ON THE SPACE OF FUNCTIONS WHOSE n^{th} DIFFERENCES SATISFY LIPSCHITZ'S CONDITION

Thesis

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CHAPTER 1 : INTRODUCTION

CHAPTER 1

INTRODUCTION

In this thesis, we study the properties of the space of complex valued continuous functions on the real line R with period 1, whose one sided nth difference satisfies the Lipschitz condition.

Let $0 < \alpha < n$, where n is a positive integer. Let $\Lambda(\alpha, n)$ denote the class of all complex valued continuous functions f on the real line R, with period 1 such that there exists a constant K satisfying the condition

$$1) \quad \sup_{x \in R} \left| \Delta_t^n f(x) \right| \leq K |t|^{\alpha} \quad \text{as } t \to 0$$

where the one sided nth difference $\Delta_t^n f(x)$ is given by

$$\Delta_t^n f(x) = f(x + nt) - \binom{n}{1} f(x + \overline{n-1}\, t) + \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots + (-1)^r \binom{n}{r} f(x + \overline{n-r}\, t) + \ldots\ldots + (-1)^n f(x)$$

We denote by $\lambda(\alpha, n)$, the sub set of $\Lambda(\alpha, n)$ consisting of those functions satisfying the relation

$$2) \quad \sup_{x \in R} \left| \frac{\Delta_t^n f(x)}{t^{\alpha}} \right| \to 0 \quad \text{as } t \to 0$$

Set

$$\|f\|_{\infty} = \sup_{x \in R} |f(x)|$$

$$\|f\|_{\alpha} = \sup_{x, t \in R} \left| \frac{\Delta_t^n f(x)}{t^{\alpha}} \right|$$

and define

$$\|f\| = \text{Max}\left\{\|f\|_\alpha , \|f\|_\alpha\right\}$$

When n=1, the spaces are denoted by Lip α and lip α respectively. K. de Leeuw [7] discussed this case and specifically showed that Lip α is canonically isometrically isomorphic to the second dual of lip α . He also obtained the extreme points of the unit sphere of the dual of lip α .

Motivated by his work, we have discussed the general case and showed that the properties proved by de Leeuw are actually valid for the spaces $\Lambda(\alpha, n)$ and $\lambda(\alpha, n)$. As the nth differences occur in many situations particularly in Approximations and Interpolations, we have also initiated the problem of lacunary interpolation for entire functions.

Chapter 1 is of preliminary nature. In Chapter 2, we introduce the spaces $\Lambda(\alpha, n)$ and $\lambda(\alpha, n)$ and obtain the elementary properties thereof. In particular, we show that $\Lambda(\alpha, n)$ is a Banach space and $\lambda(\alpha, n)$ is a closed linear sub space of $\Lambda(\alpha, n)$. If $0 < \alpha < \alpha' \leq n$, then the inclusion relation

$$\Lambda(\alpha', n) \subset \lambda(\alpha, n)$$

is established and the relation among the norms are also obtained. We have also included a nontrivial example of a function of the class $\lambda(\alpha, 2)$ to exhibit the computational aspect in its verification. In the second part of this Chapter, we discuss continuously translating sub spaces. An element f in $\Lambda(\alpha, n)$ is said to translate continuously if the mapping

$$x \longrightarrow T_x f$$

is continuous from R into the Banach space $\Lambda(\alpha, n)$. We prove that the set $\mathcal{C}\Lambda(\alpha, n)$ of all continuously translating functions in $\Lambda(\alpha, n)$ is precisely $\lambda(\alpha, n)$. As a consequence we deduce that $\lambda(\alpha, n)$ is the closed linear span of trigonometric polynomials in $\lambda(\alpha, n)$.

In Chapter 3, we prove that $\Lambda(\alpha, n)$ is canonically isometrically isomorphic to the second dual of $\lambda(\alpha, n)$. This is done by a series of auxilary results. In this process, we also obtain the representation of continuous linear functionals of $\lambda(\alpha, n)$ by constructing an isometric imbedding of $\lambda(\alpha, n)$ into a space of continuous functions supplied with sup. norm.

In Chapter 4, we study the extreme points of the unit sphere of $\lambda(\alpha, 2)$ when $0 < \alpha < 1$. Unlike de Leeuw's result, the problem of finding the extreme points of the unit sphere of the dual of $\lambda(\alpha, n)$ is quite complicated and we have included the result only in the case of $\lambda(\alpha, 2)$. By the isometric imbedding of $\lambda(\alpha, n)$ into a space of continuous functions supplied with sup. norm, our consideration reduces to the corresponding problem for a linear space of continuous functions under the sup. norm.

If X is a locally compact topological space and $C_0(X)$ the space of complex valued continuous functions on X that vanish at infinity, a closed linear space A of $C_0(X)$ is a Banach space under the sup. norm. Given a point $x \in X$, a function h in A is said to peak at x relative to A if

$$|h(x)| = 1 \quad \text{and}$$

$$|h(y)| \leq 1 \quad y \in X, y \neq x$$

with equality holding only for those y in X that satisfy either

$$g(y) = g(x) \quad \text{for all } g \text{ in } A$$

or

$$g(y) = -g(x) \quad \text{for all } g \text{ in } A$$

After constructing the peaking functions, it is shown that a functional Φ in $(\lambda(\alpha, 2))^*$ is an extreme point of the unit sphere of $(\lambda(\alpha, 2))^*$ if and only if it is either of the form

$$\Phi(f) = \epsilon\, f(\rho) \qquad f \in \lambda(\alpha, 2),\ \rho \in R$$

and ϵ is a complex number with $|\epsilon| = 1$ or of the form

$$\Phi(f) = \epsilon \cdot \frac{\Delta_t^2 f(x)}{t^\alpha} \qquad f \in \lambda(\alpha, 2)$$

$x \in R$, $0 < t \leq \frac{1}{2}$ and ϵ is a complex number with $|\epsilon| = 1$. We have next discussed the isometries of $\lambda(\alpha, 2)$. If Ext. S^* denote the set of all extreme points of the unit sphere of $(\lambda(\alpha, 2))^*$ we have proved that a function f in $\lambda(\alpha, 2)$ is a constant function if and only if

$$\{\Phi(f) : \Phi \in \text{Ext. } S^*\}$$

consists of atmost two numbers. If T is a linear isometry of $\lambda(\alpha, 2)$ it is shown that there is a complex number η with $|\eta| = 1$ such that

$$\{T^* \Phi_x : x \in R\} = \{\eta \Phi_x : x \in R\}$$

In Chapter 5, we have considered a couple of problems wherein the nth differences occur. The first one is a multiplier problem. Let $L^1[0,1]$ denote the space of periodic complex valued Lebesgue measurable functions on $[0,1]$ and $L^\infty[0,1]$ consists of those

functions in L^1 which are essentially bounded. $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denote the norms in L^1 and L^∞ respectively. Let $U(\alpha, n)$ denote the sub class of L^1 consisting of those functions f whose nth difference satisfy the integral Lipschitz condition

$$\int_0^1 |\Delta_\delta^n f(x)| \, dx \leq B.\delta^\alpha$$

where B is a constant depending only on the function f, while $V(\alpha, n)$ is a sub class satisfying the condition

$$\int_0^1 |\Delta_\delta^n f(t)| \, dt = O(\delta^\alpha)$$

The multiplier class $(L^\infty, \Lambda(\alpha, n))$ (respy. $(L^\infty, \lambda(\alpha, n))$) is the collection of all those f in L^1 such that $f * g \in \Lambda(\alpha, n)$ (respy. $\lambda(\alpha, n)$ for each $g \in L^\infty$. We have shown that

$$(L^\infty, \Lambda(\alpha, n)) = U(\alpha, n)$$

and

$$(L^\infty, \lambda(\alpha, n)) = V(\alpha, n)$$

Next is the problem of lacunary interpolation of entire functions. While the problem of lacunary interpolation for polynomials and trigonometric polynomials have drawn the attention of many a mathematician for the past few decades, in the case of entire functions of exponential type, being the natural counterpart for approximants in the case of infinite interval , only the problem of interpolation, where the values of the function and the consecutive derivatives are given at prescribed points is so far known. We shall now initiate the study of lacunary interpolation for entire functions. The most elementary case of (0,2) interpolation is being persued at the moment, the remaining ones being retained for further study.

