Projective Bundle and Blow-up

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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LIST OF PUBLICATIONS ARISING FROM THE THESIS

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Dedicated to my parents

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Contents

Summary

This thesis is divided into two parts. We know that a projective space blow-up along a linear subspace has a projective bundle structure over some other projective space. In the first part of the thesis, we give some example of non-linear subvariety of a projective space such that blow up will give a projective bundle structure. Let $i : \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ be the Segre embedding and $i\big(\mathbb{P}^1 \times \mathbb{P}^2\big) = X_0$. Then we prove that $\mathbb{P}^5$ blown up along the closed subscheme X_0 is a $\mathbb{P}^3$-bundle over $\mathbb{P}^2$. We also give a description of this particular rank four bundle of $\mathbb{P}^2$. If H is a general hyperplane in $\mathbb{P}^5$, then $H \cap X_0 = X_1$ be a degree three surface in $\mathbb{P}^4$. We prove that $\mathbb{P}^4$ blow up along X_1 also has a projective bundle structure over $\mathbb{P}^2$. Similarly, consider H_1 be general hyperplane of $\mathbb{P}^4$ and $H_1 \cap X_1 = X_2$ is a degree three curve i.e, twisted cubic in $\mathbb{P}^3$. Now the blow up of $\mathbb{P}^3$ along the twisted cubic also has a projective bundle structure over $\mathbb{P}^2$. Finally, we prove that if C is a non-linear subvariety of $\mathbb{P}^3$ and blow up of $\mathbb{P}^3$ along C has a projective bundle structure then C has to be a twisted cubic in $\mathbb{P}^3$.

In the second part of the thesis, we prove some geometric aspects of $\mathbb{P}^2$ blow-up at seven points. We know that $\mathbb{P}^2$ blow-up at 6 general points can be embedded as a cubic surface in $\mathbb{P}^3$ given by the anti-canonical divisor (see [H]). Unlike the case of $\mathbb{P}^2$ blown-up at six general points, we prove that the anti-canonical divisor of $\mathbb{P}^2$ blown-up seven general points gives a finite degree two map to $\mathbb{P}^2$. We prove that $\mathbb{P}^2$ blown up at seven general points has conic bundle structures over $\mathbb{P}^1$ and we give the

list of all linear systems which give these conic bundle structures over $\mathbb{P}^1$. It is well known that $\mathbb{P}^2$ blown up at six general points is isomorphic to a smooth cubic in $\mathbb{P}^3$ and the embedding is given by the anti-canonical divisor. Conversely, any smooth cubic of $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^2$ blown up at six general points. Motivated by this, in the second part of this thesis, we prove that $\mathbb{P}^2$ blown up at seven general points can be embedded as a $(2,2)$ divisor in $\mathbb{P}^1 \times \mathbb{P}^2$ as well as in $\mathbb{P}^5$ by the very ample divisors $4\pi^*H - 2E_i - \sum_{j=1, j\neq i}^{7} E_j$. Conversely, any smooth surface in the complete linear system $|\ (2,2)\ |$ of $\mathbb{P}^1 \times \mathbb{P}^2$ can be obtained as an embedding of blowing up of $\mathbb{P}^2$ at seven points. We know that $\mathbb{P}^2$ blown up at six general points has 27 lines, when we see it as a cubic in $\mathbb{P}^3$. Also, we know that this lines are all (-1) curves and all (-1) curves are lines. Similarly, $\mathbb{P}^2$ blown up at seven general points has 56 (-1) curves. When we see it as a degree six surface in $\mathbb{P}^5$, then it has 12 lines and all of them are (-1) curves.

Any smooth surface of $|\ (2,2)\ |$ of $\mathbb{P}^1 \times \mathbb{P}^2$ has negative curves and self-intersection of any curve is at least (-2). We prove that any smooth surface S of $|\ (2,2)\ |$ of $\mathbb{P}^1 \times \mathbb{P}^2$ has at most four (-2) curves. We give one example of a smooth surface S of $|\ (2,2)\ |$ of $\mathbb{P}^1 \times \mathbb{P}^2$ which has exactly four (-2) curves. Finally, we find a very ample line bundle of any smooth surface S of $|\ (2,2)\ |$ of $\mathbb{P}^1 \times \mathbb{P}^2$, which gives closed immersion of S into $\mathbb{P}^1 \times \mathbb{P}^2$ as well as into $\mathbb{P}^5$.

Notations

Symbol	**Description**
$\mathbb{Z}$	The ring of integers
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{R}_{>0}$	The set of positive real numbers
$\mathbb{R}_{\geq 0}$	The set of non-negative real numbers
$\mathbb{C}$	The fiels of complex numbers
$\Bbbk$	An algebraically closed field
$\mathbb{P}^n_{\Bbbk}$ or $\mathbb{P}^n$	Projective n-space over an algebraically closed field $\Bbbk$
$\mathcal{O}$	sheaf of rings
$\mathcal{O}_x$	local ring of a point x on a scheme X
$\mathfrak{m}_x$	maximal ideal of local ring at x
$\mathcal{O}_X(1)$	twisting sheaf of Serre on the projective space X
$\wedge(\mathcal{E})$	exterior algebra of $\mathcal{E}$
$\mathbb{P}(\mathcal{E})$	projective space bundle
$c_i(E)$	i-th Chern class of a coherent sheaf E

$\mathrm{Div}(X)$	The set of all divisors on a variety X
$\mathcal{O}_X(D)$	The line bundle associated to the divisor D on X
$[D]$	Numerical euqivalence class of a divisor D
$N^1(X)_{\mathbb{R}}$	The real Néron-Severi group of X
$\rho(X)$	The Picard rank of X
$\mathrm{Pic}(X)$	The Picard group of X
$\mathrm{mult}_x C$	multiplicity at x of a curve C passing x
$\mathcal{F}_x$	The stalk of a coherent sheaf $\mathcal{F}$ at x
$\mathcal{I}_Z$	The ideal sheaf corresponding to a closed subscheme Z
$f^{-1}\mathcal{I} \cdot \mathcal{O}_X$	inverse image ideal sheaf.
ω_X	canonical sheaf
$\mathcal{N}_{Y/X}$	normal sheaf
$H^i(X, \mathcal{F})$	i-th cohomology group
$h^i(X, \mathcal{F})$	dimension of $H^i(X, \mathcal{F})$
$R^i f_*$	higher direct image functor
$A(X)$	Chow ring
$A_i(X)$	i-dimensional cycle modulo rational equivalence
$A^i(X)$	i-codimensional cycle modulo rational equivalence

Chapter 1

Introduction

"Algebra is a written geometry, and geometry is a drawn algebra."

1.1 History and Motivation

It is always interesting to ask, under which criterion, blow up of a projective variety along a projective subvariety is isomorphic to a projective bundle over some projective variety. In general, blow up of a projective space along a projective subvariety may not be isomorphic to a projective bundle over some projective space. But we know some examples, where it happens e.g., Theorem 3.0.1. Let $Z = \widetilde{\mathbb{P}}^n_\Lambda$ be the blow up of projective space $\mathbb{P}^n$ along a linear subspace $\Lambda \simeq \mathbb{P}^{r-1}$. We have seen in the Theorem 3.0.1 that Z is the total space of a projective bundle i.e., $Z \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^{n-r}}(1) \oplus \mathcal{O}^r_{\mathbb{P}^{n-r}}$ is a locally free sheaf of rank $r+1$ on $\mathbb{P}^{n-r}$.

Motivated by this result, in the first part of this thesis, we produce some examples of blow-up of projective spaces along some non-linear subvariety which are isomorphic to a projective bundle over a projective variety.

The degree of an algebraic surface embedded in $\mathbb{P}^3$ is the degree of the defining homogeneous polynomial. In particular, degree one surfaces in $\mathbb{P}^3$ are isomorphic to the projective plane, degree two surfaces in $\mathbb{P}^3$ are called quadric, which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

In 1849, Arthur Cayley communicated to George Salmon writing that a general cubic surface in $\mathbb{P}^3$ contains finite number of lines. Salmon gave a prove that there are exactly 27 lines in general cubics. Cayley added Salmon's proof in his paper [C], where he also proved that a general cubic surface admits 45 tritangent planes which are planes in $\mathbb{P}^3$, whose intersection with general cubic is union of three lines. In the same year Salmon wrote the paper [Sal], in which he proved that not only a general cubic but also any non-singular cubic contains exactly 27 lines. This completes the proof of the famous Cayle-Salmon theorem. 27 lines in a smooth cubic was one of the first non-trivial results in the counting curve problems in algebraic geometry. Later, Heinrich Schröter showed, how to obtain these 27 lines in a cubic by Grassmann's construction in [Sch], in the year 1863. He was the first one who give the idea that any smooth cubic is isomorphic to $\mathbb{P}^2$ blow-up at six general points.

In the meantime, an Italian mathematician Pasquale del Pezzo was classifying two-dimensional Fano varieties, which by definition are smooth algebraic surfaces with the very ample anti-canonical divisor. In his papers [Pez1](1885) and [Pez2](1887), del Pezzo studied degree d surfaces embedded in $\mathbb{P}^d$, $3 \leq d \leq 9$. After the name of Pasquale del Pezzo, the class of smooth surfaces having the ample anti-canonical divisor are called *Del Pezzo surfaces.* Del Pezzo surfaces are extensively studied in the last century. Most of the work regarding Del Pezzo surfaces can be found in [M] and [D].

The degree of a Del Pezzo surface is the self-intersection number of the anti-canonical divisor. Let S_d be a Del Pezzo surface of degree $d \geq 3$. Then $S_d \hookrightarrow \mathbb{P}^d$ and $\deg(S_d) = d$ in $\mathbb{P}^d$, and the embedding map is given by the anti-canonical divisor $-K_{S_d}$ of S_d. Hence $K_{S_d} \cdot K_{S_d} = d$. Note that, degree three Del Pezzo surfaces are cubics in $\mathbb{P}^3$. Any Del Pezzo surface of degree $d > 1$ contains finitely many negative self-intersection curves. It is also known that S_d is isomorphic to $\mathbb{P}^2$ blow-up at $(9-d)$ points, where no three points are collinear and no six points lie on a conic for $3 \leq d \leq 7$. When $d = 8$ then S_8 in $\mathbb{P}^8$ is either isomorphic to $\mathbb{P}^2$ blow-up at one point or isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. When $d = 9$, then $S_9 \cong \mathbb{P}^2 \hookrightarrow \mathbb{P}^9$, and the embedding is given by the linear system $\mid \mathcal{O}_{\mathbb{P}^2}(3) \mid$.

Conic bundle structure on a variety and their automorphisms have been extensively studied in the end of the last century. For basic definitions and properties of conic bundle one can consult [S0], [S], and [Z]. It is always interesting to know, if any variety has a conic bundle structure over some lower dimensional variety. In his thesis [B], Jérémy Blanc proved that Del Pezzo surfaces have a conic bundle structure.

In the second part of the thesis, we describe geometry of $\mathbb{P}^2$ blow-up at seven points, which is a Del Pezzo surface as well as a conic bundle over $\mathbb{P}^1$.

1.2 Arrangement of the Thesis

In the Chapter 2 of this thesis, we recall some definitions and results from algebraic geometry with proper references which we use in the subsequent chapters.

In the third chapter, we give some examples of blow-up varieties which have a projective bundle structure. In the first section of this chapter we give an example, where $\mathbb{P}^5$ blown up along a subvariety has a projective bundle structure. There we prove the Theorem 3.1.1, and the Theorem 3.1.2. In the second section of this chapter, we give another example, where $\mathbb{P}^4$ blown up along a subvariety has a projective bundle structure i.e., we prove the Theorem 3.2.1. In the section three of this chapter we prove the final theorem (Theorem 3.3.5) of the first part of this thesis. There we prove that if C is a non-linear subvariety of $\mathbb{P}^3$ (i.e. C is not a single point or a line in $\mathbb{P}^3$) and $\widetilde{\mathbb{P}}^3_C$ has a projective bundle structure, then C has to be a twisted cubic . In the final section of the chapter three we have calculate the Nef cones of those varieties.

In the Chapter 4 of this thesis, we discuss some geometric aspects of $\mathbb{P}^2$ blow-up at seven points. In the first section of this chapter, we prove the Lemma 4.1.1, where we describe that if there is a generically degree two map from $\mathbb{P}^2$ blown up at seven points to $\mathbb{P}^2$, then the map is given by the anti-canonical divisor. In the second section, we prove the Theorem 4.2.1, which describes all linear systems of $\mathbb{P}^2$ blow-up at seven general points, which give conic bundle structure over $\mathbb{P}^1$. The Theorem 4.3.4 is proved in the third section of this chapter. There we provide all possible very ample divisors of $\mathbb{P}^2$ blow-up at seven general points, which correspond the embeddings into $\mathbb{P}^1 \times \mathbb{P}^2$ as a $(2,2)$ type divisor. The fourth section consists of descriptions of lines in $\mathbb{P}^2$ blow-up at seven general points as embedded in $\mathbb{P}^1 \times \mathbb{P}^2$. In the Section 5, we prove the Theorem 4.5.1 i.e., any smooth surface of the linear system $\mid (2,2) \mid$ is isomorphic to $\mathbb{P}^2$ blown up at seven points. Also, there are examples of smooth surfaces linearly equivalent to $(2,2)$ of $\mathbb{P}^1 \times \mathbb{P}^2$ which are isomorphic to $\mathbb{P}^2$ blown up at seven non-general points (see Example 4.5.6). Also, in this section we see that if $S \sim (2,2)$ be a smooth surface of $\mathbb{P}^1 \times \mathbb{P}^2$ and C be a curve in S, then $C.C \geq -2$. We show that there will be at most four curves in S which have self-intersection (-2) (see Theorem 4.5.4). Note that, as any smooth surface of $\mid (2,2) \mid$ of $\mathbb{P}^1 \times \mathbb{P}^2$ is isomorphic to $\mathbb{P}^2$ blown up at seven points. In this fifth section we find a very ample divisor of $\mathbb{P}^2$ blown up at seven points which corresponds this embedding as it was stated in Remark 4.5.11 .

1.3 Conventions

Through out this thesis, any scheme X will be a noetherian scheme over an algebraically closed field $\Bbbk$. A variety is an integral separated scheme of finite type over an algebraically closed field. In this thesis, the words " line bundle ", " invertible sheaf ", and " locally free sheaf of rank one " carry same meaning. The same is true for the words " vector bundle ", and " locally free sheaf of finite rank ". A divisor of an integral scheme means by the Cartier divisor. When X is an integral scheme, then we use the canonical isomorphism between divisor class group and Picard group, the group of isomorphism classes of line bundles. If $[D]$ is a divisor class on X, then $\mathcal{O}(D)$ is the corresponding isomorphism class of line bundle. In the case of integral scheme, the words " line bundle ", and " Cartier divisor " are used interchangeably.

Chapter 2

Preliminaries

In this chapter, we recall some basic definitions and results which we will use to prove main results of this thesis.
In the first three sections of this chapter, we describe projective bundle, blow-up of varieties, Chow rings, and Chern classes. Most of the results discussed in these three sections can be found in [H], [F], and [EH]. In the fourth section, we recall the definition and basic properties of a conic bundle from [S], and [P]. In the final section of this chapter, we discuss about the nef cone and the pseudoeffective cone of curves, and the reference of this discussion is [L].

2.1 Projective Bundle

Let X be a noetherian scheme, and $\mathcal{J}$ be a coherent sheaf of $\mathcal{O}_X$-module which has a graded $\mathcal{O}_X$-algebra structure. Thus, $\mathcal{J} = \bigoplus_{d\geq 0} \mathcal{J}_d$, where $\mathcal{J}_d$ is the degree d homogeneous part of the graded algebra $\mathcal{J}$. Furthermore, we assume that $\mathcal{J}_0 = \mathcal{O}_X$, $\mathcal{J}_1$ is coherent $\mathcal{O}_X$-module, and $\mathcal{J}$ is locally generated by $\mathcal{J}_1$ as an $\mathcal{O}_X$-algebra.

Let X can be covered by open affine subsets, $\{U_\lambda = \mathrm{Spec}(A_\lambda) \mid \lambda \in \Lambda\}$, such that $\mathcal{J}(U_\lambda) = \Gamma(U_\lambda, \mathcal{J}|_{U_\lambda})$ is a graded A_λ-algebra. Then we consider $\mathrm{Proj}(\mathcal{J}(U_\lambda))$, and the natural projection map $\pi_\lambda : \mathrm{Proj}(\mathcal{J}(U_\lambda)) \to U_\lambda$.

One can easily check that, if U_{λ_1} and U_{λ_2} are two such open affine subsets of X, then

$$\pi_{\lambda_1}^{-1}(U_{\lambda_1} \cap U_{\lambda_2}) \cong \pi_{\lambda_2}^{-1}(U_{\lambda_1} \cap U_{\lambda_2}).$$

This isomorphisms allow us to glue the schemes $\{\mathrm{Proj}(\mathcal{J}(U_\lambda)) \mid \lambda \in \Lambda\}$ to obtain another scheme $\mathbf{Proj}(\mathcal{J})$, which comes with a natural projection map $\pi : \mathbf{Proj}(\mathcal{J}) \to$ X, such that the restriction of π on each subscheme $\mathrm{Proj}(\mathcal{J}(U_\lambda))$ is π_λ. Furthermore, gluing the invertible sheaves $\mathcal{O}_{\mathrm{Proj}(\mathcal{J}(\mathrm{U}_\lambda))}(1)$ of each $\mathrm{Proj}(\mathcal{J}(\mathrm{U}_\lambda))$, we construct the invertible sheaf $\mathcal{O}_{\mathbf{Proj}(\mathcal{J})}(1)$ on $\mathbf{Proj}(\mathcal{J})$, which is also called the tautological sheaf.

Let $\mathcal{E}$ be a locally free coherent sheaf on X. Let $\mathcal{J} = S(\mathcal{E})$ be the symmetric algebra of $\mathcal{E}$ i.e., $\mathcal{J} = \bigoplus_{d \geq 0} S^d(\mathcal{E})$, which is a graded $\mathcal{O}_X$-algebra.

Definition 2.1.1. *We define the projective bundle $\mathbb{P}(\mathcal{E}) = \mathbf{Proj}(\mathcal{J})$, which comes with the projection map $\pi : \mathbb{P}(\mathcal{E}) \to X$, and the invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, which is also called the tautological sheaf of the projective bundle $\mathbb{P}(\mathcal{E})$.*

Note that, if $\mathcal{E}$ is a locally free sheaf of rank $n+1$ over X, then $\mathcal{E}$ is a free sheaf of rank $n+1$ over some open subset U of X. In that case, $\pi^{-1}(U) \cong U \times \mathbb{P}^n$.

Proposition 2.1.2. *Let X, $\mathcal{E}$, $\mathbb{P}(\mathcal{E})$ be as in the Definition 2.1.1 and rank of $\mathcal{E} \geq 2$. Then :*

(a) there is a canonical isomorphism of graded $\mathcal{O}_X$-algebra $\mathcal{J} = \bigoplus_{d \geq 0} S^d(\mathcal{E}) \cong \bigoplus_{l \in \mathbb{Z}} \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(l))$, with the grading on the right hand side is given by l.

In particular, for $l < 0$, $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(l)) = 0$; for $l = 0$, $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}) = \mathcal{O}_X$, and for $l = 1$, $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = \mathcal{E}$.

(b) there is a natural surjective morphism $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ of locally free sheaves.

Proof. See [H], II.7.11. □

Now we state the following theorem to describe the *Universal Property of Proj.*

Theorem 2.1.3. *Let X, $\mathcal{E}$, $\mathbb{P}(\mathcal{E})$ be as above and Y be any scheme. Then there is an one-to-one correspondence between a morphism $g : Y \to X$ such that the following diagram commutes,*

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, "g"'] & & \mathbb{P}(\mathcal{E}) \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

and an invertible sheaf $\mathcal{L}$ on Y such that there is a surjective map $g^*\mathcal{E} \to \mathcal{L}$ of locally free sheaves on Y.

Proof. See [H], II.7.12. □

The following proposition characterizes the schemes over X which have a projective bundle structure.

Proposition 2.1.4. *Let $\pi : Y \to X$ be a smooth morphism of projective schemes, whose fibers are all isomorphic to $\mathbb{P}^n$. Then, the following are equivalent:*
(a) $Y \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle of rank $n+1$ on X.
(b) X can be covered by some open subsets U such that $\pi^{-1}(U) \cong U \times \mathbb{P}^n$, and the transition automorphisms on $\mathrm{Spec(A)} \times \mathrm{\mathbb{P}^n}$ are given by A-linear automorphisms of the homogeneous coordinate ring $A[x_0, \cdots, x_n]$ (e.g., $x_i' = \sum a_{ij}x_j$, $a_{ij} \in A$).
(c) There exists a line bundle $\mathcal{L}$ on Y whose restriction to each fiber $Y_x \simeq \mathbb{P}^n$ of π is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(1)$.
(d) There exists a Cartier divisor $D \subset Y$ whose intersection to general fiber $Y_x \simeq \mathbb{P}^n$ of π is a hyperplane.

