

# Planar algebras, quantum information theory and subfactors

*By*

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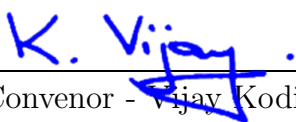
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## DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.



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## LIST OF PUBLICATIONS ARISING FROM THE THESIS

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1. *Planar algebras associated to Latin squares are of subgroup-group type*, Vijay Kodiyalam and Sruthymurali, Proc. Amer. Math. Soc. 149 (2021), no. 1, 163–172, 46L37.
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## DEDICATIONS

*To my niece **Yatra Shankhari***



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# Summary

The content of this thesis is based on the papers [11] and [13]. It mainly deals with the mathematical objects known as planar algebras and their connection to quantum information theory. We mainly focus on the construction of subfactors from some objects that arise in quantum information theory. Our objects of interest include Hadamard matrices, Latin squares, quantum Latin squares, unitary error bases and biunitary matrices which provide the mathematical foundation for the quantum computational phenomena. We construct appropriate subfactor planar algebras from these objects and in fact, the construction is described in a much more general situation.

This thesis is divided into three parts. In the first part, we define a generalized notion of a biunitary element in a planar algebra and we will identify our objects of interest in quantum information theory as some particular biunitary elements in the well known spin planar algebra.

In the second part we describe the construction of subfactor planar algebras from these objects. More generally, given a biunitary element  $u$  in a  $C^*$ -planar algebra  $P$ , to this data we will associate another  $C^*$ - planar algebra  $Q$ . For some particular biunitary elements, this  $Q$  will be a  $C^*$ - planar subalgebra of  $P$ . Surprisingly this construction yields plenty of subfactor planar subalgebras from the spin planar algebra even though the spin planar algebra is not a subfactor planar algebra.

In the final part, we examine the subfactor planar algebras associated to Latin

squares. We will prove that the subfactor planar algebra associated to a Latin square is the subgroup-group planar algebra associated to the subfactor  $R^G \subseteq R^H$ , see [4], where the pair  $H \subseteq G$  is explicitly calculated from the Latin square.

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# Chapter 1

## Introduction

### 1.1 History & Motivation

A  $II_1$ -subfactor  $N \subset M$  of a type  $II_1$  factor  $M$  is a subalgebra  $N \subset M$  which is itself a type  $II_1$  factor and contains the identity element of  $M$ . The theory of subfactors mainly originated in the seminal work of Vaughan Jones in [6]. The central object of interest is the standard invariant of a given subfactor. This invariant has many equivalent descriptions, including Ocneanu's paragroups, bimodule endomorphisms and 2- $C^*$ -tensor categories. Popa's axiomatization of the standard invariant of a finite index  $II_1$  subfactor in terms of standard  $\lambda$ -lattices, [21], was a major advance in the field. In [7] Jones introduced subfactor planar algebras, another axiomatization of the standard invariant of the subfactor which has a diagrammatic formulation and this planar algebra machinery has been a powerful tool in proving results on the combinatorial structure of the standard invariant. The fundamental theorem of Jones in [7] states that associated to any extremal subfactor with index  $[M : N] = \delta^2 < \infty$ , there exists a unique subfactor planar algebra  $P = P^{N \subset M}$  of modulus  $\delta$  and conversely any subfactor planar algebra of modulus  $\delta$  arises from an extremal subfactor of index  $\delta^2$ .

The notion of a biunitary element was introduced by Ocneanu([20]) in 1989 as a central tool in the study and classification of subfactors. Later Jones introduced these concepts via planar algebras, see [7]. The main motivation for this work comes from the beautiful results of Reutter and Vicary in [22] in which planar algebraic constructions are used to treat a variety of objects in quantum information theory. While pictorial and planar algebraic techniques are used throughout that paper, no planar algebra actually makes an appearance, leading to the question as to where these objects actually live. We define an equivalent formulation of the notion of a biunitary element in a planar algebra and some objects in quantum information theory will correspond to certain biunitary elements. Our objects of interest in quantum information theory are Hadamard matrices, quantum Latin squares, biunitary matrices and unitary error bases. They provide the mathematical foundation for an extremely rich variety of quantum computational phenomena, amongst them the study of mutually unbiased bases, quantum key distribution, quantum teleportation, dense coding and quantum error correction, see [3],[10]. This thesis mainly describes the unification of these quantum structures into a single notion of biunitary elements in the well known spin planar algebra, see [7], and the construction of subfactors out of these objects. The construction of subfactors from biunitary matrices is well known [5] [23]. Our results naturally suggest that there might be subfactors associated to our biunitary elements and it is indeed shown by constructing appropriate subfactor planar algebras.

## 1.2 About the thesis

In this section we will describe the arrangement of this thesis. The content of this thesis is divided into four chapters. Chapter 2 is devoted to a discussion of preliminary notions of planar algebras and other important terminologies which may occur later in this thesis. The main content of this thesis is contained in Chapters

3 and 4. A more detailed description of the chapters is as follows:

**Chapter 2:** The goal of this chapter is to provide the basic definitions, examples and certain properties and terminologies of subfactor planar algebras which we shall use later in this thesis. A brief description of the presentation of the spin planar algebra is also given in this chapter.

**Chapter 3:** We will divide the contents of chapter 3 into three parts. In the first part we will generalize the notion of biunitary elements and will show that the objects arising in quantum information theory such as Hadamard matrices, quantum Latin squares and unitary error bases correspond to certain biunitary elements in the spin planar algebra. In the second part, we describe the construction of subfactors from biunitary elements in a planar algebras by constructing appropriate subfactor planar algebras. In the final part we will find the planar algebra associated to the Latin squares arising from finite group tables and will prove that this is nothing but the well known group planar algebra.

**Chapter 4:** This concluding chapter is devoted to studying the subfactor associated to Latin squares. Our results show that the subfactor planar algebra associated to a Latin square is the subgroup-group planar algebra of the subfactor  $R^G \subseteq R^H$ , see [4], where the subgroup-group pair is explicitly constructed from the Latin square.



# Chapter 2

## Preliminaries

This chapter is a survey of some facts about planar algebras which we shall be using in the sequel. First we give the definition of planar tangles and then present a brief description of planar algebras and discuss some properties like sphericity, connectedness and irreducibility. Next we will define the notion of universal planar algebras and presentations by generators and relations. Finally we will briefly recall the presentation of the spin planar algebra [12].

The notion of planar algebras has been evolving since its definition in [7]. To fix the notations and definitions for the version of planar algebras that we use here, we refer to [1] or [2]. The equivalence of this new convention with the older version is shown in [2]. Other notions such as universal planar algebras are treated carefully in [14] and these cover all of the notions that are used in this thesis.

### 2.1 Planar tangles

In this section we will briefly describe notion of planar tangles, operations on tangles and some important examples.

First define the set  $Col = \{0, 1, 2, \dots\} \times \{\pm 1\}$ . We refer the pair  $(k, \epsilon)$  of elements of  $Col$  as colours where  $\epsilon$  is either  $+$  or  $-$  and it stands for  $+1$  or  $-1$ .

**Definition 2.1.1 (Planar tangle).** *A planar tangle is a subset of the plane  $\mathbb{R}^2$  defined by the following data.*

1. *It consists of one external disc  $D_0$  and a collection of finitely many non intersecting internal discs  $D_i, i = 1, \dots, k$  ( $k$  can be zero also)*
2. *Each disc  $D_i$  (external as well as internal) has an even number (again possibly 0) of marked points on its boundary circles which we refer to as distinguished points.*
3. *It has a collection of non intersecting strings (again possibly empty) such that each string is either simple closed curve or it has endpoints the distinguished points on the boundary circles  $\partial_i$  of the internal as well as external discs. Also each marked point on the discs must be the end point one of the open string and these strings intersect the marked points only transversely.*
4. *For each disc, one of its boundary arcs (connected components of the complement of the marked points on the boundary circle) is chosen as distinguished one and a  $*$  mark is placed near it.*
5. *Finally, there is a chequerboard shading of the regions (connected components of the complement of the curves) such that across any curve, the shading reverses.*

We say a disc with  $2n$  ( $n$  can be zero) marked points on its boundary has colour  $(n, +)$  or  $(n, -)$  accordingly as its distinguished  $*$ - arc is adjacent to a white or black region. We define the colour of a tangle to be the colour of its external disc. We will usually denote a tangle  $T$  of colour  $(k_0, \epsilon_0)$  with  $b$  internal discs ( $b$  may be zero) such that  $i^{th}$  internal disc has colour  $(k_i, \epsilon_i)$  by  $T_{(k_1, \epsilon_1), \dots, (k_b, \epsilon_b)}^{(k_0, \epsilon_0)}$ . The tangles are

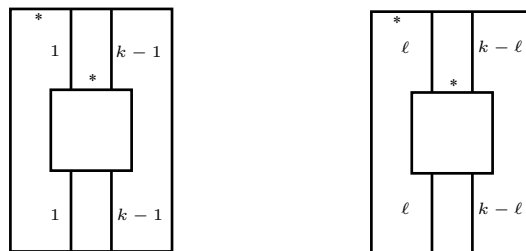


defined only upto a planar isotopy preserving all the data of the tangle given in the definition. Also we write  $D_i(T)$  to denote the  $i^{th}$  internal disc of  $T$ . A strand in a tangle with a non-negative integer, say  $\ell$ , written adjacent to it will indicate a parallel cable of  $\ell$  strands together, instead of the one and is called as the  $\ell$ -cable of that strand. We will dispense with shading figures in most cases since the shading is uniquely determined the sub- and superscripts of the tangle. Also, in most of the figures of tangles, discs are drawn as boxes.

### 2.1.2 Some examples of tangles

In this section we will give examples of several important tangles. See Figure 2.1 for some general class of basic tangles. The next example illustrates one important class of planar tangles namely rotation tangles.

**Example 2.1.3 (Rotation tangles).** For each  $k \in \mathbb{N}$ , define the rotation tangle  $R(k, \epsilon) = R(k, \epsilon)_{(k, \epsilon)}^{(k, -\epsilon)}$  and its  $\ell$ -fold iteration  $R(k, \epsilon, \ell)$  given as in Figure 2.2.



$$R(k, \epsilon) = R(k, \epsilon)_{(k, \epsilon)}^{(k, -\epsilon)} \quad R(k, \epsilon, \ell) = R(k, \epsilon, \ell)_{(k, \epsilon)}^{(k, (-1)^\ell \epsilon)}$$

Figure 2.2: Rotation tangles

In Figure 2.2 and in the sequel we adopt two conventions: (i)  $(-)^{\ell}$  denotes  $\pm$  according to the parity of  $\ell$ , and (ii) In view of the difficulty of shading diagrams which depend on the parity of  $\ell$ , we will dispense with shading figures.

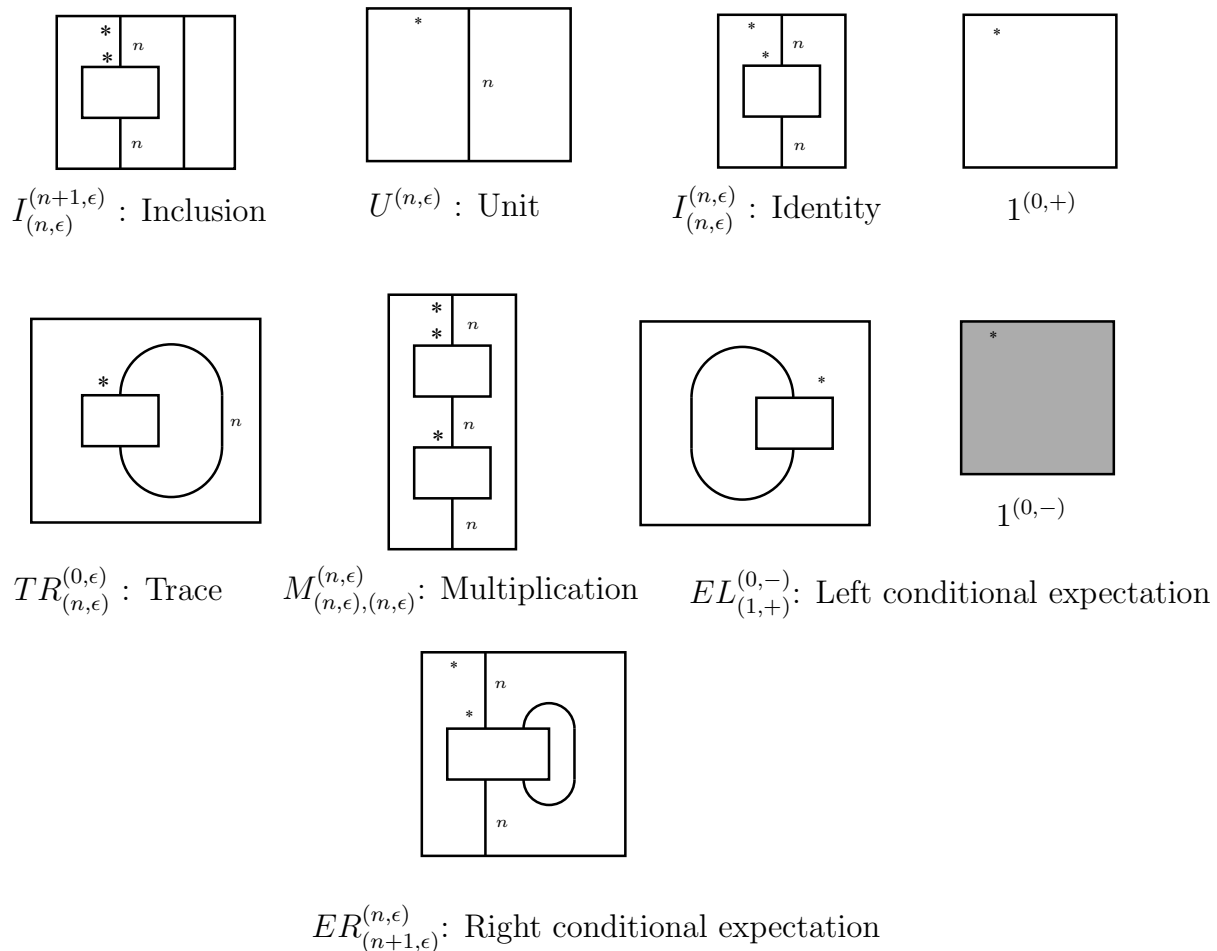


Figure 2.1: Some important tangles

## 2.1.4 Operations on tangles

Here we will describe two basic operations on tangles which produces new tangles from the old ones.

1. **Renumbering:** This operation produces new tangle just by renumbering the internal discs of the original one. More precisely we can describe it as follows. Let  $T = T_{(k_1,\epsilon_1),\dots,(k_b,\epsilon_b)}^{(k_0,\epsilon_0)}$  be a tangle. Let  $\sigma$  be a permutation in  $S_b$  which is the collection of all permutations on the set  $\{1, 2, \dots, b\}$ . Then  $\sigma(T)$  is defined to be the tangle that is same as  $T$  except the numbering of internal discs of  $\sigma(T)$  is permuted by  $\sigma$ . That is the  $i^{th}$  internal disc of  $T$  is the  $\sigma(i)^{th}$  internal disc of  $\sigma(T)$ , i.e.,  $D_i(\sigma(T)) = D_{\sigma^{-1}(i)}(T)$ .

2. **Substitution:** Consider the tangles  $T = T_{(k_1, \epsilon_1), \dots, (k_b, \epsilon_b)}^{(k_0, \epsilon_0)}$  and  $S = S_{(\tilde{k}_1, \tilde{\epsilon}_1), \dots, (\tilde{k}_{\tilde{b}}, \tilde{\epsilon}_{\tilde{b}})}^{(\tilde{k}_0, \tilde{\epsilon}_0)}$  such that the colour of the  $i^{\text{th}}$  internal disc of  $T$  is same as the colour of the external disc of  $S$  i.e.,  $(\tilde{k}_0, \tilde{\epsilon}_0) = (k_i, \epsilon_i)$ . Then we can form the composite of  $T$  and  $S$  denoted by  $T \circ_i S$  by substituting the tangle  $S$  into the  $i^{\text{th}}$  internal disc of  $T$  in such a way that the  $*$ -arc of the external disc of  $S$  agrees with the  $*$ -arc of the  $i^{\text{th}}$  internal disc of  $T$  and then deleting the boundary circle of  $S$ . Thus the composite tangle  $T \circ_i S$  is a tangle with colour  $(k_0, \epsilon_0)$  such that it has  $b + \tilde{b} - 1$  internal discs and the numbering of the internal discs is as follows. If  $\tilde{b} > 0$ , for each  $1 \leq j \leq b + \tilde{b} - 1$ , the  $j^{\text{th}}$  internal disc of  $T \circ_i S$  is the

$$\begin{cases} j - \text{th disc of } T, & \text{if } 1 \leq j \leq i - 1 \\ j - i + 1 - \text{th disc of } S, & \text{if } i \leq j \leq i - 1 + \tilde{b} \\ j - \tilde{b} + 1 - \text{th disc of } T, & \text{if } i - 1 + \tilde{b} < j \leq b + \tilde{b} - 1. \end{cases}$$

If  $\tilde{b} = 0$ , then  $T \circ_i S$  has  $b - 1$  internal discs and the  $j^{\text{th}}$  internal disc of  $T \circ_i S$  is the

$$\begin{cases} j - \text{th disc of } T & \text{if } 1 \leq j \leq i - 1 \\ j + 1 - \text{th disc of } T & \text{if } i \leq j \leq b - 1 \end{cases}$$

## 2.2 Planar algebras

In this section we will first give the definition of planar algebras and will discuss some of its important properties and terminologies which may occur in the later part of this thesis.

