Automorphisms of Riemann surfaces of genus $g \geq 2$

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Abstract

We will show that automorphism group of any Riemann surface X of genus $g \geq 2$ is finite. We will also give a bound to the cardinality of the automorphism group, depending on the genus, specifically $\mathrm{Aut}(X) \leq 84(g-1)$. This bound will be obtained by applying Hurwitz formula to the natural holomorphic map from a Riemann surface to it's quotient under action of the finite group $\mathrm{Aut}(X)$. The finiteness is proved by considering a homomorphism from $\mathrm{Aut}(X)$ to the permutation group of a finte set and showing that the kernel is finite. The finte set under consideration is the set of Weierstass points. p is a Weierstass point, if the set of integers n, such that there is no $f \in \mathcal{M}(X)$ whose only pole is p with order n, is not $\{1, \cdots, g\}$. All these are explained in Chapter 4. Riemann-Roch Theorem is heavily used which is proved in Chapter 3. Proof of Riemann-Roch Theorem requires existence of non-constant meromorphic functions on a Riemann surface, which is proved in Chapter 2. Basics are dealt with in Chapter 1.

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Contents

CHAPTER 1

Riemann surface of any genus

1. Holomorphic maps

We define the objects and morphisms of the category we are going to study.

Definition 1.1. *A Riemann surface is a connected 1 dimensional complex manifold.*

Example The easiest examples are connected open subsets of $\mathbb{C}$.

But these are not compact. Compact connected orientable smooth manifolds of dimension two can be classified as surfaces of genus g, for $g \in \mathbb{N}$. If a manifold has complex structure then it must be orientable. Can each of the genus g surfaces be given complex structure? Let us start with the genus zero case, S^2. We recall that S^2 is a smooth manifold with charts being $\mathbb{C}\setminus\{0\}$ and inverse of stereographic projections, one from the north pole and another from the south. We also recall that the transition map is $z \mapsto 1/\overline{z}$. But this is not a holomorphic map. So compose one of the stereographic projections with a conjugation map to get a new chart, so that the transition map is $z \mapsto 1/z$, which is holomorphic.

Example S^2 with the above described charts is a Riemann surface , called the *Riemann Sphere* and denoted by $\mathbb{C}_\infty$.

Next comes the genus 1 case. We know that genus 1 surface or the 2-torus can be constructed as the quotient of the properly discontinuous group action of $\mathbb{Z}^2$ on $\mathbb{R}^2$. The smooth charts in this case are the regions of $\mathbb{R}^2$ on which the quotient map π is injective, and the chart maps being π^{-1} restricted to such regions. The transition maps are just translations of $\mathbb{R}^2 \cong \mathbb{C}$ and hence holomorphic.

Example The torus with the above complex structure is called (quite unimaginatively) the *complex torus.*

Is there a complex structure for surface of genus g, for each g? Before answering that question we define what are the morphisms in the category of Riemann surfaces. In case of smooth manifolds, the morphisms are set maps which when pre and post composed with chart maps of the respective manifolds, give smooth maps between opens sets of euclidean spaces. Here we require the composition to be holomorphic between open sets of the complex plane. Such morphisms are called *holomorphic maps between Riemann surfaces.* They have a very nice local form.

Proposition 1.2. *[Local Normal Form] Let $F : X \to Y$ be a non-constant holomorphic map between two Riemann surfaces. Let $p \in X$. Then there exists a unique integer $n \geq 1$ and charts (U_1, ϕ_1) and (U_2, ϕ_2) of X and Y respectively, such that $\phi_1(p) = 0$ and $\phi_2(F(p)) = 0$ and $\phi_2 \circ F \circ \phi_1^{-1}(z) = z^n$.*

PROOF. Charts (U, ϕ), such that $\phi(p) = 0$ are called charts *centred* at p. Such charts can always be constructed by starting with any chart and composing with a suitable translation of the complex plane to make $\phi(p) = 0$. So consider two charts (V_1, ψ_1) and (U_2, ϕ_2) centred at p and $F(p)$, respectively. Then by definition the map $T := \phi_2 \circ F \circ \psi_1^{-1}(z)$ is holomorphic and hence has a power series expansion $T(w) = \sum_{i=n}^{\infty} a_i w^i$, such that $a_n \neq 0$. Note that $n \geq 1$, as $T(0) = 0$. Now T can be decomposed as $T(w) = w^n S(w)$, where S is another holomorphic function such that $S(0) \neq 0$. Therefore there exists a neighbourhood of 0 and a holomorphic function R on it, such that $R^n = S$. Then we have $T(w) = (wR(w))^n$. Define $\xi(w) = wR(w)$. Note that $\xi'(0) = R(0) \neq 0$, and hence by Inverse Function Theorem is invertible in a neighbourhood of 0. The composition $\phi_1 := \xi \circ \psi$ defines a new chart on X. Putting $z := \xi(w)$ we have,

$$\begin{aligned}\phi_2 \circ F \circ \phi_1^{-1}(z) &= \phi_2 \circ F \circ \psi_1^{-1} \circ \xi^{-1}(z)\\ &= T(\xi^{-1}(z))\\ &= T(w)\\ &= (wR(w))^n\\ &= z^n\end{aligned}$$

Uniqueness of n follows from the fact that the map $z \mapsto z^n$ has n preimages for each non-zero point. Hence so has F for each point in a neighbourhood of p and not equal to p. But this quantity is independent of the charts chosen. Hence n is unique. □

Definition 1.3. *Given a holomorphic map $F : X \to Y$ between two Riemann surfaces, and a point $p \in X$, the unique integer n such that F is locally of the form $z \mapsto z^n$, is called the multiplicity of F at the point p, and is denoted by $\mathrm{mult}_p(F)$.*

If F is locally represented by $z \mapsto z^n$ in some parametric disk (that is, image of some disk under chart map), then we note that given any non-zero point z_0 in the disk, we can find a smaller disk within the given disk, centred at z_0, which is mapped injectively by $z \mapsto z^n$. Hence no other point in the parametric disk has multiplicity greater than one. Thus points of multiplicity greater than one form a discrete set in the domain Riemann surface.

Definition 1.4. *Points with multiplicity greater than one are called ramification points and their images are called branch points.*

Thus on a compact Riemann surface there can be only finitely many ramification points. Consider the map $\phi : D \to D$ such that $z \mapsto z^n$. Any non-zero point has n preimages, each with multiplicity 1, whereas the point 0 has only 0 as it's preimage, with multiplicity n. Taking cue from this observation we have the following proposition.

Proposition 1.5. *Let $F : X \to Y$ be a non-constant holomorphic map between two compact Riemann surfaces. Then the quantity $\sum_{p \in F^{-1}(q)} \mathrm{mult}_p(F)$ is independent of $q \in Y$.*

First we prove some general results which will be used in the proof.

Lemma 1.6. *Let $F : X \to Y$ and $G : X \to Y$ be two holomorphic map between Riemann surfaces. If $F = G$ on a subset with limit point, then $F = G$.*

PROOF. Consider the set $A := \{x \in X : F(x) = G(x)\}$. We will show that this set is both open and closed in X, which is connected and so A will be the whole of X. That A is closed is clear from the fact that Y is Hausdroff and A is the preimage of the diagonal of $Y \times Y$ under the continuous map $x \mapsto (F(x), G(x))$. Now we prove that A is open. Let the subset with limit point be S and p be a limit point of S. Let (U_1, ϕ_1) and (U_2, ϕ_2) be charts of X and Y around p and $F(p) = G(p)$, respectively. Then locally F and G are represented by holomorphic functions f and g, respectively. Derivatives of each order of f and g can be calculated from their values in $\phi_1(U_1 \cap S)$, where they are equal. But these are the coefficients of power series representation of f and g in neighbourhood of $\phi_1(p)$. Hence $f = g$ in a neighbourhood of $\phi_1(p)$ and thus $F = G$ in a neighbourhood of p. Therefore A is open too. □

Lemma 1.7. *Let $F : X \to Y$ be a non-constant holomorphic map between two Riemann surfaces. Then the preimage of any point is discrete. If X is compact then the preimage is finite non-empty.*

PROOF. Let $y \in Y$ and $F(x) = y$. Let (U_1, ϕ_1) and (U_2, ϕ_2) be charts of X and Y, centred at x and y respectively. Then locally F is represented by a holomorphic function g between the $\phi_1(U_1)$ and $\phi_2(U_2)$, such that $g(0) = 0$. If $g \equiv 0$, then F is equal to a constant map on an open set, hence is constant, by previous theorem. This is a contradiction. Therefore g is non-constant and by discreteness of zeros of a holomophic function in the complex plane, there exists a punctured neighbourhood of 0, on which g is non zero. Hence F does not take the value y in a punctured neighbourhood of x. Therefore preimages of y forms a discrete set. If X is compact then any discrete set must be finite. Also F is continuous, and hence $F(X)$ is compact and so it is closed in Y. But the Open Mapping Theorem in complex analysis implies that any holomorphic map between Riemann surfaces is also open. Therefore $F(X)$ is open too. Since Y is connected, we must have $F(X) = Y$. Thus F is surjective and hence preimage of any point is non empty. □

Now we prove Proposition 1.5

PROOF. We wish to prove that the map $q \mapsto \sum_{p \in F^{-1}(q)} \mathrm{mult}_p(F)$ is locally constant. Since Y is connected, this is enough to prove that the map is constant. Consider any point $y \in Y$. We will use Proposition 1.2 to find a neighbourhood of y on which the above map is constant. Notice in the proof of Proposition 1.2, we started with charts in both domain and range, but modified only the domain chart. So we can find disjoint parametric disks $U_1, \cdots, U_n$ centred at $x_1, \cdots, x_n$, respectively, and a parametric disk V centred at y, such that near each x_i, F is represented by a map $z \mapsto z^{n_i}$, in local coordinates, from U_i to V. We know that the proposition is true for maps $D \to D$, $z \mapsto z^n$ and hence for a disjoint union of such maps, $\coprod_i D_i \to D$. Therefore it is true for F restricted to $\coprod_i U_i$. So the only thing to prove is that preimage of V is contained in $\coprod_i U_i$. We can hope to shrink V to achieve this. That shrinking will work is proved below.

Given a neighbourhoods $U_1, \cdots, U_n$ of $x_1, \cdots, x_n$ respectively, we will prove that there exists a neighbourhood V of y, such that $F^{-1}(V) \subset \coprod_i U_i$. Suppose not, then there exists a sequence $\{y_k\}$ converging to y, such that each y_k has a preimage not lying in any of the U_i's. For each y_k choose such a preimage p_k. Since X is compact, therefore $\{p_k\}$ has a convergent subsequence which converges at a point p, say. The image of this subsequence is a subsequence of y_k and hence converges to y. Therefore $F(p) = y$, hence $p = x_l$ for some $1 \leq l \leq n$. This implies that a subsequence of $\{p_k\}$, which lies outside U_l converges to x_l, which is a contradiction. □

Now we can define,

Definition 1.8. *Let $F : X \to Y$ be a non-constant holomorphic map between two compact Riemann surfaces. Then the integer $\sum_{p \in F^{-1}(q)} \mathrm{mult}_p(F)$, which is independent of q, is called the degree of F and is denoted by $\deg(F)$.*

2. Meromorphic functions

Meromorphic function on complex plane are holomorphic ones with special kind of singularities at discrete points. We can similarly define meromorphic functions on a Riemann surface as functions to $\mathbb{C}$ which near each point p, are holomorphic in a punctured neighbourhood and have either removable singularity or a pole at p. We denote the set of such functions on a Riemann surface X, by $\mathcal{M}(X)$. Corresponding to each $f \in \mathcal{M}(X)$, there exits a holomorphic function $F : X \to \mathbb{C}_\infty$, defined as

$$F(x) = \begin{cases} f(x), & \text{if } x \text{ is not a pole of } f \\ \infty, & \text{if } x \text{ is a pole of } f \end{cases}$$

We leave it to the readers to check that that F is indeed a holomorphic function. This correspondence is one to one.

We know in complex analysis that every meromorphic function g can be represented by a Laurent series near a point x and the exponent of the lowest non-zero term in the Laurent series is called the order of g at p, denoted by $\mathrm{ord}_p(g)$. The order can also be described as the only integer k, for which $\lim_{z \to p}(z-p)^{-k} g \neq 0$ or ∞. In a Riemann surface if a meromorphic function f is composed with the inverse of a chart map, then we get a meromorphic function in a region of the complex plane. Call it g. g has a Laurent series expansion. If we take a different chart then we will get a different Laurent series expansion. But we claim that the order is same in both cases. To see this note that if g_1 and g_2 are two local expressions of a meromorphic function f with respect to charts centred at p and T is the transition map between the two charts, then $g_2 = g_1 \circ T$. We have $T(0) = 0$ and $T'(0) \neq 0$. Let the local coordinate for g_1 and g_2 be z and w respectively. Then $T(w) = z$. Let $\mathrm{ord}_p(g_1) = k$. Then $\lim_{w \to 0} w^{-k} g_2(w) = \lim_{w \to 0} w^{-k} g_1 \circ T(w) = \lim_{w \to 0} w^{-k} T(w)^k (z^{-k} g_1(z)) = T'(0)^k \lim_{w \to 0} z^{-k} g_1(z) \neq 0$ or ∞. Therefore $\mathrm{ord}_p(g_2) = k$. Hence we can define

Definition 2.1. *The order of a meromorphic function at a point p, denoted by $\mathrm{ord}_p(f)$, is the order of any any local representation of it near p.*

Order of f is non-zero only at preimages of 0 or ∞. Now we observe that for a point p in the preimage of 0, $\mathrm{ord}_p(f)$ is same as multiplicity of the corresponding $F : X \to \mathbb{C}_\infty$ at p. This follows from the proof of Proposition 1.2, where we saw that the multiplicity at p is the exponent of the least non-zero term of the local representation of F with respect to charts centred at p and $F(p)$. In this case the f is such a local representation of F at the point p. Similarly if p is in the preimage of ∞, then $1/f$ is such a local represented of F. Hence $\mathrm{mult}_p(F) = \mathrm{ord}_p(1/f) = -\mathrm{ord}_p(f)$, for all p in the preimage of ∞. So suppose that $x_1, \cdots, x_n$ be the preimages of 0, and $y_1, \cdots, y_m$ be preimages of ∞. Then $\sum_{i=1}^n \mathrm{mult}_{x_i}(F) = \deg(F) = \sum_{j=1}^m \mathrm{mult}_{y_j}(F)$. Therefore

$$\begin{aligned} \sum_p \mathrm{ord}_p(f) &= \sum_i \mathrm{ord}_{x_i}(f) + \sum_j \mathrm{ord}_{y_j}(f) \\ &= \sum_i \mathrm{mult}_{x_i}(F) - \sum_j \mathrm{mult}_{y_j}(F) \\ &= 0 \end{aligned}$$

Therefore we have proved the following Proposition.

Proposition 2.2. *Let X be a compact Riemann surface and $f \in \mathcal{M}(X)$ be non constant. Then $\sum_p \mathrm{ord}_p(f) = 0$.*

3. Hurwitz formula

Compact Riemann Surfaces are also smooth 2-manifolds. Therefore we can make sense of genus of a compact Riemann surface. We have seen that a non-constant holomorphic map from a compact Riemann surface forces the range to be compact. Existence of such a map also puts a restriction on genus of the range. This restriction is described by the *Hurwitz Formula.*

Theorem 3.1. *Let $F : X \to Y$ be a non constant holomorphic map between two compact Riemann surfaces. Then*

$$2g(X) - 2 = \deg(F)(2g(Y) - 2) + \sum_{p \in X} [\mathrm{mult}_p(F) - 1]$$

PROOF. Consider a triangulation of Y, such that each branch point is a vertex. (That every Riemann surface can be trangulated is explained in [5], Chapter 1, Section 8, we assume this fact here.) Notice from the proof of Proposition 1.2 that $F : X\backslash\{\text{ramification points}\} \to Y\backslash\{\text{branch points}\}$ is a covering map. So we can lift the simplexes (minus some vertices) to $X \setminus \{\text{ramification points}\}$. Plugging in the ramification point by vertices we have a triangulation of X. We know that the Euler number defined as $E = v - e + f$, where v, e and f are the number of vertices, edges and and faces respectively of a triangulation, is independent of the triangulation. It is equal to $2-2g$, where g is the genus of the surface. Now we try to compute the Euler number of X in terms of that of Y. Let v, e and f be the number of vertices, edges and and faces respectively of Y and v', e' and f' be those of X. Since branch points occur only at vertices of Y, therefore the number of edges and faces of X are just $\deg(F) \cdot e$ and $\deg(F) \cdot f$ respectively. Let q be a vertex of Y, then number of it's preimages counting multiplicity is $\deg(F)$. But we have counted each point $p \in F^{-1}(q)$, $\mathrm{mult}_p(F)$ times. So we have to subtract $\mathrm{mult}_p(F) - 1$ for each p. Hence $|F^{-1}(q)| = \deg(F) - \sum_{p \in F^{-1}(q)} \mathrm{mult}_p(F) - 1$. Then

$$\begin{aligned} v' &= \sum_{q \text{ vertex of } Y} (\deg(F) - \sum_{p \in F^{-1}(q)} [\mathrm{mult}_p(F) - 1]) \\ &= \deg(F)v - \sum_{q \text{ vertex of } Y} \sum_{p \in F^{-1}(q)} [\mathrm{mult}_p(F) - 1] \\ &= \deg(F)v - \sum_{p \text{ vertex of } X} [\mathrm{mult}_p(F) - 1] \end{aligned}$$

Thus the Euler number of X can be calculated as

$$\begin{aligned} 2g(X) - 2 &= -v' + e' - f' \\ &= -\deg(F)v - \sum_{p \text{ vertex of } X} [\mathrm{mult}_p(F) - 1] + \deg(F) \cdot e - \deg(F) \cdot f \\ &= \deg(F)(2g(X) - 2) + \sum_{p \in X} [\mathrm{mult}_p(F) - 1] \end{aligned}$$

where the last equality is because all ramification points are vertices. □

4. Branched covering space theory

Now we begin to see how to construct a Riemann surface of any genus. We note from proof of Proposition 1.5, that except for finitely many points, a non-constant holomorphic map between two Riemann surfaces is a covering map. The domain along with such a map is called a *branched covering.* So we start with a familiar Riemann surface and construct a branched covering, specifying the number of branch points and the multiplicity of their preimages in such a way that the Hurwitz formula yields the desired genus for the constructed cover.

We know that given a permutation representation $\pi(X, x_0) \to S_d$, where S_d is the symmetric group of order d and $\pi(X, x_0)$ is the fundamental group of a connected, locally path connected, semilocally simply connected topological space X, we can construct a d sheeted covering space $p : \widetilde{X} \to X$ such that the group action $\pi(X, x_0) \to \mathrm{Perm}(p^{-1}(x_0))$ is exactly the permutation representation we started with. (See [1], pg. 68 for the construction.) Note that all the restrictions on X are satisfied if it is a Riemann surface, since a Riemann surface is locally euclidean and connected. Also when X is a Riemann surface, we would like the covering space $\widetilde{X}$ to be a Riemann surface again, such that, the covering map p is holomorphic. The charts of $\widetilde{X}$ are just lifts of charts of X, contained in some evenly covered neighbourhood, via p^{-1}. Compatibility of charts follows from that of charts in X. The important restriction is that $\widetilde{X}$ must be connected. Let us see how the connectivity of $\widetilde{X}$ affects the group action $\pi(X, x_0) \to \mathrm{Perm}(p^{-1}(x_0))$.

Lemma 4.1. *Let $p : \widetilde{X} \to X$ be a d sheeted covering space, where X is path connected and let the corresponding group action be $\rho : \pi(X, x_0) \to \mathrm{Perm}(p^{-1}(x_0))$. Then $\widetilde{X}$ is path connected if and only if the action ρ is transitive.*

PROOF. We recall how $\pi(X, x_0)$ acts on the set $p^{-1}(x_0)$. Let $[\gamma] \in \pi(X, x_0)$ and $\widetilde{x_0} \in p^{-1}(x_0)$. Let $\widetilde{\gamma}$ be the lift of γ starting from $\widetilde{x_0}$. Then $\rho([\gamma])(\widetilde{x_0}) = \widetilde{\gamma}(1)$. Now if $\widetilde{X}$ is path connected then given any two points $\widetilde{x_1}, \widetilde{x_2} \in p^{-1}(x_0)$, there exists a path $\widetilde{\gamma} : \widetilde{x_1} \rightsquigarrow \widetilde{x_2}$. Notice that $\gamma := p \circ \widetilde{\gamma}$ is a loop based at x_0. Then $[\gamma]$ acting on $\widetilde{x_1}$ gives us $\widetilde{x_2}$. Hence the action is transitive.

Again suppose the action is transitive. Then all the preimages of $p^{-1}(x_0)$ lie in the same component. Let $\widetilde{x} \in \widetilde{X}$. Since X is path connected, there exists a path $\gamma : p(\widetilde{x}) \rightsquigarrow x_0$. Let $\widetilde{\gamma}$ be a lift of γ starting at $\widetilde{x}$. It's end point must be a point in $p^{-1}(x_0)$. Therefore $\widetilde{x}$ lies in the same component as that of $p^{-1}(x_0)$. Thus $\widetilde{X}$ is path connected. □

This Lemma implies that whenever we have a transitive permutation representation $\pi(X, x_0) \to S_d$, we can construct a Riemann surface $\widetilde{X}$ and an unramified holomorphic map $F : \widetilde{X} \to X$, of degree d, such that, the group action $\pi(X, x_0) \to \mathrm{Perm}(F^{-1}(x_0))$ is exactly the permutation representation we started with.

Now we try to extend this result for branched cover of a Riemann surface. Let Y be a Riemann surface and choose finitely many points $b_1, \cdots, b_n$ in Y. These are going to be our branch points. Consider the Riemann surface $V = Y \setminus \{b_1, \cdots, b_n\}$. Given a permutation representation $\pi(V, y_0) \to S_d$, we can construct a Riemann surface $\widetilde{V}$ and an unramified holomorphic map $F : \widetilde{V} \to V$, of degree d, such that, the group action $\pi(V, y_0) \to \mathrm{Perm}(F^{-1}(y_0))$ is exactly the permutation representation we started with. Now we have to add points to $\widetilde{V}$, which will form preimages of the left out points $b_1, \cdots, b_n$, and also make the branched cover compact. Consider a parametrized unit disk W, with chart map ψ, centred at b_i, which does not contain any other b_j's. Let W^* be the punctured neighbourhood $W \setminus \{b_i\}$. Then $F^{-1}(W^*)$ is a cover of W^*, which is homeomorphic to punctured disk D^*. We know n-sheeted connected cover of D^* is of the form $D^* \to D^*$, $z \mapsto z^n$. Hence $F^{-1}(W^*)$ is a disjoint union of open sets U_j^*, $1 \leq j \leq m_i$, each homeomorphic to the unit punctured disk D^* via a homeomorphism ϕ_j, such that the following diagram commutes.

$$\begin{tikzcd} U_j^* \arrow[r, "\phi_j"] \arrow[d, "F|_{U_j^*}"'] & D^* \arrow[d, "z \mapsto z^{k_j}"] \\ W^* \arrow[r, "\psi|_{W^*}"] & D^* \end{tikzcd}$$

(1)

Notice that $F|_{U_j^*}$ and $\psi|_{W^*}$ are holomorphic, and $z \mapsto z^{k_j}$ is locally biholomorphic, hence ϕ_j is also holomorphic and hence can act as a chart map. Intuitively we feel there is a *hole* in this chart. A have a precise definition for *hole chart.*

Definition 4.2. *Let X be a Riemann surface. A hole chart on X is a chart $\phi : U \to V$ on X, such that, V contains an open punctured disk $B^* = \{z \in \mathbb{C} : 0 < \|z - z_0\| < \epsilon\}$ with the following properties.*

(1) $\overline{\phi^{-1}(B^*)} \subset U$
(2) $\phi(\overline{\phi^{-1}(B^*)}) = \{z \in \mathbb{C} : 0 < \|z - z_0\| \leq \epsilon\}$

We see that U_j^*'s are hole charts in $\widetilde{V}$, where B^* can be taken as the punctured disk centred at 0 and with radius 1/2. We need to "plug these holes". For that we need to "glue" a disk D onto this hole. We first make the idea of "glueing" precise. Let X and Y be two Riemann surfaces and let U and V be two non-empty open subsets of X and Y respectively. Assume that there exists a biholomorphic map $\phi : U \to V$. Then we can construct a quotient space of $X \coprod Y$ by identifying $u \sim \phi(u)$, when $u \in U$. Call this quotient topological space Z. There is a minor technicality here. Z may not be Hausdorff. But to have any hope of making it into a Riemann surface we must have Z Hausdorff. So we have to assume this condition and remember to check it every time we make such a construction.

