Thermodynamic corrections due to an invariant ultraviolet scale and its implications

By

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Dedicated To That Brahman
That Manifests In All
As Atman

॥ श्रीहरिः॥

ऋणस्वीकृतिः

प्रथमं तावत् अहं जगतः पितरौ मायाचैतन्य रूपिणौ उमामहेश्वरौ वन्दे यतः येषां एव परम कृपया इदं महद्कार्यः बहुलविघ्नान्तर गतमपि पूर्णतां गतः। तदनन्तरं अहं मम् मातापितृभ्यो - माता श्रीमति रेखा देवी तथा च श्री माताशंकर मिश्र इति नाम्ना पिता तेभ्यो प्रणमामि येषां कृपया,सहाय्येन,प्रोत्साहनेन एतत् महाप्रबन्धं मया रचितम्। न तु केवलं महाप्रबन्धरचनेषु अपि तु जीवनस्य सकल कार्येषु तेषां आशिषाः मे शं कुर्वन्ति इति आशासे।

अधुना तावत् अहं मम महाप्रबन्धस्य रचनायाः परमगुरुः श्रीः सिबाशीष घोषः तेभ्यो शतशः प्रणमामि, तेषां आभारः प्रस्तौमि। तदनन्तरं मम सखा, गुरुः एवं साहाय्यकारः श्री नितिनचन्द्र महोदयान् अहं स्मरामि यैः विना एतत् महानिबन्धं सिध्यतां कदापि नायात्। एतेषां उपकारान् स्मर्तुं अहं एकं श्लोकं तावत् प्रस्तौमि यथा-

असितगिरीसमं स्यात् कज्जलं सिन्धुपात्रे
सुरतरुवरशाखा लेखनीपत्रमूर्वी।
लिखामि यदि गृहीत्वा सर्वदा सर्वकालं
तदपि तवगुणानां हेगुरो पारं न यामि॥

तत् पश्चात् ममोज्ज्वल जीवनदीक्षाप्रदातृः श्री बालाकृष्णः शाह तथा च तेषां सहधर्मचारिणी अल्पा शाह, एवं च गुरुदेवः श्री संस्कृतानन्द महाभागान् अपि नम्रत्वेन स्मरामि, तेषां ऋणनिर्दिषं करोमि। यैः सुहृदवर्यैः एवं च कर्मकारिभिः अहं उपकृतः यथा च भ्राताः श्री अंकितः, अनिल शुक्लः, विवेक व्यासः, प्रीतम सेनः, दीपांजनः, अरिंदमः, संजयः, साग्निकः, अर्नबः, मधुसूदनः, बालाकृष्णनः, प्रतीकः, आनन्दः, प्रसन्नाः, प्रफुल्लः ओक, प्रफुल्लः कुमारः, रथुलः, स्वरूपः,प्रोसेनजीतः, प्रशान्तः, सौरवः, अमितः, मृगेन्द्रः, भार्गवः, राघवेन्द्रः, विनयः, सेबास्तिअनः, कमलः त्रिपाठी, अरविन्दः, श्रीनिवासनः, भगिन्यः ममता, जिल्मी, ऐन्वी, अंकिता एवं अन्ये मम सहाय्यकराः तान् सर्वान् सहृदयेन स्मरामि।

तत्पश्चात् ये जनाः पुणे विश्वविद्यालये, टाटा मूलभूत शोध संस्थान एवं राष्ट्रिय प्रौद्योगिकी संस्थान जमशेदपुर इत्यादयः संस्थायां कार्यरताः सन्ति तेभ्यो अपि स्मरामि तावत् येषां विना भारतीय संस्कृतेः अध्ययनं अतीव दुष्करं अभवत् तावत् ममकृते।

गणितीय विज्ञान संस्थान नाम्नी अस्मिन् संस्थायां ये कार्यकराः सन्ति एवं च सर्वे प्राध्यापकाः, छात्राः, सुरक्षाकर्मिणः एवं भोजनालये संचालकाः सन्ति यैः अस्मिन् महति

कार्ये माम् उपकृतवन्तः तान् सर्वान् मनसा शतशः धन्यवादान् प्रददामि एवं च तेषां ऋणी भवामि इति शम्। एतत् ऋणस्वीकृतिः लेखः आचार्य श्रीनीलाम्बर देवता महोदयेन सानुग्रहेण स्वल्पकालेनैव सम्यक् अनुवादितं तेभ्योऽपि प्रणमाम्यहम् ।

॥ ॐ शान्तिः शान्तिः शान्तिः॥

ABSTRACT

The thesis mainly deals with the effect of the invariant ultraviolet scale on the thermodynamics of the ideal gases such as photon and degenerate Fermi gas. Incorporating another invariant scale say κ in the known relativistic theory gives us an ultraviolet cut-off on energy/momentum of the particle and a modified dispersion relation. When such an effective theory was used to study the equilibrium properties of the photon gas we found the various thermodynamic quantities to be lower than the usual SR value. We get the modified Stefan-Boltzmann law and the equation of state. We have also considered the most general modification in the phase-space at Planck scale where new exotic spacetime is expected to appear. This leads to the modified Planck's energy density distribution and modified Wien's law. As expected, from the correspondence principle, we get the usual results in the SR limit. Very peculiarly we noted that the results are nonperturbative in the SR limit, which seems to be the general feature of the theories with an ultraviolet cut-off. As possible implications of the obtained results we have applied it near Big Bang and found the modified age of the Universe. We have also suggested a table top experiment to test the results at low energy scales. We looked into the low and high temperature limit of all the results as well. Next we studied the thermodynamics of the degenerate Fermi gas and obtained the modified degenerate pressure and energy. We have comparatively studied the mass and number density dependence of the degenerate pressure. Surprisingly, we found the results to be perturbative in the SR limit as opposed to the expectation. We have then studied the example of a white dwarf star and obtained the mass-radius relation and the modified Chandrasekhar limit, which gets a positive correction. The modified luminosity of the star has also been explored in detail using the radiative envelope and in both the relativistic and non-relativistic cases. We noted that since the correction in pressure is negative for a given mass and temperature and so is the correction in the luminosity as well. The correction to the luminosity of a white dwarf is nonperturbative in SR limit as expected because of the presence of an ultraviolet energy cut-off.

Contents

Synopsis

In this thesis, we study the effective theory formed out of the modification of the known relativistic theory which incorporates another invariant ultraviolet scale also known as Doubly/Deformed Special Relativity or DSR. The first chapter consists of the Introduction. In chapter 2 we look into the modified thermodynamics of Ideal photon gases in such a deformed relativistic theory and its possible realization at low energy and near Big Bang. In Chapter 3 we explore the modification in the thermodynamics of the degenerate ideal Fermi gas and as an example we look into the modification in the dynamics of the white dwarf stars. In Chapter 4 we conclude the whole thesis and give future directions.

0.1 Deformed/Doubly Special Relativity: An Effective Theory

Almost all the theories in physics are effective theories i.e, they are valid till a certain scale. This scale may be a length scale or conversely an energy scale. There are many ways to see the effects of an effective theory. One of the ways to realise it is by putting an ultraviolet cut-off on the energy of a relativistic particle. This cut-off acting as an effective scale has to be invariant as different inertial observers must observe the same effective scale till which the theory is valid. This leads to the modification of the known relativistic theory. Now, the question that naturally one encounters is that how one goes about modifying the relativistic theory and what might be the possible implications?

Although as stated before the formalism is valid at any scale provided that scale is invariant. Earlier motivation of such a modification appeared as the range of theories called Doubly/Deformed Special Relativity or DSR [1] [2] [3] [4] [5]. The motivation of DSR theories is also derived from the observation of interesting effects such as deformation of dispersion relation etc. at very high energy scales [1,6–9]. The introduction of an observer independent energy scale in DSR formulation, say κ, leads to such a modification in the dispersion relation of a free particle [1–4]. The energy threshold also acts as a cut-off on the highest possible energy value in the physical (sub-Planckian) world [4, 10].

All the candidate Quantum Gravity (QG) theories encounter a natural scale where the effects of the theory become visible. This is the Planck scale usually donated by Planck energy ($E_p = \kappa$) or Planck length (L_p). This scale acts as a threshold where a new description of spacetime is expected to appear. But the existence of such a, let's say, length scale is in conflict with the equivalence principle as different observers will see different scales i.e they won't agree on the same L_p. It is possible to have the equivalence principle by deforming special relativity (SR) in order to incorporate this threshold as an invariant quantity under a relativistic transformation. In such a theory apart from the constancy of the speed-of-light , the Planck length, equivalently, Planck energy is also constant under coordinate transformation from one inertial frame to another. The modification puts an ultraviolet cut-off on the energy of the particle and gives us a modified dispersion relation. Such modifications in the SR will have implications in the known physics. But all these modifications are only the remnants of the Quantum Gravity theory. Therefore in the limit $\kappa \to \infty$, as expected, we get the usual SR.

We can achieve the above condition by modifying the Poincare algebra. There are many ways to modify the algebra. One possible way is to modify the generators in-order to incorporate the scale κ as an invariant scale. Various ways of modifying the generators leading to modified Poincare algebra are the known as different bases. Such a modification in the algebra is also known as κ-Poincare algebra [11–16]. This modification

naturally happens in the momentum space, so we will only consider that. Although the whole phase-space modification is still a topic under study. The modification we will particularly be considering was the one suggested by Magueijo and Smolin (MS) [3] [4], where we keep the rotation generator as it is but modify the boost generator. Such a modification keeps the Lorentz sector of the algebra unmodified but modifies the action on the 4-momentum part of the Poincare algebra i.e, we get modification in the action of the Lorentz group on the momentum space representation. Preserving the Lorentz group/algebra keeps the theory simple and intuitive.

The introduction of an observer independent energy scale κ leads to a modification in the dispersion relation of a free particle and gives a modified invariant mass. The energy threshold also acts as a cut-off on the highest possible energy value in the physical (sub-Planckian) regime. Similar cut-offs appearing in other candidate quantum gravity theories like noncommutative geometry, string theory, loop quantum gravity and GUP (Generalised Uncertainty Principle) etc [2, 17–28]. This sub-Planckian regime which is characterised by the energy $E \leq \kappa$ and the momentum $p \leq \kappa$ is the result of the choice of the U-map (this is a map between the standard Lorentz generators and the modified ones resulting in the modification of the Poincare algebra keeping the Lorentz sector intact). This choice of U-map is in sync with the expectation of the emergence of the granular structure of spacetime at Planck scale.

Such a theory can also be realised as curved momentum Ads and/or ds-space with curvature given as κ.

With such a modified relativistic theory, the natural question one asks is about its possible implications on the known physics. Although various implications of DSR has been extensively studied in the literature, the thermodynamics of the relativistic particles is a special concern that one may look into. There are earlier studies of the thermodynamics and its implication on compact objects in literature [10, 12, 29–34].

This thesis arises exactly in the above mentioned contexts and comprises of two parts: the

first part deals with the corrections to the thermodynamics of photons and its implications and the second part discusses the correction to the thermodynamics of the degenerate Fermi gas, with white dwarf as a toy example. In the subsequent discussions, we will concentrate only on such modifications and its implications.

0.2 Correction to Photon thermodynamics and its implications

One of the easiest thing one may look into is the modification or correction to the massless particle i.e, photons. Since these are massless the modification appears only as ultraviolet cut-off as the dispersion relation is the same as the usual SR.

0.2.1 Modified ideal photon gas thermodynamics

We started by considering a model of an ideal photon gas obeying Bose-Einstein statistics in grand canonical ensemble and went on calculating various thermodynamic quantities such as energy density, pressure, entropy, specific heat and equilibrium number of photons with such an ultraviolet cut-off. We found one to one correspondence between the behaviour of photons with an ultraviolet cut-off and the acoustic phonons in the Debye theory. The Stefan-Boltzmann law got modified which will give correction to the dynamics of many stellar objects such as Luminosity etc. We found that the non-perturbative nature of the thermodynamic quantities in the SR limit is a general feature of the theory with an ultraviolet energy cut-off. We also noted that the values of all the thermodynamic quantities are less than the SR values because of this cut-off. The leading behaviour for $T \to 0$ and $T \to \kappa$ have been analysed. We start getting the deviation of the results obtained in the modified case from the SR case at $\frac{T}{\kappa} \sim 10^{-1}$.

Note that all the thermodynamic quantities reduce to the usual SR result in $\kappa \to \infty$ limit

in accordance with the correspondence principle.

0.2.2 Modified Measure and Photon gas thermodynamics

Another aspect of such Planck scale modification is that we also expect exotic spacetime to emerge at Planck scale. This will reflect as the modification of the phase-space measure at such scale.To list a few examples where similar modification appears, we note that in noncommutative physics, which is one of the quantum gravity candidate, the change in phase space appears as a change in the density of states as discussed in [35]. Another candidate of the quantum gravity theories, namely Loop Quantum Gravity also predicts the change in phase space measure/density of states at very high energies [36–39]. We have considered the most general modification of the phase-space by assuming the spacetime to be isotropic and to be Taylor series expandable in the powers of $\left(\frac{1}{r\kappa}\right)$ and $\left(\frac{\epsilon}{\kappa}\right)$.

We first considered the example of classical ideal gas for illustration. We found that the classical ideal gas in case of modified phase space measure has a non-trivial volume dependence in its expression for the partition function leading to the modification in the thermodynamic quantities like pressure accordingly. We went on calculating the possible change in the thermodynamic quantities due to the change in the phase space measure. Because of this modification, Planck's energy density distribution and the Wien's displacement law got modified. We observed that in the case of modified phase space measure the values of the thermodynamic quantities might be less than, equal to or greater than the SR values depending on the choice of expansion parameter $a_{n,n'}$.

We found in case of modified Wien's law that λ_{max} is a monotonically decreasing function of T. Also that λ_{max} for modified phase space measure decreases more rapidly with increasing T than the case of unmodified measure. The significant change in the values occur only if the order of the change in temperature is non-negligible with respect to κ. That is why in SR limit gives the standard Wien's displacement law.

Note that again that all the thermodynamic quantities reduce to the usual SR result in $\kappa \to \infty$ *limit and with* $a_{n,n'} = 1$.

0.2.3 Effects in Big Bang and astrophysics and cosmology

The Friedmann equations (more specifically FRW equations) and its relation is standard and well studied cosmology subject [40]. We considered the radiation dominated epoch where we have the modified energy momentum tensor $T_{\mu\nu}$, this leads to the modified energy conservation equation and hence we get modified Hubble parameter H, which characterises the rate of expansion of the Universe.

We found that $\frac{H}{H_{SR}} < 1$ always, which implies that the expansion of the Universe was at a slower rate in the radiation dominated era than the rate of expansion without such modifications. Because of the slower expansion, all the epoch would eventually get delayed resulting in the modification in the age of the known Universe.

Next we may consider such photons at an effective lower scale due to other parameters in the theory like mass, number density etc. These parameters may effectively lower the Planck scale such that its effects can be observed in very high temperature and high density regimes. To probe such effects we need to observe the stellar objects with very high temperatures and densities. For example, the astronomical data from gamma ray burst during the merging of neutron stars (which has the core temperature of $T = 10^{12}$ K) may give a bound on effective κ value. We can also explore the Chandrasekhar limits and its possible modifications which we will discuss later. It will also be interesting to see if one gets a better bound on κ in case of luminosity calculation of neutron stars using the results obtained for the blackbody radiation.

Another aspect is the possible application of this theory in bouncing loop cosmology. In “bouncing" loop quantum cosmology theories (see [41] [42] [43], we normally consider specific modifications to the spacetime geometry which effectively puts a bound on the

curvature and in this way the Big Bang singularity can be avoided. But for such “bouncing" models, we cannot use the perturbation technique at the curvature saturation, as the energy density of the cosmic fluid diverges. What one can do to still avoid the Big Bang singularity is to consider an inflation model where we can safely use the perturbation theory. We have obtained the energy density of the cosmic fluid which saturates to the Planck energy, which of course is finite. Then we may combine both the results obtained and the “bouncing" loop quantum cosmology to study the possible way out to avoid the Big Bang singularity.

In the SR limit the modified Hubble parameter becomes nearly equal to the normal SR one, as κ is very large, so the correction terms go to zero as expected. Note that the modification in the age of the Universe and possibility of using the results obtained by us to avoid the Big Bang singularity.

0.3 Correction to Degenerate Fermi gas and White Dwarf Dynamics

Since the thermodynamics of classical ideal gas and the gas of ideal photons, in such a formalism, have also been studied in detail, studying Fermi gas becomes the next immediate thing.

We have a modified dispersion relation (MDR) and a cut-off on the highest single particle energy/momentum, this in effect will give a correction to the thermodynamics of the degenerate Fermi gas which becomes nontrivial. As we have a new invariant mass, the study of thermodynamics for massive particles gets separated in three different cases $m < \kappa$, $m = \kappa$ and $m > \kappa$. The modified the degenerate pressure and energy is calculated in all the three cases. We can also make a contour plot of the pressure and energy with the parameters mass and number density and κ to see the possible sectors of the observable

effects of such modification. The degenerate pressure comes out to be perturbative in the SR limit which is unexpected result for such theories.

As is well known, the model of degenerate Fermi gas is used to study the dynamics of many compact stars such as white dwarf stars, neutron stars etc. We will consider a simple model of white dwarf stars to study the possible implications of the obtained results. The correction to the Chandrasekhar limit for white dwarf in all the three cases is looked into. For the usual particle i.e, in case of $m < \kappa$, we found that the equilibrium degeneracy pressure is greater than, equal to and less than the SR value for particular masses of the considered compact object such as white dwarf. Also for given masses the value of the radius of the white dwarf is found to be less than, equal to and greater than the usual SR value. Since energy density is related directly to mass density so denser compact objects are expected to show better measurable correction due to such a modification. One of the major predictions of our theory is that it makes an attempt to explain the observed lower radius white dwarfs and also predicts the white dwarfs having radius greater than that predicted by the present SR theory. The correction is purely perturbative in the SR limit which is quite unusual for a theory having an ultraviolet energy cut-off. Therefore we conclude that this correction is solely because of the modified dispersion relation. In the other two cases we do not get any limit on the white dwarf mass as expected. Note that the presence of observed white dwarfs having radius lesser than the SR case may find an explanation if they are modelled using a modified dispersion relation. This result has also been found using the modified Lane-Emden equation assuming radial density distribution.

Amongst many results presented in blackbody radiation case, we note that the Stefan-Boltzmann law gets modified in such theories and this result can be used to study various stellar objects. We consider the model of degenerate core with a radiative envelope of non-degenerate matter and use it to calculate the luminosity of the star, which gets a negative and nonperturbative correction. We anticipate the nonperturbativity in SR limit due to the

presence of ultraviolet cut-off.

Note that the modified Chandrasekhar limit and Lane-Emden equations are the important results obtained along with possibility of the model explaining the anomalous white dwarf behaviour.

List of Figures

1 Introduction

Almost all the theories in physics are effective theories i.e, they are valid till a certain scale. This scale may be a length scale or conversely an energy scale. There are many ways to see the possible effects of such an effective theory. One of the ways to realize it is by putting an ultraviolet cut-off on the energy of a relativistic particle. This cut-off acting as an effective scale has to be invariant, as different inertial observers must observe the same effective scale till which the theory is valid. This leads to the modification of the known relativistic theory. Now, the question that naturally one encounters is that how one goes about modifying the relativistic theory and what might be the possible implications? This requires us to rewrite the known relativistic theory keeping in mind that we should be able to retrieve the original relativistic theory in some limit.

1.1 Doubly/Deformed Special Relativity

One such method of the modification of the relativistic theory was first proposed by G. Amelino-Camelia [1, 2] and was later studied and extended by Lee Smolin and Glikman and others. The proposition is simply to modify the known relativistic algebra i.e, the Poincare algebra, in order to incorporate along with the speed of light c another invariant scale namely κ. When this scale is Planck scale then such theories are called Doubly/Deformed Special Relativity (DSR). The general modification of this kind is called κ-Poincare algebra and which is related to the well studied subject in mathematics called

Quantum Groups. In next section we will see the motivation behind the DSR theories.

1.1.1 Motivation behind DSR theories

Almost all attempts to combine gravity with quantum mechanics i.e, in all candidate quantum gravity (QG) theories a natural scale emerges and that is the Planck length $\left(l_P = \sqrt{\frac{Gc}{\hbar^3}}\right)$ which acts as the scale at which the quantum effects of gravity show up [20, 22, 44–52].This scale acts as a threshold where a new description of spacetime is expected to appear. Since it being a length scale, the existence of such a scale is in conflict with the equivalence principle as different observers in different inertial frames would not agree on the same l_P due to the Lorentz-Fitzgerald contraction. Another way to understand the same is that if the Planck scale is to play the role of the threshold for the discrete spacetime or the scale where QG effects show up, it may arise in the quantities involving the metric (or for that matter the action). Now for physics at that scale to be independent of the inertial frames the action and so the Planck scale has invariant under the relativistic transformation.

It has been shown that it is possible to hold the equivalence principle by deforming or modifying the known Special Relativity (SR). As stated before, such classes of theories fall under the name Doubly/Deformed Special Relativity [2, 5, 53, 54]. Since DSR is a more general theory therefore, according to the correspondence principle, in some limit it must reduce to the usual SR. Such a limit, also known as the special relativistic limit is when Planck length/energy $\rightarrow 0/\infty$ and the symmetry group goes to usual the Lorentz group.

The motivation of DSR theories is also derived from the observation of interesting effects such as deformation of dispersion relation etc. at very high energy scales [1, 6–9]. Such a modification in the dispersion relation is one of the features of the DSR.

There are various ways to modify the algebra, in the next section we will how to rewrite

relativity and see some of the consequences of modified algebra.

1.1.2 Rewriting Special Relativity and the Modified/Deformed Algebra

With enough motivation of introducing another invariant scale κ, next question is about the mathematical structure. As stated before the Poincare algebra is no longer valid but is modified to κ-Poincare algebra with deformation parameter κ as an invariant scale. Such algebra is more generally called quantum Hopf algebras. DSR has also been explored from the point of view of $\kappa-$Poincare algebra by many (see for example [11–16]). Briefly, given a Lie algebra we can construct what is known as quantum deformed algebra. For example, given Lie algebra $SO(3,1)$ we have deformed algebra $SO_q(3,1)$, where q is the deformation parameter, whose particular limit gives the usual undeformed algebra. The general procedure in the usual Lie algebra is to define the quantities called the generator of the algebra and write the commutation relation of the generators, whose right hand side is some linear combination of the generators and is called the basis. For quantum deformed algebra the right hand side of the commutation relation are some analytic function of the generators and therefore we can use any analytic function of the generators to define a new basis of the quantum deformed algebra. These different choice of bases has given rise to different DSR theories such as Bi-crossproduct basis, the Magueijo Smolin (MS) basis and the classical basis. In all these constructions the common feature is that the Lorentz sector of the κ-Poincare algebra remains unchanged and only Poincare sector gets deformed.

Another question naturally arises is why to deform only the Poincare part not the whole algebra including the Lorentz part as well. The first and the foremost reason is that it keep the theory simple and intuitive. Also, if the generators of boosts and rotations are interpreted physically in such a way that they give us the rules to transform measurements made by different inertial observers into each other, then they must exponentiate to a

group as the group properties follow directly from the physical principle of equivalence of inertial frames. But if we deform the Lorentz sector as well, then the exponentiation would not give us a group but a quasi-group i,e, some of the group properties may be constrained (see [55] and [2]).

Of the various formulations corresponding to different deformations of the algebra, for this thesis we will focus on the modification suggested by Magueijo and Smolin (MS basis) [3,4], the corresponding deformations and its possible implications. In order to obtain the necessary modification let us first look at the set of principles required to develop such a theory. As described in [4], we will rewrite the usual SR principles to get the DSR principles by adding two more, as follows :

1. **The Relativity of the Inertial Frame :** Ignoring the gravitational effects, all the observers in free and inertial motion are equivalent. Therefore, there is no preferred state of motion and so the velocity is a purely relative quantity.

2. **The Equivalence Principle :** Under gravitational effect, freely falling observers are all equivalent to each other and to the inertial observers.

3. **The Observer Independence of the scale κ :** All observers agree on an invariant energy scale κ.

4. **The Correspondence Principle :** At energy scale which is much much smaller than κ, the usual SR is valid (in our case this limit is $\kappa \to \infty$).

So, according to the above principles the symmetry group is still Lorentz group but now it acts non-linearly on the momentum space. We want to achieve this such that it leaves κ invariant. As DSR naturally formulates in the momentum space, for this formulation we will work in momentum space only with the four momentum given as ($p_\alpha \equiv p_0 = \varepsilon; p_1 = p_x; p_2 = p_y; p_3 = p_z$). There have been some attempts to study the representation on the position space as well [56–59]. But it is non-trivial and open to debate as there is still no

agreement on its relation to the momentum space representation. In this thesis as usual, we have used the momentum space representation only. Unless stated explicitly we will use the metric signature $(+,-,-,-)$ and the Greek numerals take values $(0,1,2,3)$ while the Roman numerals take the values $(1,2,3)$. Equipped with the above understanding we will now look into the modified/deformed algebra.

The standard Lorentz generators $L_{\alpha\beta}$ are,

$$L_{\alpha\beta} = p_\alpha \frac{\partial}{\partial p_\beta} - p_\beta \frac{\partial}{\partial p_\alpha} \tag{1.1}$$

As is obvious from the transition from Galilean to Lorentz, the trick is to keep the rotation generators $M_i \equiv \epsilon_{ijk} L_{ij}$ the same, but modify the boost term by adding a dilatation term as,

$$N^i = L_0^i + \frac{p^i D}{\kappa} \tag{1.2}$$

where D is the dilatation generator $D = p_\alpha \frac{\partial}{\partial p_\alpha}$ and satisfying $D\, o p_\alpha = p_\alpha$. These modified generators give the usual Lie algebra for the Lorentz sector,

$$[M^i, M^j] = i\,\epsilon^{ijk} M_k, \quad [M^i, N^j] = i\,\epsilon^{ijk} N_k, \quad [N^i, N^j] = i\,\epsilon^{ijk} M_k. \tag{1.3}$$

The action on the momentum generators p_α, which gives the rest of the Poincare algebra, gets modified to,

$$\left[N^i, p^j\right] = i\left(\delta^{ij} p^0 - \frac{1}{\kappa} p^i p^j\right), \tag{1.4}$$

and

$$\left[N^i, p^0\right] = i\left(1 - \frac{p^0}{\kappa}\right) p^i. \tag{1.5}$$

Note that other commutators are trivially satisfied. One easily obtains the quadratic

Casimir for this algebra and that has the following form

$$m^2 = \frac{p_0^2 - \vec{p}^2}{\left(1 - \frac{p_0}{\kappa}\right)^2}, \tag{1.6}$$

which in turn leads to the modified dispersion relation

$$\varepsilon^2 - p^2 = m^2 \left(1 - \frac{\varepsilon}{\kappa}\right)^2. \tag{1.7}$$

As speed of light c was the cut-off on the maximum speed of the particle in physical regime when we go from Galilean to SR. Here also κ acts as cut-off on the maximum energy/momentum of the relativistic particle in DSR i.e, the physical world lives in 0 to κ or the sub-Planck regime. Therefore, the main features of the DSR theories that emerge out of the above formulation is the presence of ultraviolet cut-off on energy/momentum and the modified dispersion relation (MDR). In the next section we will look into the details of the non-linear nature of the modified transformations.

1.1.3 Non-linear Transformation Laws

The immediate question that comes to ones mind is how do the modified Lorentz transformations look like corresponding to the above defined modified algebra. We take the cue from the standard SR formulation which marks the transition from Galilean to Lorentz transformations. The Galilean transformations are linear in velocity and therefore do not keep any velocity scale as invariant. In order to have an invariant velocity scale we must use the Lorentz transformations which are nonlinear in velocity. Thus, if we want to make Planck length/energy scale as an invariant scale then the relativistic transformation has to be nonlinear in position/momentum space. The symmetry group must act nonlinearly in the position/momentum space representation to keep Planck length/energy invariant. As stated before J. Magueijo and L. Smolin have argued that it is possible to keep the symmetry group for the DSR transformations to be the Lorentz group [3, 4], but will act

nonlinearly on the position/momentum space to give us another invariant scale κ. They defined an energy dependent U-map as a map between the standard Lorentz generators and the modified ones resulting in the modification of the usual Poincare algebra keeping the Lorentz sector intact.

