Higgs bundles on ruled surfaces
and
Nef and Pseudoeffective cones of certain projective varieties.

By

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Dedicated to my parents

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Contents

Summary

This thesis is divided into two parts. In the first part, we consider a ruled surface $X = \mathbb{P}_C(E)$ over a smooth irreducible projective curve C defined over an algebraically closed field k of characteristic 0 , and let $\pi : X = \mathbb{P}_C(E) \longrightarrow C$ be the ruling. We fix a polarization L on X. The numerical equivalence class of a divisor D on X will be denoted by $\big[D\big]$. We show that pullback $\pi^*(\mathcal{V})$ of a (semi)stable Higgs bundle $\mathcal{V} = (V, \theta)$ on C under the map π is an L-(semi)stable Higgs bundle on X. Conversly, if (V, θ) is an L-(semi)stable Higgs bundle on X with first Chern class $c_1(V) = \pi^*\big([\mathbf{d}]\big)$ for some divisor $\mathbf{d}$ of degree d on C and second Chern class $c_2(V) = 0$, then there exists a (semi)stable Higgs bundle $\mathcal{W} = (W, \psi)$ of degree d on C such that $\mathcal{V} \cong \pi^*(\mathcal{W})$. Consequently, we prove that the corresponding moduli spaces $\mathcal{M}_X^{Higgs}\big(r, df, 0, L\big)$ and $\mathcal{M}_C^{Higgs}\big(r, d\big)$ are isomorphic, where $\mathcal{M}_X^{Higgs}\big(r, df, 0, L\big)$ deontes the moduli space of semistable Higgs bundles $\mathcal{V} = (V, \theta)$ of rank r on X having first Chern class $c_1(V) = \pi^*\big([\mathbf{d}]\big)$ for some divisor $\mathbf{d}$ of degree d on C and second Chern class $c_2(V) = 0$, and $\mathcal{M}_C^{Higgs}\big(r, d\big)$ denotes the semistable Higgs bundles of rank r and degree d on C.

In the second part of this thesis, we consider the fibre product $\mathbb{P}_C(E_1) \times_C \mathbb{P}_C(E_2)$ of two projective bundles $\mathbb{P}_C(E_1)$ and $\mathbb{P}_C(E_2)$ over a smooth irreducible complex projective curve C. Consider the following commutative fibred diagram :

$$\begin{array}{ccc} X = \mathbb{P}_C(E_1) \times_C \mathbb{P}_C(E_2) & \xrightarrow{p_2} & \mathbb{P}_C(E_2) \\ \Big\downarrow p_1 & & \Big\downarrow \pi_2 \\ \mathbb{P}_C(E_1) & \xrightarrow{\pi_1} & C \end{array}$$

We fix the following notations in the real Néron-Severi groups,

$$\eta_1 = \left[\mathcal{O}_{\mathbb{P}(E_1)}(1)\right] \in N^1\big(\mathbb{P}(E_1)\big)_{\mathbb{R}} \quad , \quad \eta_2 = \left[\mathcal{O}_{\mathbb{P}(E_2)}(1)\right] \in N^1\big(\mathbb{P}(E_2)\big)_{\mathbb{R}},$$

$$\zeta_1 = p_1^*(\eta_1) \quad , \quad \zeta_2 = p_2^*(\eta_2) \in N^1(X)_{\mathbb{R}}.$$

Let F be the numerical equivalence classes of the fibres of the map $\pi_1 \circ p_1 = \pi_2 \circ p_2$. We calculate nef cone $\text{Nef}(X)$ and pseudoeffective cone $\overline{\text{Eff}}(X)$ of the fibre product $X = \mathbb{P}_C(E_1) \times_C \mathbb{P}_C(E_2)$ in the following three cases :

Case I : Assume both E_1 and E_2 are slope semistable bundles of rank r_1 and r_2 respectively with slopes μ_1 and μ_2 . Then

$$\text{Nef}(X) = \overline{\text{Eff}}(X) = \big\{a\lambda_1 + b\lambda_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\},$$

where $\lambda_1 = \zeta_1 - \mu_1 F$ and $\lambda_2 = \zeta_2 - \mu_2 F$.

Case II : Assume neither E_1 nor E_2 is slope semistable, and both E_1 and E_2 are normalized rank 2 bundles. Then

$$\text{Nef}(X) = \big\{a\tau_1 + b\tau_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\},$$

$$\overline{\text{Eff}}(X) = \big\{a\zeta_1 + b\zeta_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\},$$

where $l_1 = \deg(E_1)$, $l_2 = \deg(E_2)$ and $\tau_1 = \zeta_1 - l_1 F$ and $\tau_2 = \zeta_2 - l_2 F$.

Case III : Assume E_1 is slope semistable with slope μ_1 and E_2 is not slope semistable, and both E_1 and E_2 are normalized rank 2 bundles. Then

$$\text{Nef}(X) = \big\{a\gamma_1 + b\gamma_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\},$$

$$\overline{\text{Eff}}(X) = \big\{a(\zeta_1 - \mu_1 F) + b\zeta_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\},$$

where $l_2 = \deg(E_2)$, $\gamma_1 = \zeta_1 - \mu_1 F$ and $\gamma_2 = \zeta_2 - l_2 F$.

In particular, if both E_1 and E_2 are of rank 2 bundles on C, then nef cone and pseudoeffective cone of $X = \mathbb{P}_C(E_1) \times_C \mathbb{P}_C(E_2)$ both coincide if and only if both E_1 and E_2 are slope semistable.

Notations

Symbol	Description
$\mathbb{Z}$	The ring of integers
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{R}_{>0}$	The set of positive real numbers
$\mathbb{R}_{\geq 0}$	The set of non-negative real numbers
$\mathbb{C}$	The field of complex numbers
$\mathbb{P}^n_k$	Projective n-space over an algebraically closed field k
$\mathcal{O}_{\mathbb{P}^n_k}(1)$	Serre bundle on $\mathbb{P}^n_k$
$c_i(E)$	i-th Chern class of a coherent sheaf E
$\mathrm{Div}(X)$	The set of all divisors on a variety X
$\mathcal{O}_X(D)$	The line bundle associated to the divisor D on X
$[D]$	Numerical equivalence class of a divisor D
$N^1(X)_{\mathbb{R}}$	The real Néron-Severi group of X
$\rho(X)$	The Picard rank of X
$\mathrm{Pic}(X)$	The Picard group of X

$\mathrm{Pic}^0(X)$	The subgroup of $\mathrm{Pic}(X)$ of numerically trivial line bundles
$\mathrm{mult}_x C$	multiplicity at x of a curve C passing through x
$\mathcal{F}_x$	The stalk of a coherent sheaf $\mathcal{F}$ at x
$\mathrm{End}(V)$	The set of endomorphisms of a vector bundle V
$H^i(X, \mathcal{F})$	i-th cohomology group of a coherent sheaf $\mathcal{F}$
$R^i f_*$	Higher direct image functor
Ω^1_X	The cotangent sheaf on X
$\Omega^1_{X\|C}$	The relative cotangent sheaf of a curve C in a surface X.

Chapter 1

Introduction

In this chapter, we give a brief description of the historical background of the central topics discussed in this thesis. We also state our main results in the subsequent section and outline the arrangement of this thesis.

1.1 Historical Background

A Higgs bundle on a smooth irreducible projective variety defined over an algebraically closed field k, is a pair (V, θ) where V is a vector bundle on X and $\theta \in H^0\big(X, \mathrm{End}(V) \otimes \Omega^1_X\big)$ is a section satisfying $\theta \wedge \theta = 0$. Higgs bundles on Riemann surfaces were first introduced by Nigel Hitchin in [Hit] to study the Yang-Mills equation on Riemann surfaces. Later, in [Sim1],[Sim2],[Sim3], Simpson extended the notion of Higgs bundle to higher dimensional varieties and related them to the fundamental group of the base variety. Consequently, a lot of applications of these objects are found in many areas of Mathematics and Theoretical Physics. Higgs bundle comes with a natural stability condition (see Section 2.4 for the definition), which allows one to study the moduli problems. Semistabilty of Higgs bundles over smooth projective varieties has been studied by many authors (see

[Hit],[Sim1],[Nit],[FPN], [Sch]). Given a Higgs bundle $\mathcal{E} = (E, \phi)$ of rank r on a smooth polarized projective variety (X, L) of dimension n, one can associate a numerical invariant, its discriminant $\triangle(E) := 2rc_2(E) - (r-1)c_1^2(E)$. If $\mathcal{E} = (E, \theta)$ is L-semistable, then $\triangle(E) \cdot L^{n-2} \geq 0$. This is called the Bogomolov's inequality. It is interesting to classify all such semistable Higgs bundles $\mathcal{E} = (E, \theta)$ with $\triangle(E) = 0$. In the first part of this thesis, we classify some particular type of semistable Higgs bundles $\mathcal{E} = (E, \theta)$ over ruled surfaces with $\triangle(E) = 0$.

The study of divisors on a projective variety X is a very important tool in algebraic geometry to understand the geometry of X. From the last century, the sheaf-theoretic approach brought the importance of ample divisors. In the last few decades, a number of notions of positivity have been used to understand the geometry of the higher dimensional projective varieties. Most of the developments in this direction has been successfully summarized in [Laz1], [Laz2].

We denote the intersection product of a divisor D on a smooth projective variety X and a curve $C \subseteq X$ by $D{\cdot}C$. Two divisors D_1 and D_2 on X are said to numerically equivalent, denoted by $D_1 \equiv D_2$, if $D_1 \cdot C = D_2 \cdot C$ for all irreducible curves $C \subseteq X$. It is a fact that $\mathrm{Div}(X)/\equiv$ is a free abelian group of finite rank (see Proposition 1.1.16 in [Laz1]). Consider the finite dimensional real vector space

$$N^1(X)_{\mathbb{R}} := \big(\mathrm{Div}(X)/\equiv\big) \otimes \mathbb{R}$$

In this space, one can talk about different convex cones, each corresponding to different notion of positivity. Let us consider an element $D = \sum_i m_i\big[D_i\big] \in N^1(X)_{\mathbb{R}}$, $m_i \in \mathbb{R}$, $D_i \in \mathrm{Div}(X)$. We say :

(i) D is ample if $a_i > 0$ and D_i is ample for each i (i.e., for each i , there is a positive integer l_i such that l_iD_i gives a projective embedding of X).

(ii) D is big if each $a_i > 0$ and D_i is big for each i (i.e., Kodaira dimension of each D_i is equal to dimension of X).

The open convex cone generated by ample classes in $N^1(X)_{\mathbb{R}}$ is called the ample cone, denoted by $\mathrm{Amp}(X)$ and its closure is called the nef cone, denoted by $\mathrm{Nef}(X)$. Similarly, the open convex cone generated by big classes in $N^1(X)_{\mathbb{R}}$ is called the big cone, denoted by $\mathrm{Big}(X)$ and its closure is called the pseudoeffective cone, denoted by $\overline{\mathrm{Eff}}(X)$. These cones are related by the following commutative diagram :

$$\begin{array}{ccc} \mathrm{Amp}(X) & \hookrightarrow & \mathrm{Nef}(X) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Big(X)} & \hookrightarrow & \overline{\mathrm{Eff}}(X) \end{array}$$

The nef cone $\mathrm{Nef}(X) \subseteq N^1(X)_{\mathbb{R}}$ of divisors of a projective variety X is an important invariant that encodes information about all the projective embeddings of X. The nef cones and pseudoeffective cones of different smooth irreducible projective varieties has been studied by many authors in the last few decades (see [Laz1] (Section 1.5), [Miy], [Fulg], [BP], [MOH] for more details). In his paper [Miy], Miyaoka found that in characteristic 0, the nef cone of $\mathbb{P}_C(E)$ is determined by the smallest slope of any nonzero torsion free quotient of E. He also gave a numerical criterion for semistability of E in terms of nefness of the normalized hyperplane class λ_E on $\mathbb{P}_C(E)$, and showed that E is a semistable bundle over the curve C if and only if the nef cone and pseudoeffective cone of $\mathbb{P}_C(E)$ coincide. Later in [BR], more generally, it is shown that a vector bundle E with vanishing discriminant over a smooth projective variety X is slope-semistable if and only if the normalized hyperplane class in $\mathbb{P}_C(E)$ is nef. [Fulg] generalized Miyaoka's result to arbitrary codimension cycles showing that the effective cones of cycles (and their duals) on $\mathbb{P}_C(E)$ are determined by the numerical data in the Harder-Narasimhan filtration of E. [BP] studied the nef cone of divisors on Grassmann bundles $\mathrm{Gr}_s(E)$ and flag bundles over smooth curves and extended Miyaokas result to characteristic p. More specifically, in [BP] and [BHP], the nef cone and pseudoeffective cone of the Grassmannian bundle $\mathrm{Gr}_s(E)$ parametrizing all the s dimensional quotients of the fibres of E where $1 \leq s \leq r-1$, has been studied. However, in most of these cases, the Picard rank of the space X is 2, and hence the nef cones are generated by two extremal rays in a two dimensional space. When the Picard rank is at least 3, there are very few examples where the nef cone and the pseudoeffective cone are computed.

1.2 About this thesis

In Chapter 2 of this thesis, we recall some definitions, basic and known results, which are used to prove our main results in the subsequent chapters.

In Chapter 3 of this thesis, we consider a smooth irreducible projective curve C of genus $g(C) \geq 0$ over an algebraically closed field k of characteristic 0, and a ruled surface $\pi : X = \mathbb{P}_C(E) \longrightarrow C$. We fix a polarization L on X. Let σ and f be the numerical classes of a section and a fibre of the ruling $\pi : X \longrightarrow C$ respectively. We discuss about the (semi)stability of pullback of a Higgs bundle on C under the ruling π. Our main results are the following :

Theorem 1.2.1. Let $\pi : X \longrightarrow C$ be a ruled surface with a fixed polarization L on X. Let $\mathcal{V} = (V, \theta)$ be a semistable Higgs bundle of rank r on C. Then, the pullback $\pi^*(\mathcal{V}) = \big(\pi^*(V), d\pi(\theta)\big)$ is L-semistable Higgs bundle on X.

Theorem 1.2.2. Let L be a fixed polarization on a ruled surface $\pi : X \longrightarrow C$. Let $\mathcal{V} = (V, \theta)$ be an L-semistable Higgs bundle of rank r on X with $c_1(V) = \pi^*\big([\mathbf{d}]\big)$, for some divisor $\mathbf{d}$ of degree d on C, then $c_2(V) \geq 0$ and $c_2(V) = 0$ if and only if there exists a semistable Higgs bundle $\mathcal{W} = (W, \psi)$ on C such that $\pi^*(\mathcal{W}) = \big(\pi^*(W), d\pi(\psi)\big) \cong \mathcal{V}$ on X.

Theorem 1.2.3. Let L be a fixed polarization on a ruled surface $\pi : X \longrightarrow C$. Then, for any stable Higgs bundle $\mathcal{W} = (W, \psi)$ on C, the pullback Higgs bundle $\pi^*(\mathcal{W})$ is L-stable Higgs bundle on X. Conversely, if $\mathcal{V} = (V, \theta)$ is an L-stable

Higgs bundle on X with $c_1(V) = \pi^*\big([\mathbf{d}]\big)$ for some divisor $\mathbf{d}$ of degree d on C and $c_2(V) = 0$, then $\mathcal{V} \cong \pi^*(\mathcal{W})$ for some stable Higgs bundle $\mathcal{W} = (W, \psi)$ on C.

According to Simpson (See [Sim2], [Sim3]), the moduli space of S-equivalence classes of semistable Higgs bundles of rank n with vanishing Chern classes on any complex projective variety X can be identified with the space of isomorphism classes of representations of $\pi_1(X, *)$ in $GL(n, C)$. For a ruled surface $\pi : X = \mathbb{P}_C(E) \longrightarrow C$ over C, there is an isomorphism of fundamental groups $\pi_1(X, *) \cong \pi_1(C, *)$. Hence, in this case, we have a natural algebraic isomorphism of the corresponding moduli of semistable Higgs bundles on X and C respectively with vanishing Chern classes (when the base field is $\mathbb{C}$).

As an application of Theorem 1.2.1, Theorem 1.2.2 and Theorem 1.2.3, we prove a similar kind of algebraic isomorphism between the corresponding moduli spaces of Higgs bundles when the Chern classes are not necessarily vanishing and the base field is any algebraically closed field k of characteristic 0. More precisely, we prove that,

Theorem 1.2.4. The moduli spaces $\mathcal{M}_X^{Higgs}\big(r, df, 0, L\big)$ and $\mathcal{M}_C^{Higgs}\big(r, d\big)$ are isomorphic as algebraic varieties, where $\mathcal{M}_C^{Higgs}\big(r, d\big)$ denotes the moduli space of S-equivalence classes of semistable Higgs bundles of rank r and degree d on C and $\mathcal{M}_X^{Higgs}\big(r, df, 0, L\big)$ denotes the moduli space of S-equivalence classes of L-semistable Higgs bundles $\mathcal{V} = (V, \theta)$ of rank r on X, having first Chern class $c_1(V) = \pi^*\big([\mathbf{d}]\big)$ for some divisor $\mathbf{d}$ of degree d on C, and second Chern class $c_2(V) = 0$.