A planar algebra is a collection  $\{P_{(k, \epsilon)} : (k, \epsilon) \in Col\}$  of vector spaces over a field  $K$  (we usually assume it to be  $\mathbb{C}$ ) which has an action by tangle. More precisely given any tangle  $T = T_{(k_1, \epsilon_1), (k_2, \epsilon_2), \dots, (k_b, \epsilon_b)}^{(k_0, \epsilon_0)}$ , there is an associated linear map  $Z_T$  also

called as the partition function from:

$$\begin{cases} P_{(k_1, \epsilon_1)} \otimes P_{(k_2, \epsilon_2)} \otimes \cdots \otimes P_{(k_b, \epsilon_b)} & \longrightarrow P_{(k_0, \epsilon_0)} \text{ if } b > 0, \\ K & \longrightarrow P_{(k_0, \epsilon_0)} \text{ if } b = 0. \end{cases}$$

such that the following three axioms are satisfied.

1. Compatibility with composition of tangles: Let  $T = T_{(k_1, \epsilon_1), (k_2, \epsilon_2), \dots, (k_b, \epsilon_b)}^{(k_0, \epsilon_0)}$  and  $S = S_{(\tilde{k}_1, \tilde{\epsilon}_1), (\tilde{k}_2, \tilde{\epsilon}_2), \dots, (\tilde{k}_{\tilde{b}}, \tilde{\epsilon}_{\tilde{b}})}^{(\tilde{k}_0, \tilde{\epsilon}_0)}$  be two planar tangles such that colour of the  $i$ -th internal disc of  $T$  is same as the colour of the external disc of  $S$  i.e.,  $(\tilde{k}_0, \tilde{\epsilon}_0) = (k_i, \epsilon_i)$ . Consider the composite  $T \circ_i S$ . Then the compatibility axiom states that the following diagram should commute.

When  $\tilde{b} > 0$ :

$$\begin{array}{ccc} (\otimes_{j=1}^{i-1} P_{(k_j, \epsilon_j)}) \otimes (\otimes_{j=1}^{\tilde{b}} P_{(\tilde{k}_j, \tilde{\epsilon}_j)}) \otimes (\otimes_{j=i+1}^b P_{(k_j, \epsilon_j)}) & & \\ (\otimes_{j=1}^{i-1} Id_{P_{(k_j, \epsilon_j)}}) \otimes Z_S \otimes (\otimes_{j=i+1}^b Id_{P_{(k_j, \epsilon_j)}}) \downarrow & \searrow^{Z_{T \circ_i S}} & \\ (\otimes_{j=1}^b P_{(k_j, \epsilon_j)}) & \xrightarrow{Z_T} & P_{(k_0, \epsilon_0)} \end{array}$$

Figure 2.3: Substitution axiom when  $\tilde{b} > 0$

When  $\tilde{b} = 0$ :

$$\begin{array}{ccc} (\otimes_{j=1}^{i-1} P_{(k_j, \epsilon_j)}) \otimes \mathbb{C} \otimes (\otimes_{j=i+1}^b P_{(k_j, \epsilon_j)}) & \xrightarrow{\cong} & \otimes_{\substack{j=1 \\ j \neq i}}^b P_{(k_j, \epsilon_j)} \\ (\otimes_{j=1}^{i-1} Id_{P_{(k_j, \epsilon_j)}}) \otimes Z_S \otimes (\otimes_{j=i+1}^b Id_{P_{(k_j, \epsilon_j)}}) \downarrow & & \downarrow Z_{T \circ_i S} \\ (\otimes_{j=1}^b P_{(k_j, \epsilon_j)}) & \xrightarrow{Z_T} & P_{(k_0, \epsilon_0)} \end{array}$$

Figure 2.4: Substitution axiom when  $\tilde{b} = 0$

2. Compatibility with renumbering of the internal discs: Let  $T = T_{(k_1, \epsilon_1), (k_2, \epsilon_2), \dots, (k_b, \epsilon_b)}^{(k_0, \epsilon_0)}$  be a tangle with  $b > 0$  and  $\sigma$  be a permutation on the set  $\{1, 2, \dots, b\}$ . Consider the tangle  $\sigma(T)$ . The renumbering axioms says that the following diagram

should commute.

$$\begin{array}{ccc}
P_{(k_1, \epsilon_1)} \otimes P_{(k_2, \epsilon_2)} \otimes \cdots \otimes P_{(k_b, \epsilon_b)} & \xrightarrow{U_\sigma} & P_{(k_{\sigma^{-1}(1)}, \epsilon_{\sigma^{-1}(1)})} \otimes \cdots \otimes P_{(k_{\sigma^{-1}(b)}, \epsilon_{\sigma^{-1}(b)})} \\
\downarrow Z_T & & \swarrow Z_{\sigma(T)} \\
P_{(k_0, \epsilon_0)} & & 
\end{array}$$

Figure 2.5: Renumbering compatibility

where  $U_\sigma : \otimes_{j=1}^b P_{(k_j, \epsilon_j)} \longrightarrow \otimes_{j=1}^b P_{(k_{\sigma^{-1}(j)}, \epsilon_{\sigma^{-1}(j)})}$  is the linear isomorphism defined by  $U_\sigma(\otimes_{j=1}^b x_j) = \otimes_{j=1}^b x_{\sigma^{-1}(j)}$  for  $\otimes_{j=1}^b x_j \in \otimes_{j=1}^b P_{(k_j, \epsilon_j)}$ .

3. Non-degeneracy axiom: This axiom states that if  $I_{(k, \epsilon)}^{(k, \epsilon)}$  denotes the identity tangle given in Figure 2.1, then  $Z_{I_{(k, \epsilon)}^{(k, \epsilon)}}^P = Id_{P_{(k, \epsilon)}}$ , for all  $(k, \epsilon) \in Col$ .

Shortly,

**Definition 2.2.1** (Planar algebra). *A planar algebra is a family of vector spaces  $P = (P_{(n, \pm)})_{n \in \mathbb{N} \cup \{0\}}$  indexed by the colours  $(n, \epsilon)$ , called  $n$ -box spaces on which the planar tangle acts as multilinear maps.*

Let us now define the notion of morphism between planar algebras.

**Definition 2.2.2.** *If  $P, Q$  are planar algebras, a morphism from  $P$  to  $Q$  is a collection  $\{\phi_{(k, \epsilon)} : P_{(k, \epsilon)} \longrightarrow Q_{(k, \epsilon)}\}_{(k, \epsilon) \in Col}$  of linear maps between the vector spaces such that given any tangle  $T = T_{(k_1, \epsilon_1), (k_2, \epsilon_2), \dots, (k_b, \epsilon_b)}^{(k_0, \epsilon_0)}$ , the following diagram commutes:*

$$\begin{array}{ccc}
\otimes_{j=1}^b P_{(k_j, \epsilon_j)} & \xrightarrow{Z_T^P} & P_{(k_0, \epsilon_0)} \\
\otimes_{j=1}^b \phi_{(k_j, \epsilon_j)} \downarrow & & \downarrow \phi_{(k_0, \epsilon_0)} \\
\otimes_{j=1}^b Q_{(k_j, \epsilon_j)} & \xrightarrow{Z_T^Q} & Q_{(k_0, \epsilon_0)}
\end{array}$$

Figure 2.6: Planar algebra morphism

Further, the morphism  $\phi$  is said to be a planar algebra isomorphism if the maps  $\phi_{(k,\epsilon)}$  are all linear isomorphisms. An automorphism of a planar algebra  $P$  is a  $*$ -preserving planar isomorphism from  $P$  into itself and the set of all automorphisms of a planar algebra  $P$  will be denoted by  $Aut(P)$ .

Now the following proposition illustrates the unital algebra structure on each vector space  $P_{(k,\epsilon)}$  of a planar algebra  $P$ . Please refer to Figure 2.1 for the tangles mentioned in the following proposition.

**Proposition 2.2.3.** *Let  $P$  be a planar algebra. Then for every colour  $(k, \epsilon) \in Col$ , the vector space  $P_{(k,\epsilon)}$  has the natural structure of an associative unital algebra with multiplication specified by the  $(k, \epsilon)$ -multiplication tangle  $M_{(k,\epsilon),(k,\epsilon)}^{(k,\epsilon)}$  and unit given by  $Z_{U^{(k,\epsilon)}}(1)$  where  $U^{(k,\epsilon)}$  is the unit tangle. Furthermore, inclusion tangles induce homomorphisms of unital algebras.*

## 2.2.4 Some properties of planar algebras

Now we will briefly describe certain important properties of planar algebras.

- **Connectedness:** A planar algebra  $P$  is said to be connected if  $dim P_{(0,\pm)} = 1$ . If the planar algebra  $P$  is connected, then we can canonically identify  $P_{(0,\pm)} \cong \mathbb{C}$ .
- **Modulus:** A connected planar algebra  $P$  is said to have modulus  $\delta$  if there is a scalar  $\delta$  such that  $Z_{T^{(0,\pm)}}^P = \delta Id_{\mathbb{C}}$  where  $T^{(0,+)}$  (resp.,  $T^{(0,-)}$ ) denotes the  $(0, +)$  (resp.,  $(0, -)$ ) tangle having no internal discs and a single closed loop.
- **Finite-dimensionality:** A planar algebra  $P$  is said to be finite dimensional if  $dim P_{(k,\epsilon)} < \infty$ , for all  $(k, \epsilon) \in Col$ .
- **Sphericity:** A connected planar algebra  $P$  is said to be spherical if for any  $(0, \pm)$  tangle  $T$ ,  $Z_T$  depends only on the isotopy class of  $T$  on the 2-sphere

compactification of  $\mathbb{R}^2$ .

We have some remarks regarding the properties defined above.

- Remark 2.2.5.**
1. If a connected planar algebra has nonzero modulus  $\delta$ , then inclusion tangles  $I_{(n,\epsilon)}^{(n+1,\epsilon)}$  induce injective maps from  $P_{(n,\epsilon)}$  to  $P_{(n+1,\epsilon)}$ .
  2. Now if  $P$  is a connected planar algebra with a positive modulus  $\delta > 0$ , then  $\tau = \delta^{-k} Z_{TR^{(k,\epsilon)}^0}$  gives a normalized trace map from  $P_{(k,\pm)} \rightarrow P_{(0,\epsilon)} \cong \mathbb{C}$ , where  $TR^{(0,\epsilon)}$  are the trace tangles given in 2.1.
  3. A connected planar algebra is spherical if and only if  $Z_{EL_{1,+}^{0,-}}^P = Z_{TR_{1,+}^{0,+}}^P$  (see figure 2.1 for definitions of  $EL_{1,+}^{0,-}$  and  $TR_{1,+}^{0,+}$ ), where both  $Z_{EL_{1,+}^{0,-}}^P$  and  $Z_{TR_{1,+}^{0,+}}^P$  are regarded as linear functionals on  $P_{(1,+)}$ .

**Definition 2.2.6.** For any planar tangle  $T$ , we define the ‘adjoint of  $T$ ’ denoted by  $T^*$  to be the planar tangle obtained by applying an orientation reversing diffeomorphism (such as reflection) of  $\mathbb{R}^2$  to  $T$  and all its data.

**Definition 2.2.7 (\*- Planar algebra).** A \*-planar algebra is a planar algebra  $P$  with each  $P_{(k,\epsilon)}$  equipped with a conjugate linear involution map  $*$  such that for any tangle  $T$  with  $b$  input discs with colours  $(k_1, \epsilon_1), (k_2, \epsilon_2), \dots, (k_b, \epsilon_b)$  and output disc of colour  $(k_0, \epsilon_0)$ ,

$$(Z_T(x_1 \otimes \dots \otimes x_n))^* = Z_{T^*}(x_1^* \otimes \dots \otimes x_n^*), \quad \text{for } x_i \in P_{(k_i, \epsilon_i)}$$

**Definition 2.2.8 ( $C^*$ - planar algebra).** A \*-planar algebra  $P$  is said to be a  $C^*$ -planar algebra if there exist positive normalized traces  $\tau_{\pm} : P_{(0,\pm)} \rightarrow \mathbb{C}$  such that all the traces  $\tau_{\pm} \circ Z_{TR^{(0,\pm)}}$  defined on  $P_{(k,\pm)}$  are faithful and positive. Thus all the spaces equipped with the trace innerproduct are Hilbert spaces.

Now let us define the most important class of planar algebras:

**Definition 2.2.9 (Subfactor planar algebra).** *Let  $P$  be a finite dimensional, connected, spherical  $*$ -planar algebra with positive modulus  $\delta > 0$ . Then  $P$  is said to be a subfactor planar algebra if the normalized trace map  $\tau = \delta^{-k} Z_{TR_{(k,\pm)}^0}$  is positive and faithful, i.e.,  $\tau(x^*x) \geq 0$  and  $\tau(x) = 0$  if and only if  $x = 0$ .*

## 2.3 Universal planar algebras and presentations

In this section let us briefly describe the presentation of a planar algebra by generators and relations.

Let  $L = \coprod_{(k,\epsilon) \in Col} L_{(k,\epsilon)}$  ( $L_{(k,\epsilon)}$  can very well be empty) be a ‘label set’. We define an  $L$ -labelled tangle to be a pair  $(T, f)$  such that  $T = T_{(k_1,\epsilon_1), (k_2,\epsilon_2), \dots, (k_b,\epsilon_b)}^{(k_0,\epsilon_0)}$  is a planar tangle and  $f$  is a function from  $\{D_1(T), D_2(T), \dots, D_b(T)\}$  to  $L$  such that  $f(D_i(T)) \in L_{(k_i,\epsilon_i)}$  for all  $i$ . Thus an  $L$ -labelled tangle is simply a usual planar tangle each of its internal discs of colour  $(k, \epsilon)$  consists of an input from  $L_{(k,\epsilon)}$ . Hence if  $L_{(k,\epsilon)} = \phi$  for some colour  $(k, \epsilon)$ , then no  $L$ -labelled tangle can have an internal disc of colour  $(k, \epsilon)$ .

The universal planar algebra on  $L$ , denoted by  $P(L)$ , is described as follows. For each colour  $(k, \epsilon)$ , we have the vector space  $P(L)_{(k,\epsilon)}$  with basis consists of all  $L$ -labelled  $(k, \epsilon)$  tangles with the action of a planar tangle on a tensor product of basis vectors is given by the obvious  $L$ -labelled tangle obtained by substituting these basis vectors into the appropriate internal discs to get another basis vector.

Let us now introduce the notion of a planar ideal in a planar algebra.

**Definition 2.3.1.** *A planar ideal  $I$  of a planar algebra  $P$  is a collection  $\{I_{(k,\epsilon)} : (k, \epsilon) \in Col\}$  of vector spaces where each  $I_{(k,\epsilon)}$  is a linear subspace of  $P_{(k,\epsilon)}$  such that given any tangle  $T = T_{(k_1,\epsilon_1), (k_2,\epsilon_2), \dots, (k_b,\epsilon_b)}^{(k_0,\epsilon_0)}$ ,  $Z_T(\otimes_{j=1}^b x_j) \in I_{(k_0,\epsilon_0)}$  whenever  $x_j \in I_{(k_j,\epsilon_j)}$  for some  $j$ ,  $1 \leq j \leq b$ .*



Let  $P$  be a planar algebra  $P$  and let  $R = \{R_{(k,\epsilon)} : (k,\epsilon) \in Col\}$  be a subset of  $P$  (i.e., each  $R_{(k,\epsilon)}$  is a subset of  $P_{(k,\epsilon)}$ ). Then the planar ideal generated by  $R$ , denoted by  $I(R)$ , is the smallest planar ideal in  $P$  containing  $R$ . Or equivalently, if we set  $I_{(k,\epsilon)}$  to be the span of all  $Z_T(x_1 \otimes \cdots \otimes x_b)$  where  $T$  is a  $(k,\epsilon)$ -tangle, say  $T = T_{(k_1,\epsilon_1),(k_2,\epsilon_2),\dots,(k_b,\epsilon_b)}^{(k,\epsilon)}$ , and at least one  $x_i \in R$ , then  $I = \{I_{(k,\epsilon)} : (k,\epsilon) \in Col\}$  is the planar ideal generated by  $R$ .

Given a planar ideal  $I$  in a planar algebra  $P$ , there is a natural planar algebra structure on the quotient  $P/I = \{(P/I)_{(k,\epsilon)} := P_{(k,\epsilon)}/I_{(k,\epsilon)} : (k,\epsilon) \in Col\}$  together with a surjective planar algebra morphism from  $P$  to  $P/I$ . We now describe the notion of planar algebra presented with generators and relations.

**Definition 2.3.2 (Presentation of a planar algebra).** *Given a label set  $L = \amalg_{(k,\epsilon) \in Col} L_{(k,\epsilon)}$ , consider the universal planar algebra  $P(L)$  on  $L$ . Let  $R$  be a subset of  $P(L)$  and suppose  $I(R)$  denotes the planar ideal generated by  $R$ . The quotient planar algebra  $P(L)/I(R)$  is said to be the planar algebra presented with generators  $L$  and relations  $R$  and is usually denoted by  $P(L, R)$ .*

The following simple lemma allows us to define a  $*$ -structure on a planar algebra with a presentation by defining it on the label set. That is,

**Lemma 2.3.3.** *Let  $P = P(L, R)$  for some label set  $L$  and some set of relations  $R$  in  $P(L)$ . Suppose that  $L$  is equipped with an involution  $*$  (by which we mean each  $L_{(k,\epsilon)}$  is) such that for every relation in  $R$ , its adjoint is also in the planar ideal generated by  $R$ . Then  $P$  has a natural  $*$ -planar algebra structure.*

## 2.4 Spin planar algebra

The spin planar algebra is well known from the very first paper of Jones [7] on planar algebras. Recently, it was shown in [12] that the spin planar algebra has a

presentation in terms of generators and relations. The purpose of this section is to briefly describe this presentation by stating the main results without giving the proofs.

