ON SOME METHODS OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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P R E F A C E

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C O N T E N T S

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CHAPTER-I

INTRODUCTION

1. Preliminary remarks

Stochastic theory plays a vital role in the modelling and analysis of a variety of disciplines ranging from social sciences to biological studies, besides its being an integral part of the modelling of many physical processes. The evolution of system is described by means of a random function defined on a suitable parameter space. In many of the evolving processes, a certain regularity is found in the statistical features of the random function. This important fact enables us to extend the methods employed in deterministic systems with suitable modifications and with appropriate criteria, to the problems of random functions.

Thus the random functions obey difference or differential or integrodifferential equations, characteristic of the evolutionary phenomena they describe. Such random equations arise in the investigation of a variety of phenomena, besides the popular Brownian motion. The following are some of the situations in which random equations play a significant role: wave propagation in random media, electrical circuit theory, turbulence theory, astrophysical problems, analysis of neuronal networks, population studies, statistical mechanics of continuous media. A lucid and systematic analysis of random equations can be found in the survey article by Ramakrishnan (1959). The reader can study with profit the book by Srinivasan and Vasudevan (1971).

Generally, coefficients, parameters of initial conditions in a classical differential equation can be random. Also the forcing term in the differential equation may be a random function. A simple example is

$$\dot{X}(t) = f(X(t), \eta(t), t), \quad X(t_0) = C \tag{1.1}$$

with the random function η (t), the random initial value C. If the functions are well behaved, satisfying some regularity properties, we can consider eq.(1.1) as a family of classical problems for the individual sample functions. The classical methods are available for treating such equations.

But, we have a different picture if the random functions are of the 'white noise' type. Then eq.(1.1) is written formally as

$$\dot{X}(t) = f(X(t), t) + G(X(t), t)\, W(t)$$
$$X(t_0) = C \tag{1.2}$$

where W(t) is the white noise. The white noise is described as a stationary Gaussian process with mean zero and constant spectral density. Such a process does not exist in the usual sense; but it is a very useful mathematical idealisation for describing random influences.

Langevin (1908) introduced the white noise as the force exerted on the Brownian particle by the molecular collisions. In the Langevin's equation, the white noise W(t) is introduced additively and hence the probability distributions of the solution process can be calculated.

But in the model described by eq.(1.2), the white noise enters the equation in a multiplicative way and a new theory is necessary for studying this equation. Even though, the white noise process bristles with difficulties when one tries to rigorously formulate the integrals over such functions, it is identified to be the formal derivative of the Brownian motion process $B(t)$. Hence eq.(1.2) can be written as

$$dX(t) = f(X(t),t)\,dt + G(X(t),t)\,dB(t)$$
$$X(t_0) = C \tag{1.3}$$

This can be written as the integral equation

$$X(t) = C + \int_{t_0}^{t} f(X(\tau),\tau)\,d\tau + \int_{t_0}^{t} G(X(\tau),\tau)\,dB(\tau) \tag{1.4}$$

The first integral on the right hand side of eq.(1.4) can be defined in the usual way. But the second integral cannot be defined as a Riemann-Stieltjes integral with respect to the sample functions of $B(t)$, because the value depends on the intermediate points in the approximating sums. In 1951, Ito defined integrals of the above type and put the theory of stochastic differential equations on a solid footing. It may be pointed out that Ito's theory is established as a self consistant theory and not as an extension of ordinary differential equations. The rules of Ito calculus differ from the rules of ordinary calculus, mainly due to the peculiar properties of the Brownian motion process $B(t)$. The fact that

$E\left[\{B(t+\Delta t) - B(t)\}^2\right]$ is proportional to Δt plays an important role in the Ito's rule for determining the differentials.

In 1964, Stratonovich introduced another type of stochastic integrals and equations with applications in various modelling problems. The rules of Stratonovich calculus agree with those of ordinary calculus.

Methods have been developed to write the Fokker Planck equations for the Langevin equation (1.3) described with Ito and Stratonovich rules. The difference in the two Fokker Planck equations is a term in the drift velocity, appearing in the Stratonovich but not in the Ito's method. The difference is due to different assumptions made by Ito and Stratonovich about the correlations between the system variables and the fluctuations. Whenever the fluctuations are additive, the two approaches are equivalent. But when they appear as multiplicative term, the correlation between the fluctuating function and the system variables is finite in Stratonovich rule, while it is assumed to be zero in Ito's rule.

Recently there has been a growing interest in systems described by stochastic differential equations in which the fluctuations depend multiplicatively on the system variables. Naturally we have to consider the two approaches as described above and study the implications. Similar attempts have been made in a variety of contexts. Examples of such phenomena include electromagnetic waves propagating in a random medium Tatarski (1961), magnetic resonance (Kubo 1966) Kubo line broadening (West et al 1979).

The main theme of this thesis is to study the Langevin equations arising in different physical contexts in the light of

Ito and Stratonovich theories. The areas of investigation broadly include fluctuation dissipation relations, stability problems, applications of path integral techniques and smoothing approximation methods. Also stochastic differential equations driven by point processes are considered. The point processes are characterised in terms of certain point functions known as cumulants and product densities. (Ramakrishnan 1950, 1959). We relate them to new concept of combinants and the relevance of Bell polynomials is high lighted. In the next section we give short summarise of the various chapters in this thesis.

2. Chapter Summaries

This thesis consists of six chapters.

In Chapter II, we give a short account of stochastic differential equations as formulated by Ito and Stratonovich. The mathematical aspects of the two prescriptions are brought out. The symmetric ⚲ -multiplication method of Ito (1950) and Hasegawa (1980) is explained and Ito's chain rule is derived. The solution process of Ito stochastic differential equation is Markovian and hence we can write the Fokker Planck equations for the evolution of the probability density of the solution process. Even before introducing his concept of stochastic integrals, Stratonovich (1963) derived the Fokker-Planck equation taking into account the correlation between the fluctuating functions and the system variables. We give a short account of his derivation due to its importance in our future applications. We have obtained the Fokker-Planck equation of the Langevin equation in the two senses, by considering the evolution of the characteristic function of the solution process. This new method (R.Vasudevan and K.V.Parthasarathy, 1981a) entirely depends on calculating the infinitesimal generator (Wong 1971) for the system. This method is equally applicable for stochastic systems driven by both continuous and point processes.

We have given a short account of 'unified calculus' stressing canonical extension method of McShane (1974) and its generalisations of Marcus (1978). The concept of Lie series is introduced in the investigations.

In the final section of this chapter we have our new results on finding the Fokker-Planck equation for the stochastic system driven by random telegraph noise (R.Vasudevan and K.V.Parthasarathy 1981b). The stochastic Liouville's equation method of Van Kampen (1976) is used and the evolution of the probability density is found using cumulant expansion techniques (Mukamel 1978 etc). The steady state solution of the process is found by a new method of applying a suitable operator formalism. The approach to white noise limit and the relevance of Stratonovich theory are indicated. Our method (R.Vasudevan and K.V.Parthasarathy, 1981b) differs from the usual technique by Kitahara et al (1979) for coloured noise.

Chapter III deals with the new results on fluctuation - dissipation relations. Starting from the Langevin equation, Einstein established the relation between the frictional coefficient and the fluctuating force correlation. Taking a multiplicative term in the forcing function in the Langevin equation as random, we have obtained modified result on first fluctuation -

dissipation relation. We have also modified the generalised Langevin equation of Kubo (1966) by introducing a fluctuating force in multiplicative way. This gives a significant result in a modified form for the second fluctuation-dissipation theorem. We have also found out the time evolution of the averages of $\overline{L}$ and $\overline{L}^2$, $\overline{L}$ being the angular momentum. An observation on the virial theorem is noted and the relevance of Stratonovich rule is emphasised (R.Vasudevan and K.V.Parthasarathy, 1981c).

Chapter IV deals with the new results on stability theory, again discussed in the light of Stratonovich theory. Moment stability properties of the harmonic oscillator with fluctuating frequencies are widely discussed (West et al 1980). In dealing with non-linear oscillator problems, it is generally difficult to get closed sets of solutions of moment equations. Hence methods are devised to obtain the Fokker-Planck equation for a new variable like the energy of the system. The energy envelope method of Stratonovich is a powerful technique in this approach. We exploit another invariant of the system and the auxiliary equations, introduced by Lewis (1968) in his discussion on time dependent harmonic oscillator. The motion of the harmonic oscillator on a straight line can be considered as the projection of a two-dimensional motion of a particle under an attraction to the same centre. Using this fact, the equation of a different envelope of the motion is studied for the stability in mean and mean square. Due to the complexity of the Fokker-Planck equations, we use approximations to linearise the system,and obtain stability results similar to the results of West et al (1980) (R.Vasudevan and K.V.Parthasarathy, 1981d).

Chapter V, consists of three important sections. In the initial section, we obtain the mean and correlation function of the solution process of the stochastic differential equation $L^{x}\, x(t) = K(t)$, L^{x} being the stochastic differential operator of the form $L^{x} = \frac{\partial}{\partial t} + \gamma_0 + \gamma_1 Y(t)$, γ_0, γ_1 are constants, $Y(t)$ is the random process. This occurs in the description of the polarisation of the dielectric media (Mazur 1975). If we take several approximate solutions of the equations we can arrive at the smoothing approximation (Frisch 1968, Keller 1964, Adomian 1970) and other higher types of approximations of the equations. We compare these approximations with the solution corresponding to the Stratonovich formulation (R.Vasudevan and K.V.Parthasarathy, 1981f).

In the subsequent sections, we give some results on the path integration techniques. We follow the transformation rules given by Van Kampen (1980) for the Ito and Stratonovich approaches. Choosing a suitable transformation we express the equations in the transformed variables with additive fluctuating force. Following Haken (1976), we give the path integral solution for the probability density of the process. (Vasudevan, R. and Parthasarathy, K.V. 1981e).

Next we introduce the new concept of Routhian path integral.

In Feynman's approach, the propagator is described by a path integral depending upon the Lagrangian of the system. In recent years, more attention has been paid on the Hamiltonian path integrals in phase space (Garrod 1966). In classical mechanics (Goldstein 1980), Routh's procedure of solving a problem is well known, when some of the coordinates are cyclic. We use the picture of Routhian in the most general form, combining the features of both the Lagrangian and the Hamiltonian formulations. The ~~advantages of the~~ path integral defined through the Routhian has many advantages. (K.V.Parthasarathy, 1981).

In Chapter VI, we study how the combinants (Kauffmann, Gynlassy, 1978) can be related to the product densities (Ramakrishnan 1959). To characterise the probability of n-events occuring in a given interval, it is useful to describe p(n), the desired probability, in terms of its deviations from the Poisson. Writing the generating function as

$$F(\lambda) = \sum_{n=0}^{\infty} \exp\left[C(k)(\lambda^{k}-1)\right] \qquad (2\cdot1)$$

We obtain a characterisation of the probability by the quantity C(K) called the combinants. The cumulants of the distribution can be directly obtained from $C(k)$. To study point processes Ramakrishnan (1950) introduced the powerful tool of product densities. The product densities can be related to the factorial moments $\left\langle \frac{n}{\lfloor n-r} \right\rangle$ of the events occuring in an interval by proper integration over the continuous state variables. The moments of p(n) are related to those integrals by using the well known C_{r}^{n} coefficients. In a similar manner the combinants can be related to appropriate sum of integrals over cluster functions. Hence we note that the combinants play the same role in calculating the cumulants, as probabilities do in finding moments. Doubly stochastic point processes are studied using the method of combinants. As an application to branching phenomena, we have used the method of combinants to find the statistics of the population in the n^{th} generation.

The Bell polynomials (Riordon, 1958) are used as an effective tool in deriving the relations between C(k)'s and $\frac{P(k)}{P(0)}$ and vice versa. The problem of compound Poisson process is also discussed , using Bell Polynomials (R.Vasudevan, P.R.Vittal and K.V.Parthasarathy, 1981).

This approach to study the point process is new and is very useful to analyse ~~multiple and correlated, point functions in physical sciences.~~ physical phenomena, involving multiple production and highly correlated random processes

CHAPTER II

STOCHASTIC DIFFERENTIAL EQUATIONS

1. Introduction

Mathematical equations play a central role in the modelling analysis and the prediction of the various phenomena that arise in physics, biology, engineering, economics and other social sciences. These equations involve several parameters and coefficients; for instance, the diffusion coefficient in heat conduction, refractive index in wave propagation, volume scattering coefficient in underwater acoustics and growth rate in population studies. The magnitudes of these coefficients are experimentally determined and the mean of a set of experimental values is used for a particular coefficient. The average values of the coefficient or the parameter will serve the purpose in certain cases; but in many cases the dispersion may be so large, that the deterministic methods fail to provide suitable models for the problem. In other words, the coefficients are stochastic in nature. Hence, models based on stochastic formulations are necessary for the scientific investigations. By taking into account the various random effects, we get a random equation governing a scientific phenomenon. An elegant and detailed discussion of the basic principles and solutions of such random equations is found in the excellent article by Ramakrishnan(1959).

In a series of three classic papers, Ramakrishnan (1950, 1953, 1956) has described his original ideas regarding the integrals of random functions.

Random differential equations are differential equations involving random elements. In general, the randomness enters the equation through (i) the initial conditions, (ii) the forcing term and (iii) the coefficients. We follow the mean-square theory to study such equations. This approach is simple and well-developed and it is defined in terms of distributions and moments of the processes with which we are interested.

Random differential equations have been the subject of intensive study right from the beginning of this century. In the study of the motion of suspended pollen particles, Albert Einstein (1905, 1906) and Smoluchowsky (1916) introduced random equations and showed that the solutions of such equations can be obtained indirectly by the results of random walk theory. Langevin (1908) modelled the motion of Brownian particles by a differential equation with a random forcing term. A special ~~stocha~~ stochastic integral was defined by Campbell (1909), to deal with the the fluctuations in the electron stream, which was used to obtain the general results for the cumulative response to point events. In the mean time, Uhlenbeck and Ornstein (1930) gave a complete analysis of the theory of Brownian motion, by considering a second order linear differential equation with a random forcing term with Gaussian white characteristics. This was followed by another paper, by Van Lear and Uhlenbeck (1934), dealing with a second order partial differential equation with inhomogeneous random forcing term. These attempts had a profound impact on physicists and engineers, drawing their attention to the problems of stochastic processes. In this context, we refer to the Survey articles of Chandrasekhar (1943), and Wang and Uhlenbeck (1945), highlighting the major problems of physics discussed by the methods of stochastic theory. These works are of immense use in stochastic modelling problems encountered in physical and engineering sciences.

The formal mathematical analysis of Brownian motion and the consequent development of the theory of stochastic integrals were undertaken by Wiener (1920, 1921, 1930), anticipating Kolmogorov's formalisation of probability. The concept of convergence in the mean and the mean square and the development of stationary processes, highlighted by Wiener (1930), Khintchine (1934) and Karhunen (1950) continued to attract the attention of mathematicians, physicists and electrical engineers. In a comprehensive survey, Moyal (1949) gave a detailed account of the theory of random integrals and equations with illustrations from different disciplines of physics. Bartlett (1966) incorporated most of these results in his monograph. This was followed by a physical approach to stochastic processes and in particular to stochastic integrals by Ramakrishnan (1959), who gave an excellent account of the subject in his survey articles, published in Handbuch der Physik. Different types of random differential equations have been analysed in Ramakrishnan et. al. (1956a), (1956b), (1960). Since then, there has been tremendous activity in this area among physicists and engineers, apart from the parallel and systematic attempts by mathematicians (Doob 1953, Ito 1951, Gikhman and Skorokhod 1968, 1971).

Wong and Zakai (1965) discussed the relation between ordinary and stochastic differential equations with special reference to Ito's work. It was pointed out clearly that the theory of stochastic differential equations was established by Ito as a self-consistent theory and not as an extension or limit of ordinary differential equations. The main object in the study of such stochastic differential equations was to provide a construction of Markov processes, corresponding to the system of differential equations. But due to the inherent non-uniqueness present in this approach, some physical scientists (for example, Caughey and Gray, 1965) came forward with other definitions of integrals to be used in modelling. The work of Stratonovich (1964) merits special mention, since it provides an alternative and fruitful approach to stochastic integrals. Stochastic differential equation written in the symmetrized form of Stratonovich can be interpreted as the limit of equations written for non-Markov processes.

The ambiguity in the values of the stochastic integrals mentioned so far, may be eliminated to a great extent by employing the sample path approach developed by Wiener. Cameron and Martin (1947) published a series of papers to evaluate Wiener integrals of a class of functionals. Feynman (1948) made explicit use of Wiener's concept in his reformulation of non-relativistic quantum mechanics in terms of an integral over the space of paths. Ramakrishnan (1956) proposed a phenomenological interpretation of integrals of a class of random functions. It extensively uses the idea of the probability of sample paths and incorporates the concept of path integral in a different manner from that of Wiener measure. Later on, Srinivasan and Vasudevan (1971) gave a heuristic account of the estimation of probabilities of sample paths, in the context of integrals of random processes, which included point processes as well as the Gaussian white noise process. Their monograph lays emphasis on specific phenomena and the mathematical problems that arise from them, have been solved to a reasonable degree of satisfaction. Recently the stochastic calculus developed by McShane (1974) provides a reasonable framework for the development of stochastic models leading to unambiguous results. Having thus traced the development of the study of random differential equations, we give a brief account of stochastic integrals of Ito and Stratonovich.

The layout of this chapter is as follows. Section 2 introduces the necessary preliminary concepts of stochastic processes. We give a short summary of specific processes like

Markov processes and independent-increment processes. Then the concepts regarding the analysis of ~~stochastic~~ Stochastic processes such as continuity, differentiation and integration are presented. The section concludes with a brief review of the Wiener Process and the White Noise Process. In Section 3, we introduce the stochastic integrals as formulated by Ito and Stratonovich. We pointed out the important fact of the Brownian motion process $B(t)$, i.e. $E\left\{(\Delta B(t))^2\right\}$ is of first order in Δt. This fact influences the definition of Ito integral and the Ito differentiation rule. Hence the rules of the Ito calculus differ from those of ordinary calculus. The relationship between the two integrals is stated. Also the symmetric Q-multiplication method of Ito is discussed and as an application, Ito's chain rule is given. Section 4 provides a short account of Fokker-Planck equations corresponding to the two types of Langevin's equation. Even before introducing his concept of stochastic integrals, Stratonovich (1963) derived the Fokker-Planck equation taking into account the correlation between solution and fluctuating functions. We give a short account of his derivation. This is followed by another method of deriving the Fokker-Planck equation in section 5. It is derived from the equation describing the time evolution of the characteristic function of the process subjected to both continuous and point processes as inputs. Section 6 deals with a short account of stochastic differential equations by point processes. In section 7, we present our new results on stochastic systems subjected to real noise processes. We take the random telegraph noise as the input and obtain the Fokker-Planck equation describing the evolution of the probability density using cumulant expansion techniques. The stationary solution is also obtained. Our method of derivation is entirely different from the techniques adopted by others.

2. Basic Review of Stochastic Processes

In this section, we present a short review of stochastic processes, which are needed for our further study. An excellent survey can be obtained from the book by Soong (1973).

Let Ω be the sample space, ω a sample point and T, the parameter space, associated with the stochastic process $X(t, \omega)$. For convenience, we write $X(t)$ instead of $X(t, \omega)$. The mean value of the process $X(t)$ is defined by

$$m(t) = E\{x(t)\}, \tag{2.1}$$

E being the mathematical expectation. It is thus a function of time. Higher moments are defined similarly.

The Covariance function of the process $X(t)$ is defined by

$$\begin{aligned} R(s,t) &= \mathrm{Cov}(X(s), X(t)) \\ &= K(X(s), X(t)) \\ &= E\{X(s)X(t)\} - E\{X(s)\}E\{X(t)\}. \end{aligned} \tag{2.2}$$

by The class of all stochastic processes is too general for many purposes, but by considering a subclass with specified properties, we can simplify the problem to a great extent. A specific class of processes that is quite useful is that of Markov processes.

The Markov Process

This important class of stochastic processes was initiated by A.A.Markov in 1906. Markov properties of stochastic processes are discussed extensively in the books of Doob (1953) and Bharucha-Reid (1960).

A stochastic process $X(t)$, $t \in T$ is called a Markov Process, if for every n and for $t_1 < t_2 < \ldots < t_n$ in T we have

$$F\{x_n, t_n / x_{n-1}, t_{n-1}; x_{n-2}, t_{n-2}; \ldots ; x_1, t_1\}$$
$$= F\{x_n, t_n / x_{n-1}, t_{n-1}\}, \tag{2.3}$$

F being the conditional distribution function of $X(t)$. This is equivalent to

$$f\{x_n, t_n / x_{n-1}, t_{n-1}; x_{n-2}, t_{n-2}; \ldots ; x_1, t_1\}$$
$$= f\{x_n, t_n / x_{n-1}, t_{n-1}\}, \tag{2.4}$$

f being the conditional density function of $X(t)$. Equation (2.3) or (2.4) implies that a Markov process represents a collection of trajectories whose conditional probability distribution at given instant, given all past observations, depends only upon the latest past. This is the probabilistic analog of the deterministic theory in mechanics where the trajectory at a given time t is completely determined by its state at some $t' < t$ requiring no information of its states at times prior to t'. A Markov process is completely specified by its first and second distribution functions.

Independent - Increment Processes

Consider a stochastic process $X(t), t \geq 0$. The random variable $X(t_2) - X(t_1)$, $0 \leq t_1 < t_2$ is called an increment of $X(t)$ on $[t_1, t_2)$. If for all $t_1 < t_2 < \ldots < t_n$, the increments $X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1})$ are mutually independent, the stochastic process $X(t), t \geq 0$ is called an independent - increment stochastic process.

In the above definition, if the probability distributions of $X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ depend only on the parameter differences $t_2 - t_1, t_3 - t_2, \ldots, t_n - t_{n-1}$, the process $X(t)$ is said to have stationary independent increments.

It may be noted that a continuous-parameter independent increment stochastic process is Markovian.

Analysis of Stochastic Processes

In order to discuss the behaviour of dynamical systems whose inputs are stochastic processes, it is necessary to develop the analysis of stochastic processes. Hence, we develop the concepts of continuity, derivative and integral of a stochastic process in the mean square sense. The analysis of such concepts, ultimately depends upon the behaviour of deterministic functions like the mean and the covariance of the process $X(t)$. This is the main advantage of mean square calculus. We consider a particular class of stochastic processes $X(t)$, for which

$$E\left\{[X(t)^2]\right\} < \infty \tag{2.5}$$

throughout our study on stochastic integration.

Continuity

A stochastic process $X(t), t \in T$ is said to be continuous at t, in the mean square sense if

$$\lim_{\Delta t \to 0} E\left\{[X(t+\Delta t) - X(t)]^2\right\} = 0 \tag{2.6}$$

The process $X(t)$, $t \in T$ is mean square continuous at t, if and only if, its covariance function $R(s,t)$ is continuous at (t,t).

If $X(t)$ is wide-sense stationary, we have

$$R(s,t) = R(s-t) = R(\tau) \tag{2.7}$$

Hence, a wide-sense stochastic process $X(t)$ is mean square continuous at all $t \in T$, if and only if $R(\tau)$ is continuous at $\tau = 0$.

<u>Differentiability</u>

A stochastic process $X(t)$, $t \in T$ is said to be differentiable at t in the mean square sense, if

$$\lim_{\Delta t \to 0} \frac{X(t+\Delta t) - X(t)}{\Delta t} = \dot{X}(t) \tag{2.8}$$

exists in the sense of mean-square convergence, i.e. if

$$\lim_{\Delta t \to 0} E\left\{\left[\frac{X(t+\Delta t) - X(t)}{\Delta t} - \dot{X}(t)\right]^2\right\} = 0 \tag{2.9}$$

If a process $X(t)$ is differentiable for all $t \in T$, it is said to be a differentiable stochastic process.

Again, it is mean square differentiable at t , if and only if the second order derivative of its covariance function,

$$\frac{\partial^2 R(s,t)}{\partial s\,\partial t} \quad \text{exists at } (t,t) .$$

Mean Square Integration

In the development of integrals of stochastic processes, we are interested in Riemann integral and Riemann-Stieltjes integral in the mean square sense.

Let $X(t)$, $t \in T$, be a stochastic process. Consider an interval $[a,b] \in T$. Let $a = t_0 < t_1, \ldots, < t_n = b$ as a partition of $[a,b]$. Consider the sum

$$I_n = \sum_{k=1}^{n} X(\tau_k)(t_k - t_{k-1}) \tag{2.10}$$

where $t_{k-1} \leq \tau_k \leq t_k$.

The process $X(t)$, $t \in T$ is said to be Riemann integrable, if I_n converges to a limit in the mean square sense as $n \to \infty$, in such a way that $\rho = \max_k |t_k - t_{k-1}| \to 0$. This limit is called the mean square Riemann integral of $X(t)$ over $[a,b]$ and is denoted by

$$I = \int_a^b X(t)dt \tag{2.11}$$

Further, the above integral exists, if the integrals

$$\int_a^b m(t)\,dt \quad \text{and} \quad \int_a^b \int_a^b R(s,t)\,ds\,dt$$

exist.

Also

$$E\left\{\int_a^b X(t)\,dt\right\} = \int_a^b E\{X(t)\}\,dt = \int_a^b m(t)\,dt. \tag{2.12}$$

$$E\left\{\int_a^b \int_a^b X(t)\,X(s)\,dt\,ds\right\} = \int_a^b \int_a^b E\{X(t)X(s)\}\,ds\,dt$$

$$= \left\{\int_a^b m(t)\,dt\right\}^2 + \int_a^b \int_a^b R(s,t)\,ds\,dt. \tag{2.13}$$

Next, we consider stochastic Riemann-Stieltjes integrals of the types

$$V_1 = \int_a^b f(t)\,dX(t). \tag{2.14}$$

$$V_2 = \int_a^b X(t)\,d\,f(t). \tag{2.15}$$

where $X(t)$ is a stochastic process defined on $[a,b] \in T$, and $f(t)$ is an ordinary function on the same interval for t,. Keeping the definitions given earlier in the case of Riemann integrals, we form the random variables

$$V_{1n} = \sum_{k=1}^{n} f(\tau_k)\left[X(t_k) - X(t_{k-1})\right] \tag{2.16}$$

If $\underset{\substack{n \to \infty \\ \rho \to 0}}{\text{l.i.m}} V_{1n} = V_1$ exists, then V_1 is called the Riemann-Stieltjes integral of $f(t)$, on the interval $[a,b]$ with respect to $X(t)$; It exists, if and only if the ordinary double Riemann-Stieltjes integral

$$\int_a^b \int_a^b f(t)f(s) \, dd\, R(t,s) \text{ exists and is finite} \tag{2.17}$$

The definition and existence of V_2 can be developed on similar lines.

In section (3), we will develop the concept of integration with respect to Brownian motion process.

The Wiener Process

The Wiener Process or the Brownian motion process $B(t)$ is of vital importance in the modelling of stochastic systems.

A rigorous mathematical analysis of this process was given by Wiener in 1923.

The Wiener process is defined as follows: A continuous parameter stochastic process $B(t), t \geq 0$ is called a Wiener Process if

(i) $$P\left\{B(o) = 0\right\} = 1 . \tag{2.18}$$

(ii) The probability distribution of $\left\{B(t) - B(s)\right\}$ is Gaussian with

$$E\left\{\left[B(t) - B(s)\right]\right\} = 0 \tag{2.19}$$

$$E\left\{\left[B(t) - B(s)\right]^2\right\} = 2D\,(t-s), \tag{2.20}$$

The quantity D is a physical constant. Further, it was shown that $B(t), t \geq 0$ is a stationary independent-increment process; Since the Wiener process is Gaussian, it is completely specified by the mean and covariance functions. In general, if $B(t) = B(t + \Delta t) - B(t)$, we have

$$E\left\{\left[\Delta B(t)\right]^k\right\} = 0, \qquad k \text{ odd} \tag{2.21a}$$

$$E\left\{\left[\Delta B(t)\right]^k\right\} = 1 \cdot 3 \cdot 5 \cdot \ldots (k-1)(2D\Delta t)^{\frac{k}{2}}, \qquad k \text{ even} \tag{2.21b}$$

The fact that $E\left\{\left[\Delta B(t)\right]^2\right\}$ is of first order in Δt is what causes the peculiarities in the Ito calculus, to be described later in Section 3.

Now we give the vital properties of the Brownian motion process.

The covariance of the Wiener Process

The Wiener process $B(t)$, $t \geq 0$ has

$$E\left\{B(t)\right\} = 0 \tag{2.22}$$

$$E\left\{\left[B(t)\right]^2\right\} = 2\,Dt \tag{2.23}$$

from equations (2.19) and (2.20).

Considering $o < s < t$, we have $\left[B(t) - B(s)\right]$ and $\left[B(s) - B(0)\right]$ are independent processes. Hence,

$$E\left\{\left[B(t) - B(s)\right]\left[B(s) - B(0)\right]\right\}$$

$$= E\left\{\left[B(t) - B(s)\right]\right\} \cdot E\left\{\left[B(s) - B(0)\right]\right\}$$

$$= 0$$

using (2.19).

i.e. $E\left\{\left[B(t)\,B(s)\right] - \left[B(s)\right]^2\right\} = 0$, since $B(o) = 0$.

This gives

$$E\left\{\left[B(t)\,B(s)\right]\right\} = E\left\{\left[B(s)\right]^2\right\} = 2\,Ds \text{ using (2.23)}$$

$$= 2D \min(s,t).$$

Thus, the Wiener process has the covariance function

$$R(s,t) = 2D \min(s,t). \qquad (2.24)$$

The Wiener Process is mean square continuous

Now from equation (2.20)

$E\left\{\left[B(t + \Delta t) - B(t)\right]^2\right\} = 2D\Delta t$, and this tends to zero as $\Delta t \to 0$. Hence from the definition of mean square continuity (eq.2.6), the mean square continuity of Wiener Process follows.

The Wiener Process is not differentiable

Again, for the Wiener Process $B(t)$, we have

$$E\left\{\left[\frac{B(t+\Delta t) - B(t)}{\Delta t}\right]^2\right\} = \frac{2D}{\Delta t} \qquad (2.25)$$

using eq.(2.20).

The above expectation diverges, in the limit $\Delta t \to 0$. Hence the Wiener process is not differentiable.

We mention two important points about the Wiener process. The first is the mean square continuity of the Wiener Process is useful in the sense that when driving a system with such a process, the output may also be continuous.

The second main fact is that the Wiener process is not of bounded variation (McGarty 1974) and thus is not differentiable. We know that the common Riemann-Stoiltjes integral is defined only for functions of bounded variation. Thus, integrals with respect to Wiener process, do not exist in the ordinary sense. We develop this concept in section 3 in detail.

White Noise

This is one of the widely used concepts in stochastic processes modelled as an approximation ~~in many~~ in many physical problems.

We define a white noise process $W(t)$ as a zero mean process with correlation function

$$E\{[W(t)W(s)]\} = 2D\,\delta(t-s), \tag{2.26}$$

where $\delta(t)$ is the Dirac delta function.

A Representation of the White Noise

We show that the white noise can be written *formally* as

$$W(t) = \frac{dB(t)}{dt} \tag{2.27}$$

Since $B(t)$ is Gaussian, $\dot{B}(t)$ is also Gaussian. The covariances of $X(t)$ and $\dot{X}(t)$ denoted by $R_X(s,t)$ and $R_{\dot{X}}(s,t)$ are connected by the relation (Papoulis 1965)

$$R_{\dot{X}}(s,t) = \frac{\partial^2}{\partial s \partial t} R_X(s,t) \tag{2.28}$$

Hence,
$$\begin{aligned} R_{\dot{B}}(s,t) &= \frac{\partial^2}{\partial s \partial t}\left(R_B(s,t)\right) \\ &= 2D\frac{\partial^2}{\partial s \partial t}\{\min(s,t)\} && \text{from Eq.(2.24)} \\ &= 2D\frac{\partial}{\partial s} H(s-t) \\ &= 2D\,\delta(s-t) && \text{from Eq.(2.24)} \end{aligned}$$

(In the above $H(s-t)$ is Heaviside unit step function defined as

$$\begin{aligned} H(s-t) &= 0 \quad \text{if} \quad s < t \\ &= 1 \quad \text{if} \quad s > t) \end{aligned} \tag{2.29}$$

This means the formal derivative $\dot{B}(t)$, $t \geq 0$ has the properties of white noise. Thus we have the relation (2.27).

Section 3

The Ito and Stratonovich Theory on Stochastic Integrals

Let us consider the basic system of diffdrential equations, represented by

$$\frac{d\,\overline{X}(t)}{dt} = f\left(\overline{X}(t), \overline{Y}(t), t\right), t \geq 0 \tag{3.1}$$

where f is linear or non-linear real n-vector function, $\overline{X}(t)$

the n-dimensional vector state and $\overline{Y}(t)$ is the m-vector random disturbance at time t. The initial conditions for (3.1) can be a fixed constant or a random variable $\overline{X}(t_0) = \overline{X}_0$ with a specified distribution. The probability law for the process $\overline{Y}(t)$, $t \in T$ is assumed to be specified.

Now, if the fuction f and the $\overline{Y}(t)$ process are suitably defined such that the integral

$$\int_{t_0}^{t} f(\overline{X}(\tau), \overline{Y}(\tau), \tau)\, d\tau \tag{3.2}$$

exists in the mean square sense, the derivative $\frac{dX(t)}{dt}$ exists in the mean square sense, and,

$$\overline{X}(t) - \overline{X}(t_0) = \int_{t_0}^{t} f(\overline{X}(\tau), \overline{Y}(\tau), \tau)\, d\tau \tag{3.3}$$

Equations (3.1) and (3.3) are equivalent and we are interested in the density function of $\overline{X}(t)$.

A special class of random differential equations is one where $\overline{Y}(t)$ in eq. (3.1) has only white noise components. More specifically, we mean equations of the form

$$\dot{\overline{X}}(t) = f(\overline{X}(t),t) + G(\overline{X}(t),t)\,\overline{W}(t),\ t \in T$$
$$\overline{X}(t_0) = \overline{X}_0 \tag{3.4}$$

where $\overline{W(t)}$ is an m-dimensional vector stochastic process whose components belong to a class called white noise, $G(\overline{X(t)}, t)$ is an $n \times m$ matrix function and it is allowed to be a function of $\bar{X}$ and t, to take into account the possibility that the noise may depend on the state of the system, $\overline{X_0}$ is independent of $\overline{W(t)}$, $t \in T$. This equation is also called the Langevin equation for the process.

The above equations (3.4) plays an important role in modelling phenomena, due to two important reasons. The first is that the solution process generated by the eq.(3.4) is Markovian and powerful analytical techniques exist to study the solution. The second is, although white noise is a mathematical artifice, it approximates closely the behaviour of many noise processes in engineering, biomedical and electronic systems. Books dealing with these applications include Aoki (1967), Stratonovich (1968), Jazwinski (1970) and Kushner (1967), Ramakrishnan (1959) Srinivasan and Vasudevan (1971), Arnold (1973) Van Kampen (1976), Mortensen (1968). Soong (1973).

The noise in a stochastic system enters the differential equation representing the system in two ways - additive and multiplicative. Let the dynamical system be represented by the equation

$$\dot{X}(t) = f(X,t) + G(X(t), \xi(t)) \qquad (3.5)$$

where $\xi(t)$ is a fluctuating function, $F(x)$, $G(X, \xi)$ are arbitrary functions. If

$$G(X(t), \xi(t)) = \xi(t) \tag{3.6}$$

then the system (3.5) is said to have additive noise, i.e. the noise enters additively. It is not multiplied by any function of the solution process. If

$$G(X(t), \xi(t)) = g(X(t))\, \xi(t), \tag{3.7}$$

then the system is said to have multiplicative noise, i.e. the noise has a coefficient which is a function of the dependent variable.

