STUDIES IN FIRST PASSAGE PROBLEMS AND APPLICATIONS

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P R E F A C E

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CHAPTER - I

INTRODUCTION

1. Preliminary Remarks.

Stochastic theory is playing an ever increasing and vital role in the modelling and analysis of a variety of practical disciplines, since many natural phenomena are characterised by random events. It has contributed to a wide spectrum of subjects ranging from number theory to physics. Nay, more it has penetrated decisively into our way of scientific thinking. The natural role of such processes in the study of dams, inventories, collective risks, population growth, reliability, neuronal discharge activity, delay in communications theory and queues has been the subject of a vast number of investigations.

In the description of such processes, certain features of the problems like first level crossing probability etc. have been the study of special interest. A systematic analysis on the first passage problems for processes with state space and parameters being discrete and continuous has been treated in detail by Ramakrishnan (1959). In the case of additive stochastic processes the first passage distributions are related to (1) in queueing theory as probability distribution of the first busy period, (2) in storage theory as the probability distribution of first emptiness of a reservoir, (3) in collective risk theory as the first passage time to ruin, (4) in modelling of neuronal spikes, as the interval between two successive spikes etc.

Abstracting the main idea, from these studies, we can state the problem in a succinct way as below. If a process X(t)

starts from $X = x_0$ at time t=o the central quantity of interest is the passage time $t = T$, when a boundary x=k is first either reached or crossed:

i.e $$T = \inf \left\{ t : X(t) \geq K \right\} \tag{1.1}$$

The real problem is to determine $M(x_0, k, t)$, the probability that x(t) crosses or reaches as the situation may be the barrier x=k, for the first time. The distribution

$$F(x,t) = P\left[X(t) \leq x \ , \ T > t \right] \tag{1.2}$$

is also to be found. In the case of processes with two barriers, the probability distributions for the time when one of the barriers is reached or crossed for the first time, before the other barrier is reached or crossed are of practical interest. In these cases the barriers are treated as absorbing barriers. There are realistic situations wherein one barrier can be taken as an absorbing barrier and the other as a reflecting barrier. Such problems, for example, occur in dam models and neuronal discharge activity etc. For good discussions of the F.P.D. one can refer to the works by Cox&Miller (1965) and Prabhu (1965).

The basic markov process in continuous time which is spatially and temporally homogeneous is a pure jump process defined by

$$X_J(t) = \sum_{i=1}^{N(t)} Z_i \tag{1.3}$$

where N(t) denotes the number of Poisson jumps in the interval (o,t) and $\{Z_i\}$ is a sequence of independent and identically distributed random variables. A trivial extension of (1.3) which is of great practical importance is the addition of a linear drift upwards (or downwards) to the process given by (1.3) and is defined by

$$X_\alpha(t) = \sum_{i=1}^{N(t)} Z_i + \alpha \tag{1.4}$$

where is the drift parameter. The process defined by (1.3) is usually referred to as homogeneous drift process.

Another extension of (1.3) which also serves as a practically useful model in neuronal discharge activity, dams, collective risk problems etc. is the process with exponential decay. This process is defined by

$$X(t) = \sum_{n=1}^{\infty} Z_n \exp[-\alpha(t-t_n)] \tag{1.5}$$

where α is the exponential decay parameter $t_1, t_2, \ldots$ are the times at which the jumps of magnitudes $Z_1, Z_2, \ldots$ occur.

Sophistications are introduced in this type of analysis by including both exponential and linear deterministic change of the level x(t). Also of great importance, and mathematically exciting are cases in which the barriers move with time. In

biological contexts the moving end conditions are of importance indeed. Also, the jump processes described in (1.3) and (1.5) need not be simple at all. The jump at any time t, may be dependent on the existing level x(t) and these jumps may be in both the directions. These are the types of problems that are investigated herein with the emphasis on obtaining closed solutions with a view to validate the models by comparisons with experimental results.

2. Methods.

A number of techniques are available in the literature for studying the first passage times in different situations. Wald obtained the Wald identity (1947) which is used for studying various first passage problems of bounded processes. Kemperman (1961), Keilson (1965) and many others used the Wiener-Hopf techniques for barrier problems. A good perspective of the Wiener-Hopf technique is given in Cohen (1967). Lindley considered Wiener-Hopf integral equations for the single server queue problems (1952). Pollaczek's method (1957) for studying the waiting time process has been widely used in queuing theory. Takacs (1967) has used combinatorial methods. Keilson (1965) has used the technique of compensation functions for studying bounded processes.

Wald Identity: Wald's fundamental identity in sequential analysis (1947) has been widely used for various applications apart from sequential sampling. Bartlett (1955) has used it for

the insurance risk problem and a gambler's ruin problem. Phatarford (1963) used it for studying the periods of first emptiness and first overflow in a finite dam. This identity for a discrete process is

$$E\left[e^{-sX_N} M(s)^{-N}\right] = 1 \tag{2.1}$$

where N is the time of absorption by the barrier and M(s) is the moment generating function (m.g.f) of the random variables representing the jump which are independent and identically distributed and the process starts from x=o. The relation (2.1) is applicable for a single barrier as well as two barriers. For a continuous time process the identity is

$$E\left[e^{-sX_T} e^{-\ell T}\right] = 1 \tag{2.2}$$

where t=T is the time of absorption at the barrier. In the case of the particle reaching the barrier, exact results for the first passage probability is obtained. But if the particle has to cross the barrier by a random jump the identity leads to approximate solutions. This identity has been derived in a different form by Kemperman (1961) for discrete and continuous time process. For a discrete process,

$$\left[1 - Z M(s)\right] \bar{Q}_0(s,z) = 1 - \bar{Q}_1(s,z) - \bar{Q}_2(s,z) \tag{2.3}$$

where $\bar{Q}_0$, $\bar{Q}_1$, $\bar{Q}_2$ are the double transforms of $Q_0(x,n)$ $Q_1(x,n)$ and $Q_2(x,n)$ which are the joint probability functions for the position x at the nth step, in the regions (a,b) $(-\infty, a)$ and (b, ∞) respectively where a and b are the barriers. Kemperman (1963) also obtained a similar identity for a continuous time process. He has used the Wiener-Hopf technique and the above identity to obtain the first passage density in terms of Wiener-Hopf factors.

Bellman (1957) and Phatarford (1971) extended the Wald identity when the increments of the stochastic process have Markovian dependence. Phatarford used this to obtain some approximate results in storage models.

Lindley's method: Lindley (1952) has considered a single server queuing system and has obtained an integral equation for the waiting time distribution. Here x=o is taken as a barrier. The solution of this equation has been obtained by Wiener-Hopf techniques.

Combinatorial Methods: Takacs (1967) has used combinatorial analysis in studying the busy period of a single server queue and the wet period of a finite dam for similar applications in other fields. He has used in an elementary way the generalisation of the classical Ballot theorem for different situations.

Compensation Functions: The compensation functions developed by Keilson (1962, 1963, 1964, 1965) have been found extraordinarily useful for studying bounded processes. The

philosophy of this method lies in converting a bounded process into an unbounded process by introducing the compensation functions in the usual integro differential equation for the process. The compensation function acts as a source and takes care of the boundary effects. The advantages of this method are (1) it enables the bounded process to be discussed in terms of the Green's function of the unbounded process (i.e.) processes with natural boundaries and homogeneous boundary conditions. This procedure often enables us to calculate the desired first passage densities fairly easily (2). The classical method of solving the equation of Lindley's process is by the Wiener-Hopf method, an exercise in complex analysis usually bristling with difficulties. The compensation method is more easily tractable, leading to results expressible in an elegant manner (Keilson 1965).

Inspite of all these techniques there are a number of problems remaining unsolved due to complications of real situations. Especially for a stochastic process with random jumps in both directions and exponential decay, even in the case of Poisson inputs and outputs we have not come across closed solutions. We have succeeded in arriving at closed solutions in a number of such cases employing the powerful method of imbedding and other sophisticated analysis.

<u>Imbedding method:</u> The philosophy of this technique is to imbed the given process into a class of similar processes and

arrive at a general solution which holds valid for a continuous set of problems by enlarging the parameter class. This is achieved by investigating what happens in an initial interval of the space or time parameter and obtaining equations quite similar to backward equations. Detailed methodology is found in the book by Bellman and Wing (1976). Reference to backward equations can be found in Ramakrishnan (1959), Barucha Reid (1960) and Harris (1963). First passage problems for different situations are obtained by suitably defining a functional of the underlying variables and writing imbedding equations for these. Closed solutions are obtained in the cases mentioned earlier. Physical features of the process like mean and moments of the first passage times and answers to other type of questions are obtained. We have also made contact with other types of investigations (existing in the literature) and results obtained therein, such as (1) the method of Wald identity (2) the method of compensation functions etc.

This thesis consists of six chapters and summaries of each chapter is given in the following section.

3. Chapter Summaries.

In Chapter II we study stochastic models for the spike discharge activity of neurons. Model I treats only excitatory impulses occurring as Poisson events with exponential density and in the absence of these events the subthreshold potential decays exponentially. A closed form solution for the Laplace transform

of the renewal density of the discharge is obtained by the imbedding method. This result in the analytical closed form has not been derived so far. In model II we consider both excitatory and inhibitory impulses with exponential decay of the subthreshold potential in between the jumps. Analytical closed form solution for the first passage problem is obtained in this case also. The mean interval time between the spikes and the stationarity product density of firing are obtained in a general form. Hence the moments of the spikes can be obtained easily and the model can be compared with experimental findings. The results of Tsurui and Osaki (1976) can be obtained from our analysis; also the integral equation deduced by them in equation (3.1) of their paper has been shown by us to be the solution of the forward equation with compensation function (of Keilson (1965)) included to take care of the boundaries as shown by our equation (5.11) of Chapter II in our text.

In Chapter III, a class of first passage problems for time dependent barriers is studied. We start with the determination of the first passage time for a homogeneous drift process to cross a constant barrier. Using the compensation technique the solution is obtained in terms of the Green's function of the unbounded process (Keilson 1963). Here we show that our results are valid if we take the density of jumps in the positive direction in a separable form, the density in the negative direction being described in an arbitrary manner. By

proceeding to the limit as described by Keilson (1963) we easily obtain the first passage density for the diffusion process of the Wiener type (Ricciardi 1977, Sugiyama et. al. 1970) starting from a two way discrete jump process with exponential densities. Considering a barrier moving linearly with time the first passage time density is obtained which is equivalent to a constant barrier problem with the particle sliding with a linear drift in a direction opposite to that of the direction of motion of the barrier. Of course the solution of this problem can be obtained in terms of the corresponding unbounded Green's function by using the compensation function method (Keilson 1965).

Here again we go to the diffusion limit by adopting the procedure as described by Keilson (1963) and derived an explicit result for the first passage time for the diffusion process with a moving barrier.

For a discrete jump process with linear drift Wald's identity is derived in an interesting fashion. Starting from a two way jump processes and writing an integral equation by an intutive argument, we obtain the Wald identity by proceeding to the limit as t tends to infinity. From this identity we can go to the diffusion limit in a way similar to that given by Keilson. Here we make the intensity of Poisson jumps ν as $\gamma^2\nu$ and ascribe special properties to the mean and second moment of the jump densities. From this procedure one obtains the diffusion process. It is found that on applying this

procedure we arrive at the Wald identity of the diffusion process, which coincides with what Clay and Goel (1973) obtained.

It has been suggested by Fuortes and Mantegazzini (1962) and Tuckwell (1979) that the input impulse in a neuron, should alter the membrane potential in some way depending upon the existing subthreshold potential gathered till that time by the neuron. This aspect has been studied in this Chapter by taking the density of jumps as "Pareto" distribution. Here we have taken the boundary as constant as well as moving exponentially. Using a transformation of the independent variable, we convert the process with exponential decay into a process with linear decay. By adopting the compensation method solutions are obtained in terms of the Green's function of the unbounded process. This solution can be written down in terms of the original variables also. In the case of constant boundary similar result is obtained by the imbedding techniques for this type of jump processes easily.

For homogeneous process with or without drift first passage time densities for either reaching or crossing a barrier moving exponentially down with time are obtained. Our method is different from that of Kryukov (1976). Equations for the mean can be given by an integral equation which can be solved upto any desired approximation.

In Chapter IV the imbedding technique is used as a powerful tool to study a host of problems in storage theory.

We consider a dam model described by the equation

$$X(t) = \sum_{n_1=1}^{N_1(t)} Z_{n_1}^{+} e^{-\alpha(t-t_{n_1})} - \sum_{n_2=1}^{N_2(t)} Z_{n_2}^{-} e^{-\alpha(t-t_{n_2})} \tag{2.4}$$

where $X(0) = i\ u$. The inputs into the dam are taken as Poisson events with intensity ν_1. There are two types of outputs. One is a random event with Poisson intensity ν_2 and the other is a deterministic release. The deterministic release is taken as proportional to the content at that time and this occurs only in the absence of random events. For the process described above, we have obtained explicit expression for the following probability densities.

(1) The passage time for first overflow before emptiness.

(2) The first passage time density for overflow with any number of emptiness.

(3) The passage time density for first emptiness before overflow.

(4) The first passage time density for emptiness with any number of overflows.

(5) Lastly we have also obtained the expected amount of water overflowed in a given time before emptiness.

A number of well known results are deduced. Analytical result for these entities are from the solution of a third order equation with suitable boundary conditions. By using imbedding analysis (Bellman 1976) we write integro-differential equations

for two classes of first passage density functions M^+ and M^- as in equations (2.3) of Chapter IV, of the text. M^+ refers to processes starting with a positive jump wherever a jump occurs, and M^- is similarly defined. This device enables one to arrive at the equation for the full p.d.f.

In Chapter V, we study the collective risk theory with a view to compute ruin probabilities. We consider a model for the risk process when the claims are Poisson events and when the income to the company consists of two components (1) coming at a constant rate β and (2) those proportional to the risk reserve at any time. We further assume that there is an upper limit K to the risk reserve and thereby the process is considered as a stochastic process with two barriers one at x=o and the other at x=k. We have derived analytical expressions for the L.T. of the following density functions.

(1) The probability density function for the ruin at time t before risk reserve reach the value k.

(2) The p.d.f. for ruin when X=K is a reflecting barrier. This means that the risk reserve can reach the value k. Once it reaches k, it stops at k until there is a claim. After the claim the process repeats.

(3) The p.d.f. for the risk reserve to reach the limit k before ruin take place. Using the results of (1) and (3) we have verified that within infinite time the sum of these two probabilities is unity. Other quantities of interest are also studied.

We have obtained the solutions in a closed analytical form, and in the limit when t goes to the infinity this coincides with the result obtained by Segerdahl (1959).

In Chapter VI we study how the combinants (Kauffmann, Gyulassy, 1978) can be related to the product densities (Ramakrishnan 1959). To characterise the probability of n events occurring in a given interval it is useful to describe p(n) , the desired probability, in terms of its deviations from the Poisson. Writing down the generating function

$$F(\lambda) = \sum_{n=0}^{\infty} \lambda^n p(n) \tag{2.5}$$

of a given distribution as

$$F(\lambda) = \sum_{n=0}^{\infty} \exp\left\{ C(K)\ (\lambda^K - 1)\right\} \tag{2.6}$$

we obtain a characterisation of the probability by the quantity $C(K)$ called the combinants. These can be directly related to the cumulants of the distributions. To study point process Ramakrishnan (1950) introduced the powerful tool of product density. They can be related to the factorial moments $\left\langle n!/(n-r)! \right\rangle$ of the events occuring in an interval by proper integration over the continuous state variables. The moments of p(n) are related to those integrals by using the well known C_s^r coefficients.

Similarly we show that the combinants can be related to appropriate sum of integrals over cluster functions. Hence we note that the combinants play the same role in calculating the cumulants, as probabilities do in finding moments. Some statistical features of a doubly stochastic process are described.

C H A P T E R - II

NEURONAL SPIKE TRAINS WITH EXPONENTIAL DECAY*

1. Introduction.

Sequences of neuronal firings referred to as spike trains arise from the so-called spontaneous activity of response of the neurons to external stimuli. Spike trains and the corresponding interval histograms of spontaneous firings of a single neuron have been recorded experimentally using microelectrodes (Redman and Lambard 1968a). Many mathematical and statistical models have been proposed to reproduce the neuronal activity to fit the data. Neuronal cells with random inputs have been studied experimentally (Redman and Lambard 1968a, Redman et. al. 1968b, Faganel 1970). Detailed surveys of theoretical models of single neurons are given in (Feinberg 1974, Holden 1976, Srinivasan and Sampath 1977, Yang and Chen 1978). An important physiological detail we must include in our models is the fact that the neuron is a leaky integrator (i.e.) sub-threshold potentials decay exponentially towards the resting membrane potential as found by Eccles (1964).

In modelling the activity of a single neuron we have to assume its random nature due to random variations inherent in the input. Due to the indistinguishability (Perkel et. al. 1967a, 1967b) of the firings (which constitute a point process) we think of this stochastic process as stationary. For an excellent survey of point process the reader is

*Based on the paper by R.Vasudevan, P.R.Vittal and A.Vijayakumar submitted for publication.

referred to the article by Ramakrishnan (1959). Hence we want to determine the interspike interval distribution under different assumptions. There have been a number of probabilistic models describing the firing activity. In one such model which we call as the continuous state model the incoming impulse are treated as potentials transferred through the synaptic junctions in a given neuron. These input events are random with a poisson distribution and the amount of excitatory impulse gathered by the neuron in these independent events are governed by an exponential distribution. Inhibitory events as such are ignored in some treatments like in the reference (Tsurui and Osaki, 1976).

In many other situations such as the problem relating to the content of a reservoir and the first overflow when the release is exponential (Moran, 1954) we have identical questions to be answered. Also in pharmacokenitics the steady state distribution of the concentration of a drug in the blood stream under various dosing regimens when the concentration follows an exponential decline relates to problems of similar kind (Brill and Moon, 1978). Some storage models of dams and integrated models of shot noise problems belong to this category (Keilson, 1959).

In the first part of our treatment of the neuronal problems we will assume that the inhibitory and excitatory events are somehow integrated to be replaced by the EPSP's only which are governed by the exponential inputs occuring

as poisson events in time. In the absence of external events the state variable expressing the potential accumulating in the neuron decays exponentially. For the neuron to fire this cumulative process should collect a total potential value $x = z$ where z exceeds the threshold value K for the given neuron. Just after each firing the neuron is reset to its resting value with $x = o$ and the whole process starts all over in an identical fashion. The refractory period of the neuron which introduces complications of a moving boundary is ignored usually and is done here also. The first passage time for the state variable to cross the threshold K is the theme of this Chapter when there are upward and downward jumps. Even when there are upward jumps only, it has been found difficult to get analytical expressions in closed form for the first passage probability density function (p.d.f.) with exponential decay as pointed out in (Srinivasan and Sampath, 1975). Tsurui and Osaki (1976) first gave the result of this type of model in a cumbersome manner.

We have here adopted a different route to treat this problem and that is the method of imbedding (Bellman and Wing, 1976) to arrive at the second order differential equation directly for the transform of the first passage probability density function. We arrive at elegant analytical results which make contact with the conclusions of reference (Tsurui and Osaki, 1976). Of course this problem with exponential decay is not so difficult in the diffusion approximation as

found in the reference (Sugiyama et. al., 1970) for a constant threshold.

In the later portion of this Chapter we treat a more realistic model in which the input stimulus is both excitatory and inhibitory, the amount of inputs being governed by exponential distributions. By imbedding techniques we arrive at a third order differential equation for the transform of the level crossing density. We obtain analytical expressions for the transform and hence the mean first passage time in this case also. This has not been derived so far. Complications arising due to the refractory period of the neuron will be dealt with in a later Chapter.

2. <u>The first passage time differential equations for Model I.</u>

Recently the authors (R.Vasudevan et. al. 1978) considered the problem of the motion of a particle on a real line in continuous time, with the particle moving with upward jumps and downward drift. The first passage time relating to this particle crossing the barrier K with jumps in both the directions and with linear drift either up or down was completely analysed. The authors also studied the same type of process with two barriers (R.Vasudevan et. al., 1979). In this Chapter we consider the random motion of the particle with upward jumps and exponential sliding which can represent the many features of the phenomenon of neuronal spiking.

The neuronal potential at any instant is represented by the random variable X(t) which is subjected to excitatory

impulses occuring as poisson events in time with intensity. The inputs (upward jumps) are governed by the exponential distribution a(z). In the neuron the random membrane potential at each instant is the resultant of the potentials gathered through its synaptic junctions which may be both excitatory and inhibitory. The resting potential of the neuron is taken as the zero point, x=o for our model and the instantaneous potential is the stochastic variable which is pictured as the position x(t) of a moving particle on the real line. The exponential sliding of the particle towards zero is the continuous decay of the neuronal potential with time. Also the resultant synaptic inputs at any instant are taken as Poisson events leading to increments in the potential. When the potential reaches the threshold value x(t)=k the spike discharge occurs and the neuron goes back to its resting state. In our model the first passage time t=T to cross the barrier will represent the neuronal spike interval time. Hence equation for the stochastic process x(t) is:-

$$X(t) = U + \sum_{n=1}^{N(t)} Z_n \, e^{-\alpha(t-t_n)} \, . \qquad (2.1)$$

where $N(t)$ is the number of Poisson jumps in the interval $(0,t)$ and Z_n are independent and identically distributed random variables expressing the magnitude of the nth jump of the particle and t_n denotes the time of occurrence of the nth jump. Z_n are non-negative random variables with a common

density $a(z)$.

Define:

$p(u,k,t)$ = the probability that the particle crosses the barrier for the first time between time t and $t+\Delta$ given that it was at u at time $t=0$.

We now want to imbed the process into a class of similar processes. Hence we consider the time intervals $(0,t)$ and $(0,t+\Delta)$ and relate the $P(\cdot)$ for these two time intervals. This leads to an integro differential equation when we proceed to the limit as Δ tends to zero. Taking into account the different mutually exclusive possibilities during the initial time interval $(0,\Delta)$ we have after proceeding to the limit as $\Delta \to 0$,

$$\frac{\partial p}{\partial t} + \alpha u \frac{\partial p}{\partial u} + \nu p = \nu \int_0^{k-u} a(z)\, p(u+z,k,t)\, dz + \delta(t)\, \nu \int_{k-u}^{\infty} a(z)\, dz \tag{2.2}$$

where $\delta(t)$ is the Dirac delta function.

Define the Laplace transform of $p(u,k,t)$ with respect to t as

$$\bar{p}(u,k,\ell) = \int_0^{\infty} e^{-\ell t}\, p(u,k,t)\, dt \tag{2.3}$$

The equation (2.2) becomes

$$\alpha u \frac{\partial \bar{p}}{\partial u} + (\ell+\nu)\bar{p} = \nu \int_0^{k-u} a(z)\, \bar{p}(u+z, k, \ell)\, dz + \nu \int_{k-u}^{\infty} a(z)\, dz \tag{2.4}$$

Let us choose the density function for the magnitude of jump as:

$$a(z) = \eta e^{-\eta z} \quad , \quad z > 0$$
$$= 0 \text{ otherwise} \tag{2.5}$$

Then the equation (2.4) becomes

$$e^{-\eta u}\left[\alpha u \frac{\partial \bar{p}}{\partial u} + (\ell+\nu)\bar{p}\right] = \nu\eta \int_u^k e^{-\eta y}\, \bar{p}(y, k, \ell)\, dy + \nu e^{-\eta k} \tag{2.6}$$

Differentiating (2.6) with respect to the variable u we get,

$$\alpha u \frac{\partial^2 \bar{p}}{\partial u^2} + \left[\ell + \nu + \alpha - \eta\alpha u\right]\frac{\partial \bar{p}}{\partial u} - \eta \ell \bar{p} = 0 \tag{2.7}$$

Letting $\eta u = v$, the equation (2.7) transforms into

$$v \frac{\partial^2 \bar{p}}{\partial v^2} + \left[\frac{\ell+\nu}{\alpha} + 1 - v\right]\frac{\partial \bar{p}}{\partial v} - \frac{\ell}{\alpha}\bar{p} = 0 \tag{2.8}$$

The general solution of this equation is

$$\bar{p}(u, k, \ell) = A(k,\ell)\, {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha} + 1, \eta u\right) + B(k,\ell)(\eta u)^{-\frac{\ell+\nu}{\alpha}}\, {}_1F_1\left(-\frac{\nu}{\alpha}, 1 - \frac{\ell+\nu}{\alpha}, \eta u\right) \tag{2.9}$$

where $A(K,\ell)$ and $B(K,\ell)$ are the arbitrary constants which are to be determined using boundary conditions. The Kummer's confluent hypergeometric function which occurs in the above solution is given by

$$ {}_1F_1(a,c,z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} $$

where $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$ (2.10)

The solution (2.9) is valid for initial levels u of the variable X only if $u \neq 0$. Choosing $u=0$ we note that the second term in the RHS of the solution (2.9) becomes infinite for $[(\ell+\nu)/\alpha] > 0$. Since α and ν are positive and real $\ell > 0$, we have to choose $B(K,\ell)=0$ if the solution is to be valid both for $u=0$ and $u \neq 0$ i.e.

$$ \bar{P}(u,K,\ell) = A(K,\ell)\, {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}+1, \eta u\right) $$

$$ \text{for} \quad 0 \leq u \leq K \qquad (2.11) $$

Now the function $A(K,\ell)$ is determined by substituting the solution (2.11), in the first order integro differential equation (2.4) which serves as the second boundary condition.

Feeding the solution (2.12) into the equation (2.7) we get

$$e^{-\eta u}\left\{\alpha u A \frac{\partial}{\partial u}\,{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}+1, \eta u\right)\right.$$
$$\left. + (\ell+\nu) A\,{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}+1, \eta u\right)\right\}$$
$$= \nu \eta A \int_u^K e^{-\eta y}\,{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}+1, \eta y\right) dy + \nu e^{-K} \tag{2.12}$$

We will scale all lengths in the above equation by the factor η for convenience; in otherwords we can put $\eta = 1$ throughout. We also recall the standard properties of the confluent hypergeometric function (Abramowitz and Stegun, 1968).

$$\int e^{-x}\,{}_1F_1(a,b,x)\,dx = e^{-x}\frac{(b-1)}{1+a-b}\,{}_1F_1(a,b-1,x)$$
$$+ \text{Constant}$$
$$(\text{for } b-a \neq 1) \tag{2.13}$$

Hence the equation (2.6) becomes

$$e^{-u} A\left\{\alpha u \frac{\partial}{\partial u}\,{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}+1, u\right)\right.$$
$$\left. + (\ell+\nu)\left[{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}+1, u\right) - {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}, u\right)\right]\right\}$$
$$= e^{-K}\left[\nu - A(\ell+\nu)\,{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}, K\right)\right] \tag{2.14}$$

The LHS is zero because of the identities

$$(b-1)\,{}_1F_1(a,b-1,z) = (b-1)\,{}_1F_1(a,b,z)$$
$$+ z\frac{d}{dz}\,{}_1F_1(a,b,z)$$

(2.15)

Hence $A(K,\ell)$ is given by

$$A(K,\ell) = \frac{\nu}{\ell+\nu} \frac{1}{{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}, K\right)} \tag{2.16}$$

Thus the Laplace transform of the first passage density to cross the barrier K, starting from an initial position u at time $t=0$ is

$$\bar{p}(u,K,\ell) = \frac{\nu}{\ell+\nu} \frac{{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}+1, u\right)}{{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}, K\right)} \tag{2.17}$$

if we want the solution to be valid for all values of u including $u=0$.

If the process starts from the origin at time $t=0$, the first passage density is given by

$$\bar{p}(0,K,\ell) = \frac{\nu}{(\ell+\nu)} \frac{1}{{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}, K\right)} \tag{2.18}$$

If we assume that K, the barrier is made to come down and coincide with zero the first jump from $X=0$ will result in the crossing of the barrier. Hence as K tends to zero, from (2.18) we have

$$\lim_{K\to 0} \bar{p}(0,K,\ell) = \nu/(\ell+\nu) \tag{2.19}$$

which is the Laplace transform of first jump probability

$$\nu e^{-\nu t} \int_0^\infty e^{-z} \, dz$$

3. Comparison of results.

When the process is a pure jump process we can take $\alpha = 0$. Then from (2.18), we find

$$\left[\bar{P}(0,K,\ell)\right]_{\alpha=0} = \frac{\nu}{\ell+\nu} e^{-\ell K/(\ell+\nu)} \tag{3.1}$$

Inverting this we get

$$\left[P(0,K,t)\right]_{\alpha=0} = \nu e^{-K} e^{-\nu t} I_0(2\sqrt{K\nu t})$$

$$\sim \nu e^{-\eta K} e^{-\nu t} I_0(2\sqrt{\eta K \nu t}) \tag{3.2}$$

where I_0 is the modified Bessel function of order zero. This result agrees with that of C.R.Cox (1962) who treated the jump problem without decay.

