THE TEMPERLEY-LIEB ALGEBRA

A Thesis submitted in partial fulfilment of the requirements for the award of the degree of

Master of Science

by

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BONAFIDE CERTIFICATE

Certified that this dissertation titled **The Temperley-Lieb Algebra** is a bonafide record of work of **Mr. S. Sundar** who carried out the project under my supervision.

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Preface

The main aim of this thesis is to determine the maximal $C^\star$ quotient of the Temperley-Lieb algebra $T_n(\tau)$.

In chapter 1, we define $T_n(\tau)$ for every $n \in \mathbb{N}$ and for every *non zero* complex number τ. The algebra $T_n(\tau)$ is defined as the universal $\mathbb{C}$ algebra generated by $1, e_1, e_2, \cdots e_{n-1}$ satisfying the following relation:

$$\begin{aligned} e_i^2 &= e_i \quad \text{ for } i \in \{1, 2, \cdots, n-1\} \\ e_i e_j &= e_j e_i \;\text{ if }\; |i-j| \geqq 2 \\ e_i e_j e_i &= \tau e_i \;\text{ if }\; |i-j| = 1 \end{aligned}$$

We prove that $T_n(\tau)$ is a $\star$ algebra by identifying $T_n(\tau)$ with the diagram algebra $D_n(\beta)$ when $\tau = \frac{1}{\beta^2}$.

In chapter 2, Jones- Wenzl idempotents are defined. Wenzl's theorem, which states that if $TL(\tau) = \cup_{k=1}^{\infty} T_k(\tau)$ admits a non-trivial $C^\star$ representation then $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4}sec^2(\frac{\pi}{n+1}) : n \geq 2\}$, is proved.

In chapter 3, we obtain $C^\star$ representations of $TL(\tau)$ when the parameter $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4}sec^2(\frac{\pi}{n+1}) : n \geq 2\}$. Jones' basic construction for inclusion $N \subset M$ of finite dimensional $C^\star$ algebras together with a faithful trace is explained. When the trace is Markov of modulus τ, we can repeat the Jones' basic construction and obtain a tower of finite dimensional C* algebras called the Jones tower and a sequence of projections e_n^J called the Jones projections and consequently a sequence of quotients $J_n(\tau)$ for $T_n(\tau)$.

In chapter 4, we obtain the maximal $C^\star$ quotient of $T_k(\tau)$. If $\tau \leq \frac{1}{4}$, the quotient map $\phi : T_k(\tau) \longrightarrow J_k(\tau)$ is $\star$ algebra isomorphism. When the parameter $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$, the map $\phi : T_k(\tau) \longrightarrow J_k(\tau)$ is an isomorphism for $1 \leq k \leq n-1$. For $k \geq n$, Let $\tilde{1} : T_k(\tau) \longrightarrow \mathbb{C}$ be the trivial map for which $\tilde{1}(e_i) = 0$. Then we prove that $(J_k(\tau) \oplus \mathbb{C}, \phi \oplus \tilde{1})$ is the maximal $C^\star$ quotient of $T_k(\tau)$ when $k \geq n$. Much of the material in this thesis can be found in [Jon].

Acknowledements

I thank V.S.Sunder for suggesting me this topic. The discussions that I had with him helped me in understanding the topic better. I also thank him for making me think and write mathematics better.

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SUNDAR

Contents

Chapter 1

The Temperley-Lieb Algebra

1.1 The Temperley-Lieb algebra $T_n(\tau)$

We consider only $\mathbb{C}$ algebras. Let τ be a nonzero complex number.

Definition 1. *For $n \geq 2$, let $T_n(\tau)$ be the $\mathbb{C}$ algebra generated by $1, e_1, e_2 \cdots e_{n-1}$ subject to the following relations :*

$$\begin{aligned} e_i^2 &= e_i \quad \textit{for } i \in \{1, 2, \cdots, n-1\} \\ e_i e_j &= e_j e_i \quad \textit{if} \quad |i-j| \geq 2 \\ e_i e_j e_i &= \tau e_i \quad \textit{if} \quad |i-j| = 1 \end{aligned}$$

$T_n(\tau)$ has the following universal property. Let A be a unital $\mathbb{C}$ algebra. Let $f_1, f_2, \cdots, f_{n-1} \in A$ be such that

$$\begin{aligned} f_i^2 &= f_i \quad \text{for } i \in \{1, 2, \cdots, n-1\} \\ f_i f_j &= f_j f_i \quad \text{if} \quad |i-j| \geq 2 \\ f_i f_j f_i &= \tau f_i \quad \text{if} \quad |i-j| = 1 \end{aligned}$$

Then there exists a unique algebra homomorphism $\phi : T_n(\tau) \to A$ such that $\phi(e_i) = f_i$ and $\phi(1) = 1_A$ where 1_A denotes the multiplicative identity of A.

We now proceed to prove that $T_n(\tau)$ is finite dimensional. By a word on $1, e_1, e_2, \cdots, e_{n-1}$ we mean a product $e_{i_1} e_{i_2} \cdots e_{i_p}$. By convention empty product denotes 1. Note that words on $1, e_1, e_2, \cdots, e_{n-1}$ span $T_n(\tau)$.

Lemma 1. *Let w be a word on $1, e_1, e_2 \cdots, e_{n-1}$. Then*

$$w = \tau^k (e_{i_1} e_{i_1 - 1} \cdots e_{j_1})(e_{i_2} e_{i_2 - 1} \cdots e_{j_2}) \cdots (e_{i_p} e_{i_p - 1} \cdots e_{j_p})$$

where $k \in \mathbb{N} \cup \{0\}$ *and*

$$1 \le i_1 < i_2 < \cdots i_p \le n-1$$
$$1 \le j_1 < j_2 < \cdots j_p \le n-1$$
$$i_1 \ge j_1, i_2 \ge j_2, \cdots, i_p \ge j_p$$

*Proof.*The proof can be found in [Jon]. We prove this by induction on n. Clearly the result is true for $n = 2$. Now assume that any word in $1, e_1, e_2, \cdots, e_{n-1}$ is of the required form. Let w be a word in $1, e_1, e_2, \cdots, e_n$. If w does not contain e_n then we are done. So suppose that w contains e_n.

Assertion. $w = \tau^k w_1 e_n w_2$ where w_1, w_2 are words in $1, e_1, e_2, \cdots, e_{n-1}$.

w has the form $v_1 e_n v e_n v_2$ where v_1, v_2 are words in $1, e_1, e_2, \cdots, e_n$ and v is a word in $1, e_1, e_2, \cdots, e_{n-1}$.
If v does not contain e_{n-1} then e_n commutes with v and hence $w = v_1 v e_n v_2$.
If v contains e_{n-1} then by induction hypothesis $v = \tau^r u_1 e_{n-1} u_2$ where u_1, u_2 are words in $1, e_1, e_2, \cdots, e_{n-2}$. Now

$$w = \tau^r v_1 u_1 e_n e_{n-1} e_n u_2 v_2$$
$$w = \tau^{r+1} v_1 u_1 e_n u_2 v_2$$

In any case w is τ^l multiple of a word which has one e_n less. Repeating this process proves the assertion.

Hence $w = \tau^k w_1 e_n w_2$ where w_1, w_2 are words in $1, e_1, e_2, \cdots, e_{n-1}$. By induction hypothesis

$$w_2 = \tau^l v_2 (e_{n-1} e_{n-2} \cdots, e_{j_p})$$

where v_2 is a word in $1, e_1, e_2, \cdots, e_{n-2}$. (The product $(e_{n-1} e_{n-2} \cdots e_{j_p})$ could be empty). Hence

$$w = \tau^s w_1 v_2 (e_n e_{n-1} \cdots e_{j_p})$$

where $w_1 v_2$ is a word in $1, e_1, e_2, \cdots, e_{n-1}$

Hence by induction hypothesis,

$$w = \tau^k (e_{i_1} e_{i_1 - 1} \cdots e_{j_1})(e_{i_2} e_{i_2 - 1} \cdots e_{j_2}) \cdots (e_{i_p} e_{i_p - 1} \cdots e_{j_p})$$

where $k \in \mathbb{N} \cup \{0\}$ and

$$1 \le i_1 < i_2 < \cdots i_p \le n-1$$
$$i_1 \ge j_1, i_2 \ge j_2, \cdots, i_p \ge j_p$$

Hence we have written w in the form needed with $i's$ increasing. Now consider such an expression which has the least length. Then we claim that $j's$ are also increasing. Let

$$w = \tau^k(e_{i_1}e_{i_1-1}\cdots e_{j_1})(e_{i_2}e_{i_2-1}\cdots e_{j_2})\cdots(e_{i_p}e_{i_p-1}\cdots e_{j_p})$$

be such an expression. Suppose $j_1 \geq j_2$. Then

$$\begin{aligned}
w &= \tau^k(e_{i_1}e_{i_1-1}\cdots e_{j_1})(e_{i_2}e_{i_2-1}\cdots e_{j_2})\cdots(e_{i_p}e_{i_p-1}\cdots e_{j_p})\\
w &= \tau^k(e_{i_1}e_{i_1-1}\cdots e_{j_1+1})(e_{i_2}\cdots e_{j_1}e_{j_1+1}e_{j_1}\cdots e_{j_2})\cdots(e_{i_p}e_{i_p-1}\cdots e_{j_p})\\
w &= \tau^{k+1}(e_{i_1}e_{i_1-1}\cdots e_{j_2})(e_{i_2}e_{i_2-1}\cdots e_{j_1+2})\cdots(e_{i_p}e_{i_p-1}\cdots e_{j_p})
\end{aligned}$$

which has length decreased by one which is a contradiction. Hence $j_1 < j_2$. Similarly $j_r < j_{r+1}$. This completes the proof. □

Now we consider the following combinatorial problem. Consider $\mathbb{Z}^2 \subset \mathbb{R}^2$. Consider paths on $\mathbb{Z}^2$. The only allowed moves are either up or right i.e. from (a, b) one can go to either $(a+1, b)$ or $(a, b+1)$.

Proposition 1. *The number of paths from $(0,0)$ to (n,n) where $n \in \mathbb{N}$ which lie in the region $y \leq x$ is $\frac{1}{n+1}\binom{2n}{n}$. Let $p_n = \frac{1}{n+1}\binom{2n}{n}$. Then p_n satisfy the following recurrence*

$$\begin{aligned}
p_1 &= 1\\
p_n &= \sum_{i=1}^{n} p_{i-1}p_{n-i}, \text{for } n \geq 2.
\end{aligned}$$

For a proof,we refer to [GHJ]. □

The relevance of proposition 1 in our context is as follows:
Given $(i_1, i_2, \cdots, i_p)$ and $(j_1, j_2, \cdots, j_p)$ such that

$$1 \leq i_1 < i_2 < \cdots i_p \leq n-1,\ 1 \leq j_1 < j_2 < \cdots j_p \leq n-1,\ i_1 \geq j_1, i_2 \geq j_2, \cdots, i_p \geq j_p$$

one can associate the path from $(0,0)$ to (n,n) given by

$$(0,0) \to (i_1, 0) \to (i_1, j_1) \to (i_2, j_1) \to \cdots (i_p, j_p) \to (n, j_p) \to (n, n)$$

This is clearly a bijection from the set of paths from $(0,0)$ to (n,n) to the set of ordered pairs $((i_1, i_2, \cdots, i_p), (j_1, j_2, \cdots, j_p))$ which satisfies the following condition.

$$1 \leq i_1 < i_2 < \cdots i_p \leq n-1,\ 1 \leq j_1 < j_2 < \cdots j_p \leq n-1,\ i_1 \geq j_1, i_2 \geq j_2, \cdots, i_p \geq j_p$$

Hence we get an onto map from the set of paths from $(0,0)$ to (n,n) to

$\{(e_{i_1}e_{i_1-1}\cdots e_{j_1})(e_{i_2}e_{i_2-1}\cdots e_{j_2})\cdots(e_{i_p}e_{i_p-1}\cdots e_{j_p}):$
$1\leq i_1<i_2<\cdots i_p\leq n-1; 1\leq j_1<j_2<\cdots j_p\leq n-1; i_1\geq j_1, i_2\geq j_2,\cdots,i_p\geq j_p\}$

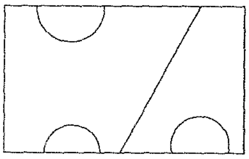
which spans $T_n(\tau)$ by Lemma 1. Hence we have proved the following result.