CHAPTER 2 : THE SPACES $\Lambda(\alpha, n)$ AND $\lambda(\alpha, n)$ AND THEIR PROPERTIES

1. THE SPACES $\Lambda(\alpha, n)$ AND $\lambda(\alpha, n)$

Let $0 < \alpha < n$, where n is a positive integer. Let $\Lambda(\alpha, n)$ denote the class of continuous complex valued functions f on the real line R with period 1, such that there exists a constant $K > 0$ satisfying the condition

$$(1) \quad \sup_{x \in R} |\Delta_t^n f(x)| \leq K |t|^\alpha \quad \text{as } t \to 0$$

where the one sided nth difference Δ_t^n is defined by

$$(2) \quad \Delta_t^n f(x) = f(x+nt) - \binom{n}{1} f(x+\overline{n-1}\, t) + \dots + (-1)^n f(x)$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(x + \overline{n-r}\, t)$$

We denote by $\lambda(\alpha, n)$ the subset of $\Lambda(\alpha, n)$ consisting of those functions f which satisfy the condition

$$(3) \quad \sup_{x \in R} |\Delta_t^n f(x)| = o(|t|^\alpha) \quad \text{as } t \to 0$$

Set

$$\| f \|_\infty = \sup_{x \in R} |f(x)|$$

$$\| f \|_\alpha = \sup_{x, t \in R} \frac{|\Delta_t^n f(x)|}{|t|^\alpha}$$

and define

$$(4) \quad \| f \| = \max \{ \| f \|_\infty , \| f \|_\alpha \}$$

for each $f \in \Lambda(\alpha, n)$. Then it is easy to see that (4) defines a norm on $\Lambda(\alpha, n)$.

THEOREM 1.1. $\Lambda(\alpha, n)$ <u>is a Banach space with</u> $\|\cdot\|$ <u>and</u> $\lambda(\alpha, n)$ <u>is a closed linear sub-space of</u> $\Lambda(\alpha, n)$.

PROOF. It is easy to see that $\Lambda(\alpha, n)$ is a vector space over the field of complex numbers with the usual definition of addition and scalar multiplication of functions and we have a normed linear space in $\Lambda(\alpha, n)$. We shall only prove the completeness here. Let $\{f_p\}_{p=1}^{\infty}$ be a Cauchy sequence in $\Lambda(\alpha, n)$. Then $\|f_m - f_p\| \to 0$ as $m, p \to \infty$. This implies that $\|f_m - f_p\|_\infty \to 0$ and $\|f_m - f_p\|_\alpha \to 0$ as $m, p \to \infty$. Now for each $x \in R$, we have

$$|f_m(x) - f_p(x)| \leq \|f_m - f_p\|_\infty \to 0$$

so that $\{f_p(x)\}_{p=1}^{\infty}$ is a Cauchy sequence of complex numbers and hence there exists $f(x)$ such that $f_p(x) \to f(x)$ as $p \to \infty$. We define f by setting

$$f(x) = \lim_{p \to \infty} f_p(x)$$

We now assert that $f \in \Lambda(\alpha, n)$ and that $\|f_p - f\| \to 0$ as $p \to \infty$.

(i) Since each f_p has period 1, it follows that f is also periodic and has period 1.

(ii) We first observe that every element in $\Lambda(\alpha, n)$ is uniformly continuous on R. This is because each element in $\Lambda(\alpha, n)$ is a continuous function with period 1 and every continuous function on a closed and bounded interval is uniformly continuous. Now by virtue of the uniform continuity of f_p, we have

$$\sup_{x \in R} |f_p(x) - f(x)| = \sup_{x \in R} \lim_{m \to \infty} |f_p(x) - f_m(x)|$$

$$= \lim_{m \to \infty} \sup_{x \in R} |f_p(x) - f_m(x)|$$

$$\leq \lim_{m \to \infty} \| f_p - f_m \|_\infty$$

which tends to zero as $p \to \infty$. Thus

(5) $$\| f_p - f \|_\infty \to 0 \quad \text{as } p \to \infty$$

(iii) If $x, y \in R$, then

(6) $$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_p(x)| + |f_p(x) - f_p(y)| \\ &\quad + |f_p(y) - f(y)| \\ &\leq 2 \| f - f_p \|_\infty + |f_p(x) - f_p(y)| \end{aligned}$$

Now given $\epsilon > 0$, we can choose an integer p, such that $\| f - f_p \|_\infty < \epsilon/3$ and using the uniform continuity of f_p, we can find a $\delta > 0$, such that $|x - y| < \delta$ implies that $|f_p(x) - f_p(y)| < \epsilon/3$. Then for this δ, we have by virtue of (6)

$$\begin{aligned} |f(x) - f(y)| &\leq 2 \| f - f_p \|_\infty + |f_p(x) - f_p(y)| \\ &\leq 2\, \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

This proves that f is in fact uniformly continuous.

(iv) To see that $f \in \Lambda(\alpha, n)$, it remains only to show that $\| f \|_\alpha < \infty$. In fact, by the uniform continuity again,

$$\sup_{x \in R} |\Delta_t^n f(x)| = \sup_{x \in R} \lim_{p \to \infty} |\Delta_t^n f_p(x)|$$

(7) $$= \lim_{p \to \infty} \sup_{x \in R} |\Delta_t^n f_p(x)|$$

$$\leq \lim_{p \to \infty} \| f_p \|_\alpha \cdot |t|^\alpha$$

Since $\{f_p\}$ is a Cauchy sequence in $\Lambda(\alpha, n)$, there exists a constant $K > 0$ such that

$$\| f_p \| \leq K \quad \text{for all } p$$

Hence

$$\| f_p \|_\alpha \leq \| f_p \| \leq K$$

As a consequence, we have, by (7),

(8) $$\| f \|_\alpha \leq K$$

(v) We now claim that $\| f_p - f \| \to 0$ as $p \to \infty$. To this end, we have by virtue of uniform continuity,

$$\sup_{x \in R} |\Delta_t^n f(x) - \Delta_t^n f_p(x)|$$

$$= \sup_{x \in R} \lim_{m \to \infty} |\Delta_t^n f_m(x) - \Delta_t^n f_p(x)|$$

$$= \lim_{m \to \infty} \sup_{x \in R} |\Delta_t^n (f_m - f_p)(x)|$$

$$\leq \lim_{m \to \infty} \| f_m - f_p \|_\alpha \cdot |t|^\alpha$$

so that

$$\| f_p - f \|_\alpha \leq \lim_{m \to \infty} \| f_m - f_p \|_\alpha$$

from which it follows that

(9) $\| f_p - f \|_\alpha \to 0$ as $p \to \infty$

From (5) and (9), we obtain

(10) $\| f_p - f \| \to 0$ as $p \to \infty$

We have thus proved that $\Lambda(\alpha, n)$ is a complete normed linear space. It is clear that if $f_p \in \lambda(\alpha, n)$, so does f which implies that $\lambda(\alpha, n)$ is a closed linear subspace of $\Lambda(\alpha, n)$. This completes the proof.

THEOREM 1.2. <u>Let</u> $0 < \alpha < \alpha' < n$. <u>Denote by</u> $\| \cdot \|$ <u>and</u> $\| \cdot \|'$ <u>the norms in</u> $\Lambda(\alpha, n)$ <u>and</u> $\Lambda(\alpha', n)$ <u>respectively. Then</u>

(i) $\Lambda(\alpha', n) \subset \lambda(\alpha, n)$

(ii) $\| f \| \leq 2^{n \cdot \frac{\alpha' - \alpha}{\alpha}} \| f \|' \; : f \in \Lambda(\alpha', n)$

PROOF. Let $f \in \Lambda(\alpha', n)$ and let $x \in R$. Suppose $|t|^\alpha < 2^n$. Then

(11) $$\frac{|\Delta_t^n f(x)|}{|t|^\alpha} = \frac{|\Delta_t^n f(x)|}{|t|^{\alpha'}} \cdot |t|^{(\alpha' - \alpha)} \leq \| f \|_{\alpha'} \cdot 2^{n \frac{\alpha' - \alpha}{\alpha}} \leq 2^{n \frac{\alpha' - \alpha}{\alpha}} \cdot \| f \|'$$

on the other hand, if $|t|^\alpha \geq 2^n$, then

(12) $$\frac{|\Delta_t^n f(x)|}{|t|^\alpha} \leq \frac{1}{2^n} \cdot |\Delta_t^n f(x)| \leq \| f \|_\infty \leq \| f \|'$$

Since $1 < 2^{n\frac{\alpha'-\alpha}{\alpha}}$, we may combine (11) and (12) to obtain

(13) $\|f\|_{\alpha} \leq 2^{n\frac{\alpha'-\alpha}{\alpha}} \|f\|'$

Since $\|f\|_{\alpha} \leq 2^{n\frac{\alpha'-\alpha}{\alpha}} \|f\|'$, we conclude that

$$\|f\| \leq 2^{n\frac{\alpha'-\alpha}{\alpha}} \|f\|'$$

Now from equation (11), we have

$$\sup_{x \in R} \frac{|\Delta_t^n f(x)|}{|t|^{\alpha}} \leq \|f\|_{\alpha'} |t|^{(\alpha'-\alpha)}$$

Letting $t \longrightarrow 0$, we see that $f \in \lambda(\alpha, n)$.

