

The Multiset Partition Algebra

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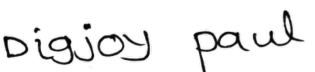
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Dedicated to my teachers

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Summary

In this thesis, we introduce and study a new diagram algebra, its representation theory, and related combinatorics. A large variety of diagram algebras arose from the classical Schur–Weyl duality and its variants. A quintessential example is the partition algebra, which appeared independently in the work of Martin and Jones, stemming from a study of the Potts models in statistical mechanics. Jones proved Schur–Weyl duality between partition algebra and the centralizer algebra of Symmetric group acting on tensor products of its permutation representation. The representation theory and structural properties, like semisimplicity, have been studied by Martin and others. Later, C. Xi proved that partition algebras are cellular in the sense of Graham and Lehrer. One advantage of being cellular is the description of the irreducible representations of partition algebra, even in non-semisimple case. The combinatorial representation theory of partition algebras has been the center of attention for decades to mathematicians like Arun Ram, Tom Halverson, and Georgia Benkart.

Given a vector λ of nonnegative integers, we introduce the multiset partition algebra $\mathcal{MP}_\lambda(\xi)$ over $F[\xi]$, where F is a field of characteristic 0, based on specific multiset partitions. When ξ is specialized to a positive integer n , we establish the Schur–Weyl duality between the actions of the resulting algebra $\mathcal{MP}_\lambda(n)$ and the symmetric group S_n on the tensor products of symmetric powers $\text{Sym}^\lambda(F^n)$. We show that $\mathcal{MP}_\lambda(\xi)$ embeds inside the partition algebra $\mathcal{P}_{|\lambda|}(\xi)$ and also recover the

latter as a particular case. Using this embedding, we prove that $\mathcal{MP}_\lambda(\xi)$ is a cellular algebra over F , and that it is semisimple when ξ is not an integer or ξ is an integer such that $\xi \geq 2|\lambda| - 1$. We also study an important subalgebra, the *balanced* multiset partition algebra, which is the centralizer algebra of monomial matrices acting on $\text{Sym}^\lambda(F^n)$.

A combinatorial rule for the multiplicity of the irreducible representation of S_n in the restriction of irreducible polynomial representation of $GL_n(F)$ remains an open problem. This problem is known as the restriction problem and the multiplicities as restriction coefficients. It is well known that specific semistandard multiset tableaux count the multiplicity of an irreducible representation of S_n in $\text{Sym}^\lambda(F^n)$. We obtain a generating function for these tableaux in terms of a plethysm of Schur functions. As a consequence, not only do we obtain an indexing set for the irreducible representations of $\mathcal{MP}_k(n)$, but also we give an alternative proof of Littlewood's formula for the restriction coefficients.

We extend the Robinson–Schensted–Knuth correspondence for multisets to offer a bijection between multiset partition diagrams and pairs of semistandard multiset tableaux, which corresponds to the Wedderburn decomposition of Multiset partition algebra. This bijection opens a door for a possible Gelfand model for $\mathcal{MP}_\lambda(n)$ analogous to partition algebras and its subalgebra.

Notation

Symbol	Description
F	Field of characteristic 0.
$[n]$	The set $\{1, 2, \dots, n\}$.
S_n	Symmetric group on n letters.
FS_n	Symmetric group algebra.
$GL_n(F)$	General linear group over F .
$F[\xi]$	Polynomial ring.
$(F^n)^{\otimes k}$	Set of k -fold tensor product of F^n .
$\text{Sym}^k(F^n)$	Set of k -th symmetric tensors.
$\nu \vdash n$	Integer partition of n .
$\lambda = (\lambda_1, \dots, \lambda_s)$	Vector of nonnegative integers.
$ \lambda $	$\lambda_1 + \lambda_2 + \dots + \lambda_s$
V_ν	Specht module.
W_ν	Weyl module.
$SSYT$	Semistandard Young tableaux.
$SSMT$	Semistandard multiset tableaux.
$\text{End}_G(V)$	Centralizer algebra of group G acting on V .
$\mathcal{P}_k(\xi)$	The Partition algebra over $F[\xi]$.
$\mathcal{MP}_\lambda(\xi)$	The multiset partition algebra over $F[\xi]$.
$\mathcal{MP}_\lambda^{bal}$	The balanced multiset partition algebra.

Chapter 1

Introduction

1.1 A brief history and motivation

The symmetric group S_k acts on the k -fold tensor space $(\mathbf{C}^n)^{\otimes k}$ by permuting its tensor factors. The space $\mathbf{C}^n$ is the defining representation of $GL_n(\mathbf{C})$, and $GL_n(\mathbf{C})$ acts on $(\mathbf{C}^n)^{\otimes k}$ diagonally. These two actions commute and each generates the centralizers of the other. Further, as a $(GL_n(\mathbf{C}), S_k)$ -bimodule

$$(\mathbf{C}^n)^{\otimes k} \cong \bigoplus_{\mu} W_{\mu} \otimes V_{\mu},$$

where μ runs over partitions of k with at most n parts, W_{μ} is an irreducible polynomial representation (Weyl module) of $GL_n(\mathbf{C})$ and V_{μ} is a Specht module of S_n . This phenomenon, discovered by Schur [29] and later popularized by Weyl [31], is called the classical Schur–Weyl duality and is a cornerstone of representation theory.

Brauer [4] restricted the action of $GL_n(\mathbf{C})$ on $(\mathbf{C}^n)^{\otimes k}$ to the orthogonal group $O_n(\mathbf{C})$ and showed Schur–Weyl duality between the actions of the Brauer algebra and $O_n(\mathbf{C})$ on $(\mathbf{C}^n)^{\otimes k}$. Jones [12] and Martin [19], independently, further

restricted the action of $O_n(\mathbf{C})$ to the symmetric group S_n . This led to the definition of the partition algebra $\mathcal{P}_k(\xi)$ over the polynomial ring $F[\xi]$, where F is a field of characteristic 0. The algebra $\mathcal{P}_k(\xi)$ is a free module over $F[\xi]$ generated by partitions of a set of cardinality $2k$. A partition of such a set can be pictorially represented as a graph, called the partition diagram. In particular, $\mathcal{P}_k(\xi)$ is a diagram algebra in the sense of Martin [20]. Martin and Saluer [21] showed that $\mathcal{P}_k(\xi)$ is semisimple over F unless ξ is an integer such that $\xi < 2k - 1$.

Jones [12] showed that, when ξ is evaluated at a positive integer n , then $\mathcal{P}_k(n)$ maps onto the commutant of S_n acting on $(F^n)^{\otimes k}$. Moreover, when $n \geq 2k$,

$$(1.1) \quad \mathcal{P}_k(n) \cong \text{End}_{S_n}((F^n)^{\otimes k}).$$

In particular, this isomorphism gives a diagrammatic interpretation of the centralizer algebra $\text{End}_{S_n}((F^n)^{\otimes k})$.

Martin [19] classified the irreducible representations of $\mathcal{P}_k(\xi)$ and a combinatorial description of the same was given by Halverson and Ram [7]. For $n \geq 2k$, Benkart and Halverson [2] constructed a basis for the irreducible representations of $\mathcal{P}_k(n)$ in terms of set-partition tableaux. Orellana and Zabrocki [24] studied the combinatorics of the multiset tableaux, which generalize set-partition tableaux. Multiset tableaux play a central role in the representation theory of the algebra we define in this paper.

Motivated by the Kazhdan–Lusztig basis of Hecke algebras, Graham and Lehrer [6] defined the notion of a cellular algebra in terms of the existence of a special kind of basis, called cellular basis. A cellular basis allows one to construct the irreducible representations of the cellular algebra. König and Xi [15] gave a basis-free definition of cellular algebra. Prominent examples of cellular algebras include Ariki–Koike Hecke algebras, Brauer algebras, and partition algebras.

1.2 Aim of the thesis

The defining action of $GL_n(F)$ on F^n extends functionally to the k th symmetric power $\text{Sym}^k(F^n)$. For a vector $\lambda = (\lambda_1, \dots, \lambda_s)$ of non-negative integers, the representation

$$\text{Sym}^\lambda(F^n) = \text{Sym}^{\lambda_1}(F^n) \otimes \cdots \otimes \text{Sym}^{\lambda_s}(F^n),$$

along with the tensor product of exterior powers of F^n , are the building blocks for constructing the irreducible polynomial representations of $GL_n(F)$. By restriction to permutation matrices, $\text{Sym}^\lambda(F^n)$ is also a representation of S_n .

Now one can ask what plays the role of the partition algebra in this setting?

Group	$\curvearrowright$	Vector Space	$\curvearrowright$	Diagram algebra	Centralizer algebra
S_n	$\curvearrowright$	$(F^n)^{\otimes k}$	$\curvearrowright$	$\mathcal{P}_k(n)$	$\cong \text{End}_{S_n}((F^n)^{\otimes k})$
S_n	$\curvearrowright$	$\text{Sym}^\lambda(F^n)$	$\curvearrowright$	??	$\cong \text{End}_{S_n}(\text{Sym}^\lambda(F^n))$

We answer the above question comprehensively in this thesis by constructing a new diagram algebra, based on certain multiset partitions, which is a generalization of the partition algebra. We explore some structural properties like cellularity and semisimplicity. In the next chapter we give details of representation theoretic results and relevant combinatorial essence of this new algebra.

1.3 Overview of the results obtained

We study, in spirit of the isomorphism (1.1), the centralizer algebra $\text{End}_{S_n}(\text{Sym}^k(F^n))$ by constructing a new diagram algebra. We define a unital associative algebra $\mathcal{MP}_k(\xi)$ over $F[\xi]$ in terms of a basis indexed by the multiset partitions of the multiset $\{1^k, 1'^k\}$. Pictorially these multiset partitions can be represented by certain bipartite multigraphs (Equation (3.1)). The structure

constants of $\mathcal{MP}_k(\xi)$ with respect to this basis are polynomials in ξ (Equation (3.3)) such that these polynomials give integer values when evaluated at a positive integer. In particular, the algebra $\mathcal{MP}_k(n)$ obtained by evaluating ξ to any positive integer n may be defined over the ring of integers. We call $\mathcal{MP}_k(\xi)$ the *multiset partition algebra*.

The Schur–Weyl duality between the actions of $\mathcal{MP}_k(n)$ and S_n on $\text{Sym}^k(F^n)$ is demonstrated in Theorem 3.1.5. This provides a diagrammatic interpretation of $\text{End}_{S_n}(\text{Sym}^k(F^n))$, thereby answering a question in [9, p. 22].

For a partition ν of n , let a_ν^λ denote the multiplicity of the irreducible representation of S_n corresponding to ν in $\text{Sym}^\lambda(F^n)$. These multiplicities are interpreted as a count of certain multiset tableaux in [24] and in [9, Proposition 3.11]. In Theorem 4.0.2 we obtain the ordinary generating function for $\{a_\nu^\lambda\}_\lambda$ as the plethysm $s_\nu[1 + h_1 + h_2 + \cdots]$. As a result, we explicitly describe the irreducible representations of S_n occurring in $\text{Sym}^\lambda(F^n)$ when $\lambda = (k)$ in Theorem 4.0.3. Due to Schur–Weyl duality between $\mathcal{MP}_k(n)$ and S_n , this is also an indexing set of the irreducible representations of $\mathcal{MP}_k(n)$. As another application of this generating function we give a proof, in Theorem 4.1.1, of a formula by Littlewood for the multiplicity of a Specht module in the restriction of an irreducible polynomial representation of $GL_n(F)$ to S_n . We also obtain a generating function for those multiplicities.

A bijection between the set of bipartite multigraphs and certain equivalence classes of partition diagrams is described in Theorem 5.1.2. By utilizing this bijection we construct an embedding of $\mathcal{MP}_k(\xi)$ in $\mathcal{P}_k(\xi)$ (Theorem 5.1.5). Moreover, the image of the embedding is $eP_k(\xi)e$ for an idempotent $e \in P_k(\xi)$. As a result of this, we obtain that $\mathcal{MP}_k(\xi)$ is a cellular algebra over F , and $\mathcal{MP}_k(\xi)$ is semisimple when ξ is not an integer or ξ is integer such that $\xi \geq 2k - 1$.

In Chapter 6, we generalize the construction of $\mathcal{MP}_k(\xi)$ to any vector λ of

non-negative integers. The algebra so obtained has a basis indexed by the multiset partitions of $\{1^{\lambda_1}, \dots, s^{\lambda_s}, 1'^{\lambda_1}, \dots, s'^{\lambda_s}\}$, which we pictorially represent by certain bipartite multigraphs. Schur–Weyl duality exists between the actions of $\mathcal{MP}_\lambda(n)$ and S_n on $\text{Sym}^\lambda(F^n)$. In particular, for $n \geq 2|\lambda|$, $\mathcal{MP}_\lambda(n)$ is isomorphic to the centralizer algebra $\text{End}_{S_n}(\text{Sym}^\lambda(F^n))$. We also call $\mathcal{MP}_\lambda(\xi)$ the multiset partition algebra. We define an embedding of $\mathcal{MP}_\lambda(\xi)$ inside $\mathcal{P}_{|\lambda|}(\xi)$. The image of the embedding is $eP_{|\lambda|}(\xi)e$ for an idempotent $e \in P_{|\lambda|}(\xi)$. As an application of this embedding, we obtain that $\mathcal{MP}_\lambda(\xi)$ is cellular, and semisimple when ξ is not an integer or ξ is an integer such that $\xi \geq 2|\lambda| - 1$. The aforementioned embedding is an isomorphism when $\lambda = (1^k)$, so that

$$\mathcal{P}_k(\xi) \cong \mathcal{MP}_{(1^k)}(\xi).$$

Thus the partition algebra is a special case of the multiset partition algebra.

In [9, Proposition 4.1], the action of $GL_n(F)$ on $(F^n)^{\otimes k}$ is restricted to the subgroup consisting of monomial matrices. This prompts the definition of balanced partition algebra which is a subalgebra of $\mathcal{P}_k(\xi)$. In chapter 6.3, we define balanced multiset partition algebra $\mathcal{MP}_\lambda^{bal}$ whose structure constants do not depend on ξ . Theorem 6.3.3 gives Schur–Weyl duality between the actions of $\mathcal{MP}_\lambda^{bal}$ and the group of monomial matrices on $\text{Sym}^\lambda(F^n)$.

In [5, Section 5], Colmenarejo, Orellana, Saliola, Schilling, and Zabrocki modified the Robinson–Schensted–Knuth correspondence to give a bijection between ordered multiset partitions and pairs of tableaux, where the insertion tableau is a standard Young tableau, and the recording tableau is a semistandard multiset tableau. We provide a representation-theoretic interpretation of this bijection in Equation (7.1). In Theorem 7.0.1, we give a bijection between multiset partitions and pairs of semistandard multiset tableaux which corresponds to the decomposition of $\mathcal{MP}_\lambda(n)$ as $\mathcal{MP}_\lambda(n) \times \mathcal{MP}_\lambda(n)$ -module. We conclude the thesis

with an open problem about the possibility of a Gelfand model for $\mathcal{MP}_\lambda(n)$.

1.4 Recent interests

The terms multiset partition algebra and balanced multiset partition algebra were coined by Harman in [9, Section 4], where he proposed to study the corresponding centralizer algebras in the future. A recent article by Orellana and Zabrocki [25] describes the centralizer algebra of the symmetric group acting on a certain space of polynomials that arises in the context of Howe duality. Their algebra, also indexed by certain multisets, contains an isomorphic copy of the multiset partition algebra introduced in this thesis.

1.5 Convention

Throughout this thesis, F denotes a field of characteristic 0. In order to avoid an excessive amount of notation, most of the results of this thesis are proved for $\lambda = (k)$. The proofs of the analogous results in the general case follow by minor modifications that we outline where required.

Chapter 2

Preliminaries

In this chapter, we give an overview of multiset tableaux, centralizer algebras, permutation representations of a finite group, and partition algebras, which are used in this thesis.

2.1 Multiset tableaux

In this section, we define and introduce notation for various terms related to multisets, partitions of multisets, and tableaux. For more detail description, a standard reference is [5].

A multiset is a collection of possibly repeated objects. For example, $\{1, 1, 1, 2, 2\}$ is a multiset with three occurrences of the element 1 and two occurrences of the element 2. More precisely

Definition 2.1.1. Let S be a set. A multiset drawn from S is a non-negative integer-valued function on S , which we call a multiplicity function. Let $\mathcal{A}_S$ denotes the set of all multisets with elements drawn from S .

Given $S = \{1, 2, \dots, n\}$ and a multiplicity function f , we shall denote

$$(S, f) = \{1^{f(1)}, 2^{f(2)}, \dots, n^{f(n)}\}.$$

Here $f(i)$ indicates the number of times i appears in the multiset. Thus the multiset $\{1, 1, 1, 2, 2\}$ would be denoted $\{1^3, 2^2\}$. If (S, f) and (S, g) are multisets drawn from S , their disjoint union is the multiset $(S, f + g)$, where $f + g$ denotes the pointwise sum of f and g .

Definition 2.1.2. A multiset partition of a given multiset M is a multiset of multisets whose disjoint union equals M .

For example, $\pi = \{\{1, 2\}, \{2\}, \{1^2\}\}$ is a multiset partition of $\{1^3, 2^2\}$.

Given an ordered alphabet S , to define the notion of semistandard tableaux with entries as multisets in $\mathcal{A}_S$, we fix a total order on $\mathcal{A}_S$.

Definition 2.1.3. Let λ be a partition, and S be an ordered alphabet. Given a total order on multisets in $\mathcal{A}_S$, a semistandard multiset tableau (SSMT) of shape λ is a filling of the cells of Young diagram of λ with the entries from $\mathcal{A}_S$ such that:

- the entries increase strictly along each column,
- the entries increase weakly along each row.

The content of a SSMT P is the multiset obtained by taking the disjoint union of entries of P . Let $\text{SSMT}(\lambda, C)$ denote the set of all semistandard multiset tableau of shape λ and content C .

Definition 2.1.4. A SSMT is said to be set partition tableau if its content is a set, and semistandard Young tableau if all its entries are multisets of size 1.