If α tends to infinity we get from (2.18)

$$\lim_{\alpha \to \infty} \bar{P}(0,K,\ell) = \frac{\nu e^{-K}}{\ell + \nu e^{-K}} \tag{3.3}$$

This on inverting with respect to ℓ gives

$$\lim_{\alpha\to\infty} p(0,K,t) = \nu\, e^{-K} \exp[-\nu\, e^{-K} t]$$
$$\sim \nu\, e^{-\eta K} \exp[-\nu\, e^{-\eta K} t] \tag{3.4}$$

For the special case $\nu = 1$ and $\eta = 1$ the above result (3.4) was also deduced by Tsurui and Osaki from their result for $\bar{p}(0,K,\ell)$. Let us now make contact with the rather cumbersome expression (4.6) of the paper of Tsurui and Osaki by comparing our result for another parametric value $\alpha = 1$ The result obtained from the equation (4.2) of their paper reduces to

$$g(\ell)\, e^{-K} = [\bar{p}(0,K,\ell)]_{\alpha=1}$$
$$= \frac{(\ell+K)\, e^{-K}}{\ell(\ell+1) + K\, e^{-K}\, {}_1F_1(\ell,\ell+2,K)} \tag{3.5}$$

According to our calculation (equation 2.18) for $\nu = 1$ and $\eta = 1$ becomes when we put $\alpha = 1$.

$$[\bar{p}(0,K,\ell)]_{\alpha=1} = 1 \big/ (\ell+1) \cdot {}_1F_1(\ell,\ell+1,K) \tag{3.6}$$

The two expressions for $[\bar{p}(0,K,\ell)]_{\alpha=1}$, (3.5) and (3.6) are seen to be identical if one realises that for the Kummer's confluent hypergeometric function we have the relation

$$b(b-1)\,{}_1F_1(a,b-1,z)+b(1-b-z)\,{}_1F_1(a,b,z)$$
$$+z(b-a)\,{}_1F_1(a,b+1,z)=0 \tag{3.7}$$

4. Region of validity.

In the previous section we have obtained the second order equation (2.3) for the first passage time density function and the two independent solution are

$${}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\nu}{\alpha}+1,u\right) \text{ and } u^{-\frac{\ell+\nu}{\alpha}}\,{}_1F_1\left(-\frac{\nu}{\alpha},1-\frac{\ell+\nu}{\alpha},u\right)$$

As seen in the previous section, the second order equation (2.8) has the total solution (2.9) which is a linear combination of two independent solutions with coefficients $A(K,\ell)$ and $B(K,\ell)$. $B(K,\ell)$ is argued to be zero since for real $\ell>0$, the second solution diverges for $u=0$. Hence we obtained $\bar{P}(0,K,\ell)=A$ as in equation (2.18). However one can see that when $u\neq 0$ the second solution does not blow up, hence B need not be zero for real $\ell>0$. One should emphasize the fact that a particle starting at an initial level $u\neq 0$, will never come down to touch zero at any finite time. Hence let us now take the solution (2.9) without putting $B=0$ and substitute this in the first order equation (2.6) to obtain

$$e^{-u} A \Big[\alpha u \frac{\partial}{\partial u} {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}+1, u\right)$$
$$+(\ell+\nu)\, {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}+1, u\right)$$
$$-(\ell+\nu)\, {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}, u\right)\Big]$$
$$+ e^{-u} B \Big[\alpha u \frac{\partial}{\partial u} u^{-\frac{\ell+\nu}{\alpha}} {}_1F_1\left(-\frac{\nu}{\alpha}, 1-\frac{\ell+\nu}{\alpha}, u\right)$$
$$-(\ell+\nu)\, u^{-\frac{\ell+\nu}{\alpha}} {}_1F_1\left(-\frac{\nu}{\alpha}, 1-\frac{\ell+\nu}{\alpha}, u\right)$$
$$-\frac{\alpha\nu}{\ell+\nu-\alpha} u^{1-\frac{\ell+\nu}{\alpha}} {}_1F_1\left(1-\frac{\nu}{\alpha}, 2-\frac{\ell+\nu}{\alpha}, u\right)\Big]$$
$$= e^{-K} \Big[-A\,(\ell+\nu)\, {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}, K\right)$$
$$-\frac{B\nu\alpha}{\ell+\nu-\alpha} K^{1-\frac{\ell+\nu}{\alpha}} {}_1F_1\left(1-\frac{\nu}{\alpha}, 2-\frac{\ell+\nu}{\alpha}, K\right) + \nu\Big] \tag{4.1}$$

The terms in the brackets multiplying the coefficients A and B on the L.H.S. of equation (4.1), vanish individually and identically. Hence we have one equation connecting A and B, given by

$$AL + \frac{B\nu\alpha}{(\ell+\nu)(\ell+\nu-\alpha)} K^{1-\frac{\ell+\nu}{\alpha}} M = \frac{\nu}{\ell+\nu} \tag{4.2}$$

where $L = {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}, K\right)$

and

$$M = {}_1F_1\left(1-\frac{\nu}{\alpha}, 2-\frac{\ell+\nu}{\alpha}, K\right) \tag{4.3}$$

Let us now picture how $\bar{p}(0,K,\ell)$ can be connected with $\bar{p}(u,K,\ell)$ by a physical reasoning to obtain a second relation connecting A and B. Starting from $u=0$ the particle can jump to a level u in time r, $(r<K)$ and $u<K$ for the first time and then cross over K from the initial level u in the rest of the time $t-r$. Or it can jump directly into the region (K,∞) by an initial jump after waiting for time t. Hence taking Laplace transform of entities involved with respect to time we have

$$\bar{p}(0,K,\ell) = \frac{\nu}{\ell+\nu}\int_0^K e^{-u}\,\bar{p}(u,K,\ell)\,du + \frac{\nu}{\ell+\nu}\,e^{-K} \tag{4.4}$$

This is the equation one gets by putting $u=0$ in (2.4). Substituting for $\bar{p}(u,K,\ell)$ in the expression (2.10) we land on the following relations for A and B

$$-AL + Ae^{K} - \frac{B\nu\alpha}{(\ell+\nu)(\ell+\nu-\alpha)}\,K^{1-\frac{\ell+\nu}{\alpha}}\,M = -\frac{\nu}{\ell+\nu} + e^{K}\,\bar{p}(0,K,\ell) \tag{4.5}$$

Thus the entity $\bar{P}(0,K,\ell)$ serves as the second boundary condition for the differential equation (2.9), while the first order equation is the first condition. Hence we find after solving the simultaneous equations (4.2) and (4.5)

$$A = \bar{P}(0,K,\ell)$$

$$B = \left\{ \frac{\nu - (\ell+\nu)\,{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}, K\right)\bar{P}(0,K,\ell)}{{}_1F_1\left(1-\frac{\nu}{\alpha}, 2-\frac{\ell+\nu}{\alpha}, K\right)} \right\} \frac{\ell+\nu-\alpha}{\nu\alpha} K^{\frac{\ell+\nu}{\alpha}-1} \tag{4.6}$$

which is valid for all the region $u>0$. However as we have pointed out earlier that for ℓ, to be the transform variable we should have real $\ell>0$. Also in deriving the equation (4.5) from (4.4) we should have $1-\frac{\ell+\nu}{\alpha}>0$, i.e, $\ell+\nu<\alpha$ which means $Re(\ell+\nu)<\alpha$, α and ν being both positive. If however $\alpha=\nu$, B has to be necessarily zero. If $\bar{P}(0,K,\ell)$ is given by equation (2.18) B automatically vanishes. Hence solution for $u \neq 0$ is given by (2.17).

5. Mean Passage Time.

One can obtain all the moments of the distribution from the Laplace transform $\bar{P}(0,K,\ell)$. The mean passage time is given by

$$E(T) = \lim_{\ell \to 0} \bar{P}(0,K,\ell) \tag{5.1}$$

From (2.18)

$$E(T) = \lim_{\ell \to 0} \left[\frac{1}{\nu} + \frac{d}{d\ell} \, {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}, K\right) \right] \tag{5.2}$$

where $d/d\ell$ denotes the total derivative of the function with respect to ℓ. It is well known that (Slatee) for the hypergeometric functions ${}_1F_1(a,b,z)$

$$\frac{\partial}{\partial a} \, {}_1F_1(a,b,z) = \sum_{r=1}^{\infty} \frac{(a)_r}{(b)_r} \frac{z^r}{r!} \sum_{n=1}^{r} \frac{1}{a+n-1} \tag{5.3}$$

and

$$\frac{\partial}{\partial b} \, {}_1F_1(a,b,z) = -\sum_{r=1}^{\infty} \frac{(a)_r}{(b)_r} \frac{z^r}{r!} \sum_{n=1}^{r} \frac{1}{b+n-1} \tag{5.4}$$

$$E(T) = \frac{1}{\nu} + \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{\Gamma(\nu/\alpha)}{\Gamma(n+\nu/\alpha)} \frac{K^n}{n}$$

$$\text{Where } \Gamma(n) = n\,\Gamma(n-1) \tag{5.5}$$

This result also agrees with the equations (5.12) and (5.13) when $\nu = 1$ and $\alpha = 1$ which correspond to equation (5.2) of Tsurui et. al. When $\nu = 1$, and $\alpha = 1/2$ we obtain from (4.6) the same equation as (5.3) of Tsurui and Osaki.

We will now show that the integral equation obtained by Tsurui and Osaki (1976) for a particle crossing the barrier K can be derived from the compensation technique of Keilson (1962). Defining $\pi(x,t\,|\,u,0)$ as the transition probability density for the particle to be at $X=x$ at time t given that it was at $X=u$ at time t=o, the forward equation for the unrestricted process (2.1) is given by

$$\frac{\partial \pi}{\partial t} - \alpha \frac{\partial}{\partial x}(x\pi) + \nu\pi = \nu \int \pi(x',t)\, a(x-x')\, dx'$$

$$\text{with} \quad \pi(x,0) = \delta(x-u) \tag{5.6}$$

For this process x=o serves as a natural barrier. To give effect to the boundary at x=k we introduce the compensation function into the equation (5.6). This boundary current acts as a source and takes care of the boundary effects. Hence the equation

$$\frac{\partial \pi}{\partial t} - \alpha \frac{\partial}{\partial x}(x\pi) + \nu\pi$$

$$= \nu \int_0^\infty \pi(x',t)\, a(x-x')\, dx' - f(x,t) \tag{5.7}$$

can be considered as an equation for this solution process with no boundary. The compensation function $f(x,t)$ is defined as(Keilson (1964))

$$f(x,t)=\nu\int_0^K \pi(x',t)\,a(x-x')\,dx' \;, \quad x>K$$
$$= 0 \;, \qquad \text{otherwise}. \tag{5.8}$$

If $f(x,t)$ was not included in (5.7) we have the integro differential equation for the free process whose green's function $G(x,t|u,0)$ can be easily found. Utilising this Green's function the solution of (5.7) can be written as

$$\pi(x,t|u,0) = G(x,t|u,0) - \int_0^t dt' \int_K^\infty G(x-x',t-t')\,f(x',t')\,dx' \tag{5.9}$$

From (5.8) and (5.9) we have

$$\nu\int_0^K \pi(x'',t)\,a(z-x'')\,dx''$$
$$= \nu\int_0^K G(x'',t|u,0)\,a(z-x'')\,dx''$$
$$-\nu\int_0^K dx''\int_0^t dt'\int_K^\infty G(x''-x',t-t')\,f(x',t')\,a(z-x'')\,dx'$$
$$\text{for } z>K \tag{5.10}$$

Since we require $z>K$, identifying L.H.S. of (5.10) as $f(z,t)$ which is also the probability that the system point jumps out of the barrier k into the interval $(z, z+dz)$ at time t. Hence (5.10) constitute an integral equation for f.

$$f(z,t) = \nu \int_0^K G(x',t)\, a(z-x')\, dx'$$
$$- \nu \int_0^K dx'' a(z-x'') \int_0^t dt' \int_K^\infty dx'$$
$$\times \left\{ G(x''-x', t-t')\, f(x',t') \right\} \qquad (5.11)$$

We at once see that this is the same integral equation (3.1) of Tsurui and Osaki. Since $f(z,t) = e^{-z}\, g(t)$ as given in that reference we can obtain the integral for $g(t)$ which is the same as equation (4.3) of Tsurui and Osaki.

Let us define $h(K,t)$ as the probability that there is a spike between time t and $t+\Delta$ irrespective of what happened earlier. This is the product density introduced by Ramakrishnan (1950). We have been using the renewal density $p(0,K,t)$ which is the inverse of $\bar{p}(0,K,\ell)$ of equation (2.19)

$$h(K,t) = p(0,K,t) + \int_0^1 h(K,\tau)\, p(0,K,t-\tau)\, d\tau \qquad (5.12)$$

Hence

$$\bar{h}(K,\ell) = \frac{\bar{p}(0,K,\ell)}{1-\bar{p}(0,K,\ell)} \qquad (5.13)$$

When we go to the limit as t tends to infinity the stationarity density of the spike event is given by

$$\lim_{t\to\infty}\left\{\frac{1}{-\bar{p}'(0,k,\ell)\Big|_{\ell=0}}\right\} = \frac{1}{E(T)} \tag{5.14}$$

6. Excitatory and inhibitory impulses - Model II.

In this section we consider the motion of a particle subjected to random jumps in both the directions and exponential sliding in the downward direction. The position $x(t)$ of the particle will correspond to the cumulative membrane potential gathered by the neuron in time t. This potential decays in between the jumps. The jump in the positive direction corresponds to an excitatory impulse and the jump in the negative direction represents the inhibitory impulse. When the membrane potential crosses the threshold value K the neuron fires and the interval between the successive firings of the neuron is the first passage time to cross the barrier K. After such firing the neuron is reset to its initial value $X=0$. This may be considered as a stationary stochastic process. Let the magnitude of the jumps be governed by the density $a(z)$. The membrane potential cannot go below the zero level. Therefore this random motion can be treated as a process with two barriers one at $X=0$ and the other at $X=K$. For the neuron to fire, we are interested in the study of the first passage problem of the potential $X(t)$ crossing the level K at time t for the first time when there is an artificial barrier at $x=o$.

In section 2 we analysed the motion with only upward jumps and exponential decay. In that case the random variable $X(t)$ once it starts at $X(0) = u>0$ can never reach zero at all in a finite time. Even if it starts at $X(0)=0$ after its first jump away from zero there is no possibility of reaching the zero level in finite time. Therefore the level $X=0$ serves as a natural barrier. In this section we consider also inhibitory impulses which means that $X(t)$ can be reduced by jumps also along with decay. However the probability of $X(t)<0$ is assumed to be zero, for all $t \geq 0$ as a boundary condition. This model is also similar to the study of the first passage time of overflow in a finite dam at a given time after any number of emptiness.

Define $Q(u,K,t)$ as the probability that the particle starting from $X=u$ at time $t=0$ crosses the barrier K for the first time between time t and $t+\Delta$, when $X=0$ is a reflecting barrier. That is $X_n(t)=0$ for $X_{n-1}+Z_n \leq 0$ where $n(t)$ represents the nth jump which occurs at t. Further we decompose $Q(u,K,t)$ as

$$Q(u,K,t) = Q^{+}(u,K,t) + Q^{-}(u,K,t) \tag{6.1}$$

here

$Q^{+}(u,K,t)$ = the probability that the particle starting from $X=u$ at time $t=0$ crosses the barrier $X=K$ for the first time between time t and $t+\Delta$ with the initial jump in the positive direction (upwards) whenever it occurs.

$\bar{Q}(u,K,t)$ = the probability that the particle starting from $X=u$ at time $t=0$ crosses the barrier K for the first time between time t and $t+\Delta$ with the initial jump in the negative direction (downwards) whenever it occurs.

By the imbedding arguments we see that $Q^{+}(u,K,t)$ satisfies:

$$\frac{\partial Q^{+}}{\partial t}+\alpha u\frac{\partial Q^{+}}{\partial u}+\nu Q^{+}$$
$$=\nu\int_{0}^{K-u} a(z)\,Q(u+z,K,t)\,dz+\delta(t)\,\nu\int_{K-u}^{\infty} a(z)\,dz \tag{6.2}$$

Defining $\bar{Q}$, $\bar{Q}^{+}$ as the Laplace transforms of Q and Q+ respectively with respect to the time variable t, the equation (6.2) becomes

$$\alpha u\frac{\partial \bar{Q}^{+}}{\partial u}+(\ell+\nu)\frac{\partial \bar{Q}^{+}}{\partial u}$$
$$=\nu\int_{0}^{K-u} a(z)\,\bar{Q}(u+z,K,\ell)\,dz+\nu\int_{K-u}^{\infty} a(z)\,dz \tag{6.3}$$

Working with the p.d.f. of the jumps $a(Z)$ defined by

$$a(z) = \tfrac{1}{2}\eta\, e^{-\eta z} \quad , \quad z > 0$$

$$= \tfrac{1}{2}\eta\, e^{\eta z} \quad , \quad z < 0 \tag{6.4}$$

We are led to the second order equation:-

$$\alpha u \frac{\partial^2 \bar{Q}^+}{\partial u^2} + \left[\ell + \nu + \alpha - \eta \alpha u\right] \frac{\partial \bar{Q}^+}{\partial u} - \eta(\ell+\nu)\bar{Q}^+$$

$$= -\frac{\nu\eta}{2}\bar{Q} \tag{6.5}$$

By imbedding arguments for $Q^-(u,k,t)$ we arrive at

$$\frac{\partial Q^-}{\partial t} + \alpha u \frac{\partial Q^-}{\partial u} + \nu Q^-$$

$$= \nu \int_{-u}^{0} a(z)\, Q(u+z,k,t)\, dz$$

$$+ \nu \int_{-\infty}^{-u} a(z)\, dz \cdot Q(0,k,t) \tag{6.6}$$

where $Q(0,k,t)$ is the probability of crossing the barrier K starting from $X(0)=0$ at $t=0$. This leads to the second order differential equation for $\bar{Q}^-(u,k,\ell)$:

(6.7)

Eliminating $\bar{Q}^+$ and $\bar{Q}^-$ from equations (6.5) and (6.7) we arrive at the equation for $\bar{Q}$ the total level crossing density commencing with any type of initial jump as:-

$$\alpha u \frac{\partial^3 \bar{Q}}{\partial u^3} + (\ell + \nu + 2\alpha)\frac{\partial^2 \bar{Q}}{\partial u^2} - \eta^2 \alpha u \frac{\partial \bar{Q}}{\partial u} - \eta^2 \ell \bar{Q} = 0 \tag{6.8}$$

The solution of this third order differential equation when $\alpha = 1$ and $\eta = 1$ is given by (Kemke 1959)

$$\bar{Q}(u,K,\ell) = c_1 \int_0^1 t^{\ell-1} (1-t^2)^{\nu/2} e^{tu}\, dt + c_2 \int_0^1 t^{\ell-1} (1-t^2)^{\nu/2} e^{-tu}\, dt + c_3 \int_1^\infty t^{\ell-1} (t^2-1)^{\nu/2} e^{-tu}\, dt \tag{6.9}$$

where C_1, C_2 and C_3 are the three independent constants to be determined by the boundary conditions.

7. <u>Solution for Model II</u>.

In this section by utilising the boundary conditions we determine the arbitrary functions C_1, C_2 and C_3 which are independent of u and hence obtain a closed solution for $\bar{Q}(u,K,\ell)$. The function C_3 can be shown to be zero by studying the behaviour of the three independent solutions in different regions of convergence for ℓ. Let us

specifically be concerned with the problem in which the initial position of the particle is $u=0$. In that case the third independent solution remains finite for values of ℓ such that

$$\ell+\nu<0 \tag{7.1}$$

This means ℓ has to be negative since ν , the poisson intensity is positive. However the other two solutions can be finite only for values of ℓ such that

$$\ell>0 \tag{7.2}$$

since the first two integrals for $u=0$ lead to Beta functions (Gradshteyn and Ryzhik 1965) which can exist only if $\ell>0$ Hence the region of ℓ for hich the first two solutions converge is disparate from the region of ℓ for which the third solution converges. Hence for this initial condition we should choose $C_3=0$. However we will see that the same solution holds both for u equal to zero and u not equal to zero. This can be physically seen since even when we start from u not equal to zero, the potential can come down to zero due to the negative jumps any number of times, before it crosses K.

The first order equation for $\bar{Q}(u,K,\ell)$, if written down taking into account, both types of jumps and decay

will serve as the boundary condition for this problem for any starting initial value of $X(0) = u$. The integro differential equation for $Q(u, K, t)$ is

$$\frac{\partial Q}{\partial t} + \alpha u \frac{\partial Q}{\partial u} + \nu Q = \frac{\nu\eta}{2} \int_0^{K-u} e^{-\eta z} \, Q(u+z, K, t) \, dz$$

$$+ \delta(t) \frac{1}{2} \nu \eta \int_{K-u}^{\infty} e^{-\eta z} \, dz$$

$$+ \frac{1}{2} \nu \eta \int_{-u}^{0} e^{\eta z} \, Q(u+z, K, t) \, dz$$

$$+ \frac{1}{2} \nu \eta \int_{-\infty}^{-u} e^{\eta z} \, dz \;\; Q(0, K, t) \tag{7.3}$$

Taking the Laplace transform with respect to the variable t, the equation (7.3) becomes

$$e^{-\eta u} \left[\alpha u \frac{\partial \bar{Q}}{\partial u} + (\ell + \nu) \bar{Q} \right] = \frac{1}{2} \nu \eta \int_u^K e^{-\eta y} \, \bar{Q}(y, K, \ell) \, dy$$

$$+ \frac{1}{2} \nu \eta \, e^{-2\eta u} \int_0^u e^{\eta y} \, \bar{Q}(y, K, \ell) \, dy$$

$$+ (\nu/2) \, e^{-\eta K} + \frac{1}{2} \nu \, e^{-2\eta u} \, \bar{Q}(0, K, \ell) \tag{7.4}$$

Here again we take $\alpha = 1$ and $\eta = 1$. Substituting the general solution (6.9) in (7.4) and collecting the terms containing e^{-u} to one side equation (7.4) reduces to

$$
\begin{aligned}
e^{-u}\Big\{ & c_1\Big[u\int_0^1 t^{\ell}(1-t^2)^{\nu/2}\, e^{tu}\, dt \\
& \quad +(\ell+\nu)\int_0^1 t^{\ell-1}(1-t^2)^{\nu/2}\, e^{tu}\, dt \\
& \quad -\nu\int_0^1 t^{\ell-1}(1-t^2)^{\frac{\nu}{2}-1}\, e^{tu}\, dt\Big] \\
& + c_2\Big[-u\int_0^1 t^{\ell}(1-t^2)^{\nu/2}\, e^{-tu}\, dt \\
& \quad +(\ell+\nu)\int_0^1 t^{\ell-1}(1-t^2)^{\nu/2}\, e^{-tu}\, dt \\
& \quad -\nu\int_0^1 t^{\ell-1}(1-t^2)^{\frac{\nu}{2}-1}\, e^{tu}\, dt\Big] \\
& + c_3\Big[-u\int_1^{\infty} t^{\ell}(t^2-1)^{\nu/2}\, e^{-tu}\, dt \\
& \quad +(\ell+\nu)\int_1^{\infty} t^{\ell-1}(t^2-1)^{\nu/2}\, e^{-tu}\, dt \\
& \quad -\nu\int_1^{\infty} t^{\ell-1}(t^2-1)^{\frac{\nu}{2}-1}\, e^{-tu}\, dt\Big]\Big\} \\
& +\tfrac{1}{2}\nu\, e^{-2u}\Big[c_1\int_0^1 t^{\ell-1}(1-t^2)^{\frac{\nu}{2}-1}(1-t)\, dt \\
& \quad +c_2\int_0^1 t^{\ell-1}(1-t^2)^{\frac{\nu}{2}-1}(1+t)\, dt \\
& \quad +c_3\int_1^{\infty} t^{\ell-1}(t^2-1)^{\nu/2-1}(1+t)\, dt \\
& \quad -(c_1+c_2)\int_0^1 t^{\ell-1}(1-t^2)^{\nu/2}\, dt - c_3\int_1^{\infty} t^{\ell-1}(t^2-1)^{\frac{\nu}{2}}\, dt\Big]
\end{aligned}
$$

$$= -\frac{\nu}{2}\Big[C_1 \int_0^1 t^{\ell-1}(1-t^2)^{\frac{\nu}{2}-1}(1+t)\,e^{k(t-1)}\,dt$$
$$+C_2 \int_0^1 t^{\ell-1}(1-t^2)^{\frac{\nu}{2}-1}(1-t)\,e^{-k(1+t)}\,dt$$
$$+C_3 \int_1^\infty t^{\ell-1}(t^2-1)^{\frac{\nu}{2}-1}(1-t)\,e^{-k(1+t)}\,dt\Big]$$
$$+\frac{\nu}{2}\,e^{-k} \tag{7.5}$$

The term containing $Q(0,k,t)$ leads to the integral of the type

$$\int_1^\infty t^{\ell-1}(1-t^2)^{\frac{\nu}{2}}\,dt = \beta\left(-\frac{\ell+\nu}{2},\ \frac{\nu}{2}+1\right) \tag{7.6}$$

where $\beta(\cdot\,,\cdot)$ is the Beta function. This term can exist only if $\ell+\nu<0$. Thus we are forced to take C_3 equal to zero and allow C_1 and C_2 since the Laplace transform variable ℓ is taken to be positive. It is easily seen that the coefficients of C_1 and C_2 in L.H.S of equation (7.5) vanish separately. Further we observe that the reduced equation is an identity true for all values of u. Hence we equate to zero the coefficient of the term containing e^{-2u} and these independent of u separately. Thus we arrive at the pair of equations for the functions C_1 and C_2

$$C_1\left[\int_0^1 t^{\ell-1}(1-t^2)^{\nu/2}dt-\int_0^1 t^{\ell-1}(1-t^2)^{\frac{\nu}{2}-1}(1-t)\,dt\right]$$

$$+C_2\left[\int_0^1 t^{\ell-1}(1-t^2)^{\nu/2}dt-\int_0^1 t^{\ell-1}(1-t^2)^{\frac{\nu}{2}-1}(1+t)\,dt\right]$$

$$=0 \tag{7.7}$$

$$C_1\left[\int_0^1 t^{\ell-1}(1-t^2)^{\frac{\nu}{2}-1}(1+t)\,e^{tK}\,dt\right]$$

$$+C_2\left[\int_0^1 t^{\ell-1}(1-t^2)^{\frac{\nu}{2}-1}(1-t)\,e^{-tK}\,dt\right]$$

$$=1 \tag{7.8}$$

Using the standard result (Gradshteyn and Ryzhik 1965)

$$\int_0^1 x^{2m-1}(1-x^2)^{n-1}e^{\mu x}\,dx$$

$$=\tfrac{1}{2}\beta(m,n)\;{}_1F_2\left(m,\tfrac{1}{2},m+n,\mu^2/4\right)$$

$$+\tfrac{\mu}{2}\beta\left(m+\tfrac{1}{2},n\right){}_1F_2\left(m+\tfrac{1}{2},\tfrac{3}{2},m+n+\tfrac{1}{2};\mu^2/4\right) \tag{7.9}$$

and defining

$$X=\int_0^1 t^{\ell-1}(1-t^2)^{\nu/2}dt=\tfrac{1}{2}\beta\left(\tfrac{\ell}{2},\tfrac{\nu}{2}+1\right)$$

$$Y=\int_0^1 t^{\ell-1}(1-t^2)^{\frac{\nu}{2}-1}dt=\tfrac{1}{2}\beta\left(\tfrac{\ell}{2},\tfrac{\nu}{2}\right) \tag{7.10}$$

(7.11)

$$Z = \int_0^1 t^{\ell} (1-t^2)^{\frac{\nu}{2}-1} dt = \frac{1}{2}\beta\left(\frac{\ell+1}{2}, \frac{\nu}{2}\right) \tag{7.12}$$

$$W = \frac{1}{2}\beta\left(\frac{\ell}{2}+1, \frac{\nu}{2}\right) \tag{7.13}$$

$$L = {}_1F_2\left(\frac{\ell}{2}, \frac{1}{2}, \frac{\ell+\nu}{2}, K^2/4\right) \tag{7.14}$$

$$M = {}_1F_2\left(\frac{\ell+1}{2}, \frac{3}{2}, \frac{\ell+\nu+1}{2}, K^2/4\right) \tag{7.15}$$

$$N = {}_1F_2\left(\frac{\ell+1}{2}, \frac{1}{2}, \frac{\ell+\nu+1}{2}, K^2/4\right) \tag{7.16}$$

$$R = {}_1F_2\left(\frac{\ell}{2}+1, \frac{3}{2}, \frac{\ell+\nu}{2}+1, K^2/4\right) \tag{7.17}$$

equation (7.6) and (7.7) become

$$C_1[X-Y+Z] + C_2[X-Y-Z] = 0 \tag{7.18}$$

and

$$C_1[YL + KZM + ZN + KWR] + C_2[YL - KZM - ZN + KWR] = 1 \tag{7.19}$$

Hence the complete solution for $\bar{Q}(u,K,\ell)$ can be given analytically in a close form as:

$$\bar{Q}(u,K,\ell) = \frac{1}{2}(c_1+c_2)\beta\left(\frac{\ell}{2},\frac{\nu}{2}+1\right){}_1F_2\left(\frac{\ell}{2},\frac{1}{2},\frac{\ell+\nu}{2}+1,\frac{u^2}{4}\right)$$

$$+\frac{1}{2}(c_1-c_2)\,u\,\beta\left(\frac{\ell+1}{2},\frac{\nu}{2}+1\right){}_1F_2\left(\frac{\ell+1}{2},\frac{3}{2},\frac{\ell+\nu+3}{2},\frac{u^2}{4}\right) \tag{7.20}$$

In particular if we are interested in the renewal density for the initial position $u=0$

$$\bar{Q}(0,K,\ell) = \frac{1}{2}(c_1+c_2)\,\beta\left(\frac{\ell}{2},\frac{\nu}{2}+1\right) \tag{7.21}$$

On proceeding to the limit as ℓ tends to zero we clearly see that $\bar{Q}(u,K,\ell)$ tends to unity. Here we observe that, as ℓ tends to zero

i) $c_1 = c_2$ (ii) $W = \ell Y/(\ell+\nu) = 0$

ii) $X = (\nu/\ell+\nu)\,Y = Y$

we assume that the barrier K is made to come down and incide with $x=0$, the first jump in the upward direction l result in crossing the barrier. Here we have from the egral equation for

$Q(0, K, t)$, as K tends to zero,

$$\bar{Q}(0,0,\ell) = \frac{\nu}{2(\ell+\nu)} + \frac{\nu}{2(\ell+\nu)}\bar{Q}(0,0,\ell) \tag{7.22}$$

$$\bar{Q}(0,0,\ell) = \nu / 2\ell+\nu \tag{7.23}$$

which is also the result we get from (7.21) when we go to the limit as K tends to zero.

Then as detailed in Model I the Laplace transform of $h(K,t)$ the product density of spike at any time t, can be found from $\bar{Q}(0,K,\ell)$. The stationarity value of $h(K,t)$ as t tends to infinity is given by

$$\lim_{t\to\infty} h(K,t) = 1/E(T) \tag{7.24}$$

where

$$E(T) = -\lim_{l\to 0} \frac{d}{dl}\bar{Q}(0,K,l) \tag{7.25}$$

$E(T)$ is the expectation of the first passage time for crossing the threshold.

In conclusion, we point out that we have carried out the analysis for the first passage time probability density for the neuronal firings in two cases. (a) Only excitatory poisson inputs and with exponential p.d.f. for the magnitude of the potential gathered at each spoch. (b) Both excitatory and inhibitory impulses occuring as poisson events, transfering membrane potentials, governed by the exponential p.d.f. In both the cases the membrane potential decayed with time between the poisson epochs, in an exponential manner. The firing threshold K(constant), and the resting potential zero, formed the two barriers of the problem. Analytical expressions for the transform of the p.d.f. of the first passage times to cross the barrier in both the cases, have been obtained by imbedding techniques. The mean time for crossing and the stationarity density of the neuronal firing at any time is also found in both the models. To make the model more realistic, the case of the moving threshold is being studied and will be presented in a later chapter.

CHAPTER - III

TIME DEPENDENT BARRIERS AND FIRST PASSAGE TIMES - DIFFERENT TYPES OF NEURONAL MODELS*

1. Introduction.

In the previous Chapter we have considered spike trains in single neurons by assuming two types of random models in which excitatory and inhibitory jumps occur in a poisson fashion. The neuron is taken to be a leaky integrator. This corresponds to an additive stochastic process with a constant barrier and an exponential or linear deterministic decay. Closed expressions for first passage probability density were arrived at. In the present Chapter we will be concerned with the barrier problems where the barriers are moving with deterministic drift. Also the process may be subjected to deterministic motion either exponential or linear. It is generally accepted that the neuronal spike trains can be taken as stationary stochastic process and that renewal theory can be used. For an elaborate description of stationary processes and renewal theory the reader is referred to Ramakrishnan (1959) and Cox (1962).

The modelling of neuronal spikes can be analysed from the point of view of first passage problems considered in detail by Ramakrishnan (1959) and Cox and Miller (1965). The membrane potential gets moved up or down in a random fashion by impulses arriving in a Poisson manner. When the potential

*Based on the paper by R.Vasudevan and P.R.Vittal to be submitted for publication.

crosses the threshold value the neuron fires. Once the neuron fires the membrane potential returns to its resting value and the interval distribution between two successive firings is the one we are seeking for. Very good descriptions of mathematical models in this connection have been elaborated in Holden (1976), Ricciardi (1977) and Sampath and Srinivasan(1977).

Homogeneous additive processes on a semi infinite or finite intervals have been studied in many forms. Walds' identity for the first passage processes (Wald 1947, Cox and Miller 1965) have been obtained in a variety of situations like in the theory of queues, dams (Lindlay 1952, Miller 1961). Wiener-Hopf methods also have been used to study these problems by Smith (1953), Kemperman (1961), Keilson (1961, 1962), Borovkov (1970).