Proposition 2. *The algebra $T_n(\tau)$ is finite dimensional and it's dimension is atmost $\frac{1}{n+1}\binom{2n}{n}$.*

1.2 Diagram algebra $D_n(\beta)$

Fix a non-zero complex number β. Let m,n be nonegative integers such that $m-n$ is even. By an (m,n) **Kauffman** diagram we mean a rectangle in the plane with m points on the top and n points on the bottom and $\frac{n+m}{2}$ curves which connect pairs of points such that the curves do not intersect.

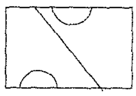
A $(3,5)$ diagram is shown below

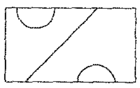


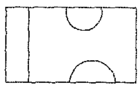
Let a be an (m,n) diagram and b be an (n,p) diagram. Let $b\odot a$ denote the (m,p) diagram obtained by placing a on the top and b on the bottom and removing the loops. Define

$$ba=\beta^r b\odot a$$

where r denotes the number of loops removed.
For example,

$a =$ 

$b =$ 

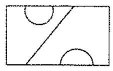
$ba = \beta$ 

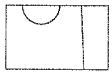
Let $Hom(m,n)$ denote the $\mathbb{C}$ vector space with (m,n) Kauffman diagrams as basis. The 'multiplication' that we have defined on diagrams extends to a bilinear map

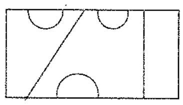
$$Hom(m,n) \times Hom(n,p) \rightarrow Hom(m,p)$$

which is associative.

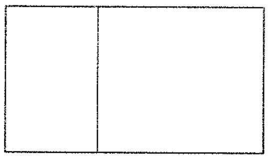
For a an (m,n) diagram and b a (p,q) diagram, $a \otimes b$ denote the $(m+p, n+q)$ diagram obtained by horizontal juxtaposition.
For example,

$a =$ 

$b =$ 

$a \otimes b =$ 

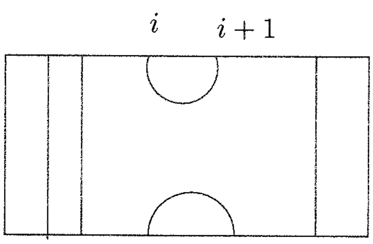
Let $1 \in Hom(1,1)$ denote the $(1,1)$ diagram shown below:

$1 =$ 

Let $1_n = 1 \otimes 1 \otimes 1 \cdots \otimes 1$, the (n,n) diagram with all strands coming vertically down.

Define $D_n(\beta) = Hom(n,n)$. Then $D_n(\beta)$ is a unital $\mathbb{C}$ algebra with 1_n as the multiplicative identity. The map $a \rightarrow a \otimes 1$ is an embedding of $D_n(\beta)$ into $D_{n+1}(\beta)$. With this embedding in mind, we write $D_n(\beta) \subset D_{n+1}(\beta)$.

Let E_i denote the following diagram in $D_n(\beta)$



Then we have the following relations:

$$\begin{aligned} E_i^2 &= \beta E_i \quad \text{for } i \in 1, 2, \cdots, n-1 \\ E_i E_j &= E_j E_i \quad if\ |i-j| \geqq 2 \\ E_i E_j E_i &= E_i \quad if\ |i-j| = 1 \end{aligned}$$

Let $e_i^D = \frac{1}{\beta}E_i$.
Then we have the following relations:

$$\begin{aligned} (e_i^D)^2 &= (e_i^D) \quad \text{for } i \in 1, 2, \cdots, n-1 \\ e_i^D e_j^D &= e_j^D e_i^D \quad if \ |i-j| \geqq 2 \\ e_i^D e_j^D e_i^D &= \frac{1}{\beta^2} e_i^D \quad if \ |i-j| = 1 \end{aligned}$$

For $0 \neq \tau \in \mathbb{C}$, a nonzero complex number, let β be such that $\beta^2 = \frac{1}{\tau}$. Then by the universal property of $T_n(\tau)$, there exists a unique unital homomorphism $\phi : T_n(\tau) \to D_n(\beta)$ such that $\phi(e_i) = e_i^D$. We now proceed to prove that ϕ is an isomorphism.

Lemma 2. *The dimension of $D_n(\beta)$ is $\frac{1}{n+1}\binom{2n}{n}$.*

Proof. Let p_n denote the number of (n, n) Kauffman diagrams. Think of an (n, n) Kauffman diagram as a disk with $2n$ points on the boundary with n curves connecting pairs of points without any intersection. Then we have the following recurrence relation

$$\begin{aligned} p_0 &= p_1 = 1 \\ p_n &= \sum_{i=1}^{n} p_{i-1}p_{n-i}, \text{for } n \geq 2. \end{aligned}$$

Hence, by proposition 1, $p_n = \frac{1}{n+1}\binom{2n}{n}$. □

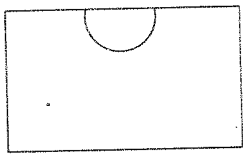
Lemma 3. $\{1, E_i : i = 1, 2, \cdots, n-1\}$ *generate the algebra* $D_n(\beta)$

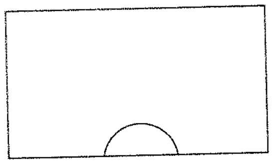
Proof. We prove this result by induction on n. If $n = 2$ the result is clear. Let a be an (n, n) Kauffman diagram. If that a has a strand that comes straight down then $a = b \otimes 1 \otimes c$ with $b \in D_r(\beta)$ and $c \in D_s(\beta)$ with $r, s < n$. Hence by induction hypothesis a can be written as a scalar multiple of E_i' s and we are done. Now we consider two cases.

Case 1. a has a through string i.e a string which joins a top point with a bottom point. Let us call a strand that comes vertically down a vertical string. Pick the rightmost through string. Let $\nu(a)$ be the number of vertices to the right of the rightmost through string of a(inclusive of the vertices that the rightmost through string joins).
We prove that a can be written as a scalar multiple of a product of $E_i's$ by induction on $\nu(a)$. If $\nu(a) = 2$ then the rightmost through string is vertical and we are through. Assume that it slants from right to left. Then

$a = b \otimes 1 \otimes c \otimes d$ with $b \in Hom(l,k)$, $c \in Hom(0,2r)$, $d \in Hom(t,s)$ for some non negative integers l,k,r,s,t with $r > 0$.

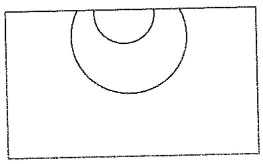
Let $\cup \in Hom(2,0)$ and $\cap \in Hom(0,2)$ be the following diagrams.





Let $\cup^r = \cup \otimes \cup \otimes \cdots \otimes \cup$ (r *times*). Similarly $\cap^r$ is defined. Note that $1 \otimes c = (1 \otimes \cup^r \otimes c)(\cap^r \otimes 1)$. Let $\bar{b} = 1_k \otimes 1 \otimes \cup^r \otimes c \otimes 1_s$ and $\bar{c} = b \otimes \cap^r \otimes 1 \otimes d$. Then $a = \bar{b}\bar{c}$ where $\bar{b}$ has a vertical string and $\nu(\bar{c}) < \nu(a)$. Hence by induction a can be written as a scalar multiple of a product of $E_i' s$. The proof is similar when the rightmost through string slants from left to right.

Case 2. a has no through strings. By a concentric loop we mean a Kauffman diagram which is either $\cup^r \circ (1 \otimes \alpha \otimes \cap^{r-1} \otimes 1)$ where α is a $(2r-2, 0)$ Kauffman diagram $(r \geq 2)$ or $(1 \otimes \gamma \otimes \cup^{2s-2} \otimes 1) \circ \cap^s$ where γ is a $(0, 2s-2)$ Kauffman diagram $(s \geq 2)$. An example of a concetric loop is given below:



If a does not have a concentric loop, then $a = E_1 E_3 \cdots$. Hence assume that a has concentric loops. Then $a = b \otimes c \otimes d$ where c is a concetric loop in $Hom(2k+2, 0)$ (assuming c is on top) and where $b \in Hom(r,s)$ and $d \in Hom(p,q)$ for some nonegative integers p,q,r,s,k with $k > 0$. Then $c = \cup^{k+1}(1 \otimes a \otimes \cap^k) \otimes 1)$. Let $\bar{c} = 1_r \otimes 1 \otimes a \otimes \cap^k \otimes 1 \otimes 1_p$. Let $\bar{b} = b \otimes \cup^{k+1} \otimes d$. Then $a = \bar{b}\bar{c}$ where both $\bar{b}$, $\bar{c}$ has one concentric loop less than that of a. Therefore, by induction on the number of concetric loops that a has, it follows that a can be written as a product of diagrams which have no concentric loop. Hence a is a product of $E_i's$. This completes the proof. □

Theorem 1. *Let β be a nonzero complex number. Let $\tau = \frac{1}{\beta^2}$. Then the unique unital algebra homomorphism $\phi : T_n(\tau) \to D_n(\beta)$ such that $\phi(e_i) = e_i^D$ is an isomorphism.*

Proof. By Lemma 3, ϕ is onto. By rank-nullity theorem,

$$rank(\phi) + nullity(\phi) = dim\ T_n(\tau) \leq \frac{1}{n+1}\binom{2n}{n}$$
$$\frac{1}{n+1}\binom{2n}{n} + nullity(\phi) \leq \frac{1}{n+1}\binom{2n}{n}$$

Hence nullity(ϕ) = 0. Thus ϕ is one-one. Therefore ϕ is an isomorphism. □

From now on we will identify $T_n(\tau)$ with $D_n(\beta)$ when $\tau = \frac{1}{\beta^2}$ and e_i with e_i^D. Note that the natural map $i : T_n(\tau) \to T_{n+1}(\tau)$ is injective since $\phi(ia) = \phi(a) \otimes 1$ for $a \in T_n(\tau)$.

1.3 Trace and Conditional expectation on $D_n(\beta)$

Definition 2. *Let $N \subset M$ be unital $\mathbb{C}$ algebras such that $1_N = 1_M$. A linear map $E : M \to N$ is said to be a conditional expectation if*

1. *$E(nm) = nE(m)$ and $E(mn) = E(m)n$ $\forall n \in N, m \in M$*
2. *$E(n) = n$ $\forall n \in N$*

Now we describe a conditional expectation $\epsilon_n : D_{n+1}(\beta) \to D_n(\beta)$ as follows: Let $\tilde{\epsilon_n} : D_{n+1}(\beta) \to D_n(\beta)$ be defined by $\tilde{\epsilon_n}(a) = (1_n \otimes \cup)(a \otimes 1)(1_n \otimes \cap)$. If a is an $(n+1, n+1)$ diagram, then $\tilde{\epsilon_n}(a)$ is obtained by just closing up the last strand. Hence if $a \in D_n(\beta)$ then $\tilde{\epsilon_n}(a) = \beta a$. Let $\epsilon_n(a) = \frac{1}{\beta}\tilde{\epsilon_n}(a)$ for $a \in D_n(\beta)$. Then ϵ_n is a conditional expectation.

Definition 3. *Let M be a unital $\mathbb{C}$ algebra. Let $\rho : M \to \mathbb{C}$ be linear. Then ρ is said to be a trace if $\rho(ab) = \rho(ba) \forall a, b \in M$. The functional ρ is said to be unital if $\rho(1) = 1$.*

Let $tr_n : D_n(\beta) \to \mathbb{C}$ be defined by $tr_n(a) = (\epsilon_1\epsilon_2 \cdots \epsilon_{n-1})(a)$. Note that $tr_n(a) = tr_{n+1}(a)$ if $a \in D_n(\beta)$. Hence we can and will denote tr_n by tr. If a is a diagram, let $c(a)$ be the number of loops one gets when one closes all the strands. Then $tr(a) = \beta^{c(a)-n}$

$tr : D_n(\beta) \to \mathbb{C}$ is a unital trace and satisfy the following properties:

1. $tr(x) = tr(\epsilon_n(x))$ $\forall\ x \in D_{n+1}(\beta)$.
2. $e_n x e_n = \epsilon_{n-1}(x)e_n$ $\forall\ x \in D_n(\beta)$.
3. $tr(e_i) = \tau$ where $\tau = \frac{1}{\beta^2}$.

1.4 $\star$ structure on $D_n(\beta)$

Definition 4. *Let M be a $\mathbb{C}$ algebra. A $\star$ structure on M is a function $\star : M \to M$ (We write $\star(a) = a^\star$) such that the following holds*

1. $(a+b)^\star = a^\star + b^\star \ \forall \ a, b \in M$
2. $(\alpha a)^\star = \bar{\alpha} a^\star \ \forall \ a \in M, \alpha \in \mathbb{C}$
3. $(ab)^\star = b^\star a^\star \ \forall \ a, b \in M$
4. $(a^\star)^\star = a \ \forall \ a \in M$

A $\star$ algebra is a $\mathbb{C}$ algebra together with a $\star$ structure.