For example, the modified boost generator is given as

$$N^i = U^{-1}(\varepsilon)L_0^i U(\varepsilon) \tag{1.8}$$

where $U(\varepsilon) = exp(\frac{\varepsilon D}{\kappa})$ and D is the dilatation operator defined before. The nonlinear representation is generated by $U(\varepsilon)$ such that,

$$U(\varepsilon)o\vec{p} = \frac{\vec{p}}{\left(1 - \frac{\varepsilon}{\kappa}\right)} \tag{1.9}$$

Also, note that $U(\varepsilon)$ is not unitary and therefore there is no unitary equivalence also that it is singular at $\varepsilon = \kappa$, a signal of emergence of a new invariant scale κ. Therefore, the exponentiation now gives the nonlinear representation of the Lorentz transformation (in this case the transformation corresponds to the boost along z-axis) on momentum space as,

$$\Lambda(v) : \varepsilon \to \varepsilon' = \frac{\gamma(\varepsilon - vp_z)}{1 + (\gamma - 1)\frac{\varepsilon}{\kappa} - \gamma v\frac{p_z}{\kappa}} = \frac{\gamma(\varepsilon - vp_z)}{\alpha} \tag{1.10}$$

$$\Lambda(v) : p_z \to p'_z = \frac{\gamma(p_z - v\varepsilon)}{1 + (\gamma - 1)\frac{\varepsilon}{\kappa} - \gamma v\frac{p_z}{\kappa}} = \frac{\gamma(p_z - v\varepsilon)}{\alpha} \tag{1.11}$$

$$\Lambda(v) : p_x \to p'_x = \frac{p_x}{1 + (\gamma - 1)\frac{\varepsilon}{\kappa} - \gamma v\frac{p_z}{\kappa}} = \frac{p_x}{\alpha} \tag{1.12}$$

$$\Lambda(v) : p_y \to p'_y = \frac{p_y}{1 + (\gamma - 1)\frac{\varepsilon}{\kappa} - \gamma v\frac{p_z}{\kappa}} = \frac{p_y}{\alpha} \tag{1.13}$$

with $\gamma = \frac{1}{\sqrt{1-v^2}}$, $c = 1$ and $\alpha = 1 + (\gamma - 1)\frac{\varepsilon}{\kappa} - \gamma v\frac{p_z}{\kappa}$. Note that in compact form it looks very similar to the SR case. Another interesting fact is that they also satisfy the group

multiplication law of the usual Lorentz group, i.e.,

$$\Lambda(u)\Lambda(v) = \Lambda\left(\frac{u+v}{1+uv}\right) \tag{1.14}$$

With the above modified transformation laws it is easy to see that κ is preserved. If we boost in z-direction with velocity v then $(\kappa, 0, 0, 0)$ goes to $(\kappa, -v\kappa, 0, 0)$ and under boost in the photon's direction of motion we have $(\kappa, \kappa, 0, 0) \rightarrow (\kappa, \kappa, 0, 0)$. Since the modified Casimir diverges at $E = \kappa$ so $E = \kappa$ has to be invariant from the invariance of (1.6).

Another offshoot of this formalism is that the transformation gives us an *invariant mass* m,

$$m = \frac{\left(1 - \frac{\varepsilon}{\kappa}\right)}{\sqrt{\varepsilon^2 - p^2}} \tag{1.15}$$

This is not the same as the rest mass energy m_0 (which is obtained by putting $p = 0$ in the dispersion relation), but is related to the same by following transformation,

$$m_0 = \frac{m}{1 + \frac{m}{\kappa}} \qquad \Longrightarrow m = \frac{m_0}{1 - \frac{m_0}{\kappa}} \tag{1.16}$$

where, $0 \leq m_0 \leq \kappa$ and $0 \leq m \leq \infty$.

Therefore, the momentum space representation of the modified algebra is given by (1.10) – (1.13) which in turn leads to the modification in the dispersion relation discussed above and given by (1.7). The space of 3-momenta is no more the flat R^3 manifold, but is maximally symmetric space with constant curvature κ [60]. If the curvature of the space is κ then it is de Sitter space of momenta and when the curvature is $-\kappa$, we have Anti-de Sitter space.

This whole formalism, in turn, gives us three sections as in case of SR and the energy threshold κ acts as a cut-off on the highest possible energy value in the physical (sub-Planckian) world as stated before. This sub-Planckian regime which is characterized by the energy $E \leq \kappa$ and the momentum $p \leq \kappa$ is the result of the choice of the $U-$map.

This choice of $U-$map is in sync with the expectation of the emergence of the granular structure of spacetime at Planck scale.

Although in this thesis we have only explored the MS formalism of DSR. But, similar cut-offs in momentum and/or energy and MDR are seen in other DSR formalisms as well [11]. Also, DSR is not the only theory where such cut-off appearing, but we see similar cut-offs appearing in other candidate quantum gravity theories like noncommutative geometry, string theory, loop quantum gravity and GUP (Generalized Uncertainty Principle) etc. [2, 17–28]. The existence of such a cut-off is also suggested by black hole physics [50,61,62]. Similarly, the existence of MDR and its possible implications has been seen in many candidate QG theories as well, as in low energy relativistic theories [32, 63–72]. DSR formalism can also be extended to curved spacetime. One such extension was proposed by MS and has since then been studied from various perspectives [73–77].

1.1.4 Relation to Non-Commutative Geometry

As stated before the κ-Poincare quantum algebra is an algebra that describes only the energy-momentum sector of the DSR theory, there is yet no coherent formulation and interpretation of whole configuration space formulation of DSR. One of the many ways proposes that, in order to go from momentum space to configuration space we define co-algebra, which is an additional structure of the quantum Hopf algebra. The form of the commutator algebra on the whole of the phase space can be derived using the co-algebra. The co-algebra is dual to the algebra we derived in the previous section. J. Kowalski-Glikman in the article [59] gives the step by step construction of the whole phase space using the co-algebra. DSR can be promoted to the usual κ-Poincare quantum Hopf algebra by a homomorphism and we can then use the co-product of this algebra and the Heisenberg double construction of κ-deformed phase space to get the non-commutative spacetime structure and the whole phase-space of DSR. The obtained commutation rela-

tions are as follows,

$$[x^0, x^i] = -\frac{i}{\kappa}\, x^i, \quad [x^i, x^j] = 0 \tag{1.17}$$

$$[p^0, x^i] = -\frac{i}{\kappa} p^i, \quad [p^0, x^0] = i\left(1 - \frac{2p^0}{\kappa}\right), \quad [p^i, x^j] = -i\,\delta_{ij}, \quad [p^i, x^0] = -\frac{i}{\kappa}\, p^i. \tag{1.18}$$

Another fact to note is that this algebra satisfies the Jacobi identity. It is obvious from (1.17) that we have non-commutative spacetime and such a non-commutative space-time is called κ-Minkowski. This seems to be related to the theory with non-commutative spacetime which was proposed by Snyder [78, 79]. The authors in [80] discuss the canonical phase space approach to the deformed algebra. The relation of DSR with non-commutative geometry is still not fully formulated and is a topic under study.

1.2 Modified Thermodynamics of Ideal gases

The consequences of the modified dispersion relations and the ultraviolet cut-off on the thermodynamics are being studied extensively to infer the effect of Planck scale physics [29, 32, 81–85]. The effect of modified dispersion relations in loop-quantum-gravity on black hole thermodynamics was studied in [81]. As is true with many high energy theories such as Noncommutativity etc. which finds application at lower energy scales as well [86] [87] [88], the same is true with DSR. Although it was originally formulated to study the effects at the Planck scale but the formalism can be valid at any scale depending on the choice of the modification parameter κ. Choosing the scale κ gives us an effective theory and such scale has to be invariant, as all observers must agree on the same effective scale. This, in turn, means that the effects of such a modification can be observed not only at high energies but effectively at low energies as well (see for example [89]). DSR as a Lorentz violating phenomena on the thermodynamics of macroscopic systems (like white dwarfs) [32, 82, 83] and as a noncommutative phenomena on cosmology and astrophysical systems [84, 85] have also been studied. Moreover, photon gas thermodynamics in the context of modified dispersion relations [29] and DSR [34] are being investigated. In this

thesis, we study the photon gas thermodynamics and its implications in MS formalism. We will see its implications with regards to both high and low energy physics. We will also explore the effects of the modification of the phase-space at Planck scale on photon gas thermodynamics. Along with that we will also study the thermodynamics of degenerate Fermi gas and the effects in case of white dwarf stars as an example.

The organization of the thesis is as follows: In the second chapter we study various equilibrium thermodynamic properties of blackbody radiation (i.e. a photon gas) with an ultraviolet energy cut-off. We start with calculating various modified thermodynamic quantities. We show that the energy density, specific heat etc. follow usual acoustic phonon dynamics as have been well studied by Debye. Other thermodynamic quantities like pressure, entropy etc. have also been calculated. The find that the usual Stefan-Boltzmann law gets modified. We observe and note that the values of the thermodynamic quantities with the energy cut-off is lower than the corresponding values in the theory without any such scale. The phase-space measure is also expected to get modified for an exotic spacetime appearing at Planck scale, which in turn leads to the modification of Planck energy density distribution and the Wien's displacement law. We find that the non-perturbative nature of the thermodynamic quantities in the SR limit (for both the case with ultraviolet cut-off and the modified measure case), due to nonanalyticity of the leading term, is a general feature of the theory accompanied with an ultraviolet energy cut-off. We have also discussed the possible modification in the case of Big Bang and the Stellar objects. We discuss both the low and high temperature limit of the thermodynamic quantities and have suggested a table top experiment for verification in effective low energy case.

The third chapter discusses the invariant ultraviolet scale correction to the thermodynamics of the ideal Fermi gas. We have considered the modified dispersion relation and the cut-off to the maximum possible momentum/energy (Planck energy) of the non-interacting ideal degenerate Fermi gas particles. With such a modification the expression for the degenerate pressure and the total energy gets modified accordingly. We discuss

the number density n and mass m dependence of the degenerate pressure. We found that the degenerate pressure is perturbative in the SR limit which is quite unusual for a theory having an ultraviolet energy cut-off. We then take the example of white dwarfs to explore the possible implications. Using this modified degenerate pressure, we calculate the possible modification to the Chandrasekhar limit for white dwarfs using the Magueijo-Smolin (MS) modified dispersion relation. The mass-radius M-R plot shows that the modified/corrected radius of the white dwarf can be greater than, equal to and smaller than the usual special relativity (SR) value for particular masses. We found that the Chandrasekhar mass limit gets a positive correction i.e, the maximum possible mass for white dwarf increases in this formalism. We note that the presence of observed white dwarfs having radius smaller than the SR Chandrasekhar limit may find an explanation if they are modeled using a modified dispersion relation. The correction, as stated before, is purely perturbative in the SR limit. Therefore this correction is solely because of the modified dispersion relation. The value of the obtained degenerate pressure for a given mass is found to be greater than, equal to and smaller than the usual special relativity (SR) value for particular masses as expected. It is shown by Mishra et al [89], that the Stefan-Boltzmann law gets a correction in such a theory with an ultraviolet cut-off. Using this result we have calculated the luminosity of the white dwarf by taking the model of partially degenerate gas and considering the modified radiative envelope equation. In such an analysis we observe that the pressure for a given mass and temperature value is less than that predicted by the usual SR theory. The luminosity also gets a negative correction. The correction to luminosity is nonperturbative as expected for such a theory.

We summarise our obtained results in the last chapter and give further future directions.

2 Photon Gas Thermodynamics

The materials presented in this chapter are the result of an original research done in collaboration with Nitin Chandra and Vinay Vaibhav, and these are based on the published article [89].

2.1 Prologue

In this chapter we will try to explore the modification in the photon gas physics given an ultraviolet cut-off in the relativistic theory and its implications. Interestingly as explained in detail in the introduction, as has been studied by many, it is possible to keep the Lorentz group/algebra intact for the DSR theories. On the other hand, the representation of the Lorentz group/ algebra becomes non-linear to accommodate the invariant energy/length scale (for example see [3, 4] and the references therein). Preserving the Lorentz group/algebra keeps the theory simple and intuitive. As stated before for our present study, we will follow the DSR formulation developed in [3, 4] by MS where we have an ultraviolet cut-off on energy/momentum and the dispersion relation modifies to,

$$\varepsilon^2 - p^2 = m^2 \left(1 - \frac{\varepsilon}{\kappa}\right)^2 \tag{2.1}$$

The Special Relativistic (SR) limit, i.e. $\kappa \to \infty$ gives the usual dispersion relation,

$$\varepsilon^2 - p^2 = m^2 \tag{2.2}$$

We will stick to the natural units ($\hbar = 1, c = 1, k_B = 1$) if not stated explicitly.

As mentioned in the introduction, it is obvious that the modification in dispersion relation and the presence of ultraviolet cut-off in energy introduced in the DSR theory will affect the thermodynamics of many well studied systems [10, 12, 29–34]. In [10] an extensive study of classical ideal gas thermodynamics has been done using MS formalism. On the other hand [34] studies the photon gas thermodynamics in the same DSR formalism. It should be noted that the study in [34] contains flaws. They have considered the photon gas as a canonical ensemble obeying classical (Maxwell-Boltzmann) statistics. On the other hand, it is a well known fact that photon gas follows a grand canonical ensemble (due to non-conservation of the photon number) and obeys the quantum (Bose-Einstein) statistics. Because of this error in their formalism, the results obtained do not match with the usual photon gas thermodynamics (for example see section 7.3 of [90]). Surprisingly, they match their results to the massless limit of the classical ideal gas thermodynamics in SR. For a photon, the mass being zero, dispersion relation remains same as in the case of SR, i.e. $\varepsilon = p$. The DSR effect for the equilibrium properties of blackbody radiation is basically due to the ultraviolet cut-off κ in energy. We model the blackbody radiation in equilibrium as a grand canonical ensemble of photons obeying Bose-Einstein statistics as usually done. We have also considered the most general possible modification in the phase space measure for exotic spacetimes appearing at Planck scale. To list a few examples where similar modification appears, we note that in noncommutative physics, which is one of the quantum gravity candidate, the change in phase space appears as a change in the density of states as discussed in [35]. Another candidate of the quantum gravity theories, namely Loop Quantum Gravity also predicts the change in phase space measure/density of states at very high energies [36–39]. We, in this chapter, being non specific will proceed with the most generalized (isotropic and Taylor series expandable) modification in phase space integration measure. In [10], a momentum dependent measure has been considered which is a special case of our generalized approach. There have been many interesting attempts to incorporate the additivity of energy and momentum of composite systems in

DSR [61, 91–96]. In this chapter, we will not follow any particular prescription as the issue is still not well settled.

The present chapter starts with calculating various thermodynamic quantities such as energy density, pressure, entropy etc. with an ultraviolet energy cut-off. Next, we study the possible changes due to the change in phase space measure at Planck scale. We then go on calculating various thermodynamic quantities as energy density, pressure, entropy etc. with such a modified measure. The possible realizations in case of Big Bang and Stellar objects have also been discussed. We have then analysed the low and high temperature limits of all the thermodynamic quantities in both cases. Finally, we have also discussed the possible physical realizations of the results obtained in effective low energy case. In doing so we have suggested a very simple and intuitive table top experiment to test our results. We, therefore, have discussed the DSR effects in three scenarios. The general development with an ultraviolet energy cut-off is discussed in Section 2.2. The modification at Planck scale, as a change in phase space measure, is discussed in section 2.3. The Planck scale effects and effective Planck scale effects are studied in section 2.4. And finally, the low energy effective cut-off effects has been explored in section 2.6. We have also summarized the whole chapter at the end. Some of the results are listed in the appendix, in order not to break the continuity of the chapter.

2.2 Equilibrium properties of blackbody radiation with an ultraviolet cut-off

In this section, we will see the possible changes in thermodynamic quantities of photon gas with an ultraviolet energy cut-off. The model contains an ideal gas of identical and indistinguishable quanta namely, photons, [90]. There are n_ω number of photons each with energy $\varepsilon = \omega$. The mean value of n_ω is,

$$\langle n_\omega \rangle = \frac{\sum_{n_\omega=0}^{\infty} n_\omega e^{-\frac{n_\omega \omega}{T}}}{\sum_{n_\omega=0}^{\infty} e^{-\frac{n_\omega \omega}{T}}} = \frac{1}{e^{\frac{\omega}{T}} - 1} \tag{2.3}$$

giving mean energy as,

$$\langle \varepsilon \rangle = \omega \langle n_\omega \rangle = \frac{\omega}{e^{\frac{\omega}{T}} - 1} \tag{2.4}$$

In the large volume limit, the volume of the phase space can be used to find the number of modes between the range ω and $\omega + d\omega$ which are given by,

$$a(\omega)d\omega = \frac{2}{(2\pi)^3}\left(4\pi p^2 dp\right)\int d^3x = \frac{V_{ac}\omega^2 d\omega}{\pi^2} \tag{2.5}$$

Note that photons obey the dispersion relation $\omega = \varepsilon = p$. Factor 2 comes due to the 2 transverse polarizations of a photon. It is also to be noted that the above expression will get modified when we consider the change of the phase space measure in case of DSR. The energy density distribution therefore becomes,

$$u(\omega)d\omega = \frac{a(\omega)d\omega}{V_{ac}}\langle \varepsilon \rangle = \frac{1}{\pi^2}\frac{\omega^3 d\omega}{e^{\frac{\omega}{T}} - 1} \tag{2.6}$$

This is the usual Planck energy density distribution.

2.2.1 Energy Density

Integrating (2.6) from $\omega = 0$ to $\omega = \kappa$ we get the energy density of the photon gas as,

$$u \equiv \frac{U}{V_{ac}} = \int_0^{\kappa} u(\omega)d\omega = \frac{T^4}{\pi^2}\int_0^{\frac{\kappa}{T}} \frac{x^3 dx}{e^x - 1} = \frac{6T^4}{\pi^2}\left[Z_4(0) - Z_4\left(\frac{\kappa}{T}\right)\right] \tag{2.7}$$

Here we have changed the variable to $x = \frac{\omega}{T}$. Note that at finite and non-zero T, as $\kappa \to \infty$ this expression reduces to the one given in (12) on page 203 in [90], giving usual law. Also, $Z_n(x)$ is the incomplete zeta functions or “Debye functions"(refer to section 27.1 of [98]), and is given as,

$$Z_n(x) = \frac{1}{\Gamma(n)}\int_x^{\infty} \frac{t^{n-1}}{e^t - 1}dt \tag{2.8}$$

We note that $Z_n(0) = \zeta(n)$ where $\zeta(z)$ is the Riemann-Zeta function, in particular $Z_4(0) = \zeta(4) = \frac{\pi^4}{90}$. It is remarkable that (2.7) is exactly same as in the case of acoustic phonons [99] with the replacements $\kappa \to \Theta_D$ (Debye temperature), 2 (number of photon polarizations) $\to$ 3 (number of acoustic modes in monoatomic Bravais lattice) and the velocity of acoustic phonons has to be taken to be equal to 1 for correct matching as we are working in natural units [1]. In case of acoustic phonons the cut-off on the possible frequencies comes due to the finiteness of first Brillouin zone which itself is restricted by the number density of ions in the lattice. On the other hand the energy cut-off κ in (2.7) comes from the quantum gravity considerations. We expect the specific heat $C_V = \left(\frac{\partial U}{\partial T}\right)_{V_{ac}} = T\left(\frac{\partial S}{\partial T}\right)_{V_{ac}}$ for a photon gas with such an ultraviolet energy cut-off to follow the behaviour of C_V as in the case of acoustic phonons. For a mathematically rigorous treatment of Debye theory

[1] Note that there are many theories that predict phonon like behaviour like Non-commutative geometry (see for example [84]) etc., but we have explicitly derived and shown that any theory with an ultraviolet cut-off will have this one to one correspondence in the thermodynamics of photons and usual acoustic phonons.

see [100]. Debye functions $Z_n(z)$ are related to the polylogarithm function $Li_n(z)$ by (see (16.2) in [101])

$$Z_n(z) = \sum_{k=0}^{n-1} Li_{n-k}(e^{-z})\frac{z^k}{k!} \tag{2.9}$$

for $n > 0$. Especially $Z_n(0) = Li_n(0)$. Here polylogarithm functions themselves can be series expanded for $|z| < 1$ as (see (8.1) in [101])

$$Li_n(z) = \sum_{a=1}^{\infty} \frac{z^a}{a^n}. \tag{2.10}$$

The integral representation of $Li_n(z)$ is, for $Re(n) > 0$, as follows (see (1) in [101])

$$Li_n(z) = \frac{z}{\Gamma(n)} \int_0^{\infty} \frac{t^{n-1}}{e^t - z} dt \tag{2.11}$$

In particular $Li_1(z) = -\ln(1 - z)$ (see (6.1) in [101]). Thus the energy density can be written in terms of $Li_n(z)$ as given below

$$u = \frac{\pi^2 T^4}{15} - \left[\left(\frac{6T^4}{\pi^2}\right)Li_4\left(e^{-\frac{\kappa}{T}}\right) + \left(\frac{6\kappa T^3}{\pi^2}\right)Li_3\left(e^{-\frac{\kappa}{T}}\right) + \left(\frac{3\kappa^2 T^2}{\pi^2}\right)Li_2\left(e^{-\frac{\kappa}{T}}\right) + \left(\frac{\kappa^3 T}{\pi^2}\right)Li_1\left(e^{-\frac{\kappa}{T}}\right)\right]. \tag{2.12}$$

Note that the first term corresponds to the usual Stefan-Boltzmann law. All the other terms modify the law which in turn, will give a correction to the temperature measurements of different stellar objects. These correction terms vanish in the SR limit. Note that the SR limit $\frac{1}{\kappa} \to 0$ is nonanalytic in nature and hence the energy density cannot be

perturbatively expanded in a Taylor series around this limit. This observation has also been seen in case of classical ideal gas with an invariant energy scale [10]. As the only contribution for a photon gas is due to the ultraviolet cut-off introduced, it is clear that the non-perturbative nature of the modified thermodynamics is a consequence of this cut-off. Also for all possible temperatures, the argument of the polylogarithm in (2.12) i.e. $e^{-\frac{\kappa}{T}}$ is a positive quantity making (2.9) a positive number which leads to the correction term in the expression of energy density being negative. This fact is clearly visible from the plot of energy density (see figure 2.2a on page 41) where the plot with modified energy density is always lower than the corresponding SR plot. This fact can also be understood from the integral expression in (2.7) where the integrand is always a positive quantity and a positive contribution $\frac{6T^4}{\pi^2}Z_4\left(\frac{\kappa}{T}\right) = \frac{T^4}{\pi^2}\int_{\frac{\kappa}{T}}^{\infty}\frac{x^3dx}{e^x-1}$ has been removed from the SR value to get the corresponding modified value.

2.2.2 Specific heat

We put $\frac{U}{T} = u\frac{V_{ac}}{T}$ and use (2.12) along with using the derivatives of polylogarithm given by (see (4.1) in [101]) $\frac{\partial}{\partial\mu}[Li_n(e^{\mu})] = Li_{n-1}(e^{\mu})$ and obtain the expression for specific heat as,

$$\begin{aligned} C_V &= \left(\frac{\partial U}{\partial T}\right)_{V_{ac}} \\ &= -\frac{\kappa^4 V_{ac}}{\pi^2 T}\frac{1}{(e^{\frac{\kappa}{T}}-1)} + \left[\frac{4\pi^2T^3V_{ac}}{15} - \left(\frac{24T^3V_{ac}}{\pi^2}\right)Li_4\left(e^{-\frac{\kappa}{T}}\right) - \left(\frac{24\kappa T^2V_{ac}}{\pi^2}\right)Li_3\left(e^{-\frac{\kappa}{T}}\right)\right. \\ &\left. -\left(\frac{12\kappa^2TV_{ac}}{\pi^2}\right)Li_2\left(e^{-\frac{\kappa}{T}}\right) - \frac{4V_{ac}\kappa^3}{\pi^2}Li_1\left(e^{-\frac{\kappa}{T}}\right)\right] = -\frac{\kappa^4V_{ac}}{\pi^2T}\frac{1}{(e^{\frac{\kappa}{T}}-1)} + 4\left(\frac{U}{T}\right). \end{aligned} \tag{2.13}$$

As we have already seen that the energy density u in the modified case has one to one correspondence with that of the acoustic phonon modes in Debye model, therefore the specific heat $C_V = V_{ac}\left(\frac{\partial u}{\partial T}\right)_{V_{ac}} = \left(\frac{T}{\kappa}\right)^3\frac{\kappa^3V_{ac}}{\pi^2}\int_0^{\frac{\kappa}{T}}\frac{x^4e^xdx}{(e^x-1)^2}$ will also follow the same correspon-

dence (see (17), (18), (19) etc. of section 7.4 in [90]). Again the SR limit gives the usual result $C_V = 4U/T$ (note that $V_{ac} \to V$ in this limit). The extra negative contribution in (2.13) is non-perturbative in the SR limit along with the non-perturbative contributions from the term $4U/T$. Obviously, overall C_V takes a lower value than the corresponding SR values. This fact is visible from the plot also (see figure 2.2d on page 41). The behaviour of C_V for the full range of $\frac{T}{\kappa} \in [0, 1]$ is also shown in the figure 2.2f on page 41 which certainly mimics the Debye theory. In the Debye theory, however, T may go up to infinity in which case the specific heat goes to a constant value.

2.2.3 Radiation Pressure

The grand canonical partition function for the photon gas (with fugacity $z = 1$) is [90] $Q(V_{ac}, T) = \prod_\varepsilon \frac{1}{1-e^{-\frac{\varepsilon}{T}}}$ leading to the expression for q-potential as $q \equiv \frac{PV_{ac}}{T} \equiv \ln Q(V_{ac}, T) = -\sum_\varepsilon \ln(1 - e^{-\frac{\varepsilon}{T}})$. In the large volume limit doing integration by parts we obtain,

$$\begin{aligned} P &= -\frac{T\kappa^3}{3\pi^2} ln(1 - e^{-\frac{\kappa}{T}}) + \frac{T^4}{3\pi^2} \int_0^{\frac{\kappa}{T}} \frac{x^3 dx}{e^x - 1} \\ &= -\frac{T\kappa^3}{3\pi^2} ln(1 - e^{-\frac{\kappa}{T}}) + \frac{1}{3} u. \end{aligned} \tag{2.14}$$

Thus the equation of state for the blackbody radiation field, i.e., the relation between the pressure and the energy density got modified and goes to the correct SR limit $P_{SR} = \frac{1}{3}(u)_{SR} = \frac{\pi^2 (T_{SR})^4}{45}$. The explicit temperature dependence of the radiation pressure is given by,

$$P = \frac{\pi^2 T^4}{45} - \left[\left(\frac{2T^4}{\pi^2} \right) Li_4 \left(e^{-\frac{\kappa}{T}} \right) + \left(\frac{2\kappa T^3}{\pi^2} \right) Li_3 \left(e^{-\frac{\kappa}{T}} \right) + \left(\frac{\kappa^2 T^2}{\pi^2} \right) Li_2 \left(e^{-\frac{\kappa}{T}} \right) \right]. \tag{2.15}$$

The correction term in the above expression is negative (see figure 2.2b on page 41) and non-perturbative in nature (see discussion in section 2.2.1).

2.2.4 Entropy

The Helmholtz free energy is given by (the chemical potential $\mu = 0$),

$$A = \mu N - PV_{ac} = -PV_{ac} = \frac{T\kappa^3 V_{ac}}{3\pi^2} ln(1 - e^{-\frac{\kappa}{T}}) - \left(\frac{U}{3}\right). \tag{2.16}$$

The entropy becomes,

$$S = \frac{U - A}{T} = -\frac{\kappa^3 V_{ac}}{3\pi^2} ln(1 - e^{-\frac{\kappa}{T}}) + \frac{4}{3}\left(\frac{U}{T}\right). \tag{2.17}$$

In SR limit, the first term vanishes and the expression goes to the correct result $S_{SR} = \frac{4}{3}\left(\frac{U}{T}\right)_{SR}$ (see (19) of section 7.3 in [90]). As done in case of radiation pressure if we write the explicit temperature dependence of the entropy, the negative non-perturbative contribution will be very apparent which can be seen from the plot as well (see figure 2.2c on page 41). The decrease in the entropy value for the modified case can be explained by the presence of ultraviolet cut-off which restricts the number of available microstates to the system.

2.2.5 Equilibrium number of photons

The equilibrium number of photons can be obtained by integrating the product of mean number of photons (2.3) and the volume of the phase space (2.5),

$$\bar{N} = \int_0^{\kappa} \frac{V_{ac}}{\pi^2} \frac{\omega^2 d\omega}{e^{\frac{\omega}{T}} - 1} = \frac{2V_{ac}T^3}{\pi^2} \left[Z_3(0) - Z_3\left(\frac{\kappa}{T}\right) \right] \tag{2.18}$$

Here $Z_3(0) = \zeta(3)$ is also called Apery's constant. This with the proper replacements corresponds to the equilibrium number of acoustic phonons in the Debye theory. In the $\kappa \to \infty$ limit $Z_n(\frac{\kappa}{T}) \to 0$ and $V_{ac} \to V$ and we get the usual SR result $(\bar{N})_{SR} = \frac{T^3 V}{\pi^2}(2\zeta(3))$ as given in (23) of section 7.3 in [90]. Like other thermodynamic quantities the equilibrium number of photons also gets a negative non-perturbative correction. The decrease in the $\bar{N}$ value for the modified case is also due to the cut-off which restricts the number of available normal modes.