In [Var], similar kind of questions are discussed for a relatively minimal non-isotrivial elliptic surfaces $\varphi : X \longrightarrow C$ over a smooth curve C of genus $g(C) \geq 2$ defined over $\mathbb{C}$.

Next, in Chapter 4 of this thesis, we consider two vector bundles E_1 and E_2 of rank r_1 and r_2 respectively over a smooth irreducible curve C defined over C, and the fibre product $X = \mathbb{P}_C(E_1) \times_C \mathbb{P}_C(E_2)$. Note that, in this case, the cones are 3-dimensional while the literature abounds with 2-dimensional examples (e.g. $\mathbb{P}_C(E)$, $\mathrm{Gr}_s(E)$ etc.). Consider the following commutative diagram :

$$\begin{array}{ccc} X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2) & \xrightarrow{p_2} & \mathbb{P}(E_2) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{P}(E_1) & \xrightarrow{\pi_1} & C \end{array}$$

We first fix the following notations :

$$\eta_1 = \left[\mathcal{O}_{\mathbb{P}(E_1)}(1)\right] \in N^1\big(\mathbb{P}(E_1)\big)_{\mathbb{R}} \quad , \quad \eta_2 = \left[\mathcal{O}_{\mathbb{P}(E_2)}(1)\right] \in N^1\big(\mathbb{P}(E_2)\big)_{\mathbb{R}},$$

$$\zeta_1 = p_1^*(\eta_1) \quad , \quad \zeta_2 = p_2^*(\eta_2) \in N^1(X)_{\mathbb{R}}.$$

Let F be the numerical equivalence classes of the fibres of the map $\pi_1 \circ p_1 = \pi_2 \circ p_2$.

In Chapter 4, we calculate nef cone and pseudoeffective cone of X in the following three cases :

Case I : Assume both E_1 and E_2 are slope semistable bundles of rank r_1 and r_2 respectively with slopes μ_1 and μ_2 . Then

$$\mathrm{Nef}(X) = \overline{\mathrm{Eff}}(X) = \big\{a\lambda_1 + b\lambda_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\},$$

where $\lambda_1 = \zeta_1 - \mu_1 F$ and $\lambda_2 = \zeta_2 - \mu_2 F$.

Case II : Assume neither E_1 nor E_2 is slope semistable, and both E_1 and E_2 are normalized rank 2 bundles. Then

$$\mathrm{Nef}(X) = \big\{a\tau_1 + b\tau_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\}$$

$$\overline{\mathrm{Eff}}(X) = \big\{a\zeta_1 + b\zeta_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\}$$

where $l_1 = \deg(E_1)$, $l_2 = \deg(E_2)$ and $\tau_1 = \zeta_1 - l_1 F$ and $\tau_2 = \zeta_2 - l_2 F$.

Case III : Assume E_1 is slope semistable with slope μ_1 and E_2 is not slope semistable, and both E_1 and E_2 are normalized rank 2 bundles. Then

$$\mathrm{Nef}(X) = \left\{a\gamma_1 + b\gamma_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\right\}$$

$$\overline{\mathrm{Eff}}(X) = \left\{a(\zeta_1 - \mu_1 F) + b\zeta_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\right\}$$

where $l_2 = \deg(E_2)$, $\gamma_1 = \zeta_1 - \mu_1 F$ and $\gamma_2 = \zeta_2 - l_2 F$.

These results show that if both E_1 and E_2 are of rank 2 bundles on C, then nef cone and pseudoeffective cone of $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ both coincide if and only if both E_1 and E_2 are slope semistable.

1.3 Conventions

Throughout this thesis, a variety is always assumed to be reduced and irreducible . A divisor on a projective variety X will mean a Cartier divisor on X. We will use the canonical isomorphism between the divisor class group and the Picard group of a projective variety X over an algebraically closed field. For a divisor D on X, $\mathcal{O}_X(D)$ will denote the corresponding line bundle on X. Also, The words vector bundles and locally free sheaf will be used interchangeably, and so is the case with line bundle and invertible sheaf.

Chapter 2

Preliminaries

2.1 Chern classes of Coherent sheaves

In this section, we recall the definition and well known properties of Chern classes of a coherent sheaf on a smooth projective variety. See [Ful] for more details.

Let X be a nonsingular projective variety of dimension s over an arbitrary field k. A l-cycle on X is a finite formal sum $\sum_i n_i W_i$, where W_i's are l-dimensional subvarieties of X and $n_i \in \mathbb{Z}$ for all i. The group of l-cycles on X will be denoted by $\mathcal{Z}_l(X)$. The group of l-cycles modulo rational equivalence will be denoted by $A_l(X)$. We define, $A^l(X) := A_{s-l}(X)$. Since X is smooth, the intersection product

$$A^m(X) \times A^n(X) \longrightarrow A^{m+n}(X) \quad \text{for all} \quad 0 \leq m, n \leq s,$$

gives a graded ring structure on $A^*(X) := \bigoplus_{l=0}^{s} A^l(X)$.

Let V be a vector bundle of rank r on X. Then, $\wedge^r V$ is a line bundle on X corresponding to a divisor class, say D on X. The first Chern class of V is defined as

$$c_1(V) := c_1(\wedge^r V) := D \in A^1(X).$$

In general, to define higher Chern classes $c_i(V) \in A^i(V)$, one first defines it for a vector bundle V of rank r, which admits a filtration by subbundles

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq \cdots\cdots\cdots \subsetneq V_m = V$$

such that $M_i := V_i/V_{i-1}$ is a line bundle for each i. In this case,

$$c(V) := 1 + c_1(V) + c_2(V) + \cdots + c_r(V) = \prod_{i=1}^{r}\big(1 + c_i(M_i)\big)$$

and the actual formula for $c_i(V) \in A^i(X)$ is obtained by equating the terms lying in $A^i(X)$ from both sides. $c(V)$ is called the total Chern class of V.

2.1.1 Splitting Principle

More generally, if V is any vector bundle of rank r on X, then there is a projective variety Y together with a flat morphism $f : Y \longrightarrow X$ such that

(i) $f^* : A^*(X) \longrightarrow A^*(Y)$ is injective, and

(ii) $f^*(V)$ has a filtration by subbundles

$$0 = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots\cdots \subsetneq W_{l-1} \subsetneq W_l = f^*(V)$$

with $L_i := W_i/W_{i-1}$ is a line bundle for each i. We then define $c_i(f^*V)$ as above, and one can show that these classes are in the image of the map f^*. Then, one define $c_i(V)$ to be the unique class in $A^i(X)$ such that $f^*\big(c_i(V)\big) = c_i\big(f^*(V)\big)$.

The Chern classes constructed as above are functorial, i.e., if $g : X \longrightarrow Y$ is a morphism, then $c_i(g^*V) = g^*\big(c_i(V)\big)$ for each vector bundle V on Y, and for each i. They also satisfy the Whitney product formula :

If

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

is an exact sequence of vector bundles on X, then $c(V_2) = c(V_1) \cdot c(V_3)$.

One can also define the Chern classes of any coherent sheaf $\mathcal{F}$ on X. By a theorem due to Serre, every coherent sheaf $\mathcal{F}$ admits a finite resolution of locally free sheaves

$$0 \longrightarrow V^n \longrightarrow V^{n-1} \longrightarrow \cdots\cdots \longrightarrow V^0 \longrightarrow \mathcal{F} \longrightarrow 0$$

(see 1.1.17 in [HL] or Chapter III, Ex. 68,69 in [Har]). We then define the Chern classes of $\mathcal{F}$ by the formula

$$c(\mathcal{F}) = \prod_i c(V^i)^{(-1)^i}$$

One can show that this definition is independent of the choice of the resolution, and the Chern classes so defined satisfy the Whitney product formula.

2.2 Positive Cones in $N^1(X)_{\mathbb{R}}$

In this section, we discuss about different notions of positivity of a line bundle on a projective variety, and various convex cones in its real Néron Severi group, corresponding to each of these notions of positivity. We follow [Laz1],[Laz2] for the notations in this section.

Definition 2.2.1. A line bundle L on a projective variety X over a field k is said to be *very ample* if $L = \phi^*\big(\mathcal{O}_{\mathbb{P}^N}(1)\big)$ for some closed embedding $\phi : X \hookrightarrow \mathbb{P}^N$ for some positive integer N . A line bundle L is called *ample* if some integral multiple

$L^{\otimes m}$ of it is very ample. A divisor D on X is said to be *ample* (resp. *very ample*), if the corresponding line bundle $\mathcal{O}_X(D)$ is *ample* (resp. *very ample*).

There are very few numerical criteria to check very ampleness or ampleness of a given line bundle on a projective variety. We mention a few of them here. However, all these criteria may not be easy to check in practical situations.

Theorem 2.2.2. (Nakai-Moishezon-Kleiman criterion) Let L be a line bundle on a projective variety X. Then L is ample if and only if $L^{\dim V} \cdot V > 0$ for every positive-dimensional irreducible subvariety $V \subseteq X$.

Theorem 2.2.3. (Seshadri's criterion) A line bundle L on a projective variety X is ample if and only if there exists a positive number $\epsilon > 0$ such that

$$\frac{L \cdot C}{\operatorname{mult}_x C} \geq \epsilon$$

for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x.

Two divisors D_1 and D_2 on X are said to be numerically equivalent, denoted by $D_1 \equiv D_2$, if $D_1 \cdot C = D_2 \cdot C$ for every irreducible curve C in X. The *Néron Severi group* of X is the quotient $N^1(X)_{\mathbb{Z}} := \operatorname{Div}(X)/\equiv$. It is a basic fact that the Néron-Severi group $N^1(X)_{\mathbb{Z}}$ is a free abelian group of finite rank (see Proposition 1.1.16 in [Laz1]). The rank of $N^1(X)_{\mathbb{Z}}$ is called the *Picard number* of X, and is denoted by $\rho(X)$. The real vector space $\operatorname{Div}_{\mathbb{R}}(X) := \operatorname{Div}(X) \otimes \mathbb{R}$ is called the space of $\mathbb{R}$-divisors on X. An element $D = \sum_i c_i A_i \in \operatorname{Div}_{\mathbb{R}}(X)$ is called numerically trivial if $\sum_i c_i(A_i \cdot C) = 0$ for all irreducible curve $C \subseteq X$. Let $\operatorname{Div}^0_{\mathbb{R}}(X)$ be the subspace of $\operatorname{Div}_{\mathbb{R}}(X)$ consisting of numerically trivial $\mathbb{R}$-divisors on X. The *real Néron Severi group* , denoted by $N^1(X)_{\mathbb{R}}$ is defined as follows :

$$N^1(X)_{\mathbb{R}} := \operatorname{Div}_{\mathbb{R}}(X)/\operatorname{Div}^0_{\mathbb{R}}(X)$$

Note that there is an isomorphism $N^1(X)_{\mathbb{R}} \cong N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}$ as real vector spaces. Thus, $N^1(X)_{\mathbb{R}}$ is a finite dimensional real vector space. We view $N^1(X)_{\mathbb{R}}$ as a topological space equipped with its standard Euclidean topology. Also, the numerical equivalence class of an integral divisor D on X will be denoted by $\big[D\big] \in N^1(X)_{\mathbb{R}}$.

Definition 2.2.4. An $\mathbb{R}$-divisor D on X is *ample* if it can be expressed as a finite sum $D = \sum_i c_i A_i$, where each $c_i > 0$ is a positive real number and each A_i is an ample integral divisor.

It can be shown that if two $\mathbb{R}$-divisors D_1 and $D_2 \in \mathrm{Div}_{\mathbb{R}}(X)$ are numerically equivalent, then D_1 is ample if and only if D_2 is ample. Hence, one can talk about an ample class in $N^1(X)_{\mathbb{R}}$. The convex cone of all ample classes in $N^1(X)_{\mathbb{R}}$ is called the *Ample cone* , and is denoted by $\mathrm{Amp}(X) \subseteq N^1(X)_{\mathbb{R}}$.

2.2.1 Nef Cone and Pseudoeffective Cone

Definition 2.2.5. A line bundle L over a projective variety X is called *numerically effective* or *nef* , if $L \cdot C \geq 0$ for every irreducible curve $C \subseteq X$. A Cartier divisor D on X is called *nef* if the corresponding line bundle $\mathcal{O}_X(D)$ is nef. Similarly, an $\mathbb{R}$-divisor D on X is called *nef* , if $D \cdot C \geq 0$ for all irreducible curve $C \subseteq X$.

The intersection product being independent of numerical equivalence class, one can talk about nef classes in the Néron-Severi group $N^1(X)_{\mathbb{Z}}$ and $N^1(X)_{\mathbb{R}}$. The convex cone of all nef classes in $N^1(X)_{\mathbb{R}}$ is called the *Nef cone* , and is denoted by $\mathrm{Nef}(X) \subseteq N^1(X)_{\mathbb{R}}$. We have the following characterization of nef cone due to S.L.Kleiman (see [K]).

Theorem 2.2.6. (see Theorem 1.4.23 in [Laz1]) $\overline{\mathrm{Amp}(X)} = \mathrm{Nef}(X)$ and $int\big(\mathrm{Nef}(X)\big) = \mathrm{Amp}(X)$ in $N^1(X)_{\mathbb{R}}$.

Definition 2.2.7. An integral divisor D on X is called *big* if there is an ample divisor A on X, a positive integer $m > 0$ and an effective divisor N on X such that mD is numerically equivalent to $A + N$.

An $\mathbb{R}$-divisor D on X is *big* if it can be expressed as a finite sum $D = \sum c_i A_i$, where each $c_i > 0$ is a positive real number and each A_i is a big integral divisor. One can show that bigness of an $\mathbb{R}$-divisor only depends on its numerical equivalence class. The convex cone of all big classes in $N^1(X)_{\mathbb{R}}$ is called the *Big cone* , and is denoted by $\mathrm{Big}(X) \subseteq N^1(X)_{\mathbb{R}}$.

Definition 2.2.8. A numerical equivalence class c in $N^1(X)_{\mathbb{R}}$ is called *effective* if $c = \big[\mathcal{O}_X(D)\big] \in N^1(X)_{\mathbb{R}}$ for some effective divisor D on X.

The *Pseudoeffective cone* , $\overline{\mathrm{Eff}}(X) \subseteq N^1(X)_{\mathbb{R}}$ is the closure of the convex cone generated by the set of all effective classes in $N^1(X)_{\mathbb{R}}$.

Theorem 2.2.9. (see Theorem 2.2.26 in [Laz1]) $\mathrm{Big}(X) = int\big(\overline{\mathrm{Eff}}(X)\big)$ and $\overline{\mathrm{Eff}}(X) = \overline{\mathrm{Big(X)}}$ in $N^1(X)_{\mathbb{R}}$.

As a corollary to Theorem 2.2.6 and Theorem 2.2.9, we get, $\mathrm{Nef}(X) \subseteq \overline{\mathrm{Eff}}(X)$.

2.2.2 Duality Theorems

A finite formal sum $\gamma = \sum\limits_i a_i \cdot C_i$, where $a_i \in \mathbb{R}$ and $C_i \subseteq X$ is an irreducible curve in X, is called a real 1-cycle on X, and the $\mathbb{R}$-vector space of real 1-cycles on X is denoted by $\mathcal{Z}_1(X)_{\mathbb{R}}$. Two 1-cycles $\gamma_1, \gamma_2 \in \mathcal{Z}_1(X)_{\mathbb{R}}$ are called numerically equivalent, denoted by $\gamma_1 \equiv \gamma_2$ if $(D \cdot \gamma_1) = (D \cdot \gamma_2)$ for every $D \in \mathrm{Div}_{\mathbb{R}}(X)$. Let $N_1(X)_{\mathbb{R}} := \mathcal{Z}_1(X)_{\mathbb{R}}/\equiv$. We also have a perfect pairing

$$N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \longrightarrow \mathbb{R} \quad ; \quad (\delta, \gamma) \longmapsto (\delta \cdot \gamma) \tag{2.1}$$

The *cone of curves* , $\mathrm{NE}(X) \subseteq N_1(X)_{\mathbb{R}}$ is the cone spanned by the classes of all effective real 1-cycles on X i.e.,

$$\mathrm{NE}(X) = \Big\{ \sum_i a_i \cdot \big[C_i\big] \ \mid \ C_i \subseteq X \text{ an irreducible curve}, a_i \in \mathbb{R}_{\geq 0} \Big\}$$

Its closure $\overline{\mathrm{NE}}(X) \subseteq N_1(X)_{\mathbb{R}}$ is the *closed cone of curves* on X.

