Holographic and exact RG beta function computations of the Sine-Gordon model

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DECLARATION

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Abstract

The exact renormalization group is used to study the RG flow of quantities in field theories. The basic idea is to write an evolution operator for the RG flow and evaluate it in perturbation theory. This is easier than directly solving the differential equation. This is illustrated by reproducing known results in the four dimensional ϕ^4 field theory and the two dimensional Sine-Gordon theory. It is shown that the calculation of beta function is somewhat simplified. The technique is also used to calculate the c-function in two dimensional Sine-Gordon theory. This agrees with other prescriptions for calculating c-functions in the literature. If one extrapolates the connection between central charge of a CFT and entanglement entropy in two dimensions, to the c-function of the perturbed CFT, then one gets a value for the entanglement entropy in Sine-Gordon theory that is in exact agreement with earlier calculations. Next, the Sine Gordon theory is generalized to include many scalar fields and several cosine terms. This is similar to the world sheet description of a string propagating in a tachyon background. This model is studied as a (boundary) 2d euclidean field theory and also using an AdS_3 holographic bulk dual. The beta functions for the cosine vertex of this modified theory are first computed in the boundary using techniques based on the exact RG. The beta functions are also computed holographically using position space and momentum space techniques. The results are in agreement with each other and with earlier computations. The cosine perturbation is of the form $\cos bX$. Due to wave function renormalisation the parameter b, and thus the dimension of the cosine, get renormalised. The beta

function for this parameter is thus directly related to the anomalous dimension of the X field. We compute this beta function in position space. They match with the earlier results in [22].

Contents

SYNOPSIS

Introduction

The exact renormalisation group (ERG), first written down by Wilson [1, 2, 3] has been an object of much study. It has been developed further [4] and different versions suitable for different purposes have been written down since then [5, 6, 7].

In the first paper we studied the application of the ERG to the Sine-Gordon model. Many calculations are tractable in two dimensions and therefore two dimensional theories are a good laboratory to try new ideas. The Sine-Gordon is one such model. Therefore we apply the ERG to two dimensional field theories- the emphasis being on a simple way of writing down the solution to the ERG in terms of an evolution operator. We reproduced some known results of Amit [8] and also obtained a new result on the flow of the c-function in the Sine-Gordon theory.

An attempt has been made in [9] to make a connection between the ERG and the holographic RG by showing that an ERG evolution operator in a boundary theory can be mapped directly to a scalar field action in AdS space time without ever invoking the AdS-CFT correspondence. Here, a specific transformation is chosen to map the ERG evolution operator to the free scalar action in AdS. Some suggestions for how the interactions should work out were given there. To do this for the more complicated case of composite operators it is important to understand RG equations in the boundary theory and obtain them from some bulk computations. Then the precise connection between these equations and what is called "holographic RG" - which is really a radial evolution equation of the bulk theory - could be understood better. A step towards this goal: take a specific composite operator, (we choose the Sine-Gordon theory), invoke the AdS-CFT correspondence and compute its beta functions in as many ways as you can. This was the goal of the second paper. Use this fully worked out example to make some precise statements about the direct transformations for the more general cases. This computation also serves as a check on the AdS-CFT correspondence and it shows the agreement between the ERG and the holographic RG.

The Sine-Gordon theory is interesting for many reasons. Since it is two dimensional many calculations are tractable. It has an interesting and non trivial RG flow - the famous Kosterlitz-Thouless flow which is of great interest in condensed matter physics. It also has relevance in string theory. It describes a bosonic string propagating in specific tachyonic background. The model can be generalized to include several cosines so that it describes a more general tachyonic background. The beta functions of this theory are in fact the space-time equations of motion of the tachyon. A calculation was done by [10] to compute the equations of motion upto a cubic term. In the second paper we do a calculation to reproduce the beta function of the tachyon at the cubic order using both ERG and holographic techniques. This computation is a good check on the ERG.

Now we will introduce the ERG.

ERG

The original Wilsonian form of the ERG had the following structure ($\dot{G} = \frac{\partial G}{\partial \tau}$):

$$\frac{\partial \psi}{\partial \tau} = -\frac{1}{2}\dot{G}\frac{\partial}{\partial y}(\frac{\partial}{\partial y} + 2G^{-1}y)\psi \tag{0.0.1}$$

We separate out the kinetic and interaction part by substituting $\psi = e^{-\frac{1}{2}G^{-1}y^2}\psi'$ in the above equation to get the Polchinski equation:

$$\frac{\partial \psi'}{\partial \tau} = -\frac{1}{2}\dot{G}\frac{\partial^2 \psi'}{\partial y^2} \tag{0.0.2}$$

(2.1.29) can be written as (using t instead of τ)

$$\frac{\partial \psi}{\partial t} = -H\psi \tag{0.0.3}$$

with $H = \frac{1}{2}\dot{G}\frac{\partial^2}{\partial X^2}$. (We relabel $\psi' \to \psi$.)

The formal solution is

$$\psi(X,t) = e^{-\int_0^t dt' H}\psi(X,0) = e^{-\frac{1}{2}(G(t)-G(0))\frac{\partial^2}{\partial X^2}}\psi(X,0) \tag{0.0.4}$$

$$\psi(X,t) = e^{-\frac{1}{2}(F(t))\frac{\partial^2}{\partial X^2}}\psi(X,0) \tag{0.0.5}$$

The evolution operator $e^{-\frac{1}{2}\int d^2x_1 d^2x_2 F_{x_1x_2t}\frac{\delta}{\delta X_1}\frac{\delta}{\delta X_2}}$ acting on $\psi(X,0)$ upto some scale t, gives $\psi(X,t)$, thus implementing the RG. Here, $F_{x_1x_2t} = -\frac{1}{2}ln\frac{(x_1-x_2)^2+a(t)^2}{(x_1-x_2)^2+a(0)^2}$ is the ERG high energy "propagator". t is the scale upto which you are doing the RG transformations. $a(0)$ is the UV cutoff, $a(t)$ is the IR cutoff. $a(t) = a(0)e^t$. Which implies $t = ln(a(t)/a(0)$, the log of the ratio of the scales whose coefficient is the beta function.

The Sine-Gordon theory

The action for the theory is

$$S = \frac{1}{4\pi} \int \frac{d^2x}{a(0)^2} \Big((\partial X)^2 + m^2 X^2 + F \cos(bX) \Big) \tag{0.0.6}$$

Computing beta functions

To compute the beta function we bring down appropriate powers of cosine from the exponential and act on it with the ERG operator. The calculation can then be organised as the ERG operator acting on a power series of cosines.

We want corrections to the $\cos bX$ term. The ERG operator acting on the power series reduces to

$$(c1 + \ c2 + \ c3)t \int \frac{d^2x_1}{a(t)^2} \cos bX(x_1) \tag{0.0.7}$$

c1, c2 and c3 are the coefficients obtained after the ERG operator acts on the power series term by term.

The derivative of the coefficient w.r.t to t is the beta function.

The Beta functions

The ERG operator acting on cosine gives the leading term of the beta function. The sub-leading term is given by the ERG operator acting on the appropriate powers of cosines. The leading term in the β-function for F is,

$$-2\delta F \tag{0.0.8}$$

$\delta = b^2/4 - 1$ is the anomalous dimension of the cosine. At marginality $b^2 = 4$. As the theory starts to flow δ becomes non-zero and this is the leading contribution to the beta function for F. The sub-leading contribution to the Sine-Gordon comes from a third order term. The third order term is made of two positive exponentials and one negative one or vice versa and there are three such terms that can combine to give a cosine as the leading term in the OPE. Therefore the contribution to the beta function from the third order term is

$$-\frac{F^3}{8} \tag{0.0.9}$$

The full β_F

The full beta function is

$$\beta_F = -2F\delta - \frac{F^3}{8} \tag{0.0.10}$$

Beta function for δ

The ERG operator acting on the term $(cos(bX(x_1))cos(bX(x_2)))_c$, where the c is for connected, gives the beta function for the parameter δ. This is a correction to the kinetic term therefore we pick out the coefficient of the term $\frac{1}{4\pi}\int d^2x_1\ \partial_a X \partial^a X(x_1)$. The beta function is given by

$$\beta_\delta = -\left(\frac{\delta+1}{8}\right) F^2 \tag{0.0.11}$$

The generalized Sine-Gordon model

The action for the generalized theory is

$$\begin{aligned} S_{boundary} =& \frac{1}{4\pi}\int d^2x[(\partial_\mu \vec{X}).(\partial^\mu \vec{X}) + m^2\vec{X}.\vec{X} + \frac{F}{a(0)^2}\cos\ (\vec{b}_1.\vec{X}) \\ &+ \frac{G}{a(0)^2}\cos\ (\vec{b}_2.\vec{X}) + \frac{H}{a(0)^2}\cos\ (\vec{b}_3.\vec{X})] \end{aligned} \tag{0.0.12}$$

This is in euclidean d=2. There are three cosines with three different parameters $\vec{b_i}$'s instead of the usual Sine-Gordon where we have one b. Powers of $a(0)$, the UV cutoff, have been added so that the engineering dimension of the cosine term is zero. All b_i's and X's are vectors, in the d-dimensional target space. The quantum scaling dimension of the cosine is $b_i^2/2$. $b_i^2/2 = 2$ for a marginal cosine.

We can view this model as the world sheet action for a string and the $e^{i\vec{b_i}.\vec{X}}$, therefore, can be interpreted as the tachyon vertex operator with a definite momentum $\vec{b_i}$. Here, the metric g_{MN} (for the d-dimensional space in which the string theory resides) in the dot product $b^M b^N g_{MN}$ has Minkowski signature. By doing this we have an additional freedom to tune the norm of the vector b^M to the required value by modifying the individual components of the vector. This is important, for example, for the massless string modes(when $b^2 = 0$). This is an additional structure in this model. Another advantage of generalizing the model is that the sub-leading term for the beta function of F appears at the quadratic order, as against, in the usual case, where it appears at the cubic order. Although the generalized model is more complicated, the computational complexity of getting the beta functions is greatly reduced. This might help significantly when we try to get the direct transformations, as stated in the introduction.

Beta functions

The beta functions of the generalized Sine-Gordon are a dual power series in F and $\vec{b}_1$.

The leading term in the β-function for F,

$$-2\delta F \tag{0.0.13}$$

As before.

To calculate the sub-leading contribution we will bring down one power of $\frac{G}{a(0)^2}\cos\ b_2.X(x)$ and $\frac{H}{a(0)^2}\cos\ b_3.X(x)$ each. We act with the ERG operator, impose $b_2 + b_3 = b_1$ and exponentiate(taking only the connected pieces). The beta function for the coupling F is the t-derivative of the coefficient of the $cos b_1.X$ term. The contribution at this order is

$$\frac{GH}{4} \tag{0.0.14}$$

A similar calculation was done by [10]. This result is a good check on the

ERG.

The full beta function is

$$\beta_F = -(2F\delta - \frac{GH}{4}) \tag{0.0.15}$$

The beta function for the $\vec{b}_1$ parameter is the same as for b.

C-function of the Sine Gordon

Lagrangians have a set of parameters(couplings). Generically, when various parameters in the Lagrangian of a theory are set to zero one finds that the symmetries of the theory are enhanced. If one reintroduces the parameters, this enhanced symmetry breaks explicitly to the actual symmetry of the theory. To get the enhanced symmetry, instead of setting the parameters to zero, we can always assign transformation rules to the coupling constants such that, if the field transformations are accompanied by transformations of the coupling constants, the full enhanced symmetry is preserved. Some of these extended symmetries of the Lagrangian could have quantum anomalies. Then, under a simultaneous transformation of the fields and the coupling constants by this extended symmetry one picks up only the anomaly of the symmetry transformations. Then, if we path integrate over the dynamical fields and remain with a functional of the background parameters, then this functional of the background parameters reproduces the anomaly. In our case the anomaly comes from the violation of the conformal symmetry from some quantum effects.

We consider a 2d RG flow from a CFT in the UV(with central charge

C_{UV}) to a CFT in the IR(with central change C_{IR}). In a general CFT the trace anomaly under Weyl transformations has the form

$$\delta_\sigma S = \frac{C_{UV}}{24\pi}\int d^2x\sqrt{g}\sigma R \tag{0.0.16}$$

Here $g_{\mu\nu} \to e^{2\sigma}g_{\mu\nu}$ is the Weyl transformation of the background metric $g_{\mu\nu}$. To make a general theory Weyl invariant we introduce a background field τ such that under Weyl transformations $\tau \to \tau + \sigma$. Then $e^{-2\tau}g_{\mu\nu}$ is Weyl invariant. We also replace every mass scale M, that breaks Weyl invariance, by $Me^{-\tau}$. After doing all this the full theory is invariant under Weyl transformations. Thus we have extended the symmetries of the theory. The anomaly, $\delta_\sigma S = \frac{C_{UV}}{24\pi}\int d^2x\sqrt{g}\sigma R$, is a property of the full quantum theory, it must be reproduced at all scales. In the deep IR one obtains the contribution C_{IR} to the anomaly from CFT_{IR}. Therefore after one flows to the IR, the rest of the anomaly has to come from an explicit functional of the form

$$\frac{C_{UV} - C_{IR}}{24\pi}\int d^2x(\partial\tau)^2 \tag{0.0.17}$$

Because the simultaneous variation of this action and the effective IR action, add up together to reproduce the anomaly of the UV theory. This term gives us a notion of the central charge. We calculate this term.

The interaction term is

$$S = \int \frac{d^2x}{a(0)^2}F\ cos\ bX$$

We act with the ERG operator on it and introduce a dilaton field ϕ to restore conformal invariance. The action is invariant when $t \to t + \xi$ and $\phi \to \phi - \xi$ are done simultaneously. Now we go to $\frac{1}{2!} < S^2 >_c$. Here we operate the ERG operator on $O(\cos^2 bX)$ term. The coefficient of $-\frac{1}{24\pi} \int d^2x \, \partial_a \phi(x) \partial^a \phi(x)$ is the central charge.

$$\Delta c = 3\pi^2 F^2 \delta$$

Beta functions for the generalized theory from the bulk–position space calculation

[11] argued that for near marginal operators in d=2 the coefficient of the leading logarithmic deviation from $1/R^4$ scaling behaviour for correlation functions is the beta function. The leading term for the beta function of F comes from the two point function of the cosine term. We compute this correlator from the bulk. For the sub-leading term, we take the two point function, insert another operator and look at the deviation in the scaling behaviour of this object from the $1/R^4$ behaviour. This gives us the sub-leading term in the beta function for F. We compute a suitable three point correlator from the bulk.

Bulk action

To construct an action that reproduces the appropriate two and three point correlators we take bulk scalar fields such that their masses match the scaling behaviour of the cosine operators on the boundary. We also have to introduce an interaction vertex in the bulk theory to reproduce the three point boundary correlator.

The bulk action:

$$S_{bulk} = \int d^3x\sqrt{g}[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(m_\phi\phi)^2 + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}(m_\chi\chi)^2 \\ + \frac{1}{2}(\partial\gamma)^2 + \frac{1}{2}(m_\gamma\gamma)^2 - \lambda_3\phi\gamma\chi] \quad (0.0.18)$$

ϕ, γ and χ are massive scalar bulk fields($m^2 = \Delta(\Delta - d)$) with boundary conditions

$\phi(z_0, \vec{z}) = 0$ for $z_0 \to \infty$ and $\phi(z_0, \vec{z}) \to z_0^{d-\Delta}\phi_0(\vec{z})$ as $z_0 \to 0$.

Δ's are the mass dimension of the dual operators(in our case the cosine's). $\Delta = b^2/2$. Similar relations hold for γ and χ. The kinectic term plus the mass term give the two point boundary correlator. The $\lambda_3\phi\gamma\chi$ interaction vertex is chosen so that the boundary three point correlator of the three cosines is reproduced. λ_3 is fixed by computing the three point correlator from both sides. ϕ_0, γ_0 and χ_0 are related to the boundary couplings F, G and H by a relative normalization. This is fixed by computing two point correlators from both sides.

Beta function for F

The generating function for the two point function can be obtained from the kinetic plus the massive term. The generating function for the three point function can be obtained from the $\lambda_3 \phi\gamma\chi$ vertex. Substitute the solution to the free equation of motion in these. Fix the relations between ϕ_0, γ_0, χ_0 and F,G,H and compute the value of λ_3 by comparing the appropriate two and three point functions from the bulk and in the boundary theory. Substitute these and pick out the log divergent part. We get,

$$\beta_F = -\left(2\delta F - \frac{GH}{4}\right) \tag{0.0.19}$$

which matches our result from the boundary calculation.

Beta function for δ

To compute the beta function for δ we look at

$$\begin{aligned} S_{bulk} = \int d^3x\sqrt{g}[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(m_\phi \phi)^2 \\ + \frac{1}{2}(\partial\sigma)^2 + \frac{1}{2}m_\sigma^2\sigma^2 - \frac{1}{2}\lambda_\sigma \sigma \bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi] \end{aligned} \tag{0.0.20}$$

To compute this beta function we want to calculate the generating function for the three point function from the vertex $-\frac{1}{2}\lambda_\sigma\sigma\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Putting in all the relative normalizations and the value of λ_σ, the log divergent term gives us the beta function

$$\beta_\delta = \frac{-F^2(1+\delta)}{8} \tag{0.0.21}$$

This matches with earlier results of Amit et al.

Beta functions for the generalized theory from the bulk–momentum space calculation

We start with the bulk action with the term $\Phi\chi\gamma$ (here we relabel the field ϕ from earlier to Φ for this section for notational clarity).

$$S_{bulk} = \int d^3x\sqrt{g}[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}(m_\Phi\Phi)^2 + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}(m_\chi\chi)^2 + \frac{1}{2}(\partial\gamma)^2 + \frac{1}{2}(m_\gamma\gamma)^2 - \lambda_3\Phi\gamma\chi] \tag{0.0.22}$$

For small λ_3, we expand the field Φ in power of λ_3, and obtain the equations of motion order by order in λ_3. γ and χ have similar expansions. We fourier transform along all directions parallel to the boundary at $z = 0$. We solve the equations of motion upto $O(\lambda_3)$ and then pick out the log divergent parts of the solutions. To renormalize these the source terms of the bulk fields start running. These terms cancel off the divergent parts and give the beta function. The beta function is

$$\beta_F = -(2\delta F - \frac{GH}{4}) \tag{0.0.23}$$

which matches all earlier results.

Conclusion

In this thesis wwe study the Beta fucntions of the Sine-Gordon model. We also calculate the central charge. We show that this quantity is consistent with calculations that others have done using standard techniques and also holographically. Then we construct a bulk AdS_3 dual which reproduces some specific correlators and use this to compute the beta functions of the Sine-Gordon on the boundary theory holographically. We do this in position and momentum space. All results are in agreement.

Plan of the thesis

An approximate plan of the thesis is as follows.

1. We start with a detailed discussion of the ERG technique and how to use it for computing beta functions.

2. In the second chapter we will describe the Sine-Gordon model, the generalized model and show the beta function computations for both.

3. In the third chapter we will describe the C-function calculation and obtain a form for the C-function of the Sine-Gordon term using the Exact RG.

4. In the fourth chapter we will construct a bulk action that reproduces specific correlators which we will use to compute beta functions using position and momentum space techniques.

1 Introduction

The exact renormalisation group (ERG), first written down by Wilson [1, 2, 3] has been an object of much study. It has been developed further [4] and different versions suitable for different purposes have been written down since then [5, 6, 7]. There are a large number of good reviews [13, 14, 15, 16]. A lot of work has been done on the RG of the Sine-Gordon model over the last few years and many computations have been carried out analytically and numerically [56, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84].

This thesis studies the application of ERG mainly to two dimensional field theories - the emphasis being on a simple way of writing down the solution to the ERG in terms of an evolution operator. Some known results are reproduced and a new result is obtained on the flow of the c-function in Sine-Gordon theory. While the main application of the ERG has been in the study of critical phenomena - to obtain the numerical value of critical exponents, our motivation comes from string theory. In the context of string theory the renormalisation group has been used as a formal tool. Recently, the ERG was used to obtain the equations of motion for the fields of the string somewhat as in string field theory[17]. In this approach string propagation in a general background is described as a completely general two dimensional field theory - all relevant, irrelevant and marginal terms are included. This is a natural generalization of the idea that a conformal field theory describes a consistent string background[1]. The condition of conformal invariance is

[1]See [17] for references to earlier papers on this topic.

imposed on the action. Thus the exact renormalisation group equation for this two dimensional theory is written down and the fixed point equations for the couplings are identified with the space time equations of motion of the background fields. One has to further generalise the original RG approach to obtain equations that are gauge invariant. The new ingredient is the use of loop variables. In order to make the equations gauge invariant the two dimensional field theory is written in terms of loop variables [18]. Loop variables have also been incorporated into the ERG - and gauge invariant and interacting equations have been written down. Furthermore these equations are background independent [17].

Again within string theory, but now in the context of the AdS/CFT correspondence the idea of the renormalisation group has emerged in the guise of holographic RG [19, 20]. The RG flow of quantities has been equated with the evolution of the holographic dual bulk fields in the radial direction. Thus a flow of renormalised coupling constants in the boundary is compared to the flow of the bulk field, which also requires renormalization. Many details of this comparison have been worked out in [21]. In field theory there are quantities such as the c-function of Zamolodchikov in two dimensions[50] and the c and a-functions in four dimensions [52] that are monotonic along the flow. There have been attempts to find analogous quantities in the holographic dual in the bulk. One such quantity is the entanglement entropy which has also been shown to be monotonic along the flow - both in the field theory and its holographic dual. (Although the precise connection with Zamolodchikov's c-function is not established.)

Besides being interesting due to the connection with string theory, two dimensional models have some advantages as an arena where these ideas can be developed. They are simpler to work with and their holographic dual AdS_3 equations are often exactly solvable. This motivates us to explore two dimensional field theories using ERG. An interesting and very non trivial field theory is the Sine-Gordon theory. The Sine-Gordon β-functions in fact are closely related to equations of motion of the bosonic string tachyon [10]. The equations of motion of the generalized version of the Sine-Gordon theory describes the bosonic string propagating in a tachyonic background and the β-function equations are proportional to the tachyon equation of motion. This is a special case of the connection to string theory mentioned above.

In the context of critical phenomena also this model has been related to a very interesting two dimensional model - the X-Y model. The X-Y model has an interesting phase transition first noticed by Kosterlitz and Thouless. It is possible to rewrite the X-Y model as a Sine-Gordon model. This theory has been studied in great detail in [8] who obtained the phase diagram as well as the Kosterlitz-Thouless flow equations using continuum field theory techniques. They also showed that the model is renormalizable when physical quantities are written as a power series in terms of two coupling constants[2].