Proof. See [EH], Proposition 9.4. □

The projection map $\pi : \mathbb{P}(\mathcal{E}) \to X$ induces the injective map in Picard groups level,

$$\pi^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\mathbb{P}(\mathcal{E}))$$

It is also known that,

$$\mathrm{Pic}(\mathbb{P}(\mathcal{E})) \cong \mathrm{Pic}(X) \times \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$$

In the next corollary, we will see when two different vector bundles give rise to the same projective bundle up to isomorphism.

Corollary 2.1.5. *Let X be a scheme. Two projective bundles $\pi : \mathbb{P}(\mathcal{E}) \to X$ and $\pi' : \mathbb{P}(\mathcal{E}) \to X$ are isomorphic as X-schemes if and only if there is a line bundle $\mathcal{L}$ on X such that $\mathcal{L} \otimes \mathcal{E}' = \mathcal{E}$. In this case, the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ corresponds to the line bundle $\pi'^*(\mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$ under this isomorphism in Picard groups level.*

Proof. See [EH], Corollary 9.5. □

2.2 Blow-up

Let X be a noetherian scheme, and let $\mathcal{I}$ be a coherent sheaf of ideals on X. Consider the sheaf

$$\mathcal{G} = \bigoplus_{d \geq 0} \mathcal{I}^d$$

which has a graded $\mathcal{O}_X$-algebra structure, where $\mathcal{I}^d$ is the d-th power of the ideal $\mathcal{I}$ and $\mathcal{I}^0 = \mathcal{O}_X$. Now we can consider $\widetilde{X} = \mathbf{Proj}(\mathcal{G})$ as the blow-up of X with respect to the coherent sheaf of ideals $\mathcal{I}$. Let Y be the closed subscheme of X corresponding to the ideal sheaf $\mathcal{I}$. Then $\widetilde{X}$ is also called the blow-up of X along the closed subscheme Y. $\pi : \widetilde{X} \to X$ is the natural projection map. Inverse image of the ideal sheaf $\mathcal{I}$ along the map π is an invertible sheaf $\widetilde{\mathcal{I}} = (\pi^{-1}\mathcal{I}) \cdot \mathcal{O}_{\widetilde{X}}$ of $\widetilde{X}$. Moreover, π is a birational map such that $\pi : \pi^{-1}(X - Y) \to X - Y$ is an isomorphism.

If X is a smooth variety, then the invertible sheaf $\widetilde{\mathcal{I}}$ corresponds to an effective divisor, which is called the exceptional divisor. If $\mathcal{I}$ corresponds to the closed subscheme Y, then the exceptional divisor $\pi^{-1}(Y) = E_Y$ is corresponded by the invertible sheaf $\widetilde{\mathcal{I}}$. Also, we can describe E_Y as a projective bundle over Y as follows,

$$E_Y = \mathbf{Proj}(\mathcal{G} \otimes \mathcal{O}_{\mathrm{X}}/\mathcal{I}) = \mathbf{Proj}(\mathcal{O}_{\mathrm{X}}/\mathcal{I} \oplus \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \cdots)$$

Note that, $\mathcal{I}/\mathcal{I}^2$ is the dual of the normal sheaf $\mathcal{N}_{Y/X}$ of Y in X, i.e. $\mathcal{I}/\mathcal{I}^2 = \mathcal{N}^*_{Y/X}$. Then,

$$E_Y = \mathbf{Proj}(\mathcal{N}^*_{\mathrm{Y/X}}) = \mathbb{P}(\mathcal{N}^*_{\mathrm{Y/X}})$$

If X is any smooth variety, and Y is any subvariety of codimension at least two, then we denote the blow-up of X along Y by $\widetilde{X}_Y$. Hence we have the following commutative diagram,

$$\begin{tikzcd}
E_Y = \mathbb{P}(\mathcal{N}^*_{Y/X}) \arrow[r, hook, "j"] \arrow[d, "\pi_Y"] & \widetilde{X}_Y = \mathbf{Proj}(\mathcal{G}) \arrow[d, "\pi"] \\
Y \arrow[r, hook, "i"] & X
\end{tikzcd}$$

Proposition 2.2.1. *With the same notations as described above, the normal sheaf of $E_Y = \mathbf{Proj}(\mathcal{N}^*_{\mathrm{Y/X}})$ in $\widetilde{X}_Y$ is*

$$\mathcal{N}_{E_Y/\widetilde{X}_Y} = \mathcal{O}_{\mathbb{P}(\mathcal{N}^*_{Y/X})}(-1),$$

where $\zeta_{E_Y} = \mathcal{O}_{\mathbb{P}(\mathcal{N}^*_{Y/X})}(1)$ is the tautological sheaf on $\mathbb{P}(\mathcal{N}^*_{Y/X})$.

Proof. See, [EH], Proposition 13.11. □

Now we state the *Universal Property of Blowing Up*,

Proposition 2.2.2. *Let X be a noetherian scheme, $\mathcal{I}$ be a coherent sheaf of ideals, and $\pi : \widetilde{X} \to X$ is the blowing-up map with respect to $\mathcal{I}$. If $f : Z \to X$ is any* *morphism such that* $(f^{-1}\mathcal{I}) \cdot \mathcal{O}_Z$ *is an invertible sheaf of ideals on* Z*, then there exists a unique morphism* $g : Z \to \widetilde{X}$ *factoring* f*,*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "f"'] & \widetilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

Proof. See [H], II.7.14. □

Definition 2.2.3. *Let* Y' *be any subvariety of* X *other than* Y*. The strict transformation or proper transformation of* Y' *is to be the closure of* $\pi^{-1}(Y' \cap (X - Y))$ *in* $\widetilde{X}_Y$*, which is denoted by* $\widetilde{Y'}$*.*

In the next theorem, we will see how blow-up resolves base loci of rational map to projective space.

Theorem 2.2.4. *Let* X *be a scheme,* $\mathcal{L}$ *be a line bundle on* X*, and* $s_0, s_1, \cdots, s_n$ *be sections of* $\mathcal{L}$*, which corresponds the linear sub-system* $\mathfrak{b}$ *of* $\mathcal{L}$*. Consider* Y *be the base locus of* $\mathfrak{b}$*. Let* $\phi : X \dashrightarrow \mathbb{P}^n$ *be a rational map corresponding to the linear system* $\mathfrak{b}$*. Then the rational map* ϕ *can be extended uniquely from* $\widetilde{X}_Y$*, i.e.* $\widetilde{\phi} : \widetilde{X}_Y \to \mathbb{P}^n$ *and this morphism* $\widetilde{\phi}$ *corresponds to the invertible sheaf* $\pi^*(\mathcal{L}) \otimes \mathcal{O}(-E_Y)$*, where* $\pi : \widetilde{X}_Y \to X$ *is the natural blow-up map.*

Proof. See [V], 22.4.L, page no. 598. □

There is another blow-up extension theorem on surface, which is called "elimination of indeterminacy":

Theorem 2.2.5. *Let* $g : S \dashrightarrow X$ *be a rational map from a smooth projective surface to a projective variety. Then there exists a surface* S'*, and a morphism* $\pi : S' \to S$ *which is the compositions of finite number of blow-ups such that* g *can be extended from* S' *as* $\widetilde{g} : S' \to X$*, i.e., the following diagram commutes,*

$$\begin{tikzcd} & S' \arrow[dl, "\pi"'] \arrow[dr, "\widetilde{g}"] & \\ S \arrow[rr, dashed, "g"] & & X \end{tikzcd}$$

Proof. See [BE], Theorem II.7. ☐

2.2.1 Blow-up of surface at points

In this section, X be a smooth projective surface and Y be a finite set of points in X. Let $Y = \{P_1, P_2, \cdots, P_r\}$ and we denote the blow-up of X at $P_1, P_2, \cdots, P_r$ by $\widetilde{X}_{P_1P_2\cdots P_r}$ or $\widetilde{X}_r$. Here $\pi : \widetilde{X}_{P_1P_2\cdots P_r} \to X$ is the blowing-up map. Note that, $\widetilde{X}_{P_1P_2\cdots P_r} - \pi^{-1}\{P_1, P_2, \cdots, P_r\} \cong X - \{P_1, P_2, \cdots, P_r\}$ and $\pi^{-1}(P_i) = E_{P_i}$ is the exceptional curve with $E_{P_i} \cong \mathbb{P}^1$ for each $i \in \{1, \cdots, r\}$.

As X is a smooth variety, there is a isomorphism between the set of isomorphism classes of line bundles $\mathrm{Pic}(X)$, and the set of linear equivalence classes of divisors $\mathrm{Cl}(X)$. Any curve C in X corresponds to the class $[C]$ of an effective divisor in $\mathrm{Cl}(X)$, and the isomorphism map from $\mathrm{Cl}(X)$ to $\mathrm{Pic}(X)$ is given by $[C] \to \mathcal{O}_X(C)$. If C_1 and C_2 are two distinct irreducible curves in X and $x \in C_1 \cap C_2$, then we define the *intersection multiplicity* of C_1 and C_2 at x by

$$m_x(C_1 \cap C_2) = \dim\big(\mathcal{O}_{x,X}/(f_1, f_2)\big),$$

where f_1 and f_2 are local equations of C_1 and C_2 in $\mathcal{O}_{x,X}$ respectively. In particular, if $m_x(C_1 \cap C_2) = 1$, then C_1 and C_2 intersect transversely at x.

Definition 2.2.6. *Let C_1 and C_2 be two irreducible curves in X. The intersection number $C_1 \cdot C_2$ is defined by:*

$$C_1 \cdot C_2 = \sum_{x \in C_1 \cap C_2} m_x(C_1 \cap C_2)$$

We can define a bilinear form on Pic(X), using intersection number as defined above.

$$\mathcal{O}_X(C_1)\cdot\mathcal{O}_X(C_2)=C_1\cdot C_2$$

Lemma 2.2.7. *Let C be a non-singular irreducible curves on X and L be any line bundle of X. Then,*

$$\mathcal{O}_X(C)\cdot L=deg(L|_C)$$

Proof. See [BE], I.6 □

We have the natural blow-up map $\pi:\widetilde{X}_{P_1P_2\cdots P_r}\to X$, which induces the pullback map between Picard groups, $\pi^*:Pic(X)\to Pic(\widetilde{X}_{P_1P_2\cdots P_r})$. Moreover,

$$Pic(\widetilde{X}_{P_1P_2\cdots P_r})\cong\pi^*Pic(X)\bigoplus_{i=1}^{r}\mathbb{Z}\mathcal{O}(E_{P_i})$$

The intersection products of generators of $Pic(\widetilde{X}_{P_1P_2\cdots P_r})$ is described in the following proposition:

Proposition 2.2.8. *Let X be a smooth surface and $\widetilde{X}_{P_1P_2\cdots P_r}$ be as above. Then:*
(i) if $C_1,C_2\in$ Pic(X), then $(\pi^*C_1)\cdot(\pi^*C_2)=C_1\cdot C_2$;
(ii) if $C\in$Pic(X), then $(\pi^*C)\cdot\mathcal{O}(E_{P_i})=0$, for $i=1,\cdots,r$;
(iii) $E_{P_i}^2=-1$, for $i=1,\cdots,r$;
(iv) Let us denote $\pi_*:Pic(\widetilde{X}_{P_1P_2\cdots P_r})\to\pi^*Pic(X)$ as the first projection of the isomorphism $Pic(\widetilde{X}_{P_1P_2\cdots P_r})\simeq\pi^*Pic(X)\bigoplus_{i=1}^r\mathbb{Z}\mathcal{O}(E_{P_i})$. Now, if $C\in Pic(X)$ and $C'\in\widetilde{X}_{P_1P_2\cdots P_r}$, then $(\pi^*C)\cdot C'=C\cdot(\pi_*C')$.

Proof. See [H], V.3.2. □

We can also give the description of the canonical divisor of the blow-up variety;

Proposition 2.2.9. *The canonical divisor of* $\widetilde{X}_{P_1P_2\cdots P_r}$ *is given by*

$$K_{\widetilde{X}_{P_1P_2\cdots P_r}} = \pi^* K_X + \sum_{i=1}^{r} E_{P_i},$$

where K_X *is the canonical divisor of* X*. Clearly,* $K^2_{\widetilde{X}_{P_1P_2\cdots P_r}} = K_X^2 - r$.

Proof. See [H], V.3.3. □

Now we will see what happens to an effective curve under the blow-up map $\pi : \widetilde{X}_{P_1P_2\cdots P_r} \to X$. Let C be an effective curve on X, and $\widetilde{C}$ be the strict transformation of C, i.e., $\widetilde{C}$ is the closure of $\pi^{-1}\big(C \cap (X - \{P_1, \cdots, P_r\})\big)$ in $\widetilde{X}_{P_1P_2\cdots P_r}$. Also, it is clear that $\widetilde{C}$ is the closure of $\pi^*(C) \cap (\widetilde{X}_r - \sum_{i=1}^{r} E_i)$ in $\widetilde{X}_{P_1P_2\cdots P_r}$.

Definition 2.2.10. *Let* C *be an effective curve on the surface* X*,* P *be a point of* X*, and* $f \in \mathcal{O}_{P,X}$ *be a local equation of* C *at* P*. Also consider* $\mathfrak{m}_{P,X}$*, the maximal ideal of* $\mathcal{O}_{P,X}$*. The multiplicity of* C *at* P *is defined by,*

$$\mu_P(C) = max\{r \in \mathbb{N} \cup 0 \mid f \in \mathfrak{m}^r_{P,X}\}$$

Remark 2.2.11. *If* $P \notin C$*, then* $\mu_P(C) = 0$*. Moreover,* C *has a singularity at* P *if and only if* $\mu_P(C) > 1$*. We will see that the the intersection multiplicity of* E_P *and* $\widetilde{C}$ *depends on the multiplicity of* C *at the point* P*.*

Proposition 2.2.12. *Let* C *be an effective curve on* X*, and let the multiplicity of* C *at the point* P_i *be* r_i*, for* $i = 1, \cdots, r$*. Then under the blow-up map* $\pi : \widetilde{X}_{P_1P_2\cdots P_r} \to X$*,*

$$\pi^* C = \widetilde{C} + \sum_{i=1}^{r} r_i E_{P_i}$$

.

Proof. See [H], V.3.6. □

Corollary 2.2.13. *With the same hypothesis of the Proposition 2.2.12, we have* $\widetilde{C} \cdot E_{P_i} = r_i$ *for each* i*, and the arithmetic genus of the curve* $\widetilde{C}$*,*

$$g(\widetilde{C}) = g(C) - \sum_{i=1}^{r} \frac{1}{2} r_i(r_i - 1),$$

where $g(C)$ *is the genus of the curve* C*.*

Proof. See [H], V.3.7 and V.3.9.2. □

Let us consider X as the projective plane $\mathbb{P}^2$. Let P_1, P_2, P_3 be any three points of $\mathbb{P}^2$. The *quadratic transformation* centered at P_1, P_2, P_3 is a birational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ which is defined by a linear subsystem $\mathfrak{d}$ of $|\ \mathcal{O}_{\mathbb{P}^2}(2)\ |$, such that ϕ is defined on the open set $\mathbb{P}^2 - \{P_1, P_2, P_3\}$. For example, if $P_1 = (1,0,0)$, $P_2 = (0,1,0)$ and $P_3 = (0,0,1)$, then the linear system $\mathfrak{d}$ is spanned by x_1x_2, x_0x_2, and x_0x_1, where x_0, x_1, x_2 are homogeneous coordinates of $\mathbb{P}^2$. For more details, see [H], V.4.2.3.

Let $P_1, \cdots, P_r$ be a finite set of r-points of $\mathbb{P}^2$. We say that these r-points are in *general position* if no three are collinear and after finite number of quadratic transformations, the new set of r-points also has no three collinear.

If $r \leq 8$, then $P_1, \cdots, P_r$ are in general position if and only if no three are collinear and no six lie on a conic.

Theorem 2.2.14. *Let* $P_1, \cdots, P_r$ *be any set of* r*-points in* $\mathbb{P}^2$ *and* P_1, P_2, P_3 *are not collinear. Take a quadratic transformation* ϕ *centered at* P_1, P_2, P_3 *and we get a new set of* r*-points in* $\mathbb{P}^2$*, which is* $\{Q_1, Q_2, Q_3, \phi(P_i) = P_i' \mid i = 4, \cdots, r\}$*. Then the following diagram commutes,*

$$\begin{array}{ccc} \widetilde{P}^2_{P_1P_2P_3\cdots P_r} & \xrightarrow{j} & \widetilde{P}^2_{Q_1Q_2Q_3\cdots P_r'} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathbb{P}^2 & \overset{\phi}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

where π *and* π' *are corresponding blow-up maps and* j *is an isomorphism. Moreover,* $d\pi^*(H) - \sum_{i=1}^{r} a_i E_i \sim j^*(d'\pi'^*(H) - \sum_{i=1}^{r} a_i' E_i')$*, where* $d' = 2d - a_1 - a_2 - a_3$*,*

$a'_1 = d - a_2 - a_3$, $a'_2 = d - a_1 - a_3$, $a'_3 = d - a_1 - a_2$, *and* $a'_k = a_k$, *for* $k = 4, \cdots, r$. *(H be a hyperplane section of* $\mathbb{P}^2$*).*

Proof. See [H], V.4. $\square$

Proposition 2.2.15. *Let* $P_1, \cdots, P_r$ *be any set of r-points in* $\mathbb{P}^2$*. Then;*

(a) $Pic(\widetilde{\mathbb{P}^2}_r) \cong \mathbb{Z}^{r+1}$, *which is generated by* $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, $\mathcal{O}(E_i)$, *for* $i = 1, \cdots, r$;

(b) $\mathcal{O}(K_{\widetilde{\mathbb{P}^2}_r}) \sim \pi^*(\mathcal{O}_{\mathbb{P}^2}(-3)) \otimes \mathcal{O}(\sum E_i)$

(c) Let D be an effective divisor of $\widetilde{\mathbb{P}^2}_r$*. Then,*

$$D \sim aL - \sum_{i=1}^{r} b_i E_i,$$

for some $a, b_i \in \mathbb{Z}$ *and* $a \geq 0$, *where L is the divisor class of* $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$.

If D be a irreducible curve and $a > 0$, *then* $b_i \geq 0$.

(d) The arithmetic genus of D is

$$g(D) = \frac{1}{2}(a-1)(a-2) - \frac{1}{2}\sum_{i=1}^{r} b_i(b_i - 1)$$

.

2.3 Chern Classes and Chow Ring

The group of cycles on X is denoted by $Z(X)$, which is the free abelian group generated by the set of subvarieties of X. Let $Z_k(X)$ be the set of formal linear combinations of k-dimensional subvarieties of X which are called k-cycles for $k \geq 0$. Hence,

$$Z(X) = \bigoplus_{k=0}^{dim(X)} Z_k(X)$$

The Chow group of X, $A(X)$ is the group of cycles of X modulo rational equivalence i.e.,

$$A(X) = Z(X)/\sim$$

Moreover, the Chow group of X is graded by dimension i.e.,

$$A(X) = \bigoplus_{k=0}^{dim(X)} A_k(X)$$

where $A_k(X)$ is the group of rational equivalence classes of k-cycles. For more details about the chow group and the rational equivalence, see [F].

When X is an equidimensional variety, we may define the codimension of a subvariety $Y \subseteq X$ by dim$X-$ dim$Y$. When X is a smooth variety, then we may grade the Chow group by the codimension, where the group $A_{dim(X)-c}(X)$ corresponds to $A^c(X)$, the group of codimension c-cycles up to rational equivalence.

Note that $A(X)$ has the induced product structure. If A and B are two subvarieties of X, then $[A]\cdot[B] = [A \cap B]$. This binary operation makes

$$A(X) = \bigoplus_{c=0}^{dim(X)} A^c(X)$$

as an associative, commutative ring, and graded by codimension, which is also called the Chow ring of X.

Let $\mathcal{L}$ be a line bundle with a rational section σ on X, and $\dim(X) = n$. Line bundle with a rational section (independent of choice) corresponds to a divisor class up to linear equivalence.

$$c_1 : \mathrm{Pic}(X) \to \mathrm{Cl}(X)$$
$$(\mathcal{L}, \sigma) \to \mathrm{Div}(\sigma)$$

Note that, the rational equivalence on $(n-1)$-dimensional cycles is same as the linear equivalence on divisors on X i.e., $\mathrm{Cl}(X) \cong A_{n-1}(X)$.

Proposition 2.3.1. *Let X be a variety of dimension n. Then the map $c_1 : Pic(X) \to$*

$Cl(X)$ is a group homomorphism. If X is smooth, then c_1 is an isomorphism.