Theorem 2.2.10. (Kleiman ; 1966) The closed cone of curves is dual to Nef cone, i.e.

$$\overline{\mathrm{NE}}(X) = \Big\{ \gamma \in N_1(X)_{\mathbb{R}} \mid (\delta \cdot \gamma) \geq 0 \text{ for all } \delta \in \mathrm{Nef(X)} \Big\}.$$

(see Proposition 1.4.28. in [Laz1] for the proof)

Definition 2.2.11. A class $\gamma \in N_1(X)_{\mathbb{R}}$ is called a *movable curve*, if there exists a projective bi-rational mapping $\mu : X' \to X$, together with ample classes $a_1, ..., a_{n-1} \in N^1(X')_{\mathbb{R}}$ such that $\gamma = \mu_*(a_1 \cdot a_2 \cdot \cdots \cdot a_{n-1})$ where $\dim(X) = n$.

The *movable cone* of X is the closed convex cone spanned by all movable classes in $N^1(X)_{\mathbb{R}}$, and is denoted by $\mathrm{Mov}(X)$.

Note that if X is a 3 - fold and $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Nef}(X)$, then $\mathcal{L}_1 \cdot \mathcal{L}_2 \in \overline{\mathrm{Mov}}(X)$.

Theorem 2.2.12. (Boucksom-Demailly-Paun-Peternall ; 2004) Movable cone of curves is dual to Pseudoeffective cone, i.e.

$$\overline{\mathrm{Mov}}(X) = \Big\{ \gamma \in N_1(X)_{\mathbb{R}} \mid (\delta \cdot \gamma) \geq 0 \text{ for all } \delta \in \overline{\mathrm{Eff}}\mathrm{(X)} \Big\}.$$

(see Theorem 11.4.19. in [Laz2] for the proof).

2.3 Stable and Semistable sheaves

In this section, we recall some basic and well known results on stability and semistability of torsion free sheaves on a smooth irreducible projective variety from [HL],[Fri], and [OSS].

Let X be a smooth irreducible projective variety over an arbitrary field k of dimension s. If $\mathcal{F}$ is a coherent sheaf on X, then rank of $\mathcal{F}$ is defined as the rank of the $\mathcal{O}_\xi$-vector space $\mathcal{F}_\xi$, where ξ is the unique generic point of X. Moreover, if $\mathcal{F}$ is torsion-free i.e. $\mathcal{F}_x$ is torsion-free $\mathcal{O}_{X,x}$-module for every $x \in X$, then there is an open dense subset $U \subseteq X$ containing all points of codimension 1, such that $\mathcal{F}|_U$ is locally free. The rank of $\mathcal{F}$, in that case, is equal to the rank of $\mathcal{F}|_U$. Recall that, if the invertible sheaf $\wedge^r(\mathcal{F}|_U)$ corresponds to a divisor class D on U , then the first Chern class of $\mathcal{F}|_U$ on U is given by

$$c_1(\mathcal{F}|_U) := c_1\big(\wedge^r(\mathcal{F}|_U)\big) = D.$$

As divisors are determined at points of codimension 1, one can consider D as a divisor class on whole of X. Using the functorial property of Chern classes applied to the open immersion $U \hookrightarrow X$, we define

$$c_1(\mathcal{F}) := c_1(\mathcal{F}|_U) = D.$$

By a polarization on X, we mean, a ray $\mathbb{R}_{>0} \cdot L$, where L is in the ample cone inside the real Néron-Severi group $N^1(X)_{\mathbb{R}}$. We will say (X, L) is a polarized projective variety if we fix a particular polarization L on X. Now, we define the degree of a nonzero torsion-free coherent sheaf $\mathcal{F}$ of rank r with respect to a fixed polarization L on X, by

$$\deg_L(\mathcal{F}) := \big(c_1(\mathcal{F}) \cdot L^{s-1}\big)$$

and, the slope of $\mathcal{F}$ with respect to L is defined by

$$\mu_L(\mathcal{F}) := \frac{c_1(\mathcal{F}) \cdot L^{s-1}}{r}.$$

Definition 2.3.1. A non-zero torsion free sheaf $\mathcal{F}$ is said to be *slope L-semistable* (resp. *slope L-stable*) if for any coherent subsheaf $\mathcal{G}$ of $\mathcal{F}$ with $0 < \mathrm{rank}(\mathcal{G}) < \mathrm{rank}(\mathcal{F})$, $\mu_L(\mathcal{G}) \leq \mu_L(\mathcal{F})\big($ resp. $\mu_L(\mathcal{G}) < \mu_L(\mathcal{F})\big)$.

Lemma 2.3.2. (see [Fri], Chapter 4, Lemma 2) Let (X, L) be a polarized smooth projective variety over a field k and

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

be an exact sequence of non-zero coherent torsion-free sheaves on X. Then,

$$min\big\{\mu_L(\mathcal{F}_1), \mu_L(\mathcal{F}_3)\big\} \leq \mu_L(\mathcal{F}_2) \leq max\big\{\mu_L(\mathcal{F}_1), \mu_L(\mathcal{F}_3)\big\}$$

and equality holds if and only if $\mu_L(\mathcal{F}_1) = \mu_L(\mathcal{F}_2) = \mu_L(\mathcal{F}_3)$.

Lemma 2.3.3. (see [8], Chapter 4, Lemma 6) Let

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

be an exact sequence of non-zero coherent torsion-free sheaves on a polarized smooth projective variety (X, L) with $\mu_L(\mathcal{F}_1) = \mu_L(\mathcal{F}_2) = \mu_L(\mathcal{F}_3)$. Then $\mathcal{F}_2$ is slope L-semistable if and only if both $\mathcal{F}_1$ and $\mathcal{F}_3$ are slope L-semistable. In particular, $\mathcal{F}_2$ is slope L-semistable if $\mathrm{rank}(\mathcal{F}_1) = \mathrm{rank}(\mathcal{F}_3) = 1$. Also, in this situation, $\mathcal{F}_2$ is never slope L-stable.

Lemma 2.3.4. Let $\mathcal{F}_1, \mathcal{F}_2$ be two coherent torsion-free sheaves on a polarized smooth projective variety (X, L). If both $\mathcal{F}_1, \mathcal{F}_2$ are slope L-semistable and $\mu_L(\mathcal{F}_1) > \mu_L(\mathcal{F}_2)$, then $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) = \{0\}$.

Proof. Let $\phi \in \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2)$ be a non-zero homomorphism and let $\mathcal{G}$:=Image(ϕ). Since $\mathcal{F}_1$ and $\mathcal{F}_2$ both are L-semistable, $\mu_L(\mathcal{F}_1) \leq \mu_L(\mathcal{G}) \leq \mu_L(\mathcal{F}_2)$, which contradicts the given assumption. This proves the lemma. □

2.4 Higgs bundles and their moduli

In this section, we recall the definition of semistability of a Higgs bundle on a smooth irreducible projective variety X over an algebraically closed field k, and the functorial representation of their moduli spaces. More details can be found in [Sim1],[Sim2],[Sim3].

2.4.1 Semistability of Higgs bundles

Definition 2.4.1. A *Higgs sheaf* $\mathcal{E}$ on X is a pair (E, θ), where E is a coherent sheaf on X and $\theta : E \longrightarrow E \otimes \Omega^1_X$ is a morphism of $\mathcal{O}_X$-module such that $\theta \wedge \theta = 0$, where Ω^1_X is the cotangent sheaf to X and $\theta \wedge \theta$ is the composition map

$$E \longrightarrow E \otimes \Omega^1_X \longrightarrow E \otimes \Omega^1_X \otimes \Omega^1_X \longrightarrow E \otimes \Omega^2_X.$$

A *Higgs bundle* is a Higgs sheaf $\mathcal{V} = (V, \theta)$ such that V is a locally-free $\mathcal{O}_X$-module.

If $\mathcal{E} = (E, \phi)$ and $\mathcal{G} = (G, \psi)$ are Higgs sheaves, a morphism $f : (E, \phi) \longrightarrow (G, \psi)$ is a morphism of $\mathcal{O}_X$-modules $f : E \longrightarrow G$ such that the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & G \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ E \otimes \Omega^1_X & \xrightarrow{f \otimes id} & G \otimes \Omega^1_X \end{array}$$

$\mathcal{E}$ and $\mathcal{G}$ are said to isomorphic if there is a morphism $f : \mathcal{E} \longrightarrow \mathcal{G}$ such that f as an $\mathcal{O}_X$-module map is an isomorphism.

Definition 2.4.2. A Higgs sheaf $\mathcal{E} = (E, \theta)$ is said to be *L-semistable* $\big($ resp. *L-stable* $\big)$ if E is a non-zero torsion-free coherent sheaf and for every θ-invariant subsheaf G of E $\big($ i.e. $\theta(G) \subseteq G \otimes \Omega^1_X\big)$ with $0 < \text{rank}(G) < \text{rank}(E)$, one has $\mu_L(G) \leq \mu_L(E)\big($ resp. $\mu_L(G) < \mu_L(E)\big)$.

Remark 1. (i) Moreover, when X is a smooth projective curve or a surface, in the definition of semistability(resp. stability) for a Higgs bundle $\mathcal{V} = (V, \theta)$, it is enough to consider θ-invariant subbundles G of V with $0 < \text{rank}(G) < \text{rank}(V)$ for which the quotient V/G is torsion-free.

(ii) It is clear from the definition that, for a L-semistable (resp. stable) Higgs bundle $\mathcal{E} = (V, \theta)$ with zero Higgs field (i.e. $\theta = 0$), the underlying vector bundle V itself slope L-semistable (resp. stable). Also, a slope L-semistable (resp. stable) vector bundle on X is Higgs semistable (resp. stable) with respect to any Higgs field θ defined on it.

(iii) If X is a smooth projective curve, then for a torsion-free sheaf $\mathcal{F}$ of rank r, $\mu_L(\mathcal{F})$ is independent of the choice of the polarization L. Hence, whenever (semi)stability of bundles is considered on a curve, the polarization will not be mentioned.

For a smooth map $\phi : X \longrightarrow Y$ between two smooth projective varieties X and Y, and a Higgs bundle $\mathcal{V} = (V, \theta)$ on Y, its pullback $\phi^*(\mathcal{V})$ under ϕ is defined as the Higgs bundle $\big(\phi^*(V), d\phi(\theta)\big)$, where $d\phi(\theta)$ is the composition map

$$\phi^*(V) \longrightarrow \phi^*(V) \otimes \phi^*(\Omega^1_Y) \longrightarrow \phi^*(V) \otimes \Omega^1_X$$

Lemma 2.4.3. (see Lemma 3.3 in [BR]) If $\phi : X \longrightarrow Y$ is a finite separable morphism of smooth projective curves defined over an algebraically closed field of characteristic 0, then a Higgs bundle $\mathcal{V}$ is semistable on Y if and only if $\phi^*(\mathcal{V})$ is semistable on X.

2.4.2 Moduli of Higgs bundles

For an L-semistable Higgs bundle (V,θ) on X, there is a filtration of θ-invariant Higgs subsheaves

$$0=(V_0,\theta_0)\subsetneq(V_1,\theta_1)\subsetneq(V_2,\theta_2)\subsetneq\cdots\cdots\subsetneq(V_{n-1},\theta_{n-1})\subsetneq(V_n,\theta_n)=(V,\theta)$$

called the *Jordan-Hölder filtration* , where for each $i\in\{1,....,n\}$, $\mu_L\big(V_i/V_{i-1}\big)=\mu_L(V)$ and the induced Higgs sheaves $\big(V_i/V_{i-1},\overline{\theta}_i\big)$ are L-stable. This filtration is not unique, but the graded Higgs sheaf $\mathrm{Gr}_L(V,\theta):=\Big(\bigoplus_{i=1}^{n}V_i/V_{i-1},\bigoplus_{i=1}^{n}\overline{\theta}_i\Big)$ is unique up to isomorphism. Two L-semistable Higgs bundles are said to be *S-equivalent* if they admit filtrations such that their corresponding graded sheaves are isomorphic. Hence, two L-stable Higgs bundles are S-equivalent if and only if they are isomorphic as Higgs bundles.

We denote the moduli of S-equivalence classes of L-semistable Higgs bundles $\mathcal{V}=(V,\theta)$ of rank r on X with Chern classes $c_1,c_2,....,c_s$ by $\mathcal{M}_X^{Higgs}\big(r,c_1,c_2,...,c_s,L\big)$. There is a functorial representation for these moduli spaces. We denote the category of finite type schemes over Spec(k) by *Sch* , and the category of sets by *Sets* . Let $T\in\mathrm{ob}(Sch)$. A family of L-semistable Higgs bundles on X of rank r and with Chern classes $c_1,c_2,....,c_s$, parametrized by T is a pair $\big(\mathcal{V},\psi\big)$, where $\mathcal{V}$ is a coherent sheaf on $X\times T$, flat over T and $\psi\in\mathrm{Hom}\big(\mathcal{V},\mathcal{V}\otimes_{\mathcal{O}_{X\times T}}p_1^*(\Omega_X^1)\big)$. (Here $p_1:X\times T\longrightarrow X$ is the first projection map) such that, for each closed point $t\in T$, under the natural embedding $t:X\longmapsto X\times T$, the pair $\big(\mathcal{V}_t,\psi_t\big):=\big(t^*\mathcal{V},t^*\psi\big)$ is an L-semistable Higgs bundle on X of rank r having Chern classes $c_1,c_2,.....,c_s$ respectively. Moreover, two such families $\big(\mathcal{V}_1,\psi_1\big)$ and $\big(\mathcal{V}_2,\psi_2\big)$ are said to be equivalent if there is a line bundle N on T, and an isomorphism of vector bundles on $X \times T$

$$\Phi : \mathcal{V}_1 \longrightarrow \mathcal{V}_2 \otimes p_2^*(N)$$

such that $\psi_1 = \Phi^*\big(\psi_2 \otimes id_{p_2^* N}\big)$. (Here $p_2 : X \times T \longrightarrow T$ is the second projection map). Now, consider the following functor :

$$M_X^{Higgs}\big(r, c_1, c_2, ..., c_s, L\big) : Sch \longrightarrow Sets$$

given by

$$M_X^{Higgs}\big(r, c_1, c_2, ..., c_s, L\big)(T) = \left\{ \begin{array}{l} \text{Equivalent classes of families of} \\ L\text{-semistable Higgs bundles} \\ \text{of rank } r \text{ having Chern classes} \\ c_1, c_2, ..., c_s \text{ and parametrized by } T. \end{array} \right\}$$

Theorem 2.4.4. (see [HL]) The moduli space $\mathcal{M}_X^{Higgs}\big(r, c_1, c_2, ..., c_s, L\big)$ corepresents the functor $M_X^{Higgs}\big(r, c_1, c_2, ..., c_s, L\big)$.

2.5 Projective Bundle

In this section, we recall the definition and basic properties of projective bundle on a projective variety from [Har], Chapter II, Section 7. We also discuss about the known results on nef cone and pseudoeffective cone of projective bundles over a smooth curve.

Let X be an irreducible projective variety over a field k and $\mathcal{E}$ be a vector bundle of rank r on X. Consider the symmetric algebra of $\mathcal{E}$, $Sym(\mathcal{E}) := \bigoplus_{m \geq 0} Sym^m(\mathcal{E})$. Then, $Sym(\mathcal{E})$ is a sheaf of graded $\mathcal{O}_X$-algebras such that $Sym^0(\mathcal{E}) = \mathcal{O}_X, Sym^1(\mathcal{E})$ is a coherent $\mathcal{O}_X$-module, and $Sym(\mathcal{E})$ is locally generated by $Sym^1(\mathcal{E})$ as an $\mathcal{O}_X$-

algebra. The projective bundle $\mathbb{P}_X(\mathcal{E})$ associated to $\mathcal{E}$ over X is defined as

$$\mathbb{P}_X(\mathcal{E}) := \mathbf{Proj}\big(\mathrm{Sym}(\mathcal{E})\big)$$

together with a projection morphism $\pi : \mathbb{P}_X(\mathcal{E}) \longrightarrow X$, and a line bundle $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ such that $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ restricted to each fibre of the map over a closed point of X is isomorphic to $\mathbb{P}_k^{r-1}$. Moreover, if $\mathcal{E}|_U \cong \mathcal{O}_U^r$ over an affine open subset $U \cong \mathrm{Spec}(A)$ of X, then $\pi^{-1}(U) \cong \mathbb{P}_U^{r-1}$, so $\mathbb{P}_X(\mathcal{E})$ is relative projective space over X. If $\mathrm{rank}(\mathcal{E}) > 1$, then $Sym(\mathcal{E}) \cong \bigoplus_{n\in\mathbb{Z}} \pi_*\big(\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(n)\big)$ as graded $\mathcal{O}_X$-algebras. (Here, the grading in the right hand side is given by n). In particular,

$$\pi_*\big(\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(n)\big) = \begin{cases} 0 & \text{if } n < 0 \\ Sym^n(\mathcal{E}) & \text{if } n \geq 0 \end{cases}$$

We will use the short hand notation $S^n(\mathcal{E})$ for $Sym^n(\mathcal{E})$.