We can define a complex structure on Z as follows. Let $i_X : X \to Z$ and $i_Y : Y \to Z$ be inclusions of X and Y in Z. Note $Z = i_X(X) \cup i_Y(Y)$ and both i_X and i_Y are homeomorphisms onto their image. For every chart (U, h) in X we specify that $(i_X(U), h \circ i_X^{-1})$ is a chart of Z. Similar charts are constructed for points in $i_Y(Y)$. Now we have to check compatibility. Given a pair of charts there are two cases. One in which both of them are constructed using charts of the same space, either X or Y; and the other in which the one chart is constructed from that of X and the other from that of Y. In the first case we can assume without loss of generality that the two charts are of the form $(i_X(U), h \circ i_X^{-1})$ and $(i_X(V), k \circ i_X^{-1})$. Then the transformation map is $k \circ i_X^{-1} \circ (h \circ i_X^{-1})^{-1} : h(U) \cap k(V) \to h(U) \cap k(V)$. But $k \circ i_X^{-1} \circ (h \circ i_X^{-1})^{-1} = k \circ i_X^{-1} \circ i_X \circ h^{-1} = k \circ h^{-1}$, which being a transformation map for charts in X, is holomorphic. In the second case the two charts are of the form $(i_X(U), h \circ i_X^{-1})$ and $(i_Y(V), k \circ i_Y^{-1})$. Then the transformation map simplifies to $k \circ i_Y^{-1} \circ i_X \circ h^{-1}$. But $i_Y^{-1} \circ i_X | U = \phi$, which is holomorphic map between U and V. Hence $k \circ i_Y^{-1} \circ i_X \circ h^{-1}$ is also holomorphic. Therefore the charts are all compatible. So we have a complex structure on Z.

One more condition for Z to be a Riemann surface is that it should be connected. X and Y are connected. i_X and i_Y are continuous. Therefore $i_X(X)$ and $i_Y(Y)$ are also connected subsets of Z. Also $i_X(X) \cap i_Y(Y) \neq \emptyset$. Hence Z is also connected. Therefore Z is a Riemann surface.

Returning to the case of hole charts U_j^*, in the covering space $\widetilde{V}$, we glue $\widetilde{V}$ and the disk B centred at 0 and of radius 1/2, via the map $\phi_j|_{\phi_j^{-1}(B^*)} : \phi_j^{-1}(B^*) \to B^*$. That the new space is Hausdorff will follow from the second condition of Definition 4.2. We do this for every hole chart in the preimage of a punctured parametric disk around each b_i. Let X be the new Riemann surface obtained by "plugging" all these "holes". We claim that X is compact. If the neighbourhoods W of b_i, for each i, is removed from Y, then the resulting Riemann surface remains compact, since it is a closed subset of compact Y. Now X is the union of the preimage of this Riemann surface under F and the closures of the images of the finitely many glued disks. The first set in the union is a finite cover of a compact set, hence compact. Others are also compact. Hence X being a finite union of compact sets, is itself compact.

We wish to extend F to a map $F' : X \to Y$. Let a_j be a newly added point. Then $a = i_D(0)$ and $i_D(D)$ is a chart of X containing a_j. Also if U_j^* was the hole chart corresponding to this glueing, then $i_D(D) \setminus \{0\} = U_j^*$. So rename $i_D(D)$ as U_j. From diagram (1), we see that $\phi_j^{-1} \circ F \circ \psi|_{W^*} = i_D|_{D^*}^{-1} \circ F \circ \psi|_{W^*} : D^* \to D^*$, maps z to z^{k_j}. Therefore this map can be extended holomorphically to D, by sending 0 to 0. Basically this means that a is sent to the branch point b_i which is contained in W. Doing this for each new point induces a holomorphic extension of F to X.

Notice that all the b_i's may not be branch points. If a_j is in the preimage of b_i, then F' locally looks like $z \mapsto z^{k_j}$, where k_j may be *equal to 1*. If this happens for every a_j in the preimage of b_i, then b_i is not a branch point. We can at most say that the branch points are a subset of $\{b_1, \cdots, b_n\}$. There is a way to determine whether b_i is a branch point.

Let W^* be the punctured neighbourhood of b_i as before. W^* is homeomorphic to a unit disk D^*. Consider the loop $t \mapsto \frac{1}{2}e^{2\pi i t}$, $0 \le t \le 1$ in D^*. Let β be its preimage in D^* starting at $q \in W^*$. Y is path connected. Then, there exists a path $\alpha : y_0 \rightsquigarrow q$ where y_0 is the base point of the fundamental group of Y, not equal to any of the b_i's. Then, the loop $\alpha\beta\alpha^{-1}$ represents an element of $\pi_1(Y, y_0)$. Call such a loop *small loop* around b_i and denote it by γ. Notice that the action of $[\gamma]$ on $F^{-1}(y_0)$ is the same as the action of $[\beta]$ on $F^{-1}(q)$, where the identification of $F^{-1}(y_0)$ and $F^{-1}(q)$ is via the end points of lifts of γ. As before, $F^{-1}(W^*) = \coprod_{j=1}^m U_j^*$ such that (1) holds. Then, β induces a cyclic permutation of order k_j for the preimages of q in U_j^*, for all j. Thus, the cyclic structure of the permutation induced by β is $(k_1, \ldots, k_m)$. Therefore, the same is true that for that of γ. Thus, we arrive at the following Lemma:

Lemma 4.3. *Let $F : X \to Y$ be a branched cover of a compact Riemann surface Y. Let $b \in Y$ be a branch point. Let $F^{-1}(b) = \{a_1, \ldots, a_m\}$, such that $\mathrm{mult}_{a_j}(F) = k_j$. Then, the cycle structure of the permutation induced by a small loop around b is $(k_1, \ldots, k_m)$.*

This lemma implies that in the construction of branched cover corresponding to a transitive permutation representation of $\pi_1(Y \setminus \{b_1, \cdots, b_n\}, x_0)$, a point b_i is not a branch point if and only if the action of a small loop around b_i has the cyclic structure $(1, \cdots, 1)$.

5. Riemann surface of any genus

Take the simplest compact Riemann surface, the Riemann sphere. Given any $g \in \mathbb{N}$, we wish to construct a cover $F : X \to \mathbb{C}_\infty$, such that, genus of X will be g. Then the Hurwitz formula tells us

$$2g - 2 = -2\deg(F) + \sum_{p \in X} [\mathrm{mult}_p(F) - 1] \tag{2}$$

If we take $\deg(F) = 2$, then a branch point can have only one preimage with multiplicity 2. Thus if there are n branch points, then from (2), we have $2g - 2 = -4 + n$ or $n = 2g + 2$. So choose $2g + 2$ points $\{b_1, \cdots, b_{2g+2}\}$ in $\mathbb{C}_\infty$. Let $V = \mathbb{C}_\infty \setminus \{b_1, \cdots, b_{2g+2}\}$. The $\pi_1(V, x_0)$ is the quotient of the free group generated by small loops γ_i around each b_i, with the relation $[\gamma_1] \cdots [\gamma_{2g+2}] = 1$. We define a homomorphism $\rho : \pi_1(V, x_0) \to S_2$ by specifying $[\gamma_i] = (12)$, for each i. Since $2g + 2$ is an even number, therefore ρ respects the relation $[\gamma_1] \cdots [\gamma_{2g+2}] = 1$, and hence is well defined. Clearly ρ is a transitive action. Then we construct a degree 2 branched cover X of $\mathbb{C}_\infty$ corresponding to the permutation representation ρ as in the previous section. Note that small loop γ_i around each b_i has cyclic structure (2). Hence b_i's are all branch points. Thus X is a Riemann surface of genus g.

Definition 5.1. *If there exists a non-constant holomorphic map $F : X \to \mathbb{C}_\infty$ of degree 2, then X is called a hyperelliptic Riemann surface.*

See [6], pg. 60, for a concrete construction of hyperelliptic surface. A hyperelliptic surface has a canonical automorphism which interchanges the two point in the fibre of any non-branch point of $F : X \to \mathbb{C}_\infty$ and fixes the ramification points. Clearly this automorphism has order 2. It is called *hyperelliptic involution* and is generally denoted by σ. Hyperelliptic involution will turn up later in proving finiteness of automorphism groups Riemann surfaces of genus $g \geq 2$.

CHAPTER 2

Existence of Meromorphic Functions

1. Holomorphic and Harmonic Differentials

We wish to prove the existence of a non-constant meromorphic function on a Riemann surface. We will assume familiarity with differential forms and integration on a manifold. (Chapter 2 and 4 of [**2**] is a good reference.) In general a chain complex of smooth differential forms are considered. But here we will be more lenient and allow any kind of differential forms. If differential operator is used on a form, it will be specified that it belongs to class C^k for some $k \geq 1$.

In local coordinates (x, y) a 1-form ω looks like $pdx + qdy$. We define conjugate operation denoted by $*$ on ω, as follows. In (x, y) coordinates $*\omega$ is given by $-qdx + pdy$. We need to check that these local 1-forms piece together to give a global one. That is we need to check that if different local coordinates are chosen then the two local 1-forms represent the same form. Suppose (x', y') is another local coordinate and let $\phi = u + iv$ be the holomorphic transformation between them. Let $p'(x', y')dx' + q'(x', y')dy'$ be the the representation of ω in these new coordinates. Then we know

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \begin{pmatrix} p \circ \phi \\ q \circ \phi \end{pmatrix}$$

Since $u_x = v_y$ and $u_y = -v_x$, by Cauchy-Riemann equations, we can write

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} u_x & -u_y \\ u_y & u_x \end{pmatrix} \begin{pmatrix} p \circ \phi \\ q \circ \phi \end{pmatrix} \tag{3}$$

This implies that

$$\begin{pmatrix} -q' \\ p' \end{pmatrix} = \begin{pmatrix} u_x & -u_y \\ u_y & u_x \end{pmatrix} \begin{pmatrix} -q \circ \phi \\ p \circ \phi \end{pmatrix}$$

Thus $*\omega$ is well defined. We note that the conjugation operation is function linear and $**\omega = -\omega$. We know that differentials of the form df, for some $f \in C^1(X)$, are called *exact differentials*. We say differentials of the form $*df$ are *coexact differentials*.

Definition 1.1. *A 1-form ω on a Riemann surface X is called a holomorphic differential if it is locally given by df, where f is holomorphic.*

Suppose ω is a holomorphic differential. In local coordinates (x, y), ω is the differential of a holomorphic map $f(x, y) = u(x, y) + iv(x, y)$, and hence looks like $df = du + idv$. Now by Cauchy-Riemann $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x} = *du$. Hence $df = du + i * du$. Applying conjugate operation on both sides $*df = *du - idu = -idf$. Therefore $*\omega = -i\omega$. In fact this criterion along with the property of being closed (which is same as being locally exact, by Poincare lemma) is necessary and sufficient for a 1-form to be holomorphic.

Theorem 1.2. *$\omega \in C^1$ is holomorphic $\Leftrightarrow$ $d\omega = 0$ and $*\omega = -i\omega$.*

Suppose $\omega = pdx + qdy$ in some local coordinate (x, y), then $*\omega = -qdx + pdy$ and $-i\omega = -ipdx - iqdy$. Hence the conditions $*\omega = -i\omega$, in local coordinate translates to the fact that $q = ip$. thus we have $\omega = pdx + ipdy$. The closure condition then implies $d(pdx + ipdy) = 0$ or $\frac{\partial p}{\partial y}dy \wedge dx + i\frac{\partial p}{\partial x}dx \wedge dy = 0$, which implies $\frac{\partial p}{\partial y} = i\frac{\partial p}{\partial x}$. If $p = u + iv$, then this implies that $u_x = v_y$ and $u_y = -v_x$. Therefore p is a holomorphic function. Now define $dz := dx + idy$, then every holomorphic differential is locally of the form pdz, where p is holomorphic. Also any differential of the form pdz with p holomorphic is a holomorphic differential since a holomorhic function always has a primitive in a simply connected region. Notice from (3), if we have a new complex coordinate $w = \phi(z)$, then $\omega = p'(w)dw = p(z)\frac{dz}{dw}dw$.

Now if we ask for p meromorphic instead of holomorphic, then we will get a holomorphic 1-form in X minus a discrete set. Such a differential is called a *meromorphic differential* on X. The points at which the local meromorphic functions have singularities, are called singularities of ω. Given two distinct meromorphic differentials ω_1 and ω_2, we can make sense of a meromorphic function ω_1/ω_2, which just

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Now if we ask for p meromorphic instead of holomorphic, then we will get a holomorphic 1-form in X minus a discrete set. Such a differential is called a *meromorphic differential* on X. The points at which the local meromorphic functions have singularities, are called singularities of ω. Given two distinct meromorphic differentials ω_1 and ω_2, we can make sense of a meromorphic function ω_1/ω_2, which just means that if in a coordinate chart $\omega_1 = p_1 dz$ and $\omega_2 = p_2 dz$, then in that chart the meromorphic function is p_1/p_2. Suppose we select another complex local coordinate w and express ω_1 and ω_2 as $p_1'(w)dw$ and $p_2'(w)dw$ respectively. Then we know $p_1'(w) = p_1(z)\frac{dz}{dw}$ and $p_1'(w) = p_1(z)\frac{dz}{dw}$. Then $p_1(z)/p_2(z) = p_1'(w)/p_2'(w)$, and hence the meromorphic function ω_1/ω_2 is well defined. So if we can prove the existence of two distinct meromorphic functions then existence of a non-constant meromorphic function will immediately follow.

If $f = u + iv$ is a holomorphic function on the complex plane then by Cauchy-Riemann equations, the real part u and the complex part v, satisfy $u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$. The operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the Laplacian and is denoted by $\triangle$. Thus we have $\triangle u = 0 = \triangle v$. A function $f \in C^2$ satisfying this condition at each point in the domain of definition is called a *harmonic function*. Though here u and v are real valued, there is no need for this restriction in a general definition of harmonic function and we allow complex valued functions. We follow the definition of holomorphic 1-form to define

Definition 1.3. *A 1-form $\omega \in C^1$ on a Riemann surface X is called a harmonic differential if it is locally given by df, where f is harmonic.*

Like in case of holomorphic differentials, a similar characterization of harmonic differentials exist.

Theorem 1.4. *$\omega \in C^1$ is harmonic $\Leftrightarrow d\omega = 0$ and $d * \omega = 0$, that is, ω is both exact and coexact.*

First we note that $d * df = \triangle f dx \wedge dy$. This is because

$$\begin{aligned} d * df &= d * (\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy) \\ &= d(-\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy) \\ &= -\frac{\partial^2 f}{\partial y^2}dy \wedge dx + \frac{\partial^2 f}{\partial x^2}dx \wedge dy \\ &= \triangle f dx \wedge dy \end{aligned}$$

Now if ω is harmonic, then locally $\omega = df$, where f is harmonic. Therefore locally $d\omega = ddf = 0$ and $d * \omega = d * df = \triangle f dx \wedge dy = 0$. Conversely if $d\omega = 0$, then locally $\omega = df$. Then $d * \omega = 0$ implies locally $d * df = \triangle f dx \wedge dy = 0$ and hence $\triangle f = 0$, proving ω is a harmonic differential.

As we did for holomorphic differential, let us analyze what these two conditions mean in a local setting. Let $\omega = pdx + qdy$ in local coordinates (x, y). Then $d\omega = 0$ implies $d(pdx + qdy) = (\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y})dx \wedge dy = 0$ and $d * \omega = 0$ implies $d * (pdx + qdy) = d(-qdx + pdy) = (\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y})dx \wedge dy = 0$. Hence we have $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ and $\frac{\partial p}{\partial x} = -\frac{\partial q}{\partial y}$. If ω is real harmonic, that is p and q are real then, this is just the Cauchy-Riemann equations for $p - iq$. Now notice that locally $\omega + i * \omega = (p - iq)dx + i(p - iq)dy = (p - iq)dz$. Hence $\omega + i * \omega$ is holomorphic differential. So to lay hands on a holomorphic differential on any domain, it is enough to find a real harmonic differential. Since real and imaginary part of harmonic differential are also harmonic, therefore it is enough to find a harmonic differential.

2. The Hilbert Space of 1-forms

Given any measure space $(X, \mathcal{M}, \mu)$ we can define an inner product on the space of complex valued measurable functions as follows

$$\langle f, g\rangle := \int f\overline{g}$$

Then the space

$$L^2(X, \mathcal{M}, \mu) := \{f : X \to \mathbb{C} \mid f \text{ is measurable and } \|f\| < \infty\}$$

is a Hilbert space provided we identify two functions which differ only on a measure zero set. Here we have 2-forms as integrable objects and can define an inner product on the space of measurable 1-forms as

$$\langle \omega, \gamma\rangle := \int \omega * \overline{\gamma}$$

where the complex conjugate $\overline{\gamma}$ of a 1-form γ is defined locally as follows. If in local coordinates (x, y), γ is represented by $pdx + qdy$, then $\overline{\gamma}$ is represented by $\overline{p}dx + \overline{q}dy$. It is clear that such local 1-forms piece together to form a well defined global 1-form. Then locally $\omega * \overline{\gamma}$ looks like

$$\begin{aligned}&(pdx + qdy) \wedge *(\overline{p'}dx + \overline{q'}dy)\\ =&(pdx + qdy) \wedge (-\overline{q'}dx + \overline{p'}dy)\\ =&(p\overline{p'} + q\overline{q'})dx \wedge dy\end{aligned}$$

Now we can easily check that this gives an inner product, except the condition that $\langle \omega, \omega \rangle = 0$ implies that $\omega = 0$. But this is easily rectified by identifying two differentials which differ only by a measure zero set in any chart. Of course we have to restrict to 1-forms which are locally of the form $pdx + qdy$, with p and q measurable. We call such 1-forms measurable. Thus the wedge product of two measurable 1-forms is integrable. Now define the space

$$L^2(X) := \{f : X \to \mathbb{C} \mid \omega \text{ is measurable and } \|\omega\| < \infty\}$$

The proof of completeness is analogous to the proof that L^p spaces are complete and we omit it here. (See [3], pg. 182, for the complete proof.)

Now we state a fact which is a direct consequence of the Stokes' theorem. (See [2] pg. 148, for the satement and proof.)

Theorem 2.1. *Suppose f is a C^1 function and ω is a C^1 1-form in a Riemann surface X. If either f or ω has compact support in X, then*

$$\iint_X d(f \wedge \omega) = \iint_X f \wedge d\omega - \iint_X \omega \wedge df = 0$$

□

Applying this to a harmonic form $\omega \in L^2(X)$ and $\overline{f}$ for any $f \in C^1(X)$ with compact support we get

$$\iint_X \omega \wedge d\overline{f} = \iint_X \overline{f} \wedge d\omega = 0 \tag{4}$$

$$\text{Hence, } \langle \omega, *df \rangle = -\iint_X \omega \wedge d\overline{f} = 0 \tag{5}$$

The space of C^1 functions with compact supports is denoted by C_0^1. If ω is harmonic then so is $\overline{\omega}$. Again applying Theorem 2.1 on $*\overline{\omega}$ for $\omega \in L^2(X)$ harmonic and $f \in C_0^1(X)$, we get

$$\iint_X *\overline{\omega} \wedge df = \iint_X f \wedge d * \overline{\omega} = 0 \tag{6}$$

$$\text{Hence, } \langle df, \omega \rangle = -\iint_X *\overline{\omega} \wedge df = 0 \tag{7}$$

Let the space of harmonic differentials in $L^2(X)$ be denoted by H. It is perpendicular to both spaces of exact and coexact differentials in $L^2(X)$. In fact it is perpendicular to the closures of such spaces. This follows from the simple Hilbert Space fact

Lemma 2.2. *If $x_n \to x$ and $y_n \to y$ in a Hilbert Space $\mathcal{H}$, then* $\langle x, y \rangle = \lim_{n\to\infty} \langle x_n, y_n \rangle$.

We define,

$$E = \text{the closure in } L^2(X) \text{ of differentials of the form } df, \text{ where } f \in C_0^2$$
$$E^* = \text{the closure in } L^2(X) \text{ of differentials of the form } *df, \text{ where } f \in C_0^2$$

We required a differentiability of order 2 from f because we want the exact and coexact differentials to be available for application of the differential operator d. This helps us to prove

Proposition 2.3. *E and E^* are orthogonal subspaces of $L^2(X)$*

PROOF. Let $\omega \in E$ and $\gamma \in E^*$. Then there exists sequences of C_0^2 functions f_n and g_n, such that $df_n \to \omega$ and $*dg_n \to \gamma$ in $L^2(X)$. Then by Lemma 2.2, it is enough to prove that $\langle df_n, *dg_n \rangle = 0$ for all n. This a simple application of Theorem 2.1:

$\langle df_n, *dg_n \rangle = -\iint_X df_n \wedge d\overline{g_n} = -\iint_X f_n \wedge dd\overline{g_n} = 0$ □

We have seen that $H \subset E^\perp \cap E^{*\perp}$. Are they equal? Let $\omega \in E^\perp \cap E^{*\perp}$. If further we assume it to be C^1, then ω is harmonic. This follows from the lemma below.

Lemma 2.4. *If ω is C^1, then*

(a) $d\omega = 0 \Leftrightarrow \omega \in E^{*\perp}$

(b) $d * \omega = 0 \Leftrightarrow \omega \in E^\perp$

PROOF. We will only prove (a), (b) will follow similarly. ($\Rightarrow$) follows immediately from the calculation we did in (4) and (5) and Theorem 2.2. For ($\Leftarrow$), note that $\omega \in C^1 \cap E^{*\perp}$ implies that for all $f \in C_0^2$,

$$0 = \langle \omega, *df \rangle = -\iint_X \omega \wedge d\overline{f} = -\iint_X \overline{f} \wedge d\omega \tag{8}$$

Suppose $d\omega \neq 0$ at a point $p \in X$. Let $Adx \wedge dy$ represent $d\omega$ in the local coordinates. Then either ReA or ImA is not equal to zero near 0. Say Re$A > 0$ in a parametric unit disk U centred at p. Let $\phi : B(0,1) \to U$ be the chart map. Construct a smooth bump function in $B(0,1)$ which is 1 at 0 and vanishes outside $B(0,1/2)$. Let this function composed with ϕ^{-1} be called f. f can be extended to whole of X by specifying $f \equiv 0$ in $X \setminus \phi(B(0,1))$. Then $f \in C_0^2(X)$. Therefore by (8), we must have $\langle \omega, *df \rangle = 0$. But again

$$\begin{aligned} Re\langle \omega, *df \rangle &= -Re \iint_X \omega \wedge df \\ &= -Re \iint_X f \wedge d\omega \\ &= -Re \iint_X f A dx \wedge dy \\ &= -\iint_X f ReA dx \wedge dy < 0 \end{aligned}$$

This is a contradiction □

3. Weyl's Lemma

We saw that if a differential in $E^\perp \cap E^{*\perp}$ is C^1, then it is harmonic. Are all such differentials C^1? This is the question addressed by the Weyl's lemma. First we note that being C^1 is a local property. So enough to show that ω is C^1 in every chart. But that means we can as well work in a region in the complex plane. Say the unit ball denoted by D. Also, a ball of radius r will be denoted by D_r. Before going into Weyl's lemma, we will discuss a technique by which any 1-forms can be made C^1, without losing it's essential properties. We concentrate on 1-forms in D. A 1-form looks like $\omega = pdx + qdy$. So ω is C^1 means that p and q are C^1. So we will define a smoothing operator on functions first, and it can be carried over to forms by applying the operator on the coefficients of dx and dy. Since $\omega \in L^2(X)$, functions will be assumed to be integrable. Let $\rho < 1$. Define a function

$$s_\rho : D \to \mathbb{R}$$

$$s_\rho(x,y) = \begin{cases} k(\rho^2 - x^2 - y^2)^2, & x^2 + y^2 < \rho^2 \\ 0, & x^2 + y^2 \geq \rho^2 \end{cases}$$

where k is chosen such that $\iint_{D_\rho} s_\rho(x,y)dxdy = 1$. Calculating we find $k = 3/(\pi\rho^6)$. Note that s_ρ is C^1. Now define the action of a smoothing operator M_ρ on a $L^2(D)$ function f as follows:

$$M_\rho f(x,y) = \int_0^{2\pi} \int_0^\rho f(x + r\cos\theta, y + r\sin\theta) s_\rho(r\cos\theta, r\sin\theta) r dr d\theta \tag{9}$$

Note that $M_\rho f$ is defined in $D_{1-\rho}$. Change of variable will yield the following two useful forms of (9).

$$M_\rho f(x,y) = \iint_{D_\rho} f(x+u, y+v) s_\rho(u,v) dudv \tag{10}$$

$$M_\rho f(x,y) = \iint_D f(u,v) s_\rho(u-x, v-y) dudv \tag{11}$$

The operator M_ρ acting on the subspace $L^2(D)$ of the space of 1-forms on D, has the following properties

P1: $M_\rho(\omega)$ is C^1
P2: ω is harmonic in $D \Rightarrow M_\rho\omega = \omega$ in $D_{1-\rho}$
P3: $\lim_{\rho\to 0}\|\omega - M_\rho\omega\| = 0$
P4: If supp$(\gamma) \in D_{1-\rho}$, then supp$(M_\rho\gamma) \in D$ and

$$\langle M_\rho\omega, \gamma \rangle_{D_{1-\rho}} = \langle \omega, M_\rho\gamma \rangle_D$$

In the last property **P4**, $M_\rho\gamma$ can be defined over whole of D, since γ can be extended to the whole plane by setting $\gamma \equiv 0$ outside D.