Now onwards, we assume the order on multisets to be the graded lexicographic order L (see [5, Section 2.4]). Given multisets $M_1 = \{a_1, \dots, a_r\}$ with

$a_1 \leq \cdots \leq a_r$, and $M_2 = \{b_1, \dots, b_s\}$ with $b_1 \leq \cdots \leq b_s$, we say $M_1 <_L M_2$ in this order if

1. $r < s$, or
2. if $r = s$, then there exists a positive integer $i \leq r$ such that $a_j = b_j$ for all $1 \leq j < i$ and $a_i < b_i$.

Example 2.1.5. Let $S = \{1, 2, 3, 4\}$ with usual order on integers. Assume graded lexicographic order on $\mathcal{A}_S$. Three multiset tableaux are follows.

$\{1\}$	$\{1\}$	$\{1\}$	$\{1, 2\}$	$\{1\}$	$\{2, 3\}$
$\{2\}$	$\{4\}$	$\{2\}$	$\{2, 2\}$	$\{4\}$	
$\{3\}$		$\{1, 1\}$			

The first one is semistandard Young tableau. The second one is a semistandard multiset tableau of shape $(2, 2, 1)$ with content $\{1^4, 2^4\}$. The third tableau is a set partition tableau as its content is a set $\{1, 2, 3, 4\}$.

2.2 Centralizer algebra and permutation representations

Let $\mathcal{A}$ be an associative, finite-dimensional algebra over a field F with characteristic zero. Let V be a finite dimensional $\mathcal{A}$ -module. Let $\rho : \mathcal{A} \rightarrow \text{End}_F(V)$ be the corresponding representation. Then V is said to be semisimple if $V = \bigoplus V_\lambda$ for some family $\{V_\lambda\}_\lambda$ of simple submodules of V . The algebra $\mathcal{A}$ is said to be semisimple if every $\mathcal{A}$ -module is semisimple. One important example of semisimple algebra is group algebra. Given a finite group G , let FG denote the group algebra,

having G as the underlying basis of the vector space and whose product is defined by linearly extending the multiplication of group G . The group representations of G are equivalent to the algebra representations of FG . Maschke's theorem says that if G is finite, then FG is semisimple.

The *centralizer algebra* of $\mathcal{A}$ on V , denoted by $\text{End}_{\mathcal{A}}(V)$, defined by

$$\text{End}_{\mathcal{A}}(V) := \{T \in \text{End}_F(V) \mid T\rho(a) = \rho(a)T, \forall a \in \mathcal{A}\}.$$

When V is a simple $\mathcal{A}$ -module, Schur's lemma says that $\text{End}_{\mathcal{A}}(V)$ is a division ring. A semisimple F -algebra $\mathcal{A}$ is called *split* semisimple if $\text{End}_{\mathcal{A}}(V) = F1_V$ for each simple $\mathcal{A}$ -module V .

Assume that $\mathcal{A}$ is semisimple. Let $\{A_\lambda \mid \lambda \in \hat{\mathcal{A}}\}$ be the set of all irreducible $\mathcal{A}$ -modules. Then

$$(2.1) \quad V \cong \bigoplus_{\lambda \in \hat{\mathcal{A}}_V} A_\lambda^{\oplus m_\lambda}$$

where $\hat{\mathcal{A}}_V \subset \hat{\mathcal{A}}$ indexes the irreducible $\mathcal{A}$ -modules that appear in V and m_λ is called multiplicity of A_λ in V .

Now, by definition, the actions of $\mathcal{A}$ and $\text{End}_{\mathcal{A}}(V)$ on V commute with each other. Therefore we may view V as a bimodule for $\mathcal{A} \otimes \text{End}_{\mathcal{A}}(V)$. The following theorem tells us that $\text{End}_{\mathcal{A}}(V)$ is a semisimple algebra when $\mathcal{A}$ is a split semisimple. See [7, Theorem 5.4] for the proof.

Theorem 2.2.1. *Let $\mathcal{A}$ be split semisimple algebra. Assume the decomposition 2.1. Then*

1. *As an $\mathcal{A}$ -module,*

$$\text{End}_{\mathcal{A}}(V) \cong \bigoplus_{\lambda \in \hat{\mathcal{A}}_V} \mathcal{M}_{m_\lambda}(K)$$

2. As an $\mathcal{A} \otimes \text{End}_{\mathcal{A}}(V)$ bimodule,

$$V \cong \bigoplus_{\lambda \in \hat{\mathcal{A}}_V} A_{\lambda} \otimes B_{\lambda}$$

where $\{B_{\lambda} \mid \lambda \in \hat{\mathcal{A}}_V\}$ is a complete set of representatives of the isomorphism classes of simple $\text{End}_{\mathcal{A}}(V)$ -modules. Moreover, $\dim(B_{\lambda}) = m_{\lambda}$ for each λ .

Remark 1. The symmetric group algebra FS_n is split semisimple since $\dim_{\mathbb{Q}} \text{End}_{S_n}(V_{\lambda})$ is equal to one, for every irreducible representation V_{λ} , $\lambda \vdash n$, of S_n .

Let us recall a combinatorial treatment of the centralizer algebra $\text{End}_G(V)$ when V is a permutation representation of a finite group G . See [27, Section 2.4] for details. For a finite set X , let $F[X]$ denote the space of F -valued functions on X . By a G -set X , we mean a set X with a left G -action. Then G acts on $F[X]$ as follows

$$(2.2) \quad g \cdot f(x) = f(g^{-1}x), \text{ for } x \in X, g \in G, \text{ and } f \in F[X].$$

The space $F[X]$ is called the permutation representation of G associated to X . Suppose that X and Y are finite G -sets. Given a function $\phi : X \times Y \rightarrow F$, the *integral operator* $\zeta_{\phi} : F[Y] \rightarrow F[X]$ associated to ϕ is defined as

$$(2.3) \quad (\zeta_{\phi}f)(x) = \sum_{y \in Y} \phi(x, y)f(y), \text{ for } f \in F[Y].$$

If Z is another G -set and $\phi : X \times Y \rightarrow F$ and $\psi : Y \times Z \rightarrow F$ are functions, then

$$\zeta_{\phi} \circ \zeta_{\psi} = \zeta_{\phi * \psi},$$

where $\phi * \psi : X \times Z \rightarrow F$ is the *convolution product*

$$(2.4) \quad \phi * \psi(x, z) = \sum_{y \in Y} \phi(x, y) \psi(y, z).$$

From [27, Theorem 2.4.4] we have:

Theorem 2.2.2. *Let X be a finite G -set. For each orbit O in the orbit space $(G \setminus X \times X)$ define*

$$\phi_O(x, y) = \begin{cases} 1 & \text{if } (x, y) \in O, \\ 0 & \text{otherwise.} \end{cases}$$

Write $\zeta_O = \zeta_{\phi_O}$. Then the set

$$\{\zeta_O \mid O \in (G \setminus X \times X)\}$$

is a basis for $\text{End}_G(F[X])$. Consequently,

$$\dim \text{End}_G(F[X]) = |G \setminus (X \times X)|.$$

2.3 Partition algebra

Let k be a positive integer. A set partition of $\{1, 2, \dots, k, 1', 2', \dots, k'\}$ is of the form $\{B_1, B_2, \dots, B_l\}$, where $B_1, B_2, \dots, B_l$ are mutually disjoint sets such that

$$\sqcup_{i=1}^l B_i = \{1, 2, \dots, k, 1', 2', \dots, k'\}.$$

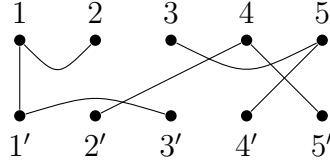
A set partition can be drawn as a graph, called a partition diagram, which has vertices on two rows. The vertices of the top and the bottom rows are $\{1, 2, \dots, k\}$ and $\{1', 2', \dots, k'\}$ respectively. For $i, j \in \{1, 2, \dots, k, 1', 2', \dots, k'\}$, there is an edge between i and j if only if $i, j \in B_s$, for $1 \leq s \leq l$. The set B_s (if non-empty) is

called a block in the corresponding partition diagram.

Example 2.3.1. The partition

$$\{\{1, 2, 1', 3'\}, \{3, 5, 4'\}, \{4, 2', 5'\}\}$$

of $\{1, 2, 3, 4, 5, 1', 2', 3', 4', 5'\}$ corresponds to the following partition diagram:



Let A_k denote the set of all partition diagrams on $\{1, 2, \dots, k, 1', 2', \dots, k'\}$. Let $\mathcal{P}_k(\xi)$ denote the free module over the polynomial ring $F[\xi]$ with basis A_k . Given $d_1, d_2 \in A_k$, let $d_1 \circ d_2$ denote the concatenation of d_1 with d_2 , i.e., place d_2 on the top of d_1 , and then identify the bottom vertices of d_2 with the top vertices of d_1 and remove any connected components that lie completely in the middle row.

(Borrowing analogy from the composition of maps, in the composition $d_1 \circ d_2$, we apply d_2 followed by d_1 , this differs from the standard convention but it has been adapted, for example, in Bloss [3, p. 694].) For the basis elements in A_k of $\mathcal{P}_k(\xi)$, define the multiplication as follows:

$$(2.5) \quad d_1 d_2 := \xi^l d_1 \circ d_2$$

where l is the number of connected components that lie entirely in the middle row while computing $d_1 \circ d_2$.

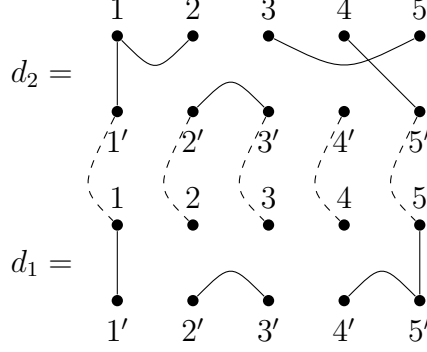
Example 2.3.2. The multiplication of

$$d_1 = \{\{1, 1'\}, \{2\}, \{3\}, \{4\}, \{2', 3'\}, \{5, 4', 5'\}\}$$

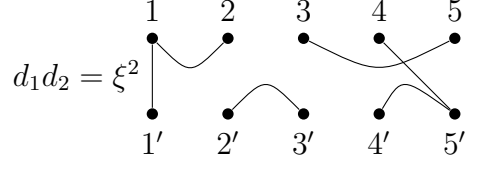
with

$$d_2 = \{\{1, 2, 1'\}, \{3, 5\}, \{2', 3'\}, \{4'\}, \{4, 5'\}\}$$

is illustrated below:



In the above, there are exactly two connected components which lie entirely in the middle, so



By linearly extending the multiplication (2.5), $\mathcal{P}_k(\xi)$ becomes an associative unital algebra over $F[\xi]$, and it is called the partition algebra.

Now we define another basis of $\mathcal{P}_k(\xi)$, called the orbit basis, which is particularly essential for this paper. Given $d, d' \in A_k$, we say d' is coarser than d , denoted as $d' \leq d$, if whenever i and j are in the same block in d , then i and j are in the same block in d' .

For a partition diagram d , define the element $x_d \in \mathcal{P}_k(\xi)$ by setting:

$$(2.6) \quad d = \sum_{d' \leq d} x_{d'}.$$

It can be easily seen that the transition matrix between $\{d\}$ and $\{x_d\}$ is

unitriangular. Thus $\{x_d\}$ is also a basis of the partition algebra $\mathcal{P}_k(\xi)$.

The structure constants of $\mathcal{P}_k(\xi)$ with respect to the orbit basis $\{x_d\}$ is given in [2, Theorem 4.8]. To state this result, we need the following definitions.

Definition 2.3.3. For a positive integer l and $f(\xi) \in F[\xi]$, the falling factorial polynomial is $(f(\xi))_l = f(\xi)(f(\xi) - 1) \cdots (f(\xi) - l + 1)$. We assume $(f(\xi))_l = 1$ when $l = 0$.

Definition 2.3.4. Given a partition diagram $d = \{B_1, \dots, B_n\} \in A_k$, we define

$$B_j^u := B_j \cap \{1, \dots, k\} \text{ and } B_j^l := B_j \cap \{1', \dots, k'\} \text{ for } 1 \leq j \leq n.$$

Then we can rewrite $d = \{(B_1^u, B_1^l), \dots, (B_n^u, B_n^l)\}$. Also define $d^u := \{B_1^u, \dots, B_n^u\}$ and $d^l := \{B_1^l, \dots, B_n^l\}$.

For $d_1, d_2 \in A_k$, we say $d_1 \circ d_2$ matches in the middle if the set partition d_1^u is same as the set partition obtained from d_2^l after ignoring the primed numbers in d_2^l .

Denote $[d_1 \circ d_2]$ by the number of connected components that lies entirely in the middle row while computing $d_1 \circ d_2$. For a partition diagram $d \in A_k$, let $|d|$ denote the number of blocks in d . From [22, Lemma 3.1], we recall, in the following theorem, the structure constants of $\mathcal{P}_k(\xi)$ with respect to the basis $\{x_d\}$.

Theorem 2.3.5. For $d_1, d_2 \in A_k$, we have

$$(2.7) \quad x_{d_1} x_{d_2} = \begin{cases} \sum_d (\xi - |d|)_{[d_1 \circ d_2]} x_d & \text{if } d_1 \circ d_2 \text{ matches in the middle,} \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is over all the coarsenings d of $d_1 \circ d_2$ which are obtained by connecting a block of d_2 , which lie entirely in the top row of d_2 , with a block of d_1 , which lie entirely in the bottom row of d_1 .

2.3.1 Schur–Weyl duality

By specializing ξ to a positive integer n , we recall Schur–Weyl duality between actions $\mathcal{P}_k(n)$ and the symmetric group S_n on $(F^n)^{\otimes k}$. Let F^n be the n -dimensional vector space over F with standard basis $e_1, e_2, \dots, e_n$. The group S_n acts on a basis element e_i as:

$$\sigma e_i := e_{\sigma(i)}, \text{ where } \sigma \in S_n.$$

So S_n acts on the k -fold tensor product $(F^n)^{\otimes k}$ diagonally.

Let $I(n, k)$ denote the set of k -tuples with entries from the set $\{1, 2, \dots, n\}$. For $\underline{i} = (i_1, i_2, \dots, i_k) \in I(n, k)$ and $\sigma \in S_n$, define

$$\sigma \underline{i} := (\sigma(i_1), \dots, \sigma(i_k)).$$

So $I(n, k)$ is a S_n -set. The space $(F^n)^{\otimes k}$ is isomorphic to the permutation representation $F[I(n, k)]$ of S_n . Using Theorem 2.2.2, the centralizer algebra

$$(2.8) \quad \text{End}_{S_n}((F^n)^{\otimes k}) = \text{End}_{S_n}(F[I(n, k)])$$

has a basis given by the integral operators, which are indexed by the orbits of S_n acting on $I(n, k) \times I(n, k)$. Given $(i_1, i_2, \dots, i_k)$ and $(i_{1'}, i_{2'}, \dots, i_{k'})$ in $I(n, k)$, define

$$B_s = \{r \mid i_r = s, r \in \{1, 2, \dots, k, 1', 2', \dots, k'\}\}, \text{ for } 1 \leq s \leq n.$$

The set $\{B_1, B_2, \dots, B_n\}$ (some B_s possibly could be empty) is a partition diagram with at most n blocks. It is easy to see that two elements in $I(n, k) \times I(n, k)$ are in the same orbit if and only if their corresponding partition diagrams are the same. Thus the orbits of S_n acting on $I(n, k) \times I(n, k)$ are parameterized by the partition diagrams with at most n blocks.

For $d \in A_k$, define

$$(d)_{i_1', \dots, i_{k'}}^{i_1, \dots, i_k} = \begin{cases} 1 & \text{if } i_r = i_s \text{ if and only if } r \text{ and } s \\ & \text{are in the same block of } d, \\ 0 & \text{otherwise.} \end{cases}$$

Define a map $\phi_k : \mathcal{P}_k(n) \rightarrow \text{End}_F((F^n)^{\otimes k})$ as follows:

$$(2.9) \quad \phi_k(x_d)(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{i_1', \dots, i_{k'}} (d)_{i_1', \dots, i_{k'}}^{i_1, \dots, i_k} e_{i_1'} \otimes \cdots \otimes e_{i_{k'}}.$$

The Schur–Weyl duality between the actions of the group algebra FS_n and the partition algebra $\mathcal{P}_k(n)$ on $(F^n)^{\otimes k}$ is given in [12] and in [18]. We state, in the following theorem, the Schur–Weyl duality from [7, Theorem 3.6], which is more applicable in this paper.

Theorem 2.3.6. *1. The image of map $\phi_k : \mathcal{P}_k(n) \rightarrow \text{End}_F((F^n)^{\otimes k})$ is $\text{End}_{S_n}((F^n)^{\otimes k})$. The kernel of ϕ_k is spanned by*

$$\{x_d \mid d \text{ has more than } n \text{ blocks}\}.$$

In particular, when $n \geq 2k$, $\mathcal{P}_k(n)$ is isomorphic to $\text{End}_{S_n}((F^n)^{\otimes k})$.

2. The group S_n generates the centralizer algebra $\text{End}_{\mathcal{P}_k(n)}((F^n)^{\otimes k})$.

It is well known that the partition algebra $\mathcal{P}_k(\xi)$ is semisimple unless ξ is an integer such that $\xi \in \{0, 1, \dots, 2k - 2\}$ [7, Theorem 3.27]. For a comprehensive treatment of irreducible representations of $\mathcal{P}_k(\xi)$ see [19]. Also, see [7] for a combinatorial description. Given a partition $\nu = (\nu_1 \geq \nu_2 \geq \cdots)$ of n , let V_ν denote the irreducible representation of S_n . We have the following decomposition

of $(F^n)^{\otimes k}$ as an S_n -module

$$(F^n)^{\otimes k} \cong \bigoplus_{\nu \vdash n} V_\nu^{\oplus m_{\nu,k}}$$

where $m_{\nu,k}$ is the number of multiset tableaux of shape ν and content $\{1^k\}$. See [2, Theorem 3.18] for more equivalent interpretations.