The compensation functions developed for barrier problems (Keilson 1962, 1963) have been found to be extraordinarily useful. The philosophy of the method is to arrive at the first passage probabilities in terms of the Green's function for an unbounded process. However this method has been used to obtain explicit result only in the case of a barrier at $x = 0$. All the attempts have been made to cross or reach a barrier at $x=0$. In an earlier attempt Vasudevan et. al. (1979) have arrived at the compensation function for the barrier at $x=0$ and another at $x = k$ for jump process. In our previous chapter (Vasudevan et. al. 1980) we have used the imbedding techniques (Bellman and Wing 1976) to write down the equations for the

first passage probabilities and final solutions for exponential jump process were related to the results obtained by compensation function method (Vasudevan et. al. 1978, 1979).

Starting with a jump process and writing down the first passage equation one can go to the diffusion equation of either the Wiener process or the O.U. type process by a limiting procedure used by Keilson (1963). This means that one can solve the diffusion equation with boundary conditions starting from a discrete jump process, obtaining the compensation functions and then going to the limit valid for the diffusion equation. The boundary may be a moving boundary for the diffusion process. In this case it is found quite useful to find the first passage densities or their transforms for the jump process and go to the diffusion limit later. In this chapter the moving boundary is taken to simulate the refractory period for the neuron after each firing. Many models have been constructed in which the threshold is a function of time. Linear decay of threshold has been proposed by Calvin and Stevens (1965). Weiss (1964) used exponentially decaying thresholds. Other types of threshold functions exponentially decreasing in time starting from a high value have been considered by Fuortes and Mantegazzini (1962) and Hagiwara (1954) to account for experimental findings. Some of the results for these models are in agreement with experimental results of Goldberg, Adrian and Smith (1964). A number of reviews of this problem for first passage density for the diffusion equation with constant boundaries exist in

the literature (Sugiyama et. al. 1970, Siegert 1951, Ricciardi 1977). A good description of this theme is given in the reference (Holden 1976, Ricciardi 1977). For time dependent barrier cases suitable transformations of time and space variables have been used to obtain a Wiener equation with constant barrier (Ricciardi 1977). Also Clay and Goel (1973) have treated varying thresholds for the diffusion model and arrived at Wald identity for the first passage problem. For discrete jump process there is not much existing literature except that of Kryukov (1976) for the exponentially moving barrier.

In this chapter we consider moving boundaries for jump processes and arrive at the solution for the diffusion models in a straight forward fashion. Corresponding Wald identities are also obtained when we go to the diffusion limit.

In a realistic approach the input impulses should not really be independent and identically distributed exponential functions as described in these models. According to Fuortes and Mantegazzini (1962) and Tuckwell (1976) the excitatory impulses arriving at any time should really depend upon the existing sub-threshold potentials gathered till that time by the neuron. This is expected to simulate local and short term facilitation. This has been treated in this Chapter with constant boundary and boundaries with linear motion. Also thresholds varying with time in an exponential manner of different types have been analysed. This is accomplished by using a change

of space variables. Solutions have been obtained for discrete models in terms of unbounded free Green's functions which can be obtained very easily. Conversion to the original variables is straight forward. From these results one goes to the diffusion equation when we apply the usual limiting procedure described herein. We are not aware of the treatment of such problems in the existing literature.

In Section 2 we obtain the first passage density for a constant barrier when the motion of the discrete jump process $X(t)$ in subthreshold region has a deterministic linear drift. We go to the diffusion equation to obtain the first passage density automatically by a limiting procedure. This can be achieved for a two way jump process when the forward jump probabilities are known to be separable. The negative jump process can be of any type. In Section 3, we treat a moving barrier and obtain the first passage density for the jump process as well as for a process described by a suitable diffusion equation. In Section 4, we obtain the Wald identities by physical arguments for the jump process. We also go to the diffusion limit and obtain the results of Clay and Goyal (1973). We can obtain these identities for either reaching or crossing the barrier both for discrete jump process and their diffusion analogues. Exponentially moving barriers have been treated in Sections 5 & 6 both for crossing and reaching. In Section 7 the new type of jump probabilities depending on the existing

potential have been treated with exponentially moving barrier of two different types suggested by experimentalists. Results in the diffusion limit are obtained. Closed expressions for the mean passage times are conveniently arrived at.

2. First passage for constant barrier and diffusion approximation.

To investigate the first passage time process let us consider the well treated problem of the motion of a particle on a real line in continuous time. Let the particle be moving with random jumps and linear drift. We shall study the first passage time to cross the constant barrier $X = k$ in a finite time. Let the particle start from $X = X_0$ at time t=o. We assume that the random jumps occur with poisson frequency ν and are governed by the density function a(z). Keilson (1963) studied this problem using the compensation technique. For him the barrier was the origin $X = 0$.

We will compute $M(x_0, K, t)$, the first passage time density to cross the barrier at $x = k$ between time t and $t + dt$. This passage density will be obtained in terms of the Green's function of the unrestricted process using the compensation method. In the next section this result will be used to obtain for the first passage density for crossing the boundary moving linearly.

Let $X(t)$ be the cumulative poisson process (Keilson 1965) defined by

$$X(t) = X_0 - \alpha t + \sum_{i=1}^{N(t)} Z_i \tag{2.1}$$

where N(t) is the number of Poisson jumps occurring with intensity ν in the interval $(0,t)$, Z_i are independent and identically distributed random variables with a common density $a(z_i)$ representing the size of the ith jump and α is the linear downward drift parameter.

Define $M(x_0,k,t)$ = the probability that the particle crosses the barrier $x = k$ for the first time between time t and $t+dt$ and $f(x,t/x_0)$ the transition probability that the particle is between x and $x+\Delta x$ at time t given that it was at $x = x_0$ at time $t=0$.

Denoting $f(x,t/x_0)$ as $f(x,t)$ the forward equation is:

$$\frac{\partial f(x,t)}{\partial t} - \alpha \frac{\partial f(x,t)}{\partial x} + \nu f(x,t) = \nu \int_{-\infty}^{\infty} f(x',t)\, a(x-x')\, dx'$$

$$\text{With } f(x,0) = \delta(x-x_0) \tag{2.2}$$

The transition function $f(x,t)$ is easily found if there is no boundary. To give effect to the boundary condition at $x = k$ we shall introduce the compensation function into the differential equation (2.2). The equation can now be treated as an equation with free boundaries. Hence we write

$$\frac{\partial f}{\partial t} - \alpha \frac{\partial f}{\partial x} + \nu f$$
$$= \nu \int_{-\infty}^{\infty} f(x',t)\, a(x-x')\, dx' + C(x,t) \tag{2.3}$$

The compensation function $C(x,t)$ is given by

$$C(x,t) = -\nu H(x-k) \int_{-\infty}^{k} f(x',t)\, a_{+}(x-x')\, dx' \tag{2.4}$$

where $H(x)$ is the Heaviside function defined by

$$H(x) = 1 \quad \text{for} \quad x > 0$$
$$= 0 \quad \text{otherwise} \tag{2.5}$$

and $a_{+}(x)$ is the p.d.f. for the positive jumps. With the introduction of the compensation function for the source term in equation (2.3) which gives a boundary current it can be seen that $f(x,t)$ for values of $x \geq k$ (outside the barrier) becomes zero for all time t if the particle has started from x_0 which is inside the boundary K. We will now demonstrate that the evaluation of the function $C(x,t)$ will result in the determination of $M(x_0, K, t)$.

We take the double sided laplace transform with respect to the variable x and one sided transform with respect to the variable t, and we define:

$$\bar{f}(s,\ell) = \int_{-\infty}^{\infty} dx \int_{0}^{\infty} e^{sx} e^{-\ell t} f(x,t)\, dt \tag{2.6}$$

$$\bar{c}(s,\ell) = \int_{-\infty}^{\infty} dx \int_{0}^{\infty} e^{sx} e^{-\ell t} c(x,t)\, dt \tag{2.7}$$

$$\bar{a}(s) = \int_{-\infty}^{\infty} e^{sx} a(x)\, dx \tag{2.8}$$

Hence from (2.3)

$$\left[\ell + \alpha s + \nu(1-\bar{a}(s))\right] \bar{f}(s,\ell) = e^{sx_0} + \bar{c}(s,\ell) \tag{2.9}$$

If σ_ℓ is a root of the equation

$$\ell + \alpha s + \nu\left(1 - \bar{a}(s)\right) = 0 \tag{2.10}$$

we have

$$\bar{c}(\sigma_\ell, \ell) = -e^{\sigma_\ell x_0} \tag{2.11}$$

It is evident from the definition of $c(x,t)$ and $M(t) = M(x_0, k, t)$ that

$$M(t) = -\int_{k}^{\infty} c(x,t)\, dx \tag{2.12}$$

Taking Laplace transform with respect to the variable t we easily see that

$$\bar{C}(0,\ell) = -\bar{M}(\ell) \tag{2.13}$$

Hence putting $s=0$ in (2.9) we arrive at

$$\ell\,\bar{f}(0,\ell) = 1 - \bar{M}(\ell) \tag{2.14}$$

$$\text{i.e,} \quad M(t) = -\frac{d}{dt}\int_{-\infty}^{k} f(x,t)\,dx \tag{2.15}$$

where $f(x,t)$ is the solution for the equation (2.3). We will now see that by choosing the density

$$a(z) = a_{+}(z) + a_{-}(z) \tag{2.16}$$

where

$$a_{+}(z-z') = \tilde{a}_1(z)\,\tilde{a}_2(z')\,H(z-z') \tag{2.17}$$

and $a_{-}(z)$ any arbitrary density, M(t) can be then expressed in terms of the Green's function $g(x,t)$ of the unbounded process.

From the definition of the compensation function we have

$$M(t) = -\int_{K}^{\infty} C(x,t)\,dt = \hat{g}(K)\,S(K,t) \tag{2.18}$$

where

$$S(K,t) = -\nu \int_{-\infty}^{K} f(x',t)\,\tilde{a}_2(x')\,dx' \tag{2.19}$$

and

$$\hat{g}(K) = \int_{K}^{\infty} \tilde{a}_1(x)\,dx \tag{2.20}$$

If we take

$$\bar{a}_+(z) = H(z)\,\lambda\,e^{-\eta z} \tag{2.21}$$

we obtain

$$M(t) = -(\lambda/\eta)\,e^{-\eta K}\,S(K,t) \tag{2.22}$$

$$C_1(s,t) = (\lambda/\eta - s)\,e^{-\eta K}\,e^{sK}\,S(K,t) \tag{2.23}$$

where $C_1(s,t)$ is the L.T. of $C(x,t)$ with respect to x only. Hence we have

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$$C_1(s,t) = -(\eta/\eta-s)\, e^{sK}\, M(t) \tag{2.24}$$

Taking Laplace transform with respect to t,

$$\bar{c}(s,\ell) = -(\eta/\eta-s)\, e^{sK}\, \bar{M}(\ell) \tag{2.25}$$

or

$$\bar{M}(\ell) = -((\eta-s)/\eta)\, e^{-sK}\, \bar{c}(s,\ell) \tag{2.26}$$

$$= \left[1-\sigma_\ell/\eta\right] e^{\sigma_\ell (x_0-K)} \tag{2.27}$$

in view of equation (2.11), where σ_ℓ is a root of the equation (2.10).

We will now show that $e^{\sigma_\ell(x_0-K)}$ can be expressed in terms of the Green's function $\xi(x,t)$ of the unbounded process. Denoting $\bar{\xi}(s,\ell)$ as the double transform of $\xi(x,t)$ one can easily see that

$$\bar{\xi}(s,\ell) = 1\Big/\left\{\ell+\alpha s+\nu(1-\bar{a}(s))\right\} \tag{2.28}$$

Using the standard technique of inversion we have

$$\xi_1(x,\ell) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{e^{-sx}\Big/\left[\ell+\alpha s+\nu(1-\bar{a}s)\right]\right\}ds \tag{2.29}$$

It has been pointed out by Keilson (1963) that there is only one root $s=\sigma_\ell$, being positive in the upper half plane for the equations (2.10) for $\mathrm{Re}\, s>0$

$$\xi(x,\ell) = -e^{-\sigma_\ell x} \frac{d\sigma_\ell}{d\ell} \tag{2.30}$$

or

$$\xi_1(-x_0,\ell) = -e^{\sigma_\ell x_0} \frac{d\sigma_\ell}{d\ell} = -\frac{1}{x_0}\frac{d}{d\ell} e^{\sigma_\ell x_0} \tag{2.31}$$

From the standard formula for inversion we get

$$\mathcal{L}\left\{ \frac{|x|}{t}\, \xi(-x,t)\right\} = e^{\sigma_\ell x} \tag{2.32}$$

The term $|x|$ appears since RHS is positive for all x. Hence inverting (2.27) we get

$$M(t) = \left(1-\frac{1}{\eta}\frac{d}{dx_0}\right)\frac{|x_0-K|}{t}\,\xi_\alpha(K-x_0,t) \tag{2.33}$$

where $\xi_\alpha(x,t)$ is the Green's function of the unbounded negative drift process. If $\xi_0(x,t)$ is the Green's function of the process without any drift,

$$\xi_\alpha(x,t) = \xi_0(x+\alpha t, t) \tag{2.34}$$

Hence

$$M(t) = \left(1 - \frac{1}{\eta}\frac{d}{dx_0}\right)\frac{|x_0 - K|}{t}\, g_0(K - x_0 + \alpha t, t) \tag{2.35}$$

When $K=0$ (2.35) reduces to Keilson's result for the barrier at x=o.

We shall now deduce the Cox's result for the first passage time for a pure jump process without drift. Let the jumps be governed by a(z) such that

$$\begin{aligned} a_+(z) &= \eta\, e^{-\eta z}\, H(z) \\ a_-(z) &= 0 \end{aligned} \tag{2.36}$$

The Green's function of the process is

$$g(x,t) = e^{-\nu t}\,\delta(x) + \sum_{n=1}^{\infty} e^{-\nu t}\,\frac{(\nu t)^n}{n!}\, a^n(x) \tag{2.37}$$

where $a^n(x)$ is the n fold convolution of a(x) with itself. It can be easily shown that for a(x) chosen above

$$a^n(x) = \eta^n\, x^{n-1}\, e^{-\eta x}\,\frac{1}{(n-1)!}\, H(x) \tag{2.38}$$

$$g(x,t) = e^{-\nu t}\,\delta(x) + \left\{\sum_{n=1}^{\infty} e^{-\nu t}\,\frac{(\nu t)^n}{n!}\,\frac{\eta^n x^{n-1}}{(n-1)!}\, e^{-\eta x}\right\} H(x) \tag{2.39}$$

Using the power series expansion for modified Bessel function

$$I_n(z) = (z/2)^n \sum_{r=0}^{\infty} (z^2/4)^r / r!\,(n+r)! \quad , n>0 \tag{2.40}$$

the Green's functions can be expressed in the form

$$g(x,t) = e^{-\nu t}\,\delta(x) + \eta\nu t\, e^{-\nu t}\, e^{-\eta x}\, \frac{I_1(2\sqrt{\eta\nu t x})}{\sqrt{\eta\nu t x}} \tag{2.41}$$

Since $|x_0 - K|$ is not zero when we calculate $M(t)$ we can drop the first delta function term and get

$$M(t) = \left[1 - \frac{1}{\eta}\frac{d}{dx_0}\right] \frac{|x_0 - K|}{t}\, \eta\nu t\, e^{-\nu t}\, e^{-\eta(K-x_0)}$$

$$\times \left\{ I_1\left(2\sqrt{\eta\nu t(K-x_0)}\right) \Big/ \sqrt{\eta\nu t(K-x_0)} \right\} \tag{2.42}$$

Let us go to the variable y defined by

$$\sqrt{y} = 2\sqrt{\eta\nu t\,(K-x_0)} \tag{2.43}$$

Using the standard property of the Bessel function

$$\frac{d}{dy}\left(y\, I_1(y)\right) = y\, I_0(y) \tag{2.44}$$

we find that

$$M(t) = \nu e^{-\nu t} e^{-\eta(K-x_0)} I_0\left(2\sqrt{\eta\nu t(K-x_0)}\right) \tag{2.45}$$

If the process starts from $x=0$ we have

$$M(t) = \nu e^{-\nu t} e^{-\eta K} I_0\left(2\sqrt{\eta\nu t K}\right) \tag{2.46}$$

This agrees with the Cox's result (equation (12), page 99,1962).

We shall now establish that the first passage time transform $\overline{M}(\ell)$ can be determined in a more general way by taking the density of the jumps $a_+(x,x')$ in a separable form (i.e.):

$$a_+(x,x') = \tilde{a}_1(x)\,\tilde{a}_2(x') \tag{2.47}$$

In this case,

$$M(t) = -h(K)\,S(K,t) \tag{2.48}$$

where

$$h(K) = \int_K^{\infty} \tilde{a}_1(x)\,dx \tag{2.49}$$

$$S(K,t) = -\nu\int_{-\infty}^{K} f(x',t)\,\tilde{a}_2(x')\,dx' \tag{2.50}$$

We also find

$$\bar{c}(s,\ell) = a_1(K,s)\, s(K,\ell) \tag{2.51}$$

where

$$a_1(K,s) = \int_K^\infty e^{sx}\, \tilde{a}_1(x)\, dx \tag{2.52}$$

We have found in equation (2.11)

$$\bar{c}(s,\ell) = -e^{s x_0} \tag{2.53}$$

where s is chosen to be a root of (2.10). Evidently from equation (2.9) it can be inferred that s and ℓ have to be related so that we write

$$\bar{M}(\ell) = -\frac{h(K)}{a_1(K, s(\ell))}\, \bar{c}(s(\ell),\ell) \tag{2.54}$$

$$= \frac{h(K)}{a_1(K,\sigma_\ell)}\, e^{\sigma_\ell x_0} \tag{2.55}$$

Let us now take the original equation for the process with free boundaries

$$\frac{\partial f}{\partial t} + \nu f = \nu \int_{-\infty}^{\infty} f(x',t)\, a(x-x')\, dx' \tag{2.56}$$

Here we have just ignored the drift term for the sake of simplicity. Taking laplace transform with respect to x,

$$\frac{\partial f_1(s,t)}{\partial t} = -\nu \left[1 - \bar{a}(s) \right] f_1(s,t) \tag{2.57}$$

Let us now go to the diffusion limit of this equation by the following method. Let the density a(x) be changed to $\gamma\, a(\gamma x)$ whose L.T. is given by $\bar{a}(s/\gamma)$. Let the frequency ν be taken as $\gamma^2 \nu$. Also we impose the condition that the mean of the increament jumps goes to zero i.e. $\int_{-\infty}^{\infty} y\, a(y) = 0$. This is guaranteed since we have jumps on both the directions. Another important condition for the limiting process is given by

$$\nu \int_{-\infty}^{\infty} y^2\, a(y)\, dy = D \text{ , a constant}$$

In the limit as $\gamma \to \infty$ we can easily see that the jump process goes over to the diffusion process governed by

$$\frac{\partial}{\partial t} f_1(s,t) = \frac{D s^2}{2} f_1(s,t) \tag{2.58}$$

when we put $s = i r$ (r being the transform variable) we have the solution

$$f(r,t) = \exp\left[-\tfrac{1}{2} r^2 D t\right] \tag{2.59}$$

Equation (2.58) on conversion goes to the diffusion equation

$$\frac{\partial}{\partial t} f(x,t) = \frac{1}{2} D \frac{\partial^2 f(x,t)}{\partial x^2} \tag{2.60}$$

whose solution is

$$f(x,t) = (2\pi D t)^{-\frac{1}{2}} \exp\left[-x^2/2Dt\right] \tag{2.61}$$

which is the well known Wiener p.d.f. for the diffusion process with no drift and no barrier.

We shall now adopt the same limiting procedure to the first passage density $M(t)$ to show that in the limit it goes to the first passage density for reaching the barrier at $X = k$, for the Wiener process with zero drift. The function $\tilde{a}_1(x)$ in (2.47) is changed into $\gamma\, \tilde{a}_1(\gamma x)$. To arrive at the limiting form for equation (2.54) we consider the function with the changes incorporated as

$$\frac{h(k)}{a_1(k, s(\ell))} = \frac{\int_{k\gamma}^{\infty} \tilde{a}_1(y)\, dy}{\int_{k\gamma}^{\infty} \exp(sy/\gamma)\, \tilde{a}_1(y)\, dy} \tag{2.62}$$

Proceeding to the limit as $\gamma \to \infty$ and using the L' Hospital rule we find that

$$\lim_{\gamma \to \infty} \frac{h(k)}{a_1(k, s(\ell))} = e^{-s(\ell)k} \tag{2.63}$$

In this limit

$$\overline{M}(\ell) = e^{-s(\ell)k} \, e^{s(\ell)x_0} \tag{2.64}$$

where $s(\ell)$ is the solution of the equation (2.10) when $\alpha = 0$ i.e.

$$\overline{M}(\ell) = e^{\sigma_\ell (x_0 - k)} \tag{2.65}$$

The inversion of $\overline{M}(\ell)$ is

$$M(t) = \frac{|x_0 - k|}{t} \, g(k - x_0, t) \tag{2.66}$$

The Green's function $g(x,t)$ in equation (2.66) is the solution of the Wiener equation which is the diffusion limit of the equation (2.3) with $\alpha = 0$. Here we want to point that if we have started with the equation (2.3) we obtain for a separable form of $a_+(x)$ the solution for $\overline{M}(\ell)$ given

by the equation (2.27) for crossing the barrier. Of course this has been obtained for a particular form for the separable function $a_+(x)$ taken therein. When we go to the diffusion limit the term containing $1/\eta$ in the R.H.S. of equation (2.27) vanishes leaving the solution in the limit as

$$\overline{M}(\ell) = \exp\left[\sigma_\ell(x_0 - k)\right]$$

and this result before going to the limit is the first passage time transform for reaching the barrier $x = k$ when the <u>positive direction is skip free.</u> To obtain the diffusion limit f.p.d. we have not taken any form for a_+ but have only assumed that $a_+(Z-Z')$ is separable and that the moments of the total jump density $a(Z)$ have stipulated properties. Thus for a diffusion process with a boundary at $X = K$, the first passage time is given by

$$M_d(t) = \frac{|x_0 - k|}{t} \frac{1}{\sqrt{2\pi D t}} e^{-\frac{(k-x_0)^2}{2Dt}} \tag{2.67}$$

$$= \frac{|x_0 - k|}{\sqrt{2\pi D}} t^{-3/2} e^{-\frac{(k-x_0)^2}{2Dt}} \tag{2.68}$$

Hence we have solved the diffusion equation with a boundary at $x = k$ starting from the jump process $X(t)$ and arriving

at the usual Wald identity for such processes.

3. <u>First passage for crossing a linear barrier</u>.

In many situations the barrier may not be stationary. For example in the Stefan's problem, the boundaries of the diffusion equation may be moving in time due to changes in the thickness of the ice layers (Bellman , 1976). Also in the neuronal case the phenomenon of refractory period can be simulated by a moving threshold boundary which may be very high just after a firing. Then it may move down to a lower value with a time constant. These have been analysed by Ricciardi (1977), Clay and Goyal (1973) etc. Let us take the boundary to be at $X = k(t)$ where $k(t) = a + bt$. For processes with only jumps we have $X(t)$ described by

$$X(t) = x_0 + \sum_{i=1}^{N(t)} Z_i \tag{3.1}$$

and $X(0) = x_0$. The equation (3.1) has the same meaning as equation (2.1) except that $\alpha = 0$.

The forward equation for the transition density $f(x,t)$ is

$$\frac{\partial f}{\partial t} + \nu f = \nu \int_{-\infty}^{\infty} f(x',t)\, a(x-x')\, dx'$$

$$\text{with } f(x,0) = \delta(x - x_0) \tag{3.2}$$

We will now restrict the path of the particle on the real line to the part of the real line $X(t) \leq k(t)$. The bounded process can be described by an equation similar to (2.3) with the inclusion of compensation function $C(x,t)$ to take care of the boundary effects.

$$\frac{\partial f}{\partial t} + \nu f = \nu \int_{-\infty}^{\infty} f(x',t)\, a(x-x')\, dx' + C(x,t) \tag{3.3}$$

where

$$C(x,t) = -\nu H(x-(a+bt)) \int_{-\infty}^{\infty} f(x',t)\, a_+(x-x')\, dx' \tag{3.4}$$

Here again in order to express $a_+(x-x')$ in a separable form we take

$$a_+(x) = \lambda \exp(-\eta x) \quad H(x) \tag{3.5}$$

$a_-(x)$ may have any arbitrary form. Then we find

$$\begin{aligned} C(x,t) &= -\nu H(x-(a+bt))\, \lambda e^{-\eta x} \\ &\quad \times \int_{-\infty}^{a+bt} f(x',t)\, e^{\eta x'}\, dx' \\ &= H(x-(a+bt))\, \lambda \exp(-\eta x)\, S(t) \end{aligned} \tag{3.6}$$

Taking L.T. with respect to x, (3.6) becomes

$$C_1(s,t) = (\lambda/\eta-s)\, e^{-(\eta-s)(a+bt)} \quad S(t) \tag{3.7}$$

where $C_1(s,t)$ is the L.T. of $C(x,t)$ with respect to x. Defining $M(t) = M(x_0, k(t), t)$ the probability that the particle crosses the barrier $k(t) = a+bt$ for the first time between time t and $t+dt$,

$$M(t) = -\int_{a+bt}^{\infty} C(x,t)\, dx \tag{3.8}$$

$$= -(\lambda/\eta)\, e^{-\eta(a+bt)} \quad S(t) \tag{3.9}$$

Substituting (3.9) in (3.7) we get

$$C_1(s,t) = -(\eta/\eta-s)\, e^{s(a+bt)} \quad M(t) \tag{3.10}$$

This on taking laplace transform with respect to t, becomes

$$\bar{C}(s,\ell) = -(\eta/\eta-s)\, e^{sa} \quad M(\ell-bs) \tag{3.11}$$

Hence

Hence

$$M(\ell - bs) = -\left[(\eta - s)/\eta\right] \bar{c}(s, \ell) \tag{3.12}$$

Let us look at the equation

$$\bar{f}(s,\ell)\left\{\ell + \nu\left[1 - \bar{a}(s)\right]\right\} = e^{sx_0} + \bar{c}(s,\ell) \tag{3.12a}$$

Putting $\ell = \ell' + bs$ this equation becomes

$$\bar{f}(s,\ell')\left[\ell' + bs + \nu\left[1 - \bar{a}(s)\right]\right] = e^{sx_0} + \bar{c}(s,\ell') \tag{3.13}$$

For this equation (3.13) we have

$$\bar{c}(\sigma_{\ell'}, \ell') = -e^{\sigma_{\ell'} x_0} \tag{3.14}$$

where $\sigma_{\ell'}$ is the root of the equation

$$\ell' + bs + \nu\left[1 - \bar{a}(s)\right] = 0 \tag{3.15}$$

Hence

$$\bar{M}(\ell') = \left[1 - \sigma_{\ell'}/\eta\right] e^{\sigma_{\ell'}(x_0 - k)} \tag{3.16}$$

The root $\sigma_{\ell'}$ now refers to an equation for the process x(t)

with a deterministic drift b downwards per unit time. Hence the Green's function which is to be arrived at by inverting $e^{\sigma_{\ell'}(x_0-k)}$ will refer to the Green's function of the unbounded process defined by the equation

$$\bar{f}(s,\ell)\left\{\ell+bs+\nu\left(1-\bar{a}(s)\right)\right\}=0 \tag{3.17}$$

For this process the motion is governed by random jumps in either directions and a linear drift downwards with a rate b per unit time. Hence the first passage density is

$$M(t)=\left[1-\frac{1}{\eta}\frac{d}{dx_0}\right]\frac{|x_0-k|}{t}\,\xi_b(k-x_0,t) \tag{3.18}$$

where ξ_b is the Green's function of the unbounded process with a linear drift b downwards as described by the equation

$$\frac{\partial f}{\partial t}-b\frac{\partial f}{\partial x}+\nu f=\nu\int_{-\infty}^{\infty}f(x',t)\,a(x-x')\,dx'$$

$$\text{with } f(x,0)=\delta(x) \tag{3.19}$$

The L.T. of (3.19) is

$$\frac{\partial}{\partial t}f_1(s,t)=-\left[bs+\nu\left(1-\bar{a}(s)\right)\right]f_1(s,t) \tag{3.20}$$

This when we go to the diffusion limit as described earlier yields the solution (when $s = ir$)

$$f(r,t) = e^{-(irb + Dr^2/2)t} \tag{3.21}$$

which when inverted gives the Green's function

$$\xi(x,t) = (2\pi Dt)^{-1/2} e^{-(x+bt)^2/2Dt} \tag{3.22}$$

and the corresponding diffusion equation is

$$\frac{\partial f}{\partial t} = b\frac{\partial f}{\partial x} + \frac{D}{2}\frac{\partial^2 f}{\partial x^2} \tag{3.23}$$

Thus in this case by adopting the limiting procedure we have gone from the jump process to the diffusion limit with the only condition that $a_+(x-x')$ in general can be of a separable form. We have arrived at the first passage density for the diffusion process as

$$M_D(t) = \frac{|x_0 - k|}{\sqrt{2\pi D}} t^{-3/2} \exp\left[-\frac{(a+bt-x_0)^2}{2Dt}\right] \tag{3.24}$$

This means that starting from a jump process we have arrived at the first passage time density of a diffusion process with

a moving barrier with linear dependence on time. This result agrees with the result of Ricciardi (equation 2.19 page 70,1977).

If originally we have started with a drift process with drift rate α downward and the barrier moving up we can easily see that the solution corresponds to the case when the drift rate is $\alpha + b$ per unit time with a constant barrier at $X = a$. This remark applies also to the diffusion limit situation described by the Wiener equation with a drift and a moving boundary.