THEOREM 1.3. <u>If</u> $f \in \Lambda(\alpha, n)$, <u>then</u>

$$\sup_{x,h \in R} \frac{|\Delta_h^n f(x)|}{|h|^{\alpha}} = \sup_{\substack{0 \leq x \leq 1 \\ 0 < h \leq \frac{1}{2}}} \frac{|\Delta_h^n f(x)|}{h^{\alpha}}$$

PROOF. Let $x,h \in R$. If $|h| > \frac{1}{2}$, we can find an integer k such that $h = k + h_1$ where $|h_1| \leq \frac{1}{2}$. Then

$$\frac{|\Delta_h^n f(x)|}{|h|^{\alpha}} = \frac{\left|\sum_{r=0}^{n} (-1)^r \cdot \binom{n}{r} f(x + \overline{n-r}\, h)\right|}{|h|^{\alpha}}$$

$$= \frac{\left|\sum_{r=0}^{n} (-1)^r \binom{n}{r} f(x + \overline{n-r}\, k + \overline{n-r}\, h_1)\right|}{|h|^{\alpha}}$$

$$= \frac{\left|\sum_{r=0}^{n} (-1)^r \cdot \binom{n}{r} f(x + \overline{n-r}\, h_1)\right|}{|h_1|^{\alpha}} \left|\frac{h_1}{h}\right|^{\alpha}$$

$$\leq \frac{|\Delta_{h_1}^n f(x)|}{|h_1|^\alpha}$$

$$\leq \sup_{\substack{x \in R \\ |h_1| \leq \frac{1}{2}}} \frac{|\Delta_{h_1}^n f(x)|}{|h_1|^\alpha}$$

so that

$$\sup_{\substack{x \in R \\ h \in R}} \frac{|\Delta_h^n f(x)|}{|h|^\alpha} \leq \sup_{\substack{x \in R \\ |h_1| \leq \frac{1}{2}}} \frac{|\Delta_{h_1}^n f(x)|}{|h_1|^\alpha} \tag{14}$$

If $-\frac{1}{2} \leq h < 0$, set $h_1 = -h$, so that $|h| = |h_1|$ and $0 < h_1 \leq \frac{1}{2}$. Then

$$\frac{|\Delta_h^n f(x)|}{|h|^\alpha} = \frac{\left|\sum_{r=0}^{n} (-1)^r \binom{n}{r} f(x + \overline{n-r}\, h)\right|}{|h|^\alpha}$$

$$= \frac{\left|\sum_{r=0}^{n} (-1)^r \binom{n}{r} f(x - \overline{n-r}\, h_1)\right|}{|h_1|^\alpha}$$

$$= \frac{\left|\sum_{r=0}^{n} (-1)^r \binom{n}{r} f(y + r h_1)\right|}{|h_1|^\alpha}$$

with $y = x - n h_1$

$$= \frac{\left|\sum_{r=0}^{n} (-1)^{n-r} \binom{n}{n-r} f(y + r h_1)\right|}{|h_1|^\alpha}$$

$$= \frac{|\Delta_{h_1}^{n} f(y)|}{|h_1|^{\alpha}}$$

$$\leq \sup_{y \in R,\, 0 < h_1 \leq \frac{1}{2}} \frac{|\Delta_{h_1}^{n} f(y)|}{|h_1|^{\alpha}}$$

from which we deduce that

$$(15) \quad \sup_{x \in R,\, |h| \leq \frac{1}{2}} \frac{|\Delta_h^n f(x)|}{|h|^{\alpha}} \leq \sup_{x \in R,\, 0 < h \leq \frac{1}{2}} \frac{|\Delta_h^n f(x)|}{h^{\alpha}}$$

where by the periodicity of f, we have

$$\Delta_1^n f(x) = 0 \quad \text{for every} \quad x \in R$$

Since f has period 1, it follows that

$$(16) \quad \sup_{x \in R} |\Delta_h^n f(x)| = \sup_{0 \leq x \leq 1} |\Delta_h^n f(x)|$$

Thus from (14), (15) and (16), we conclude that

$$(17) \quad \sup_{\substack{x \in R \\ h \in R}} \frac{|\Delta_h^n f(x)|}{|h|^{\alpha}} \leq \sup_{\substack{0 \leq x \leq 1 \\ 0 < h \leq \frac{1}{2}}} \frac{|\Delta_h^n f(x)|}{h^{\alpha}}$$

The opposite inequality being trivial, the proof of our theorem is complete.

REMARK 1.4. If $f \in \Lambda(\alpha, n)$, then $f \in \Lambda(\alpha, k)$ for every $k > n$. In fact

$$\Delta_h^k f(x) = \Delta_h^{k-n} (\Delta_h^n f(x))$$

so that

$$|\Delta_h^k f(x)| \leq 2^{k-n} |\Delta_h^n f(x)|$$

REMARK 1.5. If f has its nth derivative continuous, then $f \in \Lambda(n, n)$ and hence $f \in \lambda(\alpha, n)$ for every $\alpha < n$.

We shall now give a nontrivial example of a function of the class $\lambda(\alpha, 2)$ to exhibit the computational aspect involved in its verification.

THEOREM 1.6. <u>Let</u> $0 < x_0 < \frac{1}{2}$. <u>If</u>

$$g_1(x) = \sin^2 \frac{\pi x}{2x_0}$$

<u>and</u>

$$g_2(x) = \sin^2 \frac{\pi(x - 2x_0)}{1 - 2x_0}$$

<u>then the function</u> F <u>defined by</u>

$$F(x) = \begin{cases} c_1 g_1(x) & \text{for } 0 \leq x \leq 2x_0 \\ c_2 g_2(x) & \text{for } 2x_0 \leq x \leq 1 \end{cases}$$

<u>and</u>

$$F(x+1) = F(x)$$

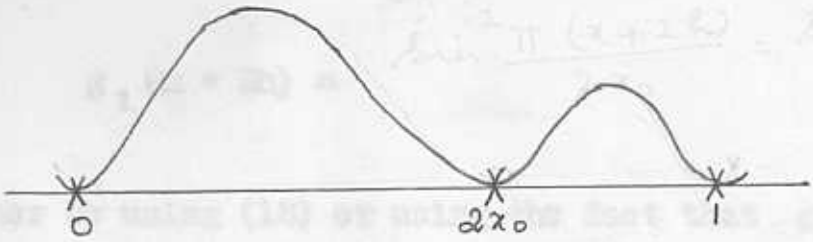
<u>where</u> c_1, c_2 <u>are constants belongs to the class</u> $\Lambda(2,2)$. <u>In particular</u> $F \in \lambda(\alpha, 2)$ <u>for</u> $\alpha < 2$.

PROOF. We first observe that both g_1 and g_2 are infinitely differentiable, non-negative and periodic with periods $2x_0$ and $1 - 2x_0$ respectively. All their zeros are of multiplicity 2. Since $g_1(2x_0) = g_2(2x_0) = 0$ and $g_1(0) = 0 = g_2(1)$ it is clear that F is well defined, periodic with period 1 and

non-negative if C_1, $C_2 > 0$. However F is differentiable only once at 0, $2x_0$, 1 in the periodic interval. We shall show that

$$\sup_{0 \leq x \leq 1} \left| \Delta_h^2 F(x) \right| = O(h^2)$$

Without loss of generality, we can assume that $h > 0$



CASE I. $0 \leq x < x + h < x + 2h \leq 2x_0$

or

$2x_0 < x < x + h < x + 2h \leq 1$

In this case, all the three points lie either on the first curve or on the second curve. Since either component of F is infinitely differentiable in these intervals and every twice continuously differentiable function has its second difference as $O(h^2)$, it follows that

$$\left| \Delta_h^2 F(x) \right| = O(h^2)$$

CASE II. $0 \leq x < x + h \leq 2x_0 \leq x + 2h \leq 1$

Here

$$\Delta_h^2 F(x) = C_2 g_2(x + 2h) - 2C_1 g_1(x + h) + C_1 g_1(x)$$
$$= C_2 g_2(x + 2h) - C_1 g_1(x + 2h) + C_1 \Delta_h^2 g_1(x)$$