As an application of Theorem 2.3.6 along with Theorem 2.2.1, we have the following theorem.

Theorem 2.3.7. *For $n \geq 2k$, as $(\mathcal{P}_k(n), S_n)$ -bimodule, the tensor space has multiplicity free decomposition*

$$(F^n)^{\otimes k} \cong \bigoplus_{\nu \in L_{k,n}} P_\nu^k \otimes V_\nu$$

where $L_{k,n} = \{\nu \vdash n \mid \nu_2 + \nu_3 + \cdots \leq k\}$ and P_ν^k denote the irreducible representation of $\mathcal{P}_k(n)$ corresponding to the partition ν , of dimension $m_{\nu,k}$.

Chapter 3

The multiset partition algebra

For a positive integer k , each multiset partition of $\{1^k, 1'^k\}$ can be represented by the equivalence class of certain graphs by a procedure described below (see Equation (3.1)). We use these graphs to define the multiset partition algebra $\mathcal{MP}_k(\xi)$.

Let Γ be a bipartite multigraph with a bipartition of vertices into

$$\{0, 1, \dots, k\} \sqcup \{0, 1, \dots, k\}$$

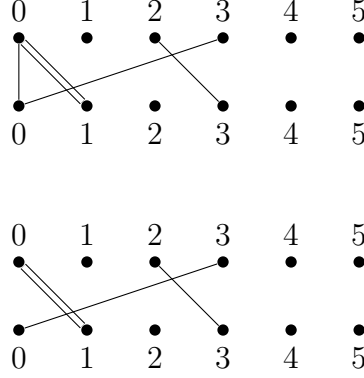
and edge multiset E_Γ . We arrange the vertices in two rows, with edges joining a vertex in the top row to one in the bottom row. We define a weight function $w : E_\Gamma \rightarrow \mathbb{Z}_{\geq 0}^2$ by setting $w(e) = (i, j)$ for every edge e from the vertex i in the top row to the vertex j in the bottom row. The weight of Γ is $w(\Gamma) := \sum_{e \in E_\Gamma} w(e)$.

Let $\mathcal{B}_k$ be the set of all bipartite multigraphs Γ with the weight $w(\Gamma) = (k, k)$. Given an edge e of $\Gamma \in \mathcal{B}_k$, we say e is non-zero weighted if $w(e) \neq (0, 0)$. Two bipartite multigraphs are said to be equivalent if they have the same non-zero weighted edges. Denote by $\tilde{\mathcal{B}}_k$ the set of all equivalence classes in $\mathcal{B}_k$.

Define the rank of Γ , denoted $\text{rank}(\Gamma)$, to be the number of non-zero weighted

edges of Γ . Since all graphs in the same equivalence class have the same rank, the rank of the equivalence class $[\Gamma]$ is defined as $\text{rank}([\Gamma]) = \text{rank}(\Gamma)$.

Example 3.0.1. The following two graphs in $\mathcal{B}_5$ have rank four, and they are equivalent.



The set $\tilde{\mathcal{B}}_k$ is in bijection with the set of multiset partitions of the multiset $\{1^k, 1'^k\}$. Explicitly, for a class $[\Gamma] \in \tilde{\mathcal{B}}_k$, let $E_\Gamma = \{(a_1, b_1), \dots, (a_n, b_n)\}$ be the collection of non-zero edges of Γ then we have the following bijective correspondence:

$$(3.1) \quad [\Gamma] \leftrightarrow \{\{1^{a_i}, 1'^{b_i}\} \mid i = 1, 2, \dots, n\}.$$

Thus graphs in $\tilde{\mathcal{B}}_k$ are a diagrammatic interpretation of multiset partitions of $\{1^k, 1'^k\}$. For example, the multiset partition associated with the class of graphs in Example 3.0.1 is $\{\{1'\}, \{1'\}, \{1^2, 1'^3\}, \{1^3\}\}$.

Let $\mathcal{MP}_k(\xi)$ denote the free module over $F[\xi]$ with basis $\tilde{\mathcal{B}}_k$ and $\mathcal{U}_k$ be the subset of $\tilde{\mathcal{B}}_k$ consisting of equivalence classes of graphs whose edges are of the form (i, i) for $i \in \{0, 1, \dots, k\}$. Rest of the section is devoted to proving the following theorem. is the main theorem of this chapter.

Theorem 3.0.2. For $[\Gamma_1], [\Gamma_2]$ in $\tilde{\mathcal{B}}_k$ we define the operation:

$$(3.2) \quad [\Gamma_1] * [\Gamma_2] = \sum_{[\Gamma] \in \tilde{\mathcal{B}}_k} \Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma]}(\xi)[\Gamma]$$

where $\Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma]}(\xi) \in F[\xi]$ is given in Equation (3.3). The linear extension of this operation makes $\mathcal{MP}_k(\xi)$ an associative unital algebra over $F[\xi]$, with the identity element $id = \sum_{[\Gamma] \in \mathcal{U}_k} [\Gamma]$.

Let $[\Gamma_1], [\Gamma_2] \in \tilde{\mathcal{B}}_k$ and pick the respective representatives Γ_1, Γ_2 with n edges, where $n \geq \max\{\text{rank}(\Gamma_1), \text{rank}(\Gamma_2)\}$. Place the graph Γ_2 above Γ_1 and identify the vertices in the bottom row of Γ_2 with those in the top row of Γ_1 . A *path* on this diagram is an ordered triple (a, b, c) such that (a, b) is an edge of Γ_2 and (b, c) is an edge of Γ_1 .

Definition 3.0.3 (Configuration of paths). A configuration of paths with respect to an ordered pair $(\Gamma_1, \Gamma_2) \in \mathcal{B}_k \times \mathcal{B}_k$ is a multiset $P = \{p_1, \dots, p_n\}$ of n paths, $p_i = (a_i, b_i, c_i)$, such that the covering condition (C) holds:

$$(C) \quad \begin{aligned} E_{\Gamma_2} &= \{(a_i, b_i) \mid 1 \leq i \leq n\}, \text{ and} \\ E_{\Gamma_1} &= \{(b_i, c_i) \mid 1 \leq i \leq n\}. \end{aligned}$$

Let $\text{Supp}^n(\Gamma_1, \Gamma_2)$ denote the set of all configurations of paths with respect to (Γ_1, Γ_2) .

For $P \in \text{Supp}^n(\Gamma_1, \Gamma_2)$, let Γ_P denote the multigraph with edge multiset $E_{\Gamma_P} = \{(a_1, c_1), \dots, (a_n, c_n)\}$. Let $D = \{(s_1, t_1), \dots, (s_j, t_j)\}$ be the set of distinct edges of Γ_P , labelled in lexicographically increasing order. For $(s, t) \in D$, define the multiset $P_{st} = \{b \mid (s, b, t) \in P\}$. Let p_{st} denote the cardinality of P_{st} , and for $r = 0, 1, \dots, k$ denote the multiplicity of the path (s, r, t) occuring in P by $p_{st}(r)$. Let $[\Gamma] \in \tilde{\mathcal{B}}_k$ with $n \geq \text{rank}(\Gamma)$. Let $\text{Supp}_\Gamma^n(\Gamma_1, \Gamma_2)$ be the set of path configurations

P in $\text{Supp}^n(\Gamma_1, \Gamma_2)$ such that $\Gamma_P = \Gamma$.

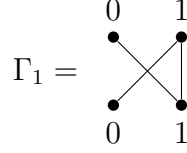
The structure constant of $[\Gamma]$ in the product $[\Gamma_1] * [\Gamma_2]$ is the following:

$$(3.3) \quad \Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma]}(\xi) = \sum_{P \in \text{Supp}_{\Gamma}^{2k}(\Gamma_1, \Gamma_2)} \mathcal{K}_P \cdot (\xi - \text{rank}(\Gamma))_{[P^{\Gamma_1 \circ \Gamma_2}]}, \text{ where}$$

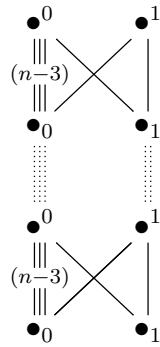
$$\mathcal{K}_P = \frac{1}{p_{00}(1)! \cdots p_{00}(k)!} \prod_{(s,t) \in D \setminus \{(0,0)\}} \binom{p_{st}}{p_{st}(0), \dots, p_{st}(k)},$$

and $[P^{\Gamma_1 \circ \Gamma_2}] = \sum_{i=1}^k p_{00}(i)$, the number of paths $(0, i, 0)$ in P for all $i \geq 1$.

Example 3.0.4. Let $k = 2$. Consider the following graph:



Let $\Gamma_2 = \Gamma_1$. We wish to find $[\Gamma_1] * [\Gamma_2]$, so we pick representatives Γ_1 and Γ_2 of the classes with $n = 4$ edges each. We do so by adding the requisite number $(n - 3)$ of extra $(0, 0)$ edges. Then we identify the bottom row of Γ_2 with the top row of Γ_1 to obtain:



Thus $\text{Supp}^{2k}(\Gamma_1, \Gamma_2)$ comprises four elements $\{P_1, P_2, P_3, P_4\}$ where

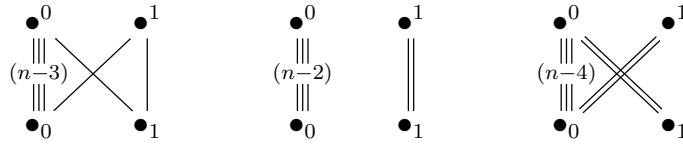
$$P_1 = \{(0, 0, 0)^{n-4}, (0, 1, 0), (0, 0, 1), (1, 0, 0), (1, 1, 1)\},$$

$$P_2 = \{(0, 0, 0)^{n-3}, (0, 1, 1), (1, 1, 0), (1, 0, 1)\},$$

$$P_3 = \{(0, 0, 0)^{n-3}, (0, 1, 0), (1, 0, 1), (1, 1, 1)\},$$

$$P_4 = \{(0, 0, 0)^{n-4}, (0, 0, 1), (0, 1, 1), (1, 1, 0), (1, 0, 0)\}.$$

The graphs Γ_{P_1} , Γ_{P_3} and Γ_{P_4} , corresponding to configurations P_1, P_3 and P_4 respectively, represented by:



Observe that $\Gamma_{P_2} = \Gamma_{P_1} = \Gamma_1$ and hence the set $\text{Supp}_{\Gamma_1}^{2k}(\Gamma_1, \Gamma_2)$ consists of P_1, P_2 .

By Equation (3.3), $\mathcal{K}_{P_1} \cdot (\xi - \text{rank}(\Gamma_1))_{[P_1 \Gamma_1 \circ \Gamma_2]} = \xi - 3$ as $[P_1^{\Gamma_1 \circ \Gamma_2}] = 1$ and

$\mathcal{K}_{P_1} = 1$. Again, $\mathcal{K}_{P_2} \cdot (\xi - \text{rank}(\Gamma_1))_{[P_2 \Gamma_1 \circ \Gamma_2]} = 1$. Therefore, $\Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma_1]} = \xi - 2$.

A similar computation for P_3, P_4 yields $\Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma_{P_3}]}$ and $\Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma_{P_4}]}$ Thus finally $[\Gamma_1] * [\Gamma_2]$ is equal to:

$$(\xi-2) \begin{array}{cc} \bullet^0 & \bullet^1 \\ \parallel & \parallel \\ \bullet_0 & \bullet_1 \end{array} + 2(\xi-2) \begin{array}{cc} \bullet^0 & \bullet^1 \\ \parallel & \parallel \\ \bullet_0 & \bullet_1 \end{array} + 4 \begin{array}{cc} \bullet^0 & \bullet^1 \\ \parallel & \parallel \\ \bullet_0 & \bullet_1 \end{array}$$

Remark 2. Let $n \geq \max\{\text{rank}([\Gamma]), \text{rank}([\Gamma_1]), \text{rank}(\Gamma_2)\}$. Assume $n \leq 2k$. Then the assignment

$$(3.4) \quad \beta : \text{Supp}_{\Gamma}^n(\Gamma_1, \Gamma_2) \rightarrow \text{Supp}_{\Gamma}^{2k}(\Gamma_1, \Gamma_2)$$

$$P \mapsto P \sqcup \{(0, 0, 0)^{2k-n}\}$$

is a bijection. Moreover, $\mathcal{K}_P = \mathcal{K}_{\beta(P)}$ and $[P^{\Gamma_1 \circ \Gamma_2}] = [\beta(P)^{\Gamma_1 \circ \Gamma_2}]$. The case when $n \geq 2k$ can be done likewise by adding $\{(0, 0, 0)^{n-2k}\}$ to every configuration in $\text{Supp}_{\Gamma}^{2k}(\Gamma_1, \Gamma_2)$.

The polynomial defined in Equation (3.3) is integer valued when evaluated at integers, and $\mathcal{MP}_k(n)$ is an algebra over $\mathbf{Z}$. Moreover, we have the following theorem:

Theorem 3.0.5. *Let $[\Gamma_1], [\Gamma_2], [\Gamma] \in \tilde{\mathcal{B}}_k$ and the respective representatives $\Gamma_1, \Gamma_2, \Gamma$ have n edges. Define*

$$(3.5) \quad C_{\Gamma_1 \Gamma_2}^{\Gamma}(n) := \{(b_1, \dots, b_n) \mid \{(a_1, b_1, c_1), \dots, (a_n, b_n, c_n)\} \in \text{Supp}_{\Gamma}^n(\Gamma_1, \Gamma_2)\}.$$

Then

$$\Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma]}(n) = |C_{\Gamma_1 \Gamma_2}^{\Gamma}(n)|.$$

Proof. Note that $n \geq \max\{\text{rank}(\Gamma), \text{rank}(\Gamma_1), \text{rank}(\Gamma_2)\}$. If $\text{Supp}_{\Gamma}^n(\Gamma_1, \Gamma_2)$ is empty, then $C_{\Gamma_1 \Gamma_2}^{\Gamma}(n) = \emptyset$ and by Remark 2, $\text{Supp}_{\Gamma}^{2k}(\Gamma_1, \Gamma_2) = \emptyset$. We have

$$\Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma]}(n) = |C_{\Gamma_1 \Gamma_2}^{\Gamma}(n)| = 0.$$

When $\text{Supp}_{\Gamma}^n(\Gamma_1, \Gamma_2) \neq \emptyset$, for $P \in \text{Supp}_{\Gamma}^n(\Gamma_1, \Gamma_2)$, let D and P_{st} be as defined on page 41. Let $\text{Perm}(P_{st})$ be the set of distinct permutations of the elements of P_{st} . Then $\text{Perm}(P) := \text{Perm}(P_{s_1 t_1}) \times \dots \times \text{Perm}(P_{s_j t_j})$ is a subset of $C_{\Gamma_1 \Gamma_2}^{\Gamma}(n)$.

From Equation (3.5), we have

$$C_{\Gamma_1 \Gamma_2}^{\Gamma}(n) = \bigsqcup_{P \in \text{Supp}_{\Gamma}^n(\Gamma_1 \Gamma_2)} \text{Perm}(P).$$

Note that, $|Perm(P_{st})| = \binom{p_{st}}{p_{st}(0), \dots, p_{st}(k)} = \frac{p_{st}!}{p_{st}(0)!p_{st}(1)! \cdots p_{st}(k)!}$. Hence

$$|Perm(P)| = \prod_{(s,t) \in D} \binom{p_{st}}{p_{st}(0), \dots, p_{st}(k)}.$$

Also, $p_{00} = n - \text{rank}(\Gamma) = p_{00}(0) + [P^{\Gamma_1 \circ \Gamma_2}]$, thus

$$\begin{aligned} \binom{p_{00}}{p_{00}(0), \dots, p_{00}(k)} &= \binom{p_{00}}{p_{00}(0)} \binom{p_{00} - p_{00}(0)}{p_{00}(1), \dots, p_{00}(k)} \\ &= \frac{(n - \text{rank}(\Gamma))_{[P^{\Gamma_1 \circ \Gamma_2}]}}{[P^{\Gamma_1 \circ \Gamma_2}]!} \binom{p_{00} - p_{00}(0)}{p_{00}(1), \dots, p_{00}(k)} \\ &= \frac{(n - \text{rank}(\Gamma))_{[P^{\Gamma_1 \circ \Gamma_2}]}}{p_{00}(1)! \cdots p_{00}(k)!}. \end{aligned}$$

Recall $\mathcal{K}_P$ from Equation (3.3), then we have

$$|Perm(P)| = \mathcal{K}_P \cdot (n - \text{rank}(\Gamma))_{[P^{\Gamma_1 \circ \Gamma_2}]}.$$

Thus $|C_{\Gamma_1 \Gamma_2}^\Gamma(n)| = \sum_{P \in \text{Supp}_\Gamma^n(\Gamma_1 \Gamma_2)} \mathcal{K}_P \cdot (n - \text{rank}(\Gamma))_{[P^{\Gamma_1 \circ \Gamma_2}]}$ and from Remark 2, we conclude $\Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma]}(n) = |C_{\Gamma_1 \Gamma_2}^\Gamma(n)|$. $\square$

3.1 Centralizer algebra and Schur–Weyl duality

The defining action of $GL_n(F)$ on F^n extends uniquely to the k th symmetric power $\text{Sym}^k(F^n)$. So the subgroup S_n of $GL_n(F)$ acts on $\text{Sym}^k(F^n)$ by the restriction. Let

$$M(n, k) = \{\underline{a} := (a_1, \dots, a_n) \mid \sum_{i=1}^n a_i = k, a_i \in \mathbf{Z}_{\geq 0}\}.$$

The set $\{e^{\underline{a}} = e_1^{a_1} \cdots e_n^{a_n} \in \text{Sym}^k(F^n) \mid \underline{a} = (a_1, \dots, a_n) \in M(n, k)\}$ is a basis of $\text{Sym}^k(F^n)$. For $w \in S_n$ and $\underline{a} \in M(n, k)$ define an action of S_n on $M(n, k)$ by

$$w.\underline{a} = (a_{w^{-1}(1)}, \dots, a_{w^{-1}(n)}).$$

The S_n -linear map $e^{\underline{a}} \mapsto 1_{\underline{a}}$, where $1_{\underline{a}}$ is the indicator function of $\underline{a} \in M(n, k)$, determines the isomorphism:

$$(3.6) \quad \text{Sym}^k(F^n) \cong F[M(n, k)].$$

Thus $\text{Sym}^k(F^n)$ is a permutation representation of S_n . We have the following algebra isomorphism

$$(3.7) \quad \text{End}_{S_n}(\text{Sym}^k(F^n)) \cong \text{End}_{S_n}(F[M(n, k)]).$$

We express an element $(\underline{a}, \underline{b}) \in M(n, k) \times M(n, k)$ as the biword

$\begin{bmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{bmatrix}$. The action of S_n on $(\underline{a}, \underline{b})$ corresponds to permuting the columns of its biword. Define $\tilde{\mathcal{B}}_{k,n} := \{[\Gamma] \in \tilde{\mathcal{B}}_k \mid \text{rank}([\Gamma]) \leq n\}$. Consider the map

$$(3.8) \quad S_n \backslash (M(n, k) \times M(n, k)) \rightarrow \tilde{\mathcal{B}}_{k,n}$$

$$\text{defined by } \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{bmatrix} \mapsto [\Gamma_{\underline{a}, \underline{b}}]$$

where $\Gamma_{\underline{a}, \underline{b}}$ is the graph with multiset of edges $\{(a_i, b_i) \mid i = 1, 2, \dots, n\}$.