PROOF. OF **P1** Enough to prove this for functions instead of differentials. So let $f \in L^2(D)$ be a function on D. We claim that

$$\frac{\partial M_\rho f(x,y)}{\partial x} = \iint_D f(u,v)\frac{\partial s_\rho}{\partial x}(u-x, v-y)dudv$$

We wish to apply the Dominated Convergence Theorem on the sequence of functions

$$\{f(u,v)\frac{s_\rho(u-x-h_n, v-y) - s_\rho(u-x, v-y)}{h_n}\}, \text{ where } h_n \to 0 \text{ as } n \to \infty$$

these functions converge pointwise to

$$f(u,v)\frac{\partial s_\rho}{\partial x}(u-x, v-y)$$

Now

$$\begin{aligned}
&|f(u,v)\frac{s_\rho(u-x-h_n, v-y) - s_\rho(u-x, v-y)}{h_n}| \\
=&|f(u,v)||\frac{1}{h_n}\int_{u-x}^{u-x-h_n}\frac{\partial s_\rho}{\partial t}(t, v-y)dt| \\
\leq&|f(u,v)|(\frac{1}{h_n}\int_{u-x-h_n}^{u-x}|\frac{\partial s_\rho}{\partial t}(t, v-y)|dt) \\
\leq&|f(u,v)|M
\end{aligned}$$

where M is the upper bound for continuous $\partial s_\rho/\partial t$ in $\overline{D}$. $|f(u,v)|M$ is integrable and hence DCT applies. Therefore we have

$$\begin{aligned}
\frac{\partial M_\rho f(x,y)}{\partial x} &= \lim_{n\to\infty}\frac{M_\rho f(x+h_n) - M_\rho f(x)}{h_n} \\
&= \lim_{n\to\infty}\iint_D f(u,v)\frac{s_\rho(u-x-h_n, v-y) - s_\rho(u-x, v-y)}{h_n} \\
&= \iint_D f(u,v)\frac{\partial s_\rho}{\partial x}(u-x, v-y)dudv
\end{aligned}$$

Now we want to prove continuity of $\partial M_\rho f(x,y)/\partial x$. That is, given any $\epsilon > 0$ and $(x,y) \in D_{1-\rho}$, we want to find a $\delta > 0$, such that

$$|\frac{\partial M_\rho f}{\partial x}(x', y') - \frac{\partial M_\rho f}{\partial x}(x,y)| < \epsilon, \text{ whenever } |(x',y') - (x,y)| < \delta$$

Now

$$\begin{aligned}
&|\frac{\partial M_\rho f}{\partial x}(x', y') - \frac{\partial M_\rho f}{\partial x}(x,y)| \\
\leq &\iint_D |f(u,v)||\frac{\partial s_\rho}{\partial x}(u-x', v-y') - \frac{\partial s_\rho}{\partial x}(u-x, v-y)|dudv
\end{aligned}$$

$\partial s_\rho/\partial x$ is continuous in $\overline{D}$ and hence uniformly continuous. Hence there exists $\delta > 0$, such that

$$|\frac{\partial s_\rho}{\partial x}(u-x', v-y') - \frac{\partial s_\rho}{\partial x}(u-x, v-y)| < \frac{\epsilon}{\iint_D |f(u,v)|dudv}$$

whenever $|(x',y') - (x,y)| < \delta$. Then this is our required δ. The proof for $\partial M_\rho f(x,y)/\partial y$ follows similarly. □

PROOF. OF **P2** ω is harmonic and D is simply connected. Therefore there exists a single function $f \in C^1(D)$, such that $\omega = df$. Then $M_\rho\omega = M_\rho df$. Does M_ρ commute with d ? That is, we wish to prove that

$$M_\rho\frac{\partial f}{\partial x}dx + M_\rho\frac{\partial f}{\partial y}dy = \frac{\partial M_\rho f}{\partial x}dx + \frac{\partial M_\rho f}{\partial y}dy$$

We will to show that $M_\rho \frac{\partial f}{\partial x} = \frac{\partial M_\rho f}{\partial x}$, the proof for coefficients of dy follows similarly. Since s_ρ has an upper bound in D and f is C^1, we can differentiate under the integral sign to get

$$\begin{aligned}\frac{\partial M_\rho f}{\partial x}(x,y) &= \frac{\partial}{\partial x}\iint_{D_\rho} f(x+u, y+v)s_\rho(u,v)dudv \\ &= \iint_{D_\rho} \frac{\partial}{\partial x} f(x+u,y+v)s_\rho(u,v)dudv \\ &= M_\rho \frac{\partial f}{\partial x}(x,y)\end{aligned}$$

Now we have $M_\rho \omega = M_\rho df = dM_\rho f$. So now it is enough to prove that $M_\rho f = f$ in D_ρ. Using the definition (9) of $M_\rho f(x,y)$ and the definition of the function s_ρ we have,

$$\begin{aligned}M_\rho f(x,y) &= \int_0^{2\pi}\int_0^\rho f(x + r\cos\theta, y + r\sin\theta)s_\rho(r\cos\theta, r\sin\theta)rdrd\theta \\ &= \int_0^{2\pi}\int_0^\rho f(x+r\cos\theta, y+r\sin\theta)k(\rho^2-r^2)^2 rdrd\theta \\ &= \int_0^\rho k(\rho^2-r^2)^2 r\int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)d\theta dr\end{aligned}$$

By Mean Value Theorem of harmonic functions we have

$$\int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)d\theta = 2\pi f(x,y)$$

Thus

$$M_\rho f(x,y) = 2\pi f(x,y)\int_0^\rho k(\rho^2-r^2)^2 rdr = f(x,y)$$

since we had chosen k such that $2\pi\int_0^\rho k(\rho^2-r^2)^2 rdr = \iint_{D_\rho} s_\rho(x,y)dxdy = 1$. □

PROOF. OF **P3** Suppose $\omega = pdx + qdy$. Then

$$\|\omega - M_\rho\omega\|^2_{D_{1-\rho}} = \iint_{D_{1-\rho}} (|p - M_\rho p|^2 + |q - M_\rho q|^2)dxdy$$

We wish to show that this quantity goes to zero as $\rho \to 0$. We will show that $\iint_{D_{1-\rho}} |p - M_\rho p|^2 dxdy \to 0$ as $\rho \to 0$, the other part will follow similarly. Further since $p = p_1 + ip_2$, $\iint_{D_{1-\rho}} |p - M_\rho p|^2 dxdy = \iint_{D_{1-\rho}} (|p_1 - M_\rho p_1|^2 + |p_2 - M_\rho p_2|^2)dxdy$. So enough to show that $\iint_{D_{1-\rho}} |p_1 - M_\rho p_1|^2 dxdy \to 0$ as $\rho \to 0$. So may assume that p is real. We know that simple functions are dense in the function space $L^2(D)$ and since we are in a Borel σ-algebra, therefore continuous functions are dense in the simple ones. Thus continuous functions are dense in $L^2(D)$. Therefore given $\epsilon > 0$ there exists g continuous in $\overline{D}$ such that $\|p - g\|_D < \epsilon$. Then

$$\begin{aligned}(\iint_{D_{1-\rho}} |p - M_\rho p|^2 dxdy)^{1/2} &= \|p - M_\rho p\|_{D_{1-\rho}} \\ &\le \|p - g\|_{D_{1-\rho}} + \|g - M_\rho g\|_{D_{1-\rho}} + \|M_\rho g - M_\rho p\|_{D_{1-\rho}}\end{aligned}$$

The first term is already less than ϵ. We have to show that the other two are also less than ϵ for sufficiently small ρ. Using definition (10) of $M_\rho g$, we have

$$\begin{aligned}|M_\rho g(x,y) - g(x,y)| &= |\iint_{D_\rho} (g(x+u,y+v) - g(x,y))s_\rho(u,v)dudv| \\ &\le \iint_{D_\rho} |g(x+u,y+v) - g(x,y)|s_\rho(u,v)dudv\end{aligned}$$

We know that g is continuous on compact $\overline{D}$, hence uniformly continuous in D. therefore there exists $\delta > 0$, such that

$$|g(x',y') - g(x,y)| < \epsilon, \text{ whenever } |(x',y') - (x,y)| < \delta$$

Therefore for $\rho < \delta$,

$$|M_\rho g(x,y) - g(x,y)| \le \epsilon\iint_{D_\rho} s_\rho(u,v)dudv = \epsilon$$

Therefore

$$\|g - M_\rho g\|_{D_{1-\rho}} = (\iint_{D_{1-\rho}} |M_\rho g(x,y) - g(x,y)|^2 dxdy)^{1/2} \le \sqrt{\pi}\epsilon$$

whenever $\rho < \delta$. So it only remains to prove that $\|M_\rho g - M_\rho p\|_{D_{1-\rho}} < \epsilon$. Again using definition (10)

$$|M_\rho g - M_\rho p|^2 = |\iint_{D_\rho} (g(x+u, y+v) - p(x+u, y+v))s_\rho(u,v)dudv|^2$$

Applying Scwartz inequality on $g(x+u,y+v) - p(x+u,y+v))\sqrt{s_\rho(u,v)}$ and $\sqrt{s_\rho(u,v)}$ we have

$$\begin{aligned}
&|\iint_{D_\rho} (g(x+u,y+v) - p(x+u,y+v))s_\rho(u,v)dudv|^2 \\
&\le \iint_{D_\rho} |g(x+u,y+v) - p(x+u,y+v)|^2 s_\rho(u,v)dudv \times \iint_{D_\rho} s_\rho(u,v)dudv \\
&= \iint_{D_\rho} |g(x+u,y+v) - p(x+u,y+v)|^2 s_\rho(u,v)dudv
\end{aligned}$$

Therefore

$$\begin{aligned}
\|M_\rho g - M_\rho p\|^2_{D_{1-\rho}} &= \iint_{D_{1-\rho}} |M_\rho g - M_\rho p|^2 dxdy \\
&\le \iint_{D_{1-\rho}} (\iint_{D_\rho} |g(x+u,y+v) - p(x+u,y+v)|^2 s_\rho(u,v)dudv)dxdy
\end{aligned}$$

Applying Fubini on RHS,

$$\begin{aligned}
&\|M_\rho g - M_\rho p\|^2_{D_{1-\rho}} \\
&\le \iint_{D_\rho} s_\rho(u,v)(\iint_{D_{1-\rho}} |g(x+u,y+v) - p(x+u,y+v)|^2 dxdy)dudv
\end{aligned}$$

But $\iint_{D_{1-\rho}} |g(x+u,y+v) - p(x+u,y+v)|^2 dxdy = \|p-g\|^2_{D_{1-\rho}} < \epsilon^2$ for all $(u,v) \in D_\rho$. Therefore

$$\|M_\rho g - M_\rho p\|^2_{D_{1-\rho}} \le \epsilon^2 \iint_{D_\rho} s_\rho(u,v)dudv = \epsilon^2$$

□

PROOF. OF **P4** First we show that $\text{supp}(M_\rho\gamma) \subset D$. For any $\tilde{x} \notin D$, $B(\tilde{x},\rho) \cap D_{1-\rho} = \varnothing$, hence $M_\rho\gamma(\tilde{x}) = 0$. Therefore $M_\rho\gamma$ is non-zero only in D. But support means the *closure* of the non-zero region. So we will show that infact $\text{supp}(\gamma) \subset D_{1-\rho-\delta}$, for some $\delta > 0$, and hence the non zero region will be inside $D_{1-\delta}$, proving that $\text{supp}(M_\rho\gamma) \subset D$. $\text{supp}(\gamma) \subset D_{1-\rho}$, where $\text{supp}(\gamma)$ is closed and $D_{1-\rho}$ is open. Let $2\delta := d(\text{supp}(\gamma), D^c_{1-\rho}) > 0$. Then $\text{supp}(\gamma) \subset D_{1-\rho-\delta}$.

Now we wish to prove that $\langle M_\rho\omega, \gamma\rangle_{D_{1-\rho}} = \langle \omega, M_\rho\gamma\rangle_D$. Let $\omega = pdx + qdy$ and $\gamma = adx + bdy$. Then

$$\begin{aligned}
\langle M_\rho\omega, \gamma\rangle_{D_{1-\rho}} &= \iint_{D_{1-\rho}} M_\rho p(x,y)\overline{a(x,y)} + M_\rho q(x,y)\overline{b(x,y)}dxdy \\
\langle \omega, M_\rho\gamma\rangle_D &= \iint_D p(x,y)\overline{M_\rho a(x,y)} + q(x,y)\overline{M_\rho b(x,y)}dxdy
\end{aligned}$$

Using definition (11) of $M_\rho p(x,y)$ and applying Fubini thereafter, we have

$$\begin{aligned}
&\iint_{D_{1-\rho}} M_\rho p(x,y)\overline{a(x,y)}dxdy \\
&= \iint_{D_{1-\rho}} \overline{a(x,y)}(\iint_D p(u,v)s_\rho(u-x, v-y)dudv)dxdy \\
&= \iint_D p(u,v)(\iint_{D_{1-\rho}} \overline{a(x,y)}s_\rho(u-x,v-y)dxdy)dudv \\
&= \iint_D p(u,v)(\iint_D \overline{a(x,y)}s_\rho(u-x,v-y)dxdy)dudv \\
&= \iint_D p(u,v)\overline{M_\rho a(x,y)}dudv
\end{aligned}$$

Similarly,

$$\iint_{D_{1-\rho}} M_\rho q(x,y)\overline{b(x,y)}dxdy = \iint_D q(u,v)\overline{M_\rho b(x,y)}dudv$$

Thus $\langle M_\rho\omega, \gamma\rangle_{D_{1-\rho}} = \langle \omega, M_\rho\gamma\rangle_D$. □

Now we state and prove the Weyl's Lemma.

Lemma 3.1. *(Weyl's lemma) Let D be the unit disk in the complex plane. Let $\omega \in E^{\perp}(D) \cap E^{*\perp}(D)$. Then ω is C^1 almost everywhere.*

PROOF. The idea is to show that $\omega = M_\rho\omega$ a.e. in $D_{1-\rho}$, for all $0 < \rho < 1$. By **P1**, $M_\rho\omega$ is C^1, hence ω is C^1 a.e. in $D_{1-\rho}$, for all ρ, and hence in all of D.

First we prove that $M_\rho\omega$ is independent of ρ, that is, if $\rho, \sigma < 1/2$, then $M_\rho\omega = M_\sigma\omega$ in $D_{1-\rho-\sigma}$. **P2** tells us that for ω harmonic we have $M_\rho\omega = \omega$ in $D_{1-\rho}$. We wish to use this to show that $M_\rho\omega = M_\sigma M_\rho\omega = M_\rho M_\sigma\omega = M_\sigma\omega$ in $D_{1-\rho-\sigma}$. The middle equality requires a proof, but first we have to prove that $M_\rho\omega$ is harmonic, for all ρ, so that **P2** can be applied. We recall that if a 1-form belongs to $E^{\perp} \cap E^{*\perp}$ *and* is C^1, then it is harmonic. By **P1**, $M_\rho\omega$ is C^1. We will prove that $M_\rho\omega \in E^{\perp}(D_{1-\rho}) \cap E^{*\perp}(D_{1-\rho})$. Let $f \in C_0^2(D_\rho)$, by **P4** we have,

$$\langle M_\rho\omega, df\rangle_{D_{1-\rho}} = \langle \omega, M_\rho df\rangle_D = \langle \omega, dM_\rho f\rangle_D$$
$$\langle M_\rho\omega, *df\rangle_{D_{1-\rho}} = \langle \omega, M_\rho * df\rangle_D = \langle \omega, *dM_\rho f\rangle_D$$

where the last step in each is due to the fact that M_ρ commutes with both d and $*$. This proves our claim and hence $M_\rho\omega$ is harmonic in $D_{1-\rho}$.

Proving $M_\sigma M_\rho\omega = M_\rho M_\sigma\omega$ is just an exercise in Fubibi. Let $\omega = pdx + qdy$. Using definition (10) of $M_\rho p$, we have,

$$\begin{aligned}
M_\sigma M_\rho p &= \iint_{D_\sigma} M_\rho p(x+u, y+v)s_\sigma(u,v)dudv \\
&= \iint_{D_\sigma} s_\rho(u,v)(\iint_{D_\rho} p(x+u+u', y+v+v')s_\rho(u',v')du'dv')dudv \\
&= \iint_{D_\rho} s_\rho(u',v')(\iint_{D_\sigma} p(x+u+u', y+v+v')s_\sigma(u,v)dudv)du'dv' \\
&= M_\rho M_\sigma p
\end{aligned}$$

Similarly $M_\sigma M_\rho q = M_\rho M_\sigma q$ and we have proved that $M_\rho\omega$ is independent of ρ. Now fix a ρ. We wish to prove that $\omega = M_\rho\omega$ a.e. in $D_{1-\rho}$. We know by property **P3**, that $\lim_{\sigma\to 0}\|\omega - M_\sigma\omega\|_{D_{1-\sigma}} = 0$. We have

$$\|\omega - M_\sigma\omega\|_{D_{1-\sigma-\rho}} \leq \|\omega - M_\sigma\omega\|_{D_{1-\sigma}}$$

Taking limit $\sigma \to 0$ on both sides,

$$\lim_{\sigma\to 0}\|\omega - M_\sigma\omega\|_{D_{1-\sigma-\rho}} = 0$$

Now in $D_{1-\sigma-\rho}$, $M_\sigma\omega = M_\rho\omega$. Hence

$$\lim_{\sigma\to 0}\|\omega - M_\rho\omega\|_{D_{1-\sigma-\rho}} = 0$$

Now for all $\sigma < \delta$,

$$\|\omega - M_\rho\omega\|_{D_{1-\rho-\delta}} \leq \|\omega - M_\rho\omega\|_{D_{1-\rho-\sigma}}$$

Taking limit $\sigma \to 0$ on both sides,

$$\|\omega - M_\rho\omega\|_{D_{1-\rho-\delta}} \leq \lim_{\sigma\to 0}\|\omega - M_\rho\omega\|_{D_{1-\rho-\sigma}} = 0$$

Therefore $\|\omega - M_\rho\omega\|_{D_{1-\rho-\delta}} = 0$ for all $\delta > 0$. That is $\omega = M_\rho\omega$ a.e in $D_{1-\rho-\delta}$ for all $\delta > 0$. Let $A = \{x \in D_{1-\rho} : \omega(x) \neq M_\rho\omega\}$. Let μ be the measure. Then $\mu(A \cap D_{1-\rho-1/n}) = 0$ for all large n. Taking union over these sets $\mu(A \cap (\cup_n D_{1-\rho-1/n})) = 0$, that is $\mu(A \cap D_{1-\rho}) = 0$ or $\mu(A) = 0$. □

4. Decomposition of a 1-form into orthogonal components

Weyl's lemma tells us that the space $E^{\perp} \cap E^{*\perp}$ is equal to the space of harmonic differentials H. But $E^{\perp} \cap E^{*\perp} = (E \oplus E^*)^{\perp}$. Now E and E^* are orthogonal subspaces of Hilbert space $L^2(X)$. Therefore $E \oplus E^*$ is also a subspace. Therefore $H = (E \oplus E^*)^{\perp}$ is also subspace and $L^2(X) = (E \oplus E^*) \oplus H$. We have the following theorem

Theorem 4.1. *$L^2(X) = E \oplus E^* \oplus H$ and hence every $\omega \in L^2(X)$ can be written as $\omega = \gamma + \pi + \omega_h$ a.e., where $\gamma \in E$, $\pi \in E^*$ and ω_h is a harmonic differential.*

We wish to know a condition on ω such that in the decomposition $\omega = \gamma + \pi + \omega_h$ a.e., the γ is exact and π is coexact. First we prove that,

Lemma 4.2. *$\gamma \in E$ is C^1. Then it is exact.*
$\pi \in E^*$ is C^1. Then it is coexact.

Before going to the proof of this lemma, we establish a checkable criterion for a closed differential to be exact.

Theorem 4.3. *A closed differential ω is exact $\Leftrightarrow$ For any piecewise differentiable closed curve α in X, $\int_\alpha \omega = 0$.*

This is just Morera's theorem in Complex Analysis and we leave the proof to the reader.

PROOF. **of Lemma 4.2** $\gamma \in E \subset E^{*\perp}$. Since γ is C^1, by Lemma 2.4, $d\gamma = 0$, that is γ is closed. So in light of the previous theorem, it is enough to show that integration of γ over any piecewise differentiable closed curve is 0. So let $\alpha : [0,1] \to X$ be such a curve. We wish to simplify the picture and work with simple closed curve, instead of a highly entangled closed curve. Consider a triangulation of the Riemann surface such that each triangle is inside a parametric disk and each edge is a differentiable curve. Since $\alpha(I)$ is compact, it is contained in finitely many such triangles. By slightly shifting α at places, if necessary, we can find a point in the interior of each of these triangles which is not in the image of α. We can then project the intersection of $\alpha(I)$ with the interior of this triangle, to the edges of the triangle. Each triangle lie in a parametric disk and $d\gamma = 0$, hence there exists a holomorphic function f such that $df = \gamma$ locally. Hence integration of γ over α inside the triangle is same as that over the new curve. Call this new curve α from now on. Notice that this allows us to concentrate only on simple piecewise differentiable closed curve, since finitely many application of the result $\int_\beta \gamma = 0$, for simple β, will yield the result for general α. Hence we assume that α is a simple piecewise differentiable closed curve.

The idea is to show that $\int_\alpha \gamma = \langle \gamma, \omega \rangle$ for some $\omega \in E^{\perp}$. We know by Lemma 2.4, if ω is C^1, then $\omega \in E^{\perp} \Leftrightarrow d * \omega = 0$. So we look for η which is C^1 and closed, so that we can take $\omega = *\eta$. The easiest way to construct a closed differential is to start with a function and take it's differential. So assume $\eta = df$ for some f. Since we are going to construct f by hand, using bump functions, therefore we may as well assume that f is real and $\mathrm{supp}(\eta)$ is contained in a region R with nice boundary. Now let us try to find the conditions on f such that $\int_\alpha \gamma = \langle \gamma, *\eta \rangle$.

$$\langle \gamma, *\eta \rangle = -\iint_R \gamma \wedge df = \int_{\partial R} f \wedge \gamma - \iint_R f \wedge d\gamma = \int_{\partial R} f\gamma \tag{12}$$

Thus ∂R should have α as one of it's components, on which f should have value 1, while f should be zero on the other components. Before starting the construction we quote this result from [2], pg. 11:

Proposition 4.4. *Let M be a manifold. Let A and G be closed and open in M, respectively, such that, $A \subset U$. Then there exists a smooth $\phi : M \to \mathbb{R}$ such that*

(1) $0 \leq \phi(p) \leq 1$, *for all* $p \in M$
(2) $\phi(p) = 1$, *if* $p \in A$
(3) $\mathrm{supp}(\phi) \subset G$

For every $t \in I$, consider a chart (U_t, ϕ_t) centred at $\alpha(t)$. For small enough ϵ, $\phi_t^{-1}(D_\epsilon) \cap \alpha(I)$ has only one component. Call $B_t = \phi_t^{-1}(D_{\epsilon/2})$ and $G_t = \phi_p^{-1}(D_\epsilon)$. $\{B_t : t \in I\}$ is an oven coper of compact $\alpha(I)$. Let $\{B_i : 1 \leq i \leq n\}$ be a fine subcover. Define $G := \cup_{i=1}^n G_i$ and $B := \cup_{i=1}^n \overline{B_i}$. Note $\alpha(I) \in G$ divides G into two components. Let the part of G to the right of $\alpha(I)$ in G be R and left of that be L. Now $\alpha(I) \subset B \subset G$, where B is closed and G is open. Therefore by Proposition 4.4, there exists $g : X \to \mathbb{R}$ which is 0 on B and has support inside G. Now define

$$f : X \to \mathbb{R}$$
$$f(p) := \begin{cases} g(p), & p \in R \\ 1, & p \in L \cup \alpha(I) \\ 0, & p \in X \setminus G \end{cases}$$

This function is not differentiable throughout X. But df is defined within G and if we extend it by specifying it's value to be zero outside G, then this extended differential is C_0^1. This is our η and it has support within R, where it is equal to df. Now we see that (12) hold for the function f, differential η and the region R that we constructed. Therefore

$$0 = \langle \gamma, \eta \rangle = \int_{\partial R} f\gamma = \int_{\alpha} \gamma$$

since on the other component of ∂R, f is zero. Therefore γ is exact.