Let  $S = \{s_1, s_2, \dots, s_n\}$  be a finite set. We define an abstract planar algebra over  $\mathbb{C}$  by generators and relations, carefully study various aspects of its structure and finally identify it with the spin planar algebra. Begin with the label set  $L = L_{(0,-)} = S$  equipped with the identity involution  $*$ . Consider the quotient  $P = P(L, R)$  of the universal planar algebra  $P(L)$  by the set  $R$  of relations in Figures 2.7 and 2.8 (where  $\delta_{ij}$  denotes the Kronecker delta).

Figure 2.7: The white and black modulus relations

Figure 2.8: The multiplication relation and the black channel relation

**Lemma 2.4.1** (Lemma 2 of [12]). *The relations in Figure 2.9 and Figure 2.10 hold in  $P$ .*

Figure 2.9: The unit and modulus relations

Next theorem gives various properties of the planar algebra  $P$  defined above. This is the main result of [12].

$$\sum_i \left[ \begin{array}{|c|c|} \hline * & \square s_i \\ \hline \end{array} \right] = \left[ \begin{array}{|c|c|} \hline * & \square \\ \hline \end{array} \right]$$

Figure 2.10: Another unit relation

**Theorem 2.4.2.** (Theorem 1 of [12]). *The planar algebra  $P$  is a finite-dimensional  $C^*$ -planar algebra with modulus  $\sqrt{n}$  and such that  $\dim(P_{(0,+)}) = 1$  and  $\dim(P_{(0,-)}) = n$ . For  $k > 0$ ,  $\dim(P_{(k,\pm)}) = n^k$  with bases as in Figures 2.11 and 2.12 for  $k$  even and odd respectively.*

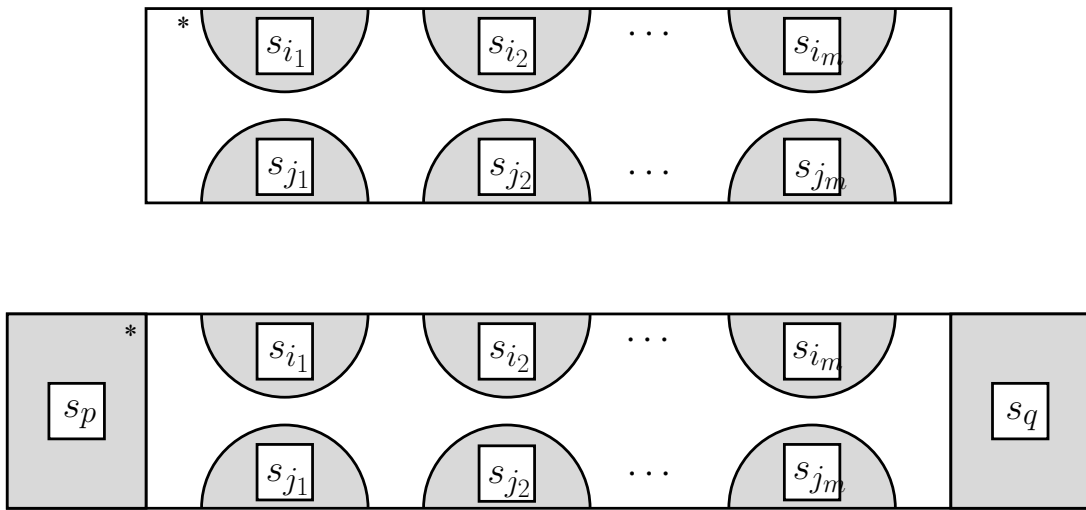


Figure 2.11: Bases  $\mathcal{B}_{(2m,+)}$  for  $m \geq 1$  and  $\mathcal{B}_{(2m+2,-)}$  for  $m \geq 0$

We will need some notation for the normalized version of the bases for  $P_{(k,\pm)}$  for  $k = 1, \dots$  given in Figures 2.11 and 2.12. We denote  $(\sqrt{n})^m$  times the elements on the top and bottom in Figure 2.11 by  $e_{j_1 \dots j_m}^{i_1 \dots i_m}$  and  $e[p]_{j_1 \dots j_m}^{i_1 \dots i_m}(q)$  respectively. Similarly, we denote  $(\sqrt{n})^m$  times the elements on the top and bottom in Figure 2.12 by  $e_{j_1 \dots j_m}^{i_1 \dots i_m}(q)$  and  $e[p]_{j_1 \dots j_m}^{i_1 \dots i_m}$ . The multiplication relation among these elements is given in the following lemma and justifies the notation.

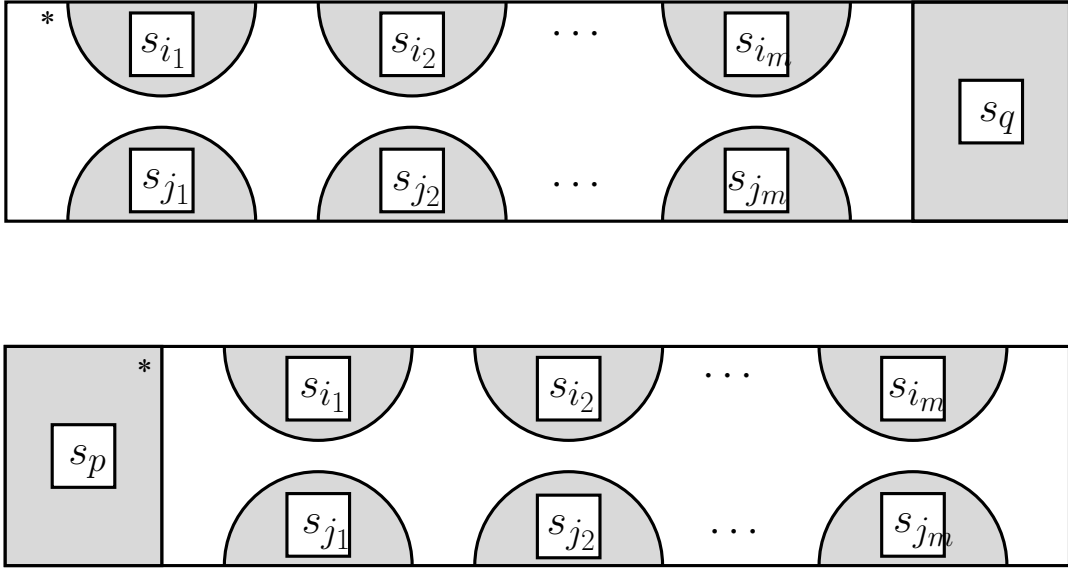


Figure 2.12: Bases  $\mathcal{B}_{(2m+1, \pm)}$  for  $m \geq 0$

**Lemma 2.4.3.** (Lemma 10 of [12]). *The following relations hold in  $P$ .*

$$\begin{aligned}
e_{j_1 \dots j_m}^{i_1 \dots i_m} \cdot e_{l_1 \dots l_m}^{k_1 \dots k_m} &= \delta_{j_1 k_1} \cdots \delta_{j_m k_m} e_{l_1 \dots l_m}^{i_1 \dots i_m} \\
e[p]_{j_1 \dots j_m}^{i_1 \dots i_m}(q) \cdot e[r]_{l_1 \dots l_m}^{k_1 \dots k_m}(s) &= \delta_{pr} \delta_{j_1 k_1} \cdots \delta_{j_m k_m} \delta_{qs} e[p]_{l_1 \dots l_m}^{i_1 \dots i_m}(s) \\
e_{j_1 \dots j_m}^{i_1 \dots i_m}(q) \cdot e_{l_1 \dots l_m}^{k_1 \dots k_m}(s) &= \delta_{j_1 k_1} \cdots \delta_{j_m k_m} \delta_{qs} e_{l_1 \dots l_m}^{i_1 \dots i_m}(q) \\
e[p]_{j_1 \dots j_m}^{i_1 \dots i_m} \cdot e[r]_{l_1 \dots l_m}^{k_1 \dots k_m} &= \delta_{pr} \delta_{j_1 k_1} \cdots \delta_{j_m k_m} e[p]_{l_1 \dots l_m}^{i_1 \dots i_m}.
\end{aligned}$$

Action of the rotation tangle  $R(k, \epsilon)$  given in Figure 2.2 on the basis elements given in Figures 2.11 and 2.12 is stated in the following lemma:

**Lemma 2.4.4.** (Lemma 16 of [13]). *With notation as above,*

$$\begin{aligned}
Z_{R(2m, +)}(e_{j_1 \dots j_m}^{i_1 \dots i_m}) &= \sqrt{n} e[j_1]_{j_2 \dots j_m}^{i_1 \dots i_{m-1}}(i_m) \\
Z_{R(2m+2, -)}(e[p]_{j_1 \dots j_m}^{i_1 \dots i_m}(q)) &= \frac{1}{\sqrt{n}} e_{j_1 \dots j_m}^{p i_1 \dots i_m} \\
Z_{R(2m+1, +)}(e_{j_1 \dots j_m}^{i_1 \dots i_m}(q)) &= e[j_1]_{j_2 \dots j_m}^{i_1 \dots i_m} \\
Z_{R(2m+1, -)}(e[p]_{j_1 \dots j_m}^{i_1 \dots i_m}) &= e_{j_1 \dots j_m}^{p i_1 \dots i_{m-1}}(i_m).
\end{aligned}$$

In the following final result, the planar algebra  $P$  is identified with the planar algebra of the bipartite graph  $\Gamma$  with one even vertex and one odd vertex given in figure 2.13 which in turn can be identified with the spin planar algebra, see Example 4.2 in [9].

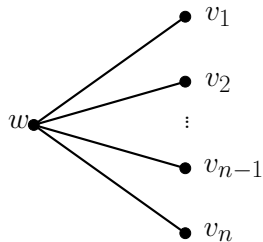


Figure 2.13: The bipartite graph  $\Gamma$

**Proposition 2.4.5.** (*Proposition 13 of [13]*). *Let  $\Gamma$  be the bipartite graph in Figure 2.13. With  $S = \{v_1, \dots, v_n\}$ , the planar algebra  $P$  of Theorem 2.4.2 is isomorphic to  $P(\Gamma)$  by the map that takes  $v_i \in P_{(0,-)}$  to the loop of length 0 based at  $v_i$  in  $P(\Gamma)_{(0,-)}$ .*



# Chapter 3

## Planar algebras, quantum information theory and subfactors

This chapter is devoted to studying the relation between subfactor planar algebras and quantum information theory. We will first formulate the notion of a biunitary element in a planar algebra and establish a correspondence between certain biunitary elements in the spin planar algebra and some objects in quantum information theory. In the later section, we will describe the construction of subfactor planar algebras from biunitary elements and in particular, this naturally associates subfactors - say via the GJS construction - to these objects of interest.

### 3.1 Biunitary elements in a planar algebra

In this section we define the notion of biunitary elements in a planar algebra and some objects of interest in quantum information theory. A more generalized notion of biunitarity involving annular tangles will be defined in Section 3.2. Recall the rotation tangle defined in Figure 2.2.

**Definition 3.1.1.** *Let  $P$  be a  $*$ -planar algebra and let  $u \in P_{(k,\epsilon)}$ . For  $0 < \ell < k$ ,*

the element  $u$  is said to be a  $\{0, \ell\}$ -biunitary element if the elements  $u \in P_{(k, \epsilon)}$  and  $Z_{R(k, \epsilon, \ell)}(u) \in P_{(k, \epsilon(-)^\ell)}$  are both unitary.

**Lemma 3.1.2.** (Lemma 4 of [13]). The element  $u \in P_{(k, \epsilon)}$  is  $\{0, \ell\}$ -biunitary if and only if the relations in Figure 3.1 hold in  $P_{(k, \epsilon)}$ .

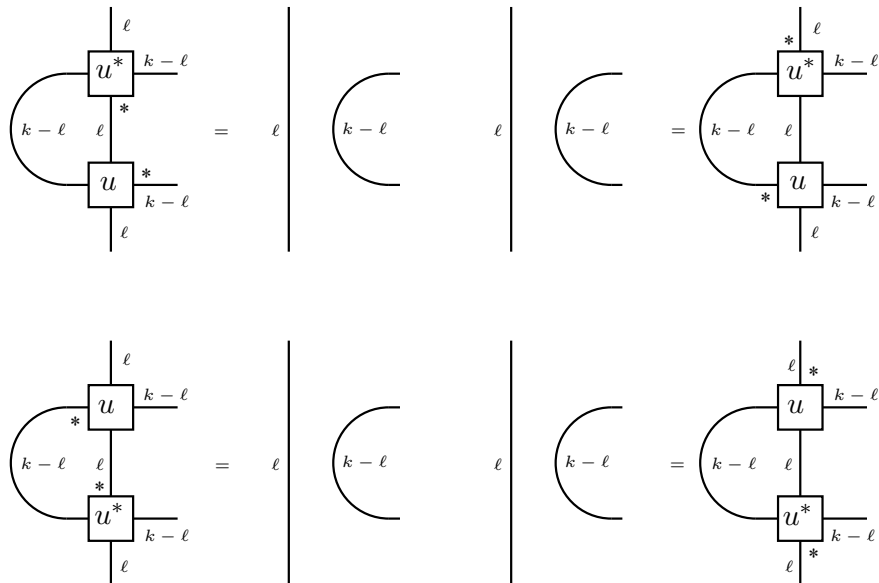


Figure 3.1:  $\{0, \ell\}$ -biunitarity relations

*Proof.* After choosing the external  $*$ -arc appropriately, the relations on top in Figure 3.1 are equivalent to the unitarity of  $u$  while the relations on the bottom are equivalent to the unitarity of  $Z_{R(k, \epsilon, \ell)}(u)$ .  $\square$

**Remark 3.1.3.** Observe that if  $u \in P_{(k, \epsilon)}$  is a  $\{0, \ell\}$ -biunitary element, then so are  $Z_{R(k, \epsilon, k)}(u) \in P_{(k, \epsilon(-)^k)}$  and  $Z_{R(k, \epsilon, -\ell)}(u^*) \in P_{(k, \epsilon(-)^\ell)}$ .

In the rest of this section we will define certain objects which arise in quantum information theory.

**Definition 3.1.4 (Hadamard matrix).** An  $n \times n$  complex matrix  $H = ((h_{ij}))$  is said to be a complex Hadamard matrix if  $HH^* = nI$  and  $|h_{ij}| = 1$  for each  $i, j$ .



**Example 3.1.5.** Let  $\omega \in \mathbb{C}$  be a primitive  $n^{\text{th}}$ -root of unity. The matrix

$$H = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \cdots & (\omega^2)^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \omega^{n-1} & \cdots & (\omega^{n-1})^{n-1} \end{bmatrix}$$

is a complex Hadamard matrix which is a multiple of the so-called Fourier matrix.

**Definition 3.1.6 (Latin Squares).** A Latin square is an  $n \times n$  array filled with  $n$  different symbols, each occurring exactly once in each row and in each column.

**Example 3.1.7.** The multiplication table of a finite group is a Latin square. The smallest example which is not (equivalent to one) of this type is of size 5 and is given by:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 5 & 3 & 2 & 1 & 4 \end{bmatrix}.$$

**Definition 3.1.8 (Quantum Latin Squares).** A quantum Latin square of size  $n$  is an  $n \times n$  matrix of vectors in  $\mathbb{C}^n$  such that each row and each column is an orthonormal basis for  $\mathbb{C}^n$ .

**Example 3.1.9.** Any Latin square gives a quantum Latin square in the following simple-minded way by choosing the symbol set to be  $\{e_1, \dots, e_n\}$ , the standard orthonormal basis of  $\mathbb{C}^n$ . Consider the  $5 \times 5$  Latin square in the Example 3.1.6. It

gives the following quantum Latin square:

$$\begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ e_2 & e_4 & e_1 & e_5 & e_3 \\ e_3 & e_5 & e_4 & e_2 & e_1 \\ e_4 & e_1 & e_5 & e_3 & e_2 \\ e_5 & e_3 & e_2 & e_1 & e_4 \end{bmatrix}.$$

But not every quantum Latin square arises in this way. For more on quantum Latin squares and non-trivial examples see [19].

**Definition 3.1.10 (Biunitary Matrix).** A matrix  $U = ((u_{kl}^{ij})) \in M_{n^2}(\mathbb{C})$  (for  $i, j, k, l \in \{1, \dots, n\}$ ) is said to be a biunitary matrix if both  $U$  and its block transpose, say  $V = ((v_{kl}^{ij}))$ , defined by  $v_{kl}^{ij} = u_{il}^{kj}$ , are unitary matrices.

**Example 3.1.11.** For examples of biunitary matrices of size 9 which are, in addition, permutation matrices, and their applications in subfactor theory see [16].

**Definition 3.1.12 (Unitary error basis).** A unitary error basis for  $M_n(\mathbb{C})$  is a collection of  $n^2$  unitary matrices which form an orthonormal basis with respect to the normalized trace inner product given by  $\langle A|B \rangle = \frac{\text{Tr}(B^*A)}{n}$ , where  $\text{Tr}$  is the usual matrix trace.