Proposition 2.5.1. Let $\mathcal{E}$ be a vector bundle of rank r on X, and $\pi : \mathbb{P}_X(\mathcal{E}) \longrightarrow X$ be the corresponding projective bundle. Then, there is a natural 1-1 correspondence between invertible sheaves $\mathcal{E} \longrightarrow \mathcal{M} \longrightarrow 0$ of $\mathcal{E}$ and sections of π (i.e., morphisms $s : X \longrightarrow \mathbb{P}_X(\mathcal{E})$ such that $\pi \circ s = \mathrm{id}_X$).

We recall the following well-known facts from [Ful] or [EH].

(1) $A^*\big(\mathbb{P}_X(\mathcal{E})\big)$ is a free $A^*(X)$-module generated by $1, \xi, \xi^2, \ldots\ldots\ldots, \xi^{r-1}$, where $\xi \in A^1\big(\mathbb{P}_X(\mathcal{E})\big)$ is the class of the divisor corresponding to $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$. More specifically, $A^*\big(\mathbb{P}_X(\mathcal{E})\big) \cong \frac{A^*(X)[\xi]}{<f(\xi)>}$, where $f(\xi) = \sum_{i=0}^{r} (-1)^i \pi^* c_i(\mathcal{E}) \cdot \xi^{r-i}$.

(2) $\mathrm{Pic}\big(\mathbb{P}_X(\mathcal{E})\big) = \pi^*\big(\mathrm{Pic}(X)\big) \bigoplus \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$.

(3) $\rho\big(\mathbb{P}_X(\mathcal{E})\big) = \rho(X) + 1$.

2.5.1 Nef and pseudoeffective cones of Projective bundle

Let $\mathcal{E}$ be a vector bundle of rank r on a smooth irreducible projective curve C, and $\pi : \mathbb{P}_C(\mathcal{E}) \longrightarrow C$ be the corresponding projective bundle. We denote the numerical equivalence classes of the tautological line bundle $\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ and a fibre of the map π by η and f respectively. The normalized hyperplane class, denoted by $\lambda_{\mathcal{E}}$, is defined as $\lambda_{\mathcal{E}} := \eta - \mu(\mathcal{E})f$, where $\mu(\mathcal{E})$ is the slope of the vector bundle $\mathcal{E}$. In [Miy], Miyaoka studied the nef cone $\text{Nef}\big(\mathbb{P}_C(\mathcal{E})\big)$ under the assumption that the base field has characteristic 0. More precisely, he showed the following ,

Theorem 2.5.2. (see Theorem 3.1 in [Miy]) Let $\pi : \mathbb{P}_C(\mathcal{E}) \longrightarrow C$ be a projective bundle on a smooth irreducible projective curve C over a field k of characteristic 0. Then, the following are equivalent :

(i) $\mathcal{E}$ is slope semistable.

(ii) The normalized hyperplane class $\lambda_{\mathcal{E}}$ is nef.

(iii) $\text{Nef}\big(\mathbb{P}_C(\mathcal{E})\big) = \big\{a\lambda_{\mathcal{E}} + bf \mid a, b \in \mathbb{R}_{\geq 0}\big\}$.

(iv) $\overline{\text{NE}}(X) = \big\{a\lambda_{\mathcal{E}}^{r-1} + b\lambda_{\mathcal{E}}^{r-2} \mid a, b \in \mathbb{R}_{\geq 0}\big\}$, where r is the rank of $\mathcal{E}$.

(v) Every effective divisor on $\mathbb{P}_C(\mathcal{E})$ is nef, i.e., $\text{Nef}\big(\mathbb{P}_C(\mathcal{E})\big) = \text{Eff}\big(\mathbb{P}_C(\mathcal{E})\big)$.

Later, in his paper [Fulg], Fulger completed the calculation of $\text{Nef}\big(\mathbb{P}_C(\mathcal{E})\big)$ without any restriction on $\mathcal{E}$. For every vector bundle $\mathcal{E}$ over C, there is a unique filtration

$$0 = \mathcal{E}_l \subsetneq \mathcal{E}_{l-1} \subsetneq \cdots\cdots \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_0 = \mathcal{E}$$

called the Harder-Narasimhan filtration, such that $\mathcal{E}_i/\mathcal{E}_{i+1}$ is slope semistable for each $i \in \{0, 1, ..., l-1\}$ and $\mu(\mathcal{E}_i/\mathcal{E}_{i+1}) > \mu(\mathcal{E}_{i-1}/\mathcal{E}_i)$ for all $i \in \{1, 2, ..., l-1\}$.

Theorem 2.5.3. (see Lemma 2.1 in [Fulg]) Let $\mathcal{E}$ be a vector bundle of rank r

on a smooth irreducible complex projective curve C having the Harder-Narasimhan filtration as above. Then,

$$\text{Nef}\big(\mathbb{P}_C(\mathcal{E})\big) = \big\{a(\eta - \mu_1 f) + bf \mid a, b \in \mathbb{R}_{\geq 0}\big\}$$

where $\mu_1 = \mu(\mathcal{Q}_1) = \mu(\mathcal{E}_0/\mathcal{E}_1)$.

Theorem 2.5.4. (see Lemma 2 in [MSC] or Proposition 1.3 in [Fulg]) Let $\mathcal{E}$ be a vector bundle of rank r on a smooth irreducible complex projective curve C having the Harder-Narasimhan filtration as above. Futher, assume that $\text{rank}(\mathcal{Q}_1) := \text{rank}(\mathcal{E}_0/\mathcal{E}_1) = r - 1$, so that we can consider $\mathbb{P}_C(\mathcal{Q}_1)$ as a divisor in $X = \mathbb{P}_C(\mathcal{E})$. Let $q_1 = \big[\mathbb{P}_C(\mathcal{Q}_1)\big] \in N^1(X)_{\mathbb{R}}$. Then,

$$\overline{\text{Eff}}\big(\mathbb{P}_C(\mathcal{E})\big) = \big\{aq_1 + bf \mid a, b \in \mathbb{R}_{\geq 0}\big\}$$

Sometimes, we will write $\mathbb{P}(\mathcal{E})$ instead of $\mathbb{P}_X(\mathcal{E})$ whenever the base space X is clear from the context.

2.5.2 Ruled Surface

Definition 2.5.5. Let C be a smooth projective algebraic curve of genus g over an algebraically closed field k. A *geometrically ruled surface* or simply a *ruled surface* , is a smooth projective surface X, together with a surjective morphism $\pi : X \longrightarrow C$ such that the fibre X_y is isomorphic to $\mathbb{P}^1_k$ for every closed point $y \in C$, and π admits a section (i.e. a morphism $\sigma : C \longrightarrow X$ such that $\pi \circ \sigma = id_C$).

Ruled surfaces are characterized by the following theorem.

Theorem 2.5.6. (see Proposition 2.2, Chapter V in [Har]) If $\pi : X \longrightarrow C$ is a ruled surface, then $X \cong \mathbb{P}_C(E)$ over C for some vector bundle E of rank 2 on C. Conversely, every such $\mathbb{P}_C(E)$ is a ruled surface over C. Moreover, E_1 and E_2

are two vector bundles of rank 2 on C such that $X \cong \mathbb{P}_C(E_1) \cong \mathbb{P}_C(E_2)$ as ruled surfaces over C if and only if there is a line bundle N on C such that $E_1 \cong E_2 \otimes N$.

Let σ and f be the numerical class of a section and a fibre of the ruling $\pi : X \longrightarrow C$. Then,

(i) $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot \mathcal{O}_X(1) \oplus \pi^*\big(\mathrm{Pic}(C)\big)$, and

(ii) $N^1(X) \cong \mathbb{Z} \cdot \sigma \oplus \mathbb{Z} \cdot f$ satisfying $\sigma \cdot f = 1, f^2 = 0$.

We mention the following theorem from [Har] (Chapter V, Section 2).

Theorem 2.5.7. (see Proposition 2.8, Chapter V in [Har]) If $\pi : X \longrightarrow C$ is a ruled surface, then there is a vector bundle F of rank 2 on C such that $X \cong \mathbb{P}_C(F)$ as ruled surfaces over C, and F have the property : $H^0(C, F) \neq 0$, but $H^0(F \otimes L) = \{0\}$ for all line bundles L on C with $\deg(L) < 0$.

In this case, $e = -\deg(F)$ is an invariant of X. Further, in this case there is a section $\sigma_0 : C \longrightarrow X$, called the normalized section, such that $\mathrm{Image}(\sigma_0) = C_0$ and $\mathcal{O}_X(C_0) \cong \mathcal{O}_X(1)$.

Motivated by the above theorem, one define the following :

Definition 2.5.8. A vector bundle E of rank 2 on a smooth irreducible projective curve C is said to be *normalized* if $H^0(E) \neq 0$, but $H^0(E \otimes N) = \{0\}$ for all line bundle N on C with $\deg(N) < 0$.

In this context, we recall the following theorem from [Har].

Theorem 2.5.9. (see Proposition 2.12, Chapter V in [Har]) Let $\pi : X \longrightarrow C$ be a ruled surface over a smooth irreducible curve C of genus g, determined by a normalized vector bundle E.

(i) If E is decomposable (i.e., a direct sum of two line bundles), then $E =$

$\mathcal{O}_C \oplus M$ for some line bundle M with $\deg(M) \geq 0$. Therefore, $e = -\deg(E) \geq 0$. In this case, e can take all non-negative integral value.

(ii) If E is indecomposable, then $-2g \leq e \leq 2g - 2$.

As a corollary to the above theorem, we get the following :

Corollary 2.5.10. In the above theorem, if $g = 0$, then $e \geq 0$, and for each $e \geq 0$, there is exactly one rational ruled surface $\mathbb{P}_{\mathbb{P}^1}\big(\mathcal{O}_C \oplus \mathcal{O}(-e)\big)$ with invariant e, over $C = \mathbb{P}^1_k$.

When the base curve C of a ruled surface X is an elliptic curve (i.e., a smooth curve of genus 1), there are further restriction on the possible values of the invariant e.

Theorem 2.5.11. (see Proposition 2.15, Chapter V in [Har]) If X is a ruled surface over an elliptic curve C, determined by an indecomposable normalized rank 2 bundle E, then $e = 0$ or -1, and there is exactly one such ruled surface over C corresponding to each of these two values of e.

(i) When $e = 0$, E is the unique nonsplit extension of $\mathcal{O}_C$ by $\mathcal{O}_C$.

(ii) When $e = 1$, E is the unique nonsplit extension of the form

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow \mathcal{O}_C(p) \longrightarrow 0$$

for some closed point $p \in C$.

Chapter 3

Higgs Bundle on Ruled Surfaces

In this chapter, we consider a smooth irreducible projective curve C over an algebraically closed field k of characteristic 0, and a ruled surface $\pi : X = \mathbb{P}_C(E) \longrightarrow C$. We fix a polarization L on X, and discuss about the (semi)stability of pullback of a Higgs bundle on C under the ruling π (see [Mis]). From now onwards, we will write $\mathbb{P}(E)$ instead of $\mathbb{P}_X(E)$ whenever the base space X is clear from the context.

3.1 Change of Polarization

Let $\mathcal{F}$ be a coherent sheaf on a smooth irreducible projective variety X over an algebraically closed field k of dimension $n \geq 2$, having Chern classes c_i and rank r. The *discriminant* of $\mathcal{F}$ by definition is the characteristic class

$$\triangle(F) = 2rc_2 - (r-1)c_1^2$$

Let $\mathcal{E} = (E, \theta)$ be a semistable Higgs sheaf with respect to a fixed polarization L on X. Then, *Bogomolov's inequality* says that $\triangle(E) \cdot L^{n-2} \geq 0$. In his paper [Sim1] (see Proposition 3.4 in [Sim1]), Simpson proved Bogomolov's inequality for semistable

Higgs bundle over a smooth complex irreducible projective variety. More generally, later in [Lan], it is shown that Bogomolov's inequality holds true for semistable Higgs sheaves on smooth projective varieties defined over an algebraically closed field k of characteristic 0 as well.

For a Higgs bundle $\mathcal{V} = (V, \theta)$ of rank r on a smooth irreducible complex projective variety X, let $Gr_m(\mathcal{V})$ be the schemes parametrizing locally free quotients of V of rank m whose kernels are θ-invariant. Let $\pi_m : Gr_m(\mathcal{V}) \longrightarrow X$ be the projection maps, and $\mathcal{Q}_m = (Q_m, \Phi_m)$ be the universal quotient Higgs bundle on them. We define,

$$\lambda_{m,\mathcal{V}} = c_1(\mathcal{O}_{\mathbb{P}Q_m}) - \frac{1}{r}\pi_m^* c_1(V)$$

We recall the following result from [BR]

Theorem 3.1.1. (see Theorem 1.3 in [BR]) Let $\mathcal{V} = (V, \theta)$ be a Higgs bundle of rank r on a smooth irreducible complex projective variety. Then, the following are equivalent

(i) All classes $\lambda_{m,\mathcal{V}}$ are nef, for $0 < m < r$.

(ii) $\mathcal{V}$ is semistable and $\triangle(V) = 0$.

Moreover, if $\mathcal{V} = (V, \theta)$ is semistable with $\triangle(V) = 0$, then for every smooth curve C and for every smooth morphism $\phi : C \longrightarrow X$, $\phi^*(\mathcal{V})$ is Higgs semistable on C. However, the converse is not true in general (see [BG]). Note that nefness does not depend on the choice of a polarization on X, hence by the Theorem 3.1.1, the semistability of $\mathcal{V} = (V, \theta)$ is independent of the choice of the polarization if it is semistable with $\triangle(V) = 0$.

We prove a similar result for Higgs bundles on a smooth irreducible algebraic surface over any algebraically closed field of characteristic 0.

Proposition 3.1.2. If $\mathcal{V} = (V, \theta)$ is L_1-semitable Higgs bundle on a smooth irreducible algebraic surface X defined over an algebraically closed field k of characteristic 0, with a fixed polarization L_1 on X and $\triangle(V) = 0$, then the semistability of the Higgs bundle $\mathcal{V} = (V, \theta)$ is independent of the polarization chosen.

Proof. Suppose there is a polarization L_2 such that $\mathcal{V} = (V, \theta)$ is not L_2-semistable Higgs bundle. Then there exist a saturated θ-invariant subsheaf $V_0 \subseteq V$ with $\mu_{L_2}(V_0) > \mu_{L_2}(V)$. Let V' is any θ-invariant saturated subsheaf with this property. We define

$$r(V') := \frac{\mu_{L_1}(V) - \mu_{L_1}(V')}{\mu_{L_2}(V') - \mu_{L_2}(V)}.$$

Then $\mu_{L_1+r(V')L_2}(V') = \mu_{L_1+r(V')L_2}(V)$. We note that $L_0 := L_1 + r(V_0)L_2$ is a polarization on X. If $r(V') < r(V_0)$, then $\mu_{L_0}(V') > \mu_{L_0}(V)$. By Grothendieck's Lemma (See Lemma in 1.7.9 in [HL]) , the family of saturated subsheaves V' with $\mu_{L_0}(V') > \mu_{L_0}(V)$ is bounded. So, there are only finitely many numbers $r(V')$ which are smaller than $r(V)$. We can further choose V_0 in such a way that $r(V_0)$ is minimal. Then, V and V_0 are L_0-Higgs semistable with $\mu_{L_0}(V_0) = \mu_{L_0}(V)$. So, we have an exact sequence of torsion free Higgs sheaves

$$0 \longrightarrow V_0 \longrightarrow V \longrightarrow V_1 \longrightarrow 0 \tag{3.1}$$

with $\mu_{L_0}(V_0) = \mu_{L_0}(V) = \mu_{L_0}(V_1)$. Let $\overline{\theta} : V_1 \longrightarrow V_1 \otimes \Omega^1_X$ be the induced map.

Our claim is that $(V_1, \overline{\theta})$ is also a L_0-semistable Higgs sheaf. Now, let $\overline{W}_1$ be a $\overline{\theta}$-invariant subsheaf of V_1. Then we have an exact sequence of torsion-free sheaves

$$0 \longrightarrow V_0 \longrightarrow W_1 \longrightarrow \overline{W}_1 \longrightarrow 0 \tag{3.2}$$

where W_1 is θ-invariant subsheaf of V containing V_0. Using the L_0-semistability of

V, we get, $\mu_{L_0}(W_1) \leq \mu_{L_0}(V) = \mu_{L_0}(V_0)$. From the exact sequence (3.2) (using Lemma 2, Chapter 4 in [Fri]), we also have

$$min\{\mu_{L_0}(V_0), \mu_{L_0}(\overline{W}_1)\} \leq \mu_{L_0}(W_1) \leq max\{\mu_{L_0}(V_0), \mu_{L_0}(\overline{W}_1)\}$$

Hence, $\mu_{L_0}(\overline{W}_1) \leq \mu_{L_0}(W_1) \leq \mu_{L_0}(V_0) = \mu_{L_0}(V) = \mu_{L_0}(V_1)$. This proves our claim. Therefore, by Bogomolov's inequality, $\triangle(V_0) \geq 0$, $\triangle(V_1) \geq 0$.