2.3 Photon gas thermodynamics at Planck scale for exotic spacetimes

In this section, we will discuss the possible modifications in the known thermodynamic quantities if we consider a change in phase space measure along with an invariant ultraviolet cut-off. Almost all the thermodynamic quantities for well studied systems encounter the large volume limit where discrete summation over energy values goes to the integration over phase space i.e. $\sum_\varepsilon \to \frac{1}{(2\pi)^3} \int\int d^3x d^3p$. But for exotic spacetimes appearing at Planck scale, we expect the phase-space to modify (see for example [10]) as $\sum_\varepsilon \to \frac{1}{(2\pi)^3} \int\int d^3x d^3p f(\vec{x}, \vec{p})$. Here we have considered the most general possible modification. The thermodynamic quantities are derivable from the partition function of the form $\frac{1}{(2\pi)^3} \int\int d^3x d^3p F(\varepsilon)$ which due to the change in phase space measure modifies to $\frac{1}{(2\pi)^3} \int\int d^3x d^3p f(\vec{x}, \vec{p}) F(\varepsilon)$. Assuming the spacetime to be isotropic and $f(\vec{x}, \vec{p}) = f(r, p)$ to be Taylor series expandable in the powers of $(\frac{1}{r\kappa})$ and $(\frac{\varepsilon}{\kappa})$ we get,

$$f(r,p) = \sum_{n=0,n'=0}^{\infty} \frac{a_{n,n'}}{n!n'!} \left(\frac{\varepsilon}{\kappa}\right)^n \left(\frac{1}{r\kappa}\right)^{n'}, \tag{2.19}$$

with $a_{0,0} = 1$ as for $\kappa \to \infty$ we expect $f(r,p) \to 1$. This expansion is valid only when $\frac{\varepsilon}{\kappa}, \frac{1}{r\kappa} < 1$ throughout the integration range, this requires $\varepsilon < \kappa$ and $r > \frac{1}{\kappa}$. Thus κ acts as highest energy cut-off while $\frac{1}{\kappa}$ acts as the lowest length cut-off. Finally the integral changes to,

$$\frac{1}{(2\pi)^3} \sum_{n=0,n'=0}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^{n+n'}} \int_{r=\frac{1}{\kappa}}^{R} \int_{p=0}^{\kappa} d^3x d^3p \ \varepsilon^n \left(\frac{1}{r}\right)^{n'} F(\varepsilon), \tag{2.20}$$

R being the radius of the spherical volume considered. Here we have interchanged the double summation and the integration which is allowed if (see appendix A),

$$\sum_{n=0,n'=0}^{\infty} \frac{|a_{n,n'}|}{n!n'!\kappa^{n+n'}} \int_{r=\frac{1}{\kappa}}^{R} \int_{p=0}^{\kappa} d^3x d^3p \ \varepsilon^n \left(\frac{1}{r}\right)^{n'} |F(\varepsilon)| < \infty. \tag{2.21}$$

Performing the integration over the coordinate space we obtain

$$\begin{aligned} &\tfrac{1}{(2\pi)^3} \int\int d^3x d^3p f(r,p) F(\varepsilon) \\ &= \tfrac{1}{(2\pi)^3} \sum_{\substack{n=0,n'=0 \\ n'\neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^{n+3}} \frac{4\pi}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}} - 1\right] \int_{p=0}^{\kappa} d^3p \ (\varepsilon)^n F(\varepsilon) \\ &\quad + \tfrac{1}{(2\pi)^3} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^{n+3}} \left(\frac{4\pi}{3}\right) \ln\left(\frac{3V\kappa^3}{4\pi}\right) \int_{p=0}^{\kappa} d^3p \ (\varepsilon)^n F(\varepsilon) \end{aligned} \tag{2.22}$$

where $V = \frac{4}{3}\pi R^3$ is the volume of the spherical ball of radius R. The accessible part of the volume for the particle is $V_{ac} = V - \frac{4\pi}{3\kappa^3}$. For the large volume limit the minimum length $\frac{1}{\kappa} << R$ implying $V\kappa^3 >> 1$ which in turn implies $V_{ac} \approx V$. Note that a small volume $\frac{4\pi}{3\kappa^3}$

is inaccessible to each particle. This inaccessible volume can be extracted out at any point in the space as volume being large, all the space points are equivalent. We have extracted out this volume at the centre $r = 0$. $n' \geq 4$ as the powers of $\frac{1}{V\kappa^3}$ increases.

2.3.1 Classical Ideal gas in canonical ensemble: an example

Let us take a particular example of $F(\varepsilon)$ to illustrate this further. We consider the classical ideal gas in canonical ensemble obeying Maxwell-Boltzmann statistics with the partition function [90],

$$Z_N\left(V_{ac}, T\right) = \sum_E \exp[-\beta E] = \frac{1}{N!}[Z_1\left(V_{ac}, T\right)]^N, \tag{2.23}$$

where $Z_1(V_{ac}, T)$ is the single particle partition function, N is the total number of constituent particles, $\beta = \frac{1}{T}$ and the total energy E of the system is $E = \sum_\varepsilon n_\varepsilon \varepsilon$. Here n_ε is the number of particles corresponding to the single particle energy ε and satisfies $\sum_\varepsilon n_\varepsilon = N$. The single particle partition function is given by $Z_1\left(V_{ac}, T\right) = \sum_\varepsilon \exp[-\beta(\varepsilon - m_0)]$ In the large volume limit using (2.22) for $F(\varepsilon) = \exp(-\beta(\varepsilon - m_0))$ and following the arguments given in [10] we obtain,

$$\begin{aligned} Z_1\left(V_{ac}, T\right) &= \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^n}\left(\frac{3}{(3-n')(\kappa^3 V_{ac})}\right)\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}} - 1\right]\left(m_0 - \frac{\partial}{\partial\beta}\right)^n Z_1^0\left(V_{ac}, T\right) \\ &+ \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!\kappa^n}\left(\frac{4\pi}{18\kappa^3 V_{ac}}\right)\ln\left(\frac{3V\kappa^3}{4\pi}\right)\left(m_0 - \frac{\partial}{\partial\beta}\right)^n Z_1^0\left(V_{ac}, T\right) \end{aligned} \tag{2.24}$$

where $Z_1^0\left(V_{ac}, T\right)$ is the single particle partition function with the unmodified measure,

$$Z_1^0\left(V_{ac}, T\right) = \frac{V_{ac}}{(2\pi)^3}\int_{p=0}^{\kappa} d^3p \exp(-\beta(\varepsilon - m_0)). \tag{2.25}$$

The expression for $Z_1(V_{ac}, T)$ has now non-trivial dependence on V unlike in the case of $Z_1^0(V_{ac}, T)$. With this modification the value of thermodynamic quantities, especially pressure, changes. Let's not digress anymore and continue with the study of the photon gas thermodynamics.

We will now consider the change in phase space as described above. This leads to the modification of the energy density distribution as well as the q-potential which in effect modifies all the thermodynamic quantities.

2.3.2 Modified Planck's energy distribution and Wien's law

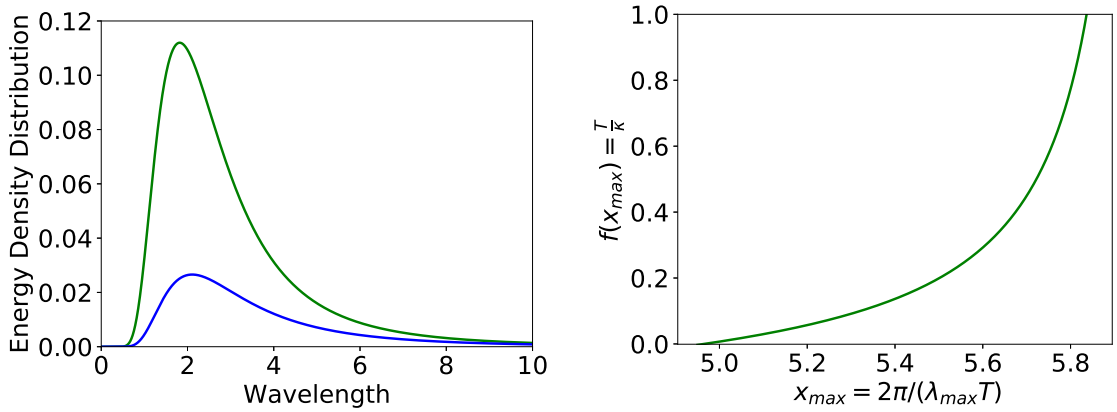
It is clear that the change in phase space is going to modify the Planck distribution for the energy density of the blackbody radiation. In such a scenario (2.5) modifies to,

$$\begin{aligned} a(\omega)d\omega = & \frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!\kappa^{n+3}}\frac{4\pi}{(3-n')}\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\omega^{n+2}d\omega \\ & + \frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!\kappa^{n+3}}\left(\frac{4\pi}{3}\right)\ln\left(\frac{3\kappa^3 V}{4\pi}\right)\omega^{n+2}d\omega \end{aligned} \tag{2.26}$$

Now the Planck energy density distribution (2.6) changes to,

$$\begin{aligned} u(\omega)d\omega = & \frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!\kappa^{n}}\frac{4\pi}{(3-n')}\left(\frac{1}{V_{ac}\kappa^3}\right)\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\frac{\omega^{n+3}d\omega}{e^{\frac{\omega}{T}}-1} \\ & + \frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!\kappa^{n}}\left(\frac{4\pi}{3\kappa^3 V_{ac}}\right)\ln\left(\frac{3\kappa^3 V}{4\pi}\right)\frac{\omega^{n+3}d\omega}{e^{\frac{\omega}{T}}-1} \end{aligned} \tag{2.27}$$

A typical plot of the modified energy density distribution in comparison to the usual Planck distribution is shown in figure 2.1a on page 38. Let us first express the above distribution in terms of wavelength λ. The energy density between ω and $\omega + d\omega$ or the



(a) Modified Planck's Distribution

(b) Modified Nature Of Wien's Law

Figure 2.1: The plot (a) shows the Planck energy distribution as a function of wavelength λ for both modified and unmodified measures. Here the blue and the green colour correspond to the cases of unmodified and the modified measures respectively. We have taken $a_{0,0} = 1.0$, $a_{0,1} = a_{1,0} = 0.2$ and all other a's are zero, temperature is $T = 0.6$, volume is 10^{35} and $\kappa = 1$ in Planck units. The plot (b) shows the modification in the Wien's displacement law. The usual Wien's law would have given us a constant value of x_{max} corresponding to the point where $f(x_{max}) = 0$. We can clearly see from the plot that this value is $x_{max} \approx 4.965$. The coefficient $a_{1,0}$ has been taken to be 1. We have chosen the range such that $\frac{T}{\kappa}$ lies between 0 and 1. Although to decide the actual range of the possible T-values one needs more thorough and careful thinking, on the other hand in this chapter for our purpose we have discussed the only cases where T lies between 0 and $\kappa = 1$.

corresponding λ and $\lambda + d\lambda$ is $u(\lambda)d\lambda = u(\omega)d\omega$ which implies $u(\lambda) = u(\omega)\frac{d\omega}{d\lambda} = -\frac{\omega^2 u(\omega)}{2\pi}$, where ω and λ are related by $\omega = \frac{2\pi}{\lambda}$. From (2.27) we can write $u(\omega) = \sum_{n=0}^{\infty} A_n \frac{\omega^{n+3}}{e^{\frac{\omega}{T}}-1}$ where,

$$A_n = \frac{1}{\pi^2}\sum_{n'=0,n'\neq 3}^{\infty}\frac{a_{n,n'}}{n!n'!\kappa^n}\left(\frac{4\pi}{3-n'}\right)\frac{1}{\kappa^3 V_{ac}}\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}} - 1\right] + \frac{1}{\pi^2}\frac{a_{n,3}}{n!3!\kappa^n}\left(\frac{4\pi}{3\kappa^3 V_{ac}}\right)\ln\left(\frac{3V\kappa^3}{4\pi}\right) \tag{2.28}$$

is a constant and is independent of both λ and T. We then have,

$$u(\lambda) = -\sum_{n=0}^{\infty}\frac{(2\pi)^{n+4}A_n}{\lambda^{n+5}\left(e^{\frac{2\pi}{\lambda T}} - 1\right)}. \tag{2.29}$$

Differentiating with respect to λ we get

$$\frac{du(\lambda)}{d\lambda} = \frac{1}{\lambda^6\left(e^{\frac{2\pi}{\lambda T}}-1\right)}\sum_{n=0}^{\infty}\frac{(2\pi)^{n+4}A_n}{\lambda^n}\left[n+5-\frac{\left(\frac{2\pi}{\lambda T}\right)}{1-e^{-\frac{2\pi}{\lambda T}}}\right]. \tag{2.30}$$

Note that in the case of unmodified measure, we have $A_0 = 1/\pi^2, A_1 = A_2 = ... = 0$ and the above expression reduces to

$$\frac{du(\lambda)}{d\lambda} = \frac{1}{\lambda^6\left(e^{\frac{2\pi}{\lambda T}}-1\right)}\frac{(2\pi)^4}{\pi^2}\left[5-\frac{\left(\frac{2\pi}{\lambda T}\right)}{1-e^{-\frac{2\pi}{\lambda T}}}\right]. \tag{2.31}$$

Thus $u(\lambda)$ is maximum at $\lambda = \lambda_{max}$ which can be found by the extremum condition $\left.\frac{du(\lambda)}{d\lambda}\right|_{\lambda_{max}} = 0$ giving $5 = \frac{x_{max}}{1-e^{-x_{max}}}$ where $x_{max} = \frac{2\pi}{\lambda_{max}T}$. The above equation can be numerically solved to get $x_{max} = \frac{2\pi}{\lambda_{max}T} \approx 4.965, \Rightarrow \lambda_{max}T \approx 1.266$. This behaviour of λ_{max} on temperature T is called *Wien's displacement law*. Now for the case of modified measure the extremum condition becomes,

$$\sum_{n=0}^{\infty} T^n x_{max}^n A_n\left[n+5-\frac{x_{max}}{1-e^{-x_{max}}}\right] = 0. \tag{2.32}$$

It is obvious that the solution of x_{max} is now dependent on T. Thus the value of $x_{max} = \frac{2\pi}{\lambda_{max}T}$ is no more constant, but a function of T. To understand the behaviour in a better way, we keep the leading order terms in $\frac{T}{\kappa}$ and $\frac{1}{V^{1/3}\kappa}$ and neglect all the higher order terms i.e. $A_0 \approx 1/\pi^2, \quad A_1 \approx \frac{a_{1,0}}{\pi^2\kappa}, \quad A_2 \approx A_3 \approx ... \approx 0$. The extremum condition then becomes

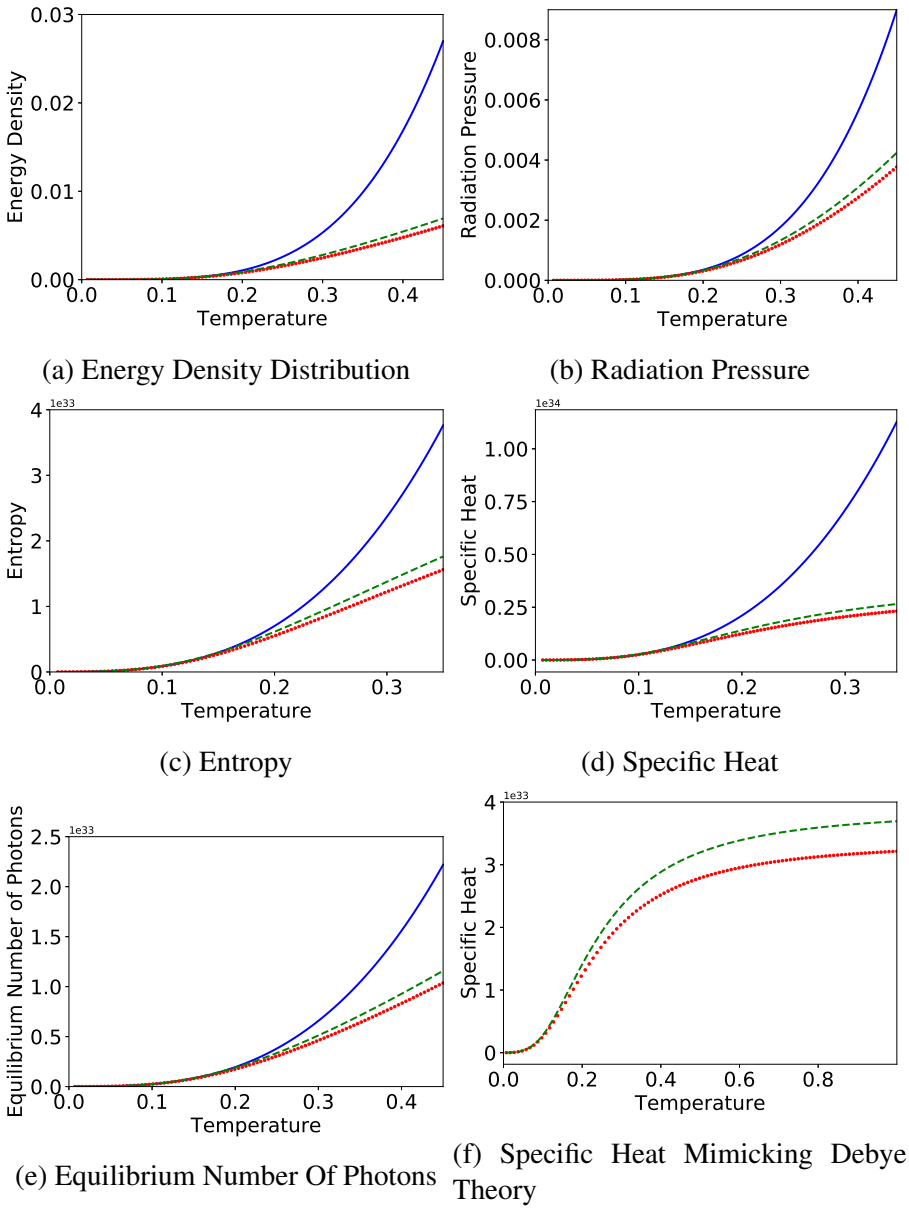
$$\frac{T}{\kappa} = -\frac{1}{x_{max}a_{1,0}}\left(\frac{5-\frac{x_{max}}{1-e^{-x_{max}}}}{6-\frac{x_{max}}{1-e^{-x_{max}}}}\right) = f(x_{max}). \tag{2.33}$$

We have plotted this function with respect to x_{max} (see figure 2.1b on page 38). For a fixed value of $y-$axis, i.e., a fixed $\frac{T}{\kappa}-$ value the corresponding value of x_{max} can be obtained from the plot. As visible from the plot $x_{max} = \frac{2\pi}{\lambda_{max}T}$ is a monotonically increasing function of T, i.e., $f^{-1}(\frac{T}{\kappa})$. This implies $\lambda_{max} = \frac{2\pi}{Tf^{-1}(\frac{T}{\kappa})}$ is a monotonically decreasing function of T. Note that λ_{max} for modified phase space measure decreases more rapidly with increasing T than the case of unmodified measure where $x_{max} = f^{-1}(\frac{T}{\kappa})$ takes a constant value.

The significant change in the values of x_{max} occurs only if the order of the change in temperature is non-negligible with respect to κ. That is why in SR limit, i.e., $\frac{T}{\kappa} \to 0$, the $x_{max} = \frac{2\pi}{\lambda_{max}T}$ is almost constant giving the standard Wien's displacement law. The extremum condition for the unmodified measure corresponds to $f(x_{max}) = 0$. As it is visible in figure 2.1b on page 38 this gives the usual value $x_{max} \approx 4.965$. Note that the value of $f^{-1}\left(\frac{T}{\kappa}\right)$ is always greater than the SR value 4.965. Hence $\frac{(\lambda_{max})_{DSR}}{(\lambda_{max})_{SR}} = \frac{4.965}{f^{-1}\left(\frac{T}{\kappa}\right)} \leq 1$. Thus, the frequency at which the energy density distribution of blackbody radiation at a given temperature peaks, gets a positive correction. Now, suppose we demand at least 1% correction i.e., $\frac{(\lambda_{max})_{DSR}}{(\lambda_{max})_{SR}} = \frac{4.965}{f^{-1}\left(\frac{T}{\kappa}\right)} = \frac{99}{100}$ then we get $f^{-1}\left(\frac{T}{\kappa}\right) = 5.015$. The corresponding $\frac{T}{\kappa}$ from the plot is 0.01. So, to get an observable effect of DSR using modified Wien's displacement law one needs to consider a system having temperature in the range of 100th part of the effective κ value. Note that some exotic phenomenon in the semi-classical regime of quantum gravity may reduce the effective value of the energy cut-off in certain specific systems. A similar reduction in the effective value of high energy cut-off has been suggested in a simple quantum mechanical table top experiment in section 2.6.

2.3.3 Various thermodynamic quantities with modified measure

We have calculated various thermodynamic quantities with the modified measure. The exact results in form of lengthy expressions are listed in B.1 and here we proceed further with physical analysis only. Note that in this case all the thermodynamic quantities reduce to the unmodified case for $n = 0$ and $n' = 0$. Also in the SR limit, we get the usual SR result as expected. The value of the thermodynamic quantities, in this case, can either be less than or equal to (for certain T-values only) or greater than both the SR value and the values in case with only an ultraviolet energy cut-off, depending on the choice of $a_{n,n'}$. To plot these quantities we have chosen the $a_{n,n'}$ values in such a way that the value of the modified case is more than the unmodified case and less than the SR case. Though it is not visible in the plot because of the chosen $a_{n,n'}$ values, but it is a fact that for certain choices



(a) Energy Density Distribution

(b) Radiation Pressure

(c) Entropy

(d) Specific Heat

(e) Equilibrium Number Of Photons

(f) Specific Heat Mimicking Debye Theory

Figure 2.2: The plots show the variation of energy density, radiation pressure, entropy, specific heat and equilibrium number of photons with temperature for thermodynamic quantities with an ultraviolet energy cut-off and also with the modified measure. The blue solid, the green dotted and the red dashed lines correspond to the SR, the case with the ultraviolet energy cut-off and the case with the modified measure respectively. As is visible from the plots it matches with SR at low T and with increasing T it deviates from SR significantly. Here $\kappa = 1$ and $V = 10^{35}$ in Planck units and $a_{0,0} = 1$, $a_{0,1} = a_{1,0} = 0.2$ and all other a's are taken to be zero. Note that all the quantities above become approximately linear near $T \to \kappa$. In the low temperature regime, energy density u and radiation pressure P follow $\sim T^4$ behaviour, while the entropy S, the specific heat C_V and the equilibrium number of photons $\bar{N}$ follow $\sim T^3$ behaviour. The behaviour of C_V for the full range of $\frac{T}{\kappa} \in [0, 1]$ is shown in the figure at bottom right corner, which certainly mimics the Debye theory. In the Debye theory however T may go up to infinity in which case the specific heat goes to a constant value.

of $a_{n,n'}$ the modified DSR value becomes equal to the value of SR at some temperatures and can even overshoot the SR curve. The nonanalytic nature in the SR limit for the case of modified measure is similar as in the unmodified case. The leading order behaviour in the low and high temperature limits are discussed in the next section.

2.4 Effects of DSR in Big Bang and cosmology

In this section we will explore the possible effects of the behaviour of DSR photons near the Planck scale with an invariant ultraviolet energy cut-off. To see the physical applicability of the results with such an invariant ultraviolet energy cut-off one has to, in general, probe near the Planck scale. The results can then be used to study the early Universe thermodynamics especially Big Bang cosmology. Since we have the modified energy density u and Pressure P, therefore we have a modified energy-momentum tensor $T_{\mu\nu}$. In general, we should use the modified metric when we are exploring the early Universe near Planck scale. In DSR, Smolin has suggested one such metric called the Rainbow metric [73] [55] [102–104], but we will not attempt to discuss this here. With the above quantities at hand, we can then solve the Friedmann equations (more specifically FRW equations) and see the possible modification in the known results of the expansion of the Universe after Big Bang at such a scale. This is very involved and a more detailed study can be done separately in future. But we can still consider a scenario where we can see the possible modification near the Big Bang. The FRW and its relation is a standard and well studied cosmology subject. We, for our analysis, will follow chapter 8 of [40]. We will consider the radiation dominated epoch where the modified energy density and pressure is given by (2.7) and (2.14) respectively. With such a modification of $T_{\mu\nu}$, the energy conservation equation (8.54) in section 8.3 of [40] gets modified to

$$\frac{\dot{u}}{u} = -\left(4 - \frac{\frac{T\kappa^3}{\pi^2} ln(1 - e^{-\frac{\kappa}{T}})}{u}\right)\frac{\dot{a}}{a}. \tag{2.34}$$

We then express u in terms of T to get,

$$H = \frac{\dot{a}}{a} = -\frac{\dot{T}}{T}\left[\frac{24\left[Z_4(0) - Z_4\left(\frac{\kappa}{T}\right)\right] - \left(\frac{\kappa}{T}\right)^4 Li_0\left(e^{\frac{-\kappa}{T}}\right)}{24\left[Z_4(0) - Z_4\left(\frac{\kappa}{T}\right)\right] + \left(\frac{\kappa}{T}\right)^3 Li_1\left(e^{\frac{-\kappa}{T}}\right)}\right] \tag{2.35}$$

Here a is the dimensionless scale factor and H is the Hubble parameter which characterizes the rate of expansion of the Universe. It is easy to see that the numerator is always less than the denominator. Therefore $\frac{H}{H_{SR}} < 1$ always, where $H_{SR} = -\frac{\dot{T}}{T}$, which implies that the expansion of the Universe was at a slower rate in the radiation dominated era than the rate of expansion without such modifications. Because of the slower expansion, all the epochs would eventually get delayed resulting in the modification in the age of the known Universe. In the SR limit the modified Hubble parameter becomes nearly equal to the normal SR one, as $\frac{\kappa}{T}$ is very large, so the correction terms go to zero as expected.

We can also see its application in case of “bouncing" loop quantum cosmology theories (see [41] [42] [43] and the references therein), where normally we consider specific modifications to the spacetime geometry which effectively puts a bound on the curvature and in this way the Big Bang singularity can be avoided. But for such “bouncing" models, we cannot use the perturbation technique at the curvature saturation, as the energy density of the cosmic fluid diverges. What one can do to still avoid the Big Bang singularity is to consider an inflation model where we can safely use the perturbation theory. Here we have obtained the energy density of the cosmic fluid which saturates to the Planck energy which of course is finite. Then we can combine both the results obtained in this chapter and the “bouncing" loop quantum cosmology to study the possible way out to avoid the Big Bang singularity.

Next we consider the DSR photons at an effective lower scale due to other parameters in the theory like mass, number density etc. These parameters may effectively lower the Planck scale such that its effects can be observed in very high temperature and high density regimes. To probe such DSR effects we need to observe the stellar objects with

very high temperatures and densities. For example, the astronomical data from gamma ray burst during the merging of neutron stars (which has the core temperature of $T = 10^{12}$ K) may give a bound on effective κ value. We can also explore the Chandrasekhar limits and its possible modifications. The application of the theory developed here has been explored in detail for white dwarfs in [105] and will be discussed in detail in the next chapter.

2.5 The leading behaviour for $T \to 0$ and $T \to \kappa$

We have plotted various thermodynamic quantities as a function of temperature (see figure 2.2 on page 41). Let us now analyse the behaviour near $T = 0$ and $T = \kappa$. In the low temperature regime we take $\frac{T}{\kappa} = \epsilon << 1$. The low temperature behaviour is as follows,

$$u \approx \frac{\pi^2\kappa^4\epsilon^4}{15} - \frac{\kappa^4}{\pi^2}\epsilon e^{-\frac{1}{\epsilon}} \approx \frac{\pi^2 T^4}{15}, \quad P \approx \frac{\pi^2 T^4}{45},$$
$$S \approx \frac{4V_{ac}\pi^2 T^3}{45}, \quad C_V \approx \frac{4V_{ac}\pi^2 T^3}{15}, \quad \bar{N} \approx \frac{2V_{ac}\zeta(3)T^3}{\pi^2}. \tag{2.36}$$

Here we have used the fact that $Li_n(z) \to z$ as $z \to 0$. In the expression for energy density, we have neglected the second term with respect to the first. We can see this by putting $x = \frac{1}{\epsilon}$ and as $x \to \infty$ the ratio of the second term to the first, in the above equation goes to zero. Note that the second term $\epsilon e^{-\frac{1}{\epsilon}}$ is the nonanalytic piece which makes this limit non-perturbative i.e. this expression cannot be Taylor series expanded in the low temperature limit. Let us consider the energy density relation in the $\frac{T}{\kappa} = \epsilon << 1$ limit, given by $u \approx \frac{\pi^2\kappa^4\epsilon^4}{15} - \frac{\kappa^4}{\pi^2}\epsilon e^{-\frac{1}{\epsilon}}$. Now assuming that we get at least 1% correction i.e.,

$$\frac{\frac{\kappa^4}{\pi^2}\epsilon e^{-\frac{1}{\epsilon}}}{\frac{\pi^2\kappa^4\epsilon^4}{15}} \geq \frac{1}{100} \tag{2.37}$$

which, in turn, gives a bound on ϵ as $0.10 \leq \epsilon \leq 2.1$. But since we have taken $\epsilon << 1$, therefore the equality holds at $\epsilon = 0.10$. This fact is also visible from the plots of the thermodynamic quantities in which the modified behaviour starts deviating from the SR result at $\frac{T}{\kappa} \sim 10^{-1}$. Another point to note is that the value of ϵ, for at least 1% correction, in case of modified Wien's displacement law came out to be around 0.01 (See section 2.3.2). Thus, the modified Wien's displacement law starts giving an observable correction for the systems having temperature one order less compared to the systems used in case of modified thermodynamic quantities. For pressure we do the similar analysis where the first term in (2.14) is nothing but $\left(\frac{\kappa^3 T}{3\pi^2}\right) Li_1(e^{\frac{-\kappa}{T}})$. A similar analysis follows for other thermodynamic quantities as well. Thus in low temperature regime, energy density u and radiation pressure P follow $\sim T^4$ behaviour, while the entropy S, the specific heat C_V and the equilibrium number of photons $\bar{N}$ follow $\sim T^3$ behaviour. The nonanalyticity in this limit is a general feature of all the thermodynamic quantities. For high temperature, $T \approx \kappa(1-\epsilon)$ such that $\epsilon << 1$ which gives $\frac{\kappa}{T} \approx 1+\epsilon$. We will expand all the quantities to the leading order in ϵ and finally put $\epsilon = 1 - \frac{T}{\kappa}$ to get the leading high temperature behaviour. The results are listed in B.2. Note that to get the linear dependence of C_V on T in (B.12) by differentiating the high T behaviour of U, we need to expand U up to T^2 order. All these linear behaviours for $T \to \kappa$ are very clearly visible in the plots. For the modified measure we get essentially the similar behaviour for both the limits. The results of the leading behaviour in case of modified measure are listed in B.3 and B.4.