In this thesis we use the ERG to obtain the β function equations of Sine-Gordon theory using the ERG. A particular form of the ERG due to Polchinski is used here. This ERG is reformulated as a linear evolution operator. Although this reformulation has been noticed [5, 6], it has not received much attention in practice. We show that it is very convenient

[2] In the string theory context one of these couplings corresponds to the tachyon and the other to the dilaton.

to work out the flow of objects in a systematic perturbation series. In the usual continuum calculations β-functions are calculated as a byproduct of the renormalization program. As first explained by Wilson [2], when one obtains the flow of a marginal coupling, in the limit that the UV cutoff is taken to infinity, the β-function has the property that it depends only on the value of the coupling and not explicitly on the scale. This also implies that the logarithmic divergence has the information about the β-function and higher orders in the logarithm are determined by the coefficient of the leading logarithmic divergence. Thus if we have an evolution equation one needs to only evaluate the leading divergence. Furthermore this is different from actually solving the ERG equation which gives a coupled differential equation involving an infinite number of couplings. The process of eliminating the irrelevant couplings and solving for the marginal coupling is automatically implemented during the perturbative evaluation of the evolution operator.

Thus mathematically one can imagine a set of coupled recursion equations [2] for a marginal coupling g_l, a relevant coupling μ_l and an irrelevant coupling w_l obtained in a blocking transformation that implements the RG. We reproduce a summary of the discussion in [2](The factors of 4 and 1/4 are illustrative):

$$
\begin{aligned}
g_{l+1} &= g_l + N_g[g_l, \mu_l, w_l] \\
\mu_{l+1} &= 4\mu_l + N_\mu[g_l, \mu_l, w_l] \\
w_{l+1} &= \frac{1}{4} w_l + N_w[g_l, \mu_l, w_l]
\end{aligned}
\qquad (1.0.24)
$$

Here N_g, N_w, N_μ are the nonlinear terms. As explained in [2] one can reorganize the equations and solve them iteratively so that it depends on w_0 (initial condition for w_l) and μ_L (final value of μ_l) and then one finds that on solving this iteratively, and when $1 << l << L$ is very large, so the memory of the initial conditions have been lost, one can set $\mu_L = w_0 = 0$ and obtain a recursion equation for g_l alone

$$g_{l+1} = V(g_l) \tag{1.0.25}$$

The crucial point is that in this limit $V(g_l)$ has no explicit dependence on l. One can now extract from this a β-function $\beta_g = \frac{dg}{dt}$ (l is replaced by a continuous variable t) which depends only on $g(t)$.

Now imagine using the evolution equation to obtain $g(t+\tau)$ starting from $g(t)$. One obtains a series of the form

$$g(t+\tau) = g(t) + \tau\frac{dg(t)}{dt} + \frac{\tau^2}{2!}\frac{d^2g(t)}{dt^2} + .. \tag{1.0.26}$$

Now $\frac{dg(t)}{dt} = \beta(g(t))$. Thus

$$\frac{d^2g(t)}{dt^2} = \frac{d\beta(g(t))}{dt} = \frac{d\beta(g(t))}{dg}\frac{dg(t)}{dt} = \frac{d\beta(g(t))}{dg}\beta(g(t))$$

Thus when $t = 0$, $\tau = \ln\frac{\Lambda_0}{\Lambda}$ is what we call the logarithmic divergence in perturbation theory. What we are seeing is that the leading term decides the β-function and the higher powers of τ are fixed in terms of the leading term.[3] The application of the evolution operator in powers of the evolution

[3]This is the counterpart of the statement for dimensional regularization, that the $\frac{1}{\epsilon}$ pole determines the beta function, when only marginal couplings are present. The higher order pole residues are fixed in terms of the leading residue.

Hamiltonian, gives us a series as above in τ. It automatically gives the solution of the ERG recursion equations and one can extract a power series for the evolution of the marginal coupling. Thus β -functions are obtained in a simple way without worrying about the technicalities of renormalization.

We illustrate this method with some examples such as the computation of the central charge of a free scalar field theory and calculating the flow of the coupling in ϕ^4 theory in four dimensions. We then apply it to the more interesting case of the Sine-Gordon theory. We find that the equations obtained are consistent with those obtained in [8]. While the precise coefficients are not the same, the combination of coefficients identified in [8] as being universal, matches exactly. In addition to flow of couplings, one can study the flow of the c-function [50, 51, 49, 53]. In particular we do the calculation of the c-function for the Sine-Gordon theory.

Recently the entanglement entropy of this theory has been calculated both in the field theory and in the holographic dual and the answers are shown to agree to lowest order [55]. The central charge calculation done here also gives results in exact agreement with these calculations - if we assume that the relation between entanglement entropy and central charge function persists at least to lowest non trivial order away from the fixed point. This computation is a first attempt towards developing an understanding of a precise connection between the ERG in the boundary theory and Holographic RG in the bulk.

Another interesting computation, to understand better the holographic RG, would be to reproduce the β- functions for the Sine-Gordon model holographically. It has been shown in [9] that an ERG equation in a boundary theory can be mapped to a scalar field action in AdS space time. The main results are for a free theory. Some suggestions for how the interactions should work out were given there. To understand these issues better it is important to understand RG equations in the boundary theory and obtain them from some bulk computations. The precise connection between these equations and what is called "holographic RG" - which is really a radial evolution equation of the bulk theory - needs to be understood better. These computations are a step towards that goal. There is extensive literature on the AdS/CFT correspondence and holographic RG, [23]–[46], [85]–[92] to name a few.

The boundary theory is a free CFT perturbed by some composite (cosine) operators. The bulk theory that reproduces the leading two and three point correlators is a cubic theory. Of course there are any number of composite operators with definite scaling dimension and so the bulk theory should have a field of definite mass corresponding to each of these - we are assuming that an AdS dual exists for the free scalar theory in 2 dimensions. One can study the RG flow of this theory and one should be able to reproduce the β - function of the cosine operator of the boundary theory. We do this calculation also in this thesis.

However, motivated by the string theory tachyon connection we consider a generalized Sine-Gordon theory. In string theory, instead of one scalar field, there are D scalar fields ($D = 26$ for the bosonic string). The tachyon perturbation is of the form

$$\int d^2z \int_k \phi(k) e^{i\vec{k}.\vec{X}}$$

$\phi(k)$ is the tachyon field in momentum space. We can consider a continuum of values of $\vec{k}$. For each value of k it corresponds to a Sine-Gordon like theory. In [10] this theory was considered and shown to reproduce the leading non linear terms in the tachyon - dilaton system equations of motion in string theory.

Holographic techniques in position space are well suited for calculating correlation functions. In [11] a proper time method was used to evaluate the tachyon equation of motion starting from two point functions. We will use this technique here. For near marginal operators the two point function has the form

$$\langle O_i(R) O_j(0) \rangle = \frac{G_{ij}}{R^4} + \frac{H_{ij}}{R^4} \ln \frac{R}{a}$$

G_{ij} is the Zamoldchikov metric. A similar formula exists for the open string boundary CFT, with R^4 replaced by R^2. In [11] it was shown (in the context of the open string) that

$$H_{ij} \phi^j = 0$$

is the tachyon equation of motion to all orders in perturbation theory. Furthermore, it was argued by Polyakov [56] (for closed strings) that the equation of motion and β-function are related simply:

$$\frac{\partial \Gamma[\phi]}{\partial \phi^i} = G_{ij} \beta^j$$

This was also shown to all orders in perturbation theory in the open string context in [11]. Thus we can conclude [12] that

$$H_{ij}\phi^j = G_{ij}\beta^j$$

Thus to extract the beta function we can compute the two point function, corrected by interactions, and obtain the leading logarithmic deviation from the $\frac{1}{R^4}$ scaling to obtain the β function. In the position space holographic calculation we employ this technique.

Once the perturbation is turned on it is no longer a CFT. This should reflect itself in the bulk deviations from AdS. This requires taking into account the gravitational back reaction. This back reaction in the bulk can be seen to manifest itself in the field strength renormalization of the boundary scalar fields. This gives us the beta function for the field strength renormalization. To compute this we look at the fluctuations of the graviton about the AdS. This contribution comes from another cubic vertex in the bulk. This is also equivalent to the dilaton equation in the string theory context.

This thesis is organised as follows. Chapter 2 introduces the ERG and gives some background material on it. We first obtain the specific form of the Polchinski equation that we use to implement the ERG. Next we describe the evolution operator approach and illustrate it with a calculation of the c-function in a free massive scalar field theory. Then we illustrate the method by calculating the β-function for ϕ^4 field theory in four dimensions. In Chapter 3 we start with a brief overview of the Sine-Gordon model. We fix propagators and other normalizations. Finally the β-functions for the Sine-Gordon theory are calculated. It is shown that the results are in agreement with those of [8]. In Chapter 4 we motivate and detail the modification of the Sine-Gordon model and compute the sub-leading(quadratic) term for the generalized theory. This concludes our boundary calculation for the Sine-Gordon beta functions. In Chapter 5 the central charge calculation is described and the c-function for the sine Gordon theory is calculated.

In Chapter 6 we give a brief overview of AdS-CFT computations using position space techniques. Then we compute the leading and sub-leading terms for the beta function from the bulk. In Chapter 7 we start by briefly introducing computational techniques in AdS-CFT in momentum space. Then we calculate beta functions for the tachyon using momentum space techniques. All calculations are found to be in agreement with the boundary calculations. In Chapter 8 we compute the beta function for the running of the field strength renormalization. This calculation is done in position space. This is found to be in agreement with previous results[8, 22].

The calculations done in Chapters 2,3 and 5 are done in [22]. The calculations done in Chapters 4,6,7 and 8 can be found in [26].

2 Exact Renormalization Group

2.1 The Polchinski Equation

Renormalization Group is integrating out high momentum modes leaving an effective theory of the low momentum modes. This is what is called "incomplete integration". Wilson observed that the equation($\dot{G} = \frac{\partial G}{\partial t}$)

$$\frac{\partial \psi(X,t)}{\partial t} = -\frac{1}{2}\dot{G}\frac{\partial}{\partial X}(\frac{\partial}{\partial X} + 2G^{-1}X)\psi(X,t) \tag{2.1.27}$$

realizes the notion of incomplete integration. The heat kernel of this equation gives a smooth interpolation of $\psi(X,t)$ between a completely un-integrated function $\psi(X,0)$ and its completely integrated form. Thus the equation is a possible candidate for an exact RG equation.

Substituting $\psi = e^{-S}$ in the above equation, we get

$$\frac{\partial S}{\partial t} = -\frac{1}{2}\dot{G}[\frac{\partial^2 S}{\partial X^2} - (\frac{\partial S}{\partial X})^2 + 2G^{-1}X\frac{\partial S}{\partial X}] + \underbrace{\dot{G}G^{-1}}_{fld\ indep} \tag{2.1.28}$$

From here on we drop all field independent terms as they contain no dynamics and are vacuum bubles. Adding such terms shifts the energy level of the Lagrangian, like a cosmological constant, and does not change anything till you couple it to gravity.

If we substitute $\psi = e^{-\frac{1}{2}G^{-1}X^2}\psi'$ we get an equation:

$$\frac{\partial \psi'}{\partial t} = -\frac{1}{2}\dot{G}\frac{\partial^2 \psi'}{\partial X^2} \tag{2.1.29}$$

Here $\psi' = e^{-S_{int}}$, S_{int} is the interaction part of the action. Again in terms of

S_{int} it becomes an equation in the form first written by Polchinski [20]

$$\frac{\partial S}{\partial t} = -\frac{1}{2}\dot{G}[\frac{\partial^2 S}{\partial X^2} - (\frac{\partial S}{\partial X})^2] \tag{2.1.30}$$

In these equations one can replace X by $\phi(p)$ and easily generalise to field theory. In a field theory RG t is the logarithm of the ratio of scales: the short distance cutoff $a(0)$ is changed to $a(0)e^t$. In a field theory action $\frac{1}{2}G^{-1}X^2$ would stand for the kinetic term (and G for the Green function) and then S would be the interaction part of the action. Polchinski's equation is usually used in the form (2.1.30) (or in the form (2.1.28) for the full action).

In this paper however we use it in the form (2.1.29). This is a *linear* equation and is just a free particle Schroedinger equation. The formal solution of this equation in terms of an evolution operator can easily be written down. Writing a formal solution in this form is useful in some situations: The ERG as is usually written down is an infinite number of equations that give the β-function of one coupling parameter in terms of all the other infinite number of coupling parameters. The usual continuum beta function involves only a few of the parameters involving the lower dimensional operators. To go from the first form to the second form one has to solve these infinite number of equations iteratively [2]. The evolution operator method does this operation in a convenient way (as will be shown). It thus acts as a bridge between the ERG and the continuum field theoretic β-function.

2.2 Free Theory

Let us understand the connection between the ERG equation and the evolution operator by considering the free theory as a pedagogical exercise. The first step is to construct the field theoretic version of Polchinski's ERG:

2.2.1 ERG and β-function

The ERG acting on Ψ is:

$$\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\int dz \int dz' \, \dot{G}(z,z',t)\frac{\delta^2 \Psi}{\delta X(z)\delta X(z')} \equiv -H(t)\Psi(t) \tag{2.2.31}$$

with $\Psi = e^{-\int du \; L(u,t)}$. This can be written as an ERG for L.

Let us write the Wilson interaction as $S \equiv -\int du \; L(u,t)$. We get

$$\frac{\partial S}{\partial t} = -\frac{1}{2}\int dz_1 \int dz_2 \; \dot{G}(z_1,z_2,t)[\frac{\delta^2 S}{\delta X(z_1)\delta X(z_2)} + \frac{\delta S}{\delta X(z_1)}\frac{\delta S}{\delta X(z_2)}] \tag{2.2.32}$$

We could start with a local bare action:

$$S = -\int du \; \frac{1}{2}\delta m^2(u) X(u)^2$$

where $\delta m^2(u) = (e^{2\phi(u)}-1)m^2$ is a position dependent coupling(mass), but in general even if we start with a local action, after one iteration of the RG it becomes non-local. So we start with a non-local action

$$S = -\int du \int dv \; \frac{1}{2} z(u,v,t)X(u)X(v) - m_0(t) \tag{2.2.33}$$

Substituting this in (2.2.32) we get

$$
\begin{aligned}
\dot{m}_0(t) &= -\frac{1}{2}\int_{z_1}\int_{z_2}\dot{G}(z_1,z_2,t)z(z_1,z_2,t) \\
\dot{z}(u,v,t) &= \int_{z_1}\int_{z_2}\dot{G}(z_1,z_2,t)z(z_1,u,t)z(z_2,v,t)
\end{aligned}
\tag{2.2.34}
$$

The set of β-function equations (2.2.34) is exact. But the simplicity is a little misleading because $z(u,v,t)$ is a function of two locations u,v and actually represents an infinite number of local (position dependent) coupling functions, which can be defined by Taylor expansions. Note that even for the free field case we get a non local Wilson action.

2.2.2 Evolution Operator

(2.1.29) can be written as

$$\frac{\partial\psi}{\partial t} = -H\psi \tag{2.2.35}$$

with $H = \frac{1}{2}\dot{G}\frac{\partial^2}{\partial X^2}$, for which the solution is formally

$$\psi(X,t) = e^{-\int_0^t dt' H}\psi(X,0) = e^{-\frac{1}{2}(G(t)-G(0))\frac{\partial^2}{\partial X^2}}\psi(X,0) = e^{-\frac{1}{2}(F(t))\frac{\partial^2}{\partial X^2}}\psi(X,0) \tag{2.2.36}$$

Consider the Schroedinger equation:

$$\frac{\partial\psi}{\partial T} = -\frac{1}{2}F(t)\frac{\partial^2}{\partial X^2}\psi$$

which is solved formally as

$$\frac{d\psi}{\psi} = -\frac{1}{2}\Big(\int dT\Big)F(t)\frac{\partial^2}{\partial X^2}$$

$$\log\psi(X,t,T) = -\frac{1}{2}TF(t)\frac{\partial^2}{\partial X^2} + \log\psi(X,0)$$

$$\psi(X,t,T) = e^{-T\frac{1}{2}F(t)\frac{\partial^2}{\partial X^2}}\psi(X,0)$$

With $T = 1$ we get our solution (2.2.36). The solution to the schrodinger equation is known in terms of a kernel

$$\psi(X,t,T) = \int dX'\ e^{\frac{1}{2F(t)}\frac{(X-X')^2}{T}}\psi(X',0)$$

So setting $T = 1$ we get the solution to our original problem:

$$\psi(X,t) = \int dX'\ e^{\frac{1}{2F(t)}(X-X')^2}\psi(X',0) \tag{2.2.37}$$

If we write $\psi = e^{-S}$ we get

$$e^{-S(X,t)} = \int dX'\ e^{\frac{1}{2F(t)}(X-X')^2}e^{-S(X',0)} \tag{2.2.38}$$

which can also be written in a well known standard form as [58, 5, 6, 16]

$$e^{-S(X,t)} = \int dX'\ e^{\frac{1}{2F(t)}X'^2}e^{-S(X+X',0)} \tag{2.2.39}$$

We can convert the above solution to a field theoretic case and in the free theory, obtain an exact form of the solution to ERG evolution. Working in momentum space, all we need to do is to replace X by $X(p)$. The integral over X' becomes a functional integral over $X(p)$ and in the action we need to sum over all p. The "propagator" $F(t)$ becomes $F(p,t) = G(p, a(0)e^t) - G(p, a(0))$:

$$\int \mathcal{D}X' e^{\frac{1}{2}\int \frac{d^2p}{(2\pi)^2} F^{-1}(p)X'(p)X'(-p)} e^{-S[X+X']} \tag{2.2.40}$$

In this form it looks a free particle (field) calculation where the propagator is $F(p,t) \equiv G(p,a(t)) - G(p,a(0))$ with $a(t) = a(0)e^t$ the moving cutoff. Thus the propagator only propagates the modes that are being integrated out. So, for e.g., when $t = 0$ it vanishes because no integration has been done.

2.2.3 Free Field Theory: Exact Solution of ERG

In the case of the free field the integrations can be carried out exactly.

$$\int \mathcal{D}X' e^{\frac{1}{2}\int \frac{d^2p}{(2\pi)^2} F^{-1}(p)X'(p)X'(-p)} e^{-\frac{1}{2}\int z(p)(X+X')(p)(X+X')(-p)} \tag{2.2.41}$$

$$= \int \mathcal{D}X' e^{\frac{1}{2}\int \frac{d^2p}{(2\pi)^2}(F^{-1}(p)-z(p))X'(p)X'(-p)} e^{-\frac{1}{2}\int_p z(p)X(p)X(-p)+2z(p)X(p)X'(-p)} \tag{2.2.42}$$

$$= \int \mathcal{D}X' e^{\frac{1}{2}\int \frac{d^2p}{(2\pi)^2}(F^{-1}(p)-z(p))X'(p)X'(-p)} e^{-\frac{1}{2}\int_p z(p)X(p)X(-p)+2z(p)X(p)X'(-p)} \tag{2.2.43}$$

$$= \int \mathcal{D}X' e^{-\frac{1}{2}\int \frac{d^2p}{(2\pi)^2} \underbrace{\left(z(p) - F^{-1}(p)\right)}_{\mathcal{F}^{-1}(p)} X'(p)X'(-p)} e^{-\frac{1}{2}\int_p z(p)X(p)X(-p)+2z(p)X(p)X'(-p)} \tag{2.2.44}$$

Completing squares

$$\frac{1}{2}\left(X'^2\sqrt{\mathcal{F}^{-1}} + 2zXX'\right) = \frac{1}{2}\left(X'^2\sqrt{\mathcal{F}^{-1}} + 2zXX' + z^2X^2\mathcal{F} - z^2X^2\mathcal{F}\right)$$

$$= \frac{1}{2}\left(Y'\mathcal{F}^{-1}Y'\right) - \frac{1}{2}Xz\mathcal{F}zX$$

where

$$Y' = X' + zX\mathcal{F}$$

and

$$\mathcal{F} = \frac{F}{Fz - 1}$$

So for (2.2.44) we get

$$= Det^{\frac{1}{2}}[\mathcal{F}] \exp[\frac{1}{2}\int_p X(p)(z\mathcal{F}z - z)X(-p)] \tag{2.2.45}$$

$$= Det^{\frac{1}{2}}[\mathcal{F}] \exp[-\frac{1}{2}\int_p X(p) \underbrace{(\frac{z}{1 - Fz})}_{-z(t)} X(-p)] \tag{2.2.46}$$

Here

$$z(t) = -\frac{z}{1 - Fz}$$

Thus we have an exact solution for the Wilson action.

Furthermore,

$$\frac{dz}{dt} = z^2 \frac{dF}{dt} = z^2 \frac{dG}{dt}$$

which is the second eqn in (2.2.34). We thus make contact with the differential version of ERG.

2.3 β-function of ϕ^4 theory in four dimensions

Now we illustrate the method of calculating the β function for the ϕ^4 theory using the ERG. We use the Polchinski equation with all the corrections to kinetic term being put into the interactions. Since we are only integrating modes with $p > \Lambda$ we do not need a mass as a regulator. So we can put $m^2 = 0$.