Now $\pi \in E^*$ implies there exists a sequence of C_0^2 functions f_n, such that $\lim_{n\to\infty} \|\pi - *df_n\| = 0$. Now note that for any $\omega_1, \omega_2 \in L(X)$,

$$\langle *\omega_1, *\omega_2 \rangle = -\iint_X *\omega_1 \wedge \overline{\omega_2} = \iint_X \overline{\omega_2} \wedge *\omega_1 = \overline{\langle \omega_2, \omega_1 \rangle} = \langle \omega_1, \omega_2 \rangle$$

Hence $\lim_{n\to\infty} \| * \pi + df_n\| = 0$. Therefore $*\pi \in E$. Thus $*\pi$ is exact and hence π is coexact. □

We will use the following result PDE result:

Lemma 4.5. *Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be a C_0^2 function. Then $\triangle\psi = \phi$ has a solution ψ which is C^2.*

Lemma 4.6. *Let X be a Riemann surface . If a 1-form ω in X, is C^3, then locally $\omega = df + *dg$, where f, g are local C^2 functions.*

Proof. Let $p \in X$. Let (U, ϕ) be a chart in centred at p, such that the unit disk D is contained in $\phi(U)$. Let $A := \phi^{-1}(\overline{D_{1/2}})$ and $G := \phi^{-1}(D)$. Then applying Proposition 4.4, there exists a function s which is smooth and takes value 1 on A and 0 outside G. Now let $\gamma = s\omega$. Then it is enough to show that $\gamma = df + d * g$ in G, since $\omega = \gamma$ in $\phi^{-1}(D_{1/2})$. Let $\gamma = pdx + qdy$ in local coordinates (x, y). then $d\gamma = ((\partial q/\partial x) - (\partial p/\partial y))dx \wedge dy$. the function $(\partial q/\partial x) - (\partial p/\partial y)$ is C^2 and has a compact support in D, and thus is extendible to the whole complex plane. Applying Lemma 4.5, the exists a C^2 function g such that $\triangle g = (\partial q/\partial x) - (\partial p/\partial y)$. That is $d * dg = d\gamma$, hence $d(\gamma - *dg) = 0$. Hence there exists a C^2 function f such that $df = \gamma - *dg$. This proves our assertion. □

Theorem 4.7. *Let X be a Riemann surface . If $\omega \in L(X)$ is C^3, then $\omega = df + *dg + \omega_h$ a.e, where f, g are C^2 functions in X and ω_h is harmonic.*

Proof. We already know that $\omega = \gamma + \pi + \omega_h$ a.e, where $\gamma \in E$, $\pi \in E^*$ and ω is harmonic. By Lemma 4.2, it is enough to prove that both γ and π are C^1. Being C^1 is a local property. So let us concentrate on a parametric disk D. By Lemma 4.6, there exists C^2 functions f, g in D such that $\omega = df + *dg$. Then we have $\gamma + \pi + \omega_h = df + *dg$ a.e in D. Rearranging we have $\gamma + \omega_h - df = -\pi + *dg$. We call this quantity θ. Note that if θ is C^1, then so are γ and π. To prove θ is C^1, we use Weyl's lemma. Let h be any C_0^2 function on D. We wish to show that $\langle \theta, dh \rangle_D = 0$ and $\langle \theta, *dh \rangle_D = 0$. Now,

$$\langle \theta, dh \rangle_D = \langle -\pi + *dg, dh \rangle_D = -\langle \pi, dh \rangle_D + \langle *dg, dh \rangle_D$$

We can extend h to the whole of X by setting $h \equiv 0$ outside D. Then $\langle \pi, dh \rangle_D = \langle \pi, dh \rangle_X$. But $\langle \pi, dh \rangle_X = 0$, since $\pi \in E^*$ and $dh \in E$. Therefore we need to prove that $\langle *dg, dh \rangle_D = 0$. But this is just an application of Theorem 2.1,

$$\langle *dg, dh \rangle_D = \overline{\langle dh, *dg \rangle_D} = -\overline{\iint_D dh \wedge d\overline{g}} = \overline{\iint_D h \wedge dd\overline{g}} = 0$$

Hence $\langle \theta, dh \rangle_D = 0$. Again

$$\langle \theta, *dh \rangle_D = \langle \gamma + \omega_h - df, *dh \rangle_D = \langle \gamma, *dh \rangle_D + \langle \omega_h, *dh \rangle_D - \langle df, *dh \rangle_D$$

As before $\langle \gamma, *dh \rangle_X = \langle \gamma, *dh \rangle_D$, but $\gamma \in E$ and $*dh \in E^*$. Hence $\langle \gamma, *dh \rangle_D = 0$. Similarly $\langle \omega_h, *dh \rangle_D = 0$. A similar application of Theorem 2.1, shows that $\langle df, *dh \rangle_D = 0$. Hence $\langle \theta, *dh \rangle_D = 0$. Thus by Weyl's lemma, θ is C^1, and hence so are γ and π. Therefore γ is exact and π is coexact. □

5. Existence of meromorphic functions

Let p_1 and p_2 be two distinct point in X. Our aim is to construct meromorphic function with a pole at p_1 and a zero at p_2. As we have discussed before, we will go about doing this by taking quotient of two meromorphic differentials, say ω_1/ω_2. First we define what we mean by *order* of a meromorphic differential ω at a point p.

Definition 5.1. *Let $\omega = fdz$ in a local coordinate z centred at p. Then define* $\mathrm{ord}_p(\omega) := \mathrm{ord}_p(f)$.

To prove well definedness, consider another local coordinate w centred at p and let the transition function be $T : w \mapsto z$. Then $\omega = f \circ T(w)T'(w)dw$. T is an analytic isomorphism, therefore $\mathrm{ord}_p(f) = \mathrm{ord}_p(f \circ T)$. Also $\mathrm{ord}_p(T) = 1$. Therefore, $\mathrm{ord}_p((f \circ T) \cdot T') = \mathrm{ord}_p(f \circ T) \cdot \mathrm{ord}_p(T') = \mathrm{ord}_p(f)$. Hence $\mathrm{ord}_p(\omega)$ is well defined.

Now we can say that if $f = \omega_1/\omega_2$, then $\mathrm{ord}_p f = \mathrm{ord}_p(\omega_1) - \mathrm{ord}_p(\omega_2)$. Thus the condition of f having a simple pole at p_1 and a simple zero at p_2, translates to the requirement: $(-1)^i = \mathrm{ord}_{p_i}(\omega_1) - \mathrm{ord}_{p_i}(\omega_2)$ for $i = 1, 2$. Meromorphic differentials are obtained via construction of harmonic ones. We will construct harmonic differentials with specified singularity at a point.

Theorem 5.2. *Let X be a Riemann surface , and p be a point in X. Let z be a local coordinate centred at p. Let n be an an integer greater than 0. Then there exists a 1-form ω, such that*

(1) ω is harmonic in $X \setminus \{p\}$.
(2) $\omega - d(1/z^n)$ is harmonic in a punctured neighbourhood N of p.

PROOF. Let (U, ϕ) be a chart in centred at p, such that the unit disk D is contained in $\phi(U)$. The singularity in local coordinate z is $d(1/z^n)$. We will extend this to a 1-form in $X \setminus \{p\}$, with the help of a bump function. Let $A := \phi^{-1}(\overline{D_{1/2}})$ and $G := \phi^{-1}(D)$. Then applying Proposition 4.4, there exists a function s which is smooth and takes value 1 on A and 0 outside G. Define a differential ψ on $X \setminus \{p\}$ as follows

$$\psi := \begin{cases} d(\frac{s(z)}{z^n}), & \text{in } G \\ 0, & \text{in } X \setminus G \end{cases}$$

Let $N := \phi^{-1}(D_{1/2})$. Then $\psi = d(1/z^n)$ in $N \setminus \{p\}$. Therefore ψ is holomorphic in $N \setminus \{p\}$. Hence by Theorem 1.2, $\psi - i * \psi \equiv 0$ in $N \setminus \{p\}$. Therefore $\psi - i * \psi$ is almost every where smooth. Hence Theorem 4.7 can be applied, yielding $\psi - i * \psi = df + *dg + \omega_h$ a.e, where f, g are C^2 functions in X and ω_h is harmonic. Rearranging, we get $\psi - df = i * \psi + \omega_h + *dg$ a.e.. Let $\omega = \psi - df$ be a 1-form in $X \setminus \{p\}$. We will show that ω satisfies (1) and (2).

First we show that ω is harmonic in $N \setminus \{p\}$. By Theorem 1.4, it enough to show that $d\omega = 0 = d{*}\omega$. First note that ω is C^1, since ω_h is harmonic and f is C^2. Now $d\omega = d(\psi - df) = d\psi - ddf = 0$, since ψ is exact in $X \setminus \{p\}$. Again $d * \omega = d * (i * \psi + \omega_h + *dg) = -id\psi + d * \omega_h - ddg = 0$ a.e.. Since ω is C^1, therefore $d * \omega$ is continuous and it is zero in a dense set (complement of a measure zero set), hence $d * \omega = 0$. Therefore ω is harmonic in $N \setminus \{p\}$.

In $N \setminus \{p\}$, $\psi = d(1/z^n)$, hence $\omega - d(1/z^n) = -df$. Also $\psi = i * \psi$ in $N \setminus \{p\}$, which implies $\omega = \omega_h + *dg$ a.e.. Therefore $d(\omega - d(1/z^n)) = -ddf = 0$ and $d * (\omega - d(1/z^n)) = d * \omega_h - ddg = 0$ a.e.. By same continuity argument as before we conclude that $d * (\omega - d(1/z^n)) = 0$. Therefore $\omega - d(1/z^n)$ is harmonic in N. □

Let ω be as in the theorem and let γ be it's real part. Then $\gamma + i * \gamma$ a meromorphic differential. $\omega - d(1/z^n)$ is harmonic in N, taking the real part, $\gamma - Re(d(1/z^n))$ is harmonic in N. Again $*(\omega - d(1/z^n))$ is harmonic in N and consider it's real part $Re(*(\omega - d(1/z^n)))$. This quantity is almost everywhere equal to $Re * \omega - Re * d(1/z^n)$, and hence by continuity equal. Now $Re * \omega = *Re\, \omega = *\gamma$. For the other part we do a general calculation for any $f = u + iv$ holomorphic in place of $1/z^n$.

$$\begin{aligned} Re(*df) &= Re * (\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy) \\ &= Re(-\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy) \\ &= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \\ &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \\ &= Im(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy) \\ &= Im(df) \end{aligned}$$

Therefore $i * \gamma - i\ Im(1/z^n)$ is harmonic in N. Hence $\gamma + i * \gamma - d(1/z^n)$ is holomorphic in N. If $\xi = \gamma + i * \gamma$, then ξ is a meromorphic differential with $\mathrm{ord}_p(\xi) = -(n+1)$. Thus we have obtained the following corollary.

Corollary 5.3. *Let X be a Riemann surface and p be any point in it. Let n be an integer greater than 1. Then there exists a meromorphic differential which has order $-n$ at p and is holomorphic in $X \setminus \{p\}$.*

Now we state and prove the existence theorem of meromorphic functions.

Theorem 5.4. *Let X be a Riemann surface and p_1, p_2 be any two distinct points in it. Then there exists a meromorphic function f, such that, $ord_{p_1}(f) = 1$ and $ord_{p_2}(f) = -1$.*

PROOF. By Corollary 5.3, there exists meromorphic differentials γ_i, which are holomorphic in $X\setminus\{p_i\}$ and have $\mathrm{ord}_{p_i}(\gamma_i) = -(1 + 2^{i-1})$, for $i = 1, 2$. Then $\omega_1 := \gamma_1 + \gamma_2$ is a meromorphic differential which is holomorphic in $X \setminus \{p_1, p_2\}$ and has $\mathrm{ord}_{p_i}(\omega_1) = -(1 + 2^{i-1})$ for $i = 1, 2$. Similarly there exits a meromorphic differential ω_2, which is holomorphic in $X \setminus \{p_1, p_2\}$ and has $\mathrm{ord}_{p_i}(\omega_2) = -(2^i + (-1)^{i-1})$ for $i = 1, 2$. Now consider the meromorphic function $f = \omega_1/\omega_2$. f is holomorphic in $X \setminus \{p_1, p_2\}$ and has $\mathrm{ord}_{p_i}(f) = \mathrm{ord}_{p_i}(\omega_1) - \mathrm{ord}_{p_i}(\omega_2) = (-1)^{i-1}$. $\square$

CHAPTER 3

Riemann-Roch Theorem

1. The Mittag-Leffler Problem

Given a compact Riemann surface X and points $p_1, \cdots, p_k \in X$ and $n_1, \cdots, n_k \in \mathbb{Z}$, can we find $f \in \mathcal{M}(X)$ such that $\mathrm{ord}_{p_i}(f) = n_i$ for all $i \in 1, \cdots, k$? In fact we can be more specific and ask for a meromorphic function with given Laurent tails in fixed local coordinates at each of finitely many points. A Laurent tail here means a Laurent polynomials which is the tail of a Laurent series. Let us do this for one point.

Suppose the given Laurent tail is $r(z) = \sum_{i=n}^{m} c_i z^i$ with $c_n \neq 0 \neq c_m$. There are $m - n + 1$ terms. We will proceed by induction on number of terms. Suppose there is only one term, that is, $r(z) = cz^n$. We know the existence of a $g \in \mathcal{M}(X)$ such that $\mathrm{ord}_p(g) = 1$. Then g^n multiplied by a suitable constant will give us our desired function. Now suppose that r has more than one term. There exists $h \in \mathcal{M}(X)$ with Laurent tail $c_n z^n$. Let s be the Laurent tail of $h - r$ upto z^m term. Since s has fewer terms than r, by induction there exists $g \in \mathcal{M}(X)$ having s as Laurent tail. Therefore $h - g$ has r as Laurent tail at p.

Now let us generalize this to k points $p_1, \cdots, p_k \in X$ with Laurent tails $r_i(z_i)$, where z_i are fixed local coordinates centred at p_i. For convenience we will assume, by adding zero coefficient terms if necessary, that $r_i(z_i) = \sum_{j=n}^{m} c_j^i z_i^j$ for fixed m and n. By previous result we know for each i, there exists $g_i \in \mathcal{M}(X)$ with $g_i = r_i +$terms of order higher than m. We have to somehow combine these meromorphic functions together, so that their Laurent tails at each p_i is r_i. Consider a combination of the form $f = \sum_{i=1}^{k} g_i h_i$ for $h_i \in \mathcal{M}(X)$. For each i, if we can arrange to make $h_i = 1 +$terms of degree higher than $m - n$, at p_i and $h_i =$ terms of degree higher than $m - n$, at p_j, for $i \neq j$, then f will have the desired Laurent tails at each of the p_is. That means we want $\mathrm{ord}_{p_i}(h_i - 1) > m - n$ and $\mathrm{ord}_{p_j}(h_i) > m - n$.

So our aim is to construct meromorphic function of the above form. For convenience write h in place of h_i. We have information about order of $h - 1$ at one point and about order of h itself at others. We wish to combine these into information on orders of a single meromorphic function. The crucial point to note here is that addition of a constant to a meromorphic function messes up it's zeros but not it's poles. Here if we consider $h - 1$, then the information about zeros of h will be lost, but if we consider $H = \frac{1}{h} - 1$, then we have

$$\begin{aligned}
\mathrm{ord}_{p_j}(H) &= \mathrm{ord}_{p_j}(\frac{1}{h} - 1) \\
&= \mathrm{ord}_{p_j}(\frac{1}{h}) \text{ (since } \frac{1}{h} \text{ has pole at } p_j) \\
&= -\mathrm{ord}_{p_j} h \\
&< n - m
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{ord}_{p_i}(H) &= \mathrm{ord}_{p_i}(\frac{1}{h} - 1) \\
&= \mathrm{ord}_{p_i}(\frac{1-h}{h}) \\
&= \mathrm{ord}_{p_i}(1 - h) \text{ (since } \mathrm{ord}_{p_i}(h) = 0) \\
&> m - n
\end{aligned}$$

So we have to construct a meromorphic function with a zero of order greater than $m - n$ at one point and poles of order greater than $m - n$ at others. Note that specification on order of zeros or poles does not matter, if we can find a meromorphic function with zeros and poles at specified points, then raising by power $m - n + 1$ will satisfy the desired condition on order at each point. This is what we will do.

We proceed by induction on number of poles. In the previous chapter we have constructed a meromorphic function with a zero and a pole at two specified points, say p and q. Suppose there exists a meromorphic function which has a zero at p and poles at $q_1, \cdots, q_{n-1}$. Call it g. If g has a pole also at

q_n, then we are done. Otherwise, let $h \in \mathcal{M}(X)$ has a zero at p and a pole at q_n. If we simply combine these two as $f = g + h$, then clearly we have a zero at p and a pole at q_n, but the poles of g and h may cancel each other at the q_is. So we take $f = g + h^l$, with l so large that even if h has a pole at one of the q_is, for $i \in \{1, \cdots, n-1\}$, then $\operatorname{ord}_{p_i}(h) < \operatorname{ord}_{p_i}(g)$, making sure that f indeed has poles at q_i for $i \in \{1, \cdots, n-1\}$.

For future reference we record the weaker version of this result as a lemma:

Lemma 1.1. *Given a compact Riemann surface X and points $p_1, \cdots, p_k \in X$ and $n_1, \cdots, n_k \in \mathbb{Z}$, there exists $f \in \mathcal{M}(X)$ such that $\operatorname{ord}_{p_i}(f) = n_i$ for all $i \in 1, \cdots, k$.*

We have constructed meromorphic functions with given Laurent tails at finitely many points, but in doing so we had no control of it's behaviour at the other points. A meromorphic function has zeros and poles only at finitely many points. So specifying Laurent tails at finitely many points *and* demanding that the function be holomorphic at others is a natural problem. Here we are demanding restriction on two different properties of the meromorphic function, one it's order at each point and the other it's Laurent tail at finitely many points. The order at each point of a non zero meromorphic function can be described by the formal sum $\sum_{p \in X} \operatorname{ord}_p(f) \cdot p$. Note that all but finitely many coefficients are zero. Making this into a formal definition we have:

Definition 1.2. *A divisor is the free abelian group generated by points of a compact Riemann surface.*

Group of divisors of a compact Riemann surface is denoted by $\operatorname{Div}(X)$. And an element in it is generally denoted by D. We often view D as a function from X to $\mathbb{Z}$, that is, if $D = \sum_{p \in X} n_p \cdot p$, then $D(p) = n_p$, for all $p \in X$.

Definition 1.3. *A divisor of the form $\sum_{p \in X} \operatorname{ord}_p(f) \cdot p$, where $f \in \mathcal{M}(X)$ is called a principal divisor.*

We denote it by $\operatorname{div}(f)$. We can also define *divisor of zeros of f*, as $\operatorname{div}_0(f) := \sum_{\operatorname{ord}_p(f) > 0} \operatorname{ord}_p(f) \cdot p$ and *divisor of poles of f*, as $\operatorname{div}_\infty(f) := \sum_{\operatorname{ord}_p(f) < 0} (-\operatorname{ord}_p(f)) \cdot p$. Then $\operatorname{div}(f) = \operatorname{div}_0(f) - \operatorname{div}_\infty(f)$ with $\operatorname{div}_0(f)$ and $\operatorname{div}_\infty(f)$ having disjoint support. In fact given any $D \in \operatorname{Div}(X)$, we can write $D = P - N$, where both P and N takes non-negetive values for all $p \in X$ and have disjoint support. Just like with meromorphic functions, we can also attach a divisor with meromorphic 1-forms.

Definition 1.4. *If ω is a meromorphic 1-form, then we define $\operatorname{div}(\omega) = \sum_{p \in X} \operatorname{ord}_p(\omega) \cdot p$. Such a divisor is called canonical divisor.*

Similarly we can collect all Laurent tail divisors at finitely many points to form a group:

Definition 1.5. *A Laurent tail divisor is a finite formal sum $\sum_p r_p(z_p) \cdot p$, where r_p is a Laurent polynomial in pre-chosen local coordinate z_p, centred at p.*

The group of Laurent tail divisors of a compact Riemann surface is denoted by $\mathcal{T}(X)$. Now given $\sum_p r_p(z_p) \cdot p \in \mathcal{T}(X)$ we attach a divisor to it in the following way. $D(p) =$ one more than degree of top term in r_p, if p appears in the finite sum, and 0 otherwise. Then we are looking for a meromorphic function which satisfies the following property at each point p:

Look at it's Laurent series expansion in terms of z_p. Consider it's tail consisting of terms of degree less than $D(p)$. If p does not appear in the finite sum of the Laurent tail divisor, then this tail is non-existent, otherwise it is r_p.

In the above we started with a Laurent tail divisor, and then constructed a divisor from it. Going the other way, that is, choosing $D \in \operatorname{Div}(X)$ first, we can define the following subgroup of $\mathcal{T}(X)$:

$$\begin{aligned}\mathcal{T}[D](X) := \{\sum_p r_p \cdot p \in \mathcal{T}(X) :\ &\text{top term of } r_p \text{ has degree strictly less than} \\ &-D(p)\ \text{, whenever } p \text{ appears in the sum}\}\end{aligned}$$

Now we define a map $\alpha_D : \mathcal{M}(X) \to \mathcal{T}[D](X)$ sending a meromorphic function f to $\sum_p r_p \cdot p$, where r_p is the truncation of the Laurent series of f at p in terms of z_p, removing all terms of degree $-D(p)$ and higher. We immediately notice that if $D_1 \leq D_2$, that is $D_1(p) \leq D_2(p)$ for all $p \in X$, then there is a natural map $t_{D_1}^{D_2} : \mathcal{T}[D_1](X) \to \mathcal{T}[D_2](X)$ by sending $\sum_p r_p \cdot p$ to $\sum_p s_p \cdot p$ where s_p is the truncation of

r_p, by removing all terms of degree $-D_2(p)$ and higher. Then we have a commuting diagram:

$$\begin{tikzcd} \mathcal{M}(X) \arrow[r, "\alpha_{D_1}"] \arrow[dr, "\alpha_{D_2}"'] & \mathcal{T}[D_1](X) \arrow[d, "t_{D_1}^{D_2}"] \\ & \mathcal{T}[D_2](X) \end{tikzcd} \tag{13}$$

Now, given an element of $\mathcal{T}[D](X)$, our original question was: Does there exist a preimage under α_D? This is called the Mittag-Leffler Problem. Surjectivity of α_D is too much to expect. Consider a complex torus X and any point p in it. Suppose $\frac{1}{z_p} \cdot p \in \mathcal{T}[0](X)$ has a preimage $f \in \mathcal{M}(X)$. Let $F : X \to \mathbb{C}_\infty$ be the corresponding holomorphic map. Then since f has a simple pole at p and no other poles, therefore deg $F = 1$. Hence F is an isomorphism. But torus and sphere are not even homeomorphic, so this is absurd. Hence α_D is not surjective in this case.

Note that $\mathcal{T}[D](X)$ is also a complex vector space and so is $\mathcal{M}(X)$ (in fact it is a field extension). Then the map α_D is a $\mathbb{C}$ linear map. Hence coker α_D is also a vector space. We call it $H^1(D)$. This space is a measure of obstruction in solving the Mittag-Leffler Problem. We wish to prove that this space is finite dimensional.