In the additive type, the fluctuations jiggle the particles irrespective of their positions. Examples of this additive type occur, for instance, in electrical networks with noise sources, fluctuations of electro-magnetic fields in continuous media (Landau and Lifshitz, 1960). But there exist processes, where the fluctuations do depend on the values of the macroscopic variables. Examples for the later type of equations are found in varied disciplines, for instance - supharmonic generation (Graham, 1973), Raman scattering and the laser in the regime of quantum optics, autocatalytic chemical reactions, like those proposed by Schlogl (Schlogl 1972, Horsthemke and Malak-Mansour, 1976, Horsthemko and Lefever, 1977), the verhulst model in the field of population dynamics (Horsthemko and Malak-Mansour, 1976, Nitzan et. al. 1974), random magnetic field (Bourret, 1965) and wave propagation in a medium with random refractive index (Keller 1964), etc.

Having stressed the importance of multiplicative noise system, we proceed in study eq. (3.4) in greater detail.

We eliminate the white noise $\widetilde{W}(t)$ in eq. (3.4) by using its formal representation as the derivative of the Brownian Process (eq. 2.27). Hence, concentrating on the scalar case, eq. (3.4) is formally equivalent to

$$dX(t) = f(X(t),t) + G(X(t),t)\,dB(t),\ t\in T$$
$$X(t_0) = X_0 \tag{3.8}$$

This eq. (3.8) can be converted into the integral equation

$$X(t) = X(t_0) + \int_{t_0}^{t} f(X(\tau),\tau)\,d\tau + \int_{t_0}^{t} G(X(\tau),\tau)\,dB(\tau),\ t\in T$$
$$X(t_0) = X_0 \tag{3.9}$$

where $X(t_0)$ is independent of the increment $d\,B(t)$, $t \in T$.

The first integral can be defined as a mean square Riemann integral. The second integral is not defined in the mean square sense as a Riemann Stieltjes integral, since $B(t)$ is of unbounded variation as stated earlier. If want the integral

$$Y(t) = \int_a^b X(t)\, dB(t) \tag{3.10}$$

We define the random variable Y_n by

$$Y_n = \sum_{k=1}^{n} X(\tau_k)\left[B(t_k) - B(t_{k-1})\right], \quad \tau_k \in [t_{k-1}, t_k] \tag{3.11}$$

This sequence of random variables does not converge in the mean square sense to a unique limit. The limit depends on the particular particular choice of τ_k. Therefore, the integral (3.11) does not exist as a mean square integral in the usual sense.

We now study eq.(3.8) in the mean square sense following the interpretation of Ito (1961).

Let $B(t)$, $t \in T = [a,b]$ be a Wiener Process with

$$E\{B(t)\} = 0 \tag{2.19}$$

$$E\{[B(t) - B(s)]^2\} = 2D(t-s) \tag{2.20}$$

Let $X(t)$ be mean square continuous on T. At any time $t \in T$, the stochastic process $X(t)$ is independent of the increment $\left[B(t_{k+1}) - B(t_k)\right]$ for all t_k, t_{k+1} satisfying $a \le t \le t_k \le t_{k+1} \le b$. Let $\rho = \max_k |t_{k+1} - t_k|$

Consider a sequence of finite subdivisions of T and let us form the random variable

$$Y_n = \sum_{k=0}^{n-1} X(t_k)\left[B(t_{k+1}) - B(t_k)\right] \qquad (3.12)$$

if $\underset{\substack{n \to \infty \\ \rho \to 0}}{\text{l.i.m}}\ Y_n = Y$ exists, the random variable Y is called the Ito stochastic integral of $X(t)$ with respect to $B(t)$ over the interval T and is denoted by

$$\int_a^b X(t)\, d\, B(t) \qquad (3.13)$$

It may be noted that the values of $X(t)$ in (3.12) are not taken at arbitrary points in the interval $\left[t_k, t_{k+1}\right]$ but at the points t_k. Thus the definition given above is not one of a mean square integral in the usual sense.

We show that the mean square limit of the sequence Y_n given by (3.12) depends on the choice of t_k at which $X(t)$ takes the values.

We take the process $X(t)$ as $B(t)$ itself and define

$$Y_n = \sum_{k=0}^{n-1} B(t_k)\left[B(t_{k+1}) - B(t_k)\right] \tag{3.14}$$

$$Z_n = \sum_{k=0}^{n-1} B(t_{k+1})\left[B(t_{k+1}) - B(t_k)\right] \tag{3.15}$$

where $a = t_0 \angle t_1 \angle \dots \angle t_n = b.$

We show that

$$\underset{\substack{n \to \infty \\ \rho \to 0}}{\text{l.i.m}} (Z_n - Y_n) = \underset{\substack{n \to \infty \\ \rho \to 0}}{\text{l.i.m}} \sum_{k=0}^{n-1} \left[B(t_{k+1}) - B(t_k)\right]^2$$
$$= 2D(b-a) \tag{3.16}$$

and hence the sequences $\{Y_n\}$ and $\{Z_n\}$ converge in mean square to different limits.

<u>Proof.</u> Taking

$$\Delta(B(t_k)) = B(t_{k+1}) - B(t_k) = B_{k+1} - B_k \tag{3.17}$$

$$\Delta(t_k) = t_{k+1} - t_k, \tag{3.18}$$

and noting, from (2.21b)

$$E\left\{[\Delta B_k]^2\right\} = 2D\Delta t_k \tag{3.19}$$

$$E\left\{[\Delta B_k]^4\right\} = 3(2D\Delta t_k)^2 \tag{3.20}$$

We have

$$E\left[\left\{\sum_k (\Delta B_k)^2\right\} - 2D(\Delta t_k)\right]^2$$

$$= E\left[\sum_k \left\{(\Delta B_k)^2 - 2D(\Delta t_k)\right\}\right]^2$$

$$= \sum_k E\left[\left\{(\Delta B_k)^2 - 2D(\Delta t_k)\right\}^2\right]$$

$$= \sum_k E\left[(\Delta B_k)^4 + 4D^2(\Delta t_k)^2 - 4D(\Delta t_k)(\Delta B_k)^2\right]$$

$$= \sum_k \left[3(2D\Delta t_k)^2 + 4D^2(\Delta t_k)^2 - 4D(\Delta t_k)(2D\Delta t_k)\right]$$

$$= \sum_k 8D^2(\Delta t_k)^2 \leq 8D^2\rho \sum_k (\Delta t_k) = 8D^2\rho(b-a) \to 0 \text{ as } n \to \infty, \ \rho \to 0$$

Hence the eq.(3.16).

We now enlist some important results on Ito integrals.

(1) The Ito integral is linear and additive in the domain of integration.

(2) Let $g(t)$ be a random function satisfying the conditions

(a) $g(t)$ is independent of $B(t_k) - B(t_\ell)$, $(t \leq t_\ell \leq t_k \leq b)$, for all $t \in T$ and

$$\int_T E\left\{[g(t)]^2\right\} dt < \infty \tag{3.21}$$

Let f(t) be another random function similarly defined. Then

$$E\left\{\int_T g(t)\, dB(t)\right\} = 0 \tag{3.22}$$

$$E\left\{\int_T g(t)\, dB(t) \cdot \int_T f(t)\, dB(t)\right\} = 2D \int_T E\left\{g(t)\, f(t)\right\} dt. \tag{3.23}$$

2D being the variance parameter of B(t).

(3) If $X(t) = \int_a^t g(t)\, d\, B(t),\ t \in T$ (3.24)

then X(t) is mean square continuous on T.

Another interesting extension of Ito's stochastic integral is Ito's differentiation rule (Ito 1961). We just give the statement of the theorem and evaluate $\int_a^t \left\{B(t) - B(a)\right\} dB(t)$ and point out the significance of this integral.

Let X(t) be the unique solution of the vector Ito stochastic differential equation (3.4), which can be written as

$$d\bar{X}(t) = f(\bar{X}(t),t) + G(\bar{X}(t),t)\,d\bar{B}(t), \quad t \in T \tag{3.25}$$

Let $\phi(X(t),t)$ be a scalar valued real function continuously differentiable in t and having continuous second mixed partial derivatives with respect to X. Then the stochastic differential $d\phi$ of ϕ is

$$d\phi = \phi_t\,dt + \phi_X^T dX + \frac{1}{2}\,\mathrm{tr}\,G D G^T \phi_{XX}\,dt \tag{3.26}$$

where

$$\phi_t = \frac{\partial \phi}{\partial t}, \quad \phi_X^T = \left[\frac{\partial \phi}{\partial x_1}, \cdots\cdots \frac{\partial \phi}{\partial x_n}\right]$$

$$\phi_{XX} = \begin{bmatrix} \frac{\partial^2 \phi}{\partial x_1^2}, & \frac{\partial^2 \phi}{\partial x_1 \partial x_2}, & \cdots & \frac{\partial^2 \phi}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 \phi}{\partial x_n \partial x_1}, & \frac{\partial^2 \phi}{\partial x_n \partial x_2}, & \cdots & \frac{\partial^2 \phi}{\partial x_n^2} \end{bmatrix} \tag{3.27}$$

and

$$E[\,d\bar{B}(t)(d\bar{B}(t))^T\,] = 2D\,dt$$

where D denotes the $m \times m$ matrix whose ij^{th} element is D_{ij}.

The equation (3.26) is a stochastic differential equation whose solution is a function of the solution of eq. (3.25).

To evaluate $\int_a^t \{B(t) - B(a)\}\, d\,B(t)$

we take

$$\phi = \frac{1}{2}\left[X(t) - B(a)\right]^2 - D(t-a) \tag{3.28}$$

and apply eq. (3.26).

Let $d\,X(t) = d\,B(t)$.

Then we have,

$$d\phi = -D\,dt + (X(t) - B(a))\,d\,X(t) + \frac{1}{2}(2D)\,dt$$

$$= (X(t) - B(a))\,d\,X(t)$$

$$= (B(t) - B(a))\,d\,B(t)$$

Hence

$$\int_a^t \{B(t) - B(a)\}\,dB(t) = \frac{1}{2}\left[B(t) - B(a)\right]^2 - D(t-a) \tag{3.29}$$

2D being the variance parameter of $B(t)$.

The above result (3.29) is due to Doob (1953) and this illustrates the important consequence of Ito integration. It does not agree with the ordinary rules of integration, using which we get only the first term on the right hand side of

(3.29). Intuitively the presence of the term $D(t-a)$ can be viewed as a correction term which ensures $E(I \int_a^t [B(t)-B(a)] dB(t)) = 0$ which is in agreement with (3.22).

We next show that the solution process $X(t)$ of the eq. (3.8) is Markovian. We write the eq. (3.8) as

$$X(t+\Delta t)-X(t)=f(X(t),t)\,\Delta t+G(X(t),t)\big(B(t+\Delta t)-B(t)\big) \tag{3.30}$$

Then given $X(t)$, $X(t+\Delta t)$ depends only on the Brownian increment $B(t+\Delta t)-B(t)$. The Brownian motion increments are independent and by assumption $B(t)$ is independent of $X(\tau)$, $\tau \leq t$. Thus the $X(t)$ process is Markovian.

Stochastic Integral of Stratonovich

Recently Stratonovich (1966) proposed a new (symmetric) definition of a stochastic integral $\int_a^b G(B(t),t)\, d\, B(t)$.

Let the scalar random function $G(B(t),t), t \in T = [a,b]$ be an explicit function of $B(t)$, where $B(t)$, $t \in T$ is a scalar Brownian motion process with variance parameter 2D. Let $a = t_0 < t_1 < \dots < t_n = b$ be a partition of $[a,b]$ and $\rho = \max_k (t_{k+1} - t_k)$. Then the Stratonovich Stochastic integral is defined by

$$S\int_a^b G(B(t),t)\,dB(t)$$

$$= \underset{\substack{n\to\infty \\ \rho\to 0}}{l.i.m} \sum_{k=0}^{n-1} G\left(\frac{B(t_k)+B(t_{k+1})}{2}, t_k\right)\left(B(t_{k+1})-B(t_k)\right) \tag{3.31}$$

Equivalently, $G\left(\frac{1}{2}\{B(t_k)+B(t_{k+1})\}, \frac{1}{2}\{t_k+t_{k+1}\}\right)$ can be used instead of $G\left(\frac{1}{2}\{B(t_k)+B(t_{k+1})\}, t_k\right)$. Gray and Caughey (1965) noted that $\frac{1}{2}\left[G(B(t_k),t_k)+G(B(t_{k+1}),t_{k+1})\right]$ is also an equivalent choice. Stratonovich has shown that if $G(B(t), t)$ is continuous in t and has a continuous partial derivative $\frac{\partial G}{\partial B}$ and further satisfies the condition

$$\int_a^b E\left\{|G(B(t),t)|^2\right\}dt < \infty \tag{3.32}$$

then the mean square limit in (3.31) exists and is related to the Ito integral by

$$S\int_a^b G(B(t),t)\,dB(t) = I\int_a^b G(B(t),t)\,dB(t) + D\int_a^b \frac{\partial G}{\partial B}\,dt. \tag{3.33}$$

The last integral is a well-defined Riemann integral.

We note that the earlier Stratonovich integral is defined for a much more restricted class of functions than the Ito integral. Stratonovich integral is defined only for explicit functions of B(t) . Also the Stratonovich integral no longer has zero mean.

At this stage, we point out (Van Kampen, (1981)) that equation (3.4) is just a "pre-equations" in the sense that it does not define a stochastic process X(t) . This "pre-equation" can be turned into an actual equation by supplementing it with an additional interpretation rule. The two rules are those prescribed by Ito and Stratonovich, as described in this section. But, once the prescription is given, we can writen down the equivalent equation in the other prescription, as given by Stratonovich (1964). If eq. (3.4) is supplemented with Stratonovich interpretation, the equivalent Ito equation is given by the transformation rule that the i^{th} component of the vector $\bar{f}(X(t),t)$ is modified by the addition of

$$\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{m} G_{kj}\frac{\partial G_{ij}}{\partial x_k}. \tag{3.34}$$

Now we give the relation between the Ito and Stratonovich integrals, in the scalar case following Hasegawa and Ezawa (1980) and Stratonovich (1966), in the form

$$S\int_a^b G(X(t),t)\,dB(t) = I\int_a^b G(X(t),t)\,dB(t) + \frac{1}{2}\int_a^b \frac{\partial G(X(t),t)}{\partial X}\,G(X(t),t)\,dt \tag{3.35}$$

(Taking variance parameter of $B(t)$ to be unity in (3.34) and (3.35)).

Proof. Denoting the integrands of the Ito and the Stratonovich integrals by $G\, dB$ and $G . dB$, we have from the underlying definitions of the two types of integrals;

$$G\, dB = G(X(t),t)\left[B(t+\Delta t) - B(t)\right], (\Delta t > 0) \tag{3.36}$$

and

$$G . dB = \frac{1}{2}\left[G(X(t+\Delta t), t+\Delta t) + G(X(t),t)\right]\left[B(t+\Delta t) - B(t)\right], \quad \Delta t > 0 \tag{3.37}$$

In (3.36) since $X(t)$ depends on $B(s)$ in the past $s \leq t$ only G and dB are satistically independent. Hence

$$\langle G(X(t),t)\, dB(t) \rangle = 0 \tag{3.38}$$

But nosuch simple relation exists in (3.37). But, we have

$$G . dB - G\, dB = \frac{1}{2}\left[G(X(t+\Delta t), t+\Delta t) - G(X(t),t)\right]\left[B(t+\Delta t) - B(t)\right]$$

$$= \frac{1}{2}\left[\frac{\partial G(X(t),t)}{\partial X} G(X(t),t)\right]\Delta t + 0(\Delta t)^{3/2} \tag{3.39}$$

Using $dB(t) = O(\triangle t)^{\frac{1}{2}}$, mentioned earlier. Hence we have (3.35). Here, it may be noted that we have $G(X(t),t)$ instead of $G(B(t),t)$ as in (3.33).

Hence, to state explicitly, we have that if

$$(S)\, dX(t) = f(X(t),t)\,dt + G(X(t),t)\,dB(t),\ t \in T \tag{3.40}$$

is a Stratonovich equation, the equivalent Ito equation is

$$(I)\, dX(t) = \left[f(X(t),t) + \frac{1}{2} G(X(t),t)\frac{\partial G}{\partial x}(X(t),t)\right]dt + G(X(t),t)\,dB(t),\ t \in T \tag{3.41}$$

Similarly, if (3.40) is considered as an Ito equation, the corresponding Stratonovich equation is given by

$$(S)\, dX(t) = \left[f(X(t),t) - \frac{1}{2} G(X(t),t)\frac{\partial G}{\partial x}(X(t),t)\right]dt + G(X(t),t)\,dB(t),\ t \in T \tag{3.42}$$

Having described the Langevin equation in the two approaches, we give a short account of the unified treatment of stochastic calculus, introduced by Ito (1975) by the name of symmetric Q-multiplication.

The symbol used to indicate Stratonovich definition of the product (3.37) was in fact used by Ito. The relation between the two types of multiplication of a stochastic differential by a time dependent random variable may be expressed by the general formula.

$$Y \cdot dX = Y\,dX + \frac{1}{2}\,dX\,dY \tag{3.43}$$

where $dX\,dY \propto dt$ (necessary to retain $O(dt)$) (3.44)

$$\langle Y\,dX \rangle = 0 \tag{3.45}$$

and $dX\,dY\,dZ = 0$ neglect of terms of $O(dt)^{3/2}$ (3.46)

The main advantage of the symmetric multiplication is that it satisfies all the rules of usual differential calculus. This fact arises from the identity

$$d(XY) = X\,dY + Y\,dX + dX\,dY \tag{3.47}$$

This may be easily verified, from

$$X(t+\Delta t)\,Y(t+\Delta t) - X(t)\,Y(t) = X(t)\left[Y(t+\Delta t) - Y(t)\right]$$
$$+ Y(t)\left[X(t+\Delta t) - X(t)\right] + \left[X(t+\Delta t) - X(t)\right]\left[Y(t+\Delta t) - Y(t)\right]$$

Also, from (3.43) and (3.47), we have

$$d(XY) = X \cdot dY + Y \cdot dX \tag{3.48}$$

Thus the symmetric multiplication defined by (3.43) restores the basic rule for the differential of products.

As an immediate application of (3.43), we obtain the formula for Ito's differential $d\,\phi\,(X(t))$ (3.26).

Now $d\,\phi(X(t)) = \phi'(X(t)) \cdot dX(t)$

$= \phi'(X(t)) \cdot [f(X,t)\,dt + G(X,t)\,dB]$ using (3.8)

$= \phi'(X(t)) \cdot f(X,t)\,dt + \phi'(X(t)) \cdot G(X,t)\,dB$

$= \phi'(X(t))\, f(X,t)\,dt + \frac{1}{2}\, d(\phi'(X(t))\, f(X,t)\,dt)$

$+ \phi'(X(t))\, G(X,t)\,dB + \frac{1}{2}\, d\,(\phi'(X(t))\,G(X,t)\,dB)$

$= \phi'(X(t))\, f(X,t)\,dt + \phi'(X(t))\, G(X,t)\,dB$

$+ \frac{1}{2}\, d(\phi'(X(t))\,[f(X,t)\,dt + G(X,t)\,dB]$

$= \phi'(X(t))\, f(X,t)\,dt + \phi'(X(t))\, G(X,t)\,dB$

$+ \frac{1}{2} d\phi'(X(t))\;\; dX(t).$

$= \phi'(X(t))\, f(X,t)\,dt + \phi'(X(t))\, G(X,t)\,dB$

$+ \frac{1}{2}\, \phi''(X(t)) \cdot (dX(t))^2$

$= \phi'(X(t))\, f(X,t)\,dt + \phi'(X(t))\, G(X,t)\,dB + \frac{1}{2}\phi''(X)\, G^2 (dB)^2 + 0\,(dt)^{3/2}$

$$= \Big[f(X)\phi'(X) + \frac{1}{2}\, G^2(X)\,\phi''(X)\Big]_{X=X(t)} dt + \big[\phi'(X)\,G(X)\big]_{X=X(t)}\, dB(t) \qquad (3.49)$$

in agreement with (3.26), with unit variance parameter for $dB(t)$.

Having discussed the Ito and Stratonovich types of Langevin equations, we give the equations for the time development of the transition probability density functions for the solution process in the next section.

4. The Fokker-Planck Equations

A dynamical theory is described by an equation of motion. The equation of motion in statistical theory can be either of the two types (a) an equation for the time development of the dynamical variables of the system, or (b) an equation for the time development of the probability distribution of the dynamical variables of the system.

In this section, we give the Fokker-Planck equation satisfied by the transition probability density of the solution process of Ito and Stratonovich types of Langevin equations.

Based on the work of Bartlett (1966), the first probability density function $P(x,t)$ of a stochastic process $X(t)$, $t \in T$ satisfies the differential equation

$$\frac{\partial P(x,t)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\underline{|n}} \frac{\partial^n}{\partial x^n} \left\{ \alpha_n(x,t) P(x,t) \right\} \tag{4.1}$$

where

$$\alpha_n(x,t) = \lim_{\Delta t \to 0} E\left\{[X(t+\Delta t) - X(t)]^n / X(t) = x\right\}$$

$$n = 1, 2, 3, \ldots \tag{4.2}$$

Here $\alpha_n(x,t)$ are called the ~~derivative~~ derivate moments of the stochastic process X(t) and the equation (4.1) is called the kinetic equation associated with the process X(t). Necessarily α_n's should exist in the limit $\Delta t \to 0$.

The above equation can be easily extended to the determination of the second density function $P(x,t;x_1,t_1)$. Since

$$P(x,t;x_1,t_1) = P(x,t|x_1,t_1)\,P(x_1,t_1) \tag{4.3}$$

$P(x,t|x_1,t_1)$ being the conditional density, we must calculate $P(x,t|x_1,t_1)$ knowing $P(x_1,t_1)$. By a similar procedure, the equation for $P(x,t|x_1,t_1)$ is given by

$$\frac{\partial P(x,t|x_1,t_1)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\lfloor n} \frac{\partial^n}{\partial x^n}\left[\beta_n(x,t;x_1,t_1)\,P(x,t|x_1,t_1)\right] \tag{4.4}$$

where

$$\beta_n(x,t;x_1,t_1) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left\{[X(t+\Delta t) - X(t)]^n / X(t) = x, X(t_1) = x_1\right\}$$

$$n = 1, 2, 3, \ldots \tag{4.5}$$

From the Kinetic equation, it is obvious that the above method is useful only when a few of the derivate moments are nonzero. But Pawulo (1967) has proved the theorem."If the derivate moments $\alpha_n(x,t)$ exist for all n and is zero for the some even n, then $\alpha_n(x,t) = 0$, $n \geq 3$. " In view of this theorem, we will apply the Kinetic equation to study the processes for which $\alpha_n(x,t)$ vanishes for $n \geq 3$. Physically, this means that the process can change only by small amounts in a small time interval, $\alpha_n(x,t)$ defined by (4.2) will approach zero faster than Δt as $\Delta t \to 0$ for $n \geq 3$. Under this condition the Kinetic equation has the form of the Fokker-Planck equation in the theory of Markov processes.

We know that the solution of Ito differential equation is Markovian. Hence the Fokker-Planck equation for the study of the equation (3.25) can be written immediately. In (3.25) $\overline{X}(t)$ is the n-dimensional solution process, $G(\overline{X(t)},t)$ is an n x m matrix function, and $\overline{B(t)}$ is an m-dimensional vector Wiener Process with components $B_j(t)$, j = 1,2,...m having the properties

$$E\{\Delta B_j(t)\} = E\{B_j(t+\Delta t)-B_j(t)\} = 0$$

$$E\{\Delta B_i(t)\,\Delta B_j(t)\} = 2D_{ij}\Delta t, \quad t \geq t_0$$

$$i,j = 1,2,\ldots m. \tag{4.6}$$

Then the transition probability density of $\overline{X(t)}$, $P(\overline{x}, t / \overline{x_0}, t_0)$ satisfies the Fokker-Planck equation

$$\frac{\partial P(\overline{x}, t / \overline{x_0}, t_0)}{\partial t} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left[\alpha_j(\overline{x}, t) P(\overline{x}, t / \overline{x_0}, t_0) \right] + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left[\alpha_{ij}(\overline{x}, t) \overline{P}(\overline{x}, t / \overline{x_0}, t_0) \right] \quad (4.7)$$

where

$$\alpha_j(\overline{x}, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E \left\{ \Delta x_j / \overline{X}(t) = \overline{x} \right\} \quad (4.8)$$

$$\alpha_{ij}(x, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E \left\{ \Delta x_i \Delta x_j / \overline{X}(t) = \overline{x} \right\} \quad (4.9)$$

with the initial conditions

$$P(\overline{x}, t_0 / \overline{x_0}, t_0) = \prod_{j=1}^{n} \delta(x_j - x_{0j}) \quad (4.10)$$

and the boundary conditions

$$P(\overline{x}, t / \overline{x_0}, t_0) \to 0 \text{ as } x_j \to \pm\infty \text{ for any } j. \quad (4.11)$$

Eq.(4.7) is also called the forward equation, because it is considered as a function of the forward variable, $\overline{x}$, t moving forward in time.

The vector version of the Kolmogrov equation or the backward equation is

$$\frac{\partial p}{\partial t_0}(\bar{x},t|\bar{x}_0,t_0) = -\sum_{j=1}^{n} \frac{\partial p}{\partial x_{0j}} \alpha_j(\bar{x}_0,t) - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 p}{\partial x_{0i}\partial x_{0j}} \alpha_{ij}(\bar{x}_0,t) \tag{4.12}$$

with conditions (4.10) and $P(\bar{x}, t/\bar{x}_0, t_0) \rightarrow 0$ as

$$x_{0j} \rightarrow \pm\infty \text{ for any } j \tag{4.13}$$

It can be shown (Soong 1973), that

$$\alpha_j(\bar{x},t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left\{\Delta x_j(t) / \bar{x}(t) = \bar{x}\right\} = f_j(\bar{x},t) \tag{4.14}$$

where $f_j(\bar{x},t)$ is the j^{th} component of $\bar{f}(\bar{x}, t)$, and

$$\alpha_{ij}(\bar{x},t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left\{\Delta x_i(t)\,\Delta x_j(t) / \bar{x}(t) = \bar{x}\right\} = 2\left(GDG^T\right)_{ij}, \quad i,j = 1,2,\ldots n \tag{4.15}$$

where D denotes the $m \times m$ matrix whose ij^{th} element is D_{ij}.

Hence, the Fokker-Planck equation (4.7) may be put in the form, writing P for $P(\bar{x}, t, \bar{x}_0, t_0)$.

$$\frac{\partial P}{\partial t} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left\{ f_j(\bar{x}, t) P \right\} + \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left\{ (G D G^T)_{ij} P \right\}.$$

(4.16)

with the condition (4.10) and (4.11).

The Fokker-Planck equation for eq.(3.8), in the Ito sense is given by, writing P for $P(x,t/x_0, t_0)$

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \left\{ f(x) P \right\} + D \frac{\partial^2}{\partial x^2} \left\{ G^2(x) P \right\}$$

(4.17)

The transition probability for the same eq.(3.8) in the convention of Stratonovich satisfies the equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \left[\left\{ f(x) + D G(x) \frac{\partial G}{\partial x} \right\} P \right] + D \frac{\partial^2}{\partial x^2} \left\{ G^2(x) P \right\}$$

(4.18)

with the condition

$$P(x, t_0 | x_0, t_0) = \delta(x - x_0) \qquad (4.19a)$$

and

$$P(x, t | x_0, t_0) \to 0 \text{ as } x \to \pm\infty. \qquad (4.19b)$$

Eq. (4.18) can also be written as

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left(f(x)P\right) + D\frac{\partial}{\partial x}G(x)\frac{\partial}{\partial x}G(x)P. \tag{4.20}$$

We observe that whether we take the eq.(3.8) in the sense of Ito or Stratonovich, we arrive at the same solution of long as $G(X(t), t)$ is independent of $X(t)$. When $G(X(t),t)$ is a function of $X(t)$, we obtain two distinct Markov Processes, as solutions, which differs in the systematic (drift) behaviour but not in the fluctuation behaviour.

Even before introducing his interpretation for the stochastic integral, Stratonovich (1963) has given the Fokker-Planck equation satisfied by the transition probability density for the Markovian process $X(t)$, we given below an outline of his derivation (Stratonovich 1963).

We consider the Langevin equation

$$\dot{x} = \epsilon F(x,t) \tag{4.21}$$

in the most general form, ϵ being a small parameter. F is a function involving random arguments. Taking

$$x(t_0) = x_0 \tag{4.22}$$

as the initial value at time t_0, we consider the increment

$$x(t) - x_0 = H(x_0) = H(x_0, t, t_0) \tag{4.23}$$

which depends on the initial value x_0 essentially. We write $H(x_0)$ in the form of an expansion

$$H(x_0) = \epsilon H_1(x_0) + \epsilon^2 H_2(x_0) + \cdots \tag{4.24}$$

Substituting (4.24) into (4.21), we get

$$\epsilon \dot{H}_1 + \epsilon^2 \dot{H}_2 + \cdots$$

$$= \epsilon F(x_0) + \epsilon \frac{\partial F}{\partial x}(x_0)\left[\epsilon H_1 + \epsilon^2 H_2 + \cdots\right] + \frac{\epsilon}{2}\frac{\partial^2 F}{\partial x^2}(x_0)\left[\epsilon H_1 + \epsilon^2 H_2 + \cdots\right]^2 + \cdots$$

Equating coefficients of like powers in ϵ, we get

$$\dot{H}_1(x_0) = F(x_0)$$

$$\dot{H}_2(x_0) = \frac{\partial F}{\partial x}(x_0) H_1(x_0)$$

$$\dot{H}_3(x_0) = \frac{\partial F}{\partial x}(x_0) H_2(x_0) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(x_0) H_1^2(x_0)$$

$$\cdots\cdots \tag{4.25}$$

so that

$$H_1(x_0) = \int_{t_0}^{t} F(x_0, t')\,dt'$$

$$H_2(x_0) = \int_{t_0}^{t} dt' \frac{\partial F}{\partial x}(x_0, t') \int_{t_0}^{t'} F(x_0, t'')\,dt'' \tag{4.26}$$

$$\cdots\cdots$$

Having expressed $H(x_0)$ in terms of F, to some desired accuracy, we consider the statistical characteristics of $H(x_0)$.

The characteristic function of $H(x_0)$ is given by

$$\langle e^{iu(x-x_0)} \rangle = 1 + \sum_{s=1}^{\infty} \frac{(iu)^s}{\lfloor s} \langle H^s(x_0) \rangle \qquad (4.27)$$

Inverting, we get the one-dimensional probability density

$$P(x,t/x_0,t_0) = \frac{1}{2\pi} \int e^{-iu(x-x_0)} \left[1 + \sum_{s=1}^{\infty} \frac{(iu)^s}{\lfloor s} \langle H^s(x_0) \rangle \right] du \qquad (4.28)$$

Interchanging the order of integration and summation in (4.28) we have

$$P(x,t/x_0,t_0)$$

$$= \left[1 + \sum_{s=1}^{\infty} \frac{1}{\lfloor s} \left(-\frac{\partial}{\partial x} \right)^s \langle H^s(x_0) \rangle \right] \frac{1}{2\pi} \int e^{-iu(x-x_0)} du$$

$$= \left[1 + \sum_{s=1}^{\infty} \frac{1}{\lfloor s} \left(-\frac{\partial}{\partial x} \right)^s \langle H^s(x_0) \rangle \right] \delta(x-x_0) \qquad (4.29)$$

Multiplying (4.29) by a function $f(x_0)$ from a suitable class of functions and integrating we get

$$\int P(x,t|x_0,t_0) f(x_0)\, dx_0 = f(x) + \sum_{s=1}^{\infty} \frac{1}{\lfloor s} \left(-\frac{\partial}{\partial x}\right)^s \left[\langle H^s(x)\rangle f(x)\right] \tag{4.30}$$

Introducing, the operator

$$L = \sum_{s=1}^{\infty} \frac{1}{\lfloor s} \left(-\frac{\partial}{\partial x}\right)^s \langle H^s(x)\rangle \tag{4.31}$$

eq. (4.29) can be written as

$$P(x,t|x_0,t_0) = (1+L)\,\delta(x-x_0) \tag{4.32}$$

Differentiating this we get $\dot{P} = \dot{L}\,\delta(x - x_0)$ (4.33)

Using (4.32), this equation can be written as

$$\dot{P} = \dot{L}(1+L)^{-1} P \tag{4.34}$$

where the operator

$$(1+L)^{-1} = 1 - L + L^2 - \ldots\ldots \tag{4.35}$$

is the inverse of the operator $(1+L)$. We have

$$\dot{L}' = \left(-\frac{\partial}{\partial x}\right)\langle \epsilon \dot{H}_1 + \epsilon^2 \dot{H}_2 \rangle + \left(-\frac{\partial}{\partial x}\right)^2 \langle \epsilon^2 \dot{H}_1 H_1 \rangle + O(\epsilon^3) \quad (4.36)$$

$$(1+L)^{-1} = 1 - \epsilon\left(-\frac{\partial}{\partial x}\right)\langle H_1 \rangle + O(\epsilon^2) \quad (4.37)$$

and

$$L'(1+L)^{-1} = -\epsilon\frac{\partial}{\partial x}\langle \dot{H}_1 + \epsilon \dot{H}_2 \rangle + \epsilon^2 \frac{\partial^2}{\partial x^2}\langle \dot{H}_1 H_1 \rangle - \epsilon^2 \frac{\partial}{\partial x}\langle \dot{H}_1 \rangle \frac{\partial}{\partial x}\langle H_1 \rangle + O(\epsilon^3) \quad (4.38)$$

Using eq.(4.25), this may be simplfied as

$$L'(1+L)^{-1} = -\epsilon\frac{\partial}{\partial x}\langle F \rangle - \epsilon^2 \frac{\partial}{\partial x}\left\{K\left[\frac{\partial F}{\partial x}, H_1\right]\right\} + \epsilon^2 \frac{\partial^2}{\partial x^2}\left\{K\left[F, H_1\right]\right\} + O(\epsilon^3) \quad (4.39)$$

where $K\left[\ldots,\ldots\right]$ is the covariance of the indicated arguments defined by eq. (2.2). Neglecting terms of order ϵ^3 and higher and using (4.34), eq. (4.39) reduces to the Fokker-Planck equation,

$$\frac{\partial P}{\partial t}(x,t/x_0,t_0) = -\frac{\partial}{\partial x}\left[K_1(x)P(x,t/x_0,t_0)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[K_2(x)P(x,t/x_0,t_0)\right] \tag{4.40}$$

where

$$K_1(x) = \epsilon\langle F(x)\rangle + \epsilon^2 K\left[\frac{\partial F(x)}{\partial x}, H_1(x)\right] \tag{4.41}$$

$$K_2(x) = 2\epsilon^2 K\left[F(x), H_1(x)\right] \tag{4.42}$$

Again, using (4.26) we get

$$K_1(x) = \epsilon\langle F(x)\rangle + \epsilon^2\int_{t_0-t}^{0} K\left[\frac{\partial F(x)}{\partial x}, F_\tau(x)\right]d\tau \tag{4.43}$$

$$K_2(x) = 2\epsilon^2\int_{t_0}^{t} K\left[F(x,t), F(x,t')\right]dt' = 2\epsilon^2\int_{t_0-t}^{0} K\left[F(x), F_\tau(x)\right]d\tau \tag{4.44}$$

where $F_\tau(x) = F(x, t+\tau)$. (4.45)

As $t - t_0 \to \infty$, we get

$$K_2(x) \to K(x), \quad K_1(x) \to M(x) + \frac{1}{4}K'(x). \tag{4.46}$$

where

$$M(x) = \epsilon \langle F(x) \rangle \tag{4.47}$$

$$K(x) = 2\epsilon^2 \int_{-\infty}^{0} K\left[F(x), F_\tau(x)\right] d\tau \tag{4.48}$$

$$K'(x) = 4\epsilon^2 \int_{-\infty}^{0} K\left[\frac{\partial F(x)}{\partial x}, F_\tau(x)\right] d\tau \tag{4.49}$$

Hence we get the Fokker-Planck equation in the form

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left[\left\{M(x) + \frac{1}{4}K'(x)\right\}P\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[K(x)P\right] \tag{4.50}$$

In this connection the following observation may be noted. The coefficient

$$K_1(x) = M(x) + \frac{1}{4}K'(x) \tag{4.51}$$

Is the "average derivative"

$$\lim_{\tau \to 0} \frac{\langle X(\tau) - X(0) \rangle}{\tau} = \langle \dot{x} \rangle = \langle \epsilon F[x(t), \xi(t)] \rangle = K_1(x)$$

So, if we wish to calculate the mean value of $\epsilon F[x(t), \xi(t)]$ we have to allow for the correlation between $x(t)$ and ξ (t). According to (4.51), $K_1(x)$ differs from the mean value

$$M(x) = \epsilon \langle F[x, \xi(t)] \rangle.$$

Calculated without regard for correlation between $x(t)$ and $\xi(t)$ by the quantity

$$\epsilon^2 \int_{-\infty}^{0} K\left[\frac{\partial F}{\partial x}, F_\tau\right] d\tau + O(\epsilon^3) \tag{4.52}$$

Similarly we can extend the above considerations to systems of Langevin equations involving several processes.

$$\begin{aligned} \dot{x}_1 &= \epsilon F_1(x_1, \ldots x_p, t) = \epsilon F_1(x, t) \\ &\ldots \\ \dot{x}_p &= \epsilon F_p(x_1, \ldots x_p, t) = \epsilon F_p(x, t) \end{aligned} \tag{4.53}$$

which define the random processes, $x_1(t), \ldots, x_p(t)$. If p denotes the transition probability density, we have the Fokker-Planck equation,

$$\frac{\partial p}{\partial t} = -\epsilon \sum_{l} \frac{\partial}{\partial x_l} \left\{ \left(\langle F_l \rangle + \epsilon \sum_{m} \int_{t_0 - t}^{0} K\left[\frac{\partial F_l}{\partial x_m}, F_{m\tau}\right] d\tau \right) p \right\}$$
$$+ \epsilon^2 \sum_{l,m} \frac{\partial^2}{\partial x_l \partial x_m} \left[\left\{ \int_{t_0 - t}^{0} K[F_l, F_{m\tau}] d\tau \right\} p \right] \tag{4.54}$$

Thus we observed the structure of the Fokker-Planck equations corresponding to the Langevin equation, interpreted in the Ito and Stratonovich types.