The evolution operator is

$$e^{-\frac{1}{2}\int_{x_1}\int_{x_2}(G(x_1,x_2,\Lambda(t)) - G(x_1,x_2,\Lambda(0)))\frac{\delta^2}{\delta\phi(x_1)\delta\phi(x_2)}}$$

We set

$$\psi(0) = e^{-S[\phi,0]} = e^{\underbrace{-\frac{\lambda}{4!}\int_x \phi(x)^4}_{''V''}}$$

The action of the evolution operator on e^V is,

$$e^{-\frac{1}{2}\int_{x_1}\int_{x_2} F(x_1,x_2)\frac{\delta^2}{\delta\phi(x_1)\delta\phi(x_2)}} e^{-\frac{\lambda}{4!}\int\phi^4}$$

$$= \int \mathcal{D}\phi' e^{\frac{1}{2}\int_{x_1}\int_{x_2} F^{-1}(x_1,x_2)\phi'(x_1)\phi(x_2')} e^{-\frac{\lambda}{4!}\int(\phi'+\phi)^4}$$

$$= \int \mathcal{D}\phi' e^{\frac{1}{2}\int_{x_1}\int_{x_2} F^{-1}(x_1,x_2)\phi'(x_1)\phi(x_2')} e^{-\frac{\lambda}{4!}\int(\phi^4+4\phi^3\phi'+6\phi^2\phi'^2+4\phi\phi'^3+\phi'^4)}$$

We can keep some terms in the exponent and bring down the rest:

$$= \int \mathcal{D}\phi' e^{\frac{1}{2}\int_{x_1}\int_{x_2} F^{-1}(x_1,x_2)\phi'(x_1)\phi(x_2')} e^{-\frac{\lambda}{4!}\int(\phi^4+4\phi^3\phi'+6\phi^2\phi'^2)}$$

$$[1 - \frac{\lambda}{4!}\int(4\phi\phi'^3 + \phi'^4) + \frac{1}{2!}(\frac{\lambda}{4!})^2[\int(4\phi\phi'^3 + \phi'^4)]^2 + ...]$$

Let us evaluate:

$$\int \mathcal{D}\phi' e^{\int_{x_1}\int_{x_2}\underbrace{(\frac{1}{2}F^{-1}(x_1,x_2) - \frac{\lambda}{4!}\delta(x_1-x_2)6\phi^2(x_1))}_{\frac{1}{2}H^{-1}(x_1,x_2)}\phi'(x_1)\phi(x_2') - \int_x J(x)\phi'(x)}$$

$$= Det^{-\frac{1}{2}} H^{-1}(x_1,x_2) e^{\frac{1}{2}\int_{x_1}\int_{x_2} J(x_1)H(x_1,x_2)J(x_2)}$$

$$Det^{\frac{1}{2}} H = e^{\frac{1}{2}Tr\ln[\frac{1}{F^{-1}-\frac{2\lambda}{4!}6\phi^2 I}]}$$

$$= e^{\frac{1}{2}Tr\ln F - \frac{1}{2}Tr\ln[1-\frac{\lambda}{2}\phi^2 F]}$$

Expand the log:

$$\frac{1}{2}Tr\ln[1 - \frac{\lambda}{2}\phi^2 F] =$$

$$\frac{1}{2}\Big(-\frac{\lambda}{2}\int_x \phi^2(x)F(x,x) - \frac{1}{2}(\frac{\lambda}{2})^2 \int_{x_1}\int_{x_2} \phi^2(x_1)F(x_1,x_2)\phi^2(x_2)F(x_2,x_1) + ... \Big)$$

In momentum space F can be understood as a propagator with momentum restricted in the range $\Lambda < p < \Lambda_0$. Thus

$$\phi^2(x)F(x,x) = \phi^2(x)\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2}$$

This is the usual quadratically divergent mass correction. To get the correction to the ϕ^4 term we consider the next term in $-\frac{1}{2}Tr\ln[1+\frac{\lambda}{2}\phi^2 F]$,

$$-\frac{1}{2}\times\frac{1}{2}\frac{\lambda^2}{4}\times\frac{1}{(4\pi)^2}\int_{\Lambda_0^2}^{\Lambda^2} p^2dp^2\frac{1}{p^4}\phi(0)^4$$

The external momentum is set to zero i.e. $\phi(x)$ is uniform. This is a correction to $\frac{\lambda}{4!}$ so we factor out 4! to get,

$$-\frac{4!}{4!}\times\frac{1}{2}\times\frac{1}{2}\frac{\lambda^2}{4}\times\frac{1}{(4\pi)^2}\int_{\Lambda_0^2}^{\Lambda^2} dp^2\frac{1}{p^2}$$

$$= -\frac{1}{4!}\frac{3}{2}\frac{\lambda_0^2}{(4\pi)^2}\ln\frac{\Lambda^2}{\Lambda_0^2}$$

Since $\Lambda = e^{-t}\Lambda_0$ we get

$$-\frac{1}{4!}\frac{3}{2}\frac{\lambda^2}{(4\pi)^2}(-2t)$$

Thus

$$-\lambda(t) = -\frac{\lambda}{4!}+\frac{3}{(4\pi)^2}\lambda^2 t$$

$$\dot{\lambda} = -\frac{3}{16\pi^2}\lambda^2$$

This is the well known β function of the ϕ^4 theory in four dimensions.

What about contributions to β function from $\langle \frac{\lambda}{3!}\phi\phi'^3\rangle$? For this we calculate,

$$\int \mathcal{D}\phi'[-\frac{\lambda}{3!}\phi\phi'^3]e^{\int_{x_1}\int_{x_2}\frac{1}{2}H^{-1}(x_1,x_2)\phi'(x_1)\phi(x_2')-\int_x J(x)\phi'(x)}$$

where J will be set to $\frac{\lambda}{3!}\phi^3$ in the end. Thus one evaluates

$$-\frac{\lambda}{3!}\phi\frac{\delta^3}{\delta J(x)^3}[Det^{-\frac{1}{2}}H^{-1}(x_1,x_2)e^{\frac{1}{2}\int_{x_1}\int_{x_2}J(x_1)H(x_1,x_2)J(x_2)}]$$

All terms necessarily have one factor of the form HJ. To lowest order in λ, $H = F$. When we set $J = \frac{\lambda}{3!}\phi^3$ the external momentum is zero (for constant ϕ) and thus we have an F propagator with zero momentum. This is zero because F is non zero only for momenta greater than Λ. Thus this correction is zero to lowest order.

3 The Sine-Gordon theory.

We now turn to the Sine-Gordon theory. We compute the β-functions for this theory using the ERG evolution operator.

The action for the theory is

$$S = \frac{1}{4\pi}\int \frac{d^2x}{a(0)^2}\Big((\partial X)^2 + m^2X^2 + F\cos(bX)\Big) \tag{3.0.47}$$

$a(0)$ is the UV cut-off.

3.1 The Green's Function.

The Green function for the Klein Gordon field in two dimensions in Euclidean space is

$$G(x_2,t_2;x_1,t_1) = \int_0^\infty ds\ (\frac{1}{4\pi s})e^{-m^2s}e^{-\frac{(x_2-x_1)^2+(t_2-t_1)^2}{4s}} \tag{3.1.48}$$

The small t region gets contribution from $x^2=0$ region. This is the UV. A way to regularise this is to cutoff the integral:

$$G(x_2,x_1,\epsilon) = \int_\epsilon^\infty ds\ (\frac{1}{4\pi s})e^{-m^2s}e^{-\frac{(x_2-x_1)^2}{4s}} \tag{3.1.49}$$

$$\int_0^\infty ds\ (\frac{1}{4\pi s})e^{-m^2s}e^{-\frac{(x_2-x_1)^2+(t_2-t_1)^2}{4s}} = \frac{1}{2\pi}K_0(mx) \tag{3.1.50}$$

Here $x = \sqrt{(x_2-x_1)^2+(t_2-t_1)^2}$.

$$
\begin{aligned}
&= \frac{1}{2\pi}\sqrt{\frac{\pi}{2mx}}e^{-mx} + ... : mx >> 1 \\
&= \frac{1}{2\pi}[-ln\ (mx/2)(1+\sum_{k=1}^{\infty}\frac{(mx/2)^{2k}}{(k!)^2})] + \psi(1) + \sum_{k=1}^{\infty}\frac{(mx)^{2k}}{2^{2k}(k!)^2}\psi(k+1) \quad : mx << 1
\end{aligned}
$$

We will do our calculations in the $mx << 1$ regime.

3.2 Reproducing the continuum β-functions.

We would like to reproduce the flow for F and b. For the kinetic term

$$S_{Kinetic} = \frac{1}{\alpha'}\int d^2z\ \partial_z X \partial_{\bar{z}} X = \frac{1}{2\alpha'}\int d^2x\ \partial_a X \partial^a X$$

the Green's function in complex coordinates is

$$G =< X(z)X(w) >= -\frac{\alpha'}{2\pi}\ln\frac{|z-w| + a(0)}{R}$$

and we choose $\alpha' = 2\pi$ and substitute that when we carry out calculations in the later sections. Here R is some scale.

$$< X(0)X(0) >= -\frac{\alpha'}{2\pi}\ln\ \frac{a(0)}{R}$$

In real coordinates

$$G =< X(x_1)X(x_2) >= -\frac{\alpha'}{4\pi}\ln|\frac{(x_1-x_2)^2 + a(0)^2}{R^2}| \qquad (3.2.51)$$

The evolution operator acting on the unintegrated theory gives

$$\begin{aligned}
\psi(t) &= e^{-\int_{t_0}^{t} H(t')dt'}\psi(0) \\
&= e^{-\frac{1}{2}\int_{t_0}^{t} dt' \int d^2x_1 \int d^2x_2 \ \dot{G}(x_1,x_2,t') \frac{\delta^2}{\delta X(x_1)\delta X(x_2)}} e^{-S[X,0]} \\
&= \int \mathcal{D}X'' e^{\frac{1}{2}\int d^2x_1 \int d^2x_2 \ F^{-1}(x_1,x_2)X''(x_1)X''(x_2)} e^{-S[X+X'']} \qquad (3.2.52)
\end{aligned}$$

Here

$$S[X,0] = \int \frac{d^2x}{a(0)^2}\frac{F}{4\pi}[\frac{e^{ibX}+e^{-ibX}}{2}]$$

and

$$F(x_1,x_2,t) = G(x_1,x_2,a(t)) - G(x_1,x_2,a(0)) = G(x_1,x_2,a(0)e^t) - G(x_1,x_2,a(0))$$

Thus

$$F(x_1,x_2,t) = -\frac{\alpha'}{4\pi}\ln\ [\frac{(x_1-x_2)^2+a(t)^2}{(x_1-x_2)^2+a(0)^2}] \qquad (3.2.53)$$

is like a propagator. Note $F(x_1,x_2,t)$ will also be denoted by $F_{x_1x_2t}$ which are distinct from F, which is the coupling of the $\cos b_1.X(x)$ term. Also

$$F(x,x,t) = -\frac{\alpha'}{4\pi}\ln\ \frac{a(t)^2}{a(0)^2} = -\frac{\alpha'}{2\pi}t \qquad (3.2.54)$$

Thus the evolution operator acting on the $e^{-S_{int}}$ gives

$$\psi(t) = exp[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!} < S^n >_c] \qquad (3.2.55)$$

where $\langle S^n\rangle_c$ stands for the connected part of $< S^n >$ and $< ... >$ stands for doing the X'' integral. This is the cumulant expansion for the Wilson action at scale t.

3.2.1 Leading Order β function for $F(t)$.

Let us bring down one power of $(S[X,0])_c = (\int L)_c$, where the sub-script c signifies that only the connected parts for all terms will be retained from $e^{-S[X,0]}$ and act on it with the evolution operator. Writing the cosine as a sum of exponentials, and noting that the action of the evolution operator gives the same factor for both exponentials, we get:

$$(\int L)_c \equiv \int \frac{d^2x_1}{a(0)^2} \frac{F}{4\pi} e^{\frac{1}{2}(b)^2(F(x_1,x_1,t))} cos(bX(x_1))$$

Powers of $a(0)$ have been added for dimensional consistency. We can use the form given in (3.2.54) to get

$$(\int L)_c \equiv \int \frac{d^2x_1}{a(0)^2} \frac{F}{4\pi} (\frac{a(0)^2}{a(t)^2})^{\frac{b^2}{4}} cos(bX(x_1))$$

The factor $(\frac{a(0)^2}{a(t)^2})^{\frac{b^2}{4}}$ is the effect of self contractions in a range of energies $(\Lambda, \Lambda e^{-t})$. This is also the normal ordering factor that one usually obtains which has $a(t)$ replaced by the IR cutoff $1/m$. The usual normal ordering integrates out self contractions of *all* fields, i.e up to the IR cutoff. In the ERG only some fields are integrated out and after the ERG evolution the field X only has lower momentum modes in it, and the pre-factor is the effect of integrating out the rest. One more difference is that normal ordering takes care of only self interactions. The ERG removes all interactions between high momentum modes because the modes themselves are integrated out. This is the origin of terms of the form $b^2F(x_1,x_2,t)$ in the exponent(which will be seen in the later calculations). This is like the correlator between two exponentials, but with only some modes - high momentum - participating.

This can be written as

$$\int \frac{d^2x_1}{a(t)^2}\frac{F}{4\pi}\left(\frac{a(0)^2}{a(t)^2}\right)^{\frac{b^2}{4}-1} cos(bX(x_1))$$

which shows that it is exactly marginal for $b^2=4$. If $\frac{b^2}{4}-1<<1$ we can expand

$$\left(\frac{a(0)^2}{a(t)^2}\right)^{\frac{b^2}{4}-1} \approx 1-2t\left(\frac{b^2}{4}-1\right)$$

Thus $F(t)=F(0)(1-2t(\frac{b^2}{4}-1))+...$ valid for small t. This also gives the leading term in the β-function:

$$\beta_F=\dot{F}(t)=-2(\frac{b^2}{4}-1)F_0=-2(\frac{b^2}{4}-1)F(t)=-2\delta F \tag{3.2.56}$$

where we have approximated $F(0)$ by $F(t)$ to this order in t and F. Here $\delta=b^2/4-1$ is the deviation of the mass dimension of the cosine from marginality as the theory begins to flow. Thus for $(\frac{b^2}{4}-1)>0$ it goes to zero in the infrared and for $(\frac{b^2}{4}-1)<0$ it is a relevant variable that goes to infinity in the IR. This is the lowest order K-T flow.

The third order contribution, $(O(F^3))$, to the β function is calculated in Appendix (A).

3.2.2 $\beta_\delta - \beta$ function for b

Since b multiplies X the latter flow is equivalent to field strength renormalization. So we would like to get terms on the RHS of the ERG involving $cos(bX)$ or $\partial X \partial X$. One has to bring down the term

$$\frac{1}{2!}(\int L)^2_c$$

Thus we need to evaluate the action of the ERG operator on

$$(cos(bX(x_1))cos(bX(x_2)))_c = \frac{1}{4}(e^{ibX(x_1)} + e^{-ibX(x_1)})(e^{ibX(x_2)} + e^{-ibX(x_2)})_c \tag{3.2.57}$$

It is clear that the product can only give terms whose leading term is 1 or e^{2ibX}. The anomalous dimension of e^{2ibX} is $4b^2/2 = 2b^2$. For $\cos bX$ to be marginal, b^2 has to be set to 4. This gives $2b^2 \approx 8$. For the operator $\cos 2bX$, the deviation from marginality is given by $2b^2 - 2 \approx 6$.(The marginality condition for $\cos bX$ is $b^2/2 - 2 \approx 0$). Thus it is a highly irrelevant operator. The term starting with 1 can have terms involving the marginal $\int d^2x\, \partial X \partial X$. This corrects the kinetic term which gives the flow for the b parameter in terms of δ.

The action of ERG evolution operator on the marginal combination gives

$$\frac{1}{2!}\frac{F^2}{4(4\pi)^2}\int \frac{d^2x_1}{a(0)^2}\int \frac{d^2x_2}{a(0)^2}\, e^{b^2F(x_1,x_2)+\frac{b^2}{2}(F(x_1,x_1)+F(x_2,x_2))}(e^{ibX(x_1)-ibX(x_2)} + e^{-ibX(x_1)+ibX(x_2)})$$

where only the contributing terms have been retained.

Replacing $x_2 - x_1 = y$ we get

$$-\frac{b^2}{16}\frac{F^2}{(4\pi)^2}(\frac{a(0)^2}{a(t)^2})^{\frac{b^2}{2}-2}\int\frac{d^2x_1}{a(t)^2}\int\frac{d^2y}{a(t)^2}(\frac{y^2+a(t)^2}{y^2+a(0)^2})^{\frac{b^2}{2}}y^2(\partial X)^2$$

We are interested in the logarithmically divergent part in order to match with the continuum calculation. (In the above equation one can also replace $a(0)$ by $a(t_0)$ and pick terms proportional to $ln\ (\frac{a(t)}{a(t_0)})$.) We also take the limit $a(0) \to 0$ so that all powers of $a(0)$ can be set to zero. But in the limit $a(0) \to 0$ there is translation invariance in time ($t = \ln(\frac{a(t)}{a(0)})$)in the evolution equation and as explained in the introduction the beta function cares only about the linear term in t. Furthermore if we assume that $a(t) \approx \frac{1}{m}$, which is the IR cutoff, we can replace $y^2 + a(t)^2$ by $a(t)^2$. Thus we get for the y integral:

$$\pi(a(t)^2)^{\frac{b^2}{2}-2}\int d(y^2)(\frac{1}{y^2+a(0)^2})^{\frac{b^2}{2}}y^2$$

Putting back the prefactors:

$$= -(\frac{\delta+1}{16})\frac{F^2}{4\pi}[\frac{(\frac{a(t)^2}{a(0)^2})^{-2\delta}-1}{-2\delta} - \frac{(\frac{a(t)^2}{a(0)^2})^{-2\delta-1}-1}{-2\delta-1}]\int d^2x_1\ \partial_a X\partial^a X(x_1) \tag{3.2.58}$$

Let us take the limit $\delta \to 0$ and keep leading terms:

$$= -(\frac{\delta+1}{16})\frac{F^2}{4\pi}[2t+O(t^2\delta)+((\frac{a(0)^2}{a(t)^2})(1-4t\delta+...)-1)(1-2\delta+...)]\int d^2x_1\ \partial_a X\partial^a X(x_1)$$

If we now take $a(0) \to 0$ we get only the first term. The beta function only cares about the leading logarithm, which is the linear term in t. This is a correction to the kinetic term $\frac{1}{4\pi}\int d^2x_1\ \partial_a X \partial^a X(x_1)$. Therefore the beta function is

$$\beta_\delta = -(\frac{\delta+1}{8})F^2$$

3.3 The Beta functions.

Collecting all the beta functions we get

$$\beta_F = -2F\delta - \frac{F^3}{8} \tag{3.3.59}$$

$$\beta_\delta = -\frac{F^2}{8}(1+\delta) \tag{3.3.60}$$

The $O(F^3)$ piece for β_F is calculated in Appendix (A).

3.4 Comparing with Amit *et al* [8].

In their notation $\frac{\beta^2}{8\pi} = \frac{b^2}{4}$. Thus $\frac{\beta^2}{8\pi} = \delta + 1$. This is the same δ that they use. $\frac{F_A}{\beta^2} = \frac{F}{4\pi}$ where F_A is the variable used in [8]. Thus we have

$$F = \frac{F_A}{2(\delta+1)}$$

If we write $F = \frac{F_A}{2}$, (which is not quite the same as $\frac{F_A}{2(1+\delta)}$) we get the beta functions in their notation

$$\beta_{F_A} = -2F_A\delta - \frac{F_A^3}{32} \tag{3.4.61}$$

$$\beta_{\delta_A} = -\frac{F_A^2}{32}(1+\delta) \tag{3.4.62}$$

to first order in δ. We can compare this with the beta functions obtained by Amit *et al.*

$$\beta_F = 2F_A\delta + \frac{5F_A^3}{64} \tag{3.4.63}$$

$$\beta_\delta = \frac{F_A^2}{32}(1-2\delta) \tag{3.4.64}$$

(Their beta functions are given by the flow to the UV and have the opposite sign.)

The zero-eth order terms agree with [8]. The first order terms are not universal. It is shown in their paper that $B+2A$ is a universal quantity where A and B are the non-leading coefficients. $B+2A=\frac{5}{32}-\frac{2}{32}=\frac{3}{32}$. It can be checked that we get the same $(\frac{2+1}{32})$.

4 The ERG beta function calculation of the generalized Sine-Gordon model.

4.1 The generalized Sine-Gordon model

The action for the generalized theory is

$$\begin{aligned} S_{boundary} =& \frac{1}{4\pi}\int d^2x[(\partial_\mu \vec{X}).(\partial^\mu \vec{X}) + m^2\vec{X}.\vec{X} + \frac{F}{a(0)^2}\cos\ (\vec{b}_1.\vec{X}) \\ &+ \frac{G}{a(0)^2}\cos\ (\vec{b}_2.\vec{X}) + \frac{H}{a(0)^2}\cos\ (\vec{b}_3.\vec{X})] \end{aligned} \tag{4.1.65}$$

in euclidean d=2. Powers of $a(0)$, the UV cutoff, have been added so that the engineering dimension of the action is zero. The mass term acts like an IR regulator in the propagator. In our calculations we cut off all integrals in the IR by a moving scale, therefore we encounter no IR divergences. At marginality all $b_i^2 = 4$. This can be viewed as a world sheet action for a string in the presence of a background tachyon field with some definite momentum [10]. The marginality condition is the "on-shell" condition for the tachyon. In that case the metric, g_{MN}, for the dot product of $b^M b^N g_{MN}$ has Minkowski signature. By doing this we have an additional freedom to tune the norm of the vector to the required value by modifying the individual components of the vector. This is important for the massless and higher string modes though it is not required for the tachyon. From here on all b_i's and X's are understood to be vectors - b_i^M, X^M, in the N-dimensional target space where the string is propagating. We will drop all arrows on the top and suppress the vector index.

We want to calculate beta functions for the flow of F and b_1. Due to wave function renormalisation the parameter b_1, and thus the dimension of the cosine, get renormalised. The beta function for this parameter is thus directly related to the anomalous dimension of the X field. This is the same calculation as was done in the previous section and the beta functions for b_1 is the same as before with $\delta = \frac{b_1^2}{4} - 1$ in this case. F gets corrections from the self interaction of the cosine and corrections from higher order terms. We have calculated the leading order term in an earlier section. From the string point of view, we are computing scattering amplitudes for the zeroeth mode of the closed strings with momenta b_i at position $X(x)$, where $\exp ib_i.X(x)$ is the vertex operator for the tachyon. $\exp ib_i X(x)$ is a tachyon vertex operator for a distinct closed string, each with momentum b_i, where now instead of b being continuous, as in the introduction, we choose a discrete set of b_i's. We will choose $b_1 + b_2 + b_3 = 0$ for reasons that will become clear. It has been shown in [8] that the Sine-Gordon theory is renormalizable with a well defined expansion in F and a parameter $\delta = \frac{b^2}{4} - 1$.

We will also reproduce the beta functions of the Sine-Gordon model from the bulk. In the boundary theory we will look at the action of the generalized Sine-Gordon theory and use that to compute the beta function. In the $\cos b.X$ term, the dot product is over an N dimensional vector space and as such the b's and the X's all are vectors under some Lie group, such as SO(N). With N scalar fields, the central charge, c, of the free CFT is N. In AdS_3 a large c-expansion plays the role of large N in the more familiar AdS_5 case. We could do an expansion of the boundary theory in this N. Thus we can invoke the AdS-CFT correspondence to, as explained in the introduction, compute the appropriate multi-point boundary correlators from the bulk, to get the beta functions. These correlators diverge when some or all of these points coincide. Thus, to extract the beta function we compute the leading logarithmic deviation from the $\frac{1}{R^4}$ scaling of these correlators. In the position space holographic calculation we employ this technique.

When $b_1 + b_2 + b_3 = 0$, the first non vanishing higher point correlator is the cubic one involving all three cosines. From the point of view of the string theory tachyon, this constraint on the b_i's is just momentum conservation. From the CFT viewpoint, this comes from integrating over the zero mode of $X(x)$. In this case the beta function for F starts at quadratic order

$$\beta_F \approx O(GH)$$

This is the *cubic* term in the tachyon equation of motion [10]. At higher orders the four point correlator is always non zero and there is a contribution of $O(F^3)$. (This is the first sub-leading term in the usual Sine-Gordon model that we computed earlier.)

4.2 The propagator, other preliminaries.

We start with the kinetic term

$$S_{Kinetic} = \frac{1}{2\alpha'} \int d^2x \; \partial_\mu X \partial^\mu X \tag{4.2.66}$$

α' is like the string tension. The propagator is

$$G^{MN}(x_1,x_2) =< X^M(x_1)X^N(x_2) >= -g^{MN}\frac{\alpha'}{2\pi}\ln\frac{|\vec{x}_1-\vec{x}_2|}{L} \qquad (4.2.67)$$

Set $\alpha' = 2\pi$. L is an arbitrary scale to make the argument of log dimensionless.