Now we make $D_n(\beta)$ a $\star$ algebra. The $\star$ structure is defined on the level of diagrams (and then extends conjugate linearly) as follows:
For a diagram a, $a^\star$ denotes the diagram obtained by reflecting along the horizontal middle line. Then $E_i^\star = E_i$. If β is real, then $(e_i^D)^\star = e_i^D$. Thus for $\tau > 0$, $T_n(\tau)$ is a $\star$ algebra with e_i selfadjoint.

Chapter 2

C* representations of $TL(\tau)$

In this chapter we will prove Wenzl's result. It characterises the values of τ for which $TL(\tau)$ admits a nontrivial $C^\star$ represntation.

Definition 5. *Let M be a $\star$ algebra. By a $C^\star$ representation of M we mean an algebra homomorphism $\pi : M \to A$ where A is a $C^\star$ algebras such that $\pi(a^\star) = (\pi(a))^\star$.*

By a **non-trivial reprsentation** of $T_n(\tau)$ we mean a $C^\star$ representation π such that $\pi(e_i) \neq 0$ for some $i \in \{1, 2, \cdots, n-1\}$.

First we define **Jones-Wenzl idempotents** in $T_n(\tau)$. See [Wen].

Define a sequence of polynomials recursively by

$$P_0(\lambda) = 1 = P_1(\lambda)$$
$$P_k(\lambda) = P_{k-1}(\lambda) - \lambda P_{k-2}(\lambda), \text{for } k \geq 2$$

The basic properties of $P_k(\lambda)$ are summarised in the following proposition.

Proposition 3. *Let k be a non-negative integer and let $m = [\frac{k}{2}]$. Then*

1. *The polynomial P_k is of degree m. It's leading coefficient is $(-1)^m$ if $k = 2m$ and $(-1)^m(m+1)$ if $k = 2m+1$.*

2. *The polynomial P_k has m distinct roots given by $\{\frac{1}{4}\sec^2(\frac{\pi j}{k+1}) : j = 1, 2, \cdots, m\}$.*

3. *Assume $k \geq 1$. Let $\lambda \in \mathbb{R}$ be such that $\frac{1}{4}\sec^2(\frac{\pi}{k+2}) < \lambda < \frac{1}{4}\sec^2(\frac{\pi}{k+1})$. Then $P_i(\lambda) > 0$ for $i \in \{1, 2, \cdots, k\}$ and $P_{k+1}(\lambda) < 0$*

Proof. For a proof, we refer to [GHJ]. □

Let $TL(\tau) = \bigcup_n T_n(\tau)$. Then $TL(\tau)$ is a $\star$ algebra generated by $1, e_1, e_2, \ldots$. When $\tau > 0$, e_i's are self adjoint.

Proposition 4. *Let τ be a nonzero complex number such that $P_k(\tau) \neq 0$ for $k = 1, 2, \cdots, n$. Define f_k in $TL(\tau)$ recursively as follows.*

$$f_0 = 1 = f_1$$
$$f_{k+1} = f_k - \frac{P_{k-1}(\tau)}{P_k(\tau)} f_k e_k f_k, \ 1 \leq k \leq n.$$

Then,

1. $f_k \in T_k(\tau)$ *for* $1 \leq k \leq n+1$.
2. $1 - f_k$ *is in the algebra generated by* $\{e_1, e_2, \cdots, e_{k-1}\}$ *for* $2 \leq k \leq n+1$.
3. $(e_k f_k)^2 = \frac{P_k(\tau)}{P_{k-1}(\tau)} e_k f_k$, $(f_k e_k)^2 = \frac{P_k(\tau)}{P_{k-1}(\tau)} f_k e_k$ *for* $1 \leq k \leq n+1$.
4. f_k *is an idempotent for* $1 \leq k \leq n+1$.
5. $f_k e_i = 0$, $e_i f_k = 0$ *if* $i \leq k-1$ *where* $1 \leq k \leq n+1$
6. $tr(f_k) = P_k(\tau)$ *for* $1 \leq k \leq n+1$.

When $\tau > 0$, f_k *is selfadjoint.*

Proof. This is due to Wenzl and we include a proof here for completeness. The proof is by induction on k. $1, 2 \cdots, 6$ are clearly true for $k \leq 2$. Now assume that $1, 2 \cdots, 6$ are true for $1 \leq k \leq l$ where $l \geq 2$. We will show the result is true for $k = l+1$.

Since f_l is in the algebra generated by $1, e_1, e_2, \cdots, e_{l-1}$ by definition it follows that f_{l+1} is in the algebra generated by $1, e_1, e_2, \cdots, e_l$. Hence $f_{l+1} \in T_{l+1}(\tau)$. Since $1 - f_l$ is in the algebra genrated by $e_1, e_2, \cdots, e_{l-1}$, by definition, it follows that $1 - f_{l+1}$ is in the algebra generated by $e_1, e_2, \cdots, e_l$.

Now note that $f_{l+1} f_l = f_{l+1}$ and $f_l f_{l+1} = f_{l+1}$ since f_l is an idempotent. Since $f_l \in T_l(\tau)$, e_{l+1} commutes with f_l. Hence we have,

$$\begin{aligned} e_{l+1} f_{l+1} e_{l+1} &= e_{l+1} f_l - \frac{P_{l-1}(\tau)}{P_l(\tau)} f_l e_{l+1} e_l e_{l+1} f_l \\ &= \frac{P_{l+1}(\tau)}{P_l(\tau)} e_{l+1} f_l \end{aligned}$$

Hence $(e_{l+1} f_{l+1})^2 = \frac{P_{l+1}(\tau)}{P_l(\tau)} e_{l+1} f_{l+1}$.

The proof that $(f_{l+1}e_{l+1})^2 = \frac{P_{l+1}(\tau)}{P_l(\tau)} f_{l+1}e_{l+1}$ is similar. Now

$$\begin{aligned} f_{l+1}^2 &= f_l^2 - 2\frac{P_{l-1}(\tau)}{P_l(\tau)} f_l e_l f_l + \left(\tfrac{P_{l-1}(\tau)}{P_l(\tau)}\right)^2 f_l e_l f_l e_l f_l \\ &= f_l^2 - 2\frac{P_{l-1}(\tau)}{P_l(\tau)} f_l e_l f_l + \left(\tfrac{P_{l-1}(\tau)}{P_l(\tau)}\right)^2 \frac{P_l(\tau)}{P_{l-1}(\tau)} f_l e_l f_l \\ &= f_l - \frac{P_{l-1}(\tau)}{P_l(\tau)} f_l e_l f_l = f_{l+1} \end{aligned}$$

Hence f_{l+1} is an idempotent. Since $f_{l+1}e_i = f_{l+1}f_l e_i$, it follows that $f_{l+1}e_i = 0$ if $i \leq l-1$. Now $f_{l+1}e_l = f_l e_l - \frac{P_{l-1}(\tau)}{P_l(\tau)}(f_l e_l)^2$. But $(f_l e_l)^2 = \frac{P_l(\tau)}{P_{l-1}(\tau)} f_l e_l$. Hence $f_{l+1}e_l = 0$. Hence $f_{l+1}e_i = 0$ for $i \leq l$. Similarly $e_i f_{l+1} = 0$. Now

$$\begin{aligned} tr(f_{l+1}) &= tr(f_l) - \frac{P_{l-1}(\tau)}{P_l(\tau)} tr(f_l e_l f_l) \\ &= tr(f_l) - \frac{P_{l-1}(\tau)}{P_l(\tau)} tr(\epsilon_l(f_l e_l f_l)) \\ &= tr(f_l) - \frac{P_{l-1}(\tau)}{P_l(\tau)} tr(f_l \epsilon_l(e_l) f_l) \\ &= tr(f_l) - \frac{P_{l-1}(\tau)}{P_l(\tau)} tr(\tau f_l) \\ &= P_l(\tau) - \tau P_{l-1}(\tau) = P_{l+1}(\tau) \end{aligned}$$

If $\tau > 0$ then $P_k(\tau)$ is real. Hence by induction it follows that $f_k's$ are self-adjoint. □

The idempotents described in the previous proposition are called **Jones-Wenzl idempotents**.

Let τ be positive. The following result due to Wenzl restricts the values of τ for which $TL(\tau)$ has a nontrivial $C^\star$ representation. The proof can be found in [Wen]. We include the proof for completeness.

Theorem[Wenzl]. *Let τ be a positive real number. If $TL(\tau)$ has a non-trivial $C^\star$ representation, then $\tau \leq \frac{1}{4}$ or $\tau = \frac{1}{4}\sec^2(\frac{\pi}{n+1})$ for some $n \geq 2$.*

We begin the proof with the following lemma.

Lemma 4. *Let τ be such that $\frac{1}{4}sec^2(\frac{\pi}{n+2}) < \tau < \frac{1}{4}sec^2(\frac{\pi}{n+1})$ for some $n \in \mathbb{N}$, with $n \geq 2$. Suppose $\pi : TL(\tau) \to B(H)$ be a $\star$ homomorphism, where H is a Hilbert space. Let e_i^T denote the idempotents in $TL(\tau)$. Then the Jones-Wenzl idempotents f_k^T's are defined for $k = 1, 2, \cdots n+2$. Suppose $f_k = \pi(f_k^T)$ for $k \leq n+2$. Then*

(1) $1 - f_k = e_1 \vee e_2 \vee \cdots e_{k-1}$ *for* $k \leq n+2$.

(2) $e_{n+1} f_{n+1} = 0$.

(3) e_{n+1} *is orthogonal to* f_n.

Proof. Note that $P_k(\tau) > 0$ for $k = 1, 2, \cdots n$ and $P_{n+1}(\tau) < 0$. Hence the Jones-Wenzl idempotents are defined for $k = 1, 2, \cdots n+2$.

By proposition 4, it follows that $f_k e_i = 0$ for $i \leq k-1$. Hence we have $e_1 \vee e_2 \vee \cdots \vee e_{k-1} \leq 1 - f_k$. Since $1 - f_k$ is in the algebra generated by $e_1, e_2, \cdots, e_{k-1}$, it follows that $1 - f_k \leq e_1 \vee e_2 \vee \cdots e_{k-1}$. This proves (1).

Observe that $e_{n+1} f_{n+1} e_{n+1} = \frac{P_{n+1}(\tau)}{P_n(\tau)} e_{n+1} f_n$. But $e_{n+1} f_{n+1} e_{n+1}$ is positive and $e_{n+1} f_n$ is a projection. Since $P_{n+1}(\tau) < 0$, it follows that $e_{n+1} f_n = 0$ and $(f_{n+1} e_{n+1})^\star f_{n+1} e_{n+1} = 0$. Hence $f_{n+1} e_{n+1} = 0$ and e_{n+1} is orthogonal to f_n. By taking adjoints, we get $e_{n+1} f_{n+1} = 0$. This proves (2) and (3). □

Proposition 5. *Let H be a Hilbert space. Suppose $e_1, e_2, \cdots$ is a sequence of non-zero projections in $B(H)$ satisfying the following relation :*

$$\begin{aligned} e_i^2 &= e_i = e_i^\star \\ e_i e_j &= e_j e_i = 0 \quad && \text{if } |i-j| \geq 2 \\ e_i e_j e_i &= \tau e_i && \text{if } |i-j| = 1 \end{aligned}$$

Then $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4} sec^2(\frac{\pi}{n+1}) : \ n \geq 2\}$.

Proof. There exists a nontrivial $C^\star$ representation of $TL(\tau)$ say π which is unital and for which $\pi(e_i^T) = e_i$ where e_i^T denote the idempotents in $TL(\tau)$. By taking norms on the third relation, it follows that $\tau \leq 1$. Suppose that τ is not in the set given in the proposition. Then there exists $n \geq 2$ such that $\frac{1}{4} sec^2(\frac{\pi}{n+2}) < \tau < \frac{1}{4} sec^2(\frac{\pi}{n+1})$. Then $P_k(\tau) > 0$ for $k = 1, 2, \cdots n$ but $P_{n+1}(\tau) < 0$. Hence, the Jones Wenzl idempotents f_k^T's are defined for $k = 1, 2, \cdots n+2$. Let $f_k = \pi(f_k^T)$ for $k \leq n+2$.