It is well known that if $0 \leq x \leq \frac{\pi}{2}$, then

(18) $$\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1$$

Now

$$g_2(x+2h) = \sin^2 \frac{\pi(x+2h-2x_0)}{1-2x_0}$$

and

$$g_1(x+2h) = \sin^2 \frac{\pi(x+2h)}{2x_0} = \sin^2 \frac{(x+2h-2x_0)\pi}{2x_0}$$

Either by using (18) or using the fact that g_1 and g_2 have double zeros at $2x_0$, we see that

$$g_1(x+2h) = O\left((x+2h-2x_0)^2\right)$$

and

$$g_2(x+2h) = O\left((x+2h-2x_0)^2\right)$$

But

$$x+2h-2x_0 \leq (x+2h)-(x+h) = h$$

so that

$$g_1(x+2h) = O(h^2)$$

$$g_2(x+2h) = O(h^2)$$

Thus

$$|\Delta_h^2 F(x)| = O(h^2)$$

CASE III. $0 \leq x \leq 2x_0 \leq x + h < x + 2h \leq 1$

Here

$$\Delta_h^2 F(x) = C_2 g_2(x + 2h) - 2C_2 g_2(x + h) + C_1 g_1(x)$$

$$= C_2 \Delta_h^2 g_2(x) - C_2 g_2(x) + C_1 g_1(x)$$

using the relations

$$2x_0 - x \leq (x + h) - x = h$$

we have

$$g_1(x) = \sin^2 \frac{\pi x}{2x_0} = \sin^2 \frac{\pi(x - 2x_0)}{2x_0}$$

$$\leq \frac{\pi^2 (x - 2x_0)^2}{4x_0^2} \leq \frac{\pi^2 h^2}{4x_0^2}$$

and

$$g_2(x) = \sin^2 \frac{\pi(x - 2x_0)}{1 - 2x_0} \leq \frac{\pi^2 (x - 2x_0)^2}{(1 - 2x_0)^2}$$

$$\leq \frac{\pi^2 \cdot h^2}{(1 - 2x_0)^2}$$

Hence

$$|\Delta_h^2 F(x)| = O(h^2)$$

CASE IV. $0 \leq x < x + h \leq 2x_0 < 1 \leq x + 2h \leq 1 + 2x_0$

Here

$$2x_0 - x \leq (x + 2h) - x = 2h$$

$$2x_0 - x - h \leq (x + 2h) - (x + h) = h$$

$$x + 2h - 1 \leq (x + 2h) - (x + h) = h$$

Now

$$\Delta_h^2 F = C_1 \; g_1(x) - 2\,C_1 \; g_1(x+h) + C_1 \; g_1(x+2h-1)$$

But then

$$g_1(x) = \sin^2 \frac{\pi x}{2x_0} \qquad = \sin^2 \frac{\pi(x-2x_0)}{2x_0}$$

$$\leq \frac{\pi^2(x-2x_0)^2}{4x_0^2} \leq \frac{\pi^2 h^2}{x_0^2}$$

$$g_1(x+h) = \sin^2 \frac{\pi(x+h)}{2x_0} \qquad = \sin^2 \frac{\pi(x+h-2x_0)}{2x_0}$$

$$\leq \frac{\pi^2(x+h-2x_0)^2}{4x_0^2} \leq \frac{\pi^2 h^2}{4x_0^2}$$

and

$$g_1(x+2h-1) = \sin^2 \frac{\pi(x+2h-1)}{2x_0}$$

$$\leq \frac{\pi^2(x+2h-1)^2}{4x_0^2} \leq \frac{\pi^2 h^2}{4x_0^2}$$

CASE V. $0 \leq x \leq 2x_0 \leq x+h \leq 1 \leq x+2h \leq 1+2x_0$

Now

$$\Delta_h^2 F = C_1 \, g_1(x) - 2\,C_2 \; g_2(x+h)$$
$$+ C_1 \; g_1(x+2h-1)$$

Since

$$x+2h-1 \leq (x+2h)-(x+h) = h$$

$$1-(x+h) \leq (x+2h)-(x+h) = h$$

we see that

$$g_1(x+2h-1) = O(h^2) \text{ and}$$

$$g_2(x+h) = g_2(x+h-1+1)$$
$$= O(x+h-1)^2 = O(h^2)$$

Moreover

$$2x_0 - x \leq (x+h) - x = h, \text{ so that}$$
$$g_1(x) = g_1(x - 2x_0) = O(h^2)$$

CASE VI. $2x_0 \leq x < x+h \leq 1 \leq x+2h \leq 1+2x_0$

Now

$$\Delta_h^2 F = C_2\, g_2(x) - 2C_2\, g_2(x+h) + C_1\, g_1(x+2h-1)$$

(i) $x+2h-1 \leq (x+2h)-(x+h) = h$, so that

$$g_1(x+2h-1) = O(h^2)$$

(ii) $g_2(x) - 2g_2(x+h) = \Delta_h^2 g_2(x) - g_2(x+2h)$

But

$$x+2h-1 \leq (x+2h)-(x+h) = h$$

so that

$$g_2(x+2h) = g_2(x+2h-1+1) = O(x+2h-1)^2 = O(h^2)$$

CASE VII. $2x_0 \leq x < x+h \leq 1 < 1+2x_0 \leq x+2h \leq 2$

In this case

$$\Delta_h^2 F = C_2\, g_2(x) - 2C_2\, g_2(x+h) + C_2\, g_2(x+2h-1)$$
$$= C_2\, \Delta_h^2\, g_2(x) - C_2\, g_2(x+2h) + C_2\, g_2(x+2h-1)$$

Now

$$x+2h-1 \leq (x+2h)-(x+h) = h, \text{ so that}$$
$$g_2(x+2h) = g_2(x+2h-1+1) = O(x+2h-1)^2 \leq O(h^2)$$

and

$$g_2(x + 2h - 1) = O(x + 2h - 1)^2$$
$$= O(h^2)$$

CASE VIII. $2x_0 \leq x \leq 1 \leq x + h \leq 1 + 2x_0 \leq x + 2h \leq 2$

Here

$$\Delta_h^2 F = C_2\, g_2(x) - 2\, C_1\, g_1(x + h - 1) + C_2\, g_2(x + 2h - 1)$$

From

$$1 - x \leq x + h - x = h$$

we have

$$g_2(x) = g_2(x + 1 - 1) = O(x - 1)^2 = O(h^2)$$

since

$$x + h - 1 \leq x + h - x = h$$

we get

$$g_1(x + h - 1) = O(h^2)$$

Further

$$x + 2h - 1 - 2x_0 \leq (x + 2h) - (x + h) = h$$

implies

$$g_2(x + 2h - 1) = g_2(x + 2h - 1 + 1 - 2x_0)$$
$$= O(x + 2h - 1 - 2x_0)^2$$
$$= O(h^2)$$

CASE IX. $2x_0 \leq x \leq 1 \leq x + h < x + 2h \leq 1 + 2x_0$

Now

$$\Delta_h^2 F = C_2\, g_2(x) - 2\, C_1\, g_1(x + h - 1) + C_1\, g_1(x + 2h - 1)$$

Because

$$x + 2h - 1 - 2x_0 \leq (x + 2h) - (x + h) = h$$
$$x + h - 1 \leq (x + h) - x = h$$
$$1 - x \leq (x + h) - x = h$$

we have

$$g_2(x) = g_2(x - 1 + 1) = O(x - 1)^2 = O(h^2)$$
$$g_1(x + h - 1) = O(x + h - 1)^2 \leq O(h^2)$$
$$g_1(x + 2h - 1) = g_1(x + 2h - 1 + 1 - 2x_0)$$
$$= O(x + 2h - 1 - 2x_0)^2 = O(h^2)$$

CASE X. $2x_0 \leq x \leq 1 < 1 + 2x_0 \leq x + h < x + 2h \leq 2$

Now

$$\Delta_h^2 F = C_2\, g_2(x) - 2\, C_2\, g_2(x + h - 1) + C_2\, g_2(x + 2h - 1)$$

As in the earlier cases, we see that

$$g_2(x) = g\ (x - 1 + 1) = O(x - 1)^2 = O(h^2)$$

$$g_2(x + h - 1) = g_2(x + h - 1 + 1 - 2x_0)$$
$$= O(x + h - 1 - 2x_0)^2 = O(h^2)$$
$$g_2(x + 2h - 1) = O(x + 2h - 1)^2 = O(h^2)$$

In the remaining cases with $x \leq 1$, h will be $\geq \frac{1}{2}$. Thus we see that

$$F \in \Lambda(2, 2)$$

This completes the proof.