Therefore,

$$\langle : \cos b.X(x_1) :: \cos b.X(x_2) :\rangle = \frac{1}{2}\left(\frac{|\vec{x}_1-\vec{x}_2|}{L}\right)^{-b^2} \qquad (4.2.68)$$

The mass dimension of a marginal operator in d=2 is 2. Therefore, $b^2/2 = 2$. The beta functions are a power series expansion in the two couplings F and $\vec{b}$. F is a small number, $F \to 0$. $b^2 = 4$ is a large number, therefore we will look for a suitable expansion parameter which is small. Both parameters get corrections. When F is non zero, the theory is interacting and wave function renormalization causes δ to run.

4.3 The ERG calculation.

The ERG can be described by

$$\psi(X,t) = e^{-\frac{1}{2}\int d^2x_1 d^2x_2 F_{x_1x_2t}\frac{\delta}{\delta X(x_1)}\frac{\delta}{\delta X(x_2)}}\psi(X,0) \qquad (4.3.69)$$

Here

$F_{x_1x_2t} \equiv F(x_1,x_2,t) == -\frac{1}{2}ln\frac{(x_1-x_2)^2+a(t)^2}{(x_1-x_2)^2+a(0)^2}$ is the ERG high energy "propagator". t is the scale upto which you are doing the RG transformations,

$a(0)$ is the UV cutoff, $a(t)$ is the IR cutoff. $a(t) = a(0)e^t$. Which implies $t = ln(a(t)/a(0)$, the log of the ratio of the scales whose coefficient is the beta function.

$$\psi(X,0) = e^{-\int d^2x \frac{1}{4\pi}[\frac{F}{a(0)^2}\cos\ (\vec{b}_1.\vec{X}) + \frac{G}{a(0)^2}\cos\ (\vec{b}_2.\vec{X}) + \frac{H}{a(0)^2}\cos\ (\vec{b}_3.\vec{X})]} \tag{4.3.70}$$

is the un-integrated "partition function" of the theory and the evolution operator $e^{-\frac{1}{2}\int d^2x_1 d^2x_2 F_{x_1x_2t}\frac{\delta}{\delta X_1}\frac{\delta}{\delta X_2}}$ acting on $\psi(X,0)$ upto some scale t, gives $\psi(X,t)$, thus implementing the RG.

One can bring down appropriate powers of cosine from the exponential and act on it with the ERG operator. The calculation can then be organised as the ERG operator acting on a power series

$$\begin{aligned} & e^{-\frac{1}{2}\int d^2x_1 d^2x_2 F_{x_1x_2t}\frac{\delta}{\delta X(x_1)}\frac{\delta}{\delta X(x_2)}}[\int \frac{d^2x_1}{a(0)^2}(a1)\cos b_i.X(x_1) \qquad (4.3.71) \\ & + \int \frac{d^2x_1}{a(0)^2}\frac{d^2x_2}{a(0)^2}(a2)\cos b_i.X(x_1)\cos b_j.X(x_2) \\ & + \int \frac{d^2x_1}{a(0)^2}\frac{d^2x_2}{a(0)^2}\frac{d^2x_3}{a(0)^2}(a3)\cos b_i.X(x_1)\cos b_j.X(x_2)\cos b_k.X(x_3)] \end{aligned}$$

the ai's, most generally, being the different corresponding coefficients. We look for corrections to cos bX which is a term of the form

$$(c1+\ c2+\ c3)t\int \frac{d^2x_1}{a(t)^2}\cos b_i.X(x_1) \tag{4.3.72}$$

where c1, c2 and c3 are the coefficients obtained after the ERG operator acts on the power series term by term. The final expression can be reorganized such that $(c1 + c2 + c3)\ t$ is the correction to the coupling(the above equation) and its derivative w.r.t to t is the beta function. Further details can be found in [22] and [48].

For our case, to calculate the leading contribution we bring down one power of $F cos b_1.X$ and apply the ERG operator to it.

4.4 Leading term in β_F.

The ERG operator acting on the interaction term gives

$$\int \frac{d^2x_1}{a(0)^2}\frac{F}{4\pi}\exp(-\frac{1}{2}\int d^2x_i d^2x_j F_{x_i x_j t}\frac{\delta^2}{\delta X(x_i)\delta X(x_j)})\cos(b_1.X(x_1))$$

Simplifying we get,

$$\int \frac{d^2x_1}{a(t)^2}\frac{F}{4\pi}(1-2\delta t)\cos(b_1.X(x_1)) \tag{4.4.73}$$

The leading term in the β-function for F is,

$$\beta_F = -2\delta F \tag{4.4.74}$$

$\delta = \frac{b_1^2}{4} - 1$ is the other small expansion parameter in terms of which we will calculate beta functions. We have done this calculation in an earlier section.

4.5 The sub-leading term.

To calculate the sub-leading contribution we will bring down one power of $\frac{G}{a(0)^2}\cos\ b_2.X(x)$ and $\frac{H}{a(0)^2}\cos\ b_3.X(x)$ each.

$$\frac{GH}{(4\pi)^2}\int\frac{d^2x_1d^2x_2}{a(0)^4}\exp\left(-\frac{1}{2}\int d^2x_id^2x_jF_{x_ix_j t}\frac{\delta^2}{\delta X(x_i)\delta X(x_j)}\right)\cos b_2.X(x_1)\quad\cos b_3.X(x_2) \tag{4.5.75}$$

$$= \frac{GH}{4}t\frac{1}{(4\pi)}\int\frac{d^2x}{a(t)^2}\cos b_1.X(x) \tag{4.5.76}$$

Refer to Appendix (B) for further details.

4.6 The beta function.

We can organize the calculation as follows,

$$\left(F(1-2\delta t)+\frac{GH}{4}t\right)\frac{1}{4\pi}\int\frac{d^2x_1}{a(t)^2}\cos(b_1.X(x_1)) \tag{4.6.77}$$

Therefore, the full beta function, the t-derivative of the coefficient of the above expression, is,

$$\beta_F = -(2F\delta - \frac{GH}{4}) \tag{4.6.78}$$

5 The Central Charge

We can use the ERG to compute the central charge of a theory using the method described in [49, 53]. For completeness we review the basic ideas. Later the same ideas will be used for the sine-Gordon theory.

5.1 Discussion of Central Charge Calculation

Let $\hat{g}_{\alpha\beta} = e^{2\sigma}\delta_{\alpha\beta}$. As is well known [4]

$$\int \mathcal{D}_{\hat{g}} X e^{-\frac{1}{2}\int d^2x\ (\partial X)^2} = e^{\frac{1}{24\pi}\int d^2x (\partial\sigma)^2} \tag{5.1.79}$$

Proof:

We start with the action

$$S[g, X] = \frac{1}{4\pi\alpha'}\int d^2x \sqrt{g}\left(g^{\alpha\beta}\partial_\alpha X_\mu \partial_\beta X^\mu\right) \tag{5.1.80}$$

We will analyze how the partition function changes under Weyl rescalings. consider two metric related by the transformation

$$\hat{g}_{\alpha\beta} = e^{2\sigma} g_{\alpha\beta} \tag{5.1.81}$$

On varying σ the partition function $Z[\hat{g}]$ changes as

$$\frac{1}{Z[X,\hat{g}]}\frac{\partial Z[X,\hat{g}]}{\partial\sigma} = \frac{1}{Z[X,\hat{g}]}\int D_{\hat{g}} X e^{-S[X,\hat{g}]}\left(-\frac{\partial S[X,\hat{g}]}{\partial \hat{g}_{\alpha\beta}}\frac{\partial \hat{g}_{\alpha\beta}}{\partial\sigma}\right) \tag{5.1.82}$$

[4] See for instance [56, 57].

$$= \frac{1}{Z[X,\hat{g}]} \int D_{\hat{g}} X e^{-S[X,\hat{g}]} \left(-\frac{1}{2\pi} \sqrt{\hat{g}} T^{\alpha}_{\alpha} \right) \tag{5.1.83}$$

Since

$$T^{\alpha}_{\alpha} = -\frac{c}{12} R \tag{5.1.84}$$

$$\frac{1}{Z} \frac{\partial Z}{\partial \sigma} = \frac{c}{24\pi} \sqrt{\hat{g}} \hat{R} \tag{5.1.85}$$

For two metrics related by a Weyl transformation $\hat{g}_{\alpha\beta} = e^{2\sigma} g_{\alpha\beta}$, their Ricci scalars are related by

$$\sqrt{\hat{g}} \hat{R} = \sqrt{g} \left(R - 2\nabla^2 \sigma \right) \tag{5.1.86}$$

Therefore,

$$\frac{1}{Z} \frac{\partial Z}{\partial \sigma} = \frac{c}{24\pi} \sqrt{g} \left(R - 2\nabla^2 \sigma \right) \tag{5.1.87}$$

This is a differential equation that expresses the partition function, $Z[\hat{g}]$, defined on one worldsheet, in terms of $Z[g]$, defined on another. Solving this we get

$$Z[\hat{g}] = Z[g] \exp \left[-\frac{1}{4\pi\alpha'} \int d^2x \sqrt{g} \left(-\frac{c\alpha'}{6} (g_{\alpha\beta} \partial^{\alpha} \sigma \partial^{\beta} \sigma + R\sigma) \right) \right] \tag{5.1.88}$$

If we set $g_{\alpha\beta} = \delta_{\alpha\beta}$ and $\alpha' = \frac{1}{2\pi}$, to match with the kinetic term on the LHS of the statement(where $\sqrt{\hat{g}}$ has been suppressed throughout), then the above equation becomes

$$Z[\hat{g}] = \exp\left[-\frac{c}{24\pi}\int d^2x\,(\partial\sigma)^2\right] \tag{5.1.89}$$

where $\sqrt{g} = 1$, $c = 1$ for a single scalar and $R = 0$.

QED.

Even though g drops out of kinetic term, the information about $\hat{g}$ comes from defining the operator:

$$\Delta = \frac{1}{\sqrt{g}}\partial_\alpha\sqrt{g}g^{\alpha\beta}\partial_\beta = e^{-2\sigma}\Box$$

And what we are calculating is $Det^{-\frac{1}{2}}\Delta$. It is thus there in the measure.

It is implicit in the above that the UV cutoff is taken to infinity. Thus we can write

$$\int_{\Lambda\to\infty}\mathcal{D}_{\hat{g}}Xe^{-\frac{1}{2}\int d^2x\ (\partial X)^2} = e^{\frac{1}{24\pi}\int d^2x(\partial\sigma)^2} \tag{5.1.90}$$

On the other hand because of scale invariance, we do not have to take $\Lambda\to\infty$. We can also write

$$\int_{\Lambda\to 0}\mathcal{D}_{\hat{g}}Xe^{-\frac{1}{2}\int d^2x\ (\partial X)^2} = e^{\frac{1}{24\pi}\int d^2x(\partial\sigma)^2} \tag{5.1.91}$$

without modifying the action, i.e. it is not the Wilson action obtained by integrating out modes from (5.1.90).

In flat space we can set $\sigma = 0$ in the above to get:

$$\int_{\Lambda\to\infty}\mathcal{D}Xe^{-\frac{1}{2}\int d^2x\ (\partial X)^2} = 1 = \int_{\Lambda\to 0}\mathcal{D}Xe^{-\frac{1}{2}\int d^2x\ (\partial X)^2} \tag{5.1.92}$$

Thus we can say that for $\Lambda \to \infty$, the following statement about integration measures is true:

$$\mathcal{D}_{\hat{g}}X \; e^{-\frac{1}{2}\int d^2x \; (\partial X)^2} = \mathcal{D}X \; e^{-\frac{1}{2}\int d^2x \; (\partial X)^2} \; e^{\frac{1}{24\pi}\int d^2x (\partial\sigma)^2} \tag{5.1.93}$$

We cannot take finite values of Λ because we may have to integrate over expressions that contain a scale.

Now consider adding a mass term : $\frac{1}{2}\int d^2x \; \sqrt{g} m^2 X^2 = \frac{1}{2}\int d^2x \; e^{2\sigma} m^2 X^2$. This term explicitly violates scale invariance. We can add a dilaton to make it Weyl invariant: $\frac{1}{2}\int d^2x \; e^{2\sigma+2\phi} m^2 X^2$. So if we set $\delta\phi = -\delta\sigma$, it is invariant. Thus the invariance is spontaneously broken rather than explicitly. Because of this if we now integrate over X we expect the anomaly to remain the same. Thus we expect

$$\int_{\Lambda\to\infty} \mathcal{D}_{\hat{g}} X e^{-\frac{1}{2}\int d^2x \; (\partial X)^2 + m^2 e^{2\sigma+2\phi} X^2} = e^{\frac{1}{24\pi}\int d^2x \hat{R}\phi - (\partial\phi)^2} = e^{\frac{1}{24\pi}\int d^2x \; 2\phi\Box\sigma - (\partial\phi)^2} \tag{5.1.94}$$

Therefore on setting the variation $\delta\phi = -\delta\sigma$ we get $-\delta\sigma \frac{1}{12\pi}\Box\sigma = \delta\left(\frac{(\partial\sigma)^2}{24\pi}\right)$. Thus we have obtained the original anomaly.

For $\Lambda << m$,

$$\int_{\Lambda << m} \mathcal{D}_{\hat{g}} X e^{-\frac{1}{2}\int d^2x \; (\partial X)^2 + m^2 e^{2\sigma+2\phi} X^2} = 1 \tag{5.1.95}$$

because all the modes are frozen - effectively there is no scalar field.

Both equations in flat space ($\sigma = 0$) give:

$$\int_{\Lambda \to \infty} \mathcal{D}X e^{-\frac{1}{2}\int d^2x \ (\partial X)^2 + m^2 e^{2\phi} X^2} = e^{-\frac{1}{24\pi}\int d^2x (\partial\phi)^2} \tag{5.1.96}$$

and

$$\int_{\Lambda << m} \mathcal{D}X e^{-\frac{1}{2}\int d^2x \ (\partial X)^2 + m^2 e^{2\phi} X^2} = 1 \tag{5.1.97}$$

Here, the coefficient of the dilaton kinetic term $\frac{(\partial\phi)^2}{24\pi}$ in $e^{-\frac{1}{24\pi}\int d^2x (\partial\phi)^2}$ in (5.1.96) is the anomaly of the defining UV theory because of the Weyl violating mass term $m^2 X^2$. Under an RG flow from $\Lambda = \infty$ to $\Lambda = 0$ we should get the anomaly, that we get for the defining theory in the UV, from the Wilsonian action in the IR. Thus we should get

$$\int_{\Lambda << m} \mathcal{D}X e^{-\frac{1}{2}\int d^2x \ (\partial X)^2 + m^2 e^{2\phi} X^2 + \Delta L[\phi]} = e^{-\frac{1}{24\pi}\int d^2x \ (\partial\phi)^2} \tag{5.1.98}$$

Here $\Delta L(\phi)$ are all the additional terms in the Wilson action that are generated under an RG flow to the IR. But since effectively there is no integration and all degrees are frozen, we must have

$$\Delta L = -\frac{1}{24\pi} (\partial\phi)^2$$

This gives the expected result $\Delta c = 1$ for a free massive scalar as you flow from the UV to the IR.

5.2 Central Charge for Free Scalar:

Let us now apply the ERG evolution operator to obtain the ϕ dependence along the RG trajectory. This gives us a definition of the c-function.

We start with a non-local action

$$S = -\int d^2u \int d^2v \; \frac{1}{2} z(u,v,t) X(u) X(v) - m_0(t) \tag{5.2.99}$$

But then we choose $z(u,v,0) = \delta m^2(u)\delta(u-v)$ as our bare action at $t=0$ and then set $\delta m^2(u) = (e^{2\phi(u)} - 1)m^2$ where ϕ is the external dilaton field.

$$S[\phi] = \int d^2x \; \frac{1}{2} m^2 X^2 (e^{2\phi} - 1)$$

We act with the evolution operator on the interaction term.

$$\int \mathcal{D}X' e^{-\frac{1}{2}\int d^2x_1 \int d^2x_2 F^{-1}(x_1,x_2) X'(x_1) X'(x_2)} e^{-S[X+X']}$$

This is the integral form of the evolution operator obtained in (2.2.37). We are interested in the coefficient of $(e^{2\phi}-1)^2$ because one has to extract the coefficient of the dilaton kinetic term which gives the c function and this is the term which will contribute to the leading order. We set $X = 0$ and evaluate

$$\int \mathcal{D}X' e^{-\frac{1}{2}\int d^2x_1 \int d^2x_2 [F^{-1}(x_1,x_2) X'(x_1) X'(x_2) + m^2 X'^2 (e^{2\phi}-1)]}$$

Path integrating we get

$$e^{-\frac{1}{2}TrLn[F^{-1}+m^2(e^{2\phi}-1)]} = e^{-\frac{1}{2}(TrLn[F^{-1}]+TrLn[1+Fm^2(e^{2\phi}-1)])}$$

Expanding the logarithm one gets for the quadratic (in ϕ) term:

$$\frac{1}{4}Tr[(Fm^2(e^{2\phi}-1))^2]$$

$$= \frac{1}{4}m^4 \int d^2x_1 \int d^2x_2 \phi(x_1)(x_1-x_2)^2 \partial^2\phi(x_1)(G(x_1,x_2,a(t))-G(x_1,x_2,a(0)))^2 \tag{5.2.100}$$

$G(x_1, x_2, a(t))$ is understood to be evaluated with a cutoff equal to $a(t) = a(0)e^t$. $t \to \infty$ corresponds to $a(t) = \infty$. All modes have been integrated out. So the propagator vanishes: $G(x_1, x_2, \infty) = 0$. This is also clear from (3.1.49). When $t = 0$ we get the propagator at the UV scale $a(0)$. So we go from the completely unintegrated theory at $a(t = 0)$ to one with everything integrated out at $a(t \to \infty)$. Integrating by parts we get,

$$-\frac{1}{4}\int d^2x_1 \int d^2x_2 [G(x_1,x_2,0)]^2 m^4 (x_1-x_2)^2 (\partial\phi)^2 \tag{5.2.101}$$

Now

$$G(x_1, x_2, 0) = \frac{1}{2\pi} K_0(m\,|x_1 - x_2|)$$

Substituting in (5.2.101) we get

$$-\frac{1}{24\pi}\int d^2x\ (\partial\phi)^2 \tag{5.2.102}$$

What we have calculated is $-L(u,\infty)+L(u,0)=-\Delta L=1$. The change in c is thus 1. The final theory where the scalar field is infinitely massive has $c=0$. The initial theory therefore had $c=1$. The anomalous transformation under scale changes is provided by the $(\partial\phi)^2$ term - this is the argument used by [49, 53]. We have obtained it using the ERG.

(5.2.100) defines a c-function for any value of t along the flow. It is also clear that it is monotonic.

5.3 Central Charge of the Sine-Gordon Theory

In this section we calculate yet another flow - the c-function defined by Zamolodchikov. We calculate it using the ERG first. We also compare this with a calculation using a prescription given in [53].

5.3.1 The C-function using the ERG flow equation

Let us begin with some normalization details. The interaction vertex is,

$$\int d^2x\frac{F}{a(0)^2}cos\ bX$$

then. This term violates Weyl invariance and therefore one introduces a dilaton to restore Weyl invariance.

$$\int d^2x e^{2\phi} \frac{F}{a(0)^2} cos\ bX$$

Therefore, under $\sigma \rightarrow \sigma + \xi$ and $\phi \rightarrow \phi - \xi$ the theory is invariant. $a(0) \rightarrow a(0)e^{-\phi}$ gives the dilaton coupling. Instead of associating the dilaton with $a(0)$ we associate it to the coupling constant F or equivalently to the dimensionful operator $cos\ bX$. When we do RG evolution $a(0) \rightarrow a(0)e^t \equiv a(t)$ and as usual and there is no ϕ associated with it.

The dilaton coupling:

The normal ordered interaction term is

$$S = \int \frac{d^2x}{a(0)^2} F \left(\frac{a(0)}{a(t)}\right)^{\frac{b^2}{2}} : cos\ bX :$$

$$= \int \frac{d^2x}{a(t)^2} F \left(\frac{a(0)^2}{a(t)^2}\right)^{\delta} : cos\ bX :$$

and $\delta = \frac{b^2}{4} - 1$ as before. Now we introduce a ϕ dependence, we get

$$S = \int \frac{d^2x}{a(t)^2} F\ e^{-2\delta(t+\phi(x,t))} cos\ bX$$

To this order

$$F(t, \phi) = F(\phi)e^{-2\delta t} = F e^{-2\delta(t+\phi(x))}$$

Note that the coupling constant has become x-dependent and has to be placed inside the integral:

$$\int \frac{d^2x}{a(t)^2}\ F(t,x)\ cos\ bX(x)$$

Thus we have determined the dilaton coupling. This has the information of the contribution of the anomalous scaling behaviour of the cosine operator under an RG flow to the central charge.