2.6 Effective low energy realizations of the theory

In this section, we present the possibilities of physical realization of the results obtained with an effective cut-off for the photons such that they behave as phonons. As is clear from the description this cut-off might not be invariant which was the case in the other applications discussed above. But this is an interesting case in its own regard, as we have a way to get a bunch of photons behaving as phonons and they can be observed in a lab-

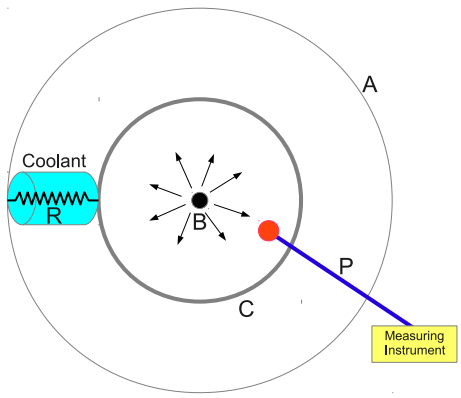


Figure 2.3: A quantum mechanical table top experiment for testing the theory. Here B is a perfect blackbody surrounded by a spherical cathode C, which is enclosed by a spherical anode A and the circuit is completed using a high resistance resistor R. All the radiations above the frequency ν_{th} corresponding to the photoelectric threshold of the cathode are absorbed and will be lost as Joule heating through R. The resistor is continuously being cooled by coolant, shown in blue colour, to avoid the melting of the resistor. The photons below the frequency ν_{th} stay inside the cavity surrounded by the cathode C and behave as acoustic phonons as shown in this chapter. The properties of these photons acting as acoustic phonons can be measured by inserting a probe P connected with the measuring instrument.

oratory as well. To start with here we will be suggesting a simple table top experiment to test the result obtained in section 2.2. Note that due to no change in dispersion relation the only effect on the thermodynamics of a photon gas is due to the high energy cut-off. If one can introduce such a cut-off on the photon energy in some experiment then the photons will start behaving like acoustic phonons. Consider a perfect blackbody (see figure 2.3 on page 46) surrounded by a spherical cathode which is enclosed by a spherical anode and the circuit being completed using a high resistance. Now, suppose the cathode has the photoelectric threshold ν_{th} such that all the radiations above frequency ν_{th} gets absorbed by the cathode. These absorbed radiations lead to the Joule heating of the resistor which is then cooled by an appropriate coolant. Since, we do not want the heating of the resistor, in any way, to affect the radiations inside the cathode, therefore the cathode may be coated with an insulating material. Another possible alternative is to drill a small hole in anode then connect the resistor outside and far away from the anode where it can be cooled. Now, we are left with photons in the cathode cavity, with energy less than $\kappa = \nu_{th}$ which is the desired cut-off in the theory. We, therefore, have generated photons inside the cathode which mimic acoustic phonons. To observe this we can now

drill a very small hole in both anode and cathode, through which a probe, connected to the measuring instrument, can be inserted to test the properties of the photons. To observe at least 1% deviation (see the discussion in section 2.5) from the usual photon thermodynamics at room temperature $T = 293$ K the material of the cathode can be selected with threshold $\nu_{th} = \frac{k_B T}{0.1h} = 6.1 \times 10^{13} Hz$ (here we have put the actual values of h and k_B). Many commercially available materials fall into this category.

2.7 Summary

In the present chapter we considered a model of a photon gas obeying Bose-Einstein statistics in grand canonical ensemble and calculated various thermodynamic quantities such as energy density, pressure, entropy, specific heat and equilibrium number of photons with such an ultraviolet cut-off. We found one to one correspondence between the behaviour of photons with an ultraviolet cut-off and the acoustic phonons in the Debye theory. The usual Stefan-Boltzmann law got modified. We found that the non-perturbative nature of the thermodynamic quantities in the SR limit is a general feature of the theory with an ultraviolet energy cut-off. We also noted that the values of all the thermodynamic quantities are less than the SR values because of this cut-off. We then studied the change in the phase space measure for exotic spacetimes at Planck scale and discussed the example of classical ideal gas for illustration. We found that the classical ideal gas in case of modified phase space measure has a non-trivial volume dependence in its expression for the partition function leading to the modification in the thermodynamic quantities like pressure accordingly. We went on calculating the possible change in the thermodynamic quantities due to the change in the phase space measure. Because of this modification, Planck's energy density distribution and the Wien's displacement law got modified. Note that all the thermodynamic quantities reduce to the usual SR result in $\kappa \to \infty$ limit. We have plotted the temperature dependence of various thermodynamic quantities. We can clearly see from the plots of various thermodynamic quantities that we start getting the

deviation of the results obtained in the modified case from the SR case at $\frac{T}{\kappa} \sim 10^{-1}$. We found that the modified Wien's law can be observed at comparatively lower temperature than the thermodynamic quantities. Next, we discussed the possible realization of the modification at Planck scale by considering its effects near the Big Bang. The effectively lower Planck scale cosmological observations and modifications of DSR have also been discussed. The leading behaviour for $T \to 0$ and $T \to \kappa$ have been analysed. We observed that in the case of modified phase space measure the values of the thermodynamic quantities might be less than, equal to or greater than the SR values depending on the choice of $a_{n,n'}$. As was seen both for the case with only an ultraviolet cut-off and the modified measures, the nonanalyticity in the special relativistic limit is a general feature of the energy cut-off introduced in the theory. In the last section, we have given the possible scenarios of the physical observation of results obtained in effective low energy by suggesting a quantum mechanical table top experiment.

3 Degenerate Fermi gas Thermodynamics

The materials presented in this chapter are the result of an original research done in collaboration with Nitin Chandra, and these are based on the communicated article [105].

3.1 Prologue

Modified Dispersion relation and effective theory with an ultraviolet cut-off are one of the aspects which gives us a possible way to explore beyond the known physics. Since the photons have been studied in the previous chapter, therefore studying Fermi gas becomes the next immediate thing. Since DSR gives us a modified dispersion relation and puts a cut-off on the highest single particle energy/momentum, this in effect will give a correction to the thermodynamics of the degenerate Fermi gas which becomes nontrivial. As is well known, the model of degenerate Fermi gas is used to study the dynamics of many compact stars such as white dwarf stars, neutron stars etc. We will consider a simple model of white dwarf stars to study the possible implications of the obtained results. The white dwarf stars are the final stage of the stellar evolution after the nuclear processes inside the star have died down. Inside a white dwarf, which has used up almost all its fuel, practically no fusion is occurring. Therefore there is no source of thermal energy to sup-

port against the huge gravitational collapse. The stability, in this case, is provided by what is known as degeneracy pressure which is the quantum pressure inside a degenerate Fermi gas. This idea was was suggested first by Fowler [106] and Chandrasekhar [107]. The degeneracy pressure at 0 K (considering relativistic quantum gas) is much much larger than non-degenerate thermal pressure at very high density of stellar medium (density keeps on increasing as the star keeps on collapsing under self gravity) and it is this degeneracy pressure that supports the white dwarf against the gravitational collapse. We, therefore, can calculate the thermodynamic degenerate pressure and equate it to the pressure due to gravity to get the mass and radius relationship at equilibrium. This in turn gives the Chandrasekhar limit for white dwarfs in the ultra-relativistic regime.

Various stellar objects and their Chandrasekhar mass limit have been studied in other such formalisms as well [82] [32] [85] [83]. The effect on a compact star core is well studied in [108]. [109] and [33] study white dwarfs and their Chandrasekhar limit in detail using GUP (Generalized Uncertainity Principle). [89] studies in detail the modification of the equilibrium properties of blackbody radiation in a theory with an ultraviolet cut-off using MS formalism. Amongst many results presented we note that the Stefan-Boltzmann law gets modified in DSR and this result can be used to study various stellar objects. It is a known fact that white dwarfs radiate much less than any other massive celestial white body and that too, is mainly a surface phenomenon. The interior is completely degenerate but the surface, which radiates, is non-degenerate matter. Since the degenerate pressure and the density goes to zero at the surface of the white dwarf, we have a thin envelope of non-degenerate gas which is responsible for the radiation instead of the whole bulk (see section 5.3 of [110]). This model of degenerate core with a radiative envelope of non-degenerate matter can then be used to calculate the luminosity of the star, which gets a negative and nonperturbative correction in DSR. We anticipate the nonperturbativity in SR limit due to the presence of ultraviolet cut-off (for details see [89] [10]).

In this chapter our aim will be to study the thermodynamics of degenerate Fermi gas with

such a modified dispersion relation and an ultraviolet cut-off. Since we are considering massive Fermi particles, the study of the thermodynamics of such a massive particle gets separated in three different cases $m < \kappa$, $m = \kappa$ and $m > \kappa$ (for details see [10]) . In section 3.2 the modified thermodynamic pressure and the total energy in all the three possible mass cases has been discussed in detail. We will especially look into the mass and number density dependence of the degenerate pressure for $m < \kappa$ case in detail, both in SR (as almost no literature discusses this dependence) and the modified case. We will also briefly discuss the two extreme non-relativistic and the ultra-relativistic limits of the degenerate pressure. In section 3.3 white dwarf is taken as an example to show one of the possible implications. The correction to the Chandrasekhar limit for white dwarf in all the three cases will be looked into. We will also study the modified Lane-Emden equation and the general modified structure equation for the white dwarf in section 3.4. And in section 3.5 we consider the radiative envelope model of white dwarf and study in detail the modified luminosity-mass relationship for white dwarfs in both the relativistic and the non-relativistic regimes separately. Finally, we will summarize the whole chapter and suggest works that may be done in future.

3.2 Thermodynamics of degenerate Fermi gas

The dispersion relation of a particle gets deformed in a relativistic theory with an invariant ultraviolet energy scale. In the MS model of the DSR [3] [4] usual dispersion relation $E^2 - p^2 = m^2$ gets modified to (2.1) i.e.,

$$E^2 - p^2 = m^2\left(1 - \frac{E}{\kappa}\right)^2 \tag{3.1}$$

Here E, p and m are the total energy, magnitude of the 3-momentum and the rest mass energy of the particle and κ is the invariant energy scale of the DSR theory ($\hbar = 1, c = 1$ and $k_B = 1$ unless or otherwise stated explicitly). Note that the parameter m is "invariant

mass" and is not the physical rest mass of the particle. To obtain the physical rest mass m_0, we put $p = 0$ in the dispersion relation (2.1). The dispersion relation (2.1) gives,

$$m_0 = \frac{m}{1 + \frac{m}{\kappa}} \qquad \Longrightarrow \; m = \frac{m_0}{1 - \frac{m_0}{\kappa}} \tag{3.2}$$

where, $0 \leq m_0 \leq \kappa$ and $0 \leq m \leq \infty$. For a detailed calculation see [10]. Note that m increases monotonically with increasing m_0. We will now proceed to see the possible corrections to the Chandrasekhar limit of a white dwarf in DSR.

We will start by considering a grand canonical ensemble of a degenerate Fermi gas composed of N relativistic electrons obeying Fermi-Dirac statistics. The total number of particles can be calculated as (here the spin degeneracy $g = 2s + 1 = 2$),

$$N = \int\int \langle n_p \rangle \frac{g d^3x d^3p}{h^3} = \frac{V_{ac}}{\pi^2} \int_0^{\kappa} \frac{p^2 dp}{z^{-1} e^{\beta E(p)} + 1} \tag{3.3}$$

Here $\langle n_p \rangle$ is the mean occupation number and fugacity $z = e^{\beta\mu}$, μ being the chemical potential of the gas. For the Fermi gas at $T = 0K$, the mean occupation number is $\langle n_p \rangle = 1$ for $E < \mu_0$ and $\langle n_p \rangle = 0$ for $E > \mu_0$. Here $\mu_0 = E_F$ is the chemical potential at $T = 0K$. In such a case the number of particles become,

$$N = \int_{V_{ac}} \int_0^{p_F} \frac{d^3x d^3p}{h^3}.2 = \frac{V_{ac} p_F^3}{3\pi^2} \tag{3.4}$$

Here the accessible volume for a particle $V_{ac} = V - \frac{4\pi}{3\kappa^3}$ [89]. This gives the Fermi momentum,

$$p_F = (3\pi^2 n)^{1/3} \tag{3.5}$$

where $n = \frac{N}{V_{ac}}$ is the electron number density of the star. The grand canonical partition function for such a gas is given by (see [90]),

$$Q(V_{ac}, T) = \prod_E \left(1 + z e^{-\frac{E}{T}}\right) \tag{3.6}$$

The q-potential therefore becomes,

$$q \equiv \frac{P_{th}V_{ac}}{T} \equiv \ln Q(z, V_{ac}, T) = \sum_E \ln\left(1 + ze^{-\frac{E}{T}}\right). \tag{3.7}$$

Taking the large volume limit we get,

$$\left(\frac{P_{th}V_{ac}}{T}\right) = \frac{gV_{ac}}{(2\pi)^3}\int_0^{p_F} \ln\left[1 + ze^{-\beta E(p)}\right] 4\pi p^2 dp \tag{3.8}$$

Note that in such a large volume limit $V_{ac} = V - \frac{4\pi}{3\kappa^3} \approx V$. The dispersion relation (2.1) for an electron gives

$$\frac{dE}{dp} = \frac{p}{\frac{m^2}{\kappa} + \left(1 - \frac{m^2}{\kappa^2}\right)E} \tag{3.9}$$

Using the above equation along with (3.3) we obtain,

$$\begin{aligned} P_{th} &= \frac{T}{\pi^2}\left[\frac{p_F^3}{3}\ln\left[1 + ze^{-\beta E_F}\right]\right] + \frac{1}{3}n\left\langle p\frac{dE}{dp}\right\rangle \\ &= \ln[1 + ze^{-\beta E_F}]\left(\frac{NT}{V_{ac}}\right) + \frac{1}{3}n\left\langle p\frac{dE}{dp}\right\rangle \end{aligned} \tag{3.10}$$

Since we are taking Fermi gas at $T = 0K$, so the first term vanishes. Using equations (3.4) and (3.9) and then changing the integration variable to energy the expression of the thermodynamic pressure P_{th} comes out to be,

$$P_{th} = \frac{1}{3}n\,\frac{\int_{V_{ac}}\int_0^{p_F}\frac{d^3xd^3p}{(2\pi)^3}.2.p\frac{dE}{dp}}{\int_{V_{ac}}\int_0^{p_F}\frac{d^3xd^3p}{(2\pi)^3}.2} = \frac{1}{3\pi^2}\int_{m_0}^{E_F} dE\ \ [f(E)]^{3/2} \tag{3.11}$$

where $f(E)$ is given by,

$$f(E) = E^2 - m^2\left(1 - \frac{E}{\kappa}\right)^2 = (E - E_1)(E - E_2) \tag{3.12}$$

with $E_1 = m_0$ and $E_2 = \frac{m}{\frac{m}{\kappa}-1} = \frac{m_0}{\frac{2m_0}{\kappa}-1}$. The total energy is the next thing that we look into.

The expression for total energy U is given as,

$$U = \sum_E E\langle n_E \rangle, \tag{3.13}$$

which in large volume limit for degenerate case becomes,

$$U = \frac{V_{ac}}{\pi^2} \int_0^{p_F} p^2 \, dp \, E \tag{3.14}$$

Expressing the above in terms of the energy E we get,

$$U = \frac{V_{ac}}{\pi^2} \int_{m_0}^{E_F} E \left[E + \frac{m^2}{\kappa} \left(1 - \frac{E}{\kappa} \right) \right] \left[E^2 - m^2 \left(1 - \frac{E}{\kappa} \right)^2 \right]^{\frac{1}{2}} dE \tag{3.15}$$

As stated before (i.e, in massive case) and also looking at the expressions of the integrals, we have three different cases namely (see for example [10])

1. $m_0 < \frac{\kappa}{2} \Leftrightarrow m < \kappa$
2. $m_0 = \frac{\kappa}{2} \Leftrightarrow m = \kappa$
3. $m_0 > \frac{\kappa}{2} \Leftrightarrow m > \kappa$

We will now consider each case separately.

3.2.1 Case I: $m_0 < \frac{\kappa}{2} \Leftrightarrow m < \kappa$

In this case we have $-\infty < E_2 < 0 < E_1 = m_0 < \frac{\kappa}{2}$. The signature of $f(E)$, therefore, changes as follows

- $f(E)$ is +ve for the regions $E < E_2$ and $E > E_1 = m_0$
- $f(E)$ is -ve for $E_2 < E < E_1 = m_0$

The integrand in equation (3.11) remains real throughout the range of integration and for this case we have

$$[f(E)]^{3/2} = \left(1 - \frac{m^2}{\kappa^2}\right)^{3/2} \left[\left(E + \frac{m^2/\kappa}{1 - \frac{m^2}{\kappa^2}}\right)^2 - \frac{m^2}{\left(1 - \frac{m^2}{\kappa^2}\right)^2}\right]^{3/2}. \tag{3.16}$$

The expression of pressure for this case becomes,

$$P_{th} = \frac{1}{3\pi^2}\left(1 - \frac{m^2}{\kappa^2}\right)^{3/2} \int_{m'}^{E'_F} dE'(E'^2 - m'^2)^{3/2}. \tag{3.17}$$

Note that P_{th} is always positive, therefore we are considering the positive root only. Here,

$$E' = E + \frac{m^2/\kappa}{1 - \frac{m^2}{\kappa^2}} \tag{3.18}$$

and

$$m' = \frac{m}{1 - \frac{m^2}{\kappa^2}} \tag{3.19}$$

We now change the variable to x such that $E' = m' \cosh x$. This change is perfectly allowed as we are considering the positive root only. Note that the limits of E' ensures $E' \geq m'$ giving $\cosh x \geq 1$ (as E_F can take values from m_0 to κ). With the above substitution we have,

$$\int_{m'}^{E'_F} dE'(E'^2 - m'^2)^{3/2} = m'^4 \int_0^{x_F} dx \ \sinh^4 x = \frac{m'^4}{4}\left(\frac{1}{8}\sinh 4x_F - \sinh 2x_F + \frac{3}{2}x_F\right) \tag{3.20}$$

Here we choose x_F as the positive value of x corresponding to the Fermi energy ($E = E_F$), but choosing the negative value of x will not change the answer either. Therefore the expression of the thermodynamic pressure becomes,

$$P_{th} = \frac{1}{3\pi^2}\left(1 - \frac{m^2}{\kappa^2}\right)^{3/2} \frac{m'^4}{4}\left(\frac{1}{8}\sinh 4x_F - \sinh 2x_F + \frac{3}{2}x_F\right) \tag{3.21}$$

Note that the dispersion relation (2.1) gives

$$p_F^2 = f(E_F) = \left(1 - \frac{m^2}{\kappa^2}\right)\left[\left(E_F + \frac{m^2/\kappa}{1 - \frac{m^2}{\kappa^2}}\right)^2 - \frac{m^2}{\left(1 - \frac{m^2}{\kappa^2}\right)^2}\right] \tag{3.22}$$

The above equation leads to the value of Fermi energy as,

$$E_F = \frac{\sqrt{p_F^2\left(1 - \frac{m^2}{\kappa^2}\right) + m^2} - \frac{m^2}{\kappa}}{1 - \frac{m^2}{\kappa^2}} \tag{3.23}$$

which in turn gives,

$$\cosh x_F = \frac{E'_F}{m'} = \frac{\sqrt{p_F^2\left(1 - \frac{m^2}{\kappa^2}\right) + m^2}}{m} \tag{3.24}$$

For the familiar version, we will now express various expressions in terms of a new variable z_F to give,

$$\sinh x_F = \sqrt{\cosh^2 x_F - 1} = \frac{p_F}{m}\left(\sqrt{1 - \frac{m^2}{\kappa^2}}\right) = z_F \tag{3.25}$$

$$\sinh 2x_F = 2\sinh x_F \cosh x_F = 2z_F\sqrt{1 + z_F^2} \tag{3.26}$$

$$\cosh 2x_F = \sinh^2 x_F + \cosh^2 x_F = 1 + 2z_F^2 \tag{3.27}$$

$$\sinh 4x_F = 2\sinh 2x_F \cosh 2x_F = 4z_F\sqrt{1 + z_F^2}\left(1 + 2z_F^2\right) \tag{3.28}$$

Note that x_F, $\cosh x_F$, $\sinh x_F$, $\cosh 2x_F$, $\sinh 2x_F$, $\sinh 4x_F$, etc are all positive valued. Putting the above expressions, the thermodynamic pressure (3.21) becomes,

$$\begin{aligned} P_{th} &= \frac{m'^4}{24\pi^2}\left(1 - \frac{m^2}{\kappa^2}\right)^{3/2}\left[z_F\sqrt{1 + z_F^2}\left(2z_F^2 - 3\right) + 3\ln\left[z_F + \sqrt{1 + z_F^2}\right]\right] \\ &= \frac{m^4}{24\pi^2}\left(1 - \frac{m^2}{\kappa^2}\right)^{-5/2} C(z_F) \end{aligned} \tag{3.29}$$

where $C(u) = \left[u\sqrt{1 + u^2}\left(2u^2 - 3\right) + 3\ln\left[u + \sqrt{1 + u^2}\right]\right]$.

Note that the it is obvious from the expression of pressure (3.29) that we have explicit

dependence on three parameters namely modification parameter κ, mass m and number density n. The dependence of the degenerate pressure of Fermi gas on mass and number density is almost absent from the literature. Therefore, for comparison let us, for this case explore the results in SR case as well. The degenerate pressure in SR case comes out to be,

$$P_{th}^{SR} = \frac{m_0^4}{24\pi^2}\left[z\sqrt{1+z^2}\left(2z^2-3\right)+3\ln\left[z+\sqrt{1+z^2}\right]\right] = \frac{m_0^4}{24\pi^2}C(z) \tag{3.30}$$

here $z = \frac{(3\pi^2 n)^{1/3}}{m_0}$. We may now plot a contour map for the pressure in mass and number density. Note that each colored line in the contour plot would represent the constant pressure value i.e, $P(m,n) = \text{Constant}$. We can also define the tangent or slope of such a curve as,

$$\frac{dn}{dm} = \frac{\frac{-\partial P}{\partial m}}{\frac{\partial P}{\partial n}} \tag{3.31}$$

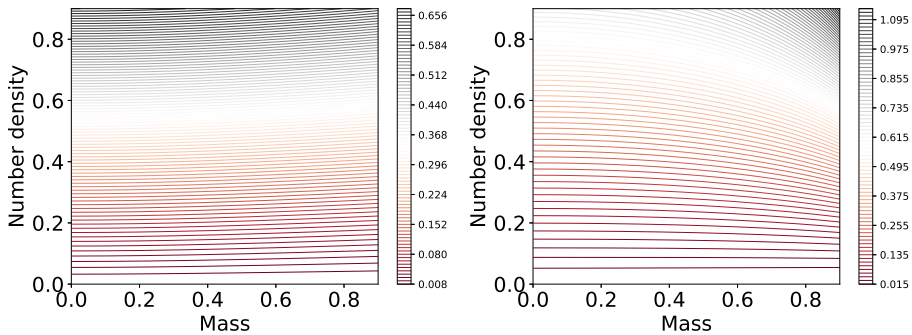
We can calculate the tangent $\frac{dn}{dm}$ for both the SR and the modified case. The slope in the SR case is,

$$\left.\frac{dn}{dm_0}\right|_{SR} = \left(\frac{-4}{m_0}\right)\frac{C(u)}{C'(u)\left(\frac{\partial u}{\partial n}\right)} - \frac{\left(\frac{\partial u}{\partial m_0}\right)}{\left(\frac{\partial u}{\partial n}\right)}, \tag{3.32}$$

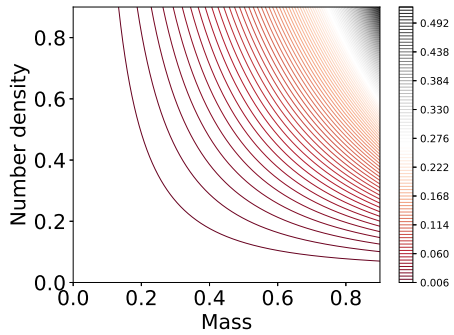
where $C'(u) = \sqrt{1+u^2}(2u^2-3) + \frac{u^2}{\sqrt{1+u^2}}(2u^2-3) + \sqrt{1+u^2}(4u^2) + \frac{3}{\sqrt{1+u^2}}$, $\left(\frac{\partial u}{\partial m_0}\right) = \left(\frac{-1}{m_0^2}\right)(3\pi^2 n)^{1/3}$ and $\left(\frac{\partial u}{\partial n}\right) = \left(\frac{\pi^2}{m_0}\right)(3\pi^2 n)^{-2/3}$. In modified case the slope comes out to be,

$$\frac{dn}{dm} = \left(\frac{-4}{m_0}\right)\frac{C(u)}{C'(u)\left(\frac{\partial u}{\partial n}\right)}\left(1-\frac{m^2}{\kappa^2}\right)^{-1/2} - \frac{\left(\frac{\partial u}{\partial m}\right)}{\left(\frac{\partial u}{\partial n}\right)}\left(1-\frac{m^2}{\kappa^2}\right)^{-1} - \left(\frac{5m}{\kappa^2}\right)\frac{C(u)}{C'(u)\left(\frac{\partial u}{\partial n}\right)}\left(1-\frac{m^2}{\kappa^2}\right)^{-3/2}, \tag{3.33}$$

with $\left(\frac{\partial u}{\partial m}\right) = \left(\frac{-1}{m^2}\right)(3\pi^2 n)^{1/3}$ and $\left(\frac{\partial u}{\partial n}\right) = \left(\frac{\pi^2}{m}\right)(3\pi^2 n)^{-2/3}$. Figure 3.1a and 3.1b shows the counter plots for the SR and the modified cases for typical values of mass and number density in Planck units. It is clearly visible from the figure 3.1 that the contours in plot



(a) Usual SR Pressure Contour Plot (b) Modified Pressure Contour Plot



(c) Difference of Modified and SR Pressure Contour Plot

Figure 3.1: The plot (a) shows contour plot of degenerate pressure of the ideal Fermi gas with mass m and number density n. The plot (b) shows the modified pressure contour plot. The plot (c) shows the $(P_m - P)$ contour plot. Each colored line here represents a constant pressure value and we have chosen the values in Planck units with $\kappa = 1$. We can clearly see that the difference is positive for this particular range of the plots. Another point to note is that contours in plot (a) have positive tangent $\frac{dn}{dm}$ but the contours in plot (b) have negative tangent. Also the correction is more for larger n and m values.

3.1a have positive slope $\frac{dn}{dm}$ whereas plot 3.1b shows that the slopes are negative. This transition may be attributed to the dominance of the extra negative term that we get in the modified case which was not present in the usual SR case, which becomes more and more dominant for the larger mass m values. Note that the plot 3.1c shows that the correction is positive for this particular range of mass and number density. But the correction can be positive or negative depending on the selected range of the m and n. And another thing to note is that the correction is more prominent for larger n and m values for the chosen scale $\kappa = 1$. Note that the value of the scale here is $\kappa = 1$ for illustration, but can be appropriately chosen for the given scenario. Another important thing that we can now look into are the two extreme non-relativistic and ultra-relativistic cases.

1. $z << 1$**: Non-relativistic SR case**

   Using equation (4.6.31) of [98], the expression for the degenerate pressure becomes (keeping only the lowest orders),

$$P_{th}^{SR} \approx \frac{m_0^4}{24\pi^2}\left(\frac{8}{5}z^5 - \frac{4}{7}z^7\right) = \frac{(3\pi)^{5/3}}{15\pi^2}\frac{n^{5/3}}{m_0} - \frac{(3\pi)^{7/3}}{42\pi^2}\frac{n^{7/3}}{m_0^3}. \tag{3.34}$$

   This clearly implies the inverse dependence on the mass.