**Example 3.1.13.** The matrices  $\{U^i V^j : 1 \leq i, j \leq n\}$ , where  $U$  is the  $n \times n$  Fourier matrix and  $V$  is the permutation matrix corresponding to the cycle  $(1\ 2\ \dots\ n)$ , form a unitary error basis.

## 3.2 Planar algebras and quantum information theory

Recall the spin planar algebra  $P = P(S)$  presented in terms of generators and relations in section 2.4. In this section we identify various biunitary elements in spin planar algebra with the objects defined in section 3.1. The main result of this section is the following theorem.

**Theorem 3.2.1.** *(Theorem 17 of [13]). There are natural 1-1 correspondences between the following sets:*

- 1)  $\{0, 1\}$ -biunitary elements in  $P_{(2,+)}$  and Hadamard matrices of size  $n \times n$ ,
- 2)  $\{0, 1\}$ -biunitary elements in  $P_{(3,+)}$  and quantum Latin squares of size  $n \times n$ , and
- 3)  $\{0, 2\}$ -biunitary elements in  $P_{(4,+)}$  and biunitary matrices of size  $n^2 \times n^2$ .

*Proof.* 1) Let  $u = \sum_{i,j} a_j^i e_j^i \in P_{(2,+)}$ . Then,

$$uu^* = 1 \Leftrightarrow \sum_j a_j^i \overline{a_j^p} = \delta_{i,p} \quad \forall i, p = 1, \dots, n, \quad (3.1)$$

and

$$Z_{R(2,+)}(u)(Z_{R(2,+)}(u))^* = 1 \Leftrightarrow n a_j^i \overline{a_j^i} = 1 \quad \forall i, j = 1, \dots, n. \quad (3.2)$$

Let  $H = ((\sqrt{n}a_j^i))$ . Then Equation 3.1 is equivalent to  $HH^* = n I$  and Equation 3.2 is equivalent to  $|\sqrt{n}a_j^i| = 1$ . Thus the association of  $u$  with  $H$  is a 1-1 correspondence between  $\{0, 1\}$ -biunitary elements of  $P_{(2,+)}$  and Hadamard matrices of size  $n \times n$ .

2) Now let  $u = \sum_{i,j,k} a_{ij}^k e_k^i(j) \in P_{(3,+)}$ . Then,

$$uu^* = 1 \Leftrightarrow \sum_k a_{ij}^k \overline{a_{pj}^k} = \delta_{i,p}, \quad \forall j, i, p = 1, \dots, n, \quad (3.3)$$

and

$$Z_{R(3,+)}(u)(Z_{R(3,+)}(u))^* = 1 \Leftrightarrow \sum_j a_{ij}^k \overline{a_{pj}^k} = \delta_{i,p}, \quad \forall k, i, p = 1, \dots, n. \quad (3.4)$$

Let  $Q = ((Q_j^i))$  where  $Q_j^i = (a_{1j}^i, a_{2j}^i, \dots, a_{nj}^i) \in \mathbb{C}^n$ . Then, Equation 3.3 is equivalent to the column vectors of  $Q$  forming an orthonormal basis for  $\mathbb{C}^n$  and Equation 3.4 is equivalent to the row vectors of  $Q$  forming an orthonormal basis for  $\mathbb{C}^n$ . Thus the association of  $u$  with  $Q$  is a 1-1 correspondence between  $\{0, 1\}$ -biunitary elements of  $P_{(3,+)}$  and quantum Latin squares of size  $n \times n$ .

3) Let  $u = \sum_{i,j,k,\ell} a_{k\ell}^{ij} e_{\ell k}^{ij} \in P_{(4,+)}$ . Then,

$$uu^* = 1 \Leftrightarrow \sum_{k,\ell} a_{k\ell}^{ij} \overline{a_{k\ell}^{pq}} = \delta_{i,p} \delta_{j,q}, \quad \forall i, j, p, q = 1, \dots, n \quad (3.5)$$

and

$$Z_{R(4,+2)}(u)(Z_{R(4,+2)}(u))^* = 1 \Leftrightarrow \sum_{j,k} a_{k\ell}^{ij} \overline{a_{ks}^{pj}} = \delta_{i,p} \delta_{\ell,s}, \quad \forall i, p, \ell, s = 1, \dots, n \quad (3.6)$$

Here let  $U = ((a_{k\ell}^{ij}))$ . Then, Equation 3.5 is equivalent to  $U$  being unitary and Equation 3.6 is equivalent to the block transpose of  $U$  being unitary. Thus the association of  $u$  with  $U$  is a 1-1 correspondence between  $\{0, 2\}$ -biunitary elements of  $P_{(4,+)}$  and biunitary matrices of size  $n^2 \times n^2$ .  $\square$

An analogous 1-1 correspondence result for unitary error bases requires a modified version of the notion of biunitary element which we will now define. Recall that a labelled annular tangle in a planar algebra  $P$  is a tangle  $A$  all of whose internal boxes except for one have been labelled by elements of the appropriate  $P_{(k,\epsilon)}$ 's. Actually we would like to consider linear extensions of the definition and of the vector

operations to linear combinations of such tangles, provided of course that all the annular tangles involved yield maps between the same spaces. We will use the term modified annular tangle for such linear combinations.

**Definition 3.2.2.** Let  $P$  be a  $*$ -planar algebra and  $A, B$  be modified annular tangles with their unlabelled box of colour  $(k, \epsilon)$ . An element  $u \in P_{(k, \epsilon)}$  is said to be an  $\{A, B\}$ -biunitary element if  $Z_A(u)$  and  $Z_B(u)$  are both unitary.

**Remark 3.2.3.** Note that a  $\{0, \ell\}$ -biunitary element in  $P_{(k, \epsilon)}$  is nothing but an  $\{I, R(k, \epsilon, \ell)\}$ -biunitary element, where  $I$  denotes the identity tangle of colour  $(k, \epsilon)$ , see Figure 2.1.

In order to state the analogue of Theorem 3.2.1 for unitary error bases we will need the modified annular tangle  $A = A_{(4, +)}^{(4, +)}$ , with labelled internal boxes coming from the spin planar algebra  $P$ , defined by Figure 3.2.

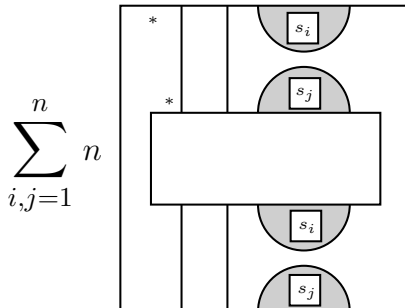


Figure 3.2: The annular tangle  $A$

Observe that  $Z_A(e_{kl}^{ij}) = e_{kj}^{il}$ .

**Proposition 3.2.4.** (Proposition 20 of [13]). There is a natural 1-1 correspondence between  $\{A, R(4, +)\}$ -biunitary elements in  $P_{(4, +)}$  and unitary error bases in  $M_n(\mathbb{C})$ .

*Proof.* Let  $u = \sum_{i, j, k, \ell} a_{k\ell}^{ij} e_{\ell k}^{ij} \in P_{(4, +)}$ . Then,

$$Z_{R(4, +)}(u) = \sum_{i, j, k, \ell} \sqrt{n} a_{k\ell}^{ij} e[\ell]_k^i(j), \text{ and}$$

$$Z_A(u) = \sum_{i, j, k, \ell} a_{k\ell}^{ij} e_{\ell j}^{ik}.$$

by Lemma 2.4.4 and the observation above. Hence

$$(Z_A(u))^* Z_A(u) = 1 \Leftrightarrow \sum_{i,k} a_{k\ell}^{ij} \overline{a_{ks}^{iq}} = \delta_{\ell s} \delta_{jq} \quad \forall \ell, s, j, q = 1, \dots, n \quad (3.7)$$

and

$$Z_{R(4,+)}(u)(Z_{R(4,+)}(u))^* = 1 \Leftrightarrow \sum_k a_{k\ell}^{ij} \overline{a_{k\ell}^{pj}} = \frac{1}{n} \delta_{ip} \quad \forall j, \ell, i, p = 1, \dots, n \quad (3.8)$$

Let  $B(j, \ell)$  be the  $n \times n$  matrix given by  $B(j, \ell) = ((\sqrt{n}a_{k\ell}^{ij}))$ . Then Equation 3.8 is equivalent to  $B(j, \ell)$  being a unitary matrix for all  $j, \ell$  and Equation 3.7 is equivalent to the collection  $\{B(j, \ell)\}_{j,\ell}$  forming an orthonormal basis for  $M_n(\mathbb{C})$ . Thus the association of  $u$  with  $\{B(j, \ell)\}_{j,\ell}$  is a 1-1 correspondence between  $\{A, R(4, +)\}$ -biunitary elements of  $P_{(4,+)}$  and unitary error bases of  $M_n(\mathbb{C})$ .  $\square$

### 3.3 From biunitary elements to subfactor planar algebras

In this section we will describe the construction of subfactor planar algebras from biunitary elements and this, in turn, associates subfactors to the objects mentioned in Section 3.1.

Throughout this section,  $P$  will be a spherical  $C^*$ -planar algebra with modulus  $\delta$  and  $u \in P_{(k,\epsilon)}$  will be a  $\{0, \ell\}$ -biunitary. To this data, we will associate a  $C^*$ -planar subalgebra  $Q$  of the  $(\ell, \epsilon)^{th}$  cabling  $(\ell, \epsilon)P$  of  $P$ . The notion of cabling that we use here is a generalized version of the one defined in [2]. A careful definition is as follows.

**Definition 3.3.1.** *Let  $(\ell, \epsilon) \in \mathbb{N} \times \{\pm\}$ . Define  $(n, \eta)^{(\ell, \epsilon)}$  to be  $(n\ell, \epsilon\eta^\ell)$ . Next, define the  $(\ell, \epsilon)$ -cable of a tangle  $T$ , denoted by  $T^{(\ell, \epsilon)}$ , as follows. Consider the tangle  $T$*

ignoring its shading and replace each of its strands (including the closed loops) by a cable of  $\ell$  parallel strands without changing the  $*$ -arcs. Introduce shading in this picture such that an  $(n, \eta)$ -box of  $T$  becomes an  $(n, \eta)^{(\ell, \epsilon)}$ -box of  $T^{(\ell, \epsilon)}$ .

It can be proved that this extends uniquely to a checkerboard shading of  $T^{(\ell, \epsilon)}$  making it a tangle and that  $T \mapsto T^{(\ell, \epsilon)}$  is an ‘operation on tangles’ in the sense of [14]. The corresponding operation on planar algebras will be denoted by  $P \mapsto^{(\ell, \epsilon)} P$ . To give an example of cabling, note that the  $(\ell, \epsilon)$ -cable of the rotation tangle  $R(n, \eta)$  of Figure 2.2 is given as in Figure 3.3 below. The shading of the  $*$ -arc of the external box is given by  $\epsilon(-\eta)^\ell$ .

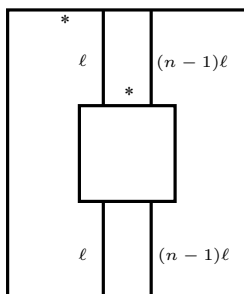


Figure 3.3: The  $(\ell, \epsilon)$ -cabling of the rotation tangle  $R(n, \eta)$

**Definition 3.3.2.** For  $(\ell, \epsilon) \in \mathbb{N} \times \{\pm\}$ , the planar algebra  $^{(\ell, \epsilon)} P$  has underlying vector spaces given by  $^{(\ell, \epsilon)} P_{(n, \eta)} = P_{(n, \eta)^{(\ell, \epsilon)}}$  with the tangle action given by  $Z_T^{(\ell, \epsilon)} P = Z_{T^{(\ell, \epsilon)}}^P$ .

Before proceeding to define subspaces  $Q_{(n, \eta)}$  of  $^{(\ell, \epsilon)} P_{(n, \eta)}$ , we begin with the inevitable notation. For  $n = 0, 1, \dots$ , define the elements  $u_{(n, \eta)} \in P_{(n\ell+k-\ell, \epsilon\eta^\ell)}$  as in Figure 3.4 - where the label in the last box is  $u$  or  $u^*$  depending on the parity of  $n$ .

**Proposition 3.3.3.** (Proposition 23 of [13]). For  $(n, \eta) \in \text{Col}$  and  $x \in P_{(n\ell, \epsilon\eta^\ell)}$ , the following three conditions are equivalent.

- (1) There exists  $y \in P_{(n\ell, \epsilon\eta^\ell(-)^{k-\ell})}$  such that the equation in Figure 3.5 holds.
- (2) There exists  $y \in P_{(n\ell, \epsilon\eta^\ell(-)^{k-\ell})}$  such that the equations in Figure 3.6 hold.
- (3) The equation in Figure 3.7 holds.

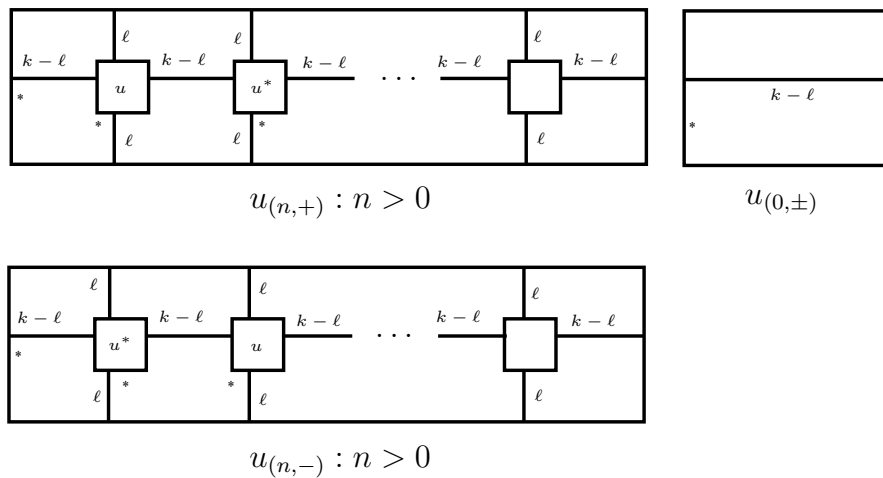


Figure 3.4: The elements  $u_{(n,\pm)}$

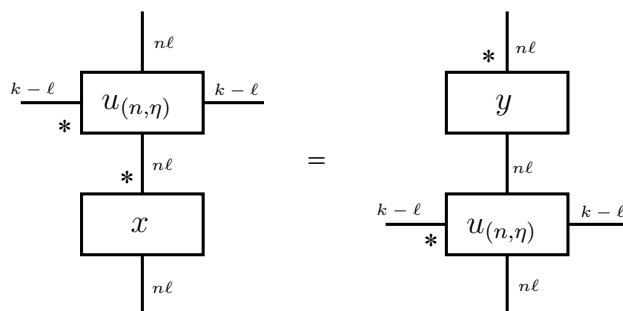


Figure 3.5: Relation between  $x \in P_{(nl,\epsilon\eta^\ell)}$  and  $y \in P_{(nl,\epsilon\eta^\ell(-)^{k-l})}$

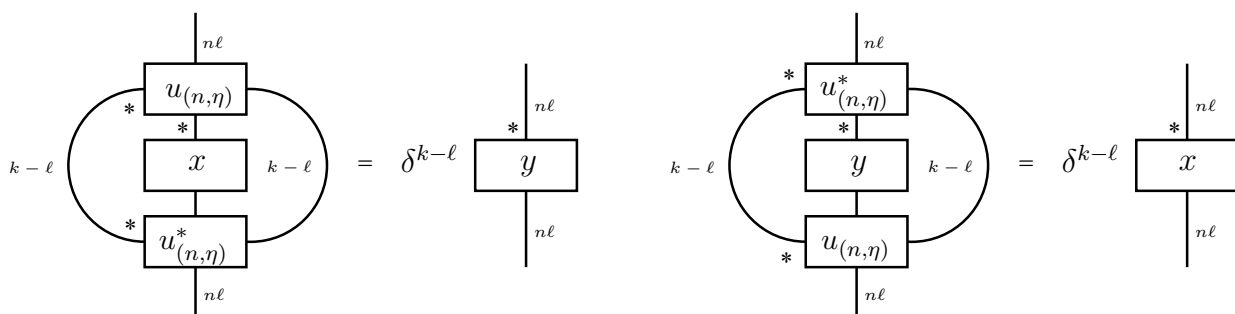


Figure 3.6: Expressions for  $x$  and  $y$  in terms of each other

The proof of Proposition 3.3.3 that we give here is an adaptation of the one in [7] with a few more details included. Our proof seems to show that the assumption of sphericity of  $P$  made there (or of a weakening) is not really required. We pave the way for the proof by defining and proving some properties of a map from  $P_{(nl,\epsilon\eta^\ell)}$  to  $P_{(nl+k-l,\epsilon\eta^\ell)}$ .