We denote $\xi \equiv \big(r \cdot c_1(V_0) - r_0 \cdot c_1(V)\big) \in N^1(X)_{\mathbb{R}}$ where r and r_0 denotes the ranks of V and V_0 respectively. Hence, $\xi \cdot L_0 = 0$ and $\xi \cdot L_2 > 0$. So, by Hodge Index Theorem, $\xi^2 < 0$. On the other hand, from the exact sequence (3.1) we have,

$$0 = \triangle(V) = \frac{r}{r_1}\triangle(V_0) + \frac{r}{r - r_1}\triangle(V_1) - \frac{\xi^2}{r_1(r - r_1)}$$

Since $\triangle(V_0) \geq 0$ and $\triangle(V_1) \geq 0$, we have, $\xi^2 \geq 0$ which is a contradiction. This proves our result. □

Remark 2. A similar argument as in Proposition 3.1.2 will prove that if $\mathcal{V} = (V, \theta)$ is an L-stable Higgs bundle on a smooth algebraic surface X with a fixed polarization L on X and $\triangle(V) = 0$, then the stability of the Higgs bundle $\mathcal{V} = (V, \theta)$ is independent of the polarization chosen.

3.2 Semistability under pullback

Theorem 3.2.1. Let $\pi : X \longrightarrow C$ be a ruled surface with a fixed polarization L on X. Let $\mathcal{V} = (V, \theta)$ be a semistable Higgs bundle of rank r on C. Then, the pullback $\pi^*(\mathcal{V}) = \big(\pi^*(V), d\pi(\theta)\big)$ is L-semistable Higgs bundle on X.

Proof. Let H be a very ample line bundle on X. By Bertini's Theorem, there exist a smooth projective curve B in the linear system $|H|$. Let us consider the induced map between the two smooth projective curves

$$\pi_B : B \hookrightarrow X \longrightarrow C.$$

Since $B \cdot f = H \cdot f > 0$, B is not contained in any fibre of the map π. Hence, π_B is a finite separable morphism between two smooth projective curves and by Lemma 3.3 in [BR], $\pi_B^*(\mathcal{V})$ is a semistable Higgs bundle on B. Now, our claim is that $\pi^*(\mathcal{V})$ is H-semistable Higgs bundle on X. If not, then there exist $d\pi(\theta)$-invariant subbundle W of $\pi^*(V)$ with $\mu_H(W) > \mu_H\big(\pi^*(V)\big)$. Hence, we have

$$\mu\big(W|_B\big) > \mu\big(\pi^*(V)|_B\big)$$

But, $W|_B$ is a $d\pi_B(\theta)$-invariant subbundle of $\pi_B^*(V) = \pi^*(V)|_B$. Thus, $\pi_B^*(\mathcal{V}) = \big(\pi_B^*(V), d\pi_B(\theta)\big)$ is not a semistable Higgs bundle on B, which is a contradiction. Therefore, $\pi^*(\mathcal{V}) = \big(\pi^*(V), d\pi(\theta)\big)$ is an H-semistable Higgs bundle on X. Now, the discriminant of $\pi^*(V)$,

$$\triangle\big(\pi^*(V)\big) = 2rc_2\big(\pi^*(V)\big) - (r-1)c_1^2\big(\pi^*(V)\big)$$

$$= 2rc_2\big(\pi^*(V)\big) - (r-1)df \cdot df \text{ , where } d = \deg\big(\wedge^r(V)\big) \text{ in } C.$$

$$= 0 \text{ , (Since } df \cdot df = d^2 \cdot f^2 = 0 \text{) .}$$

Hence, by Proposition 3.1.2, $\pi^*(\mathcal{V}) = \big(\pi^*(V), d\pi(\theta)\big)$ is an L-semistable Higgs bundle on X for any polarization L on X. □

Proposition 3.2.2. Let $\pi : X = \mathbb{P}(E) \longrightarrow C$ be a ruled surface on C. Then, the natural map, $\Omega_C^1 \xrightarrow{\eta} \pi_*(\Omega_X^1)$ is an isomorphism.

Proof. Consider the exact sequence

$$0 \longrightarrow \pi^*(\Omega_C^1) \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X|C}^1 \longrightarrow 0 \tag{3.3}$$

Applying π_* to the exact sequence (3.3), we get the following long exact sequence,

$$(3.4) \qquad 0 \longrightarrow \Omega^1_C \longrightarrow \pi_*(\Omega^1_X) \longrightarrow \pi_*(\Omega^1_{X|C}) \longrightarrow \Omega^1_C \otimes R^1\pi_*(\mathcal{O}_X) \longrightarrow ...$$

We also have

$$(3.5) \qquad 0 \longrightarrow \Omega^1_{X|C} \longrightarrow \pi^*(E) \otimes \mathcal{O}_{\mathbb{P}(E)}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(E)} \longrightarrow 0$$

Since π is a smooth map of relative dimension 1 between two nonsingular varieties, by Proposition 10.4 in [Har], $\Omega^1_{X|C}$ is a locally free sheaf of rank 1 on X. Applying π_* to exact sequence (3.5), we get

$$(3.6) \qquad 0 \longrightarrow \pi_*(\Omega^1_{X|C}) \longrightarrow \pi_*\big((\pi^*(E)) \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)\big) \longrightarrow ..$$

By the projection formula, $\pi_*\big(\pi^*(E) \otimes \mathcal{O}_{\mathbb{P}(E)}(-1)\big) = E \otimes \pi_*\big(\mathcal{O}_{\mathbb{P}(E)}(-1)\big) = 0$. Therefore, from exact sequence (3.6), we get, $\pi_*(\Omega^1_{X|C}) = 0$ and hence from exact sequence (3.3), we have, the natural map $\Omega^1_C \xrightarrow{\eta} \pi_*(\Omega^1_X)$ is an isomorphism. □

We recall the following lemma, which we will use repeatedly in the proofs of our main results.

Lemma 3.2.3. (see Lemma 2.2 in [Su]) Let $\mathcal{F}$ be a torsion free sheaf of rank r on a ruled surface $\pi : X \longrightarrow C$ and $\mathcal{F}|_f \cong \mathcal{O}_f^{\oplus r}$ for a generic fibre f of the map π. Then $c_2(\mathcal{F}) \geq 0$ and $c_2(\mathcal{F}) = 0$ if and only if $\mathcal{F} \cong \pi^*(V)$ for some vector bundle V on C.

Theorem 3.2.4. Let L be a fixed polarization on a ruled surface $\pi : X \longrightarrow C$. Let $\mathcal{V} = (V, \theta)$ be an L-semistable Higgs bundle of rank r on X with $c_1(V) = \pi^*\big([\mathbf{d}]\big)$, for some divisor $\mathbf{d}$ of degree d on C. Then $c_2(V) \geq 0$ and $c_2(V) = 0$ if and only if there exists a semistable Higgs bundle $\mathcal{W} = (W, \psi)$ on C such that $\pi^*(\mathcal{W}) = \big(\pi^*(W), d\pi(\psi)\big) \cong \mathcal{V}$ on X.

Proof. By Bogomolov's inequality, $2rc_2(V) \geq (r-1)c_1^2(V) = 0$. Hence, $c_2(V) \geq 0$.

If $c_2(V) = 0$, then $\triangle(V) = 0$. Our claim is that, in this case, $V|_f$ is a slope semistable vector bundle on a generic fibre f of the map π, and hence , $V|_f \cong \mathcal{O}_f^{\oplus r}$ (as $\deg(V|_f) = c_1(V) \cdot f = 0$). If not, then, $V|_f = \mathcal{O}_f(a_1) \oplus \mathcal{O}_f(a_2) \oplus \cdots \oplus \mathcal{O}_f(a_r)$ for some integers $a_1, a_2, .., a_r$ such that not all of them are zero. Without loss of generality, we assume that $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_r$. As $\deg(V|_f) = \sum_j a_j = 0$, one can further assume that $a_1 > 0$. Let $a_1 = a_2 = \cdots = a_i > a_{i+1}$ for some $1 \leq i < r$. Consider $W_f = \mathcal{O}_f(a_1) \oplus \cdots \oplus \mathcal{O}_f(a_i)$. Then W_f is a slope semistable vector bundle on f and $\deg(W_f) > 0$. Consider the exact sequence

$$0 \longrightarrow \pi^*(\Omega_C^1) \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X|C}^1 \longrightarrow 0$$

Restricting the above exact sequence to a generic fibre f, we get

$$0 \longrightarrow \mathcal{O}_f \longrightarrow \Omega_X^1|_f \longrightarrow \Omega_{X|C}^1|_f \longrightarrow 0$$

Now, the canonical divisor $K_X \equiv -2C_0 + (2g-2-e)f$, where C_0 is the normalized section of π (see Corollary 2.11, Chapter 5 in [Har]). Hence, $\deg(\Omega_X^1|_f) = \deg(K_X|_f) = -2$. We also have $\Omega_{X|C}^1|_f \cong \mathcal{O}_f(-2)$. Therefore, $\Omega_X^1|_f = \mathcal{O}_f \oplus \mathcal{O}_f(-2)$. Note that $\mu\big(W_f\big) > \mu\big(\mathcal{O}_f(a_l)\big)$ and $\mu\big(W_f\big) > \mu\big(\mathcal{O}_f(a_l-2)\big)$ for every l satisfying $(i+1) \leq l \leq r$.

Now, $V|_f \otimes \Omega_X^1|_f = \big\{W_f \oplus \mathcal{O}_f(a_{i+1}) \oplus \cdots \oplus \mathcal{O}_f(a_r)\big\} \otimes \Omega_X^1|_f$

$= \big\{W_f \otimes \Omega_X^1|_f\big\} \oplus \big\{\mathcal{O}_f(a_{i+1}) \otimes \Omega_X^1|_f\big\} \oplus \cdots \oplus \big\{\mathcal{O}_f(a_r) \otimes \Omega_X^1|_f\big\}$.

As W_f is slope semistable, this implies that for each i satisfying $(i+1) \leq l \leq r$, there does not exists any non-zero map from W_f to $\mathcal{O}_f(a_l) \otimes \Omega_X^1|_f$. Hence, $\theta|_f : W_f \longrightarrow W_f \otimes \Omega_X^1|_f$. We extend W_f to a θ-invariant subbundle $W \hookrightarrow V$ such that the quotient is also torsion-free. Since $\theta|_f$ preserves W_f, W is also preserved by θ. Note that, $c_1(W) \cdot f = \deg(W_f) > 0$. Hence, for a large $m \gg 0$, $\mu_{L+mf}(W) > \mu_{L+mf}(V)$. Since $\triangle(V) = 0$, this contradicts that (V, θ) is $(L + mf)$-semistable Higgs bundle. Hence, our claim is proved.

Therefore, $V|_f \cong \mathcal{O}_f^{\oplus r}$ for a generic fibre f. As $c_2(V) = 0$, by Lemma 3.2.3, $V \cong \pi^*(W)$ for some vector bundle W on C. Note that by projection formula, we have

$$H^0\big(X, \mathrm{End}(V) \otimes \pi^*(\Omega^1_C)\big) \cong H^0\big(X, \mathrm{End}(V) \otimes \Omega^1_X\big) \cong H^0\big(C, \mathrm{End}(W) \otimes \Omega^1_C\big).$$

Hence, the natural inclusion map

$$H^0\big(X, \mathrm{End}(V) \otimes \pi^*(\Omega^1_C)\big) \hookrightarrow H^0\big(X, \mathrm{End}(V) \otimes \Omega^1_X\big)$$

is also surjective i.e. every Higgs-field on V factors through $V \otimes \pi^*(\Omega^1_C)$. We consider the following Higgs-field ψ on C defined using Proposition 3.2.2 and projection formula

$$\psi := \pi_*(\theta) : \pi_*\big(\pi^*(W)\big) \cong W \longrightarrow \pi_*\big(\pi^*(W) \otimes \Omega^1_X\big) \cong W \otimes \Omega^1_C$$

Since C is a curve, The condition $\psi \wedge \psi = 0$ is automatically satisfied. Hence $\mathcal{W} := (W, \psi)$ is a well-defined Higgs bundle on C. Now, consider the following commutative diagram.

$$\begin{array}{ccccc}
V & \xrightarrow{\quad\theta\quad} & V \otimes \pi^*(\Omega^1_C) & \xrightarrow{\quad id\otimes\eta\quad} & V \otimes \Omega^1_X \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\pi^*(W) & \xrightarrow{\pi^*(\psi)} & \pi^*(W) \otimes \pi^*(\Omega^1_C) & \xrightarrow{id\otimes\eta} & \pi^*(W) \otimes \Omega^1_X
\end{array}$$

From the above commutative diagram, we have, $\pi^*(\mathcal{W}) \cong \mathcal{E}$.

Our claim is that (W, ψ) is Higgs semistable on C. If not, then there is a ψ-invariant subbundle, say, W_1 of W such that $\mu(W_1) > \mu(W)$. This implies

$$\mu_L\big(\pi^*(W_1)\big) = \mu(W_1)(L \cdot f) > \mu(W)(L \cdot f) = \mu_L(V)$$

But, $\pi^*(W_1)$ is a θ-invariant subbundle of V, and hence it contradicts that (V, θ) is L-semistable Higgs bundle. This completes the proof of the theorem. $\square$

3.3 Stability under pullback

Theorem 3.3.1. Let L be a fixed polarization on a ruled surface $\pi : X \longrightarrow C$. Then for any stable Higgs bundle $\mathcal{W} = (W, \psi)$ on C, the pullback Higgs bundle $\pi^*(\mathcal{W})$ is L-stable Higgs bundle on X. Conversely, if $\mathcal{V} = (V, \theta)$ is an L-stable Higgs bundle on X with $c_1(V) = \pi^*\big([\mathbf{d}]\big)$ for some divisor $\mathbf{d}$ on C and $c_2(V) = 0$, then $\mathcal{V} \cong \pi^*(\mathcal{W})$ for some stable Higgs bundle $\mathcal{W} = (W, \psi)$ on C.

Proof. By Theorem 3.2.4, $\pi^*(\mathcal{W})$ is an L-semistable Higgs bundle on X. If $\pi^*(\mathcal{W})$ is strictly L-semistable Higgs bundle, then there is a short exact sequence of torsion-free sheaves

$$0 \longrightarrow V_1 \longrightarrow \pi^*(W) \longrightarrow \mathcal{V}_2 \longrightarrow 0 \tag{3.7}$$

where V_1 is a $d\pi(\psi)$-invariant subbundle of $\mathcal{V}$ of rank m (say), $\mathcal{V}_2$ is a torsion-free sheaf of rank n (say) on X such that $\mathcal{V}_2$ is a locally-free sheaf V_2 on the complement a closed subscheme $Z = \{x_1, x_2, ..., x_d\}$ (say) of codimension 2 in X, and

$$\mu_L(V_1) = \mu_L\big(\pi^*(W)\big) = \mu_L(\mathcal{V}_2).$$

Restricting the above exact sequence (3.7) to a generic fibre f of the map π such that $Z \bigcap f = \emptyset$,we have

$$0 \longrightarrow V_1|_f \longrightarrow \pi^*(W)|_f \longrightarrow V_2|_f \longrightarrow 0 \tag{3.8}$$

Since $\pi^*(W)\mid_f \cong \mathcal{O}_f^{\oplus r}$, it is a slope semistable bundle of degree 0 on $f \cong \mathbb{P}^1$ and hence

$\deg(V_2 \mid_f) \geq 0$. Our claim is that $\deg(V_2|_f) > 0$. If not, let $\deg(V_2|_f) = 0$. Hence, from the above exact sequence (3.8), for a generic fibre f, we get, $\deg(V_1|_f) = 0$, $\deg(V_2|_f) = 0$ and $V_1|_f$, $V_2|_f$ are semistable on $f \cong \mathbb{P}^1$. Therefore, for a generic fibre f, $V_1|_f \cong \mathcal{O}_f^{\oplus m}$ and $V_2|_f \cong \mathcal{O}_f^{\oplus n}$. By Lemma 3.2.3, we have $c_2(V_1) \geq 0$, $c_2(\mathcal{V}_2) \geq 0$. Using the Whitney sum formula to the exact sequence (3.7), we get

$$\pi^*\big(c_2(W)\big) = c_2\big(\pi^*(W)\big) = 0 = c_2(V_1) + \mathrm{c}_2(\mathcal{V}_2)$$

which implies $c_2(V_1) = c_2(\mathcal{V}_2) = 0$. Hence by the Lemma 3.2.3, $V_1 \cong \pi^*(W_1)$ and $\mathcal{V}_2 \cong V_2 \cong \pi^*(W_2)$ for some vector bundle W_1 and W_2 on C and $Z = \emptyset$ in the exact sequence (3.7). Note that since V_1 is $d\pi(\psi)$-invariant, it will imply W_1 is ψ-invariant. Note that, in this case,

$$\mu_L(V_1) = \mu(W_1)(L \cdot f) = \mu_L\big(\pi^*(W)\big) = \mu(W)(L \cdot f)$$

This implies $\mu(W_1) = \mu(W)$ for the ψ-invariant subbundle $W_1 \longrightarrow W$, which contradicts the Higgs stability of $\mathcal{W}$. Thus, our claim is proved i.e. $\deg(V_2|_f) > 0$ and hence $\deg(V_1|_f) < 0$.

