac{p^*}{p}[N-p-pa]}}\ dx\right]^{\frac{(p+t)-p}{p^*-p}} .$$

Notice that, for $t = 0$ equality occurs. Thus, we obtain

(5.2.5)

$$\int_{\mathbb{R}^N} \left[\frac{|u|^{p+t}}{\delta_E^{N-\frac{p+t}{p}(N-p-pa)}} - \frac{|u|^p}{\delta_E^{N-\frac{p}{p}(N-p-pa)}}\right] \frac{dx}{t} \le \frac{1}{t}\left[\mathrm{A}_1^{\frac{p^*-(p+t)}{p^*-p}} \mathrm{B}_1^{\frac{(p+t)-p}{p^*-p}} - \mathrm{A}_1^{\frac{p^*-p}{p^*-p}} \mathrm{B}_1^{\frac{p-p}{p^*-p}}\right] ,$$

where $\mathrm{A}_1 = \displaystyle\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}}\ dx$ and $\mathrm{B}_1 = \displaystyle\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{\delta_E^{N-\frac{p^*}{p}[N-p-pa]}}\ dx$. Furthermore,

$$\lim_{t\to 0} \frac{1}{t}\left[\mathrm{A}_1^{\frac{p^*-(p+t)}{p^*-p}} \mathrm{B}_1^{\frac{(p+t)-p}{p^*-p}} - \mathrm{A}_1^{\frac{p^*-p}{p^*-p}} \mathrm{B}_1^{\frac{p-p}{p^*-p}}\right] = \left(\frac{1}{p^*-p}\right) \mathrm{A}_1 \log\left(\frac{\mathrm{B}_1}{\mathrm{A}_1}\right) ,$$

$$\lim_{t\to 0} \frac{1}{t}\left[\frac{|u|^{p+t}}{\delta_E^{N-\frac{p+t}{p}(N-p-pa)}} - \frac{|u|^p}{\delta_E^{N-\frac{p}{p}(N-p-pa)}}\right] = \left(\frac{1}{p}\right) \frac{|u|^p}{\delta_E^{p(a+1)}} \log\left(\delta_E^{N-p-ap}|u|^p\right) .$$

Hence, by taking limit $t \to 0$ in (5.2.5) and using Fatou's lemma, we get

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}} \log\left(\delta_E^{N-p-ap}|u|^p\right)\ dx &\le \left(\frac{p}{p^*-p}\right) \mathrm{A}_1 \log\left(\frac{\mathrm{B}_1}{\mathrm{A}_1}\right) \\ &= \frac{p^*}{p^*-p} \mathrm{A}_1 \log\left(\frac{\mathrm{B}_1^{\frac{p}{p^*}}}{\mathrm{A}_1}\right) + \mathrm{A}_1 \log \mathrm{A}_1 .\end{aligned}$$

This yields

$$\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}} \log\left(\frac{\delta_E^{N-p-ap}|u|^p}{\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}}}\right) \; dx$$

$$\leq \frac{p^*}{p^*-p}\left[\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}} \; dx\right] \log\left(\frac{\left[\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{\delta_E^{N-\frac{p^*}{p}[N-p-pa]}} \; dx\right]^{\frac{p}{p^*}}}{\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}} \; dx}\right).$$

Now, since $a \in (-\frac{(N-d)(p-1)}{p}, \frac{(N-p)(N-d)}{Np})$, we have

$$\dim_A(E) = d < \frac{p^*}{p}(N-p-pa) \quad \text{and} \quad \dim_A(E) = d < N + \frac{pa}{p-1}.$$

Hence, it follows from Lemma 5.2.1 that

$$\left[\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{\delta_E^{N-\frac{p^*}{p}[N-p-pa]}} \; dx\right]^{\frac{1}{p^*}} \leq \mathrm{C}\left[\int_{\mathbb{R}^N} \frac{|\nabla u|^p}{\delta_E^{pa}} \; dx\right]^{\frac{1}{p}}.$$

Consequently,

$$\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}} \log\left(\frac{\delta_E^{N-p-ap}|u|^p}{\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}}}\right) \; dx \leq \frac{N}{p}\left[\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}} \; dx\right] \log\left(\frac{\mathrm{C}\int_{\mathbb{R}^N} \frac{|\nabla u|^p}{\delta_E^{pa}} \; dx}{\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}} \; dx}\right).$$

By taking $\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}} \; dx = 1$ we obtain (1.4.2). □

Remark 5.2.2. In particular, if we take $a = 0$ in the above proof, then we obtain

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|\delta_E(x)|^p} \log[|\delta_E(x)|^{N-p}|u|^p] \; dx \leq \frac{N}{p}\log\left(\mathrm{C}\int_{\mathbb{R}^N} |\nabla u|^p \; dx\right), \tag{5.2.6}$$

for all $u \in \mathrm{C}_c^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^p} \; dx = 1$. Further, if we take $E = \{0\}$ and $p = 2$, then (5.2.6) coincides with (5.2.4).