5. Fokker-Planck Equation - Another Approach*

Recently an ever increasing interest is paid to the modelling of systems in terms of stochastic differential equations. The well known procedure is to describe the variable of interest in terms of a Markov process by a Langevin equation driven by white noise. A better justification of the modelling requires that the fluctuations should be described by continuous and discontinuous sample paths generated from Markovian noise. In this section, we give the method of finding the time evolution of the characteristic function of the solution process, following McCarty (1974). As an illustration of this powerful technique, we obtain the Fokker-Planck equations for the Ito and Stratonovich types of Langevin equation.

We recall that to describe a Markov process, it is sufficient to obtain the transition probability density of $\overline{X}(t)$ given $\overline{X}(s)$ for some $s < t$. With this and the first density of $\overline{X}(t)$ at some arbitrary time $t_0 < s < t$, we have the complete characterisization of the process $\overline{X}(t)$. Also the characteristic function, which is the Fourier transform of the

*Based on a paper by R.Vasudevan and K.V.Parthasarathy to be submitted for publication probability density function, would be sufficient to describe the system. We given below, in the form of a theorem, the differential equation satisfied by the transition probability density. The proof, essentially depends on the differential equation satisfied by the characteristic function of the process $\overline{X(t)}$.

We give the following main theorem

THEOREM . Let

$$d\overline{X}(t) = \overline{f}(\overline{X}(t),t)\,dt + d\overline{B}(t) + d\overline{n}_p(t), \quad t \geq t_0 \tag{5.1}$$

be the stochastic differential equation , where $\overline{X(t)}$ is the n-dimensional solution process, $B(t)$ is an n-dimensional vector Wiener Process with covariance matrix

$$E\left[d\overline{B}(t)\,(d\overline{B}(t))^T\right] = 2\overline{D}\,dt \tag{5.2}$$

$\overline{n_p(t)}$ is an n-dimensional vector generalised Poisson Process with rate vector $\overline{\lambda}(t)$ and jump probability density $p_a(\alpha)$. Let $P = p_X(\overline{x}, t / \overline{v}, s)$ be the transition probability density function for the process $\overline{X(t)}$. Then P satisfies the partial differential equation

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f_i P) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 (D_{ij} P)}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\left[P * p_{a_i} - P\right] \tag{5.3}$$

where the convolution (*) is defined by

$$p * p_{ai} = \int p_{ai}(x_i - v_i)\, p_{\bar{X}}(x_1 \cdots x_i \cdots x_n, t / \bar{X}(s))\, dv_i$$

(5.4)

The proof of this theorem follows immediately from the lemmas given below.

LEMMA 1. Let $M_{\bar{X}}(\bar{u}, t / \bar{X}(S))$ us the characteristic function of the Markov process X(t). Assume the following:

1. $M_{\bar{X}}(\bar{u}, t / \bar{X}(S))$ is continuously differentiable in t, $t \in T$

2. $$\frac{1}{\Delta t}\left| E\left\{\left(\exp\left\{i\bar{u}^T[\bar{X}(t+\Delta t) - \bar{X}(t)]\right\} - 1\right) / \bar{X}(t)\right]\right| \leq g(\bar{u}; t, \bar{X}) \qquad (5.5)$$

where $E\left[|g|\right]$ is bounded on T.

3. $$\lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left[\left(\exp\left\{i\bar{u}^T[\bar{X}(t+\Delta t) - \bar{X}(t)]\right\} - 1\right) / \bar{X}(t)\right] = \phi(\bar{u}, t; \bar{X}(t)) \qquad (5.6)$$

Then

$$\frac{\partial M_{\bar{x}}(\bar{u},t/\bar{x}(s))}{\partial t} = E\left[\exp\{i\bar{u}^T\bar{x}(t)\}\,\phi(\bar{u},t;\bar{x}(t))\Big|\bar{x}(s)\right]$$

(5.7)

where the expectation in (5.7) is over $\bar{x}(t)$. The function $\phi(\bar{u},t;\bar{x}(t))$ is called the infinitesimal generator of the Markov semigroup. (Wong 1965). Once $\phi(\bar{u},t,\bar{x}(t))$ is evaluated, all that is needed to define a stochastic system fully is known. We calculate $\phi(\bar{u},t;\bar{x}(t))$ for the system described by eq. (5.1), in the following lemma 2.

LEMMA 2. The infinitesimal generator $\phi(\bar{u},t,\bar{x}(t))$ for the system (5.1) is given by

$$\phi(\bar{u},t;\bar{x}(t)) = i\bar{u}^T f(\bar{x},t) - \frac{1}{2}\bar{u}^T \varphi \bar{u} - \sum_{i=1}^{n} \lambda_i\left[1 - M_{ai}(u_i)\right]$$

where λ_i is the i^{th} component of $\bar{\lambda}(t)$ and Mai(ui) is the characteristic function of the i^{th} jump.

We refer to McGarty (1970) for a detailed proof.

We now consider the scalar Ito stochastic differential equation given by

$$dX(t) = f(X(t), t)\,dt + G(X(t), t)\,dB(t), \quad t \geq t_0 \tag{5.8}$$

If $p = p(x, t/x_0, t_0)$ be the transition probability density of the solution process $x(t)$, the characteristic function

$M_x(u, t/x_0, t_0)$ is given by

$$M_x(u, t/x_0, t_0) = \int_{-\infty}^{\infty} e^{iu\xi}\, p(\xi, t/x_0, t_0)\, d\xi \tag{5.9}$$

Using eq.(5.6) we get

$$\phi(u, t/x_0, t_0) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left[e^{iu\Delta X(t)} - 1 \,/\, X(t) \right]$$

Now eq.(5.8) can be written in the incremental form

$$\begin{aligned} \Delta X(t) &= X(t+\Delta t) - X(t) \\ &= f(X(t), t)\Delta t + G\,\Delta B(t) + o(\Delta t) \end{aligned} \tag{5.10}$$

Using (5.10) in (5.6) we get

$$\phi(u, t/x_0, t_0) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left[e^{iu\{f\Delta t + G\Delta B(t) + o(\Delta t)\}} - 1 \,/\, X(t) \right]$$

(writing f for $f(X(t), t)$, G for $G(X(t), t)$).

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\left[\left(e^{iuf\Delta t}\right) E\left\{e^{iuG\Delta B(t)}\right\} - 1\right] \Big| x(t)\right]$$

(using the conditioning on x(t))

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big[\Big[\left\{1 + iuf\Delta t + \cdots\cdots\right\}$$
$$\times \left\{1 + iuE\left(G\Delta B(t)\right) - \frac{u^2}{2} E\left(G^2(\Delta B(t))^2\right) + \cdots\right\}$$
$$-1\Big] \Big| x(t)\Big]$$

Using, Ito's theory we have

$$E\left\{G(x(t),t)\,\Delta B(t)\right\} = 0 \qquad (5.11)$$

and

$$E\left\{G^2(\Delta B(t))^2\right\} = G^2 E(\Delta B(t))^2$$
$$= G^2\, 2D\,\Delta t. \qquad (5.12)$$

We obtain

$$\phi(u,t | x_0,t_0) = iuf - DG^2u^2. \qquad (5.13)$$

Now the evolution of the characteristic function is obtained by using (5.13) in (5.6). Hence, we get the equation

$$\frac{\partial M_x(u,t|x_0,t_0)}{\partial t} = E\left[e^{iux}\{iuf - DG^2u^2\}\right]$$

$$= iu\, E\left(e^{iux} f(x(t),t)\right) - Du^2 E\left(e^{iux_2} G^2(x,t)\right)$$

(5.14)

We know that, if the Fourier transform of a function $q(x)$ is $Q(u)$ ie.

$$Q(u) = \int_{-\infty}^{\infty} e^{iux} q(x)\,dx = \widetilde{F}(q(x))$$

(5.15)

then the Fourier transforms of $q'(x)$ and $q''(x)$ are respectively given by $-iuQ(u)$ and $-u^2Q(u)$.

Hence

$$\widetilde{F}\left\{\frac{\partial}{\partial x}\left(f(x,t)P(x,t|x_0,t_0)\right)\right\} = -iu\widetilde{F}\left(f(x,t)P(x,t|x_0,t_0)\right)$$

$$= -iu\, E\left(e^{iux} f(x,t)\right) \quad (5.16)$$

and

$$\widetilde{F}\left\{\frac{\partial^2}{\partial x^2}\left(G^2(x,t)P(x,t|x_0,t_0)\right)\right\} = -u^2\widetilde{F}\left(G^2(x,t)P(x,t|x_0,t_0)\right)$$

$$= -u^2 E\left(G^2(x,t)e^{iux}\right) \quad (5.17)$$

Hence, taking inverse Fourier transform in eq. (5.14) we get the usual Fokker Planck equation

$$\frac{\partial P(x,t/x_0,t_0)}{\partial t} = -\frac{\partial}{\partial x}(fP) + D\frac{\partial^2}{\partial x^2}(G^2 P) \tag{5.18}$$

Similarly, considering the equation (5.8) in the Stratonovich sense, we can write the equivalent Ito equation as

$$dX(t) = f(X(t),t)\,dt + DG(X(t),t)\frac{\partial G}{\partial x}(X(t),t)\,dt + G(X,t)\,dB(t).$$

Now $\langle G(X,t)\,dB(t)\rangle = 0$ and proceeding on similar lines we get the corresponding Fokker-Planck equation.

Instead of $f(x,t)$ in eq.(5.10) we have the modified form for drift term $f(X(t),t) + DG(X(t),t)\frac{\partial G}{\partial x}(X(t),t)$.

The other derivations are exactly similar and hence we get the final form as

$$\frac{\partial P}{\partial t}(x,t/x_0,t_0) = -\frac{\partial}{\partial x}\left[\left\{f(x,t) + DG(x,t)\frac{\partial G(x,t)}{\partial x}\right\}P\right] + D\frac{\partial^2}{\partial x^2}\left[G^2(x(t),t)\,P\right] \tag{5.19}$$

Thus we have exploited the method of McCarty to derive the Fokker-Planck equations.

6. Systems with point processes as inputs

In this section we proceed, with a short account of the modelling of stochastic differential equations driven by point processes. The theory of stochastic integrals with respect to point processes is well developed and it follows a parallel approach to Ito and Stratonovich integrals (Snyder 1975, Marcus 1978). Realising the limitations of Ito integrals, Mcshane (1974) has developed a 'unified calculus' which is well-suited to modelling analysis. We give below, his concept of canonical extension and the recent generalisation of Marcus (1978), applicable to more general types of noise phenomena.

Consider a dynamical system whose state $x(t)$ at time t satisfies the integral equation

$$X(t) = X(0) + \int_{t_0}^{t} f(X(\tau))\,d\tau + \sum_{i=1}^{k} \int_{t_0}^{t} G_i(X(\tau))\,dZ^i(\tau) \tag{6.1}$$

or the equivalent differential equation

$$dx(t) = f(x(t))\,dt + \sum_{i=1}^{k} G_i(x(\tau))\,dZ^i(\tau) \tag{6.2}$$

where $Z^i(t)$ are independent and smooth noise processes (Lipschitzian, for example). When $Z^i(t)$ is the Brownian motion process eq.(6.2) represents the well-known stochastic differential equation studied so far. But this model is not adequate to describe certain physical phenomena and hence other noise models are used as inputs for yielding better results.

In order to investigate such systems, Mcshane (1974,1975) developed a more general calculus which includes ordinary and Ito calculus, under certain assumptions on $Z(t)$. In his approach, eq.(6.1) is replaced by a canonical equation, which is valid for more general continuous noise processes, including the Brownian motion process. When $Z(t)$ happens to be the Brownian motion process, the canonical equation is equivalent to interpreting eq.(6.1) in the Stratonovich sense. Mcshane obtains his results on the assumption that the noise processes are sample continuous and this results in the vanishing of his Ito-beleted integrals of order three or higher. But this property is not true for point processes. Recently, a more general canonical extension is defined in (Marcus 1978) which includes the point processes as well as noise processes of Mcshane.

The generalised canonical extension is defined as follows. Let the differential operator D_i be defined by

$$D_i = \sum_{\ell=1}^{n} G_i^{\ell}(x) \frac{\partial}{\partial x^{\ell}} \qquad (6.3)$$

where G_i^{ℓ} and x^{ℓ} are the ℓ^{th} components of G_i and x

If $a(x) = (a^1(x), \ldots a^p(x))'$, where prime denotes transpose, let

$$D_i(a(X)) = \left(D_i(a^1(x)), \ldots D_i(a^p(x))\right)' \tag{6.4}$$

Powers of D_i are defined by

$$D_i^{m+1}(a(X)) = D_i\left(D_i^m(a(X))\right) \tag{6.5}$$

and

$$D_i^0(a(X)) = a(X) \tag{6.6}$$

The canonical extension of (6.1) and (6.2) is defined in terms of the operators D_i as

$$X(t) = X(0) + \int_{t_0}^{t} f(X(\tau))d\tau + \sum_{m=1}^{\infty} \sum_{i=1}^{k} \frac{1}{\lfloor m} \int_{t_0}^{t} D_i^{m-1}(G_i(X(t))(dz^i(t))^m . \tag{6.7}$$

where the last integral in (5) is the Ito-beloted integral of Mcshane et (6.7) can also be written as

$$dx(t) = f(X(t))dt + \sum_{m=1}^{\infty} \sum_{i=1}^{k} \frac{1}{\lfloor m} D_i^{m-1}(G_i(X(t))(dz^i(t))^m \tag{6.8}$$

This is the canonical extension of that defined by Mcshane in the sense, that if each $z^i(t)$ satisfies a $K \Delta t$

condition and is sample-continuous, then all the Ito-beleted integrals in (6.7) with $m > 2$ will vanish. As an example, we see that if $Z^i(t)$ are independent Brownian motion process, then eq.(6.8) becomes

$$dX(t) = \left[f(X(t)) + \frac{1}{2} \sum_{i=1}^{k} D_i \left(G_i(X(t)) \right) \right] dt + \sum_{i=1}^{k} G_i(X(t))\, dZ^i(t) \tag{6.9}$$

Further if the $Z^i(t)$ are Lipschitzian, all the integrals with $m > 1$ vanish and eq.(6.7) is same as (6.1).

We now consider the case in which $Z^1(t), \ldots Z^k(t)$ are independent point processes with constant jump sizies $\alpha_1, \alpha_2, \ldots \alpha_k$ and $N^1 \ldots N^k$ are the independent counting processes, (Snyder 1975) which count the number of jumps of $Z^1(t) \ldots Z^k(t)$. In this case the Ito-beleted integrals for $m > 3$ do not vanish. It is proved in (Mcshane 1974) that

$$\int_{t_0}^{t} h(\tau) \left(dZ^i(\tau) \right)^p = (\alpha_i)^p \int_{t_0}^{t} h(\tau)\, dN^i(\tau) = (\alpha_i)^p \sum_{n=1}^{N_t} h(\tau_n^i) \tag{6.10}$$

where τ_n^i are the jump times of N^i.

The concept of Lie series (Cap and Weil 1970) is useful to show that eq.(6.7) is well-defined. The solution of the differential equation

$$\dot{X}(t) = G_i(X(t)), \quad X(0) = X_0 \tag{6.11}$$

is given by the Lie series

$$X(t) = e^{tD_i} X(0) = X(0) + \sum_{m=1}^{\infty} \frac{t^m}{m!} D_i^{m-1}\left(G_i(X(0))\right) \tag{6.12}$$

Hence, if the Z^i are point processes, (6.10) and (6.12) imply that the canonical extension (6.7) is well defined and equivalent to the point process driven equation

$$X(t) = X(0) + \int_{t_0}^{t} f(X(\tau))\,d\tau + \sum_{i=1}^{k} \int_{t_0}^{t} \left[e^{\alpha_i D_i} X(\tau) - X(\tau)\right] dN^i_{(\tau)} \tag{6.13}$$

If $G_i(X) = G_i$ = constant for all i, then $e^{\alpha_i D_i} x = \alpha_i G_i + x$. Hence the point canonical extension is

$$\begin{aligned} X(t) &= X(0) + \int_{t_0}^{t} f(X(\tau))\,d\tau + \sum_{i=1}^{k} \int_{t_0}^{t} \alpha_i G_i \, dN^i_\tau \\ &= X(0) + \int_{t_0}^{t} f(X(\tau))\,d\tau + \sum_{i=1}^{k} \int_{t_0}^{t} G_i \, dZ^i(t) \end{aligned} \tag{6.14}$$

Hence we get the important result that the point process canonical extension of (6.1) is itself.

More recent generalisations of this concept can be found in Marcus (1981).

7. Stochastic Systems Induced by External Coloured Noise*

In recent years, Langevin equations are frequently used in various branches of physics, due to the mathematical elegance of stochastic differential equations with white noise inputs. However, these theories represent only specific models corresponding to the Markovian limit. In the formulation of the concept of random frequency modulation, Kubo (1969) called this as "narrowing limit". Further, in the works of Horsthemke et al (1976, 1977, 1978) it was established that certain non-linear systems, subjected to external white noise exhibit a series of transitions, which are not expected in the usual phenomenological point of view. This new class of non-equilibrium phase transitions has been called noise induced phase transitions. The white noise model is adequate only when the correlation time of the external noise is much shorter than the characteristic macroscopic time of the system. With these limitations on the white noise model, it became natural to study the behaviour of non linear systems subjected to more realistic

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* based on a paper by R.Vasudevan and K.V.Parthasarathy to be submitted for publication

noise. The development in this direction was concentrated mainly on two aspects (i) the influence of the correlation time of fluctuations on the macroscopic behaviour of non-linear systems, and (ii) an explicit study of the approach of a real noise to white noise ie. when the correlation time τ cor tending to zero. But arguments were put forth (Horsthemke et al, 1978) that the noise induced phase transitions are not due to the white noise idealisation but they occur for real noise with a short but non-vanishing correlation time.

With the coloured noise represented by Ornstein-Uhlenbeck process only an approximate formula for the stationary probability could be obtained (Horsthemke et al 1978).

Exact analytical results for a special case of external coloured noise ie. "dichotomous noise" were obtained in Kitahara et al (1979).

In this section we obtain the equation of the evolution for the probability density of the solution process by an entirely different approach. We start with the Liouville's equation of Van Kampen (1976) for the phase space distribution function ρ (x,t), We express the Liouville's equations in terms of suitably defined operators and connect it with the equation of evolution of probability density function with the cumulant expansion technique.

The method of finding the stationary solution is based on the new approach of applying the operator of the form $(\gamma + L_0)^{\dagger}$ defined subsequently.

Random Telegraph Noise

We consider a non linear system subjected to coloured noise I (t), which has two levels $\pm\Delta$, as the state space. This noise is often called the 'random telegraph noise' or 'the dichotomous noise'. The temporal evolution of the conditional probability $P(I,t/I_0)$, characterising the process is described by the master equation

$$\frac{d}{dt}\begin{pmatrix} P_+(t) \\ P_-(t) \end{pmatrix} = -\frac{\gamma}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} P_+(t) \\ P_-(t) \end{pmatrix} \qquad (7.1)$$

with $P_{\pm}(t) = P(I(t) = \pm\Delta)$

The stationary solution of eq.(7.1) is easily seen to be

$$P(I,\infty|I_0) = \frac{1}{2} \qquad (7.2)$$

If the random telegraph noise has this (7.2) as initial condition, then I(t) is a stationary process, with

$$E\{I(t)\} = 0 \qquad (7.3)$$

$$E\{I(t)I(t')\} = \Delta^2 e^{-\gamma|t-t'|} . \qquad (7.4)$$

Hence, regarding the mean and the correlation functions, this noise is indistinguishable from the Ornstein-Uhlenbeck process. Further it converges to the Gaussian white noise in the limit $\Delta \to \infty$, $\gamma \to \infty$ such that $\frac{\Delta^2}{\gamma} = \frac{\sigma^2}{2}$ (finite)

With these basic properties of $I(t)$, our interest is to find the equation of evolution of the probability density, $P(x,t/x_0)$ for the system, described by the equation of the multiplicative type

$$\dot{x}(t) = f(x(t)) + g(x(t))\, I(t) \tag{7.5}$$

clearly

$$P(x,t/x_0) = P(x,\Delta,t/x_0,I_0) + P(x,-\Delta,t/x_0,I_0) \tag{7.6}$$

Consider a probability flow in phase space of an ensemble of dynamically equivalent systems, described by eq.(7.5), but distinguished by the realisation of the fluctuating function $I(t)$. In this phase space picture, there exists the density of systems $\rho(x,t)$ at the point x at time t. If $x_I(t)$ is the solution of (7.5) for a particular realisation of $I(t)$, then the phase-space distribution is given by (Lax, 1966, Van Kampen 1976)

$$\rho(x,t) = \delta(x - x_I(t)) \tag{7.7}$$

The density $\rho(x,t)$ must satisfy a continuity equation

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x}\left[\dot{x}\,\rho(x,t)\right] = 0 \qquad (7.8)$$

with the initial condition

$$\rho(x,t=0) = \delta\left(x - x(0)\right) \qquad (7.9)$$

Van Kampen (1976) has proved that the conditional probability density $P(x,t/x_0)$ is obtained from averaging the phase space density function over all possible realisations of I(t), ie.

$$P(x,t/x_0) = \langle \rho(x,t) \rangle \qquad (7.10)$$

We assume that each member of the ensemble starts from the same initial condition $x(0)$ and hence we are not indexing the initial value for each realisation.

Using (7.5) in (7.8), we get

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x}\left\{\left[f(x) + g(x)I(t)\right]\rho(x,t)\right\} = 0 \qquad (7.11)$$

We introduce operators L_0 and L_1 defined on an arbitrary function $q(x,t)$ by

$$L_0\, q(x,t) = -\frac{\partial}{\partial x}\left[f(x)\, q(x,t)\right] \qquad (7.12)$$

$$L_1 q(x,t) = -\frac{\partial}{\partial x}\left[g(x) I(t) q(x,t)\right] \tag{7.13}$$

In terms of these operators, (7.1) can be written as

$$\frac{\partial \rho(x,t)}{\partial t} = L_0 \rho(x,t) + L_1 \rho(x,t) \tag{7.14}$$

By the definition of the operators, we note that L_0 is associated with the deterministic and L_1 with the stochastic evolution of the system.

We now proceed to obtain the equation satisfied by/(7.10) using cumulant expansion technique. (Van Kampen 1976, Kubo 1963, Mukamel et al 1978). In particular we follow the arguments advanced by Mukamel et al (1978).

We introduce the phase-space distribution function $\hat{\rho}(x,t)$ defined by

$$\hat{\rho}(x,t) = e^{-L_0 t} \rho(x,t) \tag{7.15}$$

Differentiation with respect to t gives

$$\frac{\partial \hat{\rho}(x,t)}{\partial t} = -L_0 e^{-L_0 t} \rho(x,t) + e^{-L_0 t} \frac{\partial \rho(x,t)}{\partial t}$$

Hence, using (7.14) we get

$$\frac{\partial \hat{\rho}(x,t)}{\partial t} + L_0 \hat{\rho}(x,t) = e^{-L_0 t}\left(L_0 \rho(x,t) + L_1 \rho(x,t)\right)$$

$$= L_0 \hat{\rho}(x,t) + e^{-L_0 t} L_1(t) e^{L_0 t} \hat{\rho}(x,t)$$

Thus we have

$$\frac{\partial \hat{\rho}(x,t)}{\partial t} = e^{-L_0 t} L_1(t) e^{L_0 t} \hat{\rho}(x,t) \qquad (7.16)$$

using the interaction representation, denoted by

$$\tilde{L}_1(t) = e^{-L_0 t} L_1(t) e^{L_0 t} \qquad (7.17)$$

eq.(7.16) becomes

$$\frac{\partial \hat{\rho}(x,t)}{\partial t} = \tilde{L}_1(t) \hat{\rho}(x,t) \qquad (7.18)$$

The formal solution of (7.18) in terms of the time ordered evolution operator

$$\tilde{U}(t,0) = \left[\exp\left(\int_0^t \tilde{L}_1(t')dt'\right)\right]_T, \qquad (7.19)$$

where T is the time ordering, is given by

$$\hat{\rho}(x,t) = \tilde{U}(t,0)\,\hat{\rho}(x,0) \qquad (7.20)$$

In (7.19), $[\ ]_T$ denotes 'time-ordering' defined by the prescription that one should first expand the exponential

$$\tilde{U}(t,0) = \Big[1 + \int_0^t \tilde{L}_1(\tau_1)\,d\tau_1 + \frac{1}{\underline{|2}}\int_0^t\int_0^t \tilde{L}_1(\tau_1)\tilde{L}_1(\tau_2)\,d\tau_1 d\tau_2 + \cdots\cdots\Big]_T \tag{7.21}$$

and subsequently, in each multiple integral reorder the operator $\tilde{L}_1$ according to decreasing values of their times

$$\tilde{U}(t,0) = \Big\{1 + \int_0^t \tilde{L}_1(\tau_1)\,d\tau_1 + \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2\, \tilde{L}_1(\tau_1)\tilde{L}_1(\tau_2) + \cdots\cdots \Big\} \tag{7.22}$$

$$= 1 + \sum_{n=1}^{\infty} M_n(t) \tag{7.23}$$

where

$$M_n(t) = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n\, \tilde{m}_n(\tau_1, \ldots \tau_n) \tag{7.24}$$

with

$$\tilde{m}_n(\tau_1, \ldots \tau_n) = \tilde{L}_1(\tau_1)\tilde{L}_1(\tau_2) \cdots \tilde{L}_1(\tau_n) \tag{7.25}$$

Taking the average over the ensemble of realisations of I(t), we have

$$\langle \hat{f}(x,t) \rangle = \langle \hat{U}(t,0) \rangle \hat{f}(x,0) \qquad (7.26)$$

since each member of the ensemble has the same initial condition.

If (7.26), we use the expansion method of Kubo (1963) to express the average of the evolution operator in terms of the cumulants. We have

$$\langle \tilde{U}(t,0) \rangle = \exp[K(t)]_T \qquad (7.27)$$

where

$$K(t) = \sum_{n=1}^{\infty} K_n(t) \qquad (7.28)$$

the $K_n(t)$ being the cumulants defined as

$$K_n(t) = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n \langle\langle \tilde{L}_1(\tau_1) \ldots \tilde{L}_1(\tau_n) \rangle\rangle \qquad (7.29)$$

where the double bracket-notation $\langle\langle \quad \rangle\rangle$ indicates a cumulant ordering.

If we define the time derivative of the cumulant operators in the interaction representation as

$$\widetilde{\dot{K}_n}(t) = e^{L_0 t} \dot{K}_n(t) e^{-L_0 t} \qquad (7.30)$$

we get the equation satisfied by $P(x,t/x_0)$ as

$$\frac{\partial P(x,t|x_0)}{\partial t} = L_0 P(x,t/x_0) + \sum_{n=1}^{\infty} \tilde{K}_n(t) P(x,t|x_0) \quad (7.31)$$

This equation simplifies in an elegant form when we expand the evolution operator upto the second order terms. The first cumulant vanishes here and the second cumulant is same as the second moment. Hence, using (7.27) and (7.28), (7.26) becomes

$$\langle \hat{P}(x,t) \rangle = \exp(K_2(t)) \hat{P}(x,0) \quad (7.32)$$

Hence

$$\frac{\partial}{\partial t} \langle \hat{P}(x,t) \rangle = \dot{K}_2(t) \exp K_2(t) \hat{P}(x,0) \quad (7.33)$$

But, from (7.15),

$$\frac{\partial}{\partial t} \langle \hat{P}(x,t) \rangle = \frac{\partial}{\partial t} \langle e^{-L_0 t} P(x,t) \rangle$$

$$= \langle -L_0 e^{-L_0 t} P \rangle + \langle e^{-L_0 t} \frac{\partial P(x,t)}{\partial t} \rangle$$

$$= -L_0 e^{-L_0 t} \langle P(x,t) \rangle + e^{-L_0 t} \langle \frac{\partial P(x,t)}{\partial t} \rangle \quad (7.34)$$

Combining (7.33) and (7.34), we have

$$e^{-L_0 t} \left\langle \frac{\partial f(x,t)}{\partial t} \right\rangle$$

$$= L_0 e^{-L_0 t} \langle f(x,t) \rangle + \dot{K}_2(t) \exp K_2(t) \hat{f}(x,0)$$

$$= L_0 e^{-L_0 t} \langle f(x,t) \rangle + \dot{K}_2(t) \langle \hat{f}(x,t) \rangle$$

by 7.32

Hence, differentiating (7.29) for n = 2, we have

$$\left\langle \frac{\partial f(x,t)}{\partial t} \right\rangle = L_0 \langle f(x,t) \rangle + e^{L_0 t} \int_0^t d\tau \langle \tilde{L}_1(t) \tilde{L}_1(\tau) \rangle \langle \hat{f}(x,t) \rangle$$

$$= L_0 \langle f(x,t) \rangle + e^{L_0 t} \int_0^t d\tau \langle e^{-L_0 t} L_1(t) e^{L_0 t} e^{-L_0 \tau} L_1(\tau) e^{L_0 \tau} \rangle \langle \hat{f}(x,t) \rangle$$

$$= L_0 P(x,t/x_0) + \int_0^t d\tau \langle L_1(t) e^{L_0(t-\tau)} L_1(\tau) e^{L_0 \tau} \rangle e^{-L_0 t} \langle f(x,t) \rangle$$

$$= L_0 P + \int_0^t d\tau \left\langle \left\{ -\frac{\partial}{\partial x} g(x) I(t) \right\} e^{L_0(t-\tau)} \left(-\frac{\partial}{\partial x} g(x) I(\tau) \right) \right\rangle e^{L_0(\tau - t)} \langle f \rangle \; (*)$$

(where $P = P(x,t/x_0)$; $f = f(x,t)$)

$$= L_0 P + \frac{\partial}{\partial x} g(x) \int_0^t d\tau \langle I(t) I(\tau) \rangle e^{L_0(t-\tau)} \frac{\partial}{\partial x} g(x) \langle f(x,\tau) \rangle .$$

$$= L_0 P + \Delta^2 \frac{\partial}{\partial x} g(x) \int_0^t d\tau \, e^{-(\gamma - L_0)(t-\tau)} \frac{\partial}{\partial x} g(x) P(x,\tau) .$$

(*) Here we can approximately replace $e^{-L_0(\tau - t)}$ by $e^{-L(\tau - t)}$ since the term we neglect is of higher order.

Hence we get the Fokker-Planck equation as (using 7.12)

$$\frac{\partial P(x,t|x_0)}{\partial t} = -\frac{\partial}{\partial x}\left\{ f P(x,t|x_0) \right\} + \Delta^2 \frac{\partial}{\partial x} g(x) \int_{-\infty}^{t} d\tau\, e^{-(\gamma + \frac{\partial f}{\partial x})(t-\tau)} \frac{\partial}{\partial x} g(x) P(x,\tau) \qquad (7.35)$$

The non-markovian character is exhibited by the memory kernal.

In the white noise limit $\Delta \to \infty, \gamma \to \infty, \frac{\Delta^2}{\gamma} = \frac{\sigma^2}{2}$ (finite) the kernel reduces to a Dirac delta function and the Fokker-Planck equation becomes

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left\{ f(x) P \right\} + \frac{\sigma^2}{2} \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) P. \qquad (7.36)$$

This is exactly the Fokker-Planck equation, corresponding to the stochastic differential equation

$$dx = f(x(t),t)\,dt + g(x(t),t)\,dB(t),$$

which has to be interpreted in the Stratonovich sense.

This illustrates the theorem that whenever a stochastic differential equation is obtained as the white noise limit of a real noise problem, the equation should be interpreted in the Stratonovich sense (Wong and Zakai, 1965).

Next, we obtain the stationary solution $P_{st}(x)$ of eq. (7.35) by the following new approach. It is given by the equation

$$f\tilde{P} = \Delta^2 g \int_{-\infty}^{t} d\tau \, e^{-(\gamma - L_0)(t-\tau)} \frac{\partial}{\partial x} g(x) \tilde{P}(x)$$

where $\tilde{P} = P_{st}(x)$ (7.37)

(ie.) $$\frac{f(x)\tilde{P}}{\Delta^2 g(x)} = \frac{1}{\gamma - L_0} \frac{\partial}{\partial x} g(x) \tilde{P}.$$

Operating both sides by $\gamma - L_0$ we get

$$(\gamma - L_0) \frac{f(x)\tilde{P}}{\Delta^2 g(x)} = \frac{\partial}{\partial x} g(x) \tilde{P}$$

(ie) $$\frac{\gamma f(x)\tilde{P}}{\Delta^2 g(x)} + \frac{\partial}{\partial x}\left(\frac{f(x) f(x) \tilde{P}}{\Delta^2 g(x)}\right) = \frac{\partial}{\partial x} g(x) \tilde{P}$$

Carrying out the differentiation and grouping we get

$$\tilde{P}\left[\frac{\gamma f(x)}{\Delta^2 g(x)} - \frac{f^2(x) g'(x)}{\Delta^2 g^2(x)} + \frac{2 f(x) f'(x)}{\Delta^2 g(x)} - g'(x)\right]$$
$$= \frac{\partial \tilde{P}}{\partial x}\left[g(x) - \frac{f^2(x)}{\Delta^2 g(x)}\right]$$

Now $\frac{\partial \tilde{P}}{\partial x} = \frac{d\tilde{P}}{dx}$

or, $$\left[\frac{\gamma f(x) + 2f(x)f'(x) - \Delta^2 g(x)g'(x) - f^2(x)\dfrac{g'(x)}{g(x)}}{\Delta^2 g^2(x) - f^2(x)}\right] = \frac{d\tilde{p}}{\tilde{p}}.$$

(i.e) $$\left[\frac{\gamma f(x)}{\Delta^2 g^2(x) - f^2(x)} + \frac{2f(x)f'(x) - 2\Delta^2 g(x)g'(x)}{\Delta^2 g^2(x) - f^2(x)} + \frac{\Delta^2 g(x)g'(x) - f^2(x)\dfrac{g'(x)}{g(x)}}{\Delta^2 g^2(x) - f^2(x)}\right] = \frac{d\tilde{p}}{\tilde{p}}$$

Integrating,

$$\int^{x} \frac{\gamma f(x')\,dx'}{\Delta^2 g^2(x') - f^2(x')} - \log\left(\Delta^2 g^2(x) - f^2(x)\right) + \log g = \log\tilde{p} + \text{Constant}$$

Hence

$$\tilde{p} = P_{st}(x)$$

$$= N\frac{g(x)}{\Delta^2 g^2(x) - f^2(x)} \exp\left[-\gamma\int^{x}\frac{f(x')\,dx'}{\Delta^2 g^2(x') - f^2(x')}\right]. \quad (7.38)$$

Thus, we recover the expression derived by a different method. (Kitahara, 1979)

The influence of coloured noise in concrete applications is worthy of mention particularly in the study of population growth and chemical reaction schemes. (Schlogl 1972, Keizer and Fox 1974, Glansdorff et al 1974). The size u of an isolated population varying with time is often described by the nonlinear Malthus equation (Lotka 1921)

$$\dot{u} = g(u) - \delta u^2. \tag{7.39}$$

This is also the equation describing the Verhulst model. Both the coefficients depend on the environments and may be considered as random. In particular, we may take $g = 1$, $\delta = 1 + \alpha I(t)$ where I is the random telegraph noise. We can use the Fokker-Planck equation and the steady state solution derived above and obtain results similar to those of Van Kampen (1976).