Now choose a positive integer $i \gg 0$ such that $L_i := L + if$ is ample. We then have, for all $d\pi(\psi)$-invariant subbundle $0 \longrightarrow M \longrightarrow \pi^*(W)$ of $\pi^*(\mathcal{W})$,

$$\mu_{L_i}(M) < \mu_{L_i}\big(\pi^*(W)\big)$$

Hence, $\pi^*(\mathcal{W})$ is L_i-stable Higgs bundle and the discriminant of $\pi^*(W)$ being 0, by Remark 2, $\pi^*(\mathcal{W})$ is L-stable Higgs bundle on X for any polarization L on X.

Conversely, if $\mathcal{V} = (V, \theta)$ is a L-stable Higgs bundle on X with $c_1(V) = \pi^*\big([\mathbf{d}]\big)$ for some divisor $\mathbf{d}$ on C and $c_2(V) = 0$, then by Theorem 3.3.1, $\mathcal{V} \cong \pi^*(\mathcal{W})$ for some semistable Higgs bundle $\mathcal{W} = (W, \psi)$ on C. If $\mathcal{W}$ is strictly semistable Higgs bundle, then there is an exact sequence of ψ-invariant subbundle of W

$$0 \longrightarrow W_1 \longrightarrow W \longrightarrow W_2 \longrightarrow 0 \tag{3.9}$$

such that $\mu(W_1) = \mu(W) = \mu(W_2)$. The exact sequence (3.9) will then pullback to an exact sequence

$$0 \longrightarrow \pi^*(W_1) \longrightarrow V \longrightarrow \pi^*(W_2) \longrightarrow 0$$

of θ-invariant subbundles of $\mathcal{V}$ such that $\mu_L\big(\pi^*(W_1)\big) = \mu_L(V) = \mu_L\big(\pi^*(W_2)\big)$. This contradicts the is L-stability of the Higgs bundle $\mathcal{V}$. Therefore, $\mathcal{W}$ is stable Higgs bundle on C such that $\pi^*(\mathcal{W}) \cong \mathcal{V}$. □

As a corollary to Theorem 3.3.1, we have the following result which generalizes the results of Takemoto and Marian Aprodu for rank 2 ordinary vector bundles (see Proposition 3.4, Proposition 3.6 in [Tak] and Corollary 3 in [AB]).

Corollary 3.3.2. Let L be a fixed polarization on a ruled surface $\pi : X \longrightarrow C$. Then, for any stable bundle W of rank r on C, the pullback bundle $\pi^*(W)$ is slope L-stable bundle on X . Conversely, if V is a slope L-stable vector bundle of rank r on X with $c_1(V) = \pi^*\big([\mathbf{d}]\big)$ for some divisor $\mathbf{d}$ on C and $c_2(V) = 0$, then $V \cong \pi^*(W)$ for some slope stable vector bundle W on C.

3.4 An Example

Consider the following example from [BG]. Let C be a smooth complex projective curve of genus $g(C) \geq 2$. Let $E = K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}$, where $K_C^{\frac{1}{2}}$ is a square root of the canonical bundle K_C. Note, $K_C^2 \cong \mathrm{Hom}(K_C^{-\frac{1}{2}}, K_C^{\frac{1}{2}} \otimes K_C)$. Then, we obtain a Higgs field ψ on E by setting,

$$\psi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$$

where $\omega \neq 0 \in \text{Hom}\ (C, K_C^2)$ and 1 is the identity section of trivial bundle $\text{Hom}(K_C^{\frac{1}{2}}, K_C^{-\frac{1}{2}} \otimes K_C)$. Now, (E, ψ) is a stable Higgs bundle, as $K_C^{\frac{1}{2}}$ is not ψ-invariant and there is no subbundle of positive degree which is preserved by ψ. However, E is not slope semistable. Let $\pi : X \longrightarrow C$ be a ruled surface. In such cases, by Theorem 3.3.1, the pullback of these non-trivial stable Higgs bundle on C will prove the existence of non-trivial stable Higgs bundles on the ruled surface X whose underlying vector bundles are not slope stable.

3.5 Isomorphism of Moduli spaces

Let $\mathbf{d}$ be a degree d divisor on a curve C and $\pi : X \longrightarrow C$ be a ruled surface on C with a fixed polarization L on X. Recall that the moduli space of S-equivalence classes of Higgs L-semistable bundles $\mathcal{V} = (V, \theta)$ of rank r on X, having $c_1(V) = \pi^*\big([\mathbf{d}]\big)$ and $c_2(V) = 0$, is denoted by $\mathcal{M}_X^{Higgs}\big(r, df, 0, L\big)$. We also denote the moduli space of S-equivalence classes of semistable Higgs bundles of rank r and degree d on C by $\mathcal{M}_C^{Higgs}\big(r, d\big)$.

We have the following theorem which is a corollary to the theorems proved in previous sections in this chapter.

Theorem 3.5.1. The moduli spaces $\mathcal{M}_X^{Higgs}\big(r, df, 0, L\big)$ and $\mathcal{M}_C^{Higgs}\big(r, d\big)$ are isomorphic as algebraic varieties.

Proof. Let $M_X^{Higgs}\big(r, df, 0, L\big)$ and $M_C^{Higgs}\big(r, d\big)$ denote the moduli functors whose corresponding coarse moduli spaces are $\mathcal{M}_X^{Higgs}\big(r, df, 0, L\big)$ and $\mathcal{M}_C^{Higgs}\big(r, d\big)$ respectively. For a given finite-type scheme T over k, $M_X^{Higgs}\big(r, df, 0, L\big)(T)$ is the set of equivalence classes of flat families of L-semistable Higgs Bundles on X of rank r with $c_1(V) = \pi^*\big([\mathbf{d}]\big)$ and $c_2(V) = 0$ parametrized by T. A family parametrized by T corresponding to $M_X^{Higgs}\big(r, df, 0, L\big)$ is a pair $(\mathcal{F}, \psi)$ where $\mathcal{F}$ is a coherent sheaf on $X \times T$, flat over T and $\psi \in \mathrm{Hom}\big(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_{X\times T}} p_1^*(\Omega_X^1)\big)$, where p_1 denotes the first projection map from $X \times T$ to X. Further, for every closed point $t \in T$, we have for the natural embedding $t : X \hookrightarrow X \times T$, the pair $\big(F_t, \psi_t\big) := \big(t^*(F), t^*(\psi)\big)$ is an L-semistable Higgs bundle of rank r with $c_1\big(F_t\big) = \pi^*\big([\mathbf{d}]\big)$ and $c_2\big(F_t\big) = 0$. Let $\pi_T := \pi \otimes id_T : X \times T \longrightarrow C \times T$. Then, from Theorem 3.2.4, we get a flat family $\big(\mathcal{G}, \phi\big) := \big((\pi_T)_*(\mathcal{F}), (\pi_T)_*(\psi)\big)$ parametrized by T corresponding to $M_C^{Higgs}\big(r, d\big)$ such that $\big(\mathcal{F}_t, \psi_t\big) \cong \big(\pi^*(\mathcal{G}_t), d\pi(\phi_t)\big)$ with $\deg\big(\mathcal{G}_t\big) = d$ and $\big(\mathcal{G}_t, \phi_t\big)$ is a semistable Higgs bundle for every closed point $t \in T$. $\big[$ Here for every closed point $t \in T$ and for the natural embedding $\tilde{t} : C \longrightarrow C \times T$, we define $\big(\mathcal{G}_t, \phi_t\big) := \big(\tilde{t}^*(\mathcal{G}), \tilde{t}^*(\phi)\big)\ \big]$. So, we get a natural transformation of functors

$$\pi_* : M_X^{Higgs}\big(r, df, 0, L\big) \longrightarrow M_C^{Higgs}\big(r, d\big)$$

Similarly, starting from a flat family $(\mathcal{G}, \phi)$ of semistable Higgs bundles parametrized by T with $\deg\big(\mathcal{G}_t\big) = d$ and $\mathrm{rank}\big(\mathcal{G}_t\big) = r$ for every closed point t of T, and by using Theorem 3.2.1, we can get a flat family $(\mathcal{F}, \psi)$ of L-semistable Higgs bundles on X parametrized by T such that for every closed point t in T, $c_1\big(\mathcal{F}_t\big) = \pi^*\big([\mathbf{d}]\big)$, $c_2\big(\mathcal{F}_t\big) = 0$ and $\big(\mathcal{F}_t, \psi_t\big) \cong \big(\pi^*(\mathcal{G}_t), d\pi(\phi_t)\big)$. So, we get a natural transformation of functors

$$\pi^* : M_C^{Higgs}\big(r, d\big) \longrightarrow M_X^{Higgs}\big(r, df, 0, L\big)$$

By construction, $\pi_* \circ \pi^*$ and $\pi^* \circ \pi_*$ are identity transformations on $M_C^{Higgs}\big(r, d\big)$ and $M_X^{Higgs}\big(r, df, 0, L\big)$ respectively. Hence, the corresponding coarse moduli spaces are also isomorphic. $\square$

Chapter 4

Nef cone and Pseudoeffective cone of the fibre product

4.1 Geometry of products of projective bundles over curves

In this chapter, we compute the nef cones $\text{Nef}\big(\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)\big)$ and pseudoeffective cones $\overline{\text{Eff}}\big(\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)\big)$ under the assumption that both E_1 and E_2 are slope semistable, and in a few other cases, e.g.; rank(E_1) =rank(E_2) = 2 (see [KMR]).

Let E_1 and E_2 be two vector bundles over a smooth curve C of rank r_1, r_2 and degrees d_1, d_2 respectively. Let $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$ be the associated projective bundle together with the projection morphisms $\pi_1 : \mathbb{P}(E_1) \longrightarrow C$ and $\pi_2 : \mathbb{P}(E_2) \longrightarrow C$ respectively. Let $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ be the fibre product over C. Consider the following commutative diagram:

$$\begin{array}{ccc} X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2) & \xrightarrow{p_2} & \mathbb{P}(E_2) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{P}(E_1) & \xrightarrow{\pi_1} & C \end{array}$$

Note that, $X \cong \mathbb{P}\big(\pi_1^*(E_2)\big) \cong \mathbb{P}\big(\pi_2^*(E_1)\big)$. Let f_1, f_2, g_1, g_2 and F be the numerical equivalence classes of the fibres of the maps π_1, π_2, p_1, p_2 and $\pi_1 \circ p_1 = \pi_2 \circ p_2$ respectively. We first fix the following notations for the numerical equivalence classes,

$$\eta_1 = \big[\mathcal{O}_{\mathbb{P}(E_1)}(1)\big] \in N^1\big(\mathbb{P}(E_1)\big)_{\mathbb{R}} \quad , \quad \eta_2 = \big[\mathcal{O}_{\mathbb{P}(E_2)}(1)\big] \in N^1\big(\mathbb{P}(E_2)\big)_{\mathbb{R}} \ ,$$

$$\xi_1 = \big[\mathcal{O}_{\mathbb{P}(\pi_1^*(E_2))}(1)\big] \quad , \quad \xi_2 = \big[\mathcal{O}_{\mathbb{P}(\pi_2^*(E_1))}(1)\big] \in N^1(X)_{\mathbb{R}} \ ,$$

$$\zeta_1 = p_1^*(\eta_1) \quad , \quad \zeta_2 = p_2^*(\eta_2) \in N^1(X)_{\mathbb{R}} \ .$$

It is a well known fact that (see [Har], Chapter II)

$$\mathrm{Pic}\big(\mathbb{P}(E_1)\big) = \pi_1^*\big(\mathrm{Pic}(C)\big) \oplus \mathbb{Z}\mathcal{O}_{\mathbb{P}(E_1)}(1) \ , \ \mathrm{Pic}\big(\mathbb{P}(E_2)\big) = \pi_2^*\big(\mathrm{Pic}(C)\big) \oplus \mathbb{Z}\mathcal{O}_{\mathbb{P}(E_2)}(1),$$

$$N^1\big(\mathbb{P}(E_1)\big)_{\mathbb{R}} = \mathbb{R}\eta_1 \oplus \mathbb{R}f_1 \ , \ N^1\big(\mathbb{P}(E_2)\big)_{\mathbb{R}} = \mathbb{R}\eta_2 \oplus \mathbb{R}f_2.$$

From these facts, it can be easily concluded that

$$\mathrm{Pic}(X) = p_1^*\big(\mathrm{Pic}(\mathbb{P}(E_1))\big) \oplus \mathbb{Z}\mathcal{O}_X(1) = p_1^*\big(\pi_1^*((\mathrm{Pic}(C))\big) \oplus \mathbb{Z}p_1^*\big(\mathcal{O}_{\mathbb{P}(E)}(1)\big) \oplus \mathbb{Z}\mathcal{O}_X(1),$$

$$N^1(X)_{\mathbb{R}} = \mathbb{R}\xi_1 \oplus \mathbb{R}\zeta_1 \oplus \mathbb{R}F.$$

From the Chern polynomial of ζ_1 , we get

$$\text{(4.1)} \qquad \eta_1^{r_1} - \big(\deg(E_1)\big)f_1 \cdot \eta_1^{r_1-1} = 0 \ , \ \Longrightarrow \zeta_1^{r_1} = \big(\deg(E_1)\big)F \cdot \zeta_1^{r_1-1}$$

Similarly, from the Chern class of ζ_2 , we get

$$\text{(4.2)} \qquad \eta_2^{r_2} - \big(\deg(E_2)\big)f_2 \cdot \eta_2^{r_2-1} = 0 \ , \ \Longrightarrow \zeta_2^{r_2} = \big(\deg(E_2)\big)F \cdot \zeta_2^{r_2-1}$$

We also have $\mathcal{O}_{\mathbb{P}(E_1)}(1)|_{\pi_1^{-1}(x)} \cong \mathcal{O}_{\mathbb{P}^{r_1-1}}(1)$ and $\mathcal{O}_{\mathbb{P}(E_2)}(1)|_{\pi_2^{-1}(x)} \cong \mathcal{O}_{\mathbb{P}^{r_2-1}}(1)$ for any $x \in C$. Note that each fibre Γ of the map $\pi_1 \circ p_1 = \pi_2 \circ p_2$ is isomorphic to

$\mathbb{P}^{r_1-1} \times \mathbb{P}^{r_2-1}$. Hence, $\text{Pic}(\Gamma) \cong \mathbb{Z} \oplus \mathbb{Z}$, $N^1(\Gamma)_{\mathbb{Z}} \cong \mathbb{Z} \oplus \mathbb{Z}$, and

$$p_1^*\big(\mathcal{O}_{\mathbb{P}(E_1)}(1)\big)|_\Gamma \cong (1,0) \in \text{Pic}(\Gamma) \quad , \quad p_2^*\big(\mathcal{O}_{\mathbb{P}(E_2)}(1)\big)|_\Gamma \cong (0,1) \in \text{Pic}(\Gamma)$$

Note that according to our notations, the numerical equivalence class of each fibre, $[\Gamma] = F \in N^1(X)_{\mathbb{R}}$. Let $i : \Gamma \hookrightarrow X$ be the inclusion. Then, $i^*(\zeta_1) = \big[i^*\big(p_1^*(\mathcal{O}_{\mathbb{P}(E_1)}(1)\big)\big] = \big[p_1^*\big(\mathcal{O}_{\mathbb{P}(E_1)}(1)\big)|_\Gamma\big] = \big[(1,0)\big] \in N^1(\Gamma)_{\mathbb{Z}}$. Similarly, $i^*(\zeta_2) = \big[(0,1)\big] \in N^1(\Gamma)_{\mathbb{Z}}$.

From (4.1), we get,

$$\zeta_1^{r_1} \cdot \zeta_2^{r_2-1} = \big(\deg(E_1)\big)F \cdot \zeta_1^{r_1-1} \cdot \zeta_2^{r_2-1} = \deg(E_1)\big(i^*(\zeta_1^{r_1-1})\big) \cdot \big(i^*(\zeta_2^{r_2-1})\big),$$

$$= \deg(E_1)\big[(1,0)\big]^{r_1-1} \cdot \big[(0,1)\big]^{r_2-1} = \deg(E_1)$$

Similarly,$\zeta_2^{r_2} \cdot \zeta_1^{r_1-1} = \deg(E_2)$, and $F^2 = F \cdot F = p_1^*(f_1) \cdot p_1^*(f_1) = p_1^*(f_1 \cdot f_1) = p_1^*(f_1^2) = 0$.