2. Algebraic formulation

Let us expand the map $\mathcal{T}[D](X) \twoheadrightarrow H^1(D)$ into an exact sequence. For that we need to find out the kernel of the map α_D. Let $f \in$ Ker α_D. Then at $p \in X$ the Laurent series of f has no terms less than or equal to $-D(p)$. That means at each p, $\text{div}(f)(p) \geq -D(p)$. Thus the kernel of α_D is the space:

$$L(D) := \{f \in \mathcal{M}(X) : \text{div}(f) \geq -D\}$$

We note in passing that a similar space can be defined for meromorphic 1-forms.

$$L^{(1)}(D) := \{\omega \in \mathcal{M}^{(1)}(X) : \text{div}(\omega) \geq -D\}$$

$L(D)$ is a $\mathbb{C}$ vector space. So we have the exact sequence:

$$0 \to L(D) \to \mathcal{M}(X) \to \mathcal{T}[D](X) \to H^1(D) \to 0$$

We can make this into a short exact sequence as:

$$0 \to \mathcal{M}(X)/L(D) \to \mathcal{T}[D](X) \to H^1(D) \to 0 \tag{14}$$

We first claim that $L(D)$ is finite dimensional for all $D \in$ Div(X). To see this we first notice that $L(0)$ is the set of holomorphic functions. But only holomorphic functions on a compact Riemann surface are the constant ones. Therefore dim$L(0) = 1$. A divisor D differs from the divisor 0 by a finite sum, so an induction argument seems feasible here. The natural candidate that present itself for applying induction on, is what we call *degree of a divisor.*

Definition 2.1. *The degree of a divisor D on a compact Riemann surface is*

$$\deg(D) = \sum_{p \in X} D(p)$$

Note this is a group homomorphism. Also for any $f \in \mathcal{M}(X)$, with X compact, we have $\deg(\text{div}(f)) = 0$. The precise result on dimension of $L(D)$ is the following.

Lemma 2.2. *Let X be a compact Riemann Surface, and $D \in$ Div(X). Write D as $D = P - N$, where P and N are non-negative divisors with disjoint support. Then* dim$L(D) \leq 1+$ deg(P).

PROOF. We will apply induction on degree of the positive part P of D. $\deg(P) = 0$ implies $P = 0$. Therefore dim$L(P) = 1$. Also note that $D \leq P$ implies $L(D) \subset L(P)$. So we have dim$L(D) \leq$ dim$L(P) = 1 = 1+$ deg(P) as required. Now suppose the statement is true for deg$(P) = k - 1$. Let D has positive part P whose degree is k. Choose a point p in the support of P such that $P(p) > 0$. Then the positive part of $D - p$, which is $P - p$ has degree $k - 1$. By induction hypothesis dim$L(D - p) \leq 1+$deg$(P - p)=$ deg(P). We only need to show that dim$L(D) \leq 1+$dim$L(D - p)$. This will be an application of rank-nullity theorem. We wish to find a linear transformation from $L(D)$ to a one dimensional vector space over $\mathbb{C}$, whose kernel is $L(D - p)$. Define a local coordinate z centred at p. The Laurent series of a meromorphic function $f \in L(D)$ is of the form $f = cz^{-D(p)}+$ higher order terms. Define a function $\phi : L(D) \to \mathbb{C}$, sending f to c, in the above notation. This is then our desired linear transformation as the kernel is clearly $L(D-p)$. So now rank-nullity implies $\dim L(D) \leq 1+\dim L(D-p)$ and we are done. $\square$

For future use we prove the following Corollary.

Corollary 2.3. *Let $P \in \mathrm{Div}(X)$ be a positive divisor. Then* $\dim L(P) = 1+\deg(P)$ *implies* $X \cong \mathbb{C}_\infty$.

PROOF. Let $\deg(P) = d$. Let $P = \sum_{i=1}^{d} p_i$, where p_i's may not be distinct. Consider the sequence

$$L(0) \subset L(p_1) \subset L(p_1+p_2) \subset \cdots \subset L(\sum_{i=1}^{d} p_i)$$

$\dim L(0) = 1$ and $\dim L(\sum_{i=1}^{d} p_i) = d+1$. Thus the dimension increases from 1 to $d+1$ in d steps. Also by Lemma 2.2, the increase in dimension in each step can be atmost one. Therefore the increase of dimension in each step is exactly 1. Hence $\dim L(p_1) = 2$. Therefore there exists a non constant meromorphic function f in $L(p_1)$. Then the corresponding holomorphic function to $\mathbb{C}_\infty$ has degree one, and hence is an isomorphism. $\square$

Since $\mathbb{C}$ is algebraically closed and $\mathcal{M}(X)$ is a non-trivial field extension of $\mathbb{C}$, therefore transcendence degree of $\mathcal{M}(X)$ over $\mathbb{C}$ is atleast one, hence $\mathcal{M}(X)$ is an infinite dimensional complex vector space. Therefore so is $\mathcal{M}(X)/L(D)$. And clearly is $\mathcal{T}[D](X)$ is infinite dimensional too. So there is no hope of getting any information about dimension of $H^1(D)$ directly from the short exact sequence (14). But the the truncation map $t_{D_1}^{D_2}$ gives a way to compare $H^1(D_1)$ and $H^1(D_2)$, whenever $D_1 \leq D_2$. We can have the following chain map:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}(X)/L(D_1) & \xrightarrow{\overline{\alpha_{D_1}}} & \mathcal{T}[D_1](X) & \xrightarrow{\pi_1} & H^1(D_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow t_{D_1}^{D_2} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}(X)/L(D_2) & \xrightarrow{\overline{\alpha_{D_1}}} & \mathcal{T}[D_2](X) & \xrightarrow{\pi_2} & H^1(D_2) & \longrightarrow & 0
\end{array}$$

where the middle vertical map is $t_{D_1}^{D_2}$, the left vertical map is defined by starting with the quotient map $\mathcal{M}(X) \to \mathcal{M}(X)/L(D_2)$ and noting that $L(D_1) \subset L(D_2)$ implies $L(D_1)$ is in it's kernel. The left hand square then commutes because of (13). For the right vertical map we send $[\sum_p r_p \cdot p]$ to $\pi_2 \circ t_{D_1}^{D_2}(\sum_p r_p \cdot p)$. To check well definedness we note that $[\sum_p r_p \cdot p] = [\sum_p s_p \cdot p]$ implies that $\sum_p r_p \cdot p - \sum_p s_p \cdot p = \alpha_{D_1}(f)$ for some $f \in \mathcal{M}(X)$. Applying $t_{D_1}^{D_2}$ on both sides we see, $t_{D_1}^{D_2}(\sum_p r_p \cdot p) - t_{D_1}^{D_2}(\sum_p s_p \cdot p) = t_{D_1}^{D_2} \circ \alpha_{D_1}(f) = \alpha_{D_2}(f)$. Hence $\pi_2 \circ t_{D_1}^{D_2}(\sum_p r_p \cdot p) = \pi_2 \circ t_{D_1}^{D_2}(\sum_p s_p \cdot p)$. By definition the right hand square also commutes.

Note that the left vertical map is surjective, hence by Snake Lemma, we have a short exact sequence of kernels of the vertical maps. Kernel of the left vertical map is $L(D_2)/L(D_1)$. Since $L(D)$'s are finite dimensional,

$$\dim(L(D_2)/L(D_1)) = \dim L(D_2) - \dim L(D_1) \tag{15}$$

Kernel of $t_{D_1}^{D_2}$ consists of Laurent tail divisors $\sum_p r_p \cdot p$ such that the top term of r_p is less than $-D_1(p)$ and the bottom term greater than or equal to $-D_2(p)$. The basis of $\mathcal{T}[D_1](X)$ consists of z_p^n with $p \in X$ and $n < -D_1(p)$. Therefore basis of the kernel consists of z_p^k for which $-D_2(p) \leq k < -D_1(p)$. $D_2(p) - D_1(p)$ basis elements for each $p \in X$, therefore summing up the total dimension is:

$$\dim \ker(t_{D_1}^{D_2}) = \sum_{p \in X} (D_2(p) - D_1(p)) = \deg(D_2) - \deg(D_1) \tag{16}$$

The kernel of the third vertical map which we denote by $H^1(D_1/D_2)$ can now be computed from the short exact sequence

$$0 \to L(D_2)/L(D_1) \to \ker(t_{D_1}^{D_2}) \to H^1(D_1/D_2) \to 0$$

We record this result as a lemma:

Lemma 2.4. *D_1 and D_2 are arbitrary divisors on a compact Riemann surface X, with $D_1 \leq D_2$. Then*

$$\dim H^1(D_1/D_2) = [\deg(D_2) - \dim L(D_2)] - [\deg(D_1) - \dim L(D_1)] \tag{17}$$

We have mentioned that the map α_D may not in general be surjective. Can we find atleast one D for which α_D is surjective? It is same as asking for a divisor D, for which $H^1(D) = 0$. We start with an arbitrary divisor A. Suppose $H^1(A) \neq 0$. Then there exists $\mathcal{R} \in \mathcal{T}[A](X)$, such that $[\mathcal{R}] \neq 0$ in $H^1(A)$. Increase A to a divisor B, such that $t_A^B(\mathcal{R}) = 0$. Then $\pi_2 \circ t_A^B(\mathcal{R}) = 0$ and hence $[\mathcal{R}] \in H^1(A/B)$.

Therefore $H^1(A/B) \neq 0$. Then by Lemma 2.4, $1 \leq \dim H^1(A/B) = [\deg(B) - \dim L(B)] - [\deg(A) - \dim L(A)]$. Hence $\deg(B) - \dim L(B) > \deg(A) - \dim L(A)$. Now if $H^1(B)$ is also not equal to zero then we can find a $C \in \mathrm{Div}(X)$, such that $\deg(C) - \dim L(C) > \deg(B) - \dim L(B)$. Thus we see that the quantity $\deg(A) - \dim L(A)$ will continue to strictly increase as long as we do not hit upon a $A \in \mathrm{Div}(X)$ with $H^1(A) = 0$. So question is, does this quantity have any upper bound?

3. Upper bound for deg(A) – dim($L(A)$)

$\deg(A)$ has no upper bound. So we have to look for a lower bound of $\dim L(A)$. That is, we wish to find a minimum number of linearly independent meromorphic functions in $L(A)$. But we do not know if we have even one for an arbitrary divisor A. So let us start with a meromorphic function and create a divisor in whose L space it belongs. Fix a non-zero meromorphic function f and define $D = \mathrm{div}_\infty(f)$. Then $f \in L(D)$. Note that $f \in L(mD)$ for all $m > 0$, infact $1, f, \cdots, f^m \in L(mD)$. Let us restrict our attention at the moment to divisors of the form mD, for $m \in \mathbb{N}$. We will find a lower bound for $\dim L(mD)$, for large m. We already have $m+1$ linearly independent functions in $L(mD)$, but that gives us an inequality: $\deg(mD) - \dim L(mD) \leq m\deg(D) - m - 1$, which is not independent of m, unless $\deg(D) = 1$, which we can have only if $X = \mathbb{C}_\infty$. So let us try to find some more meromorphic function in $L(mD)$. Start with an arbitrary non-zero meromorphic function h. We first try to remove the poles of h that do not coincide with that of f. If there is no such then note that $h \in L(mD)$, for some m. Otherwise let $p_1, \cdots, p_k$ be such points. Then the meromorphic function $g := h \cdot \prod_{i=1}^k (f - f(p_i))^{-\mathrm{ord}_{p_i}(h)}$ has poles only at poles of f, that is, only at poles of f we may have $\mathrm{ord}_p(g) < 0$. Hence there exists $m > 0$ such that $g \in L(mD)$. Now suppose we take n many such different $h_i \in \mathcal{M}(X)$ and apply the same procedure to get $g_i := h_i r_i(f) \in L(mD)$ where r_i are polynomials with complex coefficients, for large m. We want these to be linearly independent. Suppose they are not. Then we will have a complex linear combination of such g_i's equal to zero. Absorbing the complex coefficients in the polynomials $r_i(f)$, we have an equation of the form:

$$r_1(f)h_1 + \cdots + r_n(f)h_n = 0$$

This means that the meromorphic functions h_i are linearly dependent as vectors over the field $\mathbb{C}(f)$. So we have to pick only those h_is that are linearly independent over $\mathbb{C}(f)$. We can have $[\mathcal{M}(X) : \mathbb{C}(f)]$ many of them. So we wish to find this number or atleast a lower bound for it.

Proposition 3.1. *Let f be a non-constant meromorphic function on a compact Riemann surface X and $D = \mathrm{div}_\infty(f)$. Then*

$$[\mathcal{M}(X) : \mathbb{C}(f)] \geq \deg(D)$$

PROOF. Let $D = \sum_{i=1}^m n_i p_i$. By Lemma 1.1, we can construct a meromorphic function g_{ij} which has a pole of order j at p_i and no zero or pole at any of the other p_k's. We claim that the set $\{g_{ij} : 1 \leq i \leq m, 1 \leq j \leq n_i\}$ is linearly independent over $\mathbb{C}(f)$. Suppose not. Then there exists $\mathbb{C}(f)$-linear combination of such functions which is equal to 0.

$$\sum_{i,j} c_{ij}(f) g_{ij} = 0$$

c_{ij}'s are rational functions of f. By clearing denominator we may assume that they are infact polynomial functions of f. Let $c_{i_0 j_0}$ has the maximum degree among these polynomials. Renumber such that $i_0 = 1$, then divide the above expression throughout by $c_{1 j_0}$ to get

$$\sum_{i,j} d_{ij}(f) g_{ij} = 0 \tag{18}$$

where $d_{1j_0} = 1$. Since c_{ij} is polynomial function in f, therefore it has poles only at poles of f and infact it has a pole of order exactly $\deg(c_{ij})n_k$ at p_k. Now,

$$\begin{aligned}\mathrm{ord}_{p_k}(d_{ij}) &= \mathrm{ord}_{p_k}(c_{ij}) - \mathrm{ord}_{p_k}(c_{1j_0}) \\ &= (-\deg(c_{ij}) + \deg(c_{1j_0}))n_k \\ &\geq 0\end{aligned}$$

Let us consider the Laurent series of the LHS of (18) in some local coordinates centred at p_1. For the terms with $i \neq 1$, $\mathrm{ord}_{p_1}(g_{ij}) = 0$ and $\mathrm{ord}_{p_1}(d_{ij}) \geq 0$, so they do not contribute to the negative exponent part of the Laurent series. For terms with $i = 1$, $\mathrm{ord}_{p_1}(g_{1j}) = -j$, with $1 \leq j \leq n_1$ and $\mathrm{ord}_{p_1}(d_{ij})$ is a non-negative multiple of n_1. Therefore only way $\mathrm{ord}_{p_1}(d_{1j}(f)g_{1j}) < 0$, is if $\mathrm{ord}_{p_1}(d_{1j}) = 0$ and in this case $\mathrm{ord}_{p_1}(d_{1j}(f)g_{1j}) = -j$. Note that the j's are all distinct. For $j = j_0$, we have such a term:

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$\text{ord}_{p_1}(d_{1j_0}(f)g_{1j_0}) = \text{ord}_{p_1}(g_{1j_0}) = -j_0$. Consider all terms with $\text{ord}_{p_1}(d_{1j}) = 0$ and pick the one with maximum j. Then this term contributes a negative exponent term in the Laurent series of LHS of (18) which is not cancelled by any other terms. But the RHS of (18) is 0. Hence contradiction. □

Infact $[\mathcal{M}(X) : \mathbb{C}(f)] = \deg(D)$. We do not need this, but the interested reader will find the proof of the other side inequality in [6], pg. 176. Coming back to the problem of finding an upper bound for $\dim L(mD)$, we have the following result.

Lemma 3.2. *X is a compact Riemann surface and $f \in \mathcal{M}(X)$. Let $D = \text{div}_\infty(f)$. Then there exists $m_0 > 0$, such that for all $m \geq m_0$,*

$$\dim L(mD) \geq (m - m_0 + 1)\deg(D)$$

PROOF. Let $\deg(D) = k$. Then we can find meromorphic functions $h_1, \cdots, h_k$, which are linearly independent over $\mathbb{C}(f)$. Then by the procedure described above we can find $g_i := h_i r_i(f) \in L(m_0 D)$, where r_i are polynomials with complex coefficients and a large enough m_0. Since h_i's are linearly independent over $\mathbb{C}(f)$, therefore g_is are independent over $\mathbb{C}$. Now note that for $m > m_0$, $f^j g_i$ also belong to $L(mD)$, for $0 \leq j \leq m - m_0$. We claim that the set $\{f^j g_i \in \mathcal{M}(X) : 1 \leq i \leq k, 0 \leq j \leq m - m_0\}$ is linearly independent over $\mathbb{C}$. Suppose not. Then there is a $\mathbb{C}$ linear combination of such function which is identically equal to zero.

$$\begin{aligned} &\sum_{i,j} c_{ij} f^j g_i = 0 \\ \Rightarrow &\sum_{i,j} c_{ij} f^j r_i(f) h_i = 0 \\ \Rightarrow &\sum_i ((\sum_j c_{ij} f^j) r_i(f)) h_i = 0 \end{aligned}$$

h_i's are linearly independent over $\mathbb{C}(f)$, therefore we must have

$$\begin{aligned} &(\sum_j c_{ij} f^j) r_i(f) = 0 \text{ , for all } 0 \leq i \leq k \\ \Rightarrow &\sum_j c_{ij} f^j = 0 \text{ , since } r_i \neq 0 \text{ for any } i \end{aligned}$$

Since $\mathbb{C}$ is algebraically closed, this means that f is a constant function, which is a contradiction. Therefore $\{f^j g_i \in \mathcal{M}(X) : 1 \leq i \leq k, 0 \leq j \leq m - m_0\}$ is linearly independent over $\mathbb{C}$, and hence for all $m \geq m_0$

$$\dim L(mD) \geq (m - m_0 + 1)\deg(D)$$

□

Now let us apply this result to the problem of finding an upper bound for $\deg(mD) - \dim L(mD)$, for D of the form $D = \text{div}_\infty f$, for a fixed $f \in \mathcal{M}(X)$. We have, for a large enough m,

$$\begin{aligned} \deg(mD) - \dim L(mD) &\leq m \deg(D) - (m - m_0 + 1)\deg(D) \\ &= (m_0 - 1)\deg(D) \end{aligned}$$

How do we generalize this to find a lower bound for $\deg(A) - \dim L(A)$, for an arbitrary divisor A? For this we have to study divisors a little more deeply. The set of principal divisors form a subgroup of $\text{Div}(X)$. This follows from the following lemma.

Lemma 3.3. *Let f and g be non zero meromorphic functions on Riemann surface X. Then,*
(a) $\text{div}(fg) = \text{div}(f) + \text{div}(g)$
(b) $\text{div}(1/f) = -\text{div}(f)$

PROOF. (a) For any $p \in X$,

$$\begin{aligned} \text{div}(fg)(p) &= \text{ord}_p(fg) \\ &= \text{ord}_p(f) + \text{ord}_p(g) \\ &= \text{div}(f)(p) + \text{div}(g)(p) \end{aligned}$$

(b) For any $p \in X$,

$$\begin{aligned}\mathrm{div}(1/f)(p) &= \mathrm{ord}_p(1/f)\\ &= -\mathrm{ord}_p(f)\\ &= -\mathrm{div}(f)(p)\end{aligned}$$

□

The subgroup of principal divisors is denoted by PDiv(X). Similarly the canonical divisors form a subgroup denoted by KDiv(X). Now we can consider the quotient Div(X)/PDiv(X). We had noticed in the last chapter that, if ω_1 and ω_2 are two meromorphic 1-forms, then there exists $f \in \mathcal{M}(X)$ such that $\omega_1 = f\omega_2$. Then it follows that KDiv(X) is a coset of PDiv(X). The cosets have the following two properties, when X is compact:

(1) $D_1 \sim D_2$, implies $\deg(D_1) = \deg(D_2)$
(2) $D_1 \sim D_2$, implies $\dim L(D_1) = \dim L(D_2)$

The first property is obvious. We now prove the second one. $D_1 \sim D_2$ implies there exists $h \in \mathcal{M}(X)$ such that $D_1 + D_2 = \mathrm{div}(h)$. Then define a map

$$\begin{aligned}\mu_h : L(D_1) &\to L(D_2)\\ f &\mapsto hf\end{aligned}$$

hf indeed belongs to $L(D_2)$ since div(hf) =div(h)+div(f) $\geq$div(h) $- D_1 = D_2$. Similarly $\mu_{1/h}$ maps $L(D_2)$ to $L(D_1)$ and is inverse of μ_h. Therefore μ_h is an isomorphism.

We return to denoting, for a fixed non zero meromorphic function f, $\mathrm{div}_\infty(f)$ by D. Now if we can show that any divisor $A \sim mD$ for some m, then $\deg(A) - \dim L(A) = \deg(mD) - \dim L(mD)$. Infact we require less. We had already noted that, by Lemma 2.4, whenever $D_1 \leq D_2$, we have $\deg(D_1) - \dim L(D_1) \leq \deg(D_2) - \dim L(D_2)$. So it is enough to show that $A \sim B$, such that $B \leq mD$ for some m. That is we wish to find $g \in \mathcal{M}(X)$ such that $A-$div(g) $\leq mD$, for some $m > 0$. But then we essentially did the same thing at the beginning of this section, while constructing new meromorphic functions for $L(mD)$, starting from arbitrary $h \in \mathcal{M}(X)$. There we constructed a polynomial $r(f)$ such that div($r(f)$)+div(h) $\geq -mD$, for some m. We wish to replace div(h) by $-A$ in this inequality. Following the same procedure we first list the points $p_1, \cdots, p_k$, for which $A(p_i) > 0$, but $D(p) = 0$, that is, f has no pole at p_i's. Then the function $r(f) = \prod_{i=1}^k (f - f(p_i))^{A(p_i)}$ has no poles other than that of f and has $\mathrm{ord}_{p_i}(r(f)) \geq A(p_i)$ for each i. Thus $\mathrm{ord}_p(r(f)) - A(p)$ =div($r(f)$)(p) $- A(p) \leq 0$ only if p is a pole of f, that is $D(p) < 0$. Therefore there exists $m > 0$, such that div($r(f)$) $- A \geq -mD$. Summing it all up, we have proved,

Lemma 3.4. *Let X be a compact Riemann surface. Then for all $A \in$Div(X), there exists $M \in \mathbb{Z}$, such that*

$$\deg(A) - \dim L(A) \leq M$$

4. Finite dimensionality of $H^1(D)$

Recall the discussion at the end of Section 2. We were looking for a divisor A, for which $\alpha_A : \mathcal{M}(X) \to \mathcal{T}[A](X)$ is surjective and found out that we can have a sequence of divisors with strictly increasing value of the quantity $\deg(A) - \dim L(A)$, unless one of the divisors in the sequence satisfied our desired property. But in the previous section we produced a uniform upper bound for this quantity and hence the sequence of divisors with strictly increasing $\deg(A) - \dim L(A)$ cannot go on, but has to yield a divisor A_0 for which α_{A_0} is surjective, that is $H^1(A_0) = 0$. Now that we have produced atleast one divisor whose H^1 space is finite dimensional and we already know the finite dimensionality of $H^1(A/B)$, for $A \leq B$, it seems we can prove finite dimensionality of $H^1(A)$, for any divisor A, by comparing it with the right divisor. Precisely, we have the following proposition.