2. CONTINUOUSLY TRANSLATING SUBSPACE

Let X denote the real numbers modulo one. Suppose F is some translation invariant Banach space of (complex, periodic, integrable) functions on X. Then F will be a subspace of $L^1[0,1]$. We call F a translating space if its norm obeys

$$\left| \int_0^1 f(x)\, e^{2\pi i x}\, dx \right| \leq \text{Const.} \|f\|$$

for each $f \in F$. The Plessner problem is to find the set $\mathcal{C}F$ of all continuously translating functions in F, an $f \in F$ being said to translate continuously if the mapping $x \to T_x f$ is continuous from X into the Banach space F, where the translate $T_x f$ is defined by

$$T_x f(y) = f(x+y)$$

It is enough to know that this mapping is continuous at a single point x_0 or that the numerical function $x \to \|T_x f - f\|$ is continuous at 0.

We now collect below the various properties of translating spaces.

1. Each translation T_x is a bounded operator from $F \to F$. [9, Lemma 3.1]

2. The set $\mathcal{C}F$ of all continuously translating $f \in F$ is a closed subspace of F. [9, Lemma 3.2], Also see [15]

3. There exists an equivalent norm on $\mathcal{C}F$ that makes all translations isometric. [9, Lemma 3.3], Also see [15]

4. The norm on a translating space F must satisfy

$$\| f \| \geq \text{Const.} \int_0^1 |f(x)| \, dx$$

[9, Cor. 3.5]

5. Let F be a translation invariant space of (periodic, integrable) functions. Then among all norms $\| f \|$ larger than L^1 norm, there exists at most one (upto equivalence) that will make F complete. [9, Cor. 3.6]

6. The continuously translating subspace $\mathcal{C}$ F is precisely the closure in F of $P \cap F$ where P is the set of all trigonometric polynomials. [9, Lemma 3.8], Also see [15]

We shall now show that the continuously translating subspace $\mathcal{C} \Lambda(\alpha, n)$ is $\lambda(\alpha, n)$.

PROPOSITION 2.1. <u>Let</u> $f \in \lambda(\alpha, n)$. <u>Then the mapping</u> $y \to T_y f$ <u>is continuous from</u> R <u>into</u> $\Lambda(\alpha, n)$.

PROOF. We have to prove that $\| T_y f - f \| \to 0$ as $y \to 0$. By the definition of norm it is enough to show that

(i) $\| T_y f - f \|_\infty \to 0$ as $y \to 0$

and

(ii) $\| T_y f - f \|_\alpha \to 0$ as $y \to 0$

Since f is uniformly continuous on R, (i) is trivial. To prove (ii), let

$D = [0,1] \times (0, \frac{1}{2}]$ and define a function F by

$$F(x, s) = \frac{\Delta_s^n f(x)}{s^\alpha} \qquad (x, s) \in D$$

Since $f \in \lambda(\alpha, n)$ we see that

$$\lim_{s \to 0} F(x, s) = 0 \quad \text{for all } x.$$

Hence F is a continuous function vanishing at infinity on the locally compact Hausdorff space D and hence it is uniformly continuous on D. Given $\epsilon > 0$, there exist therefore $\delta > 0$ such that for all $(x, s), (x', s') \in D$, $|x - x'| < \delta$, $|s - s'| < \delta$ imply

$$|F(x, s) - F(x', s')| < \epsilon$$

In particular, this gives for all $(x, s) \in D$,

$$|F(x + y, s) - F(x, s)| < \epsilon$$

if $|y| < \delta$. This is the same as saying that

$$\frac{|\Delta_s^n T_y f(x) - \Delta_s^n f(x)|}{s^\alpha} = \frac{|T_y \Delta_s^n f(x) - \Delta_s^n f(x)|}{s^\alpha}$$

for all x and s if $|y| < \delta$. This gives

$$\|T_y f - f\|_\alpha \to 0 \quad \text{as } y \to 0$$

which is (ii).

PROPOSITION 2.2. $\|\Delta_s^n p\|_\infty = o(s^\alpha)$ <u>for every trigonometric polynomial.</u>

PROOF. Since every trigonometric polynomial is infinitely differentiable and a fortiori has its nth derivative continuous, it follows that

$$\|\Delta_s^n p\|_\infty = O(s^n)$$

and since $\alpha < n$, our result follows.

THEOREM 2.3. <u>The continuously translating subspace of</u> $\Lambda(\alpha, n)$ <u>is</u> $\lambda(\alpha, n)$.

PROOF. By Proposition 2.1, $\lambda(\alpha, n) \subset \mathcal{C}\Lambda(\alpha, n)$. By the property 6 stated above,

$$\mathcal{C}\Lambda(\alpha, n) = \overline{P \cap \Lambda(\alpha, n)} \text{ in } \Lambda(\alpha, n)$$

But by Proposition 2.2, $P \subset \lambda(\alpha, n)$ and since $\lambda(\alpha, n)$ is closed, we have

$$\overline{P} \subset \lambda(\alpha, n)$$

Hence

$$\mathcal{C}\Lambda(\alpha, n) = \overline{P \cap \Lambda(\alpha, n)} = \overline{P} \subset \lambda(\alpha, n)$$

This completes the proof.

REMARK 2.4. $\lambda(\alpha, n)$ is the closed linear span of trigonometric polynomials in $\Lambda(\alpha, n)$.

CHAPTER 3 : $\Lambda(\alpha, n)$ AS SECOND DUAL OF $\lambda(\alpha, n)$

3. EVALUATION FUNCTIONALS

Our object now is to obtain the representation of a bounded linear functional on the space $\lambda(\alpha, n)$. This is done by identifying $\lambda(\alpha, n)$ with a closed linear subspace of the space of continuous functions which vanish at infinity on a locally compact Hausdorff space. First we introduce evaluation functionals and some of their properties.

DEFINITION 3.1. For each $x \in R$, the <u>evaluation functional</u> φ_x is defined by

$$\varphi_x(f) = f(x) \qquad f \in \Lambda(\alpha, n)$$

PROPOSITION 3.2. <u>For each integer</u> n <u>and each</u> $x \in R$, <u>we have</u>

$$\varphi_{x+n} = \varphi_x .$$

PROOF. This follows immediately from the definition, since each f in $\Lambda(\alpha, n)$ has period 1.

PROPOSITION 3.3. <u>For each</u> $x \in R$, φ_x <u>is a bounded linear functional on</u> $\Lambda(\alpha, n)$ <u>with</u> $\|\varphi_x\| \leq 1$.

PROOF. It is easy to see that φ_x is a linear functional on $\Lambda(\alpha, n)$. If $f_1, f_2 \in \Lambda(\alpha, n)$ and c is a complex number then

$$\varphi_x(f_1 + f_2) = (f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$= \varphi_x(f_1) + \varphi_x(f_2)$$

$$\varphi_x(cf) = (cf)(x) = c\,f(x)$$
$$= c.\,\varphi_x(f)$$

Further, if $f \in \Lambda(\alpha, n)$, then

$$|\varphi_x(f)| = |f(x)| \leq \|f\|_\alpha \leq \|f\|$$

Hence

$$\|\varphi_x\| \leq 1$$

DEFINITION 3.4. If $L_1, L_2, \ldots L_p$ are linear functionals on a vector space E and $C_1, C_2, \ldots C_p$ are scalars, then the <u>linear combination</u> $\sum_{i=1}^{p} C_i L_i$ is defined by

$$\left(\sum_{i=1}^{p} C_i L_i\right)(f) = \sum_{i=1}^{p} C_i L_i(f) \qquad f \in E$$

It will then follow that $\sum_{i=1}^{p} C_i L_i$ is also a linear functional on E.

PROPOSITION 3.5. <u>For each</u> $x, t \in R$, <u>we have</u>

$$(1) \quad \left\| \varphi_{x+nt} - \binom{n}{1}\varphi_{x+\overline{n-1}t} + \cdots\cdots + (-1)^n \varphi_x \right\| \leq |t|^\alpha$$

PROOF. For each $f \in \Lambda(\alpha, n)$ we have, by definition

$$\left(\varphi_{x+nt} - \binom{n}{1}\varphi_{x+\overline{n-1}t} + \cdots\cdots + (-1)^n \varphi_x\right)(f)$$

$$= f(x+nt) - \binom{n}{1} f(x+\overline{n-1}\,t) + \ldots + (-1)^n f(x)$$

$$= \Delta_t^n f(x)$$

so that

$$(2) \quad \left|\left(\sum_{r=0}^{n} (-1)^r \binom{n}{r} \varphi_{x+\overline{n-r}t}\right)(f)\right| = |\Delta_t^n f(x)| \leq \|f\| \, |t|^\alpha$$

since (2) holds for every $f \in \Lambda(\alpha, n)$, we conclude that

$$\left\| \sum_{r=0}^{n} (-1)^r \binom{n}{r} \varphi_{x+\overline{n-r}t} \right\| \leq |t|^{\alpha}$$

which is (1).