Definition 2.3.2. *$c_1(\mathcal{L})$ is called the first Chern class of the line bundle $\mathcal{L}$.*

Similarly we can define the higher Chern classes of a vector bundle. Let $\mathcal{E}$ be a globally generated vector bundle of rank r on X and $\tau_0, \tau_1, \cdots, \tau_{r-i}$ be global sections of $\mathcal{E}$, where $i \leq r$. Let $D = V(\tau_0 \wedge \cdots \wedge \tau_{r-i})$ be the degeneracy locus where $\tau_0, \tau_1, \cdots, \tau_{r-i}$ are dependent. Then $c_i(\mathcal{E}) = [D] \in A^i(X)$ is the *i-th Chern class* of $\mathcal{E}$.

To define Chern classes of any vector bundle, we use the *Splitting construction.*

Lemma 2.3.3. *Let X be any smooth variety and let $\mathcal{E}$ be a vector bundle of rank r on X. Then there exists a smooth variety Y with a flat morphism $\phi : Y \to X$ such that following properties hold:*
(a) The induced pullback map $\phi^* : A(X) \to A(Y)$ is injective.
(b) The pullback bundle $\phi^*\mathcal{E}$ on Y has a following filtration,

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \mathcal{E}_{r-1} \subset \mathcal{E}_r = \phi^*\mathcal{E}$$

with successive quotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ are line bundle.

Proof. See [EH], Lemma 5.12. □

Now,

$$c_t(\mathcal{E}) = \Pi_{i=1}^r (1 + c_1(\mathcal{E}_i/\mathcal{E}_{i-1})t)$$

Then $c_i(\mathcal{E}) =$ the coefficient of t^i of the above equation. Note that $c_i(\mathcal{E}) \in A^i(X)$ and the above equation is called the *Chern polynomial* of $\mathcal{E}$. In other words,

$$c_t(\mathcal{E}) = 1 + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \cdots + c_r(\mathcal{E})t^r$$

Example 2.3.4. *Let $\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$ be a vector bundle on X where $\mathcal{L}_i$ are line bundles for $i = 1, \cdots, r$. We can use the Lemma 2.3.3, to calculate Chern classes of $\mathcal{E}$. Here we consider $Y = X$ and $\mathcal{E}_i/\mathcal{E}_{i-1} = \mathcal{L}_i$.*

2.3.1 Functoriality

Let $f : X \to Y$ be a proper morphism of schemes. If V be a subvariety of X, then $f(V)$ will be a subvariety of Y. But $\dim(f(V)) \leq \dim V$. Now set,

$$deg\big(V/f(V)\big) = \begin{cases} \big[\Bbbk(V) : \Bbbk(f(V))\big], & \text{if } \dim(f(V)) = \dim(V). \\ 0, & \text{if } \dim(f(V)) < \dim(V)., \end{cases}$$

where $\Bbbk(V)$ and $\Bbbk(f(V))$ are the field of rational functions of V and $f(V)$ respectively. Now, we define a linear homomorphism between $Z_k(X)$ and $Z_k(Y)$ by pushing forward the k-th cycles;

$$f_* : Z_k(X) \to Z_k(Y),$$

such that $f_*(V) = deg\big(V/f(V)\big)f(V)$.

Moreover, the pushforward map f_* preserves the rational equivalence.

Theorem 2.3.5. *Let $f : X \to Y$ be a proper morphism of schemes, and let α be a k-cycle on X which is rationally equivalent to zero. Then $f_*(\alpha)$ is also rationally equivalent to zero on Y. Hence the map $f_* : Z_k(X) \to Z_k(Y)$ induces a group homomorphism $f_* : A_k(X) \to A_k(Y)$, as well as $f_* : A(X) \to A(Y)$.*

Proof. See [F], Theorem 1.4. □

Let $f : X \to Y$ be a flat morphism of relative dimension n, and V be a subvariety of dimension k of Y. Define,

$$f^*(V) = f^{-1}(V)$$

This can be extended as a pullback homomorphism,

$$f^* : Z_k(Y) \to Z_{k+n}(X).$$

This pullback map, f^* preserves rational equivalence.

Theorem 2.3.6. *Let $f : X \to Y$ be a flat morphism of relative dimension n, and α be a k-cycle on Y which is rationally equivalent to zero. Then $f^*(\alpha)$ is also rationally equivalent to zero in $Z_{k+n}(X)$. Hence the map $f^* : Z_k(Y) \to Z_{k+n}(X)$ induces a group homomorphism $f^* : A_k(Y) \to A_{k+n}(X)$, as well as $f^* : A(Y) \to A(X)$.*

Proof. See [F], Theorem 1.7. □

In general, we can also define the pullback morphism which is induced from any morphism $f : X \to Y$ between two varieties using the excess intersection formula. More details on excess intersection can be found in [F], Chapter 6.

Theorem 2.3.7. *Let $f : Y \to X$ be a morphism between two varieties Y and X, and $Z \subseteq X$ be a smooth subvariety of X. Also let $W = \pi^{-1}(Z)$ and assume that W is smooth, and covered by the connected components W_k of dimension d_k. Consider the inclusion maps, $i : Z \hookrightarrow X$ and $i'_k : W_k \hookrightarrow Y$, and $f_k : W_k \to Z$ be the restriction of f to W_k. Then for any $\beta \in A_r(Z)$,*

$$f^*(i_*\beta) = \sum_k (i'_k)_* \{f_k^*(\beta c(\mathcal{N}_{Z/X}))s(W_k, Y)\}_{r+dim(Y)-dim(X)}$$

Proof. See [F], Chapter 6.3. □

2.3.2 Chow Ring of Projective Bundle

In the next theorem, we see that the Chow ring of a projective bundle in terms of the Chow ring of the base variety and the Chern classes of the vector bundle;

Theorem 2.3.8. *Let $\mathcal{E}$ be a vector bundle of rank $n+1$ on a smooth projective scheme X, let $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the projection map, and let $\zeta = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \in A^1(\mathbb{P}(\mathcal{E}))$. Then the pullback map between Chow rings, $\pi^* : A(X) \to A(\mathbb{P}(\mathcal{E}))$ is an injective ring homomorphism and we have the following isomorphism;*

$$A(\mathbb{P}(\mathcal{E})) \cong A(X)[\zeta]/\big(\zeta^{n+1} - \pi^* c_1(\mathcal{E})\zeta^n + \cdots + (-1)^{n+1}\pi^* c_{n+1}(\mathcal{E})\big).$$

In particular,

$$A(X)^{n+1} \cong A(\mathbb{P}(\mathcal{E}))$$

as a group, and the homomorphism is give by $(\alpha_1, \alpha_2, \cdots, \alpha_r) \to \sum \zeta^i \pi^*(\alpha_i)$.

Proof. See [EH], Theorem.9.6. □

2.3.3 Chow Ring of Blow-up Variety

In the next theorem, we describe generators of the Chow group of blow-up varieties. We use same notations of blow-up space as it is defined in the section 2.2 and in the Proposition 2.2.1.

Proposition 2.3.9. *Let $\pi : \widetilde{X}_Y \to X$ be the blow-up of X along Y, and $E_Y = \pi^{-1}(Y)$ be the exceptional divisor in $\widetilde{X}_Y$. Then the Chow ring $A(\widetilde{X}_Y)$ is generated by pullback cycles from X i.e., $\pi^* A(X)$ and the image of the Gysin homomorphism i.e., $j_* A(E_Y)$, induced by the inclusion $j : E_Y \hookrightarrow \widetilde{X}_Y$. The rules for the multiplication described in the following,*

$$\begin{aligned} \pi^*\alpha \cdot \pi^*\beta &= \pi^*(\alpha \cdot \beta) && \textit{for } \alpha, \beta \in A(X). \\ \pi^*\alpha \cdot j_*\gamma &= j_*(\gamma \cdot \pi_Y^* i^* \alpha) && \textit{for } \alpha \in A(X),\ \gamma \in A(E_Y) \\ j_*\gamma \cdot j_*\delta &= -j_*(\gamma \cdot \delta \cdot \zeta) && \textit{for } \gamma, \delta \in A(E_Y). \end{aligned}$$

Proof. See [EH], Proposition 13.12. □

Non-trivial cases arises when we blow-up a subscheme of dimension greater than one. Let us consider the first nontrivial case, $\mathbb{P}^3$ blow-up along a smooth curve. Let C be a curve of genus g and degree d in $\mathbb{P}^3$. We have the following diagram,

$$\begin{tikzcd} E_C = \mathbb{P}(\mathcal{N}^*_{C/\mathbb{P}^3}) \arrow[r, hook, "j"] \arrow[d, "\pi_C"] & \widetilde{\mathbb{P}^3}_C \arrow[d, "\pi"] \\ C \arrow[r, hook, "i"] & \mathbb{P}^3 \end{tikzcd}$$

π, π_C, i, j, E_C are same as described in the section 2.2.

Also consider, $h \in A^1(\mathbb{P}^3)$ as the class of a hyperplane section and $e = [E_Y]$.

Corollary 2.3.10. *Considering the notations as above,* $A(\widetilde{\mathbb{P}^3}_C) = \bigoplus_{i=0}^3 A^i(\widetilde{\mathbb{P}^3}_C)$. *Then,*

$A^0(\widetilde{\mathbb{P}^3}_C) = \mathbb{Z}$, *generated by the class* $[\widetilde{\mathbb{P}^3}_C]$

$A^1(\widetilde{\mathbb{P}^3}_C) = \mathbb{Z}^2$, *generated by* e *and* $\pi^*(h)$

$A^2(\widetilde{\mathbb{P}^3}_C)$ *is generated by* $e^2 = j_*(\zeta)$, $j_*\pi_C^*(D)$ *for* $D \in A^1(C)$, *and* $\pi^*(h)^2$

$A^3(\widetilde{\mathbb{P}^3}_C) = \mathbb{Z}$, *generated by the class of a point.*

The intersection product among these generators are

$$deg(e \cdot \pi_C^*(D)) = deg(D), \qquad \pi_C^*(D) \cdot \pi^*(h) = 0, \qquad deg(\pi^*(h)^3) = 1,$$
$$deg(\pi^*(h)^2 \cdot e) = 0, \qquad deg(\pi^*(h) \cdot e^2) = -d, \qquad deg(e^3) = -4d - 2g + 2.$$

Proof. See [EH], Proposition 13.13. □

2.4 Conic Bundle

Definition 2.4.1. *A conic bundle is a tuple* (V, X, π), *together with a regular morphism* $\pi : V \to X$ *between two smooth varieties* V *and* X *whose generic fiber is an irreducible rational curve. We call that* V *is a conic over* X.

Definition 2.4.2. *Two conic bundles* (V, X, π) *and* (V, X', π') *are called equivalent if there exists a commutative diagram,*

$$
\begin{array}{ccc}
V & \xrightarrow{id} & V \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\
X & \xrightarrow{\phi} & X'
\end{array}
$$

such that the lower horizontal map ϕ is birational. An equivalence class of a conic bundle is called a conic bundle structure on the variety V.

Definition 2.4.3. *A conic bundle (V, X, π) is called regular, if π is a flat morphism.*

Definition 2.4.4. *A regular conic bundle (V, X, π) is called standard, if the preimage of an irreducible divisor under the map π is also an irreducible divisor .*

Let (V, X, π) be a regular conic. Hence $\pi : V \to X$ is a flat morphism. Let us denote f_x as the fiber of π at the point $x \in X$, and note that $f_x \cong \mathbb{P}^1$. So the normal bundle $\mathcal{N}_{f_x/V}$, of the fiber is a trivial bundle on f_x, because the normal bundle $\mathcal{N}_{f_x/V}$ is isomorphic to the tangent space of X at the point x i.e., $\mathcal{N}_{f_x/V} \simeq \pi_x^* \mathcal{T}_{x,X}$, where π_x is the restriction of π on the fiber f_x.
Using general *adjunction formula*, we get

$$\omega_{f_x} = \omega_V \otimes \wedge^r \mathcal{N}_{f_x/V},$$

this implies $\mathcal{O}_{\mathbb{P}^1}(-2) \simeq \omega_{f_x} = \omega_V \otimes \mathcal{O}_{f_x}$, where ω_{f_x} and ω_V are canonical sheaves of the fiber f_x and V respectively. Then clearly,

$$deg(-K_V \mid_{f_x}) = deg(-K_{f_x}) = 2.$$

where K_V and K_{f_x} are canonical divisors of V and f_x.
Recall a result from semicontinuity,

Theorem 2.4.5. *Let $f : Y \to X$ be a projective morphism of varieties, and let $\mathcal{F}$ be a coherent sheaf on Y, flat over X. Consider that*

$$h^i(x, \mathcal{F}) = dim_{\Bbbk(x)} H^i(Y_x, \mathcal{F}_x)$$

is a constant function on X, for $x \in X$. Then $R^i f_*(\mathcal{F})$ is a locally free sheaf on X, and for every $x \in X$ the natural maps

$$R^i f_*(\mathcal{F}) \otimes \Bbbk(x) \to H^i(Y_x, \mathcal{F}_x)$$

is an isomorphism. Hence, the rank of $R^i f_*(\mathcal{F})$ is $h^i(x, \mathcal{F})$.

Proof. See [H], III.12.9. □

Applying the above theorem in our case $\mathcal{E} = \pi_* \mathcal{O}_V(-K_V)$ is a locally free sheaf of rank 3. Also the sheaf $\mathcal{O}_V(-K_V)$ is relatively very ample on the variety V which defines an embedding of V in $\mathbb{P}(\mathcal{E})$. Then we have the following diagram,

$$\begin{tikzcd} V \arrow[rr, hook, "i"] \arrow[dr, "\pi"'] & & \mathbb{P}(\mathcal{E}) \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

Furthermore, $\pi^{-1}(x) \subset \pi'^{-1}(x) \cong \mathbb{P}^2$, where $\pi^{-1}(x)$ is a conic in $\mathbb{P}^2$ for each $x \in X$.

Definition 2.4.6. *A regular conic (V, X, π) is called honest if pullback of an integral divisor is reduced i.e., if D is an integral divisor in X, then $\pi^{-1}(D)$ is a reduced divisor.*

In this thesis, we will see that $\mathbb{P}^2$ blow-up at general seven points has conic bundle structure over $\mathbb{P}^1$.

2.5 Nef Cone and Pseudoeffective Cone of Curves

In this section, we discuss about the nefness of a line bundle on a projective variety as well as the nef cone of the projective variety, the pseudoeffective cone of curves, and the duality theorem.

Definition 2.5.1. *A line bundle $\mathcal{L}$ on a projective variety X is said to be very ample if $\mathcal{L} = \phi^*(\mathcal{O}_{\mathbb{P}^N}(1))$ for some closed embedding $\phi : X \hookrightarrow \mathbb{P}^N$ for some N. A line bundle $\mathcal{L}$ is called ample if some integral multiple $\mathcal{L}^{\otimes m}$ is very ample.*
A divisor D on X is said to be ample (resp. very ample), if the corresponding line bundle $\mathcal{O}_X(D)$ is ample (resp. very ample).

There are very few numerical criterions to check ampleness or very ampleness of a given line bundle on a projective variety. One of such interesting criterions is the following "Nakai-Moishezon-Kleiman criterions".

Theorem 2.5.2. *Let $\mathcal{L}$ be a line bundle on a projective variety X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{dimV} \cdot V > 0$ for every positive dimensional irreducible subvariety $V \subseteq X$.*

Two divisors D_1 and D_2 on X are said to be numerically equivalent, if $D_1 \cdot C = D_2 \cdot C$ for every irreducible curve C in X, and it is denoted by $D_1 \equiv D_2$. The *Neron Severi group* of X is the quotient group $N^1(X)_{\mathbb{Z}} = Div(X)/\equiv$, which is a free abelian group of finite rank (see Proposition 1.1.16, [L]). The rank of $N^1(X)_{\mathbb{Z}}$ is called the *Picard number* of X, and is denoted by $\rho(X)$. The real vector space $Div(X)_{\mathbb{R}} = Div(X) \otimes \mathbb{R}$ is called the space of $\mathbb{R}$-divisors on X. Let $D = \sum c_i D_i \in Div(X)_{\mathbb{R}}$, and $D \cdot C = \sum c_i(D_i \cdot C)$ for any curve C in X. Let

$$Div^0(X)_{\mathbb{R}} = \{D \in Div(X)_{\mathbb{R}} \mid D \cdot C = 0, \text{ for all curve } C \subseteq X\}.$$

$Div^0(X)_{\mathbb{R}}$ is a subspace of $Div(X)_{\mathbb{R}}$. The *real Neron Severi group*, denoted by $N^1(X)_{\mathbb{R}}$ is

$$N^1(X)_{\mathbb{R}} = Div(X)_{\mathbb{R}}/Div^0(X)_{\mathbb{R}}.$$

Note that, there is an isomorphism $N^1(X)_{\mathbb{R}} \cong N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}$ as a real vector space and the dimension is $\rho(X)$. We can view $N^1(X)_{\mathbb{R}}$ as a topological space equipped with the standard Euclidean topology. Also, we denote $[D] \in N^1(X)_{\mathbb{R}}$ as the numerical equivalence class of an integral divisor D on X.

Definition 2.5.3. *An $\mathbb{R}$-divisor D on X is ample if it can be expressed as a finite sum $D = \sum c_i D_i$, where $c_i > 0$ and D_i is an ample integral divisor.*

Ampleness does not depend on the representative of the numerical class of the divisor. Hence we can talk about ample class in $N^1(X)_{\mathbb{R}}$. The convex cone of all ample classes in $N^1(X)_{\mathbb{R}}$ is called the *Ample Cone*, and is denoted by $\mathrm{Amp}(X) \subseteq N^1(X)_{\mathbb{R}}$.

Definition 2.5.4. *A line bundle $\mathcal{L}$ over a projective variety X is called numerically effective, or nef, if $\mathcal{L} \cdot C \geq 0$ for every irreducible curve $C \subseteq X$. A Cartier divisor D on X is called nef if the corresponding line bundle $\mathcal{O}_X(D)$ is nef. Similarly, an $\mathbb{R}$-divisor D on X is called nef, if $D \cdot C \geq 0$ for all irreducible curve $C \subseteq X$.*

As the intersection product is independent of the representative of the numerical equivalence class, one can talk about nef classes in $N^1(X)_{\mathbb{Z}}$ as well as $N^1(X)_{\mathbb{R}}$. The convex cone of all nef classes in $N^1(X)_{\mathbb{R}}$ is called the *Nef cone*, and is denoted by $\mathrm{Nef}(X) \subseteq N^1(X)_{\mathbb{R}}$. We have the following relation between the nef cone and the ample cone of projective variety due to S. L. Kleiman (see [K]).

Theorem 2.5.5. *The closure of the ample cone is the nef cone i.e. $\overline{Amp(X)} = Nef(X)$ in $N^1(X)_{\mathbb{R}}$. The interior of the nef cone is the ample cone i.e. $int(Nef(X)) = Amp(X)$ in $N^1(X)_{\mathbb{R}}$.*

We denote the real one cycle of a scheme X by $Z_1(X)_{\mathbb{R}} = \{\gamma = \sum a_i C_i \mid a_i \in \mathbb{R}$, and $C_i \subseteq X$ is an irreducible curve $\}$. Two one cycles $\gamma_1, \gamma_2 \in Z_1(X)_{\mathbb{R}}$ are numerically equivalent i.e. $\gamma_1 \equiv \gamma_2$ if $D \cdot \gamma_1 = D \cdot \gamma_2$ for every $D \in Div(X)_{\mathbb{R}}$. Let $N_1(X)_{\mathbb{R}} = Z_1(X)_{\mathbb{R}} / \equiv$. We also have a perfect pairing:

$$N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}\ ;\ (\delta, \gamma) \longmapsto (\delta \cdot \gamma)$$

The *cone of curves*, $\mathrm{NE}(X) \subseteq N_1(X)_{\mathbb{R}}$ is the cone spanned by the classes of all effective one cycles on X i.e.,

$$\mathrm{NE}(X) = \{\textstyle\sum_i a_i[C_i] \mid C_i \subseteq X \text{ an irreducible curve, } a_i \in \mathbb{R}_{>0}\}$$

The closure $\overline{\mathrm{NE}}(X) \subseteq N_1(X)_{\mathbb{R}}$ is called the *pseudoeffective cone of curves*.

Theorem 2.5.6. *The closed cone of curves is dual to the nef cone, i.e.*

$$\overline{NE}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} \mid (\delta \cdot \gamma) \geq 0 \textit{ for all } \delta \in Nef(X)\}.$$

Proof. See Proposition 1.4.28 in [L]. □

Chapter 3

Examples of blown up varieties having projective bundle structures

It is well known that the projective space blown up along a projective subspace always has a projective bundle structure.