2. $z >> 1$**: Ultra-relativistic SR case**

   Here again, using equation (4.6.31) of [98], the expression for the degenerate pressure becomes (keeping only the highest order),

$$P_{th}^{SR} \approx \frac{m_0^4}{24\pi^2}2z^4 = \frac{(3\pi)^{4/3}}{12\pi^2}n^{4/3}. \tag{3.35}$$

   Note that there is no mass dependence in this case.

We may now play the same game in modified case as well using the expressions (3.29) and (3.24). The corresponding expressions in modified case are as follows,

1. $z_F << 1$**: Non-relativistic SR case**

The expression for the degenerate pressure becomes (keeping only the lowest orders),

$$P_{th} \approx \frac{m^4}{24\pi^2}\left(1-\frac{m^2}{\kappa^2}\right)^{-5/2}\left(\frac{8}{5}z_F^5-\frac{4}{7}z_F^7\right)=\frac{(3\pi)^{5/3}}{15\pi^2}\frac{n^{5/3}}{m}-\frac{(3\pi)^{7/3}}{42\pi^2}\frac{n^{7/3}}{m^3}\left(1-\frac{m^2}{\kappa^2}\right). \tag{3.36}$$

Note that the correction comes in the next order.

2. $z_F >> 1$**: Ultra-relativistic SR case**

   The expression for the degenerate pressure becomes (keeping only the highest order),

$$P_{th} \approx \frac{m^4}{24\pi^2}2\left(1-\frac{m^2}{\kappa^2}\right)^{-5/2} z_F^4 = \frac{(3\pi)^{4/3}}{12\pi^2}\left(1-\frac{m^2}{\kappa^2}\right)^{-1/2} n^{4/3}. \tag{3.37}$$

   Note that there is a mass dependence in this case as opposed to the SR one. It is obvious that the SR limit gives the usual SR results given before.

Another very important point to note at this juncture is to see whether the expression for pressure (3.29) is nonperturbative or perturbative in the SR limit as we expect the expression to be nonperturbative in theories with an ultraviolet cut-off (see for example [10] and [89]). To see such a behavior we expand the above expression in m^2/κ^2 to get,

$$P_{th} = P_{th}^{SR} + \frac{m^2}{\kappa^2}\left[\frac{5}{2}P_{th}^{SR} - \frac{m^4}{24\pi^2}\left\{(1+u^2)^{-1/2}\left(u^4+\frac{3}{2}u^3+\frac{3}{2}u\right)+(1+u^2)^{1/2}\left(3u^2+\frac{3}{2}u\right)\right\}\right] + O\left(\frac{m^4}{\kappa^4}\right). \tag{3.38}$$

This expression clearly shows that the the expression in the SR limit in perturbative as oppose to the expectation. In typical DSR case the SR limit takes the upper limit of the integral over the single particle energy, describing the thermodynamic quantities, to infinity which is nonperturbative in nature. On the other hand no such thing happens for a degenerate Fermi gas. The Fermi energy E_F remains the upper limit of the integral in both

SR and DSR. This explains the absence of nonperturbativity. Therefore, what separates the results of DSR and SR in case of degenerate Fermi gas is the presence of the modified dispersion relation.

The expression of total energy U can be similarly calculated using (3.15) as,

$$\begin{aligned}
U &= \frac{V_{ac}}{\pi^2}\left(1-\frac{m^2}{\kappa^2}\right)^{5/2}\int_{m'}^{E'_F} dE'E'^2(E'^2-m'^2)^{1/2} \\
&- \frac{V_{ac}}{3\pi^2}\left(1-\frac{m^2}{\kappa^2}\right)^{3/2}\left(\frac{m^2}{\kappa}\right)\int_{m'}^{E'_F} dE'E'(E'^2-m'^2)^{1/2} \\
&= \frac{V_{ac}}{\pi^2}\left(1-\frac{m^2}{\kappa^2}\right)^{5/2} m'^4\left[\int_0^{x_F} dx\ (\cosh\ x-1)\ \sinh^2 x\ \cosh\ x\right] \\
&= \frac{V_{ac}}{\pi^2}\left(1-\frac{m^2}{\kappa^2}\right)^{5/2} m'^4\left(\frac{1}{32}\sinh 4x_F-\frac{1}{3}\sinh^3 x_F-\frac{1}{8}x_F\right) \\
&= \frac{V_{ac}}{24\pi^2}\left(1-\frac{m^2}{\kappa^2}\right)^{5/2} m'^4\left[8z_F^3\left(\sqrt{1+z_F^2}-1\right)-C(z_F)\right] = \frac{m^4V_{ac}}{24\pi^2}\left(1-\frac{m^2}{\kappa^2}\right)^{-3/2} D(z_F)
\end{aligned} \tag{3.39}$$

where $D(u) = \left[8u^3\left(\sqrt{1+u^2}-1\right)-C(u)\right]$ and $C(u)$ is as defined above.

3.2.2 Case II: $m_0 = \frac{\kappa}{2} \Leftrightarrow m = \kappa$

For this case the $f(E)$ becomes,

$$f(E) = 2\kappa\left(E-\frac{\kappa}{2}\right) \tag{3.40}$$

Putting the above in (3.11) we get the expression of the pressure as

$$P_{th} = \frac{2(2\kappa)^{3/2}}{15\pi^2}\left(E_F-\frac{\kappa}{2}\right)^{5/2} \tag{3.41}$$

Now using the dispersion relation (2.1) and (3.5), in this case we have

$$E_F-\frac{\kappa}{2} = \frac{p_F^2}{2\kappa} = \frac{(3\pi^2 n)^{2/3}}{2\kappa} \tag{3.42}$$

The expression of the total energy, therefore, in this case becomes,

$$U = \frac{\sqrt{2}V_{ac}\kappa^{3/2}}{3\pi^2}\left[\frac{2}{5}\left(E_F - \frac{\kappa}{2}\right)^{5/2} - \frac{\kappa}{3}\left(E_F - \frac{\kappa}{2}\right)^{3/2}\right] \tag{3.43}$$

3.2.3 Case III: $m_0 > \frac{\kappa}{2} \Leftrightarrow m > \kappa$

In this case we have $0 < \frac{\kappa}{2} < E_1 = m_0 < \kappa < E_2 < \infty$. The signature of $f(E)$ therefore changes as follows

- $f(E)$ is -ve for the regions $E < E_1 = m_0$ and $E > E_2$
- $f(E)$ is +ve for $m_0 = E_1 < E < E_2$

Again the integrand in equation (3.11) remains real throughout the range of integration and is given by

$$[f(E)]^{3/2} = \left(\frac{m^2}{\kappa^2} - 1\right)^{3/2}\left[\frac{m^2}{\left(\frac{m^2}{\kappa^2} - 1\right)^2} - \left(E - \frac{m^2/\kappa}{\frac{m^2}{\kappa^2} - 1}\right)^2\right]^{3/2} \tag{3.44}$$

Thus the pressure therefore becomes

$$P_{th} = \frac{1}{3\pi^2}\left(\frac{m^2}{\kappa^2} - 1\right)^{3/2}\int_{-m''}^{-E''_F} dE'(m''^2 - E'^2)^{3/2} \tag{3.45}$$

where

$$E''_F = -E'_F \tag{3.46}$$

and

$$m'' = -m', \tag{3.47}$$

where E'_F is corresponding to (3.18) and m' is given by (3.19). Here, E''_F and m'' take only positive values and $0 \leq E''_F \leq m''$. Now we again make the change of variable to y such

that

$$E' = -m'' \cos y \tag{3.48}$$

Limits of E' ensures that $0 \geq E' \geq -m'' \Rightarrow 0 \leq \cos y \leq 1$. Using this substitution we get,

$$\int_{-m''}^{-E''_F} dE'(m''^2 - E'^2)^{3/2} = m''^4 \int_0^{y_F} dy \ \sin^4 y = \frac{m''^4}{4}\left(\frac{1}{8}\sin 4y_F - \sin 2y_F + \frac{3}{2}y_F\right) \tag{3.49}$$

Here y_F is the value of y corresponding to the Fermi energy $E = E_F$ and $0 \leq y_F \leq \frac{\pi}{2}$. The expression of the thermodynamic pressure therefore becomes,

$$P_{th} = \frac{1}{3\pi^2}\left(\frac{m^2}{\kappa^2} - 1\right)^{3/2} \frac{m''^4}{4}\left(\frac{1}{8}\sin 4y_F - \sin 2y_F + \frac{3}{2}y_F\right) \tag{3.50}$$

Now, the dispersion relation (2.1) gives

$$p_F^2 = f(E_F) = \left(\frac{m^2}{\kappa^2} - 1\right)\left[\frac{m^2}{\left(\frac{m^2}{\kappa^2} - 1\right)^2} - \left(E_F - \frac{m^2/\kappa}{\frac{m^2}{\kappa^2} - 1}\right)^2\right]. \tag{3.51}$$

Rearranging the above equation we have,

$$E'_F = -\frac{\sqrt{m^2 - p_F^2\left(\frac{m^2}{\kappa^2} - 1\right)}}{\frac{m^2}{\kappa^2} - 1} \tag{3.52}$$

Note that in this case E'_F is $-ve$. The Fermi Energy E_F is therefore given by

$$E_F = \frac{\frac{m^2}{\kappa} - \sqrt{m^2 - p_F^2\left(\frac{m^2}{\kappa^2} - 1\right)}}{\frac{m^2}{\kappa^2} - 1}. \tag{3.53}$$

Note that for all values of $p_F \in [0, \kappa]$, the expression inside the square-root is always positive. But the value of p_F can at most be κ and in this case $m > \kappa$, therefore we only get the non-relativistic particles or at most relativistic particles. Physically this means that the particles are so heavy that their ultra-relativistic motion is not possible. The

above value of Fermi energy E_F gives us,

$$\cos y_F = -\frac{E'_F}{m''} = \frac{\sqrt{m^2 - p_F^2\left(\frac{m^2}{\kappa^2} - 1\right)}}{m} \tag{3.54}$$

As seen before, we will express the various quantities in terms of variable q_F as,

$$\sin y_F = \sqrt{1 - \cos^2 y_F} = q_F \tag{3.55}$$

$$\sin 2y_F = 2\sin y_F \cos y_F = 2q_F\sqrt{1 - q_F^2} \tag{3.56}$$

$$\cos 2y_F = \cos^2 y_F - \sin^2 y_F = 1 - 2q_F^2 \tag{3.57}$$

$$\sin 4y_F = 2\sin 2y_F \cos 2y_F = 4q_F\sqrt{1 - q_F^2}\left(1 - 2q_F^2\right) \tag{3.58}$$

In this case also y_F, $\cos y_F$, $\sin y_F$, $\cos 2y_F$, $\sin 2y_F$, $\sin 4y_F$, etc are all positive valued. Putting the above values in (3.49) we get,

$$\int_{-m''}^{-E''_F} dE'(m''^2 - E'^2)^{3/2} = \frac{m''^4}{8}\left[3\sin^{-1}(q_F) - q_F\sqrt{1 - q_F^2}\left(2q_F^2 + 3\right)\right] \tag{3.59}$$

Hence, the thermodynamic pressure (3.50) becomes,

$$P_{th} = \frac{m''^4}{24\pi^2}\left(\frac{m^2}{\kappa^2} - 1\right)^{3/2}\left[3\sin^{-1}(q_F) - q_F\sqrt{1 - q_F^2}\left(2q_F^2 + 3\right)\right] = \frac{m^4}{24\pi^2}\left(\frac{m^2}{\kappa^2} - 1\right)^{-5/2} J(q_F) \tag{3.60}$$

where $J(u) = \left[3\sin^{-1}(u) - u\sqrt{1 - u^2}\left(2u^2 + 3\right)\right]$. Using (3.15) the total energy in this case can be easily calculated as,

$$\begin{aligned} U &= \frac{V_{ac}}{\pi^2}\left(\frac{m^2}{\kappa^2} - 1\right)^{5/2}\int_{-m''}^{-E''_F} dE'E'^2(m''^2 - E'^2)^{1/2} \\ &+ \frac{V_{ac}}{3\pi^2}\left(\frac{m^2}{\kappa^2} - 1\right)^{3/2}\left(\frac{m^2}{\kappa}\right)\int_{-m''}^{-E''_F} dE'E'(m''^2 - E'^2)^{1/2} \\ &= \frac{V_{ac}}{\pi^2}\left(\frac{m^2}{\kappa^2} - 1\right)^{5/2} m''^4\left[\int_0^{y_F} dx\ (\cos y - 1)\ \sin^2 y\ \cos y\right] \end{aligned}$$

$$= \frac{V_{ac}}{\pi^2}\left(\frac{m^2}{\kappa^2}-1\right)^{5/2} m''^4\left(-\frac{1}{32}\sin 4y_F - \frac{1}{3}\sin^3 y_F + \frac{1}{8}y_F\right)$$
$$= \frac{V_{ac}}{24\pi^2}\left(\frac{m^2}{\kappa^2}-1\right)^{5/2} m''^4\left[8q_F^3\left(\sqrt{1-q_F^2}-1\right)+C(q_F)\right] = \frac{m^4 V_{ac}}{24\pi^2}\left(1-\frac{m^2}{\kappa^2}\right)^{-3/2} K(z_F) \tag{3.61}$$

where $K(u) = \left[8u^3\left(\sqrt{1-u^2}-1\right)+J(u)\right]$ and $J(u)$ is as defined above.

The discussion till now is a modification in the thermodynamics of ideal relativistic Fermi gas in an effective theory with invariant ultraviolet cut-off. These results are valid at any energy depending on the choice of the of the parameters n, m and the scale κ. One of the places which finds direct application of the degenerate Fermi gas is the study of the dynamics of the white dwarf stars. In the next section we explore the modified dynamics of such a typical white dwarf star as one of the examples of this formalism.

3.3 White Dwarf: An Example

A typical model of a white dwarf star consists of N free electrons and $\frac{N}{2}$ helium nuclei. The mass of the star is given by ($m_n \simeq m_p$)

$$M = Nm_e + \frac{N}{2}\left(2m_n + 2m_p\right) \simeq Nm_e + 2Nm_p = N(m_e + 2m_p). \tag{3.62}$$

Here m_e and m_p are the rest masses of the electron and the proton respectively. In such stars the pressure support is given by the non-interacting (ideal) gas of degenerate electrons and the mass density is mainly non-degenerate carbon or helium ions [110] [111]. We may neglect the presence of the helium nuclei contribution to pressure as they do not contribute significantly to the dynamics of the problem but only the mass. The internal temperature of the white dwarf is of the order of 10^7 K which is obviously not enough to hold the star against the self gravitational collapse. It is easy to see that the Fermi energy of the electrons E_F is of the order of 10^9 K, which is higher than the average kinetic en-

ergy of the electron (T). Hence the degeneracy condition holds and the gas of electrons can be approximated as a zero-temperature Fermi gas [110–113]. For a similar reason the effect of the radiation as well can be neglected as we choose $T = 0K$. Since all the levels upto the Fermi level are filled (system is in the ground state) and therefore there is no radiation effect showing up in the dynamics as a first approximation. Note, however, that it is an observed fact that the white dwarfs radiate and so have a finite luminosity that leads to its cooling. Therefore, they must have a thin radiative envelope along with a degenerate core. Thus, the complete degenerate matter approximation is valid except at the thin non-degenerate surface envelope responsible for the finite luminosity of the white dwarf star. This we will discuss in detail in the luminosity section later in this chapter.

We will take the star to be spherical in shape and therefore the change in the total thermodynamic energy of the star (dE_{th}) due to infinitesimal change in the radius of the star (dR) is given by

$$dE_{th} = P_{th}dV = 4\pi R^2 P_{th}dR \tag{3.63}$$

Whereas the change in the gravitational energy is given by,

$$dE_g = \alpha\frac{GM^2}{R^2}dR \tag{3.64}$$

where α is of the order 1. The exact value of α will depend on the spatial variation of n. At equilibrium we have $dE_{th} = dE_g$, which leads to

$$P_{th} = \frac{\alpha GM^2}{4\pi R^4} \tag{3.65}$$

3.3.1 Case I: $m_0 < \frac{\kappa}{2} \Leftrightarrow m < \kappa$

Using (3.62), (3.5), (3.18) and (3.19) in (3.24)we get

$$\cosh x_F = \sqrt{1 + A^2\frac{M^{2/3}}{R^2}} \tag{3.66}$$

where

$$A = \frac{1}{\sqrt{mm'}}\left(\frac{9\pi}{4m_e + 8m_p}\right)^{1/3} = \frac{\left(1 - \frac{m_e}{\kappa}\right)\left[1 - \left(\frac{m_e}{\kappa - m_e}\right)^2\right]^{1/2}}{m_e}\left(\frac{9\pi}{4m_e + 8m_p}\right)^{1/3} \tag{3.67}$$

Note that here $A\frac{M^{1/3}}{R} = \frac{p_F}{m}\left(\sqrt{1 - \frac{m^2}{\kappa^2}}\right)$ gives relation between A and p_F, which in the SR limit gives the correct expected result. We will now express various expressions in terms of $A\frac{M^{1/3}}{R}$ to give,

$$\sinh x_F = \sqrt{\cosh^2 x_F - 1} = A\frac{M^{1/3}}{R} \tag{3.68}$$

$$\sinh 2x_F = 2\sinh x_F \cosh x_F = 2A\frac{M^{1/3}}{R}\sqrt{1 + A^2\frac{M^{2/3}}{R^2}} \tag{3.69}$$

$$\cosh 2x_F = \sinh^2 x_F + \cosh^2 x_F = 1 + 2A^2\frac{M^{2/3}}{R^2} \tag{3.70}$$

$$\sinh 4x_F = 2\sinh 2x_F \cosh 2x_F = 4A\frac{M^{1/3}}{R}\sqrt{1 + A^2\frac{M^{2/3}}{R^2}}\left(1 + 2A^2\frac{M^{2/3}}{R^2}\right) \tag{3.71}$$

Putting the above expressions the thermodynamic pressure (3.21) becomes,

$$\begin{aligned} P_{th} = \frac{m'^4}{24\pi^2}\left(1 - \frac{m^2}{\kappa^2}\right)^{3/2}&\left[A\frac{M^{1/3}}{R}\sqrt{1 + A^2\frac{M^{2/3}}{R^2}}\left(2A^2\frac{M^{2/3}}{R^2} - 3\right)\right. \\ &\left. +3\ln\left[\left(A\frac{M^{1/3}}{R}\right) + \sqrt{1 + \left(A^2\frac{M^{2/3}}{R^2}\right)}\right]\right] \end{aligned} \tag{3.72}$$

Equating equation (3.65) with (3.72) we get the mass-radius relationship of the white dwarf star in DSR as,

$$\begin{aligned} \frac{m'^4}{24\pi^2}\left(1 - \frac{m^2}{\kappa^2}\right)^{3/2}&\left[A\frac{M^{1/3}}{R}\sqrt{1 + A^2\frac{M^{2/3}}{R^2}}\left(2A^2\frac{M^{2/3}}{R^2} - 3\right)\right. \\ &\left. +3\ln\left[\left(A\frac{M^{1/3}}{R}\right) + \sqrt{1 + \left(A^2\frac{M^{2/3}}{R^2}\right)}\right]\right] = \frac{\alpha G M^2}{4\pi R^4} \end{aligned} \tag{3.73}$$

In the SR limit ($\kappa \to \infty$) we have $m' \to m_e$ (see equation (3.19)), $A \to \frac{1}{m_e}\left(\frac{9\pi}{4m_e + 8m_p}\right)^{1/3}$ (see equation (3.67)) and hence the above expression reduces to the correct relationship

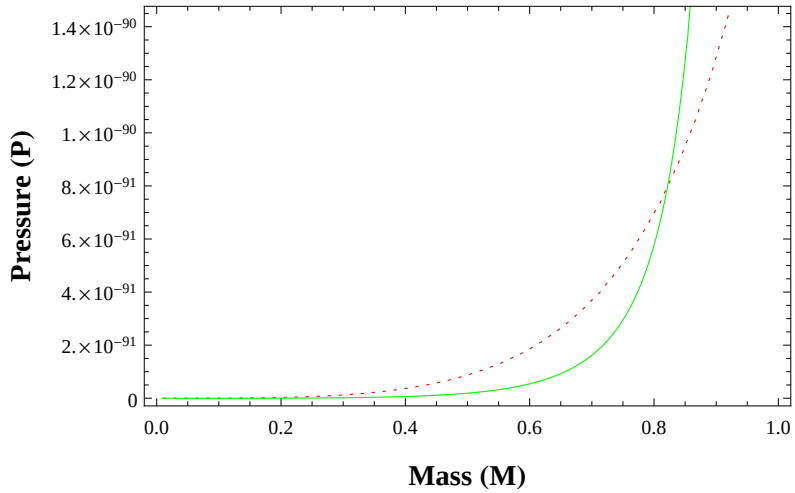


Figure 3.2: Figure showing the variation of degeneracy pressure (P) with mass (M). Here also the mass is expressed in terms of the limiting mass M_0^{SR} and the radius in terms of the Characteristic length of the order 10^{42} in Planck units. In this case too the value of the parameter κ is $\kappa = 10^{-22}$. Here green line denotes the SR and the red dotted denotes the modified relation. Note that the value of the modified pressure is greater than the SR value for certain masses, equals and crosses to become less than the SR value for certain masses of the white dwarf as expected.

as given in section 8.5 of [90], with the identification that $x = A\frac{M^{\frac{1}{3}}}{R}$. The M-R plot is shown in figure 3.3. It is clearly obvious from figure 3.3 that the modified M-R relation is below the SR one for some values, equals the SR value at the crossing point and is greater than the SR value for certain masses of the white dwarfs. And figure 3.2 shows that the equilibrium degeneracy pressure is greater than the SR value for certain masses, equals and crosses to become less than the SR value for certain masses. This is easy to see as for a given mass value the thermodynamic pressure is inversely related to radius and since the radius for modified case is lesser than, equal to and greater than the SR case and so the modified pressure is expected to be greater than, equal to and less than corresponding the SR value. Since energy density is related directly to mass density as Mass Density $= \frac{\text{Energy Density}}{c^2}$, therefore denser compact objects are expected to show better measurable correction due to such a modification. Note that in the plots, the scale κ is chosen to be $\kappa = 10^{-22}$ for illustration. Similar scale is noted by many who attempt to study the corrections to the dynamics of compact objects with modified dispersion relations [114] [115] [33]. This is the effective semi-classical scale where the effects starts to show up for considered white dwarf model. Note that this scale is of the order

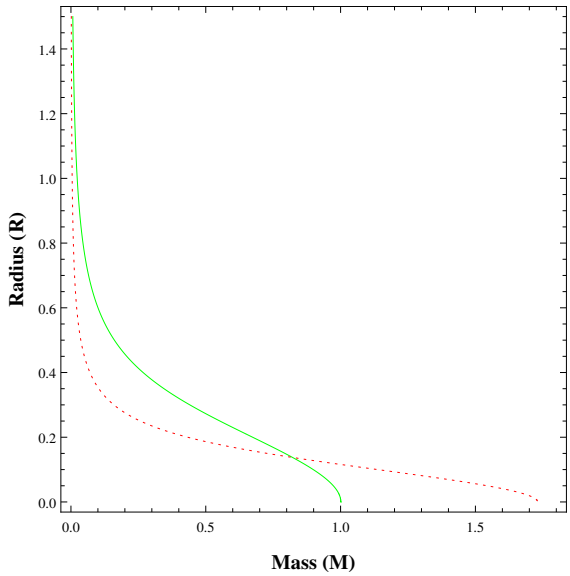


Figure 3.3: Figure showing the Mass-Radius relationship. The mass is expressed in terms of the limiting mass M_0^{SR} and the radius in terms of the Characteristic length of the order 10^{42} in Planck units. The value of the parameter κ is $\kappa = 10^{-22}$. Here green line denotes the SR and the red dotted denotes the modified relation. Note that the modified M-R relation is below the SR one for some values of mass, equals the SR value at the crossing point and is greater than the SR value for certain masses of the white dwarfs. Note that the fact that the modified Chandrasekhar mass is greater than the SR mass is clearly visible from the plot.

of the energy of the constituent particles as expected. In future compact stars which are much denser can be studied where this scale might appropriately near the Planck scale. We will now look at both the non-relativistic and ultra-relativistic limits of the above obtained result.

1. $R >> AM^{1/3}$**: Non-relativistic case**

   In such a limit, as stated before, using equation (4.6.31) of [98], the pressure (3.72) becomes,

$$P_{th} \simeq \frac{m'^4}{15\pi^2}\left(1-\frac{m^2}{\kappa^2}\right)^{3/2}\left(A^5\frac{M^{5/3}}{R^5}\right), \tag{3.74}$$

   and the mass-radius relationship becomes

$$\frac{m'^4 A^5}{5}\frac{M^{5/3}}{R^5} \simeq \frac{3\pi\alpha G}{4}\left(1-\frac{m^2}{\kappa^2}\right)^{-3/2}\frac{M^2}{R^4} \tag{3.75}$$

This in turn gives

$$R \simeq \frac{4m'^4A^5}{15\pi\alpha G}\left(1-\frac{m^2}{\kappa^2}\right)^{3/2} M^{-1/3} \tag{3.76}$$

The SR limit gives the expected result (see for example equation (21) of section 8.5 of [90]).

2. $R << AM^{1/3}$**: Ultra-relativistic case**

Here again using equation (4.6.31) of [98], in this limit the pressure (3.72) becomes

$$P_{th} \simeq \frac{m'^4}{12\pi^2}\left(1-\frac{m^2}{\kappa^2}\right)^{3/2}\left[A^4\frac{M^{4/3}}{R^4}-A^2\frac{M^{2/3}}{R^2}\right], \tag{3.77}$$

and the relationship gives

$$\frac{m'^4A^4}{4}\frac{M^{4/3}}{R^4}\left(1-\frac{R^2}{A^2M^{2/3}}\right) \simeq \frac{3\pi\alpha G}{4}\left(1-\frac{m^2}{\kappa^2}\right)^{-3/2}\frac{M^2}{R^4} \tag{3.78}$$

We therefore have,

$$R \simeq AM^{1/3}\left[1-\left(\frac{M}{M_0}\right)^{2/3}\right]^{1/2} \tag{3.79}$$

where

$$M_0 = \frac{(m'A)^6}{(3\pi\alpha G)^{3/2}}\left(1-\frac{m^2}{\kappa^2}\right)^{9/4} \tag{3.80}$$

which is the modified Chandrasekhar mass limit, which is the maximum stable mass for a white dwarf in this Chandrasekhar model (see [116]).

Therefore, using (3.19) and (3.67) the relation between SR and DSR case is,

$$M_0 = \frac{M_0^{SR}}{\left[1-\frac{m^2}{\kappa^2}\right]^{\frac{3}{4}}} \tag{3.81}$$

Here also this gives the correct SR limit equations (see equation (22) and (23) of section 8.5 of [90]) as expected. Since in this case $m < \kappa$, so the denominator of the above equation is less than 1 , which implies that the Chandrasekhar mass limit of the white dwarf has actually increased. This fact is clearly visible from the figure 3.3. Interestingly, the

SR limit of the above relation is purely perturbative and has no nonperturbative signature as was discussed before. The DSR correction to the Chandrasekhar limit comes solely because of the modification in the dispersion relation as the energy cut-off does not effect the calculation.

3.3.2 Case II: $m_0 = \frac{\kappa}{2} \Leftrightarrow m = \kappa$

In this case, for a spherical model of star we have $n = \frac{N}{V} = \frac{3M}{8\pi m_p R^3}$ and therefore the thermodynamic pressure becomes

$$P_{th} = \frac{1}{15\pi^{1/3}\kappa}\left(\frac{9}{4m_e + 8m_p}\right)^{5/3}\frac{M^{5/3}}{R^5} \tag{3.82}$$

Equating the above to (3.65) we get the mass-radius relationship as

$$R = \frac{4}{15\pi\alpha\kappa G}\left(\frac{9\pi}{4m_e + 8m_p}\right)^{5/3} M^{-1/3} \tag{3.83}$$

We conclude that there is no Chandrasekhar limit in this case for the masses of the white-dwarf stars.