4. <u>First passage time for reaching a linear barrier.</u>

In this section let us consider, as in the previous cases, a jump process with a deterministic drift in either upward or downward direction. Let the jumps be confined only to the direction opposite to that of the drift. We examine the stochastic process

$$X(t) = x_0 + \alpha t - \sum_{i=1}^{N(t)} Z_i \tag{4.1}$$

The above equation means that the particle moves up linearly and the downward jumps are the random variables of value Z_i governed by a density function a(z). They are considered as independent and identically distributed random variables and the jumps occurrin a Poissonian way with intensity ν .
If $f(x,t)$ is the transition density of the process then the

forward equation of motion is

$$\frac{\partial f}{\partial t} + \alpha \frac{\partial f}{\partial x} + \nu f = \nu \int_{-\infty}^{+\infty} f(x',t)\, a_-(x-x')\, dx'$$

$$\text{With} \quad f(x,0) = \delta(x-x_0) \tag{4.2}$$

and a_- represents the density for jumps in the downward direction with $a_+(x) = 0$ for $x > 0$. Since there is a drift in the positive direction the barrier will be exactly reached at some time t. We shall find the probability density function $M(x_0, K, t)$ for reaching the barrier. x_0 is the position of the particle at time $t = 0$ and $X = k$ is the barrier. If the barrier is moving and is given by $k(t) = a + bt$ we can find the first passage density $M(t)$ for this time dependent barrier, following an analysis similar to the one adopted in the previous section for the case of crossing the barrier. In the present case the barrier is an absorbing barrier for our process. We shall introduce the compensation function so that the bounded process can be treated as an unbounded one with a source term at the barrier. Hence we write

$$\frac{\partial f}{\partial t} + \alpha \frac{\partial f}{\partial x} + \nu f = \nu \int_{-\infty}^{+\infty} f(x',t)\, a_-(x-x')\, dx' + c(t)\, \delta(x-(a+bt)) \tag{4.3}$$

The above equation can now be treated with free boundary conditions for the unrestricted process defined by (4.1). However if the last term is absent and if there is no boundary the solution is the usual Green's function of the process with drift up and jumps down. Taking L.T. with respect to x, (4.3) becomes

$$\frac{\partial f_1(s,t)}{\partial t} - \alpha s f_1(s,t) + \nu f_1(s,t)$$

$$= \nu f_1(s,t)\,\bar{a}_+(s) - c(t)\exp[s(a+bt)] \tag{4.4}$$

where $f_1(s,t)$ is the L.T. of $f(x,t)$ with reference to x. Again taking L.T. with respect to the variable t,

$$\bar{f}(s,\ell)\left[\ell - \alpha s + \nu(1-\bar{a}_-(s))\right] = e^{sx_0} + \bar{c}(\ell - bs) \tag{4.5}$$

Changing the variable ℓ to ℓ' by the relation $\ell' = \ell - bs$ we get

$$\bar{f}(s,\ell)\left[\ell' - (\alpha - b)s + \nu(1-\bar{a}_-(s))\right]$$

$$= e^{sx_0} + \bar{c}(\ell') \tag{4.6}$$

One can easily identify $f(x,t)$ as the transition density for a process with the rate of drift per unit time as $\alpha - b$ in

a direction determined by the relative values of α and b. $C(t)$ is the compensation function for the process. Hence we observe that the first passage time to the process with drift parameter α to reach the barrier $a+bt$ is equivalent to the first passage time for the process with drift parameter $\alpha-b$ to reach the constant barrier $x=a$. Hence if $s=\sigma_{\ell'}$ is the root in the upper half plane for the equation

$$\ell'-(\alpha-b)s+\nu\left[1-\bar{a}(s)\right]=0 \tag{4.7}$$

We have from (4.6)

$$\bar{c}(\ell')=-\exp\left[\sigma_{\ell'}(x_0-a)\right] \tag{4.8}$$

Further it can be easily shown that

$$M(t)=M(x_0,a,t)=-c(t) \tag{4.9}$$

where $M(x_0,a,t)$ is the first passage time to reach the barrier $x=a$ between time t and $t+dt$; starting from $x=x_0$ at time t=o. From (4.8) and (4.9) one gets

$$\bar{M}(x_0,a,\ell')=\exp\left[\sigma_{\ell'}(x_0-a)\right] \tag{4.10}$$

It is easily seen from (4.10) that the first passage density of the process $x(t)$ with drift $\alpha - b$ is

$$M(x_0, a, t) = (|x_0 - a|/t)\, g(a - x_0 - (\alpha - b)t, t) \tag{4.11}$$

We can also arrive at the above result by the following reasoning. Let us consider a point y above K(t) where K(t) = a+bt is the barrier of the process. Define $\phi(y, t/x_0)$ as the probability that the particle at time t is in the interval $(-\infty, y)$ given that at time t=0 it was at $x = x_0$. Also define $F_y(k(t), t/x_0)$ as the probability that the particle is in the interval $(-\infty, k(t))$ where $k(t) \leq y$ at any time t. Let $M(x_0, k(t), t)$ be the probability that the process reaches the barrier K(t) at time t and $t + dt$ starting from $x = x_0$. Then the function ϕ, F and M are connected by the following integral equation

$$\phi(y, t/x_0) = F_y(k(t), t/x_0) + \int_0^t M(k(t'), t'/x_0)\, \phi(y, t-t'/k(t'))\, dt' \tag{4.12}$$

where F_y indicates that it is the distribution function of the process $x(t)$ with a barrier at y, to reflect the condition

$k(t) \leq y$. $f_y(k(t), t/x_0)$ is the solution of the equation of the type

$$\frac{\partial f}{\partial t} + \alpha \frac{\partial f}{\partial x} + \nu f = \nu \int_{-\infty}^{y} f(x',t)\, a_-(x-x')\, dx' \tag{4.13}$$

with the boundary condition

$$f(x,t \mid x_0) = 0 \qquad \text{for} \quad x \geq y$$

and

$$F_y(k(t), t/x_0) = \int_{-\infty}^{k(t)} f_y(x,t|x_0)\, dx \tag{4.14}$$

Differentiating with respect to y the equation (4.12) becomes

$$\frac{\partial \phi(y,t|x_0)}{\partial y} = \frac{\partial F_y}{\partial y} + \int_0^t M(k(t'), t'|x_0) \frac{\partial}{\partial y} \phi(y, t-t' \mid k(t'))\, dt' \tag{4.15}$$

$(\partial\phi/\partial y)$ is the unbounded solution of equation (4.2) with free boundaries. When t becomes infinite K(t) becomes larger than y and the first term F_y in (4.12) vanishes.

$$\frac{\partial \phi(y,t|x_0)}{\partial y} = \int_0^t M(k(t'), t'|x_0) \frac{\partial \phi(y, t-t' \mid k(t'))}{\partial y}\, dt' \tag{4.16}$$

Taking the laplace transform of (4.16) with respect to y we get, as t tends to infinity,

$$1 = \int_0^\infty M(k(t'), t' \mid x_0) \exp\left[-\alpha s t' + \nu\left[1 - \bar{a}_-(s)\right] t'\right] \times \exp\left[s k(t') - s x_0\right] dt' \tag{4.17}$$

This is the Wald identity for a moving barrier, with the $Re(s)$ being taken positive. We can easily identify RHS integral in (4.17) as the L.T. of $M(k(t'))$ with respect to t' where the laplace variable is

$$q = (\alpha - b) s - \nu(1 - \bar{a}_-(s)) \tag{4.18}$$

Hence we have

$$1 = \bar{M}(q)\, e^{-s(x_0 - a)} \tag{4.19}$$

If we consider the equation for a drift process with rate $\alpha - b$ and with free boundaries we have the equation

$$\left\{\ell - (\alpha - b) s + \nu(1 - \bar{a}_-(s))\right\} \bar{f}(s, \ell) = 0 \tag{4.20}$$

for the free process with the transform variables s and ℓ. If we take the value $s = \sigma_\ell$ which satisfies the equation

$$\ell - (\alpha - b)s + \nu\left[1 - \bar{a}_-(s)\right] = 0 \tag{4.21}$$

We know that $\exp[\sigma_\ell(x_0 - x)]$ when inverted yields the result

$$\mathcal{L}^{-1}\left\{e^{\sigma_\ell(x_0 - x)}\right\} = \frac{|x_0 - x|}{t}\, g\left\{x - x_0 - (\alpha - b)s,\, t\right\} \tag{4.22}$$

where the Green's function corresponds to the free equation with drift $\alpha - b$. Hence if in the equation (4.19) if we choose $s = \sigma_\ell$ then we identify $q = \ell$ and hence we have

$$\overline{M}(\ell) = \exp\left[\sigma_\ell(x_0 - a)\right] \tag{4.23}$$

This can be computed by inversion since g is well known.

So far we have taken a(z) = a_(z). Let us now consider the jumps in both the directions. It is now possible to go to the diffusion limit since we have both $a_+(z)$ and a_(z) and hence the first moment of a(z) can be made to be zero. Adopting the limiting procedure as done earlier we arrived at the result

$$1 = \int_0^\infty M(t', a, b \mid x_0)\, e^{-s(x_0 - a)}\, e^{-\left[(\alpha - b)s + \frac{Ds^2}{2}\right]t'}\, dt' \tag{4.24}$$

This is the Wald identity when there is a barrier $k(t) = a + bt$ for the diffusion process, $\mathrm{Re}\, s$ being positive definite. The result (4.24) is also derived by Clay and Goyal (1973).

5. <u>First passage for crossing an exponential barrier</u>.

Here again we consider the pure jump process defined in section (3) and take the density of jumps as .

$$a(z) = a_+(z) + a_-(z)$$

where

$$a_+(z) = \lambda \exp(-\eta z)\, H(z)$$

and $a_-(z)$ is in any arbitrary form. The equation of continuity for this process is

$$\frac{\partial f}{\partial t} + \nu f = \nu \int_{-\infty}^{\infty} f(x',t)\, a(x-x')\, dx' \qquad (5.1)$$

where $f(x,t)$ is the usual transition density. For this process let us introduce the time dependent boundary

$$k(t) = a + b\, e^{-\beta t} \quad , \quad a, b > 0$$

The differential equation for the unbounded process after introducing the appropriate source term is

$$\frac{\partial f}{\partial t} + \nu f = \nu \int_{-\infty}^{\infty} f(x',t)\, a(x-x')\, dx' + c(x,t) \qquad (5.2)$$

where

$$C(x,t) = H\left(x-(a+b\,e^{-\beta t})\right)\lambda\, e^{-\eta x}\; S(t) \tag{5.3}$$

and

$$S(t) = -\nu \int_{-\infty}^{a+b\,e^{-\beta t}} f(x',t)\, e^{\eta x'}\, dx'$$

The L.T. of $C(x,t)$ with respect to x is

$$C_1(s,t) = \frac{\lambda}{\eta-s}\, e^{-(\eta-s)(a+b\,e^{-\beta t})}\; S(t) \tag{5.4}$$

Defining $M(t) = M(x_0, k(t), t)$ as the first passage time density for crossing the exponential barrier, K(t) between time t and $t+dt$ we easily see the relation connecting $C(x,t)$ and M(t) in the form

$$C_1(s,t) = -\frac{\eta}{\eta-s}\, e^{s(a+b\,e^{-\beta t})}\; M(t) \tag{5.5}$$

$$= -\frac{\eta}{\eta-s}\, e^{sa} \sum_{n=0}^{\infty} \frac{(sb)^n}{n!}\, e^{-n\beta t}\; M(t) \tag{5.6}$$

Taking again L.T. with respect to t,

$$\bar{c}(s,\ell) = -\frac{\eta}{\eta-s}\, e^{sa} \sum_{n=0}^{\infty} \frac{(sb)^n}{n!}\, M(\ell+n\beta) \tag{5.7}$$

It is evident for the process we considered, in equation (5.1)

$$\bar{c}(\sigma_\ell, \ell) = -e^{\sigma_\ell x_0} \tag{5.8}$$

where $s = \sigma_\ell$ is the root in the upper half plane for the equation $\ell + \nu[1 - \bar{a}(s)] = 0$. Hence the L.T. of the first passage time density is given by

$$\sum_{n=0}^{\infty} \frac{(\sigma_\ell b)^n}{n!} M(\ell + n\beta) = \left(1 - \frac{\sigma_\ell}{\eta}\right) e^{\sigma_\ell (x_0 - a)}$$

$$= \left(1 - \frac{1}{\eta}\frac{d}{dx_0}\right) e^{\sigma_\ell (x_0 - a)} \tag{5.9}$$

when $\beta = 0$ this result reduces to the result (2.27) we arrived at for a constant boundary. Byvconsidering the process with drift defined by (2.1) the first passage density is given by the equation (5.9) with the modification that $s = \sigma_\ell$ is the root of the equation

$$\ell + \alpha s + \nu[1 - \bar{a}(s)] = 0$$

6. <u>First passage time for reaching an exponential barrier.</u>

Let us now consider the process

$$X(t) = x_0 + \alpha t - \sum_{i=1}^{N(t)} Z_i \tag{6.1}$$

In this case the barrier can exactly be reached since the motion

in the upward direction is continuous. The differential equation to be studied with the compensation function is

$$\frac{\partial f}{\partial t} + \alpha \frac{\partial f}{\partial x} + \nu f = \nu \int_{-\infty}^{\infty} f(x',t)\, a_-(x-x')\, dx' + c(t)\, \delta(x-(a+b e^{-\beta t})) \tag{6.2}$$

with free boundary values. Taking L.T. with respect to x,

$$\frac{\partial f_1(s,t)}{\partial t} = \left[\alpha s - \nu(1-\bar{a}_-(s))\right] f_1(s,t) + c(t) \exp\left[s(a+b e^{-\beta t})\right] \tag{6.3}$$

where $f_1(s,t)$ is L.T. of $f(x,t)$. Again taking L.T. with respect to t we get

$$\bar{f}(s,\ell)\left\{\ell - \alpha s + \nu\left[1-\bar{a}_-(s)\right]\right\} = e^{s x_0} + e^{sa} \sum_{n=0}^{\infty} \frac{(sb)^n}{n!} c(\ell + n\alpha) \tag{6.4}$$

Let $s = \sigma_\ell$ satisfy the equation

$$\ell - \alpha s + \nu\left[1-\bar{a}_-(s)\right] = 0$$

Then we find the Wald identity as

$$\sum_{n=0}^{\infty} \frac{(\sigma_\ell b)^n}{n!} c(\ell + n\beta) = -\exp\left[\sigma_\ell (x_0 - a)\right] \tag{6.5}$$

The compensation function $c(t)$ is connected with $M(t)$ by the relation

$$c(t) = -M(t) \tag{6.6}$$

Hence we get from (6.5) and (6.6)

$$\sum_{n=0}^{\infty} \frac{(\sigma_\ell b)^n}{n!} M(\ell + n\beta) = e^{\sigma_\ell (x_0 - a)} \tag{6.7}$$

This result has been derived by Kryukov (1976) in a different manner. We also see by taking $K(t) = a + b\,e^{-\beta t}$ in equation (4.17) and proceeding to the limit as t tends to infinity the result (6.7) is obtained.

7. Time dependent thresholds for a type of neuron model.

In this section we shall consider a neuron firing problem with the density function of the jumps modelled according to the idea expounded in the experimental work of Fuortes and Mantegazzini (1962) according to which the jumps may not be independent. Tuckwell (1979) also supports the view that the jumps are not independent, as was taken in earlier sections, but should in some way depend upon the potential gathered by the neuron just before the excitation or inhibition comes in. This may correspond to the fact that as the existing potential level approaches nearer to the threshold jump magnitudes are larger. This is in some way equivalent to some kind of short

term local facilitation. Hence if one takes the random variable R which is the magnitude of the ratio of the jump to the existing potential at any time, then it can be postulated that $p(R)\,dR$ the probability density function for the occurrance of this ratio is given by

$$p(R)\,dR = \eta \frac{1}{(1+|R|)^{\eta+1}}\,dR \tag{7.1}$$

Let us now confine ourselves to positive jumps only. Let $a(x, x')\,dx$, the jump p.d.f. be taken as

$$a(x,x') = a_+(x,x') + a_-(x,x') \tag{7.2}$$

where

$$\begin{aligned} a(x,x') &= a_+(x,x') \qquad x \geq x' \\ &= a_-(x,x') = 0 \;, \quad x < x' \end{aligned} \tag{7.3}$$

Expressing (7.1) as $a(x,x')$ we find

$$a_+(x,x') = \eta\, x'^{\eta} / x^{\eta+1} \tag{7.4}$$

Equation (7.4) is known as Pareto distribution well known in the modelling of income distribution in economics (Mood et. al. 1974). Let us consider the case when X the potential level is always above zero which is the case when the resting

value is taken as $X_0 > 0$. We are here concerned with the positive region of X . The excitatory jumps starting from the resting potential X_0 at time t=o occur with a poisson frequency ν . So the stochastic process is defined by

$$X(t) = X_0 + \sum_{n=1}^{N(t)} Z_n \, e^{-\alpha(t-t_n)} \tag{7.5}$$

The exponential parameter α and the coefficient η in the definition of $a(x,x')$ are taken to be positive. The forward equation of such a process is

$$\frac{\partial f}{\partial t} - \alpha \frac{\partial (x f)}{\partial x} + \nu f = \nu \int f(x',t)\, a(x-x')\, dx' \tag{7.6}$$

$$\text{with} \quad f(x,0) = \delta(x-x_0) \tag{7.6a}$$

Due to the nature of the jumps which relate to only positive ratios and due to the deterministic exponential decay the point $X=0$ is a natural absorbing barrier in this problem. The threshold value for a given neuron serves as another absorbing barrier. The problem of the first passage time to cross $X=K$ can easily be solved in closed form if one considers the transformation of the variable X by Y where

$$x = e^{y} \tag{7.7}$$

When this transformation is effected we find that $a(x,x')\,dx$ becomes $b(y,y')\,dy$ where

$$b(y,y') = \eta\, e^{-\eta(y-y')} \tag{7.8}$$

Also for the transition density function $f(x,t)$ we have

$$f(x,t) = \pi(y,t)\left|\frac{dy}{dx}\right| \tag{7.9}$$

Hence by the transformation (7.7) the forward equation (7.6) becomes

$$\frac{\partial \pi(y,t)}{\partial t} - \alpha\frac{\partial \pi(y,t)}{\partial y} + \nu\,\pi(y,t) = \nu\int_{-\infty}^{\infty}\pi(y',t)\,e^{-\eta(y-y')}\,dy' \tag{7.10}$$

Thus the effect of the transformation is that the problem with exponential decay with a parameter α is converted into a linear drift problem with the drift coefficient α. The initial condition (7.6a) is to be replaced by

$$\pi(y,0) = \delta(y-y_0) \tag{7.11}$$

where $y_0 = \log x_0$ is the starting point in the y-space.

Also the constant barrier $x = k$ becomes $\bar{k} = \log k$ in the new space. This is exactly the case described by the equation (2.2) of section 2. We can obtain the solution for this case by bringing in the compensation function of Keilson (1963) and proceeding as in section 2. The transform of the first passage density $\bar{M}_y(\ell)$ in the y space as in equation (2.27) is

$$\bar{M}_y(\ell) = (1 - \sigma_\ell/\eta)\, e^{\sigma_\ell (y_0 - \bar{k})} \tag{7.12}$$

where σ_ℓ is the root of the equation

$$\ell - \alpha s + \nu [1 - \bar{b}(s)] = 0 \tag{7.13}$$

$\bar{b}(s)$ being the double sided laplace transform of $b(y)$ as in equation (2.8) of section 2. We are automatically led to the solution

$$M_y(t) = \left[1 - \frac{1}{\eta}\frac{d}{dy_0}\right] \frac{|y_0 - \bar{k}|}{t}\, \mathcal{G}_{oy}(\bar{k} - y_0 + \alpha t, t) \tag{7.14}$$

similar to the equation (2.35) of section 2. $\mathcal{G}_{oy}$ is the well known Green's function of the unbounded jump process with jump p.d.f given by $b(y)$ and with no barriers. This is easily

derived and hence one can go back to the original space by retransforming the variable y to x suitably. Hence we get

$$M(t) = \left(1 - \frac{x_0}{\eta}\frac{d}{dx_0}\right)\frac{1}{t}\log\left(\frac{K}{x_0}\right)\mathcal{G}_y\left(\log\frac{K}{x_0} + \alpha t, t\right) \tag{7.15}$$

By $\mathcal{G}_y$ we mean the same Green's function in the y space with corresponding variables in x space. Hence equation (7.15) is the solution to the constant boundary problem with the jump density defined by (7.2), (7.3) and (7.4).

The first passage problem in this case can also be solved by writing down the imbedding equation for $M(u,k,t)$ the probability for first crossing K between t and $t+dt$ starting from u as was done in our earlier paper for different problem (Vasudevan et. al. 1980). The imbedding equation for M for a constant barrier can be arrived at as

$$\frac{\partial M}{\partial t} + \alpha u \frac{\partial M}{\partial u} + \nu M = \nu \int_u^K M(z,k,t)\, a(z,u)\, dz + \delta(t)\, \nu \int_K^\infty a(z,u)\, dz \tag{7.16}$$

with the initial condition $M(u,k,0) = 0$. On taking laplace transform (7.16) becomes

$$\alpha u \frac{\partial \bar{M}}{\partial u} + (\ell+\nu)\bar{M} = \nu \int_u^K \bar{M}(z,K,\ell)\,\eta\,\frac{u^\eta}{z^{\eta+1}}\,dz + \nu\eta \int_K^\infty \frac{u^\eta}{z^{\eta+1}}\,dz \tag{7.17}$$

where $a(x,x')$ is given by (7.2), (7.3) and (7.4).

Transforming to the variable $u = e^y$ we arrive at an equation for $\bar{M}(\ell)$ corresponding to a motion with linear drift and exponential jump density in the y space

$$e^{-\eta y}\left[\alpha \frac{\partial \bar{M}}{\partial y} + (\ell+\nu)\bar{M}\right] = \nu \int_y^{\bar{K}} \bar{M}(y',K,\ell)\,e^{-\eta y'}\,dy' + \nu\, e^{-\eta \bar{K}} \tag{7.18}$$

Differentiating with respect to y we easily see that $\bar{M}_y(\ell)$ satisfies the second order differential equation

$$\alpha \frac{\partial^2 \bar{M}(y,\bar{K},\ell)}{\partial y^2} + (\ell+\nu-\eta\alpha)\frac{\partial \bar{M}}{\partial y}(y,\bar{K},\ell) - \eta\ell\,\bar{M}(y,K,\ell) = 0 \tag{7.19}$$

The range of y will be $-\infty$ to $\bar{K}$ resulting from the range of the variables in the x space from 0 to K. Since $\bar{M}_y$ has to be finite the solution of equation (7.19) is given only by

$$\bar{M}(y,\bar{K},\ell) = A\,\exp(m_1 y) \tag{7.20}$$

where m_1 is the positive root of the equation

$$\alpha m^2 + (\ell + \nu - \eta\alpha) m - \eta \ell = 0 \tag{7.21}$$

A the only constant independent of y is to be determined. This can be done by substituting the solution into the first order equation (7.18). We find

$$A = \left(1 - \frac{m_1}{y}\right) e^{-m_1 K} \tag{7.22}$$

Thus the solution in the y space is

$$\bar{M}(y, \bar{K}, \ell) = \left(1 - \frac{m_1}{\eta}\right) e^{m_1 (y - \bar{K})} \tag{7.23}$$

It can be easily seen that σ_ℓ of the equation (2.10) is the same as m_1 of the equation (7.21). Hence $M(\ell)$ in y space is given by

$$\bar{M}(y, \bar{K}, \ell) = \left(1 - \frac{\sigma_\ell}{\eta}\right) e^{\sigma_\ell (y - \bar{K})} \tag{7.24}$$

When inverted this is expressed in terms of the free Green's function in the y space as given by the equation (7.10). If one wants to express $\bar{M}(\ell)$ in x space we have

$$\bar{M}(x_0, K, \ell) = \left(1 - \frac{\sigma_\ell}{\eta}\right) (u/K)^{\sigma_\ell} \tag{7.25}$$

We can easily verify that this solution satisfies first order equation (7.17) in the x space.

We can also find the moments of M(t) from (7.25) as we know that n^{th} moment is

$$E(T^n) = (-1)^n \frac{d^n}{d\ell^n} \bar{M}(\ell)\Big|_{\ell=0} \tag{7.26}$$

The mean passage time is given by

$$E(T) = \left(1-\eta \log(u/k)\right)/(\nu-\eta\alpha) \tag{7.27}$$

Since E(T) has always to be positive we should have the condition that $(\nu/\eta) > \alpha$. In the y space we know that η^{-1} is the average value of the jump magnitude. The average of the Poisson jumps per unit time is ν. Hence (ν/η) represents the average upward increase in y space per unit time and α is the deterministic downward drift per unit time. Hence $\nu > \eta\alpha$ is a necessary condition for the particle to cross the barrier in any time.

The second moment of T about the origin in

$$E(T^2) = \frac{1}{\eta}\left(1-\eta \log(u/k)\right)\left\{\frac{1-\log(u/k).\eta}{\nu-\eta\alpha} + \frac{2\nu\eta}{\nu-\eta\alpha}\right\} \tag{7.28}$$

and the variance is $E(T^2) - [E(T)]^2$.

In the above model we consider a moving barrier as given by the equation

$$K(t) = k\, e^{-\beta t} \tag{7.29}$$

K being very high. This means there is a refractory period after each spike. This type of boundary has been treated in Clay and Goyal (1973); and Ricciardi (1977). This barrier in y space is $\bar{y}(t) = \log k - \beta t$. This is a linear barrier moving down in the y plane. But this is equivalent to the problem of finding the first passage time density with constant barrier $\bar{k} = \log k$ and a downward linear drift $\alpha - \beta$. The solution of this problem is

$$M_y(t) = \left(1 - \frac{1}{\eta}\frac{d}{dy_0}\right) \frac{|y_0 - \bar{k}|}{t}\, g_y\left(\bar{k} - y_0 + (\alpha-\beta)t, t\right) \tag{7.30}$$

Transforming to the X space we can find the first passage density as

$$M_x(t) = \left(1 - \frac{x_0}{\eta}\frac{d}{dx_0}\right)\frac{1}{t}\log(k/x_0)\, g_y\left(\log\frac{k}{x_0} + (\alpha-\beta)t, t\right) \tag{7.31}$$

Another type of model simulating a refractory threshold was introduced by Hagiwara (1954). Here the boundary K(t)

is defined by

$$K(t) = K\, e^{\beta/t} \tag{7.32}$$

This boundary in y space is $\bar{K} + \beta/t$ where $\bar{K} = \log K$.

The forward equation for the density function $f(y,t)$ in the y space is

$$\frac{\partial f(y,t)}{\partial t} - \alpha \frac{\partial f(y,t)}{\partial y} + \nu f(y,t)$$

$$= \nu \int_{-\infty}^{\infty} f(y',t)\, b(y-y')\, dy' + C(y,t)$$

$$\text{with} \quad f(y,0) = \delta(y - y_0) \tag{7.33}$$

$C(y,t)$ is the compensation function responsible for the boundary effects. It is given by

$$C(y,t) = -\nu\, H(y - (\bar{K} + \beta/t)) \int_{-\infty}^{\bar{K}+\frac{\beta}{t}} f(y',t)\, e^{-\eta(y-y')}\, dy' \tag{7.34}$$

$$= H(y - (\bar{K} + \tfrac{\beta}{t}))\, \eta\, e^{-\eta y}\, S(t) \tag{7.35}$$

$M_y(t)$ the first passage time density is given by

$$M_y(t) = -\int_{\bar{K}+\beta/t}^{\infty} C(y,t)\, dy \tag{7.36}$$

$$= \exp\left[-\eta(\bar{K} + \beta/t)\right] S(t) \tag{7.37}$$

As seen in a number of previous cases the laplace transform of $C(y,t)$ is

$$C_1(s,t) = -\frac{\eta}{\eta - s} \exp\left[s\left(\bar{K} + \frac{\beta}{t}\right)\right] M(t) \tag{7.38}$$

Taking laplace transform with respect to t, equation (7.38) becomes

$$\bar{C}(s,\ell) = -\frac{\eta}{\eta - s} e^{s\bar{K}} \mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{(s\beta)^n}{n!} \frac{M(t)}{t^n}\right\} \tag{7.39}$$

where $\mathcal{L}$ is the laplace transform operator. In other words

$$-\frac{\eta - s}{s} e^{-s\bar{K}} \bar{C}(s,\ell) = \bar{M}(\ell) + \mathcal{L}\left\{\sum_{n=1}^{\infty} \frac{(s\beta)^n}{n!} \frac{M(t)}{t^n}\right\} \tag{7.40}$$

Using the results (Spregel 1965, Hildebrand 1965)

$$\mathcal{L}\left\{M(t)/t\right\} = \int_{\ell}^{\infty} \bar{M}(\xi)\, d\xi \tag{7.41}$$

and

$$\int_a^x dx_n \int_a^{x_n} dx_{n-1} \cdots \int_a^{x_2} f(x_1)\, dx_1 = \frac{1}{(n-1)!} \int_a^x (x-\xi)^{n-1} f(\xi)\, d\xi \tag{7.42}$$

we obtain a series for $\bar{M}(\ell)$ as

$$\left(1-\frac{s}{\eta}\right) e^{-s\bar{K}}\ \bar{c}(s,\ell) = \bar{M}(\ell)$$

$$+\sum_{n=1}^{\infty} \frac{(s\beta)^n}{n!} \frac{(-1)^{n-1}}{(n-1)!} \int_{\ell}^{\infty} (\ell-\xi)^{n-1}\ \bar{M}(\xi)\,d\xi \tag{7.43}$$

Going back to the equation (7.33) one finds that

$$\bar{c}(s,\ell) = -\ e^{sY_0} \tag{7.44}$$

where s is chosen as the solution of the equation

$$\ell + \alpha s + \nu\left\{1-\bar{a}(s)\right\} = 0 \tag{7.45}$$

with a root $s=\sigma_\ell$ in the upper half plane. Hence the Wald identity in this case is

$$\left(1-\frac{1}{\eta}\frac{d}{dy_0}\right) e^{\sigma_\ell(Y_0-\bar{K})}$$

$$= \bar{M}(\ell) + \sum_{n=1}^{\infty} \frac{(\sigma_\ell\beta)^n}{n!} \frac{(-1)^{n-1}}{(n-1)!} \int_{\ell}^{\infty} (\ell-\xi)^{n-1}\ \bar{M}(\xi)\,d\xi \tag{7.46}$$

$$= \bar{M}(\ell) + \sigma_\ell\beta \int_{\ell}^{\infty} \sum_{r=0}^{\infty} \frac{(\sigma_\ell\beta)^r}{r!} \frac{(-1)^r}{(r+1)!} (\ell-\xi)^r\ \bar{M}(\xi)\,d\xi \tag{7.47}$$

The coefficient of $\bar{M}(\ell)$ in the integrand of (7.47) can be expressed in terms of J functions as defined by Mathews and Walker (1964)

$$J_m(x) = \frac{x}{2} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(r+1)!} \left(\frac{x}{2}\right)^{2r} \tag{7.48}$$

Hence we can rewrite the Wald identity (7.46) as an integral equation relating to the transform of M(t).

$$\left(1 - \frac{1}{\eta}\frac{d}{dy_0}\right) e^{\sigma_\ell (y_0 - \bar{K})} = \bar{M}(\ell) + 2\sigma_\ell \beta \int_\ell^\infty \frac{J_1\left(\sqrt{2(\ell+\xi)\sigma_\ell \beta}\right)}{2\sqrt{(\ell-\xi)\sigma_\ell \beta}} \bar{M}(\xi)\, d\xi \tag{7.49}$$

If $\ell = 0$ we observe that both sides of (7.49) becomes unity since $\sigma_\ell = 0$ at $\ell = 0$. σ_ℓ can be computed for given density function $b(y)$ from the equation (7.45).