From lemma 4, it follows that e_{n+1} is orthogonal to f_n. But e_{n+1} is orthogonal to $e_1 \vee e_2 \vee \cdots e_{n-1}$ which is, again by lemma 4, $1 - f_n$. Hence $e_{n+1} = e_{n+1} f_n + e_{n+1}(1 - f_n) = 0$ which is a contradiction. This completes the proof. □

Now we will prove the previous conclusion without the orthogality assumption of $e_i' s$.

Proposition 6. *Let H be a Hilbert space. Suppose $e_1, e_2, \cdots$ is a sequence of non-zero projections in $B(H)$ satisfying the following relation :*

$$\begin{aligned} e_i^2 &= e_i = e_i^\star \\ e_i e_j &= e_j e_i \quad \text{if } |i-j| \geq 2 \\ e_i e_j e_i &= \tau e_i \quad \text{if } |i-j| = 1 \end{aligned}$$

Then $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4} sec^2(\frac{\pi}{n+1}) : n \geq 2\}$.

Proof. Suppose that τ is not in the set described above. Then there exists $n \geq 2$ such that $\frac{1}{4} sec^2(\frac{\pi}{n+2}) < \tau < \frac{1}{4} sec^2(\frac{\pi}{n+1})$. From lemma 4, it follows that $e_{n+1} f_{n+1} = 0$. Also $e_i f_{n+1} = 0$ for $i \leq n$. Hence $f_{n+1} \leq 1 - e_1 \vee e_2 \vee \cdots \vee e_{n+1} = f_{n+2}$. But $f_{n+2} \leq f_{n+1}$. Hence $f_{n+1} = f_{n+2}$. Let k be the least element in $\{2, 3, \cdots, n\}$ for which $f_{k+1} = f_{k+2}$. Let $g_i = e_{k+i} f_{k-1}$ for $i \geq 0$. We will derive a contradiction by showing that $g_i' s$ satisfy the hypothesis of proposition 5.

Since e_{k+i} commutes with f_{k-1} for $i \geq 0$, it follows that g_i's are projections. For the same reason, $g_i' s$ satisfy the third relation of proposition 5. First, we show that $g_0 \neq 0$. By the choice of k, $f_k \neq f_{k+1}$. Hence $f_k e_k f_k \neq 0$. Since $f_k \leq f_{k-1}$, it follows that $f_{k-1} e_k = g_0 \neq 0$.

Now we show that $g_i g_j = 0$ if $|i-j| \geq 2$. We begin by showing $g_0 g_2 = 0$. Observe that since $f_{k+1} = f_{k+2}$, we have

$$e_{k+1} f_k = e_{k+1}(f_k - f_{k+1}) e_{k+1} = e_{k+1}(\frac{P_{k-1}(\tau)}{P_k(\tau)} f_k e_k f_k) e_{k+1} = \tau \frac{P_{k-1}(\tau)}{P_k(\tau)} e_{k+1} f_k.$$

Since $P_{k+1}(\tau) \neq 0$, it follows that $e_{k+1} f_k = 0$. By premultiplying and postmultiplying by e_{k+2}, we see that $e_{k+2} f_k = 0$. Hence we have,

$$\begin{aligned} g_0 g_2 &= e_k e_{k+2} f_{k-1} \\ &= e_k e_{k+2}(f_{k-1} - f_k) e_{k+2} e_k \\ &= e_{k+2} e_k (f_{k-1} - f_k) e_k e_{k+2} \\ &= e_{k+2} e_k (\frac{P_{k-2}(\tau)}{P_{k-1}(\tau)} f_{k-1} e_{k-1} f_{k-1}) e_k e_{k+2} \\ &= \tau \frac{P_{k-2}(\tau)}{P_{k-1}(\tau)} g_0 g_2 \end{aligned}$$

Since $P_k(\tau) \neq 0$, it follows that $g_0 g_2 = 0$. Let $i \geq 2$. Let us consider the partial isometry $w = (\frac{1}{\tau})^{i-1} e_{k+i} e_{k+i-1} \cdots e_{k+2}$. Since w commutes with e_k and f_{k-1}, $w e_k f_{k-1}$ is a partial isometry. Note that $(w e_k f_{k-1})^\star w e_k f_{k-1} = g_0 g_2 = 0$. Thus, $g_i g_0 = w e_k f_{k-1} (w e_k f_{k-1})^\star = 0$. Hence $g_i g_0 = 0$ if $i \geq 2$. Let i, j be such that $j \geq i+2$. Now let $u = (\frac{1}{\tau})^{i+1} e_{k+i} e_{k+i-1} \cdots e_k$. Then u is a partial isometry which commutes with f_{k-1} and e_{k+j}. Let $v = u e_{k+j} f_{k-1}$. Then v is a partial isometry such that $v^\star v = g_0 g_j$ and $vv^\star = g_i g_j$. Since $v^\star v = 0$, it follows that $vv^\star = 0$. Thus $g_i g_j = 0$. Therefore g_i 's satisfy the assumptions of proposition 5. Hence we have a contradiction. This completes the proof. □

Now Wenzl's theorem follows from proposition 6.

Chapter 3

Existence of C* representations of $T_n(\tau)$

In this chapter we will describe C* representations of $T_n(\tau)$ when the parameter $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4}sec^2(\frac{\pi}{m+1}) : m \geq 2\}$. First we describe the basic construction for a pair of finite dimensional C* algebras due to Jones. We refer to [Jon] for most of the material in this chapter. But first let us recall some basic facts about finite dimensional C* algebras.

3.1 Finite dimensional C* algebras

Let M be a finite dimensional C* algebra. Then M is unital. Let $\{p_1, p_2, \cdots, p_s\}$ be the set of minimal central projections of M.
Let $p_iMp_i = \{x \in M : p_ix = xp_i = x\}$ and $\mu_i = \sqrt{dim\ p_iMp_i}$.
Then M is isomorphic to $M_{\mu_1}(\mathbb{C}) \oplus \cdots \oplus M_{\mu_s}(\mathbb{C})$ as C* algebras. The algebra M is called a **factor** if it's center is trivial. Let $\vec{\mu} = (\mu_1, \mu_2, \cdots, \mu_s)$. The vector $\vec{\mu}$ is called the dimenstion vector of M.

Definition 6. *Let M be a C* algebra. A linear functional $\rho : M \longrightarrow \mathbb{C}$ is said to be a trace if $\rho(ab) = \rho(ba)\ \ \forall a, b \in M$. The functional ρ is said to be positive if $\rho(x^\star x) \geq 0\ \ \forall x \in M$ and faithful if $\rho(x^\star x) = 0$ implies $x = 0$. If M is unital then ρ is said to be unital if $\rho(1) = 1$.*

Any trace on $M_n(\mathbb{C})$ is just a multiple of the usual matrix trace i.e. if $\rho : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$ is a trace then $\rho((a_{ij})) = \lambda \sum_{i=1}^{n} a_{ii}$. If p is a minimal projetion in $M_n(\mathbb{C})$ then $\rho(p) = \lambda$. Hence ρ is determined by it's value on any minimal projection.

Let M be a finite dimensional C* algebra. Let $\{p_1, p_2, \cdots, p_s\}$ be the set of minimal central projections of M and let $\vec{\mu}$ be the dimension vector of M. Suppose $\rho : M \longrightarrow \mathbb{C}$ is a trace. Suppose e_i is a minimal projection in p_iMp_i

and let $t_i = \rho(e_i)$. Let $\vec{t} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix}$. Then $\vec{t}$ is calles the trace vector associated to ρ. Then ρ is positive if and only if $t_i \geq 0 \ \ \forall i$. The trace ρ is faithful if and only if $t_i > 0 \ \ \forall i$ and it is unital if and only if $\vec{\mu}.\vec{t} = 1$.

Let N and M be finite dimensional C* algebras such that $N \subset M$. We always assume that the inclusion is unital i.e. $1_N = 1_M$. Let $\{p_1, p_2, \cdots, p_s\}$ and $\{q_1, q_2, \cdots, q_r\}$ be the minimal central projections of M and N respectively. Then $q_i p_j M q_i p_j$ and $q_i p_j N q_i p_j$ are factors. Define $\Lambda_{ij} = \sqrt{\frac{dim\ q_i p_j M q_i p_j}{dim\ q_i p_j N q_i p_j}}$ if $p_j q_i \neq 0$. If $p_j q_i = 0$ then define $\Lambda_{ij} = 0$. Then Λ is an $r \times s$ matrix such that $\vec{\mu} = \vec{\nu}.\Lambda$. The matrix Λ is called the inclusion matrix for the inclusion $N \subset M$.

Let $N \subset M$ be a unital inclusion with inclusion matrix Λ. Let ρ_M be a trace on M with trace vector $\vec{t}$ and ρ_N be a trace on N with trace vector $\vec{s}$. Then $\rho_M \mid_N = \rho_N$ if and only if $\Lambda.\vec{t} = \vec{s}$.

The inclusion $N \subset M$ can also be described by it's **Bratelli diagram.** Let $N \subset M$ be a unital inclusion of finite dimensional C* algebras with inclusion matrix Λ. Let $\{q_1, q_2, \cdots, q_r\}$ and $\{p_1, p_2, \cdots, p_s\}$ be the minimal central projections of N and M respectively.The Bratelli diagram for the pair $N \subset M$ is a bipartite graph with verices $\{q_1, q_2, \cdots q_r\} \coprod \{p_1, p_2, \cdots, p_s\}$ where p_j is joined to q_i with Λ_{ij} bonds.

Let us recall the finite dimensional version of von Neumann's double commutant theorem whose proof can be found for instance in [GHJ]. Let H be a Hilbert space. Let $B(H)$ denote the space of bounded linear operators on H. For $S \subset B(H)$, it's commutant denoted by $S^{'}$ is defined as follows:

$$S^{'} := \{x \in B(H) : xs = sx \ \forall s \in S\}.$$

Note that $S \subset S^{''}$.

Theorem [von Neumann]. *Let H be a finite dimensional Hilbert space. Let $M \subset B(H)$ be a $\star$ closed algebra such that M contains the identity operator. Then $M^{''} = M$. If M is a factor then $M \otimes M^{'}$ is isomorphic to $B(H)$ and Hence $dimM \ dimM' = (dim \ H)^2$.*

We end this section with the following lemma. Let $M \subset F$ be a unital inclusion of finite dimensional C* algebras with F as factor. Then the commutant of M in F is denoted by $C_F(M)$.

Lemma 5. *Let $M \subset F$ be a unital inclusion of finite dimensional C^* algebras. Assume that F is a factor. Suppose $q \in M \cup C_F(M)$ is a nonzero projection. Then*

(1) qFq is a factor.

(2) $C_{qFq}(qMq) = qC_F(M)q$.

Suppose $N \subset M$ be a unital inclusion of finite dimensional C^* algebras with the inclusion matrix Λ. Then the inclusion matrix for $C_F(M) \subset C_F(N)$ is Λ^t.

Proof. If $F = B(H)$ for some finite dimensional Hilbert space then $qFq = B(qH)$. Hence (1) is true.

Let us first consider the case when $q \in M$. Let $x \in M$ and $y \in C_F(M)$. Then $(qxq)(qyq) = qxyq = qyxq = (qyq)(qxq)$. Hence $qC_F(M)q \subset C_{qFq}(qMq)$. Now let $s \in C_{qFq}(qC_F(M)q)$ be given. Then $sq = qs = s$. Let $t \in C_F(M)$. Then $st = sqqt = sqtq = qtqs = tqqs = ts$. Hence $s \in C_F(C_F(M)) = M$. Hence $C_{qFq}(qC_F(M)q) \subset qMq$. Hence taking commutants and using von-Neumann's double commutant theorem $C_{qFq}(qMq) \subset qC_F(M)q$. Hence $C_{qFq}(qMq) = qC_F(M)q$. The case $q \in C_F(M)$ follows from von Neumann's double commutant theorem.