3.3.3 Case III: $m_0 > \frac{\kappa}{2} \Leftrightarrow m > \kappa$

Playing the same game, using (3.62) and (3.5) we have

$$\cos y_F = -\frac{E'_F}{m''} = \sqrt{1 - B^2\frac{M^{2/3}}{R^2}} \tag{3.84}$$

where

$$B = \frac{1}{\sqrt{mm''}}\left(\frac{9\pi}{4m_e + 8m_p}\right)^{1/3} \tag{3.85}$$

As seen before, we will express the various quantities in terms of $B\frac{M^{1/3}}{R}$ as,

$$\sin y_F = \sqrt{1-\cos^2 y_F} = B\frac{M^{1/3}}{R} \tag{3.86}$$

$$\sin 2y_F = 2\sin y_F \cos y_F = 2B\frac{M^{1/3}}{R}\sqrt{1-B^2\frac{M^{2/3}}{R^2}} \tag{3.87}$$

$$\cos 2y_F = \cos^2 y_F - \sin^2 y_F = 1 - 2B^2\frac{M^{2/3}}{R^2} \tag{3.88}$$

$$\sin 4y_F = 2\sin 2y_F \cos 2y_F = 4B\frac{M^{1/3}}{R}\sqrt{1-B^2\frac{M^{2/3}}{R^2}}\left(1-2B^2\frac{M^{2/3}}{R^2}\right) \tag{3.89}$$

In this case also y_F, $\cos y_F$, $\sin y_F$, $\cos 2y_F$, $\sin 2y_F$, $\sin 4y_F$, etc are all positive valued. Putting the above in (3.49)

$$\begin{aligned}\int_{-m''}^{-E''_F} dE'(m''^2 - E'^2)^{3/2} = \frac{m''^4}{8}\Bigg[&3\sin^{-1}\left(B\frac{M^{1/3}}{R}\right)\\ &-B\frac{M^{1/3}}{R}\sqrt{1-B^2\frac{M^{2/3}}{R^2}}\left(2B^2\frac{M^{2/3}}{R^2}+3\right)\Bigg]\end{aligned} \tag{3.90}$$

Hence, the thermodynamic pressure (3.50) is given by

$$\begin{aligned}P_{th} = \frac{m''^4}{24\pi^2}\left(\frac{m^2}{\kappa^2}-1\right)^{3/2}\Bigg[&3\sin^{-1}\left(B\frac{M^{1/3}}{R}\right)\\ &-B\frac{M^{1/3}}{R}\sqrt{1-B^2\frac{M^{2/3}}{R^2}}\left(2B^2\frac{M^{2/3}}{R^2}+3\right)\Bigg]\end{aligned} \tag{3.91}$$

Now at equilibrium we equate the above expression of pressure to (3.65) and get the mass-radius relationship of the white dwarf star as,

$$\frac{m''^4}{24\pi^2}\left(\frac{m^2}{\kappa^2}-1\right)^{3/2}\left[3\sin^{-1}\left(B\frac{M^{1/3}}{R}\right) - B\frac{M^{1/3}}{R}\sqrt{1-B^2\frac{M^{2/3}}{R^2}}\left(2B^2\frac{M^{2/3}}{R^2}+3\right)\right] = \frac{\alpha G M^2}{4\pi R^4} \tag{3.92}$$

The equation (3.86) tells us that $0 \leq B\frac{M^{1/3}}{R} \leq 1 \Rightarrow R \geq BM^{1/3}$ as $0 \leq y_F \leq \frac{\pi}{2}$. The asymptotic behavior $R >> BM^{1/3}$ of the mass-radius relationship, using equation (4.4.40)

of [98], is given by

$$R \simeq \frac{4m''^4 B^5}{15\pi\alpha G}\left(\frac{m^2}{\kappa^2}-1\right)^{3/2} M^{-1/3} \tag{3.93}$$

Again, there is no limit for the masses of the white-dwarf stars as expected. Note that the Chandrasekhar limit only in ultra-relativistic case which is not possible.

We conclude this section by noting that we only get the Chandrasekhar limit in ultra-relativistic limit of the $m < \kappa$ case, which apparently is the physical case. We also note that this correction is actually positive. Therefore the mass- radius relation has changed and so the radius of white dwarf is lower, equal to and greater than the SR values for given masses. The decrease in radius is theoretically predicted [109] [33] and experimentally observed as well [117] [118] [119] [120]. This analysis may not be the only explanation of the observed decrease but is surely an attempt. In future we may observe the white dwarfs with radius greater than that predicted by the present SR theory.

3.4 Modified Structure equations

In previous sections we considered the matter density to be constant. In this section we will consider variable matter density and obtain the static structure of the white dwarf using stellar structure equations [113]. As is obvious from the analysis of the previous section that the Chandrasekhar limit is obtained only for case I i.e., $m < \kappa$. We will, therefore, consider that case only for the present analysis.

3.4.1 Lane-Emden equation and the Chandrasekhar mass

For the stellar model we first need to express the expressions in terms of the matter density ρ instead of number density. We will use the mean molecular weight $\mu = \frac{\rho}{nm_H}$, where $\frac{1}{m_H} = N_A$ as the gas constant (Avogadro's Number), $m_H = 1.6605 \times 10^{-27}$ Kg is atomic mass constant (equal to mass of hydrogen atom for all practical purposes) and n is number

density as usual. Therefore, for $m < \kappa$ case the Fermi momentum is given as,

$$p_F = \left(A\frac{M^{1/3}}{R}\right)\frac{m}{\sqrt{1-\frac{m^2}{\kappa^2}}}$$
$$= (3\pi^2 n)^{1/3} = \left(3\pi^2\frac{\rho}{\mu m_H}\right)^{1/3} \tag{3.94}$$

The pressure in non-relativistic case becomes,

$$P_{th} \simeq \frac{m'^4}{15\pi^2}\left(1-\frac{m^2}{\kappa^2}\right)^{3/2}\left(A^5\frac{M^{5/3}}{R^5}\right) = C_1\rho^{5/3}, \tag{3.95}$$

where $C_1 = \frac{(3\pi^2)^{2/3}}{5m}\frac{1}{(\mu m_H)^{5/3}}$. Similarly, the pressure in ultra-relativistic case becomes,

$$P_{th} \simeq \frac{m'^4}{12\pi^2}\left(1-\frac{m^2}{\kappa^2}\right)^{3/2}\left(A^4\frac{M^{4/3}}{R^4}\right) = C_2\rho^{4/3}, \tag{3.96}$$

where $C_2 = \frac{1}{4}\left(1-\frac{m^2}{\kappa^2}\right)^{-1/2}(3\pi^2)^{1/3}\frac{1}{(\mu m_H)^{4/3}}$. Therefore, we conclude that both in non-relativistic and ultra-relativistic case the degenerate pressure depends on the density of matter and the degenerate electron gas behaves as a perfect gas with the polytropic equation of state. Then assuming the mass density only depends on pressure, not on temperature, we can solve the structure equations. The general polytrope is of the form,

$$P = K\rho^{\gamma} \tag{3.97}$$

where $\gamma = 1 + \frac{1}{n}$ and n is called polytropic index. Note that here the κ dependence is in the coefficient K. We will then use this in the Poisson equation to get the usual Lane-Emden equation and which further gives the expressions for the radius R and mass M. For a non-rotating fluid we have the hydrostatic equilibrium structure equations as (see [110] [111] [113]),

$$\frac{dm(r)}{dr} = 4\pi r^2\rho \tag{3.98}$$

and

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2}. \tag{3.99}$$

Combining the above two equations we get,

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \tag{3.100}$$

Using (3.97) and the boundary conditions $\rho(r = 0) = \rho_0$ and $\frac{d\rho}{dr}\Big|_{(r=0)} = 0$, we can obtain $\rho(r)$ by solving (3.100). We will, instead, scale this to the dimensionless form using

$$\rho(r) = \rho_0\theta^n(r) \quad \text{and} \quad r = \left(\frac{(n+1)K{\rho_0}^{(1-n)/n}}{4\pi G}\right)^{1/2}\xi = a\xi, \tag{3.101}$$

here θ and ξ are the dimensionless density and radius respectively and a is the scale factor. With the above substitution in (3.100) we get the Lane-Emden equation for polytrope index n as,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{3.102}$$

Using the boundary conditions $\theta(r = 0) = 1$ and $\frac{d\theta}{dr}\Big|_{r=0} = 0$, the above equation can be integrated numerically, starting from $\xi = 0$ for a particular choice of κ. One can see that for $n < 5$, the solutions decrease monotonically with the zero at $\xi = \xi_1$ i.e. $\theta(\xi_1) = 0$ or $\rho(r_1 = a\xi_1) = 0$. This $r_1 = a\xi_1 = R$ is the radius of the star,

$$R = \left(\frac{(n+1)K{\rho_0}^{(1-n)/n}}{4\pi G}\right)^{1/2}\xi_1 \tag{3.103}$$

Using (3.101) and (3.102) we get,

$$M = \int_0^R 4\pi r^2\rho dr = 4\pi\left(\frac{(n+1)K}{4\pi G}\right)^{3/2}{\rho_0}^{(3-n)/2n}\xi_1^2|\theta'(\xi_1)|. \tag{3.104}$$

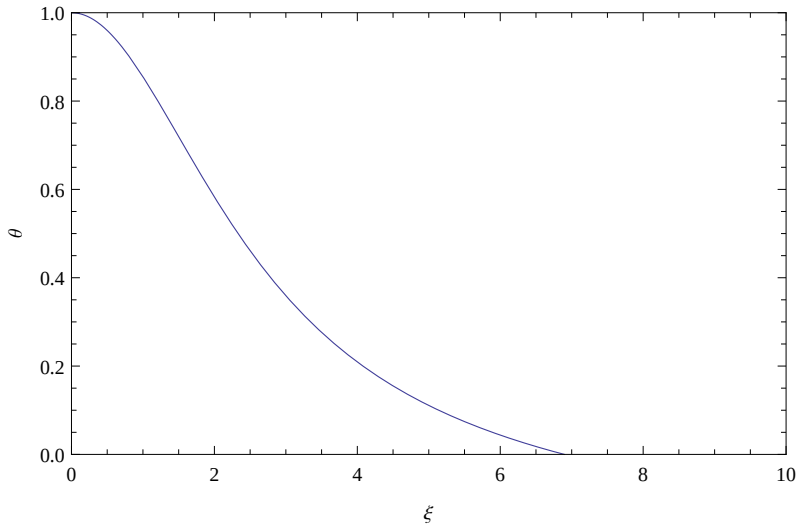


Figure 3.4: Figure showing the numerical solution of (3.102) for $n = 3$, i.e. the variation of $\theta(\xi)$ as a function of ξ. It can be clearly seen from the plot that the value of $\xi_1 = 6.89$ corresponding to $\theta = 0$.

We get the mass-radius relation by eliminating the ρ_0 between R and M relation as,

$$M = 4\pi\left(\frac{(n+1)K}{4\pi G}\right)^{n/(n+1)} \xi_1^{(n+3)/(1-n)}\xi_1^2|\theta'(\xi_1)|R^{(3-n)/(1-n)}. \tag{3.105}$$

The interesting case is the ultra-relativistic case with $\gamma = \frac{4}{3}$ or $n = 3$ and the corresponding values are $\xi_1 = 6.89$ and $\xi^2|\theta'(\xi_1)| = 2.02$ (refer to [116]). The numerical solution is plotted in figure 3.4. The mass then becomes,

$$M_0 = \left[\frac{(3\pi)^{1/2}}{m_H{}^2G^2}\left(1 - \frac{m^2}{\kappa^2}\right)^{-3/4}\right]\left(\frac{2.02}{2\mu^2}\right) = \frac{M_0^{SR}}{\left[1 - \frac{m^2}{\kappa^2}\right]^{3/4}} \tag{3.106}$$

which is exactly the mass M_0 obtained in section 3.1.2 above, with the proper substitutions. Note that the white dwarfs which are predominantly made up of ^{12}C or ^{16}O, the value of $\mu \simeq 2$. The value can be written in terms of the mass of the sun with proper units introduced as,

$$M_0 = \frac{5.7}{\mu^2\left[1 - \frac{m^2}{\kappa^2}\right]^{3/4}} M_\odot, \tag{3.107}$$

where $M_\odot$ is the mass of the sun. Note that this mass is independent of radius R and the central density ρ_0. The radius can then be written as,

$$R = \left[\frac{(3\pi)^{1/2}}{m_e m_H \sqrt{G}}\left(1 - \frac{m^2}{\kappa^2}\right)^{-1/4}\right]\left(\frac{6.89}{2\mu}\right)\left(\frac{\rho_c}{\rho_0}\right)^{1/3} \tag{3.108}$$

where $\rho_c = \frac{m_H \mu {m_e}^3}{3\pi^2}$ is the critical density which defines a rough partition between non-relativistic and and relativistic regimes. Such that, $\rho << \rho_c$ corresponds to non-relativistic case and $\rho >> \rho_c$ corresponds to ultra-relativistic case.

3.4.2 General structure equation

In the previous section we assumed a particular polytropic form, with polytropic index n, of the density dependence of pressure. The non-relativistic and ultra-relativistic cases can be understood by taking a particular values of n and solving the differential equation. But we will now take a more general approach than just taking the two extreme cases. We first express the number density in terms of matter density as $n = \frac{\rho}{\mu m_H}$ in (3.94) to give the expression of density as,

$$\rho = \frac{\mu m_H m^3}{3\pi^2}\left(1 - \frac{m^2}{\kappa^2}\right)^{-3/2} A^3 \frac{M}{R^3} = C_1 \mu x^3 \tag{3.109}$$

where $C_1 = \frac{m_H m^3}{3\pi^2}\left(1 - \frac{m^2}{\kappa^2}\right)^{-3/2} = \frac{m_H m_e^3}{3\pi^2\left(1-\frac{m_e}{\kappa}\right)^3}\left[1 - \left(\frac{m_e}{\kappa - m_e}\right)^2\right]^{-3/2}$ and $x = A\frac{M^{1/3}}{R}$ i.e, $x = \left[\frac{\left(1-\frac{m_e}{\kappa}\right)\left[1-\left(\frac{m_e}{\kappa-m_e}\right)^2\right]^{1/2}}{m_e}\left(\frac{9\pi}{4m_e+8m_p}\right)^{1/3}\right]\frac{M^{1/3}}{R}$. The pressure is given by (3.72) as,

$$P_{th} = C_2 F(x), \tag{3.110}$$

where $C_2 = \frac{m^4}{24\pi^2}\left(1 - \frac{m^2}{\kappa^2}\right)^{-5/2} = \frac{m_e^4}{24\pi^2\left(1-\frac{m_e}{\kappa}\right)^4}\left[1 - \left(\frac{m_e}{\kappa - m_e}\right)^2\right]^{-5/2}$ and $F(x) = x(2x^2 - 3)(1 + x^2)^{1/2} + 3\sinh^{-1}(x)$. Note that the dependence of κ is in the coefficients C_1 and C_2. The spherically symmetric fluid in equilibrium with gravitational force is given by (3.100). In

order to cast it into a convenient form, we will make a change of variable as $z^2 \equiv (x^2 + 1)$ and find that

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{8C_2}{C_1\mu}\left(\frac{dz}{dr}\right) \tag{3.111}$$

Using the above equation in (3.100) we get,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dz}{dr}\right) = -\frac{\pi}{2}\left(\frac{G{C_1}^2\mu^2}{C_2}\right)(z^2 - 1)^{3/2} \tag{3.112}$$

Let us take the value of z at $r = 0$ as z_c. As was done previously, we will rescale the variables as

$$z = Qz_c \quad \text{and} \quad r = \sqrt{\frac{2C_2}{\pi G}}\left(\frac{1}{C_1\mu z_c}\right)\xi = a\xi. \tag{3.113}$$

Using the above scaling (3.112) can be written in terms of new variables Q and ξ as,

$$\frac{dQ^2}{d\xi^2} + \frac{2}{\xi}\frac{dQ}{d\xi} + \left(Q^2 - \frac{1}{z_c^2}\right)^{3/2} = 0 \tag{3.114}$$

with the boundary conditions $Q(\xi = 0) = 1$ (by definition) and $\frac{dQ}{d\xi}\Big|_{\xi=0} = 0$ (by assuming that the gradient of pressure at the origin vanishes). Given $z_c > 1$, this can be numerically solved for the given boundary conditions outwards from $\xi = 0$. The density can as well be written in terms of the new variables as,

$$\rho = C_1\mu x^3 = C_1\mu(z^2 - 1)^{3/2} = C_1\mu z_c^3\left(Q^2 - \frac{1}{z_c^2}\right)^{3/2}. \tag{3.115}$$

For any star, $\rho = 0$ at the surface (i.e. at $r = R$ or $\xi = \xi_1$). Therefore at surface we have,

$$x_1 = 0, \quad z_1 = 1, \quad Q_1 = \frac{1}{z_c} \quad \text{at} \quad \xi = \xi_1. \tag{3.116}$$

The expression of radius R is,

$$R = a\xi_1 = \sqrt{\frac{2C_2}{\pi G}}\left(\frac{1}{C_1\mu z_c}\right)\xi_1 \tag{3.117}$$

The total mass M of the system is,

$$\begin{aligned} M &= \int_0^R 4\pi r^2 \rho dr = 4\pi\left(C_1\mu a^3 z_c^3\right)\int_0^{Q_1}\xi^2\left(Q^2 - \frac{1}{z_c^2}\right)^{3/2} d\xi = 4\pi\left(C_1\mu a^3 z_c^3\right)\left(-\xi^2\frac{dQ}{d\xi}\right)_1 \\ &= \frac{4\pi}{(C_1\mu)^2}\left(\frac{2C_2}{\pi G}\right)^{3/2}\left(-\xi^2\frac{dQ}{d\xi}\right)_1 \end{aligned} \tag{3.118}$$

As before (3.112) can be numerically solved for various values of $\frac{1}{z_c} = 0$ to $\frac{1}{z_c} = 1$ (i.e. from $x_c = \infty$ to $x_c = 0$). Let us consider the two extreme cases:

1. $\mathbf{\frac{1}{z_c}=0}$: This case corresponds to the fully relativistic degenerate one. The numerical integration gives the values as $\xi_1 = 6.89$ and $\left(-\xi^2\frac{dQ}{d\xi}\right)_1 = 2.02$. Therefore, the the corresponding values are $x_c = \infty$, $\rho_0 = \infty$, $R = 0$.

2. $\mathbf{\frac{1}{z_c}=1}$: This case corresponds to the non-relativistic degenerate one. The numerical integration gives the values as $\xi_1 = \infty$ and $\left(-\xi^2\frac{dQ}{d\xi}\right)_1 = 0$. In this case the the values are $x_c = 0$, $\rho_0 = 0$, $R = \infty$.

The central density $\rho_0 = C_1\mu x_c$ decreases with decreasing x_c i.e. the relativistic compact objects are denser compared to the non-relativistic ones. Another point to note is that the radius of the system decreases with increasing x_c, i.e. the massive white dwarfs are smaller in size.

3.5 Luminosity of a white dwarf

In this section we will explore the possible correction to the luminosity of a white dwarf in DSR. Till now we saw the model of white dwarf where the whole white dwarf is assumed

to be made up of degenerate gas, such that whole star is supported against the gravity crunch by the degenerate pressure. But we know that a star has a non-uniform density distribution i.e, the density goes to zero as radial distance becomes R i.e, at the surface of the star. Hence the complete degenerate description of white dwarf is inapplicable (see section 5.3 of reference [110], also see [111] [112] [113]). Therefore, we must consider the situation where the white dwarf is composed of partial degenerate matter. We wish to calculate the luminosity of a white dwarf using such a composition. We will start by describing the model for such a white dwarf. The detailed calculation for the luminosity will be shown thereafter. The assumptions involved will be stated explicitly.

3.5.1 The model

As stated above our model is that of a partial degenerate matter constituting the white dwarf. What we mean is that upto certain radius r_0 from the centre the constituents of the white dwarf behave as degenerate gas and beyond that it behaves as non-degenerate matter. We have seen in the preceding sections that the degenerate pressure which holds the white dwarf against the gravity is calculated using the Fermi gas with $T = 0$ K. On the other hand we know that the white dwarf star has some non zero finite temperature. To consider the radiative processes which leads us to understand the luminosity of the star we must have the non-degenerate radiative envelope. It is a standard observation that almost all the stars have a radiative envelope (see [110] [111] [112] [113] [121]). In our model we will also consider a non-degenerate radiative envelope encompassing the degenerate core of star such that up-to some r_0 from the centre of star we have degenerate Fermi gas and above which we have classical non-degenerate ideal gas till the surface. The model is shown in figure 3.5. The transition point $T = T_0$ represents the transition from quantum ideal degenerate Fermi gas to a classical ideal gas (note that this transition is smooth). Due to very high conductivity of the degenerate gas, the interior of the white dwarf upto the $r = r_0$ is isothermal with temperature T_0. The temperature gradient starts

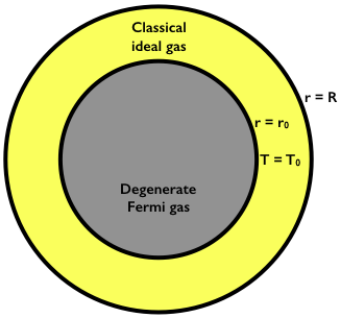


Figure 3.5: Figure showing the partial degenerate model of white dwarf with radiative envelope. We consider the transition from quantum to classical is occurring at $r = r_0$ and $T = T_0$. Below $r = r_0$ the gas is quantum and the pressure is degenerate pressure and above $r = r_0$ we have the radiative envelope and the gas is classical ideal gas. Therefore, above $r = r_0$ the pressure is classical thermodynamic pressure $P = \frac{N_A}{\mu}\rho T$.

as we enter the non-degenerate part and it is this part that is responsible for the cooling of the white dwarf. As we move towards the surface of the star the temperature decreases making a finite surface temperature lower than T_0. We will consider both the relativistic and non-relativistic cases for this partial degenerate model in the DSR theory.

3.5.2 The assumptions

Usually modeling a star is very complicated and for many aspects of the star we still do not have a proper theory. But as is usually done we will take the standard set of assumptions and proceed further.

1. We have a spherically symmetric star in steady state with all the physical variables depending only on radial coordinate r.

2. The radiative envelope is in local thermal equilibrium, such that energy density is given by Planck's spectrum B_ν corresponding to local temperature (see section 2.4 and exercise 2.9 of [110] and also see section 9.2 of [112]).

3. The stellar fluid is stable against convection and so the entire flux is transferred through radiative process only.

4. Only one of the various processes of scattering are dominant making the polytrope

to follow the power law for the opacity in terms of density and temperature (see section 2.4.1 of [110]). The opacity γ is given by $\gamma = \gamma_0 \rho^n T^{-s}$ (see section 2.2 of [110]). The proportionality constant γ_0 depends on the composition of the star while the power indices n and s depend on the nature of the dominant scattering processes.

5. Pressure P follows ideal gas equation.

3.5.3 Variation of temperature and pressure and the expression of pressure in radiative envelope

With the above model and assumptions in mind we will now attempt to calculate the luminosity of a white dwarf. As was shown in [89], the energy density and pressure get a modification in DSR and so does the energy flux etc. We note that at such a local thermal equilibrium we have, for a given frequency, the following relations (see section 6.8 of [111])

1. Intensity, $I_\nu \approx B_\nu = \frac{4\pi\nu^3}{e^{\frac{2\pi\nu}{T}}-1}$
2. Energy density, $U_\nu \approx 4\pi B_\nu$
3. Luminosity, $L_\nu = \left(4\pi R^2\right)\frac{U_\nu}{4\pi}$
4. Pressure, $P^{\alpha\beta} = P_\nu \delta^\alpha_\beta$

Here P_ν is the radiation pressure for frequency ν. Under such an approximation the energy flux is given by (refer to equations (6.165)-(6.174) of [111]),

$$F^\alpha \approx \frac{-1}{\rho\gamma_\nu}\frac{\partial P^{\alpha\beta}}{\partial x^\beta} \approx \frac{-1}{\rho\gamma_\nu}\frac{\partial}{\partial x^\beta}(P_\nu \delta^\alpha_\beta), \tag{3.119}$$

where γ_ν is the opacity for a particular frequency ν . Note that the above equation is true for a frequency ν. The actual equation is found by integrating over the frequency (refer to

the C.1). Therefore, using the modified pressure and energy density relation in [89] and the derivatives of polylogarithm given by (see (4.1) in [101]) $\frac{\partial}{\partial\mu}[Li_n(e^{\mu})] = Li_{n-1}(e^{\mu})$, the required energy flux is given by (see section 2.2 of [110]),

$$F(r) = \frac{L(r)}{4\pi r^2} = \left[\left(\frac{1}{\rho\gamma_\kappa}\right)\left(\frac{-dP_{rad}}{dr}\right)\right]_{\text{boundary term}} + \left[\left(\frac{1}{\rho\gamma}\right)\left(\frac{-dP_{rad}}{dr}\right)\right] \tag{3.120}$$

The expression for the boundary term is given by,

$$\left(\frac{1}{3\rho\gamma_\kappa}\right)\left[\frac{\kappa^3}{\pi^2}\ln\left(1-e^{-\frac{\kappa}{T}}\right) - \frac{\kappa^4}{\pi^2 T}\frac{1}{e^{\frac{\kappa}{T}}-1}\right]\frac{dT}{dr} \tag{3.121}$$

and the other term is given by,

$$\begin{aligned}\left(\frac{1}{3\rho\gamma}\right)\Bigg[&-\frac{4\pi^2T^3}{15} + \frac{24T^3}{\pi^2}Li_4\left(e^{\frac{-\kappa}{T}}\right) + \frac{24\kappa T^2}{\pi^2}Li_3\left(e^{\frac{-\kappa}{T}}\right)\\ &+\frac{12\kappa^2 T}{\pi^2}Li_2\left(e^{\frac{-\kappa}{T}}\right) - \frac{4\kappa^3}{\pi^2}\ln\left(1-e^{\frac{-\kappa}{T}}\right) + \frac{\kappa^4}{\pi^2 T}\frac{1}{e^{\frac{\kappa}{T}}-1}\Bigg]\frac{dT}{dr}\end{aligned} \tag{3.122}$$

Here γ is the mean radiative opacity of the star, $\frac{1}{\gamma_\kappa} = \int_0^\kappa \frac{d\nu}{\gamma_\nu}$ is the mean radiative opacity corresponding to the boundary term (note that such a term is not present in SR case) and ρ is the matter density of the star. The motivation for the above expression is that if energy flux is $F(r)$ then momentum flux is also $F(r)$ (as $c = 1$), therefore the momentum scattered per second per unit volume will be $(n\sigma)F(r) = (\rho\gamma)F(r)$ (here σ is the cross section and n is the number density of scatterers) and this is force per unit volume due to radiation on matter and is related to $-\nabla P_{rad}$. The above equation can also be understood in terms of molecules in a room which are moving randomly but as soon as we open the windows we get a flow of air outside or inside depending on the pressure difference. Similarly, in a star the photons are moving randomly because of collisions. But since in a star the temperature decreases outward and the radiation pressure is smaller at greater distances from the center. This gradient in the radiation pressure is responsible for the net

movement of photons toward the surface of the star that carries the radiative flux.