To derive some meaningful results from the expression (7.49) let us assume that β is small which is not a bad assumption if we realise that the threshold from a very high value just after a spike gets down to values near K quite rapidly. This will correspond to the fact that the refractory period is about 0.7 to 2 m sec only. The interval between firings indicate a mean rate of 8 min. Hence we now approximate the J_1 function (7.49) by its appropriate value for small arguments and write an approximate integral equation as

$$\left(1-\frac{1}{\eta}\frac{d}{dy_0}\right)e^{\sigma_\ell(y_0-\bar{K})} = \bar{M}(\ell) + 2\sigma_\ell\beta\int_\ell^\infty \bar{M}(\xi)\,d\xi$$
$$-c\,\sigma_\ell^2\beta^2\int_\ell^\infty(\ell-\xi)\,\bar{M}(\xi)\,d\xi \tag{7.50}$$

where c is a constant. When ℓ is large we see from the solution of the equation (7.45)

$$\sigma_\ell = \frac{1}{2\alpha}\left\{-(\ell+\nu-\eta\alpha) + \left[(\ell+\nu-\eta\alpha)^2+4\eta\alpha\ell\right]^{1/2}\right\}$$
$$\cong \eta \tag{7.51}$$

This means that we are interested in small time region. For this approximation we find that the inverse of the equation (7.50) yields the result

$$\left(1-\frac{1}{\eta}\frac{d}{dy_0}\right)\frac{|y_0-\bar{K}|}{t}\,g(y_0-\bar{K}+\alpha t,\,t)$$
$$= M(t) + \eta\beta M(t)/t - c\eta^2\beta^2 M(t)/t^2 \tag{7.52}$$
$$+ \text{ higher order terms}$$

Here g is the well known Green's function for the unrestricted process in y space. We can easily convert the above result into the original x space variable. Since ℓ is taken to be large the solution corresponds to small values of t. Even if the above result happens to be an approximation for short time, we have an analytical expression to compare with experi-

mental data.

We will now demonstrate that an exact integral equation for the average value of M(t) can be obtained from equation (7.49) by differentiating (7.49) with respect to ℓ and using the following properties of J functions.

(1) $$\frac{d}{dx}\left[x^{-1} J_1\right] = -x^{-1} J_2$$

(2) As x tends to zero, (7.53)

$x^{-1} J_2(x)$ tends to zero and $x^{-1} J_1(x)$ tends to unity.

It can also be easily seen that

$$\frac{d}{d\ell}\,\sigma_\ell\Big|_{\ell=0} = \eta/(\nu-\eta\alpha) \tag{7.54}$$

$$-\frac{d}{d\ell}\,M(\ell)\Big|_{\ell=0} = E(T) \tag{7.55}$$

Here T is the random variable representing the time interval between two spikes. Hence differentiating (7.49) with respect to ℓ and using the above results we get when $\ell=0$

$$\frac{1}{\nu-\eta}\left[1+\eta(\bar{K}-y)\right] + \frac{\eta\beta}{\nu-\eta\alpha}\int_0^\infty \frac{M(t)}{t}\,dt$$

$$= \int_0^\infty t\, M(t)\, dt \tag{7.56}$$

8. Conclusion.

In conclusion we want to highlight the results obtained in various sections which are not found in the literature. In section 2 we have obtained the first passage time density to cross a constant barrier at $x=k$ for a two way discrete jump process. Since we are using the compensation function to arrive at the result $M(x_0, k, t)$, the first passage density, it is enough if we take the jumps in the directions of the barrier in a separable form (7.2)

$$a_+(x,x') = \tilde{a}_1(x)\,\tilde{a}_2(x')$$

When we proceed to the limit in this case as described by Keilson (1963) we easily obtain the first passage density for the diffusion process as given by (2.65)

$$\bar{M}(\ell) = \exp\left[\sigma_\ell(x_0-k)\right]$$

Herein we have not specified the type of density for backward jump at all. It is also required that the first moment of the total jump process is zero and the second moment multiplied by the frequency goes to a constant limit.

Starting with two way discrete jump process with exponential densities and considering a barrier moving linearly with time we derive in section 3 the first passage time density in terms of an unbounded Green's function with constant barrier

for a particle moving with linear drift. This is obtained in the equation (3.18)

$$\overline{M}(\ell) = \left(1 - \frac{1}{\eta}\frac{d}{dx_0}\right) \frac{|x_0 - K|}{t} \; g_0(a - x_0 + bt, t)$$

which in the diffusion limit goes over to equation (3.24)

$$M_D(t) = \frac{|x_0 - K|}{\sqrt{2\pi D}} \; t^{-3/2} \exp\left\{-\frac{(a + bt - x_0)^2}{2Dt}\right\}$$

for the diffusion process (3.23) with a moving barrier.

For a discrete jump process Wald identity is derived in an interesting fashion in section 4, as given in (4.16) which when t tends to infinity becomes (4.17)

$$1 = \int_0^\infty M(K(t'), t \mid x_0) \exp\left[-\alpha s t' - \nu(1 - \bar{a}_-(s))\right] e^{sK(t') - sx_0} \, dt'$$

This in the diffusion limit goes over to the equation (4.16)

$$1 = \int_0^\infty M(t', a, b \mid x_0) e^{-s(x_0 - a)} \exp\left[-\left((\alpha - b)s + \frac{Ds^2}{2}\right)t'\right] dt'$$

as obtained by Clay and Goel (1973) for positive definite real part of s.

For a special type of jump density in the upward direction only given by 7.4, the pareto distribution which can be supposed to simulate short term facilitation, we obtain the solution in the transformed variable y

$$M_y(t) = \left(1 - \frac{1}{\eta}\frac{d}{dy_0}\right) \frac{|y_0 - \bar{k}|}{t} \xi_{oy}(\bar{k} - y_0 + \alpha t, t)$$

This can be inverted into the original variable as in equation (7.15). It is easy to see that we can obtain $\bar{M}(x_0 k, \ell)$ given simply by (7.26)

$$\bar{M}(x_0, k, \ell) = \left(1 - \frac{\sigma_\ell}{\eta}\right)\left(\frac{x_0}{k}\right)^{\sigma_\ell}$$

and the mean first passage time (7.28) is

$$E(T) = \left\{1 - \eta \log(x_0/k)\right\}(\nu - \eta\alpha)^{-1}$$

and the second moment is given by the equation (7.29).

For an exponentially moving barrier the method gives the solutions (7.31), (5.9) and (6.7) very easily for crossing and reaching the barriers both in the discrete jump and diffusion limit. Kryukov (1976) has only considered the problem of reaching the barrier only.

Similarly for a type of moving barrier given by (7.33)

$$k(t) = k\, e^{\beta/t}$$

we obtain a closed integral equation for the transform given by the equation (7.57)

$$\left(1-\frac{1}{\eta}\frac{d}{dy_0}\right)\exp\left(\sigma_\ell[y_0-\bar{K}]\right)=$$

$$=\bar{M}(\ell)+2\sigma_\ell\beta\int_\ell^\infty J_1\left[2\sqrt{(\ell-\xi)\sigma_\ell\beta}\right]\frac{\bar{M}(\xi)}{2\sqrt{(\ell-\xi)\sigma_\ell\beta}}\,d\xi$$

This has not been treated by Kryukov. It is easy to go from this result to obtain the mean first passage time (7.56),

$$\frac{1}{\nu-\eta}\left\{1+\eta(\bar{K}-y_0)\right\}+\frac{\eta\beta}{\nu-\eta\alpha}\int_0^\infty\frac{M(t)}{t}\,dt$$

$$=\int_0^\infty t\,M(t)\,dt.$$

C H A P T E R - IV

DAM PROBLEMS IN CONTINUOUS TIME WITH RANDOM INPUTS AND OUTPUTS*

1. Introduction.

There is a lot of literature connecting with the studies of dam theory, queueing theory and storage models. The problems met with in these fields can be analysed by identical techniques. Moran (1954) has initiated the studies of storage system by working out solutions for discrete time models. Further contributions in this field have been made by Gani (1955), Prabhu (1964) and others. The content of an infinite dam with Poisson inputs have been studied in considerable detail and a survey of this interesting field is given by Prabhu (1964). However a dam with finite boundaries poses difficult problems. For Poisson inputs, Tackas (1967) adopted the combinatorial techniques to study the fluctuations of the content of a finite dam. The extension to continuous time version of Moran's discrete model was carried out by Moran (1956) and Downton(1957) using some limiting methods.

A systematic version of the continuous time model has been formulated by Kendall (1957). He has obtained an elegant result for the wet period. Cochen (1969) has made use of Pollaczek's (1957) integral equation for various models of general storage theory. A number of time dependent results for some of these models have been given by Saaty (1961),

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Yeo (1961), Chover and Yeo (1965) and others. In all these cases the input is a Poisson or renewal process and the amount of input is governed by independent and identically distributed random variable. The concept of first passage densities for a compound Poisson process and ideas of renewal theory and product densities which are used in these studies are elaborated in an excellant review in Handbuch der Physik (Ramakrishnan 1959). The epochs of inputs are assumed to constitute a stationary renewal point process in Srinivasan (1974), Phaterford (1971). The method of using the backward integral equation described by Bellman and Harris (1948) has been used by Srinivasan (1974). Phaterford (1963) has used Wald identity for studying the wet period of a finite dam.

Regarding the release policy different models have been thought of. The release has been considered as a deterministic process with a constant rate (Gaver and Miller 1962; Srinivasan 1974). The general type of deterministic release problems have been studied by Cinlar and Pinsky (1972). Exponential release rule has been considered by Yeo (1974) for a finite dam model and Keilson and Mermin (1959) for studying shot noise problems.

This type of modelling also has been applied for the reception of light on the retina of eye when the light is switched on in a dark room. Impulses are received in a Poisson manner and between impulses exponential loss of light occurs while one can 'see' when the level of light on the retina reaches a

threshold K. The classical theory of quantal response assays can be found in books, like Finney (1947, 1952) and Bliss (1952). Plackett and Hewlett (1967) have tried to produce a unified theory of quantal response to mixtures of drugs. Mathematical modelling in this field has been done by Puri and Senthuria (1971).

Karlin and Fabens (1962) used the renewal process to permit certain inter dependence between successive inputs for discrete time models in the theory of stationary inventory models. In the warehouse model problems, the demand for the storage occurs in a Poisson manner by outputs governed by independent and identically distributed random variables. When the storage falls below a certain specified recorder level, if orders are received they are not refused but kept on record and filled in later. These have been described by Prabhu (1965). All these problems pose the same type of questions. The quantities of interest which are studied in these problems are similar to the computation of First Passage Density for the overflow or the emptiness for a finite dam.

In all the existing literature regarding dams the release is governed by a deterministic policy which may or may not depend upon the existing level of the dam. A search of the literature reveals that there are not enough contributions regarding continuous time models in which not only inputs but also releases are allowed to vary in a random manner. The case of an infinite depth dam with Poisson inputs and Poisson release

has been considered by Puri and Senthuria (1975). They have not included any type of deterministic release.

In this contribution we will be concerned with a finite dam with Poisson inputs and Poisson release having different Poisson intensities. The amounts of inputs and overflows are governed by independent and identically distributed random variables. There is also a deterministic release either of exponential or linear type. The first passage time to emptiness of the dam and first overflow have been explicitly obtained by using the imbedding technique of Bellman and Wing (1976). This method leads to a third order differential equation for the first passage density which is explicitly solved. The moments of excess overflow and the wet period and other relevant features are obtained. The method adopted is very different from that of Puri and Senthuria (1975), who considered the p.d.f. for the content at any time t and obtained the limiting distribution as t tends to infinity. By straight forward arguments important aspects relating to a model including random withdrawals along with deterministic release for a finite dam have been studied in detail.

In Section 2, we study the L.T. for the first passage time density for overflow before emptiness. The first passage time density for overflow with any number of emptiness is considered in Section 3. Section 4 is devoted to the study of the first passage time for emptiness before overflow. In Section 5, we analyse the first passage time density for emptiness with any

number of overflows. Lastly in Section 6 we obtain the expected overflow before emptiness in a given time.

2. Imbedding equation for first passage time for overflow.

In this section we consider a stochastic model for dams in order to study the first passage time for overflow. Let $X(t)$ be the random variable representing the content of the dam at time t with an initial content u at time $t=0$. Let the inputs into the dam be random events occurring according to a Poisson process with intensity $\nu(1-p)=\nu_1$ where p is positive and always less than or equal to unity. Let there be a sequence of outputs occurring as another Poisson process with intensity $\nu p=\nu_2$. Thereby we mean that in a small interval of time dt there is either a jump up or jump down in the level of the dam with frequencies ν_1 and ν_2 respectively. We also assume that there is a continuous release from the dam proportional to the content $X(t)$ at time t. The dam is of finite capacity K.

The equation describing the process is

$$X(t) = u + \sum_{n_1=1}^{N_1(t)} Z_{n_1}^{+} e^{-\alpha(t-t_{n_1})} - \sum_{n_2=1}^{N_2(t)} Z_{n_2}^{-} e^{-\alpha(t-t_{n_2})} \qquad (2.1)$$

where $N_1(t)$ is the number of Poisson jumps upwards, $N_2(t)$ is the number of Poisson jumps downwards and $N_1(t)$ and $N_2(t)$ are non-zero integers. The initial content of the dam is $X(0)=u$

There is a continuous exponential release from the dam

$Z^{+}_{n_1}$ and $Z^{-}_{n_2}$ are two sequences of independent and identically distributed random variables having probability density functions $a_{+}(Z)$ and $a_{-}(Z)$ respectively. Since the dam is of finite capacity K the process X(t) has two barriers one at x=o and another at X=K. We are interested in the finding of the passage times for first overflow. Equally well we are interested in finding the passage time for first emptiness. To find these we define

$M_1(u,k,t)$ = the probability that the overflow occurs for the first time between time t and t+dt without emptiness occurring in the interval (o,t) and that $X(o)=u$ is the initial content at time t=o.

This means that if the initial level is $u \neq 0$, the overflow occurs between time t and t+dt with this probability and with no downward jump bringing the level below zero, in the time interval (o,t). However if the process starts with $u=0$ it does not return back to zero before it overflows first. The exponential decay cannot bring the level from any value $X=x$ to zero in any finite time. However the downward jump can make the level to cross zero. This we prevent by imposing the boundary conditions. In our previous Chapter $X=0$ was a natural boundary since we did not consider the downward jump for the first problem we studied in Chapter II.

We now decompose $M_1(u,k,t)$ into two mutually exclusive probabilities $M_1^+(u,k,t)$ and $M_1^-(u,k,t)$ where

$M_1^+(u,k,t)$ = the probability that the dam overflows for the first time without emptiness between time t and t+dt given that the first random jump whenever it occurs corresponds to an input into the dam

and

$M_1^-(u,k,t)$ = the probability that the dam overflow for the first time without any emptiness between time t and t+dt given that the first random jump whenever it occurs is an output from the dam.

Hence

$$M_1(u,k,t) = M_1^+(u,k,t) + M_1^-(u,k,t) \quad (2.2)$$

We have adopted this device of considering M_1^+ and M_1^- apparently to obtain equations for M_1. In our earlier work (Vasudevan et. al. 1978, 1979) we have considered only either pure jumps in both the directions or jumps with deterministic linear decay.

Considering the different independent possibilities which are mutually exclusive in the interval $(0, dt)$ and proceeding to the limit as dt tends to zero we arrive at the

imbedding equation (Bellman and Wing 1976) for $M_1^+(u,k,t)$ as

$$\frac{\partial M_1^+}{\partial t} + \alpha u \frac{\partial M_1^+}{\partial u} + \nu M_1^+$$

$$= \nu_1 \int_0^{k-u} M_1^+(u+z,k,t)\, a_+(z)\, dz + \delta(t) \int_{k-u}^{\infty} a_+(z)\, dz \tag{2.3}$$

Defining the following Laplace transforms with respect to t

$$\bar{M}_1(u,k,\ell) = \int_0^{\infty} e^{-\ell t}\, M_1(u,k,t)\, dt \tag{2.4}$$

$$\bar{M}_1^+(u,k,\ell) = \int_0^{\infty} e^{-\ell t}\, M_1^+(u,k,t)\, dt \tag{2.5}$$

The equation (2.3) gets converted into

$$\alpha u \frac{\partial \bar{M}_1^+}{\partial u} + (\ell+\nu)\bar{M}_1^+ = \nu_1 \int_0^{k-u} \bar{M}_1(u+z,k,\ell)\, a_+(z)\, dz + \nu_1 \int_{k-u}^{\infty} a_+(z)\, dz \tag{2.6}$$

A tractable solution can be obtained by taking the densities $a_+(z)$ and $a_-(z)$ as exponential (i.e.)

$$a_+(z) = \eta\, e^{-\eta z}\, H(z) \tag{2.7}$$

$$a_-(z) = \eta\, e^{\eta z}\, H(-z) \tag{2.8}$$

The equation (2.6) becomes

$$\alpha u \frac{\partial \bar{M}_1^+}{\partial u} + (\ell+\nu)\bar{M}_1^+ = \nu_1 \eta \int_0^{K-u} \bar{M}_1(y,K,\ell)\, e^{-\eta z}\, dz + \nu_1 e^{-\eta(K-u)} \tag{2.9}$$

We put $y = u + z$ in the integral in RHS of (2.9) and then dividing the equation (2.9) throughout by $e^{\eta u}$ we have

$$e^{-\eta u}\left[\alpha u \frac{\partial \bar{M}_1^+}{\partial u} + (\ell+\nu)\bar{M}_1^+\right] = \nu_1 \eta \int_u^K \bar{M}_1(y,K,\ell)\, e^{-\eta y}\, dy + \nu_1 e^{-\eta K} \tag{2.10}$$

Differentiating with respect to u and simplifying we get from (2.10)

$$\alpha u \frac{\partial^2 \bar{M}_1^+}{\partial u^2} + \left[\ell+\nu+\alpha-\eta\alpha u\right]\frac{\partial \bar{M}_1^+}{\partial u} - \eta(\ell+\nu)\bar{M}_1^+ = -\nu_1 \eta \bar{M}_1 \tag{2.11}$$

If we write down the imbedding equation for $M_1^-(u,K,t)$ and adopt the same procedure as was done in the case of M_1^+ function we obtain the second order equation for $\bar{M}_1^-$ given by

$$\alpha u \frac{\partial^2 \bar{M}_1^-}{\partial u^2} + \left[\ell+\nu+\alpha+\eta\alpha u\right]\frac{\partial \bar{M}_1^-}{\partial u} + \eta(\ell+\nu)\bar{M}_1^- = \nu_1 \eta \bar{M}_1 \tag{2.12}$$

From the second order equations (2.11) and (2.12) we eliminate $\bar{M}_1^+$ and $\bar{M}_1^-$ and obtain a third order equation for the total function $\bar{M}_1(u,\kappa,\ell)$ which is L.T. of the p.d.f. for first overflow before emptiness as

$$\alpha u \frac{\partial^3 \bar{M}_1}{\partial u^3} + (\ell+\nu+2\alpha)\frac{\partial^2 \bar{M}_1}{\partial u^2}$$

$$+\eta\left[\nu(1-2p)-\alpha u\right]\frac{\partial \bar{M}_1}{\partial u} - \eta^2 \ell \bar{M}_1 = 0 \qquad (2.13)$$

If $\alpha = 0$ we obtain the second order equation which is the same as the equation in our earlier work (Vasudevan et. al. 1978) and this equation was easily solved.

Let us now assume for simplicity that $\eta = 1$ without loss of generality. Then we have the equation (from 2.13)

$$u\frac{\partial^3 \bar{M}_1}{\partial u^3} + \left(\frac{\ell+\nu}{\alpha}+2\right)\frac{\partial^2 \bar{M}_1}{\partial u^2}$$

$$+\left\{\frac{\nu(1-2p)}{\alpha} - u\right\}\frac{\partial \bar{M}_1}{\partial u} - \frac{\ell}{\alpha}\bar{M}_1 = 0 \qquad (2.14)$$

. Solutions for X(t).

From the equation (2.14) we can extract the solution for the transform $\bar{M}_1$ with respect to t. It can be shown (see Appendix A) that the solution of a third order differential equation of the type

$$x\frac{d^3y}{dx^3} + (a+b)\frac{d^2y}{dx^2} + (c-x)\frac{dy}{dx} - ay = 0 \qquad (2.15)$$

where a, b, c are constants is

$$y = \sum_{i=1}^{3} C_i \int_{\alpha_i}^{\beta_i} e^{-tx} \, |1-t^2|^{\frac{b}{2}-1} \, |t|^{a-1} \left| \frac{1+t}{1-t} \right|^{\frac{c}{2}} dt \qquad (2.16)$$

We determine 3 pairs of values (α_i, β_i), $i=1,2,3$ such that we have three independent solutions of the equation (2.15) (Forsyth, Chapter VII). In our problem for the equation (2.14)

$$a = \frac{\ell}{\alpha} \quad , \quad b = \frac{\nu}{\alpha} + 2 \quad , \quad c = \frac{\nu(1-2p)}{\alpha} \qquad (2.17)$$

Hence the solution of equation (2.14) is

$$\begin{aligned} \overline{M}_1(u,k,\ell) &= C_1 \int_{-1}^{0} e^{-tu} \, |t|^{\frac{\ell}{\alpha}-1} \, |1-t|^{\frac{\nu_2}{\alpha}} \, |1+t|^{\frac{\nu_1}{\alpha}} dt \\ &\quad + C_2 \int_{0}^{1} e^{-tu} \, |t|^{\ell/\alpha} \, |1-t|^{\frac{\nu_2}{\alpha}} \, |1+t|^{\nu_1/\alpha} \, dt \qquad (2.18) \\ &\quad + C_3 \int_{1}^{\infty} e^{-tu} \, |t|^{\ell/\alpha} \, |1-t|^{\nu_2/\alpha} \, |1+t|^{\nu_1/\alpha} \, dt \\ &= C_1 \int_{0}^{1} e^{tu} \, t^{\frac{\ell}{\alpha}-1} \, (1+t)^{\frac{\nu_2}{\alpha}} \, (1-t)^{\frac{\nu_1}{\alpha}} \, dt \\ &\quad + C_2 \int_{0}^{1} e^{-tu} \, t^{\frac{\ell}{\alpha}-1} \, (1-t)^{\nu_2/\alpha} \, (1+t)^{\nu_1/\alpha} \, dt \\ &\quad + C_3 \int_{1}^{\infty} e^{-tu} \, t^{\frac{\ell}{\alpha}-1} \, (t-1)^{\nu_2/\alpha} \, (1+t)^{\nu_1/\alpha} \, dt \qquad (2.19) \end{aligned}$$

To obtain C_1, C_2 and C_3 we have to apply the boundary conditions. $M_1(u,k,t)$ is the probability for first overflow before emptiness occurs. However if we start with $u=0$ the process should not cross zero because of downward jumps. Hence we write that $\bar{M}_1(u,k,\ell)$ is finite and $\bar{M}_1(0_-,k,\ell)=0$.

If we substitute $u=0$ in the third integral on RHS of equation (2.19) we find that $\bar{M}_1(u,k,\ell)$ is infinite since

$$\int_1^\infty t^{\frac{\ell}{\alpha}-1}(t-1)^{\nu_2/\alpha}(1+t)^{\nu_1/\alpha}\,dt$$

$$= B\left(-\frac{(\ell+\nu)}{\alpha},\ \frac{\nu_2}{\alpha}+1\right)\times$$

$$\times\ \Phi_1\left[-\frac{\ell+\nu}{\alpha},-\frac{\nu_1}{\alpha},-\frac{\ell+\nu}{\alpha}+\frac{\nu_2}{\alpha}+1,-1,0\right] \tag{2.20}$$

is divergent as Beta function blows up for $\ell>0$, and Φ_1 function which is ${}_1F_1\left(-\frac{\ell+\nu}{\alpha},-\frac{\ell+\nu}{\alpha}+\frac{\nu_2}{\alpha}+1,-1\right)$ is finite. The evaluation of the integral (2.20) is from Gradshteyn and Ryzhik (page 321-3.185). Hence the coefficient C_3 should be chosen as zero. The solution becomes

$$\bar{M}_1(u,k,\ell) = C_1\int_0^1 e^{tu}\,t^{\frac{\ell}{\alpha}-1}(1-t)^{\frac{\nu_1}{\alpha}}(1+t)^{\frac{\nu_2}{\alpha}}\,dt$$

$$+C_2\int_0^1 e^{-tu}\,t^{\frac{\ell}{\alpha}-1}(1-t)^{\nu_2/\alpha}(1+t)^{\nu_1/\alpha}\,dt \tag{2.21}$$

The expressions for C_1 and C_2 which are functions of K and ℓ and are independent of u have to be determined by other boundary conditions. The boundary condition for crossing the barrier X=K is taken care of by the first order integro differential equation to be satisfied by the total $\bar{M}_1(u,K,\ell)$ of equation (2.2). To obtain this we write down the imbedding equation for $M_1(u,K,t)$:

$$\frac{\partial M_1}{\partial t} + \alpha u \frac{\partial M_1}{\partial u} + \nu M_1 = \nu_1 \int_0^{K-u} M_1(u+z,K,t) e^{-z} dz + \delta(t)\, \nu_1 \int_{K-u}^{\infty} e^{-z} dz + \nu_2 \int_{-u}^{0} M_1(u+z,K,t) e^{z} dz \tag{2.22}$$

which gets transformed to

$$e^{-u}\left[\alpha u \frac{\partial \bar{M}_1}{\partial u} + (\ell+\nu)\bar{M}_1\right] = \nu_1 \int_u^K \bar{M}_1(y,K,\ell) e^{-y} dy + \nu_1 e^{-K} + e^{-2u} \nu_2 \int_0^u \bar{M}_1(y,K,\ell) e^{y} dy \tag{2.23}$$

Substituting the solution (2.21) in (2.23) we get

$$e^{-u} c_1 \Big[\alpha u \frac{\partial}{\partial u} \int_0^1 e^{tu} t^{\frac{\ell}{\alpha}-1} (1-t)^{\nu_1/\alpha} (1+t)^{\nu_2/\alpha} dt$$

$$+ (\ell+\nu) \int_0^1 e^{tu} t^{\frac{\ell}{\alpha}-1} (1-t)^{\nu_1/\alpha} (1+t)^{\nu_2/\alpha} dt \Big]$$

$$+ e^{-u} c_2 \Big[\alpha u \frac{\partial}{\partial u} \int_0^1 e^{-tu} t^{\frac{\ell}{\alpha}-1} (1-t)^{\nu_2/\alpha} (1+t)^{\nu_1/\alpha} dt$$

$$+ (\ell+\nu) \int_0^1 e^{-tu} t^{\frac{\ell}{\alpha}-1} (1-t)^{\nu_2/\alpha} (1+t)^{\nu_1/\alpha} dt \Big]$$

$$= \nu_1 c_1 \int_0^1 t^{\frac{\ell}{\alpha}-1} (1-t)^{\frac{\nu_1}{\alpha}-1} (1+t)^{\nu_2/\alpha} \Big[e^{(t-1)u} - e^{(t-1)K} \Big] dt$$

$$+ \nu_1 c_2 \int_0^1 t^{\frac{\ell}{\alpha}-1} (1-t)^{\nu_2/\alpha} (1+t)^{\frac{\nu_1}{\alpha}-1} \Big[e^{-(t+1)u} - e^{-(1+t)K} \Big] dt$$

$$+ \nu_2 e^{-2u} c_1 \int_0^1 t^{\frac{\ell}{\alpha}-1} (1-t)^{\nu_1/\alpha} (1+t)^{\nu_2/\alpha - 1} \Big[e^{(t+1)u} - 1 \Big] dt$$

$$+ \nu_2 e^{-2u} c_2 \int_0^1 t^{\frac{\ell}{\alpha}-1} (1-t)^{\frac{\nu_2}{\alpha}-1} (1+t)^{\frac{\nu_1}{\alpha}} \Big[e^{(1-t)u} - 1 \Big] dt$$

$$+ \nu_1 e^{-K} \qquad (2.24)$$

It can be easily shown that the coefficients of $e^{-u} c_1$ and $e^{-u} c_2$ in the above equation (2.24) becomes zero (see Appendix 2). The above equation then reduces to two terms one containing e^{-2u} and the other independent of u. As the

equation is an identity for all values of u in $0 \leq u \leq K$, we get two conditions connecting C_1 and C_2 by equating to zero those terms separately. Hence we have after deduction

$$C_1 \int_0^1 e^{tK} t^{\frac{\ell}{\alpha}-1} (1-t)^{\frac{\nu_1}{\alpha}-1} (1+t)^{\frac{\nu_2}{\alpha}} dt$$

$$+ C_2 \int_0^1 e^{-tK} t^{\frac{\ell}{\alpha}-1} (1-t)^{\frac{\nu_2}{\alpha}} (1+t)^{\frac{\nu_1}{\alpha}-1} dt$$

$$= 1 \tag{2.25}$$

and

$$C_1 \int_0^1 t^{\frac{\ell}{\alpha}-1} (1-t)^{\nu_1/\alpha} (1+t)^{\frac{\nu_2}{\alpha}-1} dt$$

$$+ C_2 \int_0^1 t^{\frac{\ell}{\alpha}-1} (1-t)^{\frac{\nu_2}{\alpha}-1} (1+t)^{\frac{\nu_1}{\alpha}} dt = 0 \tag{2.26}$$

From tables of integrals (Gradshteyn and Ryzhik, page 231, 3.385, 1965) we have

$$\int_0^1 x^{m-1} (1-x)^{n-1} (1-\beta x)^{-p} e^{-\mu x} dx$$

$$= B(m,n) \; \Phi_1[m, p, m+n, \beta, -\mu] \tag{2.27}$$

where $\mathrm{Re}\, m > 0$, $\mathrm{Re}\, n > 0$, $|\arg(1-\beta x)| < \pi$. Here B is the usual beta function and Φ_1 is the degenerate hypergeometric function of two variables defined by

$$\Phi_1(\alpha,\beta,\gamma,x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m!\,n!} \tag{2.28}$$

(also see Erdilyi, page 139: (24) and page 384)

Using the result (2.27) in (2.25) and (2.26) we arrive at the equations

$$C_1 B\left[\frac{\ell}{\alpha}, \frac{\nu_1}{\alpha}\right] \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_2}{\alpha}, \frac{\ell+\nu_1}{\alpha}, -1, K\right]$$

$$+ C_2 B\left[\frac{\ell}{\alpha}, \frac{\nu_2}{\alpha}+1\right] \Phi_1\left[\frac{\ell}{\alpha}, 1-\frac{\nu_1}{\alpha}, \frac{\ell+\nu_2}{\alpha}+1, -1-K\right]$$

$$= 1 \tag{2.29}$$

and

$$C_1 B\left[\frac{\ell}{\alpha}, \frac{\nu_1}{\alpha}+1\right] \Phi_1\left[\frac{\ell}{\alpha}, 1-\frac{\nu_2}{\alpha}, \frac{\ell+\nu_1}{\alpha}+1, -1, 0\right]$$

$$+ C_2 B\left[\frac{\ell}{\alpha}, \frac{\nu_2}{\alpha}\right] \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_1}{\alpha}, \frac{\ell+\nu_2}{\alpha}, -1, 0\right]$$

$$= 0 \tag{2.30}$$

The above two equations (2.29) and (2.30) can be written as

$$C_1 L X + C_2 M Y = 1 \tag{2.31}$$

$$C_1 Z N + C_2 W P = 0 \tag{2.32}$$

where

$$X = B\left[\frac{\ell}{\alpha}, \frac{\nu_1}{\alpha}\right] \tag{2.33}$$

$$Y = B\left[\frac{\ell}{\alpha}, \frac{\nu_2}{\alpha}+1\right] = \frac{\nu_2}{\ell+\nu_2} W \tag{2.34}$$

$$Z = B\left[\frac{\ell}{\alpha}, \frac{\nu_1}{\alpha}+1\right] = \frac{\nu_1 X}{\ell+\nu_1} \tag{2.35}$$

$$W = B\left[\frac{\ell}{\alpha}, \frac{\nu_2}{\alpha}\right] \tag{2.36}$$

$$L = \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_2}{\alpha}, \frac{\ell+\nu_1}{\alpha}, -1, K\right] \tag{2.37}$$

$$M = \Phi_1\left[\frac{\ell}{\alpha}, 1-\frac{\nu_1}{\alpha}, \frac{\ell+\nu_2}{\alpha}+1, -1, -K\right] \tag{2.38}$$

$$N = \Phi_1\left[\frac{\ell}{\alpha}, 1-\frac{\nu_2}{\alpha}, \frac{\ell+\nu_1}{\alpha}+1, -1, 0\right] \tag{2.39}$$

$$= {}_1F_1\left[\frac{\ell}{\alpha}, \frac{\ell+\nu_1}{\alpha}+1, -1\right]$$

$$P = \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_1}{\alpha}, \frac{\ell+\nu_2}{\alpha}, -1, 0\right] \tag{2.40}$$

$$= {}_1F_1\left[\frac{\ell}{\alpha}, \frac{\ell+\nu_2}{\alpha}, -1\right]$$

Solving the equations (2.31) and (2.32) we get

$$C_1 = \frac{WP}{WXLP - YZMN} \tag{2.41}$$

$$C_2 = \frac{-ZN}{WXLP - YZMN} \tag{2.42}$$

Now the complete closed form solution for $\bar{M}_1(u, K, \ell)$ is

$$\bar{M}_1(u,K,\ell) = C_1 Z \; \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_2}{\alpha}, \frac{\ell+\nu_1}{\alpha}+1, -1, u\right]$$
$$+ C_2 Y \; \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_1}{\alpha}, \frac{\ell+\nu_2}{\alpha}+1, -1, -u\right] \tag{2.43}$$

where C_1 and C_2 are given by (2.41) and (2.42). Thus we have obtained a closed solution for $\bar{M}_1(u, K, \ell)$.