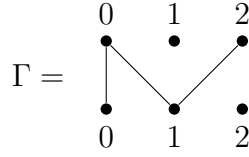
The map is well-defined because two biwords, which are the same up to a permutation of their columns, determine the same graph. For $[\Gamma] \in \tilde{\mathcal{B}}_{k,n}$, one can choose the representative Γ' with n edges. The multiset of edges of Γ' uniquely determines an S_n -orbit of $M(n, k) \times M(n, k)$. Hence the map (3.8) is a bijection.

Following Theorem 2.2.2 we define the integral operator corresponding to an S_n -orbit of $M(n, k) \times M(n, k)$.

Definition 3.1.1. For each $[\Gamma] \in \tilde{\mathcal{B}}_{k,n}$, define $T_{[\Gamma]} \in \text{End}_{S_n}(F[M(n, k)])$ by

$$T_{[\Gamma]}1_{\underline{a}} = \sum_{\{\underline{b} \in M(n, k) \mid [\Gamma_{\underline{a}, \underline{b}}] = [\Gamma]\}} 1_{\underline{b}}, \quad \text{where } \underline{a}, \underline{b} \in M(n, k).$$

Example 3.1.2. For $k = 2$ and $n = 3$, consider the following graph:



We compute the action of the integral operator $T_{[\Gamma]}$ on some basis elements of $F[M(3, 2)]$.

For $\underline{a} = (2, 0, 0)$, we must choose tuples $\underline{b}$ such that the pairs (a_i, b_i) exhaust all the edges of the graph Γ . There are two such tuples: $(1, 1, 0)$ and $(1, 0, 1)$ which can easily be verified to yield the multiset $\{(0, 0), (0, 1), (2, 1)\}$ of edges. So

$$T_{[\Gamma]}(1_{(2,0,0)}) = 1_{(1,1,0)} + 1_{(1,0,1)}.$$

For $\underline{a} = (1, 1, 0)$, there are no admissible tuples $\underline{b}$ that yield the multiset of edges as above. Hence $T_{[\Gamma]}(1_{(1,1,0)}) = 0$.

Theorem 3.1.3. *The set $\{T_{[\Gamma]} \mid [\Gamma] \in \tilde{\mathcal{B}}_{k,n}\}$ is a basis of $\text{End}_{S_n}(\text{Sym}^k(F^n))$.*

Proof. The proof follows from the isomorphism (3.7), the bijection (3.8), and Theorem 2.2.2. □

Given $[\Gamma_1], [\Gamma_2] \in \tilde{\mathcal{B}}_{k,n}$, we can express their product as follows:

$$T_{[\Gamma_1]}T_{[\Gamma_2]} = \sum_{[\Gamma] \in \tilde{\mathcal{B}}_{k,n}} \alpha_{[\Gamma_1][\Gamma_2]}^{[\Gamma]} T_{[\Gamma]},$$

where $\alpha_{[\Gamma_1][\Gamma_2]}^{[\Gamma]}$ is the structure constant.

Theorem 3.1.4. *Given graph $[\Gamma_1], [\Gamma_2], [\Gamma] \in \tilde{\mathcal{B}}_{k,n}$, the structure constant*

$$\alpha_{[\Gamma_1][\Gamma_2]}^{[\Gamma]} = |C_{\Gamma_1\Gamma_2}^\Gamma(n)|,$$

where $\Gamma, \Gamma_1, \Gamma_2$ are the representative of the respective classes with n edges and $C_{\Gamma_1\Gamma_2}^\Gamma(n)$ is defined in (3.5).

Proof. Let $(\underline{a}, \underline{c}) \in M(n, k) \times M(n, k)$ such that $[\Gamma_{\underline{a}, \underline{c}}] = [\Gamma]$. Then

$$\begin{aligned} T_{[\Gamma_1]}T_{[\Gamma_2]}(1_{\underline{a}}) &= T_{[\Gamma_1]} \left(\sum_{\{b | [\Gamma_{\underline{a}, b}] = [\Gamma_2]\}} 1_b \right) \\ &= \sum_{\{b | [\Gamma_{\underline{a}, b}] = [\Gamma_2]\}} T_{[\Gamma_1]}(1_b) \\ &= \sum_{\{b | [\Gamma_{\underline{a}, b}] = [\Gamma_2]\}} \sum_{\{z | [\Gamma_{\underline{b}, z}] = [\Gamma_1]\}} 1_z. \end{aligned}$$

So $\alpha_{[\Gamma_1][\Gamma_2]}^{[\Gamma]} = |\{b \in M(n, k) \mid [\Gamma_{\underline{a}, b}] = [\Gamma_2], [\Gamma_{\underline{b}, c}] = [\Gamma_1]\}|$ which is same as $|C_{\Gamma_1\Gamma_2}^\Gamma(n)|$. □

In the seminal paper [12], Jones proved the Schur–Weyl duality (Theorem 2.3.6) between the actions of $\mathcal{P}_k(n)$ and S_n on $(F^n)^{\otimes k}$. In this chapter we introduce an analog of the above scenario for the actions of $\mathcal{MP}_k(n)$ and S_n on $\text{Sym}^k(F^n)$.

Theorem 3.1.5 (Schur–Weyl duality for the multiset partition algebra). *Define a map*

$$(3.9) \quad \phi : \mathcal{MP}_k(n) \rightarrow \text{End}_{S_n}(\text{Sym}^k(F^n))$$

$$\phi([\Gamma]) = \begin{cases} T_{[\Gamma]} & \text{if } \text{rank}[\Gamma] \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

The map ϕ is a surjective algebra homomorphism with the kernel

$$\ker(\phi) = F\text{-span}\{[\Gamma] \in \tilde{\mathcal{B}}_k \mid \text{rank}([\Gamma]) > n\}.$$

In particular, when $n \geq 2k$, $\mathcal{MP}_k(n) \cong \text{End}_{S_n}(\text{Sym}^k(F^n))$. Moreover, S_n generates $\text{End}_{\mathcal{MP}_k(n)}(\text{Sym}^k(F^n))$.

Proof. From Theorem 3.1.3, $\{T_{[\Gamma]} \mid [\Gamma] \in \tilde{\mathcal{B}}_{k,n}\}$ is a basis of $\text{End}_{S_n}(\text{Sym}^k(F^n))$, so by the definition of ϕ it is a surjective linear map.

Let $[\Gamma_1]$ and $[\Gamma_2] \in \tilde{\mathcal{B}}_k$. If $[\Gamma_1], [\Gamma_2] \in \tilde{\mathcal{B}}_{k,n}$,

$$\begin{aligned} \phi([\Gamma_1] * [\Gamma_2]) &= \phi\left(\sum_{[\Gamma] \in \tilde{\mathcal{B}}_k} \Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma]}(n)[\Gamma]\right) \text{ by Theorem 3.0.2} \\ &= \sum_{[\Gamma] \in \tilde{\mathcal{B}}_{k,n}} \Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma]}(n)T_{[\Gamma]} \text{ by Definition 3.9} \\ &= T_{[\Gamma_1]}T_{[\Gamma_2]} \text{ from Theorem 3.1.4 and Theorem 3.0.5.} \end{aligned}$$

If $\text{rank}(\Gamma_1) > n$ or $\text{rank}(\Gamma_2) > n$ and $[\Gamma] \in \tilde{\mathcal{B}}_k$ with $\text{rank}([\Gamma]) \leq n$ appears in $[\Gamma_1] * [\Gamma_2]$, then we claim that $(n - \text{rank}(\Gamma))_{[P^{\Gamma_1 \circ \Gamma_2}]} = 0$ for all $P \in \text{Supp}_{\Gamma}^{2k}(\Gamma_1, \Gamma_2)$. Assume that the $\text{rank}(\Gamma_1) > n$, then for any $P \in \text{Supp}_{\Gamma}^{2k}(\Gamma_1, \Gamma_2)$, rewrite

$$(n - \text{rank}(\Gamma))_{[P^{\Gamma_1 \circ \Gamma_2}]} = (\text{rank}(\Gamma_1) - \text{rank}(\Gamma) - (\text{rank}(\Gamma_1) - n))_{[P^{\Gamma_1 \circ \Gamma_2}]}.$$

Note that $\text{rank}(\Gamma_1) - \text{rank}(\Gamma)$ is the set of edges of Γ_1 of the form $(i, 0)$ for $i \geq 1$ that become $(0, 0)$ edges in Γ_P . This quantity is $[P^{\Gamma_1 \circ \Gamma_2}]$. Thus the falling factorial is 0 since $\text{rank}(\Gamma) - n > 0$. A similar argument works when $\text{rank}(\Gamma_2) > n$. Thus

$$\phi([\Gamma_1] * [\Gamma_2]) = 0 = \phi([\Gamma_1]) \circ \phi([\Gamma_2]).$$

Note that $[\Gamma] \in \tilde{\mathcal{B}}_k$ has rank at most $2k$, so when $n \geq 2k$, $\ker(\phi)$ is trivial and thus

ϕ is an isomorphism. □

Corollary 3.1.6 (Associativity). *For $[\Gamma_1], [\Gamma_2]$, and $[\Gamma_3] \in \tilde{\mathcal{B}}_k$,*

$$([\Gamma]_1 * [\Gamma_2]) * [\Gamma_3] = [\Gamma_1] * ([\Gamma_2] * [\Gamma_3]).$$

Proof. For $[\Gamma] \in \tilde{\mathcal{B}}_k$, the coefficients of $[\Gamma]$ both in $([\Gamma]_1 * [\Gamma_2]) * [\Gamma_3]$ and $[\Gamma_1] * ([\Gamma_2] * [\Gamma_3])$ are polynomials in ξ . From Theorem 3.1.5, these polynomials are equal when evaluated to any positive integer $n \geq 2k$. Since characteristic of F is 0, these polynomials must be the same. □

Chapter 4

Representation theory of the multiset partition algebra

Given a partition $\nu = (\nu_1 \geq \nu_2 \geq \cdots)$ of n , the irreducible S_n -module (Specht) V_ν has a combinatorial basis known as Young's seminormal basis [33]. The elements of the basis are indexed by standard Young tableaux of shape ν . In the same spirit, we observe that the dimension of an irreducible representation of $\mathcal{MP}_k(n)$ is obtained by counting certain semistandard multiset tableaux. We find a generating function for semistandard multiset tableaux in terms of plethysm of Schur functions in Theorem 4.0.2. As an application of this identity:

1. We give an indexing set of the irreducible representations of $\mathcal{MP}_k(n)$ when $n \geq 2k$.
2. We give a proof of a formula by Littlewood for the multiplicity of a Specht module in the restriction of an irreducible polynomial representation of $GL_n(F)$ to S_n . We also obtain a generating function for those multiplicities.

We know by Theorem 3.1.5 that the irreducible representations M_ν^k are indexed by the subset $\Lambda_{k,n}$ of those partitions ν of n such that V_ν occurs with positive

multiplicity in $\text{Sym}^k(F^n)$.

For every vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ of non-negative integers, consider the following representation of S_n :

$$(4.1) \quad \text{Sym}^\lambda(F^n) := \text{Sym}^{\lambda_1}(F^n) \otimes \text{Sym}^{\lambda_2}(F^n) \otimes \cdots \otimes \text{Sym}^{\lambda_s}(F^n).$$

For every $\nu \vdash n$, let a_ν^λ denote the multiplicity of V_ν in $\text{Sym}^\lambda(F^n)$. In the following proposition, we paraphrase Proposition 3.11 of [9] to express a_ν^λ in the context of multiset tableaux.

Proposition 4.0.1. *The multiplicity a_ν^λ is the number of semistandard multiset tableaux over the alphabet $[1, 2, \dots, s]$ of shape ν and content $\{1^{\lambda_1}, 2^{\lambda_2}, \dots, s^{\lambda_s}\}$.*

Recall that the Schur function corresponding to ν is defined as

$$s_\nu(x_1, x_2, \dots) = \sum_T x^T,$$

where the sum is over all the semistandard Young tableau (SSYT) T of shape ν , and $c_i(T)$ is defined to be the number of occurrences of a positive integer i in T , and $x^T := x_1^{c_1(T)} x_2^{c_2(T)} \cdots$.

Littlewood [16] introduced the concept of plethysm of two symmetric functions (for the definition also see [17, Section I.8]). Given symmetric functions f and g , let $f[g]$ denote the plethystic substitution of g into f .

Theorem 4.0.2. *Let h_i denote the i -th complete homogeneous symmetric function. Then,*

$$s_\nu[1 + h_1 + h_2 + \cdots] = \sum_\lambda a_\nu^\lambda q^\lambda,$$

where h_i is the complete homogeneous symmetric function of degree i in the variables $q_1, q_2, \dots$, and for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$, $q^\lambda = q_1^{\lambda_1} q_2^{\lambda_2} \cdots q_s^{\lambda_s}$.

Proof. Each monomial in the Schur function $s_\nu(q_1, q_2, \dots)$ corresponds to an SSYT T , and the degree of q_i in this monomial is $c_i(T)$.

By the definition of plethystic substitution, $s_\nu[1 + h_1 + h_2 + \dots]$ can be regarded as the bijective substitution of the monomials of $1 + h_1 + h_2 + \dots$ into the variables $q_1, q_2, \dots$ of s_ν . The monomials of the expression $1 + h_1 + h_2 + \dots$ are indexed by multisets of finite size (in particular the monomials of h_k are indexed by multisets of size k), and the variables of s_ν are indexed by integers. The aforementioned substitution is a bijection, denoted ϕ , from the positive integers to the multisets of finite size.

Given a positive integer i , let $\phi(i) = \{1^{\phi(i)_1}, 2^{\phi(i)_2}, \dots\}$. Under the total order induced on multisets by ϕ , replacing each integer in the filling of a SSYT by its image yields a semistandard multiset tableau. This association is reversible. Thus

$$s_\nu[1 + h_1 + h_2 + \dots] = \sum_T (q_1^{\phi(1)_1} q_2^{\phi(1)_2} \dots)^{c_1(T)} (q_1^{\phi(2)_1} q_2^{\phi(2)_2} \dots)^{c_2(T)} \dots,$$

where the sum is over all the semistandard Young tableau (SSYT) T of shape ν .

Thus $s_\nu[1 + h_1 + h_2 + \dots]$ is the generating function for the multiset semistandard tableaux of shape ν by their content. The coefficient of q^λ in this function is thus the number of semistandard multiset tableaux of shape ν and content $\{1^{\lambda_1}, \dots, s^{\lambda_s}\}$ which is precisely a_ν^λ . $\square$

Remark 3. The above result might be restated for a finite number of variables $q_1, \dots, q_s$. Let $I(s) = \{q_1^{i_1} \dots q_s^{i_s} \mid i_1, \dots, i_s \in \mathbf{Z}_{\geq 0}\}$ denote the set of monomials of $1 + h_1 + h_2 + \dots$ in the variables $q_1, \dots, q_s$. Then we have:

$$s_\nu(I(s)) = \sum_\lambda a_\nu^\lambda q^\lambda.$$

where $s_\nu(I(s))$ is the Schur function in the set of variables $I(s)$ and the sum is over vectors of non-negative integers with at most s parts.

The following theorem describes the set $\Lambda_{k,n} = \{\nu \vdash n \mid a_\nu^k \neq 0\}$.

Theorem 4.0.3. *Let M_ν^k denote the irreducible representation of $\mathcal{MP}_k(n)$ corresponding to the partition ν of n . For $n \geq 2k$, as $S_n \times \mathcal{MP}_k(n)$ -bimodule, we have*

$$\mathrm{Sym}^k(F^n) \cong \bigoplus_{\nu \in \Lambda_{k,n}} V_\nu \otimes M_\nu^k$$

where $\Lambda_{k,n} := \{\nu \vdash n \mid \sum_{i=1}^n (i-1)\nu_i \leq k\}^1$.

Proof. Note that the decomposition of $\mathrm{Sym}^k(F^n)$ as $S_n \times \mathcal{MP}_k(n)$ -bimodule follows from the Schur–Weyl duality between S_n and $\mathcal{MP}_k(n)$ when $n \geq 2k$.

From [30, Corollary 7.21.3], we have the identity:

$$(4.2) \quad s_\nu(1, q, q^2, \dots) = \frac{q^{b(\nu)}}{\prod_{u \in \nu} (1 - q^{h(u)})}$$

where $b(\nu) = \sum_{i=1}^n (i-1)\nu_i$ and, for a cell u in the Young diagram of ν , $h(u)$ is the hook-length of u .