**Lemma 3.3.4.** (Lemma 24 of [13]). *The map  $\sigma : P_{(nl,\epsilon\eta^\ell)} \rightarrow P_{(nl+k-l,\epsilon\eta^\ell)}$  defined*



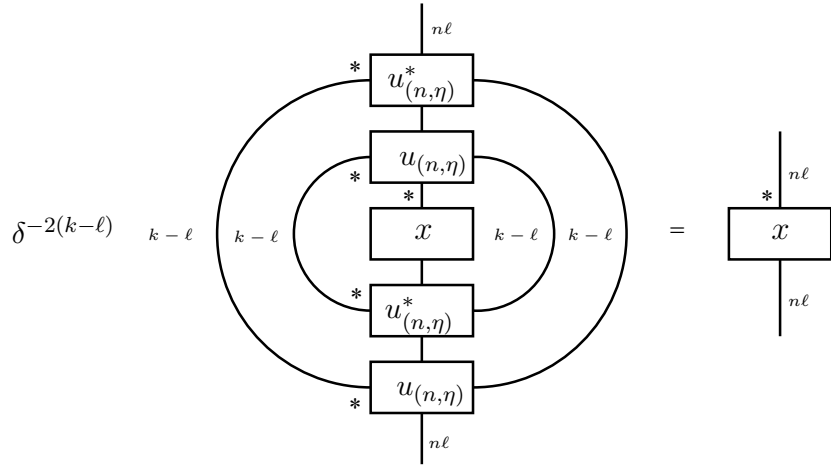


Figure 3.7: The double circle relation

as in the left of Figure 3.8 is an isometry with adjoint  $\sigma^*$  given as in the right of Figure 3.8. Further, both  $\sigma$  and  $\sigma^*$  are equivariant for the  $*$ -operations on  $P_{(nl, \epsilon\eta^\ell)}$

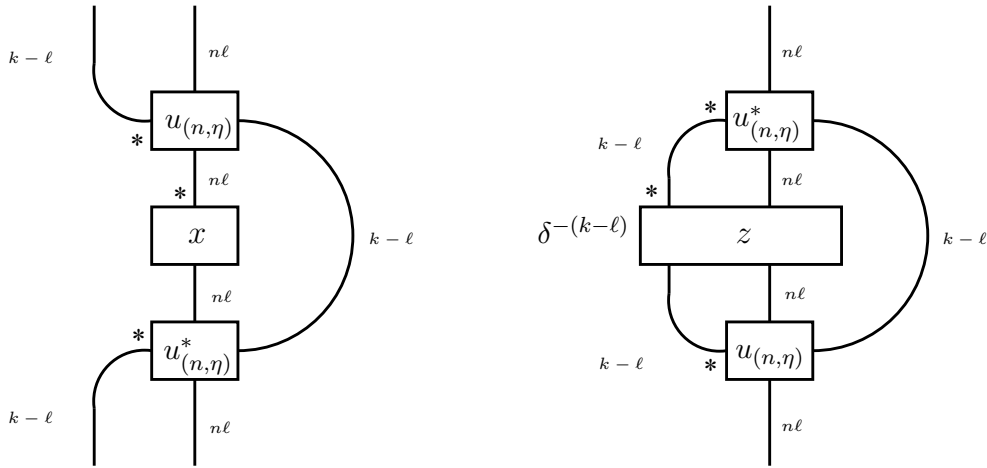


Figure 3.8:  $\sigma(x)$  and  $\sigma^*(z)$  for  $x \in P_{(nl, \epsilon\eta^\ell)}$  and  $z \in P_{(nl+k-l, \epsilon\eta^\ell)}$

and  $P_{(nl+k-l, \epsilon\eta^\ell)}$ .

*Proof.* Consider the maps  $\sigma$  and  $\sigma^*$  defined by the left and right side pictures in Figure 3.8 respectively. It is clear that they are equivariant for the  $*$ -operations on  $P_{(nl, \epsilon\eta^\ell)}$  and  $P_{(nl+k-l, \epsilon\eta^\ell)}$ . To show that they are actually adjoints of each other, it suffices to verify the equality of Figure 3.9 for arbitrary  $x \in P_{(nl, \epsilon\eta^\ell)}$  and  $z \in P_{(nl+k-l, \epsilon\eta^\ell)}$ . This is clear by isotopy. Finally, that  $\sigma^*\sigma = id$  is a simple pictorial verification using the equalities of Figure 3.1.  $\square$

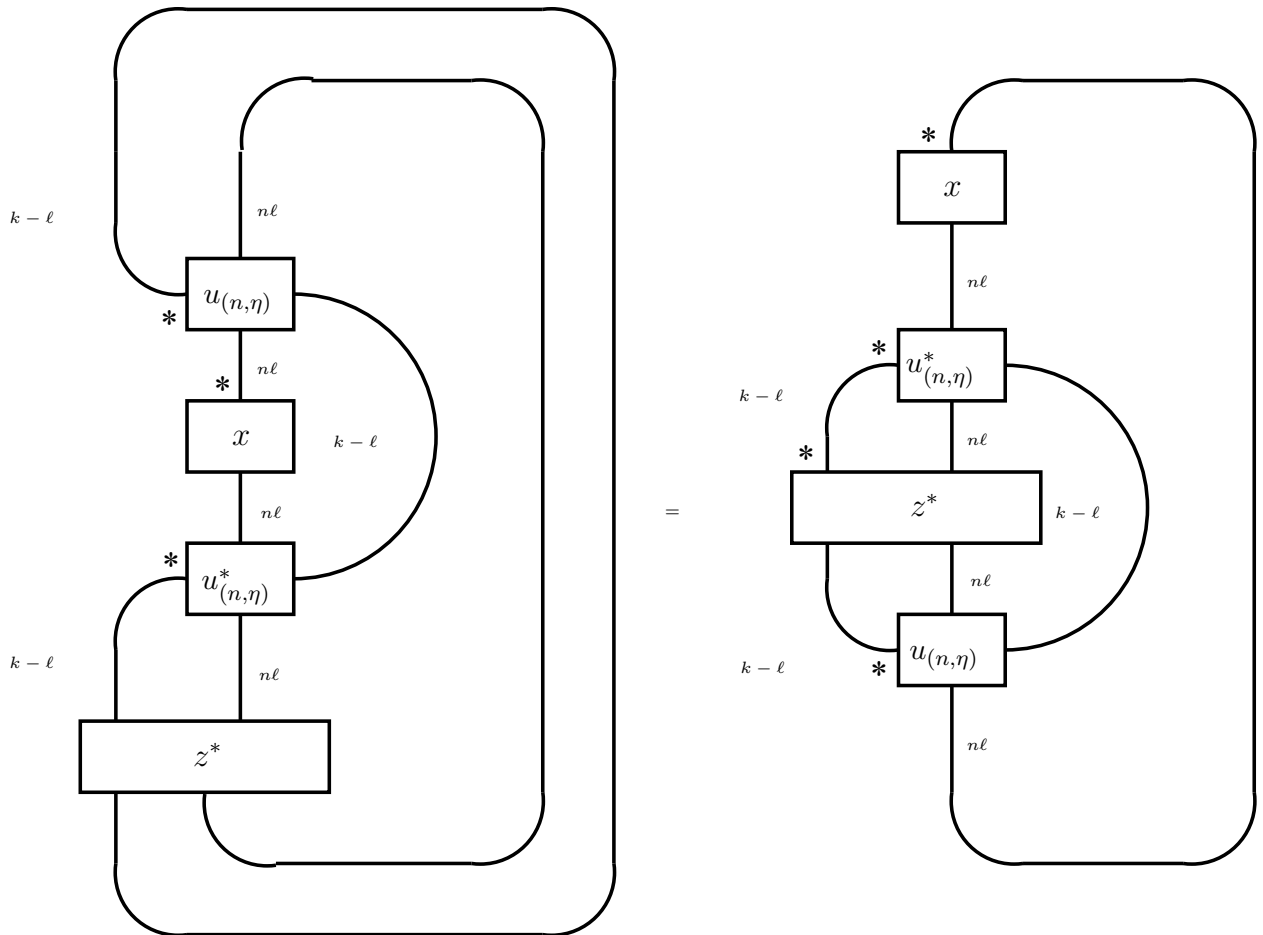


Figure 3.9: Equality to be verified

From Lemma 3.3.4 it follows that  $E = \sigma\sigma^*$  is a projection onto  $\text{ran}(\sigma) \subseteq P_{(nl+k-l, \epsilon\eta^\ell)}$  that is equivariant for the  $*$ -operations. Pictorially  $E$  is given by the picture on the left in Figure 3.10. We will also need the picture on the right in Figure 3.10 which is the projection onto the subspace  $P_{(k-l, nl+k-l, \epsilon\eta^\ell)}$  of  $P_{(nl+k-l, \epsilon\eta^\ell)}$ , which also is  $*$ -equivariant.

*Proof of Proposition 3.3.3.* (1)  $\Rightarrow$  (2) Using the relations in Figure 3.1, it is easy to see that the equation in Figure 3.5 implies those of Figure 3.6.

(2)  $\Rightarrow$  (3) This is clear.

(3)  $\Rightarrow$  (1) We need to see that the double circle relation of Figure 3.7 implies the existence of a  $y$  satisfying the equation in Figure 3.5.

Observe that the picture on the left in Figure 3.7 is given by  $\sigma^*F\sigma(x)$ . Thus

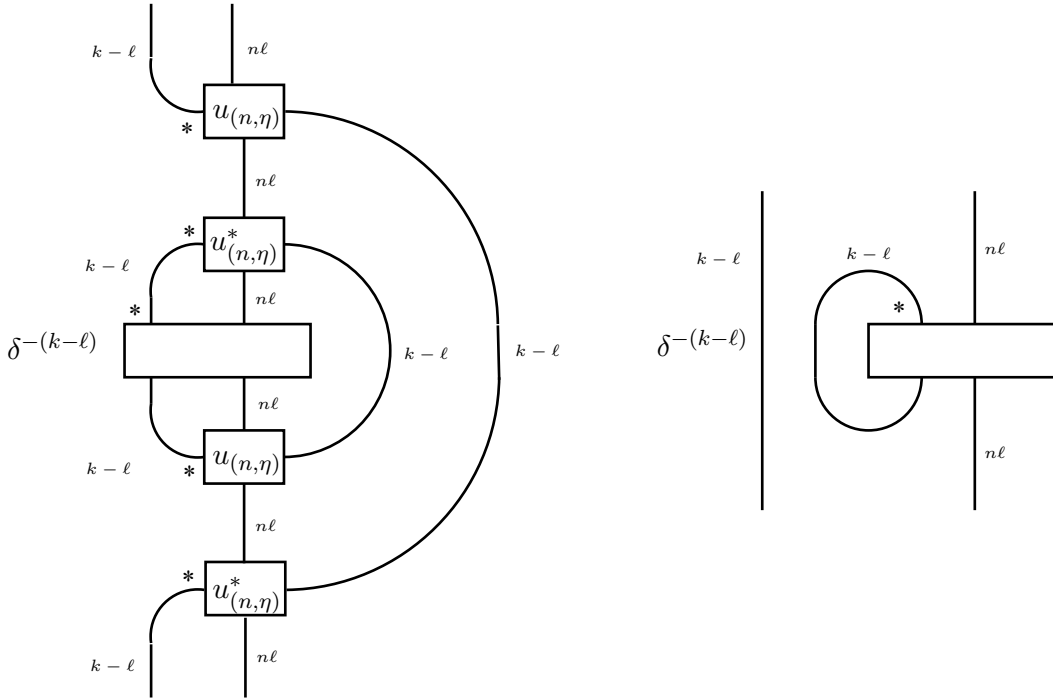


Figure 3.10: The projections  $E$  and  $F$

the double circle relation implies that  $\sigma^*F\sigma(x) = x$  and hence (applying  $\sigma$  on both sides and using the definition of  $E$ ) that  $EF\sigma(x) = \sigma(x)$ . Since  $E$  and  $F$  are projections, norm considerations imply that  $\|EF\sigma(x)\| \leq \|F\sigma(x)\| \leq \|\sigma(x)\|$ . Therefore equality holds throughout and so  $F\sigma(x) = \sigma(x)$ . Now define  $y$  by the first equality in Figure 3.6. The equation  $F\sigma(x) = \sigma(x)$  then implies that the equation on the left of Figure 3.11 holds and therefore also the equation on the right.

Finally, using the relations of Figure 3.1, this completes the proof.  $\square$

We now define subspaces  $Q_{(n, \eta)}$  of  ${}^{(\ell, \epsilon)}P_{(n, \eta)}$  by  $Q_{(n, \eta)} = \{x \in {}^{(\ell, \epsilon)}P_{(n, \eta)} = P_{(n\ell, \epsilon\eta^\ell)} : \text{The equivalent c}$

The main result of this section is the following theorem.

**Theorem 3.3.5.** (Theorem 26 of [13]). *The subspaces  $Q_{(n, \eta)}$  yield a  $C^*$ -planar subalgebra  $Q$  of  ${}^{(\ell, \epsilon)}P$ .*

*Proof.* In order to prove that  $Q$  is a  $C^*$ -planar subalgebra of  ${}^{(\ell, \epsilon)}P$  it is enough to prove that it is a planar subalgebra of  ${}^{(\ell, \epsilon)}P$  and that it is closed under  $*$ . Closure



$Z_{I^{(\ell,\epsilon)}}^P \in Q_{(n+1,\eta)}$ . Observe that the tangle  $I^{(\ell,\epsilon)}$  is the  $\ell$ -fold iterated inclusion tangle from  $P_{(n\ell,\epsilon\eta^\ell)}$  to  $P_{((n+1)\ell,\epsilon\eta^\ell)}$ . Suppose that  $y \in P_{(n\ell,\epsilon\eta^\ell(-)^{k-\ell})}$  is such that the equation in Figure 3.5 holds for  $x, y$ . Again, an easy verification shows that the equation in Figure 3.5 also holds for  $Z_{I^{(\ell,\epsilon)}}(x), Z_{\tilde{I}^{(\ell,\epsilon)}}(y)$ , where  $\tilde{I}^{(\ell,\epsilon)}$  is the  $\ell$ -fold iterated inclusion tangle from  $P_{(n\ell,\epsilon\eta^\ell(-)^{k-\ell})}$  to  $P_{((n+1)\ell,\epsilon\eta^\ell(-)^{k-\ell})}$ .

Closure under  $E = E_{(n+1,\eta)}^{(n,\eta)}$ : We need to see that if  $x \in Q_{(n+1,\eta)}$ , then  $Z_E^{((\ell,\epsilon)P)}(x) = Z_{E^{(\ell,\epsilon)}}^P(x) \in Q_{(n,\eta)}$ . Observe that the tangle  $E^{(\ell,\epsilon)}$  is the  $\ell$ -fold iterated conditional expectation tangle from  $P_{((n+1)\ell,\epsilon\eta^\ell)}$  to  $P_{(n\ell,\epsilon\eta^\ell)}$ . Take  $y \in P_{((n+1)\ell,\epsilon\eta^\ell(-)^{k-\ell})}$  such that the equation in Figure 3.5 holds for  $x, y$ . We will verify that then, the equation in Figure 3.5 also holds for  $Z_{E^{(\ell,\epsilon)}}(x), Z_{\tilde{E}^{(\ell,\epsilon)}}(y)$ , where  $\tilde{E}^{(\ell,\epsilon)}$  is the  $\ell$ -fold iterated conditional expectation tangle from  $P_{((n+1)\ell,\epsilon\eta^\ell(-)^{k-\ell})}$  to  $P_{(n\ell,\epsilon\eta^\ell(-)^{k-\ell})}$ .

First note that, it is an easy consequence of the relations in Figure 3.1 that the relations of Figure 3.12 hold for all  $(n, \eta)$ . Now, these relations, in turn, imply the

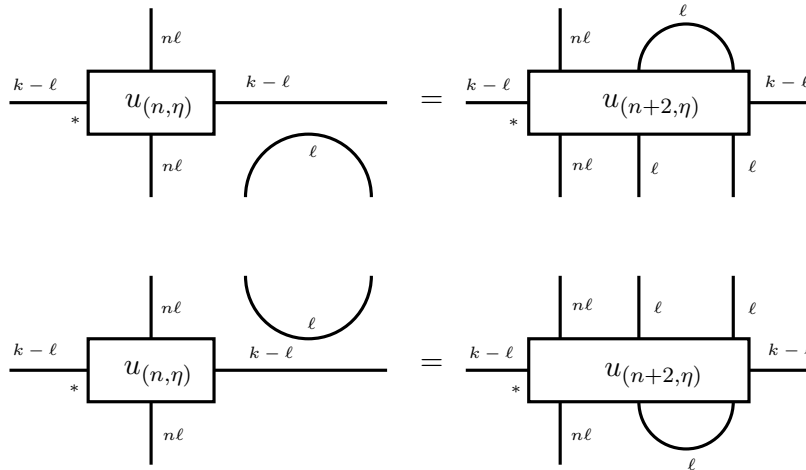


Figure 3.12: Consequence of  $\{0, \ell\}$ -biunitarity relations

equations in Figure 3.13. In this figure, the first and the third equalities follow from Figure 3.12 while the second equality is a consequence of the proof of closure under  $I$ .