DEFINITION 3.6. The <u>dual space</u> E^* of a normed linear space E is the space of all bounded linear functionals on E. For each functional F in the dual space $(\lambda(\alpha, n))^{**}$ of $(\lambda(\alpha, n))^*$ is associated a function $\hat{F}$ on R defined by

$$\hat{F}(x) = F(\varphi_x)$$

REMARK 3.7. There is a natural imbedding of any normed linear space E in its second dual E^{**} given by

$$f \to \psi_f \qquad f \in E$$

where

$$\psi_f(L) = L(f) \qquad L \in E^*$$

If $f \in \lambda(\alpha, n)$ and F_f is its image under the canonical imbedding of $\lambda(\alpha, n)$ in $\lambda(\alpha, n)^{**}$, then the function $\hat{F}_f$ is the same as f.

PROPOSITION 3.8. <u>If</u> F <u>is a functional in</u> $(\lambda(\alpha, n))^{**}$, <u>then the function</u> $\hat{F}$ <u>belongs to</u> $\Lambda(\alpha, n)$ <u>and</u>

$$\|\hat{F}\| \leq \|F\|$$

PROOF. Let $F \in (\lambda(\alpha, n))^{**}$. Since $\varphi_x \in (\lambda(\alpha, n))^*$ for each $x \in R$, $\hat{F}(x)$ is well defined. Moreover $\hat{F}(x+1) = F(\varphi_{x+1}) = F(\varphi_x) = \hat{F}(x)$ by Proposition 3.2 for each $x \in R$ which

proves that $\hat{F}$ is periodic with period 1. Further more

$$|\hat{F}(x)| = |F(\varphi_x)| \leq \|F\| \|\varphi_x\| \leq \|F\|$$

because $\|\varphi_x\| \leq 1$ by Proposition 3.3. This gives

(3) $\|\hat{F}\|_\infty \leq \|F\|$

Also, if $x, t \in R$, then

$$|\Delta_t^n \hat{F}(x)| = |F(\varphi_{x+nt}) - \binom{n}{1} F(\varphi_{x+\overline{n-1}t}) \ldots + (-1)^n F(\varphi_x)|$$

$$= |F(\sum_{r=0}^{n} (-1)^r \cdot \binom{n}{r} \varphi_{x+\overline{n-r}t}|$$

$$\leq \|F\| \, \|\sum_{r=0}^{n} (-1)^r \cdot \binom{n}{r} \varphi_{x+\overline{n-r}t}\|$$

$$\leq \|F\| \cdot |t|^\alpha \quad \text{by Proposition 3.5}$$

Thus

(4) $\|\hat{F}\|_\alpha \leq \|F\|$

combining (3) and (4), we get

(5) $\|\hat{F}\| \leq \|F\|$

To complete the proof it remains to show that $\hat{F}$ is continuous. To this end, we see by virtue of (4), because $\alpha > 0$

$$|\Delta_t^n \hat{F}(x)| \longrightarrow 0 \quad \text{as} \quad |t|^\alpha \longrightarrow 0$$

That is,

(6) $\hat{F}(x + nt) - \binom{n}{1} \hat{F}(x + n-1\ t) + \ldots + (-1)^n \cdot \hat{F}(x)$

$\rightarrow 0$ as $|t| \rightarrow 0$ for all x.

Using the relation

$$1 - \binom{n}{1} + \binom{n}{2} + \dots + (-1)^n = (1-1)^n = 0$$

we obtain from (6), letting $t \rightarrow 0$ through positive and negative values separately

(7) $\hat{F}(x) = \hat{F}(x+0)$

(8) $\hat{F}(x) = \hat{F}(x-0)$

and so

$$\hat{F}(x) = \hat{F}(x-0) = \hat{F}(x+0)$$

for each $x \in R$. This proves that $\hat{F}$ is continuous. We have thus proved that $\hat{F}$ is a continuous periodic function on R with period 1 satisfying (4) and hence belongs to $\Lambda(\alpha, n)$. Since (5) holds, the proposition is completely proved.

4. REPRESENTATION OF FUNCTIONALS

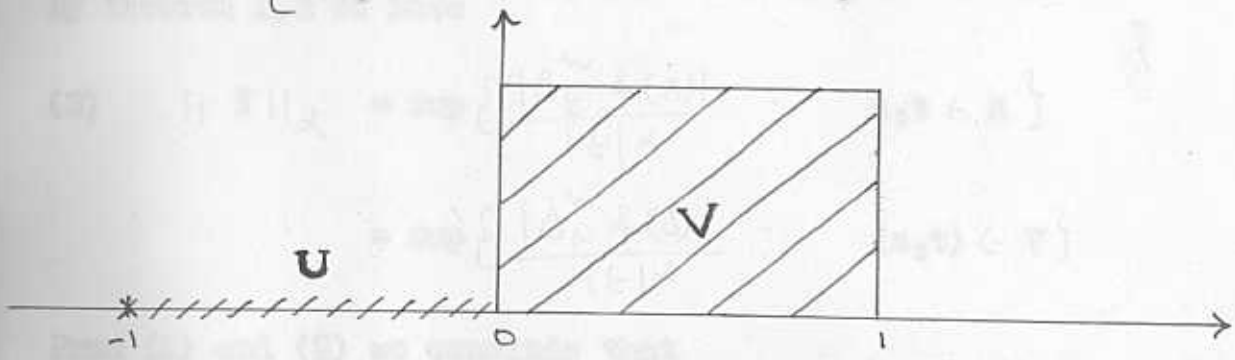
We now identify the continuous linear functionals of $\lambda(\alpha, n)$ by constructing isometric imbedding of $\lambda(\alpha, n)$ into a space of continuous functions with sup norm.

Let

$$U = \{ y \in R : -1 \leq y \leq 0 \}$$

and

$$V = \{ (s,t) \in R^2 : 0 \leq s \leq 1 , 0 < t \leq \tfrac{1}{2} \}$$



The disjoint union $U \cup V$ is denoted by W and it is a locally compact Hausdorff space. Let $C_0(W)$ denote the Banach space of complex valued continuous functions on W which vanish at infinity with the norm

$$\| g \|_W = \sup_{x \in W} |g(x)| \qquad g \in C_0(W)$$

DEFINITION 4.1. For $f \in \lambda(\alpha, n)$, define $j f = \tilde{f}$ where $\tilde{f}$ is defined by

$$\tilde{f}(u) = f(u) \qquad u \in U$$

$$\tilde{f}(v) = \tilde{f}(s, t) = \frac{\Delta_t^n f(s)}{|t|^\alpha} \qquad v = (s,t) \in V$$

PROPOSITION 4.2. j <u>is a linear isometry of</u> $\lambda(\alpha, n)$ <u>with the norm</u> $\|\cdot\|$ <u>into</u> $C_0(W)$ <u>with the sup norm</u> $\|\cdot\|_W$ <u>on</u> W.