C H A P T E R III

NEW RESULTS ON FLUCTUATION DISSIPATION THEORY*

1. Introduction

In physics and engineering, a number of problems lead to random differential equations of the so called Langevin type. In physics, these equations are encountered in the study of Brownian motion, (Wang and Uhlenbeck 1945). Even though there had been a lot of investigations of Brownian motion, it was not until 1905 that a quantitative theory, making predictions susceptible of experimental verification was put forward. Albert Einstein (1905), who showed that the kinetic theory of matter required that small particles suspended in a fluid undergo an irregular motion. This motion was too chaotic to be described in any other way than statistically and Einstein obtained some of the statistical characterisation of the motion. There have been a number of investigations along this line. Kirkwood (1946) showed the relation between the friction constant of a Brownian particle and the fluctuation of the force acting on it. Even earlier, Nyquist (1928) proved the relation between the thermal noise and impedance of a resistor, which is the first clear statement on fluctuation - dissipation theorem.

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* Based on a paper by R.Vasudevan and K.V.Parthasarathy to be submitted for publication.

The importance of this theorem has been widely recognised in the development of the statistical mechanics of irreversible processes (Callen and Welton 1951, Kubo 1957, 1966). Fluctuation dissipation theorems are used in characterising the fluctuations in the system and to derive the admittance from thermal fluctuations. The random driving force on the particle and the frictional force arise out of the random collisions of the media molecules with the particle. Since these forces arise out of a common source, it is natural to expect some relations between them. These relations are generally called fluctuation-dissipation theorems.

In most of the Langevin equations used to model physical systems, the fluctuations are additive. Recently there has been a great deal of interest in systems described by Langevin equations in which the fluctuations depend multiplicatively on the system variables. The fact that fluctuations are magnified or reduced depending on the state of the system leads to interesting behaviour, that cannot occur in the presence of only additive fluctuations.

As described earlier, whenever the noise enters the Langevin equation multiplicatively, the two approaches of Ito and Stratonovich are available. The Ito treatment of stochastic differential equations always ignores correlations and thus leads to non physical results. Hence we follow Stratonovich theory in the study of suitable Langevin equations to derive new results on fluctuation-dissipation theory.

The plan of this chapter is as follows: Section 2 gives a short account of fluctuation-dissipation relation of Einstein. We generalise the Langevin equation by taking the friction coefficient random and consider the new equation in the Stratonovich sense. By using the method of moments, based on a general law we get modified form for first fluctuation dissipation relation. We also point out significant contribution regarding the virial theorem. In section 3, we consider Kubo's form of generalised Langevin-equation with an additional multiplicative fluctuating term. We present our second fluctuation dissipation result in the Fourier transformed version. Section 4 describes another application of the general law described in section 2. Using this method new results are obtained for the time evolution of the average of the $\vec{L}$ and $\vec{L}^2$, $\vec{L}$ being the angular momentum.

In all these results, the prescription as laid down by Stratonovich brings out inherent characteristics of the stochastic system, which are suppressed in other considerations.

2. First Fluctuation - Dissipation Theorem

The Langevin equation of a free Brownian particle in one dimension is given by the phenomenological stochastic equation

$$m\dot{p} = -m\gamma p + R(t) \tag{2.1}$$

where m is the mass of the particle, moving with velocity p. The first term on the right hand side is a systematic frictional force, linearly related to the particle's velocity. The second term is the random force due to the collisions of the surrounding molecules.

For a simple and idealised model of the Brownian motion, (Uhlenbeck and Ornstein 1930, Chandrasekhar 1943, Wang and Uhlenbeck,1945, Ramakrishnan,1959), the following conditions are assumed: (i) the process R(t) is Gaussian (ii) Its correlation time is infinitely short, ie. the correlation function of R(t) has the form

$$\langle R(t) R(t') \rangle = 2D\,\delta(t-t') \tag{2.2}$$

where $\langle\ \rangle$ denotes the average over an ensemble of realisations of R(t) and (iii) The Brownian motion takes place in the medium in thermal equilibrium.

The first assumption is justified for a Brownian particle having a mass much larger than the colliding molecules, since the motion is a result of numerous successive independent collisions, which enables one to appeal to central limit theorem. The second assumption is also reasonable because the correlation time between successive impacts is short compared with the time scale of the Brownian motion.

Due to the first two assumptions the process p(t) is Gaussian and Markovian (Wang and Uhlenbeck 1945). Hence we have a complete information of the process p(t) from the transition probability $P(p,t|p_0,t_0)$, from the velocity p_0 at time to the velocity p at time t. $P(p,t|p_0,t_0)$ satisfies the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial p}(\gamma p P) + D\frac{\partial^2 P}{\partial p^2} \qquad (2.3)$$

with

$$P(p,t_0;p_0,t_0) = \delta(p-p_0) \qquad (2.4)$$

The fluctuating force and the diffusion constant are given by the relation

$$D = \frac{1}{m^2}\int_0^\infty \langle R(t_0)R(t_0+t)\rangle dt. \qquad (2.5)$$

Result (2.5) is given in the standard derivation of the Fokker-Planck equation. From assumption (iii), we have

$$\lim_{t\to\infty} P(p,t;p_0,t_0) = c\exp\left(-\frac{1}{2}\frac{mp^2}{kT}\right) \qquad (2.6)$$

ie. the stationary solution of eq.(2.3) must coincide with the Maxwellian distribution. This gives the Einstein relation between the friction and the diffusion constants

$$D = \frac{\gamma}{m} kT \tag{2.7}$$

where T is the bath temperature and k is Boltzmann's constant. Eqns. (2.5) and (2.7) give

$$\gamma = \frac{1}{mkT}\int_0^\infty \langle R(t_0) R(t_0+t)\rangle dt. \tag{2.8}$$

eq. (2.7) or (2.8) gives the relation between friction constant γ and the fluctuations of the random force. This is the concept of fluctuation-dissipation theorem. The fluctuation-dissipation relation ensures that the energy $E=\frac{1}{2}mp^2$ of the Brownian particle is canonically distributed at equilibrium

$$P_{eq}(E) = \frac{1}{kT}\exp\left(-\frac{E}{kT}\right). \tag{2.9}$$

The above discussion is equally true in the case of the Brownian motion in a potential field. (Kubo 1966, Mori 1965).

Attempts were made to generalise the Langevin equation (2.1), to get a deeper insight of the fluctuation-dissipation theory from a more general point of view. In our new attempts, we take the friction coefficient γ to be random, so that this random force enters the Langevin equation in a multiplicative manner. Following the work of L. de la Pena (1980). We can obtain important relations describing the average properties of the stochastic systems, as particular cases of a general law. As pointed out in the introduction, the multiplicative noise, interpreted in the Stratonovich sense, gives rise to a modification in first fluctuation-dissipation relation and the Virial theorem which is very useful in the kinetic theory of gases. The Langevin equation now under consideration is given by the set of equations, in phase space

$$\begin{aligned} \dot{x} &= p/m \\ \dot{p} &= -\gamma p + F(x) + R(t) \end{aligned} \tag{2.10}$$

F(x) being the external force and R(t) is Gaussian driving force with mean zero and correlation function

$$\langle R(t) R(t') \rangle = 2D\,\delta(t-t') \tag{2.11}$$

The friction coefficient γ is also a fluctuating function which we partition as

$$\gamma = \beta_0 + \lambda(t) \tag{2.12}$$

where

$$\langle \lambda(t) \rangle = 0 \tag{2.13}$$

and

$$\langle \lambda(t)\,\lambda(t')\rangle = 2\tilde{D}\,\delta(t-t') \qquad (2.14)$$

We assume that λ (t) and R(t) are independent. Our analysis is based on the evolution of the probability density P(x,p,t) for the oscillator displacement and momentum in phase space.

Following Stratonovich (1963), the Fokker-Planck equation satisfied by P(x,p,t)can be written using eq.( ) of chapter II. Denoting the right hand sides in eq.(2.16) by F_1 and F_2 respectively, we find that the following correlation functions vanish

$$\left.\begin{matrix} K(F_1, F_{1\tau}),\ K(F_1, F_{2\tau}),\ K(F_2, F_{1\tau}), \\ K\left(\frac{\partial F_1}{\partial x}, F_{1\tau}\right),\ K\left(\frac{\partial F_1}{\partial p}, F_{2\tau}\right),\ K\left(\frac{\partial F_2}{\partial x}, F_{1\tau}\right) \end{matrix}\right\}$$

where

$$\dot{x} = F_1 = p/m \qquad (2.15a)$$

$$\dot{p} = F_2 = -\beta_0 p - \lambda(t)p + F(x) + R(t) \qquad (2.15b)$$

The only two surviving correlation functions are the following:

$$K\left(\frac{\partial F_2}{\partial p}, F_{2\tau}\right) = K\left[(-\beta_0 - \lambda(t)), (-\beta_0 p - \lambda_\tau(t)p + F(x) + R_\tau(t))\right] = \langle \lambda(t)\,\lambda_\tau(t)\rangle p. \qquad (2.16)$$

$$K(F_2, F_{2\tau}) = K\left[\{-\beta_0 p - \lambda(t)p + F(x) + R(t)\}, \{-\beta_0 p - \lambda_\tau(t)p + F(x) + R_\tau(t)\}\right] = \langle \lambda(t)\,\lambda_\tau(t)\,p^2 + R(t)R_\tau(t)\rangle. \qquad (2.17)$$

Using (2.16) and (2.17), we get the evolution equation for P(x,p,t) as

$$\frac{\partial P(x,p,t)}{\partial t} = -\frac{p}{m}\frac{\partial P}{\partial x} - \frac{\partial}{\partial p}\left\{(-\beta_0 p + F(x))P\right\} - \frac{\partial}{\partial p}\left\{\tilde{D} p P\right\} + \frac{\partial^2}{\partial p^2}\left\{(\tilde{D} p^2 + D)P\right\}$$

(i.e)

$$\frac{\partial P(x,p,t)}{\partial t} = -\frac{p}{m}\frac{\partial P}{\partial x} - \frac{\partial}{\partial p}\left[\left\{(\tilde{D}-\beta_0)p + F(x)\right\}P\right] + D\frac{\partial^2 P}{\partial p^2} + \tilde{D}\frac{\partial^2}{\partial p^2}(p^2 P) \quad (2.18)$$

Let f(x,p) be an integrable, but otherwise an arbitrary function of the phase space variables. Multiplying eq.(2.18) by f(x,p) and integrating over the phase space, we get the equation giving the evolution of $\langle f(x,p)\rangle$ as follows

$$\frac{d}{dt}\langle f(x,p)\rangle = \frac{1}{m}\left\langle p\frac{\partial f}{\partial x}\right\rangle + \left\langle F(x)\frac{\partial f}{\partial p}\right\rangle + (\tilde{D}-\beta_0)\left\langle p\frac{\partial f}{\partial p}\right\rangle + D\left\langle\frac{\partial^2 f}{\partial p^2}\right\rangle + \tilde{D}\left\langle p^2\frac{\partial^2 f}{\partial p^2}\right\rangle. \quad (2.19)$$

This equation constitutes the general law for finding the evolution of the average of any dynamical variable and it is the core of our discussion. In

In (2.19), let us take successively $f(x,p) = x, p, x^2$, xp and p^2 and get the equations

$$\frac{d}{dt}\langle x \rangle = \frac{1}{m}\langle p \rangle \tag{2.20a}$$

$$\frac{d}{dt}\langle p \rangle = \langle F(x) + (\tilde{D} - \beta_0)p \rangle \tag{2.20b}$$

$$\frac{d}{dt}\langle x^2 \rangle = \frac{2}{m}\langle xp \rangle \tag{2.21a}$$

$$\frac{d}{dt}\langle xp \rangle = \frac{\langle p^2 \rangle}{m} + \langle xF(x) \rangle + (\tilde{D} - \beta_0)\langle xp \rangle \tag{2.21b}$$

$$\frac{d}{dt}\langle p^2 \rangle = 2\langle pF(x) \rangle + 2(2\tilde{D} - \beta_0)\langle p^2 \rangle + 2D. \tag{2.21c}$$

The above equations are of vital importance in characterising the evolution of the stochastic system under consideration.

Equation (2.15b) can be written as equivalent Ito equation

$$\dot{p} = -\beta_0 p + F(x) + \tilde{D}p + R(t) - \lambda(t)p. \tag{2.22}$$

Averaging we get $\frac{d}{dt}\langle p \rangle = \langle F(x) + (\tilde{D} - \beta_0)p \rangle$.

which is same as eq.(2.20b). If the equation (2.15b) itself were taken in the Ito sense, the term $\tilde{D}p$ in (2.20b) will be absent and this is equivalent to suppressing the correlation between p and $\lambda(t)$.

Considering the equilibrium state, we have from eqns. (2.20) $\langle p\rangle = 0$ and $\langle F(x)\rangle = 0$. Eq.(2.21a) gives $\langle xp\rangle = 0$ ie. x and p are uncorrelated. Using the eq.(2.21b) gives

$$\frac{1}{2m}\langle p^2\rangle = -\frac{1}{2}\langle xF(x)\rangle \qquad (2.27)$$

This can be identified as the virial theorem for the system (Goldstein 1980) $\langle xF(x)\rangle$ determines the average kinetic energy.

Next we derive the important extension on first fluctuation dissipation theorem (Kubo, Chandrasekhar, 1966, 1945).

Since x and p are uncorrelated, F(x) and p are also uncorrelated and hence $\langle p\,F(x)\rangle = 0$.

Hence eq.(2.21c) gives

$$D = (\beta_0 - 2\tilde{D})\langle p^2\rangle \qquad (2.28)$$

This is the important result of this section, throwing light on the implication of the inherent correlation between the fluctuating force and the system variable. We find this is the modified form of the famous Einstein's relation. When the friction coefficient is assumed to be random. The effect of the above correlation is to decrease, the average friction constant by an amount $2\tilde{D}$. When this correlation is ignored as in Ito's sense, we get back the original Einstein's relation. Using (2.28), (2.27) becomes

$$\frac{D}{2m(\beta_0 - 2\tilde{D})} = -\frac{1}{2}\langle x F(x)\rangle \tag{2.29}$$

which means the average kinetic energy is increased. This is the modification in the virial theorem.

Coming back to the time-dependent case, we have, from eq.(2.21b)

$$\frac{1}{2m}\langle p^2\rangle = -\frac{1}{2}\langle x F(x)\rangle + \frac{1}{2}\frac{d}{dt}\langle xp\rangle + \frac{1}{2}(\beta_0 - \tilde{D})\langle xp\rangle \tag{2.30}$$

This is the generalised form of the virial theorem in the time dependent case. In the equilibrium state x and p were uncorrelated. But now we find the difference in the virial theorem due to the correlation between x and p. Further, there is the modification in the coefficient of $\langle xp\rangle$ ie. $\frac{1}{2}(\beta_0 - \tilde{D})$. Here also we note the effect of Stratonovich approach.

Next, we have the fluctuation dissipation theorem to the time-dependent case. Eq.(2.21c) gives

$$D = \frac{1}{2}\frac{d}{dt}\langle p^2 \rangle - \langle pF(x) \rangle - (2\tilde{D} - \beta_0)\langle p^2 \rangle \qquad (2.31)$$

We find the result in a modified form due to the following reasons:

(i) the correlation between p and F(x)

(ii) the time dependence of $\langle p^2 \rangle$

(iii) the friction coefficient is random and multiplicative and the correlation between λ (t) and the phase space variable p is taken into consideration.

We conclude this section with the derivation of the modified form of the first fluctuation dissipation theorem by considering the evolution of the average energy of the classical system. Neglecting the friction and stochastic forces, the average energy is given by the Hamiltonian $H_c = \frac{p^2}{2m} + V(x)$ where $V(x)$ is the potential related to the external force F (x) through

$$F(x) = -\frac{dV}{dx} \qquad (2.32)$$

using eq.(2.19) with f (x,p) = H_c , we get

$$\frac{d}{dt}\langle H_c\rangle = \frac{1}{m}\langle p V'(x)\rangle + \langle F(x)\frac{p}{m}\rangle + (\tilde{D}-\beta_0)\langle \frac{p^2}{m}\rangle + \frac{D}{m} + \tilde{D}\langle \frac{p^2}{m}\rangle$$

$$= (2\tilde{D}-\beta_0)\frac{\langle p^2\rangle}{m} + \frac{D}{m} \qquad \text{using (2.31)}$$

At equilibrium, $\frac{d}{dt}\langle H_c\rangle = 0$ which gives

$$D = (\beta_0 - 2\tilde{D})\langle p^2\rangle \tag{2.33}$$

in agreement with (2.28).

3. Second Fluctuation-Dissipation Theorem

In the last section, we considered one aspect of generalising the Langevin equation by considering the friction coefficient as random. Kubo (1968) and Mori (1965) considered various generalisations of the Brownian motion of a particle which is not necessarily heavier than the particles interacting with it. In order to meet such requirements they introduced a frequency dependent friction instead of a constant friction. This is equivalent to the assumption that the frictional force depends on the history of the motion through an integral of the form $-\int_{t_0}^{t}\gamma(t-t')\,p(t')\,dt'$. They chose the following generalised Langevin equation

$$\dot{p}(t) = -\int_{t_0}^{t} \gamma(t-t')p(t')dt' + \frac{1}{m}R(t) + \frac{1}{m}K(t), \quad t > t_0 \qquad (3.1)$$

where $R(t)$ is the random force and $K(t)$ is the external force.

The random force satisfies the conditions

$$\langle R(t) \rangle = 0 \qquad (3.2)$$

$$\langle p(t_0) R(t) \rangle = 0 \qquad (3.3)$$

ie. $R(t)$ is not correlated withe velocity $p(t_0), t > t_0$

If there is no external force we put $K(t) = 0$. With the above assumptions we get the velocity correlation function from the equation

$$\frac{d}{dt}\langle p(t_0)p(t_0+t)\rangle = -\int_0^t \gamma(t-t')\langle p(t_0)p(t_0+t')\rangle dt'. \qquad (3.4)$$

Its Fourier-Laplace transform is found to be

$$\int_0^\infty \langle p(t_0)p(t_0+t')\rangle e^{-i\omega t} dt = \frac{1}{i\omega + \gamma(\omega)} \langle p^2(t_0)\rangle \qquad (3.5)$$

where $\gamma(\omega)$ is given by

$$\gamma(\omega) = \int_0^\infty e^{-i\omega t} \gamma(t)\, dt. \quad (3.6)$$

The admittance $\mu(\omega)$ is given by

$$\mu(\omega) = \frac{1}{m} \frac{1}{i\omega + \gamma(\omega)} \quad (3.7)$$

and hence we have

$$\mu(\omega) = \frac{1}{m\langle p^2(t_0)\rangle} \int_0^\infty \langle p(t_0)\, p(t_0+t)\rangle e^{-i\omega t}\, dt. \quad (3.8)$$

If the equipartition law

$$m\langle p^2(t_0)\rangle = kT \quad (3.9)$$

is assumed we can write (3.8) as

$$\mu(\omega) = \frac{1}{kT} \int_0^\infty \langle p(t_0)\, p(t_0+t)\rangle e^{-i\omega t}\, dt. \quad (3.10)$$

We note that (3.8) is the generalisation of (2.7).

Kubo (1966) has also derived the relation

$$m\gamma(\omega) = \frac{1}{m\langle p^2(t_0)\rangle} \int_0^\infty R(t_0)\, R(t_0+t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{kT} \int_0^\infty \langle R(t_0)\, R(t_0+t)\, e^{-i\omega t}\, dt \quad (3.11)$$

This is the relation between the frequency dependent friction and the correlation of the random force expressed in the Fourier-Laplace transformed versions. This result is often called the second fluctuation dissipation theorem.

The generalised Langevin equation (3.1) has been considered by many physicists. Like Kubo, Mori (1965) also gives the versions of the fluctuation-dissipation theorems, expressing the kernel as a continued fraction and using projection operator techniques. Kannan and Bharucha-Reid (1972) have obtained the fluctuation-dissipation relations directly using the mean square solution of the generalised Langevin equation.

We modify the generalised Langevin equation by introducing the fluctuating force $g(t)\, p(t)$. We analyse the effects of the multiplicative fluctuations in the context of Stratonovich theory. We find that the second fluctuation-dissipation relation is significantly different from that Kubo's result.

The equation under consideration is

$$\dot{p}(t) = -\int_{t_0}^{t} \gamma(t-t')\, p(t')\, dt' + R(t) + g(t)\, p(t), \quad t > t_0 \qquad (3.12)$$

For convenience we take the mass of the particle as unity. $R(t)$ is the random force with prescribed statistics. The fluctuating force $g(t)$ has zero mean and correlation given by

$$\langle g(t)\, g(t')\rangle = 2\tilde{D}\,\delta(t-t') \tag{3.13}$$

The R(t) and g(t) are assumed to be independent. Eq.(3.12) is equivalent to the equation

$$p(t) = p(t_0) - \int_{t_0}^{t}\left(\int_{t_0}^{t'} \gamma(t'-t'')\, p(t'')\, dt''\right) dt' + \int_{t_0}^{t} R(t')\, dt' + \int_{t_0}^{t} g(t')\, p(t')\, dt'. \tag{3.14}$$

Considering the last integral in eq.(3.14) in Stratonovich sense, we write the equivalent equation in the sense of Ito. Using eq.(3.35) of of Chapter II, we have

$$S\int_{t_0}^{t} g(t')\, p(t')\, dt' = I\int_{t_0}^{t} g(t')\, p(t')\, dt' + \tilde{D}\int_{t_0}^{t} p(t')\, dt'. \tag{3.15}$$

Hence eq.(3.14) transforms to

$$\dot{p}(t) = -\int_{t_0}^{t}\left[\gamma(t-t') - \tilde{D}\,\delta(t-t')\right] p(t')\, dt' + R(t) + g(t)\, p(t) \tag{3.16}$$

Now g(t) and p(t) are independent.

We make the transformation

$$p = \bar{p}\, e^{\int_{t_0}^{t} g(t')\,dt'} \tag{3.17}$$

and get

$$\dot{p} = \left(\dot{\bar{p}} + \bar{p}\, g(t)\right) e^{\int_{t_0}^{t} g(t')\,dt'}, \quad p(t_0) = \bar{p}(t_0) \tag{3.18}$$

Hence eq.(3.16) reduces to

$$R(t) + g(t)\,\bar{p}\, e^{\int_{t_0}^{t} g(t')\,dt'} = \left(\dot{\bar{p}}(t) + \bar{p}(t) g(t)\right) e^{\int_{t_0}^{t} g(t')\,dt'}$$
$$+ \int_{t_0}^{t} \left[\gamma(t-t') - \hat{D}\,\delta(t-t')\right] dt'\, \bar{p}(t')\, e^{\int_{t_0}^{t'} g(t'')\,dt''} \tag{3.19}$$

Eq.(3.19) again simplifies to

$$R(t)\, e^{-\int_{t_0}^{t} g(t')\,dt'} = \dot{\bar{p}}(t) + \int_{t_0}^{t} \left\{ \left[\gamma(t-t') - \hat{D}\,\delta(t-t')\right] \bar{p}(t') \times e^{-\int_{t'}^{t} g(t'')\,dt''} \right\} dt' \tag{3.20}$$

From eq.(3.20), we get,

$$R(t_0) = \dot{\bar{p}}(t_0) \tag{3.21}$$

$$R(t_0+t)\, e^{-\int_{t_0}^{t_0+t} g(t')dt'} = \dot{\bar{p}}(t+t_0)$$

$$+\int_{t_0}^{t_0+t} \left[\gamma(t+t_0-t') - \tilde{D}\,\delta(t+t_0-t')\right] e^{-\int_{t'}^{t+t_0} g(t''')dt'''}\, \bar{p}(t')\, dt'. \tag{3.22}$$

In the last integral, we effect a transformation $t' = t_0 + t''$ and simplify it as

$$\int_{t''=0}^{t} \left[\gamma(t-t'') - \tilde{D}\,\delta(t-t'')\right] e^{-\int_{t_0+t''}^{t_0+t} g(t''')dt'''}\, \bar{p}(t_0+t'')\,dt''. \tag{3.23}$$

Hence eq.(3.22) becomes

$$R(t_0+t)\, e^{-\int_{t_0}^{t_0+t} g(t')dt'} = \dot{\bar{p}}(t_0+t)$$

$$+\int_{t'=0}^{t} \left[\gamma(t-t') - \tilde{D}\,\delta(t-t')\right] \bar{p}(t_0+t')\, e^{-\int_{t_0+t'}^{t_0+t} g(\tau)d\tau}\, dt'. \tag{3.24}$$

From (3.21) and (3.24), we get

$$R(t_0)\,R(t_0+t)\,e^{-\int_{t_0}^{t_0+t} g(t')dt'} = \dot{\bar{p}}(t_0)\,\dot{\bar{p}}(t_0+t)$$

$$+\int_{t'=0}^{t} dt'\left[\gamma(t-t') - \tilde{D}\,\delta(t-t')\right]\dot{\bar{p}}(t_0)\,\bar{p}(t_0+t')\,e^{-\int_{t_0+t'}^{t_0+t} g(\tau)d\tau} \quad (3.25)$$

The fluctuating functions R(t) and g(t) are on equal footing, which means

$$\langle R(t)\,p(t')\rangle = 0 \qquad t' \le t$$

$$\langle g(t)\,p(t')\rangle = 0 \qquad t' \le t \qquad (3.26)$$

We use eqns. (3.26) and the fact that R(t) and g(t) and hence R(t) and a function of g(t) are independent.

Averaging on both sides of eq.(3.25), we have

$$\langle R(t_0)R(t_0+t)\rangle\left\langle e^{-\int_{t_0}^{t_0+t} g(t')dt'}\right\rangle = \langle \dot{\bar{p}}(t_0)\,\dot{\bar{p}}(t_0+t)\rangle$$

$$+\int_{t'=0}^{t}\left[\gamma(t-t') - \tilde{D}\,\delta(t-t')\right]\langle \dot{\bar{p}}(t_0)\,\bar{p}(t_0+t')\rangle\left\langle e^{-\int_{t_0+t'}^{t_0+t} g(\tau)d\tau}\right\rangle dt'. \quad (3.27)$$

From Kubo (1962) we have

$$\left\langle \exp\left\{-it\int_0^t \xi(t')\,dt'\right\}\right\rangle$$

$$= \exp\left\{\sum_{m=1}^{\infty}\frac{(-it)^m}{\lfloor m}\int_0^t\cdots\int_0^t \langle\langle \xi(t_1)\cdots\xi(t_m)\rangle\rangle\, dt_1\cdots dt_m\right. \qquad (3.28)$$

where $\langle\langle\cdots\rangle\rangle$ denotes the cumulants of $\xi(t)$.

Hence using $\langle\langle g(t_1)g(t_2)\rangle\rangle = \langle g(t_1)g(t_2)\rangle$

$= 2\tilde{D}\delta(t_1-t_2)$, the only surviving terms ~~in~~ is

$$\left\langle e^{-\int_{t_0}^{t_0+t} g(t')\,dt'}\right\rangle$$

$$= \exp\frac{1}{2}\int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t}\langle\langle g(t_1)g(t_2)\rangle\rangle\, dt_1\,dt_2$$

$$= \exp\frac{1}{2}\int_0^t\int_0^t\langle\langle g(t_0+t_1')\,g(t_0+t_2')\rangle\rangle\, dt_1'\,dt_2'$$

where $t_1 = t_0 + t_1'$, $t_2 = t_0 + t_2'$

$$= \exp\frac{1}{2}\int_0^t\int_0^t 2\tilde{D}\delta(t_2'-t_1')\, dt_1'\,dt_2'$$

$$= \exp \tilde{D}t. \qquad (3.29)$$

Similarly

$$\left\langle e^{-\int_{t_0+t'}^{t_0+t} g(\tau)\,d\tau} \right\rangle = e^{\widetilde{D}(t-t')} \tag{3.30}$$

Hence eq.(3.27) simplifies to

$$\langle R(t_0)R(t_0+t)\rangle e^{\widetilde{D}t} = \langle \dot{\bar{p}}(t_0)\dot{\bar{p}}(t_0+t)\rangle$$

$$+\int_{t'=0}^{t}\left[\gamma(t-t')-\widetilde{D}\,\delta(t-t')\right]\langle \dot{\bar{p}}(t_0)\bar{p}(t_0+t')\rangle e^{\widetilde{D}(t-t')}\,dt' \tag{3.31}$$

Putting

$$\left[\gamma(t)-\widetilde{D}\,\delta(t)\right]e^{\widetilde{D}t} = \alpha(t) \tag{3.32}$$

and taking Laplace Fourier transforms,

$$L(\gamma(t)) = \gamma(\omega) = \int_0^\infty e^{-i\omega t}\gamma(t)\,dt, \tag{3.33}$$

we got

$$L\left[\langle R(t_0)R(t_0+t)\rangle e^{\widetilde{D}t}\right] = L\left[\langle \dot{\bar{p}}(t_0)\dot{\bar{p}}(t_0+t)\rangle\right]$$

$$+L\left[\int_{t'=0}^{t}\alpha(t-t')\langle \dot{\bar{p}}(t_0)\bar{p}(t_0+t')\rangle\,dt'\right] \tag{3.34}$$

To simplify the R.H.S. of (3.34), we proceed as follows:

Let

$$L < \bar{p}(t_0)\, \bar{p}(t_0+t) > = f(\omega) \qquad (3.35)$$

Multiply both sides of eq.(3.24) by $\bar{p}(t_0)$ and taking expectations, we get

$$< \bar{p}(t_0)\dot{\bar{p}}(t_0+t) > = < R(t_0+t)\bar{p}(t_0) > < e^{-\int_{t_0}^{t_0+t} g(\tau)d\tau} >$$

$$- \int_0^t [\gamma(t-t') - \tilde{D}\,\delta(t-t')] < \bar{p}(t_0)\bar{p}(t_0+t') > < e^{-\int_{t_0+t'}^{t_0+t} g(\tau)d\tau} > dt'.$$

(i.e) $< \bar{p}(t_0)\dot{\bar{p}}(t_0+t) >$:

$$= - \int_0^t [\gamma(t-t') - \tilde{D}\,\delta(t-t')]\, e^{\tilde{D}(t-t')} < \bar{p}(t_0)\bar{p}(t_0+t') > dt'$$

$$= - \int_0^t \alpha(t-t') < \bar{p}(t_0)\bar{p}(t_0+t') > dt'.$$

Since $< R(t+t_0)\bar{p}(t_0) > = 0$ by eq.(3.26)

(i.e) $$\frac{d}{dt} < \bar{p}(t_0)\bar{p}(t_0+t) > = - \int_0^t \alpha(t-t') < \bar{p}(t_0)\bar{p}(t_0+t') > dt'.$$

Taking transforms we get

$$i\omega L\langle \bar{p}(t_0)\,\bar{p}(t_0+t)\rangle - \langle \bar{p}^2(t_0)\rangle$$

$$= -\bar{\alpha}(\omega)\, L\langle \bar{p}(t_0)\,\bar{p}(t_0+t)\rangle$$

Hence, $f(\omega) = L\langle \bar{p}(t_0)\,\bar{p}(t_0+t)\rangle$

$$= \frac{\langle p^2(t_0)\rangle}{i\omega + \bar{\alpha}(\omega)} \qquad (3.36)$$

$$(\text{Since } \bar{p}(t_0) = p(t_0))$$

Using the stationarity condition

$$\langle \dot{\bar{p}}(t_0)\,p(t_0+t)\rangle = -\langle \bar{p}(t_0)\,\dot{\bar{p}}(t_0+t)\rangle$$

we have again,

$$L\langle \dot{\bar{p}}(t_0)\,\dot{\bar{p}}(t_0+t)\rangle$$

$$= L\left\{\frac{d}{dt}\langle \dot{\bar{p}}(t_0)\,\bar{p}(t_0+t)\rangle\right\}$$

$$= -L\left\{\frac{d}{dt}\langle \bar{p}(t_0)\,\dot{\bar{p}}(t_0+t)\rangle\right\}$$

$$= -L\left\{\frac{d^2}{dt^2}\langle \bar{p}(t_0)\,\bar{p}(t_0+t)\rangle\right\}$$

$$= -\left[(i\omega)^2 f(\omega) - i\omega\langle \bar{p}^2(t_0)\rangle - \left\{\frac{d}{dt}\langle \bar{p}(t_0)\,\dot{\bar{p}}(t_0+t)\rangle\right\}_{t=0}\right.$$

$$= \omega^2 f(\omega) + i\omega \langle p^2(t_0) \rangle, \tag{3.37}$$

Since $\langle \bar{p}(t_0) \dot{\bar{p}}(t_0+t) \rangle$

$$= \langle \bar{p}(t_0) R(t_0) \rangle$$

$$= 0$$

from (3.3) and (3.21)

Further,

$$L \left[\int_{t_0}^{t} \alpha(t-t') \langle \bar{p}(t_0) \dot{\bar{p}}(t_0+t') \rangle dt' \right.$$

$$= -L \left[\int_{t_0}^{t} \alpha(t-t') \langle \dot{\bar{p}}(t_0) \bar{p}(t_0+t') \rangle dt' \right.$$

$$= -L \left[\int_{t_0}^{t} \alpha(t-t') \frac{d}{dt'} \langle \bar{p}(t_0) \bar{p}(t_0+t') \rangle dt' \right.$$

$$= -\bar{\alpha}(\omega) \left[i\omega f(\omega) - \langle p^2(t_0) \rangle \right], \tag{3.38}$$

$\bar{\alpha}(\omega)$, being the transform of $\alpha(t)$

Using (3.36), (3.37), (3.38) in (3.34) we get

$$L\left\{\langle R(t_0)R(t_0+t)\rangle e^{\tilde{D}t}\right\rangle = \langle p^2(t_0)\rangle \bar{\alpha}(\omega) \quad (3.39)$$

If $M(\omega)$ is taken to be $L\{\langle R(t_0)R(t_0+t)\rangle\}$,

$$M(\omega-\tilde{D}) = \langle p^2(t_0)\rangle L\left[\{\gamma(t)-\tilde{D}\delta(t)\}e^{\tilde{D}t}\right] \quad (3.40)$$

where $\langle p^2(t_0)\rangle = \frac{kT}{m} = kT$ here.

This is the new form of second fluctuation dissipation theorem. This modified form is attained by considering the multiplicative noise term in the Stratonovich sense. If $\tilde{D} = 0$ we get back eq.(3.11).

4. Results on Angular Momentum

In section 2, starting from the general law (eq.2.19) giving the time evolution of the average value of the dynamical variables of a stochastic system, we arrived at interesting results on Virial theorem and fluctuation-dissipation theorems. These results were based on the time-evolution of the second order moments. In this section, we consider results derived from the three-dimensional generalisations of eq.(2.19) we study the behaviour of $\langle \bar{L} \rangle$ and $\langle L^2 \rangle$ where $\langle L \rangle$ is the angular momentum vector.