Theorem 3.0.1. *([EH]. Theorem 9.3.2) Let $V' \subset V$ be an r-dimensional subspace of an $(n+1)$-dimensional vector space V. Then, $\mathbb{P}^n = \mathbb{P}(V)$ and $\mathbb{P}^{r-1} = \mathbb{P}(V')$. Let Z be the blowing-up of $\mathbb{P}^n$ along the $(r-1)$-dimensional subspace $\mathbb{P}^{r-1}$. Then, $Z \simeq \mathbb{P}(\mathcal{E}) \to \mathbb{P}^{n-r}$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^{n-r}}(1) \oplus \mathcal{O}^r_{\mathbb{P}^{n-r}}$. Under this isomorphism, the blow-up map $Z \to \mathbb{P}^n$ corresponds to the complete linear series $\mid \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \mid$.*

One has the following basic question : Is there any non-linear subvariety Y in $\mathbb{P}^n$, such that the blow-up of $\mathbb{P}^n$ along Y admits a projective bundle structure over a smaller dimensional projective space?

In this chapter, we exhibit some examples of blow-up of projective space along a non-linear subvariety which have projective bundle structure.

3.1 Blown up of $\mathbb{P}^5$ having projective bundle structure

Let $i : \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ be the Segre embedding defined by sending $[x_0, x_1] \times [y_0, y_1, y_2]$ to $[x_0y_0, x_0y_1, x_0y_2, x_1y_0, x_1y_1, x_1y_2]$. Let $\{z_i \mid i = 0, 1, ...5\}$ be the homogeneous coordinate of $\mathbb{P}^5$. If $i(\mathbb{P}^1 \times \mathbb{P}^2) = X_0$, then X_0 is defined by equations $f_0 = z_0z_4 - z_1z_3$, $f_1 = z_0z_5 - z_2z_3$, and $f_2 = z_2z_4 - z_1z_5$ in $\mathbb{P}^5$. The morphism i is defined by the very ample divisor $(1, 1)$ of $\mathbb{P}^1 \times \mathbb{P}^2$, where the Picard group of $\mathbb{P}^1 \times \mathbb{P}^2$ is $\mathbb{Z} \oplus \mathbb{Z}$. Hence, $deg(X_0) = (1, 1) \cdot (1, 1) \cdot (1, 1) = 3$ in $\mathbb{P}^5$.

Theorem 3.1.1. *$\mathbb{P}^5$ blown up along the closed subscheme X_0 is a $\mathbb{P}^3$-bundle over $\mathbb{P}^2$, i.e. $\widetilde{\mathbb{P}}^5_{X_0} \cong \mathbb{P}(E)$, where E is a rank four vector bundle of $\mathbb{P}^2$. Let $\widetilde{\phi}_0 : \widetilde{\mathbb{P}}^5_{X_0} \to \mathbb{P}^2$ be the projectivization map, and $\pi : \widetilde{\mathbb{P}}^5_{X_0} \to \mathbb{P}^5$ be the blow-up map. Then $\widetilde{\phi}_0$ is defined by the linear system $\mid 2\pi^*H - E_{X_0} \mid$, where H be the hyperplane section of $\mathbb{P}^5$, and π is given by the linear system $\mid \mathcal{O}_{\mathbb{P}(E)}(1) \mid$.*

Proof. Let us consider the linear system $\mid \mathcal{O}_{\mathbb{P}^5}(2) \otimes \mathcal{I}_{X_0} \mid$, where $\mathcal{I}_{X_0}$ is the ideal sheaf of the closed subscheme X_0. The linear system $\mid \mathcal{O}_{\mathbb{P}^5}(2) \otimes \mathcal{I}_{X_0} \mid$ consist of all degree two hypersurfaces of $\mathbb{P}^5$ containing X_0. Then the vector space $H^0(\mathcal{O}_{\mathbb{P}^5}(2) \otimes \mathcal{I}_{X_0})$ is generated by a basis, $\{f_0 = z_0z_4 - z_1z_3, f_1 = z_0z_5 - z_2z_3, f_2 = z_2z_4 - z_1z_5\}$. Hence, the linear system $\mid \mathcal{O}_{\mathbb{P}^5}(2) \otimes \mathcal{I}_{X_0} \mid$ is isomorphic to $\mathbb{P}^2$, and corresponds to the rational map $\phi_0 : \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$. Using the Theorem 2.2.4, we can extend the map ϕ_0 as $\widetilde{\phi}_0 : \widetilde{\mathbb{P}}^5_{X_0} \to \mathbb{P}^2$ such that the following diagram commutes.

$$\begin{tikzcd}
\widetilde{\mathbb{P}}^5_{X_0} \arrow[r, "\pi"] \arrow[dr, "\widetilde{\phi}_0"] & \mathbb{P}^5 \arrow[d, dashed, "\phi_0"] \\
 & \mathbb{P}^2
\end{tikzcd}$$

Moreover, the map $\widetilde{\phi}_0$ is defined by the linear system $\mid 2\pi^*H - E_{X_0} \mid$ of $\widetilde{\mathbb{P}}^5_{X_0}$.

Now, our claim is that each fiber of $\widetilde{\phi_0}$ is isomorphic to $\mathbb{P}^3$. First, we define the map ϕ_0 coordinate wise which is $\phi_0([z_0, z_1, \cdots, z_5]) = [z_0z_4 - z_1z_3, z_0z_5 - z_2z_3, z_2z_4 - z_1z_5] = [f_0, f_1, f_2]$. Then, $\phi_0^{-1}[1,0,0] = V(f_1, f_2)$, $X_0 \subseteq V(f_1, f_2)$, and $\deg(V(f_1, f_2)) = 4$. As $\deg(X_0) = 3$ in $\mathbb{P}^5$, $V(f_1, f_2) = X_0 \cup L$ and $\deg(L) = 1$ i.e., $L \cong \mathbb{P}^3$ in $\mathbb{P}^5$. Clearly, $\widetilde{\phi_0}^{-1}[1,0,0]$ is isomorphic to the strict transformation of L. Similarly, when $a_0 \neq 0$, $\widetilde{\phi_0}^{-1}[a_0, a_1, a_2]$ is the strict transformation of $\overline{V(a_0f_1 - a_1f_0, a_0f_2 - a_2f_0)\backslash X_0}$ in $\widetilde{\mathbb{P}}^5_{X_0}$. When $a_1 \neq 0$, $\widetilde{\phi_0}^{-1}[a_0, a_1, a_2]$ is the strict transformation of

$\overline{V(a_1f_0 - a_0f_1, a_2f_1 - a_1f_2)\backslash X_0}$ and when $a_2 \neq 0$, $\widetilde{\phi_0}^{-1}[a_0, a_1, a_2]$ is the strict transformation of $\overline{V(a_0f_2 - a_2f_0, a_1f_2 - a_2f_1)\backslash X_0}$. Finally we get that $\widetilde{\mathbb{P}}^5_{X_0}$ is a Brauer-Severi variety over $\mathbb{P}^2$. As the Brauer group of $\mathbb{P}^2$ is trivial, this shows that $\widetilde{\mathbb{P}}^5_{X_0}$ has a projective bundle structure over $\mathbb{P}^2$.

□

Now, $\widetilde{\mathbb{P}}^5_{X_0} \cong \mathbb{P}(E)$, where E is a rank four vector bundle over $\mathbb{P}^2$, and $\widetilde{\phi}_0^*H' = 2\pi^*H - E_{X_0}$, where H' is the class of hyperplane section of $\mathbb{P}^2$. Note that, the Picard group as well as the Neron Severi group of $\mathbb{P}(E)$ is generated by $\widetilde{\phi}_0^*H'$ and $\mathcal{O}_{\mathbb{P}(E)}(1)$. Let $\pi^*(H) = \mathcal{O}_{\mathbb{P}(E)}(n_1) \otimes \widetilde{\phi}_0^*(n_2H')$. Note that, $(\widetilde{\phi}_0^*H')^2 = (2\pi^*H - E_{X_0})^2$ is the rational equivalence class of fibers $F \cong \mathbb{P}^3$ of the map $\widetilde{\phi_0}$ which is not contained in the exceptional divisor E_{X_0}. Hence $\pi(F) \cong \mathbb{P}^3$ in $\mathbb{P}^5$. Therefore $F \cdot \pi^*(H)^3 = \pi(F) \cdot H^3 = 1$. Eventually we get the following equalities,

$$1 = \pi^*(H)^3(2\pi^*(H) - E_{X_0})^2 = (\mathcal{O}_{\mathbb{P}(E)}(n_1) + \widetilde{\phi}_0^*(n_2H'))^3\widetilde{\phi}_0^*(H')^2$$

Hence it follows that $n_1 = 1$, which implies,

$$\pi^*(H) = \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \widetilde{\phi}_0^*(n_2H')$$

If we take $E' = E \otimes n_2H'$, then $\mathbb{P}(E) \cong \mathbb{P}(E')$, and the line bundle $\mathcal{O}_{\mathbb{P}(E')}(1)$

corresponds to the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \widetilde{\phi}_0^*(n_2 H')$, in the Picard groups lavel. So with out loss of generality, we can consider a vector bundle E, such that $\widetilde{\mathbb{P}}^5_{X_0} \cong \mathbb{P}(E)$ and $\pi^*(H) = \mathcal{O}_{\mathbb{P}(E)}(1)$.

Theorem 3.1.2. *If $\mathbb{P}(E)$ is as above, then E is the cokernel of an injective homomorphism $\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^6$ i.e., we have an exact sequence,*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^6 \longrightarrow E \longrightarrow 0. \tag{3.1}$$

Proof. In this proof, we use the same notations as it is described in this section. Note that, $\mathcal{O}_{\mathbb{P}(E)}(1)$ is globally generated, and $\dim H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)) = 6$ because $\pi^*(H) \sim \mathcal{O}_{\mathbb{P}(E)}(1)$. Using the projection formula, we get $6 = \dim H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)) = \dim H^0(\mathbb{P}^2, \pi_*(\mathcal{O}_{\mathbb{P}(E)}(1))) = \dim H^0(\mathbb{P}^2, E)$. As $\mathcal{O}_{\mathbb{P}(E)}(1)$ is globally generated, E is also globally generated vector bundle of $\mathbb{P}^2$. Hence, there is a surjection from $\mathcal{O}_{\mathbb{P}^2}^6$ to E and let $\mathcal{F}$ be the kernel i.e.,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^6 \longrightarrow E \longrightarrow 0 \tag{3.2}$$

Now from the long exact sequence of cohomology, we conclude that $H^0(\mathbb{P}^2, \mathcal{F}) = 0$ and $H^1(\mathbb{P}^2, \mathcal{F}) = 0$. We claim that $\mathcal{F} = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$.

The morphism $\widetilde{\phi}_0$ is a flat and proper map. Let H' be the hyperplane section of $\mathbb{P}^2$ and H be the hyperplane section of $\mathbb{P}^5$. Then,

$$\begin{aligned} c_1(E) \cdot H' &= \widetilde{\phi}_{0*}(\mathcal{O}_{\mathbb{P}(E)}(1)^4 \cdot \widetilde{\phi}_0^*(H')) \\ &= \widetilde{\phi}_{0*}((\pi^* H)^4 \cdot (2\pi^* H - E_{X_0})) = 2 \end{aligned}$$

(see [F], chapter 3 for detail calculation). Hence, $c_1(E) \sim \mathcal{O}_{\mathbb{P}^2}(2)$ and $c_1(\mathcal{F}) \sim \mathcal{O}_{\mathbb{P}^2}(-2)$.

We know that any rank two bundle $\mathcal{F}$ of $\mathbb{P}^2$ can be written as an extension of two coherent sheaves,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(n) \otimes \mathcal{I}_Z \longrightarrow 0 \tag{3.3}$$

where Z is a zero dimensional closed subscheme of $\mathbb{P}^2$ and $\mathcal{I}_Z$ is the corresponding ideal sheaf ([Fr], Chapter.2).

In our case

$$-2H' = c_1(\mathcal{F}) = (m+n)H', \qquad \text{and} \qquad c_2(\mathcal{F}) = mn + l(Z) \tag{3.4}$$

Our claim is that, $l(Z) = 0$ and $m = n = -1$.

Let $\zeta = \mathcal{O}_{\mathbb{P}(E)}(1)$. Note that, we have the following Chern equation,

$$\zeta^4 = c_1(E)\zeta^3 - c_2(E)\zeta^2 \tag{3.5}$$

Apply ζ in the both side of the equation 3.5 and put the value $\zeta^5 = 1$ on it. Then,

$$1 = c_1(E)\zeta^4 - c_2(E)\zeta^3 \tag{3.6}$$

Substituting ζ^4 in the equation (3.6) we get

$$(c_1(E)^2 - c_2(E))\zeta^3 = 1$$
$$\Rightarrow (4H'^2 - c_2(E))\zeta^3 = 1$$
$$\Rightarrow c_2(E) = 3$$

Now, from (3.2) we have, $c(E).c(\mathcal{F}) = c(\mathcal{O}_{\mathbb{P}^2}) \Rightarrow c(\mathcal{F}) = c(E)^{-1} = (1 + 2H' + 3H'^2)^{-1} = 1 - 2H' + H'^2 \Rightarrow c_2(\mathcal{F}) = 1$. So from the equation (3.4), it is clear that either $n = m = -1$ or $m < -1$; $n \geq 0$. If $n = m = -1$, then we are done. Otherwise, assume that $m < -1$, and $n \geq 0$. Tensoring the equation (3.4) by the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ we get,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m+1) \longrightarrow \mathcal{F}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(n+1) \otimes \mathcal{I}_Z \longrightarrow 0 \tag{3.7}$$

Hence, $c_1(\mathcal{F}(1)) = 0$ and $c_2(\mathcal{F}(1)) = 0$. Then we can apply the Theorem ([Fr],4.14.(iv)), which says, $\mathcal{E}$ is rank two bundle of $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 0$. If $\mathcal{E}$ is stable, then $c_2(\mathcal{E}) \geq 2$. So in our case $\mathcal{F}(1)$ is not stable. Then from the Theorem ([Fr],4.14.(i)) we have $m + 1 \geq 0$, hence $m \geq -1$. This contradict our assumption.

So we conclude that $\mathcal{F} = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$. □

Remark 3.1.3. We know from ([EH],9.6), the Chow Ring of $\mathbb{P}(E)$ is

$$A(\mathbb{P}(E)) = \frac{A(\mathbb{P}^2)[\zeta]}{< \zeta^4 + c_1(E^*)\zeta^3 + c_2(E^*)\zeta^2 >} \tag{3.8}$$

where $\zeta \sim \pi^*(H) \sim \mathcal{O}_{\mathbb{P}(E)}(1)$. Also, we have $A(\mathbb{P}^2) = \frac{\mathbb{Z}[\alpha]}{<}\alpha^3 >$, where $\alpha \sim H'$. Using the following short exact sequence,

$$0 \longrightarrow E^* \longrightarrow \mathcal{O}_{\mathbb{P}^2}^6 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) = F \longrightarrow 0 \tag{3.9}$$

and the Whitney Sum formula, we get

$$c_t(E^*) = c_t(\mathcal{O}_{\mathbb{P}^2}^6).c_t(F)^{-1} = \frac{1}{(1+H't)^2} = 1 - 2H't + 3H'^2t^2 \tag{3.10}$$

Hence, the Chow Ring of $\widetilde{\mathbb{P}}^5_{X_0}$ or $\mathbb{P}(E)$ is $\frac{\mathbb{Z}[\alpha,\zeta]}{<\alpha^3, \zeta^4 - 2\alpha\zeta^3 + 3\alpha^2\zeta^2>}$, $\zeta, \alpha \in A^1(\widetilde{\mathbb{P}}^5_{X_0})$ and $E_{X_0} \sim 2\zeta - \alpha$.

3.2 Blown up of $\mathbb{P}^4$ having projective bundle structure

The vector bundle $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ has six independent sections and it can be generated by four sections. Let us take five independent global sections of $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, which generate this rank two bundle. Hence there is a surjection $\mathcal{O}_{\mathbb{P}^2}^5 \to \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ and E_1^* be the kernel of this map i.e., we have the following exact sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^5 \longrightarrow E_1 \longrightarrow 0. \tag{3.11}$$

Theorem 3.2.1. *Let E_1 be as above. Then $\mathbb{P}(E_1) \cong \widetilde{\mathbb{P}}^4_{X_1}$, where X_1 is the hyperplane section of X_0 in $\mathbb{P}^5$ i.e., X_1 is a cubic surface in $\mathbb{P}^4$. If $\pi_1 : \widetilde{\mathbb{P}}^4_{X_1} \to \mathbb{P}^4$ is the blown up map and $\widetilde{\phi}_1 : \mathbb{P}(E_1) \to \mathbb{P}^2$ is the projectivization map, then $\widetilde{\phi}_1^*(\mathcal{O}_{\mathbb{P}^2}(1)) \sim \pi_1^*\mathcal{O}_{\mathbb{P}^4}(2) \otimes \mathcal{O}(-E_{X_1})$.*

Proof. We have the following commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1^* & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^5 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E^* & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^6 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) & \longrightarrow & 0 \end{array} \tag{3.12}$$

As the right and the middle arrows are identity and injection respectively, the left arrow does exist and injective, which implies $E \to E_1$ is a surjective map. This corresponds to the inclusion $i_1 : \mathbb{P}(E_1) \hookrightarrow \mathbb{P}(E)$, such that $\mathcal{O}_{\mathbb{P}(E_1)}(1) \cong i_1^*\mathcal{O}_{\mathbb{P}(E)}(1)$.

Let $\widetilde{\phi}_1 : \mathbb{P}(E_1) \to \mathbb{P}^2$ be the projection map. It is clear from the short exact sequence (3.11) that E_1 is globally generated, and $h^0(\mathbb{P}^2, E_1) = 5$. Then, we have a morphism $\pi_1 : \mathbb{P}(E_1) \to \mathbb{P}^4$ given by the line bundle $\mathcal{O}_{\mathbb{P}(E_1)}(1)$. Note that the following diagram commutes,

$$\begin{tikzcd}
\mathbb{P}(E_1) \arrow[r, "i_1"] \arrow[d, "\pi_1"] & \mathbb{P}(E) \arrow[d, "\pi"] \\
\mathbb{P}^4 \arrow[r, "i"] & \mathbb{P}^5
\end{tikzcd}$$

Here i is the canonical inclusion map, such that $i^*(\mathcal{O}_{\mathbb{P}^5}(1)) = \mathcal{O}_{\mathbb{P}^4}(1)$.

In the Proposition (3.1.1), we have defined a rational map $\phi_0 : \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$, which is $\phi_0([z_0, z_1, ..., z_5]) = [z_0z_4 - z_1z_3, z_0z_5 - z_2z_3, z_2z_4 - z_1z_5]$. Now, consider $\mathbb{P}^4$ as a hyperplane of $\mathbb{P}^5$, given by the equation $z_0 = z_5$. The restriction of ϕ_0 in this hyperplane induces the map $\phi_1' : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$, which is $\phi_1'([z_0, z_1, z_2, z_3, z_5]) = [z_0z_4 - z_1z_3, z_0^2 - z_2z_3, z_2z_4 - z_1z_0]$. The map ϕ_1' is not defined on the cubic surface X_1 of $\mathbb{P}^4$ given by the equations $g_0 = z_0z_4 - z_1z_3, g_1 = z_0^2 - z_2z_3$, and $g_2 = z_2z_4 - z_1z_0$. Now, ϕ_1' can be extended to $\widetilde{\phi_1'} : \widetilde{\mathbb{P}}^4_{X_1} \to \mathbb{P}^2$, and $\mid 2\pi_1'^*\mathcal{O}_{\mathbb{P}^4}(1) - E_{X_1} \mid$ is the corresponding linear system of $\widetilde{\mathbb{P}}^4_{X_1}$ (using the Theorem 2.2.4) where $\pi_1' : \widetilde{\mathbb{P}}^4_{X_1} \to \mathbb{P}^4$ is the natural blow-up map. Also we have the following commutative diagram,

$$\begin{tikzcd}
\widetilde{\mathbb{P}}^4_{X_1} \arrow[r, "i_2"] \arrow[d, "\pi_1'"] & \widetilde{\mathbb{P}}^5_{X_0} \arrow[d, "\pi"] \\
\mathbb{P}^4 \arrow[r, "i"] & \mathbb{P}^5
\end{tikzcd}$$

where i_2 is the inclusion, π_1' and π are blow-up maps.