Suppose $N \subset M$ be a unital inclusion of finite dimensional C* algebras with the inclusion matrix Λ. Let Γ be the inclusion matrix for $C_F(M) \subset C_F(N)$. Let $q_1, q_2, \cdots q_r$ be the minimal central projections of N and $p_1, p_2 \cdots, p_s$ be that of M. Since the center of $C_F(M)$ and M are the same, it follows that $p's$ and $q's$ are the minimal central projections of $C_F(M)$ and $C_F(N)$ respectively. Suppose $p_iq_j \neq 0$. Then

$$\begin{aligned}\Gamma_{ij}^2 &= \frac{dim\ p_iq_jC_F(N)p_iq_j}{dim\ p_iq_jC_F(M)p_iq_j} \\ &= \frac{dim\ C_{p_iq_jFp_iq_j}(p_iq_jNp_iq_j)}{dim\ C_{p_iq_jFp_iq_j}(p_iq_jMp_iq_j)}\end{aligned}$$

For $X = M$ or N, Since $p_iq_jXp_iq_j$ is a factor in $p_iq_jFp_iq_j$, it follows, from von Neumann's theorem, that $dim\ C_{p_iq_jFp_iq_j}(p_iq_jXp_iq_j) = \frac{dim\ p_iq_jFp_iq_j}{dim\ p_iq_jXp_iq_j}$. Hence $\Gamma_{ij}^2 = \Lambda_{ij}^2$. Hence $\Gamma = \Lambda^t$. This completes the proof. □

3.2 Basic construction

In this section, We describe the Jones' basic construction for a unital inclusion $N \subset M$ of finite dimensional C* algebras with a faithful unital trace.

We refer to [Jon] for this section. But we include the proofs for completeness.

Let $N \subset M$ be a unital inclusion of finite dimensional $C^\star$ algebras. Suppose $tr : M \to \mathbb{C}$ is a faithful unital positive trace. Then for $x, y \in M$, define $\langle x, y \rangle = tr(y^\star x)$. Then $\langle , \rangle$ defines an inner product on M. We denote this Hilbert space by $L^2(M, tr)$. Let $E : M \to N$ be the orthogonal projection.

Proposition 7. *E is the unique trace preserving conditional expectation of M onto N. That is*

(1) $E(axb) = aE(x)b$ for $a, b \in N$ and $x \in M$.

(2) $E(n) = n$ for $n \in N$.

(3) $tr(E(x)) = tr(x)$.

Further $(1), (2)$ *and* (3) *determine E uniquely.*

Proof. Let $a, b \in N$ and $x \in M$ be given. For $n \in N$, we have

$$\begin{aligned}
\langle aE(x)b, n \rangle &= tr(n^\star aE(x)b) \\
&= tr(bn^\star aE(x)) \\
&= \langle E(x), a^\star nb^\star \rangle \\
&= \langle x, a^\star nb^\star \rangle \\
&= tr(bn^\star ax) = tr(n^\star axb) \\
&= \langle axb, n \rangle = \langle axb, E(n) \rangle \\
&= \langle E(axb), n \rangle
\end{aligned}$$

Hence $\langle aE(x)b, n \rangle = \langle E(axb), n \rangle$ for every $n \in N$. Thus $E(axb) = aE(x)b$. This proves (1). Since E is the orthogonal projection of M onto N, (2) is true. Let $x \in M$. Now $tr(E(x)) = \langle E(x), 1 \rangle = \langle x, E(1) \rangle = \langle x, 1 \rangle = tr(x)$. Hence (3) is true.

Let $E' : M \to N$ be linear such that (1), (2) and (3) are satisfied for E'. Let $x \in M$ be given. Then for $n \in N$, $\langle E'(x), n \rangle = tr(n^\star E'(x)) = tr(E'(n^\star x)) = tr(n^\star x)$. A similar calculation with E shows that $\langle E(x), n \rangle = tr(n^\star x)$. Hence $\langle E'(x), n \rangle = \langle E(x), n \rangle$ for every $n \in N$. Hence $E(x) = E'(x)$. Hence $E = E'$. □

We denote E by e_N when we think of E as an element in $B(L^2(M, tr))$. For $x \in M$, define $\pi_l(x)(y) = xy$ for $y \in M$ and $\pi_r(x)(y) = yx$ for $y \in M$. Then $\pi_l(x), \pi_r(x) \in B(L^2(M, tr))$ for $x \in M$. The map $\pi_l : M \to B(L^2(M, tr))$ is a faithful unital * homomorphism. But π_r is an anti homomorphism in the sense that $\pi_r(x^*) = (\pi_r(x))^*$ and $\pi_r(xy) = \pi_r(y)\pi_r(x)$.

Lemma 6. *The commutant of $\pi_r(M)$ in $B(L^2(M, tr))$ is $\pi_l(M)$.*

Proof. It is clear that $\pi_l(M)$ commutes with $\pi_r(M)$. Let $T \in \pi_r(M)'$. Let $x = T(1)$. Now $T(y) = T\pi_r(y)(1) = \pi_r(y)(T(1)) = xy = \pi_l(x)(y)$. Hence $T = \pi_l(x) \in \pi_l(M)$. This completes the proof. □

Henceforth we identify M with $\pi_l(M)$. Now $\pi_r(N) \subset \pi_r(M)$. Note that $\pi_l(M) = \pi_r(M)' \subset \pi_r(N)'$. Hence starting with a unital inclusion $N \subset M$ together with a unital faithful positive trace on M, we obtain another unital inclusion $M \subset \pi_r(N)'$.

Definition 7. *Suppose $N \subset M$ be a unital inclusion of finite dimensional C^* algebras. Let tr be a faithful, unital, positive trace on M. Then the inclusion $M \subset \pi_r(N)'$ is called the* **basic construction** *for the pair $(N \subset M, tr)$.*

The main properties of the basic construction are summarised in the following porposition.

Proposition 8. *Suppose $N \subset M$ be a unital inclusion of finite dimensional C^* algebras. Let tr be a faithful, unital, positive trace on M. Then,*

1. *The C^* algebra generated by M and e_N in $B(L^2(M, tr))$ is $\pi_r(N)'$.*
2. *The central support of e_N in $\pi_r(N)'$ is 1.*
3. *$e_N x e_N = E(x)e_N$ for $x \in M$.*
4. *If Λ is the inclusion matrix for $N \subset M$ then Λ^t is the inclusion matrix for $M \subset \pi_r(N)'$.*

Proof. Let $\langle M, e_N\rangle$ denote the C* algebra generated by M and e_N. We prove that the commutant of $\langle M, e_N\rangle$ is $\pi_r(N)$. Let $T \in (\langle M, e_N\rangle)'$. Since T commutes with e_N, T leaves N invariant. Let $x = T(1)$. Then $x \in N$. Now $T(y) = T\pi_l(y)(1) = \pi_l(y)T(1) = yx = \pi_r(x)(y)$. Hence $T \in \pi_r(N)$. This implies $\langle M, e_N\rangle' \subset \pi_r(N)$ On the other hand, $\pi_r(N)$ commutes with M. Since N is invariant under $\pi_r(N)$, it follows that $\pi_r(N)$ commutes with e_N. Hence $\pi_r(N)$ commutes with $\langle M, e_N\rangle$. This implies $(\langle M, e_N\rangle)' = \pi_r(N)$. By von Neumann's double commutant theorem, $(\langle M, e_N\rangle) = \pi_r(N)'$.

Let $q_1, q_2, \cdots, q_r$ denote the minimal central projections in N. Then the minimal central projections of $(\pi_r(N))'$ are $\pi_r(q_1), \pi_r(q_2), \cdots, \pi_r(q_r)$. Since $\pi_r(q_i)e_N(q_i^*) = q_i^* q_i$, we have $\pi_r(q_i)e_N \neq 0$. Thus the central support of e_N in $\langle M, e_N\rangle$ is 1.

Let $x \in M$ be given. On $N^\perp$, $e_N x e_N = 0 = E(x)e_N$. Let $n \in N$ be given. Then $e_N x e_N(n) = E(xn) = E(x)n = E(x)e_N(n)$. Hence $e_N x e_N = E(x)e_N$.

For a C* algebra A, Let A^{op} denote the C* algebra whose underlyind set and the invloution are that of A but the multiplication is changed to $x.y = yx$. Now the center of A^{op} is same as the center of A. Hence the minimal central projections of A^{op} are the same as that of A. Now $\pi_r : M^{op} \to B(L^2(M, tr))$ is a unital inclusion. Now the inclusion matrix of $N^{op} \subset M^{op}$ is the same as that of $N \subset M$ since the minimal central projections of N^{op} and M^{op} are the same as that of N and M. Now by Lemma 5, it follows that the inclusion matrix for $M = (\pi_r(M))' \subset (\pi_r(N))') = \langle M, e_N \rangle$ is Λ^t. This completes the proof. □

Definition 8. *Suppose $N \subset M$ is a unital inclusion of finite dimensional C^* algebras. Let $tr : M \to \mathbb{C}$ be a faithful, unital, positive trace on M. Let $M \subset \langle M, e_N \rangle$ be the basic construction associated to the pair $(N \subset M, tr)$. Then tr is called a* **Markov trace** *of modulus τ if there exists a positive trace $Tr : \langle M, e_N \rangle \to \mathbb{C}$ such that*

1. *$Tr(xe_N) = \tau tr(x)$ for $x \in M$.*
2. *$Tr(x) = tr(x)$ for $x \in M$.*

Proposition 9. *Let $N \subset M$ be a unital inclusion of finite dimensional C^* algebras with a faithful positive trace tr. Suppose that tr is a Markov trace of modulus τ.Then there exists a unique positive trace Tr on $\langle M, e_N \rangle$ satisfying (1) and (2) of definition 8.*

Proof. By definition, there exists a positive trace Tr on $\langle M, e_N \rangle$ such that (1) and (2) holds. Let Tr_1 be another trace for which (1) and (2) holds. Let $x, y \in M$. Now $Tr(xe_N y) = Tr(yxe_N) = \tau tr(yx) = Tr_1(yxe_N) = Tr_1(xe_N y)$. Consider the set $I = \{\sum_{i=1}^{n} x_i e_N y_i : x_i, y_i \in M \ , n \in \mathbb{N}\}$. Then proposition 8 implies that I is an ideal in $\langle M, e_N \rangle$ which contains e_N. Since the central support of e_N is 1, it follows that $I = \langle M, e_N \rangle$. The preceeding calculations show that $Tr_1 = Tr$ on I. Hence $Tr = Tr_1$. □

The following proposition determines when a trace for the pair $N \subset M$ is a Markov trace of modulus τ. Before that we need the following Lemma.

Lemma 7. *Let $N \subset M$ be a unital inclusion of finite dimensional C^* algebras with a faithful, unital, positive trace tr. Suppose $q_1, q_2, \cdots, q_r$ are the minimal central projections in N. Then $\pi_r(q_1), \pi_r(q_2), \cdots, \pi_r(q_r)$ are the minimal central projections in $\langle M, e_N \rangle$. If f is a minimal projection in $q_i N q_i$ then fe_N is minimal in $\pi_r(q_i)\langle M, e_N \rangle$.*

Proof. Since N commutes with e_N, the map $x \to xe_N$ from $N \to \langle M, e_N \rangle$ is a homomorphism. We assert that this map is 1-1 and it's range is $e_N \langle M, e_N \rangle e_N$. Suppose that $xe_N = 0$ for some $x \in N$. Then $\pi_l(x)e_N(1) = 0$.

Hence $x = 0$. Hence $x \longrightarrow xe_N$ is 1-1. Let $T \in e_N\langle M, e_N\rangle e_N$ be given. Since T commutes with e_N, T leaves N invariant. Let $x = T(1)$. Then $x \in N$. Since $T(1 - e_N) = 0$ it follows that $T = 0$ on $N^\perp$. Hence $T = xe_N$ on $N^\perp$. Since T is right N linear, it follows that for $n \in N$, $T(n) = T(1)n$. Hence $T(n) = xe_N(n)$ for $n \in N$. Hence $T = xe_N$ on N. Hence $T = xe_N$. It is clear that the map $x \longrightarrow xe_N$ has range in $e_N\langle M, e_N\rangle e_N$. This proves the assertion.

Let f be a minimal projection in q_iNq_i. Note that $\pi_r(q_i)e_N = \pi_l(q_i)e_N$. Note that $fe_N\pi_r(q_i) = fq_ie_N = fe_N$. Hence $fe_N \leq \pi_r(q_i)$. Let p be a nonzero projection in $\langle M, e_N\rangle$ such that $p \leq fe_N$. Now $p = fe_Npfe_N = e_Nfpfe_N$. Hence $p = xe_N$ for some $x \in N$. By the 1-1 ness of the map $x \longrightarrow xe_N$, it follows that x is a nonzero projection. Now $xe_N = xe_Nfe_N = xfe_N$. Thus $x = xf$. Similarly $x = fx$. Hence by the minimality of f, it follows that $x = f$ and hence $p = fe_N$. Therefore fe_N is minimal. This completes the proof. □

Proposition 10. *Suppose $N \subset M$ be a unital inclusion of finite dimensional C^* algebras with a faithful, unital, positive trace tr. Let Λ be the inclusion matrix for $N \subset M$. Let $\vec{\mu}$ and $\vec{\nu}$ be the dimension vectors for M and N respectively. Suppose $\vec{r}$ and $\vec{s}$ are the trace vectors for $tr\mid_N$ and $tr\mid_M$ respectively. Then tr is a Markov trace of modulus τ if and only if $\Lambda^t\Lambda\vec{s} = \frac{1}{\tau}\vec{s}$ and $\Lambda\Lambda^t\vec{r} = \frac{1}{\tau}\vec{r}$.*

Proof. Let tr be Markov of modulus τ and Let Tr be the corresponding trace on $\langle M, e_N\rangle$. Let $\vec{t}$ be the trace vector for Tr on $\langle M, e_N\rangle$. By lemma 7, we have $\vec{t} = \tau\vec{r}$. Since the traces are consistent, we have $\vec{r} = \Lambda\vec{s} = \Lambda\Lambda^t(\vec{t}) = \Lambda\Lambda^t(\tau\vec{r}) = \tau\Lambda\Lambda^t(\vec{r})$. Also, $\vec{s} = \Lambda^t(\vec{t}) = \Lambda^t(\tau\vec{r}) = \tau\Lambda^t\Lambda(\vec{s})$.