The Langevin equation in the cartesian component forms are easily given by

$$\dot{x}_1 = \frac{p_1}{m}, \quad \dot{x}_2 = \frac{p_2}{m}, \quad \dot{x}_3 = \frac{p_3}{m} \tag{4.1}$$

$$\dot{p}_1 = -(\beta_0 + \lambda_1(t))p_1 + F_1(x_1, x_2, x_3) + R_1(t)$$

$$\dot{p}_2 = -(\beta_0 + \lambda_2(t))p_2 + F_2(x_1, x_2, x_3) + R_2(t)$$

$$\dot{p}_3 = -(\beta_0 + \lambda_3(t))p_3 + F_3(x_1, x_2, x_3) + R_3(t) \tag{4.2}$$

Here x_i, p_i, F_i $(i=1,2,3)$ are the components of the displacement $\bar{r}$, momentum $\bar{p}$ and external force $\bar{F}(\bar{r})$ respectively. $\lambda_i(t), R_i(t), (i=1,2,3)$ are the components of the independent fluctuating functions, which have vanishing means and correlations given by

$$\langle \lambda_i(t)\lambda_j(t') \rangle = 2\tilde{D}_{ij}\,\delta(t-t') \tag{4.3}$$

$$\langle R_i(t) R_j(t') \rangle = 2D_{ij}\,\delta(t-t') \tag{4.4}$$

The equations (3.2) can be written in the equivalent Ito sense as follows:

$$\dot{p}_1 = F_1(x_1, x_2, x_3) + (\tilde{D}_{11} - \beta_0) p_1 + R_1(t) - \lambda_1(t) p_1 \qquad (4.5)$$

and two similar equations,

If $P = P(x_i, p_i, t / x_{i0}, p_{i0}, t_0)$ (i = 1,2,3),

the Fokker-Planck equation is given by (Soong 1973, Lax 1966)

$$\frac{\partial P}{\partial t} = -\frac{1}{m} \sum_{i=1}^{3} p_i \frac{\partial P}{\partial x_i} - \sum_{i=1}^{3} \frac{\partial}{\partial p_i} \left[\left\{ F_i + p_i (\tilde{D}_{ii} - \beta_0) \right\} P \right] + \sum_{i,j=1}^{3} \frac{\partial^2}{\partial p_i \partial p_j} (D_{ij} P) + \sum_{i,j=1}^{3} \frac{\partial^2}{\partial p_i \partial p_j} \left\{ (G \tilde{D} G^{-1})_{ij} P \right\} \qquad (4.6)$$

where

$$[D_{ij}] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \qquad (4.7)$$

and

$$G \tilde{D} G^T = \begin{bmatrix} \tilde{D}_{11} p_1^2 & \tilde{D}_{12} p_1 p_2 & \tilde{D}_{13} p_1 p_3 \\ \tilde{D}_{21} p_2 p_1 & \tilde{D}_{22} p_2^2 & \tilde{D}_{23} p_2 p_3 \\ \tilde{D}_{31} p_3 p_1 & \tilde{D}_{32} p_3 p_2 & \tilde{D}_{33} p_3^2 \end{bmatrix} = [\tilde{D}_{ij}] \qquad (4.8)$$

If $f(x_i, p_i)$ is any dynamical variable, the evolution of its average value is given by

$$\frac{d\langle f\rangle}{dt} = \frac{1}{m}\sum_{i=1}^{3}\left\langle p_i \frac{\partial f}{\partial x_i}\right\rangle + \sum_{i=1}^{3}\left\langle F_i \frac{\partial f}{\partial p_i}\right\rangle + \sum_{i=1}^{3}\left(\tilde{D}_{ii} - \beta_0\right)\left\langle p_i \frac{\partial f}{\partial p_i}\right\rangle + \sum_{i,j=1}^{3} D_{ij}\left\langle \frac{\partial^2 f}{\partial p_i \partial p_j}\right\rangle + \sum_{i,j=1}^{3}\left\langle \bar{D}_{ij}\frac{\partial^2 f}{\partial p_i \partial p_j}\right\rangle \quad (4.9)$$

where D_{ij}, $\bar{D}_{ij}'$ are the elements of the matrices in (4.7) and (4.8) respectively.

Taking $f = x_1 p_2$ and $x_2 p_1$ successively we get

$$\frac{d}{dt}\langle x_1 p_2\rangle = \frac{1}{m}\langle p_1 p_2\rangle + \langle x_1 F_2\rangle + (\tilde{D}_{22} - \beta_0)\langle p_2 x_1\rangle \quad (4.10)$$

$$\frac{d}{dt}\langle x_2 p_1\rangle = \frac{1}{m}\langle p_1 p_2\rangle + \langle F_1 x_2\rangle + (\tilde{D}_{11} - \beta_0)\langle p_1 x_2\rangle \quad (4.11)$$

Taking $\tilde{D}_{kl} = \tilde{D}\delta_{kl}$ and subtracting (4.11) from (4.10) we get

$$\frac{d}{dt}\langle x_1 p_2 - x_2 p_1\rangle = \langle x_1 F_2 - x_2 F_1\rangle + (\tilde{D} - \beta_0)\langle x_1 p_2 - x_2 p_1\rangle \quad (4.13)$$

Similarly writing the other two component equations, we get the equation for the evolution of $\langle\vec{L}\rangle$. Denoting $\vec{M}=\vec{r}\times\vec{F}$ the torque of the external force, we get

$$\frac{d}{dt}\langle\vec{L}\rangle=\langle\bar{M}\rangle+(\tilde{D}-\beta_0)\langle\vec{L}\rangle \tag{4.14}$$

Hence the equilibrium average angular momentum is

$$\langle\vec{L}\rangle_{eq}=\frac{\langle\bar{M}\rangle}{\beta_0-\tilde{D}} \tag{4.15}$$

whereas, for any central force, we have the general formula

$$\langle\vec{L}\rangle=\langle\vec{L}_0\rangle e^{-(\beta_0-D)t} \tag{4.16}$$

$\langle\vec{L}\rangle$ decays following the same law for all central force problems when $\beta_0>\tilde{D}$.

Next, we consider the evolution of $\langle\vec{L}^2\rangle$, for which we take

$$\begin{aligned}\vec{L}^2&=\vec{r}^2\vec{p}^2-(\vec{r}\cdot\vec{p})^2\\&=(x_1^2+x_2^2+x_3^2)(p_1^2+p_2^2+p_3^2)-(x_1p_1+x_2p_2+x_3p_3)^2\\&=(x_1p_2-x_2p_1)^2+(x_2p_3-x_3p_2)^2+(x_3p_1-x_1p_3)^2\end{aligned} \tag{4.17}$$

using Lagrange's identity.

Using eq.(4.9) and (4.12), with some calculations, we get

$$\frac{d}{dt}\langle \vec{L}^2 \rangle = 2\langle \vec{M}\cdot\vec{L} \rangle + (\tilde{D}-\beta_0)\langle \vec{L}^2 \rangle + 4D\langle \bar{r}^2 \rangle + 2\tilde{D}\langle \vec{L}^2 \rangle$$

$$= 2\langle \vec{M}\cdot\vec{L} \rangle + 4D\langle \bar{r}^2 \rangle + (3\tilde{D}-\beta_0)\langle \vec{L}^2 \rangle \qquad (4.18)$$

From this result we see that at equilibrium and for central forces,

$$\langle \vec{L}^2 \rangle = \frac{4D\langle \bar{r}^2 \rangle}{\beta_0 - 3\tilde{D}} \qquad (4.19)$$

Integrating (4.17) for all central force problems, we have

$$\langle \vec{L}^2(t) \rangle = \langle \vec{L}^2(0) \rangle e^{-(\beta_0 - 3\tilde{D})t} + 4D\int_0^t e^{-(\beta_0-3\tilde{D})(t-t')} \langle \bar{r}^2(t') \rangle dt' \qquad (4.20)$$

Thus we have obtained modified equations for $\langle \vec{L} \rangle$ and $\langle \vec{L}^2 \rangle$ valid for all times and at equilibrium. In all the areas discussed so far, we have dramatic behaviour in the results due to multiplicative noise introduced, with Stratonovich interpretation

CHAPTER IV

STRATONOVICH THEORY IN STABILITY PROBLEMS*

1. Introduction

An accurate mathematical model of a dynamical system, often requires the consideration of stochastic elements. The additive stochastic elements may represent, for instance, measurement errors and external perturbing inputs; The multiplicative stochastic elements may account e.g. for parameter uncertainities and for variations of the amplifier gains. The analysis of systems with additive noise is well developed and is not essentially different from the analysis of corresponding deterministic systems. But, the presence of multiplicative stochastic elements has an essential effect on the system properties. This is in particular true for the stability behaviour. For a beautiful account of the stability concepts we refer to Arnold (1974), Kozin (1969) and Williams (1976).

The organisation of the present chapter is as follows: In section 2, introducing the basic concepts of stability in stochastic systems, we describe the stability aspects of the Brownian motion of a simple harmonic oscillator with fluctuating frequencies. Section 3, deals with the energy envelope method of Stratonovich, which is a powerful tool for reducing the Fokker-

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Planck equation to a space with relevant constants of motion as coordinates. In section 4, we consider the invariants for non-linear equations of motion introduced by Lewis (1968). Giving a short account of Routhian procedure, of investigating a dynamical problem, we derive the anxiliary equation through the Routhian approach. Section 5, deals with our new results an stability theory in the context of Stratonovich interpretation.

2. Stability of Stochastic Oscillators

Stability of a dynamical system means insensitivity of the state of the system to small changes in the initial state or the parameters of the system. The trajectories of a stable system that are close to each other at a particular time should remain close to each other at all subsequent times.

The stability study of the moments of the solutions of random differential equations was initiated by Rosenbloom (1954). The concept of mean square stability was introduced by Samuels (1959).

We give the following definitions, useful for further study.

Let $\overline{X(t)}$ be the solution process of a system of random differential equations

Definition The system is said to be stable in the mean if

$$\lim_{t \to \infty} E|\overline{X(t)}| < \overline{C} \tag{2.1}$$

where $\overline{C}$ is a finite constant vector. This implies that every input with bounded mean gives rise to an output with bounded mean.

Definition The system is asymptotically stable in the mean if

$$\lim_{t \to \infty} E|\overline{X}(t)| \to 0. \tag{2.2}$$

(We shall always take the equilibrium solution $\overline{X}(t) = 0$ as the solution whose stability is being studied, unless otherwise stated).

Definition The system is said be mean square stable if

$$\lim_{t \to \infty} E|\overline{X}(t)\,\overline{X}(t)^T| < \overline{C} \tag{2.3}$$

Where $\overline{C}$ is a constant square matrix whose elements are finite.

Definition The system is asymptotically mean square stable if

$$\lim_{t \to \infty} E|\overline{X}(t)\,\overline{X}(t)^T| \to 0. \tag{2.4}$$

where 0 is the null matrix.

The mean square stability of a system with white noise coefficients can be studied with the differential equations satisfied by the moments of the solution processes. With this short introduction, we study the stability properties of stochastic oscillators.

The various aspects of the Brownian motion of a simple harmonic oscillator have been noted by a number of investigators. (Ramakrishnan 1959, Wang and Uhlenbeck 1945, Van Kampen 1976, Bourret et al 1973). Oscillators with fluctuating frequencies can be used as a model in a variety of dynamical systems (Frisch 1968, Tatarski 1961).

In recent years, the instability encountered in such systems has been the subject of study of West et al (1980a,1980b)

The equation under consideration is

$$\ddot{x} + 2\lambda\dot{x} + \omega^2(t)\,x + F(x) + \beta(t)\,G(x,\dot{x}) = f(t) \qquad (2.5)$$

Here $x(t)$ is the oscillator displacement, λ is the dissipative parameter. $F(x)$, $G(x,\dot{x})$ are non-linear potential functions. $f(t)$, $\omega^2(t)$, $\beta(t)$ are fluctuating functions with the following statistics

$$\overline{f(t)} = 0 \qquad (2.6)$$

$$\overline{f(t)f(t')} = 2\tilde{D}\,\delta(t-t') \qquad (2.7)$$

where the bar denotes the average over an ensemble of realisation of $f(t)$.

Taking

$$\omega^2(t) = \Omega_0^2 + \gamma(t) \qquad (2.8)$$

where $\Omega_0^2 = \langle \omega^2(t) \rangle$, we have $\langle \gamma(t) \rangle = 0$ with $\langle\ \rangle$ denoting the average over an ensemble of realisations of $w(t)$. The cumulants of $\gamma(t)$ are given by

$$\langle\langle \gamma(t_1)\gamma(t_2)\cdots\gamma(t_n)\rangle\rangle = 2^{n-1} D_n \delta(t_1-t_2)\cdots\delta(t_{n-1}-t_n) \tag{2.9}$$

Similar statistics hold for $\beta(t)$ also. The fluctuating forces are taken to be independent.

Concentrating on the linear stochastic oscillators, (setting the non linear functions $F(x)$, $G(x,\dot{x})$ in (2.5) equal to zero). We write eq.(2.5) as the set of coupled first order differential equations

$$\dot{x} = p$$
$$\dot{p} = -2\lambda p - (\Omega_0^2 + \gamma(t))x + f(t) \tag{2.10}$$

The equation of evolution for the distribution $P(x,p,t)$ is given by

$$\frac{\partial P(x,p,t)}{\partial t} = \left\{\left[-p\frac{\partial}{\partial x} + \frac{\partial}{\partial p}(2\lambda p + \Omega_0^2 x) + \tilde{D}\frac{\partial^2}{\partial p^2} + \sum_{n=2}^{\infty} D_n x^n \frac{\partial^n}{\partial p^n}\right] P\right\} \tag{2.11}$$

The moment stability properties are studied by the solution of $P(x,p,t)$ as $t \to \infty$. The average oscillator displacement and momentum are found to be identical with those of an oscillator with constant frequency Ω_0. The mean displacement and momentum are stable irrespective of the magnitude of the frequency fluctuations $\gamma(t)$.

The second moments are affected by the frequency fluctuations. When the frowuency fluctuations are too large, the second moments $\langle\overline{x^2(t)}\rangle$, $\langle\overline{p^2(t)}\rangle$ and $\langle\overline{x(t)\,p(t)}\rangle$ are unstable. The critical value $D_2 = 2\lambda\Omega_0^2$ separates stable from unstable behaviour. If $D_2 < 2\lambda\Omega_0^2$ then as $t\to\infty$ one obtains steady state values

$$\langle\overline{p^2(t)}\rangle_{SS} = \Omega_0^2\langle\overline{x^2(t)}\rangle_{SS} = \frac{\tilde{D}\,\Omega_0^2}{2\lambda\Omega_0^2 - D_2}\,; \quad \langle\overline{x(t)\,p(t)}\rangle = 0 \tag{2.12}$$

The steady state second moments are non zero in the presence of f(t) in (2.5). If $D_2 > 2\lambda\Omega_0^2$, the second moments grow exponentially in time. It may be noted that the higher order cumulants D_n, $n\geq 3$ do not appear in the results.

The moment stability results on linear oscillator were discussed based on the closed sets of transport equations for the oscillator integer moments. It is not possible, to obtain such closed set of equations for the non-linear oscillator. In such cases, we get an infinite hierarchy of moment equations. The truncating procedure is not feasible and it becomes necessary to introduce other approximation schemes.

Usually we construct an approximate reduced evolution equation for a single variable of the system explicitly and obtain the moments of the chosen variable from this exact solution. Generally we introduce as new variables a set of 'constants of motion' corresponding to the unperturbed problem. The well-known constants of motion of the systems are energy angular momentum, action variable and eccentricity. Even the 'initial conditions' provide a general and theoretically useful kind of constants (Lax 1966). Based on these lines, we have the energy envelope method described in the next section.

3. Energy Envelope Method of Stratonovich

Stratonovich (1963) constructed a single variable equation for an oscillator with constant frequency driven by a random force. The procedure is based upon describing the oscillator in terms of displacement x(t) and energy E(t), on the assumption that for small damping λ ; the average energy envelope $\overline{E(t)}$ varies much more slowly than $\overline{x(t)}$.
We consider the equation

$$\ddot{x} + \epsilon \dot{x} - f(x) = \sqrt{\epsilon}\, \xi(t) \tag{3.1}$$

where ϵ is a small parameter, $\xi(t)$ is the fluctuating force with

$$\int_{-\infty}^{\infty} \langle \xi \xi_\tau \rangle \, d\tau = k . \tag{3.2}$$

The energy

$$E = \frac{1}{2}\dot{x}^2 + u(x) \tag{3.3}$$

where

$$u(x) = -\int_{x_1}^{x} f(z)\,dz \tag{3.4}$$

is the 'potential function'.

From (3.3),

$$\dot{E} = \left(\ddot{x} + \frac{du}{dx}\right)\dot{x} \tag{3.5}$$

Hence multiplying (3.1) by $\dot{x}$, and using eq.(3.5), we get

$$\dot{E} = -\epsilon\dot{x}^2 + \sqrt{\epsilon}\,\dot{x}\,\xi(t) \tag{3.6}$$

Thus, we can write the coupled equations

$$\dot{x} = \sqrt{2[E - u(x)]} \tag{3.7}$$

$$\dot{E} = -2\epsilon[E - u(x)] + \sqrt{2\epsilon[E - u(x)]}\,\xi(t) \tag{3.8}$$

The Fokker-Planck equation for the probability density $W(x,E,t)$ is given by

$$\frac{\partial W(x,E,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\sqrt{2(E-u(x))}\,P\right] + 2\epsilon\frac{\partial}{\partial E}\left[\left(E-u-\frac{k}{4}\right)P\right] + \epsilon k\frac{\partial^2}{\partial E^2}\left[(E-u)P\right] \tag{3.9}$$

where

$$W(x,E,t)\,dx\,dE = p(x,p,t)\,dx\,dp$$

The probability distribution $W(x,E,t)$ can be written as the product

$$W(x,E,t) = W_2(x,t/E)\,W(E,t) \tag{3.10}$$

where $W_2(x,t/E)$ is the conditional probability density of x - at time t given that its energy envelope is E, and $W(E,t)$ is the probability density that the energy envelope is E at time t. If ϵ is small, the energy E is conserved during a large number of periods and the time spent at x is inversely proportional to the velocity $\dot{x}$, (ie.)

$$W_2(x,t/E) \propto \frac{1}{\sqrt{E-u(x)}}, \quad (u(x) < E) \tag{3.11}$$

The probability density $W(x,E,t)$ can be written as

$$W(x,E,t) = \frac{W(E,t)}{2\phi'(E)\sqrt{E-u(x)}} \tag{3.12}$$

where

$$\phi'(E) = \frac{1}{2}\int_R \frac{dx}{\sqrt{E-u(x)}} \tag{3.13}$$

the region of integration R is given by $u(x) < E$.

Substituting (3.12) into (3.9) and integrating with respect to x, we get the Fokker-Planck equation for the single variable E as

$$\frac{\partial W(E,t)}{\partial t} = \epsilon \frac{\partial}{\partial E}\left[\left(\frac{\phi(E)}{\phi'(E)} - \frac{k}{2}\right) W\right] + \epsilon \frac{k}{2}\frac{\partial^2}{\partial E^2}\left[\frac{\phi(E) W}{\phi'(E)}\right] \qquad (3.14)$$

where

$$\phi(E) = \int_R \sqrt{E - u(x)}\, dx. \qquad (3.15)$$

Using this method, West et al arrived at the following Fokker-Planck equation for eq.(2.10) expressed in terms of the variables x and E .

$$\frac{\partial}{\partial t} W(E,t) = \left\{\frac{\partial}{\partial E}\left[\left(2\lambda - \frac{D_2}{\Omega_0^2}\right)E - \overline{D}\right] W(E,t)\right\} + \frac{\partial^2}{\partial E^2}\left\{\left[\overline{D} E + \frac{D_2}{2\Omega_0^2} E^2\right] W(E,t)\right\} \qquad (3.16)$$

The steady state solution of eq.(3.16) is easily found and all moments $\langle E^m \rangle_{SS}$ converge for $m \leq n$ to a value proportional to $\overline{D}^m$ and diverge to ∞ for $m > n$, whenever

$$\frac{4\lambda\Omega_0^2}{n+1} \leq D_2 < \frac{4\lambda\Omega_0^2}{n} \qquad (3.17)$$

This technique is used for the stochastic non-linear problems, for which the reader is referred to West et al (1980b).

As stated earlier, we devise suitable procedures to reduce Fokker-Planck equations to a space whose coordinates are some 'relevant' constants of motion, corresponding to the unperturbed problem. The problem of finding exact invariants for time-dependent Hamiltonian systems is receiving the attention of many investigators in recent years. This general method is useful in the problem of stochastic differential equations. Hence we give a short account of this theory in the next section.

4. Invariants for Nonlinear Equations of Motion

Exact invariants for time-dependent systems are of vital importance in the study of their physical properties. The simplest example is the time dependent harmonic oscillator, described by the equation

$$\ddot{x} + \omega^2(t)\, x = 0 \tag{4.1}$$

where $\omega(t)$ is a function of t. Lewis (1968) has shown that the quantity

$$I = \frac{1}{2}\left[\frac{\hbar^2 x^2}{\rho^2} + (\rho p - \dot{\rho} x)^2\right] \tag{4.2}$$

is an exact invariant for the time-dependent oscillator. Here $\rho(t)$ is a function introduced in the discussion which satisfies the anxiliary equation

$$\ddot{\rho} + \omega^2 \rho = \frac{k^2}{\rho^3} \qquad (4.3)$$

The invariant I was used by Lewis and Riesenfeld(1969) to construct an exact quantum theory of the time dependent oscillator. The same invariant was used by Khandekar and Lawande (1979) who derived an expression for the Feymman propagator in terms of the eigen functions of I.

Several different approaches have been made in finding the invariants. The fundamental and the most fruitful approach is Lutzky's version of Noether's theorem (1978), which we outline below:

A symmetry transformation for a system is described by the group operator

$$X = \xi(\rho, t)\frac{\partial}{\partial t} + \eta(\rho, t)\frac{\partial}{\partial \rho} \qquad (4.4)$$

If (4.5) is a symmetry transformation for a system with the Lagrangian $L(\rho, \dot{\rho}, t)$, then the combination of terms

$$\xi\frac{\partial L}{\partial t} + \eta\frac{\partial L}{\partial \rho} + (\dot{\eta} - \dot{\rho}\dot{\xi})\frac{\partial L}{\partial \dot{\rho}} + \dot{\xi}L \qquad (4.5)$$

is a total time derivative of a function $f(\rho, t)$

$$\text{(ie.)}\ \xi\frac{\partial L}{\partial t} + \eta\frac{\partial L}{\partial \rho} + (\dot{\eta} - \dot{\rho}\dot{\xi})\frac{\partial L}{\partial \dot{\rho}} + \dot{\xi}L = \frac{df}{dt} \qquad (4.6)$$

If (4.4) is satisfied, then the following is an invariant for the system

$$I = (\xi \dot{\rho} - \eta)\frac{\partial L}{\partial \dot{\rho}} - \xi L + f. \qquad (4.7)$$

For example ξ = constant, $\eta = 0$, $f = 0$ correspond to the conservation of energy.

Lutzky (1978) derived the invariant (4.2), by applying Noether's theorem to the Lagrangian

$$L = \frac{1}{2}\left[\dot{\rho}^2 - \omega^2(t)\rho^2\right] \qquad (4.8)$$

A recent generalisation of the Lewis system ((4.1), (4.2),(4.3)) is due to Ray and Reid (1979), given by

$$\ddot{x} + \omega^2(t)x = \frac{g(\rho/x)}{x^2\rho} \qquad (4.9)$$

$$\ddot{\rho} + \omega^2(t)\rho = \frac{f(x/\rho)}{\rho^2 x} \qquad (4.10)$$

and

$$I = \frac{1}{2}(\rho\dot{x} - x\dot{\rho}) + \int^{x/\rho} f(\eta)\,d\eta + \int^{\rho/x} g(\eta)\,d\eta \qquad (4.11)$$

We note that eqs.(4.9) and (4.10) are coupled.

This brief review of pairs of nonlinear equations giving rise to invariants is important for our study. In particular, a physical meaning to the origin of the invariant, due to Eliezor and Gray (1976) is of immense value.

Now eq. (4.1) describes the motion of a particle in a straight line. It is instructive to consider it as the projection of a two-dimensional motion of a particle under an attraction to the same centre of force and according to the same law of force.

The equation governing the anxiliary motion is given by

$$\ddot{\bar{\gamma}} + w^2(t)\bar{\gamma} = 0 \tag{4.12}$$

where (x,y) are the cartesian components of $\bar{\gamma}$. Using polar coordinates (ρ,θ) we have the following equations of motion, corresponding to the radial and transverse directions

$$\ddot{\rho} - \rho\dot{\theta}^2 + w^2\rho = 0 \tag{4.13}$$

$$\frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\theta}) = 0 \tag{4.14}$$

On integration, (4.14) gives

$$\rho^2\dot{\theta} = h \tag{4.15}$$

where h is the angular momentum constant. Eliminating $\dot{\theta}$ between (4.13) and (4.15) we get Lewis equation (4.3) for ρ. Hence, in this case the invariant I of (4.1) is $\frac{h^2}{2}$, for, using

$$x = \rho\cos\theta,\ \dot{x} = p,$$

$$I = \frac{1}{2}\left[\frac{h^2x^2}{\rho^2} + (p\rho - \dot{\rho}x)^2\right]$$

$$= \frac{1}{2}\left[h^2\cos^2\theta + \left\{(\dot{\rho}\cos\theta - \rho\sin\theta\,\dot{\theta})\rho - \rho\dot{\rho}\cos\theta\right\}^2\right]$$

$$= \frac{1}{2}\left(h^2\right) \qquad (4.16)$$

We study the properties of a simple harmonic motion in a straight line by considering it as the projection of the uniform motion in a circle of which the straight line is a diameter. The approach described above is just a generalisation under general laws of force.

We give an alternative method of deriving eq.(4.3) by Routh's procedure which is explained below (Goldstein 1980).

Routh's Procedure

We consider a dynamical system with N degrees of freedom. It is well known that the Lagrangian is a function of q and $\dot{q}$ and can be written as

$$L = L(q_1 \cdots q_N, \dot{q}_1 \cdots \dot{q}_N, t) \qquad (4.17)$$

$q, \dot{q}$ being the generalised coordinate and velocity.

Similarly the Hamiltonian is a function of the generalised coordinates q and momenta p where $p = \frac{\partial L}{\partial \dot{q}}$ ie

$$H = H(q_1 \cdots q_N, p_1 \cdots p_N, t) \tag{4.18}$$

The advantages of the Hamiltonian may be combined with the Lagrangian procedure by a method devised by Routh. In the Routhian procedure, we have a mathematical transformation from the q, $\dot{q}$ basis to q , p basis only with respect to some of the coordinates.

If

$$L = L(q_1 \cdots q_s, q_{s+1} \cdots q_N, \dot{q}_1 \cdots \dot{q}_s, \dot{q}_{s+1} \cdots \dot{q}_N) \tag{4.19}$$

We express the Routhian as a function of the arguments given below

$$R = R(q_1 \cdots q_s, q_{s+1} \cdots q_N, p_1 \cdots p_s, \dot{q}_{s+1} \cdots \dot{q}_N) \tag{4.20}$$

The Routhian and the Lagrangian are connected by the relation

$$R = -L + \sum_{j=1}^{s} p_j \dot{q}_j \tag{4.21}$$

The Routhian behaves like the Hamiltonian with respect to $q_1 \cdots q_s$ and like the Lagrangian with respect to $\dot{q}_{s+1}, \cdots \dot{q}_N$.

This means

$$\frac{\partial R}{\partial q_i} = -\dot{p}_i \, , \; \frac{\partial R}{\partial p_i} = \dot{q}_i \, ; \; i = 1, 2, \ldots s \qquad (4.22)$$

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) = \frac{\partial R}{\partial q_i} \, , \; i = s+1, \ldots, N \qquad (4.23)$$

In particular if $q_1 \ldots q_s$ are Cyclic, $p_1, \ldots p_s$ are constants. Hence the Routhian is a function of only the non-cyclic coordinates and their generalised velocities. The Lagrange's equations for the non cyclic coordinates can be solved without any regard for the behaviour of the cyclic coordinates.

The prime example where Routh's procedure may be usefully applied is in the examination of deviations from steady motions and stability of such deviations. Standard examples are the steady motion of a particle in a circular orbit about a centre of force, uniform precession of a heavy top. In such problems the steady motion is characterised by all of the noncyclic coordinates being constant. For a detailed study we refer to Goldstein (1980).

Considering our problem eq.(4.12) the Lagrangian is given by

$$L = \frac{1}{2}\left[\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \omega^2(t)\rho^2\right] \qquad (4.24)$$

described in (ρ,θ) system. Eq.(4.21) gives the Routhian

$$R = R(\rho,\dot{\rho},\theta,p_\theta) = -L + \dot{\theta}p_\theta \qquad (4.25)$$

where $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \rho^2\dot{\theta} = h$

Hence

$$R = -\frac{1}{2}\left[\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \omega^2\rho^2\right] + \frac{h^2}{\rho^2}$$

$$= \frac{h^2}{2\rho^2} - \frac{\omega^2\rho^2}{2} - \frac{\dot{\rho}^2}{2} \quad \text{using } \dot{\theta} = \frac{h}{\rho^2}$$

The equation of motion is given by, using eq.(4.23)

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\rho}}\right) = \frac{\partial R}{\partial \rho}$$

We got

$$\ddot{\rho} + \omega^2\rho = \frac{h^2}{\rho^3}.$$

in agreement with eq.(4.3).

We conclude this section by noting that the above discussions can be extended to systems where the equation takes the form

$$\frac{d^2x}{dt^2} + P(t)\frac{dx}{dt} + Q(t)x = 0 \qquad (4.26)$$

The invariant I_1 corresponding to this equation is

$$I_1 = \frac{h^2 x^2}{\rho^2} + (\dot{\rho} x - \rho p)^2 \exp\left(2\int_0^t p(t')dt'\right). \quad (4.27)$$

where ρ is any solution of

$$\frac{d^2\rho}{dt^2} + p(t)\frac{d\rho}{dt} + Q(t)\rho = \frac{h^2}{\rho^3}\exp\left(-2\int_0^t p(t')dt'\right). \quad (4.28)$$

We use some of these concepts in our further study in the next section.

5. Stability Analysis in Two Dimensional Motion

In the previous sections, we had occasions to study the stability problems discussed in terms of certain constants of motion.

Now the stability properties of the simple harmonic oscillator equation

$$\ddot{x} + \omega^2(t)x = 0 \quad (5.1)$$

and these of corresponding random differential equation

$$\ddot{y} + (\omega^2(t) + b(t)\eta(t))y = 0, \quad (5.2)$$

$\eta(t)$, being a fluctuating force are closely inter connected. Gikhman and Skorokod (1971) have discussed this problem when η (t) is the white noise.

As described in section 4, equations (4.1) and (4.3) are connected in the Lewis system, We modify eq.(4.3) as a random equation by various methods. One such method is to introduce the random force $f(t)$ sec θ on the right hand side of eq.(4.3) ie

$$\ddot{\rho} + \omega^2 \rho = \frac{h^2}{\rho^3} + f(t) \sec\theta \tag{5.3}$$

This is, exactly the equation of motion of a particle subjected to a central force $\omega^2 \rho$ towards the origin and the fluctuating force $f(t)$ sec θ in the radial direction. The invariant used in the derivation is the angular momentum

$$\rho^2 \dot{\theta} = h \tag{5.4}$$

Further this is the envelope of the motion of a particle whose projection on the x axis

$$\ddot{x} + \omega^2 x = f(t) \tag{5.5}$$

This is easily verified by applying the transformation $x = \rho\cos\theta$ in eq.(5.5).

Similarly we consider

$$\ddot{x} + 2\lambda(t)\dot{x} + \omega^2 x = f(t) \tag{5.6}$$

using $x = \rho\cos\theta$, the above equation becomes

$$\cos\theta(\ddot{\rho}-\rho\dot{\theta}^2+k\dot{\rho}+\omega^2\rho) - \sin\theta(2\dot{\rho}\dot{\theta}+\rho\ddot{\theta}+k\rho\dot{\theta}) = f(t) \quad (5.7)$$

(where $k = 2\lambda(t)$, with

$$\rho^2\dot{\theta} = h\,e^{-\int_0^t k(t')dt'} \quad (5.8)$$

as seen from (4.27)

this equation becomes

$$\ddot{\rho}+k(t)\dot{\rho}+\omega^2\rho = \frac{h^2}{\rho^3}e^{-2\int_0^t k(t')dt'} + f(t)\sec\theta \quad (5.9)$$

In (5.9) the noise $f(t)$ enters multiplicatively. Hence, we are interested in the stability problem of such equations and the interpretations in the context of Ito and Strat•novich theory. Further we study the stability of the first and second moments of ρ and p where $p = \dot{\rho}$

The above equation (5.9) can be made more complicated by taking the damping coefficient $k(t)$ and the fluctuating frequency ω^2 also as random.

Let $k(t) = k_0 + F(t)$ where $\langle k(t)\rangle = k_0$

and the fluctuating function is delta correlated,

$$\langle F(t)F(t')\rangle = 2D_2\,\delta(t-t') \tag{5.10}$$

Similarly $\omega^2(t) = \Omega_0^2 + \gamma(t)$ where

$$\langle \gamma(t)\rangle = 0$$
$$\langle \gamma(t)\gamma(t')\rangle = 2D_1\,\delta(t-t') \tag{5.11}$$

and f(t) has zero mean and

$$\langle f(t)f(t')\rangle = 2\tilde{D}\,\delta(t-t') \tag{5.12}$$

Now eq.(5.9) can be written as a pair of coupled equations

$$\mathring{\rho} = p$$
$$\mathring{p} = -(k_0 + F(t))\,p - (\Omega_0^2 + \gamma(t))\,\rho + \frac{h^2}{\rho^3}\, e^{-2k_0 t - 2\int_0^t F(t')dt'} + \frac{f(t)}{\cos\theta} \tag{5.13}$$

Using Stratonovich approach , we can write the Fokker Planck equation for P (ρ,p,t), and obtain transport equations for moments to discuss moment stability. But it is extremely difficult to solve such differential equations. Hence we adopt certain approximations and modifications in our discussions to follow. We take ω^2 to be deterministic and replace $\langle e^{-2\int_0^t F(t')dt'} \rangle$ by $e^{4D_2 t}$ (similar to eq.(3.29) of Chapter III). Then eq.(5.13) becomes

$$\dot{\rho} = p$$

$$\dot{p} = -k_0 p - F(t)p - \omega^2 \rho + \frac{h^2}{\rho^3} e^{-2(k_0 - 2D_2)t} + \frac{f(t)}{c_0 \theta} \tag{5.14}$$

To solve this equation elegantly, we consider the deterministic solution given by

$$\rho^4 = \frac{h^2}{\omega^2} e^{-2(k_0 - 2D_2)t} \tag{5.15}$$

$$\rho = \sqrt{\frac{h}{\omega}}\, e^{-\frac{1}{2}(k_0 - 2D_2)t} \tag{5.16}$$

Hence

$$p = \frac{1}{2}\sqrt{\frac{h}{\omega}}\,(2D_2 - k_0)\, e^{-\frac{1}{2}(k_0 - 2D_2)t} \tag{5.17}$$

We find that the average values of ρ and $p \to 0$ as $t \to \infty$ if $k_o > 2D_2$. Hence the region of stability is modified when $k(t)$ is made random.

The stability of higher order moments can be similarly discussed with eq.(5.14).

CHAPTER - V

STUDY OF STOCHASTIC EQUATIONS AND PATH INTEGRALS*

1. Introduction

In this Chapter we study some approximate methods of solutions of stochastic differential equations. The perturbation approach is a very powerful technique, whenever the random parametric variations are small, compared with the deterministic parametric values. One main advantage of this technique is that it transforms random coefficient problems into those with random inputs. Besides this method there are various techniques for finding approximate solutions of the averaged random ... equations like truncated hierarchy methods, cumulant and correlation discard methods etc. We follow the method of smoothing approximation ... (Frisch 1968) and compare with Stratonovich method. Also the path integral methods are extensively used for representing the solutions of Fokker-Planck equations (Haken 1976) we consider some of these aspects in this chapter.