Closure under  $R = R_{(n,\eta)}^{(n,-\eta)}$ : We need to see that if  $x \in Q_{(n,\eta)}$ , then  $Z_R^{((\ell,\epsilon)P)}(x) = Z_{R^{(\ell,\epsilon)}}^P(x) \in Q_{(n,-\eta)}$ . We illustrate how this is done when  $(n, \eta) = (3, +)$ . It should

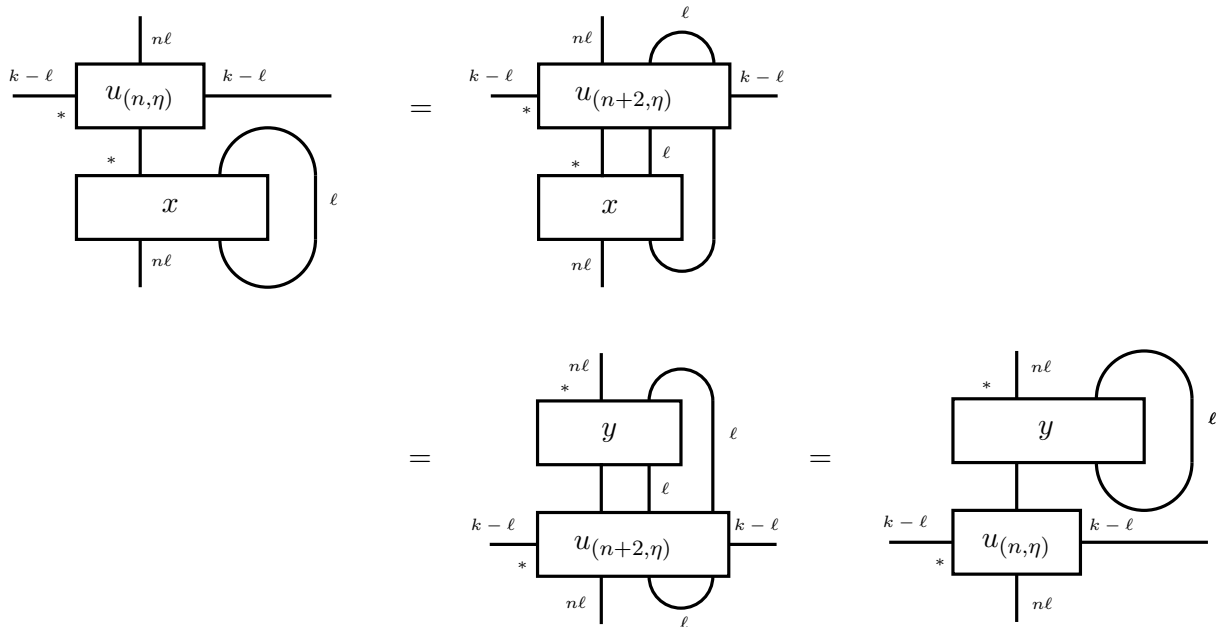


Figure 3.13: Proof of closure under  $E$

be clear that the proof of the general case is similar. Begin by observing that since  $x \in Q_{(3,+)} \subseteq P_{(3\ell,\epsilon)}$ , it satisfies the double circle relation of Figure 3.14.

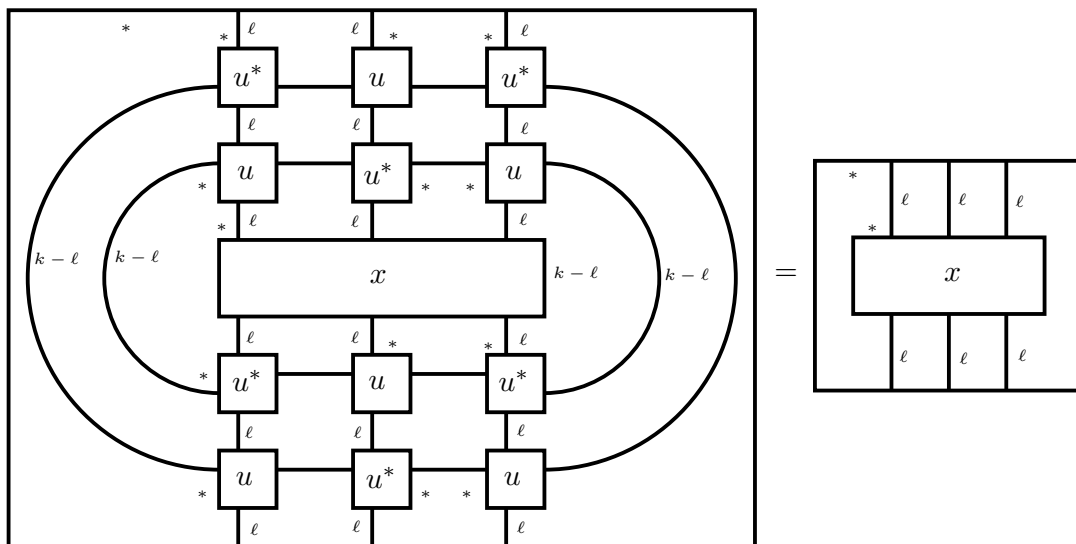


Figure 3.14: Double circle relation for  $x \in Q_{(3,+)}$

Now, moving the external  $* \ell$ -steps counterclockwise and redrawing yields the equation in Figure 3.15.

A little thought now shows that this is precisely the double circle relation for  $Z_{R(\ell,\epsilon)}^P(x) \in P_{(3\ell,\epsilon(-)\ell)}$ , establishing that  $Z_{R(\ell,\epsilon)}^P(x)$  indeed belongs to  $Q_{(3,-)}$  as desired.

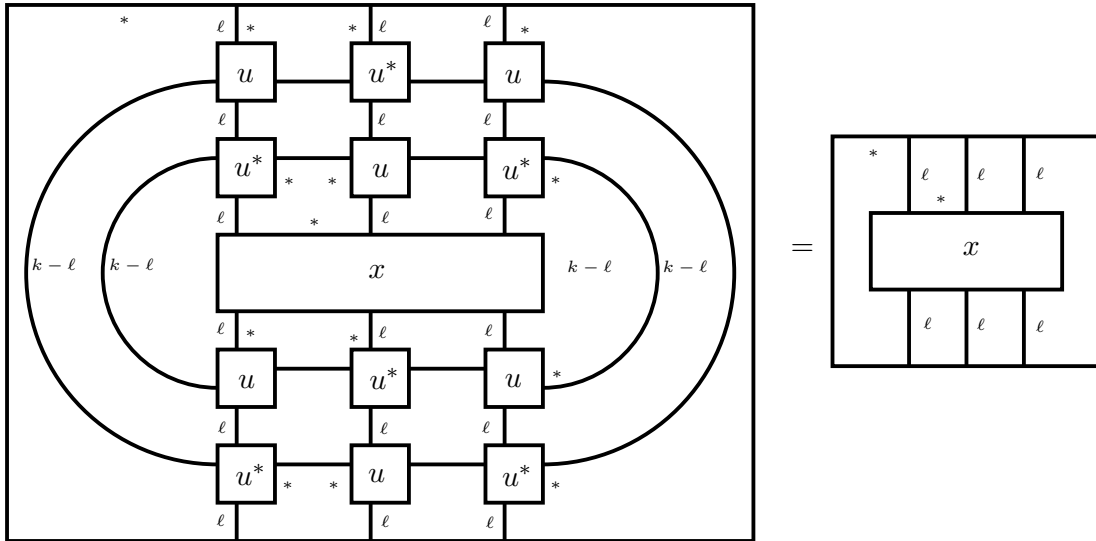


Figure 3.15: Rotated double circle relation for  $x \in Q_{(3,+)}$

□

Next we will consider conditions under which  $Q$  is a subfactor planar algebra.

**Proposition 3.3.6.** (Proposition 26 of [13]). *Let  $P$  be the spin planar algebra on  $n$  generators and  $Q$  be the planar subalgebra of  $^{(\ell,\epsilon)}P$  corresponding to a  $\{0, \ell\}$ -biunitary element  $u \in P_{(k,\epsilon)}$ . Then,  $Q$  is a subfactor planar algebra with modulus  $(\sqrt{n})^\ell$ .*

*Proof.* Given Theorem 3.3.5, what remains to be seen is that  $Q$  is connected, has modulus  $(\sqrt{n})^\ell$  and is spherical with positive definite picture trace. Since  $P$  has modulus  $\sqrt{n}$ , the cabling  $^{(\ell,\epsilon)}P$  has modulus  $(\sqrt{n})^\ell$  and so does  $Q$ . The other assertions need a little work.

Note that  $Q_{(0,\eta)}$  is a subspace of  $^{(\ell,\epsilon)}P_{(0,\eta)} = P_{(0,\epsilon\eta^\ell)}$  and so if  $\epsilon\eta^\ell = +$ , then  $Q_{(0,\eta)}$  is 1-dimensional since  $P_{(0,+)}$  is so. If  $\epsilon\eta^\ell = -$ , then,  $Q_{(0,\eta)}$  is the subspace of all  $x \in P_{(0,-)}$  such that the double circle relation of Figure 3.7 holds for  $x$ . From Theorem 2.4.2, a basis of  $P_{(0,-)}$  is given by all  $S(i)$  for  $i = 1, \dots, n$  and by the black and white modulus relations, a double circle around these gives a scalar multiple of  $1^{(0,-)}$ . It follows that  $x$  is necessarily a scalar multiple of  $1^{(0,-)}$  so that  $Q_{(0,\eta)}$  is 1-dimensional, in this case as well. Hence  $Q$  is connected.

To see that  $Q$  is spherical, observe first that on any  $P_{(n,\eta)}$  the composites of the left and right picture traces with the traces  $\tau_{\pm}$  on  $P_{(0,\pm)}$  (which specify its  $C^*$ -planar algebra structure - see Definition 2.2.8 ) are equal. This is seen by explicit computation with the bases of  $P_{(n,\eta)}$  and can be regarded as a version of sphericity for  $P$ . It is clear that this property descends to  $Q$ .

Finally observe that the picture trace on  $Q_{(n,\eta)}$  is exactly the composite of  $\tau_{\pm}$  with the picture trace on  $P_{(n\ell,\epsilon\eta^\ell)}$  and is consequently positive definite.  $\square$

**Remark 3.3.7.** *An even easier proof than that of Proposition 3.3.6 shows that if  $P$  is a subfactor planar algebra and  $Q$  is the planar subalgebra of  $^{(\ell,\epsilon)}P$  corresponding to a  $\{0,\ell\}$ -biunitary element  $u \in P_{(k,\epsilon)}$ , then,  $Q$  is a subfactor planar algebra with modulus  $(\sqrt{n})^\ell$ .*

In case  $\ell = 1$ , the planar algebra  $Q$  is even irreducible.

**Proposition 3.3.8.** *(Proposition 29 of [13]). Let  $P$  be the spin planar algebra on  $n$  generators and  $Q$  be the planar subalgebra of  $^{(1,\epsilon)}P$  corresponding to a  $\{0,1\}$ -biunitary element  $u \in P_{(k,\epsilon)}$ . Then,  $Q$  is an irreducible subfactor planar algebra.*

*Proof.* Only the irreducibility of  $Q$  needs to be seen and we will show using explicit bases computations that  $\dim(Q_{(1,+)}) = 1$ . We only consider the  $\epsilon = +$  case, the proof in the other case being similar. Thus  $Q_{(1,+)} \subseteq P_{(1,+)}$ . Begin with  $x = \sum_i \lambda_i e(i) \in Q_{(1,+)}$ . The double circle relation for  $x$  implies that the equation of Figure 3.16 holds.

Now, using the biunitarity relations of Figure 3.1 together with the black and white modulus relations, the left hand side of Figure 3.16 simplifies to  $\frac{1}{n} \sum_i \lambda_i 1_{(1,+)}$ , finishing the proof.  $\square$

**Remark 3.3.9.** *The proof of Theorem 3.3.5 relies heavily on Proposition 3.3.3, and in particular, the double circle relation, which uses the assumption that  $P$  is a*



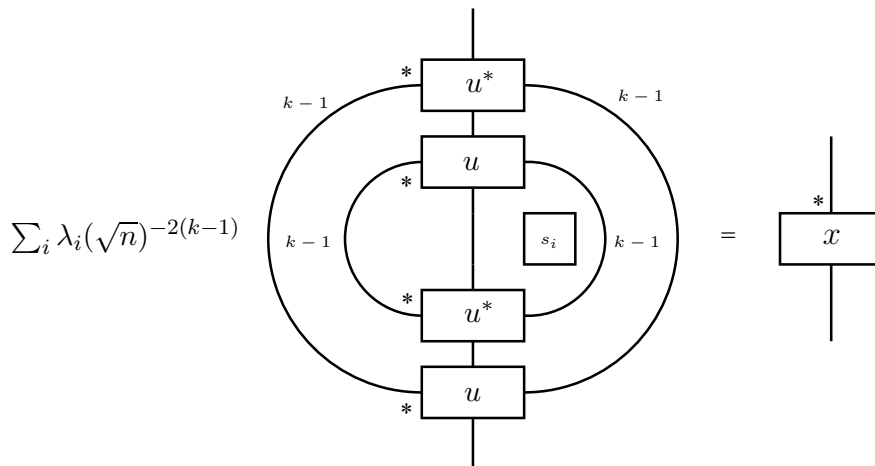


Figure 3.16: Equation satisfied by  $x$

spherical  $C^*$ -planar algebra. However, even if  $P$  is just a  $*$ -planar algebra, without the positivity conditions or sphericity holding,  $Q$  can still be shown to be a  $*$ -planar subalgebra of  $P$ . The proof is a little longer using a different larger set of generating tangles.

### 3.4 The planar algebra associated to a finite group

#### Latin square

The most familiar example of a Latin square is the multiplication table of a finite group and any Latin square gives a quantum Latin square as mentioned in Example 3.1.9 . It was shown in Theorem 3.2.1 that quantum Latin squares corresponds to  $\{0, 1\}$ -biunitary elements in  $P_{(3,+)}$ , where  $P$  is the spin planar algebra. Now, by Proposition 3.3.8 we have an irreducible subfactor planar subalgebra of the spin planar algebra. In this section we will show this planar algebra is nothing but the group planar algebra  $P(G)$  , see [17]. A more general construction of subfactor planar algebras from any Latin square is described in the next chapter.

We begin with a notational convention. The generating set for the spin planar algebra is the underlying set of the group  $G$  and we will use notation such as

$e[g]_{k_1 \dots k_m}^{h_1 \dots h_m}(\ell)$  for  $g, h_i, k_j, \ell \in G$  to denote basis elements of  $P$ . With this notation, the  $\{0, 1\}$ -biunitary element  $u \in P_{(3,+)}$  corresponding to the multiplication table of the group  $G$  is seen to be given by

$$u = \sum_{h,k} e_k^{kh}(h) \in P_{(3,+)},$$

according to Theorem 3.2.1(2).

Let  $Q$  be the planar subalgebra of  $P$  corresponding to the biunitary element  $u$  as in Proposition 3.3.8 . The next proposition identifies  $Q$  with  $P(G)$ . First let us recall the presentation of the group planar algebra  $P(G)$  in terms of generators and relations, see [15]. That is,  $P(G)$  is defined to be the planar algebra  $P(L, R)$  where

$$L_{(k,\epsilon)} = \begin{cases} G, & \text{if } (k, \epsilon) = (2, +) \\ \emptyset, & \text{otherwise} \end{cases}$$

and  $R$  being given by the set of relations in Figures 3.17 - 3.21 .

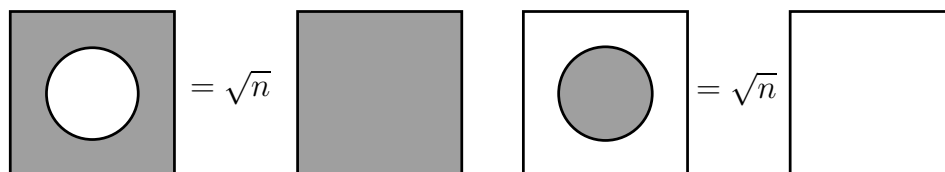


Figure 3.17: Modulus relations

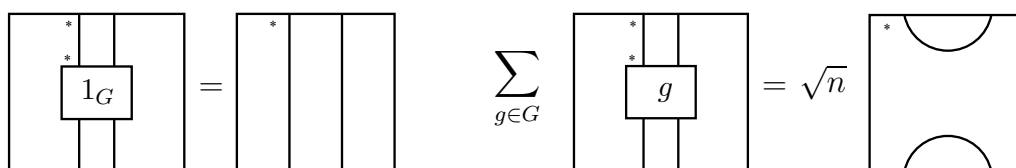


Figure 3.18: Unit and Integral relation

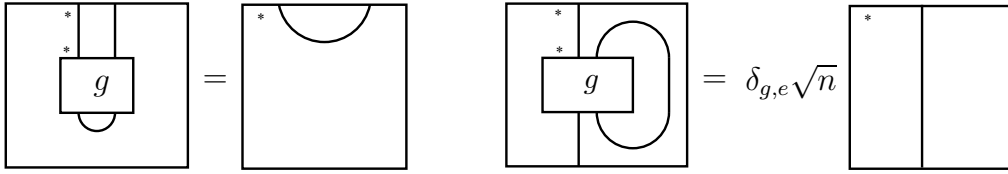


Figure 3.19: Counit and Trace relations

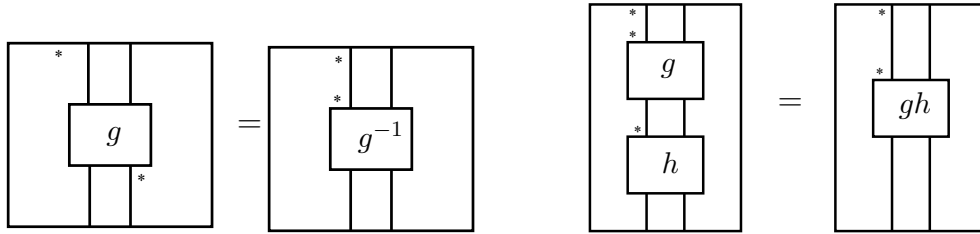


Figure 3.20: Antipode and Multiplication relations

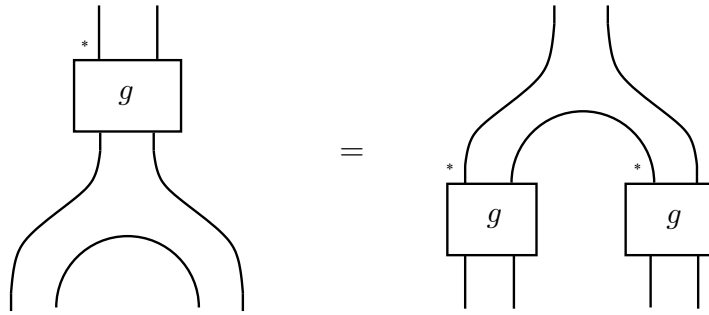


Figure 3.21: Comultiplication relation

**Proposition 3.4.1.** (Proposition 31 of [13]). *The planar algebra  $Q$  is isomorphic to  $P(G)$ .*

*Proof.* We will prove the proposition in a series of steps - (1) computing the dimensions of the spaces of  $Q$  and observing that these are equal to those of the spaces of  $P(G)$ , (2) by specifying a map from the universal planar algebra on  $L = L_{(2,+)} = G$  to  $Q$  and checking that the relations hold in  $Q$ , thereby yielding a planar algebra map from  $P(G)$  to  $Q$  and (3) verifying that this map is surjective.