Suppose the inclusion matrix satisfies the condition in the proposition. Define Tr on $\langle M, e_N\rangle$ by letting it's trace vector be $\vec{t} = \tau\vec{r}$. Then $\Lambda^t(\vec{t}) = \tau\Lambda^t(\vec{r}) = \tau\Lambda^t\Lambda\vec{s} = \vec{s}$. Hence $Tr(x) = tr(x)$ for $x \in M$. Also by definition of Tr, it follows that $Tr(pe_N) = \tau tr(p)$ for every minimal projection p in N and hence $Tr(xe_N) = \tau tr(x)$ for $x \in N$. Let $x \in M$. Now $Tr(xe_N) = Tr(e_Nxe_N) = Tr(E(x)e_N) = \tau tr(E(x)) = \tau tr(x)$. This proves that tr is a Markov trace of modulus τ. □

Corollary 1. *Let $N \subset M$ be a unital inclusion of finite dimensional C^* algebras with a faithful, unital, positive trace tr. Suppose that tr is a Markov trace of modulus τ. Then the unique trace Tr on $\langle M, e_N\rangle$which extends tr and for which $Tr(xe_N) = \tau tr(x)$ is a Markov trace of modulus τ for the pair $M \subset \langle M, e_N\rangle$.*

Proof. Let $\vec{r}, \vec{s}, \vec{t}$ be as in proposition 10. Let Λ be the inclusion matrix for the pair $N \subset M$. Then $\vec{t} = \tau\vec{r}$. Now $\Lambda\Lambda^t\vec{t} = \tau\Lambda\Lambda^t\vec{r} = \tau\frac{1}{\tau}(\vec{r}) = \frac{1}{\tau}(\vec{t})$. Hence by proposition 10, it follows that Tr is a Markov trace of modulus τ. $\square$

We end this section with a lemma which characterises the basic construction for a pair $N \subset M$ whose proof can be found in [JS].

Lemma 8. *Let $A \subset B$ be a unital inclusion of finite dimensional $C^\star$ algebras with a faithful, unital, positive trace tr. Let E be the unique trace preserving conditional expectation of B onto A. Let $B_1 = \langle B, e\rangle$ denote the result of the basic construction. Let $B \subset C$ be a unital inclusion of finite dimensional $C^\star$ algebras. Suppose C contains a projection f satisfying*

(1) $C = \langle B, f\rangle$;

(2) $fbf = E(b)f$ for $b \in B$; and

(3) f commutes with A and $a \to af$ is an injective $*$ homomorphism of A into C.

(4) The central support of f in C is 1.

Then there exists a unique isomorphism $\Psi : B_1 \to C$ such that $\Psi(b) = b$ for $b \in B$ and $\Psi(e) = f$.

3.3 Jones Tower

Let $N \subset M$ be a unital inclusion of finite dimensional C* algebras with a faithful, unital, positive trace tr. Suppose that tr is Markov of modulus τ. Then there exists a unique faithful, positive trace which extends tr which we continue to denote by tr such that $tr(xe_N) = \tau tr(x)$ for $x \in M$. Then tr is a Markov trace of modulus τ for the pair $M \subset \langle M, e_N\rangle$. Let $e_1 = e_N$.

Iterating the basic construction for the pair $M \subset \langle M, e_1\rangle$, we get a tower of finite dimensional C* algebras $N \subset M \subset \langle M, e_1\rangle \subset \langle M, e_1, e_2\rangle \subset \cdots$ with faithful, unital, positive trace on $\bigcup_n \langle M, e_1, e_2, \cdots, e_n\rangle$ which we again denote by tr. This tower is called the Jones tower. Let $M_0 = N$, $M_1 = M$ and $M_n = \langle M, e_1, e_2, \cdots, e_{n-1}\rangle$. M_{n+1} is obtained by the basic construction for the pair $(M_{n-1} \subset M_n, tr)$. Let $E_{n-1} : M_n \to M_{n-1}$ be the corresponding conditional expectation. Then we have the following,

(1) $tr(x) = tr(E_{n-1}(x))$ if $x \in M_n$.

(2) $tr(xe_n) = \tau tr(x)$ if $x \in M_n$.

(3) e_n commutes with M_{n-1}.

(4) $e_n x e_n = E_{n-1}(x) e_n$ if $x \in M_n$.

Now $tr(E_n(e_n)x) = tr(E_n(e_nx)) = tr(e_nx) = \tau tr(x) = tr(\tau x)$ for $x \in M_n$. Since tr is faithful, $E_n(e_n) = \tau$.

The next proposition says that the sequence of projections e_n satisfy the TL relations.

Proposition 11. *Suppose $N \subset M$ is a unital inclusion of finite dimensional $C^\star$ algebras and Let tr be a Markov trace of modulus τ. If $\{e_n\}$ denote the sequence of projections in the Jones tower, then*

$$\begin{aligned} e_i^2 = e_i &= e_i^\star \quad \forall\ i \in \mathbb{N} \\ e_ie_j &= e_je_i \qquad if \quad |i-j| \geq 2 \\ e_ie_je_i &= \tau e_i \qquad if \quad |i-j| = 1 \end{aligned}$$

Proof. Only the third relation requires proof. Let $n \in \mathbb{N}$ be given. Now $e_{n+1}e_ne_{n+1} = E_n(e_n)e_{n+1} = \tau e_{n+1}$. Consider the previous relation in M_{n+2}. Then, $\frac{e_{n+1}e_n}{\sqrt{\tau}}$ is a partial isometry.
Hence $(\frac{e_{n+1}e_n}{\sqrt{\tau}})^\star \frac{e_{n+1}e_n}{\sqrt{\tau}} = \frac{e_ne_{n+1}e_n}{\tau}$ is a projection. Clearly $\frac{e_ne_{n+1}e_n}{\tau} \leq e_n$. Now $tr(\frac{e_ne_{n+1}e_n}{\tau}) = tr(e_n)$. Since tr is faithful, it follows that $\frac{e_ne_{n+1}e_n}{\tau} = e_n$. This completes the proof. □

3.4 Jones quotient

We will describe a $C^\star$ quotient for $TL(\tau)$ called the Jones quotient for every $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4}sec^2(\frac{\pi}{m+1} : m \geq 2\}$.

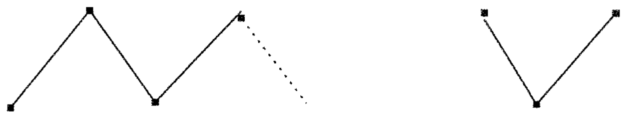
First we show that for $\tau \in \{\frac{1}{4}sec^2(\frac{\pi}{m+1} : m \geq 2\}$ there exists an inclusion $N \subset M$ of finite dimensional $C^\star$ algebras which admits a Markov trace of modulus τ. We need the following proposition for that. We say that the inclusion $N \subset M$ is connected if the Bratelli diagram for the inclusion $N \subset M$ is connected.

Proposition 12. *Let $N \subset M$ be a unital inclusion which is connected. Then there exists a unique Markov trace of modulus τ if and only if $\tau = \| \Lambda \|^{-2}$.*

For a proof we refer to [GHJ] □

Let $\tau = \frac{1}{4}sec^2\frac{\pi}{n+1}$. It is enough to exhibit a Bratelli diagram or a bipartite graph whose corresponding matrix Λ satisfies $||\Lambda|| = \frac{1}{\sqrt{\tau}}$. First suppose that n is even, say $n = 2l$. Note that the norm of a matrix won't change by changing rows and columns. Consider the following bipartite graph with

$2l = l + l$ vertices.



Let Λ be the corresponding matrix. Let $Y = \begin{pmatrix} 0 & \Lambda \\ \Lambda^t & 0 \end{pmatrix}$. Then Y is the adjacency matrix of the following path with $2l$ vertices.

Then

$$Y = \begin{pmatrix} 0 & 1 & 0 & . & 0 & 0 \\ 1 & 0 & 1 & . & 0 & 0 \\ 0 & 1 & 0 & . & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & . & 0 & 1 \\ 0 & 0 & 0 & . & 1 & 0 \end{pmatrix}$$

For $j = 1, 2, \cdots n$, one checks that $Y\xi_j = \lambda_j \xi_j$ where $\lambda_j = 2cos(\frac{j\pi}{n+1})$, $\xi_j = \left(sin(\frac{jk\pi}{n+1})\right)_{1 \leq k \leq l}$. Since Y is symmetric, it follows that $||Y|| = 2cos(\frac{\pi}{n+1})$. Now note that $YY^t = \begin{bmatrix} \Lambda\Lambda^t & 0 \\ 0 & \Lambda^t\Lambda \end{bmatrix}$. Hence $||Y||^2 = ||YY^t|| = ||\Lambda||^2$. Hence $||\Lambda||^2 = \frac{1}{\tau}$.

When n is odd say $n = 2l + 1$, considering the following bipartite graph with $2l + 1 = l + (l + 1)$ vertices and arguing as above will do the job.

We now define the Jones quotient $J_n(\tau)$ for $\tau \in \{\frac{1}{4}sec^2(\frac{\pi}{m+1} : m \geq 2\}$. Suppose $\tau \in \{\frac{1}{4}sec^2(\frac{\pi}{m+1} : m \geq 2\}$. Let $N \subset M$ be an inclusion of finite dimensional $C^\star$ algebras which admits a Markov trace of modulus τ. Let $M_0 \subset M_1 \subset M_2 \subset \cdots$ be the Jones tower. Let $J_n(\tau) \subset M_n$ be the $C^\star$ algebra generated by $1, e_1, e_2, \cdots, e_{n-1}$. We set $J_i(\tau) = \mathbb{C}$ for $i = 0, 1$. Then $E_{n-1}(J_n(\tau)) \subset J_{n-1}(\tau)$. Then we have a tower $J_n(\tau) \subset J_{n+1}(\tau)$ of finite dimensional $C^\star$ algebras and a faithful unital positive trace on $\bigcup_n J_n(\tau)$. We refer to [Jon] for the Bratelli diagram of the tower $J_n(\tau) \subset J_{n+1}(\tau)$. From the Bratelli diagram it follows that the tower $J_n(\tau) \subset J_{n+1}(\tau)$ together with the conditional expectations E_{n-1} and the trace depends only on τ and is independent of the initial inclusion $N \subset M$.

Let $\tau < \frac{1}{4}$. It is shown in [Jon] that, in this case, there exists a unital inclusion of type II_1 factors with index τ^{-1}, and that here too, just as in the finite dimensional case, one may, by iterated basic construction, obtain the Jones' tower $N \subset M \subset \langle M, e_1 \rangle \subset \langle M, e_1, e_2 \rangle$ of type II_1 factors and conditional expectations $E_n : M_{n+1} \to M_n$ where $M_0 = N$,$M_1 = M$ and $M_n = \langle M, e_1, e_2, \cdots e_{n-1} \rangle$. The tower $M_n \subset M_{n+1}$ has a faithful positive trace tr on $\bigcup_n M_n$.

Then we have the following,

(1) $tr(x) = tr(E_{n-1}(x))$ if $x \in M_n$.

(2) $tr(xe_n) = \tau tr(x)$ if $x \in M_n$.

(3) e_n commutes with M_{n-1}.

(4) $e_n x e_n = E_{n-1}(x)e_n$ if $x \in M_n$.

Also the $e_n's$ satisfy the TL relations. Now $J_n(\tau)$ is defined as in the finite dimensional case. As in the finite dimensional case, the tower $J_n(\tau) \subset J_{n+1}(\tau)$ together with the conditional expectations $E_n : J_{n+1}(\tau) \to J_n(\tau)$ and the trace depends only on τ and is independent of the initial inclusion $N \subset M$. We refer to [JS] for the definition of type II_1 factors and the basic construction for type II_1 factors.