We organise this chapter as follows: In section 2, we consider a particular stochastic equation often made use of in analysis of the dielectric properties of a fluctuating fluid (Mazur, 1975). In Section 3, we give the method of writing down the path integral solution of the Fokker-Planck equation. This is based upon the general method of transformation rule given by Ito (1950), Hasegawa (1980). In section 4, we define the new path integral basing the notion of Routhian employed in classical mechanics.

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* Based on a paper by R.Vasudevan and K.V.Parthasarathy, to be submitted for publication.

2. Stratonovich theory and smoothing approximation method

In this section, we consider some approximate methods of solutions of stochastic differential equations. One of the most powerful approximation techniques is the perturbation approach. Perturbation methods can be applied whenever the random parametric variations are small compared with the deterministic parametric values and these cover a class of problems of physical importance. We consider the basic vector differential equation

$$L\left[\overline{A}(t),t\right]\overline{X}(t)=\overline{Y}(t),\ t\geq 0 \qquad (2.1)$$

where $L(\overline{A(t)},t)$ is an $n\times n$ square matrix, whose elements are random differential operators, $\overline{A}(t)$ represents a vector random coefficient process, $\overline{Y}(t)$ is an n-dimensional random input process. The initial conditions associated with the vector solution process $\overline{x}(t)$ are assumed to be given.

We express the coefficient process $\overline{A}(t)$ in the form

$$\overline{A}(t)=\overline{a}_0(t)+\epsilon_1\overline{A}_1(t)+\epsilon^2\overline{A}_2(t)+\cdots \qquad (2.2)$$

where ϵ is a small parameter. The leading term $\overline{a}_0(t)$ is deterministic and it represents the unperturbed part. The coefficients $\overline{A_1}(t)$, $\overline{A_2}(t)$ are stochastic processes, giving the perturbed part.

Hence, we can also put the differential operator in the form

$$L(A(t),t)=L_0(a_0(t),t)+\epsilon_1 L_1(t)+\epsilon^2 L_2(t)+\cdots \qquad (2.3)$$

where L_0 is the corresponding deterministic operator. The differential operators $L_1(t)$, $L_2(t)$, ... are in general functions of $\overline{A_1}(t)$, $\overline{A_2}(t)$ .... and hence they are stochastic operators. Hence eq.(2.1) can be written as

$$\left[L_0(t)+\epsilon_1 L_1(t)+\epsilon^2 L_2(t)+\cdots\right]\bar{X}(t)=\bar{Y}(t) \quad (t\geq 0) \tag{2.4}$$

In the perturbation scheme, we seek a solution of the eq.(2.4) in the form

$$\bar{X}(t)=\bar{X}_0(t)+\epsilon\bar{X}_1(t)+\epsilon^2\bar{X}_2(t)+\cdots \tag{2.5}$$

Using eq.(2.5) in eq.(2.4), and equating coefficients of the same order of ϵ, we get the system of differential equtions

$$L_0(t)\bar{X}_0(t)=Y(t)$$

$$L_0(t)\bar{X}_1(t)=-L_1(t)\bar{X}_0(t)$$

$$\cdots\cdots$$

$$L_0(t)X_j(t)=-\left[L_1(t)\bar{X}_{j-1}(t)+L_2(t)X_{j-2}(t)+\cdots+L_j(t)\bar{X}_0(t)\right] \quad j=1,2,\ldots \tag{2.6}$$

Thus we get a system of random differential equations with random inputs only and these inputs, are known by solving eq. (2.6) successively. Thus the perturbation technique transforms random coefficient problems into equations with random inputs.

When the differential operator $L_0(t)$ is linear with constant coefficients, its inverse operator, $L_0^{-1}(t)$ is well defined. Then we can write the solution of eq.(2.6) in the form

$$\bar{X}_0(t) = L_0^{-1}(t)\,\bar{Y}(t)$$

$$\bar{X}_j(t) = -L_0^{-1}(t)\sum_{i=1}^{j} L_i(t)\bar{X}_{j-i}(t), \qquad j=1,2,\dots \qquad (2.7)$$

Hence the solution process eq.(2.5) can be written as

$$\bar{X}(t) = \Big[I - \epsilon L_0^{-1}(t)L_1(t) - \epsilon^2 L_0^{-1}(t)\big(-L_1(t)L_0^{-1}(t)L_1(t) + L_2(t)\big) + \cdots\Big] L_0^{-1}(t)\bar{Y}(t) \qquad (2.8)$$

We find that in eq.(2.8) the right hand side is an explicit function of the coefficient process $\bar{A}(t)$ and the input process $\bar{Y}(t)$. We can only evaluate the first few terms, since the evaluation of higher order terms is extremely difficult.

But, we can obtain the equations satisfied by the moments. These moment equations often throw some light on the solution process. For a detailed discussion, we refer the reader to the papers by Keller (1964), Adomian (1970).

The ~~moment~~ equation for the first moment of the solution process is given by, taking averages as both sides of eq.(2.8). Assuming statistical independence of $\bar{Y}(t)$ and $\bar{A}(t)$, We get

$$\langle \bar{X} \rangle = \langle \bar{X}_0 \rangle - \epsilon L_0^{-1}\langle L_1 \rangle \langle \bar{X}_0 \rangle + \epsilon^2 L_0^{-1}\left(\langle L_1 L_0^{-1} L_1 \rangle + \langle L_2 \rangle\right)\langle \bar{X}_0 \rangle + O(\epsilon^3) \qquad (2.9)$$

where $\langle\ \rangle$ stands for the mathematical expectation.

Eq.(2.9) gives

$$\langle \bar{X}_0 \rangle = \langle \bar{X} \rangle + \epsilon L_0^{-1} \langle L_1 \rangle \langle X_0 \rangle + O(\epsilon^2)$$
$$= \langle \bar{X} \rangle + \epsilon L_0^{-1} \langle L_1 \rangle \langle X \rangle + O(\epsilon^2)$$

(2.10)

Using eq.(2.10) in (2.8) we get, with some simplifications

$$\left[L_0 + \epsilon \langle L_1 \rangle + \epsilon^2 \left(\langle L_1 \rangle L_0^{-1} \langle L_1 \rangle - \langle L_1 L_0^{-1} L_1 \rangle + \langle L_2 \rangle \right) + O(\epsilon^3) \right] \langle \bar{X} \rangle = \langle \bar{Y} \rangle$$

(2.11)

eq.(2.11) gives the mean equation upto the second order terms. Similarly we can obtain the deterministic equations of higher-order moments. We can find, for instance the correlation function (Adomian 1970)

$$\langle X(t_1) X(t_2) \rangle = \Big\{ 1 + \epsilon^2 L_0^{-1}(t_1) L_0^{-1}(t_2) \big[\langle L_1(t_1) L_1(t_2) \rangle - \langle L_1(t_1) \rangle \langle L_1(t_2) \rangle \big] \Big\} \langle X(t_1) \rangle \langle X(t_2) \rangle + O(\epsilon^3)$$

(2.12)

This equation, together with the knowledge of the mean solution, gives the correlation function directly.

Recently, stochastic differential operator equations, both linear and nonlinear, are receiving the attention of many investigators (Adomian and Vasudevan 1981). They have developed convergent methods of successive approximations for the determination of the first and second order statistics $\langle x \rangle$ and $\langle x(t_1)\ x(t_2) \rangle$.

It may be mentioned that the approximate methods [illegible] used to solve stochastic differential equations, generally called [illegible] hierarchy techniques. (Soong 1973). If the equation under consideration is, for example

Continued in Page 152
equations (2.13) to 2.16 dropped.

$$[L_0(t) + a A(t)]\, x(t) = 0, \quad t \geq 0 \qquad (2.13)$$

with $x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0$

the hierarchy is constructed by successively multiplying eq.(2.13) by the coefficient process A(t) and averaging, we get a sequence of equation

$$L_0(t)\langle x(t)\rangle + a\langle A(t)x(t)\rangle = 0$$

$$L_0(t)\langle A(t_1)x(t)\rangle + a\langle A(t_1)A(t)x(t)\rangle = 0$$

$$L_0(t)\langle A(t_1)A(t_2)x(t)\rangle + a\langle A(t_1)A(t_2)A(t)x(t)\rangle = 0 \qquad (2.14)$$

Hence to solve for $\langle x \rangle$, we have to use certain truncation procedures. Two such procedures are correlation discard, cumulant discard. For instance, using the technique of correlation discard, where $\langle A(t_1)A(t_2) \ldots A(t_{n-1})A(t)x(t)\rangle$ is replaced by $\langle A(t_1)A(t_2) \ldots A(t_{n-1})A(t)\rangle\langle x(t)\rangle$

Having described the perturbation method, we consider the stochastic differential equation of a particular class given by

$$L^{*} x(t) = K(t) \tag{2.17}$$

Here $x(t)$ is a random process. $K(t)$ is some external force, L^{*} is a stochastic differential operator of the form

$$L^{*} = \frac{\partial}{\partial t} + \gamma_0 + \gamma_1 y(t) \tag{2.18}$$

where γ_0, γ_1 are constants and $y(t)$ is a stationary random process with mean zero and given stochastic properties.

Such stochastic differential equations occur in the study of the polarisation of the dielectric properties of a fluctuating fluid (Mazur 1975).

The formal solution of eq.(2.17) is

$$x(t) = L^{*-1} K(t) = G^{*} K(t) \tag{2.19}$$

where $\overset{*}{L}^{-1}$ is the inverse of $\overset{*}{L}$ and $\overset{*}{G} = \overset{*}{L}^{-1}$, the random Green function operator of the problem. Averaging (2.19) we get $\langle x(t)\rangle = \overset{*}{G}\ K(t)$, $\overset{*}{G}$ is the propagator for the averaged equation

Defining

$$L = G^{-1} \tag{2.20}$$

we get

$$L\langle x(t)\rangle = K(t) \tag{2.21}$$

with

$$L = \langle \overset{*}{L}^{-1}\rangle^{-1} \tag{2.22}$$

We define an operator γ' as follows

$$L = \frac{\partial}{\partial t} + \gamma_0 + \gamma' = L_0 + \gamma' \tag{2.23}$$

so that

$$L = L_0(1 + L_0^{-1}\gamma') = L_0(1 + G_0\gamma') \tag{2.24}$$

with

$$G_0 = L_0^{-1} \tag{2.25}$$

But, on the other hand

$$\overset{*}{L} = L_0(1 + \gamma_1 G_0 y(t)) \tag{2.26}$$

so that

$$\langle \overset{*}{L}^{-1}\rangle^{-1} = L_0 \langle (1 + \gamma_1 G_0 y)^{-1}\rangle^{-1} \tag{2.27}$$

Using (2.23), (2.27) in (2.22) we get

$$\gamma' = \gamma_1 < y(1+\gamma_1 G_0 y)^{-1} > < (1+\gamma_1 G_0 y)^{-1} >^{-1} \quad (2.28)$$

From (2.28) we find that the macroscopic operator γ' is given in terms of the correlation functions of $y(t)$. If γ_0 is called as the bare kinetic coefficient and $\gamma_0 + \gamma_1 y(t)$ the random kinetic coefficient, then $\gamma_0 + \gamma'$ gives the macroscopic 'renormalised' kinetic coefficient. For, from eq.(2.21) and (2.23) we get the average of X as given by

$$\frac{\partial \langle X(t) \rangle}{\partial t} = -(\gamma_0 + \gamma')\langle X(t)\rangle + K(t) \quad (2.29)$$

γ' is still an operator, which turns out to be a convolution operator in time, eq.(2.29) can be put in the form

$$\frac{\partial \langle X(t) \rangle}{\partial t} = -\gamma_0 \langle X(t) \rangle - \int_{-\infty}^{t} \gamma'(t-t')\langle X(t')\rangle dt' + K(t) \quad (2.30)$$

We will solve for the moment of the solution process of eq.(2.17) by Stratonovich theory and the smoothing approximation technique (Adomian 1970, Keller 1964).

We give a short note on the smoothing approximation method below. Closely following Frisch (1968)

We consider the equation

$$\Psi = G^0 f - G^0 L_1 \Psi \quad (2.31)$$

Let us consider ~~the~~ The projection operator P satisf~~ies~~ ying the conditions

$$PG^0 = G^0 P, \quad PL_1P = 0, \quad Pj = j$$
$$PL_0 = L_0 P, \quad P\psi(0) = \psi(0). \tag{2.32}$$

The mean field is given by $P\psi = \langle\psi\rangle$ (2.33)

and the fluctuating field is given by

$$\delta\psi = (1-P)\psi \tag{2.34}$$

Applying P and (1-P) to eq.(2.31) we get

$$\begin{aligned} P\psi &= PG^0 j - PG^0 L_1\psi \\ &= G^0 Pj - G^0 PL_1\psi \\ &= G^0 j - G^0 PL_1(1-P)\psi \\ &= G^0 j - G^0 PL_1\delta\psi \end{aligned}$$

Hence

$$P\psi = \langle\psi\rangle = G^0 j - G^0 PL_1\delta\psi \tag{2.35}$$

Also

$$\begin{aligned} (1-P)\psi = \delta\psi &= (1-P)G^0 j - (1-P)G^0 L_1\psi \\ &= G^0(1-P)j - G^0(1-P)L_1\psi \\ &= G^0 j - G^0 j - G^0(1-P)L_1[(1-P)\psi + P\psi] \end{aligned}$$

(i.e) $\delta\psi = -G^0(1-P)L_1[\delta\psi + \langle\psi\rangle]$.

or $[1+G^0(1-P)L_1]\delta\psi = -G^0(1-P)L_1\langle\psi\rangle$

or $$\delta\psi = [1+G^0(1-P)L_1]^{-1}(-G^0(1-P)L_1)\langle\psi\rangle \qquad (2.36)$$
$$= \sum_{n=1}^{\infty}[-G^0(1-P)L_1]^n\langle\psi\rangle$$

Eliminating the fluctuating field between (2.35) and (2.36) we get

$$\langle\psi\rangle = G^0 j - G^0 P L_1 \delta\psi$$
$$= G^0 j + G^0\left\{-PL_1\sum_{n=1}^{\infty}(-G^{(0)}(1-P)L_1)^n\right\}\langle\psi\rangle$$

$$= G^0 j + G^0 M\langle\psi\rangle \qquad (2.37)$$

where
$$M = -\sum_{n=1}^{\infty} PL_1[-G^{(0)}(1-P)L_1]^n$$

is called the mass operator. Retaining the first two terms in eq.(2.37) we get the smoothing approximation. We consider eq.(2.17) as

$$(L_0+L_1)X(t) = K(t) \quad \text{where} \quad L_0 = \frac{\partial}{\partial t} + \gamma_0, \quad L_1 = \gamma_1 y(t)$$

Proceeding on similar lines,

$$\langle X\rangle = PX = L_0^{-1}K(t) - L_0^{-1}PL_1\delta X \qquad (2.38)$$

Similarly

$$(1-P)\,X = \delta X = -L_0^{-1}(1-P)L_1\left[\langle X\rangle + \delta X\right]$$

Hence

$$\left[1 + L_0^{-1}(1-P)L_1\right]\delta X = -L_0^{-1}(1-P)L_1\langle X\rangle$$

giving,

$$\delta X = \sum_{n=1}^{\infty}\left(-L_0^{-1}(1-P)L_1\right)^n\langle X\rangle$$
$$= M\langle X\rangle \tag{2.39}$$

This gives

$$\langle X\rangle = L_0^{-1}K - L_0^{-1}PL_1M\langle X\rangle$$

or $(1 + L_0^{-1}PL_1M)\langle X\rangle = L_0^{-1}K$

i.e. $\langle X\rangle = (1 + L_0^{-1}PL_1M)L_0^{-1}K$

$$= \sum_{n=0}^{\infty}\left(-L_0^{-1}PL_1M\right)^n L_0^{-1}K \tag{2.40}$$
$$= L_0^{-1}K + L_0^{-1}\langle L_1 L_0^{-1} L_1\rangle L_0^{-1}K \quad \text{(upto } n=1 \text{ term)}$$

This is the equation for giving the mean under smoothing approximation. For our equation

$$\left(\frac{\partial}{\partial t} + \gamma_0 + \gamma_1 y\right)x(t) = K(t)$$

L_0^{-1} is given by the Green's function

$$g(t-t') = e^{-\gamma_0(t-t')} \tag{2.41}$$

Hence

$$\langle X\rangle=\int_0^t e^{-\gamma_0(t-t')}K(t')\,dt'$$

$$+\left\langle\int_0^t e^{-\gamma_0(t-t''')}dt'''\,\gamma_1 y(t''')\int_0^{t'''} e^{-\gamma_0(t'''-t'')}dt''\,\gamma_1 y(t'')\int_0^{t''} e^{-\gamma_0(t''-t')}K(t')\,dt'\right\rangle.$$

we simplify the second term on right side.

$$=\gamma_1^2\int_0^t e^{-\gamma_0(t-t''')}dt'''\int_0^{t'''} e^{-\gamma_0(t'''-t'')}\langle y(t''')\,y(t'')\rangle dt''\int_0^{t''} e^{-\gamma_0(t''-t')}K(t')\,dt'$$

$$=\gamma_1^2\int_0^t e^{-\gamma_0(t-t''')}dt'''\,\sigma^2\int_0^{t'''} e^{-\gamma_0(t'''-t'')}\delta(t''-t''')\,dt''\int_0^{t''} e^{-\gamma_0(t''-t')}K(t')\,dt'.$$

since $\langle y(t)y(t')\rangle=\sigma^2\delta(t-t')$

$$=\gamma_1^2\sigma^2\int_0^t\int_0^{t'''} e^{-\gamma_0(t-t''')}dt'''\,e^{-\gamma_0(t'''-t')}K(t')\,dt'\,dt'''!$$

$$=\gamma_1^2\sigma^2\int_0^t dt'''\int_0^{t'''} e^{-\gamma_0(t-t')}K(t')\,dt'$$

$$=\gamma_1^2\sigma^2\int_0^t (t-t')\,e^{-\gamma_0(t-t')}K(t')\,dt'$$

using $\int_a^x \int_a^{x_n} \cdots \int_a^{x_2} f(x_1)\, dx_1 \cdots dx_{n-1}\, dx_n = \frac{1}{\lfloor n-1} \int_a^x (x-\xi)^{n-1} f(\xi)\, d\xi.$

Hence

$$\langle X \rangle = \int_0^t e^{-\gamma_0(t-t')} K(t')\, dt' + \gamma_1^2 \sigma^2 \int_0^t (t-t') e^{-\gamma_0(t-t')} K(t')\, dt'. \quad (2.42)$$

We consider the equation (2.18) in the Stratonovich sense. The equivalent Ito equation,

$$\frac{dX}{dt} = -\gamma_0 X - \gamma_1 Y X + \frac{\sigma^2}{2}\gamma_1^2 X + K(t) \quad (2.43)$$

σ^2 being the variance parameter of $Y(t)$

Averaging, we get

$$\frac{d}{dt}\langle X \rangle = -\gamma_0 \langle X \rangle - \gamma_1 \langle Y \rangle \langle X \rangle + \frac{\sigma^2}{2}\gamma_1^2 \langle X \rangle + K(t)$$

$$= -\left(\gamma_0 - \frac{\sigma^2}{2}\gamma_1^2\right)\langle X \rangle + K(t)$$

Hence

$$\langle X \rangle = \int_0^t e^{-\left(\gamma_0 - \frac{\sigma^2 \gamma_1^2}{2}\right)(t-t')} K(t')\, dt'$$

$$= \int_0^t e^{-\gamma_0(t-t')} e^{\frac{\sigma^2 \gamma_1^2}{2}(t-t')} K(t')\, dt'$$

$$= \int_0^t e^{-\gamma_0(t-t')} \left(1 + \frac{\sigma^2 \gamma_1^2}{2}(t-t') + \cdots\right) K(t')\, dt'$$

$$\therefore \langle X \rangle = \int_0^t e^{-\gamma_0(t-t')} K(t')\, dt' + \frac{\sigma^2 \gamma_1^2}{2} \int_0^t e^{-\gamma_0(t-t')} (t-t') K(t')\, dt'. \quad (2.44)$$

Comparing (2.41) and (2.44) we see the difference of a factor $\frac{1}{2}$ in the second term.

From (2.28), we have

$$\gamma' = \gamma_1 \langle y - \gamma_1 y G_0 y \rangle \langle 1 - \gamma_1 G_0 y + \cdots \rangle^{-1}$$

$$= -\gamma_1^2 \langle y G_0 y \rangle, \text{ retaining } \gamma_1^2 \text{ term only}$$

$$\langle y(t) \rangle = 0$$

$$= -\gamma_1^2 \int_0^t e^{-\gamma_0 (t-t')} \langle y(t) y(t') \rangle dt'$$

$$= -\gamma_1^2 \int_0^t e^{-\gamma_0 (t-t')} \sigma^2 \delta(t-t')\, dt' = -\gamma_1^2 \sigma^2.$$

Hence from eq. (2.29), we have the equation satisfied by $\langle x \rangle$ given by

$$\frac{\partial \langle x(t) \rangle}{\partial t} = -\left[\gamma_0 - \gamma_1^2 \sigma^2\right] \langle x(t) \rangle + K(t) \qquad (2.45)$$

whose solution is same as (2.44).

Similarly we consider the smoothing approximation solution to evaluate $\langle x(t)\, x(t_1) \rangle$

We have, indicating the quantity evaluated at time t as $\overset{t}{\frown}$

$$x(t) = (1 - \overbrace{L_0^{-1} L_1}^{t} + \overbrace{L_0^{-1} L_1 L_0^{-1} L_1}^{t})(\overbrace{L_0^{-1} K}^{t})$$

$$x(t_1) = (1 - \overbrace{L_0^{-1} L_1}^{t_1} + \overbrace{L_0^{-1} L_1 L_0^{-1} L_1}^{t_1})(\overbrace{L_0^{-1} K}^{t_1}), \quad L_0^{-1} K = x_0$$

Hence

$$
\begin{aligned}
\langle x(t)\, x(t_1)\rangle
&= \langle x_0(t)\, x_0(t_1)\rangle - \langle x_0(t)\, \overline{L_0^{-1} L_1 x_0}^{\,t_1}\rangle \\
&\quad + \langle x_0(t)\, \overline{L_0^{-1} L_1 L_0^{-1} L_1 x_0}^{\,t_1}\rangle \\
&\quad - \langle \overline{L_0^{-1} L_1 x_0}^{\,t}\, x_0(t_1)\rangle + \langle \overline{L_0^{-1} L_1 x_0}^{\,t}\, \overline{L_0^{-1} L_1 x_0}^{\,t_1}\rangle \\
&\quad + \langle \overline{L_0^{-1} L_1 L_0^{-1} L_1 x_0}^{\,t}\, x_0(t_1)\rangle + \text{higher order terms in } \gamma_1^2
\end{aligned}
$$

Since $\langle L_1\rangle = 0$, we have

$$
\begin{aligned}
\langle x(t)\, x(t_1)\rangle &= \langle x_0(t)\, x_0(t_1)\rangle + \langle x_0(t)\, \overline{L_0^{-1} L_1 L_0^{-1} L_1 x_0}^{\,t_1}\rangle \\
&\quad + \langle \overline{L_0^{-1} L_1 x_0}^{\,t}\, \overline{L_0^{-1} L_1 x_0}^{\,t_1}\rangle + \langle \overline{L_0^{-1} L_1 L_0^{-1} L_1 x_0}^{\,t}\, x_0(t_1)\rangle
\end{aligned}
$$

+ higher order terms.

We can evaluate the respective averages and after simplification, we get

$$
\langle x(t)\, x(t_1)\rangle = \langle x_0(t)\, x_0(t_1)\rangle + \underline{\gamma_1^2}\, \sigma^2 L_0^{-1}(t)\, L_0^{-1}(t_1)\, (L_0^{-1} k)^2 , \tag{2.46}
$$

3. Path Integral in Langevin Equation

In this section we give a method of finding the path integral solution of the Fokker-Planck equations.

We consider the Langevin equation

$$\dot{X}(t) = f(X) + G(X)\,W(t) \tag{3.1}$$

Whenever an equation of this type is encountered we may be interested in a transformation to a new variable $\bar{X} = \phi(X)$

When we effect such a transformation by the usual procedures of calculus eq.(3.1) is transformed into another equation. Only when W(t) is smooth the solutions of this new equation are the transforms into $\bar{X}$ coordinate of the solutions of the original equation. But when W(t) is the white noise and the integral is an Ito integral, this is not necessarily the case.

Following Ito (1950) and Van Kampen (1980), we give the transformation rule in the Ito and Stratonovich prescriptions.

Under the transformation

$$\bar{X} = \phi(X) \tag{3.2}$$

Eq.(3.1) in the Stratonovich sense transforms to

$$\dot{\bar{X}}(t) = \bar{f}(\bar{X}) + \bar{G}(\bar{X})\,W(t) \tag{3.3}$$

where

$$\bar{f}(\bar{X}) = f(X)\,\phi'(X) \tag{3.4}$$

$$\bar{G}(\bar{x}) = G(x)\phi'(x) \tag{3.5}$$

This can be easily checked with the Fokker-Planck equation. The Fokker-Planck equation (with variance parameter unity) for the p.d.f. $p(x,t)$ is

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left(f(x)P\right) + \frac{1}{2}\frac{\partial}{\partial x}G(x)\frac{\partial}{\partial x}G(x)P \tag{3.6}$$

Under the transformation (3.2), becomes

$$\frac{\partial \bar{P}}{\partial t} = -\frac{\partial}{\partial \bar{x}}\left(\bar{f}(\bar{x})\bar{P}\right) + \frac{1}{2}\frac{\partial}{\partial \bar{x}}\bar{G}(\bar{x})\frac{\partial}{\partial \bar{x}}\bar{G}(\bar{x})\bar{P} \tag{3.7}$$

where $\bar{f}$ and $\bar{G}$ are connected with f and G by the relations (3.4) and (3.5). Also $\bar{P}(\bar{x},t)\,\phi'(x) = P(x,t)$

Hence, if $x(t)$ is defined by (3.1) in the Stratonovich sense, the process $\bar{X}(t) = \phi(X(t))$ is the one defined by (3.3) in the Stratonovich sense.

However, if we transform the Fokker-Planck equation corresponding to Ito's sense, under eq.(3.2) we get

$$\frac{\partial \bar{P}}{\partial t} = -\frac{\partial}{\partial \bar{x}}\left(\tilde{f}(\bar{x})\bar{P}\right) + \frac{1}{2}\frac{\partial^2}{\partial \bar{x}^2}\left\{\{\tilde{G}(\bar{x})\}^2\bar{P}\right\} \tag{3.8}$$

if

$$\bar{f}(\bar{x}) = f(x)\,\phi'(x) + \frac{1}{2}(G(x))^2\phi''(x) \qquad (3.9)$$

$$\bar{G}(\bar{x}) = G(x)\phi'(x) \qquad (3.10)$$

which are different from (3.4) and (3.5). Hence, if $x(t)$ is defined by (3.1) in the Ito's sense $\bar{x}(t)$ is defined by (3.3) in the Ito's sense with (3.9) and (3.10) instead of (3.4) and (3.5). With these new transformation formulae, we can freely transform the variables in the Ito scheme.

We use this transformation scheme to express the path integral solution for the physical processes with additive and multiplicative fluctuations. (Dashen 1979, Feynman and Hibbs 1965, Flatte et al 1979).

When eq.(3.1) is interpreted in the Stratonovich sense, we effect a transformation

$$\bar{X} = \phi(X) = \int \frac{dX}{G(X)} \qquad (3.11)$$

so that $\phi'(X) = \dfrac{1}{G(X)}$, $\phi''(X) = -\dfrac{G'(X)}{\{G(X)\}^2}$

Hence

$$\bar{f}(\bar{X}) = \frac{f(X)}{G(X)}, \quad \bar{G}(\bar{X}) = G(X) \cdot \frac{1}{G(X)} = 1 \qquad (3.12)$$

and the equation transforms to

$$\dot{\bar{x}}(t) = \bar{f}(\bar{x}) + W(t) \tag{3.13}$$

ie. an equation with additive fluctuations.

The Fokker-Planck equation corresponding to (3.13) is

$$\frac{\partial}{\partial t}\bar{P}(\bar{x},t) = -\frac{\partial}{\partial \bar{x}}\left[\bar{f}(\bar{x})\bar{P}(\bar{x},t)\right] + D\frac{\partial^2}{\partial \bar{x}^2}\bar{P}(\bar{x},t) \tag{3.14}$$

Following Haken (1976), (3.14) can be replaced by the integral equation

$$\bar{P}(\bar{x},t+\Delta t) = \frac{1}{\sqrt{4\pi D\Delta t}}\int_{-\infty}^{\infty} d\bar{x}' \exp\left\{-\frac{1}{4D\Delta t}\left[\bar{x}-\bar{x}'-\Delta t\left(\alpha \bar{f}(\bar{x})+\beta \bar{f}(\bar{x}')\right)\right]^2 - \alpha\Delta t\frac{\partial \bar{f}(\bar{x}')}{\partial \bar{x}'}\right\} P(\bar{x}',t) \tag{3.15}$$

with the conditions $\alpha+\beta=1,\ 0\le\alpha\le1,\ 0\le\beta\le1.$

This integral correctly gives the evolution of the probability density in the time interval $(t, t+\Delta t)$ to $o(\Delta t)$

Considering an arbitrary time interval $N\Delta t$, we iterate along a path N times in increments of $\bar{x}$ corresponding to time steps Δt. In the limit $N \to \infty$, $\Delta t \to 0$, $N\Delta t = t$, of the iterated form of eq.(3.15), we obtain a functional integral over the trajectories. Hence the path integral solution for the probability density $\bar{P}(\bar{x},t)$ is given by

$$\bar{P}(\bar{x},t) = \int \mathcal{D}(\bar{x}) \exp\left\{ -\int_0^t \mathcal{L}[\dot{\bar{x}}(t'), \bar{x}(t')] dt' \right\} \bar{P}(\bar{x},0) \tag{3.16}$$

where $\mathcal{D}(\bar{x})$ indicates integration over a path and the exponential is the probability density to have a continuous path $\bar{x}(t)$ in the space under consideration.

Similarly, we consider eq.(3.1) in the Ito sense and effect the transformation (eq.3.11), we get

$$\tilde{f}(\bar{x}) = \frac{f(x)}{\phi(x)} + \frac{1}{2}(G(x))^2 \left(\frac{-G'(x)}{G^2(x)} \right)$$

$$= \frac{f(x)}{\phi(x)} - \frac{1}{2} G'(x) \tag{3.17}$$

$$\tilde{G}(\bar{x}) = 1 \tag{3.18}$$

Hence the equation transforms to

$$\dot{\bar{x}}(t) = \tilde{f}(\bar{x}) + w(t) \tag{3.19}$$

an equation similar to (3.13). Hence the path integral solution for $\widetilde{P}(\widetilde{x},t)$ can be written similar to eq.(3.15) and eq.(3.16).

Hence, using the transformations for the two approaches we are able to write the path integral solution of the probability density in the transformed variable.

4. Routhian Path Integral

The concept of an integral over a function space was introduced by Norbert Wiener (1923) in his study of Brownian motion. In his new formulation of non-relativistic quantum mechanics, Feynman (1948) described the propagator of Schrodinger wave function by an integral over the space of all possible trajectories of the system. Ramakrishnan (1955,1956), used the idea of the probability of 'trajectories' in investigating the integrals associated with a class of random functions. The idea of path integrals was also used by Chandrasekhar (1943) in random walk problems. Kac (1951,1959) was the first to give a systematic proof for path integrals as solutions of partial differential equations.

Feynman's formulation of non-relativistic quantum mechanics had the following features.

(i) It concentrated on the propagator of the Schrodinger wave function rather than on the wave function itself, expressing the propagator as an integral over all possible paths from one given point to another.

(ii) It was directly connected with the Lagrangian rather than the Hamiltonian classical dynamics.

Garrod (1966) closely followed Feynman's approach retaining the first feature. Instead of the second feature, he used the description of the classical system in terms of canonical variables. An advantage of his Hamiltonian path integral formulation is that there is no need to introduce special normalisation constants to maintain the unitarity of the propagator. In the conversion from the Feynman's approach to the Hamiltonian formulation, the normalisation constants appear automatically.

In classical mechanics (Goldstein 1980), Routh's procedure of solving a problem is well known, when some of the generalised coordinates of a dynamical system are cyclic. We use the picture of the Routhian in the most general form, combining the features of both the Lagrangian and the Hamiltonian formulations. The path integral defined through the Routhian has many advantages, which are discussed below.

<u>The Feynman Method</u>

Feynman gave the following recipe for calculating the propagator $K(q'', q', t)$.

Let Λ represent the set of continuous, piecewise differentiable functions $q(t)$, satisfying the conditions

$q(t') = q', \; q(t'') = q''.$

The trajectory of a particle moving from q' to q'' in the time interval $t''-t'$ is represented by a function in Ω. The action integral is given by

$$A[q(t)] = \int_{t'}^{t''} L\,dt = \int_{t'}^{t''} \left[\frac{1}{2} m \dot{q}^2 - V(q(t))\right] dt \qquad (4.1)$$

The propagator is given by

$$K(q'', q', t''-t') = \int_{\Omega} \exp\left\{2\pi i A[q(t)]\right\} d[q(t)] \qquad (4.2)$$

where the integration is over a set of functions Ω.

According to Hamilton's principle in classical mechanics, we calculate the action for all paths which satisfy the given boundary conditions, to select that unique path for which $\delta A = 0$. That is the class of varied paths in completely discarded at the end of this procedure. But in Feynman's approach of quantum mechanics, we integrate over all conceivable motions including those which are classically discarded.

We define the function space integral as follows. Let us consider a sequence of partitions of the interval $t' \leq t \leq t''$ The N^{th} partition is given by the (N+1) points : $t_0 = t' < t_1 < t_2 < \cdots < t_{N-1} < t_N = t''$. For the N^{th} partition we consider the piecewise linear function which has the value x_j at t_j .

(ie.) $$q(t) = q_{j-1}\left(\frac{t_j - t}{t_j - t_{j-1}}\right) + q_j\left(\frac{t - t_{j-1}}{t_j - t_{j-1}}\right)$$

$$t_{j-1} \le t \le t_j \tag{4.3}$$

where $q_0 = q'$ and $q_N = q''$.

In the N^{th} approximation to the functional integral we integrate over all such functions with a normalisation constant so that K_N approaches a unitary kernal.

Thus,

$$K(q'', q', t''-t')$$

$$= \lim_{N\to\infty} \left\{\prod_{j=1}^{N} \left(\frac{-im}{t_j - t_{j-1}}\right)\right\}^{3/2} \int d^3q_1 \dots d^3q_{N-1}\, exp(2\pi i A(q)) \tag{4.4}$$

<u>Hamiltonian formulation</u>

Let $q_1 \dots q_N$ (collectively denoted by Q) be the coordinates, $p_1 \dots p_N$ (denoted by P) be the canonical momenta of a system. We give the following prescription for calculating $K(Q', Q, t''-t')$ as integral over the set of paths. The N^{th} approximation is defined as before by the

(N + 1) points, $t_0 = t' < t_1 < \ldots < t_{N-1} < t_N = t''$.