Extracting the anomaly:

As discussed before, the anomaly is the coefficient of the dilaton kinetic term. One has to go to over to the $\frac{1}{2!} < S^2 >_c$ term, where the subscript c signifies taking only the connected parts, to extract the anomaly. We act on this term by the ERG operator and extract the dilaton kinetic term. The calculation proceeds as follows,

$$\frac{1}{2!}\int \mathcal{D}X'' e^{\frac{1}{2}\int d^2x_1\ \int d^2x_2\ F^{-1}(x_1,x_2,t)X''(x_1)X''(x_2)}$$

$$F^2\int_x [\frac{e^{ibX(x)+ibX''(x)}+e^{-(ibX(x)+ibX''(x))}}{2}]\int_y [\frac{e^{ibX(y)+ibX''(y)}+e^{-(ibX(y)+ibX''(y))}}{2}]$$

Here $\int_x = \int \frac{d^2x}{a(0)^2}$. We replace F by $F(t,\phi)$ as before. a(t) is the IR scale for the action. The propagator has an exponential fall off beyond the IR scale. So when $|y-x| \approx a(t)$ the propagator is highly damped. So we are justified in assuming that $a < |x-y| < a(t)$. Thus the total contribution is (letting $z = y - x$)

$$= 2\frac{F^2}{8}\int \frac{d^2x}{a(t)^2}\ \int \frac{d^2z}{a(t)^2} e^{-2\delta[2t+\phi(x,t)+\phi(y,t)]} e^{\frac{b^2}{2}ln\ \left(\frac{a(t)^2}{z^2+a(0)^2}\right)}(1+izb\partial X...)$$

Now,

$$e^{-2\delta(\phi(x)+\phi(y))} = 1 - 2\delta(\phi(x)+\phi(y)) + 2\delta^2(\phi(x)+\phi(y))^2 + ...$$

The relevant part is

$$2\delta^2 2\phi(x)\phi(y) = 2\delta^2\phi(x)(y-x)^a(y-x)^b\partial_a\partial_b\phi(x) \tag{5.3.103}$$

Inserting (5.3.103) for $e^{-2\delta(\phi(x)+\phi(y))}$ we get for the term in the Wilson action involving $\phi\Box\phi$:

$$= \int d^2x\ \frac{F^2(t)}{4}\delta^2\phi(x)\partial^2\phi(x)(a(t)^2)^{2\delta}\int d^2z\ z^2(z^2+a(0)^2)^{\frac{-b^2}{2}} \tag{5.3.104}$$

Here we have used rotational symmetry to replace $z^a z^b$ by $z^2\frac{\delta^{ab}}{2}$. The integral is log divergent and the divergent piece can be extracted by introducing the regulator $a(0)$ in the limits rather than in the integrand: (The IR end is cutoff anyway by $a(t)$.)

$$\int d^2z\ z^2(z^2+a(0)^2)^{\frac{-b^2}{2}} = \pi\frac{[(a(0)^2)^{-2\delta} - (a(t)^2)^{-2\delta}]}{-2\delta}$$

Inserting in (5.3.104) and expanding for small δ we get

$$-\pi\int d^2x\ \frac{F^2}{2}\delta^2\phi(x)\partial^2\phi(x)t \tag{5.3.105}$$

The answer depends on the logarithmic range t. The calculation can be improved if we realise that F is a function of t. We assume that the range of RG evolution t is infinitesimal - dt. Then we can replace $t \to \int_0^t dt$ and acknowledge the functional dependence of F on t explicitly, $F(t)$. Then we can write (5.3.105) as

$$-\pi \int d^2x \; \delta^2 \phi(x) \partial^2 \phi(x) \int dt \frac{F^2(t)}{2}$$

Noting that $\frac{dF}{dt} = -2\delta F$ we can write $dt = -\frac{dF}{2\delta F}$ to get

$$= -3\pi^2 F^2 \delta \times \frac{1}{24\pi} \int d^2x \; \partial_a \phi(x) \partial^a \phi(x)$$

The coefficient of $-\frac{1}{24\pi} \int d^2x \; \partial_a \phi(x) \partial^a \phi(x)$ gives the change in the central charge. Thus

$$\Delta c = c(F(0)) - c(F(t)) = 3\pi^2 F^2 \delta$$

Here $c(F(0))$ if the central charge of the UV theory. $c(F(t))$ is the central charge of the IR theory. When $\delta > 0$ we have an irrelevant operator - F flows to zero under an RG evolution. So $c(F(0)) > c(F(t))$ - which is correct.

5.3.2 A confirmation with a result from Entanglement Entropy

At the conformal point the entanglement entropy for a single interval is related to the central charge of the CFT[54] by

$$EE = \frac{c}{3} \ln(l/\epsilon) \tag{5.3.106}$$

where l is the length of the interval and ϵ is the short distance cutoff. If you identify a(t) with l and a(0) with ϵ and then analyze the behaviour of this expression as an RG flow, then to leading order, one would expect the change in entanglement entropy when one goes slightly away from the fixed point to be

$$\Delta EE = \frac{\Delta c}{3} \ln \left(\frac{a(t)}{a(0)} \right) + H.O.T. \tag{5.3.107}$$

Substituting the Δc above we get,

$$\Delta EE = \pi^2 F^2 \delta \ln \left(\frac{a(t)}{a(0)} \right) \tag{5.3.108}$$

If we set $F = \frac{\lambda}{8\pi}$ and $2\delta = \Delta - 2$ we get

$$\Delta EE = \frac{\lambda^2}{128}(\Delta - 2) \ \ln \left(\frac{a(t)}{a(0)} \right)$$

ΔEE has recently been calculated holographically in [55]. We show that this expression is in agreement with their results.

5.3.3 The c-function from Komargodski's prescription

Now we will calculate the c-function of the Sine-Gordon using a technique by Komargodski and show that the results match with our earlier calculation. This is a check on both techniques. The interaction term for the Sine-Gordon action is

$$S = \int \frac{d^2x}{a(0)^2} F cos(bX(x)) \tag{5.3.109}$$

$$S = \int \frac{d^2x}{a(0)^2} F : cos(bX(x)) : (ma(0))^{(b^2/2)} \tag{5.3.110}$$

Here the scale dependence to the lowest order from normal ordering has been explicitly factored out. m will be identified with $\frac{1}{a(t)}$, where $a(t)$ is the UV cutoff after several RG transformations have been performed and is thus the IR scale.

$$S = \int m^2 d^2xF : cos(bX(x)) : (ma(0))^{(b^2/2-2)} \tag{5.3.111}$$

If under scaling $a(0) \to \lambda a(0)$, then under scaling, a dilaton $\exp(\phi)$, would transform as $\exp(\phi) \to \frac{\exp(\phi)}{\lambda}$, thus leaving the action invariant under scaling. We introduce the dilaton in the action and take its effect under scalings into account,

$$S = \int m^2 d^2xF : cos(bX(x)) : (ma(0)\exp(\phi))^{(b^2/2-2)} \tag{5.3.112}$$

The Green's function for a massive scalar is

$$G(x_1, x_2) = -\ln\left(m^2((x^2 + a(0)^2)\right) \tag{5.3.113}$$

where $x = x_1 - x_2$. The trace of the Energy-Momentum tensor, $T = T^a_a$

is

$$T = \frac{\delta S}{\delta \phi} = \int m^2 d^2 x F : cos(bX(x)) : (ma(0)\exp(\phi))^{(b^2/2-2)}(b^2/2-2) \tag{5.3.114}$$

$$< T(y)T(0) >= F^2\big(b^2/2-2\big)^2 (ma(0))^{2(b^2/2)} < \cos bX(y) \cos bX(0) > \tag{5.3.115}$$

Komargodski's prescription [53] for the change in the central charge under an RG flow gives,

$$\begin{aligned}\Delta c &= -3\pi \int d^2 y y^2 < T(y)T(0) > \\ &= \frac{3}{2}\pi^2 F^2 (ma(0))^{b^2-4}(b^2/2-2)\end{aligned} \tag{5.3.116}$$

Substituting $b^2/2 - 2 = 2\delta$ and identifying $m^{-1} \to a(t)$,

$$\Delta c = 3/2\pi^2 F^2 (a(0)/a(t))^{4\delta} 2\delta \tag{5.3.117}$$

Then,

$$F^2(a(t)/a(0))^{-4\delta} 2\delta = F^2 2\delta \Big(1 + (-4\delta)\ln(a(t)/a(0))\Big) \tag{5.3.118}$$

Now we can write $a(t) = a(0)e^t$ where $a(0)$ is the UV cut-off. So we can write $\ln(a(t)/a(0))$ as $\int_0^t dt$ for t infinitesimal and then promote $F \to F(t)$.

We get

$$F^2 2\delta(-4\delta)\ln(a(t)/a(0)) \to -8\delta^2 \int_0^t F(t)^2 dt \tag{5.3.119}$$

Therefore we get

$$\Delta c = -12\pi^2\delta^2 \int F^2(t)dt \tag{5.3.120}$$

The Beta function is given by,

$$\frac{dF}{dt} = \beta_F = -2F\delta \tag{5.3.121}$$

to leading order. Substituting this in (5.3.119) we get

$$\Delta c = 3\pi^2 F^2 \delta \tag{5.3.122}$$

as before.

5.4 Higher order terms for Δc

Under a change in renormalization δ goes to(equation 7.6[8])

$$\delta = \delta_0 + aF^2 \tag{5.4.123}$$

Therefore

$$\beta_\delta = \frac{d\delta}{dt} = 2Fa\frac{dF}{dt} = 2Fa\beta_F \tag{5.4.124}$$

We know

$$\beta_\delta = \frac{d\delta}{dt} = -\frac{F^2}{32} + \frac{F^2\delta}{16} \tag{5.4.125}$$

and

$$\beta_F = -2F\delta - \frac{5}{64}F^3 \tag{5.4.126}$$

where β_F and β_δ are the β functions as obtained in [8]. Substitute (5.4.126), (5.4.125) and (5.4.123) in (5.4.124) we get

$$\frac{F^2}{32} - \frac{F^2\delta_0}{16} = 2Fa(2F\delta_0 + \frac{5}{64}F^3) \tag{5.4.127}$$

Comparing coefficients

$$a = \frac{-1}{64} \tag{5.4.128}$$

So

$$\delta = \delta_0 - \frac{1}{64}F^2 \tag{5.4.129}$$

where the F dependence of δ has been determined to leading order. So,

$$\int \delta^2 F^2 dt = -\int \frac{\delta^2 F dF}{2\delta + \frac{5}{64}F^2} \tag{5.4.130}$$

where (5.4.126) has been substituted in the above. Substituting (5.4.129) in the above expression, simplifying and resubstituting this expression in

(5.3.120) we get

$$\Delta c = 12\pi^2\left(\frac{\delta_0 F^2}{4} - \frac{7F^4}{1024} + H.O.T.\right) \tag{5.4.131}$$

To lowest order (5.4.131) matches (5.3.122).

6 AdS/CFT and holographic RG - a brief review.

6.1 The AdS/CFT correspondence.

The AdS/CFT correspondence relates field theories on AdS spacetimes to CFT's. The AdS/CFT is an important realisation of the holographic principle. This principle states that in a gravitational theory, the number of degrees of freedom in a given volume V scales as the surface area of that volume [96]. The theory of quantum gravity involved in the AdS/CFT correspondence is defined on a manifold of the form AdS x M, where M is a compact manifold. The QFT may be thought of as being defined on the conformal boundary of this AdS space. The most prominent example relates $\mathcal{N} = 4$ Super Yang–Mills theory in 3+1 dimensions and type IIB superstring theory on AdS_5xS_5.

The strongest form of the correspondence AdS_5/CFT_4 [23], states that the $\mathcal{N} = 4$ SYM with the gauge group SU(N) and the Yang-Mills coupling g_{YM} is dynamically equivalent to the type IIB superstring theory with string length $l_s = \sqrt{\alpha'}$ and coupling constant g_s on AdS_5xS^5 with radius of curvature L. The free parameters on the field theory side, g_{YM} and N, are related to the free parameters, g_s and $L/\sqrt{\alpha'}$, on the string theory side by $g^2_{YM} = 2\pi g_s$ and $2g^2_{YM}N = L^4/\alpha'^2$. The information of the five dimensional theory obtained from Kaluza-Klein reduction of the type IIB string theory on the S^5 is mapped to a four-dimensional theory which lives on the conformal boundary of the five-dimensional spacetime. The correspondence therefore states that the two different theories describe the same physics from two different perspectives.

Although the two theories are equivalent for arbitrary values of the parameters, it is very difficult to do these calculation in the strong coupling regime. We, therefore, restrict ourselves to specific coupling regimes to make the calculations more tractable. One can use this duality to obtain new insights into the strong coupling dynamics of one theory from the computable weak coupling behaviour of the other. String theory is currently best understood in the perturbative regime, therefore we could take the weak coupling limit, $g_s << 1$ and $L/\sqrt{\alpha'}$ constant, in which case at the leading order in g_s, the AdS side reduces to a classical string theory, in the sense that we take only tree level diagrams. All higher order terms in the genus expansion are dropped.

For the CFT side, since $g_s << 1$, therefore $g_{YM} << 1$, while $g_{YM}^2 N$ could be kept finite. Thus we have to take the 't Hooft limit where $N \to \infty$ for a fixed λ. Thus a 1/N expansion on the field theory side can be mapped to a genus expansion of the worldsheet since $1/N \propto g_s$ for fixed λ. Thus the leading term from the string theory side are tree level string scattering amplitudes. All terms higher order in g_s are higher genus(loop level) terms in the string amplitudes.

If in addition to $N \to \infty$ we want to take $\lambda \to \infty$, that corresponds to $L/\sqrt{\alpha'} \to 0$. This is the point particle limit of the type IIB string theory which is the type IIB SUGRA on AdS_5xS^5 [93, 94, 95]. We do our calculations in this limit.

Precise forms of the conjecture have been stated and prescriptions for computations have been given in [23, 27, 33, 39].

6.2 Bulk computations.

We look at the Euclidean continuation of AdS_{d+1} which is the $Y_{-1} > 0$ sheet of the hyperboloid

$$-Y_{-1}^2 + Y_0^2 + \Sigma_{i=1}^d Y_i^2 = -\frac{1}{a^2} \tag{6.2.132}$$

It has curvature $R = -d(d+1)a^2$. The change of coordinates

$$z_i = \frac{Y_i}{a(Y_0 + Y_{-1})} \tag{6.2.133}$$

$$z_0 = \frac{1}{a^2(Y_0 + Y_{-1})} \tag{6.2.134}$$

brings the induced metric to the form of the Lobachevsky upper half plane

$$ds^2 = \frac{1}{a^2 z_0^2}\left(\Sigma_{\mu=0}^d dz_\mu^2\right) = \frac{1}{a^2 z_0^2}\left(dz_0^2 + \Sigma_{i=0}^d dz_\mu^2\right) = \frac{1}{a^2 z_0^2}\left(dz_0^2 + d\vec{z}^2\right) \tag{6.2.135}$$

This is the AdS metric in the Poincare patch

$$ds^2 = \frac{1}{z^2}[dz^2 + d\vec{x}^2] \tag{6.2.136}$$

$\vec{x}$ are boundary euclidean coordinates and z is the radial coordinate. a has been set to 1.

According to the AdS-CFT correspondence

$$\int D\Phi \exp(-S[\Phi]) = \left\langle \exp\left(-\int_{\partial AdS} \phi_0 O\right)\right\rangle \tag{6.2.137}$$

which to leading order is

$$S_{bulk}[\phi_0] = -W_{QFT}[\phi_0] \tag{6.2.138}$$

where $S_{bulk}[\phi_0]$ is the bulk action and $W_{QFT}[\phi_0]$ is the connected generating functional of the boundary theory.

The correlation funtions from the bulk are calculated by taking variations w.r.t ϕ_0 on both sides. A general n point correlation function is given by

$$\langle O_1(x_1)...O_n(x_n)\rangle = (-1)^{n+1}\frac{\delta^n S_{bulk}}{\delta\phi_0(x_1)...\delta\phi_0(x_n)}|_{\phi_0=0} \tag{6.2.139}$$

To compute beta functions from the bulk we start with the bulk action with the $\phi\chi\gamma$ term,

$$\begin{aligned} S_{bulk} = \int d^3x\sqrt{g}[&\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(m_\phi\phi)^2 + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}(m_\chi\chi)^2 \\ &+ \frac{1}{2}(\partial\gamma)^2 + \frac{1}{2}(m_\gamma\gamma)^2 - \lambda_3\phi\gamma\chi] \end{aligned} \tag{6.2.140}$$

The ϕ, γ and χ correspond to $\cos b_1X$, $\cos b_2X$, $\cos b_3X$ respectively.

The free equation of motion is

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right) - m^2\phi = 0 \tag{6.2.141}$$

with boundary conditions

$\phi(z_0, \vec{z}) = 0$ for $z \to \infty$ and $\phi(z_0, \vec{z}) \to z_0^{d-\Delta}\phi_0(\vec{z})$ as $z_0 \to 0$

The normalized bulk to boundary Green's function is

$$K_\Delta(z_0, \vec{z}, \vec{x}) = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - d/2)}\left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2}\right)^\Delta \tag{6.2.142}$$

The solution to (6.2.141) is

$$\phi(z_0, \vec{z}) = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - d/2)}\int d^2x\left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2}\right)^\Delta \phi_0(\vec{x}) \tag{6.2.143}$$

6.3 Holographic RG.

In holographic RG, one identifies the bulk radial coordinate with the RG scale via, as we will see $t = \log\frac{R}{x_0}$. Here $t = \log\frac{a(t)}{a(0)}$ as before. $R \to \infty$ and will denote the deep interior. $x_0 \to 0$ and will be interpreted as the UV cutoff [97].

There are many approaches to implement the holographic RG. One approach is to look at domain wall flows which interpolate between the stationary points of the potential of some supergravity theory in the bulk, [98, 99]. At the stationary points, the potential reduces to the cosmological constant of the AdS and the metric becomes AdS. We want to obtain an RG equation as a gradient flow equivalent to the supergravity EoM. These will give us the first order equations which can be interpreted as beta functions. We consider a toy model which has a scalar field and a graviton, both dynamical.

$$S = \int d^{d+1}x\sqrt{-g}\left(\frac{R}{16\pi G} - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right) \tag{6.3.144}$$

The potential $V(\phi)$ is chosen so that it has one or more stationary points ($V'(\phi) = 0$). The equations of motion for ϕ and g_{mn} are

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) - V'(\phi) = 0 \tag{6.3.145}$$

and

$$R_{\mu\nu} - \frac{Rg_{\mu\nu}}{2} = 8\pi G\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 - g_{\mu\nu}V(\phi)\right) \equiv 8\pi GT_{\mu\nu} \tag{6.3.146}$$

At stationary points $\phi(r) = \phi_i$ the solution to the scalar equation of motion is trivial and the Einstein's equation reduces to

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi Gg_{\mu\nu}V(\phi_i) \tag{6.3.147}$$

If we identify $\Lambda_i = 8\pi GV(\phi_i) = -\frac{d(d-1)}{L_i^2}$, then this is the Einstein's equation of AdS space. Then the constant scalar fields with AdS_{d+1} geometry of scale L_i are the solutions which correspond to CFT's at the RG fixed points. A more general ansatz for the metric, with a warp factor A(r), is

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 \quad , \quad \phi = \phi(r) \tag{6.3.148}$$

This is called the domain wall ansatz. For $A(r) = r/L$ we recover the AdS metric. For constant ϕ we get the bulk dual of the CFT at a fixed point. We want to solve the equations of motion for this metric and ϕ as before. One obtains a system of differential equations which are solved by considering an auxiliary function $W(\phi)$ which is called the superpotential. Substituting this in the system of differential equations obtained above one gets two first order flow equations whose solution can shown to be the solution to the equations of motion. This is the domain wall solution that interpolates between an AdS space of radius L_{UV} at the boundary and another AdS space of radius L_{IR} in the deep interior. The scalar ϕ flows from a constant ϕ_{UV} in the UV to a constant ϕ_{IR} in the IR. This domain wall solution is expected to be dual to the boundary RG flow.

[20] have developed parallels between holographic RG and the Wilsonian RG of the boundary theory. This has been further fleshed out by [21].

Another approach [19], is to mimic the conventional Wilsonian paradigm. Here formulating a holographic Wilsonian flow involves integrating out the bulk degrees of freedom between a UV hypersurface on which our field theory is defined and an IR hypersurface which explores the energy scales of interest. The specifics of doing this are as follows. A boundary theory defined with a cutoff scale Λ_0 is identified with the bulk theory defined in the spacetime region $z > \epsilon_0$ for some ϵ_0. Integrating out degrees of freedom in the boundary theory from Λ_0 to some lower scale Λ' is then identified with integrating out the bulk degrees of freedom between $z = \epsilon_0$ and some $z = \epsilon' > \epsilon_0$. Integrating out the bulk degrees of freedom in the region $\epsilon_0 < z < \epsilon'$ results in a boundary action $S_B(z = \epsilon')$ at the $z = \epsilon'$ hypersurface. S_B provides boundary conditions for bulk modes in the region $z > \epsilon'$ and can be considered as specifying a "boundary state" for the bulk theory in that region. This effective action can be identified with the Wilsonian effective action of the boundary theory at the scale Λ'. Requiring that physical observables be independent of the choice of the cut-off scale then determines a flow equation for the Wilsonian action and associated couplings. This gives a flow equation for the boundary action S_B. One should contrast this with similar approaches proposed by others, for example [98, 99], as discussed earlier. Here one gets flow equations for the bulk S_{cl}, as against for S_B in the approach discussed.

In an approach by Bzowski et al [100], they present a scheme for holographic RG in which the boundary renormalization scheme is dimensional regularisation.

Skenderis et al [30], use momentum space techniques to compute beta functions from the bulk. This method is more closely related to the conventional way of obtaining beta functions, namely, picking out the log divergence, introducing a renormalization scale and extracting the beta function from the derivative w.r.t the scale. They obtain a flow equation in the boundary values of the bulk fields which correspond to couplings of operators(composite and otherwise) on the boundary. This technique will be used to compute the beta functions for the tachyons in the generalized theory in a later chapter.

[12] have proposed another approach where one starts with an ERG flow equation which is similar to the Schroedinger equation and then a transformation, akin to a coordinate change, transforms the ERG flow equation in the boundary to an evolution equation in the bulk fields. This is done without invoking the AdSCFT correspondence and is an attempt to establish a direct connection between the ERG and the holographic RG. They have some

results for the free boundary theory. The purpose of the papers in this thesis is to compute beta functions for some composite operators in the boundary and from the bulk. Then, as was mentioned earlier, one can use this worked out example to extend this approach for writing direct transformations of the ERG boundary composite operator to the bulk theory.

To extract the beta function using position space techniques from the bulk, we first regulate the generating function for the correlators of the boundary theory to be calculated from the bulk by inserting x_0 which acts as the UV cutoff. Then we compute the generating function for the two point function with one particle offshell and then, as was described in the introduction, obtain the leading logarithmic deviation from the $\frac{1}{R^4}$ scaling which comes, at the leading order, from taking the particle offshell. Then we extract terms which are logarithmically divergent in terms of x_0, the UV scale. All these ideas will become clear in the calculations below.