For $\lambda = (k)$, by Remark 3, we have that the multiplicity of the Specht module V_ν in $\mathrm{Sym}^k(F^n)$ is the coefficient of q^k in $\frac{q^{b(\nu)}}{\prod_{u \in \nu} (1 - q^{h(u)})}$. Each term $(1 - q^{h(u)})$ in the denominator may be expanded as $1 + q^{h(u)} + (q^{h(u)})^2 + \dots$. Thus the lowest exponent of q in $\frac{q^{b(\nu)}}{\prod_{u \in \nu} (1 - q^{h(u)})}$ is $b(\nu)$. If $b(\nu) > k$ then the coefficient of q^k is zero, and thus $\nu \notin \Lambda_{k,n}$.

Conversely if $b(\nu) \leq k$, pick a corner cell u_0 with hook-length 1. Then

$$\frac{1}{1 - q^{h(u_0)}} = 1 + q + q^2 + \dots, \text{ and}$$

$$\frac{q^{b(\nu)}}{\prod_{u \in \nu} (1 - q^{h(u)})} = q^{b(\nu)} (1 + q + q^2 + \dots) \prod_{u \neq u_0} (1 + q^{h(u)} + (q^{h(u)})^2 + \dots).$$

Choosing the term 1 from each of the infinite sums corresponding to cells $u \neq u_0$

¹The description of $\Lambda_{k,n}$ does not require the condition $n \geq 2k$.

and the term $q^{k-b(\nu)}$ from the infinite sum corresponding to the cell u_0 yields an instance of q^k . Since the expression above is a positive sum, demonstrating a single instance is sufficient to prove the positivity of the coefficient of q^k . Thus $a_\nu^k \neq 0$ if and only if $b(\nu) \leq k$. $\square$

4.1 Restriction from the general linear group to the symmetric group

The irreducible polynomial representations W_λ of $GL_n(F)$ of degree d are indexed by the partitions λ of d with at most n parts. We know the character of W_λ is the Schur polynomial $s_\lambda(x_1, x_2, \dots, x_n)$. Since S_n embedded in $GL_n(F)$ as subgroup of permutation matrices, we may consider the decomposition of the restriction of W_λ to S_n into Specht modules:

$$\text{Res}_{S_n}^{GL_n(F)} W_\lambda = \bigoplus_{\nu \vdash n} V_\nu^{\oplus r_\nu^\lambda}.$$

A positive combinatorial rule for the *restriction coefficient* r_ν^λ remains a long standing open problem. Littlewood [16, Theorem XI] gave a formula (see Theorem 4.1.1) in terms of plethysm. A recent approach to study the restriction problem can be found in [23].

Scharf and Thibon [28, Corollary 5.3] proved Littlewood's formula using Hopf algebra techniques. Stanley [30, Exercise 7.74] outlines a proof using symmetric function theory. Here we give a combinatorial proof as an application of Theorem 4.0.2.

Theorem 4.1.1 (Littlewood's formula). *Let $\langle -, - \rangle$ denote the Hall inner product*

on the ring of symmetric functions. Then

$$r_\nu^\lambda = \langle s_\nu[1 + h_1 + h_2 + \cdots], s_\lambda \rangle,$$

Proof. The character of S_n -representation $\text{Res}_{S_n}^{GL_n(F)} W_\lambda$ at a permutation matrix with eigenvalues $\theta_1, \dots, \theta_n$ is given by $s_\lambda(\theta_1, \dots, \theta_n)$. Denote this character by $s_\lambda \downarrow$. Using expression of the Schur function in terms of complete symmetric function, we get

$$(4.3) \quad s_\lambda \downarrow = \sum_{\alpha \vdash |\lambda|} K_{\alpha, \lambda}^{-1} h_\alpha \downarrow$$

where $h_\alpha \downarrow$ is the character of the S_n -module $\text{Sym}^\alpha(F^n)$ and $K_{\alpha, \lambda}^{-1}$ is the inverse Kostka numbers. If χ_ν denote the character of V_ν , then we have $a_\nu^\alpha = \langle h_\alpha \downarrow, \chi_\nu \rangle$, as a_ν^α is the multiplicity of V_ν in $\text{Sym}^\alpha(F^n)$.

Now,

$$\begin{aligned} r_\nu^\lambda &= \langle s_\lambda \downarrow, \chi_\nu \rangle \\ &= \sum_{\alpha \vdash |\lambda|} K_{\alpha, \lambda}^{-1} \langle h_\alpha \downarrow, \chi_\nu \rangle \\ &= \sum_{\alpha \vdash |\lambda|} K_{\alpha, \lambda}^{-1} a_\nu^\alpha. \end{aligned}$$

Finally, using Theorem 4.0.2 and the expansion of monomial symmetric function

m_α in terms of Schur function, we have

$$\begin{aligned}
\langle s_\nu[1 + h_1 + h_2 + \dots], s_\lambda \rangle &= \left\langle \sum_{\alpha \text{ is a partition}} a_\nu^\alpha m_\alpha, s_\lambda \right\rangle \\
&= \sum_{\alpha \vdash |\lambda|} a_\nu^\alpha \langle m_\alpha, s_\lambda \rangle \\
&= \sum_{\alpha \vdash |\lambda|} a_\nu^\alpha \langle K_{\alpha, \mu}^{-1} s_\mu, s_\lambda \rangle \\
&= r_\nu^\lambda.
\end{aligned}$$

□

We conclude the chapter with a generating function for the integers r_ν^λ .

Theorem 4.1.2. *For every partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of d with at most n parts and every partition ν of n , r_ν^λ is the coefficient of $q^\lambda := q_1^{\lambda_1} \dots q_n^{\lambda_n}$ in*

$$s_\nu(\{q_1^{i_1} q_2^{i_2} \dots q_n^{i_n}\}) \prod_{i < j} (1 - q_j/q_i).$$

where recall from Remark 3 that $s_\nu(\{q_1^{i_1} q_2^{i_2} \dots q_n^{i_n}\})$ is the Schur function in the set of variables $\{q_1^{i_1} q_2^{i_2} \dots q_n^{i_n}\}$.

Proof. By Theorem 4.0.2,

$$\sum_{\mu} a_\nu^\mu q^\mu = s_\nu(\{q_1^{i_1} q_2^{i_2} \dots q_n^{i_n}\}) \quad \text{where } \mu \text{ is a composition of } n.$$

The Jacobi–Trudi identity can be expressed at the level of the Grothendieck ring of the category of polynomial representations of $GL_n(F)$ (see [1]) as:

$$(4.4) \quad W_\lambda = \det(\text{Sym}^{\lambda_i + j - i}(F^n)),$$

so

$$\begin{aligned}
r_\nu^\lambda &= \sum_{w \in S_n} \text{sgn}(w) a_\nu^{w \cdot \lambda} \\
&= \sum_{w \in S_n} \text{sgn}(w) [q^{w \cdot \lambda}] s_\nu(\{q_1^{i_1} q_2^{i_2} \cdots q_n^{i_n}\}) \\
&= [q^\lambda] \sum_{w \in S_n} \text{sgn}(w) s_\nu(\{q_1^{i_1} q_2^{i_2} \cdots q_n^{i_n}\}) \prod_{i=1}^n q_i^{i-w(i)} \\
&= [q^\lambda] s_\nu(\{q_1^{i_1} q_2^{i_2} \cdots q_n^{i_n}\}) \sum_{w \in S_n} \text{sgn}(w) \prod_{i=1}^n q_i^{i-w(i)} \\
&= [q^\lambda] s_\nu(\{q_1^{i_1} q_2^{i_2} \cdots q_n^{i_n}\}) \det(q_i^{i-j}).
\end{aligned}$$

where $w \cdot \lambda$ is the tuple whose i th coordinate is $\lambda_i + w(i) - i$, and $[q^\lambda]f(q_1, \dots, q_n)$ is the coefficient of q^λ in the function $f(q_1, \dots, q_n)$.

Recognizing the determinant on the last line as a Vandermonde determinant that evaluates to $\prod_{i < j} (q_j^{-1} - q_i^{-1})$ and simplifying gives Theorem 4.1.2. $\square$

Chapter 5

Structural properties of the multiset partition algebra

Cellular algebras were introduced by Graham and Lehrer [6] motivated by Kazhdan and Lusztig's basis of Hecke algebras. For an equivalent definition of a cellular algebra see [15, Definition 3.2]. Xi [32] proved that partition algebras are cellular with respect to an involution. The goal of this chapter is to prove the multiset partition algebras are also cellular. We achieve this by exhibiting that $\mathcal{MP}_k(\xi)$ is isomorphic to $eP_k(\xi)e$ for an idempotent $e \in P_k(\xi)$ which is fixed by the involution. We also obtain that $\mathcal{MP}_k(\xi)$ is semisimple when ξ is not an integer or when ξ is an integer such that $\xi \geq 2k - 1$.

5.1 An embedding of $\mathcal{MP}_k(\xi)$ into $\mathcal{P}_k(\xi)$

We first give a bijection between $\tilde{\mathcal{B}}_k$ and certain class of partition diagrams in A_k . We begin with describing a partition diagram associated to given element in $\tilde{\mathcal{B}}_k$.

Given $\Gamma \in \mathcal{B}_k$ with n edges, associate to it the canonical biword

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{bmatrix}$$

by arranging the edges of Γ in weakly increasing lexicographic order. For

$1 \leq i \leq n$, we associate the edge (a_i, b_i) to a subset of $\{1, 2, \dots, k, 1', 2', \dots, k'\}$ with a_i elements from $\{1, 2, \dots, k\}$ and b_i elements from $\{1', 2', \dots, k'\}$ as follows:

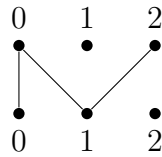
$$(5.1) \quad B_i = \left\{ \left(\sum_{r=1}^{i-1} a_r \right) + 1, \dots, \sum_{r=1}^i a_r, \left(\left(\sum_{r=1}^{i-1} b_r \right) + 1 \right)', \dots, \left(\sum_{r=1}^i b_r \right)' \right\}.$$

Note that the set B_i is empty if and only if the pair $(a_i, b_i) = (0, 0)$. Then

$$(5.2) \quad d_\Gamma := \{B_1, \dots, B_n\}$$

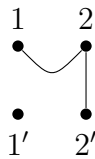
gives a set partition of $\{1, 2, \dots, k, 1', 2', \dots, k'\}$. Note that the partition diagram d_Γ is independent of choice of representatives of $[\Gamma] \in \tilde{\mathcal{B}}_k$, henceforth, we call d_Γ the canonical partition diagram associated to $[\Gamma]$.

Example 5.1.1. Let Γ be the following graph:



Then the corresponding canonical biword of Γ is $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$. By Equation (5.1), we

have $B_1 = \emptyset$, $B_2 = \{1'\}$, $B_3 = \{1, 2, 2'\}$, and so d_Γ is as follows:

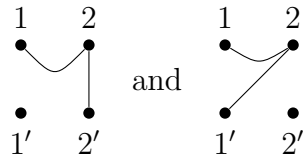


If we consider the biword of edges of $\Gamma \in \mathcal{B}_k$ in any other order and perform the above procedure then the partition diagram so obtained is in the orbit of d_Γ under the following action of $S_k \times S_k$ on A_k . For $(\sigma, \tau) \in S_k \times S_k$ and $d \in A_k$, the permutations σ and τ act by the value permutation of the vertices of the top row and the bottom row of d respectively. Note that this action preserves the number of blocks of d , and also preserves the number of elements of the top row and bottom row occurring in each block. Thus the orbit O_{d_Γ} uniquely determines the $[\Gamma] \in \tilde{\mathcal{B}}_k$, and so we obtain the following bijection.

Theorem 5.1.2. *Let $(S_k \times S_k) \setminus A_k$ denote the set of orbits of $S_k \times S_k$ on A_k . For $[\Gamma] \in \tilde{\mathcal{B}}_k$, let O_{d_Γ} denote the $S_k \times S_k$ -orbit of d_Γ . Then the following map is a bijection,*

$$(5.3) \quad \begin{aligned} \psi : \tilde{\mathcal{B}}_k &\rightarrow (S_k \times S_k) \setminus A_k \\ \psi([\Gamma]) &= O_{d_\Gamma}. \end{aligned}$$

Example 5.1.3. Under the above bijection, the graph in Example 5.1.1 is mapped to the orbit consisting of the following partition diagrams:



Definition 5.1.4. Given $d \in A_k$, let $d^l = \{B_1, \dots, B_n\}$ (for the definition of d^l see Definition 2.3.4), and $|B_i| = b_i$ for $i = 1, \dots, n$. Then

$$\alpha_d = \frac{k!}{b_1! \cdots b_n!}.$$

Now utilizing the bijection in Theorem 5.1.2 we give an embedding of $\mathcal{MP}_k(\xi)$ inside $\mathcal{P}_k(\xi)$ in the following theorem.

Theorem 5.1.5. *The following map is an injective algebra homomorphism*

$$(5.4) \quad \begin{aligned} \tilde{\eta}_k : \mathcal{MP}_k(\xi) &\rightarrow \mathcal{P}_k(\xi) \\ \tilde{\eta}_k([\Gamma]) &= \frac{1}{\alpha_{d_\Gamma}} \sum_{d \in O_{d_\Gamma}} x_d. \end{aligned}$$

Before we offer a proof, we need to set up some notation.

The canonical projection map $\pi_k : (F^n)^{\otimes k} \rightarrow \text{Sym}^k(F^n)$ has a section

$\tau_k : \text{Sym}^k(F^n) \rightarrow (F^n)^{\otimes k}$, which is defined as

$$(5.5) \quad \tau_k(e_{j_1} \cdots e_{j_k}) = \frac{1}{k!} \sum_{\sigma \in S_k} (e_{j_{\sigma(1)}} \otimes \cdots \otimes e_{j_{\sigma(k)}}) \text{ for } 1 \leq j_1 \leq \cdots \leq j_k \leq n.$$

We have,

$$(5.6) \quad \pi_k \circ \tau_k = id_{\text{Sym}^k(F^n)}.$$

Recall that both $(F^n)^{\otimes k}$ and $\text{Sym}^k(F^n)$ are permutation modules, and using the following definitions we interpret the maps π_k and τ_k at the level of permutation modules.

Definition 5.1.6. We represent an element $e_{j_1} \otimes \cdots \otimes e_{j_k}$ in $(F^n)^{\otimes k}$ as the indicator functor corresponding to an ordered set partition of $\{1, \dots, k\}$. That is,

$$e_{j_1} \otimes \cdots \otimes e_{j_k} \leftrightarrow 1_{(A_1, \dots, A_n)}$$

where $A_m = \{l \mid e_{j_l} = e_m, 1 \leq l \leq k\}$ and $\sqcup_{m=1}^n A_m = \{1, \dots, k\}$.

Definition 5.1.7. For $\underline{a} = (a_1, \dots, a_n) \in M(n, k)$, define

$$D(\underline{a}) = \{S := (S_1, \dots, S_n) \mid \sqcup_{i=1}^n S_i = \{1, \dots, k\}, |S_i| = a_i\}.$$

Example 5.1.8. The element $e_2 \otimes e_1 \otimes e_1$ in $(F^4)^{\otimes 3}$ is represented by the indicator function corresponding to the tuple $(\{2, 3\}, \{1\}, \emptyset, \emptyset)$. For $\underline{a} = (2, 1) \in M(2, 3)$, $D(\underline{a}) = \{(\{1, 2\}, \{3\}), (\{1, 3\}, \{2\}), (\{2, 3\}, \{1\})\}$.

Using Definitions 5.1.6 and 5.1.4, the maps π_k and τ_k are described as follows:

$$(5.7) \quad \pi_k(1_{(A_1, \dots, A_n)}) = 1_{(|A_1|, \dots, |A_n|)},$$

$$(5.8) \quad \tau_k(1_{\underline{a}}) = \frac{1}{|D(\underline{a})|} \sum_{S \in D(\underline{a})} 1_S.$$

Definition 5.1.9. For a set X and a tuple $r = (r_1, \dots, r_n)$ with $r_i \in X$, define

$$(5.9) \quad \text{set}(r) := \{r_1, \dots, r_n\}.$$

Let $A = (A_1, \dots, A_n)$ and $C = (C_1, \dots, C_n)$ then using Definition 2.3.4, the map (2.9) can be written as

$$(5.10) \quad x_d(1_A) = \begin{cases} \sum_{\{C | \{(A_1, C_1), \dots, (A_n, C_n)\} = d\}} 1_C & \text{if } \text{set}(A) = d^u, \\ 0 & \text{otherwise.} \end{cases}$$

Now we are ready to give a proof of Theorem 5.1.5.

Proof. The structure constants for both $\mathcal{MP}_k(\xi)$ and $\mathcal{P}_k(\xi)$ are polynomials in ξ therefore it is sufficient to prove $\tilde{\eta}_k$ is an algebra homomorphism when ξ is evaluated at any positive integer. For this define the following map, which is an algebra homomorphism,

$$\eta_k : \text{End}_{S_n}(\text{Sym}^k(F^n)) \rightarrow \text{End}_{S_n}((F^n)^{\otimes k})$$

$$\eta_k(f) = \tau_k \circ f \circ \pi_k.$$

for $f \in \text{End}_{S_n}(\text{Sym}^k(F^n))$, and consider the following diagram:

$$\begin{array}{ccc} \mathcal{MP}_k(n) & \xrightarrow{\tilde{\eta}_k} & \mathcal{P}_k(n) \\ \downarrow & & \downarrow \\ \text{End}_{S_n}(\text{Sym}^k(F^n)) & \xrightarrow{\eta_k} & \text{End}_{S_n}((F^n)^{\otimes k}) \end{array},$$

where the leftmost and the rightmost vertical maps are as in Theorem 3.1.5 and Theorem 2.3.6 respectively. By Schur–Weyl duality the vertical maps are algebra isomorphisms for $n \geq 2k$. Therefore it is enough to prove

$$(5.11) \quad \eta_k(T_{[\Gamma]}) = \frac{1}{\alpha_{d_\Gamma}} \sum_{d \in O_{d_\Gamma}} x_d, \text{ for } [\Gamma] \in \tilde{\mathcal{B}}_{k,n},$$

where α_{d_Γ} is given in Definition 5.1.4. Equation (5.11) as this ensures that the above diagram commutes.