Let $\mathcal{I}_{X_1}$ be the ideal sheaf of X_1 in $\mathbb{P}^4$. Note that, $\pi_1^{-1}\mathcal{I}_{X_1} \cdot \mathcal{O}_{\mathbb{P}(E_1)}$ is an invertible sheaf of ideal on $\mathbb{P}(E_1)$, because $\pi^{-1}\mathcal{I}_{X_0}.\mathcal{O}_{\mathbb{P}(E)}$ is an invertible sheaf of ideal on $\mathbb{P}(E) \cong \widetilde{\mathbb{P}}^5_{X_0}$, hence $\pi_1^{-1}\mathcal{I}_{X_1}.\mathcal{O}_{\mathbb{P}(E_1)} = i_1^{-1}(\pi^{-1}\mathcal{I}_{X_0}.\mathcal{O}_{\mathbb{P}(E)})$ is also an invertible sheaf on $\mathbb{P}(E_1)$. By the *Universal Property of Blowing Up* ([H] II.7), we have the unique morphism $\mathbb{P}(E_1) \to \widetilde{\mathbb{P}}^4_{X_1}$ over $\mathbb{P}^4$ i.e.,

$$\begin{tikzcd}
\mathbb{P}(E_1) \arrow[rr] \arrow[dr, "\pi_1"] & & \widetilde{\mathbb{P}}^4_{X_1} \arrow[dl, "\pi_1'"] \\
& \mathbb{P}^4 &
\end{tikzcd}$$

Note that, $\widetilde{\phi_1'}^* E_1$ and $\pi_1'^*(\mathcal{O}_{\mathbb{P}^4}(1))$ both are globally generated vector bundle of $\widetilde{\mathbb{P}}^4_{X_1}$ with global section dimension five. Hence, there is a surjection, $\widetilde{\phi_1'}^* E_1 \to \pi_1'^*(\mathcal{O}_{\mathbb{P}^4}(1))$. This corresponds to the unique morphism $\widetilde{\mathbb{P}}^4_{X_1} \to \mathbb{P}(E_1)$ over $\mathbb{P}^2$ (using the *Universal property of projective bundle* [H] II.7.12), i.e.

$$\begin{tikzcd}
\widetilde{\mathbb{P}}^4_{X_1} \arrow[rr] \arrow[dr, "\widetilde{\phi_1'}"'] & & \mathbb{P}(E_1) \arrow[dl, "\widetilde{\phi_1}"] \\
& \mathbb{P}^2 &
\end{tikzcd}$$

Therefore we have the following commutative diagram,

$$\begin{tikzcd}
\mathbb{P}(E_1) \arrow[r] \arrow[d] & \widetilde{\mathbb{P}}^4_{X_1} \arrow[r] \arrow[d] & \mathbb{P}(E_1) \arrow[d] \\
\mathbb{P}(E) \arrow[r] & \widetilde{\mathbb{P}}^5_{X_0} \arrow[r] & \mathbb{P}(E)
\end{tikzcd} \tag{3.13}$$

The composition of the lower horizontal arrows is identity and vertical maps are inclusion, this implies $\mathbb{P}(E_1) \to \widetilde{\mathbb{P}}^4_{X_1} \to \mathbb{P}(E_1)$ is also identity. Similarly, $\widetilde{\mathbb{P}}^4_{X_1} \to \mathbb{P}(E_1) \to \widetilde{\mathbb{P}}^4_{X_1}$ also identity. Hence, it proves that $\widetilde{\mathbb{P}}^4_{X_1} = \mathbb{P}(E_1)$, and $\pi_1 = \pi_1'$, and $\widetilde{\phi_1'} = \widetilde{\phi_1}$ up to isomorphism.

$\square$

Remark 3.2.2. $A(\widetilde{\mathbb{P}}^4_{X_1}) = A(\mathbb{P}(E_1)) = \frac{\mathbb{Z}[\alpha,\zeta]}{<\alpha^3, \zeta^3 - 2\alpha\zeta^2 + 3\alpha^2\zeta>}$ where $\zeta \sim \mathcal{O}_{\mathbb{P}(E_1)}(1)$ and $\alpha \sim \widetilde{\phi_1}^*(H')$ $\zeta, \alpha \in A^1(\mathbb{P}(E_1))$ and $E_{X_1} \sim 2\zeta - \alpha$.

3.3 $\mathbb{P}^3$ blow-up along twisted cubic

Corollary 3.3.1. *If we have the following short exact sequence,*

$$0 \longrightarrow E_2^* \longrightarrow \mathcal{O}^4_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0, \tag{3.14}$$

then $\widetilde{\mathbb{P}}^3_{X_2} \cong \mathbb{P}(E_2)$*, where* X_2 *is a twisted cubic in* $\mathbb{P}^3$*, which can be obtained by cutting down the cubic surface in* $\mathbb{P}^4$ *by a hyperplane.*

Proof. The proof will follow similar argument as done in the Theorem (3.2.1). $\square$

Remark 3.3.2. The Chow ring of $\mathbb{P}^3$ blown up along twisted cubic is $A(\widetilde{\mathbb{P}}^3_{X_2}) = A(\mathbb{P}(E_2)) = \frac{\mathbb{Z}[\alpha,\zeta]}{<\alpha^3, \zeta^2 - 2\alpha\zeta + 3\alpha^2>}$, where $\zeta \sim \mathcal{O}_{\mathbb{P}(E_2)}(1) \sim \pi_2^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and $\alpha \sim \widetilde{\phi_2}^*(H')$ $\zeta, \alpha \in A^1(\mathbb{P}(E_2))$ and $E_{X_2} \sim 2\zeta - \alpha$. Here, $\widetilde{\phi_2} : \mathbb{P}(E_2) \to \mathbb{P}^2$ is the projectivization map and $\pi_2 : \widetilde{\mathbb{P}}^3_{X_2} \to \mathbb{P}^3$ is the blowing up map.

Theorem 3.3.3. *Let* C *be an irreducible subvariety of* $\mathbb{P}^3$ *other than linear subspaces of* $\mathbb{P}^3$*. If* $\widetilde{\mathbb{P}}^3_C$ *has a projective bundle structure, then* $C = V(f_0, f_1, f_2)$*, where* f_i *are irreducible homogeneous polynomials,* $deg(f_i) = deg(f_j) = d$ *and* $d_1 = deg\ (C) = d^2 - 1$ *in* $\mathbb{P}^3$ *.*

Proof. Let $\widetilde{\mathbb{P}}^3_C$ has a projective bundle structure i.e., $\widetilde{\mathbb{P}}^3_C \simeq \mathbb{P}(E) \to \mathbb{P}^n$, where ϕ is the projectivization map and $n \leq 2$.

Let $C = V(\{g_i \mid i = 0, 1, \cdots r\})$. If $\deg(g_i) \neq \deg(g_j)$ for $i \neq j$, then we can construct f_i such that $C = V(\{f_i \mid i = 0, 1, \cdots r\})$ and $f_i = g_i^{n_i}$ for some positive integer n_i such that degree of f_i's are same.

Consider the following diagram,

$$
\begin{tikzcd}
\widetilde{\mathbb{P}}^3_C \arrow[r, "I"] \arrow[d, "\pi"] & \mathbb{P}(E) \arrow[d, "\phi"] \\
\mathbb{P}^3 & \mathbb{P}^n
\end{tikzcd}
$$

I is the isomorphism and π is the blow-up map. Let $\psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^n$ be the rational map which is given by the linear system $\mid \mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_C \mid$, where $\mathcal{I}_C$ is the ideal sheaf of the curve C.

Case-I $(n = 1)$

$\widetilde{\mathbb{P}}^3_C \cong \mathbb{P}(E) \xrightarrow{\phi} \mathbb{P}^1$, rank$(E) = 3$ and $C = V(f_0, f_1)$. Clearly, $\psi^{-1}([a_0, a_1]) = V(a_0 f_1 - a_1 f_0)$ and then $\pi^{-1}(\psi^{-1}([a_0, a_1])) = \overline{V(a_0 f_1 - a_1 f_0)\backslash C} \cup E_C$, where $\overline{V(a_0 f_1 - a_1 f_0)\backslash C}$ is the strict transformation of $\overline{V(a_0 f_1 - a_1 f_0)\backslash C}$ and E_C is the exceptional divisor corresponding to the blow-up map π. This gives $\phi^{-1}(a) = \overline{V(a_0 f_1 - a_1 f_0)\backslash C} \cong S_d$ which is a degree $d > 1$ hypersurface in $\mathbb{P}^3$, and is a not isomorphic to $\mathbb{P}^2$. Hence, there is a contradiction i.e., $\widetilde{\mathbb{P}}^3_C \ncong \mathbb{P}(E)$ for any vector bundle E on $\mathbb{P}^1$.

Case-II $(n = 2)$

$\widetilde{\mathbb{P}}^3_C \cong \mathbb{P}(E) \xrightarrow{\phi} \mathbb{P}^2$, rank$(E) = 2$ and $C = V(f_0, f_1, f_2)$. Clearly, $\psi^{-1}([a_0, a_1, a_2]) = V(a_0 f_1 - a_1 f_0, a_0 f_2 - a_2 f_0)$. As C is an irreducible curve, C becomes an irreducible component of $\psi^{-1}(a)$, where $a = [a_0, a_1, a_2]$. Then $\psi^{-1}(a) = C \cup C_a$. Hence $\pi^{-1}(\psi^{-1}(a)) = E_C \cup \widetilde{C_a}$, where E_C is the exceptional divisor corresponding to the blowing up map π and $\widetilde{C_a}$ is the strict transformation of the curve C_a. This implies $\phi^{-1}(a) = \widetilde{C_a}$. As $\widetilde{\mathbb{P}}^3_C$ is projective bundle over $\mathbb{P}^2$, $\widetilde{C_a}$ will be isomorphic to $\mathbb{P}^1$. Then $\deg(C_a) = 1$ in $\mathbb{P}^3$. So finally we conclude that each f_i is a reduced polynomial (otherwise C_a will become non-reduced curve for some $a \in \mathbb{P}^2$) and $\deg(C) = d^2 - 1$ (as $\deg(V(a_0 f_1 - a_1 f_0, a_0 f_2 - a_2 f_0)) = d^2$).

Hence the Theorem is proved. □

Proposition 3.3.4. *Considering the same notations of the Theorem 3.3.3, $\widetilde{\mathbb{P}}^3_C$ has a projective bundle structure only when C is a genus zero curve.*

Proof. Let $\widetilde{\mathbb{P}}^3_C \cong \mathbb{P}(E)$, where E is a rank 2 vector bundle over $\mathbb{P}^2$.

Let $A^i(X)$ be the group of rational equivalence classes of codimension i cycles of the scheme X. Here,

$$A^1(\mathbb{P}(E)) = \mathbb{Z}\mathcal{O}_{\mathbb{P}(E)}(1) \oplus \mathbb{Z}\phi^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$A^2(\mathbb{P}(E)) = \mathbb{Z}\mathcal{O}_{\mathbb{P}(E)}(1) \cdot \phi^*\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathbb{Z}(\phi^*\mathcal{O}_{\mathbb{P}^2}(1) \cdot \phi^*\mathcal{O}_{\mathbb{P}^2}(1)) \cong \mathbb{Z} \oplus \mathbb{Z}$$

Now consider the following commutative diagram,

$$\begin{array}{ccc} \mathbb{P}(N) & \xrightarrow{j} & \widetilde{\mathbb{P}}^3_C \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\ C & \xrightarrow{i} & \mathbb{P}^3 \end{array}$$

where N is the normal bundle of C in $\mathbb{P}^3$, $\mathbb{P}(N) = E_C$ is the exceptional divisor corresponding to the blowing up map ϕ, and i and j are inclusions. Let $\widetilde{h} = \pi^*\mathcal{O}_{\mathbb{P}^3}(1) \in A^1(\widetilde{\mathbb{P}}^3_C)$ and $e = [E_C] \in A^1(\widetilde{\mathbb{P}}^3_C)$.

From the Corollary 2.3.10, we have,

$$A^1(\widetilde{\mathbb{P}}^3_C) = \mathbb{Z}e \oplus \mathbb{Z}\widetilde{h}$$

$$A^2(\widetilde{\mathbb{P}}^3_C) \text{ is generated by } e^2 = -j_*(\mathcal{O}_{\mathbb{P}(N)}(1)),\ \widetilde{h}^2 \text{ and } j_*(F_D)$$

$$\text{for } D \in A^1(C) \text{ and } F_D = \pi^*(D).$$

$A^2(\widetilde{\mathbb{P}}^3_C) \cong \mathbb{Z} \oplus \mathbb{Z}$ if and only if any two point on the curve C is rationally equivalent, i.e. $\mathrm{Pic}(C) = \mathbb{Z}$. Hence, C is a genus zero curve.

$\square$

Theorem 3.3.5. *$\widetilde{\mathbb{P}}^3_C$ has a projective bundle structure if and only if C is a twisted cubic in $\mathbb{P}^3$.*

Proof. Using the same notations as in Proposition 3.3.4, we have

$$\mathrm{Pic}(\widetilde{\mathbb{P}}^3_C) = \mathbb{Z}e \oplus \mathbb{Z}\widetilde{h}$$

$$\mathrm{Pic}\ (\mathbb{P}(E)) = \mathbb{Z}\mathcal{O}_{\mathbb{P}(E)}(1) \oplus \mathbb{Z}\phi^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$\text{Rename } \zeta_E = \mathcal{O}_{\mathbb{P}(E)}(1) \text{ and } H = \phi^*\mathcal{O}_{\mathbb{P}^2}(1)$$

$$H \sim d\widetilde{h} - e \text{ and } \zeta_E \sim \widetilde{h} \text{ in the Picard group}$$

We know that $\zeta_E^2 \cdot H = c_1(E)$. Also we have $\zeta_E^2 \cdot H = \widetilde{h}^2(d\widetilde{h} - e)$. This implies,

$$d = c_1(E), \tag{3.15}$$

(using intersection products from the Corollary 2.3.10.) Also, $e^3 = -4d_1 - 2g + 2$, where from the Theorem 3.3.3, the degree of C in $\mathbb{P}^3$, $d_1 = d^2 - 1$ and from the Proposition 3.3.4, genus of C, $g = 0$. Then,

$(d\zeta_E - H)^3 = e^3 = -4(d^2 - 1) + 2$

$\Rightarrow d^3\zeta_E^3 - 3d^2\zeta_E^2 H + 3d\zeta_E H^2 - H^3 = -4d^2 + 6$

$\Rightarrow -2d^3 + 3d = -4d^2 + 6$ (As $\zeta_E^3 = 1$, $\zeta_E^2 H = c_1(E) = d$, $\zeta_E H^2 = 1$ and $H^3 = 0$)

$\Rightarrow 2d^3 - 4d^2 - 3d + 6 = 0 \Rightarrow d = 2$ or $\sqrt{\frac{3}{2}}$.

Hence, in our case $d = 2$. So C is degree three curve in $\mathbb{P}^3$ which is either a cubic in $\mathbb{P}^2$ or a twisted cubic in $\mathbb{P}^3$. As C is not a complete intersection curve, C is not a cubic curve in $\mathbb{P}^2$.

Therefore, C is a twisted cubic, given by the equations, $f_0 = Z_0Z_2 - Z_1^2$, $f_1 = Z_1Z_3 - Z_2^2$, and $f_3 = Z_0Z_3 - Z_1Z_2$, in $\mathbb{P}^3$. □

Remark 3.3.6. We know that, linear subspace is always degree one complete intersection variety. But in the above three examples, we have seen none of X_0, X_1, X_2 are complete intersection variety. So if blow up of projective space along a projective variety is isomorphic to a projective bundle, then this projective variety need not be a complete intersection.

3.4 Nef Cone of varieties

Now, it is interesting to know, what are the nef cones of $\mathbb{P}(E)$, $\mathbb{P}(E_1)$, $\mathbb{P}(E_2)$.

X_2 is a twisted cubic in $\mathbb{P}^3$. X_2 is the image of $\mathbb{P}^1$ into $\mathbb{P}^3$ by the map $[u, v] \to [u^3, u^2v, uv^2, v^3]$, where u, v are homogeneous coordinate of $\mathbb{P}^1$. Also, we can consider the map via $\mathbb{P}^1 \times \mathbb{P}^1$ i.e., $\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, where $\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is given by $[u, v] \to [u^2, v^2] \times [u, v]$ and $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is the Segre Embedding. Pic($\mathbb{P}^1 \times \mathbb{P}^1$)=$\mathbb{Z} \times \mathbb{Z}$, then $X_2 \sim (2, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. We have the blowing up map $\pi_2 : \widetilde{\mathbb{P}}^3_{X_2} \to \mathbb{P}^3$. The Neron Severi group, $N^1(\widetilde{\mathbb{P}}^3_{X_2})$ is generated by $H = \pi_2^* \mathcal{O}_{\mathbb{P}^3}(1)$ and the exceptional divisor E_{X_2}. The numerical equivalence class of one cycle, $N_1(\widetilde{\mathbb{P}}^3_{X_2})$ is generated by the pullback of general line from $\mathbb{P}^3$, $l = \pi_2^* l_{\mathbb{P}^3}$ and an exceptional curve e, as described in the Proposition 2.3.9. Then,

$$l \cdot H = 1, l \cdot E_{X_2} = 0, e \cdot H = 0, e \cdot E_{X_2} = -1 \tag{3.16}$$

The effective cone of curves $\overline{NE}(\widetilde{\mathbb{P}}^3_{X_2}) \subset N_1(\widetilde{\mathbb{P}}^3_{X_2})$ is generated by e and $\widetilde{C} \sim al - be$ where $\widetilde{C}$ is the strict transformation of a degree a curve in $\mathbb{P}^3$, which intersects X_2 at b points with b/a is maximum. Claim that $\widetilde{C} \sim l - 2e$. Assume the claim is not true, then $\frac{b}{a} > 2$. Let S be the image of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ containing the twisted cubic X_2 and $S \sim \mathcal{O}_{\mathbb{P}^3}(2)$. The strict transformation of S is $\widetilde{S} \sim 2H - E_{X_2}$ in $\widetilde{\mathbb{P}}^3_{X_2}$. Then $\widetilde{C} \cdot \widetilde{S} = 2a - b < 0$, which implies $C \subset S$. Let $C \sim (\alpha, \beta)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Then deg$C = \alpha + \beta = a$ in $\mathbb{P}^3$, and $C \cdot X_2 = 2\alpha + \beta = b$ in $\mathbb{P}^3$. Note that, $b = 2\alpha + \beta < 2(\alpha + \beta) = 2a < b$. Here is the contradiction. So our claim is true i.e., $\overline{NE}(\widetilde{\mathbb{P}}^3_{X_2})$ is generated by e and $\widetilde{C} \sim l - 2e$. For more general calculation see ([BL]).

Corollary 3.4.1. *The nef cone of $\widetilde{\mathbb{P}}^3_{X_2}$ is generated by $\pi_2^* \mathcal{O}_{\mathbb{P}^3}(1)$ and $2\pi_2^* \mathcal{O}_{\mathbb{P}^3}(1) - E_{X_2}$.*

Proof. We know from the Theorem 2.5.6 that the nef cone is the dual of the pseudoeffective cone of curves. Then the result follows from the above discussion. $\square$

Corollary 3.4.2. *The nef cone of* $\widetilde{\mathbb{P}}^4_{X_1}$ *is generated by* $\pi_1^*\mathcal{O}_{\mathbb{P}^4}(1)$ *and* $2\pi_1^*\mathcal{O}_{\mathbb{P}^4}(1) - E_{X_1}$.

Proof. We know that $\widetilde{\mathbb{P}}^4_{X_2} \cong \mathbb{P}(E_1) \xrightarrow{\widetilde{\phi_1}} \mathbb{P}^2$ and the projection map is defined by the linear system $2\pi_1^*\mathcal{O}_{\mathbb{P}^4}(1) - E_{X_1} \sim \widetilde{\phi_1}^*\mathcal{O}_{\mathbb{P}^2}(1)$ (Theorem 3.2.1), which is nef because the pullback of ample is nef. Here our claim is that this is a boundary of the nef cone. This is an effective divisor but not big as the highest power self-intersection is zero. So our claim is proved.

Now, the claim is that the another generator of the nef cone is $\pi_1^*\mathcal{O}_{\mathbb{P}^4}(1) \sim \mathcal{O}_{\mathbb{P}(E_1)}(1)$. We know that there is a surjection $E_1 \to E_2$ (see the Corollary 3.3.1). Then we have the short exact sequence of vector bundles in $\mathbb{P}^2$,

$$0 \longrightarrow \mathcal{L} \longrightarrow E_1 \longrightarrow E_2 \longrightarrow 0 \tag{3.17}$$

As $\deg(E_1) = \deg(E_2)$, $\deg(\mathcal{L}) = 0$, i.e $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}$. As a extension of two nef bundles is nef, E_1 is a nef vector bundle. Also, $\mathcal{O}_{\mathbb{P}(E_1)}(1)$ is a nef vector bundle. This is not ample because quotient of ample is ample, but E_2 is not ample. Hence the result follows. $\square$

Corollary 3.4.3. *The nef cone of* $\widetilde{\mathbb{P}}^5_{X_0}$ *is generated by* $\pi^*\mathcal{O}_{\mathbb{P}^5}(1)$ *and* $2\pi^*\mathcal{O}_{\mathbb{P}^5}(1) - E_{X_0}$.