From now on, Let $e_1^T, e_2^T, \cdots e_{n-1}^T$ denote the idempotents in $T_n(\tau)$ and $e_1^J, e_2^J, \cdots e_{n-1}^J$ denote the 'Jones' projections in $J_n(\tau)$. Suppose ϵ_n^T and ϵ_n^J denote the corresponding conditional expectation and let $T_i(\tau) = \mathbb{C}$ for $i = 0, 1$. By the universal property of $T_n(\tau)$ there exists a unique map $\phi_n : T_n(\tau) \to J_n(\tau)$ such that ϕ_n is unital and $\phi_n(e_i^T) = e_i^J$. Note that $\phi_{n+1}(a) = \phi_n(a)$ if $a \in T_n(\tau)$. Hence we can and will denote the maps ϕ_n by ϕ. The algebra $J_n(\tau)$ is called the Jones quotient of $T_n(\tau)$

Note the following properties of ϕ:

(1) The map ϕ is $*$ preserving.

(2) $\phi(\epsilon_n^T(a)) = \epsilon_n^J(\phi(a))$ if $a \in T_{n+1}(\tau)$.

(3) $\phi(tr^T(a)) = tr^J(\phi(a))$ if $a \in T_n(\tau)$.

(1),(2) and (3) can be proved by induction on n and by noting the fact that $\{x + \sum_{i=1}^r x_i e_n^T y_i \ : x, x_i, y_i \in T_n(\tau) \ and \ r \in \mathbb{N}\} = T_{n+1}(\tau)$.

Recall the polynomials $P_k(\lambda)$ and the Jones Wenzl projections f_k^T defined in chapter 2. Let $f_k^J = \phi(f_K^T)$.

Proposition 13. *If $P_k(\tau) \neq 0$ for $k = 1, 2, \cdots, n-1$ then $f_k^J = 1 - \bigvee_{i=1}^{k-1} e_i$ for $2 \leq k \leq n$.*

Proof. Let $k \geq 2$. Since $f_k^J e_i^J = 0$ for $i \in \{1, 2, \cdots, k-1\}$, it follows that $1 - f_k^J \geq e_1^J \vee e_2^J \vee \cdots \vee e_{k-1}^J$. But $1 - f_k^J$ is in the algebra generated by $e_1, e_2, \cdots e_{k-1}$. Thus $1 - f_k^J \leq e_1^J \vee e_2^J \vee \cdots \vee e_{k-1}^J$.
Hence $1 - f_k^J = e_1^J \vee e_1^J \vee \cdots \vee e_{k-1}^J$. This completes the proof. □

We refer to [Jon] for the following proposition.

Proposition 14. *If $P_k(\tau) \neq 0$ for $k = 1, 2, \cdots n-1$ then dim $J_k(\tau) = \frac{1}{k+1}\binom{2k}{k}$ for $k = 1, 2, \cdots, n-1$. Hence $\phi : T_k(\tau) \rightarrow J_k(\tau)$ is an isomorphism for $k = 1, 2, \cdots, n-1$.*

Hence if $\tau \leq \frac{1}{4}$, any C* representation of $T_k(\tau)$ is a C* representation of $J_k(\tau)$. In the next chapter, we will prove that if $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$, any C* representation π for which $\pi(e_1^T) \vee \pi(e_2^T) \cdots \vee \pi(e_{k-1}^T) = 1$ factors through $J_k(\tau)$ when $k \geq n$.

Let us recall the Murray von Neumann equivalence. Let M be a finite dimensional C* algebra. Let p, q be projections in M. We say p is Murray von Neumann equivalent to q if there exists $w \in M$ such that $w^\star w = p$ and $ww^\star = q$. Note that in $J_n(\tau)$ all the $e_i^{'}s$ are Murray von Neumann equivalent.

Let $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$. Then $P_k(\tau) \neq 0$ for $k = 1, 2, \cdots n-1$ but $P_n(\tau) = 0$. Note that $tr^J(f_n^J) = P_n(\tau) = 0$. Since tr is faithful, $f_n^J = 0$. Hence $e_1^J \vee e_2^J \vee \cdots \vee e_{k-1}^J = 1$ in $J_k(\tau)$ for $k \geq n$. We will prove in the next chapter that the kernel of the map $\phi : T_k(\tau) \rightarrow J_k(\tau)$ is the ideal generated by f_n^T in $T_k(\tau)$ for $k \geq n$. We need the following proposition for that.

Proposition 15. *Let $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ for some $n \geq 2$. Then $J_{k+1}(\tau)$ together with e_k^J is the basic construction of the pair $(J_{k-1}(\tau) \subset J_k(\tau), tr)$ for $k \geq n-1$. That is, if $\langle J_k(\tau), e\rangle$ denotes the basic construction then there exists a unique isomorphism $\Psi : \langle J_k(\tau), e\rangle \rightarrow J_{k+1}(\tau)$ such that $\Psi(a) = a$ if $a \in J_k(\tau)$ and $\Psi(e) = e_k^J$.*

Proof. Let $k \geq n-1$ be given. We apply Lemma 8 with $f = e_k^J$ to prove this. e_{k-1}^J is the unique trace preserving conditional expectation of $J_k(\tau)$

onto $J_{k-1}(\tau)$. Clearly $(1),(2)$ of lemma 8 are true. Also, e_k^J commutes with $J_{k-1}(\tau)$. Now let $xe_k^J = 0$ for some $x \in J_{k-1}(\tau)$. Then $yxe_k^J = 0$ for every $y \in J_{k-1}(\tau)$. Hence for $y \in J_{k-1}(\tau)$, $\tau tr(yx) = tr(yxe_k^J) = 0$. Hence $tr(yx) = 0$ for every $y \in J_{k-1}(\tau)$. Since tr is faithful, it follows that $x = 0$. Hence (3) of lemma 8 is satisfied.

Let p be a central projection in $J_{k+1}(\tau)$ such that $p \geq e_k^J$. Let $i \in \{1, 2, \cdots, k\}$ be given. Let $w \in J_{k+1}(\tau)$ be such that $w^*w = e_k^J$ and $ww^* = e_i^J$. Now $e_i^J p = ww^*p = wpw^* = we_k^J pw^* = we_k^J w^* = ww^* = e_i^J$. Hence $p \geq e_i^J$ for every $i \in \{1, 2, \cdots, k\}$. Hence $p \geq e_1^J \vee e_2^J \vee \cdots \vee e_k^J \geq 1 - f_n^J = 1$ by the observation preceeding this proposition. Hence (4) of lemma 8 is satisfied. The proof is complete by applying lemma 8. □

Chapter 4

Maximal C* quotient of $T_n(\tau)$

4.1 Maximal C* qoutient of a $\star$ algebra

Let A be a unital $\mathbb{C}$ algebra. For $a \in A$, it's spectrum, denoted $\sigma_A(a)$ is defined by $\sigma_A(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \textit{ is not invertible in } A\}$. Let B be a unital finite dimensional $\mathbb{C}$ algebra. Let $\pi : A \to B$ be a unital algebra homomorphism. Then $\sigma_B(\pi(a)) \subset \sigma_A(a)$ for $a \in A$.

Suppose A is a unital finite dimensional $\mathbb{C}$ algebra. For $a \in A$, let $\pi_l(a)$ be defined by $\pi_l(a)(b) = ab$. Let $End(A)$ denote the space of $\mathbb{C}$ linear endomorphisms of A. Then $\pi_l : A \to End(A)$ is a unital algebra homomorphism which is 1-1. Since $\sigma_{End(A)}(\pi_l(a))$ is nonempty, it follows that $\sigma_A(a)$ which contains $\sigma_{End(A)}(\pi_l(a))$ is nonempty. Now we will show that $\sigma_A(a)$ is finite by showing $\sigma_A(a)$ is contained in the set of zeros of the characteristic polynomial of $\pi_l(a)$.

Lemma 9. *Let A be a unital finite dimensional $\mathbb{C}$ algebra. Let $a \in A$. Then $\sigma_A(a)$ is nonempty and finite.*

Proof. We have already shown that $\sigma_A(a)$ is nonempty. Now for a polynomial $p(x)$ over $\mathbb{C}$, $p(\pi_l(a)) = \pi_l(p(a))$. Since $\pi_l(a)$ satisfies it's characteristic polynomial, it follows that $\exists$ a polynomial $p(x)$ over $\mathbb{C}$ such that $p(a) = 0$. Now we show that $\lambda \in \sigma_A(a)$ implies $p(\lambda) = 0$. Let $\lambda \in \mathbb{C}$ be such that $p(\lambda) \neq 0$. Then $p(x) - p(\lambda) = (x - \lambda)q(x)$ for some polynomial q. Now $-p(\lambda) = p(a) - p(\lambda) = (a - \lambda)q(a) = q(a)(a - \lambda)$. Hence $\frac{-q(a)}{p(\lambda)}$ is the inverse of $a - \lambda$. Thus $\lambda \notin \sigma_A(a)$. Therefore $\sigma_A(a)$ is contained in the zero set of p. As a result we conclude that $\sigma_A(a)$ is finite. □

Let A be a finite dimensional unital $\star$ algebra. Let $\pi : A \to B$ be a C* representation where B is a C* algebra. Then for $a \in A$,

$$\begin{aligned}||\pi(a)||^2 = ||\pi(a^\star a)|| &\leq sup\{|\lambda| : \lambda \in \sigma_B(\pi(a^\star a))\} \\ &\leq sup\{|\lambda| : \lambda \in \sigma_A(a^\star a)\} \quad \text{since } \sigma_B(\pi(a^\star a)) \subset \sigma_A(a^\star a)\ .\end{aligned}$$

For $a \in A$, define

$||a|| := sup\{||\pi(a)|| : \pi : A \to B$ is a $*$ algebra homomorphism where B is a C* algebra$\}$

Then $||a|| < \infty \ \forall a \in A$. Let $I = \{a \in A : ||a|| = 0\}$. Then I is an ideal in A.
For $a \in A$, note that $||a + I|| = ||a||$ depends only on $a + I$. Then A/I becomes a C* algebra with the above norm. Let $q : A \to A/I$ be the qoutient map.
A/I has the following universal property:

Let B be a C* algebra and let $\pi : A \to B$ be a $\star$ homomorphism. Then $\exists$ a unique $\star$ homomorphism $\tilde{\pi} : A/I \to B$ such that $\tilde{\pi} \circ q = \pi$.

Definition 9. *Let A be a unital finite dimensional $\star$ algebra. A C* algebra B together with a $\star$ algebra homomorphism $q : A \to B$ is said to be a maximal C* quotient of A if it has the following universal property:*
Given a $\star$ homomorphism $\pi : A \to C$ where C is a C* algebra, $\exists$ a unique $\star$ homomorphism $\tilde{\pi} : B \to C$ such that $\tilde{\pi} \circ q = \pi$.

Note that maximal C* quotient of a unital finite dimensional $\star$ algebra exists and is unique upto a unique isomorphism.

Let $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4}sec^2(\frac{\pi}{n+1}) : n \geq 2\}$. Now if $P_k(\tau) \neq 0$ for $k = 1, 2, \cdots n-1$ then the natural map $\phi : T_k(\tau) \to J_k(\tau)$ is a $\star$ isomorphism. Hence if $P_k(\tau) \neq 0$ for $k = 1, 2, \cdots, n-1$ then $(J_k(\tau), \phi)$ is the maximal C* quotient of $T_k(\tau)$ for $k = 1, 2, \cdots, n-1$. In particular, if $\tau \leq \frac{1}{4}$ then $(J_k(\tau), \phi)$ is the maximal C* quotient of $T_k(\tau)$ $\forall$ $k \geq 1$.

Let $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$. Let $\tilde{1} : T_k(\tau) \to \mathbb{C}$ be the $*$ homomorphism defined by $\tilde{1}(e_i^T) = 0$ for $i \leq k-1$ and $\tilde{1}(1) = 1$ (which exists by the universal property of $T_k(\tau)$). We will prove that $(J_k(\tau) \oplus \mathbb{C}, \phi \oplus \tilde{1})$ is the maximal C* quotient of $T_k(\tau)$ when $k \geq n$. This requires the determination of the kernel of the map $\phi : T_k(\tau) \to J_k(\tau)$ when $k \geq n$. We need the following lemma for that.