The function Q (t) is approximated by a piecewise linear function going from Q_{j-1} to Q_j in the interval t_{j-1} to t_j

The function P(t), in the same interval, approximated by a constant P_j. For any functional $F[Q(t), P(t)]$ of the phase-space trajectory the integral over paths is defined as

$$\int F[Q(t), P(t)]\, dQ(t)\, dP(t)$$

$$= \lim_{N\to\infty} \int F[Q, P]\, dQ_1 \ldots dQ_{N-1}\, dP_1 \ldots dP_N . \tag{4.5}$$

It is not necessary to introduce any normalisation factor. Following this definition, the propagator $K(Q'', Q', t''-t')$ is given by

$$K(Q'', Q', t''-t')$$

$$= \int \left\{ \exp 2\pi i \left(\int_{Q'}^{Q''} P\, dQ - \int_{t'}^{t''} H\, dt \right) \right\} d[Q(t)]\, d[P(t)] \tag{4.6}$$

where

$$\int_{Q'}^{Q''} p\, dQ = \int_{t'}^{t''} \left(\sum_{j=1}^{n} p_j \dot{q}_j \right) dt \tag{4.7}$$

As an illustration, we consider the case when Q is a rectangular and Hamiltonian

$$H = \frac{p^2}{2m} + V(q) \tag{4.8}$$

The action integral for such a path is

$$I = \int_{t'}^{t''} \left[p\dot{q} - \frac{1}{2m} p^2 - V(q) \right] dt$$

$$= \sum_{j=1}^{N} \left\{ p_j (q_j - q_{j-1}) - \frac{1}{2m} p_j^2 (t_j - t_{j-1}) - \int_{t_{j-1}}^{t_j} V\, dt \right\} \tag{4.9}$$

Putting

$$\Delta q_j = q_j - q_{j-1}, \quad \Delta t_j = t_j - t_{j-1}$$

and using

$$p \Delta q_j - \frac{p_j^2 \Delta t_j}{2m} = \frac{-\Delta t_j}{2m} \left(p_j - \frac{m \Delta q_j}{\Delta t_j} \right)^2 + \frac{m}{2} \frac{(\Delta q_j)^2}{\Delta t_j} \tag{4.10}$$

and

$$\int_{-\infty}^{\infty} d^3q \exp\left[2\pi i\left(-\frac{\Delta t_j}{2m} q^2\right)\right] = \left(\frac{-im}{\Delta t_j}\right)^{3/2} \tag{4.11}$$

eq.(4.9) becomes

$$I = \sum_{j=1}^{N}\left[\frac{-\Delta t_j}{2m}\left(p_j - \frac{m}{\Delta t_j}\Delta q_j\right)^2 + \frac{m}{2}\frac{(\Delta q_j)^2}{\Delta t_j} - \int_{t_{j-1}}^{t_j} V\,dt\right] \tag{4.12}$$

Hence $K(Q'', Q', t'' - t')$

$$= \int \exp\left[2\pi i(p\dot{q} - H)\right] d^3q_1 \ldots d^3q_{N-1}\, d^3p_1 \ldots d^3p_N$$

$$= \int e^{2\pi i\left[\sum_{j=1}^{N}\left\{-\frac{\Delta t_j}{2m}\left(p_j - \frac{m}{\Delta t_j}\Delta q_j\right)^2 + \frac{m}{2}\frac{(\Delta q_j)^2}{\Delta t_j} - \int_{t_{j-1}}^{t_j} V\,dt\right]}\, d^3q_1 \ldots d^3q_{N-1}\, d^3p_1 \ldots d^3p_N$$

Integrating with respect to p's, using (4.11), the above may be simplified as

$$\prod_{j=1}^{N}\left(\frac{-im}{t_j-t_{j-1}}\right)^{3/2}\int \exp 2\pi i\left\{\sum_{j=1}^{N}\left[\frac{m}{2}\frac{(\Delta q_j)^2}{\Delta t_j}-\int_{t_{j-1}}^{t_j} V\,dt\right]\right\}$$
$$\times d^3q_1 \ldots\ldots d^3q_{N-1}.$$

$$=\prod_{j=1}^{N}\left(\frac{-im}{t_j-t_{j-1}}\right)^{3/2}\int \exp 2\pi i\,[A(q(t))\,d^3q_1 \cdot\cdot d^3q_{N-1}. \quad (4.13)$$

Thus the N^{th} order approximations to the functional integral are the same in the two methods.

Routhian formulation

We present our new approach of path integral defined in terms of the Routhian of the system.

As defined earlier in Chapter IV, the Routhian R is related to the Lagrangian L by

$$R = -L + \sum_{r=1}^{s} p_r \dot{q}_r \quad (4.14)$$

Hence

$$L = R(q_1 q_2 \cdots q_N, p_1 \cdots p_s, \dot{q}_{s+1} \cdots \dot{q}_N) + \sum_{j=1}^{s} p_j \dot{q}_j$$

The action integral $I = \int L\,dt$

$$= \int \left\{\sum_{j=1}^{s} p_j \dot{q}_j - R(q_1 \cdots q_N, p_1 \cdots p_s, \dot{q}_{s+1} \cdots \dot{q}_N)\right\} dt \quad (4.15)$$

We consider the system with the Lagrangian

$$L = \frac{p^2}{2m} - V(q) \tag{4.16}$$

as before. The propagator in terms of the Routhian R is given by

$$\int \exp 2\pi i \left[\int_{Q'}^{Q''} P\,dQ - \int_{t'}^{t''} R\,dt \right] dQ(t)\,dP(t) \tag{4.17}$$

where

$$\int_{Q'}^{Q''} P\,dQ = \int_{t'}^{t''} \left(\sum_{j=1}^{s} p_j \dot{q}_j \right) dt \tag{4.18}$$

Since R = R ($q_1 \cdots q_s, q_{s+1} \; q_N, p_1 \cdots p_s, \dot{q}_{s+1} \; \dot{q}_N$)

$$\int_{t'}^{t''} R\,dt = \sum_{j=1}^{s} \frac{p_j^2}{2m}(t_j - t_{j-1}) - \sum_{j=s+1}^{N} \frac{m}{2} \frac{(q_j - q_{j-1})^2}{t_j - t_{j-1}} + \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} V\,dt$$

eq.(4.17) becomes

$$\int e^{2\pi i \left[\sum_{j=1}^{s} p_j (q_j - q_{j-1}) - \sum_{j=1}^{s} \frac{p_j^2}{2m}(t_j - t_{j-1}) + \sum_{j=s+1}^{n} \frac{m}{2} \frac{(q_j - q_{j-1})^2}{t_j - t_{j-1}} - \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} V\,dt \right]} d^3q_1 \cdots d^3q_{N-1}\, d^3p_1 \cdots d^3p_s \cdot \tag{4.19}$$

As before, this may be simplified as

$$\int e^{2\pi i\left\{\left[\sum_{j=1}^{s}\left(\frac{-\Delta t_j}{2m}\right)\left(p_j-\frac{m(\Delta q_j)}{\Delta t}\right)^2+\frac{m}{2}\frac{(\Delta q_j)^2}{\Delta t_j}\right]\right.} \times e^{2\pi i\left\{\sum_{j=s+1}^{N}\frac{m}{2}\frac{(\Delta q_j)^2}{\Delta t_j}-\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} V\,dt\right\}} d^3q_1\ldots d^3q_{N-1}\, d^3p_1\ldots d^3p_s.$$

Integrating w.r.t. $p_1, p_2, \ldots p_s$, using (4.11) we get this simplifying to

$$\prod_{j=1}^{s}\left(\frac{-im}{t_j-t_{j-1}}\right)^{3/2}\int e^{2\pi i\left[\sum_{j=1}^{s}\frac{m}{2}\frac{(\Delta q_j)^2}{\Delta t}+\sum_{j=s+1}^{N}\frac{m}{2}\frac{(\Delta q_j)^2}{\Delta t_j}\right]} \times e^{-2\pi i\sum\int V\,dt} \, d^3q_1\ldots d^3q_{N-1}$$

We introduce $\prod_{j=s+1}^{N}\left(\frac{-im}{t_j-t_{j-1}}\right)^{3/2}$ as the normalisation factor corresponding to $(N-s)$, $\dot{q}$'s ie $\dot{q}_{s+1}\ldots\dot{q}_N$. Hence we get eq (4.17), simplifying to

$$\prod_{j=1}^{N}\left(\frac{-im}{t_j-t_{j-1}}\right)^{3/2}\int e^{2\pi i[A\,q(t)]}\, d^3q_1\ldots d^3q_{N-1}. \qquad (4.20)$$

in agreement with eq.(4.2).

The underlying idea in the connection between the Lagrangian and the Routhian is based upon the partial Legendre transformation , (Courant and Hilbert 1953). When we express the path integral in the Hamiltonian formulation, we eliminate $\dot{q}_1 \dots \dot{q}_m$ from the Lagrangian using the equations $p_r = \frac{\partial L}{\partial \dot{q}_r} \left(r = 1, 2, \dots n \right)$. Further the integration is to be carried out with respect to $q_1 \dots q_{N-1}, p_1 \dots p_N$

But when we follow the Routhian procedure, only s of $\dot{q}$'s are to be eliminated. While evaluating the functional integral, we integrate with respect to all q's and $p_1, p_2, \dots p_s$ (s momentum coordinates) only. Hence the number of times of integration is less. This is due to the underlying Legendre transformation. There has been a lot of investigation on Hamiltonian path integrals in phase space (Garrod 1966), whose equivalence to the Lagrangian path integral has been shown only in Cartesian coordinates. But, the Hamiltonian path integral, on any coordinate basis would yield a desired propagator if an appropriate effective Hamiltonian is chosen. (~~Akua~~ Akira Inomata et al 1979). Following this scheme the Routhian path integral can always be used effectively.

CHAPTER VI

COMBINANTS AND BELL POLYNOMIALS*

1. Introduction

In the previous chapters, we studied some aspects of stochastic differential equations driven by continuous and point processes. To motivate the study of random equations driven by point processes, we consider the Langevin description of the motion of a particle immersed in a fluid, given by the equation

$$m\frac{dp}{dt} = -m\gamma p + R(t) \tag{1.1}$$

as described by eq.(2.1) of Chapter III. If it is supposed that the time of contact during a collision is very short, the resulting force may be assumed to be impulsive. In this case, the velocity will not be differentiable and eq.(1.1) must be replaced by

$$m\,dp = -m\gamma p\,dt + dn(t) \tag{1.2}$$

where $n(t), t \geq t_0$ may be supposed to be a compound Poisson process. Such point processes are widely used in the areas like optical communications, biomedicine and insurance. For an excellent survey of point processes the reader is referred to

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* Based on the paper by R.Vasudevan, P.R.Vittal and K.V.Parthasarathy to be submitted for publication.

the article by Ramakrishnan (1959).

A point process can be studied with the properties relating to a specified interval of the state space. Such a process is also called a 'Counting Process'. New techniques have been developed for a proper formulation of such problems in terms of certain point functions known as cumulant functions and product densities. The product densities introduced by Ramakrishnan (1950,1959) provide a complete statistical characterisation of the point processes. Further, they enjoy a unique privileged position since they have direct probabilistic interpretation and hence satisfy elegant differential or integral equations.

Recently, the new method of combinants was introduced by Kauffmann and Gyulassy (1978), in their study of theoretical models for created boson multiplicities. It is the purpose of this chapter to study the relationship between the combinants and other cluster correlation functions. Such relations are further brought to light through the role of Bell polynomials, introduced by Bell when assessing some combinatorial problems (Riordon, 1958, Comtot, 1974). Here we give a mathematical framework, not found in Kauffmann and Gyulassy (1978)

In section 2, we give a brief account of product densities. Section 3 introduces the basic developments of combinants. Starting with different types of generating functions, the relations between the combinants, and other cluster correlation functions are established in section 4.

In section 5, we obtain the representation of the distribution of doubly stochastic processes in terms of combinants. As an application of the method of combinants in branching phenomena, section 6 describes the evaluation of the mean and the variance of a population at any required generation. Section 7 gives a short account of Bell polynomials, necessary for future applications. In section 8, we give alternative proofs for some of the results of sections 3 and 4, using Bell polynomials in an elegant manner. This chapter concludes with an application of Bell polynomials to compound Poisson process in section 9.

2. Product Densities

The product densities introduced by Ramakrishnan (1950, 1959) are powerful tools to deal with stochastic point processes. This relates to the distribution of a discrete number of entities in continuous infinity of states. We follow the exposition of product densities by Ramakrishnan.

The central quantity of interest in this situation is $dN(t)$, the number of entities occurring in the continuous interval t and $t+dt$. We assume that the probability that there is one entity between t and $t+dt$ is proportional to dt, while the probability that there is more than one entity is of order smaller than dt. Hence the mean number in dt is

$$E[dN(t)] = f_1(t)\,dt, \tag{2.1}$$

while

$$E\left[\{dN(t)\}^r\right] = f_1(t)\,dt \tag{2.2}$$

where $f_1(t)$ is called the product density of the first order. We see that all the moments of the stochastic variable $dN(t)$ are equal to the probability that the stochastic variable assumes the value 1. Product densities of higher order express all the correlations of the stochastic variable $dN(t)$ existing at various times.

$$E\left[dN(t_1)\,dN(t_2)\right] = f_2(t_1, t_2)\,dt_1\,dt_2 \tag{2.3}$$

$$E\left[dN(t_1)\,dN(t_2)\ldots dN(t_r)\right] = f_r(t_1, t_2, \ldots t_r)\,dt_1 \ldots dt_r. \tag{2.4}$$

where f_r are the higher order product densities. The mean number of entities in a given range of the parameter t, is given by

$$E\left[N(b) - N(a)\right] = E\int_a^b dN(t)\,dt = \int_a^b f_1(t)\,dt. \tag{2.5}$$

Similarly, the mean square number of the entities in the range a to b of t is

$$E\left[\{N(b)-N(a)\}^2\right]=\iint E\left[dN(t_1)\,dN(t_2)\right]dt_1\,dt_2$$

$$=\int_a^b f_1(t)\,dt+\int_a^b\int_a^b f_2(t_1,t_2)\,dt_1\,dt_2 \tag{2.6}$$

Equation (2.6) brings out the singular behaviour of the random variables $dN(t_1)\,dN(t_2)$ when t_1 and t_2 coalesce. Ramakrishnan (1950) has proved a very useful result for the calculation of the r^{th} moment of the number of entities in the desired range. It runs as

$$E\left[\{N(b)-N(a)\}^r\right]$$

$$=\sum_{s=1}^{r} {}^{r}C_s \int_a^b dt_1\int_a^b dt_2 \cdots \int_a^b dt_s\, f_s(t_1,t_2,\ldots t_s) \tag{2.7}$$

where ${}^{r}C_s$ denotes the number of various confluences of (r-s) infinitesimal intervals, the maximum order of any confluence being (r-s). The ${}^{r}C_s$ coefficients being independent of the f functions can be derived from the following formula (Ramakrishnan, 1950), when the total number of entities is N.

$$N^r = \sum_{s=1}^{r} C_s^r \, N(N-1)\cdots(N-s+1) \tag{2.8}$$

a set of relations valid for $N = 1,2, \ldots$

An alternative expression for these coefficients obtained by Kuznetsov, Stratonovich and Tikhonov (1965) is just the same equation (2.8) in another closed form

$$C_s^r = \frac{1}{\lfloor s}\left[\frac{d^r}{dw^r}\left(e^w - 1\right)^s\right]_{w=0} \tag{2.9}$$

It can be proved also, that (Vasudevan, 1969)

$$C_s^r = \frac{1}{\lfloor s}\sum_{k=0}^{s}\binom{s}{k} k^r (-1)^{s-k} \tag{2.10}$$

These numbers are also called Stirling numbers of second kind.

Recently, in counting statistics of photons obtained in photo electric emissions, apart from the moments and cumulants of the distribution, another quantity called the combinants were introduced by Kauffman and Gyulassy (1979).

Whenever particles are produced and if one is interested in $P(n)$, the probability that n particles are produced, it was found useful to characterise the general $P(n)$ in terms of its deviation from the Poisson. If the first combinant alone exists, which is also the mean, the process is Poisson. In the case of Brown and Twiss experiment, (1954,1957) where the nature of photon bunching comes into play, we cannot expect the electron emission to be a Poisson and hence we expect the other combinants to be different from zero. It is the purpose of this Chapter to relate these combinants to the known product density functions f and the cluster functions g . It is interesting to see that the combinants play a similar role as the probability distributions themselves. In calculating the moments we sum n^r over the probability distributions, while in calculating the cumulants we sum ℓ^r over the combinants. Much of this analysis is found in the book (Kuznetsov, Stratonovich and Tikhonov, 1965), though explicit attention has not been drawn to the combinants.

3. Combinants

It is useful to characterise the general $P(n)$ in terms of its deviations from the Poisson. Traditionally, this is done by considering deviations from the usual moments of possessed in the Poisson case. The probability generating function for the Poisson distribution is given by

$$F(\lambda) = \sum_{n=0}^{\infty} \lambda^n P(n) = \exp(\lambda-1)\bar{n} \qquad (3.1)$$

We see that $\log F(\lambda)$ is a first degree polynomial. If however, we employ higher degrees, we can write

$$\log F(\lambda) = \log P(0) + \sum_{k=1}^{\infty} C(k)\lambda^{k} \qquad (3.2)$$

which means,

$$F(\lambda) = \exp\left[\sum_{k=1}^{\infty} C(k)(\lambda^{k}-1)\right] \qquad (3.3)$$

The expansion coefficients $C(1), C(2), \ldots$ thus completely characterise $P(n)$. Every $C(n)$ is found in terms of $P(n)$'s upto that order and $C(k)$'s are expressible in terms of the first k probability ratios $\frac{P(1)}{P(0)}, \frac{P(2)}{P(0)}, \ldots \frac{P(k)}{P(0)}$.

This stands in stark contrast to 'ordinary' probability coefficients such as moments and cumulants, each of which involves every single one of the infinite number of $P(n)$ in its definition. It should be however noted that the condition $P(0)>0$ is necessary to the existence of the $C(k)$ defined as above.

It there are N_i independently distributed variables, each distributed according to $P_i(n)$, which has generating function $F_i(\lambda)$, we find that the coefficients $C(k)$ of $N\ (= \sum_i N_i)$ satisfy the additive property

$$C(k) = \sum_i C_i(k), \; k = 1, 2, \ldots \tag{3.4}$$

It is easy to see from (3.2) that,

$$P(0) = \exp\left(-\sum_k C(k)\right) \tag{3.5}$$

The expression for C(k)'s in terms of $\left(\frac{P(k)}{P(0)}\right)'s$ are given as follows:

$$C(1) = \frac{P(1)}{P(0)} \tag{3.6a}$$

$$C(2) = \frac{P(2)}{P(0)} - \frac{1}{2}\left(\frac{P(1)}{P(0)}\right)^2 \tag{3.6b}$$

$$C(3) = \frac{P(3)}{P(0)} - \left(\frac{P(1)}{P(0)}\right)\left(\frac{P(2)}{P(0)}\right) + \frac{1}{3}\left(\frac{P(1)}{P(0)}\right)^2 \tag{3.6c}$$

These are derived in a systematic way in section 8 by our method.

Similarly $P(n)'s$ are given in terms of C(k)'s as

$$P(n) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \left[\prod_{k=1}^{\infty} \frac{(C(k))^{n_k} e^{-C(k)}}{\lfloor n_k}\right] \delta\left(n, \sum_{r=1}^{\infty} r n_r\right) \tag{3.7}$$

which may be noted to be a convoluted form of the multiple Poisson distribution with combinants $C(k)$ as the "means". Kauffmann and Gyulassy (1979) . From the relation,

$$\exp \sum_{k=1}^{\infty} C(k)\lambda^{k} = 1 + \tilde{P}_1\lambda + \tilde{P}_2\lambda^2 + \cdots\cdots \tag{3.8}$$

where
$$\tilde{P}_n = \frac{P(n)}{P(0)} \tag{3.9}$$

we obtain
$$\frac{\partial \tilde{P}_r}{\partial C(k)} = \tilde{P}_{r-k} . \tag{3.10}$$

This can be checked with equation (3.6).

Now, it is our purpose to relate the combinants with the product densities and the cluster functions.

4. Cluster correlation functions:

Let us take the moment generating function of $P(n)$ as

$$Q_m(t) = \sum_{n=0}^{\infty} e^{nt} P(n) \tag{4.1}$$

Then it has been shown in (Kuznetsov, Stratonovich and Tikhonov, 1965) that

$$Q_m(t) = \sum_{s=0}^{\infty} \frac{(e^t-1)^s}{\underline{|s}} \int_R \cdots \int_R f_s(x_1 \ldots x_s)\, dx_1 \ldots dx_s \tag{4.2}$$

where $f_s(x_1, x_2, \ldots x_s)$ are the s^{th} order product densities defined earlier (Ramakrishnan 1950 or 1959). It can be easily shown that $P(0)$, the probability that there is no entity in the region R is given by

$$P_R(0) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\lfloor s} \int_R \cdots \int_R f_s(x_1, \ldots x_s)\, dx_1 \ldots dx_s. \tag{4.3}$$

Also, the probability $P_R(n)$ of n entities in R is given by Kuznetsov, Stratonovich and Tikhonov (1965) as

$$P_R(n) = \frac{1}{\lfloor n} \sum_{s=0}^{\infty} \frac{(-1)^s}{\lfloor s} \int_R \cdots \int_R f_{n+s}(x_1 \ldots x_{n+s}) \} dx_1 \ldots dx_{n+s}. \tag{4.4}$$

the points $(x_1, x_2, \ldots x_n)$ being the continuous set of points in the region R.

The factorial moment generating function $Q_{fm}(t')$ is given by replacing t by $\log(1+t')$ in $Q_m(t)$ i.e.

$$Q_{fm}(t') = \sum_{s=0}^{\infty} \frac{t'^s}{\lfloor s} \int_R \cdots \int_R f_s(x_1, x_2, \ldots x_s)\, dx_1 \ldots dx_s. \tag{4.5}$$

Therefore

$$\left\{\frac{\partial^r}{\partial t'^r}\left[Q_{fm}(t')\right]\right\}_{t'=0} = \left\langle \frac{\lfloor n}{\lfloor n-r} \right\rangle \tag{4.6}$$

$$= \int_R \cdots \int_R f_r(x_1, x_2, \ldots x_r)\, dx_1 \ldots dx_r .$$

The actual moments are related to the factorial moments as given by (2.7).

The cumulant generating function is given by

$$Q_{cf}(t) = \log Q_m(t)$$

$$= \sum_{s=1}^{\infty} \frac{(e^t-1)^s}{\lfloor s} \tau_s \tag{4.7}$$

where

$$\tau_s = \int_R \cdots \int_R g_s(x_1, x_2, \ldots x_s)\, dx_1 \ldots dx_s . \tag{4.8}$$

g_s are the cluster functions of order s and relating to the point process giving rise to $P(n)$ (Kuznetsov, Stratonovich and Tikhonov, 1965). The cluster functions $g_s(x_1, x_2, \ldots x_s)$ describe the irreducible clusters (which cannot be split into functions of any of its arguments) relating to the occurrence of the point process in the region R.

The cumulants K_r of the P(n) process are obtained from $Q_{cf}(t)$ by the relation,

$$K_r = \frac{\partial^r}{\partial t^r}\left(Q_{cf}(t)\right)_{\text{at } t=0.} \tag{4.9}$$

using (4.7), (4.9) can be written as

$$K_r = \frac{\partial^r}{\partial t^r}\left(\sum_{s=1}^{\infty} \frac{(e^t-1)^s}{\underline{|s}} \tau_s\right) \tag{4.10}$$

$$\text{ie} \quad K_r = \sum_{s=1}^{r} C_s^r \tau_s \tag{4.11}$$

Thus the cumulants of the process are related to the integrals of the cluster functions of order s over the region R, by the same coefficients as the moments of the distribution are related to the product densities. Compare results (4.11) and (2.7). It is well known that the ordinary moments of P(n) are the averages $\langle n^r \rangle$ summed over the probabilities P(n) . Have we an analogous situation in the case of cumulants? The answer is 'yes' and the combinants play the same role as the probabilities P(n) in calculating the cumulants. We display

$$\sum_{l=1}^{\infty} l^r c(l) = K_r \tag{4.12}$$

just as

$$\sum_{n=0}^{\infty} n^r P(n) = \mu_r \quad (r^{th} \text{ moment}) \tag{4.13}$$

To see this, let us start from the moment generating function (4.1)

$$Q_m(t) = \sum_{n=0}^{\infty} e^{nt} P(n) \tag{4.13a}$$

Replacing e^t by t' and using (3.3) we get

$$\tilde{Q}_m(t') = \sum_{n=0}^{\infty} t'^n P(n)$$
$$= \exp \sum_{k=1}^{\infty} C(k)(t'^k - 1) \tag{4.14}$$
$$= P_0 \exp \sum_{k=1}^{\infty} C(k) t'^k \quad \text{, using (3.5)} \tag{4.15}$$

In $\tilde{Q}_m(t')$, if we replace t' by e^w we get back $Q_m(w)$

Therefore we get

$$P_0 \exp \sum_{k=1}^{\infty} C(k) e^{wk} = Q_m(w)$$
$$= \sum_{s=0}^{\infty} \frac{(e^w - 1)^s}{\lfloor s} \int_R \cdots \int_R f_s(x_1 \cdots x_s) dx_1 \cdots dx_s.$$

(using (4.2)) (4.16)

Taking logarithm both sides in (4.16) and using (4.7) we get,

$$\log P(0) + \sum_{k=1}^{\infty} C(k) e^{wk} = \sum_{s=1}^{\infty} \frac{(e^{w}-1)^{s}}{\lfloor s} \tau_s \tag{4.17}$$

Differentiating both sides with respect to ω , r times and using (4.10) and (4.11) we get

$$\sum_{l=1}^{\infty} l^{r} C(l) = K_r = \sum_{s=1}^{r} C_s^{r} \tau_s \tag{4.18}$$

Thus we see that the r^{th} cumulants are obtained by taking the average $\langle l^{r} \rangle$ over all the combinants $C(l)$ and the cumulants in turn are related to the cluster integrals τ_s by the C_s^{r} coefficients in a way analogous to the moments relationship with the integrals over the product densities.

Let us go one step further in the cumulant generating function $Q_{cf}(t)$ in (4.7) by replacing t by $\log(1+t')$ We get

$$\tilde{Q}_{cf}(t') = \sum_{s=1}^{\infty} \frac{t'^{s}}{\lfloor s} \tau_s \tag{4.19}$$

Therefore

$$\frac{\partial^{r}}{\partial t'^{r}} \left(\tilde{Q}_{cf}(t') \right) = \tau_r \tag{4.20}$$

The τ_r's given in (4.20) can be called as factorial cumulants. In (4.17), if we put $\omega = \log(1+t)$ we easily see that

$$\sum_{\ell} \frac{\lfloor \ell}{\lfloor \ell - r} C(\ell) = \tau_r \tag{4.21}$$

Thus τ_r is the r^{th} factorial cumulant with respect to $C(\ell)$. This is analogous to the factorial moments

$$\int_R \cdots \int_R f_r(x_1, \ldots x_r)\, dx_1 \ldots dx_r = \sum_n \frac{\lfloor n}{\lfloor n-r} P(n) \tag{4.22}$$

It is also easily seen that

$$C(\ell) = \frac{1}{\lfloor \ell} \sum_{s=1}^{\infty} \frac{(-1)^s}{\lfloor s} \tau_{s+\ell} . \tag{4.23}$$

This is exactly the same as the relation between P(n) and the factorial moments given by (4.4). Thus we have seen that the cumulants play very much the same role as the probabilities themselves. We compute the cumulants and the factorial cumulants with respect to $C(\ell)$ in the same manner, as we compute moments and factorial moments with respect to P(n). Since it is expected that for many types of distributions the number of $C(\ell)$'s are much less than the infinite number of P(n)'s available, it is easier to deal with the combinants. This is attributed to the fact that the $C(\ell)$'s have existence essentially due to the cluster correlations in the system regarding the occurrence of the point process.

5. Doubly stochastic process

Many interesting types of doubly stochastic processes have been met with in various situations, as in the case of the counting statistics of the electrons emitted by light falling on a sensitive material. This leads to the study of the phenomenon of intensity correlations, inaugurated by the Brown and Twiss experiments (1954,1957). A simple summary is given about doubly stochastic processes in Saleh (1978). An analysis of Barkhausan Noise in magnetism also belongs to a class of doubly stochastic process (R.Vasudevan and S.K.Srinivasan 1966). This may be called either a second order process or a doubly stochastic process. This term was first introduced by Cox(1955).

A doubly stochastic Poisson point process (DSP,PP) is a Poisson Point Process (PPP) whose rate density function $\lambda(t)$ is itself a stochastic process. For each realisation of $\lambda(t)$, the resulting PP is a Poisson point process. Therefore, the statistical properties of a DSP-PP are completely specified if the statistical properties of the $\lambda(t)$ of the original process are given. The moments and the moment generating functions can be determined by averaging the corresponding moments and the moment generating functions over the different realisation of $\lambda(t)$.

The final Poisson process for the random variable n has p.d.f. given by

$$P(n) = \frac{W^{n} e^{-W}}{\lfloor n} \tag{5.1}$$

where W is the rate of events integrated over the counting interval, i.e.

$$W = \int_{t_0}^{t_0+T} \lambda(\theta)\, d\theta \tag{5.2}$$

From (4.1) and (5.1) we have the moment generating function

$$Q^{n}_{mf}(t) = \langle e^{W(e^{t}-1)} \rangle_{W} \tag{5.3}$$

for each realisation W. The moment generating function of a DSP-PP can be obtained from those of P.PP by averaging over the realisation of the λ process which means averaging over the W process. Therefore

$$Q^{n}_{mf}(t) = \langle e^{W(e^{t}-1)} \rangle_{W} = Q_W(e^{t}-1) \tag{5.4}$$

Let $f^{n}_{s}(x_1 \cdots x_s)$ represent the product densities of the final process n. Then from (4.2) we have

$$Q^{n}_{mf}(t) = \sum_{s=0}^{\infty} \frac{(e^{t}-1)^{s}}{\lfloor s} \int_R \cdots \int_R f^{n}_{s}(x_1 \cdots x_s)\, dx_1 \cdots dx_s \tag{5.5}$$

But by (5.4), we have

$$Q_W(e^{t}-1) = \sum_{s=0}^{\infty} \frac{\left[e^{e^{t}-1}-1\right]^{s}}{\lfloor s} \int_R \cdots \int_R f^{W}_{s}(x_1 \cdots x_s)\, dx_1 \cdots dx_s. \tag{5.6}$$

where f^{W}_{s} are the product densities of the W process. As described in Section 4, we obtain the factorial moments of the final process n by replacing f_r by f^{n}_{r} in (4.6) viz.

$$\left\{\frac{\partial^{r}}{\partial t'^{r}}\left(Q^{n}_{fm}(t')\right)\right\}_{t'=0} = \left\langle \frac{\lfloor n}{\lfloor n-r} \right\rangle$$

$$= \int_R \cdots \int_R f^{n}_{r}(x_1 \ldots x_r)\, dx_1 \ldots dx_r. \tag{5.7}$$

Making a similar transformation in the expression for W process, we get

$$\widetilde{Q^{n}_{fm}}(t') = \sum_{s=0}^{\infty} \frac{(e^{t'}-1)^{s}}{\lfloor s} \int_R \cdots \int_R f^{W}_{s}(x_1 \cdots x_s)\, dx_1 \cdots dx_s. \tag{5.8}$$

Hence

$$\left\{ \frac{\partial^r}{\partial t'^r} \left(\tilde{Q}^{n}_{fm}(t') \right) \right\}_{t'=0} = \sum_{s=1}^{r} C^r_s \int_R \cdots \int_R f^W_s(x_1 \cdots x_s)\, dx_1 \cdots dx_s \qquad (5.9)$$

Therefore, we find that the factorial moments of the final process n are the usual moments of the original process W. This fact can be applied to second order processes such as photoelectron emission etc.

If the original W process is a simple Poisson with cluster correlation

$$g_s = 0 \qquad \text{for all } s > 1 \qquad (5.10a)$$

$$g_1 = \overline{W} \qquad \text{(the mean of W)} \qquad (5.10b)$$

then we find that the moment generating function of the resultant process n is given by, using (5.5) and (5.6)

$$Q^{n}_{mf}(t) = \sum_{s=0}^{\infty} \frac{(e^t - 1)^s}{\lfloor s} \int_R \cdots \int_R f^n_s(x_1 \cdots x_s)\, dx_1 \cdots dx_s.$$

$$= \sum_{s=0}^{\infty} \frac{(e^{e^t-1}-1)^s}{\lfloor s} \int_R^W \cdots \int_R f_s(x_1 \ldots x_s)\, dx_1 \ldots dx_s$$

$$= \exp\left(e^{e^t-1} - 1\right) \overline{W} \qquad (5.11)$$

From the moment generating function $Q_{mf}^{n}(t)$ we get the probability generating function $\sum_n \lambda^n P(n)$ by replacing e^t by λ i.e. the probability generating function is

$$Q_{mf}^{n}(\log \lambda) = \exp\left(e^{\lambda-1} - 1\right) \overline{W}$$

here (5.12)

This result can also be expressed in terms of the combinants. Therefore we get

$$\exp\left(e^{\lambda-1} - 1\right) \overline{W} = \exp \sum_{K=1}^{\infty} C(k)(\lambda^k - 1) \qquad (5.13)$$

We see that, the l.h.s.

$$\exp\left(e^{\lambda-1} - 1\right) \overline{W} = \exp\left[e^{-1} e^{\lambda} \overline{W} - \overline{W}\right]$$

$$= \exp\left\{\left(e^{-1}\overline{W} - \overline{W}\right) + \sum_{K=1}^{\infty} e^{-1} \overline{W} \frac{\lambda^k}{\lfloor k}\right\}$$

equating coefficient of λ^k both sides, of (5.13) we get

$$C(k) = \frac{\bar{W} e^{-1}}{\lfloor k} \tag{5.14}$$

and equating constant terms we get

$$\exp\left(-\sum_{k=1}^{\infty} C(k)\right) = \exp \bar{W}(e^{-1}-1)$$
$$= P^n(0) \quad \text{of the process } n \tag{5.15}$$

Since C(k) exists for all values of k, the process cannot be a Poisson. However the C(k)'s go down to zero very fast.

In conclusion, for the production multiplicity of particles produced according to some distributions, the final distribution of n particles can be expressed in terms of combinants, because the combinants are additive if each type of multiplicity is produced independently. This can be tested in many ways.

6. Combinants in Branching Phenomena

In this section, we use the method of combinants in the study of branching phenomena. We study the problem of population growth, whose evolution can be pictured as a tree branching out from the initial individual, with the assumption that the individual in any generation reproduces independently of the individuals in the present and past generations. A typical question asked of such a process is "what is the distribution of the total size of the population after a specified number of generations?" Such problems can be solved by using appropriate generating function techniques.

We give an important theorem below, which will be used in further study.

If a parameter of a probability distribution is altered to behave as a random variable, the resulting distribution is said to be compound. An important compound distribution is that of the sum of a random number of random variables.

Theorem Let $\{X_k\}$ be a sequence of independent and identically distributed random variables with common probability generating function (p.g.f)

$$g(\lambda) = E(\lambda^{X_i}), \quad i=1,2,3,\ldots \tag{6.1}$$

and let

$$Z_N = X_1 + X_2 + \ldots + X_N \tag{6.2}$$

where N is also a random variable, with p.g.f

$$h(\lambda) = E(\lambda^N) \tag{6.3}$$

Denote the p.g.f. of the compound distribution of Z_N by

$$G(\lambda) = E(\lambda^{Z_N}) \tag{6.4}$$

Then $G(\lambda) = h[g(\lambda)]$ (6.5)

Proof

Since $E(\lambda^{Z_N}) = E[E(\lambda^{Z_N}/N)]$

and $E(\lambda^{Z_N}/N) = E(\lambda^{X_1+X_2+\cdots+X_N}/N)$

$$= \{g(\lambda)\}^N$$

We have

$$G(\lambda) = E[\{g(\lambda)\}^N] = h[g(\lambda)]$$

Hence eq. (6.5)

This simple functional relation between the p.g.f. $G(\lambda)$ of the compound distribution of Z_N and the p.g.f's $h(\lambda)$ and $g(\lambda)$ is quite useful in branching phenomena.

The concept of branching processes was first introduced by Francis Galton in 1874. Their mathematical model and its generalisations have been extensively used in the study of epidemics (Neyman and Scott 1964), nuclear chain reactions and similar problems. An elegant mathematical treatment of the subject is given in Harris (1963).

The basic mechanism of a branching process is as follows: An individual (the 0^{th} generation) is capable of producing 0,1,2, ... offspring to form the first generation; each of his offspring, in turn produces offspring , which together constitute the second generation; and so on. We suppose the number of individuals in the n^{th} generation is Z_n .