Proof. The argument of this proof is same as Corollary 3.4.2 $\square$

Chapter 4

Geometry of projective plane blow-up at seven points

It is well known that $\mathbb{P}^2$ blown up at six general points is isomorphic to a smooth cubic in $\mathbb{P}^3$ and the embedding is given by the anti-canonical divisor. Conversely, any smooth cubic of $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^2$ blown up at six general points. Motivated by this, in the fourth chapter of this thesis, we prove that $\mathbb{P}^2$ blown up at seven general points can be embedded as a $(2,2)$ divisor in $\mathbb{P}^1 \times \mathbb{P}^2$ as well as in $\mathbb{P}^5$ by the very ample divisors $4\pi^*H - 2E_i - \sum_{j=1, j\neq i}^{7} E_j$. Conversely, any smooth surface in the complete linear system $\mid (2,2) \mid$ of $\mathbb{P}^1 \times \mathbb{P}^2$ can be obtained as an embedding of blowing up of $\mathbb{P}^2$ at seven points.

We also prove that $\mathbb{P}^2$ blown-up at seven general points has conic bundle structures over $\mathbb{P}^1$ and the anti-canonical divisor of $\mathbb{P}^2$ blown up at seven general points corresponds to the degree two map to $\mathbb{P}^2$. Moreover, we show that if S is a smooth surface in $\mathbb{P}^1 \times \mathbb{P}^2$ of type $(2,2)$ and C is an irreducible curve in S, then $C \cdot C \geq -2$. Finally, we show that any smooth surface of type $(2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ can contain at most four irreducible (-2) curves.

In sections 1, 2, 3, and 4 we denote $\mathbb{P}^2$ blow-up at r *general points*by $\widetilde{\mathbb{P}^2_r}$ or

$\widetilde{\mathbb{P}}^2_{P_1P_2\cdots P_r}$,where $P_1, P_2, \cdots, P_r \in \mathbb{P}^2$ unless stated otherwise . For the basic notations and definitions of blow-up of surface, see the subsection 2.2.1.

4.1 $\mathbb{P}^2$ blow-up at seven general points as a double cover of $\mathbb{P}^2$

Lemma 4.1.1. *Let $\widetilde{\mathbb{P}^2_7}$ be $\mathbb{P}^2$ blow-up at seven general points. If there is a degree two finite map from $\widetilde{\mathbb{P}^2_7}$ to $\mathbb{P}^2$, then this map can only be given by the anti-canonical divisor, which is $\mid 3\pi^*H - \sum_{i=1}^{7} E_i \mid$, where H is a hyperplane section of $\mathbb{P}^2$.*

Proof. First, we claim that any finite degree two map from $\widetilde{\mathbb{P}^2_7}$ to $\mathbb{P}^2$ is defined by a complete linear system. Assume that the claim is not true. Let $\mathfrak{d}$ be a sub-linear system of the complete linear system $\mid D \mid$, which defines a finite degree two map $i_{\mathfrak{d}} : \widetilde{\mathbb{P}^2_7} \to \mathbb{P}^2$. As $\mathfrak{d}$ is base point free, $\mid D \mid$ is also base point free. Hence $\mid D \mid$ induces the map $i_{|D|} : \widetilde{\mathbb{P}^2_7} \to \mathbb{P}^N$, where $h^0(\widetilde{\mathbb{P}^2_7}, \mid D \mid) = N+1$. Note that, $2 \geq \dim(i_{|D|}(\widetilde{\mathbb{P}^2_7})) \geq \dim(i_{\mathfrak{d}}(\widetilde{\mathbb{P}^2_7})) = 2$. This implies $i_{|D|}(\widetilde{\mathbb{P}^2_7})$ is a non-degenerated surface in $\mathbb{P}^N$, $i_{|D|}$ is generically finite map and $i^*_{|D|}\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}_{\widetilde{\mathbb{P}^2_7}}(D)$. As $\mathfrak{d} \subseteq \mid D \mid$ corresponds to the degree two map $i_{\mathfrak{d}}$, $D^2 = 2$. Hence,

$$D^2 = i^*_{|D|}\mathcal{O}_{\mathbb{P}^N}(1) \cdot i^*_{|D|}\mathcal{O}_{\mathbb{P}^N}(1),$$
$$D^2 = deg(i_{|D|})\big(\mathcal{O}_{\mathbb{P}^N}(1) \mid_{i_{|D|}(\widetilde{\mathbb{P}^2_7})} \cdot \mathcal{O}_{\mathbb{P}^N}(1) \mid_{i_{|D|}(\widetilde{\mathbb{P}^2_7})}\big),$$
$$2 = deg(i_{|D|}) \cdot deg(i_{|D|}(\widetilde{\mathbb{P}^2_7})).$$

Hence, either $deg(i_{|D|}) = 2$, or $deg(i_{|D|}) = 1$.

If $deg(i_{|D|}) = 2$, then $deg(i_{|D|}(\widetilde{\mathbb{P}^2_7})) = 1$ in $\mathbb{P}^N$. If $Y \subset \mathbb{P}^N$ is an irreducible non-degenerated surface of degree d, then $2 + d - 1 \geq N$(See [GH] page 173). In our case $d = 1$, hence $N = 2$. Therefore, $\dim\mathfrak{d} = \dim\mid D \mid$, and hence the linear system $\mathfrak{d}$ is a complete linear system.

If $deg(i_{|D|}) = 1$, then $deg(i_{|D|}(\widetilde{\mathbb{P}^2_7})) = 2$ in $\mathbb{P}^N$. Using the same result as referred in the above paragraph, we get $N \leq 3$. The case $N = 2$ is not possible, because there is no degree 2 surface in $\mathbb{P}^2$. Now if $N = 3$, then $i_{|D|} : \widetilde{\mathbb{P}^2_7} \to \mathbb{P}^3$ is a degree one map and the image is degree two surface in $\mathbb{P}^3$. Up to isomorphism, there are two irreducible degree two surfaces in $\mathbb{P}^3$, one are smooth quadrics which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and the other are cone over plane conic curve which have a singularity at the vertex. Let $P \in \mathbb{P}^3 \backslash i_{|D|}(\widetilde{\mathbb{P}^2_7})$ and we take a projection from P to a hyperplane in $\mathbb{P}^3$, $p : \mathbb{P}^3 \backslash \{P\} \to \mathbb{P}^2$ such that we have $i_{\mathfrak{d}} = p \circ i_{|D|}$, where $\widetilde{\mathbb{P}^2_7} \xrightarrow{i_{|D|}} i_{|D|}(\widetilde{\mathbb{P}^2_7}) \xrightarrow{p} \mathbb{P}^2$. As $i_{\mathfrak{d}}$ is a finite map of degree two, $i_{|D|}$ is a finite map of degree one and p is a finite degree two map. But there is not any finite degree one map from $\widetilde{\mathbb{P}^2_7}$ to either $\mathbb{P}^1 \times \mathbb{P}^1$ or from cone over plane conic curve. Hence we conclude that, there doesn't exist any such degree one map $\widetilde{\mathbb{P}^2_7} \xrightarrow{i_{|D|}} \mathbb{P}^3$. Our claim is proved i.e., any finite degree two map from $\widetilde{\mathbb{P}^2_7}$ to $\mathbb{P}^2$ is determined by a complete linear system.

Now, let $\phi : \widetilde{\mathbb{P}^2_7} \to \mathbb{P}^2$ be a degree two map defined by the complete linear system $\mid D \mid$ where $D = a\pi^*H - \sum_{i=1}^{7} b_i E_i$, $a > 0$, $b_i \geq 0$ and at least one $b_i > 0$. As ϕ is a degree two map, $D^2 = 2$ which implies $a^2 - \sum_{i=1}^{7} b_i^2 = 2$. We know that expected dimension, expdim$\mid D \mid \leq$ dim$\mid D \mid = 2$.

expdim$\mid D \mid = \frac{(a+1)(a+2)}{2} - 1 - \sum_{i=1}^{7} \frac{b_i(b_i+1)}{2} \leq 2$

$\Rightarrow a^2 + 3a + 2 - 2 - \sum b_i^2 - \sum b_i \leq 4$

$\Rightarrow 3a + 2 - \sum b_i \leq 4$ (as $a^2 - \sum_{i=1}^{7} b_i^2 = 2$)

$\Rightarrow 3a - 2 \leq \sum b_i$

If $\{x_i\}$ and $\{y_i\}$ are two real sequences, then by the Schwarz's inequality,

$\mid \sum_i x_i y_i \leq \mid \sum_i x_i^2 \mid . \mid \sum_i y_i^2 \mid$.

We replace $x_i = 1, y_i = b_i$ for $i = 1, 2, \cdots, 7$ and $x_i = 0, y_i = 0$ for i >7. Then we have

$(\sum b_i)^2 \leq 7(\sum b_i^2)$

$\Rightarrow (3a - 2)^2 \leq 7(a^2 - 2)$

$\Rightarrow a^2 - 6a + 9 \leq 0$

$\Rightarrow (a-3)^2 \leq 0$

$\Rightarrow a = 3.$

Hence the only possibility of D is $3\pi^*H - \sum_{i=1}^{7} E_i$. So the degree two map ϕ is given by the divisor $D = 3\pi^*H - \sum_{i=1}^{7} E_i$. This gives the proof of the statement of the theorem.

□

4.2 Conic bundle structure of $\widetilde{\mathbb{P}^2_7}$ over $\mathbb{P}^1$

For the basic definitions and properties of the conic bundle, see the section 2.4.

Theorem 4.2.1. *If $\widetilde{\mathbb{P}^2_7}$ admits a conic bundle structure over $\mathbb{P}^1$, then it is given by a complete linear system $\mid D \mid$. Moreover, D will have one of the following forms;*

(I) $\pi^*H - E_i$, $1 \leq i \leq 7$;

(II) $2\pi^*H - \sum_{i=1}^{4} E_{l_i}$, where l_i's are distinct and $1 \leq l_i \leq 7$;

(III) $3\pi^*H - 2E_i - \sum_{j=2}^{6} E_{k_j}$, where i and k_j's are distinct and $1 \leq i, k_j \leq 7$;

(IV) $4\pi^*H - \sum_{j=1}^{4} E_{k_j} - \sum_{i=5}^{7} 2E_{l_i}$ where k_j's and l_i's are distinct and $1 \leq l_i, k_j \leq 7$ and;

(V) $5\pi^*H - E_i - \sum_{j=2}^{7} 2E_{k_j}$ where i and k_j's are distinct and $1 \leq i, k_j \leq 7$.

Proof. We claim that any conic bundle structure of $\widetilde{\mathbb{P}^2_7}$ over $\mathbb{P}^1$ is defined by a complete linear system. Assume that the claim is not true. Let $\mathfrak{b}$ be a sub-linear system of the complete linear system $\mid D \mid$ which gives a conic bundle map $j_{\mathfrak{b}} : \widetilde{\mathbb{P}^2_7} \to \mathbb{P}^1$. Now, we define a morphism $j_{|D|} : \widetilde{\mathbb{P}^2_7} \to \mathbb{P}^n$, where $h^0(\widetilde{\mathbb{P}^2_7}, \mid D \mid) = n+1$. Note that, $2 \geq \dim(j_{|D|}(\widetilde{\mathbb{P}^2_7})) \geq \dim(j_{\mathfrak{b}}(\widetilde{\mathbb{P}^2_7})) = 1$.

If $\dim j_{|D|}(\widetilde{\mathbb{P}^2_7}) = 2$, then $j_{|D|}$ is a generically finite map between two surfaces. We know that $D^2 = \deg(j_{|D|}) \cdot \deg(j_{|D|}(\widetilde{\mathbb{P}^2_7}))$. Degree of a finite map and degree of a variety in $\mathbb{P}^n$ are always strictly positive i.e., $\deg(j_{|D|}) > 0$ and $\deg(j_{|D|}(\widetilde{\mathbb{P}^2_7})) > 0$. But we have $j_{\mathfrak{b}}^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}(D)$ and $j_{\mathfrak{b}}$ is a conic bundle map. Hence the fibers of $j_{\mathfrak{b}}$ are disjoint curves in $\widetilde{\mathbb{P}^2_7}$, which are linearly equivalent to D. Hence $D^2 = 0$, which contradict the strictly positivity of both $\deg(j_{|D|})$ and $\deg(j_{|D|}(\widetilde{\mathbb{P}^2_7}))$.

The other possibility is $\dim j_{|D|}(\widetilde{\mathbb{P}^2_7}) = 1$. We can project repeatedly from outside the image of $j_{|D|}$ and can get the following commutative diagram.

$$\begin{tikzcd}
\widetilde{\mathbb{P}^2_7} \arrow[r, "j_{|D|}"] \arrow[dr, "j_{\mathfrak{b}}"'] & j_{|D|}(\widetilde{\mathbb{P}^2_7}) \arrow[d, "q"] \\
& \mathbb{P}^1
\end{tikzcd}$$

where q is a finite map between two curves. As $j_{\mathfrak{b}}$ is a conic bundle map, fibers are generically connected. Then it follows that $\deg(q) = 1$.

Now, let $\psi : \widetilde{\mathbb{P}^2_7} \to \psi(\widetilde{\mathbb{P}^2_7}) \subseteq \mathbb{P}^n$ be a conic bundle map over a curve given by the divisor D. So a general element of $\mid D \mid$ is a smooth rational curve in $\widetilde{\mathbb{P}^2_7}$. Any two fibre do not intersect each other. Hence, D^2=0 and $\dim \mid D \mid = n \geq 1$. Let $D = a\pi^*H - \sum_{i=1}^{7} b_iE_i$ where $a > 0$ and $b_i \geq 0$.

$D^2 = 0 \Rightarrow a^2 = \sum b_i^2$

As the genus of any smooth curve of $\mid D \mid$ is zero,

$\Rightarrow \frac{(a-1)(a-2)}{2} - \sum_{i=1}^{7} \frac{b_i(b_i-1)}{2} = 0$ (using the Proposition 2.2.15)

$\Rightarrow a^2 - 3a + 2 - \sum b_i^2 + \sum b_i = 0$

$\Rightarrow 3a - \sum b_i = 2$

Schwarz's inequality: $\mid \sum_i x_iy_i \leq \mid \sum_i x_i^2 \mid . \mid \sum_i y_i^2 \mid$

Here we replace $x_i = 1, y_i = b_i$ for $i = 1, 2, ..., 7$ and $x_i = 0, y_i = 0$ for $i > 7$. So we have

$\Rightarrow (\sum b_i)^2 \leq 7(\sum b_i^2)$

$\Rightarrow (3a - 2)^2 \leq 7a^2$

$\Rightarrow a^2 - 6a + 2 \leq 0$

$\Rightarrow a < 6$

So possible values of a are 1, 2, 3, 4, 5.

Case I ($a = 1$)

If $a = 1$, then $b_i = 1$ and $b_j = 0$ where $i \neq j$. So with out loss of generality, take $b_1 = 1$, then $D = \pi^*H - E_1$. First, we need to check $\mid D \mid$ gives a map to $\mathbb{P}^1$. We have $1 = \text{expdim}\mid D \mid \leq \dim(\mid D \mid)$. Now we claim $\dim(\mid D \mid) = 1$. But this is clear because, any curve of $\mid D \mid$ corresponds to a line passing through P_1 in $\mathbb{P}^2$. So $\mid D \mid$ gives a map to $\mathbb{P}^1$ and the generic fiber is an irreducible rational curve. Hence $(\widetilde{\mathbb{P}^2_7}, \mathbb{P}^1, \pi^*H - E_i)$ is a conic over $\mathbb{P}^1$.

Case II ($a = 2$)

If $a = 2$, only possibilities of b_i are b_{k_j}=1 where $j = 1, 2, 3, 4$ and others are zero. Then with out loss of generality, we can consider $D = 2\pi^*H - \sum_{i=1}^{4} E_i$. Here also $\dim(\mid D \mid) = 1$, because any curve of $\mid D \mid$ corresponds to a conic in $\mathbb{P}^2$ passing through P_1, P_2, P_3, P_4 which are in general position. Hence, $D = 2\pi^*H - \sum_{i=1}^{4} E_i$ gives a map to $\mathbb{P}^1$, where the generic fiber is an irreducible rational curve. Then $(\widetilde{\mathbb{P}^2_7}, \mathbb{P}^1, 2\pi^*H - \sum_{i=1}^{4} E_i)$ is also a conic bundle over $\mathbb{P}^1$.

Case III ($a = 3$)

If $a = 3$, then only possibilities of b_i's are $b_{k_j} = 1$, where $j = 1, 2, 3, 4, 5$ and among the other two b_i's, one is two and the other is zero. So with out loss of generality, we consider $D = 3\pi^*H - 2E_1 - \sum_{i=2}^{6} E_i$ and by similar argument described as above, $\dim\mid D \mid = 1$.

So this D gives a map to $\mathbb{P}^1$ where the generic fiber is an irreducible rational curve. Then $(\widetilde{\mathbb{P}^2_7}, \mathbb{P}^1, 3\pi^*H - 2E_1 - \sum_{i=2}^{6} E_i)$ is also a conic bundle over $\mathbb{P}^1$.

Case IV ($a = 4$)

If $a = 4$, then using the above equations $3a - \sum b_i = 2$ and $a^2 = \sum b_i^2$, we have $10 = \sum_{i=1}^{7} b_i$ and $16 = \sum_{i=1}^{7} b_i^2$ respectively. Only possibilities of b_i's are $b_{i_1} = b_{i_2} = b_{i_3} = b_{i_4} = 1$ and $b_{j_5} = b_{j_6} = b_{j_7} = 2$, where i_k and j_l are distinct. So with out loss of generality, consider $D = 4\pi^*H - \sum_{j=1}^{4} E_j - \sum_{i=5}^{7} 2E_i$. Now take a quadratic transformation $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of $\mathbb{P}^2$ centered at P_5, P_6, P_7 . Then $\phi(P_i) = P_i'$ for $i = 1, 2, 3, 4$. Here π is the blow up map at the points $P_1, P_2, ..., P_7$ of $\mathbb{P}^2$ and L_{ij} is strict transformation of the line joining P_i and P_j in $\widetilde{\mathbb{P}^2}_{P_1P_2...P_7}$ for each i, j. Let π' be the blow-up map of $\mathbb{P}^2$ at the points $P_1', P_2', P_3', P_4', Q_5, Q_6, Q_7$, where $\pi'(L_{56}) = Q_7, \pi'(L_{67}) = Q_5, \pi'(L_{57}) = Q_6$. Then $\widetilde{\mathbb{P}^2}_{P_1P_2...P_7} = \widetilde{\mathbb{P}^2}_{P_1'P_2'P_3'P_4'Q_5Q_6Q_7}$, and $D \sim D'$ where $D' = 2\pi'^*H' - \sum_{i=1}^{4} E_i'$ (See the Theorem 2.2.14). Hence $\mid D \mid = \mid D' \mid$, and we have proved in the Case II that $\mid D' \mid$ gives a map to $\mathbb{P}^1$ and the generic fiber of that map is an irreducible rational curve.

$$
\begin{tikzcd}
\widetilde{\mathbb{P}^2_7} \arrow[dr, "\theta"] \arrow[drr, bend left, "\pi'"] \arrow[ddr, bend right, "\pi"'] & & \\
& \widetilde{\mathbb{P}^2_3} \arrow[r, "p"] \arrow[d, "q"] & \mathbb{P}^2 \\
& \mathbb{P}^2 &
\end{tikzcd}
$$

Then similarly $(\widetilde{\mathbb{P}^2_7}, \mathbb{P}^1, 4\pi^*H - \sum_{j=1}^{4} E_j - \sum_{i=5}^{7} 2E_i)$ is also a conic bundle over $\mathbb{P}^1$

Case V $(a = 5)$

Calculation of this case goes similar as in Case IV. If $a = 5$, then we have $13 = \sum_{i=1}^{7} b_i$ and $25 = \sum_{i=1}^{7} b_i^2$. Only possibilities of b_i's are $b_i = 1, b_{j_k} = 2$, where $k = 1, 2, ..., 6$ and $i \neq j_k$. In particular, $D = 5\pi^*H - E_1 - \sum_{i=2}^{7} 2E_i$. Now we take a quadratic transformation of $\mathbb{P}^2$ centered at P_2, P_3, P_4. After quadratic transformation we get $\mid D' \mid = \mid D \mid$ where $D' = 4\pi'^*H' - \sum_{j=1}^{4} E_j' - \sum_{i=5}^{7} 2E_i'$ (See the Theorem 2.2.14). Then using the Case IV, we get that $(\widetilde{\mathbb{P}^2_7}, \mathbb{P}^1, 5\pi^*H - E_1 - \sum_{i=2}^{7} 2E_i)$

is a conic bundle over $\mathbb{P}^1$.