<u>Deductions:</u> In the description of our model (2.1) let us now consider the output (release) as purely deterministic. This means that we are considering the case $\nu_2 = 0$ and $\nu_1 = \nu$ itself. The integral occurring as coefficient of C_2 in the solution exists only for the case ν_2 not equal to zero. This is reflected in the fact that in the expression for C_2, W appears in the denominator and W blows up when $\nu_2 = 0$ while all other terms in C_2 are finite for $\nu_2 = 0$. Hence $C_2 = 0$ and the solution is

$$\bar{M}_1(u,k,\ell)\Big|_{\nu_2=0} = C_1 B\left[\frac{\ell}{\alpha},\frac{\nu}{\alpha}+1\right]\Phi_1\left[\frac{\ell}{\alpha},0,\frac{\ell+\nu}{\alpha}+1,-1,u\right] \tag{2.44}$$

It can be easily seen from the series expansion of Φ_1 given by (2.28)

$$\Phi_1\left(\frac{\ell}{\alpha},0,\frac{\ell+\nu}{\alpha}+1,-1,u\right) = {}_1F_1\left[\frac{\ell}{\alpha},\frac{\ell+\nu}{\alpha}+1,u\right] \tag{2.45}$$

The only quantity which remains to be determined can be evaluated by the use of (2.29). This can be seen from the representation of ${}_1F_1$ given in Gradshteyn and Ryzhik (page 318, 3.383, 1965)

$$C_1 B\left[\frac{\ell}{\alpha},\frac{\nu}{\alpha}\right] {}_1F_1\left[\frac{\ell}{\alpha},\frac{\ell+\nu}{\alpha},k\right] = 1 \tag{2.46}$$

and hence

$$\bar{M}_1(u,k,\ell)\Big|_{\nu_2=0} = \frac{\nu}{\ell+\nu}\,\frac{{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\nu}{\alpha}+1,u\right)}{{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\nu}{\alpha},k\right)} \tag{2.47}$$

This is the result we arrived at as the L.T. of the interval density for the first passage time to cross the barrier at $X=K$, starting from $X(0)=u$, when there was only jumps upwards (equivalent to $\nu_2=0$, $\nu_1=\nu$) and exponential

deterministic decay as (see equation (2.17) of Chapter II).

3. First passage for overflow with any number of emptiness.

In the previous section we considered a two barrier problem and studied the first passage time to cross the barrier at X=K as the conditional probability that the barrier X=0 is not crossed, during the time (0,t) before it overflows at time t+dt. In this section we define the probability $M_2(u,k,t)$ for the first overflow regardless of the number of times of emptiness occurs in the time interval (0,t).

Define

$M_2(u,k,t)$ = the probability that the overflow occurs for the first time between time t and t+dt given that there can be any number of emptiness in (0,t) and $X(0)=u$.

$M_2^+(u,k,t)$ = the probability that the overflow occurs for the first time between time t and t+dt given that the first random jump whenever it occurs corresponds to an input into the dam.

$M_2^-(u,k,t)$ = the probability that the overflow occurs for the first time between time t and t+dt given that the first random jump whenever occurs corresponds to an output from the dam.

Clearly

$$M_2(u,k,t) = M_2^+(u,k,t) + M_2^-(u,k,t) \tag{3.1}$$

Considering the possibilities in the initial interval $(0, dt)$ and proceeding to the limit as dt tends to zero, as is customary in the imbedding methods (Bellman and Wing 1976) we obtain

$$\frac{\partial M_2^+}{\partial t} + \alpha u \frac{\partial M_2^+}{\partial u} + \nu M_2^+ = \nu_1 \int_0^{K-u} M_2(u+z, K, t)\, a_+(z)\, dz + \delta(t)\nu_1 \int_{K-u}^{\infty} a_+(z)\, dz \tag{3.2}$$

As before ν_1 is the Poisson intensity for positive jumps, ν_2 is the Poisson intensity for negative jump and $\nu = \nu_1 + \nu_2$ for any jump. Defining $\bar{M}_2(u, K, \ell)$ and $\bar{M}_2^+(u, K, \ell)$ as the L.T's of $M_2(u, K, t)$ and $M_2^+(u, K, t)$ respectively we have from (3.2)

$$\alpha u \frac{\partial \bar{M}_2^+}{\partial u} + (\ell + \nu)\bar{M}_2^+ = \nu_1 \int_0^{K-u} \bar{M}_2(u+z, K, \ell)\, a_+(z)\, dz + \nu_1 \int_{K-u}^{\infty} a_+(z)\, dz \tag{3.3}$$

Taking $a_+(z)$ as defined in section 2 (2.7) and change the variable z to y by $u + z = y$ we have

$$e^{-\eta u}\left[\alpha u \frac{\partial M_2^+}{\partial u} + (\ell + \nu)\bar{M}_2^+\right] = \nu_1 \eta \int_u^K \bar{M}_2(y, K, \ell)\, e^{-\eta y}\, dy + \nu_1 e^{-\eta K} \tag{3.4}$$

Differentiating with respect to u and simplifying we get

$$\alpha u \frac{\partial^2 \bar{M}_2^+}{\partial u^2} + \left[\ell + \nu + \alpha - \eta \alpha u\right] \frac{\partial \bar{M}_2^+}{\partial u} - \eta(\ell+\nu)\bar{M}_2^+ = -\nu_1 \eta \bar{M}_2 \tag{3.5}$$

The imbedding equation for $\bar{M}_2^-(u,k,t)$ is

$$\frac{\partial M_2^-}{\partial t} + \alpha u \frac{\partial M_2^-}{\partial u} + \nu M_2^- = \nu_2 \int_{-u}^{0} M_2(u+z,k,t)\, a_-(z)\, dz + \nu_2 \int_{-\infty}^{-u} a_-(z)\, dz\, M_2(0,k,t) \tag{3.6}$$

$M_2(0,k,t)$ represents the probability that starting from zero the dam overflows between time t and t+dt for the first time with any number of emptiness in $(0,t)$. This weight has to be added in the last term of equation (3.6) to account for the barrier at $x = 0$. Taking L.T. with respect to the variable t and substituting for $a_-(z)$ the exponential density given by (2.8) the equation (3.6) becomes

$$e^{\eta u}\left[\alpha u \frac{\partial \bar{M}_2^-}{\partial u} + (\ell+\nu)\bar{M}_2^-\right] = \nu_2 \eta \int_0^u \bar{M}_2(y,k,\ell)\, e^{\eta y}\, dy + \nu_2 \bar{M}_2(0,k,\ell) \tag{3.7}$$

This on differentiation with respect to u becomes after

simplification

$$\alpha u \frac{\partial^2 \bar{M}_2^-}{\partial u^2} + \left[\ell + \nu + \alpha + \eta \alpha u \right] \frac{\partial \bar{M}_2}{\partial u}$$

$$+ \eta (\ell + \nu) \bar{M}_2 = \nu \eta \bar{M}_2 \tag{3.8}$$

Eliminating $\bar{M}_2^+$ and $\bar{M}_2^-$ from equations (3.5) and (3.8) we get the third order differential equation

$$\alpha u \frac{\partial^3 \bar{M}_2}{\partial u^3} + \left[\ell + \nu + 2\alpha \right] \frac{\partial^2 \bar{M}_2}{\partial u^2}$$

$$+ \eta \left[\nu (1 - 2p) - \alpha u \right] \frac{\partial \bar{M}_2}{\partial u} - \eta \ell \bar{M}_2 = 0 \tag{3.9}$$

Let us put $\eta = 1$ without loss of generality. The solution of equation (3.9) as in the last section is

$$\bar{M}_2(u, K, \ell) = D_1 \int_0^1 e^{tu} t^{\frac{\ell}{\alpha} - 1} (1-t)^{\nu_1/\alpha} (1+t)^{\nu_2/\alpha} dt$$

$$+ D_2 \int_0^1 e^{-tu} t^{\frac{\ell}{\alpha} - 1} (1-t)^{\nu_2/\alpha} (1+t)^{\nu_1/\alpha} dt \tag{3.10}$$

There is of course 3 different solution for the third order equation as was found in the case of $\bar{M}_1$ in section 2. The last integral to be usually added to equation (3.10) is

$$D_3 \int_1^\infty e^{-tu} t^{\frac{\ell}{\alpha} - 1} (t-1)^{\nu_2/\alpha} (1+t)^{\nu_1/\alpha} dt$$

Since the process can come down to zero even if starts at $X(0)=u$ we have to investigate the appearance of the term with coefficient D_3 when $u=0$. As argued earlier in section 2 this term will diverge. Hence we put $D_3=0$. The first order equation for $M_2(u,K,t)$ serves as the boundary condition for determining D_1 and D_2. Hence we have

$$e^{-u}\left[\alpha u \frac{\partial \bar{M}_2}{\partial u} + (\ell+\nu)\bar{M}_2\right]$$
$$= \nu_1 \int_u^K \bar{M}_2(y,K,\ell)\, e^{-y}\, dy + \nu_1 e^{-K}$$
$$+ e^{-2u} \nu_2 \int_0^u \bar{M}_2(y,K,\ell)\, e^{y}\, dy$$
$$+ e^{-2u} \nu_2 \bar{M}_2(0,K,\ell) \qquad (3.11)$$

Notice the additional term with $\bar{M}_2(0,K,\ell)$ appearing in equation (3.11) to take care of the boundary conditions. A term of this type does not occur in the first order equation for $\bar{M}_1$ since it refers to first overflow before emptiness.

Substituting the solution (3.10) in (3.11) we can easily see that as in the case of $\bar{M}_1$ (section 2) the coefficients of e^{-u} taken together vanish and equation (3.11) reduces to two terms, one containing e^{-2u} and the other independent of u. As the resulting equation is true for all values of u, $(0 \leq u \leq k)$ we get two equations connecting D_1 and D_2. Hence we have

$$D_1 B\left[\frac{\ell}{\alpha}, \frac{\nu_1}{\alpha}\right] \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_2}{\alpha}, \frac{\ell+\nu_1}{\alpha}, -1, K\right]$$
$$+ D_2 B\left[\frac{\ell}{\alpha}, \frac{\nu_2}{\alpha}+1\right] \Phi_1\left[\frac{\ell}{\alpha}, 1-\frac{\nu_1}{\alpha}, \frac{\ell+\nu_2}{\alpha}+1, -1, -K\right]$$
$$= 1 \qquad (3.12)$$

and

$$D_1 B\left[\frac{\ell}{\alpha}, \frac{\nu_1}{\alpha}+1\right] \Phi_1\left[\frac{\ell}{\alpha}, 1-\frac{\nu_2}{\alpha}, \frac{\ell+\nu_1}{\alpha}+1, -1, 0\right]$$

$$+ D_2 B\left[\frac{\ell}{\alpha}, \frac{\nu_2}{\alpha}\right] \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_1}{\alpha}, \frac{\ell+\nu_2}{\alpha}, -1, 0\right]$$

$$- D_1 B\left[\frac{\ell}{\alpha}, \frac{\nu_1}{\alpha}+1\right] \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_2}{\alpha}, \frac{\ell+\nu_1}{\alpha}+1, -1, 0\right]$$

$$- D_2 B\left[\frac{\ell}{\alpha}, \frac{\nu_2}{\alpha}+1\right] \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_1}{\alpha}, \frac{\ell+\nu_2}{\alpha}+1, -1, 0\right]$$

$$= 0. \tag{3.13}$$

Notice that the differences in (3.13) with the equation (2.30) for C_1 and C_2 in section 2. The above equations (3.12) and (3.13) can be written as

$$D_1 L X + D_2 M Y = 1 \tag{3.14}$$

$$D_1 [ZN - ZQ] + D_2 [WP - YR] = 0 \tag{3.15}$$

where X,Y,Z,L,M,N,P are given in (2.34) to (2.40) and

$$Q = \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_2}{\alpha}, \frac{\ell+\nu_1}{\alpha}+1, -1, 0\right] \tag{3.16}$$

$$R = \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_1}{\alpha}, \frac{\ell+\nu_2}{\alpha}+1, -1, 0\right] \tag{3.17}$$

Solving equations (3.14) and (3.15) we get

$$D_1 = (WP-YR) \Big/ \{LX(WP-YR) - MY(ZN-ZQ)\} \tag{3.18}$$

$$D_2 = -Z(N-Q) \Big/ \{LX(WP-YR) - MY(ZN-ZQ)\} \tag{3.19}$$

Hence the complete solution for $\bar{M}_2(u,K,\ell)$ is known since D_1 and D_2 are given in terms of known functions. When we proceed to the limit as ℓ tends to zero we find that $\bar{M}_2$ tends to unity. This is the result one should expect since $\bar{M}_2$ involves crossing the barrier X=K with any number of emptiness.

We shall now consider the case $\nu_2 = 0$. The arguments given in section 2 can be repeated here when there are no downward jumps. It is easily seen that only M_2^+ exists and it is identical with M_1^+ when $\nu_2 = 0$. This is because when $u \neq 0$ the barrier X=0 is never reached in a finite time because of the exponential decay in the absence of jumps. However if the process starts from $u = 0$, after first jump it can never reach again the barrier X=0. Hence

$$\bar{M}_2(u,K,\ell)\Big|_{\nu_2=0} = \frac{\nu}{\ell+\nu} \; \frac{{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}+1, u\right)}{{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\nu}{\alpha}, K\right)} \tag{3.20}$$

where $\nu_1 = \nu$.

4. <u>First passage time to emptiness before overflow</u>.

In this section we determine the first passage time to emptiness before overflow for the dam model described in section 2. Define

$N_1(u,k,t)$ = the probability that the dam becomes empty for the first time without any overflow between time t and t+dt when $X(0) = u$ is the initial content of the dam of capacity K.

As was done in earlier sections, we split $N_1(u,k,t)$ as

$$N_1(u,k,t) = N_1^+(u,k,t) + N_1^-(u,k,t) \quad (4.1)$$

where the super fixes (+) and (-) have the same interpretations as for the cases relating to M_1 and M_2. We arrive at the imbedding equation for $N_1^+(u,k,t)$ as the integro differential equation

$$\frac{\partial N_1^+}{\partial t} + \alpha u \frac{\partial N_1^+}{\partial u} + \nu N_1^+ = \nu_1 \int_0^{k-u} N_1(u+z,k,t)\, a_+(z)\, dz \quad (4.2)$$

Taking L.T. with respect to t and choosing the density $a_+(z)$ as in equation (2.7) we get

$$e^{-\eta u}\left[\alpha u \frac{\partial \bar{N}_1^+}{\partial u} + (\ell+\nu)\bar{N}_1^+\right] = \nu_1 \eta \int_u^k \bar{N}_1(y,k,\ell)\, e^{-\eta y}\, dy \quad (4.3)$$

where $\bar{N}$ and $\bar{N}_1^+$ as the L.Ts of N_1 and N_1^+ respectively. This leads to the differential equation

$$\alpha u \frac{\partial^2 \bar{N}_1^+}{\partial u^2} + \left[\ell+\nu+\alpha-\eta\alpha u\right] \frac{\partial \bar{N}_1^+}{\partial u}$$

$$-\eta(\ell+\nu)\bar{N}_1^+ = -\nu_1\eta\,\bar{N}_1 \tag{4.4}$$

The imbedding equation for $\bar{N}_1^-(u,K,\ell)$, the L.T. of $N_1^-(u,K,t)$ is given by

$$\alpha u \frac{\partial \bar{N}_1^-}{\partial u} + (\ell+\nu)\bar{N}_1^- =$$

$$= \nu_2 \int_{-u}^{0} \bar{N}_1(u+z,K,\ell)\, a_-(z)\,dz + \nu_2 \int_{-\infty}^{-u} a_-(z)\,dz \tag{4.5}$$

Proceeding on the same lines as for $\bar{N}_1^+$ we get the second order differential equation

$$\alpha u \frac{\partial^2 \bar{N}_1^-}{\partial u^2} + \left[\ell+\nu+\alpha+\eta\alpha u\right] \frac{\partial \bar{N}_1^-}{\partial u}$$

$$+\eta(\ell+\nu)\bar{N}_1^- = \nu_2\eta\,\bar{N}_2 \tag{4.6}$$

Eliminating $\bar{N}_1^+$ and $\bar{N}_1^-$ from (4.4) and (4.6) we get the third order differential equation

$$\alpha u \frac{\partial^3 \bar{N}_1}{\partial u^3} + \left[\ell+\nu+2\alpha\right]\frac{\partial^2 \bar{N}_1}{\partial u^2} + \eta\left[\nu(1-2p)-u\right]\frac{\partial \bar{N}_1}{\partial u}$$

$$-\frac{\eta}{\alpha}\ell\,\bar{N}_1 = 0. \tag{4.7}$$

whose solution when $\eta = 1$ is

$$\bar{N}_1(u,K,\ell) = E_1 B\left[\frac{\ell}{\alpha}, \frac{\nu_1}{\alpha}+1\right] \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_2}{\alpha}, \frac{\ell+\nu_1}{\alpha}+1, -1, u\right]$$

$$+E_2 B\left[\frac{\ell}{\alpha}, \frac{\nu_2}{\alpha}+1\right] \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_1}{\alpha}, \frac{\ell+\nu_2}{\alpha}+1, -1, u\right] \tag{4.8}$$

As was done in the previous sections the functions E_1 and E_2 which are independent of u can be determined by substituting the solution $\bar{N}_1$ in the first order equation for the total $\bar{N}_1$. The integro-differential equation for $\bar{N}_1$ which takes care of the boundary is

$$e^{-u}\left[\alpha u \frac{\partial \bar{N}_1}{\partial u} + (\ell+\nu)\bar{N}_1\right] = \nu_1 \int_u^K \bar{N}_1(y,K,\ell)\, e^{-y}\, dy$$

$$+ e^{-2u} \nu_2 \int_0^u \bar{N}_1(y,K,\ell)\, e^{y}\, dy + e^{-2u} \nu_2 \tag{4.9}$$

Substituting the solution (4.8) in (4.9) and simplifying we get two equations connecting E_1 and E_2 by arguing out in the same fashion as in sections 2 and 3. These equations are

$$E_1 L X + E_2 M Y = 0 \tag{4.10}$$

$$E_1 Z N + E_2 W P = 1 \tag{4.11}$$

Solving (4.10) and (4.11) we get

$$E_1 = MY/\{YZMN - XWPL\} \tag{4.12}$$

$$E_2 = -LX/\{YZMN - XWPL\} \tag{4.13}$$

In view of the equation (4.8), (4.12) and (4.13) we are in possession of the analytical solution in closed form for $\bar{N}_1(u, K, \ell)$.

Deductions:

The case $\nu_2 = 0$ corresponds to the process with jumps only in the upward direction and deterministic exponential decay downwards. This means that there is no random jumps in the downward direction when we put $\nu_2 = 0$. As we have already seen in Chapter II (Vasudevan et. al. 1980) if the process starts from a level $u \neq 0$ it can never come down to zero. This result agrees with the well known fact that there is zero probability for the dam to become empty in a finite time if there is an exponential release rule and jumps in the positive direction.

For a dam like system with finite capacity K, the level can either come down to emptiness or overflows the barrier K if sufficient time is allowed. This total probability will tend to unity as time approaches infinity. In otherwords the

sum $\bar{M}_1(u,k,\ell)+\bar{N}_1(u,k,\ell)$ tends to unity as t tends to infinity. As was demonstrated in the previous sections it can be easily shown that

$$\lim_{\ell\to 0} \bar{M}_1(u,k,\ell) = \nu_1/(\nu_1+\nu_2)$$

and

$$\lim_{\ell\to 0} \bar{N}_1(u,k,l) = \nu_2/(\nu_1+\nu_2)$$

Hence $\int_0^\infty M_1(u,k,t)\,dt + \int_0^\infty N_1(u,k,t)\,dt$, the total probability for getting out of the system in infinite time is unity.

5. <u>First emptiness with overflow.</u>

In this section we consider the probability density function for first emptiness time regardless of the number of times the system has overflowed before the first emptiness. For the dam model in the earlier sections we define,

$N_2(u,k,t)$ = the probability that the dam becomes empty for the first time between time t and t+dt by allowing any number of overflows during this interval of time (0,t) with X(0)= u .

Also

$$N_2(u,k,t) = N_2^+(u,k,t) + N_2^-(u,k,t) \tag{5.1}$$

where (+) and (-) signs have the usual significances as seen in earlier sections. The imbedding equation for $N_2^+(u,k,t)$ is

$$\frac{\partial N_2^+}{\partial t} + \alpha u \frac{\partial N_2^+}{\partial u} + \nu N_2^+ = \nu_1 \int_0^{k-u} N_2(u+z,k,t)\, a_+(z)\, dz + \nu_1 \int_{k-u}^{\infty} a_+(z)\, dz \,.\, N_2(k,k,t) \tag{5.2}$$

From this imbedding equation we arrive at for the L.T. of $\bar{N}_2^+(u,k,\ell)$ as :-

$$\alpha u \frac{\partial \bar{N}_2^+}{\partial u} + (\ell + \nu + \alpha - \eta \alpha u) \frac{\partial \bar{N}_2^+}{\partial u} - \eta(\ell+\nu)\bar{N}_2^+ = -\nu_1 \eta \bar{N}_2 \tag{5.3}$$

The second order differential equation for $\bar{N}_2^-(u,k,\ell)$ is

$$\alpha u \frac{\partial^2 \bar{N}_2^-}{\partial u^2} + (\ell + \nu + \alpha + \eta \alpha u) \frac{\partial \bar{N}_2^-}{\partial u} + \eta(\ell+\nu)\bar{N}_2^- = \nu_2 \eta \bar{N}_2 \tag{5.4}$$

The third order differential equation for $\bar{N}_2(u,k,\ell)$ derived from (5.3), (5.4) is

$$\alpha u \frac{\partial^3 \bar{N}_2}{\partial u^3} + (\ell + \nu + 2\alpha) \frac{\partial^2 \bar{N}_2}{\partial u^2} + \eta\left[\nu(1-2p) - \alpha u\right] \frac{\partial \bar{N}_2}{\partial u} - \eta \ell \bar{N}_2 = 0 \tag{5.5}$$

When η is taken as unity for simplicity the solution of equation (5.5) is

$$\bar{N}_2(u,K,\ell) = F_1 B\left[\frac{\ell}{\alpha}, \frac{\nu_1}{\alpha}+1\right] \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_2}{\alpha}, \frac{\ell+\nu_1}{\alpha}+1, -1, u\right]$$

$$+ F_2 B\left[\frac{\ell}{\alpha}, \frac{\nu_2}{\alpha}+1\right] \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_1}{\alpha}, \frac{\ell+\nu_2}{\alpha}+1, -1, -u\right] \tag{5.6}$$

F_1 and F_2 are determined from the first order equation for $\bar{N}_2(u,K,\ell)$ for the total process. The integro differential equation for $\bar{N}_2(u,K,\ell)$ is:

$$e^{-u}\left[\alpha u \frac{\partial \bar{N}_2}{\partial u} + (\ell+\nu)\bar{N}_2\right] = \nu_1 \int_u^K \bar{N}_2(y,K,\ell)\, e^{-y}\, dy$$

$$\bar{N}_2(K,K,\ell) + \nu_1 e^{-K} + \nu_2 e^{-2u} \int_0^u \bar{N}_2(y,K,\ell)\, e^{y}\, dy + \nu_2 e^{-2u} \tag{5.7}$$

This integral equation serves as the boundary condition to determine F_1 and F_2. Substituting the solution (5.6) in (5.7) and proceeding exactly on the same lines as in the previous sections we get

$$F_1(Q_1 Z - L X) + F_2(Y R_1 - M Y) = 0 \tag{5.8}$$

and

$$F_1 Z N + F_2 W P = 1 \tag{5.9}$$

where

$$Q_1 = \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_2}{\alpha}, \frac{\ell+\nu_1}{\alpha}+1, -1, K\right] \tag{5.10}$$

$$R_1 = \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_1}{\alpha}, \frac{\ell+\nu_2}{\alpha}+1, -1, -K\right] \tag{5.11}$$

Solving these equations we get

$$F_1 = \frac{YR_1 - MY}{ZN(YR_1 - MY) - WP(Q_1 Z - LX)} \tag{5.11a}$$

$$F_2 = \frac{-(Q_1 Z - LX)}{ZN(YR_1 - MY) - WP(Q_1 Z - LX)} \tag{5.12}$$

The analytical solution for $\overline{N}_2(u, K, L)$ is completely determined. One can also easily verify that as ℓ tends to zero $\overline{N}_2(u, K, \ell)$ tends to unity as could be expected.

6. Excess overflow.

Another relevant feature to be investigated for this dam model is the excess of the water overflown in time t before emptiness occurs. This means the excess of the level of the process X(t) would have reached in time t over the barrier level K if the process is not chopped off at K. This is conditional on the fact that the level does not go to emptiness before time t. This aspect of our probabilistic model can also

be studied by the approach used in the previous sections. Define

$f(u,K,t)$ = the expected amount of water that has overflowed in time t+dt given that X(o)=u is the initial content of the dam at time t=o and that the dam has not become empty in the time interval (o,t).

Let us decompose f as before into f^+ and f^-

$$f(u,K,t) = f^+(u,K,t) + f^-(u,K,t) \tag{6.1}$$

where f^+ and f^- have the similar meanings as in earlier sections with regard to the notations (+) and (-).

Considering the dynamics of the process in the initial interval of time (0,dt) we have

$$\begin{aligned} f^+(u,K,t+dt) = {} & (1-\nu dt)\, f(u-\alpha u\, dt, K, t) \\ & + \nu_1 dt \int_0^{K-u} f(u+z, K, t)\, a_+(z).dz \\ & + \nu_1 dt \int_{K-u}^{\infty} a_+(z)\,[z-(K-u)]\, dz.\delta(t) \\ & + \nu_1\, dt \int_{K-u}^{\infty} a_+(z)\, dz\; f(K,K,t) \end{aligned} \tag{6.2}$$

This on proceeding to the limit as dt tends to zero becomes

$$\frac{\partial f^+}{\partial t} + \alpha u \frac{\partial f^+}{\partial u} + \nu f^+ =$$
$$= \nu_1 \int_0^{K-u} f(u+z, K, t)\, a_+(z)\, dz$$
$$+ \delta(t)\, \nu_1 \int_{K-u}^{\infty} a_+(z) \left[z-(K-u)\right] dz$$
$$+ \nu_1 \int_{K-u}^{\infty} a_+(z)\, dz\; f(K, K, t) \tag{6.3}$$

Defining $\bar{f}$ and $\bar{f}^+$ as the L.Ts. with respect to t of f and f^+ respectively and taking the density $a_+(z)$ as in (2.7) the equation (6.3) becomes after changing the variable to y by $z+u=y$,

$$e^{-\eta u}\left[\alpha u \frac{\partial \bar{f}^+}{\partial u} + (\ell+\nu)\bar{f}^+\right] = \nu_1 \eta \int_u^K e^{-\eta y}\, \bar{f}(y, K, \ell)\, dy$$
$$+ \nu_1 \eta \int_K^{\infty} e^{-\eta y} (y-K)\, dy$$
$$+ \nu_1 \eta \int_K^{\infty} e^{-\eta y}\, dy\; \bar{f}(K, K, \ell) \tag{6.4}$$

Differentiating this with respect to u and cancelling $e^{-\eta u}$ throughout we get the second order differential equation for $\bar{f}^+$ as

$$\alpha u \frac{\partial^2 \bar{f}^+}{\partial u^2} + \left[\ell+\nu+\alpha-\eta\alpha u\right] \frac{\partial \bar{f}^+}{\partial u} - \eta(\ell+\nu)\bar{f}^+ = -\nu_1 \eta \bar{f} \tag{6.5}$$

The imbedding equation for f^- is

$$\frac{\partial f^-}{\partial t} + \alpha u \frac{\partial f^-}{\partial u} + \nu f^- = \nu_2 \int_{-u}^{0} f(u+z, K, t)\, a_-(z)\, dz \tag{6.6}$$

Proceeding on the same lines as in $\bar{f}^{+}$ we get the differential equation for $\bar{f}^{-}$ as

$$\alpha u \frac{\partial^2 \bar{f}^-}{\partial u^2} + \left[\ell + \nu + \alpha + \eta \alpha u\right] \frac{\partial \bar{f}^-}{\partial u} + \eta(\ell+\nu)\bar{f}^- = \nu_2 \eta \bar{f} \tag{6.7}$$

The differential equation satisfying the total $\bar{f}$ can be obtained as

$$\alpha u \frac{\partial^3 \bar{f}}{\partial u^3} + (\ell + \nu + 2\alpha) \frac{\partial^2 \bar{f}}{\partial u^2}$$

$$+ \eta \left[\alpha(1-2p) - \alpha u\right] \frac{\partial \bar{f}}{\partial u} - \eta \ell \bar{f} = 0 \tag{6.8}$$

The solution for f is given by

$$\bar{f} = A_1 B\left[\frac{\ell}{\alpha}, \frac{\nu_1}{\alpha}+1\right] \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_2}{\alpha}, \frac{\ell+\nu_1}{\alpha}+1, -1, u\right]$$

$$+ A_2 B\left[\frac{\ell}{\alpha}, \frac{\nu_2}{\alpha}+1\right] \Phi_1\left[\frac{\ell}{\alpha}, -\frac{\nu_1}{\alpha}, \frac{\ell+\nu_2}{\alpha}+1, -1, -u\right] \tag{6.9}$$

The first order equation for $\bar{f}(u, K, \ell)$ which takes care of the boundary condition when $\eta = 1$ is

$$e^{-u}\left[\alpha u \frac{\partial \bar{f}}{\partial u} + (\ell+\nu)\bar{f}\right] = \nu_1 \int_u^K e^{-y} \bar{f}(y, K, \ell)\, dy$$

$$+ \nu_1 \int_K^\infty e^{-y}(y-K)\, dy + \nu_1 \int_K^\infty e^{-y}\, dy\, \bar{f}(K, K, \ell)$$

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$$+ e^{-2u} \nu_2 \int_0^u e^{y} f(y,k,\ell)\, dy \tag{6.10}$$

Feeding the solution (6.9) in (6.10) and carrying the usual analysis as done in earlier sections we arrive at the coupled equations for A_1 and A_2 as

$$A_1(LX - Q_1 Z) + A_2(MY - YR_1) = 1 \tag{6.11}$$

$$A_1 Z N + A_2 W P = 0 \tag{6.12}$$

Solving these equations we get

$$A_1 = \frac{WP}{WP(LX - ZQ_1) - ZN(MY - YR_1)} \tag{6.13}$$

$$A_2 = \frac{-ZN}{WP(LX - ZQ_1) - ZN(MY - YR_1)} \tag{6.14}$$

Thus $\bar{f}(u,k,\ell)$ is analytically determined.