We assume the simple structure of reproduction namely, (i) that the number X of offspring produced by an individual has the probability distribution

$$P_r\{X=k\}=p_k,\ k=0,1,\cdots \tag{6.6}$$

which is the same for each individual in a given generation, (ii) that this probability distribution remains fixed from generation to generation; and (iii) that individuals produce offspring independently of each other. Thus we are dealing with independent and identically distributed random variables.

Let

$$F(\lambda)=\sum_{k=0}^{\infty} p_k \lambda^k. \tag{6.7}$$

be the p.g.f of X and let $F_n(\lambda)$ be the p.g.f. of $Z_n,\ n=1,2,3,$. Since $Z_0=1$, the size of the first generation Z_1 has the same probability distribution as X,

$$Pr\{Z_1 = k\} = p_k \qquad (6.8)$$

and the same p.g.f $g(\lambda)$. The second generation consists of the direct descendants of the Z_1 members of the first generation, so that Z_2 is the sum of Z_1 independent random variables, each of which has the probability distribution $\{p_k\}$ and p.g.f $F(\lambda)$. Therefore Z_2 has a compound distribution with p.g.f. obtained from formula (6.5)

$$F_2(\lambda) = F(F(\lambda)) \qquad (6.9)$$

Similarly, the $(n+1)^{th}$ generation consists of the direct descendants of the Z_n members of the n^{th} generation, so that Z_{n+1} is the sum of the Z_n independent random variables, each with p.g.f. $F(\lambda)$. Hence by (6.5) the p.g.f. of Z_{n+1} is

$$F_{n+1}(\lambda) = F_n[F(\lambda)] \qquad (6.10)$$

But the $(n+1)^{th}$ generation consists of the n^{th} generation descendants of the Z_1 members of the first generation. Hence the p.g.f. of Z_{n+1} can also be written as

$$F_{n+1}(\lambda) = F[F_n(\lambda)] \qquad (6.11)$$

In spite of the iterative nature of the probability generating function, discussed above, it is fairly easy to find the mean and the variance of the population at any generation using cumulant generating function. (Bailey 1964). We obtain such results by expressing the generating function in terms of the combinants.

In this section, we use the following notations.

$\overset{\{N\}}{F}(\lambda)$ = the probability generating function for the N^{th} generation

$\overset{\{N\}}{C}(k)$ = the combinants corresponding to the N^{th} generation

$\overset{\{N\}}{M}_{[r]}$ = the r^{th} factorial moment of the N^{th} generation

$$= \sum_{n} n(n-1)\cdots(n-r+1)P(n)$$

$$= \left\{\frac{d^r}{d\lambda^r}\overset{\{N\}}{F}(\lambda)\right\}_{\lambda=1}$$

$\overset{\{N\}}{K}_{[r]}$ = the r^{th} factorial cumulant of the N^{th} generation with respect to $\overset{\{N\}}{C}(k) = \left[\frac{d^r}{d\lambda^r}\left\{\log \overset{\{N\}}{F}(\lambda)\right\}\right]_{\lambda=1}$

$$= \sum_{k}\overset{\{N\}}{C}(k)(k)(k-1)\cdots(k-r+1)$$

We have

$$\log\left[\overset{\{N\}}{F}(\lambda)\right] = \sum_{k=1}^{\infty}\overset{\{N\}}{C}(k)(\lambda^{k}-1) \qquad (6.12)$$

$$\left\{\frac{d}{d\lambda}\log\left[F^{\{N\}}(\lambda)\right]\right\}_{\lambda=1} = \left\{\sum_{k=1}^{\infty} C^{\{N\}}(k)\, k\, \lambda^{k-1}\right\}_{\lambda=1} = K_{[1]}^{\{N\}} \tag{6.13}$$

$$\left\{\frac{d^2}{d\lambda^2}\log\left[F^{\{N\}}(\lambda)\right]\right\}_{\lambda=1} = \left\{\sum_{k-1}^{\infty} C^{\{N\}}(k)\, k\,(k-1)\,\lambda^{k-2}\right\}_{\lambda=1} = K_{[2]}^{\{N\}} \tag{6.14}$$

etc.

Again $F^{\{N\}}(\lambda) = F\left[F^{\{N-1\}}(\lambda)\right] = \exp\sum_k C_k^{1}\left[\left\{F^{\{N-1\}}(\lambda)\right\}^{k} - 1\right]$

gives

$$\log F^{\{N\}}(\lambda) = \sum_k C_k^{1}\left[\left(F^{\{N-1\}}(\lambda)\right)^{k} - 1\right] \tag{6.15}$$

$$\left[\frac{d}{d\lambda}\log F^{\{N\}}(\lambda)\right]_{\lambda=1}$$

$$= \left[\sum_k C^{1}(k)\, k\left(F^{\{N-1\}}(\lambda)\right)^{k-1}\right]\frac{dF^{\{N-1\}}(\lambda)}{d\lambda} \text{ at } \lambda=1$$

$$= K_{[1]}^{\{1\}}\, M_{[1]}^{\{N-1\}} \tag{6.16}$$

$$\left[\frac{d^2}{d\lambda^2}\left(\log F^{\{N\}}(\lambda)\right)\right]_{\lambda=1} = \left\{\left[\sum_k c'(k)\, k\left(F^{\{N-1\}}(\lambda)\right)^{k-1}\right]\frac{d^2 F^{\{N-1\}}(\lambda)}{d\lambda^2}\right.$$

$$\left. + \left[\sum_k c'(k)\, k(k-1)\left(F^{\{N-1\}}(\lambda)\right)^{k-2}\right]\left(\frac{d F^{\{N-1\}}(\lambda)}{d\lambda}\right)^2\right\}_{\lambda=1}$$

$$= K^{\{1\}}_{[1]} M^{\{N-1\}}_{[2]} + K^{\{1\}}_{[2]}\left(M^{\{N-1\}}_{[1]}\right)^2 \qquad (6.17)$$

etc.

(It is easily seen that $F^{\{N\}}(\lambda)$, and its derivatives are equal to unity for all. values of N and n at $\lambda = 1$ under consideration). Comparing (6.13) and (6.16). we get

$$K^{\{N\}}_{[1]} = K^{\{1\}}_{[1]} M^{\{N-1\}}_{[1]} \qquad (6.18)$$

But the first factorial moment or first factorial cumulant is same as the mean. If the mean of the first generation is m, we have eq (6.18) , giving

$$m^{\{N\}} = m \cdot m^{\{N-1\}}, \; m^{\{N\}}$$ being the mean of the N^{th} generation

Proceeding recursively, we get

$$m^{\{N\}} = m^{N} \qquad (6.19)$$

Again comparing (6.14) and (6.17), we have

$$K_{[2]}^{\{N\}} = m^{\{1\}} M_{[2]}^{\{N-1\}} + K_{[2]}^{\{1\}} \left(m^{\{N-1\}}\right)^2 \qquad (6.20)$$

Using the relation, the second factorial moment

$$M_{[2]} = \langle X(X-1)\rangle = \text{Var } X + (\text{mean } X)^2 - \text{mean } X$$
$$= K_2 + (\text{mean } X)^2 - \text{mean } X$$

(K_2, being the second cumulant

$$K_{[2]}^{\{1\}} = \sigma^2 - m, \ldots$$

We get

$$K_2^{\{N\}} + (m^{\{N\}})^2 - (m^{\{N\}})$$
$$= m\left[K_2^{\{N-1\}} + (m^{\{N-1\}})^2 - m^{\{N-1\}}\right]$$
$$+ (\sigma^2 - m)(m^{\{N-1\}})^2 \qquad (6.21)$$

σ^2, being the variance of the first generation.

Hence

$$K_2^{\{N\}} + m^{2N} - m^N = m\left[K_2^{\{N-1\}} + m^{2N-2} - m^{N-1}\right] + \sigma^2 m^{2N-2} - m^{2N-1}$$

(i.e)

$$K_2^{\{N\}} = m K_2^{\{N-1\}} + \sigma^2 m^{2(N-1)} \qquad (6.22)$$

This is the recursive relation between the second cumulants (=variances) of the N^{th} and $(N-1)^{th}$ generations.

Hence, proceeding recursively, we get

$$K_2^{\{N\}} = (m^{n-1} + m^n + \dots + m^{2n-2})\sigma^2$$
$$= \frac{m^{n-1}(m^n - 1)}{m-1}\sigma^2 \quad (m \neq 1) \qquad (6.23)$$

If $m=1$, we get $K_2^{\{N\}} = N\sigma^2$, $m=1$, $\qquad$ (6.24)

in agreement with Bailey (1964)

An interesting problem in this context, is the question of the ultimate extinction of the population. The probability that a population starting with a single ancestor will become extinct at or before the n^{th} generation is

$$q_n = P_r\left[Z_n = 0\right] = F_n(0) \tag{6.25}$$

The limit of q_n as $n \rightarrow \infty$ is of importance. If $p_0 = 1$, the population will never start and if $p_0 = 0$, it will never become extinct. Hence we assume $0 < p_0 < 1$ Hence the generating function $F(\lambda)$ in (6.7) is a strictly monotonic increasing function of λ. It can be shown that the probability of ultimate extinction equals 1 if and only if the mean number m of offspring per individual is not greater than 1. We also find that $m > 1$, from (6.19). We expect a geometrical increase in population size and (6.29) gives the divergence in the variance. Further details can be had in Bailey (1964).

7. Bell Polynomials

In this section, we introduce Bell polynomials (Riordan 1958) using the Faa di Bruno formula for the higher-order derivatives of a composite function. We consider the function $F(t) = f[g(t)]$ and evaluate its n^{th} order derivative $F_n(0)$, We assume $f(0) = g(0) = 0$ and write the functions $f(u)$ and $g(u)$ by the formal series

$$f(u) = \sum_{\ell=1}^{\infty} \frac{f_\ell}{\underline{|\ell}} u^{\ell} \tag{7.1}$$

$$g(t) = \sum_{\ell=1}^{\infty} \frac{g_\ell}{\underline{|\ell}} t^{\ell}. \tag{7.2}$$

The coefficients f_ℓ and g_ℓ are the derivatives of the functions f and g at respective origins. The first few orders of F_n are given by

$$F_1 = f_1 g_1$$

$$F_2 = f_1 g_2 + f_2 g_1^2$$

$$F_3 = f_1 g_3 + f_2 (3 g_1 g_2) + f_3 g_1^3$$

$$F_4 = f_1 g_4 + f_2 (4 g_1 g_3 + 3 g_2^2) + f_3 (6 g_1^2 g_2) + f_4 g_1^4.$$

.... etc. In general, we have

$$F_n = \sum_{l=1}^{n} f_l \, B_{nl}(g_1, g_2, \ldots g_{n-l+1}) \tag{7.3}$$

The numbers B_{nl} are polynomials in the g_i's and are independent of f(t) and its derivatives. They are called Bell polynomials denoted by

$$B_{nl}[g(t)] = B_{nl}(g_1, g_2, \ldots) = B_{nl}[g_1, g_2, \ldots g_{n-l+1}]$$

The F_n are called complete Bell polynomials and are denoted by F_n (f;g). Obviously,

$$F(t) = \sum_{n=1}^{\infty} \frac{t^n}{\lfloor n} F_n(f;g) \tag{7.4}$$

Since B_{nl} are independent of $f(u)$, they can be computed by choosing any convenient function. For instance, if $f(u) = \exp au - 1$, we obtain the multinominal theorem

$$\frac{1}{\lfloor l}\left(\sum_{j=1}^{\infty} \frac{g_j}{\lfloor j} t^j\right)^l = \sum_{n=l}^{\infty} \frac{t^n}{\lfloor n} B_{nl}[g(t)] \tag{7.5}$$

This gives immediately

$$B_{n\ell}[g(t)] = \frac{1}{\lfloor \ell} \left[\frac{d^n}{dt^n} (g(t))^\ell \right]_{t=0} \qquad (7.6)$$

Their exact expression (Abramowitz and Stegun 1965) is

$$B_{n\ell}[g(t)] = \sum_{\{\nu_i\}}'' \frac{\lfloor n \; g_1^{\nu_1} g_2^{\nu_2} \cdots g_n^{\nu_n}}{\prod_{j=1}^{n} \lfloor \nu_j \; (\lfloor j)^{\nu_j}} \qquad (7.7)$$

where the double prime denotes that the summation is to be done over all sets $\{\nu_i\}$ on non-negative integers ν_i subject to the conditions

$$\sum_{j=1}^{n} \nu_j = \ell \quad \text{and} \quad \sum_{j=1}^{n} j\nu_j = n \qquad (7.8)$$

The following properties of $B_{n\ell}[g(t)]$ can be immediately obtained.

$$g_n = B_{n1}[g(t)] \qquad (7.9)$$

$$g_1^n = B_{nn}[g(t)] \qquad (7.10)$$

$$B_{n\ell}(ag_1, ag_2, ag_3, \ldots) = a^\ell B_{n\ell}(g_1, g_2, \ldots) \qquad (7.11)$$

$$B_{n\ell}(ag_1, a^2g_2, a^3g_3, \ldots) = a^n B_{n\ell}(g_1, g_2, g_3 \ldots) \quad (7.12)$$

Next, we give the results on Stirling numbers in terms of Bell polynomials. We define two functions $f(u)$ and $g(t)$ as inverse to each other if

$$f[g(t)] = t \quad (7.13)$$

Taking, $g(t) = e^t - 1$, the multinomial theorem gives

$$\frac{1}{\underline{|\ell}} (\exp t - 1)^\ell = \sum_{n=\ell}^{\infty} \frac{t^n}{\underline{|n}} B_{n\ell}(1,1,1,\ldots) \quad (7.14)$$

This is precisely one of the generating functions of the Stirling numbers $S_n^{(\ell)}$ of the second kind. (Abramowitz and Stegun, 1965), usually defined by

$$x^n = \sum_{\ell=0}^{n} S_n^{(\ell)} x(x-1) \cdots (x-\ell+1) \quad (7.15)$$

So, we get

$$B_{n\ell}(1,1,1,\ldots) = S_n^{(\ell)} \quad (7.16)$$

The function inverse to the above is $f(u) = \log(1+u)$

Now (7.5) gives,

$$\frac{1}{\lfloor \ell}\left[\log(1+u)\right]^{\ell} = \sum_{n=\ell}^{\infty} \frac{u^n}{\lfloor n} B_{n\ell}(\lfloor 0, -\lfloor 1, \lfloor 2, -\lfloor 3, \cdots)$$

where we have the generating function for the (signless) Stirling numbers $s_n^{(\ell)}$ of the first kind, introduced through

$$x(x-1)\cdots(x-\ell+1) = \sum_{j=0}^{\ell} s_\ell^{(j)} x^j \tag{7.17}$$

Consequently

$$B_{n\ell}(\lfloor 0, -\lfloor 1, \lfloor 2, \cdots) = s_n^{(\ell)} \tag{7.18}$$

From (7.15) and (7.17), we get the well known fact that the matrices formed by these numbers are inverse to each other.

Next, we concentrate on some results on Matrix calculus and the inversion problem through Bell polynomials. Since in $B_{n\ell}, n \geqslant \ell$, they can be considered as elements of a left-triangular infinite matrix $B[g]$. But only their sections n x n are relevant to any consideration upto order n. The determinant reduces to the product of the diagonal terms

$$= B_{11} B_{22} \cdots B_{nn}$$
$$= g_1^1 g_1^2 \cdots g_1^n = g_1^{\frac{n(n+1)}{2}}$$

Since $g_n = B_{n1}[g(t)]$, the first column of $B[g]$ gives precisely the coefficients g_ℓ. The other elements are fixed polynomials in these coefficients.

It is well known that non-singular left-triangular matrices form a group under matrix multiplication. The formal series $\left(f(t) = \sum_{\ell=1}^{\infty} \frac{f_\ell}{\lfloor \ell} t^\ell, f_1 \neq 0\right)$ also form a group (Henrici 1974) under the operation of composition. We show below that the latter group can be represented in the former.

Let us take $f[g(t)]$ in the left hand side of the multinomial theorem (7.5):

$$\frac{1}{\lfloor \ell}\left\{f[g(t)]\right\}^\ell = \frac{1}{\lfloor \ell}\left[\sum_{i=1}^{\infty} \frac{f_i}{\lfloor i} g^i(t)\right]^\ell$$
$$= \sum_{n=\ell}^{\infty} \frac{g^n(t)}{\lfloor n} B_{n\ell}[f(u)]$$
$$= \sum_{n=\ell}^{\infty} \sum_{j=n}^{\infty} \frac{t^j}{\lfloor j} B_{jn}[g(t)] B_{n\ell}[f(u)]$$
$$= \sum_{j=\ell}^{\infty} \frac{t^j}{\lfloor j} \sum_{n=\ell}^{j} B_{jn}[g(t)] B_{n\ell}[f(u)]$$

Hence,

$$B_{jl}\{f(g(t)\} = \sum_{n=l}^{j} B_{jn}[g(t)]B_{nl}[f(u)] \qquad (7.19)$$

or $$B[f \circ g] = B[g]\,B[f]$$

$$(7.20)$$

ie. $$B[F(t)] = B[f(g(t))] = B[g(t)]\,B[f(u)].$$

This means that the matrix corresponding to the composition $f \circ g$ is the product of the matrices corresponding to each function in the reverse order. So the composition operation between formal series corresponds to the right product of their respective matrices.

We are particularly interested in the problem of series inversion in the sense of (7.13). If f and g are inverse to each other, using (7.13) and (7.20) we have

$$g[f(t)] = t, \; B[f]\,B[g] = I \text{ and } B[g]\,B[f] = I.$$

So the matrix of the inverse is the inverse matrix. In particular, using (7.19) we have the following important results

$$g_n = B_{n1}[g(t)] = \sum_{\ell=1}^{n} B_{n\ell}(F_1, F_2, \cdots)\, B_{\ell 1}^{-1}[f(u)]. \quad (7.21)$$

and

$$B_{\ell 1}^{-1}[\exp u - 1] = B_{\ell 1}[\log(1+t)] \quad (7.22)$$
$$= S_\ell^{(1)} = (-1)^{\ell-1} \lfloor \ell - 1$$

Results (7.21) and (7.22) are widely used in the subsequent sections.

8. Results on Combinants - another approach

In sections 3 and 4 we derived the relations between combinants and other types of cluster correlation functions. These results can be derived in an elegant manner, using Bell polynomials. The results (7.21) and (7.22) play a vital role in the derivation of such results. This approach is entirely new compared to other usual methods.

We have, as before

$$F(\lambda) = \sum_{n=0}^{\infty} \lambda^n P(n) = \exp\left[\sum_{k=1}^{\infty} C(k)(\lambda^k - 1)\right]$$
$$= P(0) \exp\left[\sum_{k=1}^{\infty} C(k)\lambda^k\right] \quad (3.3)$$

where $$P(0) = \exp\left[-\sum_{k} c(k)\right] \qquad (3.5)$$

Taking

$$f(t) = e^{t} - 1, \quad g(\lambda) = \sum_{k=1}^{\infty} c(k)\lambda^{k}$$

we have

$$\frac{F(\lambda)}{P(0)} = \exp \sum_{k=1}^{\infty} c(k)\lambda^{k}.$$

$$\text{or, } \tilde{F}(\lambda) = \frac{F(\lambda)}{P(0)} - 1 = f[g(\lambda)] = e^{\sum_{k=1}^{\infty} c(k)\lambda^{k}}. \qquad (8.1)$$

we also have

$$f(0) = g(0) = 0$$

$$f_1(0) = f_2(0) = \cdots = 1$$

$$g_k = c(k)\,\lfloor k. \qquad (8.2)$$

Again

$$\tilde{F}(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^{n} P(n)}{P(0)} - 1 = \sum_{n=1}^{\infty} \lambda^{n} \frac{P(n)}{P(0)}$$

$$= \sum_{n=1}^{\infty} \frac{\lambda^{n}}{\lfloor n} \sum_{l=1}^{n} B_{nl}(g_1, g_2, \cdots g_{n-l+1}) \quad \text{using (7.4)}, \qquad (8.3)$$

Hence

$$\frac{P(n)}{P(0)} = \frac{1}{\lfloor n} \sum_{l=1}^{n} B_{nl}(g_1, g_2, \cdots g_{n-l+1}) \qquad (8.4)$$

Thus, as a simple illustration, we exhibit $P(n)$ in terms of Bell polynomials, as

$$P(n) = \frac{P(0)}{\lfloor n} \sum_{l=1}^{n} B_{nl}\left(C(1), \lfloor 2\, C(2), \lfloor 3\, C(3), \ldots\right) \qquad (8.5)$$

We now express $C(k)$'s in terms of $\frac{P(k)}{P(0)}$ using Bell polynomials (using eq.7.2). We enlist below the values of the required quantities. From eq. (7.6) and (8.3), we have

$$B_{nl}[F(\lambda)] = \frac{1}{\lfloor l} \left[\frac{d^n}{d\lambda^n} \left(\sum_{r=1}^{\infty} \lambda^r \frac{P(r)}{P(0)} \right)^l \right]_{\lambda=0} \qquad (8.6)$$

$$B_{11}[F(\lambda)] = \left\{ \frac{d}{d\lambda} \left[\sum_{r=1}^{\infty} \lambda^r \frac{P(r)}{P(0)} \right] \right\}_{\lambda=0} = \left[\sum_{r=1}^{\infty} r \lambda^{r-1} \frac{P(r)}{P(0)} \right]_{\lambda=0} = P(1)/P(0) \qquad (8.7)$$

$$B_{21}[F(\lambda)] = \left[\frac{d^2}{d\lambda^2} \left(\sum_{r=1}^{\infty} \lambda^r \frac{P(r)}{P(0)} \right) \right]_{\lambda=0}$$

$$= \left[\sum_{r=1}^{\infty} r(r-1) \lambda^{r-2} \frac{P(r)}{P(0)} \right]_{\lambda=0} = 2\,\frac{P(2)}{P(0)} \qquad (8.8)$$

$$B_{22}[\tilde{F}(\lambda)] = \frac{1}{\lfloor 2}\left\{\frac{d^2}{d\lambda^2}\left(\sum_{r=1}^{\infty}\lambda^r\frac{P(r)}{P(0)}\right)^2\right\}_{\lambda=0}$$

$$= \left[\frac{d}{d\lambda}\left\{\left(\sum_{r=1}^{\infty}\lambda^r\frac{P(r)}{P(0)}\right)\left(\sum_{r=1}^{\infty}r\lambda^{r-1}\frac{P(r)}{P(0)}\right)\right\}\right]_{\lambda=0}$$

$$= \left[\left(\sum_{r=1}^{\infty}r\lambda^{r-1}\frac{P(r)}{P(0)}\right)^2 + \left(\sum_{r=1}^{\infty}\lambda^r\frac{P(r)}{P(0)}\right)\left(\sum_{r=1}^{\infty}r(r-1)\lambda^{r-2}\frac{P(r)}{P(0)}\right)\right]_{\lambda=0}$$

$$= \left(\frac{P(1)}{P(0)}\right)^2 \qquad (8.9)$$

Similarly we have

$$B_{31}[\tilde{F}(\lambda)] = \left[\frac{d^3}{d\lambda^3}\left\{\sum_{r=1}^{\infty}\lambda^r\frac{P(r)}{P(0)}\right\}\right]_{\lambda=0} = \lfloor 3\,\frac{P(3)}{P(0)} \qquad (8.10)$$

$$B_{32}[\tilde{F}(\lambda)] = \frac{1}{\lfloor 2}\left[\frac{d^3}{d\lambda^3}\left(\sum_{r=1}^{\infty}\lambda^r\frac{P(r)}{P(0)}\right)^2\right]_{\lambda=0} \qquad (8.11)$$

$$= 6\left(\frac{P(1)}{P(0)}\right)\left(\frac{P(2)}{P(0)}\right).$$

$$B_{33}\left[\tilde{F}(\lambda)\right] = \frac{1}{\lfloor 3}\left[\frac{d^3}{d\lambda^3}\left(\sum_{r=1}^{\infty}\lambda^r\frac{P(r)}{P(0)}\right)^3\right]_{\lambda=0}$$

$$= \left(\frac{P(1)}{P(0)}\right)^3 \qquad (8.12)$$

..... etc.

Further,

$$B_{\ell 1}^{-1}\left[f(u)\right] = B_{\ell 1}^{-1}\left(e^u - 1\right) = B_{\ell 1}\left[\log(1+u)\right]$$

$$= \left[\frac{d^\ell}{du^\ell}\log(1+u)\right]_{u=0}$$

$$= (-1)^{\ell-1}\lfloor \ell-1 \qquad (8.13)$$

$\log(1+u)$ being the inverse of $e^u - 1$.

Now, from eq. (7.21)

$$g_n = B_{n1}\left[g(t)\right] = \sum_{\ell=1}^{n} B_{n\ell}\left[\tilde{F}(u)\right]B_{\ell 1}^{-1}\left[f(u)\right]$$,we get

Putting $n = 1, 2, \cdots$ in (7.21) we get the required relations.

$n=1$, gives, $c_1 = g_1 = B_{11}\left[\tilde{F}(u)\right] B_{11}^{-1}\left[f(u)\right] = \frac{P(1)}{P(0)}.1$

(i.e) $$c_1 = \frac{P(1)}{P(0)} \qquad (8.14)$$

$n=2$ gives,

$$\lfloor 2 \; c(2) = g_2 = \sum_{\ell=1}^{2} B_{2\ell}[F(u)] \, B_{\ell 1}^{-1}[f(u)]$$

$$= 2 \cdot \frac{P(2)}{P(0)} \cdot 1 + \left(\frac{P(1)}{P(0)}\right)^2 (-1)$$

$$\text{(i.e)} \quad c(2) = \frac{P(2)}{P(0)} - \frac{1}{2}\left(\frac{P(1)}{P(0)}\right)^2 \qquad (8.15)$$

$n=3$ gives,

$$\lfloor 3 \; c(3) = g_3 = \sum_{\ell=1}^{3} B_{3\ell}[F(u)] \, B_{\ell 1}^{-1}[f(u)]$$

$$= \lfloor 3 \; \frac{P(3)}{P(0)} + 6 \, \frac{P(2)}{P(0)} \frac{P(1)}{P(0)} (-1) + \left(\frac{P(1)}{P(0)}\right)^3 (2)$$

$$\text{(i.e)} \quad c(3) = \frac{P(3)}{P(0)} - \frac{P(1)}{P(0)} \frac{P(2)}{P(0)} + \frac{1}{3}\left(\frac{P(1)}{P(0)}\right)^3 \qquad (8.16)$$

etc. in agreement with eqns.(3.6)

In a similar fashion, we express the ratios $\frac{P(k)}{P(0)}$ in terms of $c(k)$'s using (7.19). We calculate below the various $B_{n\ell}$ required.

Now $$B_{j\ell}[f(u)] = \frac{1}{\lfloor \ell}\left[\frac{d^j}{du^j}(e^u - 1)^\ell\right]_{u=0} = \frac{1}{\lfloor \ell}\left[\frac{d^j}{du^j}\left(\sum_{r=1}^{\infty} \frac{u^r}{\lfloor r}\right)^\ell\right]_{u=0}$$

from (7.6)

$$B_{11}[f(u)] = \left[\frac{d}{du}\left(\sum_{r=1}^{\infty}\frac{u^r}{\lfloor r}\right)\right]_{u=0} = 1 \tag{8.17}$$

$$B_{21}[f(u)] = \left[\frac{d^2}{du^2}\left(\sum_{r=1}^{\infty}\frac{u^r}{\lfloor r}\right)\right]_{u=0} = 1 \tag{8.18}$$

$$B_{31}[f(u)] = \left[\frac{d^3}{du^3}\left(\sum_{r=1}^{\infty}\frac{u^r}{\lfloor r}\right)\right]_{u=0} = 1 \tag{8.19}$$

As before,

$$B_{n1}[g(\lambda)] = g_n = \lfloor n \, C(n) \tag{8.20}$$

$$B_{nn}[g(\lambda)] = g_1^{\,n} = (C(1))^n \tag{8.21}$$

$$B_{32}[g(\lambda)] = \frac{1}{\lfloor 2}\left[\frac{d^3}{d\lambda^3}\left(\sum_{k=1}^{\infty} C(k)\lambda^k\right)^2\right]_{\lambda=0}$$

On calculating the derivatives, this simplifies at $\lambda = 0$, to

$$B_{32}[g(\lambda)] = 6C(1)\,C(2) \tag{8.22}$$

Using $B_{nl}[\tilde{F}(\lambda)] = \sum_{j=l}^{n} B_{nj}[g] B_{jl}[f]$,

$n = l = 1$ gives

$$B_{11}[\tilde{F}(\lambda)] = B_{11}[g] B_{11}[f]$$

$$\text{(i.e)} \quad \frac{P(1)}{P(0)} = C(1) \tag{8.23}$$

Similarly, $B_{21}(\tilde{F}) = \sum_{j=1}^{2} B_{2j}[g] B_{j1}[f]$

$$\text{(i.e)} \quad 2\frac{P(2)}{P(0)} = \lfloor 2\, C(2) + (C(1))^2$$

$$\text{(i.e)} \quad \frac{P(2)}{P(0)} = C(2) + \frac{1}{2}(C(1))^2 \tag{8.24}$$

Further,

$$B_{31}(\tilde{F}) = \sum_{j=1}^{3} B_{3j}[g] B_{j1}[f]$$

$$\text{or,} \quad \frac{P(3)}{P(0)} = C(3) + C(1)C(2) + \frac{(C(1))^3}{6}. \tag{8.25}$$

As another application of Bell Polynomials we derive the relation between the cumulants and combinants. The cumulant generating function $K(\theta)$ is given by $\log F(e^{\theta})$

Hence

$$K(\theta) = \sum_{k=1}^{\infty} C(k)(e^{\theta k} - 1)$$

$$= f(g(\theta)) \tag{8.26}$$

where $f = \sum_{k=1}^{\infty} c(k)\left(e^{\theta k}-1\right)$

$g(\theta) = \theta$, with $f(0) = g(0) = 0$.

$$B_{n1}[K(\theta)] = \left[\frac{d^n}{d\theta^n}\left(\sum_{r=1}^{\infty} \frac{K_r \theta^r}{\lfloor r}\right)\right]_{\theta=0} \quad (8.27)$$

$$= K_n \quad (n^{th} \text{ Cumulant})$$

$$B_{j1}[f] = \left[\frac{d^j}{d\theta^j}\left\{\sum_{k=1}^{\infty} c(k)\left(e^{\theta k}-1\right)\right\}\right]_{\theta=0} \quad (8.28)$$

$$= \sum_{k=1}^{\infty} c(k)\, k^j.$$

$$B_{nj}[g] = \frac{1}{\lfloor j}\left[\frac{d^n}{d\theta^n}\left(\theta^j\right)\right]_{\theta=0} = \delta_{jn} \quad (8.29)$$

Hence using

$$B_{n1}[K(\theta)] = \sum_{j=1}^{n} B_{nj}(g)\, B_{j1}(f)$$

$$= \sum_{j=1}^{n} \delta_{jn}\left(\sum_{k=1}^{\infty} c(k)\, k^j\right)$$

$$= \sum_{k=1}^{\infty} c(k)\left(\sum_{j=1}^{n} \delta_{jn}\, k^j\right) = \sum_{k=1}^{\infty} c(K)\, K^n.$$

Hence $$K_n = \sum_{k=1}^{\infty} c(k)\, k^n. \quad (8.30)$$

Other results can also be obtained in a similar fashion.

9. Compound Poisson Process

The compound Poisson Process occupies a central place in the theory of Point Processes, due to its potential value in applications from such diverse areas as physics, geology, nuclear medicine, insurance claims and geography. A stochastic process $\{Y(t), t \geq 0\}$ is called a compound Poisson Process if it admits a representation of the form

$$Y(t) = \sum_{n=1}^{X(t)} \xi_n \tag{9.1}$$

where $X(t)$ is a Poisson Process and $\{\xi_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables such that $\{X(t), t \geq 0\}$ and $\{\xi_n, n \geq 1\}$ are independent processes.

We now show that the characteristic function of the compound Poisson Process is given by

$$\phi_{Y(t)}(u) = \exp\{\lambda t(\phi_\xi(u) - 1)\} \tag{9.2}$$

where $\phi_\xi(u)$ is the common characteristic function of the random variables ξ_n and λ is the rate of occurrence of the Poisson events.

Now

$$\phi_{Y(t)}(u) = E\left[e^{iuY(t)}\right]$$

$$= E\left[E\left\{e^{iuY(t)} \mid X(t)\right\}\right]$$

$$= \sum_{n=0}^{\infty} E\left[e^{iuY(t)} \mid X(t+s)-X(t)=n\right] P\left[X(t+s)-X(s)=n\right]$$

$$= \sum_{n\geq 0} \left[\phi_\xi(u)\right]^n e^{-\lambda t} \frac{(\lambda t)^n}{\lfloor n} = e^{-\lambda t} \cdot e^{\lambda t \phi_\xi(u)}$$

Hence eq. (9.2)

We now give the method of finding the moments of the process ξ_n given the moments of the compound Poisson Process $Y(t)$, using Bell polynomials. We denote the n^{th} moments of these processes ξ and Y by $E(\xi^n)$ and $E(Y^n)$ respectively (Snyder 1974).

The characteristic function of $Y(t)$ is $e^{\lambda t[\phi_\xi(u)-1]}$

Writing $\tilde{F}(u) = \phi_Y(u)-1$, $f(u) = e^{\lambda t u}-1$ and $g(u) = \phi_\xi(u)-1$, we have $\tilde{F}(u) = f(g(u))$ with

$$f(0) = g(0) = 0 \tag{9.3}$$

The γ^{th} moments are given by

$$E(Y^n) = B_{n1}[F(u)] \tag{9.4}$$

$$E(\xi^n) = B_{n1}[g(u)] \tag{9.5}$$

The function $G(u) = \dfrac{\log(1+u)}{\lambda t}$ (9.6)

is the inverse to $f(u)$, since

$$f[G(u)] = \exp\left\{\frac{\lambda t \log(1+u)}{\lambda t}\right\} - 1 = u$$

Hence

$$B_{l1}^{-1}[f(u)] = B_{l1}\left[\frac{\log(1+u)}{\lambda t}\right] = \frac{1}{\lambda t} B_{l1}[\log(1+u)]$$

$$= \frac{1}{\lambda t}(-1)^{l-1} \lfloor l-1 \quad \text{by } (7.22)$$

From eq.(7.21) we have.

$$B_{n1}[g(u)] = \sum_{l=1}^{n} B_{nl}[F(u)]\, B_{l1}^{-1}[f(u)]$$

$$= \sum_{l=1}^{n} B_{nl}[F(u)] \frac{1}{\lambda t}(-1)^{l-1} \lfloor l-1$$

$$= \frac{1}{\lambda t} \sum_{l=1}^{n} (-1)^{l-1} \lfloor l-1 \, B_{nl}[F(u)].$$

This formula gives the moments of the process ξ.

Putting $l=1, n=1$

$$B_{11}[g(u)] = \frac{1}{\lambda t} B_{11}[F(u)]$$

$$(\text{i.e})\quad E(\xi) = \frac{1}{\lambda t} E(Y). \tag{9.7}$$

Similarly

$$\begin{aligned} B_{21}[g(u)] &= E(\xi^2) \\ &= \frac{1}{\lambda t}\left[B_{21}[F(u)] - B_{22}[F(u)]\right] \\ &= \frac{1}{\lambda t}\left[E(Y^2) - (F_1(0))^2\right] \\ &= \frac{1}{\lambda t}\left[E(Y^2) - (E(Y))^2\right] \\ &= \frac{1}{\lambda t} \operatorname{Var} Y(t) \end{aligned}$$

$$(\text{i.e})\quad E(\xi^2) = \frac{1}{\lambda t} \operatorname{Var} Y(t). \tag{9.8}$$

In this way we can find the moments of the process ξ, upto the desired order.

In conclusion, this chapter deals with the relations of combinants with the other well known cluster point functions. The important fact that the combinants play the same role as the probabilities in calculating the cumulants is brought out. In deriving our new results, the Bell polynomials play a vital part in an elegant and systematic way.

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