7 Position space calculation of the beta function from the bulk.

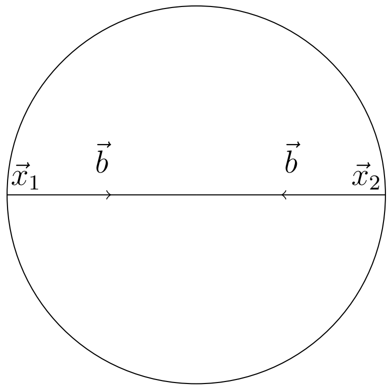

Fig. 1.
The leading order Witten diagram

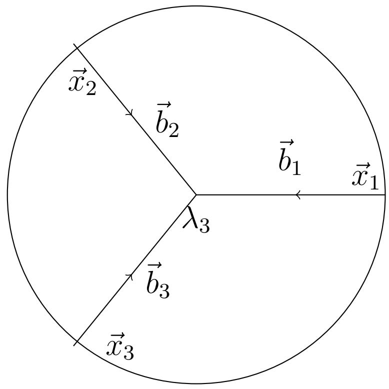

Fig. 2.
Witten diagram for
the sub-leading contribution

7.1 Leading order

The bulk action for the free massive scalar is

$$S = \int d^{d+1}y\sqrt{g}\Big[\big(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi(y)\partial_\nu\phi(y) + \frac{1}{2}m^2\phi^2(y)\big)\Big] \qquad (7.1.149)$$

Evaluating the free action on-shell we get,

$$\frac{1}{2}\phi_0^2\int d^dx_1d^dx_3d^dydy_0\Big[\partial_\mu\left(y_0^{-d+1}K_\Delta(y_0,\vec{y};\vec{x_1})\partial^\mu K_\Delta(y_0,\vec{y};\vec{x_3})\right)\Big] \quad (7.1.150)$$

Choose the outward pointing normal along the radial direction and by the Gauss's divergence theorem, do the surface integral.

$$S = -\frac{1}{2}\phi_0^2 \int d^d x_1 d^d x_3 d^d y \left[y_0^{-d+1} K_\Delta(y_0, \vec{y}; \vec{x_1}) \partial_0 K_\Delta(y_0, \vec{y}; \vec{x_3}) \right] \Big|_{y_0=\epsilon} \tag{7.1.151}$$

where $\epsilon \to 0$, therefore y_0 is close to the boundary, $y_0 \to 0$. We identify y_0 with x_0, the UV regulator. The minus sign comes from choosing the convention for the outward pointing normal $n^\mu = (-\epsilon, \mathbf{0})$. Therefore, using

$$\lim_{x_0 \to 0} x_0^{\Delta-d} K_\Delta(x_0, \vec{y}; \vec{x}_i) \to \delta^{(d)}(\vec{y} - \vec{x}_i) \tag{7.1.152}$$

$$S = -\frac{1}{2}\phi_0^2 C_\Delta \int d^d x_1 d^d x_3 d^d y \left[\frac{\Delta \delta^{(d)}(\vec{y} - \vec{x_1})}{(x_0^2 + (\vec{y} - \vec{x}_3))^\Delta} - \frac{(2\Delta) x_0^2 \delta^{(d)}(\vec{y} - \vec{x_1})}{(x_0^2 + (\vec{y} - \vec{x}_3))^{\Delta+1}} \right] \tag{7.1.153}$$

This is the action for the free massive term. A general n-point correlation function is given by

$$\langle O_1(x_1) ... O_n(x_n) \rangle = (-1)^{n+1} \frac{\delta^n S_{bulk}}{\delta\phi_0(x_1) ... \delta\phi_0(x_n)} |_{\phi_0=0} \tag{7.1.154}$$

Therefore the generating function of a two point function will have another minus. The two point functions from the bulk and the boundary now match. The generating function for the two point function is,

$$S_2 = -(-1)\frac{1}{2}\phi_0^2 C_\Delta \int d^d x_1 d^d x_3 \left[\frac{\Delta}{(x_0^2+(\vec{x}_1-\vec{x}_3))^\Delta} - \frac{(2\Delta)x_0^2}{(x_0^2+(\vec{x}_1-\vec{x}_3))^{\Delta+1}}\right] \tag{7.1.155}$$

The log divergent term comes from the first term and is retained. The second term is x_0^2 suppressed. In the limit $x_0 \to 0$, it vanishes. We drop this term. $C_{\Delta_i} = \frac{\Gamma(\Delta_i)}{\pi^{d/2}\Gamma(\Delta_i - d/2)}$, therefore

$$S_2 = \frac{\Gamma(\Delta+1)}{\pi^{d/2}\Gamma(\Delta-d/2)} \int d^d x_1 d^d x_3 \frac{1}{(x_0^2+(\vec{x_1}-\vec{x_3})^2)^\Delta}\frac{\phi_0^2}{2!} \tag{7.1.156}$$

This is the generating functional for the two point function with the UV regulator x_0. Setting $d=2$, $\Delta = 2(1+\delta)$, x_1 to zero and x_3 to R, $R \to \infty$, the expression becomes,

$$\frac{\phi_0^2}{2!}\pi\Delta(\Delta-1)\int d\left(\frac{x_1^2}{x_0^2}\right) d\left(\frac{x_3^2}{x_0^2}\right)\frac{1}{((x_0^2+R^2)/x_0^2)^{2(1+\delta)}} \tag{7.1.157}$$

Multiply and divide by R and expand the denominator for $\delta \ll 1$ such that $(R/x_0)^{-4\delta} = 1 - 4\delta \log R/x_0$, we get

$$\phi_0^2\pi \int d\left(\frac{x_1^2}{R^2}\right) d\left(\frac{x_3^2}{R^2}\right)\left(1 - 4\delta\log\frac{R}{x_0}\right) \tag{7.1.158}$$

We will extract the leading term of the beta function from this.

7.2 Order $\phi\gamma\chi$

The beta function is the change in the couplings of the theory under scaling transformations. To determine the deviation from the canonical scaling dimension we look at the behaviour of the two point function slightly away from marginality and determine the leading term of the beta function. This was the calculation we did above. To calculate the sub-leading term we start with a two point function, insert another operator, therefore now we have a three point function, and look for the log deviation from $1/R^4$ scaling for this object. We first calculate the generating function for the three point function. To do this we start with

$$S_3 = -\lambda_3 \int d^{d+1}y\sqrt{g}\ \phi(y)\ \gamma(y)\ \chi(y) \tag{7.2.159}$$

where

$$\phi(z) = \phi(z_0, \vec{z}) = \int d^d x K_\Delta(z_0, \vec{z}, \vec{x})\phi_0(\vec{x}) \tag{7.2.160}$$

and substitute this for $\phi(y)$, $\gamma(y)$, $\chi(y)$ in S_3. Here $\sqrt{g} = y_0^{-(d+1)}$, $d = 2$, $C_{\Delta_i} = \frac{\Gamma(\Delta_i)}{\pi^{d/2}\Gamma(\Delta_i - d/2)}$. ϕ_0, γ_0 and χ_0 are the couplings of the boundary theory (boundary values of the bulk fields, they have no coordinate dependence). This becomes(details in Appendix (D)),

$$S_3 = -\lambda_3\pi\phi_0\gamma_0\chi_0 \int d\left(\frac{x_1^2}{R^2}\right) d\left(\frac{x_3^2}{R^2}\right) \log\frac{R}{x_0} \tag{7.2.161}$$

7.3 The Beta function

The full generating function can be organized as

$$S = S_2 + S_3 \tag{7.3.162}$$

$$= \pi \int dx_1^2 dx_3^2 \left(\phi_0^2 - 2\phi_0 (2\phi_0 \delta + \frac{\lambda_3 \gamma_0 \chi_0}{2}) \log \frac{R}{x_0} \right) \tag{7.3.163}$$

Substituting the relations between ϕ_0, γ_0, χ_0 and F,G,H((C.1.227), (C.1.228), (C.1.229)) and the value of λ_3(C.2.233) we get,

$$= \frac{1}{64} \int dx_1^2 dx_3^2 \left(F^2 - 2F(2\delta F - \frac{GH}{4}) \log \frac{R}{x_0} \right) \tag{7.3.164}$$

We have calculated the correction to F^2. To get the beta function we want to isolate the change in F. To do this we note,

$$F^2 \to F'^2 = F^2 + \delta(F^2) = F^2 + 2F\delta(F) \tag{7.3.165}$$

$$F^2 + (-4\delta F^2 + FGH/2) \log \frac{R}{x_0} = F^2 - 2F(2\delta F - GH/4) \log \frac{R}{x_0} \tag{7.3.166}$$

and comparing the above two expressions we see

$$\delta(F) = -\left(2\delta F - \frac{GH}{4}\right)\log\frac{R}{x_0} \tag{7.3.167}$$

In the Poincare patch (6.2.136) a physical distance s between two points is $s = s_{coord}/z_0$, where s_{coord} is the coordinate distance between the points at the boundary at $z = z_0$. x_0 is the physical UV cutoff scale on the boundary where the metric is $\delta_{\mu\nu}$. On the boundary at $z = z_0$, the coordinate distance becomes $R = x_0 z_0$. If one identifies this with $a(t) = a(0)e^t$, the moving IR scale of the boundary theory, and one further identifies x_0 with a(0), then one can naturally identify the boundary position z_0 with e^t and then moving along the z direction is the same as scaling transformations of the boundary theory, thus implying that moving along the z direction is the same as doing RG transformations. Therefore $\log\frac{R}{x_0}$ gets identified with t. Then the t-derivative of (7.3.167) gives us the beta function

$$\beta_F = -\left(2\delta F - \frac{GH}{4}\right) \tag{7.3.168}$$

which matches our result from the boundary calculation (4.6.78).

8 Beta function computation using momentum space techniques from the bulk.

We will now compute the results obtained in earlier sections using momentum space techniques. This method is more closely related to the conventional way of obtaining beta functions, namely, picking out the log divergence, introducing a renormalization scale and extracting the beta function from the derivative w.r.t the scale. Although using the previous approach, one can do computations for a larger class of bulk vertices, doing the calculation by this method is much simpler in some of the cases of interest. We will start with a brief introduction to AdS-CFT computations in momentum space.

8.1 A brief summary of AdS/CFT and holographic RG from the momentum space perspective.

The bulk action with the term $\Phi\chi\gamma$ is (here we relabel the field ϕ from earlier to Φ for this section for notational clarity),

$$S_{bulk} = \int d^3x\sqrt{g}[\frac{1}{2}(\partial\Phi)^2+\frac{1}{2}(m_\Phi\Phi)^2+\frac{1}{2}(\partial\chi)^2+\frac{1}{2}(m_\chi\chi)^2+\frac{1}{2}(\partial\gamma)^2+\frac{1}{2}(m_\gamma\gamma)^2-\lambda_3\Phi\gamma\chi] \tag{8.1.169}$$

The equation of motion is

$$(-\Box_G + m^2)\Phi = \lambda_3\gamma\chi \tag{8.1.170}$$

Φ can be expanded in powers of λ

$$\Phi = \Phi_0 + \lambda_3 \ \Phi_1 + ... \tag{8.1.171}$$

$$\Phi_0 = \phi_{00} + z^2 \phi_{02} + z^4 \phi_{04} + ... \tag{8.1.172}$$

$$\Phi_1 = \phi_{10} + z^2 \phi_{12} + z^4 \phi_{14} + ... \tag{8.1.173}$$

γ and χ have similar expansions.

The equations of motion can be solved perturbatively order by order in λ_3.

$$(-\Box_G + m^2)\Phi_0 = 0 \tag{8.1.174}$$

and

$$(-\Box_G + m^2)\Phi_1 = \gamma\chi \tag{8.1.175}$$

We fourier transform along all directions parallel to the boundary at $z = 0$. We write the Fourier transform of $\Phi(z, \vec{x})$ as $\Phi(z, \vec{p})$. The free equation of motion becomes

$$L_{d,\Delta}(z, p)\Phi(z, p) = 0 \tag{8.1.176}$$

where

$$L_{d,\Delta}(z,p) = -z^2\partial_z^2 + (d-1)z\partial_z + m^2 + z^2p^2 \tag{8.1.177}$$

The bulk to boundary propagator is given by

$$\mathcal{K}_{d,\Delta}(z,p) = \frac{2^{\frac{d}{2}-\Delta+1}}{\Gamma(\Delta-\frac{d}{2})}p^{\Delta-\frac{d}{2}}z^{d/2}K_{\Delta-\frac{d}{2}}(pz) \tag{8.1.178}$$

The bulk to bulk propagator is

$$\mathcal{G}_{d,\Delta}(z,p;\zeta) = (z\zeta)^{d/2}I_{\Delta-d/2}(pz)K_{\Delta-d/2}(p\zeta) \tag{8.1.179}$$

for $z \le \zeta$

and

$$\mathcal{G}_{d,\Delta}(z,p;\zeta) = (z\zeta)^{d/2}I_{\Delta-d/2}(p\zeta)K_{\Delta-d/2}(pz) \tag{8.1.180}$$

for $z \ge \zeta$

We saw earlier that for small λ_3, Φ can be expanded in powers of λ_3

$$\Phi = \Phi_0 + \lambda_3\ \Phi_1 + ... \tag{8.1.181}$$

Φ has a near boundary(small z) expansion

$$\Phi = (\phi_{00} + \lambda_3\phi_{10} + O(\lambda_3^3)) + z^2(\phi_{02} + \lambda_3\phi_{12} + O(\lambda_3^3)) + O(z^3) \tag{8.1.182}$$

Therefore,

$$\phi_0 = \phi_{00} + \lambda_3 \phi_{10} + \lambda_3^2 \phi_{20} + ... \tag{8.1.183}$$

should be considered the full source(all z independent terms). The solutions to the equations of motion order by order in λ_3 are,

$$\Phi_0(z, \vec{p}) = \mathcal{K}_{d,\Delta}(z, p)\phi_0 \tag{8.1.184}$$

To begin with we turn off the higher order terms in the source ϕ_0. We only keep the leading term ϕ_{00} in Φ_0 here. Later we will see that the higher orders will have to be turned on.

The solution to the second equation is

$$\begin{aligned}\Phi_1 = &\int \frac{d^d k_1 d^d k_2}{(2\pi)^{2d}} \gamma_0 \chi_0 \delta^{(d)}(k_1 + k_2 + k_3) \\ &\int_0^\infty \frac{d\zeta}{\zeta^{d+1}} \mathcal{G}_{d,\Delta}(z, k_1; \zeta) \mathcal{K}_{d,\Delta}(\zeta, k_2) \mathcal{K}_{d,\Delta}(k_3, \zeta)\end{aligned} \tag{8.1.185}$$

Further details can be found in [30].

8.2 Leading order

For $d = 2$ and $\Delta = 2 + 2\delta$, the solution to the free equation of motion is

$$\Phi_0 = p^{1+2\delta} z K_{1+2\delta}(pz)\phi_0 \tag{8.2.186}$$

We expand about $z = 0$ and then expand for $\delta \ll 1$, we get

$$\Phi_0 = (1 - 2\delta \log z)\phi_0 \tag{8.2.187}$$

At $z = x_0$ where $x_0 \to 0$, $-2\delta\phi_0 \log x_0$ is divergent. Therefore ϕ_0 would have to change to

$$\phi_0 \to \phi_0 + 2\delta\phi_0 \log \frac{x_0}{R} \tag{8.2.188}$$

to cancel the divergent term. Here R is an IR scale introduced on dimensional grounds. Thus we see that a change in the canonical scaling dimension of Φ_0, induces a flow in ϕ_0.

8.3 Order $\Phi\gamma\chi$

To get the contribution at this order we look at the solution of Φ_1.

We want to solve the integral

$$I^{\delta,<}_{d=2,\Delta=2} = \int_{x_0}^{R} \frac{d\zeta}{\zeta^{d+1}} \mathcal{G}_{d,\Delta}(z, p_1; \zeta) \mathcal{K}_{d,\Delta}(\zeta, p_2) \mathcal{K}_{d,\Delta}(p_3, \zeta) \tag{8.3.189}$$

in the near boundary region $\zeta \leq R$. x_0 is the UV cut-off and R is an IR scale.

$$\Phi_1 = \gamma_0\chi_0 \int_{x_0}^{R} \frac{d\zeta}{\zeta^{d+1}} (z\zeta)^{d/2} I_{\Delta_1 - d/2}(p_1\zeta) K_{\Delta_1 - d/2}(p_1 z) \tag{8.3.190}$$
$$\frac{2^{d/2-\Delta_2+1}}{\Gamma(\Delta_2 - d/2)} p_2^{\Delta_2 - d/2} \zeta^{d/2} K_{\Delta_2 - d/2}(p_2\zeta) \frac{2^{d/2-\Delta_3+1}}{\Gamma(\Delta_3 - d/2)} p_3^{\Delta_3 - d/2} \zeta^{d/2} K_{\Delta_3 - d/2}(p_3\zeta)$$

Since the δ has no effect at this order all Δ's are set to 2. The log divergent part of Φ_1 is,

$$\frac{1}{2}\gamma_0\chi_0 p_1 z K_1(p_1 z)(-\log x_0) \tag{8.3.191}$$

We can expand Φ in powers of λ_3. Therefore we can write the log divergent terms of Φ,

$$\Phi = \Phi_0 + \lambda_3 \frac{1}{2}\gamma_0\chi_0 p_1 z K_1(p_1 z)(-\log x_0) \tag{8.3.192}$$

This diverges as $x_0 \to 0$. $p_1 z K_1(p_1 z)$ is the solution to the leading order equation of motion Φ_0, therefore to make the full Φ finite we can turn on a subleading $O(\lambda_3)$ term in the expansion of the source ϕ_0 in Φ_0 (as mentioned before we are turning on subleading coefficients).

$$\phi_0 = \phi_{00} + \lambda_3\phi_{10} + ... \tag{8.3.193}$$

The modified source ϕ_0 is

$$\phi_0 = (1 + 2\delta \log x_0)\phi_{00} + \lambda_3\phi_{10} \tag{8.3.194}$$

set

$$\phi_{10} = \frac{1}{2}\gamma_0\chi_0(\log x_0) \tag{8.3.195}$$

The modified source becomes

$$\phi_0 = \left(1 + 2\delta \log \frac{x_0}{R}\right)\phi_0 + \lambda_3 \frac{1}{2}\gamma_0\chi_0(\log \frac{x_0}{R}) \tag{8.3.196}$$

where we have again introduced the IR length scale R.

8.4 The beta function.

As before, we make the identification $\log \frac{R}{x_0} \to t$, substitute the relations between ϕ_0, γ_0, χ_0 and F,G,H((C.1.227), (C.1.228), (C.1.229)) and the value of λ_3(C.2.233). Therefore we get the beta function

$$\beta_F = -(2\delta F - \frac{GH}{4}) \tag{8.4.197}$$

which matches all earlier results.

9 Beta function for δ.

9.1 Overview of the calculation.

As mentioned before b multiplies X inside the cosine and therefore runs with the field strength renormalization. b^2 is close to 4. This is large compared to F which is close to zero. It was mentioned that the beta functions of the Sine-Gordon are a power series expansion in two parameters, $\delta = b^2/4 - 1$ is the other appropriate small parameter in which the expansion can be carried out. From the perspective of the boundary theory the correction to this modified coupling δ comes from two cosines combining together to give $(\partial X)^2$, this is the anomalous dimension and gives this beta function. This has been calculated before in [22].

From the bulk theory the leading contribution to the beta function has to come from a vertex of the type $g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi$ so that we have a structure similar to the boundary calculation and we can see that the two ϕ's correct the $g_{\mu\nu}$ which is associated to the $(\partial X)^2$ term. Next we note that the boundary kinetic term involves only diagonal components and therefore we can attempt to model the graviton by a dilaton which takes into account only the diagonal degress of freedom and is a scalar, thus simplifying the problem enormously.

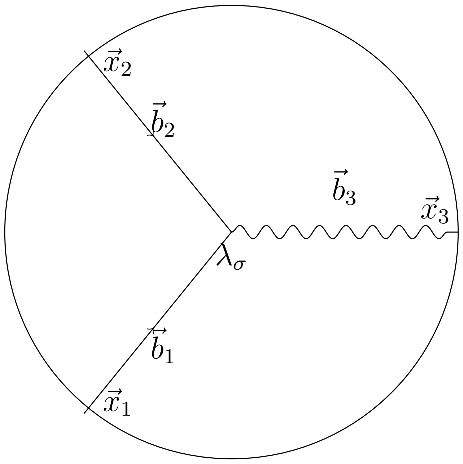

Fig. 3.
Witten diagram for the graviton-scalar-scalar vertex
for the beta function of the field strength renormalization.

9.2 Fixing the coupling of the graviton-scalar-scalar vertex in the bulk.

To compute the graviton-scalar-scalar vertex we want to look at the fluctuation about AdS

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{9.2.198}$$

$\bar{g}_{\mu\nu}$ is AdS. We want to simplify this by modeling the graviton as a dilaton. Therefore,

$$g^{\mu\nu} = e^{-\lambda_\sigma \sigma}\bar{g}^{\mu\nu} = (1 - \lambda_\sigma \sigma)\bar{g}^{\mu\nu} \tag{9.2.199}$$

Therefore the kinetic term $\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ becomes

$$\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{1}{2}(1-\lambda_\sigma\sigma)\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{9.2.200}$$

We treat the dilaton as a massive scalar with $m_\sigma \to 0$, the full action therefore becomes,

$$\begin{aligned} S_{bulk} = \int d^3x\sqrt{g}[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(m_\phi\phi)^2 + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}(m_\chi\chi)^2 \qquad (9.2.201)\\ + \frac{1}{2}(\partial\gamma)^2 + \frac{1}{2}(m_\gamma\gamma)^2 - \lambda_3\phi\gamma\chi - \frac{1}{2}\lambda_\sigma\sigma\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}(\partial\sigma)^2 + \frac{1}{2}m_\sigma^2\sigma^2] \end{aligned}$$

The kinetic term in the boundary action is modified

$$S_{boundary} = \frac{1}{4\pi}\int d^2x[(1+\xi_0)(\partial_\mu\vec{X}).(\partial^\mu\vec{X}) + m^2\vec{X}.\vec{X} + \frac{F}{a(0)^2}\cos\ (\vec{b}_1.\vec{X}) \tag{9.2.202}$$

$$+ \frac{G}{a(0)^2}\cos\ (\vec{b}_2.\vec{X}) + \frac{H}{a(0)^2}\cos\ (\vec{b}_3.\vec{X})] \tag{9.2.203}$$

ξ_0 is related to the bulk field σ whose boundary value σ_0 is equal to ξ_0 upto normalization. We vary the action with respect to ξ_0 to compute various correlators.

9.3 The 3-point correlator calculation.

To compute the beta function we want to calculate the generating functional for the three point function as before, but this time for the vertex

$-\frac{1}{2}\lambda_\sigma \sigma \bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. To do this we again start with

$$S_{\sigma 3} = -\frac{1}{2}\lambda_\sigma \int d^{d+1}y\sqrt{g}\ \bar{g}^{\mu\nu}\partial_\mu\phi(y)\ \partial_\nu\phi(y)\ \sigma(y) \tag{9.3.204}$$

We, as before, put in the expressions for the bulk to boundary propagators for all the fields and simplify. Details in Appendix (E). We get,

$$\begin{aligned} S_{\sigma 3} =& -2\pi(1+\delta)\lambda_\sigma\sigma_0\phi_0^2 \int d\left(\frac{x_1^2}{R^2}\right) d\left(\frac{x_3^2}{R^2}\right)\log\frac{R}{x_0} \\ &\left[\frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d}{2})}{\Gamma(\Delta_2)\Gamma(\Delta_3)} - \frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d+2}{2})}{\Gamma(\Delta_2)\Gamma(\Delta_3+1)}\right] \end{aligned} \tag{9.3.205}$$

$S_{\sigma 3}$ is non-zero offshell but vanishes onshell where the square bracket is zero. This is resolved when λ_σ is fixed ((C.4.250) and the comment thereafter).