Remark 5.2.3. It follows from Remark 2.4.3, that $\dim_A(\partial B_1) < N$. By taking $a = 0$ in Theorem 1.4.2 we obtain an analogue of (5.2.6) on unit ball, namely

$$\int_{B_1} \frac{|u|^p}{|\delta_{\partial B_1}(x)|^p} \log[|\delta_{\partial B_1}(x)|^{N-p}|u|^p]\, dx \le \frac{N}{p} \log\left(\mathrm{C} \int_{B_1} |\nabla u|^p\, dx \right),$$

for all $u \in \mathrm{C}_c^1(B_1)$ with $\displaystyle\int_{B_1} \frac{|u|^p}{|\delta_{\partial B_1}(x)|^p}\, dx = 1$.

Remark 2.4.3 also shows that $\dim_A(\mathbb{R}^{N-1} \times \{0\}) < N$. Thus, in a similar manner we get the following analogue of (5.2.6) in the half space

$$\int_{\mathbb{R}^N_+} \frac{|u|^p}{|x_N|^p} \log[|x_N|^{N-p}|u|^p]\, dx \le \frac{N}{p} \log\left(\mathrm{C} \int_{\mathbb{R}^N_+} |\nabla u|^p\, dx \right),$$

for all $u \in \mathrm{C}_c^1(\mathbb{R}^N_+)$ with $\displaystyle\int_{\mathbb{R}^N_+} \frac{|u|^p}{|x_N|^p}\, dx = 1$.

Next, we prove Theorem 1.4.3 which gives a second-order extension of (1.4.2).

Proof of Theorem 1.4.3. For $N \ge 3$ and $p \in (1, \frac{N}{2})$, let E be a closed set in $\mathbb{R}^N$ with $\dim_A(E) = d < \frac{N(N-2p)}{N-p}$ and $a \in (1 - \frac{(N-d)(p-1)}{p}, \frac{(N-p)(N-d)}{Np})$. Then, proceeding as in the proof of Theorem 1.4.2, we obtain

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|\delta_E(x)|^{p(a+1)}} \log[|\delta_E(x)|^{N-p-pa}|u|^p]\, dx \le \frac{N}{p} \log\left(\mathrm{C} \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{\delta_E(x)^{ap}}\, dx \right), \tag{5.2.7}$$

for all $u \in \mathrm{C}_c^2(\mathbb{R}^N)$ with $\displaystyle\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{p(a+1)}}\, dx = 1$. Since $a \in (1 - \frac{(N-d)(p-1)}{p}, \frac{(N-p)(N-d)}{Np})$, one can see that

$$d = \dim_A(E) < N - p + (1-a)p \quad \text{and} \quad d = \dim_A(E) < N - \frac{(1-a)p}{p-1}.$$

Hence, Lemma 5.2.1 gives

$$\int_{\mathbb{R}^N} \frac{|\nabla u|^p}{\delta_E^{ap}}\, dx = \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{\delta_E^{N-\frac{p}{p}[N-p-(a-1)p]}}\, dx \le C \int_{\mathbb{R}^N} \frac{|\nabla^2 u|^p}{\delta_E^{(a-1)p}}\, dx,$$

where $|\nabla^2 u| = \left(\sum_{i,j=1}^N |\frac{\partial^2 u}{\partial x_i \partial x_j}|^2\right)^{\frac{1}{2}}$. Consequently, we have (1.4.3). □

Remark 5.2.4. In particular, for $a = 1$, (1.4.3) corresponds to

$$\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{2p}} \log[\delta_E^{N-2p} |u|^p] \, dx \le \frac{N}{p} \log\left(\mathrm{C} \int_{\mathbb{R}^N} |\nabla^2 u|^p \, dx \right),$$

for all $u \in \mathrm{C}_c^2(\mathbb{R}^N)$ with $\displaystyle\int_{\mathbb{R}^N} \frac{|u|^p}{\delta_E^{2p}} \, dx = 1$, provided $\dim_A(E) < \frac{N(N-2p)}{N-p}$. Further, if $E = \{0\}$ in the above inequality then