Proposition 4.1. *Let X be a compact Riemann surface. Then for any $A \in$ Div(X), $H^1(A)$ is a finite dimensional vector space over $\mathbb{C}$.*

PROOF. We want to compare A with A_0, but they may not be comparable. So let us look at the difference $A - A_0 = P - N$, where P and N are positive divisors with compact support. We do not want the P part, so club it with A_0 to get $A_0 + P$. Note that $A_0 \leq A_0 + P$ and hence we have a surjective map from $H^1(A_0)$ to $H^1(A_0 + P)$, so that $H^1(A_0 + P) = 0$. $A \leq A_0 + P$, therefore we have a surjective map from $H^1(A)$ to $H^1(A_0 + P) = 0$. Hence $H^1(A)$ is equal to the kernel $H^1(A/A_0 + P)$, which is finite dimensional. □

Now that we have proved finite dimensionality of $H^1(A)$, we can apply rank-nullity theorem to the map $H^1(D_1) \twoheadrightarrow H^1(D_2)$, where $D_1 \leq D_2$, to get $\dim H^1(D_1/D_2) = \dim H^1(D_1) - \dim H^1(D_2)$. Substituting this in the equation of Lemma 2.4,

$$\dim H^1(D_1) - \dim H^1(D_2) = [\deg(D_2) - \dim L(D_2)] - [\deg(D_1) - \dim L(D_1)]$$
$$\Rightarrow \dim L(D_1) - \deg(D_1) - \dim H^1(D_1) = \dim L(D_2) - \deg(D_2) - \dim H^1(D_2)$$

Given any two divisors A_1 and A_2, there exists one which is greater than both, hence $\dim L(A) - \deg(A) - \dim H^1(A)$ is a constant for all $A \in \mathrm{Div}(X)$. For $A = 0$, we have $\dim L(0) - \deg(0) - \dim H^1(0) = 1 - \dim H^1(0)$. Hence we have:

Theorem 4.2. *Let A be a divisor on a compact Riemann surface X. Then*

$$\dim L(A) - \dim H^1(A) = \deg(A) + 1 - \dim H^1(0)$$

This is the preliminary version of the *Riemann-Roch Theorem.*

5. Serre Duality

First a concept from complex analysis needs to be introduced in Riemann surface context, that of *residue.* Residue of a meromorphic function at a point p in the complex plane is defined to be the coefficient of $\frac{1}{z}$ in the Laurent series of f at p. The Residue Theorem of complex analysis states that

Theorem 5.1. *Let Ω be an open set in $\mathbb{C}$ and let E be a discrete set in Ω. Let α be a closed curve in $\Omega \setminus E$ which is null homotopic in Ω. Then for any holomorphic f in $\Omega \setminus E$, the set $\{a \in E : n(\gamma, a) \neq 0)\}$, where n is the winding number of γ at the point a, is finite and*

$$\frac{1}{2\pi i}\int_\gamma f dz = \sum_{a\in E} \mathrm{res}_a(f) \cdot n(\gamma, a)$$

In a Riemann surface line integrable entities are the 1-forms. So following the complex analytic definition we define

Definition 5.2. *Let z be a local coordinate centred at $p \in X$. Let $\omega = f dz$ in this local coordinate. Then the residue of a 1-form ω at a point $p \in X$ is defined as*

$$\mathrm{res}_p(\omega) := \mathrm{res}_p(f)$$

We have to prove well definedness. This is just an application of Theorem 5.1. Let the chart whose local coordinate is z be (U, ϕ). Consider a simple loop in U, enclosing p, but not any other pole of ω. By Residue theorem, $\int_\gamma \omega = \int_{\phi\circ\gamma} f(z)dz = 2\pi i\ \mathrm{res}_p(f) = 2\pi i\ \mathrm{res}_p(\omega)$. Since the RHS is independent of the chart chosen, so is the LHS.

Now we state the residue theorem for a compact Riemann surface.

Theorem 5.3. *Let ω be a meromorphic 1-form on a compact Riemann surface X. Then,*

$$\sum_{p\in X} \mathrm{res}_p(\omega) = 0$$

PROOF. Poles of ω is a discrete set in compact X. Therefore finite. Let us call them $p_1, \cdots, p_k$. For each i, choose a simple loop α_i in a parametric disk enclosing p_i and not any other pole. By Jordan Curve Theorem, the parametric disk is divided into two disjoint components by α_i. Let U_i be the component containing p_i. By Theorem 5.1, $\int_{\alpha_i} \omega = \mathrm{res}_{p_i}(\omega)$. Let $Y := X \setminus \cup_{i=1}^n U_i$. Then as a 1-chain $\partial Y = -\sum_{1=i}^n \alpha_i$. Note that ω is holomorphic, hence closed in Y. Therefore applying Stokes Theorem,

$$0 = \iint_Y d\omega = \int_{\partial Y} \omega = -\sum_{i=1}^n \int_{\alpha_i} \omega = -\sum_{i=1}^n \mathrm{res}_{p_i}(\omega) = -\sum_{p\in X} \mathrm{res}_p(\omega)$$

□

Let ω be a meromorphic 1-form and f be a meromorphic function. Then $f\omega$ is again a meromorphic 1-form. Let z be a local coordinate centred at $p \in X$. Let $\omega = h(z)dz$ locally. Let $f = \sum_{i=-l}^\infty a_i z^i$ and

$h = \sum_{i=-k}^{\infty} c_i z^i$ be the Laurent series expansions of f and h, respectively, in terms of z. Let us calculate the residue of $f\omega$ at p.

$$\begin{aligned}\mathrm{res}_p(f\omega) &= \text{coefficient of } (1/z)dz \text{ in } (\sum_{i=-l}^{\infty} a_i z^i \cdot \sum_{j=-k}^{\infty} c_j z^j)dz \\ &= \sum_{i=-k}^{\infty} c_i a_{-1-i}\end{aligned}$$

We notice that $\mathrm{res}_p(f\omega)$ depends only on those coefficients of a_i, for which $i < k$, that is, only on the Laurent tail of degree $k-1$. Thus if $\omega \in L^{(1)}(-D)$, then we can replace $-k$ by $D(p)$ above. Then $\mathrm{res}_p(f\omega)$ depends only on the Laurent tail of f truncated at $-D(p)$, that is the residue depends only on $\alpha_D(f)$. Thus given a meromorphic 1-form $\omega \in L^{(1)}(-D)$, this leads us to define a *residue* map on $\mathcal{T}[D](X)$ as following

$$\begin{aligned}\mathrm{Res}_\omega : \mathcal{T}[D](X) &\to \mathbb{C} \\ \sum_{p\in X} r_p \cdot p &\mapsto \sum_{p\in X} \mathrm{res}_p(r_p\omega)\end{aligned}$$

This is a linear map. For $f \in \mathcal{M}(X)$, by above calculation,

$$\mathrm{Res}_\omega(\alpha_D(f)) = \sum_{p\in X} \mathrm{res}_p(r_p\omega) = \sum_{p\in X} \mathrm{res}_p(f\omega)$$

Now we know by Theorem 5.3, that $\sum_{p\in X} \mathrm{res}_p(f\omega) = 0$. Therefore we have a linear map from $\mathcal{T}[D](X)/\alpha_D(\mathcal{M}(X))$ to $\mathbb{C}$. By abuse of notation, we again call this map $\mathrm{Res}_\omega : H^1(D) \to \mathbb{C}$. Thus $\mathrm{Res}_\omega \in H^1(D)^*$. So we have a map from $L^{(1)}(-D)$ to the dual of $H^1(D)$, again called the *residue map.*

$$\begin{aligned}\mathrm{Res} : L^{(1)}(-D) &\to H^1(D)^* \\ \omega &\mapsto \mathrm{Res}_\omega\end{aligned}$$

Easy to check that this map is also linear. Now we claim ,

Theorem 5.4. *[Serre Duality] Let X be a compact Riemann surface and let $D \in \mathrm{Div}(X)$. Then $\mathrm{Res} : L^{(1)}(-D) \to H^1(D)^*$ is an isomorphism of complex vector spaces.*

Let us prove injectivity of this map. Since this is a linear map, it is enough to prove that the kernel is zero. So let $\omega \in \mathrm{Ker}(\mathrm{Res})$. Then $\mathrm{Res}_\omega(\sum_{p\in X} r_p \cdot p) = 0$, for all $\sum_{p\in X} r_p \cdot p \in \mathcal{T}[D](X)$. We wish to prove that $\omega \equiv 0$. Suppose not. Then there exists a point p, such that if $\omega = (\sum c_j z^j)dz$ in local coordinate z centred at p, then not all c_i's are zero. Now all we need to do is find a suitable Laurent polynomial r_p such that $\mathrm{Res}_\omega(r_p \cdot p) = \mathrm{res}_p(r_p\omega) \neq 0$, which will give the contradiction. Suppose k is the least integer for which $c_k \neq 0$. Take $r_p = z^{-k-1}$, then $\mathrm{res}_p(r_p\omega) = c_k \neq 0$. Hence injectivity is proved.

Now let us look at surjectivity. Consider an element $\phi : H^1(D) \to \mathbb{C}$, of $H^1(D)^*$. We will think of ϕ as a linear functional on $\mathcal{T}[D](X)$ that vanishes on $\alpha_D(\mathcal{M}(X))$. We want to find a preimage of ϕ under Res. We had noticed in the last chapter that, if ω_1 and ω_2 are two meromorphic 1-forms, then there exists $f \in \mathcal{M}(X)$ such that $\omega_1 = f\omega_2$. So we start with any meromorphic 1-form ω and try to multiply it with suitable meromorphic function to get our desired preimage. Let $K = \mathrm{div}(\omega)$. Then $\omega \in L^{(1)}(-K)$ and hence $\mathrm{Res}_\omega \in \mathcal{T}[K](X)^*$, whereas $\phi \in \mathcal{T}[D](X)^*$. To make these two comparable consider $A \in \mathrm{Div}(X)$, such that, $A \leq K, D$. Then $\mathrm{div}(\omega) \geq A$ and hence $\omega \in L^{(1)}(-A)$. Thus $\mathrm{Res}_\omega \in \mathcal{T}[A](X)^*$. Also, composing ϕ with t_A^D, we have $\phi_A := \phi \circ t_A^D \in \mathcal{T}[A](X)^*$. But they may not be equal, for that we wish to multiply ω with a suitable meromorphic function and consider it's residue map.

Given a meromorphic function f and any divisor D, we introduce a map

$$\begin{aligned}\mu_f : \mathcal{T}[D](X) &\to \mathcal{T}[D - \mathrm{div}(f)](X) \\ \sum r_p \cdot p &\mapsto \sum f r_p \cdot p \text{ truncated at } -D + \mathrm{div}(f)\end{aligned}$$

The crucial property of this map is that the following diagram commutes.

$$\begin{tikzcd} \mathcal{T}[D](X) \arrow[r, "\mu_f"] \arrow[dr, "\mathrm{Res}_{f\omega}"'] & \mathcal{T}[D - \mathrm{div}(f)](X) \arrow[d, "\mathrm{Res}_\omega"] \\ & \mathbb{C} \end{tikzcd}$$

In our case Res_ω acts on $\mathcal{T}[A](X)$. To use the above factorization we have to choose a divisor of the form $D - \mathrm{div}(f)$ on which Res_ω can act. That is, we have to choose a divisor of the form $D - \mathrm{div}(f)$ which is smaller than A. If $f \in L(C)$, then $A - C - \mathrm{div}(f)$ is such a choice. Thus we can say that the composition

$$\mathcal{T}[A - C](X) \to \mathcal{T}[A - C - \mathrm{div}(f)](X) \to \mathcal{T}[A](X) \to \mathbb{C}$$

given by $\phi_A \circ t^A_{A-C-\mathrm{div}(f)} \circ \mu_f$ is equal to $\mathrm{Res}_{f\omega}$. This motivates the following lemma which we will apply on ϕ_A and Res_ω.

Lemma 5.5. *Let $\phi_1, \phi_1 \in H^1(A)^*$ be non zero linear functionals. Then there exists a positive divisor C and non-zero $f_1, f_2 \in L(C)$ such that the following diagram commutes.*

$$\begin{tikzcd}
 & \mathcal{T}[A - C - \mathrm{div}(f_1)](X) \arrow[r, "t^A_{A-C-\mathrm{div}(f_1)}"] & \mathcal{T}[A](X) \arrow[dr, "\phi_1"] & \\
\mathcal{T}[A - C](X) \arrow[ur, "\mu_{f_1}"] \arrow[dr, "\mu_{f_2}"'] & & & \mathbb{C} \\
 & \mathcal{T}[A - C - \mathrm{div}(f_2)](X) \arrow[r, "t^A_{A-C-\mathrm{div}(f_2)}"] & \mathcal{T}[A](X) \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

PROOF. For any positive divisor C, consider the map

$$L(C) \times L(C) \to H^1(A - C)^*$$
$$(f_1, f_2) \mapsto \phi_1 \circ t^A_{A-C-\mathrm{div}(f_1)} \circ \mu_{f_1} - \phi_2 \circ t^A_{A-C-\mathrm{div}(f_2)} \circ \mu_{f_2}$$

Aim of the lemma is to show that there exists C such that for some non-zero f_1, f_2, this map is takes the value zero. Suppose (f_1, f_2) belongs to the kernel such that one of them is zero, say $f_1 = 0$. Then $\phi_2 \circ t^A_{A-C-\mathrm{div}(f_2)} \circ \mu_{f_2}$ is also zero. If f_2 is non-zero, then μ_{f_2} is invertible and $t^A_{A-C-\mathrm{div}(f_2)}$ is surjective, hence $\phi_2 = 0$, which contradicts the hypothesis. Therefore the statement that there exists non-zero f_1, f_2 for which the map takes zero value implies that the kernel is non-trivial. Suppose not. Then this map is injective. Since both the domain and range are finite dimensional, injectivity implies

$$\dim H^1(A - C) \geq 2 \dim L(C)$$

But we already have an expression for $\dim H^1(A - C)$ in Theorem 4.2. Putting it we have

$$\begin{aligned}\dim H^1(A - C) &= \dim L(A - C) - \deg(A - C) - 1 + \dim H^1(0)\\ &\leq \dim L(A) - \deg(A) - 1 + \dim H^1(0) + \deg(C),\end{aligned}$$

This implies $2 \dim L(C) \leq a + \deg(C)$, for some constant a. Again applying Theorem 4.2 to $\dim L(C)$, we have

$$\begin{aligned}\dim L(C) &= \dim H^1(C) + \deg(C) + 1 - \dim H^1(0)\\ &\geq \deg(C) + 1 - \dim H^1(0)\end{aligned}$$

This implies $2 \dim L(C) \geq b + 2\deg(C)$, for some constant b. Therefore we have

$$b + 2\deg(C) \leq a + \deg(C)$$
$$\text{or } \deg(C) \leq a - b$$

But we can take any positive divisor C, hence $\deg(C)$ cannot be bounded. This is a contradiction, proving our lemma. □

Applying this lemma to ϕ and Res_ω, there exists a positive divisor C and non-zero $f_1, f_2 \in L(C)$ such that

$$\phi_A \circ t^A_{A-C-\mathrm{div}(f_1)} \circ \mu_{f_1} = \mathrm{Res}_\omega \circ t^A_{A-C-\mathrm{div}(f_2)} \circ \mu_{f_2}$$

We have seen that the RHS is equal to $\mathrm{Res}_{f_2\omega}$. Hence we have,

$$\phi_A \circ t^A_{A-C-\mathrm{div}(f_1)} \circ \mu_{f_1} = \mathrm{Res}_{f_2\omega}$$

Now notice that μ_f is invertible, it's inverse being $\mu_{1/f}$. Composing with μ_{1/f_1} on both sides we have $\phi_A \circ t^A_{A-C-\mathrm{div}(f_1)} = \mathrm{Res}_{f_2\omega} \circ \mu_{1/f_1} = \mathrm{Res}_{(f_2/f_1)\omega}$. This implies that $\mathrm{Res}_{(f_2/f_1)\omega}$ is 0 on element which are in $\mathcal{T}[A - C - \mathrm{div}(f_1)](X)$ but not in $\mathcal{T}[A](X)$, since such elements belong to $\mathrm{Ker}(t^A_{A-C-\mathrm{div}(f_1)}) \subset \mathrm{Ker}(\phi_A \circ t^A_{A-C-\mathrm{div}(f_1)}) = \mathrm{Ker}(\mathrm{Res}_{(f_2/f_1)\omega})$. Hence $\phi_A = \mathrm{Res}_{(f_2/f_1)\omega}$ as an element of $H^1(A)^*$. Now

$\phi \circ t_A^D = \phi_A = \mathrm{Res}_{(f_2/f_1)\omega}$. A similar argument then tells us that $\phi = \mathrm{Res}_{(f_2/f_1)\omega}$ as an element of $H^1(D)^*$. This finishes the proof of Serre Duality Theorem.

6. Riemann-Roch Theorem

Serre Duality and Theorem 4.2 implies,

$$\dim L(A) - \dim L^{(1)}(-A) = \deg (A) + 1 - \dim L^{(1)}(0) \tag{19}$$

Let ω be any meromorphic 1-form and let $\mathrm{div}(\omega) = K$. Then we can construct a meromorphic 1-form belonging to $L^{(1)}(D)$ by multiplying ω with a suitable $f \in \mathcal{M}(X)$. $\mathrm{div}(f\omega) = \mathrm{div}(f) + \mathrm{div}(\omega) \geq \mathrm{div}(f) + K$; if we take $f \in L(D+K)$, then $\mathrm{div}(f\omega) \geq -D - K + K = -D$. Hence $f\omega \in L^{(1)}(D)$ if $f \in L(D+K)$. In fact these are all, that is, we have the following Lemma

Lemma 6.1. *Let ω be any meromorphic 1-form and let $\mathrm{div}(\omega) = K$. Let $D \in \mathrm{Div}(X)$, then the map*

$$\mu_\omega : L(D+K) \to L^{(1)}(D)$$
$$f \mapsto f\omega$$

is $\mathbb{C}$ linear and is an isomorphism.

PROOF. The linearity is obvious. Injectivity is also clear. For surjectivity, let $\gamma \in L^{(1)}(D)$, then there exists $f \in \mathcal{M}(X)$, such that $f\omega = \gamma$. Now $\mathrm{div}(f) = \mathrm{div}(\gamma) - \mathrm{div}(\omega) \geq -D-K$. Hence $f \in L(D+K)$. □

Thus $\dim L^{(1)}(-A) = \dim L(K-A)$ and $\dim L^{(1)}(0) = \dim L(K)$. Putting this in (19), we have

$$\dim L(A) - \dim L(K-A) = \deg(A) + 1 - \dim L(K) \tag{20}$$

If $A = K$, we have $\dim L(K) - \dim L(0) = \deg(K) + 1 - \dim L(K)$ or $2\dim L(K) = 2 + \deg(K)$. Since $\mathrm{KDiv}(X)$ is a coset of $\mathrm{PDiv}(X)$, therefore degree of a canonical divisor is constant. Therefore it is enough to find the degree of any one of them. We know that any Riemann surface X has non-constant meromorphic function. Let the corresponding map to the Riemann sphere be $F : X \to \mathbb{C}_\infty$. $\omega = dz$ is a meromorphic 1-form on $\mathbb{C}_\infty$, whose divisor $-2 \cdot \infty$. Then $F^*(\omega)$ is a 1-form on X. We wish to find the degree of this divisor. There is a general result on order of a pulled back meromorphic 1-form.

Lemma 6.2. *Let $F : X \to Y$ be a holomorphic map between two Riemann surfaces. Let ω be a meromorphic 1-form on Y. Let $p \in X$. Then*

$$\mathrm{ord}_p(F^*\omega) = (1 + \mathrm{ord}_p(\omega))\mathrm{mult}_p(F) - 1$$

PROOF. Consider the local normal form $w = z^n$ of F, in coordinates z and w centred at p and $F(p)$ respectively, where $n = \mathrm{mult}_p(F)$. ω is locally equal to $f dw$, for some local meromorphic function f. Let Laurent series of f in terms of w be $\sum_{i=k}^{\infty} a_i w^i$, where $k = \mathrm{ord}_{F(p)}(\omega)$. Then locally $F^*\omega = (\sum_{i=k}^{\infty} a_i z^{ni}) n z^{n-1} dz$. Hence $\mathrm{ord}_p(F^*\omega) = nk + n - 1 = (1+k)n + 1 = (1 + \mathrm{ord}_p(\omega))\mathrm{mult}_p(F) - 1$. □

Degree is the sum of the orders at each point. So we can apply this Lemma to calculate $\deg(F^*\omega)$, in the situation where $Y = \mathbb{C}_\infty$ and $\omega = dz$. Following computation uses Hurwitz Formula in the 5^{th} step.

$$\begin{aligned}
\deg(F^*\omega) &= \sum_{p \in X} \mathrm{ord}_p(F^*\omega) \\
&= \sum_{p \in X} [(1 + \mathrm{ord}_p(\omega))\mathrm{mult}_p(F) - 1] \\
&= \sum_{\substack{q \neq \infty \\ p \in F^{-1}(q)}} [\mathrm{mult}_p(F) - 1] + \sum_{p \in F^{-1}(\infty)} [-\mathrm{mult}_p(F) - 1] \\
&= \sum_{p \in X} [\mathrm{mult}_p(F) - 1] - \sum_{p \in F^{-1}(\infty)} 2\ \mathrm{mult}_p(F) \\
&= 2g - 2 + 2\deg(F) - 2\deg(F) \\
&= 2g - 2
\end{aligned}$$

Therefore we have $\deg(K) = 2g - 2$, hence $2 \dim L(K) = 2 + \deg(K) = 2g$ or $\dim L(K) = g$. Putting this in (20), we have the following theorem.

Theorem 6.3. *[**Riemann-Roch**] Let X be a compact Riemann surface of genus g. Then for any $A \in \mathrm{Div}(X)$ and $K \in \mathrm{KDiv}(X)$, we have*

$$\dim L(A) - \dim L(K - A) = \deg (A) + 1 - g$$

CHAPTER 4

Automorphisms for genus $g \geq 2$

1. Weierstrass points

Let X be a compact Riemann surface and $p \in X$. We have seen that if there exists $f \in \mathcal{M}(X)$ non-constant, such that f has only one pole of order 1 at p, then X is isomorphic to $\mathbb{C}_\infty$. That is $\dim L(p) > 1$ implies that $X \cong \mathbb{C}_\infty$. From now on unless mentioned otherwise we will consider Riemann surfaces of genus $g > 0$. So we ask what are all integers n, such that, there does not exists any non-constant $f \in \mathcal{M}(X)$, such that, $\operatorname{div}_\infty(f) = n \cdot p$? From Riemann-Roch Theorem we know

$$\dim L(np) - \dim L(K - np) = \deg\,(np) + 1 - g \tag{21}$$

The following lemma tells us that for n sufficiently large we can get rid of the $\dim L(K - np)$ term.

Lemma 1.1. *Let X be a compact Riemann surface and let $D \in \operatorname{Div}(X)$, with $\deg(D) < 0$. Then $L(D) = 0$.*

PROOF. Let $0 \neq f \in L(D)$. Then $\operatorname{div}(f) \geq -D$. Then we can define a non-negative divisor $E = \operatorname{div}(f) + D$. Therefore $\deg(E) \geq 0$. But $\deg(E) = \deg\,(\operatorname{div}(f)) + \deg(D) < 0$, which is a contradiction, hence $L(D) = 0$. $\square$

Thus if we can ensure that $\deg(K - np) < 0$, then $\dim L(K - np) = 0$. But $\deg(K - np) = \deg(K) - np = 2g - 2 - n$, since degree of a canonical divisor is always $2g - 2$. Thus if $n \geq 2g - 1$, $\dim L(np) = n + 1 - g$. Then $\dim L(2g \cdot p) = g + 1 \geq 2$, which means $L(2g \cdot p)$ admits non-constant meromorphic function. Now consider the sequence

$$L(0) \subset L(p) \subset \cdots \subset L(2g \cdot p)$$

$\dim L(0) = 1$ and $\dim L(2g \cdot p) = g+1$, hence the dimension increases from 1 to $g+1$ in $2g$ steps. Also we had seen in the proof of 2.2, that $\dim L(D) \leq \dim L(D-p) + 1$. Hence the increase in dimension at each step in the above sequence is atmost one. Thus there are g integers $1 = n_1 < \cdots < n_g < 2g$, such that, $L((n_i - 1)p) = L(n_i)$. These numbers are called *gaps* and the set is denoted by G_p. Note that by the formula for $\dim L(np)$, for $n \geq 2g - 1$, there are no such "gaps" for any $n > 2g - 1$. $L((n_i - 1)p) \neq L(n_i)$, means there exista a non-constant $f \in \mathcal{M}(X)$, such that $\operatorname{div}_\infty(f) = n_i \cdot p$. Hence gap numbers are those points for which there is no such meromorphic function. The complement of the set of gap numbers in $\{1, \cdots, 2g\}$ is called the set of non-gap numbers and is denoted by $1 < m_1 < \cdots < m_g = 2g$. For each i, there exists non-constant $f_i \in \mathcal{M}(X)$, such that $\operatorname{div}_\infty(f_i) = m_i \cdot p$. Now $\operatorname{div}_\infty(f_i f_j) = \operatorname{div}_\infty(f_i) + \operatorname{div}_\infty(f_j) = m_i \cdot p + m_j \cdot p$. Thus if $m_i + m_j \leq 2g$, then it is also a non-gap point.