This proves our result. $\square$

4.3 $\widetilde{\mathbb{P}^2_7}$ embedded as a (2,2) divisor of $\mathbb{P}^1 \times \mathbb{P}^2$

Theorem 4.3.1. *$\mathbb{P}^2$ blown up at seven general points can be embedded as a (2,2) divisor of $\mathbb{P}^1 \times \mathbb{P}^2$.*

Proof. We have the morphism $p_1 : \widetilde{\mathbb{P}^2_7} \to \mathbb{P}^1$ which is defined by the linear system $\mid \pi^*H - E_1 \mid$ and the morphism $p_2 : \widetilde{\mathbb{P}^2_7} \to \mathbb{P}^2$ which is defined by the linear system $\mid 3\pi^*H - \sum_{i=1}^{7} E_i \mid$. Then we can define the induced morphism $p_1 \times p_2 : \widetilde{\mathbb{P}^2_7} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^2$.

We have the Segre embedding, $\nu : \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ by the very ample divisor $\mid (1,1) \mid$ of $\mathbb{P}^1 \times \mathbb{P}^2$. Hence the morphism $\nu \circ (p_1 \times p_2) : \widetilde{\mathbb{P}^2_7} \to \mathbb{P}^5$ is given by the linear system $\mid 4\pi^*H - 2E_1 - \sum_{i=2}^{7} E_i \mid$

But we know $4\pi^*H - 2E_1 - \sum_{i=2}^{7} E_i$ is a very ample divisor of $\widetilde{\mathbb{P}^2_7}$ by Theorem 2.1 ([Ha]). Therefore, $\nu \circ (p_1 \times p_2)$ gives an embedding and $p_1 \times p_2 : \widetilde{\mathbb{P}^2_7} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is a closed immersion.

With a slight abuse of notations, we will still use $\widetilde{\mathbb{P}^2_7}$ and $\mathbb{P}^1 \times \mathbb{P}^2$ to denote their embeddings into $\mathbb{P}^5$.

As $\widetilde{\mathbb{P}^2_7}$ is a non-singular surface in a threefold, it corresponds to an element of its Weil divisor group. We know Pic($\mathbb{P}^1 \times \mathbb{P}^2$)= Pic($\mathbb{P}^1$)$\oplus$Pic($\mathbb{P}^2$) $= \mathbb{Z} \oplus \mathbb{Z}$ and the generators of the group are $\mathrm{pt}\times\mathbb{P}^2 = (1,0)$ and $\mathbb{P}^1 \times H = (0,1)$, where H is a hyperplane section in $\mathbb{P}^2$. Then assume, $\widetilde{\mathbb{P}^2_7} \sim (a,b)$ in Pic($\mathbb{P}^1 \times \mathbb{P}^2$) and note that $(a,b) = (a.pt \times \mathbb{P}^2) + (\mathbb{P}^1 \times bH)$. So generic fiber of the first projection from (a,b) is a curve of degree b and generic fiber of the second projection contains a number of points. So from Lemma(4.1.1) and Theorem(4.2.1) we have $a = 2$ and $b = 2$. Hence the result is proved. $\square$

Remark 4.3.2. *Similarly the divisors $4\pi^*H - 2E_i - \sum_{j=1, j\neq i}^{7} E_j$ also give an embedding of $\widetilde{\mathbb{P}^2_7}$ in $\mathbb{P}^1 \times \mathbb{P}^2$ as a smooth surface of $(2, 2)$ type.*

Remark 4.3.3. *Let $(\widetilde{\mathbb{P}^2}_{P_1P_2\ldots P_7}, \mathbb{P}^1, f)$ be a conic over $\mathbb{P}^1$ where the morphism f is defined by the divisor $\pi^*H - E_1$. Then each fiber is $f^*(P) \sim \pi^*H - E_1$, where $P \sim \mathcal{O}_{\mathbb{P}^1}(1)$. As $(\pi^*H - E_1) \cdot (4\pi^*H - 2E_1 - \sum_{i=2}^{7} E_i) = 2$, each fiber of f is a degree two rational curve in $\mathbb{P}^5$ i.e., a conic in some plane of $\mathbb{P}^5$.*

In the Theorem (4.3.1), we have seen that the divisor of the Lemma (4.1.1) along with divisors of Case-I of the Theorem (4.2.1) give us an embedding of $\widetilde{\mathbb{P}^2_7}$ in $\mathbb{P}^1 \times \mathbb{P}^2$. Now we investigate whether divisors of the other cases of the Theorem (4.2.1) along with the unique degree two map of the Lemma (4.1.1) will give us an embedding in $\mathbb{P}^1 \times \mathbb{P}^2$ or not.

Remark 4.3.4. Case I (7 possibilities)

$$\begin{tikzcd} \widetilde{\mathbb{P}^2}_{P_1P_2\ldots P_7} \arrow[r, "|D'|"] \arrow[dr, "|D''|"] & \mathbb{P}^1 \\ & \mathbb{P}^2 \end{tikzcd}$$

$D' = \pi^*H - E_i$ and $D'' = 3\pi^*H - \sum_{i=1}^{7} E_i$

Theorem (4.3.1) implies $D' + D''$ gives an closed immersion.

Case II (35 possibilities)

$D' = 2\pi^*H - \sum_{i=1}^{4} E_{P_{j_i}}$ and $D'' = 3\pi^*H - \sum_{i=1}^{7} E_{P_i}$. In particular consider $D' = 2\pi^*H - \sum_{i=1}^{4} E_{P_i}$. Now take a quadratic transformation centered at P_1, P_2, P_3 and we will get $\widetilde{\mathbb{P}^2}_{P_1P_2\ldots P_7} = \widetilde{\mathbb{P}^2}_{Q_1Q_2Q_3P_4'P_5'P_6'P_7'}$ and $D' \sim F'$ and $D'' \sim F''$ where $F' =$

$\pi'^* H' - E'_{P'_4}$ and $F'' = 3\pi'^* H' - \sum_{i=1}^{3} E'_{Q_i} - \sum_{i=4}^{7} E'_{P'_i}$ and $\pi' : \widetilde{\mathbb{P}^2}_{Q_1 Q_2 Q_3 P'_4 P'_5 P'_6 P'_7} \longrightarrow \mathbb{P}^2$ is the blow-up map at the points $Q_1, Q_2, Q_3, P'_4, P'_5, P'_6, P'_7$ (See the Theorem 2.2.14). From Case I we know, $F' + F''$ is very ample divisor. Hence, $D' + D''$ also a very ample divisor. So $\widetilde{\mathbb{P}^2}_{P_1 P_2 \ldots P_7} \xrightarrow{D'+D''} \mathbb{P}^1 \times \mathbb{P}^2$ is an closed immersion.

Case III (42 possibilities)

$D' = 3\pi^* H - 2E_i - \sum_{j=1, k_j \neq i}^{5} E_{k_j}$ and $D'' = 3\pi^* H - \sum_{i=1}^{7} E_{P_i}$, in particular $D' = 3\pi^* H - 2E_{P_1} - \sum_{j=2}^{6} E_{P_j}$. Now take the quadratic transformation centered at P_1, P_2, P_3, we will get $\widetilde{\mathbb{P}^2}_{P_1 P_2 \ldots P_7} = \widetilde{\mathbb{P}^2}_{Q_1 Q_2 Q_3 P'_4 P'_5 P'_6 P'_7}$ and $D' \sim F'$ and $D'' \sim F''$, where $F' = 2\pi'^* H' - E'_{Q_1} - E'_{P'_4} - E'_{P'_5} - E'_{P'_6}$ and $F'' = 3\pi'^* H' - \sum_{i=1}^{3} E'_{Q_i} - \sum_{i=4}^{7} E'_{P'_i}$ (See the Theorem 2.2.14). Now, we are in the same situation as in Case II, and we repeat the Case II. Then there exist $R_1, R_2, \ldots R_7 \in \mathbb{P}^2$ such that $\pi''^* H'' - E_{R_1} \sim F' \sim D'$ and $3\pi''^* H'' - \sum_{i=1}^{7} E_{R_i} \sim F'' \sim D''$. Similarly, $D' + D''$ also gives a closed immersion.

Case IV (35 possibilities)

$D' = 4\pi^* H - \sum_{j=1}^{4} E_{P_{k_j}} - \sum_{i=5}^{7} 2E_{P_{l_i}}$ and $D'' = 3\pi^* H - \sum_{i=1}^{7} E_{P_i}$. In particular, $D' = 4\pi^* H - \sum_{j=1}^{4} E_{P_j} - \sum_{i=5}^{7} 2E_{P_i}$. Now, take the quadratic transformation centered at P_5, P_6, P_7 and we will get $\widetilde{\mathbb{P}^2}_{P_1 P_2 \ldots P_7} = \widetilde{\mathbb{P}^2}_{P'_1 P'_2 P'_3 P'_4 Q_5 Q_6 Q_7}$. So $D' \sim F'$ and $D'' \sim F''$, where $F' = 2\pi'^* H' - \sum_{i=1}^{4} E'_{P'_i}$ and $F'' = 3\pi'^* H' - \sum_{i=1}^{4} E'_{P'_i} - \sum_{j=5}^{7} E'_{Q_j}$ (See the Theorem 2.2.14). Now, we are in the same position as in Case II, and repeat the argument of Case II again. Then there exist $R_1, R_2, \ldots, R_7 \in \mathbb{P}^2$ such that $\pi''^* H'' - E_{R_1} \sim F' \sim D'$ and $3\pi''^* H'' - \sum_{i=1}^{7} E_{R_i} \sim F'' \sim D''$. Hence, $D' + D''$ also gives an closed immersion.

Case V (7 possibilities)

$D' = 5\pi^* H - E_{P_i} - \sum_{j=1, k_j \neq i}^{6} 2E_{P_{k_j}}$ and $D'' = 3\pi^* H - \sum_{i=1}^{7} E_{P_i}$. In particular, D=$5\pi^*H - E_{P_1} - \sum_{i=2}^{7} 2E_{P_i}$. Now take the quadratic transformation centered at P_2, P_3, P_4 and we will get $\widetilde{\mathbb{P}^2}_{P_1P_2 \dots P_7} = \widetilde{\mathbb{P}^2}_{P'_1Q_2Q_3Q_4P'_5P'_6P'_7}$ and $D' \sim F'$ and $D'' \sim F''$ where $F' = 4\pi'^*H' - E_{P'_1} - \sum_{j=2}^{4} E_{Q_j} - \sum_{i=5}^{7} 2E_{P'_i}$ and $F'' = 3\pi'^*H' - E_{P'_1} - \sum_{i=2}^{4} E'_{Q_i} - \sum_{j=5}^{7} E'_{P'_j}$ (See the Theorem 2.2.14). Now we are in the same position as Case IV and repeating the argument of Case IV, there exist $R_1, R_2, ..., R_7 \in \mathbb{P}^2$ such that $\pi''^*H'' - E_{R_1} \sim F' \sim D'$ and $3\pi''^*H'' - \sum_{i=1}^{7} E_{R_i} \sim F'' \sim D''$, where $\pi'' : \widetilde{\mathbb{P}^2}_{R_1R_2 \dots R_7} \longrightarrow \mathbb{P}^2$ is the blowing up map at the points $R_1, R_2, ..., R_7$. Hence $D' + D''$ also gives an closed immersion.

This are all possible very ample divisors of $\widetilde{\mathbb{P}^2}_{P_1P_2 \dots P_7}$ such that this surface can be embedded in $\mathbb{P}^5$ inside the image of $\mathbb{P}^1 \times \mathbb{P}^2$ given by the Segre embedding. Also, the image of $p_1 \times p_2 : \widetilde{\mathbb{P}^2}_{P_1P_2 \dots P_7} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is linearly equivalent to (2,2) divisor in $\mathbb{P}^1 \times \mathbb{P}^2$.

Consider the surface S for which we have a sequence of morphism

$$S = S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} S_{n-2} \dots . \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = \mathbb{P}^2$$

where $S_i \xrightarrow{\pi_i} S_{i-1}$ is the blow-up of S_{i-1} at the point P_i. Let $\mathcal{E} = \{\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_n\}$ can be considered as a free basis of Pic(S), where $\mathcal{E}_0$ is the pullback of the class of a lines in $\mathbb{P}^2$, and $\mathcal{E}_i$ is the class of $E_i = \pi_i^{-1}(P_i)$ for $i = 1, \cdots, n$. Such collection of divisor classes $\mathcal{E}$ is called an *exceptional configuration.* Thus, there is a bijection between sequence of morphisms given above from S to $\mathbb{P}^2$ and exceptional configurations of S.

Remark 4.3.5. *Let $\mathcal{L}$ be a very ample divisor of $\widetilde{\mathbb{P}^2}_7$, which corresponds an embedding of $\widetilde{\mathbb{P}^2}_7$ into $\mathbb{P}^5$. Then it is clear from the Remark 4.3.4, that there exists an exceptional configuration $\mathcal{E} = \{\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_n\}$ of $\widetilde{\mathbb{P}^2}_7$ such that $\mathcal{L} = 4\mathcal{E}_0 - 2\mathcal{E}_1 - \sum_{i=2}^{7} \mathcal{E}_i$.*

4.4 Lines of $\mathbb{P}^2$ blow-up at seven general points

We know that $\mathbb{P}^2$ blown up at six general points has 27 lines when we see it as a cubic in $\mathbb{P}^3$ embedded by the anti-canonical divisor. Also, we know that this lines are all (-1) curves.

Here, we have seen $\mathbb{P}^2$ blown up at seven general points can be embedded in $\mathbb{P}^5$ as well as in $\mathbb{P}^1 \times \mathbb{P}^2$ using the very ample divisor $D = 4\pi^*H - 2E_1 - \sum_{i=2}^{7} E_i$ or the pair of divisors (D_1, D_2), where $D_1 + D_2 = D$, and $D_1 = \pi^*H - E_1$, $D_2 = 3\pi^*H - \sum E_i$. Recall that there are 56 (-1) curves in $\widetilde{\mathbb{P}^2_7}$; which are

- $L_{ij} = \pi^*H - E_i - E_j$, the strict transformation of the line in $\mathbb{P}^2$ containing P_i and P_j, $1 \leq i, j \leq 7$, (21 possibilities)

- $G_{ij} = \pi^*(2H) - \sum_{k \neq i,j} E_k$, the strict transformation of the conic not passing through P_i and P_j and passing through the rest of five P_k's of $P_1, P_2, ..., P_7$, $1 \leq i, j \leq 7$, (21 possibilities)

- $F_i = \pi^*(3H) - 2E_i - \sum_{j \neq i} E_j$, the strict transformation of the cubic passing through all the seven points and with a double point at P_i where $i = 1, 2, ..., 7$, (7 possibilities).

- Exceptional curves E_i, the total transformation of the points P_i, $i = 1, 2, ..., 7$, (7 possibilities)

Lemma 4.4.1. *Any line of $\widetilde{\mathbb{P}^2_7}$ in the embedding of $\mathbb{P}^5$ is a (-1) curve.*

Proof. Let $L = a\pi^*H - \sum_{i=1}^{7} b_i E_i$ be an effective curve in $\widetilde{\mathbb{P}^2_7}$. Then either $a = 0$, $b_i = 1$ for one i, and $b_j = 0$ for all $j \neq i$ or $a \geq 1$, $b_i \geq 0$ and $a \geq b_i$ $\forall i$. L is a line in the embedding of $\mathbb{P}^5$ if and only if

$$L \cdot (4\pi^*H - 2E_1 - \sum_{i=2}^{7} E_i) = 1.$$

For the first case, if L is a line, then $L = E_i$ for $i = 2, \cdots, 7$. Hence, L is a (-1) curve. For the second case,

$$4a - 2b_1 - b_2 - b_3 - b_4 - b_5 - b_6 - b_7 = 1.$$

We claim that L is a (-1) curve in this case also.

The genus of L is zero i.e $\frac{(a-1)(a-2)}{2} - \sum_{i=1}^{7} \frac{b_i(b_i-1)}{2} = 0$ (using the Proposition 2.2.15). Solving these two equations we get, $a^2 - \sum_{i=1}^{7} b_i^2 = -a + b_1 - 1$. As $b_1 \leq a$, $-a + b_1 - 1 \leq -1 \Rightarrow a^2 - \sum_{i=1}^{7} b_i^2 \leq -1 \Rightarrow L \cdot L \leq -1$.

Let D be any irreducible curve of $\widetilde{\mathbb{P}^2_7}$. The genus of D, $g(D) = \frac{1}{2}(D \cdot D - (-K_{\widetilde{\mathbb{P}^2_7}}) \cdot D) + 1 \geq 0$. As the anti-canonical of $\widetilde{\mathbb{P}^2_7}$ is an irreducible effective divisor which gives a finite map from $\widetilde{\mathbb{P}^2_7}$ to $\mathbb{P}^2$, $(-K_{\widetilde{\mathbb{P}^2_7}}) \cdot D > 0$. Then $D \cdot D \geq -1$. So we conclude that $L \cdot L = -1$. $\square$

Remark 4.4.2. We have listed all 56 (-1) curves of $\widetilde{\mathbb{P}^2_7}$. An effective divisor L is a line in $\widetilde{\mathbb{P}^2_7}$ in the embedding of $\mathbb{P}^5$ if and only if $L \cdot (4\pi^* H - 2E_1 - \sum_{i=2}^{7} E_i) = 1$. Hence, there are only 12 lines in $\widetilde{\mathbb{P}^2_7}$, which are $E_2, E_3, E_4, E_5, E_6, E_7, L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{17}$. Note that, these are also lines in $\mathbb{P}^1 \times \mathbb{P}^2$. Let $A_1(\mathbb{P}^1 \times \mathbb{P}^2) = \mathbb{Z} \oplus \mathbb{Z}$ be the group of 1-cycle modulo rational equivalence, which is generated by $pt \times H$ and $\mathbb{P}^1 \times pt$, where H is the class of lines in $\mathbb{P}^2$. Note that, any curve rationally equivalent to $pt \times H$ or $\mathbb{P}^1 \times pt$ is a line in $\mathbb{P}^1 \times \mathbb{P}^2$. This six pairs of lines in $\widetilde{\mathbb{P}^2_7}$ have the property that $E_i \cdot L_{1j} = \delta_{ij}, \forall i, j$, $E_i \cdot E_j = 0$ for $i \neq j$ and $L_{1i} \cdot L_{1j} = 0$ for $i \neq j$. We call (E_i, L_{1i}) as a pair. So this pair will be either of the form $(\mathbb{P}^1 \times pt_1, pt_2 \times L_1)$ where L_1 is a line passing through the point pt_1 or of the form $(pt_3 \times L_2, pt_3 \times L_3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. Our claim is that the first situation will never occur.

The second projection $p_2 : \widetilde{\mathbb{P}^2_7} \to \mathbb{P}^2$ is a degree two map (See Lemma 4.1.1). If $\mathbb{P}^1 \times pt_1$ is a line in $\widetilde{\mathbb{P}^2_7}$, then p_2 is not a degree two map which is a contradiction. Hence, the claim is proved i.e., $(E_i, L_{1i}) = (pt_i \times L_i, pt_i \times L_i')$.

Remark 4.4.3. *Note that,* $\widetilde{\mathbb{P}^2_7}$ *always has 12 lines with respect to any embedding inside* $\mathbb{P}^5$

Theorem 4.4.4. *As defined in the Remark (4.3.3) ,* $(\widetilde{\mathbb{P}^2}_{P_1P_2\ldots P_7}, \mathbb{P}^1, f)$ *is a honest conic bundle over* $\mathbb{P}^1$*.*

Proof. To show $(\widetilde{\mathbb{P}^2}_{P_1P_2\ldots P_7}, \mathbb{P}^1, f)$ is a honest conic bundle over $\mathbb{P}^1$, we only have to show each fiber is reduced. Assume that, f has some non-reduced fiber. Let F be such a non-reduced fiber. Then $F = 2L$, where L is a line in $\mathbb{P}^5$, as degree of F is two in $\mathbb{P}^5$. Furthermore, as F is a fiber of f, $F^2 = 0$. This implies $L^2 = 0$. But this contradicts the Lemma (4.4.1) i.e., the self-intersection of any line in $\widetilde{\mathbb{P}^2_7}$ is (-1). Hence, our assumption is not true. So, the each fiber of f is either an irreducible conic or union of two lines. Hence, $(\widetilde{\mathbb{P}^2}_{P_1P_2\ldots P_7}, \mathbb{P}^1, f)$ is a honest conic bundle over $\mathbb{P}^1$.