Assuming a fitting function for γ_κ and $\gamma(\rho, T)$ exists, we can then invert to get,

$$\frac{dT}{dr} = \frac{\frac{3\rho\gamma_\kappa L(r)}{4\pi r^2}}{\left[\frac{\kappa^3}{\pi^2}\ln\left(1-e^{-\frac{\kappa}{T}}\right) - \frac{\kappa^4}{\pi^2 T}\frac{1}{e^{\frac{\kappa}{T}}-1}\right]} + \frac{\frac{3\rho\gamma L(r)}{4\pi r^2}}{\left[-\frac{4\pi^2T^3}{15} + \frac{24T^3}{\pi^2}Li_4\left(e^{\frac{-\kappa}{T}}\right) + \frac{24\kappa T^2}{\pi^2}Li_3\left(e^{\frac{-\kappa}{T}}\right) + \frac{12\kappa^2 T}{\pi^2}Li_2\left(e^{\frac{-\kappa}{T}}\right) - \frac{4\kappa^3}{\pi^2}\ln\left(1-e^{\frac{-\kappa}{T}}\right) + \frac{\kappa^4}{\pi^2 T}\frac{1}{e^{\frac{\kappa}{T}}-1}\right]} \tag{3.123}$$

Note that this should be applied locally for each r. Following assumption 3, we have assumed that the fluid is stable against convection and so the entire flux is transferred through radiative process only. Also according to assumption 4, there are several sources of opacity. The actual value depends on the medium and the various processes that are occurring at relevant densities and temperatures. As stated before, we will assume that the radiative envelope is in local thermal equilibrium, such that energy density is given by Planck's spectrum B_ν corresponding to local temperature. Ignoring the convection we will first calculate the gradient as,

$$\nabla = \frac{d\ln T}{d\ln P} = \frac{\frac{1}{T}\frac{dT}{dr}}{\frac{1}{P}\frac{dP}{dr}} = \frac{\frac{3P\gamma_\kappa L(r)}{4\pi GM(r)}}{\left[-\frac{T\kappa^3}{\pi^2}\ln\left(1-e^{-\frac{\kappa}{T}}\right) + \frac{\kappa^4}{\pi^2}\frac{1}{e^{\frac{\kappa}{T}}-1}\right]} + \frac{\frac{3P\gamma L(r)}{4\pi GM(r)}}{\left[\frac{4\pi^2T^4}{15} - \frac{24T^4}{\pi^2}Li_4\left(e^{\frac{-\kappa}{T}}\right) - \frac{24\kappa T^3}{\pi^2}Li_3\left(e^{\frac{-\kappa}{T}}\right) - \frac{12\kappa^2T^2}{\pi^2}Li_2\left(e^{\frac{-\kappa}{T}}\right) + \frac{4T\kappa^3}{\pi^2}\ln\left(1-e^{\frac{-\kappa}{T}}\right) - \frac{\kappa^4}{\pi^2}\frac{1}{e^{\frac{\kappa}{T}}-1}\right]} \tag{3.124}$$

Here P is the pressure of the fluid in the white dwarf, which we wish to calculate. Note that we have used equation (3.123) and the equation of hydrostatic equilibrium of a star given by,

$$\frac{dP}{dr} = -G\frac{M(r)\rho(r)}{r^2} \tag{3.125}$$

This is true for the spherically symmetric star in steady state with all the physical variables depending only on radial coordinate r. Here $P(r)$, $\rho(r)$ and $M(r)$ are the pressure, density at radius r and mass contained within a sphere of radius r. The gas equation for non-degenerate gas is,

$$P = \frac{N}{V}k_B T = \left(\frac{Nm_H}{V\rho}\right)\left(\frac{k_B}{m_H}\right)\rho T = \frac{N_A k_B \rho T}{\mu} \tag{3.126}$$

Here $\mu = \frac{V\rho}{Nm_H}$ is the mean molecular weight, $\frac{k_B}{m_H} = N_A k_B$ is the gas constant, m_H is atomic mass constant, N_A is the Avogadro's number. Now using the dependence of opacity $\gamma = \gamma_0 \rho^n T^{-s}$ and equation (3.126) we get,

$$\frac{dT}{dP} = \frac{\frac{3\gamma_{1\kappa}P^n L}{4\pi GM}}{\left[-\frac{T^{n+s}\kappa^3}{\pi^2}\ln\left(1-e^{-\frac{\kappa}{T}}\right)+\frac{\kappa^4 T^{n+s-1}}{\pi^2}\frac{1}{e^{\frac{\kappa}{T}}-1}\right]} + \frac{\frac{3\gamma_1 P^n L}{4\pi GM}}{\left[\frac{4\pi^2 T^{n+s+3}}{15}-\frac{24T^{n+s+3}}{\pi^2}Li_4\left(e^{\frac{-\kappa}{T}}\right)-\frac{24\kappa T^{n+s+2}}{\pi^2}Li_3\left(e^{\frac{-\kappa}{T}}\right)-\frac{12\kappa^2 T^{n+s+1}}{\pi^2}Li_2\left(e^{\frac{-\kappa}{T}}\right)-\frac{4\kappa^3 T^{n+s}}{\pi^2}Li_1\left(e^{\frac{-\kappa}{T}}\right)-\frac{\kappa^4 T^{n+s-1}}{\pi^2}\frac{1}{e^{\frac{\kappa}{T}}-1}\right]} \tag{3.127}$$

Here $\gamma_1 = \gamma_0(\frac{\mu}{N_A})^n$ and $\gamma_{1\kappa} = \gamma_{0\kappa}(\frac{\mu}{N_A})^n$ and we have used $Li_1\left(e^{\frac{-\kappa}{T}}\right) = -\ln\left(1-e^{\frac{-\kappa}{T}}\right)$. Therefore we have,

$$\begin{aligned} P^n dP = {} & \frac{4\pi GM}{3L\gamma_{1\kappa}}\left[-\frac{T^{n+s}\kappa^3}{\pi^2}\ln\left(1-e^{-\frac{\kappa}{T}}\right)+\frac{\kappa^4 T^{n+s-1}}{\pi^2}\frac{1}{e^{\frac{\kappa}{T}}-1}\right]dT \\ & + \frac{4\pi GM}{3L\gamma_1}\left[\frac{4\pi^2 T^{n+s+3}}{15}-\frac{24T^{n+s+3}}{\pi^2}Li_4\left(e^{\frac{-\kappa}{T}}\right)-\frac{24\kappa T^{n+s+2}}{\pi^2}Li_3\left(e^{\frac{-\kappa}{T}}\right)-\frac{12\kappa^2 T^{n+s+1}}{\pi^2}Li_2\left(e^{\frac{-\kappa}{T}}\right)\right]dT \\ & + \frac{4\pi GM}{3L\gamma_1}\left[\frac{4\kappa^3 T^{n+s}}{\pi^2}\ln\left(1-e^{\frac{-\kappa}{T}}\right)-\frac{\kappa^4 T^{n+s-1}}{\pi^2}\frac{1}{e^{\frac{\kappa}{T}}-1}\right]dT \end{aligned} \tag{3.128}$$

Now we have to integrate this from P_b, T_b (photospheric boundary conditions) near the photosphere[1] to $P(r)$, $T(r)$ in the stellar envelope to get the radiative polytrope equation

[1]Photosphere is the deepest region of a luminous object, usually a star, that is transparent to photons of certain wavelengths, in other words it is the effective visual surface of the star. It is the region where the observed optical photons originate.

such that $P(r) \geq P_b, T(r) \geq T_b$. We will now use the expression of polylogarithm,

$$Li_n(z) = \sum_{a=1}^{\infty} \frac{z^a}{a^n}, \tag{3.129}$$

which is valid for $|z| < 1$ (see (8.1) in [101]). This for our case looks like $Li_n(e^{-\frac{\kappa}{T}}) = \sum_{a=1}^{\infty} \frac{e^{-\frac{a\kappa}{T}}}{a^n}$. We will try finding a general closed form expression by considering,

$$I = \int_{T_b}^{T(r)} \sum_{j=1}^{\infty} T^p \frac{e^{-\frac{j\kappa}{T}}}{j^a} dT = \sum_{j=1}^{\infty} \int_{T_b}^{T(r)} T^p \frac{e^{-\frac{j\kappa}{T}}}{j^a} dT, \tag{3.130}$$

provided $\sum_{j=1}^{\infty} \int_{T_b}^{T(r)} T^p \frac{e^{-\frac{j\kappa}{T}}}{j^a} dT < \infty$. Now, making change of variables as $x = \frac{j\kappa}{T}$ such that $dT = -\frac{dx}{x^2}(j\kappa)$ and $\frac{j\kappa}{T_b} = \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)} = \frac{j\kappa}{T(r)}$, therefore we have

$$\begin{aligned} I &= -\sum_{j=1}^{\infty} (j)^{p+1-a} (\kappa)^{p+1} \int_{\frac{j\kappa}{T_b}}^{\frac{j\kappa}{T(r)}} \frac{e^{-x}}{x^{p+2}} dx \\ &= -(\kappa)^{p+1} \sum_{j=1}^{\infty} (j)^{p+1-a} \left[\Gamma\left(-p-1, \frac{j\kappa}{T_b}\right) - \Gamma\left(-p-1, \frac{j\kappa}{T(r)}\right) \right] \\ &= -(\kappa)^{p+1} \sum_{j=1}^{\infty} (j)^{p+1-a} \left[\Gamma\left(-p-1, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right) \right] \end{aligned} \tag{3.131}$$

Here $\Gamma(n, x_1, x_2)$ is the generalized incomplete gamma function defined as $\Gamma(n, x_1, x_2) = \int_{x_1}^{x_2} \frac{e^{-t} dt}{t^{1-n}} = \Gamma(n, x_1) - \Gamma(n, x_2)$ and $\Gamma(n, x) = \int_{x}^{\infty} \frac{e^{-t} dt}{t^{1-n}}$ is the incomplete gamma function whose tabulated values are readily available or can be numerically calculated for a given κ and T. Remember this expression is true provided both $\frac{\kappa}{T_b} > 0$ and $\frac{\kappa}{T(r)} > 0$ (otherwise the integral diverges but this is true for all physical cases) and,

$$(\kappa)^{p+1} \sum_{j=1}^{\infty} (j)^{p+1-a} \left[\Gamma\left(-p-1, \frac{j\kappa}{T_b}\right) - \Gamma\left(-p-1, \frac{j\kappa}{T(r)}\right) \right] < \infty \tag{3.132}$$

We can easily check by ratio test that this series is convergent given a particular value of κ and T (Remember the value of incomplete gamma function $\Gamma(n, x)$ decreases as the value of x increases and therefore converges very fast with increasing x for a given n). Also in

SR limit i.e., as $\kappa \to \infty$, for finite T_b and $T(r)$, this whole term goes to zero as expected. We can now proceed and write the closed form expression for the equation (3.128) as,

$$
\begin{aligned}
P(r) = \Bigg[{P_b}^{n+1} &+ (n+1)\Bigg\{ \frac{4GM}{3\pi L\gamma_1}\left[\frac{4\pi^4 {T_b}^{n+s+4}}{15(n+s+4)} - \frac{4\pi^4 T(r)^{n+s+4}}{15(n+s+4)}\right] \\
&- \frac{4GM(\kappa)^{n+s+4}}{3\pi L\gamma_{1\kappa}}\left[\sum_{j=1}^{\infty}(j)^{n+s}\left[\Gamma\left(-n-s-1, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right)\right]\right] \\
&- \frac{4GM(\kappa)^{n+s+4}}{3\pi L\gamma_{1\kappa}}\left[\sum_{j=1}^{\infty}(j)^{n+s}\left[\Gamma\left(-n-s, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right)\right]\right] \\
&+ \frac{4GM(\kappa)^{n+s+4}}{3\pi L\gamma_1}\left[24\sum_{j=1}^{\infty}(j)^{n+s}\left[\Gamma\left(-n-s-4, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right)\right]\right] \\
&+ \frac{4GM(\kappa)^{n+s+4}}{3\pi L\gamma_1}\left[24\sum_{j=1}^{\infty}(j)^{n+s}\left[\Gamma\left(-n-s-3, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right)\right]\right] \\
&+ \frac{4GM(\kappa)^{n+s+4}}{3\pi L\gamma_1}\left[12\sum_{j=1}^{\infty}(j)^{n+s}\left[\Gamma\left(-n-s-2, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right)\right]\right] \\
&+ \frac{4GM(\kappa)^{n+s+4}}{3\pi L\gamma_1}\left[4\sum_{j=1}^{\infty}(j)^{n+s}\left[\Gamma\left(-n-s-1, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right)\right]\right] \\
&+ \frac{4GM(\kappa)^{n+s+4}}{3\pi L\gamma_1}\left[\sum_{j=1}^{\infty}(j)^{n+s}\left[\Gamma\left(-n-s, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right)\right]\right]\Bigg\}\Bigg]^{\frac{1}{n+1}}
\end{aligned}
\tag{3.133}
$$

Using integration by parts of the incomplete gamma function we have a recurrence relation as,

$$
\Gamma(n+1, x) = n\Gamma(n, x) + x^n e^{-x} \tag{3.134}
$$

Using above recurrence relation we can arrange the above expression in terms of one of the gammas. Further simplifying using the expression of the polylogarithm $Li_n(z)$ given in (3.129) we get the expression as (C.6) given in appendix C.2. The behaviour clearly depends mainly on the signs of $n+1$ and $n+s+4$. We can clearly see that the pressure of the DSR corrected partial degenerate gas is smaller than the usual SR case for a given mass and temperature. This was also noted theoretically in [82] [83]. We can now express pressure in terms of the density and then equate it to density obtained for degenerate Fermi

gas expression, which we will see in the next section.

3.5.4 Calculation of the luminosity

In this section we will try to find the actual expression for the luminosity of the white dwarf for both relativistic and the non-relativistic cases following the model described before shown in figure 3.5. The transition point $T = T_0$ represents the transition from non-degenerate to degenerate matter and so $E_F = T_0$, we now introduce $\mu_e = \frac{\rho}{nm_p}$ (where m_p is mass of proton and μ_e is the mass per electron) and substitute for n in (3.5) to get,

$$p_F = (3\pi^2)^{1/3}\left(\frac{\rho}{\mu_e m_p}\right)^{1/3} \tag{3.135}$$

which therefore gives,

$$\frac{\rho}{\mu_e} = \frac{m_p}{3\pi^2}(p_F)^3 \tag{3.136}$$

We will start by considering the relativistic case first.

1. **Relativistic case:** We will first consider the relativistic case with dispersion relation (2.1) giving ${p_F}^2 = {E_F}^2 - m^2\left(1 - \frac{E_F}{\kappa}\right)^2$. Substituting the value of p_F and $E_F = T_0$ in (3.136),

$$\rho_0 = \frac{m_p\mu_e}{3\pi^2}\left[{T_0}^2 - m^2\left(1 - \frac{T_0}{\kappa}\right)^2\right]^{\frac{3}{2}} \tag{3.137}$$

   Here ρ_0 is the mass density different from energy density calculated in [10]. Note that as stated before, the region $r < r_0$ is constant temperature (T_0) region because of very high conductivity of electrons. This is the region of highly dense and degenerate electron gas with long mean free path and therefore the whole matter upto r_0 is isothermal with constant temperature T_0. The non-degenerate envelope is mainly responsible for the luminosity of the white dwarf (see discussion in section 5.3 of [110]). We will express the envelope pressure expression (C.6) in terms of density using (3.126) and equate the corresponding density expression to

(3.137), to get the expression of luminosity L of relativistic gas for particular n and s and the chosen boundary conditions T_b and P_b as (C.7) given in appendix C.2. Since luminosity is proportional to the energy flux and for the radiative envelope we put an ultraviolet cut-off on the maximum single particle energy. Therefore, the flux and so the luminosity gets a negative correction. Because of the same reason the luminosity expression is nonperturbative in the SR limit. This gives the expected standard result in the SR limit.

2. **Non-relativistic case:** To get the result for the non-relativistic (which we will call DNR) case we first need to know the non-relativistic limit of DSR. Normally, to take the NR limit of SR we write the SR dispersion relation and consider $p << m_0$ and expand the relation in this limit. We will do the same procedure of considering the DSR dispersion relation and expanding it in the limit $p << m_0$. Although there are references as [122] and [123] which discuss the DNR limit of DSR, but they both take the expansion in the limit $p << m$. It seems natural to make the expansion in the limit $p << m_0$ as m_0 is the rest mass. On the other hand the invariant mass m has no such physical meaning (not to be confused with the word '*invariant*' as the rest mass m_0 is also invariant under a DSR transformation). Are these limits equivalent? The answer to this question is no, as a case may arise when $\frac{p}{m} << 1$ but $\frac{p}{m_0} \approx 1$, making the two limits different from each other. It is so when $p \approx \kappa$, $m_0 \approx \kappa \Leftrightarrow m \to \infty$. We will, therefore, start with the dispersion relation in (2.1) and rearrange the terms and complete the squares to get,

$$E = \frac{\frac{-m^2}{\kappa^2} \pm \sqrt{m^2 + p^2\left(1 - \frac{m^2}{\kappa^2}\right)}}{1 - \frac{m^2}{\kappa^2}} \tag{3.138}$$

Since we want the expansion in limit $p << m_0$, we substitute m by m_0 using (3.2) to get

$$E = \left(\frac{-{m_0}^2}{2m_0 - \kappa}\right) \pm \frac{m_0(m_0 - \kappa)}{2m_0 - \kappa}\sqrt{1 + \frac{p^2}{{m_0}^2}\left(\frac{\kappa - 2m_0}{\kappa}\right)} \tag{3.139}$$

We then make the following assumptions before doing the binomial expansion,

(a) $\frac{p}{m_0} << 1$. Note that this is the DNR assumption we expect physically.

(b) m_0, κ both are finite. Remember that in case of DSR this assumption is valid from the way it has originally been formulated ($0 \leq m_0 \leq \kappa$).

With above assumptions in mind we do the expansion and keep the first order terms in $\frac{p}{m_0}$ to get,

$$E \simeq \left(\frac{-{m_0}^2}{2m_0 - \kappa}\right) \pm \frac{m_0(m_0 - \kappa)}{2m_0 - \kappa}\left[1 + \frac{p^2}{2{m_0}^2}\left(\frac{\kappa - 2m_0}{\kappa}\right)\right]. \tag{3.140}$$

Considering the positive value first we get from further simplification,

$$E = m_0 + \frac{p^2}{2m_0}\left[1 - \frac{m_0}{\kappa}\right] = m_0 + \frac{p^2}{2m} \tag{3.141}$$

Here m_0 is the rest mass as expected and m is the non-relativistic inertial mass. Our result matches with the literature [122] and [123]. Let us now consider the negative value,

$$E = \frac{m_0\kappa}{2m_0 - \kappa} - \frac{p^2}{2m} = -\left[\frac{m_0\kappa}{\kappa - 2m_0} + \frac{p^2}{2m}\right] \tag{3.142}$$

These two energies correspond to the particle and the anti-particle respectively. Note that the rest mass of the particle and the anti-particle are different. But, DNR dynamics depends on m which is the same for both. Substituting the value of p_F and $E_F = T_0$ in (3.136) we get,

$$\rho_0 = \frac{m_p\mu_e}{3\pi^2}(2mT_0)^{\frac{3}{2}} \tag{3.143}$$

As done before we will express the envelope pressure expression (C.6) in terms of density using (3.126) and equate the corresponding density expression to the density above at $T = T_0$ to get the relation between M, L for non-relativistic gas for given n, s and chosen boundary conditions on T_b and P_b as (C.8) given in appendix

C.2. In this case also the luminosity gets a negative correction. The luminosity expression is as expected nonperturbative in the SR limit. This expression also gives the expected standard result in the SR limit.

So the luminosity in both the cases are lower than the usual value and nonperturbative in the SR limit.

3.6 Summary

In the present chapter we discussed the correction to the thermodynamic pressure and the total energy of the degenerate Fermi gas in all the three cases $m < \kappa$, $m = \kappa$ and $m > \kappa$. We discussed the number density n and mass m dependence of the degenerate pressure for $m < \kappa$ case. We found that the degenerate pressure is perturbative in the SR limit, a result unexpected for the theory with an ultraviolet cut-off. We also, briefly discussed the two extreme non-relativistic and the ultra-relativistic limits of the pressure. We took the white dwarf stars as an example and studied its modified dynamics. For the usual particle i.e, in case of $m < \kappa$, as is obvious from figure 3.2, we found that the equilibrium degeneracy pressure is greater than, equal to and less than the SR value for particular masses of the considered compact object such as white dwarf. Also as shown in figure 3.3, for given masses the value of the radius of the white dwarf is found to be less than, equal to and greater than the usual SR value. Since energy density is related directly to mass density so denser compact objects are expected to show better measurable correction due to such a modification. We do not get the Chandrasekhar limit in the other two cases as expected. The Chandrasekhar mass limit for a white dwarf in this case is greater than the usual SR value which is clearly visible from the plot. One of the major predictions of our theory is that it makes an attempt to explain the observed lower radius white dwarfs and also predicts the white dwarfs having radius greater than that predicted by the present SR theory. The correction (see (3.81)) is purely perturbative in the SR limit which is quite

unusual for a theory having an ultraviolet energy cut-off. Therefore we conclude that this correction is solely because of the modified dispersion relation. The other two cases $m = \kappa$ and $m > \kappa$ has also been studied where we do not get any limit on the white dwarf mass. Note that the presence of observed white dwarfs having radius lesser than the SR case may find an explanation if they are modelled using a modified dispersion relation. This result has also been found using the modified Lane-Emden equation assuming radial density distribution. General modified structure equation has also been discussed in detail. We also calculated the the luminosity of such a white dwarf in DSR both in non-relativistic and relativistic cases. We noted that since the correction in pressure is negative for a given mass and temperature and so is the correction in the luminosity as well. The correction to the luminosity of a white dwarf is nonperturbative in SR limit as expected because of the presence of an ultraviolet energy cut-off.

4 Summary

In chapter 2, we started with the DSR formalism by MS where the modified dispersion relation, in order to incorporate an invariant energy scale κ, is given by (2.1). But in the case of photon gas, it is simply $\varepsilon = p$. Since there is a cut-off on the maximum energy and the minimum length, the expression of the thermodynamic quantities changes accordingly. We started by considering a model of a photon gas obeying Bose-Einstein statistics in grand canonical ensemble and went on calculating various thermodynamic quantities such as energy density, pressure, entropy, specific heat and equilibrium number of photons with such an ultraviolet cut-off. We found one to one correspondence between the behaviour of photons with an ultraviolet cut-off and the acoustic phonons in the Debye theory. The Stefan-Boltzmann law got modified which will give correction to the dynamics of many stellar objects. We found that the non-perturbative nature of the thermodynamic quantities in the SR limit is a general feature of the theory with an ultraviolet energy cut-off. We also noted that the values of all the thermodynamic quantities are less than the SR values because of this cut-off. We then studied the change in the phase space measure for exotic spacetimes at Planck scale and discussed the example of classical ideal gas for illustration. We found that the classical ideal gas in case of modified phase space measure has a non-trivial volume dependence in its expression for the partition function leading to the modification in the thermodynamic quantities like pressure accordingly. We went on calculating the possible change in the thermodynamic quantities due to the change in the phase space measure. Because of this modification, Planck's energy density distribution and the Wien's displacement law got modified. Note that all

the thermodynamic quantities reduce to the usual SR result in $\kappa \to \infty$ limit. We have plotted the temperature dependence of various thermodynamic quantities. We can clearly see from the plots of various thermodynamic quantities that we start getting the deviation of the results obtained in the modified case from the SR case at $\frac{T}{\kappa} \sim 10^{-1}$. We found that the modified Wien's law can be observed at comparatively lower temperature than the thermodynamic quantities. Next, we discussed the possible realization of the modification at Planck scale by considering its effects near the Big Bang. The effectively lower Planck scale cosmological observations and modifications of DSR have also been discussed. The leading behaviour for $T \to 0$ and $T \to \kappa$ have been analysed. We observed that in the case of modified phase space measure the values of the thermodynamic quantities might be less than, equal to or greater than the SR values depending on the choice of $a_{n,n'}$. As was seen both for the case with only an ultraviolet cut-off and the modified measures, the nonanalyticity in the special relativistic limit is a general feature of the energy cut-off introduced in the theory. In the last section, we have given the possible scenarios of the physical observation of results obtained in effective low energy by suggesting a quantum mechanical table top experiment. Note that the present work only deals with the study of massless bosons i.e. photons, but the analysis can be extended to massive bosons and fermions as well. This will give us an insight into the invariant energy scale effects in well-known phenomenon such as Bose-Einstein condensation and the behaviour of degenerate Fermi gas. This, in turn, leads to the study of the stellar objects such as white dwarfs and neutron stars. It would certainly be interesting to see whether the nonperturbative effects appear in such massive cases too. Analysis of this chapter can be used to study early Universe thermodynamics.

In chapter 3, we started with the study of the effect of a relativistically invariant energy scale on the thermodynamics of a degenerate Fermi gas. We considered the model of Modified Dispersion Relation(MDR) with an invariant ultraviolet cut-off on the single particle energy. We found the correction to the thermodynamic pressure and the total energy of the degenerate Fermi gas in all the three cases $m < \kappa$, $m = \kappa$ and $m > \kappa$. We

discussed the number density n and mass m dependence of the degenerate pressure for $m < \kappa$ case. We found that the degenerate pressure is perturbative in the SR limit, a result unexpected for the theory with an ultraviolet cut-off. We also, briefly discussed the two extreme non-relativistic and the ultra-relativistic limits of the pressure. We took the white dwarf stars as an example and studied its modified dynamics. For the usual particle i.e, in case of $m < \kappa$, as is obvious from figure 3.2, we found that the equilibrium degeneracy pressure is greater than, equal to and less than the SR value for particular masses of the considered compact object such as white dwarf. Also as shown in figure 3.3, for given masses the value of the radius of the white dwarf is found to be less than, equal to and greater than the usual SR value. Since energy density is related directly to mass density so denser compact objects are expected to show better measurable correction due to such a modification. We do not get the Chandrasekhar limit in the other two cases as expected. The Chandrasekhar mass limit for a white dwarf in this case is greater than the usual SR value which is clearly visible from the plot. One of the major predictions of our theory is that it makes an attempt to explain the observed lower radius white dwarfs and also predicts the white dwarfs having radius greater than that predicted by the present SR theory. The correction (see (3.81)) is purely perturbative in the SR limit which is quite unusual for a theory having an ultraviolet energy cut-off. Therefore we conclude that this correction is solely because of the modified dispersion relation. The other two cases $m = \kappa$ and $m > \kappa$ has also been studied where we do not get any limit on the white dwarf mass. Note that the presence of observed white dwarfs having radius lesser than the SR case may find an explanation if they are modeled using a modified dispersion relation. This result has also been found using the modified Lane-Emden equation assuming radial density distribution. General modified structure equation has also been discussed in detail. Along with this it was shown in [89] that the Stefan-Boltzmann law gets modified in DSR and so does the luminosity. We therefore calculate the the luminosity of such a white dwarf in such a modified relativistic theory both in non-relativistic and relativistic cases. We noted that since the correction in pressure is negative for a given mass and

temperature and so is the correction in the luminosity as well. The correction to the luminosity of a white dwarf is nonperturbative in SR limit as expected because of the presence of an ultraviolet energy cut-off. In future, one can also do similar analysis of other dense and compact stars like neutron stars etc. Since neutron stars are denser than the white dwarfs, one expects more prominent signature to the DSR correction. Given the modification of degenerate fermions in the formalism discussed in this thesis, one can further explore the general modifications in the thermodynamics of a massive fermion gas (not necessarily degenerate) and also the study of massive ideal boson gas is in order. To study the thermodynamics of a black hole one needs to formulate the DSR on a curved spacetime.

A Criterion for swapping the double summation and the integral

The interchange of summation and integration signs for a series is a well known theorem (see theorem 1.38 of [124]). In this Appendix, we will generalize this theorem for a double series i.e. we will prove the equality,

$$\int_{\mathcal{M}} \sum_{n} \sum_{n'} A_{n,n'} = \sum_{n} \sum_{n'} \int_{\mathcal{M}} A_{n,n'} \tag{A.1}$$

holds if,

$$\sum_{n} \sum_{n'} \int_{\mathcal{M}} |A_{n,n'}| < \infty. \tag{A.2}$$

We take the two summations out of the integral one by one in the LHS of (A.1) to get the RHS, using theorem 1.38 of [124] which is allowed if,

$$\sum_{n} \int_{\mathcal{M}} \left| \sum_{n'} A_{n,n'} \right| < \infty \quad \text{and} \quad \sum_{\mathrm{n}'} \int_{\mathcal{M}} |\mathrm{A}_{\mathrm{n},\mathrm{n}'}| < \infty \quad \forall \mathrm{n}. \tag{A.3}$$

Let us rewrite (A.2) as,

$$\sum_n t_n < \infty \quad \text{with} \quad \mathrm{t_n} = \sum_{\mathrm{n'}} \int_{\mathcal{M}} |\mathrm{A_{n,n'}}| \quad \forall \mathrm{n}. \tag{A.4}$$

We note that $t_n \geq 0 \quad \forall n$, which along with (A.4) implies $t_n < \infty \quad \forall n$ which is nothing but the second inequality in (A.3). This further implies (because of theorem 1.38 of [124]),

$$t_n = \int_{\mathcal{M}} \sum_{n'} |A_{n,n'}| \quad \forall n. \tag{A.5}$$

Now we note that $\left|\sum_{n'} A_{n,n'}\right| \leq \sum_{n'} |A_{n,n'}| \quad \forall n$. Integrating over $\mathcal{M}$ followed by the summation over n we get,

$$\sum_n \int_{\mathcal{M}} \left|\sum_{n'} A_{n,n'}\right| \leq \sum_n \int_{\mathcal{M}} \sum_{n'} |A_{n,n'}| \quad \forall n \tag{A.6}$$

Taking (A.5) and (A.4) into account in the above inequality we get the first inequality of (A.3). Thus we have proved that (A.2) implies (A.3) and hence if (A.2) is satisfied then the equality (A.1) holds true.