Step 1: We first observe that for  $m \geq 1$ , the elements  $u_{(2m,+)}$  and  $u_{(2m+1,+)}$  are

given explicitly by:

$$\begin{aligned} u_{(2m,+)} &= \sum_{g,h_1,\dots,h_m} e_{gh_1,gh_2,\dots,gh_m,g}^{g,h_1^{-1},\dots,h_m^{-1}} \\ u_{(2m+1,+)} &= \sum_{g,h_1,\dots,h_{m+1}} e_{gh_1,gh_2,\dots,gh_{m+1}}^{g,h_1^{-1},\dots,h_m^{-1}} (h_{m+1}^{-1}] \end{aligned}$$

Now consider elements  $x, y \in P_{(2m,+)}$  given by:

$$\begin{aligned} x &= \sum_{k_1,\dots,k_m,\ell_1,\dots,\ell_m} \alpha_{(k_1,\dots,k_m,\ell_1,\dots,\ell_m)} e_{\ell_1,\dots,\ell_m}^{k_1,\dots,k_m} \\ y &= \sum_{k_1,\dots,k_m,\ell_1,\dots,\ell_m} \beta_{(k_1,\dots,k_m,\ell_1,\dots,\ell_m)} e_{\ell_1,\dots,\ell_m}^{k_1,\dots,k_m}, \end{aligned}$$

for  $\alpha_{(k_1,\dots,k_m,\ell_1,\dots,\ell_m)}, \beta_{(k_1,\dots,k_m,\ell_1,\dots,\ell_m)} \in \mathbb{C}$ . The condition that  $x, y$  satisfy the condition in Figure 3.5 is seen to imply that for all  $g \in G$ ,

$$\alpha_{(k_1,\dots,k_m,\ell_1,\dots,\ell_m)} = \alpha_{(gk_1,\dots,gk_m,g\ell_1,\dots,g\ell_m)}.$$

Conversely, if this condition holds, setting  $\beta_{(k_1,\dots,k_m,\ell_1,\dots,\ell_m)} = \alpha_{(k_1^{-1},\dots,k_m^{-1},\ell_1^{-1},\dots,\ell_m^{-1})}$ , the elements  $x$  and  $y$  are checked satisfy the condition in Figure 3.5.

Thus, a basis of  $Q_{(2m,+)}$  is given by the set of all

$$\sum_{g \in G} e_{g\ell_1,\dots,g\ell_m}^{gk_1,\dots,gk_m}$$

as  $(k_1, \dots, k_m, \ell_1, \dots, \ell_m)$  vary over the representatives of the diagonal action of  $G$  on  $G^{2m}$ . It follows that the dimension of  $Q_{(2m,+)}$  is given by  $n^{2m-1}$  and a similar proof shows that the dimension of  $Q_{(2m+1,+)}$  is given by  $n^{2m}$ .

Step 2: Next we define a map from the universal planar algebra on the label set

$L = L_{(2,+)} = G$  to  $Q$  given by sending  $g \in G$  to

$$X_g = \sum_{q \in G} e_{gq}^q \in Q_{(2,+)}.$$

It is a long but routine verification that all relations satisfied by the  $g \in G$  in the planar algebra  $P(G)$  also hold for their images  $X_g$  in  $Q$  thus giving a planar algebra map from  $P(G)$  to  $Q$ . We will do the verification of few relations omitting the rest. The relation in Figure 3.17 holds in the spin planar algebra by the definition itself and so descends to  $Q$ .

Let us consider the unit relation in Figure 3.18 . By the definition of the map, LHS goes to  $X_e = \sum_q e_q^q$  and it equals the image of the RHS by the black channel relation (see Figure 2.8) in the spin planar algebra, thus verifying the unit relation. The verification of counit relation in Figure 3.19 is just an application of the map and the unit relation given in Figure 2.9. We omit the remaining verifications.

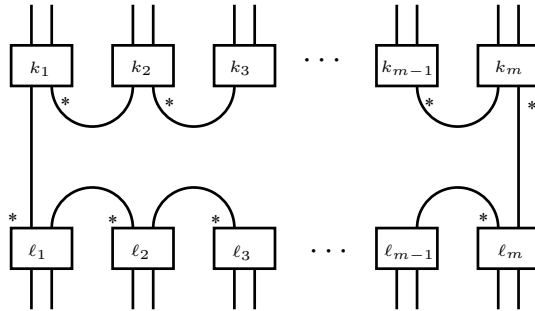


Figure 3.22: An element of  $P_{(2m,+)}$

Step 3: To verify that  $P(G) \rightarrow Q$  is surjective, it suffices to see that  $P(G)_{(2m,+)} \rightarrow Q_{(2m,+)}$  is surjective for all  $m > 0$ . A routine calculation shows that the element in Figure 3.22 goes to the basis element  $\sum_{g \in G} e_{g\ell_1, \dots, g\ell_m}^{gk_1, \dots, gk_m}$  of  $Q_{(2m,+)}$ , finishing the proof.  $\square$



# Chapter 4

## Planar algebras and Latin squares

This chapter is devoted to examining the subfactor planar algebra associated to a general Latin square. We have already seen in Section 3.4 that when the Latin square is the multiplication table of a finite group  $G$ , then the associated subfactor planar algebra is nothing but the group planar algebra  $P(G)$ . Our main result shows that the subfactor planar algebra associated to a Latin square is the one associated to a certain subgroup-group pair which is constructed explicitly from the given Latin square.

Recall first that a Latin square is an  $n \times n$  array filled with  $n$  different symbols, usually from a set  $S = \{s_1, s_2, \dots, s_n\}$ , with each occurring exactly once in every row and in every column, see Definition 3.1.6. A quantum Latin square of size  $n$  is an  $n \times n$  matrix of vectors in  $\mathbb{C}^n$  such that each row and each column is an orthonormal basis for  $\mathbb{C}^n$ , see 3.1.8 and any Latin square gives a quantum Latin square in a natural way, see 3.1.9. Recall also the presentation of the spin planar algebra  $P = P(S)$ ,  $S = \{s_1, \dots, s_n\}$  described in Section 2.4.

**Remark 4.0.1.** *Let us choose the symbols for the Latin square from the finite set  $S = \{s_1, \dots, s_n\}$ . Now if the  $(j, k)^{\text{th}}$  entry of a Latin square  $L$  is  $s_i$ , then we write  $s_i = s_j s_k$ . This makes  $L$  the multiplication table of a (not necessarily associative) binary*

operation on the set  $S$  which will be simply denoted by concatenation, appropriately bracketed.

**Definition 4.0.2.** Let  $L$  be as above and  $x, y \in S$ . Define,

1.  $x^{-1}y$  to be the unique element  $a \in S$  such that  $xa = y$ .
2.  $xy^{-1}$  to be the unique element  $b \in S$  such that  $by = x$ .

Now Theorem 3.2.1(2) implies that there is a natural 1-1 correspondence between  $\{0, 1\}$ -biunitary elements in  $P_{(3,+)}$  and quantum Latin squares of size  $n \times n$ . Let  $u$  denote the biunitary element associated to  $L$ .

Define the elements  $u_{(n,\eta)} \in P_{(n+2,\eta)}$  as in Figure 3.4 - where the label in the last box is  $u$  or  $u^*$  depending on the parity of  $n$ .

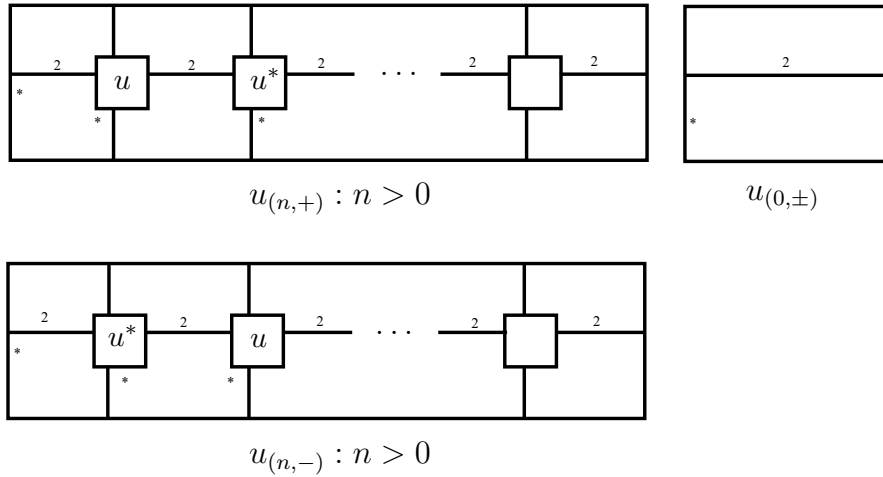


Figure 4.1: The elements  $u_{(n,\pm)}$

We now define subspaces  $Q_{(n,\eta)}$  of  $P_{(n,\eta)}$  by  $Q_{(n,\eta)} = \{x \in P_{(n,\eta)} : \text{there exists } y \in P_{((n,\eta))} \text{ satisfying the pictorial relation of Figure 4.2}\}$ .

With the foregoing notations, Proposition 3.3.8 implies that the subspaces  $Q_{(n,\eta)}$  yield a planar subalgebra  $Q$  of  $P$  which is an irreducible subfactor planar algebra.



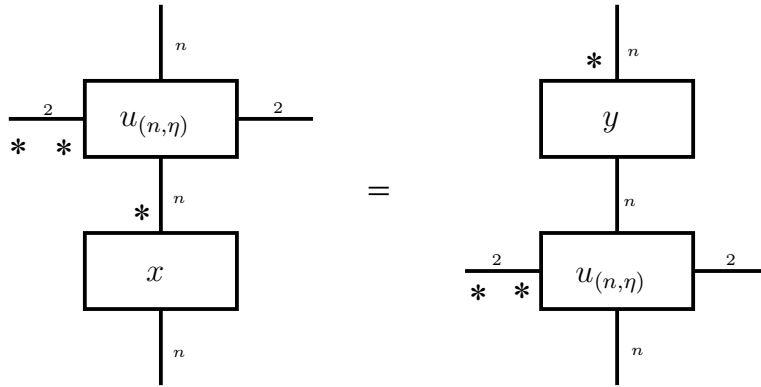


Figure 4.2: Defining relation of elements of  $Q_{(n,\eta)}$

## 4.1 Main theorem

Throughout this section, let  $L$  be a Latin square of size  $n$  and  $P = P(S)$  be the spin planar algebra on the set  $S = \{s_1, \dots, s_n\}$ . We will regard  $L$  as the multiplication table of a binary operation on the set  $S$  as described in Remark 4.0.1.

Associated to  $L$  is a  $\{0, 1\}$ -biunitary element  $u \in P_{(3,+)}$  given by  $u = \sum_{i,j,k} a_{ij}^k e_k^i(j)$  where

$$a_{ij}^k = \begin{cases} 1 & \text{if } s_i = s_k s_j, \\ 0 & \text{otherwise.} \end{cases}$$

The biunitary element  $u$  determines a planar subalgebra  $Q = P(L)$  of  $P$  as in Theorem 3.3.5. Our aim is to show that  $Q$  is a subgroup-group planar algebra, which is the planar algebra of the subgroup-subfactor  $R \rtimes G \subset R \rtimes H$ , for a pair of subgroups  $H \subseteq G$  and  $R$  is a hyperfinite  $II_1$  factor, see [4].

We will first use the Latin square  $L$  to define a pair of subgroups of the symmetric group  $S_n$ . To do this, we begin by defining an equivalence relation  $\sim$  on the  $m$ -fold Cartesian product  $S^{\times m}$  for any  $m \in \mathbb{N}$ . The relation  $\sim$  is defined to be the transitive closure of the reflexive, symmetric relation  $\approx$  on  $S^{\times m}$  defined by  $(a_1, \dots, a_m) \approx$

$(a'_1, \dots, a'_m)$  if there exists  $(t_1, \dots, t_m) \in S^{\times m}$  such that

$$\begin{aligned} a_1 t_1 &= a_2 t_2 = \dots = a_m t_m \text{ and} \\ a'_1 t_1 &= a'_2 t_2 = \dots = a'_m t_m. \end{aligned}$$

In the sequel, the following observations about the relations  $\sim$  and  $\approx$  will be used freely.

**Lemma 4.1.1.** (Lemma 10 of [11]). *Let  $\bowtie$  denote either of the relations  $\sim$  or  $\approx$  on  $S^{\times m}$ . Suppose that  $(a_1, \dots, a_m) \bowtie (a'_1, \dots, a'_m)$ . Then*

1.  $a_i = a_j$  iff  $a'_i = a'_j$ .
2. For any permutation  $\rho \in S_m$ ,  $(a_{\rho(1)}, \dots, a_{\rho(m)}) \bowtie (a'_{\rho(1)}, \dots, a'_{\rho(m)})$ .
3. For any  $k$ ,  $(b_1, \dots, b_k) \bowtie (b'_1, \dots, b'_k)$  where each  $b_i$  is some  $a_j$  and  $b'_i$  is the corresponding  $a'_j$ .

*Proof.* We will prove the above assertions when  $(a_1, \dots, a_m) \approx (a'_1, \dots, a'_m)$  and the case where  $(a_1, \dots, a_m) \sim (a'_1, \dots, a'_m)$  will follow easily.

(1) By definition there exists  $(t_1, \dots, t_m) \in S^{\times m}$  such that

$$\begin{aligned} a_1 t_1 &= a_2 t_2 = \dots = a_m t_m \text{ and} \\ a'_1 t_1 &= a'_2 t_2 = \dots = a'_m t_m. \end{aligned}$$

Now,

$$\begin{aligned} a_i = a_j &\Leftrightarrow a_i t_i = a_j t_j \\ &\Leftrightarrow t_i = t_j \text{ (Latin square condition)} \\ &\Leftrightarrow a'_i t_i = a'_j t_i \\ &\Leftrightarrow a'_i = a'_j \text{ (Latin square condition)} \end{aligned}$$

(2) It follows directly from the definition of  $\approx$  by taking the tuple  $(t_{\rho(1)}, \dots, t_{\rho(m)})$ .

(3) Since  $(a_1, \dots, a_m) \approx (a'_1, \dots, a'_m)$ , we have  $(a_1, \dots, a_k) \approx (a'_1, \dots, a'_k)$  for any  $k \leq m$  and (1) and (3) together implies (2). Now for  $k > m$ , some  $a_i$ 's are repeating and so  $a'_i$  are also repeating by (1) and this can be reduced to the case with  $k = k' \leq m$  and so the result follows.  $\square$

Now, let  $G = \{\sigma \in S_n : (s_1, s_2, \dots, s_n) \sim (s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(n)})\}$ .

**Lemma 4.1.2.** (*Lemma 11 of [11]*).  *$G$  is a subgroup of  $S_n$  and the restricted action of  $G$  on  $\{1, 2, \dots, n\}$  is transitive.*

*Proof.* Suppose that  $\sigma \in G$ . Then,  $(s_1, \dots, s_n) \sim (s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(n)})$ . Hence by Lemma 4.1.1(3), for any  $\rho \in S_n$ ,  $(s_{\rho(1)}, \dots, s_{\rho(n)}) \sim (s_{\sigma\rho(1)}, \dots, s_{\sigma\rho(n)})$ . If further  $\rho \in G$ , then  $(s_1, \dots, s_n) \sim (s_{\rho(1)}, \dots, s_{\rho(n)})$  and so by transitivity  $(s_1, \dots, s_n) \sim (s_{\sigma\rho(1)}, \dots, s_{\sigma\rho(n)})$ . Hence  $\sigma\rho \in G$  showing that  $G$  is a group.

To see that  $G$  acts transitively on  $\{1, 2, \dots, n\}$ , we need to see that given any  $k \in \{1, 2, \dots, n\}$  there is an element of  $S^{\times n}$  of the form  $(s_k, u_2, \dots, u_n)$  with all these being distinct such that  $(s_1, s_2, \dots, s_n) \sim (s_k, u_2, \dots, u_n)$ . To do this, first choose a  $t_1 \in S$  arbitrarily. Then solve the equations  $s_1 t_1 = s_2 t_2 = \dots = s_n t_n$  for  $t_2, \dots, t_n$ . Because of the Latin square condition, all  $t_i$ 's exist and are distinct. Now solve  $s_k t_1 = u_2 t_2 = \dots = u_n t_n$  for  $u_2, \dots, u_n$ . Again, by the Latin square condition, all of  $u_2, \dots, u_n$  exist and are distinct and different from  $s_k$ .  $\square$

We regard  $S_{n-1}$  as the subgroup of  $S_n$  of permutations fixing  $n$  and define  $H = G \cap S_{n-1}$ . We will refer to the pair  $H \subseteq G$  as the associated subgroup-group pair of the Latin square  $L$  and show that a planar algebra associated to this pair is isomorphic to  $Q$ . Note that since  $H$  is the stabilizer of  $n$  for the restricted (transitive) action of  $G$  on  $\{1, 2, \dots, n\}$ , the coset space  $G/H$  as a  $G$ -set may be identified with  $\{s_1, s_2, \dots, s_n\}$  with the natural restricted action of  $G$ .