PROOF. By the properties of functions in $\lambda(\alpha, n)$ it is clear that $\tilde{f} \in C_0(W)$ for each $f \in \lambda(\alpha, n)$ and j is thus a mapping of $\lambda(\alpha, n)$ into $C_0(W)$. It is easy to verify that j is a linear mapping. If $f \in \lambda(\alpha, n)$, then f has period 1; so

$$(1) \qquad \|f\|_\alpha = \operatorname{Sup}\{|f(u)| : u \in R\} = \operatorname{Sup}\{|f(u)| : u \in U\}$$

By theorem 1.3 we have

$$(2) \qquad \|f\|_\alpha = \operatorname{Sup}\left\{\frac{|\Delta_t^n f(s)|}{|t|^\alpha} : s,t \in R\right\}$$

$$= \operatorname{Sup}\left\{\frac{|\Delta_t^n f(s)|}{|t|^\alpha} : (s,t) \in V\right\}$$

From (1) and (2) we conclude that

$$\|f\| = \|\tilde{f}\|_W$$

THEOREM 4.3. <u>For every</u> $\varphi \in (\lambda(\alpha, n))^*$ <u>there is a finite measure</u> μ <u>on</u> W <u>such that</u>

(i) $\varphi(f) = \int_W \tilde{f}\, d\mu \qquad f \in \lambda(\alpha, n)$

(ii) $\|\varphi\| = \text{tot. var.}\,\mu = |\mu|(W)$

PROOF. Let $\varphi \in (\lambda(\alpha, n))^*$. Since $\lambda(\alpha, n)$ is complete its isometric imbedding $j \cdot \lambda(\alpha, n)$ is a closed linear subspace of $C_0(W)$. Treating φ as a functional on $j\lambda(\alpha, n)$, the Hahn-Banach theorem gives the existence of an extension $\Phi \in C_0(W)^*$ such that $\|\varphi\| = \|\Phi\|$. By the Riesz - representation theorem, there is a corresponding regular Borel measure μ

on W with

$$\|\Phi\| = \text{tot. var.}\,\mu$$

and

$$\Phi(f) = \Phi(\tilde{f}) = \int_W \tilde{f}\, d\mu \quad : \quad f \in \lambda(\alpha, n)$$

5. $\Lambda(\alpha, n)$ AS THE SECOND DUAL OF $\lambda(\alpha, n)$

Let $M(U)$ denote the space of regular Borel measures on U, where the sets U, V, W are defined in the previous section. Every $\mu \in M(U)$ determines a bounded linear functional φ_μ belonging to $(\lambda(\alpha, n))^*$ defined by

$$\varphi_\mu(f) = \int_U f \, d\mu \qquad f \in \lambda(\alpha, n)$$

For,

$$|\varphi_\mu(f)| \leq \| f \|_\infty \cdot |\mu|(U) \leq \| f \| \cdot |\mu|(U)$$

Let us now define two spaces of $(\lambda(\alpha, n))^*$. Let

$$E_m^* = \{\varphi_\mu \in (\lambda(\alpha, n))^* : \mu \in M(U)\}$$

and

$$E_f^* = \{\varphi_\mu \in (\lambda(\alpha, n))^* : \mu \in M(U), \text{ has finite support}\}.$$

Thus if $\varphi \in E_f^*$, then $\varphi = \sum_{i=1}^{k} \beta_i \cdot \varphi_{x_i}$ for some scalars $\beta_1, \beta_2, \ldots \beta_k$ and some $x_1, x_2, \ldots\ldots x_k$.

THEOREM 5.1. E_m^* is norm dense in $(\lambda(\alpha, n))^*$.

PROOF. Let $\varphi \in (\lambda(\alpha, n))^*$. Let μ be a measure on W such that

$$\varphi(f) = \int_W \tilde{f} \, d\mu$$

Let $\{W_r\}_{r=1}^{\infty}$ be an increasing sequence of compact sets whose union is W. That is

$$W = \bigcup_{r=1}^{\infty} W_r$$

and

$$W_1 \subset W_2 \subset W_3 \subset \quad \cdots \quad \subset W_r \subset \cdots$$

For each position integer r, define

$$\varphi_r(f) = \int_{W_r} \tilde{f}\, d\mu \qquad f \in \lambda(\alpha, n)$$

It suffices to prove that

(i) $\|\varphi - \varphi_r\| \to 0$ as $r \to \infty$

and

(ii) $\varphi_r \in E_m^*$ for each r

If $f \in \lambda(\alpha, n)$, then

$$|(\varphi - \varphi_r)(f)| = \left| \int_{W \setminus W_r} \tilde{f}.\, d\mu \right| = \|\tilde{f}\|_W \cdot |\mu|(W \setminus W_r)$$

$$= \|f\| \cdot |\mu|(W \setminus W_r)$$

This implies that

(1) $\|\varphi - \varphi_r\| \leq |\mu|(W \setminus W_r)$

Since μ is countably additive and $W = \bigcup_{r=1}^{\infty} W_r$, the right hand side of (1) tends to zero as $r \to \infty$. Hence

$$\|\varphi - \varphi_r\| \to 0 \text{ as } r \to \infty$$

This proves (i).

To see (ii), consider $f \in \lambda(\alpha, n)$. Then

(2) $$\Phi_r(f) = \int_{U \cap W_r} f(x)\, d\mu(x) + \int_{V \cap W_r} \tilde{f}(s,t) . d\mu(s,t)$$

The second integral in (2) can be written as

$$\int_{V \cap W_r} \tilde{f}(s,t) . d\mu(s,t) = \int_{V \cap W_r} \frac{\Delta_t^n f(s)}{|t|^\alpha} \cdot d\mu(s,t)$$

(3) $$= \int_{V \cap W_r} \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} f(s + \overline{n-k}\, t)}{|t|^\alpha} \cdot d\mu(s,t)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_{V \cap W_r} \frac{f(s + \overline{n-k}\, t)}{|t|^\alpha} \cdot d\mu(s,t)$$

Since $|t|^\alpha$ is bounded away from zero on $V \cap W_r$, we see that

$$\left| \int_{V \cap W_r} \frac{f(s + \overline{n-k}\, t)}{|t|^\alpha} d\mu(s,t) \right| \leq \text{const.} \| f \|_\infty$$

so that the mapping

$$f \longrightarrow \int_{V \cap W_r} \frac{f(s + \overline{n-k}\, t)}{|t|^\alpha} \cdot d\mu(s,t)$$

gives a bounded linear functional on $C(U)$, the space of continuous functions on U with sup. norm, and hence there exists a measure $\nu_k \in M(U)$ such that

(4) $$\int_{V \cap W_r} \frac{f(s + \overline{n-k}\, t)}{|t|^\alpha} d\mu(s,t) = \int_U f(s) . d\, \nu_k(s)$$

for $k = 0, 1, 2, \ldots n$.

Combining the equalities (2), (3) and (4), we obtain

$$\Phi_r(f) = \int_U f(s) \cdot d\nu(s)$$

where

$$\nu = \mu + \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \nu_k$$

Thus $\Phi_r \in E_m^*$. This is (ii). This completes the proof.

THEOREM 5.2. E_p^* <u>is norm dense in</u> $(\lambda(\alpha, n))^*$.

PROOF. Let $\Phi \in (\lambda(\alpha, n))^*$ and suppose $\epsilon > 0$ is given. By theorem 5.1, we can choose a $\mu \in M(U)$ such that

(5) $\| \Phi - \Phi_\mu \| < \epsilon/2$

Thus for every $f \in (\lambda(\alpha, n))^*$, we have

$$\Phi_\mu(f) = \int_U f \, d\mu$$

Let $\|\mu\|^*$ denote the norm of μ as an element of $C(U)^*$ and let

$$S = \{ g : g = f/\nu \, , f \in \lambda(\alpha, n), \| f \| \leq 1 \}$$

be the unit ball of $\lambda(\alpha, n)$. Since S is an equicontinuous family of bounded functions on U, S is conditionally compact in the topology of uniform convergence. Choose a finite set $T = \{g_1, g_2, \cdots g_k\}$ of functions in S such that the sphere $B(g_i, \epsilon/8\|\mu\|^*)$ cover S. Here $B(g_0, \delta) = \{ g \in S : \| g - g_0 \| < \delta \}$. The closed

$$\Sigma_\mu = \{ \eta \in C(U)^* : \|\eta\|^* \leq \|\mu\|^* \}$$

is weak* compact. By Krein-Milmann theorem, Σ_μ is the weak* closure of the convex hull of its extreme points, which are easily seen to be

$$\text{Ext.}(\Sigma_\mu) = \{ \epsilon . \|\mu\|^* . \varphi_x \quad : x \in U, \ |\epsilon| = 1 \}$$

Thus the measures with finite support are weak* dense in Σ_μ. The weak* neighbourhood $N(\mu ; g_1 ; g_2 ; \ldots ; g_k ; \epsilon/4)$ contains a measure $\eta \in \Sigma_\mu$ with finite support, that is

(i) $\|\eta\|^* \leq \|\mu\|^*$

(6) (ii) $\left| \int_U g_i \, d\mu - \int_V g_i \, d\eta \right| < \epsilon/4 \qquad g_i \in T$

(iii) $\eta = \sum_{i=1}^{k} \beta_i \varphi_{x_i}$

Denote by φ_η the functional given by

$$\varphi_\eta(f) = \int_U f \, d\mu \qquad f \in C(U)$$

Now consider $g \in S$. Choose $g_i \in T$ such that

(7) $\|g - g_i\|_\infty \leq \epsilon / 8 . \|\mu\|^*$

Then

$$|(\varphi_\mu - \varphi_\eta)(f)| = \left| \int_U g \, d\mu - \int_U g \, d\eta \right|$$

$$\leq \left| \int_U g \, d\mu - \int_U g_i \, d\mu \right| + \left| \int_U g_i \, d\mu - \int_U g_i \, d\eta \right|$$

$$+ \left| \int_U g_i \, d\eta - \int_U g \, d\eta \right|$$

$$\leq \|g - g_\iota\|_\infty \|\mu\|^* + \epsilon/4 + \|g - g_\iota\|_\infty \|\eta\|^*$$

$$< \frac{\epsilon}{8} + \frac{\epsilon}{4} + \frac{\epsilon}{8} = \frac{\epsilon}{2}$$

For any $f \in \lambda(\alpha, n)$, with $\|f\| \leq 1$, we set $g = f/\iota$. Then

$$|(\varphi - \varphi_\mu)(f)| = |(\varphi - \varphi_\mu)(g)| < \epsilon/2$$

Hence

$$\|\varphi - \varphi_\eta\| \leq \|\varphi - \varphi_\mu\| + \|\varphi_\mu - \varphi_\eta\| < \epsilon$$

Since ϵ is arbitrary, we conclude that E_p^* is norm dense in $\lambda(\alpha, n)^*$. This completes the proof.