□

Corollary 4.4.5. *Lines of* $\widetilde{\mathbb{P}^2_7}$ *are components of non-irreducible fibers of the conic bundle* $(\widetilde{\mathbb{P}^2}_{P_1P_2\ldots P_7}, \mathbb{P}^1, f)$ *over* $\mathbb{P}^1$*.*

Proof. This follows easily from the Remark (4.4.2) and the Theorem (4.4.4). □

4.5 Smooth surfaces of $\mid (2,2) \mid$ in $\mathbb{P}^1 \times \mathbb{P}^2$

As $\mid (2,2) \mid$ is a base point free linear system, by Bertini's Theorem, the generic element of the linear system is smooth. The following result may be well known,

Theorem 4.5.1. *Any smooth surface of* $\mid (2,2) \mid$ *of* $\mathbb{P}^1 \times \mathbb{P}^2$ *can be viewed as the embedding of* $\mathbb{P}^2$ *blown up at seven points.*

Proof. Let us consider $\mid D \mid = \mid (2,2) \mid$.

$$\begin{tikzcd} \mathbb{P}^1 \times \mathbb{P}^2 \arrow[r, "p_1"] \arrow[dr, "p_2"] & \mathbb{P}^1 \\ & \mathbb{P}^2 \end{tikzcd}$$

where p_1, p_2 are two projection maps. $D = p_1^*\mathcal{O}_{\mathbb{P}^1}(2) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(2)$, $-D = p_1^*\mathcal{O}_{\mathbb{P}^1}(-2) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(-2)$. Then

$R^i p_{1*}(p_1^*\mathcal{O}_{\mathbb{P}^1}(-2) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(-2))$

$= \mathcal{O}_{\mathbb{P}^1}(-2) \otimes R^i p_{1*} p_2^*\mathcal{O}_{\mathbb{P}^2}(-2)$ (Using projection formula ([H].III.8))

$= 0$ (as $R^i p_{1*} p_2^*\mathcal{O}_{\mathbb{P}^2}(-2) = 0$).

Now by the Leray spectral sequence ([H].III.8)

$H^i(\mathbb{P}^1 \times \mathbb{P}^2, p_1^*\mathcal{O}_{\mathbb{P}^1}(-2) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(-2))$

$\simeq H^i(\mathbb{P}^1, p_{1*}(p_1^*\mathcal{O}_{\mathbb{P}^1}(-2) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(-2))$

$= H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2) \otimes p_{1*} p_2^*\mathcal{O}_{\mathbb{P}^2}(-2))$

$= 0$ for all i, as $p_{1*} p_2^*\mathcal{O}_{\mathbb{P}^2}(-2) = 0$

So $h^1(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(-2,-2)) = h^2(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(-2,-2)) = 0$, also we have $h^i(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}) = 0$ for $i > 0$.

As D is an effective divisor, there is the short exact sequence,

$$0 \longrightarrow \mathcal{L}(-D) \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2} \longrightarrow \mathcal{O}_D \longrightarrow 0 \tag{4.1}$$

Then we get $h^1(\mathcal{O}_D) = h^2(\mathcal{O}_D) = 0$ from the induced long exact sequence of cohomologies. Hence the arithmetic genus of the surface D, $\rho_a(D) = 0$. This implies $\chi(\mathcal{O}_D) = 1$. We know the canonical divisor of $\mathbb{P}^1 \times \mathbb{P}^2$, $K_{\mathbb{P}^1 \times \mathbb{P}^2} = p_1^* K_{\mathbb{P}^1} + p_2^* K_{\mathbb{P}^2} = p_1^*\mathcal{O}_{\mathbb{P}^1}(-2) + p_2^*\mathcal{O}_{\mathbb{P}^2}(-3)$, where $K_{\mathbb{P}^1}$ and $K_{\mathbb{P}^2}$ are canonical divisor of $\mathbb{P}^1$ and $\mathbb{P}^2$ respectively. Note that,

$K_D = K_{\mathbb{P}^1 \times \mathbb{P}^2} + \mathcal{L}(D) \mid_D$ (using the *adjunction formula*).

$K_D = p_2^*\mathcal{O}_{\mathbb{P}^2}(-1) \mid_D$ (as $D \sim (2,2)$).

$K_D^2 = p_2^*\mathcal{O}_{\mathbb{P}^2}(-1) \cdot p_2^*\mathcal{O}_{\mathbb{P}^2}(-1) \cdot D = (\mathbb{P}^1 \times pt) \cdot (2pt \times \mathbb{P}^2 + \mathbb{P}^1 \times 2H) = 2$

By Noether's formula:

$12.\chi(\mathcal{O}_D) = \chi_{top}(D) + K_D^2$

$\Rightarrow \chi_{top}(D) = 10$

$\Rightarrow \sum_{i=0}^4 (-1)^i b_i(D) = 10$

where $b_i(D) = dim_{\mathbb{R}} H^i(D, \mathbb{R})$. As D is a surface, $b_0 = b_4 = 1$, and $b_3 = b_1$ which deduce

$$b_2(D) - 2b_1(D) = 8$$

.

We know the irregularity of surface, $q(D) = h^0(D, \Omega_D) = \frac{1}{2}b_1(D)$, where Ω_D is the sheaf of differentials on the surface D. Also we have the following short exact sequence,

$$0 \longrightarrow \mathcal{N}^*_{D/\mathbb{P}^1\times\mathbb{P}^2} \longrightarrow \Omega_{\mathbb{P}^1\times\mathbb{P}^2} \otimes \mathcal{O}_D \longrightarrow \Omega_D \longrightarrow 0 \tag{4.2}$$

where $\mathcal{N}^*_{D/\mathbb{P}^1\times\mathbb{P}^2} = \mathcal{L}(-D) \mid_D = \mathcal{O}(-2,-2) \mid_D$ is the conormal sheaf of D in $\mathbb{P}^1 \times \mathbb{P}^2$.

Note that, $\Omega_{\mathbb{P}^1\times\mathbb{P}^2} \cong (p_1^*\Omega_{\mathbb{P}^1} \oplus p_2^*\Omega_{\mathbb{P}^2})$, hence $h^0(\Omega_{\mathbb{P}^1\times\mathbb{P}^2}) = h^0(p_1^*\Omega_{\mathbb{P}^1}) + h^0(p_2^*\Omega_{\mathbb{P}^2}) = 0$. It can be calculated easily from the long exact sequence of the short exact sequence 4.1, that $h^1(\mathcal{O}(-2,-2)|_D) = 0$. Finally, we get $h^0(D, \Omega_D) = 0$ form the long exact sequence of cohomologies induced from (4.2). Hence the irregularity of the surface D, $q(D) = b_1(D) = 0$. Therefore, $b_2(D) = 8$.

The second plurigenera of D, $P_2 = h^0(D, \mathcal{O}_D(2K_D)) = h^0(p_2^*\mathcal{O}_{\mathbb{P}^2}(-2)|_D) = 0$.

So by the Castelnuovo's Rationality Criterion, [BE], D is a rational surface. We know that every rational surface is either a blow-up of $\mathbb{P}^2$ or ruled surface over $\mathbb{P}^1$ up to isomorphism ([GH], page no. 520). In Theorem 4.5.4 we observe that any smooth curve of D has self-intersection at least -2. So possibilities of D are blown-up of $\mathbb{P}^2$, $\mathbb{F}_0$, $\mathbb{F}_1$, and $\mathbb{F}_2$, where $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$. We know that $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ blow up at one point. Blow up of a particular point of $\mathbb{F}_1$ is isomorphic to blow up of a point of $\mathbb{F}_2$ ([GH], page no. 520). So $\mathbb{F}_2$ blown up at one point is isomorphic to $\mathbb{P}^2$ blown up at two points.

Now consider the short exact sequence,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_D^* \longrightarrow 0 \tag{4.3}$$

$Pic(D) \cong H^2(D, \mathbb{Z})$ as we have seen $h^2(\mathcal{O}_D) = 0$. So $\text{rank}(Pic(D)) = \text{rank} H^2(D, \mathbb{Z}) = \dim_{\mathbb{R}}(H^2(D, \mathbb{R})) = b_2(D) = 8$. We know Picard group of a ruled surface over $\mathbb{P}^1$ is always isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Hence our surface is a blow up of $\mathbb{P}^2$ when $Pic(D) = \mathbb{Z}^8$. So D is isomorphic to $\mathbb{P}^2$ blown up at seven points and

$$Pic(D) \cong Pic(\mathbb{P}^2) \oplus_{i=1}^{7} \mathbb{Z}.\mathcal{O}(E_i),$$

where E_i's are exceptional curves. □

Remark 4.5.2. *Note that, we have proved that any smooth surface of* $\mid (2,2) \mid$ *is* $\mathbb{P}^2$ *blow-up at seven points, but they may not be in general position.*

Example 4.5.3. Let S be a smooth surface in $\mathbb{P}^1 \times \mathbb{P}^2$, linearly equivalent to $(2,2)$ divisor, having the equation $y_0^2(x_0x_1 - x_2^2) + y_0y_1(x_1x_2 - x_0^2) + y_1^2(x_0x_1 - x_2^2)$, where y_0, y_1 are homogeneous coordinate of $\mathbb{P}^1$, and x_0, x_1, x_2 are homogeneous coordinate of $\mathbb{P}^2$. Clearly, $\mathbb{P}^1 \times [1,1,1]$ is a line in $\mathbb{P}^1 \times \mathbb{P}^2$ as well as in S. But in the Remark (4.4.2), we have seen that lines of $\mathbb{P}^2$ blown up at seven general points are of the form $pt \times L$, where L is a line in $\mathbb{P}^2$. Therefore, S is $\mathbb{P}^2$ blown up at seven points but seven points are not in general position.

Let S be a smooth surface and $S \sim (2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. S is a conic bundle over $\mathbb{P}^1$ and S is embedded in $\mathbb{P}(G) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{P}^1 \times \mathbb{P}^2$ by the anti-canonical divisor $-K_S$, which is a relatively ample divisor of the conic bundle. Therefore $i^*(\mathcal{O}_{\mathbb{P}(G)}(1)) = \mathcal{O}(-K_S)$, where $i : S \hookrightarrow \mathbb{P}(G)$. Note that $\mathcal{O}_{\mathbb{P}(G)}(1) \sim p_2^*(\mathcal{O}_{\mathbb{P}^2}(1))$, $p_2 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ is the second projection. So finally $\mathcal{O}(-K_S)$ gives a generically degree two map from S to $\mathbb{P}^2$. Let D be any irreducible divisor of the smooth surface S. The genus of D, $g(D) = \frac{1}{2}(D \cdot D - (-K_S) \cdot D) + 1 \geq 0$. As the anti-canonical divisor of S corresponds to a generically finite map, $(-K_S) \cdot D \geq 0$. Then $D \cdot D \geq -2$. If $D \cdot D = -2$, then $D \cdot K_S = 0$ and $g(D) = 0$ i.e., D is isomorphic to $\mathbb{P}^1$. If those seven points are not in general position, then S might have some -2 lines.

Theorem 4.5.4. *Any smooth surface S of* $\mid (2,2) \mid$ *of* $\mathbb{P}^1 \times \mathbb{P}^2$ *has at most four* (-2) *curves.*

Proof. The second projection p_2 restricted to S is a generically finite degree two map which is defined by the anti-canonical. Let L be a (-2) curve in S, then $L.(-K_S) = 0$. So the line L is contracted to a point by the morphism p_2. Then L will be of the form $\mathbb{P}^1 \times P_1$ inside $\mathbb{P}^1 \times \mathbb{P}^2$. As S is a (2,2) surface of $\mathbb{P}^1 \times \mathbb{P}^2$, the defining equation of S is $y_0^2 F_0(x_0, x_1, x_2) + y_1^2 F_1(x_0, x_1, x_2) + y_0 y_1 F_2(x_0, x_1, x_2)$, where y_0, y_1 are homogeneous co-ordinate of $\mathbb{P}^1$, x_0, x_1, x_2 are homogeneous co-ordinate of $\mathbb{P}^2$ and $\deg(F_i(x_0, x_1, x_2)) = 2$, for $i = 0, 1, 2$. If $\mathbb{P}^1 \times P_1 \subset Z(y_0^2 F_0(x_0, x_1, x_2) + y_1^2 F_1(x_0, x_1, x_2) + y_0 y_1 F_2(x_0, x_1, x_2))$, then $P_1 \in Z(F_0, F_1, F_2)$.

Conversely, if $P_1 \in Z(F_0, F_1, F_2)$, then $L = \mathbb{P}^1 \times P_1 \subset S$ and L is contracted to P_1 by the map p_2 i.e., $L \cdot (-K_S) = 0$, then $L \cdot L = -2$. Any -2 curve of S, which is always a line, will be of the form $\mathbb{P}^1 \times P$, where $P \in Z(F_0, F_1, F_2)$. But we know that $\#Z(F_0, F_1, F_2) \leq 4$, as F_i's are degree two curves in $\mathbb{P}^2$. Hence the result is proved. $\square$

Remark 4.5.5. (-2) *lines of S are reduced and in the form of* $\mathbb{P}^1 \times P_i$. *Hence, they* *are disjoint to each other.*

Example 4.5.6. *$Z(y_0^2(x_0x_2 - x_1^2) + y_1^2(x_0x_1 - x_2^2))$ is a smooth surface which is linearly equivalent to the divisor (2,2) of $\mathbb{P}^1 \times \mathbb{P}^2$. It has four (-2) lines, which are $\mathbb{P}^1 \times [1,1,1]$, $\mathbb{P}^1 \times [1,0,0]$, $\mathbb{P}^1 \times [1,\omega,\omega^2]$, $\mathbb{P}^1 \times [1,\omega^2,\omega]$, where ω is a cubic root of unity.*

Proposition 4.5.7. *Suppose that a smooth surface S of $\mid (2,2) \mid$ of $\mathbb{P}^1 \times \mathbb{P}^2$ has a (-2) curve L. Then either $L \sim E_i - E_j$ or $L \sim \pi^*H - E_i - E_j - E_k$ or $L \sim 2\pi^*H - \sum_{i=1}^{6} E_{k_i}$*

Proof. Let L be a (-2) curve, then either L is a component of some exceptional curve or $L = a\pi^*H - \sum_{i=1}^{7} b_iE_i$, where $a \geq 1$ or $b_i \geq 0$. If L is a component of an exceptional curve, then the only possibility for L is $E_i - E_j$. Otherwise, $L^2 = a^2 - \sum_{i=1}^{7} b_i^2 = -2$, and $L \cdot K_S = 0$ which implies $3a = \sum_{i=1}^{7} b_i$.

The Schwarz's inequality: $\mid \sum_i x_iy_i \mid \leq \mid \sum_i x_i^2 \mid . \mid \sum_i y_i^2 \mid$

We put the following values $x_i = 1, y_i = b_i$ for $i = 1,2,...,7$, and $x_i = 0, y_i = 0$ for $i > 7$ in the Schwarz's inequality. Hence we have,

$(\sum b_i)^2 \leq 7(\sum b_i^2)$

$\Rightarrow (3a)^2 \leq 7(a^2 + 2)$

$\Rightarrow a^2 \leq 7$

$\Rightarrow a \leq 2.$

Then using the above equations, either $L \sim \pi^*H - E_i - E_j - E_k$ or $L \sim 2\pi^*H - \sum_{i=1}^{6} E_{k_i}$. $\square$

In the section 4.3, we have seen all possible very ample divisors, by which $\mathbb{P}^2$ blown up at seven general points can be embedded as (2,2) divisor in $\mathbb{P}^1 \times \mathbb{P}^2$ as well as in $\mathbb{P}^5$. In this section, we already have seen that there are other smooth surfaces linearly equivalent to (2,2) divisor, which are isomorphic to $\mathbb{P}^2$ blown up at seven points, where those seven points are not in general position. Now, we investigate a very ample divisor which gives this embedding. (See Remark 4.5.11).

Theorem 4.5.8. *Let $\mathcal{L} \in Pic(S)$. Then $\mathcal{L}$ is very ample iff there is an exceptional configuration $\mathcal{E} = \{\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_n\}$ of S such that (i) $\mathcal{L} \cdot (\mathcal{E}_0 - \mathcal{E}_1) > 0$, (ii) $\mathcal{L} \cdot (\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2) > 0$, (iii) $\mathcal{L} \cdot (\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3) \geq 0$, (iv) $\mathcal{L} \cdot (\mathcal{E}_i - \mathcal{E}_{i+1}) \geq 0$ for $i \geq 1$, (v) $\mathcal{L} \cdot \mathcal{E}_i > 0$, (vi) $\mathcal{L} \cdot \mathcal{N} > 0$ for any (-2) curve $\mathcal{N}$, (vii) $\mathcal{L} \cdot K_S \leq -3$.*

Proof. [Ha] Theorem (2.1). □

Remark 4.5.9. Let S and S' be two surfaces, where S is $\mathbb{P}^2$ blown up at n general points and S' is $\mathbb{P}^2$ blown up at n points which are not in general position. Let $\mathcal{L}'$ be a very ample divisor of S'. By the Theorem (4.5.8), there exists an exceptional configuration $\mathcal{E}'$ such that $\mathcal{L}'$ satisfies (i)-(vii) and let $\mathcal{L}' = a\mathcal{E}'_0 - \sum_{i=1}^{7} b_i \mathcal{E}'_i$. Now, let $\mathcal{L} = a\mathcal{E}_0 - \sum_{i=1}^{7} b_i \mathcal{E}_i$ be a divisor of S with respect to a exceptional configuration $\mathcal{E}$, then $\mathcal{L}$ satisfies all the properties of the Theorem (4.5.8), which implies that $\mathcal{L}$ is also a very ample divisor of S.

The Picard group of $\widetilde{\mathbb{P}^2_r}$ is $\mathbb{Z}l \oplus_{i=1}^{r} \mathbb{Z}E_i$, where l is $\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. Let $D = al - \sum_{i=1}^{7} b_i E_i$ be a divisor of $\widetilde{\mathbb{P}^2_r}$ which we denote as $(a, b_1, b_2, ..., b_r)$ as an element of $\mathbb{Z}^{r+1}$. So by the argument of the above paragraph, if $(a, b_1, b_2, ..., b_r)$ is a very ample divisor of $\widetilde{\mathbb{P}}^2_{P_1P_2..P_r}$ for some given set of any r points of $\mathbb{P}^2$, then $(a, b_1, b_2, ..., b_r)$ is also a very ample divisor of the surface $\mathbb{P}^2$ blow-up at r general points. But the converse is not true in general because of the property (vi) of the Theorem (4.5.8).

Remark 4.5.10. In the Remark (4.3.4), we listed all possible very ample divisors of $\mathbb{P}^2$ blown up at seven general points, which give closed immersion of the surface in $\mathbb{P}^1 \times \mathbb{P}^2$ as (2,2) type divisor. Also, in the Remark (4.3.4) we noted that, if $\mathcal{L}$ is such a very ample divisor then there exists an exceptional configuration $\mathcal{E} = \{\mathcal{E}_0, \mathcal{E}_1, ..., \mathcal{E}_n\}$ such that $\mathcal{L}$ can be written as $4\mathcal{E}_0 - 2\mathcal{E}_1 - \sum_{i=2}^{7} \mathcal{E}_i$.

Remark 4.5.11. Let S be a smooth surface $S \sim (2,2)$ of $\mathbb{P}^1 \times \mathbb{P}^2$ such that $\widetilde{\mathbb{P}}^2_{P_1P_2...P_7} \cong S \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, where $P_1, P_2, ..., P_7$ are not in general position. Also, let $\mathcal{L}$ be the very ample divisor of $\widetilde{\mathbb{P}}^2_{P_1P_2...P_7}$ which gives the above closed immersion. Then by the Remarks (4.5.9) and the Remark (4.5.10) there is an exceptional configuration $\mathcal{E}' = \{\mathcal{E}'_0, \mathcal{E}'_1, ..., \mathcal{E}'_n\}$ of $\widetilde{\mathbb{P}}^2_{P_1P_2...P_7}$ such that $\mathcal{L}$ can be written as $4\mathcal{E}'_0 - 2\mathcal{E}'_1 - \sum_{i=2}^{7} \mathcal{E}'_i$.

Example 4.5.12. Let $P_1, \cdots, P_7$ be seven points of $\mathbb{P}^2$ such that P_2, P_3, P_4 are collinear, P_4, P_5, P_6 are collinear, P_2, P_7, P_5 are collinear, P_3, P_6, P_7 are collinear and P_1 is not collinear with any two P_i's and no six points lie on a conic. Let S be a surface obtained by the blown up of $\mathbb{P}^2$ at $P_1, P_2, \cdots, P_7$. Here, $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - E_2 - E_3 - E_4$, $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - E_4 - E_5 - E_6$, $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - E_2 - E_7 - E_5$ and $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - E_3 - E_7 - E_6$ are (-2) lines of S. Then, $4\pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - 2E_1 - \sum_{i=2}^{7} E_i$ is a very ample divisor of S (Theorem 4.5.8). Clearly, $3\pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - \sum_{i=1}^{7} E_i$ gives generic degree two map from S to $\mathbb{P}^2$ and $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - E_1$ gives conic bundle map from S to $\mathbb{P}^1$. Hence, S is a smooth (2,2) type surface of $\mathbb{P}^1 \times \mathbb{P}^2$ having four (-2) lines.

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