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}} \log[|x|^{N-2p} |u|^p] \, dx \le \frac{N}{p} \log\left(\mathrm{C} \int_{\mathbb{R}^N} |\nabla^2 u|^p \, dx \right), \tag{5.2.8}$$

for all $u \in \mathrm{C}_c^2(\mathbb{R}^N)$ with $\displaystyle\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}} \, dx = 1$. Notice that, this is a second order generalization of (5.2.4). Recall that, for $p = 2$,

$$\int_{\mathbb{R}^N} |\nabla^2 u|^2 \, dx \approx \int_{\mathbb{R}^N} |\Delta u|^2 \, dx, \quad \forall u \in \mathrm{C}_c^2(\mathbb{R}^N).$$

Therefore, (5.2.8) yields

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^4} \log[|x|^{N-4} |u|^2] \, dx \le \frac{N}{2} \log\left(\mathrm{C} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx \right),$$

for all $u \in \mathrm{C}_c^2(\mathbb{R}^N)$ with $\displaystyle\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^4} \, dx = 1$.

5.3 A logarithmic Lorentz-Sobolev inequality

Next, we establish a logarithmic version of Lorentz-Sobolev inequality, as stated below.

Theorem 5.3.1. *[27, Theorem 1.5] Let $N \ge 3$ and $p \in (1, N)$. Then, for all*

$u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$ *with* $\|u\|_{L^{p^*,p}} = 1$, *there exists* $C > 0$ *such that*

$$\int_0^\infty s^{\frac{p}{p^*}-1} |u^*(s)|^p \log(s^{1-\frac{p}{N}} |u^*(s)|^p)\ ds \leq \frac{N}{p} \log\left(C \int_{\mathbb{R}^N} |\nabla u|^p\ dx \right).$$

Proof. Let $r \in [p, p^*)$, we take $k = p\frac{p^*-r}{p^*-p}$. One can see that $\left[\frac{N}{p} - \frac{r}{p}[\frac{N}{p} - 1] - \frac{k}{p}\right] = 0$ and $\frac{p(r-k)}{p-k} = p^*$. Then, using the Hölder's inequality we estimate

$$\begin{aligned}
& \int_0^\infty s^{\left[\frac{p}{p^*}-1\right]\left[\frac{N}{p}-\frac{r}{p}(\frac{N}{p}-1)\right]} |u^*(s)|^r\ ds \\
= & \int_0^\infty s^{\frac{k}{p}\left[\frac{p}{p^*}-1\right]} |u^*(s)|^k s^{\left[\frac{p}{p^*}-1\right]\left[\frac{N}{p}-\frac{r}{p}(\frac{N}{p}-1)-\frac{k}{p}\right]} |u^*(s)|^{r-k}\ ds \\
\leq & \left[\int_0^\infty s^{[\frac{p}{p^*}-1]} |u^*(s)|^p\ ds\right]^{\frac{k}{p}} \left[\int_0^\infty |u^*(s)|^{p^*}\ ds\right]^{\frac{p-k}{p}},
\end{aligned}$$

for any $u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$. Since $\int_0^\infty |u^*(s)|^{p^*}\ ds = \int_{\mathbb{R}^N} |u|^{p^*}\ dx$, the above inequality yields

$$\int_0^\infty s^{\left[\frac{p}{p^*}-1\right]\left[\frac{N}{p}-\frac{r}{p}(\frac{N}{p}-1)\right]} |u^*(s)|^r\ ds \leq \left[\int_0^\infty s^{[\frac{p}{p^*}-1]} |u^*(s)|^p\ ds\right]^{\frac{k}{p}} \left[\int_{\mathbb{R}^N} |u|^{p^*}\ dx\right]^{\frac{p-k}{p}}.$$

Now, following the same arguments as in the proof of Theorem 1.3.1 we get

$$\int_0^\infty s^{\left[\frac{p}{p^*}-1\right]} |u^*|^p \log(s^{1-\frac{p}{N}} |u^*|^p)\ ds \leq \frac{N}{p} \log\left(\mathrm{C} \int_{\mathbb{R}^N} |\nabla u|^p\ dx \right),$$

for all $u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$ with $\|u\|_{L^{p^*,p}} = 1$.

□

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