7. <u>Summary of the results.</u>

In conclusion we point out that in the study of the analysis of first passage times we have considered a dam with random inputs and also random outputs apart from deterministic exponential decay of the level in the dam. Researches till now

have been centred only on deterministic outouts, either linear or exponential decay. We find only in the reference Puri and Senthuria (1975) that randomness in both inouts and outouts has been considered. However to our knowledge no work relating to random input and random output with deterministic decay, especially exponential decay has been studied.

We have derived closed form solutions for the L.T's of the probability density functions (1) M_1, the p.d.f. for first overflow before emptiness, (2) N_1, the p.d.f. for first emptiness before overflow. It has been observed that as ℓ tends to zero the sum of these functions tends to unity and this is in agreement with the mathematical fact that in infinite time the escaping of a particle having motion on the real line with two barriers one at $X=0$ and the other at X=K is a certainity. We have also derived closed expressions for $\overline{M}_2(u,k,\ell)$, the L.T. of p.d.f. for first overflow and $\overline{N}_2(u,k,\ell)$ the L.T. of the p.d.f. for first emptiness. We have noticed that each of these transforms tends to unity as ℓ tends to zero. Lastly we have obtained the expected overflow $f(u,k,t)$ in a given time in terms of $\overline{f}(u,k,\ell)$ the L.T. of $f(u,k,t)$.

In all these cases it has been possible to arrive at the solutions since we have decomposed each one of the p.d.f's as a sum of those which starts with an up kick whenever it occurs or with a down kick whenever it occurs. The imbedding equations for each of these has been written down separately. We have arrived at the total solutions for each of these functions in

terms of a third order differential equation if the deterministic decay has been exponential. However if the deterministic decay is considered as linear we would have arrived at only a second order differential equation which is simpler to solve (Vasudevan, et. al. 1978, 1979). The solutions are given in terms of Beta functions and confluent degenerate hypergeometric functions of two variables. (By considering the case $\nu_2 = 0$, which means that there is no downward jump we have obtained the solutions corresponding to upward jumps only and this agrees with the result obtained in Chapter 2 (equation 2.17).)

C H A P T E R - V

COLLECTIVE RISK THEORIES AND RUIN PROBABILITIES*

1. Introduction.

The collective theory of risk has been the object of series of investigations during the last few decades. A significant contribution to this has been made by Harald Cramer (1954), Filip Lundberg (1903), Sparre Anderson (1957), Segerdahl (1939, 1959, 1970), Olof Thorin (1970, 1971, 1974, 1975), Michael Harison (1977) and many others. The main problemsof interest in this field are (1) the probability distribution of the random variable X(t), representing the net reserve of an insurance company at time t and (2) the probability of ruin of the company in a given time t.

The classical theory of collective risk using the notion of probability was initiated by Filip Lundberg in the beginning of this century (1903). The primary model devoted to the study of this stochastic process is

$$X(t) = u + ct - \sum_{i=1}^{N(t)} Z_i \qquad (1.1)$$

In this model u represents the initial reserve at time t=o and c is the constant income to the company by means of interest and premiums. The liabilities to the company are the claims and these claims are assumed to occur in a Poissonian way

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with intensity ν . N(t) represents the number of Poisson claims in the time interval (0,t) and $N(t) > 0$. Z_i , representing the amounts of claims are assumed to be independent and identically distributed random variables with a common density function a(z). Obviously x(t) represents the net reserve at time t.

Further complications of problems (1) and (2) are due to the fact that the random claims may follow different distributions - general and special. But closed form solutions are available only in a few special cases. The study of ruin probability in general corresponds to the study of the motion of a particle with random jumps and linear upward drift with an absorbing barrier at $X = 0$. This single barrier problem has been studied by Keilson (1963) using the technique of compensation function. The classical problem of ruin probability when there is an upper limit K to the risk reserve is completely studied by Segerdahl (1970). This problem corresponds to the motion of a particle executing random walk along the real line with barrier at $X = 0$ and $X = K$. In this two barrier problem the ruin may occur after a number of crossings of the upper barrier K. Treating $X = K$ as a reflecting barrier excess of inputs cut off by the barrier $X = K$ before ruin takes place is also studied by Segerdahl (1970). In this model Segerdahl has taken the income rate to the company as a constant and the random claims to be governed by exponential density functions. One valid criticism of this classical model is the assumed independence

of the income rate c and the level of the risk reserve. Davidson suggested that the instantaneous income rate can be taken as $c(y)$ where y is taken as the risk reserve to study the realistic process

$$x(t) = u + \int_0^t c[x(t')]\,dt' - \sum_{i=1}^{N(t)} z_i \tag{1.2}$$

The stationary transition probabilities are to be obtained starting with $X(0) = u$. The jumps are Poisson with intensity ν and the jump amount are independent and identically distributed random variables. In between the jumps $X'(t) = c\,X(t)$. Segerdahl also gave an explicit expression for the ultimate ruin probability function $\psi(u)$ in terms of incomplete gamma function by taking the income process as $c(y) = \beta + \alpha y$. Here he treats the problem as a single barrier problem.

In this Chapter we consider the process (1.2) with

$$X'(t) = \beta + \alpha X \tag{1.3}$$

This means that the income to the company has two components. (1) the income through premiums coming at a constant rate β and (2) the interest earned continuously at a rate αX on the current capital. We study the process x(t) as a two barrier problem with the barrier $X = K$ as the upper limit for the risk reserve. The following probability distributions are arrived at.

(1) the probability distribution of ruin at time t given that the risk reserve has not reached its upper limit in time t. This means that the barrier $X=0$ is to be crossed at time t before the risk reserve reaches the value $X=K$.

(2) the probability distribution for ruin when $X=K$ is a reflecting barrier.

(3) the probability distribution for the risk reserve to reach the value $X=K$ before ruin takes place.

We obtain analytical solutions in closed form for above problems by assuming that the claims occur with a Poisson frequency ν and the density of the random claims as exponential.

In section 2 we obtain the Laplace transform for the ruin probability function before the risk reserve reaches the value K. We also deduce the probability function for ultimate ruin. Section 3 is devoted to the study of the ruin probability function by taking the barrier $X=K$ as a reflecting barrier. In section 4 we obtain the probability function for the risk reserve to reach the value K before ruin takes place.

2. First passage time for ruin before the risk reserve reaching the limit K.

The first passage time problems for a jump process with linear drift have been studied by Vasudevan et. al. (1978,1979) by the imbedding approach. The imbedding method is also seen to be an effective tool for studying the first passage problems for process with exponential decay (Vasudevan et. al. 1980).

(In Chapters II, III and IV we used this technique to study problems relating to neuron discharge activity and storage models) In this section we exploit this approach to study the problem of determining the probability density function for the risk reserve to reach the value $X<0$ for the first time before reaching the value $X=K$. Let X(t) be the stochastic variable representing the net capital reserve in time t. The equations defining X(t) is given by (1.2) and (1.3). (1.3) means that there is a regular income at a constant rate α in addition to the income which is proportional to risk reserve at that time. Define $P(u,K,t)$ as the probability that the insurance company starting with an initial reserve $X=u$ is ruined between time t and t+it given that the net reserve has not reached its upper limit K. Considering the different possibilities of the happenings in the initial interval (0,dt) and proceeding to the limit as dt tends to zero we arrive at the integro-differential equation

$$\frac{\partial P}{\partial t}-(\beta+\alpha u)\frac{\partial P}{\partial u}+\nu P$$
$$=\nu\int_{-u}^{0}P(u+z,K,t)\,dz+\delta(t)\,\nu\int_{-\infty}^{-u}a(z)\,dz \tag{2.1}$$

Defining $\bar{P}(u,K,\ell)$ as the Laplace transform of $P(u,K,t)$ we have from equation (2.1)

$$-(\beta+\alpha u)\frac{\partial\bar{P}}{\partial u}+(\ell+\nu)\bar{P}$$
$$=\nu\int_{-u}^{0}a(z)\,\bar{P}(u+z,K,\ell)\,dz+\nu\int_{-\infty}^{u}a(z)\,dz \tag{2.2}$$

In order to arrive at a closel form solution we consider the special case

$$a(z) = \eta e^{\eta z} H(-z) \tag{2.3}$$

and obtain after transformation the equation

$$e^{\eta u}\left[-(\beta+\alpha u)\frac{\partial \bar{P}}{\partial u} + (\ell+\nu)\bar{P}\right] = \nu\eta\int_0^u e^{\eta y}\bar{P}(y,K,\ell)\,dy + \nu \tag{2.4}$$

Differentiating (2.4) with respect to u and simplifying we get the second order differentiation equation

$$-(\beta+\alpha u)\frac{\partial^2 \bar{P}}{\partial u^2} + \left[\ell+\nu-\alpha-\eta(\beta+\alpha u)\right]\frac{\partial \bar{P}}{\partial u} + \eta\ell\bar{P} = 0 \tag{2.5}$$

Letting $\beta+\alpha u = -\alpha w/\eta$ we have from (2.5)

$$\eta w\frac{\partial^2 \bar{P}}{\partial u^2} + \left[1-\frac{\ell+\nu}{\alpha}-w\right]\frac{\partial \bar{P}}{\partial u} + \frac{\ell}{\alpha}\bar{P} = 0 \tag{2.6}$$

This equation being of Kummer's type (Slater 1960) the solution in terms of the original variable u is

$$\bar{P}(u,K,\ell) = A_1\,{}_1F_1\left[-\frac{\ell}{\alpha}, 1-\frac{\ell+\nu}{\alpha}, -\frac{\eta}{\alpha}(\beta+\alpha u)\right] + B_1\left[-\frac{\eta}{\alpha}(\beta+\alpha u)\right]^{\frac{\ell+\nu}{\alpha}}\,{}_1F_1\left[\frac{\nu}{\alpha}, 1+\frac{\ell+\nu}{\alpha}, -\frac{\eta}{\alpha}(\beta+\alpha u)\right] \tag{2.7}$$

where $A_1 = A_1(K,\ell)$ and $B_1 = B_1(K,\ell)$ are functions of K

and ℓ independent of u . These functions are to be determined using the boundary conditions. From the definition of $P(u,K,t)$ we have the boundary condition

$$\overline{P}(K,K,\ell)=0 \tag{2.8}$$

Using the boundary condition (2.8) in (2.7) we get

$$\begin{aligned} &A_1\ {}_1F_1\left[-\frac{\ell}{\alpha},1-\frac{\ell+\nu}{\alpha},-\frac{\eta}{\alpha}(\beta+\alpha K)\right] \\ &+B_1\left[-\frac{\eta}{\alpha}(\beta+\alpha K)\right]^{\frac{\ell+\nu}{\alpha}}\ {}_1F_1\left[\frac{\nu}{\alpha},1+\frac{\ell+\nu}{\alpha},-\frac{\eta}{\alpha}(\beta+\alpha K)\right] \\ &=0 \end{aligned} \tag{2.9}$$

The second boundary condition is the equation (2.4) which takes care of the crossing of the boundary at $X=0$ at time t. Hence substituting the solution (2.8) in (2.4) we get

$$\begin{aligned} &A_1e^{\eta u}\Big\{-(\beta+\alpha u)\frac{\partial}{\partial u}\,{}_1F_1\left[-\frac{\ell}{\alpha},1-\frac{\ell+\nu}{\alpha},-\frac{\eta}{\alpha}(\beta+\alpha u)\right] \\ &\quad+(\ell+\nu)\,{}_1F_1\left[-\frac{\ell}{\alpha},1-\frac{\ell+\nu}{\alpha},-\frac{\eta}{\alpha}(\beta+\alpha u)\right] \\ &\quad-(\ell+\nu)\,{}_1F_1\left[-\frac{\ell}{\alpha},-\frac{\ell+\nu}{\alpha},-\frac{\eta}{\alpha}(\beta+\alpha u)\right]\Big\} \\ &+B_1e^{\eta u}\Big\{-(\beta+\alpha u)\frac{\partial}{\partial u}\left(-\frac{\eta}{\alpha}(\beta+\alpha u)\right)^{\frac{\ell+\nu}{\alpha}}\,{}_1F_1\left[\frac{\nu}{\alpha},1+\frac{\ell+\nu}{\alpha},-\frac{\eta}{\alpha}(\beta\alpha+u)\right] \\ &\quad+(\ell+\nu)\left(-\frac{\eta}{\alpha}(\beta+\alpha u)\right)^{\frac{\ell+\nu}{\alpha}}\,{}_1F_1\left[\frac{\nu}{\alpha},1+\frac{\ell+\nu}{\alpha},-\frac{\eta}{\alpha}(\beta+\alpha u)\right] \\ &\quad+\left[-\eta(\beta+\alpha u)/\alpha\right]^{\frac{\ell+\nu}{\alpha}+1}\left(\frac{\nu\alpha}{\ell+\nu+\alpha}\right)\,{}_1F_1\left[1+\frac{\nu}{\alpha},2+\frac{\ell+\nu}{\alpha},-\frac{\eta}{\alpha}(\beta\alpha+u)\right]\Big\} \end{aligned}$$

$$= -(\ell+\nu)\, A_1 \;\; {}_1F_1\left[-\frac{\ell}{\alpha}, -\frac{\ell+\nu}{\alpha}, -\eta\beta/\alpha\right]$$

$$+ B_1 \left(-\frac{\eta\beta}{\alpha}\right)^{\frac{\ell+\nu}{\alpha}+1} \left(\frac{\nu\alpha}{\ell+\nu+\alpha}\right) {}_1F_1\left[1+\frac{\nu}{\alpha}, 2+\frac{\ell+\nu}{\alpha}, -\frac{\eta\beta}{\alpha}\right]$$

$$+\nu$$

(2.10)

Using the standard properties of ${}_1F_1$ (Slater 1960) functions it can be easily shown, as seen in earlier chapters, that the coefficients of $A_1 e^{\eta u}$ and $B_1 e^{\eta u}$ in LHS of equation (2.11) vanish. Hence we have the second condition for determining A_1 and B_1 as:-

$$A_1 \;\; {}_1F_1\left[-\frac{\ell}{\alpha}, -\frac{\ell+\nu}{\alpha}, -\eta\beta/\alpha\right]$$

$$- B_1 \left(-\eta\beta/\alpha\right)^{\frac{\ell+\nu}{\alpha}+1} \frac{\nu\alpha}{(\ell+\nu)(\ell+\nu+\alpha)} \; {}_1F_1\left[1+\frac{\nu}{\alpha}, 2+\frac{\ell+\nu}{\alpha}, -\frac{\eta\beta}{\alpha}\right]$$

$$= \nu/(\ell+\nu) \qquad (2.11)$$

Letting

$$L = {}_1F_1\left[-\frac{\ell}{\alpha}, 1-\frac{\ell+\nu}{\alpha}, -\frac{\eta}{\alpha}(\beta+\alpha K)\right] \qquad (2.12)$$

$$M = \left[-\frac{\eta}{\alpha}(\beta+\alpha K)\right]^{\frac{\ell+\nu}{\alpha}} {}_1F_1\left[\frac{\nu}{\alpha}, 1+\frac{\ell+\nu}{\alpha}, -\frac{\eta}{\alpha}(\beta+\alpha K)\right]$$

(2.13)

$$N = {}_1F_1\left[-\frac{\ell}{\alpha}, -\frac{\ell+\nu}{\alpha}, -\eta\beta/\alpha\right] \tag{2.14}$$

$$Q = \left(-\frac{\eta\beta}{\alpha}\right)^{\frac{\ell+\nu}{\alpha}+1} {}_1F_1\left[1+\frac{\nu}{\alpha}, 2+\frac{\ell+\nu}{\alpha}, -\eta\beta/\alpha\right] \tag{2.15}$$

We get from (2.9) and (2.11)

$$A_1 L + B_1 M = 0 \tag{2.16}$$

$$A_1 N - B_1 \frac{\nu\alpha Q}{(\ell+\nu)(\ell+\nu+\alpha)} = \frac{\nu}{(\ell+\nu)} \tag{2.17}$$

Solving (2.16) and (2.17)

$$A_1 = \frac{\nu(\ell+\nu+\alpha)M}{(\ell+\nu)(\ell+\nu+\alpha)MN + \nu\alpha QL} \tag{2.18}$$

$$B_1 = \frac{-\nu(\ell+\nu+\alpha)L}{(\ell+\nu)(\ell+\nu+\alpha)MN + \nu\alpha QL} \tag{2.19}$$

In the expression (2.8) since the term multiplying the coefficient of B happens to contain the term $(-1)^{\frac{\ell+\nu}{\alpha}}$ and since $\frac{\ell+\nu}{\alpha}$ can be a fraction doubt may arise whether the solutionscontains an imaginary part. This cannot be the case since ℓ can be taken as real and $\bar{P}(u,K,\ell)$ is the L.T. of a probability

density which is real. Howev r it is to be noted that M and Q contains similar terms as multiplicative factors. In view of the expressions (2.18) and (2.19) for A_1 and B_1 it is easily seen that for real ℓ, whatever be the values of ν and α, the total solution can never become complex. (2.7) together with (2.B) and (2.19) determines the complete solution for $\bar{P}(u,k,\ell)$, the L.T. of the ruin probability function $P(u,k,t)$.

Deductions. (1) The case when the income is only proportional to the level of the risk reserve: In this case $\beta = 0$. Hence we note

$$Q = 0 \quad , \qquad N = 1$$

$$A_1 = \nu/(\ell+\nu) \, , \quad B_1 = -\nu L/(\ell+\nu)M \Big|_{\beta=0} \tag{2.20}$$

$$\bar{P}(u,k,\ell)\Big|_{\beta=0} = \left(\frac{\nu}{\ell+\nu}\right) {}_1F_1\left[-\frac{\ell}{\alpha}, 1-\frac{\ell+\nu}{\alpha}, -\eta u\right]$$

$$-\left(\frac{\nu}{\ell+\nu}\right) \frac{{}_1F_1\left[-\frac{\ell}{\alpha}, 1-\frac{\ell+\nu}{\alpha}, -\eta k\right] {}_1F_1\left[\frac{\nu}{\alpha}, 1+\frac{\ell+\nu}{\alpha}, -\eta u\right]}{\left(-\frac{\eta k}{\alpha}\right)^{\frac{\ell+\nu}{\alpha}} {}_1F_1\left[\frac{\nu}{\alpha}, 1+\frac{\ell+\nu}{\alpha}, -\eta k\right]} \tag{2.21}$$

(2) The probability for ultimate ruin:

The probability for ultimate ruin is got by putting $\ell = 0$ in the solution for $\bar{P}(u,k,\ell)$

$$\bar{P}(u,k,\ell) = \int_0^\infty P(u,k,t)\,dt \tag{2.22}$$

when $\ell = 0$ we note that

$$A_1(0) = \frac{(\alpha+\nu)\, M(0)}{(\ell+\nu)\, M(0) + \alpha\, Q(0)} \tag{2.23}$$

$$B_1(0) = \frac{-(\nu+\alpha)}{(\nu+\alpha)\, M(0) + \alpha\, Q(0)} \tag{2.24}$$

and

$$\bar{P}(u,K,\ell)\Big|_{\ell=0} = A_1(0) + B_1(0)\left[-\frac{\eta}{\alpha}(\beta+\alpha u)\right]^{\nu/\alpha} {}_1F_1\left[\frac{\nu}{\alpha}, \frac{1+\nu}{\alpha}, -\frac{\eta}{\alpha}(\beta+\alpha u)\right] \tag{2.25}$$

Using the property of the confluent hypergeometric function (page 262, Abramowitz and Stegun 1965)

$${}_1F_1(a, a+1, -x) = a\, x^{-a}\, \gamma(a, x) \tag{2.26}$$

where $\gamma(a,x)$ is the incomplete gamma function defined by

$$\gamma(a,x) = \int_0^x e^{-z}\, z^{a-1}\, dz \tag{2.27}$$

we arrive at the expression for the probability of ultimate ruin,

$$\bar{P}(u,K,0) = \frac{\frac{\nu}{\alpha}\gamma\left[\frac{\nu}{\alpha}, \frac{\eta}{\alpha}(\beta+\alpha K)\right] - \frac{\nu}{\alpha}\gamma\left[\frac{\nu}{\alpha}, \frac{\eta}{\alpha}(\beta+\alpha u)\right]}{\frac{\nu}{\alpha}\gamma\left[\frac{\nu}{\alpha}, \frac{\eta}{\alpha}(\beta+\alpha K)\right] - \gamma\left[\frac{\nu}{\alpha}+1, \eta\beta/\alpha\right]} \tag{2.28}$$

When there is no upper limit to the risk reserve the probability of ultimate ruin, allowing an unlimited growth of the risk reserve, is got on proceeding to the limit as k tends to infinity. When k tends to infinity we observe that $\gamma(a,k)$ tends to $\Gamma(a)$ where

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx \tag{2.29}$$

Hence

$$\bar{P}(u,\infty,0) = P(u) = \frac{\frac{\nu}{\alpha}\Gamma(\frac{\nu}{\alpha}) - \frac{\nu}{\alpha}\gamma\left(\frac{\nu}{\alpha}, \frac{\eta}{\alpha}(\beta+\alpha u)\right)}{\frac{\nu}{\alpha}\Gamma(\frac{\nu}{\alpha}) - \gamma(\frac{\nu}{\alpha}+1, \eta\beta/\alpha)} \tag{2.30}$$

$P(u)$ being the probability of ultimate ruin. Further using the properties

$$\gamma(a+1,x) = a\,\gamma(a,x) - x^a e^{-x} \tag{2.31}$$

$$\Gamma(a,x) = \Gamma(a) - \gamma(a,x) \tag{2.32}$$

where $\Gamma(a,x) = \int_x^\infty e^{-z} z^{a-1} dz$ (2.33)

we get from (2.30)

$$P(u) = \nu\Gamma\left[\frac{\nu}{\alpha}, \frac{\eta}{\alpha}(\beta+\alpha u)\right] \Big/ \left\{\nu\Gamma\left(\frac{\nu}{\alpha}, \frac{\eta\beta}{\alpha}\right) + \alpha\left(\frac{\eta\beta}{\alpha}\right)^{\nu/\alpha} e^{-\eta\beta/\alpha}\right\} \tag{2.34}$$

This agrees with the result obtained by Segerdahl (1959). When $\beta=0$ the result (2.28) becomes

$$\bar{P}(u,K)\Big|_{\beta=0} = \frac{\Gamma(\frac{\nu}{\alpha},\eta K) - \frac{\nu}{\alpha}\gamma(\frac{\nu}{\alpha},\eta u)}{\frac{\nu}{\alpha}\gamma(\frac{\nu}{\alpha},\eta K) - \Gamma(\frac{\nu}{\alpha}+1)} \tag{2.35}$$

and the probability of ultimate ruin is

$$P(u) = \Gamma(\tfrac{\nu}{\alpha},\eta u) \Big/ \Gamma(\nu/\alpha) \tag{2.36}$$

3. <u>Ruin probability when K is a reflecting barrier</u>.

In the previous section we determined the ruin probability density function for an insurance business with the condition that the ruin is attained at time t and the risk reserve has not reached the value K in this time. In this section we shall evaluate the probability function for ruin allowing the risk reserve to reach the value K and treating $X=K$ as a reflecting barrier. This means the excess of the reserve over K is cut off as in the case of a dam of finite capacity when overflow takes place. In the language of insurance the excess over K can be considered as taken out of the risk reserve for bonus purposes, for policy holders. In such a situation how is the probability density function is different from the one discussed in the previous model is described in this section. For the stochastic process defined in (1.2) and (1.3) we shall define

$\psi(u,K,t)$ as the probability that for an insurance company with the initial reserve $X=u$ ruin takes place between time t and t+dt given that $X=K$ is a reflecting barrier.

The imbedding equation for $\psi(u,K,t)$ is given by the integro differential equation

$$\frac{\partial\psi}{\partial t}-(\beta+\alpha u)\frac{\partial\psi}{\partial u}+\nu\psi = \nu\int_0^{-u} a(z)\,\psi(u+z,K,t)\,dz + \delta(t)\,\nu\int_{-\infty}^{-u} a(z)\,dz \tag{3.1}$$

Denoting $\bar{\psi}(u,K,\ell)$ as the L.T. of $\psi(u,K,t)$ and taking the iensity of claims as in (2.3) we get

$$e^{\eta u}\left[-(\beta+\alpha u)\frac{\partial\bar{\psi}}{\partial u}+(\ell+\nu)\bar{\psi}\right] = \nu\eta\int_0^{u} e^{\eta y}\,\bar{\psi}(y,K,\ell)\,dy+\nu \tag{3.2}$$

As done in the previous section, differentiating (3.2) with respect to u and cancelling $e^{\eta u}$ throughout we get the second order differential equation

$$-(\beta+\alpha u)\frac{\partial^2\bar{\psi}}{\partial u^2}+\left[\ell+\nu-\alpha-\frac{\eta}{\alpha}(\beta+\alpha u)\right]\frac{\partial\bar{\psi}}{\partial u}+\eta\ell\bar{\psi}=0 \tag{3.3}$$

The above equation can be reduced to the Kummer's type confluent hypergeometric equation and hence the solution of equation (3.3) is

$$\bar{\psi}(u,K,\ell) = A_2\, {}_1F_1\left[-\frac{\ell}{\alpha}, 1-\frac{\ell+\nu}{\alpha}, -\frac{\eta}{\alpha}(\beta+\alpha u)\right] + B_2\left[-\frac{\eta}{\alpha}(\beta+\alpha u)\right]^{\frac{\ell+\nu}{\alpha}} {}_1F_1\left[\frac{\nu}{\alpha}, 1+\frac{\ell+\nu}{\alpha}, -\frac{\eta}{\alpha}(\beta+\alpha u)\right] \tag{3.4}$$

where A_2 and B_2 are functions of K and ℓ and they can be determined by suitable boundary conditions. As K is a reflecting barrier the risk reserve once reaches the upper value K stops at K without moving up in the absence of claims and when there is a claim it goes down the value K. As $\bar{\psi}$ is continuous at $X=K$ one expects the boundary condition for K to be a reflecting barrier as

$$\left[\frac{\partial\bar{\psi}}{\partial u}\right]_{u=K} = 0 \tag{3.5}$$

This condition can also be visibly seen by writing down the integral equation of the process starting from $X=K$ and comparing it with (3.2). Differentiating (3.4) partially with respect to u and using the condition (3.5) we get

$$A_2 L_1 \frac{\ell}{\ell+\nu-\alpha} + B_2\left[\frac{\ell+\nu}{\alpha} M_1 + \frac{\nu}{\ell+\nu+\alpha} R\right] = 0 \tag{3.6}$$

where $L_1 = {}_1F_1\left[1-\frac{\ell}{\alpha}, 2-\frac{\ell+\nu}{\alpha}, -\frac{\eta}{\alpha}(\beta+\alpha K)\right]$ (3.7)

$$M_1 = \left[-\frac{\eta}{\alpha}(\beta+\alpha K)\right]^{\frac{\ell+\nu}{\alpha}-1} {}_1F_1\left[\frac{\nu}{\alpha}, 1+\frac{\ell+\nu}{\alpha}, -\frac{\eta}{\alpha}(\beta+\alpha K)\right] \tag{3.8}$$

$$N_1 = \left[-\frac{\eta}{\alpha}(\beta+\alpha K)\right]^{\frac{\ell+\nu}{\alpha}} {}_1F_1\left[1+\frac{\nu}{\alpha}, 2+\frac{\ell+\nu}{\alpha}, -\frac{\eta}{\alpha}(\beta+\alpha K)\right] \quad (3.9)$$

The second boundary condition which will serve in getting a relation between A and B is the integral equation (3.2). As done in the earlier model substituting the solution (3.4) in (3.2) and simplifying we get the relation

$$A_2 N - \frac{B_2 \ \nu\alpha Q}{(\ell+\nu)(\ell+\nu+\alpha)} = \frac{\nu}{\ell+\nu} \quad (3.10)$$

where N and Q are given by (2.14) and (2.15). Solving (3.6) and (3.10) we arrive at

$$A_2 = \nu(\ell+\nu+\alpha) \Big/ \left\{(\ell+\nu)(\ell+\nu+\alpha)N + \frac{\nu\alpha^2 \ell(\ell+\nu+\alpha)QL}{(\ell+\nu+\alpha)\left[(\ell+\nu+\alpha)(\ell+\nu)M_1+\nu\alpha R\right]}\right\} \quad (3.11)$$

and

$$B_2 = -\nu(\ell+\nu+\alpha) \Big/ \left\{(\ell+\nu)\frac{N}{\alpha}\left[(\ell+\nu+\alpha)(\ell+\nu)M_1+\nu\alpha R\right]\frac{(\ell+\nu-\alpha)}{\ell L_1} + \nu\alpha Q\right\} \quad (3.12)$$

Here also by an argument similar to the one given in previous section we can show that the solution $\overline{\psi}$ is real for real ℓ. The complete analytical solution for $\overline{\psi}(u, K, \ell)$ is got by substituting the values of A_2 and B_2 given by (3.11) and (3.12)

in (3.4).

Deductions. (1): The probability of ultimate ruin in this case is got by putting $\ell=0$ in the solution for $\overline{\psi}(u,K,\ell)$. When $\ell=0$ we easily see from (3.11) and (3.12) that

$$A_2 = A_2(K,0) = 1 \tag{3.15}$$

$$B_2 = B_2(K,0) = 0 \tag{3.16}$$

where $A_2(K,0)$ and $B_2(K,0)$ are the values of A_2 and B_2 at $\ell=0$. Thus from the solution for $\overline{\psi}(u,K,\ell)$ we get

$$\psi(u) = \overline{\psi}(u,K,0) = \int_0^\infty \psi(u,K,t)\,dt \tag{3.17}$$

$$= 1, \text{ as could be expected.}$$

4. First passage density for risk reserve to reach the limit K before ruin.

In the previous sections we have analysed the probability of ruin when there is an upper limit to the risk reserve. The probability distributions for the risk reserve reaching the level K for the first time before ruin is also worth investigating in order to study the possibilities of giving bonus for the policy holders. We shall obtain this probability distributions in this section.

Define

$\chi(u,K,t)$ = the probability that the risk reserve reaches the level K for the first time between time t and t+dt given that the ruin has not taken place before time t.