Let Γ be the graph whose canonical biword is $\begin{bmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{bmatrix}$. Fix $\underline{a} = (a_1, \dots, a_n)$, $\underline{b} = (b_1, \dots, b_n)$. Let $A = (A_1, \dots, A_n)$ be a set partition of $\{1, \dots, k\}$.

The right hand side of Equation (5.11) is

$$\begin{aligned} \tau_k \circ T_{[\Gamma]} \circ \pi_k(1_A) &= \tau_k(T_{[\Gamma]}(1_{(|A_1|, \dots, |A_n|)})) \\ &= \tau_k\left(\sum_{\{\underline{c} | [\Gamma_{(|A_1|, \dots, |A_n|), \underline{c}}] = [\Gamma]\}} 1_{\underline{c}}\right) \\ &= \sum_{\{\underline{c} | [\Gamma_{(|A_1|, \dots, |A_n|), \underline{c}}] = [\Gamma]\}} \frac{1}{|D(\underline{c})|} \sum_{S \in D(\underline{c})} 1_S \\ (5.12) \quad &= \frac{1}{\alpha_{d_\Gamma}} \sum_{\{\underline{c} | [\Gamma_{(|A_1|, \dots, |A_n|), \underline{c}}] = [\Gamma]\}} \sum_{S \in D(\underline{c})} 1_S. \end{aligned}$$

For the last equality recall that for $\Gamma = \Gamma_{(|A_1|, \dots, |A_n|), \underline{c}}$, we have $\alpha_{d_\Gamma} = |D(\underline{c})|$.

If $\text{set}((|A_1|, \dots, |A_n|)) \neq \text{set}(\underline{a})$, then for every $\underline{c} \in M(n, k)$ we have

$$[\Gamma_{(|A_1|, \dots, |A_n|), \underline{c}}] \neq [\Gamma].$$

So $\tau_k \circ T_{[\Gamma]} \circ \pi_k(1_A) = 0$. For $d \in O_{d_\Gamma}$, the cardinality of a given block of d is equal to a_i for some $1 \leq i \leq n$. So by Equation (5.10), we get $\frac{1}{\alpha_{d_\Gamma}} \sum_{d \in O_{d_\Gamma}} x_d(1_A) = 0$.

Assume $\text{set}((|A_1|, \dots, |A_n|)) = \text{set}(\underline{a})$. Following the notation of Definition 2.3.4:

$$O_{d_\Gamma} = \left\{ d = \{(X_1, Y_1), \dots, (X_n, Y_n)\} \left| \begin{array}{l} X = (X_1, \dots, X_n) \in D(\underline{a}), \\ Y = (Y_1, \dots, Y_n) \in D(\underline{b}), \\ \Gamma_{|X|, |Y|} = \Gamma \end{array} \right. \right\}.$$

If $d \in O_{d_\Gamma}$ and $d^u \neq \text{set}(A)$ then from Equation (5.10), $x_d(1_A) = 0$

If $d \in O_{d_\Gamma}$ and $d^u = \text{set}(A)$ then $d = \{(X_1, Y_1), \dots, (X_n, Y_n)\}$, where

$$\begin{aligned} \{X_1, \dots, X_n\} &= \text{set}(A), \\ \{|Y_1|, \dots, |Y_n|\} &= \text{set}(\underline{b}) \end{aligned}$$

such that $\{(|X_1|, |Y_1|), \dots, (|X_n|, |Y_n|)\}$ is the biword associated to a graph that is in the class $[\Gamma]$.

From Equation (5.10) we know that:

$$x_d(1_A) = \sum_{\{S=(S_1, \dots, S_n) | \{(A_1, S_1), \dots, (A_n, S_n)\} = d\}} 1_S.$$

Thus we have:

$$(5.13) \quad \frac{1}{\alpha_{d_\Gamma}} \sum_{d \in O_{d_\Gamma}} x_d(1_A) = \frac{1}{\alpha_{d_\Gamma}} \sum_{\{\underline{c} | [\Gamma_{(|A_1|, \dots, |A_n|), \underline{c}}] = [\Gamma]\}} \sum_{S \in D(\underline{c})} 1_S.$$

Comparing Equation (5.13) and Equation (5.12) we see that Equation (5.11) is true. □

5.2 Cellularity of $\mathcal{MP}_k(\xi)$

The following proposition [15, Proposition 4.3] allows one to realize another cellular algebra from given a cellular algebra and an idempotent in it.

Proposition 5.2.1. *Let A be a cellular algebra with respect to an involution i . Let $e \in A$ be an idempotent such that $i(e) = e$. Then the algebra eAe is also a cellular algebra with respect to the involution i restricted to eAe .*

The involution i is defined for $d = \{(B_1^u, B_1^l), \dots, (B_n^u, B_n^l)\} \in A_k$ by:

$$(5.14) \quad i(d) = \{(B_1^l, B_1^u), \dots, (B_n^l, B_n^u)\}$$

which may be visualized as interchanging each primed element j' with unprimed element j and vice versa. This map is extended linearly to $\mathcal{P}_k(\xi)$. Xi [32] showed that partition algebras are cellular with respect to i .

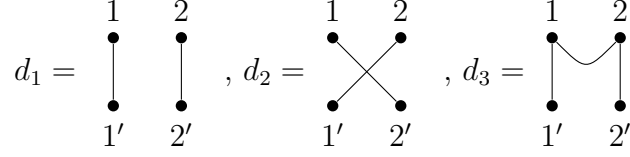
Definition 5.2.2. Let $\mathcal{Y}_k$ be the subset of A_k consisting of all partition diagram $d = \{B_1, B_2, \dots, B_n\}$ satisfying the following condition:

$$(5.15) \quad |B_j^u| = |B_j^l| \text{ for } 1 \leq j \leq n.$$

Example 5.2.3. For $k = 2$, the identity element of $\mathcal{MP}_k(\xi)$ is

$$id = \begin{array}{ccc} \begin{array}{c} 0 \\ \bullet \\ \bullet \\ 0 \end{array} & \begin{array}{c} 1 \\ \bullet \\ \parallel \\ \bullet \\ 1 \end{array} & \begin{array}{c} 2 \\ \bullet \\ \bullet \\ 2 \end{array} \\ + & & \begin{array}{c} \begin{array}{c} 0 \\ \bullet \\ \bullet \\ 0 \end{array} \quad \begin{array}{c} 1 \\ \bullet \\ \bullet \\ 1 \end{array} \quad \begin{array}{c} 2 \\ \bullet \\ \bullet \\ 2 \end{array} \end{array}.$$

The element $e = \frac{1}{2}(x_{d_1} + x_{d_2}) + x_{d_3}$ is an idempotent in $\mathcal{MP}_k(\xi)$ where



Lemma 5.2.4. *Let $e = \sum_{d \in \mathcal{Y}_k} \frac{1}{\alpha_d} x_d$. Then e is an idempotent and the embedding $\tilde{\eta}_k$ induces an isomorphism $\mathcal{MP}_k(\xi) \simeq eP_k(\xi)e$.*

Proof. We show $\tilde{\eta}_k(id) = e$, where $\tilde{\eta}_k$ is the embedding defined in Theorem 5.1.5 and $id = \sum_{[\Gamma] \in \mathcal{U}_k} [\Gamma]$ is the identity element of $\mathcal{MP}_k(n)$ and $\mathcal{U}_k$ consists of elements in $\tilde{\mathcal{B}}_k$ whose edges are of the form $(\underline{a}, \underline{a})$ for $\underline{a} \in M(n, k)$.

From the definition of embedding (5.4)

$$\tilde{\eta}_k(id) = \sum_{[\Gamma] \in \mathcal{U}_k} \frac{1}{\alpha_{d_\Gamma}} \sum_{d \in O_{d_\Gamma}} x_d.$$

For $[\Gamma] \in \mathcal{U}_k$, by construction of d_Γ we have $d_\Gamma \in \mathcal{Y}_k$, and so the orbit $O_{d_\Gamma} \subset \mathcal{Y}_k$.

Conversely, given $d \in \mathcal{Y}_k$, then under the bijection (5.3), $[\Gamma'] \in \tilde{\mathcal{B}}_k$ corresponding to the orbit O_d belongs to $\mathcal{U}_k$. So we obtain $\bigsqcup_{[\Gamma] \in \mathcal{U}_k} O_{d_\Gamma} = \mathcal{Y}_k$. Now the result follows by observing that for d_1, d_2 in the same orbit, $\alpha_{d_1} = \alpha_{d_2}$.

Since $\tilde{\eta}_k$ is an embedding, it is enough to show that image of $\tilde{\eta}_k$ is $eP_k(\xi)e$. The map $\tilde{\eta}_k$ is an algebra homomorphism and $\tilde{\eta}_k(id) = e$ therefore e is an idempotent and also the image of $\tilde{\eta}_k$ is contained in $eP_k(\xi)e$. For $d_0 \in A_k$, we show that $ex_{d_0}e$ is in the image of $\tilde{\eta}_k$. For $d = \{B_1, \dots, B_n\} \in A_k$ define $U(d) := (B_1^u, \dots, B_n^u)$ and

$L(d) := (B_1^l, \dots, B_n^l)$. We have

$$\begin{aligned}
ex_{d_0}e &= \sum_{d_1, d_2 \in \mathcal{Y}_k} \frac{1}{\alpha_{d_1} \alpha_{d_2}} x_{d_1} x_{d_0} x_{d_2} \\
&= \sum_{d_1, d_2 \in \mathcal{Y}_k, L(d_2) = U(d_0)} \frac{1}{\alpha_{d_1} \alpha_{d_2}} x_{d_1} x_{d_0 \circ d_2} \\
(5.16) \quad &= \sum_{d_1, d_2 \in \mathcal{Y}_k, L(d_2) = U(d_0), L(d_0) = U(d_1)} \frac{1}{\alpha_{d_1} \alpha_{d_2}} x_{d_1 \circ d_0 \circ d_2}
\end{aligned}$$

Since $d_1 \in \mathcal{Y}_k$ and $L(d_0) = U(d_1)$, we have $\alpha_{d_0} = \alpha_{d_1}$. Since $L(d_2) = U(d_0)$ therefore α_{d_2} is the cardinality β_{d_0} of S_k -orbit of $\text{set}(U(d_1))$.

For $d_1, d_2 \in \mathcal{Y}_k$, $L(d_2) = U(d_0)$, $L(d_0) = U(d_1)$, d_0 and $d_1 \circ d_0 \circ d_2$ are in the same orbit. Conversely, if d in the orbit of d_0 then there exist $d_1, d_2 \in \mathcal{Y}_k$ such that $d = d_1 \circ d_0 \circ d_2$.

If d_0 has exactly $i_{(a,b)}$ blocks $B_{m_1}, \dots, B_{m_{i_{(a,b)}}}$ such that $|B_{m_s}^u| = |B_{m_t}^u| = a$ and $|B_{m_s}^l| = |B_{m_t}^l| = b$ for $1 \leq s, t \leq i_{(a,b)}$ then there are $\theta_{d_0} = \prod_{(a,b) \in \mathbb{N} \times \mathbb{N}} (i_{(a,b)}!)$ partition diagrams d_1 and d_2 in $\mathcal{Y}_k$ such that $d_1 \circ d_0 \circ d_2 = d$. Then the sum (5.16) simplifies to

$$(5.17) \quad \sum_{d \in O_{d_0}} \frac{\theta_{d_0}}{\alpha_{d_0} \beta_{d_0}} x_d = \frac{\theta_{d_0}}{\beta_{d_0}} \left(\frac{1}{\alpha_{d_0}} \sum_{d \in O_{d_0}} x_d \right) = \frac{\theta_{d_0}}{\beta_{d_0}} \tilde{\eta}_k([\Gamma])$$

where $[\Gamma] \in \tilde{\mathcal{B}}_k$ corresponds to O_{d_0} under the bijection (5.3). □

Theorem 5.2.5. *The algebra $\mathcal{MP}_k(\xi)$ over F is cellular. Furthermore, $\mathcal{MP}_k(\xi)$ over F is semisimple when ξ is not an integer or when ξ is an integer such that $\xi \geq 2k - 1$.*

Proof. From Lemma 5.2.4 we have $\mathcal{MP}_k(\xi) \simeq e\mathcal{P}_k(\xi)e$ where

$$e = \sum_{d \in \mathcal{Y}_k} \frac{1}{\alpha_d} x_d.$$

We show that $e\mathcal{P}_k(\xi)e$ is a cellular algebra with respect to the involution i . The restriction of map i to $\mathcal{Y}_k$ is a bijection on $\mathcal{Y}_k$. Moreover, for $d \in \mathcal{Y}_k$, we have $i(x_d) = x_{i(d)}$. So $i(e) = \sum_{d \in \mathcal{Y}_k} \frac{1}{\alpha_d} x_{i(d)}$. By observing that for $d = \{(B_1^u, B_1^l), \dots, (B_n^u, B_n^l)\} \in \mathcal{Y}_k$, the cardinality of S_k -orbit of $\{B_1^u, \dots, B_n^u\}$ is the same as the cardinality of S_k -orbit of $\{B_1^l, \dots, B_n^l\}$, i.e, $\alpha_d = \alpha_{i(d)}$ we conclude that $i(e) = e$. By Proposition 5.2.1 we have $e\mathcal{P}_k(\xi)e$ is a cellular algebra.

Suppose ξ is not an integer or ξ is an integer such that $\xi \geq 2k - 1$. In this case, $P_k(\xi)$ is semisimple, so $\text{rad}(P_k(\xi)) = \{0\}$. Since e is an idempotent, $\text{rad}(eP_k(\xi)e) = e(\text{rad}(P_k(\xi)))e$. By Lemma 5.2.4 we have $\text{rad}(\mathcal{MP}_k(\xi)) = \{0\}$ and thus $\mathcal{MP}_k(\xi)$ is semisimple. $\square$

Chapter 6

The generalized multiset partition algebra

For every vector $\lambda = (\lambda_1, \dots, \lambda_s)$ of non-negative integers, we define a generalized multiset partition algebra $\mathcal{MP}_\lambda(\xi)$ over $F[\xi]$ with a basis indexed by the multiset partitions of $\{1^{\lambda_1}, \dots, s^{\lambda_s}, 1'^{\lambda_1}, \dots, s'^{\lambda_s}\}$. We associate such a multiset partition to a class of bipartite multigraphs by a process analogous to Section 3 by generalizing the definition of $\mathcal{B}_k$ for λ . The multiplication rule in $\mathcal{MP}_\lambda(\xi)$ is completely analogous to the case of $\mathcal{MP}_k(\xi)$. So here, we will omit the proofs.

Let Γ be a bipartite multigraph on $\mathcal{V}_\lambda \sqcup \mathcal{V}_\lambda$ where

$$(6.1) \quad \mathcal{V}_\lambda = \{I = (i_1, i_2, \dots, i_s) \in \mathbf{Z}_{\geq 0}^s \mid i_j \leq \lambda_j, 1 \leq j \leq s\}.$$

For an edge e of Γ joining $(i_1, i_2, \dots, i_s)$ in the top row with $(j_1, j_2, \dots, j_s)$ in the bottom row, the weight $w(e)$ of e is $((i_1, i_2, \dots, i_s), (j_1, j_2, \dots, j_s)) \in \mathbf{Z}_{\geq 0}^s \times \mathbf{Z}_{\geq 0}^s$.

Let $\mathcal{B}_\lambda$ denote the set of bipartite multigraphs Γ on $\mathcal{V}_\lambda \sqcup \mathcal{V}_\lambda$ such that the total weight of edges of Γ is $(\lambda, \lambda) := ((\lambda_1, \dots, \lambda_s), (\lambda_1, \dots, \lambda_s))$. For

$\mathbf{0} = (0, \dots, 0) \in \mathbf{Z}_{\geq 0}^s$, an edge e of Γ is non-zero if $w(e) \neq (\mathbf{0}, \mathbf{0})$. The $\text{rank}(\Gamma)$ is

the number of non-zero edges of Γ .

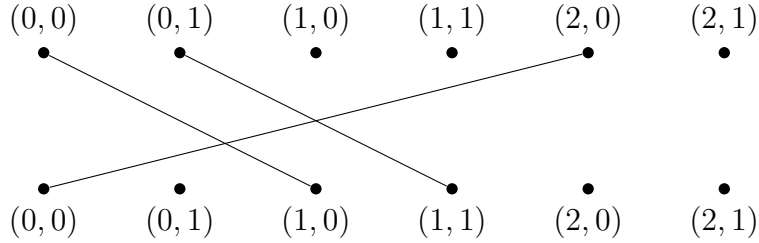
Two bipartite multigraphs in $\mathcal{B}_\lambda$ are said to be equivalent if they have the same non-zero weighted edges. Let $\tilde{\mathcal{B}}_\lambda$ denote the set of all equivalence classes in $\mathcal{B}_\lambda$. We define $\text{rank}([\Gamma]) := \text{rank}(\Gamma)$.

Let $(I, J) := ((i_1, i_2, \dots, i_s), (j_1, j_2, \dots, j_s))$ be a non-zero weighted edge of a graph in $\mathcal{B}_\lambda$. Then the following correspondence

$$(6.2) \quad (I, J) \leftrightarrow \{1^{i_1}, \dots, s^{i_s}, 1'^{j_1}, \dots, s'^{j_s}\}$$

extends to a bijection between the set $\tilde{\mathcal{B}}_\lambda$ and the set of multiset partitions of $\{1^{\lambda_1}, \dots, s^{\lambda_s}, 1'^{\lambda_1}, \dots, s'^{\lambda_s}\}$. Thus the elements of $\tilde{\mathcal{B}}_\lambda$ are the diagrammatic interpretations of the multiset partitions of $\{1^{\lambda_1}, \dots, s^{\lambda_s}, 1'^{\lambda_1}, \dots, s'^{\lambda_s}\}$.

Example 6.0.1. Consider the following graph in $\mathcal{B}_{(2,1)}$.