Putting in all the relative normalizations((C.1.227), (C.3.240)) and the value of λ_σ and we get,

$$S_{\sigma 3} = \frac{-F^2(1+\delta)}{8}\frac{\xi_0}{4\pi}\int d^2\left(\frac{x_1}{R}\right) d\left(\frac{x_3^2}{R^2}\right)\log\frac{R}{x_0} \tag{9.3.206}$$

9.4 The Beta function.

The kinetic term in the boundary theory, whose correction we are computing is

$$\frac{1+\xi_0}{4\pi}\int d^2x\ (\partial X(x))^2 \tag{9.4.207}$$

Comparing this with the expression for $S_{\sigma 3}$ above and as before, making the identification $\log \frac{R}{x_0} \to t$, we immediately see the beta function(the t-derivative) is,

$$\beta_\delta = \frac{-F^2(1+\delta)}{8} \tag{9.4.208}$$

which matches 5.2.46 in [22]. This computation is correct upto O($F^2\delta$). There are higher order corrections to it. The computation for the boundary theory using the usual field theoretic approach has been done by Amit et al [8]. In their paper(Section 7) they have a detailed analysis of the higher order corrections. These can be easily obtained by an order by order double expansion in F and δ. Therefore the next term would be O(F^3).

10 Summary and Conclusions

In this thesis we have studied the RG flow of quantities in field theories. The idea is to use Polchinski's ERG written in terms of an evolution operator. The advantage is that one can directly obtain quantities such as the beta function by looking at the linear dependence on the RG time t. In the limit of cutoff going to infinity this coefficient gives the beta function. This technique was illustrated with a few examples such as the ϕ^4 theory in four dimensions and also the Sine-Gordon theory in two dimensions - which is the main interest in the first paper [22]. We also show that another flow calculation that this method is suited for is that of the c-function. We illustrate it with the case of the free field. We then calculate it for the Sine-Gordon theory. Interestingly, if we assume, that the relation between entanglement entropy and the central charge continues to hold even for the c-function, we can evaluate the entanglement entropy of the Sine Gordon theory for small values of the perturbation. This has been done using other field theoretic and also holographic methods and there is complete agreement for the lowest order term - which is all that has been calculated [55]. For the Sine-Gordon theory the detailed results of [8] for the solution of the RG equations have been used in this paper to calculate the c-function to higher orders.

There are many open questions. It would be interesting to extend the ideas in this paper to more basic issues in holographic RG and in particular the connection with the RG on the boundary theory. In the context of entanglement entropy, it would be interesting to check the agreement between the entanglement entropy and the central charge to higher orders. Since the c-function is presumably not a universal quantity (there should be some scheme dependence) at higher orders, these checks have to be made keeping these caveats in mind.

In the second paper [26], the beta functions of a generalized Sine-Gordon theory have been calculated using a bulk holographic dual. The boundary theory is a free theory deformed by a term $F \cos b.X$. The anomalous dimension is proportional to F and goes to zero as $F \to 0$. The bulk theory is dual to a free field theory in the boundary for F=0. The bulk fields (in addition to the graviton) introduced correspond to the Cosine perturbation. The calculations have been done both in position space as well as momentum space. The boundary calculation is also done and it is shown that the results agree.

To compute the beta functions, two and three point correlation functions were computed from the bulk. We are in fact constructing the bulk dual of the free theory and including just those bulk fields and interaction vertices that are necessary to reproduce correlators of some specific boundary operators (cos b.X) which would give us the beta functions. The correlators computed from the bulk match the boundary calculations upto normalization. These normalizations and the bulk interaction vertex coupling were further fixed by comparing the two and three point correlators on both sides, which is done in the appendix. The bulk theory was constructed in such a way as to reproduce the correlators in the boundary theory, therefore agreement between results was expected.

The main motivation for doing this calculation is to understand the results of [9] better when interactions are involved. There the main example used was the free scalar theory and in this situation interactions are between composite operators. In 1+1 dimension, the cosine is one of the most interesting example of such operators and besides being related to string theory, has applications in 1+1 dimensional condensed matter systems, such as the X-Y model [8].

The model is also motivated by the first quantized description of a string propagating in a tachyon background. The beta function gives the equation of motion for the tachyon. The model also has a wave function renormalization which results in a beta function for the string theory dilaton coupling. The boundary calculation in this paper uses techniques derived from the exact RG, as used in [22] for the usual Sine-Gordon model. The main idea for the bulk calculation in position space is to identify the beta function with the coefficient of a logarithmic deviation from the canonical scaling of a two point function. This is based on the technique described in [11, 56] and is suitable for holographic computations. In the bulk momentum space calculation the technique is to first solve the fourier transformed equations of motion order by order in the coupling of the bulk interaction vertex and then identify the log divergent terms in the solutions as described in [28]. All the results agree to the order calculated.

There are many further problems that need to be addressed. One technically interesting issue of course is to go to higher orders. This should constrain the bulk dual much more. The precise bulk dual of the free scalar theory considered here, in particular, the connection to higher spin theory in AdS_3, needs to be understood better. It would be interesting if one can say something about the UV fixed point of this theory by studying the bulk.

One should remember that the underlying theory in the boundary is a free scalar theory. The interactions in the bulk involve fields dual to composite operators. There are an infinite number of them - they can be identified with the momentum modes of the string theory tachyon. One expects that there should be a corresponding simple way to package these in the bulk also. This needs to be understood better. Finally, the ERG description of composite operators and also the map to a holographic theory in the presence of these operators, is another one of many other complications. These can be studied in a controlled way in this model.

We hope to return to all these questions.

Appendix A Third Order term in the Sine-Gordon β-function

The third order term is made of two positive exponentials and one negative one or vice versa and there are three such terms that can combine to give a cosine as the leading term in the OPE:

$$3 \times \frac{1}{3!} \frac{F^3}{(4\pi)^3} \int \frac{d^2x_1}{a(0)^2} \int \frac{d^2x_2}{a(0)^2} \int \frac{d^2x_3}{a(0)^2} \frac{1}{4} e^{\frac{b^2}{2}(F(x_1,x_1)+F(x_2,x_2)+F(x_3,x_3))}$$

$$e^{-b^2(F(x_1,x_2)+F(x_1,x_3)-F(x_2,x_3))} cos\ (bX(x_1)) \tag{A.0.209}$$

We will set $b^2 = 4$ without further ado. We choose $x_1 = 0$ (by translational invariance) and for notational simplicity set $x_2 = x,\ x_3 = y$. As before we choose $a(t) \approx \frac{1}{m}$ so the integral in this approximation becomes (suppressing all prefactors $\frac{1}{8}\frac{F^3}{(4\pi)^3}$):

$$\int \frac{d^2x}{a(t)^2} \int \frac{d^2y}{a(t)^2} [\frac{a(t)^2}{x^2 + a(0)^2}]^2 [\frac{a(t)^2}{y^2 + a(0)^2}]^2 e^{2ln\ [\frac{(x-y)^2 + a(0)^2}{a(t)^2}]} \tag{A.0.210}$$

There are three regions of divergences:

1. I : $x \to 0,\ y > \Delta$

2. II:$y \to 0,\ y > \Delta$

3. III: $x, y \to 0$

   When both $x, y > \Delta$ and $x \to y$ there is a divergence, but it is of the same form as I or II and is merely a permutation of indices: In the above we have kept $x_1 = 0$ fixed, but there are other choices which will produce three similar regions and this region ($x, y > \Delta$ and $x \to y$) will be one of those.

Here Δ is some finite arbitrary length. The coefficient of the divergence cannot depend on Δ because it is an arbitrary way to split up the region of integration.

Region I: Let us Taylor expand the log in the last factor, about y^2, which is large:

$$e^{2ln\ [\frac{(x-y)^2+a(0)^2}{a(t)^2}]} = e^{2ln\ [\frac{y^2+a(0)^2+\overbrace{x^2-2x.y}^{X}}{a(t)^2}]}$$

$$= e^{2(ln\ [\frac{y^2+a(0)^2}{a(t)^2}]+\frac{X}{y^2+a(0)^2}-\frac{1}{2!}X^2\frac{1}{(y^2+a(0)^2)^2}+...)}$$

Insert this into (A.0.210) and we get:

$$\int\frac{d^2x}{a(t)^2}\int\frac{d^2y}{a(t)^2}\ [\frac{a(t)^2}{x^2+a(0)^2}]^2\{1+\underbrace{\frac{2X}{y^2+a(0)^2}}_{(i)}+\underbrace{X^2\frac{1}{(y^2+a(0)^2)^2}}_{(ii)}+..\}$$

The leading term in this expansion corresponds to a disconnected graph where x and 0 are connected and y is not connected to either of these. This has to be subtracted out since, the cumulant expansion prescription is to calculate connected graphs. So we are left with (i) and (ii). The x^4 term is finite (on doing the x integral). We get for Region I

$$-4\pi^2 ln\ [m^2\Delta^2] ln\ [\frac{\Delta^2}{a(0)^2}]$$

Region II: Gives the same as above.

Thus the total contribution from Region I and II =

$$-8\pi^2 ln\ [m^2\Delta^2] ln\ [\frac{\Delta^2}{a(0)^2}]$$

Notice that there is no Δ independent contribution to $ln\ a(0)$. That can come only when all three vertices are together. This will come from region III.

Region III:

We go back to (A.0.210) in this region of integration.

$$\int_0^\Delta \frac{d^2x}{a(t)^2} \int_0^\Delta \frac{d^2y}{a(t)^2} [\frac{a(t)^2}{x^2 + a(0)^2}]^2 [\frac{a(t)^2}{y^2 + a(0)^2}]^2 [\frac{(x-y)^2 + a(0)^2}{a(t)^2}]^2 \quad \text{(A.0.211)}$$

We expand

$$[\frac{1}{x^2 + a(0)^2}][\frac{1}{y^2 + a(0)^2}][(x-y)^2 + a(0)^2] = \frac{1}{x^2 + a(0)^2} + \frac{1}{y^2 + a(0)^2} - \frac{\overbrace{a(0)^2 + 2x.y}^{Y}}{(x^2 + a(0)^2)(y^2 + a(0)^2)}$$

Squaring it produces six terms:

(a)

$$(\frac{1}{x^2 + a(0)^2})^2$$

(b)

$$(\frac{1}{y^2 + a(0)^2})^2$$

(c)

$$\frac{Y^2}{(x^2 + a(0)^2)^2 (y^2 + a(0)^2)^2}$$

(d)

$$\frac{2}{(x^2 + a(0)^2)(y^2 + a(0)^2)}$$

(e)

$$\frac{-2Y}{(x^2 + a(0)^2)(y^2 + a(0)^2)^2}$$

(f)

$$\frac{-2Y}{(x^2 + a(0)^2)^2(y^2 + a(0)^2)}$$

Terms (a) and (b) correspond to disconnected diagrams that are subtracted out.

(c)

$$= 2\pi^2[ln^2\ (\frac{\Delta^2}{a(0)^2}) + 1 - 2ln\ (\frac{\Delta^2}{a(0)^2}) + O(a^2)]$$

(d)

$$= 2\pi^2 ln^2\ \frac{\Delta^2}{a(0)^2}$$

(e)

$$= -2\pi^2 ln\ \frac{\Delta^2}{a(0)^2}$$

(f)=(e)

$$= -2\pi^2 ln\ \frac{\Delta^2}{a(0)^2}$$

Any renormalizable theory cannot have divergences of the type $ln\ \Delta^2 ln\ a(0)^2$. Because Δ is like momentum and the counter terms would have derivative interactions to all orders. Thus the theory would be non-local.

We can now check that the coefficient of $ln\ \Delta^2 ln\ a(0)^2$ is zero.

$$8\pi^2\ (from\ I + II)\ - 4\pi^2\ (from\ (c))\ - 4\pi^2\ (from\ (d)\ = 0$$

The coefficient of $ln\ a(0)^2$ is $8\pi^2$. Thus we get putting back the prefactors

$$\frac{1}{8}\frac{F^3}{(4\pi)^3}8\pi^2 2\ ln\ \frac{a(0)}{\Delta}$$

Since any value of Δ is safe for extracting the divergence, we can extend the region of integration to its full value which is $\Delta = a(t) \approx \frac{1}{m}$.

This the modified $\frac{F}{4\pi}$. So

$$F(t) = \frac{F^3}{8}\ ln\ \frac{a(0)}{a(t)}$$

Thus the beta functions at this order is

$$\beta_F = -\frac{F^3}{8} \tag{A.0.212}$$

Appendix B The sub-leading term for β_F using ERG on the boundary

The action of the evolution operator is

$$\int d^2x_1 d^2x_2 F_{x_1x_2t} \frac{\delta^2}{\delta X_1 \delta X_2} \left[\frac{e^{ib_3.X_3} + e^{-ib_3.X_3}}{2}\right] \left[\frac{e^{ib_2.X_4} + e^{-ib_2.X_4}}{2}\right] \quad \text{(B.0.213)}$$

Here X_i means $X(x_i)$. We will drop the dot's henceforth. Keeping terms that contribute we get,

$$\int d^2x_1 d^2x_2 F_{x_1x_2t} \frac{\delta^2}{\delta X_1 \delta X_2} \left[\frac{e^{-ib_3X_3 - ib_2X_4} + e^{ib_3X_3 + ib_2X_4}}{4}\right] \quad \text{(B.0.214)}$$

In the last expression the two terms that conserve momenta have been retained. We look at the action of the evolution operator on the first term. The second term gives an identical contribution.

$$\int d^2x_1 d^2x_2 F_{x_1x_2t} \frac{\delta^2}{\delta X_1 \delta X_2} \left[e^{ib_3X_3 + ib_2X_4}\right] \quad \text{(B.0.215)}$$

$$= (-b_3^2 F_{33t} - b_2.b_3 F_{34t} - b_2.b_3 F_{34t} - b_2^2 F_{44t}) e^{i(b_3+b_2)X_4} \quad \text{(B.0.216)}$$

Here X_3 has been taylor expanded and brought to X_4. Therefore,

$$\left[\frac{e^{ib_3X_3+ib_2X_4}+e^{-ib_3X_3-ib_2X_4}}{4}\right] \tag{B.0.217}$$

becomes

$$\frac{1}{2}\cos(b_2+b_3)X_4=\frac{1}{2}cos b_1X_4 \tag{B.0.218}$$

Substituting (B.0.217) and (B.0.218) in (4.5.75) we get

$$\frac{GH}{(4\pi)^2}\int\frac{d^2x_1d^2x_2}{a^4}\exp[-\frac{1}{2}(-b_3^2-b_2^2)F_{11t}+b_2.b_3F_{12t}] \tag{B.0.219}$$
$$\frac{1}{2}\cos b_1X(x_2)$$

where $F_{12t}=-\frac{1}{2}ln\frac{(x_1-x_2)^2+a(t)^2}{(x_1-x_2)^2+a(0)^2}$

now we relabel $x_2-x_1\to y$ and $x_2\to x$ we get

$$=\frac{GH}{8}\int\frac{dy^2}{a(t)^2}e^{4t-\frac{b_3^2+b_2^2}{2}t}\left(\frac{y^2+a(t)^2}{y^2+a(0)^2}\right)^{\frac{-b_2.b_3}{2}}\frac{1}{(4\pi)}\int\frac{d^2x}{a(t)^2}\cos b_1X(x) \tag{B.0.220}$$

$a(t)$ is the IR cutoff therefore we drop y^2 from the numerator and integrate.

$$= \frac{GH}{8} e^{4t - \frac{b_3^2 + b_2^2}{2} t} a(t)^{-b_2.b_3 - 2} \frac{(a(t)^2 + a(0)^2)^{\frac{b_2.b_3}{2}+1} - a(0)^{2(\frac{b_2.b_3}{2}+1)}}{\frac{b_2.b_3}{2}+1} \tag{B.0.221}$$

$$\frac{1}{(4\pi)} \int \frac{d^2x}{a(t)^2} \cos b_1 X(x)$$

dropping $a(0)^2$ from the first term.

$$= \frac{GH}{8} e^{4t - \frac{b_3^2+b_2^2}{2} t} \frac{1 - \left(\frac{a(t)}{a(0)}\right)^{-2(\frac{b_2.b_3}{2}+1)}}{\frac{b_2.b_3}{2}+1} \frac{1}{(4\pi)} \int \frac{d^2x}{a(t)^2} \cos b_1 X(x) \tag{B.0.222}$$

For $b_2.b_3$ close to -2 and for $b_2^2 = b_3^2 = 4$ we get,

$$\frac{GH}{4} t \frac{1}{(4\pi)} \int \frac{d^2x}{a(t)^2} \cos b_1 X(x) \tag{B.0.223}$$

Appendix C Fixing relative normalization of the bulk and the boundary couplings and computing λ_3

C.1 Fixing relative normalizations between ϕ_0, γ_0, χ_0 and F, G, H.

To compute the relative normalization between the bulk and the boundary for the couplings ϕ_0 and F we compare the generating functionals of the two point functions calculated for both sides.

The generating function for the two point function for the boundary theory is

$$GF_2 = \frac{A_2}{4}\frac{F^2}{(4\pi)^2} \tag{C.1.224}$$

$$A_2 = \int d^2x_1 d^2x_2 \frac{1}{(\vec{x}_1 - \vec{x}_2)^{2\Delta}} \tag{C.1.225}$$

The generating function for the two point function for the bulk is

$$S_2 = \frac{2}{\pi}A_2\phi_0^2/(2!) \tag{C.1.226}$$

Comparing S_2 and GF_2 we get

$$\phi_0 = \frac{1}{8\sqrt{\pi}}F \tag{C.1.227}$$

Similarly,

$$\gamma_0 = \frac{1}{8\sqrt{\pi}} G \tag{C.1.228}$$

$$\chi_0 = \frac{1}{8\sqrt{\pi}} H \tag{C.1.229}$$

C.2 Computing λ_3

To compute λ_3 we compare the generating function for the three point function of the boundary theory and for the bulk theory.

The generating function for the three point function of the boundary theory is

$$GF_3 = \frac{A_3}{4} \frac{FGH}{(4\pi)^3} \tag{C.2.230}$$

$$A_3 = \int d^2x_1 d^2x_2 d^2x_3 \frac{1}{(\vec{x}_1 - \vec{x}_2)^{\Delta_{123}} (\vec{x}_2 - \vec{x}_3)^{\Delta_{231}} (\vec{x}_3 - \vec{x}_1)^{\Delta_{312}}} \tag{C.2.231}$$

Here $\Delta_{ijk} = \Delta_i + \Delta_j - \Delta_k$.

For the bulk theory

$$S_3 = -\frac{\lambda_3}{2\pi^2} A_3 \phi_0 \gamma_0 \chi_0 \tag{C.2.232}$$

Comparing GF_3 and S_3 we get

$$\lambda_3 = -4(\pi)^{1/2} \tag{C.2.233}$$

C.3 Relative normalization between σ_0 and ξ_0

To fix this we calculate the generating function of $\langle(\partial X(x_1))^2(\partial X(x_2))^2\rangle$ from the bulk and boundary and compare them.

Bulk:

$$GF_{\sigma_0^2} = \frac{1}{2}\sigma_0^2 \frac{\Gamma(\Delta+1)}{\pi^{d/2}\Gamma(\Delta - d/2)} \int d^2x_1 d^2x_2 \frac{1}{x_{12}^{2\Delta}} \tag{C.3.234}$$

$$GF_{\sigma_0^2} = \frac{\sigma_0^2}{\pi} A_2 \tag{C.3.235}$$

Boundary:

$$GF_{\xi_0^2} = \frac{1}{2!}\frac{\xi_0^2}{(4\pi)^2} \int d^2x_1 d^2x_2 \left\langle(\partial X(x_1))^2(\partial X(x_2))^2\right\rangle \tag{C.3.236}$$

$$\left\langle(\partial X(x_1))^2(\partial X(x_2))^2\right\rangle = \frac{2}{x_{12}^4} \tag{C.3.237}$$

Therefore

$$GF_{\xi_0^2} = \frac{1}{2!}\frac{\xi_0^2}{(4\pi)^2} \int d^2x_1 d^2x_2 \frac{2}{x_{12}^4} \tag{C.3.238}$$

$$GF_{\xi_0^2} = \frac{\xi_0^2}{(4\pi)^2} A_2 \tag{C.3.239}$$

Comparing,

$$\sigma_0 = \frac{\xi_0}{4\sqrt{\pi}} \tag{C.3.240}$$

C.4 Fixing λ_σ

To fix λ_σ we will compute $\langle (\partial X(x_1))^2 \cos bX(x_2) \cos bX(x_3) \rangle$ from the bulk and the boundary and compare them.

Bulk:

From (E.0.280)

$$S_{\sigma 3} = \frac{-\lambda_\sigma}{\pi^2} \sigma_0 \phi_0^2 (1+\delta) A_3 [\,] \tag{C.4.241}$$

where $[\,] = \left[\frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d}{2})}{\Gamma(\Delta_2)\Gamma(\Delta_3)} - \frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d+2}{2})}{\Gamma(\Delta_2)\Gamma(\Delta_3+1)} \right]$

Boundary:

We want to compute $\langle (\partial X(x_1))^2 \cos bX(x_2) \cos bX(x_3) \rangle$

$$\left\langle (\partial X(x_1))^2 \cos bX(x_2) \cos bX(x_3) \right\rangle_{non-vanishing} = 2/4 \left\langle (\partial X(x_1))^2 \exp ibX(x_2) \exp -ibX(x_3) \right\rangle \tag{C.4.242}$$

We change to complex coordinates. $(\partial X(x_1))^2 \to 4\partial X_1 \bar{\partial} X_1$. We get,

$$2 \left\langle \partial X_1 \bar{\partial} X_1 \exp ibX(x_2) \exp -ibX(x_3) \right\rangle \tag{C.4.243}$$

We consider the product $\exp i\alpha\partial X_1 \exp i\beta\bar{\partial} X_1$, where α and β are close to zero.

$$\left\langle \exp i\alpha\partial X_1 \exp i\beta\bar{\partial} X_1 \exp ibX(x_2) \exp -ibX(x_3)\right\rangle \tag{C.4.244}$$

$$= -\alpha\beta \left\langle \partial X_1 \bar{\partial} X_1 \exp ibX(x_2) \exp -ibX(x_3)\right\rangle \tag{C.4.245}$$

is the part to linear order in $\alpha\beta$. The coefficient of the $-\alpha\beta/2$ term will give us the correlator (C.4.242).

Now,

$$\left\langle \exp i\alpha\partial X_1 \exp i\beta\bar{\partial} X_1 \exp ibX(x_2) \exp -ibX(x_3)\right\rangle \tag{C.4.246}$$

$$= \exp\left[\frac{1}{4}\int d^2z d^2z' \Big(\delta(z-z_1)\delta(\bar{z}-\bar{z_1})\alpha\partial + \delta(z-z_1)\delta(\bar{z}-\bar{z_1})\beta\bar{\partial} \right. \tag{C.4.247}$$

$$+ \delta(z-z_1)\delta(\bar{z}-\bar{z_1})b + \delta(z-z_1)\delta(\bar{z}-\bar{z_1})(-b)\Big)\Big(ln(z-z')(\bar{z}-\bar{z}')\Big)$$

$$\Big(\delta(z-z_1)\delta(\bar{z}-\bar{z_1})\alpha\partial + \delta(z-z_1)\delta(\bar{z}-\bar{z_1})\beta\bar{\partial}$$

$$\left. + \delta(z'-z_1)\delta(\bar{z'}-\bar{z_1})b + \delta(z'-z_1)\delta(\bar{z'}-\bar{z_1})(-b)\Big)\right]$$

For $b^2 = 4(1+\delta)$ the expression becomes

$$(-\alpha\beta/2)\frac{-2b^2}{4}\frac{1}{z_{12}^2 z_{13}^2 z_{23}^2} \tag{C.4.248}$$

Therefore, the generating function from the boundary theory is,

$$GF_{\sigma 3} = \frac{1}{2!}\frac{-2\xi_0 F^2(1+\delta)}{(4\pi)^3}A_3 \tag{C.4.249}$$

Comparing (C.4.241) and (C.4.249),

$$\lambda_\sigma = \frac{4\sqrt{\pi}}{\left[\frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d}{2})}{\Gamma(\Delta_2)\Gamma(\Delta_3)} - \frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d+2}{2})}{\Gamma(\Delta_2)\Gamma(\Delta_3+1)}\right]} \tag{C.4.250}$$

Thus the square brackets cancel out in $S_{\sigma 3}$. The correlator remains finite on-shell.