The following example illustrates the explicit computation of this subgroup-group pair in a  $5 \times 5$  Latin square.

**Example 4.1.3.** *Consider the smallest non-group type Latin square. This is of size 5 and is given by the array below. This is taken from the combinatorial data page maintained by McKay - see [18].*

$$\begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ s_2 & s_4 & s_1 & s_5 & s_3 \\ s_3 & s_5 & s_2 & s_1 & s_4 \\ s_4 & s_3 & s_5 & s_2 & s_1 \\ s_5 & s_1 & s_4 & s_3 & s_2 \end{bmatrix}$$

We will compute the associated subgroup-group pair for this Latin square. First we observe from the square above that

$$s_1s_1 = s_2s_3 = s_3s_4 = s_4s_5 = s_5s_2 = s_1.$$

Next observe that

$$s_2s_1 = s_3s_3 = s_4s_4 = s_5s_5 = s_1s_2 = s_2, \quad \text{and}$$

$$s_4s_1 = s_5s_3 = s_1s_4 = s_3s_5 = s_2s_2 = s_4.$$

By definition then,  $(s_1, s_2, s_3, s_4, s_5)$  is  $\approx$  related to  $(s_2, s_3, s_4, s_5, s_1)$  and  $(s_4, s_5, s_1, s_3, s_2)$ .

This says that the associated group, say  $G$ , of this Latin square is a subgroup of  $S_5$  containing the permutations (in cycle notation)  $(1\ 2\ 3\ 4\ 5)$  and  $(1\ 4\ 3)(2\ 5)$ . Since these generate  $S_5$ , it follows that  $G = S_5$  and hence  $H = S_4$ . Explicit calculation with this example using the classification of subfactors of index 5 in [8] was the initial motivation for this paper.

We have seen in Proposition 2.4.5 that the spin planar algebra  $P = P(S)$  is isomorphic to the planar algebra of the bipartite graph given in Figure 2.13. The main result of [4] shows that the subgroup-group planar algebra associated to the subfactor  $R^G \subseteq R^H$  is the invariant planar algebra for the  $G$  action on the spin planar algebra  $P(S)$ , see Corollary 5.16 in [4]. We will denote this planar algebra by  $\tilde{Q} = P(H \subseteq G)$ . It is this which we need to see is isomorphic to  $Q$ . The way we have set things up, we will show that  $\tilde{Q} = Q$  as planar subalgebras of  $P$ . The main computational proposition is the following explicit identification of  $Q_{(2m,+)}$  as a subset of  $P_{(2m,+)}$ . We will use the following notation. An element

$$\sum_{a_1, \dots, a_{2m}} \alpha_{a_1, \dots, a_{2m}} e_{a_{m+1}, \dots, a_{2m}}^{a_1, \dots, a_m}$$

of  $P_{(2m,+)}$  is said to respect  $\sim$  if  $\alpha_{a_1, \dots, a_{2m}} = \alpha_{a'_1, \dots, a'_{2m}}$  whenever  $(a_1, \dots, a_{2m}) \sim (a'_1, \dots, a'_{2m})$  or equivalently, whenever  $(a_1, \dots, a_{2m}) \approx (a'_1, \dots, a'_{2m})$ .

We will also require the explicit identification of  $u_{(2m,+)}$  in terms of the standard basis of  $P_{(2m,+)}$ . Computing in the planar algebra  $P$  using the relations, we see that

$$u_{(2m,+)} = \sum_{p, k_1, k_2, \dots, k_m} e_{k_1, \dots, k_m, p}^{p, k_1^{-1}p, \dots, k_m^{-1}p}.$$

**Proposition 4.1.4.** (*Proposition 13 of [11]*).  $Q_{(2m,+)}$  is the set of elements of  $P_{(2m,+)}$  that respect  $\sim$ .

*Proof.* By definition  $Q_{(2m,+)} = \{x \in P_{(2m,+)} : \text{there exists } y \in P_{(2m,+)} \text{ satisfying the pictorial relation of Figure 4.3}\}$ .

In the rest of the proof,  $x$  will always denote  $\sum_{a_1, \dots, a_{2m}} \alpha_{a_1, \dots, a_{2m}} e_{a_{m+1}, \dots, a_{2m}}^{a_1, \dots, a_m}$  while  $y$  will denote  $\sum_{b_1, \dots, b_{2m}} \beta_{b_1, \dots, b_{2m}} e_{b_{m+1}, \dots, b_{2m}}^{b_1, \dots, b_m}$ .

Calculation using Lemma 2.4.3 shows that the pictures on the left and right in

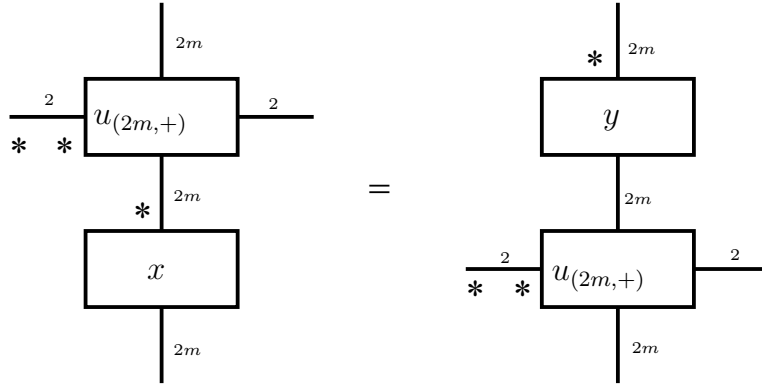


Figure 4.3: Relation between  $x, y \in P_{(2m,+)}$

Figure 3.5 evaluate to

$$\sum_{p, a_1, \dots, a_{2m}} \alpha_{a_1, \dots, a_{2m}} e_{a_{m+1}, \dots, a_{2m}, p}^{p, a_1^{-1} p, \dots, a_m^{-1} p} \quad \text{and} \quad \sum_{p, b_1, \dots, b_{2m}} \beta_{b_1, \dots, b_{2m}} e_{pb_{m+1}^{-1}, \dots, pb_{2m}^{-1}, p}^{p, b_1, \dots, b_m}$$

respectively. Thus  $x, y \in P_{(2m,+)}$  are related as in Figure 3.5 iff  $\alpha_{a_1, \dots, a_{2m}} = \beta_{b_1, \dots, b_{2m}}$  whenever  $a_1 b_1 = a_2 b_2 = \dots = a_{2m} b_{2m}$ .

Suppose that  $x \in Q_{(2m,+)}$  so that there exists  $y \in Q_{(2m,+)}$  satisfying the relation in Figure 3.5. We need to see that  $x$  respects  $\sim$ . So suppose that  $(a_1, \dots, a_{2m}) \approx (a'_1, \dots, a'_{2m})$ . Then there exist  $b_1, \dots, b_{2m}$  such that

$$\begin{aligned} a_1 b_1 &= a_2 b_2 = \dots = a_{2m} b_{2m} \quad \text{and} \\ a'_1 b_1 &= a'_2 b_2 = \dots = a'_{2m} b_{2m}, \end{aligned}$$

and so  $\alpha_{a_1, \dots, a_{2m}} = \beta_{b_1, \dots, b_{2m}} = \alpha_{a'_1, \dots, a'_{2m}}$ , as needed.

Conversely suppose that  $x \in P_{(2m,+)}$  and respects  $\sim$ . We need to construct  $y \in P_{(2m,+)}$  so as to satisfy the relation in Figure 3.5. Take  $b_1, \dots, b_{2m}$  and an arbitrary  $p \in S$  and solve for  $a_1, \dots, a_{2m}$  in  $a_1 b_1 = a_2 b_2 = \dots = a_{2m} b_{2m} = p$ . Define  $\beta_{b_1, \dots, b_{2m}} = \alpha_{a_1, \dots, a_{2m}}$ . This is easily checked to be well-defined since  $x$  respects  $\sim$ , and  $x, y$  do satisfy the relation of Figure 3.5, so that  $x \in Q_{(2m,+)}$  as needed.  $\square$

We will also need the following observation about the relation  $\sim$  on  $S^{\times m}$  and the group  $G$ .

**Lemma 4.1.5.** (Lemma 14 of [11]). For any  $m \in \mathbb{N}$ ,  $(a_1, \dots, a_m) \sim (a'_1, \dots, a'_m)$  if and only if there is a  $\sigma \in G$  such that  $(a'_1, \dots, a'_m) = \sigma(a_1, \dots, a_m)$  (with the diagonal action of  $G$ ).

*Proof.* ( $\Rightarrow$ ) We may reduce to the case  $(a_1, \dots, a_m) \approx (a'_1, \dots, a'_m)$ . If  $b_1 = a_{i_1}, \dots, b_k = a_{i_k}$  are the distinct ones among  $a_1, \dots, a_m$ , then  $(b_1, \dots, b_k) \approx (b'_1, \dots, b'_k)$ , where  $b'_1 = a'_{i_1}, \dots, b'_k = a'_{i_k}$ , by Lemma 4.1.1(2). Extend  $b_1, \dots, b_k$  to  $b_1, \dots, b_n$  such that  $\{b_1, \dots, b_n\} = \{s_1, \dots, s_n\}$ . We claim that there exist  $b'_{k+1}, \dots, b'_n \in S$  such that  $(b_1, \dots, b_n) \approx (b'_1, \dots, b'_n)$ . This is done as in the proof of Lemma 4.1.5. By assumption we have  $t_1, \dots, t_k$  such that  $b_1 t_1 = \dots = b_k t_k$  and  $b'_1 t_1 = \dots = b'_k t_k$ . Using the Latin square condition determine  $t_{k+1}, \dots, t_n$  such that  $b_1 t_1 = \dots = b_n t_n$ . Then again using the Latin square condition determine  $b'_{k+1}, \dots, b'_n$  such that  $b'_1 t_1 = \dots = b'_n t_n$ . Note that  $\{b'_1, \dots, b'_n\} = \{s_1, \dots, s_n\}$ .

Suppose  $(b_1, \dots, b_n) = (s_{\tau(1)}, \dots, s_{\tau(n)})$  and  $(b'_1, \dots, b'_n) = (s_{\rho(1)}, \dots, s_{\rho(n)})$ . Then  $(s_{\tau(1)}, \dots, s_{\tau(n)}) \approx (s_{\rho(1)}, \dots, s_{\rho(n)})$  and so  $\sigma = \rho\tau^{-1} \in G$ . Now for any  $j$  between 1 and  $n$ ,  $a'_{i_j} = b'_j = s_{\rho(j)} = \rho\tau^{-1}(s_{\tau(j)}) = \sigma(b_j) = \sigma(a_{i_j})$ . Now an appeal to Lemma 4.1.1(1) shows that  $(a'_1, \dots, a'_m) = \sigma(a_1, \dots, a_m)$ .

( $\Leftarrow$ ) Again, let  $b_1 = a_{i_1}, \dots, b_k = a_{i_k}$  be the distinct ones among  $a_1, \dots, a_m$  and extend them to  $b_1, \dots, b_n$  such that  $\{b_1, \dots, b_n\} = \{s_1, \dots, s_n\}$ . For  $j$  between 1 and  $n$ , let  $b'_j = \sigma(b_j)$ . Then  $\{b'_1, \dots, b'_n\} = \{s_1, \dots, s_n\}$ . Suppose that  $(b_1, \dots, b_n) = (s_{\tau(1)}, \dots, s_{\tau(n)})$  and  $(b'_1, \dots, b'_n) = (s_{\rho(1)}, \dots, s_{\rho(n)})$ . It follows that  $\tau = \sigma\rho$ . Since  $\sigma \in G$ ,  $(s_1, s_2, \dots, s_n) \sim (s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(n)})$ . Now, by Lemma 4.1.1(3),  $(b'_1, \dots, b'_n) \sim (b_1, \dots, b_n)$ . Finally appeal to Lemma 4.1.1(2) to conclude that  $(a'_1, \dots, a'_m) \sim (a_1, \dots, a_m)$ .  $\square$

We now prove our main result.

**Theorem 4.1.6.** (Theorem 15 of [11]). Let  $L$  be a Latin square and  $H \subseteq G$  be its associated subgroup-group pair. The planar subalgebras  $Q = P(L)$  and  $\tilde{Q} = P(H \subseteq G)$  of  $P$  are identical.

*Proof.* Since both  $Q$  and  $\tilde{Q}$  are connected planar subalgebras of  $P$ , to see that they are equal, it is enough to show that  $Q_{(2m,+)} = \tilde{Q}_{(2m,+)}$  for  $m \in \mathbb{N}$ . For then, by applying appropriate tangles it will follow that  $Q_{(m,\mu)} = \tilde{Q}_{(m,\mu)}$  for all colours  $(m, \mu)$ .

By definition  $\tilde{Q}_{(2m,+)} = \{\sum_{a_1, \dots, a_{2m}} \alpha_{a_1, \dots, a_{2m}} e_{a_{m+1}, \dots, a_{2m}}^{a_1, \dots, a_m} : \alpha_{a_1, \dots, a_{2m}} = \alpha_{a'_1, \dots, a'_{2m}} \text{ whenever } (a_1, \dots, a_{2m}) = \sigma(a'_1, \dots, a'_{2m}) \text{ for some } \sigma \in G\}$ .

But now an appeal to Proposition 4.1.4 and Lemma 4.1.5 completes the proof.  $\square$

**Remark 4.1.7.** It was shown in section 3.4 that if  $L$  is the multiplication table of a finite group  $G$ , then the associated subfactor planar algebra is the planar algebra  $P(G)$  associated to  $G$ . This is an easy special case of Theorem 4.1.6 in which the subgroup-group pair is  $\{1\} \subseteq G$ . It can be seen as follows. Here  $\approx$  itself is an equivalence relation since  $G$  is a group. Let  $\tilde{G}$  denote the group described in Lemma 4.1.2. Then by definition,  $\tilde{G} = \{\sigma \in S_n : (s_1, s_2, \dots, s_n) \approx (s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(n)})\}$  and  $\tilde{H} = \tilde{G} \cap S_{n-1}$ .

*Claim:*  $\tilde{H} = \{id\}$ . Let  $\sigma \in \tilde{H}$ . Then  $\sigma(n) = n$ . Now  $\sigma \in \tilde{G}$  implies that there exists  $(t_1, \dots, t_n) \in S^{\times n}$  such that

$$\begin{aligned} g_1 t_1 &= g_2 t_2 = \dots = g_n t_n = p \text{ (say)} & \text{and} \\ g'_1 t_1 &= g'_2 t_2 = \dots = g'_n t_n = q. \end{aligned}$$

Now  $\sigma(n) = n$  implies  $p = q$  which in turn will imply subsequently that  $\sigma(i) = i$  for all  $i < n$ , that is  $\sigma = id$ . Hence the claim. It is now easy to prove that that  $\tilde{G}$  is isomorphic to  $G$  and the subgroup-group pair is  $\{1\} \subseteq G$ .



**Remark 4.1.8.** *One important thing is to note that the subgroup-group pair certainly does not recognize the Latin square. For instance, the following non-equivalent Latin squares of size 6 all have associated subgroup-group pair to be  $S_5 \subseteq S_6$ , as can be shown by computation similar to the one in Example 4.1.3. The data of these Latin squares is taken from [18].*

$$\begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ s_2 & s_6 & s_4 & s_3 & s_1 & s_5 \\ s_3 & s_5 & s_6 & s_1 & s_2 & s_4 \\ s_4 & s_3 & s_5 & s_2 & s_6 & s_1 \\ s_5 & s_4 & s_1 & s_6 & s_3 & s_2 \\ s_6 & s_1 & s_2 & s_5 & s_4 & s_3 \end{bmatrix} \begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ s_2 & s_5 & s_6 & s_3 & s_1 & s_4 \\ s_3 & s_6 & s_2 & s_1 & s_4 & s_5 \\ s_4 & s_3 & s_5 & s_2 & s_6 & s_1 \\ s_5 & s_4 & s_1 & s_6 & s_3 & s_2 \\ s_6 & s_1 & s_4 & s_5 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ s_2 & s_4 & s_5 & s_1 & s_6 & s_3 \\ s_3 & s_6 & s_4 & s_2 & s_1 & s_5 \\ s_4 & s_3 & s_6 & s_5 & s_2 & s_1 \\ s_5 & s_1 & s_2 & s_6 & s_3 & s_4 \\ s_6 & s_5 & s_1 & s_3 & s_4 & s_2 \end{bmatrix}$$

$$\begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ s_2 & s_4 & s_5 & s_1 & s_6 & s_3 \\ s_3 & s_5 & s_6 & s_2 & s_4 & s_1 \\ s_4 & s_6 & s_1 & s_3 & s_2 & s_5 \\ s_5 & s_1 & s_4 & s_6 & s_3 & s_2 \\ s_6 & s_3 & s_2 & s_5 & s_1 & s_4 \end{bmatrix} \begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ s_2 & s_3 & s_1 & s_6 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_2 & s_6 & s_1 \\ s_4 & s_1 & s_6 & s_5 & s_2 & s_3 \\ s_5 & s_6 & s_2 & s_3 & s_1 & s_4 \\ s_6 & s_5 & s_4 & s_1 & s_3 & s_2 \end{bmatrix}$$



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