Lemma 10. *Let $N \subset M$ be a unital inclusion of finite dimensional C* algebras with a faithful, unital, positive trace tr. Then M is a $N - N$ bimodule. Let $\langle M, e_N \rangle$ denote the basic construction. Then the $M - M$ bimodule homomorphism $\Psi : M \otimes_N M \to \langle M, e_N \rangle$ defined by $\Psi(x \otimes y) = xe_Ny$ is an isomorphism.*

Proof. The map Ψ is well defined since e_N commutes with N. Consider M as a right N module. Then $\langle M, e_N \rangle$ is just the space of right N linear

60810

maps of M. Let $E : M \to N$ be the unique trace preserving conditional expectation. Let M^* dente the space of right N linear maps from M to N. Then M^* is a left N module. For $b \in M$, define $E_b(x) = E(bx)$ for $x \in M$. Then $E_b \in M^*$. Define $\theta : M \to M^*$ by $\theta(b) = E_b$. Clearly θ is left N linear.

Assertion: θ is an isomorphism.

Suppose $\theta(b) = 0$ for some $b \in M$. Then $tr(bx) = tr(E(bx)) = tr(E_b(x)) = 0\ \forall x \in M$. Since tr is faithful,we have $b = 0$. Hence θ is one one. Now let $\sigma \in M^*$ be given. Then $tr \circ \sigma$ is a linear functional on M. Since M is a Hilbert space, $\exists\ b \in M$ such that $tr \circ \sigma = \langle\ ,\ b^*\rangle$. Hence $tr(\sigma(x)) = tr(bx)\ \forall x \in M$. Hence $tr(\sigma(x)n) = tr(\sigma(xn)) = tr(bxn) = tr(E(bxn)) = tr(E(bx)n)$ for $x \in M, n \in N$. Since tr is faithful on N , $\sigma(x) = E(bx)\ \forall x \in M$. Hence $\sigma = \theta(b)$. Therefore, θ is onto. This proves the assertion.

Since C* algebras are semisimple, M as a right N module is semisimple. M is also finitely generated as an N module. Hence M is finitely genrerated projective and hence flat. Hence $id \otimes \theta : M \otimes_N M \to M \otimes_N M^*$ is an isomorphism. Since M is finitely generated and projective, the canonical map $\chi : M \otimes_N M^\star \to End_N(M)$ given by $\chi(x \otimes y^\star)(m) = xy^\star(m)$ is one one. Hence $\chi \circ id \otimes \theta$ is one one.

Assertion: $\Psi = \chi \circ (id \otimes \theta)$. Let $x, y, m \in M$ be given. Now

$$(\chi \circ (id \otimes \theta))(x \otimes y)(m) = x\theta(y)(m) = xE(ym) = xe_N y(m).$$

Hence $\chi \circ (id \otimes \theta) = \Psi$. This proves the assertion. Hence Ψ is one one.

The image of Ψ is clearly an ideal which contains e_N. Since the central support of e_N in $\langle M, e_N\rangle$ is 1, it follows that Ψ is onto. Hence Ψ is an isomorphism. □

Now We compute the kernel of the map $\phi : T_k(\tau) \to J_k(\tau)$ for $k \geq n$ when $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$. The proof of the following proposition can be found in [JR]. We include the proof for completeness.

Proposition 16. *Let $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$. Then the kernel of the natural map $\phi : T_k(\tau) \to J_k(\tau)$ for $k \geq n$ is the ideal generated by f_n^T in $T_k(\tau)$ for $k \geq n$.*

Proof. By induction, $\tilde{1}(f_k^T) = 1$ for $0 \leq k \leq n$. Hence $f_n^T \neq 0$. We will write T_k for $T_k(\tau)$.

Let $A_k = T_k$ for $0 \leq k \leq n-1$. Let $A_k = A_{k-1}e_{k-1}^T A_{k-1}$ for $k \geq n$. Then $A_k \subset T_k$.
Assertion: For every $k \geq 0$,

(1) A_k is a subalgebra of T_k.

(2) $\epsilon^T_{k-1}(A_k) \subset A_{k-1}$.

(3) A_k is a $A_{k-1} - A_{k-1}$ bimodule.

We prove this by induction on k. Clearly (1), (2) and (3) holds for $k \leq n-1$. Now assume (1), (2) and (3) holds for k. Let $x, y, z, w \in A_k$. Now $(xe_ky)(ze_kw) = x\epsilon^T_{k-1}(yz)e_kw$. Now (1), (2), (3) for A_k implies $x\epsilon^T_{k-1}(yz) \in A_k$. Hence $(xe^T_ky)(ze^T_kw) \in A_{k+1}$. Hence A_{k+1} is a subalgebra of T_{k+1}. Let $x, y \in A_k$. Then $\epsilon^T_k(xe_ky) = \tau xy \in A_k$ since A_k is a subalgebra of T_k. Hence $\epsilon^T_k(A_{k+1}) \subset A_k$. Since A_k is a subalgebra of T_k, it follows that A_{k+1} is a $A_k - A_k$ bimodule. This proves the assertion.

Assertion : The map $\phi : A_k \longrightarrow J_k$ is an isomorphism.

We prove the assertion by induction on k. The map $\phi : A_k \longrightarrow J_k$ is an isomorphism for $k \leq n-1$ is exactly proposition14. Now assume that ϕ is an isomorphism for $0 \leq l \leq k$. Let $\phi \otimes \phi$ denote the isomorphism from $A_k \otimes_{A_{k-1}} A_k$ to $J_k \otimes_{J_{k-1}} J_k$ when one identifies A_l with J_l when $l \leq k$ via ϕ. Let $\chi : A_k \otimes_{A_{k-1}} A_k \longrightarrow A_{k+1}$ be defined by $\chi(x \otimes y) = xe^T_ky$. Let Ψ be the map of Lemma 10 where $N = J_{k-1}$, $M = J_k$ and the projection $e_N = e^T_k$. Now $\Psi \circ \phi \otimes \phi = \phi \circ \chi$. By induction hypothesis, $\phi \otimes \phi$ is an isomorphism. Since Ψ is also an isomorphim, it follows that $\phi \circ \chi$ is an isomorphism. By definition, χ is onto. Hence ϕ is one-one. Since $\phi \circ \chi$ is onto, ϕ is onto. Hence $\phi : A_{k+1} \longrightarrow J_{k+1}$ is an isomorphism. This proves the assertion.

For $k \geq n$, Let I_k denote the ideal in $T_k(\tau)$ generated by f^T_n. Clearly $I_k \subset I_{k+1}$. Observe that $T_ke^T_kT_k$ is an ideal in T_{k+1} which contains e^T_k. Since $e^T_{k-1} = \frac{1}{\tau}(e^T_{k-1}e^T_ke^T_{k-1})$ it follows that $T_ke^T_kT_k$ contains e^T_{k-1}. Similarly it contains $e^T_1, e^T_2, \cdots, e^T_{k-2}$. Hence $1 - f^T_n \in T_ke^T_kT_k$ for $k \geq n-1$. Hence $I_{k+1} + T_ke^T_kT_k = T_{k+1}$ for $k \geq n-1$. We claim that $I_k + A_k = T_k$ for $k \geq n$. We prove this by induction on k. We have just proved that the claim is true for $k = n$. Now assume the claim is true for k. Since $T_{k+1} = I_{k+1} + T_ke^T_kT_k$, it is enough to show that if $x, y \in T_k$ then $xe^T_ky \in I_{k+1} + A_{k+1}$. By induction hypotheis, $\exists z, w \in I_k$ and $u, v \in A_k$ such that $x = z + u$ and $y = w + v$. Now $xe^T_ky = ze^T_kw + ue^T_kw + ze^T_kv + ue^T_kv$. Since $I_k \subset I_{k+1}$, it follows that $ze^T_kw + ue^T_kw + ze^T_kv \in I_{k+1}$. By definition $ue^T_kv \in A_{k+1}$. Hence $I_{k+1} + A_{k+1} = T_{k+1}$. Thus completes the induction and proves the claim.

Now we prove that the kernel of the map ϕ is I_k for $k \geq n$. Let $k \geq n$ be given. Since $f^J_n = 0$, it follows that $I_k \subset ker(\phi)$. Now let $x \in Ker(\phi)$ be given. Let $z \in I_k$ and $w \in A_k$ be such that $x = z + w$. Then $0 = \phi(w)$. Since $\phi : A_k \longrightarrow T_k$ is an isomorphism, it follows that $w = 0$. Hence $x \in I_k$. Thus $ker(\phi) \subset I_k$. Therefore $ker(\phi) = I_k$. This completes the proof. $\square$

Now We prove the much promised fact that when $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$, $(J_k(\tau) \oplus \mathbb{C}, \phi \oplus \tilde{1})$ is the maximal C* quotient of $T_k(\tau)$ when $k \geq n$. We begin with the following theorem.

Theorem 2. *Let $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$. Let $k \geq n$. Let A be a $C^\star$ algebra. Let $\pi : T_k(\tau) \to A$ be a $\star$ algebra homomorphism such that $\bigvee_{i=1}^{k-1} \pi(e_i) = 1$. Then $\exists$ a unique $\star$ algebra homomorphism $\tilde{\pi} : J_k(\tau) \to T_k(\tau)$ such that $\tilde{\pi} \circ \phi = \pi$.*

Proof. It is enough to show that $\pi = 0$ on $ker(\phi)$. Since $ker(\phi)$ is the ideal generated by f_n^T, it is enough to show that $\pi(f_n^T) = 0$.

Assertion: $\pi(f_n^T)\pi(e_i^T) = 0$ for $1 \leq i \leq k-1$.

Note that $f_n^T e_i^T = 0$ for $1 \leq i \leq n-1$. Hence if $k = n$ then we are done. Hence assume $k > n$. Now

$$\begin{aligned} e_n^T f_n^T e_n^T &= e_n^T f_{n-1}^T - \frac{P_{n-2}(\tau)}{P_{n-1}(\tau)} f_{n-1}^T e_n^T e_{n-1}^T e_n^T f_{n-1}^T \\ &= \frac{P_n(\tau)}{P_{n-1}(\tau)} e_n^T f_{n-1}^T \\ &= 0 \end{aligned}$$

Hence $\pi((e_n^T f_n^T)(e_n^T f_n^T)^\star)) = 0$. Hence $\pi(e_n^T f_n^T) = 0$. Hence taking adjoints $\pi(f_n^T e_n^T) = 0$. Now let i be such that $n < i \leq k$. Let $w_i = e_i^T e_{i-1}^T \cdots e_{n+1}^T$. Then $w_i e_n^T w_i^\star = \tau^{n-i} e_i^T$. But w_i commutes with T_n. Hence we have $\pi(f_n^T e_i^T) = \frac{1}{\tau^{n-i}}\pi(w_i)\pi(f_n^T e_n^T)\pi(w_i^\star) = 0$. This proves the assertion.
Since $\bigvee_{i=1}^{k-1} \pi(e_i^T) = 1$, it follows that $\pi(f_n^T) = 0$ which completes the proof.
□

Theorem 3. *Let $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$. Let $k \geq n$. Then the maximal $C^\star$ quotient of $T_k(\tau)$ is $(J_k(\tau) \oplus \mathbb{C}, \phi \oplus \tilde{1})$.*

Proof. We will show that $(J_k(\tau) \oplus \mathbb{C}, \phi \oplus \tilde{1})$ satisfies the universal property of the maximal C* quotient. Suppose A be a C* algebra and Let $\pi : T_k(\tau) \to A$ be a $\star$ algebra homomorphism. By considering the image of π, if necessary, we can assume that π is onto. Then π is unital. Let $p = \bigvee_{i=1}^{k-1} \pi(e_i^T)$. Then p is a central projection in A. Let $\pi_1 : T_k(\tau) \to pA$ be defined by $\pi_1(a) = p\pi(a)$. Then $\bigvee_{i=1}^{k-1} \pi_1(e_i^T) = 1$. Hence by Theorem 2, $\exists$ a map $\tilde{\pi_1} : T_k(\tau) \to pA$ such that $\tilde{\pi_1} \circ \phi = \pi_1$. Now define $\tilde{\pi} : J_k(\tau) \oplus \mathbb{C} \to A$ by

$\tilde{\pi}(a, \lambda) = \tilde{\pi_1}(a) + \lambda(1 - p)$. Since 1 together with nonempty reduced words form a basis for $T_k(\tau)$, it follows that $\pi(a)(1 - p) = \tilde{1}(a)(1 - p)$. Hence $\tilde{\pi} \circ (\phi \oplus \tilde{1}) = \pi$. That such a map is unique follows from the ontoness of $\phi \oplus \tilde{1}$. This completes the proof. □

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