We want to know how G_p looks like for different p's. We are looking at the difference $\dim L(np) - \dim L((n-1)p)$. But (21) tells us we can as well look at the difference $\dim L(K-(n-1)p) - \dim L(K-np)$. What we mean is, put n and $n - 1$ in place of n in (21), and subtract them to get

$$\dim L(K - (n-1)p) - \dim L(K - np) = 1 + \dim L((n-1)p) - \dim L(np)$$

Thus n is a gap number at the point p, if and only if $\dim L(K - (n-1)p) \neq \dim L(K - np)$. Now consider the vector space $L(K)$. It follows from the above discussion that n is a gap number for p, means that there exists $f \in L(K)$, such that $\operatorname{ord}_p(f) = n - 1 - K(p)$. Let $\{f_1, \cdots, f_g\}$ be a basis of $L(K)$. Choose a local coordinate z centered at p. Multiply each f_i, by $z^{K(p)}$, to make them locally holomorphic functions, say $h_i(z) := z^{K(p)} f_i(z)$. Then the power series of h_i will be as follows

$$h_i = h_i(0) + \cdots + h_i^{(n)}(0)\frac{z^n}{n!} + \cdots$$

Now consider any element $\phi \in L(K)$. Suppose $\mathrm{ord}_p(\phi) = n - 1 - K(p)$, for $1 \leq n \leq g$, then we must have $\mathrm{ord}_p(z^{K(p)}\phi) = n - 1$. Let $\phi = \sum_{i=1}^{g} a_i f_i$. Then we must have

$$\begin{pmatrix} h_1(0) & \cdots & h_g(0) \\ \vdots & \vdots & \vdots \\ h_1^{(g-1)}(0) & \cdots & h_g^{(g-1)}(0) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_g \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ * \\ \vdots \\ * \end{pmatrix} \tag{22}$$

where the right hand column vector has first $n-1$ entries zero, a non zero entry at the n^{th} place and anything after that. Call the matrix in left hand side M_p. Thus if we have a $\phi_j \in L(K)$ with $\mathrm{ord}_{p(\phi_j)} = j - 1 - K(p)$, such that $\phi_j = \sum_i a_{ij} f_i$, for each $1 \leq j \leq g$, then putting $A = (a_{ij})_{i,j}$, we note that $M_p A$ is invertible. Hence M_p is invertible. Thus M_p is invertible is equivalent to the fact that $G_p = \{1, \cdots, g\}$. If this does not happen for some $p \in X$, then p is called a *Weierstass point.* Thus

Lemma 1.2. *A point $p \in X$ is a Weirstass point* $\Leftrightarrow \det(M_p) = 0$.

It is true that the matrix M_p depends on the choice of local coordinates, but a change of coordinate just multiplies M_p with an invertible matrix and hence the determinant does not change. Therefore the condition $\det(M_p) = 0$, is independent of the coordinate chosen.

We can extend the function $p \mapsto \det M_p$ to the coordinate neighbourhood U, by sending q to

$$W_z(h_1, \cdots, h_g)(z(q)) = \det \begin{pmatrix} h_1(z(q)) & \cdots & h_g(z(q)) \\ \vdots & \vdots & \vdots \\ h_1^{(g-1)}(z(q)) & \cdots & h_g^{(g-1)}(z(q)) \end{pmatrix}$$

Note that this function is holomorphic in U. It is not clear that this function helps us to determine whether a point $q \in U^*$ is Weierstass. Notice that there exists a punctured neighbourhood $V^* := V \setminus \{p\}$ of p, such that K takes the value 0 in V^*. Thus f_i's are holomorphic in V^*. Shrink V^*, if necessary, to fit inside U. To determine whether a point $q \in U^*$ is Weierstass, we calculate the determinant of a matrix M_q, whose i^{th} column is the first $g-1$ derivatives of f_i at $z(q)$. Note that $W_z(h_1, \cdots, h_g)(z(q)) = z(q)^{gK(p)} \det M_q$. Since $z(q) \neq 0$, for all $q \in V^*$, $\det M_q \neq 0 \Leftrightarrow W_z(h_1, \cdots, h_g)(z(q)) \neq 0$. Hence $q \in V$ is a Weierstass point $\Leftrightarrow W_z(h_1, \cdots, h_g)(z(q)) \neq 0$. The function $W_z(h_1, \cdots, h_g)$ is called the *Wronskian* of the functions $h_1, \cdots, h_g$. We have the following fact about Wronskians

Lemma 1.3. *Let $h_1, \cdots, h_n$ are linearly independent holomorphic functions in neighbourhood of 0 in $\mathbb{C}$. Then the Wronskian $W_z(h_1, \cdots, h_g)$ is not identically zero near 0.*

Before giving a proof we present the consequence of this fact

Corollary 1.4. *There are finitely many Weierstrass points.*

PROOF. We wish to prove that Weierstass points are discrete. Then by compactness of the Riemann surface, we can conclude that there are finitely many Weierstass points. Suppose p is a Weierstrass point. We have proved that in a neighbourhood V of p, a point q is Weierstass if only if $W_z(h_1, \cdots, h_g)(z(q)) \neq 0$. Now $h_1, \cdots, h_g$ are linearly independent, therefore by Lemma 1.3, $W_z(h_1, \cdots, h_g)$ is a holomorphic function which is not identically zero in a neighbourhood of 0. Hence there is a punctured neighbourhood of p, where $W_z(h_1, \cdots, h_g)$ is not zero, that is p is the only Weierstass point in that neighbourhood. □

PROOF. **of Lemma 1.3** First we note that if we have a matrix function $z \mapsto A(z) := (a_{ij}(z))_{i,j}$, then the derivative of it's determinant is given by

$$\frac{d}{dz}\det(A(z)) = \sum_i \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a'_{i1} & \cdots & a'_{in} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

RHS is the sum over i of determinants of matrices which are same as A except that each entry of the i^{th} row is differentiated. Applying this on Wronskian we note that differentiation of any row, except the last, will make it equal to the next one, and hence the determinant will become zero. We will prove that $W_z(h_1, \cdots, h_n)$ is identically zero near 0 implies that the functions $h_1, \cdots, h_n$ are linearly dependent.

$W_z(h_1, \cdots, h_n)$ is identically zero near 0 means that all derivatives of $W_z(h_1, \cdots, h_n)$ at 0 is zero. First derivative is zero means the vector $(h_1^{(g)}(0) \ \cdots \ h_g^{(g)}(0))$, is linearly dependent on the first $g-1$ rows of $W_z(h_1, \cdots, h_n)(0)$. A little reflection will convince the reader that first, second, upto n^{th} derivatives are zero, will imply that the vector $(h_1^{(n+g-1)}(0) \ \cdots \ h_g^{(n+g-1)}(0))$ is linearly dependent on the first $g-1$ rows of $W_z(h_1, \cdots, h_n)(0)$. Hence $\{(h_1^{(m)}(0) \ \cdots \ h_n^{(m)}(0)) \in \mathbb{C}^g : m \in \mathbb{N}\}$ is a subset of a $n-1$ dimensional subspace. Let $(\overline{c_1} \ \cdots \ \overline{c_n})$ be perpendicular to this $n-1$ dimensional subspace with respect to the usual hermitian product. Then $\sum_i c_i h_i^{(m)}(0) = 0$, for all $m \in \mathbb{N}$. Hence $\sum_i c_i h_i$ is identically zero. Therefore $h_1, \cdots, h_n$ are linearly dependent. □

2. Weierstrass weight

We saw that Wronskian tells us whether a certain point p is Weirstrass. Suppose p is not Weirstrass. That is, the gap numbers are not the first g positive integers. Suppose $G_p = \{n_1, \cdots, n_g\}$. Can the Wronskian give some information about the n_i's? This question is addressed in the following Lemma.

Lemma 2.1. *Let $G_p = \{n_1, \cdots, n_g\}$, and $\{f_1, \cdots, f_g\}$ be a basis of $L(K)$. Let z be a local coordinate centerd at p. Then*

$$\mathrm{ord}_p(W_z(z^{K(p)} f_1, \cdots, z^{K(p)} f_g)) = \sum_{i=1}^{g} (n_i - 1)$$

Proof. First note that if we work with a different basis then the order does not change. This is because, a change in basis amounts to multiplying the matrix of the Wronskian by the constant change of basis matrix, and hence the Wronskian itself is just multiplied by the non-zero determinant of this matrix. So the order is not affected. Now $G_p = \{n_1, \cdots, n_g\}$ implies that for each i, there exists $h_i \in L(K)$, such that $\mathrm{ord}_p(h_i) = n_i - 1 - K(p)$. Infact by multiplying by appropiate scalars we can ensure that the first non-zero term in the Laurent series of each h_i has coefficient 1, that is, for each i, we have $h_i = z^{n_i - 1 - K(p)} + \cdots$. Clearly all the h_i's are linearly independent and since there are g many of them, therefore they form a basis. So we consider $W_z(z^{K(p)} h_1, \cdots, z^{K(p)} h_g)$ and look for the lowest term of it's power series. Notice that the lowest term of power series of $W_z(z^{K(p)} h_1, \cdots, z^{K(p)} h_g)$ is same as the lowest term of the power series of $W_z(z^{K(p)} \cdot z^{n_1 - 1 - K(p)}, \cdots, z^{K(p)} \cdot z^{n_g - 1 - K(p)}) = W_z(z^{n_1 - 1}, \cdots, z^{n_g - 1})$ which is equal to the determinant of

$$\begin{pmatrix} z^{n_1-1} & \cdots & z^{n_g-1} \\ (n_1-1)z^{n_1-2} & \cdots & (n_g-1)z^{n_g-2} \\ \vdots & \vdots & \vdots \\ (n_1-1)\cdots(n_1-g+1)z^{n_1-g} & \cdots & (n_g-1)\cdots(n_g-g+1)z^{n_g-g} \end{pmatrix} \tag{23}$$

The determinant of an $n \times n$ matrix $(a_{ij})_{1,j}$ can be expanded as $\sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$. In our case $a_{i\sigma(i)}$ =(some constant)$\times z^{n_{\sigma(i)} - i}$. Multiplying over i, for a fixed σ, the exponent of z that we get is $\sum_i (n_{\sigma(i)} - i) = \sum_i n_{\sigma(i)} - \sum_i i = \sum_i n_i - \sum_i i = \sum_i (n_i - i)$. Hence (23) is a monoimal, with the coefficient matrix equal to

$$\det \begin{pmatrix} 1 & \cdots & 1 \\ (n_1-1) & \cdots & (n_g-1) \\ \vdots & \vdots & \vdots \\ (n_1-1)\cdots(n_1-g+1) & \cdots & (n_g-1)\cdots(n_g-g+1) \end{pmatrix} \tag{24}$$

If we can show that (24) is non zero we are done. We can think of (24) as a polynomial in the variable n_1. Then putting $n_1 = n_i$, for any i, in (24), yeilds zero. Hence $\prod_{i=1}^{g}(n_1 - n_i)$ divides (24). We can do the same thing for each n_j. Hence $\prod_{i<j}(n_i - n_j)$ divides (24). Now thinking of (24) as a polynomial in g variables $n_1, \cdots, n_g$, we note that the degree is $g(g-1)/2$. But degree of $\prod_{i<j}(n_i - n_j)$ is also $g(g-1)/2$. Hence (24) is equal to $\pm \prod_{i<j}(n_i - n_j)$. Since n_i's are all distinct, therefore (24) is non-zero. □

The above Lemma leads us to the following definition.

Definition 2.2. *Let X be a compact Riemann surface of genus g and $p \in X$. Let $G_p = \{n_1, \cdots, n_g\}$ be the set of gap numbers at p. Then the Weierstass weight of the point p is defined as*

$$w_p = \sum_{i=1}^{g} (n_i - i)$$

We want to know how many Weierstass points are there in a Riemann surface X of genus g. But the "correct" quantitity to count is not the number of Weierstass point, but the number along with their weight. That is, we will find out the quantity $\sum_{p\in X} w_p$. Note that this sum is finite, since w_p is non-zero iff p is a Weierstass point. The strategy is to stitch up the locally defined Wronskians to a global entity, very similar to forms. We will attach a divisor to such an entity which will contain the information of the order of each of the local Wronskians. We will derive a formula for the degree of this divisor which will then lead us to the value of $\sum_{p\in X} w_p$.

The global entity we have in mind is *higher order diffentials.*

Definition 2.3. *A meromorphic n-differential on an open set $U \in \mathbb{C}$ is an expression of the form $\mu = f(z)(dz)^n$, where $f \in \mathcal{M}(U)$.*

Denote the set of meromorphic n-differentials on U by $\mathcal{M}^{(n)}(U)$. Let U and V be two open sets of $\mathbb{C}$ and let $T : U \to V$ be a holomorphic function. Then define $T^* : \mathcal{M}^{(n)}(V) \to \mathcal{M}^{(n)}(U)$ sending $\mu = f(z)(dz)^n$ to $\nu = h(w)(dw)^n$, where $h(w) = f \circ T(w)T'(w)^n$. Now we can extend this concept to Riemann surface.

Definition 2.4. *Let X be a Riemann surface. A meromorphic n-differential on X is a collection of meromorphic n-differentials $\{\mu_\phi\}$, one for each chart $\phi : U \to V$, in the variable of V, such that if $\phi_i : U_i \to V_i$, $i = 1, 2$ have overlapping domains, then on $V_1 \cap V_2$, $\mu_{\phi_1} = (\phi_2 \circ \phi_1^{-1})^*(\mu_{\phi_2})$.*

Lemma 2.5. *Let X be a compact Riemann surface and $f_1, \cdots, f_n \in \mathcal{M}(X)$. Then $W_z(f_1, \cdots, f_n)$ $(dz)^{n(n-1)/2}$ is a meromorphic $n(n-1)/2$-differential.*

PROOF. $W_z(f_1, \cdots, f_n)$ is holomorphic except at the finitely many poles of $f_1, \cdots, f_n$. At those points $W_z(f_1, \cdots, f_n)$ has pole or removable singularity. Hence $W_z(f_1, \cdots, f_n)$ is a meromorphic function. Now we just have to check compatibility. Suppose z and w are the local coordinates of two overlapping charts and let $T(w) = z$ be the change of coordinate function. By abuse of notation we think of f_i's as function of the coordinate z. Now $W_w(f_1, \cdots, f_n) = \det(d^i(f_j \circ T(w))/dw^i)_{i,j}$. By induction we can show that

$$\frac{d^i f_j \circ T(w)}{dw^i} = T'(w)^i \frac{d^i f_j(z)}{dz^i} + \sum_{k=0}^{i-1} a_{ik} \frac{d^k f_j(z)}{dz^k}$$

for some holomorphic functions a_{ik}. We know row operations do not change determinant. Thus by row operations we can convert the matrix $(d^i(f_j \circ T(w))/dw^i)_{i,j}$ to $(T'(w)^i d^i f_j(z)/dz^i)_{i,j}$. Hence

$$W_w(f_1, \cdots, f_n) = \det(T'(w)^i \frac{d^i f_j(z)}{dz^i})_{i,j} = T'(w)^{n(n-1)/2} W_z(f_1, \cdots, f_n)$$

□

Definition 2.6. *Suppose an n-differential μ is represented by $f(z)(dz)^n$ in some local coordinate z centered at p. We define the order of μ at the point p as*

$$\mathrm{ord}_p(\mu) := \mathrm{ord}_p(f)$$

We have to check well definedness. Suppose $g(w)(dw)^n$ be another representation of μ, in coordinate w centered at p. Let $T(w) = z$. Note that $T(0) = 0$ and $T'(0) \neq 0$. By compatibility $g(w) = f(T(w))T'(w)^n$. $\mathrm{ord}_p(f(z))$ is the unique integer k, for which $\lim_{z\to\infty} z^{-k} f(z) \neq 0$ or ∞. Now

$$\begin{aligned}
\lim_{w\to\infty} w^{-k} g(w) &= \lim_{w\to\infty} \frac{z^k}{w^k} z^{-k} T'(w)^n \\
&= \lim_{w\to\infty} \frac{T(w)^k}{w^k} T'(w)^n \cdot \lim_{z\to\infty} z^{-k} f(z) \\
&= T'(w)^{k+n} \cdot \lim_{z\to\infty} z^{-k} f(z) \\
&\neq 0 \text{ or } \infty
\end{aligned}$$

Therefore order of an n-differential is well defined. Now we attach a divisor to μ as $\mathrm{div}(\mu) := \sum_{p\in X} \mathrm{ord}_p(\mu) \cdot p$. Once we have divisors we can organize $\mathcal{M}^{(n)}(X)$ in partially ordered subspaces

$$L^{(n)}(D) := \{\mu \in \mathcal{M}^{(n)}(X) : \mathrm{div}(\mu) \geq -D\}$$

Lemma 2.7. *Let $f_1, \cdots, f_n \in L(D)$. Then $W_z(f_1, \cdots, f_n)(dz)^{n(n-1)/2} \in L^{(n(n-1)/2)}(nD)$.*

PROOF. Let $p \in X$ and z be a local coordinate centered at p. $\text{ord}_p f_i \geq -D(p)$, for each i, hence $z^{D(p)} f_i$ is holomorphic at p, for each i. Then $W_z(z^{D(p)} f_1, \cdots, z^{D(p)} f_n)$ is holomorphic at p. Notice that $W_z(z^{D(p)} f_1, \cdots, z^{D(p)} f_n) = z^{nD(p)} W_z(f_1, \cdots, f_n)$, since row operation does not affect the determinant. Therefore $z^{nD(p)} W_z(f_1, \cdots, f_n)$ is holomorphic at p. Hence $\text{ord}_p(W_z(f_1, \cdots, f_n)) \geq -nD(p)$. □

We know for $n = 1$, $L^{(n)}(D) \cong L(D + nK)$, where K is a canonical divisor. This is true for general n.

Lemma 2.8. *Let $\omega \in \mathcal{M}^{(1)}(X)$ and $K = \text{div}(\omega)$. If $\omega = g(z)dz$ locally then we define an n-differential ω^n which is locally $g(z)^n (dz)^n$. Now define a map*

$$\begin{aligned} \phi : L(D + nK) &\to L^{(n)}(D) \\ f &\mapsto f\omega^n \end{aligned}$$

Then ϕ is an isomorphism of vector spaces.

PROOF. First of all we have to show that $f\omega^n$ indeed belongs to $L^{(n)}(D)$. Locally $f\omega^n$ is equal to $f(z)g(z)^n(dz)^n$. Hence

$$\text{ord}_p(f\omega^n) = \text{ord}_p(f) + n\text{ord}_p(g) \geq -D(p) - nK(p) + nK(p) = -D(p)$$

Linearity and injectivity is clear. For surjectivity consider $\mu \in L^{(n)}(D)$. We have to find a meromorphic function f, such that $f\omega^n = \mu$. If z is the local coordinate in a neighbourhood then $\omega = g(z)dz$ and $\mu = h(z)(dz)^n$ locally. Define f locally as $h(z)/g(z)^n$. A change of coordinate $T(w) = z$ will result in multiplication of both numerator and denominator by non-zero $T'(w)^n$. Hence f is a well defined meromorphic function and satisfies $f\omega^n = \mu$. Hence ϕ is surjective and therefore an isomorphism. □

Now we turn our attention to the particular type of Wronskian differential which is locally given by $W_z(f_1, \cdots, f_g)(dz)^{g(g-1)/2}$, where $\{f_1, \cdots, f_g\}$ is a basis of $L(K)$. We have seen that a change of basis of $L(K)$, does not affect the order of such a differential. So we denote such a differential by $W(K)$, and it is unique upto scalar multiplication.

Corollary 2.9. $\deg(\text{div}(W(K))) = g(g-1)^2$

PROOF. By Lemma 2.8 there exists $f \in \mathcal{M}(X)$ and $\omega \in \mathcal{M}^{(1)}(X)$, such that $W(K) = f\omega^{g(g-1)/2}$. Hence

$$\begin{aligned} \deg(\text{div}(W(K))) &= \sum_p \text{ord}_p(W(K)) \\ &= \sum_p \text{ord}_p(f\omega^{g(g-1)/2}) \\ &= \sum_p \text{ord}_p(f) + \frac{g(g-1)}{2} \sum_p \text{ord}_p(\omega) \\ &= \frac{g(g-1)}{2} \times 2g - 2 \\ &= g(g-1)^2 \end{aligned}$$

since $\text{ord}_p(f) = 0$ and $\text{ord}_p(\omega) = 2g - 2$. □

Now we come to the main result of this section.

Theorem 2.10. *Let X be a compact Riemann surface of genus g. Then*

$$\sum_{p \in X} w_p = g^3 - g.$$

PROOF. Let $\{f_1, \cdots, f_g\}$ be a basis of $L(K)$. Then

$$\begin{aligned}\sum_p w_p &= \sum_p \mathrm{ord}_p((W_z(z^{K(p)}f_1, \cdots, z^{K(p)}f_g))) \text{ (by Lemma 2.1)}\\ &= \sum_p \mathrm{ord}_p(z^{gK(p)}(W_z(f_1, \cdots, f_g)))\\ &= \sum_p [gK(p) + \mathrm{ord}_p(W(K))]\\ &= g\mathrm{deg}(K) + \mathrm{deg}(\mathrm{div}(W(K)))\\ &= g(2g-2) + g(g-1)^2 \text{ (by Corollary 2.9)}\\ &= g^3 - g\end{aligned}$$

Hence proved. □

3. Bound for number of Weierstrass points

Now we are interested in counting the actual number of Weierstass points. By above theorem we can have atmost $g^3 - g$ Weierstass points for a Riemann surface of genus g. Can we get a lower bound? The idea is to find an upper bound for the weight w_p, so that dividing $\sum_p w_p = g^3 - g$ by this quantity will yeild a lower bound for number of Weierstass points. For this we need to examine non-gap points more deeply. We recall that the set of non-gap points is the complement of the set of gap points G_q in $\{1, \cdots, 2g\}$ and we denote them by $1 < m_1 < \cdots < m_g = 2g$.

Proposition 3.1. *For each $0 < i < g$, $m_i + m_{g-i} \geq 2g$.*

PROOF. We had seen that if $m_i + m_j \leq 2g$, then it is a non-gap point. Suppose $m_i + m_{g-i} < 2g$. Then for each $j \leq i$, $m_j + m_{g-i}$ is non-gap point. Thus we have atleast i non-gap points strictly between m_{g-i} and $m_g = 2g$. Hence total we have atleast $(g - i) + i + 1 = g + 1$ non-gap points. This is a contradiction. □

Proposition 3.2. *If $m_1 = 2$, then $m_i = 2i$ and $m_i + m_{g-i} = 2g$, for all $0 < i < g$.*

PROOF. If $m_1 = 2$, then $2, 4, \cdots, 2g$ are g non-gap points. Hence these are all. □

Proposition 3.3. *If $m_1 > 2$, then there exists $0 < j < g$ such that $m_j + m_{g-j} > 2g$.*

PROOF. Let [ ] denote the greatest integer function. Then $m_1, 2m_1, \cdots, [2g/m_1]m_1$ are all gap numbers. Now $m_1 > 2$ implies $2/m_1 < 1$, which implies that $[2g/m_1] < g$. Hence there exists atleast one more non-gap number outside this sequence. Let l be the least such number. There exists $1 \leq r \leq [2g/m_1] < g$ such that $rm_1 < l < (r+1)m_1$. Then the first $r + 1$ gap numbers are $m_1 < m_2 = 2m_1 < \cdots < m_r = rm_1 < m_{r+1} = l$. Suppose the claim made in the Proposition is not true, then by Proposition 3.1, $m_i + m_{g-i} = 2g$, for all $0 < i < g$. Then the last $r + 1$ non-gap numbers except $m_g = 2g$ are

$$m_{g-1} = 2g - m_1 > \cdots > m_{g-r} = 2g - rm_1 > m_{g-(r+1)} = 2g - l \tag{25}$$

Note that even if $r + 1 = g$, the last number in the sequence is m_0, which we define to be zero. Now

$$m_1 + m_{g-(r+1)} = m_1 + 2g - l = 2g - (l - m_1) > 2g - rm_1 = m_{g-r}$$

Also $2g > 2g - (l - m_1) = m_1 + m_{g-(r+1)}$, hence $m_1 + m_{g-(r+1)}$ is a non-gap number greater than m_{g-r}. Therefore it must appear in the list (25). Hence $m_1 + m_{g-(r+1)} = 2g - km_1$ for some $0 < k < r$. But this implies $l = (k-1)m_1$, which is a contradiction. □

Corollary 3.4. *$\sum_{i=1}^{g-1} m_i \geq g(g-1)$ with equality if and only if $m_1 = 2$.*

PROOF. Proposition 3.1, tells us $m_i + m_{g-i} \geq 2g$ for $0 < i < g$. Summing over i, we get $2\sum_{i=1}^{g-1} m_i \geq 2g(g-1)$. Proposition 3.2 tells us that $m_1 = 2$ implies equality and Proposition 3.3 tells us that the above inequality is strict if $m_1 > 2$. □

Now we give an upper bound for Weierstass weight w_p.