We know, $\mathcal{O}_{\mathbb{P}(\pi_1^*(E_2))}(1)\big|_{p_1^{-1}(x)} \cong \mathcal{O}_{\mathbb{P}^{r_2-1}}(1)$ for any $x \in \mathbb{P}(E_1)$. Considering the Chern polynomial of ξ_1, we get,

$$\xi_1^{r_2} - \deg(E_2)F \cdot \xi_1^{r_2-1} = 0 \tag{4.3}$$

From (4.3) we get, $\xi_1^{r_2} \cdot F = \deg(E_2)F^2 \cdot \xi_1^{r_2-1} = 0$ and using this, we have

$$\xi_1^{r_2+1} = \deg(E_2)\xi^{r_2} \cdot F = 0.$$

A similar calculation will show that, $\xi_2^{r_1} \cdot F = 0$, $\xi_2^{r_1+1} = 0$.

From (4.3) we also get, $\xi_1^{r_1+r_2-2} \cdot F = \deg(E_2)F^2 \cdot \xi_1^{r_1+r_2-3} = 0$.

Assume $\mathcal{O}_{\mathbb{P}(\pi_1^*(E_2))}(1)|_\Gamma \cong (a,1) \in \text{Pic}(\Gamma)$. Then

$\xi_1^{r_1+r_2-2} \cdot F = \big(i^*(\xi_1^{r_1-1})\big) \cdot \big(i^*(\xi_1^{r_2-1})\big)$

$$= \big[(a,1)\big]^{r_1-1} \cdot \big[(a,1)\big]^{r_2-1} = \big[(a,1)\big]^{r_1+r_2-2}.$$

Therefore, $a = 0$, $\implies \mathcal{O}_{\mathbb{P}(\pi_1^*(E_2))}(1)|_\Gamma \cong (0,1) \in \mathrm{Pic}(\Gamma)$.

A similar calculation will show that, $\mathcal{O}_{\mathbb{P}(\pi_2^*(E_1))}(1)|_\Gamma \cong (1,0) \in \mathrm{Pic}(\Gamma)$.

We here summarize all the intersection product that has been discussed above:

$$\xi_1^{r_2} \cdot F = 0\ ,\ \xi_1^{r_2+1} = 0\ ,\ \xi_2^{r_1} \cdot F = 0\ ,\ \xi_2^{r_1+1} = 0\ ,\ F^2 = 0,$$

$$\zeta_1^{r_1} = \big(\deg(E_1)\big)F \cdot \zeta_1^{r_1-1}\ ,\ \zeta_2^{r_2} = \big(\deg(E_2)\big)F \cdot \zeta_2^{r_2-1}\quad ,$$

$$\zeta_1^{r_1} \cdot \zeta_2^{r_2-1} = \deg(E_1)\quad ,\quad \zeta_2^{r_2} \cdot \zeta_1^{r_1-1} = \deg(E_2).$$

$$\zeta_1^{r_1} \cdot F = 0\ ,\ \zeta_2^{r_2} \cdot F = 0\ .$$

Theorem 4.1.1. With the notations as above, $\xi_1 = \zeta_2$ and $\xi_2 = \zeta_1$ in $N^1(X)_\mathbb{R}$.

Proof. Let $\zeta_2 = x\xi_1 + y\zeta_1 + zF$ for $x, y, z \in \mathbb{R}$.

Multiplying both sides by $F \cdot \xi_1^{r_2-2} \cdot \zeta_1^{r_1-1}$ and using the fact that $F^2 = 0$, we get

$$F \cdot \xi_1^{r_2-2} \cdot \zeta_1^{r_1-1} \cdot \zeta_2 = x\big(F \cdot \xi_1^{r_2-1} \cdot \zeta_1^{r_1-1}\big) + y\big(F \cdot \xi_2^{r_2-2} \cdot \zeta_1^{r_1}\big)\ ,$$

$$\implies i^*\big(\xi_1^{r_2-2}\big) \cdot i^*\big(\zeta_2\big) \cdot i^*\big(\zeta_1^{r_1-1}\big) = x\big\{i^*\big(\xi_1^{r_2-1}\big) \cdot i^*\big(\zeta_1^{r_1-1}\big)\big\} + y\big(\xi_2^{r_2-2} \cdot F \cdot \zeta_1^{r_1}\big),$$

$$\implies \big[(0,1)\big]^{r_2-2} \cdot \big[(0,1)\big] \cdot \big[(1,0)\big]^{r_1-1} = x\big\{\big[(0,1)\big]^{r_2-1} \cdot \big[(1,0)\big]^{r_1-1}\big\},$$

$$\implies x = 1.$$

Similarly, multiplying both sides of $\zeta_2 = x\xi_1 + y\zeta_1 + zF$ by $F \cdot \xi_1^{r_2-1} \cdot \zeta_1^{r_1-2}$ and using $F^2 = 0$, we get

$$F \cdot \xi_1^{r_2-1} \cdot \zeta_{r_1-2} \cdot \zeta_2 = x\big(F \cdot \xi_1^{r_2} \cdot \zeta_1^{r_1-2}\big) + y\big(F \cdot \xi_1^{r_2-1} \cdot \zeta_1^{r_1-1}\big),$$

$$\implies i^*\big(\xi_1^{r_2-1}\big) \cdot i^*\big(\zeta_2\big) \cdot i^*\big(\zeta_1^{r_1-2}\big) = y,$$

$$\implies \big[(0,1)\big]^{r_2} \cdot \big[(1,0)\big]^{r_1-2} = y,$$

$$\implies y = 0.$$

Now, from (4.2) we have $\zeta_2^{r_2} = \big(\deg(E_2)\big)F \cdot \zeta_2^{r_2-1}$,

$\implies \zeta_2^{r_2} \cdot \xi_2^{r_1-1} = \big(\deg(E_2)\big)F \cdot \zeta_2^{r_2-1} \cdot \xi_2^{r_1-1}$,

$\implies \zeta_2^{r_2} \cdot \xi_2^{r_1-1} = \big(\deg(E_2)\big)\big\{\big[(0,1)\big]^{r_2-1} \cdot \big[(1,0)\big]^{r_1-1}\big\} = \deg(E_2)$.

Similarly, from (4.3) we get $\xi_1^{r_2} = \big(\deg(E_2)\big)F \cdot \xi_1^{r_2-1}$,

$\implies \xi_1^{r_2} \cdot \xi_2^{r_1-1} = \big(\deg(E_2)\big)F \cdot \xi_1^{r_2-1} \cdot \xi_2^{r_1-1}$,

$\implies \xi_1^{r_2} \cdot \xi_2^{r_1-1} = \big(\deg(E_2)\big)\big\{\big[(0,1)\big]^{r_2-1} \cdot \big[(1,0)\big]^{r_1-1}\big\} = \deg(E_2)$.

Hence, we have $\zeta_2^{r_2} = \big(\xi_1 + zF\big)^{r_2} = \xi_1^{r_2} + r_2 z\big(F \cdot \xi_1^{r_2-1}\big)$,

$\implies \zeta_2^{r_2} \cdot \xi_2^{r_1-1} = \big(\xi_1^{r_2} \cdot \xi_2^{r_1-1}\big) + r_2 z\big(F \cdot \xi_1^{r_2-1}\xi_2^{r_1-1}\big)$

$\implies \deg(E_2) = \deg(E_2) + r_2 z\big\{\big[(0,1)\big]^{r_2-1} \cdot \big[(1,0)\big]^{r_1-1}\big\} = \deg(E_2) + r_2 z$,

$\implies z = 0$.

This proves our result i.e. $\zeta_2 = \xi_1 \in N^1(X)_{\mathbb{R}}$. A similar calculation will also show that $\zeta_1 = \xi_2 \in N^1(X)_{\mathbb{R}}$. □

Theorem 4.1.2. With the notations as above, the dual basis of $N_1(X)_{\mathbb{R}}$ is $\big\{\delta_1, \delta_2, \delta_3\big\}$ where $\delta_1 = F \cdot \zeta_1^{r_1-2} \cdot \zeta_2^{r_2-1}$, $\delta_2 = F \cdot \zeta_1^{r_1-1} \cdot \zeta_2^{r_2-2}$ and
$\delta_3 = \zeta_1^{r_1-1} \cdot \zeta_2^{r_2-1} - \deg(E_1)F \cdot \zeta_1^{r_1-2} \cdot \zeta_2^{r_2-1} - \deg(E_2)F \cdot \zeta_1^{r_1-1} \cdot \zeta_2^{r_2-2}$.

4.2 Nef cones of the fibre Product

Let C be a smooth irreducible curve over $\mathbb{C}$. Recall that for a vector bundle E of rank r on C, slope of E is defined as,

$$\mu(E) := \frac{\deg(E)}{r} \in \mathbb{Q}$$

A vector bundle E on C is said to be slope semistable if $\mu(F) \leq \mu(E)$ for all subbundle $F \subseteq E$. E is called slope unstable if it is not slope semistable.

4.2.1 Case I : When both E_1 and E_2 are slope semistable

Theorem 4.2.1. Let E_1 and E_2 be two slope semistable vector bundles on a smooth irreducible projective curve C over $\mathbb{C}$ with slopes μ_1 and μ_2 respectively. Consider the following commutative diagram:

$$\begin{array}{ccc} X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2) & \xrightarrow{p_2} & \mathbb{P}(E_2) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{P}(E_1) & \xrightarrow{\pi_1} & C \end{array}$$

We use the same notations as in Section 4.1. Then,

$$\mathrm{Nef}(X) = \overline{\mathrm{Eff}}(X) = \big\{ a\lambda_1 + b\lambda_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0} \big\},$$

where $\lambda_1 = \zeta_1 - \mu_1 F$ and $\lambda_2 = \zeta_2 - \mu_2 F$.

Proof. From the discussions in Section 4.1, $N^1(X)_{\mathbb{R}} = \mathbb{R}\xi_1 \oplus \mathbb{R}\zeta_1 \oplus \mathbb{R}F$. As E_1 and E_2 both are slope semistable vector bundles on C, by Theorem 3.1 in [Miy], $\nu_1 = \eta_1 - \mu_1 f_1$ and $\nu_2 = \eta_2 - \mu_2 f_2$ both are nef classes. Since pullback of a nef class under a proper map is also nef (see Example 1.4.4 in [Laz1]), $\lambda_1 = p_1^*(\nu_1) = \zeta_1 - \mu_1 F, \lambda_2 = p_2^*(\nu_2) = \zeta_2 - \mu_2 F$ and $F = p_1^*(f_1)$ are all nef classes in $N^1(X)_{\mathbb{R}}$. Hence,

$$\big\{ a\lambda_1 + b\lambda_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0} \big\} \subseteq \mathrm{Nef}(X)$$

Let $D = \mathcal{O}_{\mathbb{P}(\pi_1^*(E_2))}(l) \otimes p_1^*\mathcal{O}_{\mathbb{P}(E_1)}(m) \otimes p_1^*\big(\pi_1^*(\mathcal{M})\big)$ be an integral effective divisor on

X, where $\mathcal{M}$ is a line bundle of degree n on C. Hence,

$$H^0\big(X, \mathcal{O}_{\mathbb{P}(\pi_1^*(E_2))}(l) \otimes p_1^*\mathcal{O}_{\mathbb{P}(E_1)}(m) \otimes p_1^*(\pi_1^*(\mathcal{M}))\big) \neq 0.$$

Note that this implies $l, m \geq 0$. Now, by projection formula, we have

$$H^0\big(X, \mathcal{O}_{\mathbb{P}(\pi_1^*(E_2))}(l) \otimes p_1^*\mathcal{O}_{\mathbb{P}(E_1)}(m) \otimes p_1^*(\pi_1^*(\mathcal{M}))\big)$$

$$= H^0\big(\mathbb{P}(E_1), S^l(\pi_1^*(E_2)) \otimes \mathcal{O}_{\mathbb{P}(E_1)}(m) \otimes \pi_1^*(\mathcal{M})\big)$$

$$= H^0\big(\mathbb{P}(E_1), \pi_1^*(S^l(E_2)) \otimes \mathcal{O}_{\mathbb{P}(E_1)}(m) \otimes \pi_1^*(\mathcal{M})\big)$$

$$= H^0\big(C, S^l(E_2) \otimes S^m(E_1) \otimes \mathcal{M}\big) \neq 0.$$

Consider the following exact sequence given by a non-zero global section of $S^l(E_2) \otimes S^m(E_1) \otimes \mathcal{M}$,

$$0 \longrightarrow \mathcal{O}_C \longrightarrow S^l(E_2) \otimes S^m(E_1) \otimes \mathcal{M}$$

Since E_1 and E_2 both are slope semistable and the ground field $\mathbb{C}$ has characteristic zero, $S^l(E_2) \otimes S^m(E_1) \otimes \mathcal{M}$ is also slope semistable (see Corollary 3.2.10 in [HL]). Hence,

$\deg\big(S^l(E_2) \otimes S^m(E_1) \otimes \mathcal{M}\big) \geq 0.$

Also, note that $\operatorname{rank}\big(S^m(E_1)\big) = \binom{m+r_1-1}{r_1-1}$ and $\det\big(S^m(E_1)\big) = \det(E_1)^{\otimes\binom{m+r_1-1}{r_1}}$.

Hence, $\deg\big(S^m(E_1)\big) = \binom{m+r_1-1}{r_1} \cdot \deg(E_1) = \operatorname{rank}\big(S^m(E_1)\big)m\mu_1$.

Similarly, $\deg\big(S^l(E_2)\big) = \binom{l+r_2-1}{r_2} \cdot \deg(E_2) = \operatorname{rank}\big(S^l(E_2)\big)l\mu_2$.

Therefore, $\deg\big(S^l(E_2) \otimes S^m(E_1) \otimes \mathcal{M}\big)$

$= \operatorname{rank}\big(S^m(E_1)\big)\deg\big(S^l(E_2)\big) + \operatorname{rank}\big(S^l(E_2)\big)\big\{\deg\big(S^m(E_1)\big) \otimes \mathcal{M}\big\}$,

$= \operatorname{rank}\big(S^m(E_1)\big)\operatorname{rank}\big(S^l(E_2)\big)l\mu_2 + \operatorname{rank}\big(S^l(E_2)\big)\big\{\operatorname{rank}\big(S^m(E_1)\big)m\mu_1 + \operatorname{rank}\big(S^m(E_1)\big)\deg(\mathcal{M})\big\}$,

$= \operatorname{rank}\big(S^m(E_1)\big)\operatorname{rank}\big(S^l(E_2)\big)\big\{l\mu_2 + m\mu_1 + n\big\} \geq 0.$

Hence, we have, $l \geq 0, m \geq 0, \big(l\mu_2 + m\mu_1 + n\big) \geq 0$. Now, using Theorem 4.1.1 we also get,

$$\big[D\big] = l\xi_1 + m\zeta_1 + nF = l\zeta_2 + m\zeta_1 + nF \in N^1(X)_{\mathbb{R}},$$

$$= l(\zeta_2 - \mu_2 F) + m(\zeta_1 - \mu_1 F) + \big(l\mu_2 + m\mu_1 + n\big)F$$

$$= m\lambda_1 + l\lambda_2 + \big(l\mu_2 + m\mu_1 + n\big)F \in \operatorname{Nef}(X).$$

Hence, $\operatorname{Eff}(X) \subseteq \big\{a\lambda_1 + b\lambda_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\} \subseteq \operatorname{Nef}(X)$.
Since $\operatorname{Nef}(X) \subseteq \overline{\operatorname{Eff}}(X)$, taking closure we get,

$$\overline{\operatorname{Eff}}(X) = \big\{a\lambda_1 + b\lambda_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\} = \operatorname{Nef}(X). \tag{4.4}$$

□

Corollary 4.2.2. With the same hypothesis as in Theorem 4.2.1, the closed cone of curves in X is given by,

$$\overline{\operatorname{NE}}(X) = \big\{x\delta_1 + y\delta_2 + z\delta_3 \in N_1(X)_{\mathbb{R}} \mid x, y, z \in \mathbb{R}, (x - \mu_1 z) \geq 0, (y - \mu_2 z) \geq 0, z \geq 0\big\}$$

where $\delta_1, \delta_2, \delta_3$ are as in Theorem 4.1.2.

4.2.2 Case II : When neither E_1 nor E_2 is slope semistable and rank(E_1) = rank(E_2) = 2

Recall that a vector bundle E of rank 2 on an algebraic curve C is said to be normalized if $H^0(E) \neq 0$, but $H^0(E \otimes L) = 0$ for all line bundle L on C with $\deg(L) < 0$.

Lemma 4.2.3. Let E be a normalized vector bundle of rank 2 on a smooth irreducible complex curve C. Then, E is slope semistable if and only if $\deg(E) \geq 0$.