Appendix D Position space calculation for β_F from the bulk for the sub-leading term.

$$S_3 = -\lambda_3 \int d^d x_1 \int d^d x_2 \int d^d x_3 \int d^{d+1} y \; y_0^{-(d+1)} C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \tag{D.0.251}$$

$$\phi_0 \; \gamma_0 \; \chi_0 \frac{y_0^{\Delta_1+\Delta_2+\Delta_3}}{(y_0^2 + (\vec{y} - \vec{x_1})^2)^{\Delta_1} (y_0^2 + (\vec{y} - \vec{x_1})^2)^{\Delta_2} (y_0^2 + (\vec{y} - \vec{x_2})^2)^{\Delta_3}}$$

After Feynman parameterization we get,

$$= -\lambda_3 \int d^d x_1 \int d^d x_2 \int d^d x_3 \int d^{d+1} y \; C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \; \phi_0 \; \gamma_0 \; \chi_0 \tag{D.0.252}$$

$$\int d\alpha_1 d\alpha_2 d\alpha_3 \alpha_1^{\Delta_1 - 1} \alpha_2^{\Delta_2 - 1} \alpha_3^{\Delta_3 - 1} \delta(\Sigma_{i=1}^3 \alpha_i - 1) \frac{\Gamma\left(\Sigma_{i=1}^3 \Delta_i\right)}{\Pi_{i=1}^3 \Gamma\left(\Delta_i\right)}$$

$$\frac{y_0^{-(d+1)+\Delta_1+\Delta_2+\Delta_3}}{(y_0^2 + (\vec{y} - \Sigma_{i=1}^n \alpha_i \vec{x}_i)^2 + \Sigma_{i<j=1}^3 \alpha_i \alpha_j (\vec{x}_i - \vec{x}_j)^2)^{\Delta_1+\Delta_2+\Delta_3}}$$

We do the y_0 and $\vec{y}$ integrals

$$S_3 = -\lambda_3 \frac{\pi^{d/2} \Gamma\left(\frac{\Sigma_{i=1}^3 \Delta_i}{2}\right) \Gamma\left(\frac{\Sigma_{i=1}^3 \Delta_i - d}{2}\right)}{2 \Pi_{i=1}^3 \Gamma\left(\Delta_i\right)} \int d^d x_1 \int d^d x_2 \int d^d x_3 \; C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \; \phi_0 \; \gamma_0 \; \chi_0 \tag{D.0.253}$$

$$\int d\alpha_1 d\alpha_2 d\alpha_3 \frac{\alpha_1^{\Delta_1 - 1} \alpha_2^{\Delta_2 - 1} \alpha_3^{\Delta_3 - 1} \delta(\Sigma_{i=1}^3 \alpha_i - 1)}{(\Sigma_{i<j=1}^3 \alpha_i \alpha_j (\vec{x}_{ij})^2)^{\frac{\Delta_1+\Delta_2+\Delta_3}{2}}}$$

Now we transform from α_i's to β_i's. $\alpha_i = \beta_1 \beta_i$ for $i \geq 2$, $\alpha_1 = \beta_1$. The

Jacobian for n parameters is β_1^{n-1}. Here $n = 3$.

$$S_3 = -\lambda_3 \frac{\pi^{d/2}\Gamma\left(\frac{\Sigma_{i=1}^3 \Delta_i}{2}\right)\Gamma\left(\frac{\Sigma_{i=1}^3 \Delta_i - d}{2}\right)}{2\Pi_{i=1}^3 \Gamma\left(\Delta_i\right)} \int d^d x_1 \int d^d x_2 \int d^d x_3 \; C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \; \phi_0 \; \gamma_0 \; \chi_0 \tag{D.0.254}$$

$$\int d\beta_1 d\beta_2 d\beta_3 \frac{\beta_2^{\Delta_2-1}\beta_3^{\Delta_3-1}[\delta(\beta_1 - 1/(1+\beta_2+\beta_3))]/(1+\beta_2+\beta_3)}{\beta_1(\beta_2 x_{12}^2 + \beta_3 x_{13}^2 + \beta_2\beta_3 x_{23}^2)^{\frac{\Delta_1+\Delta_2+\Delta_3}{2}}}$$

After doing the β_1 integral we get

$$S_3 = -\lambda_3 \frac{\pi^{d/2}\Gamma\left(\frac{\Sigma_{i=1}^3 \Delta_i}{2}\right)\Gamma\left(\frac{\Sigma_{i=1}^3 \Delta_i - d}{2}\right)}{2\Pi_{i=1}^3 \Gamma\left(\Delta_i\right)} \int d^d x_1 \int d^d x_2 \int d^d x_3 \; C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \; \phi_0 \; \gamma_0 \; \chi_0 \tag{D.0.255}$$

$$\int d\beta_2 d\beta_3 \frac{\beta_2^{\Delta_2-1}\beta_3^{\Delta_3-1}}{(\beta_2 x_{12}^2 + \beta_3 x_{13}^2 + \beta_2\beta_3 x_{23}^2)^{\frac{\Delta_1+\Delta_2+\Delta_3}{2}}}$$

$$S_3 = -\lambda_3 \frac{\pi^{d/2}\Gamma\left(\frac{\Sigma_{i=1}^3 \Delta_i}{2}\right)\Gamma\left(\frac{\Sigma_{i=1}^3 \Delta_i - d}{2}\right)}{2\Pi_{i=1}^3 \Gamma\left(\Delta_i\right)} \int d^d x_1 \int d^d x_2 \int d^d x_3 \; C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \; \phi_0 \; \gamma_0 \; \chi_0 \tag{D.0.256}$$

$$\int d\beta_2 d\beta_3 \frac{\beta_2^{\Delta_2-1}\beta_3^{\Delta_3-1}}{(\beta_2(x_0^2 + x_{12}^2) + \beta_3(x_0^2 + x_{13}^2) + \beta_2\beta_3(x_0^2 + x_{23}^2))^{\frac{\Delta_1+\Delta_2+\Delta_3}{2}}}$$

Here we have introduced x_0^2's in the denominator($x_0^2 \to 0$). These act as UV regulators. We do the β_2 and β_3 integrals. Any factors of δ coming from the two beta functions from the two beta integrals contribute at $O(\delta\phi_0\gamma_0\chi_0)$.

Therefore they are dropped. We set $d = 2$ and $\Delta_2 = 2(1+\delta)$, particle 2 is offshell. We substitue C_{Δ_i}'s. We set $\vec{x}_1$ to zero using translation invariance, multiply and divide by R, the IR cut-off. Therefore, the integral simplifies to

$$S_3 = -\frac{\lambda_3}{2\pi^2} \int \frac{d^2x_1 d^2x_2 d^2x_3}{R^6} \frac{\phi_0 \gamma_0 \chi_0}{\left(\frac{x_0^2+x_2^2}{R^2}\right)^{(1+\delta)} \left(\frac{x_0^2+x_3^2}{R^2}\right)^{(1-\delta)} \left(\frac{x_0^2+x_{23}^2}{R^2}\right)^{(1+\delta)}} \tag{D.0.257}$$

We will now calculate the log divergent term. The x_2 integral is,

$$\int \frac{d^2x_2}{R^2} \frac{1}{\left(\frac{x_0^2+x_2^2}{R^2}\right)^{(1+\delta)} \left(\frac{x_0^2+x_3^2}{R^2}\right)^{(1-\delta)} \left(\frac{x_0^2+x_{23}^2}{R^2}\right)^{(1+\delta)}} \tag{D.0.258}$$

The log divergent contributions come from the two regions, when (i) $\vec{x}_2 \to \vec{x}_3$, (ii) $\vec{x}_2 \to 0$.

(i)$\vec{x}_2 \to \vec{x}_3$.

Set $\vec{y} = \vec{x}_2 - \vec{x}_3$. At $\vec{x}_2 = \vec{x}_3$, $\vec{y} = 0$.

$$\int \frac{d^2y}{R^2} \frac{1}{\left(\frac{x_0^2+(\vec{y}+\vec{x}_3)^2}{R^2}\right)^{(1+\delta)} \left(\frac{x_0^2+x_3^2}{R^2}\right)^{(1-\delta)} \left(\frac{x_0^2+y^2}{R^2}\right)^{(1+\delta)}} \tag{D.0.259}$$

We taylor expand the first term in the denominator. We get,

$$\int \frac{d^2y}{R^2} \frac{1-(1+\delta)\left(\frac{\vec{y}^2-2\vec{x}_3.\vec{y}}{x_0^2+\vec{x}_3^2}\right)}{\left(\frac{x_0^2+\vec{x}_3^2}{R^2}\right)^{(1+\delta)}\left(\frac{x_0^2+x_3^2}{R^2}\right)^{(1-\delta)}\left(\frac{x_0^2+y^2}{R^2}\right)^{(1+\delta)}} \tag{D.0.260}$$

We drop the δ term. It is higher order. We look at,

$$\int \frac{d^2y}{R^2} \frac{-\left(\frac{\vec{y}^2-2\vec{x}_3.\vec{y}}{x_0^2+\vec{x}_3^2}\right)}{\left(\frac{x_0^2+\vec{x}_3^2}{R^2}\right)\left(\frac{x_0^2+x_3^2}{R^2}\right)\left(\frac{x_0^2+y^2}{R^2}\right)} \tag{D.0.261}$$

Add and subtract x_0^2.

$$\int \frac{d^2y}{R^2} \frac{-\left(\frac{x_0^2+\vec{y}^2-x_0^2-2\vec{x}_3.\vec{y}}{x_0^2+\vec{x}_3^2}\right)}{\left(\frac{x_0^2+\vec{x}_3^2}{R^2}\right)\left(\frac{x_0^2+x_3^2}{R^2}\right)\left(\frac{x_0^2+y^2}{R^2}\right)} \tag{D.0.262}$$

The $x_0^2+\vec{y}^2$ term cancels in the numerator and the denominator. We drop that.

$$\int \frac{d^2y}{R^2} \frac{-\left(\frac{-x_0^2-2\vec{x}_3.\vec{y}}{x_0^2+\vec{x}_3^2}\right)}{\left(\frac{x_0^2+\vec{x}_3^2}{R^2}\right)\left(\frac{x_0^2+x_3^2}{R^2}\right)\left(\frac{x_0^2+y^2}{R^2}\right)} \tag{D.0.263}$$

$$\int d^2y(-2\vec{x}_3.\vec{y}) = -2\int_0^{2\pi} ydyd\theta_{3y}x_3y\cos\theta_{3y} = 0 \tag{D.0.264}$$

Therefore we drop this term. We get

$$\int \frac{d^2y}{R^2} \frac{-\left(\frac{-x_0^2}{x_0^2+\vec{x}_3^2}\right)}{\left(\frac{x_0^2+\vec{x}_3^2}{R^2}\right)\left(\frac{x_0^2+x_3^2}{R^2}\right)\left(\frac{x_0^2+y^2}{R^2}\right)} \tag{D.0.265}$$

which in the limit $x_0 \to 0$ goes to zero.

We look at

$$\left(\frac{x_0^2+y^2}{R^2}\right)^{(1+\delta)} = \left(\frac{x_0^2+y^2}{R^2}\right)\left(1+\delta\log\left(\frac{x_0^2+y^2}{R^2}\right)\right) \tag{D.0.266}$$

in the denominator. Again drop the δ term. Set $\vec{x}_3 = \vec{R}$,

$$\int \frac{d^2y}{R^2} \frac{1}{\left(\frac{x_0^2+R^2}{R^2}\right)^{(1+\delta)}\left(\frac{x_0^2+R^2}{R^2}\right)^{(1-\delta)}\left(\frac{x_0^2+y^2}{R^2}\right)^{(1+\delta)}} \tag{D.0.267}$$

The log divergent part is

$$= \pi \int_{x_0^2}^{R^2} \frac{dy^2}{R^2} \frac{1}{\left(\frac{x_0^2+y^2}{R^2}\right)^{(1+\delta)}} \tag{D.0.268}$$

$$= \pi \log\frac{R^2}{x_0^2} \tag{D.0.269}$$

A similar computation for $\vec{x}_2 \to 0$ gives an identical contribution. The total contribution from both regions is,

$$= 2\pi \log \frac{R^2}{x_0^2} \tag{D.0.270}$$

The partition function becomes

$$S_3 = -\frac{1}{2} 4\lambda_3 \pi \phi_0 \gamma_0 \chi_0 \int d\left(\frac{x_1^2}{R^2}\right) d\left(\frac{x_3^2}{R^2}\right) \log \frac{R}{x_0} \tag{D.0.271}$$

This expression corrects b_1 when $x_2 \to x_3$ and b_3 when $x_2 \to x_1$. These are both equal in magnitude. We only want the correction to b_1, therefore we divide the expression above by 2 to get the contribution of the generating functional to the beta function for $\cos b_1 X$.

$$S_3 = -\lambda_3 \pi \phi_0 \gamma_0 \chi_0 \int d\left(\frac{x_1^2}{R^2}\right) d\left(\frac{x_3^2}{R^2}\right) \log \frac{R}{x_0} \tag{D.0.272}$$

Appendix E Calculation for β_δ.

We start with

$$S_{\sigma 3} = -\frac{1}{2}\lambda_\sigma \int d^{d+1}y\sqrt{g}\ \bar{g}^{\mu\nu}\partial_\mu\phi(y)\ \partial_\nu\phi(y)\ \sigma(y) \tag{E.0.273}$$

$$= -\frac{1}{2}\lambda_\sigma\sigma_0\phi_0^2 C_{\Delta_1}C_{\Delta_2}C_{\Delta_3}\int d^{d+1}y\sqrt{g}\ y_0^2\ \partial_\mu\left(\frac{y_0}{(y_0^2+(\vec{y}-\vec{x_1})^2)}\right)^{\Delta_1} \tag{E.0.274}$$

$$\partial^\mu\left(\frac{y_0}{(y_0^2+(\vec{y}-\vec{x_2})^2)}\right)^{\Delta_2}\ \left(\frac{y_0}{(y_0^2+(\vec{y}-\vec{x_3})^2)}\right)^{\Delta_3}$$

Set $\vec{x}_1 = 0$. Under inversion [27],

$$\frac{y_0}{y_0^2+(\vec{x}-\vec{y})^2} \rightarrow \quad \vec{x}'^2\frac{y_0'}{y_0'^2+(\vec{x'}-\vec{y'})^2} \tag{E.0.275}$$

$\partial'^\mu {y_0'}^\Delta = \partial'^0 {y_0'}^\Delta = \Delta {y_0'}^{\Delta-1}$.

$$\partial'^{\mu=0}\left(\frac{y_0'}{({y_0'}^2+(\vec{y'}-\vec{x_i'})^2)}\right)^{\Delta_i} = \left(\frac{\Delta_i {y_0'}^{\Delta_i-1}}{({y_0'}^2+(\vec{y'}-\vec{x_i'})^2)^{\Delta_i}}\right) - \left(\frac{\Delta_i {y_0'}^{\Delta_i}\ 2y_0}{({y_0'}^2+(\vec{y'}-\vec{x_i'})^2)^{\Delta_i+1}}\right)$$

$$S_{\sigma 3} = -\frac{1}{2}\lambda_\sigma \sigma_0 \phi_0^2 C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \Delta_1 \Delta_2 {x'}_2^{2\Delta_2} {x'}_3^{2\Delta_3} \int d^{d+1}y' \; {y'_0}^{-(d+1)+2+\Delta_1-1+\Delta_3+\Delta_2-1} \tag{E.0.276}$$

$$\left(\frac{1}{({y'_0}^2 + (\vec{y'} - \vec{x'_3})^2)}\right)^{\Delta_3} \left(\frac{1}{({y'_0}^2 + (\vec{y'} - \vec{x'_2})^2)^{\Delta_2}}\right)$$
$$+\frac{1}{2}\lambda_\sigma \sigma_0 \phi_0^2 C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \Delta_1 \Delta_2 {x'}_2^{2\Delta_2} {x'}_3^{2\Delta_3} \int d^{d+1}y' \; {y'_0}^{-(d+1)+2+\Delta_1-1+\Delta_3+\Delta_2+1}$$
$$\left(\frac{1}{({y'_0}^2 + (\vec{y'} - \vec{x'_3})^2)}\right)^{\Delta_3} \left(\frac{2}{({y'_0}^2 + (\vec{y'} - \vec{x'_2})^2)^{\Delta_2+1}}\right)$$

Thus, setting $\vec{x}_1$ to zero and using inversion we have reduced the number of factors in the denominator from 3 to 2. This simplifies Feynman parameter integrals significantly. Now we Feynman parameterize and do y' integrals. Doing the integrals and dropping pre-factors we get,

$$\frac{\pi}{2}\frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d}{2})}{\Gamma(\Delta_3+\Delta_2)}\frac{\Gamma(\Delta_3+\Delta_2)}{\Gamma(\Delta_2)\Gamma(\Delta_3)} \tag{E.0.277}$$
$$\int d\alpha_3 \; d\alpha_2 \; \delta(\alpha_3+\alpha_2-1)\alpha_3^{\Delta_3-1}\alpha_2^{\Delta_2-1}\frac{1}{\left(\alpha_2\alpha_3 \vec{x'}_{23}^2\right)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}$$
$$-2\frac{\pi}{2}\frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d+2}{2})}{\Gamma(\Delta_3+\Delta_2+1)}\frac{\Gamma(\Delta_3+\Delta_2+1)}{\Gamma(\Delta_2)\Gamma(\Delta_3+1)}$$
$$\int d\alpha_3 \; d\alpha_2 \; \delta(\alpha_3+\alpha_2-1)\alpha_3^{\Delta_3}\alpha_2^{\Delta_2-1}\frac{1}{\left(\alpha_2\alpha_3 \vec{x'}_{23}^2\right)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}$$

The α integrals are:

The first,

$$\int d\alpha_3 \; d\alpha_2 \; \delta(\alpha_3+\alpha_2-1)\alpha_3^{\Delta_3-1}\alpha_2^{\Delta_2-1}\frac{1}{(\alpha_2\alpha_3)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}} = 1 \qquad \text{(E.0.278)}$$

The second,

$$\int d\alpha_3 \; d\alpha_2 \; \delta(\alpha_3+\alpha_2-1)\alpha^{\Delta_3}\alpha^{\Delta_2-1}\frac{1}{(\alpha_2\alpha_3)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}} = 1/2 \qquad \text{(E.0.279)}$$

Use $(\vec{x'}^{2\Delta} = 1/\vec{x}^{2\Delta})$ and
$(\vec{x'}-\vec{y'})^{2(\Delta_1+\Delta_2-\Delta_3)/2} = (\vec{x}-\vec{y})^{2(\Delta_1+\Delta_2-\Delta_3)/2}/(\vec{x}^{2(\Delta_1+\Delta_2-\Delta_3)/2}\vec{y}^{2(\Delta_1+\Delta_2-\Delta_3)/2})$.

$$\begin{aligned} S_{\sigma 3} = &-\frac{\pi}{4}\lambda_\sigma\sigma_0\phi_0^2 C_{\Delta_1}C_{\Delta_2}C_{\Delta_3}\Delta_1\Delta_2\int d^2x_1 d^2x_2 d^2x_3 \qquad \text{(E.0.280)} \\ &\frac{1}{(\vec{x}_{23}^2)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}\frac{1}{(x_2^2)^{\frac{\Delta_2-\Delta_3+\Delta_1}{2}}}\frac{1}{(x_3^2)^{\frac{-\Delta_2+\Delta_3+\Delta_1}{2}}} \\ &\left[\frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d}{2})}{\Gamma(\Delta_2)\Gamma(\Delta_3)} - \frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d+2}{2})}{\Gamma(\Delta_2)\Gamma(\Delta_3+1)}\right] \end{aligned}$$

Where now we have explicitly written integrals over the boundary coordinates(which were suppressed earlier). We insert a UV cutoff x_0^2 as before and multiply and divide by powers of R^2, we get

$$S_{\sigma 3} = -\frac{\pi}{4}\lambda_\sigma \sigma_0 \phi_0^2 C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \Delta_1 \Delta_2 \int d\left(\frac{x_1^2}{R^2}\right) d\left(\frac{x_2^2}{R^2}\right) d\left(\frac{x_3^2}{R^2}\right) \tag{E.0.281}$$
$$\frac{1}{\left(\frac{x_0^2+\vec{x}_{23}^2}{R^2}\right)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}} \frac{1}{\left(\frac{x_0^2+\vec{x}_2^2}{R^2}\right)^{\frac{\Delta_2-\Delta_3+\Delta_1}{2}}} \frac{1}{\left(\frac{x_0^2+\vec{x}_3^2}{R^2}\right)^{\frac{-\Delta_2+\Delta_3+\Delta_1}{2}}}$$
$$\left[\frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d}{2})}{\Gamma(\Delta_2)\Gamma(\Delta_3)} - \frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d+2}{2})}{\Gamma(\Delta_2)\Gamma(\Delta_3+1)}\right]$$

$C_{\Delta_i} = 1/\pi$. The square bracket vanishes. This gets renormalized when we fix λ_σ. We have taken particle 2 offshell. Therefore $\Delta_2 = 2(1+\delta)$. $\Delta_1 = 2$.

We get the same x_2 integral as before (D.0.258). The contribution from the x_2 integral from before is

$$4\pi \log\frac{R}{x_0} \tag{E.0.282}$$

The contribution to the generating function is half of this as before.

$$2\pi \log\frac{R}{x_0} \tag{E.0.283}$$

Therefore $S_{\sigma 3}$ becomes

$$S_{\sigma 3} = -2\pi(1+\delta)\lambda_\sigma \sigma_0 \phi_0^2 \int d\left(\frac{x_1^2}{R^2}\right) d\left(\frac{x_3^2}{R^2}\right) \log\frac{R}{x_0} \tag{E.0.284}$$
$$\left[\frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d}{2})}{\Gamma(\Delta_2)\Gamma(\Delta_3)} - \frac{\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1-d+2}{2})}{\Gamma(\Delta_2)\Gamma(\Delta_3+1)}\right]$$

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