The multiset partition of $\{1^2, 2, 1'^2, 2'\}$ associated to this graph is

$$\{\{1'\}, \{2, 1', 2'\}, \{1^2\}\}.$$

Define $\mathcal{MP}_\lambda(\xi)$ to be the free module over $F[\xi]$ with basis $\tilde{\mathcal{B}}_\lambda$. Let $\mathcal{U}_\lambda$ be the subset of $\tilde{\mathcal{B}}_\lambda$ consisting of equivalence classes of graphs whose edges are of the form (I, I) for $I \in \mathcal{V}_\lambda$. The following theorem gives the multiplication in $\mathcal{MP}_\lambda(\xi)$.

Theorem 6.0.2. For $[\Gamma_1], [\Gamma_2]$ in $\tilde{\mathcal{B}}_\lambda$, we define the following operation:

$$(6.3) \quad [\Gamma_1] * [\Gamma_2] = \sum_{[\Gamma] \in \tilde{\mathcal{B}}_\lambda} \Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma]}(\xi)[\Gamma],$$

where $\Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma]}(\xi) \in F[\xi]$ is given in Equation (6.4). The linear extension of the operation (6.3) makes $\mathcal{MP}_\lambda(\xi)$ an associative unital algebra over $F[\xi]$ with the identity element $id = \sum_{[\Gamma] \in \mathcal{U}_\lambda} [\Gamma]$.

In order to define $\Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma]}(\xi)$, as stated in Theorem 6.0.2, we need to give the following set-up. Given $\Gamma, \Gamma_1, \Gamma_2 \in \mathcal{B}_\lambda$ with n edges each, now by merely replacing the vertices by tuples of integers in the case of graphs in $\mathcal{B}_k$, we have analogous definitions of the configuration of paths $P = \{p_1, p_2, \dots, p_n\}$, the sets $\text{Supp}^n(\Gamma_1, \Gamma_2)$, $\text{Supp}_\Gamma^n(\Gamma_1, \Gamma_2)$ and $C_{\Gamma_1\Gamma_2}^\Gamma(n)$.

For an edge of weight $(I, J) = ((i_1, i_2, \dots, i_s), (j_1, j_2, \dots, j_s))$ in Γ_P with $P \in \text{Supp}_\Gamma^n(\Gamma_1, \Gamma_2)$, and $L \in \mathcal{V}_\lambda$, define

$$p_{IJ}(L) := \text{cardinality of the multiset } \{p_t \in P \mid p_t = (I, L, J)\}.$$

Then $\sum_{L \in \mathcal{V}_\lambda} p_{IJ}(L)$ is the multiplicity p_{IJ} of the edge (I, J) of Γ_P . Suppose $|\mathcal{V}_\lambda| = a$ and let $v_0, v_1, \dots, v_a$ be an enumeration of the elements of $\mathcal{V}_\lambda$ in the weakly increasing lexicographic order (so in particular $v_0 = \mathbf{0}$).

Let $[\Gamma_1], [\Gamma_2]$, and $[\Gamma]$ be in $\tilde{\mathcal{B}}_\lambda$. Define

$$(6.4) \quad \Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma]}(\xi) = \sum_{P \in \text{Supp}_\Gamma^{2|\lambda|}(\Gamma_1, \Gamma_2)} \mathcal{K}_P \cdot (\xi - \text{rank}(\Gamma))_{[P^{\Gamma_1 \circ \Gamma_2}]},$$

where $[P^{\Gamma_1 \circ \Gamma_2}] = \sum_{i=1}^a p_{\mathbf{0}\mathbf{0}}(v_i)$,

$$\mathcal{K}_P = \frac{1}{p_{\mathbf{0}\mathbf{0}}(v_1)! \cdots p_{\mathbf{0}\mathbf{0}}(v_a)!} \prod_{(I, J) \in D \setminus \{(\mathbf{0}, \mathbf{0})\}} \binom{p_{IJ}}{p_{IJ}(v_0), \dots, p_{IJ}(v_a)}$$

and D denotes the set of all distinct edges of Γ .

6.1 Centralizer algebra and Schur–Weyl duality

In this section, we give a basis of $\text{End}_{S_n}(\text{Sym}^\lambda(F^n))$, and also state the Schur–Weyl duality between the actions of S_n and $\mathcal{MP}_\lambda(n)$ on $\text{Sym}^\lambda(F^n)$.

Recall the definition of $\text{Sym}^\lambda(F^n)$ from (4.1). The choice of indexing set $M(n, \lambda_j)$ for a basis of each $\text{Sym}^{\lambda_j}(F^n)$ yields the following indexing set $M(n, \lambda)$ for a basis of $\text{Sym}^\lambda(F^n)$

$$M(n, \lambda) := \left\{ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \end{bmatrix} \mid a_{ij} \in \mathbf{Z}_{\geq 0}, \sum_{j=1}^n a_{ij} = \lambda_i \right\}.$$

The space $\text{Sym}^\lambda(F^n)$ has a basis $\{e^A \mid A \in M(n, \lambda)\}$, where

$$e^A := e_1^{a_{11}} \cdots e_n^{a_{1n}} \otimes e_1^{a_{21}} \cdots e_n^{a_{2n}} \otimes \cdots \otimes e_1^{a_{s1}} \cdots e_n^{a_{sn}}.$$

The symmetric group S_n acts on an element $A \in M(n, \lambda)$ by permuting columns of A . As previously (see Equation (3.6)), the following isomorphism makes $\text{Sym}^\lambda(F^n)$ a permutation representation of S_n

$$(6.5) \quad \text{Sym}^\lambda(F^n) \cong F[M(n, \lambda)].$$

As in Section 3 (see Equation (3.8)), the set of S_n -orbits of $M(n, \lambda) \times M(n, \lambda)$ is in bijection with the set $\tilde{\mathcal{B}}_{\lambda, n}$ consisting of elements of rank at most n in $\tilde{\mathcal{B}}_\lambda$. In the following definition, we give the integral operator corresponding to an S_n -orbit of $M(n, \lambda) \times M(n, \lambda)$.

Definition 6.1.1. For each $[\Gamma] \in \tilde{\mathcal{B}}_{\lambda,n}$ define $T_{[\Gamma]} \in \text{End}_{S_n}(F[M(n, \lambda)])$ by

$$T_{[\Gamma]} 1_A = \sum_{\{B \in M(n, \lambda) \mid [\Gamma_{A,B}] = [\Gamma]\}} 1_B, \quad \text{where } A, B \in M(n, \lambda).$$

Theorem 6.1.2. *The set $\{T_{[\Gamma]} \mid [\Gamma] \in \tilde{\mathcal{B}}_{\lambda,n}\}$ is a basis of $\text{End}_{S_n}(F[M(n, \lambda)])$.*

For vector $\lambda = (\lambda_1, \dots, \lambda_s)$ of non-negative integers, let $|\lambda| = \sum_{i=1}^s \lambda_i$.

Theorem 6.1.3. 1. *Define a map*

$$(6.6) \quad \phi : \mathcal{MP}_\lambda(n) \rightarrow \text{End}_{S_n}(\text{Sym}^\lambda(F^n)) \text{ by}$$

$$\phi([\Gamma]) = \begin{cases} T_{[\Gamma]} & \text{if } \text{rank}[\Gamma] \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then map ϕ is a surjective algebra homomorphism with the kernel

$$\ker(\phi) = F\text{-span}\{[\Gamma] \in \tilde{\mathcal{B}}_\lambda \mid \text{rank}([\Gamma]) > n\}.$$

In particular, when $n \geq 2|\lambda|$, $\mathcal{MP}_\lambda(n) \cong \text{End}_{S_n}(\text{Sym}^\lambda(F^n))$.

2. *The group S_n generates $\text{End}_{\mathcal{MP}_\lambda(n)}(\text{Sym}^\lambda(F^n))$.*

6.2 Cellularity of $\mathcal{MP}_\lambda(\xi)$

In this section, we give embedding of $\mathcal{MP}_\lambda(\xi)$ inside $\mathcal{P}_{|\lambda|}(\xi)$. We obtain this embedding by generalizing the bijection in Theorem 5.1.2.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ be a vector of non-negative integers such that $|\lambda| = k$.

Given $\Gamma \in \tilde{\mathcal{B}}_\lambda$ with n edges, we have a pair (P, Q) of $s \times n$ matrices that uniquely

determines the graph, where

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p_{s1} & p_{s2} & \cdots & p_{sn} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ q_{s1} & q_{s2} & \cdots & q_{sn} \end{bmatrix}.$$

Each column of P is the label for a vertex in the top row of Γ and each column of Q is the label for a vertex in the bottom row of Γ . A column of P is connected by an edge in Γ to the corresponding column of Q . The columns of P and Q are simultaneously permuted such that columns of P are in weakly increasing lexicographic order, and if $(p_{1j}, \dots, p_{sj})^T = (p_{1j+1}, \dots, p_{sj+1})^T$ then $(q_{1j}, \dots, q_{sj})^T \leq (q_{1j+1}, \dots, q_{sj+1})^T$.

We define a set partition of $\{1, \dots, k, 1', \dots, k'\}$ corresponding to the pair of matrices (P, Q) :

$$(6.7) \quad d_\Gamma = \{B_1, \dots, B_n\}, \quad \text{where}$$

$$B_j = B_{1j} \cup \cdots \cup B_{sj} \quad \text{and}$$

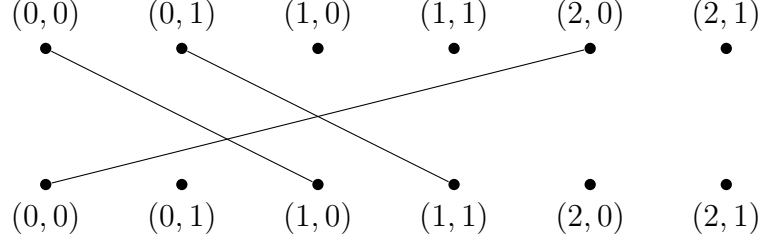
$$(6.8) \quad B_{ij} := \left\{ \left(\sum_{r=1}^{i-1} \lambda_r + \sum_{l=1}^{j-1} p_{il} \right) + 1, \dots, \left(\sum_{r=1}^{i-1} \lambda_r + \sum_{l=1}^j p_{il} \right), \right. \\ \left. \left(\sum_{r=1}^{i-1} \lambda_r + \sum_{l=1}^{j-1} q_{il} + 1 \right)', \dots, \left(\sum_{r=1}^{i-1} \lambda_r + \sum_{l=1}^j q_{il} \right)' \right\}.$$

The block of d_Γ corresponding to edge $(\mathbf{0}, \mathbf{0})$ is empty. It follows from the construction that if two graphs in $\mathcal{B}_\lambda$ are equivalent, then their corresponding partition diagrams defined as above are the same. The diagram d_Γ is called the canonical partition diagram associated to $[\Gamma] \in \tilde{\mathcal{B}}_\lambda$.

Example 6.2.1. Let $\lambda = (2, 1)$. Here

$$\mathcal{V}_{(2,1)} = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}.$$

Let Γ be the following bipartite multigraph on the vertex set $\mathcal{V}_{(2,1)} \sqcup \mathcal{V}_{(2,1)}$.



Then corresponding pair of matrices are follows:

$$P = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

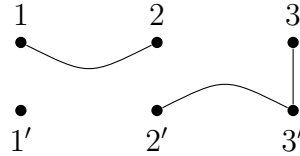
And we have:

$$\begin{aligned} B_{11} &= \{1'\}, & B_{12} &= \{2'\}, & B_{13} &= \{1, 2\}, \\ B_{21} &= \emptyset, & B_{22} &= \{3, 3'\}, & B_{23} &= \emptyset. \end{aligned}$$

Following Equation (6.7), the blocks of canonical set partition associated to Γ are:

$$B_1 = \{1'\}, B_2 = \{3, 2', 3'\}, B_3 = \{1, 2\}.$$

This canonical partition diagram d_Γ in A_3 is depicted as follows:



The subgroup $S_\lambda \times S_\lambda$ of $S_k \times S_k$ acts on A_k by the restriction. So analogously to Theorem 5.1.2 we have:

Theorem 6.2.2. *Let $(S_\lambda \times S_\lambda) \setminus A_k$ be the set of orbits of the action of $S_\lambda \times S_\lambda$ on A_k . The following map is a bijection*

$$(6.9) \quad \begin{aligned} \psi : \tilde{\mathcal{B}}_\lambda &\rightarrow (S_\lambda \times S_\lambda) \setminus A_k \\ \psi([\Gamma]) &= O_{d_\Gamma}, \end{aligned}$$

where $[\Gamma] \in \tilde{\mathcal{B}}_\lambda$ and O_{d_Γ} is the $S_\lambda \times S_\lambda$ -orbit of d_Γ .

Corollary 6.2.3. *The set $\tilde{\mathcal{B}}_{(1^k)}$ is in bijection with A_k .*

Proof. The results follows from Theorem 6.2.2, because for $\lambda = (1^k)$, the subgroup $S_\lambda \times S_\lambda$ is trivial. $\square$

Utilizing the bijection in Theorem 6.2.2, we have the following theorem.

Theorem 6.2.4. *The following map is an injective algebra homomorphism*

$$(6.10) \quad \begin{aligned} \tilde{\eta}_\lambda : \mathcal{MP}_\lambda(\xi) &\rightarrow \mathcal{P}_k(\xi) \\ \tilde{\eta}_\lambda([\Gamma]) &= \frac{1}{\alpha_{d_\Gamma}} \sum_{d \in O_{d_\Gamma}} x_d, \end{aligned}$$

where $[\Gamma] \in \tilde{\mathcal{B}}_\lambda$. In particular, for $\lambda = (1^k)$, $\mathcal{MP}_\lambda(\xi) \cong \mathcal{P}_k(\xi)$.

Theorem 6.2.5. *Let $\mathcal{Y}_\lambda$ be the set of $d = \{(B_1^u, B_1^l), \dots, (B_n^u, B_n^l)\}$ in A_k*

satisfying the following property, for $1 \leq i \leq s$ and $1 \leq j \leq n$,

$$\begin{aligned} &\left| B_j^u \cap \left\{ \left(\sum_{r=1}^{i-1} \lambda_r \right) + 1, \dots, \left(\sum_{r=1}^i \lambda_r \right) \right\} \right| \\ &= \left| B_j^l \cap \left\{ \left(\left(\sum_{r=1}^{i-1} \lambda_r \right) + 1 \right)', \dots, \left(\left(\sum_{r=1}^i \lambda_r \right) \right)' \right\} \right|. \end{aligned}$$

Let $e = \sum_{d \in \mathcal{Y}_\lambda} \frac{1}{\alpha_d} x_d$. Then e is an idempotent in $\mathcal{P}_k(\xi)$ and $\mathcal{MP}_\lambda(\xi) \cong e\mathcal{P}_k(\xi)e$.

Theorem 6.2.6. *The algebra $\mathcal{MP}_\lambda(\xi)$ over F is cellular. Furthermore, $\mathcal{MP}_\lambda(\xi)$*

over F is semisimple when ξ is not an integer or when ξ is an integer such that $\xi \geq 2k - 1$.

6.3 Balanced multiset partition algebra

In this section, we explore the balanced multiset partition algebra, a subalgebra of $\mathcal{MP}_\lambda(\xi)$, motivated by a question in [9]. Our answer to his question is the Schur–Weyl duality between the actions of the balanced multiset partition algebra and the group of monomial matrices on $\text{Sym}^\lambda(F^n)$.

We say that a multiset partition is *balanced* if each block B of the multiset partition satisfies:

$$(6.11) \quad |B \cap \{1^{\lambda_1}, \dots, s^{\lambda_s}\}| = |B \cap \{1'^{\lambda_1}, \dots, s'^{\lambda_s}\}|.$$

Denote by $\tilde{\mathcal{B}}_\lambda^{bal}$ the set of all graphs $[\Gamma]$ in $\tilde{\mathcal{B}}_\lambda$ such that their associated multiset partition is balanced. Let $\mathcal{MP}_\lambda^{bal}(\xi)$ denote the submodule of $\mathcal{MP}_\lambda(\xi)$ over $F[\xi]$ with basis $\tilde{\mathcal{B}}_\lambda^{bal}$.

Proposition 6.3.1. *The submodule $\mathcal{MP}_\lambda^{bal}(\xi)$ is a subalgebra of $\mathcal{MP}_\lambda(\xi)$ with the basis*

$$\tilde{\mathcal{B}}_\lambda^{bal} = \{[\Gamma] \in \tilde{\mathcal{B}}_\lambda \mid |I| = |J| \text{ for all edges in } \Gamma \text{ of weight } (I, J)\}$$

where for $I = (i_1, \dots, i_s)$, $|I| = \sum_{l=1}^s i_l$.

Proof. Take an edge of weight (I, J) of a graph in $\tilde{\mathcal{B}}_\lambda^{bal}$. Using Equation (6.2) it determines the multiset $\{1^{i_1}, \dots, s^{i_s}, 1'^{j_1}, \dots, s'^{j_s}\}$. Then the balanced condition (6.11) gives $|I| = |J|$.

Now we claim that the algebra structure on $\mathcal{MP}_\lambda^{bal}$ is inherited from the multiplication defined in Equation (6.3). For $[\Gamma_1], [\Gamma_2] \in \tilde{\mathcal{B}}_\lambda^{bal}$, suppose $[\Gamma] \in \tilde{\mathcal{B}}_\lambda$

occurs in $[\Gamma_1] * [\Gamma_2]$. For every edge in $[\Gamma]$ of weight (I, K) , there exists a vertex J such that an edge in $[\Gamma_2]$ of weight (I, J) is concatenated with an edge in $[\Gamma_1]$ of weight (J, K) . This implies $|I| = |J| = |K|$, thus $[\Gamma] \in \tilde{\mathcal{B}}_\lambda^{bal}$. $\square$

Remark 4. For $[\Gamma_1], [\Gamma_2]$ in $\tilde{\mathcal{B}}_\lambda^{bal}$ and $P \in \text{Supp}(\Gamma_1, \Gamma_2)$, note that the polynomial $\Phi_{[\Gamma_1][\Gamma_2]}^{[\Gamma_P]}(\xi)$ defined in Equation (6.4) is independent of ξ as $[P^{\Gamma_1 \circ \Gamma_2}]$ is 0. So $\mathcal{MP}_\lambda^{bal}(\xi)$ is independent of ξ , and henceforth we denote $\mathcal{MP}_\lambda^{bal}(\xi)$ by $\mathcal{MP}_\lambda^{bal}$.