The barrier X=K will be reached exactly in this case as there is a continuous change in the positive direction. The imbedding equation for $\bar{\chi}(u,K,\ell)$, the L.T. of $\chi(u,K,t)$ is

$$e^{\eta u}\left[-(\beta+\alpha u)\frac{\partial\bar{\chi}}{\partial u}+(\ell+\nu)\bar{\chi}\right] = \nu\eta\int_0^u e^{\eta y}\,\bar{\chi}(y,K,\ell)\,dy \tag{4.1}$$

This on differentiation with respect to u reduces to Kummer's type of confluent hypergeometric equation (Slater 1960) and hence the solution is

$$\bar{\chi}(u,K,\ell) = A_3\,{}_1F_1\left[-\frac{\ell}{\alpha},1-\frac{\ell+\nu}{\alpha},-\frac{\eta}{\alpha}(\beta+\alpha u)\right] + B_3\left[-\frac{\eta}{\alpha}(\beta+\alpha u)\right]^{\frac{\ell+\nu}{\alpha}}{}_1F_1\left[\frac{\nu}{\alpha},1+\frac{\ell+\nu}{\alpha},-\frac{\eta}{\alpha}(\beta+\alpha u)\right] \tag{4.2}$$

where A_3 and B_3 are to be determined by suitable boundary conditions. From the definition of $\chi(u,K,t)$ we have as a boundary condition

$$\bar{\chi}(K,K,\ell) = 1 \tag{4.3}$$

Hence one relation connecting A_3 and B_3 is

$$A_3 L + B_3 M = 1 \tag{4.4}$$

where L and M are as defined in (2.12) and (2.13). Another relation connecting A_3 and B_3 is got by feeding the solution (4.2) into the first order differential equation (4.1) for $\bar{\chi}(u, K, \ell)$. This equation serves as the second boundary condition. Hence we get the relation

$$A_3 N - B_3 \frac{\nu \alpha Q}{(\ell+\nu)(\ell+\nu+\alpha)} = 0 \tag{4.5}$$

where N and Q are given by (2.14) and (2.15) respectively. Solving equations (4.4) and (4.5) we get

$$A_3 = \frac{\nu \alpha Q}{(\ell+\nu)(\ell+\nu+\alpha) M N + \nu \alpha Q L} \tag{4.6}$$

$$B_3 = \frac{(\ell+\nu)(\ell+\nu+\alpha) N}{(\ell+\nu)(\ell+\nu+\alpha) M N + \nu \alpha Q L} \tag{4.7}$$

Then the complete analytical solution for $\bar{\chi}(u, K, \ell)$ is given by (4.3), (4.6) and (4.7).

<u>Deductions</u>:

(1) When $\ell = 0$ we have the probability of ever reaching the upper limit K as

$$\chi(u,K) = \int_0^\infty \chi(u,K,t)\,dt$$

$$= A_3(0) + B_3(0)\left[-\frac{\eta}{\alpha}(\beta+\alpha u)\right]^{\nu/\alpha} \times$$

$$\times\, {}_1F_1\left[\frac{\nu}{\alpha}, \frac{1+\nu}{\alpha}, -\frac{\eta}{\alpha}(\beta+\alpha u)\right] \quad (4.8)$$

where

$$A_3(0) = \frac{\alpha\, Q(0)}{(\nu+\alpha)\, M(0) + \alpha\, Q(0)} \quad (4.9)$$

$$B_3(0) = \frac{\nu+\alpha}{(\nu+\alpha)\, M(0) + \alpha\, Q(0)} \quad (4.10)$$

Here $M(0)$ and $N(0)$ stand for the expressions M and Q at $\ell=0$ (2.15 and 2.12).

(2) Further if we assume that the income to the company is only proportional to the risk reserve at time t we have $\beta=0$. In this case the probability of the risk reserve ever reaching the value K is

$$\chi(u,K)\Big|_{\beta=0} = \frac{{}_1F_1\left[\frac{\nu}{\alpha}, 1+\frac{\nu}{\alpha}, -\eta u\right]}{{}_1F_1\left[\frac{\nu}{\alpha}, 1+\frac{\nu}{\alpha}, -\eta K\right]} \quad (4.11)$$

(3) Lastly we verify that well known fact in infinite time the sum of the probabilities $P(u,K)$ and $\chi(u,K)$ is unity. When $\ell=0$ one gets from (2.23), (2.24), (4.9) and (4.10)

$$P(u,K) + \chi(u,K) = 1 \quad (4.12)$$

as could be expected.

5. Conclusion.

In our analysis of collective risk theory to study the ruin probability we have taken the claims as a Poisson while the income to the company at any time t with reserve X(t) is taken as $\beta + \alpha X$. We have obtained the L.T. for the probability ruin density assuming that there is an upper limit K to the risk reserve. We have taken the barrier X=K as an absorbing barrier in section 2 and as a reflecting barrier in section 3. We have also arrived at the results for the probability of ultimate ruin in these cases. Lastly we have studied the probability for the risk reserve reaching the value X=K before ruin takes place. We have also verified the result that

$$P(u,K) + \chi(u,K)$$

tends to unity as ℓ tends to zero.

C H A P T E R - VI

COMBINANTS

1. Introduction.

In the previous chapters, we have been dealing with first passage density problems of various kinds of systems, such as dams, neuronal firings, storage systems etc. In the case of neuronal spike discharge, they form a stationary stochastic process with the interval density as described in Chapters II and III. In Chapter II, the boundary was fixed and moving boundaries, simulating refractory periods were considered in Chapter III. Once this interval distribution is known, the probability of firing at anytime can be described by the product density functions (A.Ramakrishnan, 1950, 1959), which can be easily obtained by renewal equations (Cox 1962; Cox and Miller 1965; Snyder 1975).

The product densities introduced by Ramakrishnan (1950, 1959) are powerful tools to deal with stochastic point processes. This relates to the distribution of a discrete number of entities in continuous infinity of states.

The central quantity of interest in this situation is $dN(t)$, the number of entities occurring in the continuous interval t and $t+dt$. We assume that the probability that there is one entity between t and $t+dt$ is proportional to dt, while the probability that there is more than one entity is of order smaller than dt. Hence the mean number in dt is

$$E[dN(t)] = f_1(t)\,dt. \tag{1.1}$$

while

$$E\left[dN(t)^{\gamma}\right] = f_1(t)\,dt. \tag{1.2}$$

where $f_1(t)$ is called the product density of the first order. Product densities of higher order express all the correlations of the stochastic variable $dN(t)$ existing at various times.

$$E[dN(t_1)\,dN(t_2)] = f_2(t_1,t_2)\,dt_1\,dt_2 \tag{1.3}$$

$$E[dN(t_1)\,dN(t_2)\cdots dN(t_\gamma)] = f_\gamma(t_1,t_2,\ldots t_\gamma)\,dt_1\cdots dt_\gamma \tag{1.4}$$

where f_γ are the higher order product densities. The mean number of entities in a given range of the parameter t, is given by

$$E[N(b)-N(a)] = E\int_a^b dN(t) = \int_a^b f_1(t)\,dt. \tag{1.5}$$

Similarly, the mean square number of the entities in the range a to b of t is

$$E\left[(N(b)-N(a))^2\right] = \iint E[dN(t_1)\,dN(t_2)]$$

$$= \int_a^b f_1(t)\,dt + \int_a^b\int_a^b f_2(t_1,t_2)\,dt_1\,dt_2 \tag{1.6}$$

Equation (1.6) brings out the singular behaviour of the random variables $dN(t_1)\, dN(t_2)$ when t_1 and t_2 coalesce. Ramakrishnan (1950) has proved a very useful result for the calculation of the rth moment of the number of entities in the desired range. It runs as

$$E\left\{\left[N(b)-N(a)\right]^r\right\}$$

$$= \sum_{s=1}^{r} C_s^r \int_a^b dt_1 \int_a^b dt_2 \cdots \int_a^b dt_s\, f_s(t_1, t_2, \ldots t_s) \tag{1.7}$$

where C_s^r denotes the number of various confluences of (r-s) infinitesimal intervals, the maximum order of any confluence being (r-s). The C_s^r coefficients being independent of the f functions can be derived from the following formula (Ramakrishnan, 1950), when the total number of entities is N.

$$N^r = \sum_{s=1}^{r} C_s^r N(N-1) \cdots (N-s+1) \tag{1.8}$$

a set of relations valid for N=1,2,....

An alternative expression for these coefficients obtained by Kuznetsov, Stratonovich and Tikhonov (1965) is just the same equation (1.8) in another closed form

$$C_s^r = \frac{1}{s!}\left[\frac{d^r}{dw^r}\left(e^w - 1\right)^s\right]_{w=0} \tag{1.9}$$

It can be proved also, that (Vasudevan, 1969)

$$C_s^r = \frac{1}{s!} \sum_{K=0}^{s} \binom{s}{K} K^r (-1)^{s-K} . \tag{1.10}$$

A knowledge of these moments and other statistical features such as cumulants etc. of discharge trains can be compared with the histograms obtained from experiments to validate the various models employed.

Recently, in counting statistics of photons obtained in photo electric emissions, apart from the moments and cumulants of the distribution, another quantity called the <u>combinants</u> were introduced by Kauffman and Gyulassy (1978).

Whenever particles are produced and if one is interested in $P(n)$, the probability that n particles are produced, it was found useful to characterise the general $P(n)$ in terms of its deviation from the Poisson. If the first combinant alone exists, which is also the mean, the process is Poisson. In the case of Brown and Twiss experiment, (1954, 1957) where the nature of photon bunching comes into play, we cannot expect the electron emission to be a Poisson and hence we expect the other combinants to be different from zero. It is the purpose of this Chapter to relate these combinants to the known product density functions f and the cluster functions g. It is interesting to see that the combinants play a similar role as the probability distributions themselves. In calculating the moments we sum n^r over the probability distributions, while in calculating the

cumulants we sum ℓ^{r} over the combinants. Much of this analysis is found in the book (Kuznetsov, Stratonovich and Tikhonov, 1965) though explicit attention has not been drawn to the combinants.

2. Combinants.

It is useful to characterise the general $P(n)$ in terms of its deviations from the Poisson. Traditionally, this is done by considering deviations from the usual moments of $P(n)$ possessed in the Poisson case. The probability generating function for the Poisson distribution is given by

$$F(\lambda) = \sum_{n=0}^{\infty} \lambda^{n} P(n) = \exp(\lambda - 1)\bar{n} \tag{2.1}$$

We see that $\log F(\lambda)$ is a first degree polynomial. If however, we employ higher degrees, we can write

$$\log F(\lambda) = \log(P(0)) + \sum_{k=1}^{\infty} c(k)\lambda^{k}, \tag{2.2}$$

which means,

$$F(\lambda) = \exp\left(\sum_{k=1}^{\infty} c(k)(\lambda^{k} - 1)\right) \tag{2.3}$$

The expansion coefficients $c(1), c(2), \ldots$ thus completely characterise $P(n)$. ~~Every $C(n)$ is found in terms of $P(n)$'s upto that order and~~ $c(k)$'s are expressible in terms of the

first k probability ratios $\frac{P(1)}{P(0)}, \frac{P(2)}{P(0)}, \ldots, \frac{P(k)}{P(0)}$

This stands in stark contrast to 'ordinary' probability coefficients such as moments and cumulants, each of which involves every single one of the infinite number of $P(n)$ in its definition. It should be however noted that the condition $P(0) > 0$ is necessary to the existence of the C(k) defined as above.

If there are N_i independently distributed variables, each distributed according to $P_i(n)$, which has generating function $F_i(\lambda)$, we find that the coefficients $C(k)$ of $N(=\sum_i N_i)$ satisfy the additive property

$$C(k) = \sum_i C_i(k) \quad k = 1, 2, \ldots \tag{2.4}$$

It is easy to see from (2.2) that,

$$P(0) = \exp\left(-\sum C(k)\right) \tag{2.5}$$

The expression for C(k)'s in terms of $\left(\frac{P(n)}{P(0)}\right)$'s are given as follows:

$$C(1) = \frac{P(1)}{P(0)} \tag{2.6a}$$

$$C(2) = \frac{P(2)}{P(0)} - \frac{1}{2}\left(\frac{P(1)}{P(0)}\right)^2 \tag{2.6b}$$

$$C(3) = \frac{P(3)}{P(0)} - \left(\frac{P(1)}{P(0)}\right)\left(\frac{P(2)}{P(0)}\right) + \frac{1}{3}\left(\frac{P(1)}{P(0)}\right)^3. \tag{2.6c}$$

etc. Similarly $P(n)$'s are given in terms of C(k)'s as

$$P(n) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots\cdots \left(\prod_{k=1}^{\infty} \frac{(C(k))^{n_k} e^{-C(k)}}{n_k!} \right) \delta\left(n, \sum_{r=1}^{\infty} r n_r\right) \quad (2.7)$$

which may be noted to be a convoluted form of the multiple Poisson distribution with combinants C(k) as the "means" (Kauffmann and Gyulassy, 1978). From the relation,

$$\exp \sum_{k=1}^{\infty} C(k) \lambda^k = 1 + \tilde{P}_1 \lambda + \tilde{P}_2 \lambda^2 + \cdots \quad (2.8)$$

where $$\tilde{P}_n = \frac{P(n)}{P(0)} \quad (2.9)$$

we obtain $$\frac{\partial \tilde{P}_r}{\partial C(k)} = \tilde{P}_{r-k} \quad (2.10)$$

This can be checked with equation (2.6).

Now, it is our purpose to relate the combinants with the product densities and the cluster functions.

3. Cluster correlation functions.

Let us take the moment generating function of $P(n)$ as

$$G_m(t) = \sum_{n=0}^{\infty} e^{nt} P(n) \quad (3.1)$$

Then it has been shown in (Kuznetsov, Stratonovich and Tikhonov 1965) that

$$Q_m(t) = \sum_{s=0}^{\infty} \frac{(e^t-1)^s}{s!} \int_R \cdots \int_R f_s(x_1 \cdots x_s)\, dx_1 \cdots dx_s \quad (3.2)$$

where $f_s(x_1, \cdots x_s)$ are the sth order product densities defined earlier. It can be easily shown that $P_R(0)$, the probability that there is no entity in the region R is given by

$$P_R(0) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \int_R \cdots \int_R f_s(x_1 \cdots x_s)\, dx_1 \cdots dx_s \quad (3.3)$$

Also, the probability $P_R(n)$ of n entities in R is given by Kuznetsov, Stratonovich and Tikhonov (1965) as

$$P_R(n) = \frac{1}{\lfloor n} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \int_R \cdots \int_R f_{n+s}(x_1, x_2 \cdots x_{n+s})\, dx_1 \cdots dx_{n+s} \quad (3.4)$$

the points $(x_1, x_2, \cdots x_n)$ being the continuous set of points in the region R.

The factorial moment generating function $Q_{fm}(t')$ is given by replacing t by $\log(1+t')$ in $Q_m(t)$ i.e.

$$Q_{fm}(t') = \sum_{s=0}^{\infty} \frac{t'^s}{s!} \int_R \cdots \int_R f_s(x_1 \cdots x_s)\, dx_1 \cdots dx_s \quad (3.5)$$

Therefore,

$$\frac{\partial^r}{\partial t'^r}\left(\varphi_{fm}(t')\right)\Big|_{t'=0} = \left\langle \frac{n!}{(n-r)!} \right\rangle$$

$$= \int_R \cdots \int_R f_r(x_1 \cdots x_r)\, dx_1 \cdots dx_r \tag{3.6}$$

The actual moments are related to the factorial moments as given by (1.7).

The cumulant generating function is given by

$$\varphi_{cf}(t) = \log \varphi_m(t)$$

$$= \sum_{s=1}^{\infty} \frac{(e^t-1)^s}{s!} \tau_s \tag{3.7}$$

where

$$\tau_s = \int_R \cdots \int_R g_s(x_1, x_2, \cdots x_s)\, dx_1 \cdots dx_s \tag{3.8}$$

g_s are the cluster functions of order s and relating to the point process giving rise to $P(n)$ (Kuznetsov, Stratonovich and Tikhonov, 1965). The cluster functions $g_s(x_1 \cdots x_s)$ describe the irreducible clusters (which cannot be split into functions of any of its arguments) relating to the occurrence of the point process in the region R.

The cumulants Kr of the P(n) process are obtained from $\varphi_{cf}(t)$ by the relation,

$$K_r = \frac{\partial^r}{\partial t^r}\left(Q_{cf}(t)\right)\Big|_{t=0} \tag{3.9}$$

using (3.7), (3.9) can be written as

$$K_r = \frac{\partial^r}{\partial t^r}\left(\sum_{s=1}^{\infty} \frac{(e^t-1)^s}{s!}\tau_s\right) \tag{3.10}$$

$$= \sum_{s=1}^{r} C_s^r \tau_s \tag{3.11}$$

Thus the cumulants of the process are related to the integrals of the cluster functions of order s over the region R, by the same coefficients as the moments of the distribution are related to the product densities. Compare results (3.11) and (1.7). It is well known that the ordinary moments of P(n) are the averages $\langle n^r \rangle$ summed over the probabilities P(n). Have we an analogous situation in the case of cumulants? The answer is 'yes' and the combinants play the same role as the probabilities P(n) in calculating the cumulants. We display

$$\sum_{l=1}^{\infty} l^r c(l) = K_r \tag{3.12}$$

just as

$$\sum_{r=0}^{\infty} n^r P(n) = \mu_r \ (r^{th} \text{ moment}) \tag{3.13}$$

To see this, let us start from the moment generating function

(3.1)

$$\varphi_m(t) = \sum_{n=0}^{\infty} e^{nt} P(n)$$

Replacing e^t by t' and using (2.3) we get

$$\tilde{\varphi}_m(t') = \sum_{n=0}^{\infty} t'^n P(n)$$

$$= \exp \sum_{k=1}^{\infty} c(k)(t'^k - 1) \tag{3.14}$$

$$= P_0 \exp \sum_{k=1}^{\infty} c(k) t'^k, \text{ using } (2.5) \tag{3.15}$$

In $\tilde{\varphi}_m(t')$, if we replace t' by e^w we get back $\varphi_m(w)$

Therefore we get, by equation ( )

$$P_0 \exp \sum_{k=1}^{\infty} c(k) e^{wk} = \varphi_m(w)$$

$$= \sum_{s=0}^{\infty} \frac{(e^{w}-1)^s}{s!} \int_R \cdots \int_R f_s(x_1 \cdots x_s)\, dx_1 \cdots dx_s \tag{3.16}$$

(using (3.2))

Taking logarithm both sides in (3.16) and using (3.7) we get,

$$\log P(0) + \sum_{k=1}^{\infty} c(k) e^{wk} = \sum_{s=1}^{\infty} \frac{(e^{w}-1)^s}{s!} \tau_s \tag{3.17}$$

$c(k)(1+t')^k$

Differentiating both sides with respect to ω , r times and using (3.10) and (3.11) we get

$$\sum_{l=1}^{\infty} l^{r} c(l) = K_r = \sum_{s=1}^{r} C_s^{r} \tau_s \tag{3.18}$$

Thus we see that the rth cumulants are obtained by taking the average $\langle l^r \rangle$ over **all** the combinants $c(l)$ and the cumulants in turn are related to the cluster integrals τ_s by the C_s^r coefficients in a way analogous to the moments relationship with the integrals over the product densities.

Let us go one step further in the cumulant generating function $\varphi_{cf}(t)$ in (3.7) by replacing t by $\log(1+t')$. We get

$$\tilde{\varphi}_{cf}(t') = \sum_{s=1}^{\infty} \frac{t'^{s}}{s!} \tau_s \tag{3.19}$$

Therefore,

$$\frac{\partial^r}{\partial t'^r}\left(\tilde{\varphi}_{cf}(t')\right) = \tau_r \tag{3.20}$$

The τ_r's given in (3.20) can be called as factorial cumulants. In (3.17), if we put $\omega = \log(1+t')$ we easily see that

$$\sum_{l} \frac{l!}{(l-r)!} c(l) = \tau_r \tag{3.21}$$

Thus τ_r is the rth factorial cumulant with respect to $C(\ell)$.

This is analogous to the factorial moments

$$\int_R \cdots \int_R f_r(x_1 \cdots x_r)\, dx_1 \cdots dx_r = \sum_n \frac{n!}{(n-r)!} P(n) \qquad (3.22)$$

It is also easily seen that

$$C(\ell) = \frac{1}{\ell!} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \tau_{s+\ell}. \qquad (3.23)$$

This is exactly the same as the relation between P(n) and the factorial moments given by (3.4). Thus we have seen that the ~~cumulants~~ combinants play very much the same role as the probabilities themselves. We compute the cumulants and the factorial cumulants with respect to $C(\ell)$ in the same manner, as we compute moments and factorial moments with respect to P(n). Since it is expected that for many types of distributions the number of $C(\ell)$'s are much less than the infinite number of $P(n)$'s available, it is easier to deal with the combinants. This is attributed to the fact that the $C(\ell)$'s have existence essentially due to the cluster correlations in the system regarding the occurrence of the point process.

4. Doubly stochastic process.

Many interesting types of doubly stochastic processes have been met with in various situations, as in the case of the counting statistics of the electrons emitted by light falling on a sensitive material. This leads to the study of the phenomenon

of intensity correlations, inaugurated by the Brown and Twiss experiments (1954, 1957). A simple summary about doubly stochastic processes is given in Saleh (1978). An analysis of Barkhausan Noise in magnetism also belongs to a class of doubly stochastic process (R.Vasudevan and S.K.Srinivasan 1966). This may be called either a second order process or a doubly stochastic process. This term was first introduced by Cox (1955).

A doubly stochastic Poisson point process (DSP, PP) is a Poisson Point Process (P.PP) whose rate density function $\lambda(t)$ is itself a stochastic process. For each realisation of $\lambda(t)$, the resulting PP is a Poisson point process. Therefore, the statistical properties of a DSP-PP are completely specified if the statistical properties of the $\lambda(t)$ of the original process are given. The moments and the moment generating functions can be determined by averaging the corresponding moments and the moment generating functions over the different realisation of $\lambda(t)$.

The final Poisson process for the random variable n has p.d.f. given by

$$P(n) = \frac{W^n}{n!} e^{-W} \qquad (4.1)$$

where W is the rate of events integrated over the counting interval, i.e.

$$W = \int_{t_0}^{t_0+T} \lambda(\theta)\, d\theta \qquad (4.2)$$

From (3.1) and (4.1) we have the moment generating function

$$\varphi^{n}_{mf}(t) = \langle e^{W(e^{t}-1)} \rangle_{W} \tag{4.3}$$

for each realisation W. The moment generating function of a DSP-PP can be obtained from those of P.PP by averaging over the realisation of the λ process which means averaging over the W process. Therefore

$$\varphi^{n}_{mf}(t) = \langle e^{W(e^{t}-1)} \rangle_{W} = \varphi_{W}(e^{t}-1) \tag{4.4}$$

Let $f^{n}_{s}(x_1, \ldots, x_s)$ represent the product densities of the final process n. Then from (3.2) we have

$$\varphi^{n}_{mf}(t) = \sum_{s=0}^{\infty} \frac{(e^{t}-1)^{s}}{s!} \int_{R} \cdots \int_{R} f^{n}_{s}(x_1 \ldots x_s)\, dx_1 \cdots dx_s \tag{4.5}$$

But by (4.4), we have,

$$\varphi_{W}(e^{t}-1) = \sum_{s=0}^{\infty} \frac{(e^{t}-1)^{s}}{s!} \int_{R} \cdots \int_{R} f^{W}_{s}(x_1 \ldots x_s)\, dx_1 \cdots dx_s \tag{4.6}$$

where f^{W}_{s} are the product densities of the W process. As described in Section 3, we obtain the factorial moments of the final process n by replacing f_r by f^{n}_{r} in (3.6) viz.

$$\frac{\partial^r}{\partial t'^r}\left(\mathcal{G}^n_{fm}(t')\right)\Big/_{t'=0} = \left\langle \frac{n!}{(n-r)!} \right\rangle$$

$$= \int_R \cdots \int_R f_r^n(x_1 \cdots x_r)\, dx_1 \cdots dx_r. \tag{4.7}$$

Making a similar transformation in the expression for W process, we get

$$\widetilde{\mathcal{G}}^n_{fm}(t') = \sum_{s=0}^{\infty} \frac{(e^{t'}-1)^s}{s!} \int_R \cdots \int_R f_s^W(x_1 \cdots x_s)\, dx_1 \cdots dx_s \tag{4.8}$$

Hence

$$\frac{\partial^r}{\partial t'^r}\left(\widetilde{\mathcal{G}}^n_{fm}(t')\right)\Big/_{t'=0} = \sum_{s=1}^{r} C_s^r \int_R \cdots \int_R f_s^W(x_1, \cdots x_s)\, dx_1 \cdots dx_s. \tag{4.9}$$

Therefore, we find that the factorial moments of the final process n are the usual moments of the original process W. This fact can be applied to second order processes such as photoelectron emission etc.

If the original W process is a simple Poisson with cluster correlation

$$g_s = 0 \quad \text{for all } s > 1 \tag{4.10a}$$

$$g_1 = \bar{w} \quad (\text{the mean of } W) \tag{4.10b}$$

then we find that the moment generating function of the resultant process n is given by, using (4.5) and (4.6),

$$\mathcal{G}^{n}_{mf}(t) = \sum_{s=0}^{\infty} \frac{(e^{t}-1)^{s}}{s!} \int_{R} \cdots \int_{R} f^{n}_{s}(x_1 \cdots x_s)\, dx_1 \cdots dx_s$$

$$= \sum_{s=0}^{\infty} \frac{(e^{(e^{t}-1)}-1)^{s}}{s!} \int_{R} \cdots \int_{R} f^{W}_{s}(x_1 \cdots x_s)\, dx_1 \cdots dx_s$$

$$= \exp\left(e^{(e^{t}-1)}-1\right)\overline{W} \tag{4.11}$$

From the moment generating function $\mathcal{G}^{n}_{mf}(t)$ we get the probability generating function $\sum_{n} \lambda^{n} P(n)$ by replacing e^{t} by λ, i.e. the probability generating function is

$$\mathcal{G}^{n}_{mf}(\log \lambda) = \exp\left(e^{\lambda-1}-1\right)\overline{W} \text{ here.} \tag{4.12}$$

This result can also be expressed in terms of the combinants. Therefore we get

$$\exp\left(e^{\lambda-1}-1\right)\overline{W} = \exp \sum_{k=1}^{\infty} c(k)(\lambda^{k}-1) \tag{4.13}$$

We see that the l.h.s.

$$\exp\left(e^{\lambda-1}-1\right)\overline{W} = \exp\left[e^{-1}e^{\lambda}\overline{W} - \overline{W}\right]$$

$$= \exp\left\{\left[e^{-1}\overline{W} - \overline{W}\right] + \sum_{K=1}^{\infty} e^{-1}\overline{W}\frac{\lambda^{K}}{K!}\right\}$$

equating coefficient of λ^K both sides, of (4.13) we get

$$C(k) = \frac{\bar{w}\, e^{-1}}{k!} \tag{4.14}$$

and equating the constant terms we get,

$$\exp\left(-\sum_{k=1}^{\infty} C(k)\right) = \exp \bar{w}\left(e^{-1}-1\right) \tag{4.15}$$

$$= P_n(0), \text{ of the process } n.$$

Since C(k) exists for all values of k, the process cannot be a Poisson. However the C(k)'s go down to zero very fast.

In conclusion, for the production multiplicity of particles produced according to some distributions, the final distribution of n particles can be expressed in terms of combinants, because the combinants are additive if each type of multiplicity is produced independently. This can be tested in many ways.

Appendix A

Our object is to obtain the solution of equation (2.15) (2.9) of Chapter III (section 2)

$$x\frac{d^3y}{dx^3}+(a+b)\frac{d^2y}{dx^2}+(c-x)\frac{dy}{dx}-ay=0 \qquad (A.1)$$

where a,b,c are constants.

We can assume

$$y(t)=\int e^{-tx}\,T\,\cancel{dx}\,dt \qquad (A.2)$$

where T is a function of t (as a solution) (Forsyth Chapter VII). Then the differential equation (A.1) becomes

$$\int e^{-tx}\,T(t-t^3)\,x\,dt+\int e^{-tx}\,T\left[(a+b)t^2-ct-a\right]dt=0 \qquad (A.3)$$

and this must identically be satisfied. Hence the integration by parts of the first term in LHS ~~with~~ of (A.3) gives

$$\left[e^{-tx}\,T(t-t^3)\right]_{\alpha}^{\beta} \qquad (A.4)$$

and

$$\int_{\alpha}^{\beta} e^{-tx}\left[(1-3t^2)T+(t-t^3)\frac{dT}{dt}+T(a+b)t^2-ct-a\right]dt=0 \qquad (A.5)$$

the limits α and β are so chosen such that (A.4) is zero.

From (A.5) we see that the integrand should be zero and hence we obtain

$$T = |1-t^2|^{b/2} \; |t|^{a-1} \left|\frac{1+t}{1-t}\right|^{c/2}$$

Thus the solution of equation of (A.1) is

$$y = \sum_{i=1}^{3} A_i \int_{\alpha_i}^{\beta_i} e^{-tx} \, |1-t|^{\frac{b}{2}-1} \, |t|^{a-1} \left|\frac{1+t}{1-t}\right|^{c/2} dt$$

The 3 pairs of values of (α_i, β_i) on the real line can be chosen so as to satisfy (A.4)

Appendix B.

The purpose of this appendix is to show that the coefficients of $C_1 e^{-\eta u}$ and $C_2 e^{-\eta u}$ collected together in the equation (2.12) vanish. The coefficient of $C_1 e^{-\eta u}$ is

$$\begin{aligned}
&\alpha u \int_0^1 e^{tu} \, t^{\frac{\ell}{\alpha}-1} \, (1+t)^{\nu_2/\alpha} \, (1-t)^{\nu_1/\alpha} \, dt \\
&+(\ell+\nu) \int_0^1 e^{tu} \, t^{\frac{\ell}{\alpha}-1} \, (1+t)^{\nu_2/\alpha} \, (1-t)^{\nu_1/\alpha} \, dt \\
&-\nu_1 \int_0^1 e^{tu} \, t^{\frac{\ell}{\alpha}-1} \, (1+t)^{\nu_2/\alpha} \, (1-t)^{\frac{\nu_1}{\alpha}-1} \, dt \qquad \text{(B.1)} \\
&-\nu_2 \int_0^1 e^{tu} \, t^{\frac{\ell}{\alpha}-1} \, (1+t)^{\frac{\nu_2}{\alpha}-1} \, (1-t)^{\nu_1/\alpha} \, dt
\end{aligned}$$

Integrating by parts the first integral, we get

$$
\begin{aligned}
-\ell \int e^{tu} \Big[& \ell\, t^{\frac{\ell}{\alpha}-1} (1+t)^{\nu_2/\alpha} (1-t)^{\nu_1/\alpha} \\
& + t^{\ell/\alpha} \nu_2 (1+t)^{\nu_2/\alpha} (1-t)^{\nu_1/\alpha} \\
& - t^{\frac{\ell}{\alpha}-1} \nu_2 (1+t)^{\frac{\nu_2}{\alpha}-1} (1-t)^{\nu_1/\alpha} \\
& + t^{\frac{\ell}{\alpha}-1} \nu_1 (1+t)^{\nu_2/\alpha} (1-t)^{\nu_1/\alpha} \\
& - t^{\frac{\ell}{\alpha}-1} \nu_2 (1+t)^{\nu_2/\alpha} (1-t)^{\frac{\nu_1}{\alpha}-1} \Big] dt
\end{aligned}
\tag{B.2}
$$

Replacing (B.2) in (B.1) we find that the coefficient of $C_1 e^{-\eta u}$ is zero. On the same line we can prove that the coefficient of $C_2 e^{-\eta u}$ is also zero.

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