B List of certain results

B.1 Thermodynamic quantities with modified measure

Following what we did in section 2.2.1 along with using (2.27) we get,

$$
\begin{aligned}
u &= \frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!\kappa^n}\frac{4\pi}{(3-n')}\left(\frac{T^{n+4}}{V_{ac}\kappa^3}\right)\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\Gamma(n+4)\left[Z_{n+4}(0)-Z_{n+4}\left(\frac{\kappa}{T}\right)\right]\\
&+\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!\kappa^n}\left(\frac{4\pi T^{n+4}}{3V_{ac}\kappa^3}\right)\ln\left(\frac{3\kappa^3 V}{4\pi}\right)\Gamma(n+4)\left[Z_{n+4}(0)-Z_{n+4}\left(\frac{\kappa}{T}\right)\right]\\
&=\sum_{n=0,n'=0}^{\infty}u_{n,n'}. \qquad \text{(B.1)}
\end{aligned}
$$

Here $u_{n,n'}$ is a general term of the summation. Therefore the specific heat capacity C_V is,

$$
\begin{aligned}
C_V &= \left(\frac{\partial U}{\partial T}\right)_{V_{ac}}\\
&=\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\left[\frac{1}{(\pi)^2}\frac{a_{n,n'}}{n!n'!\kappa^{n+3}}\frac{4\pi}{(3-n')}\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\left\{\frac{1}{(1-e^{\frac{\kappa}{T}})}\frac{\kappa^{n+4}}{T}\right\}+(n+4)\left(\frac{u_{n,n'}V_{ac}}{T}\right)\right]\\
&+\sum_{n=0}^{\infty}\left[\frac{1}{(\pi)^2}\frac{a_{n,3}}{n!3!\kappa^{n+3}}\left(\frac{4\pi}{3}\right)\ln\left(\frac{3\kappa^3 V}{4\pi}\right)\left\{\frac{1}{(1-e^{\frac{\kappa}{T}})}\frac{\kappa^{n+4}}{T}\right\}+(n+4)\left(\frac{u_{n,3}V_{ac}}{T}\right)\right]. \qquad \text{(B.2)}
\end{aligned}
$$

The radiation pressure can easily be calculated in essentially the similar manner to get,

$$
\begin{aligned}
P = & \frac{1}{(\pi)^2} \sum_{\substack{n=0,n'=0 \\ n'\neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^n} \frac{4\pi}{(3-n')} \left(\frac{T}{V_{ac}\kappa^3}\right) \left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}} - 1\right] \left\{ -\ln\left(1 - e^{-\frac{\kappa}{T}}\right) \frac{\kappa^{n+3}}{n+3} \right. \\
& \left. + \frac{T^{n+3}}{(n+3)} \Gamma(n+4) \left[Z_{n+4}(0) - Z_{n+4}\left(\frac{\kappa}{T}\right) \right] \right\} \\
& + \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^n} \left(\frac{4\pi T}{3V_{ac}\kappa^3}\right) \ln\left(\frac{3\kappa^3 V}{4\pi}\right) \left\{ -\ln(1 - e^{-\frac{\kappa}{T}}) \frac{\kappa^{n+3}}{n+3} \right. \\
& \left. + \frac{T^{n+3}}{(n+3)} \Gamma(n+4) \left[Z_{n+4}(0) - Z_{n+4}\left(\frac{\kappa}{T}\right) \right] \right\}. \qquad \text{(B.3)}
\end{aligned}
$$

This can be related to the energy density as

$$
\begin{aligned}
P = & \sum_{\substack{n=0,n'=0 \\ n'\neq 3}}^{\infty} \left[\frac{1}{(\pi)^2} \frac{a_{n,n'}}{n!n'!\kappa^n} \frac{4\pi}{(3-n')} \left(\frac{T}{V_{ac}\kappa^3}\right) \left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}} - 1\right] \left\{ -\ln(1 - e^{-\frac{\kappa}{T}}) \frac{\kappa^{n+3}}{n+3} \right\} + \frac{u_{n,n'}}{(n+3)} \right] \\
& + \sum_{n=0}^{\infty} \left[\frac{1}{(\pi)^2} \frac{a_{n,3}}{n!3!\kappa^n} \left(\frac{4\pi T}{3V_{ac}\kappa^3}\right) \ln\left(\frac{3\kappa^3 V}{4\pi}\right) \left\{ -\ln(1 - e^{-\frac{\kappa}{T}}) \frac{\kappa^{n+3}}{n+3} \right\} + \frac{u_{n,3}}{(n+3)} \right]. \qquad \text{(B.4)}
\end{aligned}
$$

The Helmholtz free energy is,

$$
\begin{aligned}
A &= \mu N - PV_{ac} = -PV_{ac} \\
&= \sum_{\substack{n=0,n'=0 \\ n'\neq 3}}^{\infty} \left[\frac{1}{(\pi)^2} \frac{a_{n,n'}}{n!n'!\kappa^{n+3}} \frac{4\pi T}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}} - 1\right] \left\{ \ln(1 - e^{-\frac{\kappa}{T}}) \frac{\kappa^{n+3}}{n+3} \right\} - \frac{u_{n,n'}V_{ac}}{(n+3)} \right] \\
&+ \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^{n+3}} \left[\frac{1}{(\pi)^2} \left(\frac{4\pi T}{3}\right) \ln\left(\frac{3\kappa^3 V}{4\pi}\right) \left\{ \ln(1 - e^{-\frac{\kappa}{T}}) \frac{\kappa^{n+3}}{n+3} \right\} - \frac{u_{n,3}V_{ac}}{(n+3)} \right]. \qquad \text{(B.5)}
\end{aligned}
$$

And the entropy becomes,

$$
S = \frac{U - A}{T} =
$$

$$
\begin{aligned}
&= \sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\left[\frac{1}{(\pi)^2}\frac{a_{n,n'}}{n!n'!\kappa^{n+3}}\frac{4\pi}{(3-n')}\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\left\{-\ln(1-e^{-\frac{\kappa}{T}})\frac{\kappa^{n+3}}{n+3}\right\}\right.\\
&\left.+\frac{(n+4)}{(n+3)}\left(\frac{u_{n,n'}V_{ac}}{T}\right)\right]\\
&+\sum_{n=0}^{\infty}\left[\frac{1}{(\pi)^2}\frac{a_{n,3}}{n!3!\kappa^{n+3}}\left(\frac{4\pi}{3}\right)\ln\left(\frac{3\kappa^3V}{4\pi}\right)\left\{-\ln(1-e^{-\frac{\kappa}{T}})\frac{\kappa^{n+3}}{n+3}\right\}\right.\\
&\left.+\frac{(n+4)}{(n+3)}\left(\frac{u_{n,3}V_{ac}}{T}\right)\right].
\end{aligned}
\tag{B.6}
$$

The equilibrium number of photons in modified measure can be estimated in the same way as we did in unmodified case and using (2.26) we have,

$$
\begin{aligned}
\bar{N} &= \frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!k^{n+3}}\frac{4\pi T^{n+3}}{(3-n')}\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\Gamma(n+3)\left[Z_{n+3}(0)-Z_{n+3}\left(\frac{\kappa}{T}\right)\right]\\
&+\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!\kappa^{n+3}}\left(\frac{4\pi T^{n+3}}{3}\right)\ln\left(\frac{3\kappa^3V}{4\pi}\right)\Gamma(n+3)\left[Z_{n+3}(0)-Z_{n+3}\left(\frac{\kappa}{T}\right)\right].
\end{aligned}
\tag{B.7}
$$

B.2 The leading high temperature behaviour for unmodified case

In the high temperature ($T \to \kappa$) case we get,

$$
u \approx -\frac{18\kappa^4}{\pi^2}\left[Z_4(0)-Z_4(1)-\frac{1}{18(e-1)}\right]+\frac{24\kappa^4}{\pi^2}\left(\frac{T}{\kappa}\right)\left[Z_4(0)-Z_4(1)-\frac{1}{24(e-1)}\right], \tag{B.8}
$$

$$
P \approx -\frac{6\kappa^4}{\pi^2}\left[Z_4(0)-Z_4(1)\right]+\frac{8\kappa^4}{\pi^2}\left(\frac{T}{\kappa}\right)\left[Z_4(0)-Z_4(1)\right]-\frac{\kappa^4}{3\pi^2}\left(\frac{T}{\kappa}\right)\ln\left(1-\frac{1}{e}\right), \tag{B.9}
$$

$$S \approx -\frac{\kappa^3 V_{ac}}{3\pi^2}\ln\left(1-\frac{1}{e}\right)-\frac{16\kappa^3 V_{ac}}{\pi^2}\left[Z_4(0)-Z_4(1)-\frac{1}{48(e-1)}\right] + \frac{24\kappa^3 V_{ac}}{\pi^2}\left(\frac{T}{\kappa}\right)\left[Z_4(0)-Z_4(1)-\frac{1}{24(e-1)}\right], \tag{B.10}$$

$$\bar{N} \approx -\frac{4\kappa^3 V_{ac}}{\pi^2}\left[Z_3(0)-Z_3(1)-\frac{1}{4(e-1)}\right]+\frac{6\kappa^3 V_{ac}}{\pi^2}\left(\frac{T}{\kappa}\right)\left[Z_3(0)-Z_3(1)-\frac{1}{6(e-1)}\right] \tag{B.11}$$

and

$$C_V \approx -\frac{48\kappa^3 V_{ac}}{\pi^2}\left[Z_4(0)-Z_4(1)-\frac{(3e-2)}{48(e-1)^2}\right]+\frac{72\kappa^3 V_{ac}}{\pi^2}\left(\frac{T}{\kappa}\right)\left[Z_4(0)-Z_4(1)-\frac{(4e-3)}{72(e-1)^2}\right]. \tag{B.12}$$

B.3 The leading low temperature behaviour for modified case

The low temperature limit can be calculated as we did in case of unmodified measure to get,

$$u \approx \frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!\kappa^n}\frac{4\pi}{(3-n')}\left(\frac{T^{n+4}}{V_{ac}\kappa^3}\right)\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\Gamma(n+4)Z_{n+4}(0) + \frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!\kappa^n}\left(\frac{4\pi T^{n+4}}{3V_{ac}\kappa^3}\right)\ln\left(\frac{3\kappa^3 V}{4\pi}\right)\Gamma(n+4)Z_{n+4}(0), \tag{B.13}$$

$$P \approx \frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!\kappa^n}\frac{4\pi}{(3-n')}\left(\frac{T^{n+4}}{V_{ac}\kappa^3}\right)\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\frac{\Gamma(n+4)}{(n+3)}Z_{n+4}(0)$$
$$+\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!\kappa^n}\left(\frac{4\pi T^{n+4}}{3V_{ac}\kappa^3}\right)\ln\left(\frac{3\kappa^3 V}{4\pi}\right)\frac{\Gamma(n+4)}{(n+3)}Z_{n+4}(0), \tag{B.14}$$

$$S \approx \frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!\kappa^n}\frac{4\pi}{(3-n')}\left(\frac{T^{n+3}}{\kappa^3}\right)\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\frac{\Gamma(n+4)(n+4)}{(n+3)}Z_{n+4}(0)$$
$$+\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!\kappa^n}\left(\frac{4\pi T^{n+3}}{3\kappa^3}\right)\ln\left(\frac{3\kappa^3 V}{4\pi}\right)\frac{\Gamma(n+4)(n+4)}{(n+3)}Z_{n+4}(0), \tag{B.15}$$

$$C_V \approx \frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!\kappa^n}\frac{4\pi}{(3-n')}\left(\frac{T^{n+3}}{\kappa^3}\right)\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\Gamma(n+4)(n+4)Z_{n+4}(0)$$
$$+\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!\kappa^n}\left(\frac{4\pi T^{n+3}}{3\kappa^3}\right)\ln\left(\frac{3\kappa^3 V}{4\pi}\right)\Gamma(n+4)(n+4)Z_{n+4}(0) \tag{B.16}$$

and

$$\bar{N} \approx \frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!k^{n+3}}\frac{4\pi T^{n+3}}{(3-n')}\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\Gamma(n+3)Z_{n+3}(0)$$
$$+\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!\kappa^{n+3}}\left(\frac{4\pi T^{n+3}}{3}\right)\ln\left(\frac{3\kappa^3 V}{4\pi}\right)\Gamma(n+3)Z_{n+3}(0) \tag{B.17}$$

B.4 The leading high temperature behaviour for modified case

In high temperature ($T \to \kappa$) case the behaviour is

$$u \approx u_a + u_b T \tag{B.18}$$

where u_a and u_b is

$$\begin{aligned}
u_a = &-\frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!}\frac{4\pi}{(3-n')}\left(\frac{\kappa^4}{V_{ac}\kappa^3}\right)\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\Gamma(n+4)(n+3)\Bigg[Z_{n+4}(0)\\
&-Z_{n+4}(1)-\frac{1}{(n+3)!(n+3)(e-1)}\Bigg]\\
&-\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!}\left(\frac{4\pi\kappa^4}{3V_{ac}\kappa^3}\right)\ln\left(\frac{3\kappa^3 V}{4\pi}\right)\Gamma(n+4)(n+3)\Bigg[Z_{n+4}(0)\\
&-Z_{n+4}(1)-\frac{1}{(n+3)!(n+3)(e-1)}\Bigg]
\end{aligned} \tag{B.19}$$

$$\begin{aligned}
u_b = &\frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!}\frac{4\pi}{(3-n')}\left(\frac{\kappa^4}{V_{ac}\kappa^3}\right)\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\Gamma(n+4)\frac{(n+4)}{\kappa}\Bigg[Z_{n+4}(0)\\
&-Z_{n+4}(1)-\frac{1}{(n+3)!(n+4)(e-1)}\Bigg]\\
&+\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!}\left(\frac{4\pi\kappa^4}{3V_{ac}\kappa^3}\right)\ln\left(\frac{3\kappa^3 V}{4\pi}\right)\Gamma(n+4)\frac{(n+4)}{\kappa}\Bigg[Z_{n+4}(0)\\
&-Z_{n+4}(1)-\frac{1}{(n+3)!(n+4)(e-1)}\Bigg],
\end{aligned} \tag{B.20}$$

$$P \approx P_a + P_b T \tag{B.21}$$

where P_a and P_b is

$$P_a = -\frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!}\frac{4\pi}{(3-n')}\left(\frac{\kappa^4}{V_{ac}\kappa^3}\right)\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\Gamma(n+4)\Big[Z_{n+4}(0)-Z_{n+4}(1)\Big]$$
$$-\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!}\left(\frac{4\pi\kappa^4}{3V_{ac}\kappa^3}\right)\ln\left(\frac{3\kappa^3V}{4\pi}\right)\Gamma(n+4)\Big[Z_{n+4}(0)-Z_{n+4}(1)\Big] \tag{B.22}$$

$$P_b = \frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!}\frac{4\pi}{(3-n')}\left(\frac{\kappa^4}{V_{ac}\kappa^3}\right)\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\left\{\Gamma(n+4)\frac{(n+4)}{\kappa(n+3)}\Big[Z_{n+4}(0)-Z_{n+4}(1)\Big]\right.$$
$$\left.-\frac{1}{\kappa(n+3)}\ln\left(1-\frac{1}{e}\right)\right\}$$
$$+\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!}\left(\frac{4\pi\kappa^4}{3V_{ac}\kappa^3}\right)\ln\left(\frac{3\kappa^3V}{4\pi}\right)\left\{\Gamma(n+4)\frac{(n+4)}{\kappa(n+3)}\Big[Z_{n+4}(0)-Z_{n+4}(1)\Big]\right.$$
$$\left.-\frac{1}{\kappa(n+3)}\ln\left(1-\frac{1}{e}\right)\right\}, \tag{B.23}$$

$$S \approx S_a + S_b T \tag{B.24}$$

where S_a and S_b is

$$S_a = -\frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!}\frac{4\pi}{(3-n')}\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\frac{1}{(n+3)}\left\{\Gamma(n+4)(n+4)(n+2)\right.$$
$$\left.\Big[Z_{n+4}(0)-Z_{n+4}(1)-\frac{1}{(n+3)!(n+2)(e-1)}\Big]+\frac{1}{(e-1)}+\ln\left(1-\frac{1}{e}\right)\right\}$$
$$-\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!}\left(\frac{4\pi}{3}\right)\ln\left(\frac{3\kappa^3V}{4\pi}\right)\frac{1}{(n+3)}\left\{\Gamma(n+4)(n+4)(n+2)\right.$$
$$\left.\Big[Z_{n+4}(0)-Z_{n+4}(1)-\frac{1}{(n+3)!(n+2)(e-1)}\Big]+\frac{1}{(e-1)}+\ln\left(1-\frac{1}{e}\right)\right\} \tag{B.25}$$

$$S_b = -\frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!}\frac{4\pi}{(3-n')}\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\Bigg\{\Gamma(n+4)\frac{(n+4)}{\kappa}\Bigg[Z_{n+4}(0)-Z_{n+4}(1)$$
$$-\frac{1}{(n+3)!(n+4)(e-1)}\Bigg]\Bigg\}$$
$$-\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!}\left(\frac{4\pi}{3}\right)\ln\left(\frac{3\kappa^3V}{4\pi}\right)\Bigg\{\Gamma(n+4)\frac{(n+4)}{\kappa}\Bigg[Z_{n+4}(0)-Z_{n+4}(1)$$
$$-\frac{1}{(n+3)!(n+4)(e-1)}\Bigg]\Bigg\}, \tag{B.26}$$

$$C_V \approx C_{Va} + C_{Vb}T \tag{B.27}$$

where C_{Va} and C_{Vb} is

$$C_{Va} = -\frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!}\frac{4\pi}{(3-n')}\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\Bigg\{\Gamma(n+4)(n+4)(n+2)\Bigg[Z_{n+4}(0)-Z_{n+4}(1)$$
$$-\frac{1}{(n+3)!(n+2)(e-1)}\Bigg]+\frac{(e-2)}{(e-1)^2}\Bigg\}$$
$$-\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!}\left(\frac{4\pi}{3}\right)\ln\left(\frac{3\kappa^3V}{4\pi}\right)\Bigg\{\Gamma(n+4)(n+4)(n+2)\Bigg[Z_{n+4}(0)-Z_{n+4}(1)$$
$$-\frac{1}{(n+3)!(n+2)(e-1)}\Bigg]+\frac{(e-2)}{(e-1)^2}\Bigg\} \tag{B.28}$$

$$C_{Vb} = \frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!}\frac{4\pi}{(3-n')}\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\Bigg\{\frac{\Gamma(n+4)(n+4)(n+3)}{\kappa}$$
$$\Bigg[Z_{n+4}(0)-Z_{n+4}(1)-\frac{1}{(n+3)!(n+3)(e-1)}+\frac{1}{(n+3)!(n+4)(n+3)(e-1)^2}\Bigg]\Bigg\}$$
$$+\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!}\left(\frac{4\pi}{3}\right)\ln\left(\frac{3\kappa^3V}{4\pi}\right)\Bigg\{\frac{\Gamma(n+4)(n+4)(n+3)}{\kappa}$$
$$\Bigg[Z_{n+4}(0)-Z_{n+4}(1)-\frac{1}{(n+3)!(n+3)(e-1)}+\frac{1}{(n+3)!(n+4)(n+3)(e-1)^2}\Bigg]\Bigg\} \tag{B.29}$$

and

$$\bar{N} \approx \bar{N}_a + \bar{N}_b T \tag{B.30}$$

where $\bar{N}_a$ and $\bar{N}_b$ is

$$\begin{aligned}\bar{N}_a = &-\frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!}\frac{4\pi}{(3-n')}\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\Gamma(n+3)\Bigg\{(n+2)\Bigg[Z_{n+3}(0)-Z_{n+3}(1)\\ &-\frac{1}{(n+2)!(n+2)(e-1)}\Bigg]\Bigg\}\\ &-\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!}\left(\frac{4\pi}{3}\right)\ln\left(\frac{3\kappa^3 V}{4\pi}\right)\Gamma(n+3)\Bigg\{(n+2)\Bigg[Z_{n+3}(0)-Z_{n+3}(1)\\ &-\frac{1}{(n+2)!(n+2)(e-1)}\Bigg]\Bigg\}\end{aligned} \tag{B.31}$$

$$\begin{aligned}\bar{N}_b = &\frac{1}{(\pi)^2}\sum_{\substack{n=0,n'=0\\ n'\neq 3}}^{\infty}\frac{a_{n,n'}}{n!n'!}\frac{4\pi}{(3-n')}\left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}}-1\right]\Gamma(n+3)\Bigg\{\frac{(n+3)}{\kappa}\Bigg[Z_{n+3}(0)-Z_{n+3}(1)\\ &-\frac{1}{(n+2)!(n+3)(e-1)}\Bigg]\Bigg\}\\ &+\frac{1}{(\pi)^2}\sum_{n=0}^{\infty}\frac{a_{n,3}}{n!3!}\left(\frac{4\pi}{3}\right)\ln\left(\frac{3\kappa^3 V}{4\pi}\right)\Gamma(n+3)\Bigg\{\frac{(n+3)}{\kappa}\Bigg[Z_{n+3}(0)-Z_{n+3}(1)\\ &-\frac{1}{(n+2)!(n+3)(e-1)}\Bigg]\Bigg\}\end{aligned} \tag{B.32}$$

C Energy Flux Of Radiative Processes And Luminosity Of The White Dwarf

C.1 Energy flux for radiative processes

We have the expression of the energy flux given by

$$F^{\alpha} \approx \frac{-1}{\rho\gamma_{\nu}}\frac{\partial P^{\alpha\beta}}{\partial x^{\beta}} \approx \frac{-1}{\rho\gamma_{\nu}}\frac{\partial}{\partial x^{\beta}}(P_{\nu}\delta^{\alpha}_{\beta}) \tag{C.1}$$

Integrating over frequency and using the modified pressure and energy density relation in [89] we get,

$$\begin{aligned}\mathbf{F}_{rad} = &\left(\frac{1}{3\rho\gamma_{\kappa}}\right)\nabla\left[\frac{T\kappa^3}{\pi^2}\ln\left(1-e^{-\frac{\kappa}{T}}\right)\right]-\left(\frac{1}{3\rho\gamma_{R}}\right)\nabla\left[\frac{\pi^2T^4}{15}-\left[\left(\frac{6T^4}{\pi^2}\right)Li_4\left(e^{-\frac{\kappa}{T}}\right)\right.\right.\\ &\left.\left.+\left(\frac{6\kappa T^3}{\pi^2}\right)Li_3\left(e^{-\frac{\kappa}{T}}\right)+\left(\frac{3\kappa^2T^2}{\pi^2}\right)Li_2\left(e^{-\frac{\kappa}{T}}\right)-\left(\frac{\kappa^3T}{\pi^2}\right)\ln\left(1-e^{-\frac{\kappa}{T}}\right)\right]\right]\end{aligned} \tag{C.2}$$

This relation can be used to find the radiative force on per unit volume of matter using equation (6.167) of [111]. Here $\mathbf{F_{rad}}$ is the total radiative flux and $\frac{1}{\gamma_{\kappa}} = \int_0^{\kappa}\frac{d\nu}{\gamma_{\nu}}$, γ_R is the

modified Rosseland mean opacity defined as,

$$\frac{1}{\gamma_R} = \frac{\int_0^\kappa \frac{1}{\gamma_\nu}\left(\frac{\partial B_\nu}{\partial T}\right) d\nu}{\int_0^\kappa \left(\frac{\partial B_\nu}{\partial T}\right) d\nu} \tag{C.3}$$

where we have used the following relation,

$$\int_0^\kappa \left(\frac{\partial B_\nu}{\partial T}\right) d\nu = \frac{\partial}{\partial T}\int_0^\kappa (B_\nu) d\nu$$

$$= \frac{1}{4\pi}\frac{\partial}{\partial T}\left[\frac{\pi^2 T^4}{15} - \left[\left(\frac{6T^4}{\pi^2}\right) Li_4\left(e^{-\frac{\kappa}{T}}\right)\right.\right. \tag{C.4}$$

$$\left.\left. + \left(\frac{6\kappa T^3}{\pi^2}\right) Li_3\left(e^{-\frac{\kappa}{T}}\right) + \left(\frac{3\kappa^2 T^2}{\pi^2}\right) Li_2\left(e^{-\frac{\kappa}{T}}\right) - \left(\frac{\kappa^3 T}{\pi^2}\right)\ln\left(1 - e^{-\frac{\kappa}{T}}\right)\right]\right] \tag{C.5}$$

This expression in general can be written by considering the mean opacity as γ instead of γ_R. Now assuming the physical quantities depend on distance r, the vector equation reduce to the scalar one given in the text above.

C.2 Modified Pressure and Luminosity Expressions

The expression of the pressure becomes,

$$
\begin{aligned}
P(r) = \Bigg[& {P_b}^{n+1} + (n+1)\Bigg\{\frac{4GM}{3\pi L\gamma_1}\left[\frac{4\pi^4 {T_b}^{n+s+4}}{15(n+s+4)} - \frac{4\pi^4 T(r)^{n+s+4}}{15(n+s+4)}\right] \\
& - \frac{4GM(\kappa)^{n+s+4}}{3\pi L\gamma_{1\kappa}}\Bigg[-\Bigg(Li_4\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+4} \\
& -Li_4\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+4}\Bigg)(n+s)(n+s+2)(n+s+3) \\
& +\left(Li_3\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+3} - Li_3\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+3}\right)(n+s)(n+s+2) \\
& -\left(Li_2\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+2} - Li_2\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+2}\right)(n+s) \\
& +\left(Li_1\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_(r)}{\kappa}\right)^{n+s+1} - Li_1\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+1}\right) \\
& +\sum_{j=1}^{\infty} j^{n+s}\Gamma\left(-n-s-4, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right)(n+s)(n+s+2)(n+s+3)(n+s+4)\Bigg] \\
& + \frac{4GM(\kappa)^{n+s+4}}{3\pi L\gamma_1}\Bigg[-\Bigg(Li_4\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+4} \\
& -Li_4\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+4}\Bigg)(n+s-1)[6+n^2+s(s+3)+n(2s+3)] \\
& +\left(Li_3\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+3} - Li_3\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+3}\right)[6+n^2+s(s-1)+n(2s-1)] \\
& -\left(Li_2\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+2} - Li_2\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+2}\right)(n+s-3) \\
& +\left(Li_1\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+1} - Li_1\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+1}\right) \\
& +\sum_{j=1}^{\infty} j^{n+s}\Gamma\left(-n-s-4, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right)(n+s)(n+s+1)(n+s+2)(n+s+3)\Bigg]\Bigg\}\Bigg]^{\frac{1}{n+1}}.
\end{aligned}
\tag{C.6}
$$

The corresponding expression of luminosity in relativistic case is,

$$
\begin{aligned}
L = {} & \frac{(n+1)}{P'}\left\{\frac{4GM}{3\pi L\gamma_1}\left[\frac{4\pi^4 {T_b}^{n+s+4}}{15(n+s+4)} - \frac{4\pi^4 T(r)^{n+s+4}}{15(n+s+4)}\right]\right. \\
& - \frac{4GM(\kappa)^{n+s+4}}{3\pi L\gamma_{1\kappa}}\left[-\left(Li_4\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+4}\right.\right. \\
& \left. -Li_4\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+4}\right)(n+s)(n+s+2)(n+s+3) \\
& + \left(Li_3\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+3} - Li_3\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+3}\right)(n+s)(n+s+2) \\
& - \left(Li_2\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+2} - Li_2\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+2}\right)(n+s) + \left(Li_1\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+1}\right. \\
& \left. -Li_1\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+1}\right) \\
& \left. + \sum_{j=1}^{\infty} j^{n+s}\Gamma\left(-n-s-4, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right)(n+s)(n+s+2)(n+s+3)(n+s+4)\right] \\
& + \frac{4GM(\kappa)^{n+s+4}}{3\pi L\gamma_1}\left[-\left(Li_4\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+4}\right.\right. \\
& \left. -Li_4\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+4}\right)(n+s-1)[6+n^2+s(s+3)+n(2s+3)] \\
& + \left(Li_3\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+3} - Li_3\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+3}\right)[6+n^2+s(s-1)+n(2s-1)] \\
& - \left(Li_2\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+2} - Li_2\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+2}\right)(n+s-3) + \left(Li_1\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+1}\right. \\
& \left. -Li_1\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+1}\right) \\
& \left.\left. + \sum_{j=1}^{\infty} j^{n+s}\Gamma\left(-n-s-4, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right)(n+s)(n+s+1)(n+s+2)(n+s+3)\right]\right\},
\end{aligned}
\tag{C.7}
$$

where $P' = \left[\left(\frac{N_A T_0 m_p \mu_e\left[{T_0}^2 - m^2\left(1-\frac{T_0}{\kappa}\right)^2\right]^{\frac{3}{2}}}{3\pi^2\mu}\right)^{n+1} - P_b^{n+1}\right]$.

The expression of luminosity in nonrelativistic case is,

$$
\begin{aligned}
L = {} & \frac{(n+1)}{P'}\left\{\frac{4GM}{3\pi L\gamma_1}\left[\frac{4\pi^4 {T_b}^{n+s+4}}{15(n+s+4)} - \frac{4\pi^4 T(r)^{n+s+4}}{15(n+s+4)}\right]\right. \\
& - \frac{4GM(\kappa)^{n+s+4}}{3\pi L\gamma_{1\kappa}}\left[-\left(Li_4\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+4}\right.\right. \\
& \left. -Li_4\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+4}\right)(n+s)(n+s+2)(n+s+3) \\
& + \left(Li_3\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+3} - Li_3\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+3}\right)(n+s)(n+s+2) \\
& - \left(Li_2\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+2} - Li_2\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+2}\right)(n+s) + \left(Li_1\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_(r)}{\kappa}\right)^{n+s+1}\right. \\
& \left. -Li_1\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+1}\right) \\
& \left. + \sum_{j=1}^{\infty} j^{n+s}\Gamma\left(-n-s-4, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right)(n+s)(n+s+2)(n+s+3)(n+s+4)\right] \\
& + \frac{4GM(\kappa)^{n+s+4}}{3\pi L\gamma_1}\left[-\left(Li_4\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+4}\right.\right. \\
& \left. -Li_4\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+4}\right)(n+s-1)[6+n^2+s(s+3)+n(2s+3)] \\
& + \left(Li_3\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+3} - Li_3\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+3}\right)[6+n^2+s(s-1)+n(2s-1)] \\
& - \left(Li_2\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+2} - Li_2\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+2}\right)(n+s-3) + \left(Li_1\left(e^{\frac{-\kappa}{T_b}}\right)\left(\frac{T_b}{\kappa}\right)^{n+s+1}\right. \\
& \left. -Li_1\left(e^{\frac{-\kappa}{T(r)}}\right)\left(\frac{T(r)}{\kappa}\right)^{n+s+1}\right) \\
& \left.\left. + \sum_{j=1}^{\infty} j^{n+s}\Gamma\left(-n-s-4, \frac{j\kappa}{T_b}, \frac{j\kappa}{T(r)}\right)(n+s)(n+s+1)(n+s+2)(n+s+3)\right]\right\},
\end{aligned}
\tag{C.8}
$$

where $P' = \left[\left(\frac{N_A T_0 m_p \mu_e [2mT_0]^{\frac{3}{2}}}{3\pi^2\mu}\right)^{n+1} - P_b^{n+1}\right]$.

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