Let D be the subgroup of the diagonal matrices in $GL_n(F)$. Then $D \rtimes S_n$ is the subgroup of all the monomial matrices in $GL_n(F)$. So $D \rtimes S_n$ acts on $\text{Sym}^\lambda(F^n)$ by the restriction.

Theorem 6.3.2 (Centralizer of monomial matrices). *The set*

$$\{T_{[\Gamma]} \mid [\Gamma] \in \tilde{\mathcal{B}}_\lambda^{bal}, \text{rank}([\Gamma]) \leq n\}$$

is a basis of $\text{End}_{D \rtimes S_n}(\text{Sym}^\lambda(F^n))$.

Proof. Let $[\Gamma] \in \tilde{\mathcal{B}}_\lambda^{bal}$ such that $\text{rank}([\Gamma]) \leq n$. Let $\text{diag}(x_1, \dots, x_n)$ be a diagonal matrix in D and $A = (I_1, \dots, I_n) \in M(n, \lambda)$, where I_j is the j th column of A . Then

$$\begin{aligned} T_{[\Gamma]}(\text{diag}(x_1, \dots, x_n)e^A) &= T_{[\Gamma]}(x_1^{|I_1|} \dots x_n^{|I_n|} e^A) \\ &= x_1^{|I_1|} \dots x_n^{|I_n|} T_{[\Gamma]}(e^A) \\ &= x_1^{|I_1|} \dots x_n^{|I_n|} \sum_{[\Gamma_{A,B}]=[\Gamma]} e^B. \end{aligned}$$

On the other hand,

$$\begin{aligned} \text{diag}(x_1, \dots, x_n) T_{[\Gamma]}(e^A) &= \text{diag}(x_1, \dots, x_n) \sum_{[\Gamma_{A,B}]=[\Gamma]} e^B \\ &= \sum_{[\Gamma_{A,B}]=[\Gamma]} x_1^{|J_1|} \dots x_n^{|J_n|} e^B, \text{ if } B = (J_1, \dots, J_n). \end{aligned}$$

Since $[\Gamma] \in \tilde{\mathcal{B}}_\lambda^{bal}$ and $\Gamma_{A,B} = \Gamma$, this forces $|I_1| = |J_1|, \dots, |I_n| = |J_n|$. Thus $T_{[\Gamma]} \in \text{End}_{D \rtimes S_n}(\text{Sym}^\lambda(F^n))$.

Let $\psi \in \text{End}_{D \rtimes S_n}(\text{Sym}^\lambda(F^n))$. The symmetric group S_n is a subgroup of $D \rtimes S_n$, so we have an embedding of algebras

$$(6.12) \quad \text{End}_{D \rtimes S_n}(\text{Sym}^\lambda(F^n)) \subseteq \text{End}_{S_n}(\text{Sym}^\lambda(F^n)).$$

From Theorem 3.1.3,

$$(6.13) \quad \psi = \sum_{[\Gamma] \in \tilde{\mathcal{B}}_\lambda} \alpha_\Gamma T_{[\Gamma]}.$$

In order to prove the result it is sufficient to show that in Equation (6.13) if

$[\Gamma] \notin \tilde{\mathcal{B}}_\lambda^{bal}$ then $\alpha_\Gamma = 0$.

So let $[\Gamma'] \notin \tilde{\mathcal{B}}_\lambda^{bal}$. Then Γ' has an edge of weight (I', J') such that $|I'| \neq |J'|$. Let A' and B' in $M(n, \lambda)$ such that $\Gamma_{A', B'} = \Gamma'$. Note that there exists $1 \leq s \leq n$ such that I' and J' are the s th columns of A' and B' , respectively. Let $X = \text{diag}(x_1, \dots, x_n)$ such that $x_s = x$ and $x_i = 1$ for $i \neq s$, then

$$(6.14) \quad \begin{aligned} \psi(Xe^{A'}) &= \psi(x^{|I'|}e^{A'}) \\ &= x^{|I'|}\psi(e^{A'}) \\ &= x^{|I'|} \sum_{[\Gamma] \in \tilde{\mathcal{B}}_\lambda} \alpha_\Gamma T_{[\Gamma]}(e^{A'}) \\ &= x^{|I'|} \sum_{[\Gamma] \in \tilde{\mathcal{B}}_\lambda} \alpha_\Gamma \sum_{[\Gamma_{A', B'}] = [\Gamma]} e^B. \end{aligned}$$

On the other hand,

$$\begin{aligned}
X\psi(e^{A'}) &= X \sum_{[\Gamma] \in \tilde{\mathcal{B}}_\lambda} \alpha_\Gamma \sum_{[\Gamma_{A',B}] = [\Gamma]} e^B \\
(6.15) \quad &= \sum_{[\Gamma] \in \tilde{\mathcal{B}}_\lambda} \alpha_\Gamma \sum_{[\Gamma_{A',B}] = [\Gamma]} x^{|J_s|} e^B, \text{ where } J_s \text{ is the } s\text{th column of } B.
\end{aligned}$$

Now equating the coefficient of $e^{B'}$ in the Equations (6.14) and (6.15), we get

$$x^{|I'|} \alpha_{\Gamma'} = x^{|J'|} \alpha_{\Gamma'}, \text{ for all } x \in F.$$

Since $|I'| \neq |J'|$, we get $\alpha_{\Gamma'} = 0$. □

Theorem 6.3.3. *1. The map ϕ in Equation (6.6) restricts to a surjective algebra homomorphism from $\mathcal{MP}_\lambda^{bal}$ onto $\text{End}_{D \rtimes S_n}(\text{Sym}^\lambda(F^n))$ with kernel*

$$\ker(\phi) = F\text{-span}\{[\Gamma] \in \tilde{\mathcal{B}}_\lambda^{bal} \mid \text{rank}([\Gamma]) > n\}.$$

In particular, when $n \geq |\lambda|$, $\mathcal{MP}_\lambda^{bal} \cong \text{End}_{D \rtimes S_n}(\text{Sym}^\lambda(F^n))$.

2. The group $D \rtimes S_n$ generates $\text{End}_{\mathcal{MP}_\lambda^{bal}}(\text{Sym}^\lambda(F^n))$.

Recall from [9, p. 21] a set partition is called *balanced* if it satisfies Equation (6.11) for $\lambda = (1^k)$. Let $\mathcal{P}_k^{bal}$ denote the balanced partition algebra, i.e., the subalgebra of $\mathcal{P}_k(\xi)$ with basis consisting of balanced set partitions.

As a corollary of Theorem 5.1.5 we get:

Corollary 6.3.4. *For any vector λ of non-negative integers, the map (6.10) restricts to an embedding of algebras $\mathcal{MP}_\lambda^{bal} \hookrightarrow \mathcal{P}_k^{bal}$. In particular, when $\lambda = (1^k)$, $\mathcal{MP}_\lambda^{bal} \cong \mathcal{P}_k^{bal}$.*

Remark 5. When $\lambda = (1^k)$, Theorem 6.3.3 recovers Schur–Weyl duality ([9, Proposition 4.1]) between $\mathcal{P}_k^{bal}$ and $D \rtimes S_n$ acting on $(F^n)^{\otimes k}$.

Chapter 7

RSK correspondence for the multiset partition algebra

The Robinson–Schensted–Knuth (RSK) correspondence [13, Section 3] is a bijection from two-line arrays of positive integers onto pairs of semistandard Young tableaux of the same shape. This correspondence reflects a classical decomposition [26, Theorem 4.1], [10, Theorem 2.1.2]

$$\mathrm{Sym}^d(F^m \otimes F^n) = \bigoplus_{\lambda \vdash d} W_\lambda^n \otimes W_\lambda^m$$

as $GL_n(F) \times GL_m(F)$ bimodule. Here W_λ^n and W_λ^m are irreducible polynomial representations of $GL_n(F)$ and $GL_m(F)$ respectively.

In [5, Section 5], the authors gave a variant of the RSK correspondence by applying Schensted’s insertion algorithm on two-line arrays consisting of multiset partitions instead of positive integers. Their correspondence has the following nice interpretation in terms of multiset partition algebras.

For $n \geq 2|\lambda|$, by Theorem 6.1.3, we have, as $S_n \times \mathcal{MP}_\lambda(n)$ -bimodule,

$$(7.1) \quad \text{Sym}^\lambda(F^n) \cong \bigoplus_{\nu} V_{\nu} \otimes M_{\nu}^{\lambda}$$

where M_{ν}^{λ} denotes the irreducible representation of $\mathcal{MP}_\lambda(n)$ corresponding to the partition ν of n . Observe from Proposition 4.0.1 that the dimension of M_{ν}^{λ} is given by the number of semistandard multiset tableaux of shape ν and content $\{1^{\lambda_1}, \dots, s^{\lambda_s}\}$. We obtain the following enumerative identity [5, Section 5.3] by compairing dimensions of both sides of the decomposition (7.1)

$$\prod_{i=1}^s \binom{n + \lambda_i - 1}{\lambda_i} = \sum_{\nu \vdash n} |\text{SYT}(\nu)| \times |\text{SSMT}(\nu, \{1^{\lambda_1}, \dots, s^{\lambda_s}\})|,$$

where $\text{SYT}(\nu)$ is the set of standard Young tableaux of shpae ν .

In [5, Theorem 6.3], a bijection between the partition diagrams and pairs of set partition tableaux is exhibited, that gives a combinatorial proof of the decomposition of $\mathcal{P}_k(n)$ as $\mathcal{P}_k(n) \times \mathcal{P}_k(n)$ -module. In the following theorem, we extend their algorithm to give a combinatorial interpretation of the decomposition

$$\mathcal{MP}_\lambda(n) \cong \bigoplus_{\nu} M_{\nu}^{\lambda} \otimes M_{\nu}^{\lambda}$$

as $\mathcal{MP}_\lambda(n) \times \mathcal{MP}_\lambda(n)$ -module.

Theorem 7.0.1. *The multiset partitions of $\{1^{\lambda_1}, \dots, s^{\lambda_s}, 1'^{\lambda_1}, \dots, s'^{\lambda_s}\}$ with at most n parts are in bijection with pairs (T, S) of semistandard multiset tableaux of same shape ν , where ν is a partition of n , and S has content $\{1^{\lambda_1}, \dots, s^{\lambda_s}\}$, T has content $\{1'^{\lambda_1}, \dots, s'^{\lambda_s}\}$.*

Proof. Let $d = \{B_1, \dots, B_n\}$, some B_i 's are allowed to be empty, be a multiset partition of $\{1^{\lambda_1}, \dots, s^{\lambda_s}, 1'^{\lambda_1}, \dots, s'^{\lambda_s}\}$. Given a block B of d , we define $B^u := B \cap \{1^{\lambda_1}, \dots, s^{\lambda_s}\}$ and $B^l := B \cap \{1'^{\lambda_1}, \dots, s'^{\lambda_s}\}$.

Consider the biword of multisets

$$\begin{pmatrix} B_{i_1}^u & \cdots & B_{i_n}^u \\ B_{i_1}^l & \cdots & B_{i_n}^l \end{pmatrix}$$

satisfying

- $B_{i_j}^u \leq_L B_{i_{j+1}}^u$ for all $1 \leq j \leq n$,
- $B_{i_j}^l \leq_L B_{i_{j+1}}^l$ whenever $B_{i_j}^u = B_{i_{j+1}}^u$.

Observe that this biword uniquely determines the given multiset partition. By applying RSK insertion algorithm, we associate a pair of semistandard multiset tableaux (T, S) to the biword

$$RSK : \begin{pmatrix} B_{i_1}^u & \cdots & B_{i_n}^u \\ B_{i_1}^l & \cdots & B_{i_n}^l \end{pmatrix} \rightarrow (T, S),$$

where T is the insertion tableau and S is the recording tableau of same shape of size n . It is easy to see T, S have the required content. The inverse RSK algorithm completes the proof. $\square$

Example 7.0.2. Let $\lambda = (2, 2, 1)$ and $n = 6$. Consider the multiset partition $\{\{1, 2\}, \{1', 2'\}, \{1, 3, 1'\}, \{2, 2'\}, \{3'\}\}$ of $\{1^2, 2^2, 3^1, 1'^2, 2'^2, 3'^1\}$. We add an empty block and then arrange all six blocks in the prescribed manner to obtain the following biword:

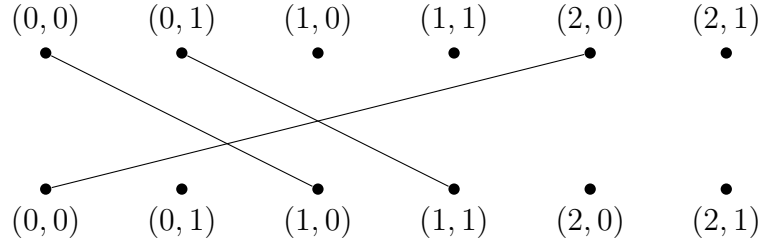
$$\begin{pmatrix} \emptyset & \emptyset & \emptyset & \{2\} & \{1, 2\} & \{1, 3\} \\ \emptyset & \{3'\} & \{1', 2'\} & \{2'\} & \emptyset & \{1'\} \end{pmatrix}.$$

By applying RSK insertion algorithm, we get

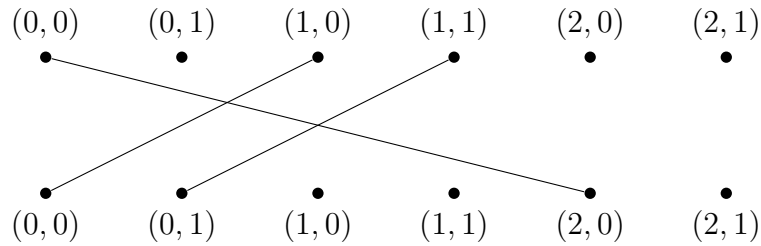
$$T = \begin{array}{|c|c|c|} \hline \emptyset & \emptyset & \{1'\} \\ \hline \{2'\} & \{1', 2'\} & \\ \hline \{3'\} & & \\ \hline \end{array}, \quad S = \begin{array}{|c|c|c|} \hline \emptyset & \emptyset & \emptyset \\ \hline \{2\} & \{1, 3\} & \\ \hline \{1, 2\} & & \\ \hline \end{array}.$$

Definition 7.0.3. Given a block B of a multiset partition, let B^t denote the block obtained from B by exchanging i by i' for all $i \in B$. A block is called *fixed* block if $B = B^t$. Given $d = \{B_1, \dots, B_n\}$, let $d^t = \{B_1^t, \dots, B_n^t\}$. We call a multiset partition d *symmetric* if $d = d^t$.

Let us understand this pictorially. Consider the multiset partition $d = \{\{1'\}, \{2, 1', 2'\}, \{1, 1\}\}$ of $\{1^2, 2, 1'^2, 2'\}$. The associated graph Γ is the following



Reflecting about the horizontal axis we get



The associated multiset partition of above is $d^t = \{\{1\}, \{2', 1, 2\}, \{1', 1'\}\}$. Hence d is not a symmetric diagram.

Corollary 7.0.4 (Symmetry). *If $d \leftrightarrow (T, S)$ via the RSK correspondence for multisets, then $d^t \leftrightarrow (S, T)$.*

Corollary 7.0.5 (Schützenberger's lemma). *Let d be a symmetric multiset partition with l fixed blocks and T be the associated multiset tableau of shape ν . Then ν has l columns of odd length.*

7.1 An open problem: Gelfand model for the multiset partition algebra?

Let $\mathcal{A}$ be a semisimple algebra over a field F and $\{A_\nu \mid \nu \in \Lambda\}$ be the set of all irreducible representation of $\mathcal{A}$. A Gelfand model for $\mathcal{A}$ is a representation M that contains each irreducible representation of $\mathcal{A}$ with multiplicity exactly one, that is $M \cong \bigoplus_{\nu \in \Lambda} A_\nu$.

For example, if $\mathcal{A} = FS_k$, then for every $\nu \vdash k$, the irreducible S_k module A_ν has a basis indexed by $|SYT(\nu)|$. The symmetry property of classical RSK correspondence gives a bijection between the set of standard Young tableaux with k entries and the set of involutions of S_k (diagrams which are fixed under horizontal symmetry). Let M be the F -span of those involutions. Then $\dim M = \sum_{\nu \vdash k} |SYT(\nu)|$. In [11], [14], the authors gave a action of symmetric group S_k on its involutions M such that M decomposes into irreducible S_k modules each with multiplicity 1. In [26, Theorem 7.5], we used the RSK correspondence to obtain a model for $GL_k(F)$ and restricted it to its (1^k) weight space to get a Gelfand model for S_k .

In the same spirit, for partition algebra and its subalgebras, Tom Halverson and Mike Reeks [8] have constructed a Gelfand model based on symmetric diagrams.

We can have an analogous Gelfand model for the new diagram algebra $\mathcal{MP}_\lambda(n)$

when $n \geq 2|\lambda|$, based on symmetric multiset partitions of $\{1_1^\lambda, \dots, s^{\lambda_s}, 1'^{\lambda_1}, \dots, s'^{\lambda_s}\}$. Let M be the free F module with basis consisting of symmetric multiset partitions. By Corollary 7.0.4,

$$\dim M = \sum_{\nu \vdash n} |SSMT(\nu, \{1^{\lambda_1}, \dots, s^{\lambda_s}\})|.$$

Since the number on the right-hand side is the total dimension of all irreducible representations of $\mathcal{MP}_\lambda(n)$, it should be possible to define an action of the algebra on the vector space M so that it decomposes into irreducible representations of $\mathcal{MP}_\lambda(n)$. We hope that this question is also tractable.

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