Theorem 3.5. *Let X be a compact Riemann surface with genus $g \geq 2$. Then for all $p \in X$, $w_p \leq g(g-1)/2$. Equality occurs only for a point p whose sequence of non-gap numbers begin with 2.*

PROOF. If $1 = n_1 < \cdots < n_g < 2g$ is the seqiuence of gap numbers and $2 \leq m_1 < \cdots < m_g = 2g$ is the sequence of non-gap numbers then,

$$\begin{aligned} w_p &= \sum_{i=1}^{g}(n_i - i) = \sum_{i=1}^{g} n_i - \sum_{i=1}^{g} i \\ &= \sum_{i=1}^{2g} i - \sum_{i=1}^{g} m_i - \sum_{i=1}^{g} i = \sum_{i=g+1}^{2g-1} i - \sum_{i=1}^{g-1} m_i \\ &\leq \frac{3g(g-1)}{2} - g(g-1) \text{ (by Corollary 3.4)} \\ &= \frac{g(g-1)}{2} \end{aligned}$$

The inequality in the above expression becomes an equality if and only if $m_1 = 2$. □

Corollary 3.6. *Let W be the number of Weierstass points on a compact Riemann surface of genus $g \geq 2$. Then $2g + 2 \leq W \leq g^3 - g$.*

We have already discussed the proof of this corollary.

Let us examine the condition $m_1 = 2$. If this is true for some point p in a compact Riemann surface X, then it means that there exists a non constant meromorphic function f in $L(2p)$. Thus the only pole of f is at p. If the order of pole at p was 1, then f would have corresponded to a a holomorphic function of degree 1, that is, an isomorphism. But since we assume that genus $g \geq 2$, the order must be 2 at p. Thus f corresponds to a holomorphic function $F : X \to \mathbb{C}_\infty$ which is of degree 2. Hence X must be a hyperelliptic surface. Again suppose X is a hyperelliptic Riemann surface of genus $g \geq 2$ and $F : X \to \mathbb{C}_\infty$ be a degree 2 holomorphic map and call the corresponding meromorphic function f. Let p be a branch point of F. If $f(p) = \infty$, then f is a non-constant meromorphic function with $\mathrm{div}_\infty(f) = 2 \cdot p$ and the first non-gap number is $m_1 = 2$. Hence p is a Weierstass point. If $f(p) \neq \infty$, then the meromorphic function $1/(f - f(p))$ has a double pole at p and no other poles. So again the first non-gap number is 2. Therefore in any case p is a Weierstass point with first non-gap number $m_1 = 2$. Hence $w_p = g(g-1)/2$. Summing over all the branch points we see that the total weight of all the branch points is $(2g-2) \times \frac{g(g-1)}{2} = g^3 - g$. But this is equal to the total weight, so branch points are all Weierstass points.

Corollary 3.7. *The number of Weierstass points is always greater than $2g + 2$, unless the Riemann surface is a hyperelliptic one, in which case the number is equal to $2g + 2$.*

4. $g(X) \geq 2$ implies Aut(X) is finite

Apart from it's intrinsic interest, the main reason we made a detailed study of Weierstass points is the following fact.

Theorem 4.1. *Automorphisms of a compact Riemann surface permutes the Weierstass points.*

We will prove this shortly, but first let us ask a natural question. Suppose $F : X \to Y$ is a holomorphic map. Then we know $F^* : \mathcal{M}(Y) \to \mathcal{M}(X)$ is a $\mathbb{C}$-linear map. Now $\mathcal{M}(Y)$ is organized in partially ordered finte dimensional subspaces $L(D)$. Then how does $F^*(L(D))$ look like? We will show that $F^*(L(D)) \subset L(D')$, for a suitably chosen $D' \in \mathrm{Div}(X)$.

Definition 4.2. *Let $F : X \to Y$ be a holomorphic map. The define $F^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ as $F^*(D)(p) := \mathrm{mult}_p(F)D(F(p))$.*

Now we show that $F^*(L(D)) \subset L(F^*(D))$. Let $f \in L(D)$ and $p \in X$. Then choosing charts (U, ϕ) and (V, ψ) centered at p and $F(p)$ respectively and contemplating the product power series of the local holomorphic representations of F and f, we see that $\mathrm{ord}_p F^*(f) = \mathrm{ord}_p(f \circ F) = \mathrm{mult}_p(F) \cdot \mathrm{ord}_p(f) \geq -\mathrm{mult}_p(F) \cdot D(p) = F^*(D)(p)$. Note if F is an isomorphism, then $L(D) \cong L(F^*(D))$. Now we proceed to proof of Theorem 4.1.

PROOF. Suppose p is a Weierstass point. This is $G_p \neq \{1, \cdots, g\}$. This means that $\dim L(kp) > \dim L((k-1)p)$ for some $k \leq g$. But this is equivalent to saying that $\dim L(gp) \geq 2$. Since F is an automorphism, $L(gp) \cong L(F^*(gp)) = L(gF(p))$. Hence $\dim L(gF(p)) \geq 2$. Thus $F(p)$ is also a Weierstass point. □

Immediately we have a group homomorphism

$$\lambda : \mathrm{Aut}(X) \to S_{|W(X)|}, \tag{26}$$

where $W(X)$ is the finite set of Weierstass points of X. If we can prove that the kernel of this map is a finite subgroup, then $\mathrm{Aut}(X)$ must also be finite. An element in $\mathrm{Ker}(\lambda)$ fixes all the Weierstass points. We have an estimate for the number of Weierstass points. So we try to find an upper bound for number of fixed point of an automorphism.

Proposition 4.3. *Suppose F is a non-identity automorphism. Then F has atmost $2g+2$ fixed points.*

Proof. Suppose h is a non-constant meromorphic function. Then $h - F^*(h) = h - h \circ F$ is also a non-constant meromorphic function unless F is the identity function. Now each fixed point of F is a zero for $h - h \circ F$. Hence the fixed points are a subset of the zeros of a non-constant meromorphic function. Therefore there can be only finitely many fixed points.

Let p be point which is not fixed by F. n is a gap number means there does not exists a non-constant meromorphic function f such that $\mathrm{div}_\infty(f) = n \cdot p$. There are g many gap numbers. Hence there exists a non-constant meromorphic function with $\mathrm{div}_\infty(f) = r \cdot p$, for some $1 \leq r \leq g+1$. Thus f has a single pole p of order r and $F^*(f)$ has a single pole $F(p)$ of order r. Since $p \neq F(p)$, $\deg(\mathrm{div}_\infty(f - F^*(f))) = 2r \leq 2g+2$. Hence there can be atmost $2g+2$ zeros of $f - F^*(f)$ and atmost that many fixed point of F. □

Now by Corollary 3.7, X always has more than $2g+2$ Weierstass point, unless it is hyperelliptic. Hence for non-hyperelliptic compact Riemann surface, λ is an injection and hence the automorphism group is finite.

What are all non-trivial automorphisms of a hyperelliptic Riemann surface X, which fixes all the Weierstass points? We know one, the hyperelliptic involution. We will show that all other automorphisms of X will have strictly less than $2g+2$ fixed points, so that hyperelliptic involution is the only non-tivial element of $\mathrm{Ker}(\lambda)$. First we show:

Lemma 4.4. *Given any two meromorphic functions f and h of degree 2 (that is, the corresponding holomorphic maps to $\mathbb{C}_\infty$ are of degree 2) on a hyperelliptic Riemann surface of genus $g \geq 1$, they are related by $h = M \circ f$, where M is a mobius transformation.*

Proof. $\mathrm{div}_\infty(f)$ is a positive divisor of degree 2. By Corollary 2.3, $\dim(\mathrm{div}_\infty(f)) < \deg(\mathrm{div}_\infty(f)) + 1 = 3$. Also $f \in L(\mathrm{div}_\infty(f))$ is non-constant, hence $\dim(\mathrm{div}_\infty(f)) \geq 2$. Therefore $\dim(\mathrm{div}_\infty(f)) = 2$ and $\{1, f\}$ is a basis. We claim that it is enough to show that $\mathrm{div}_\infty(f) \sim \mathrm{div}_\infty(h)$. This will imply that $L(\mathrm{div}_\infty(f)) \cong L(\mathrm{div}_\infty(h))$ via multiplication by some meromorphic function e. Hence $\{e, ef\}$ will be a basis of $L(\mathrm{div}_\infty(f))$. Therefore there will exist $a, b, c, d \in \mathbb{C}$, such that, $h = aef + be$ and $1 = cef + de$. Dividing we have $h = (af+b)/(cf+d)$ and we will be done.

We had shown at the end of the last section that branch points of any degree two holomorphic function $T : X \to \mathbb{C}_\infty$ are all the Weierstass points. Hence the branch points of the maps corresponging to f and h are same. Let p be one of them. We will show that $\mathrm{div}_\infty(f) \sim 2p \sim \mathrm{div}_\infty(h)$. We will only show the first equivalence, the other will then follow. If $f(p) = \infty$, then we are done. Otherwise $2p = \mathrm{div}_\infty(1/(f - f(p)))$. Now $f^{-1}(q)$, for any $q \in \mathbb{C} \cup \{\infty\}$, can be thought of as a divisor. We have $f^{-1}(\infty) \sim f^{-1}(0)$ since their difference is $\mathrm{div}(f)$. Replacing f by $f - c$, we have $f^{-1}(\infty) \sim f^{-1}(c)$, for all $c \in \mathbb{C}$. Now if A is a Mobius transformation then $(A \circ f)^{-1}(\infty) = f^{-1}(A^{-1}(\infty)) \sim f^{-1}(\infty)$. Here $1/(f - f(p)) = A \circ f$, for some Mobius transformation A. Hence $2p = \mathrm{div}_\infty(1/(f - f(p))) \sim \mathrm{div}_\infty(f)$. □

Proposition 4.5. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$. Let $\phi \in \mathrm{Aut}(X)$. Assume $\phi \neq \sigma$, where σ is the hyperelliptic involution. Then ϕ has at most four fixed points.*

Proof. Fix a degree 2 meromorphic function f on X. Given any automorphism ϕ of X, $\phi^*(f) = f \circ \phi$ is also a degree 2 meromorphic function. Hence by above lemma we must have $\phi^*(f) = M(f)$, for some Mobius transformation M. This defines a function

$$\Lambda : \mathrm{Aut}(X) \to \{\text{Mobius transformations}\}$$

This is an anti-homomorphism. Suppose $\mathrm{id} \neq \psi \in \mathrm{Ker}(\Lambda)$. Then $f \circ \psi = f$. Now f has 2 preimages for each point in $\mathbb{C} \cup \{\infty\}$, except the $2g+2$ branch points of the corresponding holomorphic function to $\mathbb{C}_\infty$. On these $2g+2$ points ψ must be constant. By Proposition 4.3, these are all. Therefore ψ interchanges the two points in the fiber of every non-branch points. Hence ψ must be the hyperelliptic involution σ. Therefore $\mathrm{Ker}(\Lambda) = \{1, \sigma\}$. So let $\phi \neq \sigma$. Let p be a fixed point of ϕ. Then

$$f(p) = f(\phi(p)) = M(f(p)),$$

where $M \neq \mathrm{id}$. Then $f(p)$ is a fixed point of M. M being a Mobius transformation, can have atmost 2 fixed points. And each of them has atmost two preimages under f. Hence ϕ can have atmost 4 fixed points. □

Thus if X is a hyperelliptic surface the kernel of the map λ in the map (26) must consist of the identity function and the hyperelliptic involution. Therefore in hyperelliptic case also the automorphism group is finite. Summing up,

Theorem 4.6. *Let X be a compact Riemann surface of genus $g \geq 2$. Then $\mathrm{Aut}(X)$ is finite.*

5. Hurwitz Theorem

We have established that automorphism group of Riemann surfaces of genus $g \geq 2$ is finite. Can we give an estimate of it's cardinality depending on the genus g? This question is answered by the Hurwitz theorem. The idea is to consider the quotient space formed by identifying points in same orbit under action of elements of the automorphism group. This can be made into a Riemann surface and then Hurwitz formula applied on the quotient map will yeild information on cardinality of the automorphism group.

A reader familiar with properly discontinuous action of a group on manifold M will know how the quotient space in that case is given a differentiable structure. The condition of properly discontinuous action, that every point p has a neighbourhood U such that $U \cap g(U) = \emptyset$, for all $g \neq \mathrm{id}$, made it possible to "project" charts of M to form a chart of M/G. We cannot expect that in general case since there can be non-tivial stabilisers of a point p. But the holomorphic nature of the action in case of Riemann surface forces the number of such points to be finite. Let X be a compact Riemann surface of genus $g \geq 2$. We will denote $\mathrm{Aut}(X)$ by G. Let $p \in X$. Let $G_p := \{g \in G : g(p) = p\}$ be the stabiliser subgroup of p.

Proposition 5.1. *The set $\{p \in X : |G_p| > 1\}$ is discrete and since X is compact, is finite.*

PROOF. Suppose p is a limit point to the above set. Then there exists a sequence of distinct points $\{p_n\}_n$ with a non trivial stabiliser g_n for each p_n, such that $p_n \to p$. Now G is a finite subgroup. Therefore there is one g_m that stabilises a subsequence of $\{p_n\}_n$. This subsequence also converges to p. g_m is continuous, hence it stabilises p too. But that means g_m is a holomorphic map that fixes a set with a limit point. Hence g_m must be identity map. This is a contradiction. Therefore the set of points with non-trivial stabiliser subgroups, is discrete and hence finite. □

The next proposition clears the way for defining charts on the quotient space X/G.

Proposition 5.2. *Given any $p \in X$, there exists a neighbourhood U of p, such that*

(1) $g(U) = U$, for all $g \in G_p$
(2) $U \cap g(U) = \emptyset$, for all $g \notin G_p$,
(3) the natural map $\alpha : U/G_p \to X/G$, induced by sending a point in U to it's orbit, is a homeomorphism onto an open subset of X/G,
(4) no point of U other than p is fixed by any element of G_p.

PROOF. Let $G \setminus G_p = \{g_1, \cdots, g_n\}$. X is Hausdorff. Therefore for each $1 \leq i \leq n$, there exists open neighbourhoods U_i of p and V_i of $g_i(p)$, such that $U_i \cap V_i = \emptyset$. Then $W_i := U_i \cap g_i^{-1}(V_i)$ is a neighbourhood of p, for each i. Let $W := \cap_{i=1}^n W_i$. Then define $U := \cap_{g \in G_p} gW$. Then $g(U) = U$ for all $g \in G_p$. This proves (1).

To prove (2), note that $W_i \cap g_i W_i = (U_i \cap g^{-1}(V_i)) \cap g_i U_i \cap V_i \subset V_i \cap U_i = \emptyset$. Therefore $U \cap g_i U = (\cap_{g \in G_p} gW) \cap (\cap_{g \in G_p} g_i g W) = \cap_{g \in G_p} g(W \cap g_i W) = \emptyset$.

Let $\pi : X \to X/G$ be the quotient map. Restrict it to U, $\pi|_U : U \to X/G$. This induces a map $\alpha : U/G_p \to X/G$. To prove injectivity let $\alpha([x]) = \alpha([y])$. This implies $y = g(x)$, for some $g \in G$. If $g \in G \setminus G_p$, then $g(x) \notin U$, but $y \in U$. Therefore $g \in G_p$. Hence $[x] = [y]$. Therefore α is injective. U/G_p has quotient topology from $\phi : U \to U/G$. To prove continuity of α, it is enough to prove, by definition of quotient topology, that the composition $\alpha \circ \phi : U \to X/G$ is continuous. But $\alpha \circ \phi = \pi|_U$, and hence continuous. Now we wish to prove that α is open. Let V be an open set in U/G_p. We want to prove that $\alpha(V)$ is also open. Since X/G has quotient topology from $\pi : X \to X/G$, it is enough to prove that $\pi^{-1}(\alpha(V))$ is open in X. We claim that $\pi^{-1}(\alpha(V)) = \cup_{g \in G} g\phi^{-1}(V)$. Let $x \in \pi^{-1}(\alpha(V))$. Then $\pi(x) \in \alpha(V)$. This implies that there exists $y \in V$, such that $\alpha(y) = \pi(x)$. Now ϕ is surjective, therefore there exists $w \in \phi^{-1}(V)$, such that $\phi(w) = y$. Therefore $\alpha(y) = \alpha \circ \phi(w) = \pi(w)$. Hence $\pi(x) = \pi(w)$. Then there exists $g \in G$, such that $x = g(w)$. This implies, $x \in g\phi^{-1}(V)$.

Therefore $\pi^{-1}(\alpha(V)) \subset \cup_{g\in G} g\phi^{-1}(V)$. Again let $gx \in g\phi^{-1}(V)$, where $x \in \phi^{-1}(V)$. Then $\phi(x) \in V$. Applying α on both sides we have $\alpha \circ \phi(x) \in \alpha(V)$ or $\pi(x) \in \alpha(V)$ or $x \in \pi^{-1}(\alpha(V))$. This proves that $\cup_{g\in G} g\phi^{-1}(V) \subset \pi^{-1}(\alpha(V))$. Therefore $\pi^{-1}(\alpha(V)) = \cup_{g\in G} g\phi^{-1}(V)$. Thus $\pi^{-1}(\alpha(V))$ is open. Therefore α is an open map and we have proved (3).

(4) follows from discreteness of set of points with non tivial stabiliser groups. Shrink U if necessary. □

Now we get down to the business of defining charts on X/G_p. From now on unless otherwise mentioned U will denote the neighbourhood described in Proposition 5.2. We will also assume, by shrinking U, if necessary, that U lies within the domain of a chart with chart map ψ. We will define a chart map on U/G_p, then it can be transferred to X/G via the homeomorphism α. Suppose $|G_p| = m$. Note that $\phi : U \to U/G$ is exactly m to 1, except at p. So the idea is to construct a function from U to $\mathbb{C}$, which will take all the m points in a single orbit under G_p action to one point, so that the induced map from U/G_p is injective. Let z be the local coordinate in U. By abuse of notation we replace $\psi^{-1}(z)$ by z. Take the function $h(z) = \prod_{g\in G_p} g(z)$. Then h is holomorphic and G_p invariant, hence induces an injective continuous $\tilde{h} : U/G_p \to \mathbb{C}$. We will prove that $\tilde{h}$ is open too. Let W be an open set in U/G_p. Therefore $\phi^{-1}(W)$ is open in U. h being holomorphic, $h(\phi^{-1}(W))$ is open. But $h(\phi^{-1}(W)) = \tilde{h}(W)$. Hence $\tilde{h}$ is a homeomorphism and our construction of chart is complete. The chart map is $\tilde{h} \circ \alpha^{-1}$.

Now we have to check compatibility. Since points with non trivial stabiliser groups are discrete we may assume that no two charts constructed two such points intersect. Note that if G_p is trivial, then $h(z) = z$ and it follows that the chart map is just $\psi \circ \phi^{-1} \circ \alpha^{-1} = \psi \circ \pi|_U^{-1}$. Thus we may assume the two chart maps to be $\tilde{h} \circ \alpha_1^{-1}$ and $\pi|_{U_2}^{-1} \circ \psi_2)$, where $\alpha_1 \circ \phi_1 = \pi|_{U_1}$. Now $\tilde{h} \circ \alpha^{-1} \circ (\psi_2 \circ \pi|_{U_2}^{-1})^{-1} = \tilde{h} \circ \alpha^{-1} \circ \pi|_{U_2} \circ \psi_2^{-1} = \tilde{h} \circ \phi_1 \circ \psi_2^{-1} = h \circ \psi_2^{-1}$ which is holomorphic. Since bijective holomorphic maps are biholomorphic, we need not check the other side. Thus the charts are all compatible. Therefore X/G is a Riemann surface.

The following proposition follows from the construction.

Proposition 5.3. *The quotient map $\pi : X \to X/G$ is holomorphic of degree $|G|$ and $\mathrm{mult}_p(\pi) = |G_p|$ for any $p \in X$.*

Let $y \in X/G$ be a branch point of π. The points in the inverse image $\pi^{-1}(y) = \{x_1, \cdots, x_s\}$ are in the same orbit and hence have conjugate stabiliser subgroups. Let the cardinality of each stabiliser subgroup be r. We know that number of points in the orbit of x_1 is $|G/G_{x_1}|$. Hence $s = |G|/r$. This leads to the following lemma.

Lemma 5.4. *For every branch point y, there exists an integer $r \geq 2$, such that $\pi^{-1}(y)$ consists of exactly $|G|/r$ points and at each of these preimages, π has multiplicity r.*

Now applying the Hurwitz formula on π, we have the following Corollary.

Corollary 5.5. *Suppose there are k branch points $y_1, \cdots, y_k$ with π having multiplicity r_i at each of the $|G|/r_i$ points above y_i. Then*

$$\begin{aligned} 2g(X) - 2 &= |G|(2g(X/G) - 2) + \sum_{i=1}^{k} \frac{|G|}{r_i}(r_i - 1) \\ &= |G|[2g(X/G) - 2 + \sum_{i=1}^{k}(1 - \frac{1}{r_i})] \end{aligned}$$

We will denote the quantity $\sum_i (1 - 1/r_i)$ by R. We note that $1 - 1/r_i \geq 1/2$, for all i, and in particular if $R \neq 0$ then $R \geq 1/2$. Now we are ready to compute an upper bound for the cardinality of $\mathrm{Aut}(X)$.

Theorem 5.6. *[Hurwitz Theorem] Let X be a Riemann surface of genus $g \geq 2$. Then* $|\mathrm{Aut}(X)| \leq 84(g-1)$.

PROOF. We have from Corollary 5.5

$$2g(X) - 2 = |G|[2g(X/G) - 2 + R] \tag{27}$$

Case 1 $g(X/G) \geq 1$: Suppose $R = 0$. The LHS of (27) is strictly positive, hence so should be the RHS. Therefore $g(X/G) \geq 2$. This implies $|G| \leq g - 1$. If $R \neq 0$, then $R \geq 1/2$. Then we have $2g(X/G) - 2 + R \geq 1/2$, therefore $|G| \leq 4(g-1)$.

Case 2 $g(X/G) = 0$**:** In this case (27) reduces to $2g(X) - 2 = |G|[-2 + R]$. LHS is strictly positive, therefore we must have $R > 2$. But R is of the form $\sum_i (1 - 1/r_i)$, where r_i's are integers. Hence we can hope that the values of R are discrete, so that there exists a minimum value of the set $\{R : R > 2\}$, which is *greater* than R. Suppose there are k many r_i's. If k = 1 or 2, then $R < 2$. If $k > 4$ then $R \geq \sum_{i=1}^{k} (1 - 1/2) = k/2 > 2$. In this case the least value is $2\frac{1}{2}$. Consider $k = 4$. If $r_i = 2$ for all $1 \leq i \leq 4$, then $R = 2$. So the very next value $3 \times (1 - \frac{1}{2}) + (1 - \frac{1}{3})$, is the least in $k = 4$ case. This value is $2\frac{1}{6} < 2\frac{1}{2}$. Now consider the case $k = 3$. Let $2 \leq r_1 \leq r_2 \leq r_3$. Suppose $r_1 \geq 3$. If all three are equal to 3, then $R = 2$. This will not serve. The next value is attained when $r_1 = 3 = r_2$ and $r_3 = 4$, which gives $R = 2\frac{1}{12} > 2\frac{1}{6}$. Putting $r_1 = 2$, $r_2 = 3$ and $r_3 = 7$, we get $R = 2\frac{1}{42} < 2\frac{1}{12}$. We claim this is the least value of R greater than 2. At least we agree that the least value must occur in the case $k = 3$ and $r_1 = 2$. Suppose the least value is assumed for $2 = s_1 < s_2 \leq s_3$. Then we must have $(1/s_2 - 1/3) + (1/s_3 - 1/7) \geq 0$. If $s_2 = 3$, the condition $R > 2$, forces $s_3 = 7$. Otherwise $s_2 \geq 4$, and hence $(1/s_2 - 1/3) \leq -1/12$. This negetive term must be ccompensated by $(1/s_3 - 1/7)$. But $s_3 \geq s_2 \geq 4$, hence $(1/s_3 - 1/7) \leq 3/28 < 1/12$. Hence we must have $s_1 = 2, s_2 = 3$ and $s_3 = 7$. Thus $R > 2$ implies $R \geq 2\frac{1}{42}$. Therefore by $2g - 2 \geq |G|(-2 + 2\frac{1}{42})$ or $|G| \leq 84(g - 1)$. $\square$

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