Proof. Let E be a slope semistable normalized rank 2 bundle on C. Hence, $H^0\big(C, E\big) \neq 0$ implies that $\mathcal{O}_C$ is a subbundle of E of slope 0 . E being slope semistable, $\deg(E) \geq \mu(E) \geq 0$.

Conversely, let E be a rank 2 normalized bundle on C with $\deg(E) \geq 0$. Suppose E is unstable bundle and it admits the following Harder-Narasimhan filtration

$$0 \subsetneq M \subsetneq E$$

where M is a sub-line bundle of E with $\deg(E/M) = \mu(E/M) < \mu(M) = \deg(M)$. Our claim is that $\deg(M) = 0$. If not, then either $\deg(M) > 0$ or $\deg(M) < 0$.

Case (i) : Let $\deg(M) > 0$ so that $\deg(M^{-1}) < 0$. Consider the following exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow E/M \longrightarrow 0$$

Tensoring the above exact sequence by M^{-1}, we get

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \otimes M^{-1} \longrightarrow \big(E/M\big) \otimes M^{-1} \longrightarrow 0$$

which implies $H^0(C, E \otimes M^{-1}\big) \neq 0$. This gives a contradiction as E is normalized.

Case (ii) : Let $\deg(M) < 0$. Then, $\deg(E/M) < \deg(M) < 0$ which implies $H^0(C, E/M) = H^0(C, M) = 0$. From the exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow E/M \longrightarrow 0$$

we then get $H^0(C, E) = 0$ which contradicts the fact that E is normalized.

Combining both the cases, we prove our claim i.e. $\deg(M) = 0$. Now, from the Harder-Narasimhan filtration of E, we have $\deg(E) = \deg(M) + \deg(E/M) < 0$

which contradicts our hypothesis. Hence, E is slope semistable. $\square$

Remark 3. By Theorem 2.5.11, any indecomposable normalized rank 2 bundle E over an elliptic curve C is one of the following type :

either (i) $0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow \mathcal{O}_C \longrightarrow 0$

or (ii) $0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow \mathcal{O}_C(p) \longrightarrow 0$ for some closed point $p \in C$.

In both cases, $\deg(E) \geq 0$. Hence, by Lemma 4.2.3, the above bundles are slope semistable normalized bundles over the elliptic curve C.

Recall that, for every vector bundle $\mathcal{E}$ over C, there is a unique filtration

$$0 = \mathcal{E}_l \subsetneq \mathcal{E}_{l-1} \subsetneq \mathcal{E}_{l-2} \subsetneq \cdots\cdots \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_0 = \mathcal{E}$$

called the *Harder-Narasimhan filtration* , such that $\mathcal{E}_i/\mathcal{E}_{i+1}$ is slope semistable for each $i \in \{0, 1, ..., l-1\}$ and $\mu(\mathcal{E}_i/\mathcal{E}_{i+1}) > \mu(\mathcal{E}_{i-1}/\mathcal{E}_i)$ for all $i \in \{1, 2, ..., l-1\}$.

Theorem 4.2.4. Let E_1 and E_2 be two normalized rank 2 vector bundles on a smooth irreducible complex projective curve C such that neither E_1 nor E_2 is semistable. Consider the following commutative diagram:

$$\begin{array}{ccc} X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2) & \xrightarrow{p_2} & \mathbb{P}(E_2) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{P}(E_1) & \xrightarrow{\pi_1} & C \end{array}$$

We use the same notations as in Section 4.1. Then

$$\mathrm{Nef}(X) = \left\{ a\tau_1 + b\tau_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0} \right\}$$

where $l_1 = \deg(E_1), l_2 = \deg(E_2)$ and $\tau_1 = \zeta_1 - l_1 F$ and $\tau_2 = \zeta_2 - l_2 F \in N^1(X)_{\mathbb{R}}$.

Proof. Let $0 \subsetneq L_i \subsetneq E_i$ be the Harder-Narasimhan filtration of E_i such that $E_i/L_i =$

Q_i for $i = 1, 2$. Since $\mathbb{P}(E_i) = \mathbb{P}(E_i \otimes L_i^{-1})$ for each i, we can consider

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_1 \longrightarrow Q_1 \longrightarrow 0 \tag{4.5}$$

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_2 \longrightarrow Q_2 \longrightarrow 0 \tag{4.6}$$

Hence, $l_i = \deg(E_i) = \deg(Q_i) < 0$. for each i. By Lemma in [Fulg],

$$\mathrm{Nef}\big(\mathbb{P}(E_i)\big) = \big\{a(\eta_i - l_i f_i) + bf_i \mid a, b \in \mathbb{R}_{\geq 0}\big\} \quad \text{for } i = 1, 2.$$

Again, using the fact that pullback of a nef class under a proper map is nef, we get $\tau_1 = p_1^*(\eta_1) - l_1 F = \zeta_1 - l_1 F$, $\tau_2 = p_2^*(\eta_2) - l_2 F = \zeta_2 - l_2 F$, F are all nef classes. Hence,

$$\big\{a\tau_1 + b\tau_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\} \subseteq \mathrm{Nef}(X).$$

For each $i \in 1, 2$, let $\mathbb{P}(Q_i) \hookrightarrow \mathbb{P}(E_i)$ be the canonical embeddings corresponding to $E_i \longrightarrow Q_i \longrightarrow 0$.

We also know that $\overline{\mathrm{NE}}\big(\mathbb{P}(E_i)\big) = \overline{\mathrm{Eff}}\big(\mathbb{P}(E_i)\big) = \big\{a_i + bf_i \mid a, b \in \mathbb{R}_{\geq 0}\big\}$ for each $i \in \{1, 2\}$. Now, consider the following numerical equivalence class of a 1-cycle in X,

$$C_1 = p_1^*(\eta_1) \cdot p_2^*(\eta_2) = \mathbb{P}(Q_1) \times_C \mathbb{P}(Q_2) \in \overline{\mathrm{NE}}(X)$$

Note that, p_1 and p_2 are proper, flat morphisms, and as the base space is smooth, p_1, p_2 are also smooth. Hence, numerical pullbacks of cycles are well defined (see [Ngu]) and preserve the pseudo-effectivity. Now, as $\tau_1 \cdot C_1 = 0$, $\tau_2 \cdot C_1 = 0$ and $F^2 = 0$ (using the intersection products discussed in Section 4.1) ; τ_1, τ_2 and F are in the boundary of $\text{Nef}(X)$.

Let $a\tau_1+b\tau_2+cF \in \text{Nef}(X) \subseteq \overline{\text{Eff}}(X)$. Then $\big(a\tau_1+b\tau_2+cF\big)\cdot C_1 = c \geq 0$. Also, $F\cdot\tau_2$ and $F\cdot\tau_1$ are intersections of nef divisor classes. So, intersecting $a\tau_1+b\tau_2+cF$ with them, we get $a \geq 0$, $b \geq 0$. Hence,

$$\text{Nef}(X) = \big\{a\tau_1 + b\tau_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\} \tag{4.7}$$

□

Corollary 4.2.5. With the same hypothesis as in Theorem 4.2.4, the closed cone of curves in X,

$$\overline{\text{NE}}(X) = \big\{x\delta_1 + y\delta_2 + z\delta_3 \in N_1(X)_{\mathbb{R}} \mid x, y, z \in \mathbb{R}, (x - l_1z) \geq 0, (y - l_2z) \geq 0, z \geq 0\big\}.$$

where $\delta_1, \delta_2, \delta_3$ are as in Theorem 4.1.2.

4.2.3 Case III : When E_1 is slope semistable and E_2 is slope unstable and rank(E_1) = rank(E_2) = 2

Theorem 4.2.6. Let E_1 is a normalized slope semistable bundle of rank 2 with slope μ_1 and E_2 is slope unstable normalized bundle of rank 2 on a smooth irreducible complex projective curve C. Consider the following commutative diagram:

$$\begin{array}{ccc} X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2) & \xrightarrow{p_2} & \mathbb{P}(E_2) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{P}(E_1) & \xrightarrow{\pi_1} & C \end{array}$$

We use the same notations as in Section 4.1. Then

$$\text{Nef}(X) = \big\{a\gamma_1 + b\gamma_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\}$$

where $l_2 = \deg(E_2)$ and $\gamma_1 = \zeta_1 - \mu_1 F$ and $\gamma_2 = \zeta_2 - l_2 F$.

Proof. Let $0 \subsetneq L_2 \subsetneq E_2$ be the Harder-Narasimhan filtration of E_2 such that $E_2/L_2 = Q_2$. Since $\mathbb{P}(E_2) = \mathbb{P}(E_2 \otimes {L_2}^{-1})$, with out loss of generality, we can consider,

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_2 \longrightarrow Q_2 \longrightarrow 0 \tag{4.8}$$

so that $l_2 = \deg(E_2) = \deg(Q_2) < 0$. Since E_1 is slope semistable, by Theorem 3.1 in [Miy],

$$\mathrm{Nef}\big(\mathbb{P}(E_1)\big) = \big\{a(\eta_1 - \mu_1 f_1) + b f_1 \mid a, b \in \mathbb{R}_{\geq 0}\big\}$$

Also, by Lemma 2.1 in [Fulg], $\mathrm{Nef}\big(\mathbb{P}(E_2)\big) = \big\{a(\eta_2 - l_2 f_2) + b f_2 \mid a, b \in \mathbb{R}_{\geq 0}\big\}$ Hence, $\gamma_1 = p_1^*(\eta_1) - \mu_1 F = \zeta_1 - \mu_1 F$, $\gamma_2 = p_2^*(\eta_2) - l_2 F = \zeta_2 - l_2 F$, $F = p_1^*(f_1)$ are all nef, so that

$$\big\{a\gamma_1 + b\gamma_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\} \subseteq \mathrm{Nef}(X)$$

Since E_1 is slope semistable, $\overline{\mathrm{NE}}\big(\mathbb{P}(E_1)\big) = \overline{\mathrm{Eff}}\big(\mathbb{P}(E_1)\big) = \big\{a(\eta_1 - \mu_1 f_1) + b f_1 \mid a, b \in \mathbb{R}_{\geq 0}\big\}$ Also, $\overline{\mathrm{NE}}\big(\mathbb{P}(E_2)\big) = \overline{\mathrm{Eff}}\big(\mathbb{P}(E_2)\big) = \big\{a\eta_2 + b f_2 \mid a, b \in \mathbb{R}_{\geq 0}\big\}$. By the same argument as in Theorem 4.2.4, we get

$$C_2 = p_1^*(\eta_1 - \mu_1 f_1) \cdot p_2^*(\eta_2) \in \overline{\mathrm{NE}}(X)$$

Also, $\gamma_1 \cdot C_2 = \deg(E_1) \cdot (\mu_1 - \mu_1) = 0$, $\gamma_2 \cdot C_2 = \deg(E_2) - l_2 = 0$. Hence, γ_1 and γ_2 are not ample, and they are in the boundary of the nef cone. Therefore,

$$\mathrm{Nef}(X) = \big\{a\gamma_1 + b\gamma_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\} \tag{4.9}$$

□

Corollary 4.2.7. With the same hypothesis as in Theorem 4.2.6, the closed cone of curves in X

$$\overline{\mathrm{NE}}(X) = \big\{x\delta_1 + y\delta_2 + z\delta_3 \in N_1(X)_{\mathbb{R}} \mid x, y, z \in \mathbb{R}, (x - \mu_1 z) \geq 0, (y - l_2 z) \geq 0, z \geq 0\big\}.$$

where $\delta_1, \delta_2, \delta_3$ are as in Theorem 4.1.2.

4.3 Pseudoeffective cones of the fibre product

Theorem 4.3.1. With the same hypothesis as in Theorem 4.2.4, the pseudoeffective cone of X

$$\overline{\mathrm{Eff}}(X) = \big\{a\zeta_1 + b\zeta_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\} \tag{4.10}$$

Proof. let $D = \mathcal{O}_{\mathbb{P}(\pi_1^*(E_2))}(l) \otimes p_1^*\mathcal{O}_{\mathbb{P}(E_1)}(m) \otimes p_1^*\big(\pi_1^*(\mathcal{N})\big)$ be an integral effective divisor on X, where $\mathcal{N}$ is a line bundle of degree n on C. Hence, $H^0(X, D) \neq 0$, which will imply $l \geq 0, m \geq 0$. A similar calculation as in Theorem 4.2.1 will show,

$$H^0\big(X, \mathcal{O}_{\mathbb{P}(\pi_1^*(E_2))}(l) \otimes p_1^*\mathcal{O}_{\mathbb{P}(E_1)}(m) \otimes p_1^*\big(\pi_1^*(\mathcal{N})\big)\big)$$
$$= H^0\big(C, S^l(E_2) \otimes S^m(E_1) \otimes \mathcal{O}_C\big(\mathcal{N}\big)\big).$$

Now, our claim is that $n \geq 0$ also. To prove this, let $L_1, L_2 \in \mathrm{Nef}(X)$. From the definition of movable cone, it is clear that $L_1 \cdot L_2 \in \overline{\mathrm{Mov}}(X)$. By Theorem 4.2.4, $\tau_1 = \zeta_1 - l_1 F$ and $\tau_2 = \zeta_2 - l_2 F \in \mathrm{Nef}(X)$. Hence, we have

$$D_1 = \tau_1 \cdot \tau_2 = \big(\zeta_1 \cdot \zeta_2\big) - l_1\big(F \cdot \zeta_2\big) - l_2\big(F \cdot \zeta_1\big) \in \overline{\mathrm{Mov}}(X).$$

But, $[D] \cdot D_1 = n \geq 0$ (by using the calculations in Section 4.1). Hence, the claim is proved. Therefore, $\overline{\mathrm{Eff}}(X) \subseteq \big\{a\zeta_1 + b\zeta_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\}$.

Conversely, let $D' = \mathcal{O}_{\mathbb{P}(\pi_1^*(E_2))}(a) \otimes p_1^* \mathcal{O}_{\mathbb{P}(E_1)}(b) \otimes p_1^*\big(\pi_1^*(\mathcal{R})\big)$ be an integral divisor on X such that $[D'] = a\zeta_1 + b\zeta_2 + cF$ where $a, b, c \geq 0$ and $\deg(\mathcal{R}) = c$. As E_i's are normalized, $\mathcal{O}_C$ is subsheaf of E_i's and hence $\mathcal{O}_C$ is also subsheaf of $S^k(E_i)$ for $k \geq 0$ and $i = 1, 2$. Also, for some divisor $\mathcal{R}$ with $\deg(\mathcal{R}) \geq 0$, $\mathcal{O}_C \hookrightarrow \mathcal{O}_C(\mathcal{R})$. This implies $\mathcal{O}_C \hookrightarrow S^l(E_2) \otimes S^m(E_1) \otimes \mathcal{O}_C(\mathcal{R})$ for $a, b, c \geq 0$. Hence,

$$H^0\big(X, \mathcal{O}_{\mathbb{P}(\pi_1^*(E_2))}(a) \otimes p_1^* \mathcal{O}_{\mathbb{P}(E_1)}(b) \otimes p_1^*\big(\pi_1^*(\mathcal{R})\big)\big),$$

$$= H^0\big(C, S^a(E_2) \otimes S^b(E_1) \otimes \mathcal{O}_C(\mathcal{R})\big) \neq 0.$$

Thus, D' is an effective divisor. This proves the result. □

Theorem 4.3.2. With the same hypothesis as in Theorem 4.2.6, the pseudoeffective cone of X

$$\overline{\mathrm{Eff}}(X) = \big\{a(\zeta_1 - \mu_1 F) + b\zeta_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\} \tag{4.11}$$

Proof. In this case, $\gamma_1 = \zeta_1 - \mu_1 F$ is one of the nef boundary. Hence, ζ_2, F, $\zeta_1 - \mu_1 F \in \overline{\mathrm{Eff}}(X)$. So,

$$\big\{a(\zeta_1 - \mu_1 F) + b\zeta_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\big\} \subseteq \overline{\mathrm{Eff}}(X).$$

By the similar argument as given in Theorem 4.2.6, if $[D] = a(\zeta_1 - \mu_1 F) + b\zeta_2 + cF$ is the numerical equivalence class of an effective divisor, then $a, b \geq 0$. Now, our claim is that $c \geq 0$. From Theorem 4.2.6, we have $\zeta_1 - \mu_1 F$, $\zeta_2 - l_2 F \in \mathrm{Nef}(X)$. Hence, $C_3 = \big(\zeta_1 - \mu_1 F\big) \cdot \big(\zeta_2 - l_2 F\big) = \big(\zeta_1 \cdot \zeta_2\big) - l_2\big(F \cdot \zeta_1\big) - \mu_1\big(\zeta_2 \cdot F\big) \in \overline{\mathrm{Mov}}(X)$, so that $C_3 \cdot [D] = c \geq 0$. This proves the result. □

Remark 4. If both E_1 and E_2 are of rank 2 bundles on C, then nef cone and pseudoeffective cone of $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ coincide if and only if both E_1 and E_2 are slope semistable.

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