COROLLARY 5.3. <u>The mapping</u> $F \longrightarrow \hat{F}$ <u>of</u> $(\lambda(\alpha, n))^{**}$ <u>into</u> $\Lambda(\alpha, n)$ <u>is one to one.</u>

PROOF. Since the mapping is linear, it is enough to consider $F \in (\lambda(\alpha, n))^{**}$ such that $\hat{F} = 0$. If $\hat{F}$ is the zero function, then F vanishes on the set of point evaluations φ_x and hence on its closed linear span $(\lambda(\alpha, n))^*$. But then F is the zero function. Thus the mapping $F \longrightarrow \hat{F}$ is one to one.

THEOREM 5.4. <u>The mapping</u> $F \longrightarrow \hat{F}$ <u>of</u> $(\lambda(\alpha, n))^{**}$ <u>into</u> $\Lambda(\alpha, n)$ <u>is onto and norm preserving.</u>

PROOF. To prove that the mapping is onto, let $g \in \Lambda(\alpha, n)$. We shall construct an F in $(\lambda(\alpha, n))^{**}$ such that $\hat{F} = g$. For this we convolute g with the Fejer's kernel

$$K_m(x) = \frac{2}{m+1}\left(\frac{\sin(m+1)\pi x}{\sin \pi x}\right)^2$$

so that the convolution $K_m * g$ is the mth $(C, 1)$ partial sum of the Fourier series of g and these converge uniformly to g. That is

(8) $$\lim_{m \to \infty} K_m * g(x) = g(x)$$

Moreover K_m is positive and

(9) $$\int_0^1 K_m(x)\, dx = 1$$

Recall that

$$(K_m * g)(x) = \int_0^1 K_m(x-t)\, g(t)\, dt$$
$$= \int_0^1 K_m(t) \cdot g(x-t)\, dt$$

so that

(10) $$\| K_m * g \|_\infty \leq \| g \|_\infty \cdot \int_0^1 K_m(t) . dt = \| g \|_\infty \leq \| g \|$$

From the relation

$$\Delta_t^n (K_m * g)(x) = K_m * \Delta_t^n g(x)$$
$$= \int_0^1 K_m(x-U) . \Delta_t^n g(U) . dU$$

we obtain

(11) $$\| K_m * g \|_\alpha \leq \| g \|_\alpha \leq \| g \|$$

(10) and (11) together yield

(12) $$\| K_m * g \| \leq \| g \|$$

This shows that $K_m * g \in \Lambda(\alpha, n)$. Now $K_m * g$ being a

trigonometric polynomial, we have

$$\sup_{x \in R} \left| \Delta_t^n (K_m * g)(x) \right| = o(|t|^n)$$

Since $\alpha < n$, we deduce that

$$\sup_{x \in R} \frac{|\Delta_t^n (K_m * g)(x)|}{|t|^\alpha} \to 0 \text{ as } t \to 0$$

This shows that $K_m * g \in \lambda(\alpha, n)$. We shall denote by F_m the functional in $(\lambda(\alpha, n))^{**}$ corresponding to $K_m * g$ under the canonical imbedding of $\lambda(\alpha, n)$ in $(\lambda(\alpha, n))^{**}$. This means that

(13) $F_m(\varphi) = \varphi(K_m * g) : \varphi \in (\lambda(\alpha, n))^*$

Since the imbedding of $\lambda(\alpha, n)$ in its second dual is an isometry, we obtain from (12)

(14) $\| F_m \| = \| K_m * g \| \leq \| g \|$

If we set $g_m = K_m * g$, we have proved that $\{g_m\}$ is a sequence of functions in $\lambda(\alpha, n)$ such that

(i) $\sup_m \| g_m \| < \infty$

(ii) $\lim_{m \to \infty} g_m(x)$ exists for each $x \in R$

Moreover F_m is canonical image of g_m in $(\lambda(\alpha, n))^{**}$. We assert now that if $\varphi \in (\lambda(\alpha, n))^*$, then $F_m(\varphi)$ is a Cauchy sequence of complex numbers. To prove our assertion, let

$M = \sup_m \| g_m \|$ and let $\epsilon > 0$ be given. Suppose $\varphi \in (\lambda(\alpha, n))^*$ choose $\varphi_p \in E_p^*$ such that $\| \varphi - \varphi_p \| \leq \epsilon/4M$. Then $\varphi_p = \sum_{i=1}^{r} \beta_i \varphi_{x_i}$ for some complex numbers $\beta_1, \beta_2, \ldots \beta_r$ and some $x_1, x_2, \ldots x_r \in R$. If k and m are two positive integers, then we have

$$
\begin{aligned}
|F_k(\varphi) - F_m(\varphi)| &= |\varphi(g_k - g_m)| \\
&\leq |(\varphi - \varphi_p)(g_k - g_m)| + |\varphi_p(g_k - g_m)| \\
&\leq \|\varphi - \varphi_p\| \|g_k - g_m\| + \left|\sum_{i=1}^{r} \beta_i [g_k(x_i) - g_m(x_i)]\right| \\
&\leq \frac{\epsilon}{4M} \cdot 2M + \left(\sum_{i=1}^{r} |\beta_i|\right) \max_{1 \leq i \leq r} |g_k(x_i) - g_m(x_i)|
\end{aligned}
$$

Choose an integer N such that $k, m \geq N$ imply

$$
|g_k(x_i) - g_m(x_i)| < \frac{\epsilon}{2\left(\sum_{i=1}^{r} |\beta_i|\right)}
$$

for $i = 1, 2, \ldots r$. Then if $k, m \geq N$ we have

$$
|F_k(\varphi) - F_m(\varphi)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
$$

Since ϵ is arbitrary, $\{F_m(\varphi)\}$ is a Cauchy sequence of complex numbers for each $\varphi \in (\lambda(\alpha, n))^*$. Define

$$
F(\varphi) = \lim_{m \to \infty} F_m(\varphi)
$$

Since

(14) $\| F \| \leq \lim_{m \to \infty} \operatorname{Sup} \| F_m \| = \lim_{m \to \infty} \operatorname{Sup} \| g_m \| \leq \| g \|$

we have $F \in (\lambda(\alpha, n))^{**}$. On the other hand, since for each $x \in R$

$$\hat{F}(x) = F(\varphi_x) = \lim_{m \to \infty} F_m(\varphi_x) = \lim_{m \to \infty} g_m(x) = g(x)$$

we see that $\hat{F} = g$. We have thus proved that the mapping $F \to \hat{F}$ is onto. From (14) it follows that $\| F \| \leq \| \hat{F} \|$. Thus to complete the proof, it remains to show that $\| \hat{F} \| \leq \| F \|$. For each $y \in R$, we have

$$|\hat{F}(y)| = |F(\varphi_y)| \leq \| F \| . \| \varphi_y \| \leq \| F \|$$

so that

(15) $\| \hat{F} \|_\infty \leq \| F \|$

Moreover for $x, t \in R$, we have

$$|\Delta_t^n \hat{F}(x)| = \left| F\left(\sum_{r=0}^{n} (-1)^r . \binom{n}{r} \varphi_{x+\overline{n-r}t} \right) \right|$$

$$\leq \| F \| . \left\| \sum_{r=0}^{n} (-1)^r . \binom{n}{r} \varphi_{x+\overline{n-r}t} \right\|$$

$$\leq \| F \| . |t|^\alpha$$

Thus

(16) $\| \hat{F} \|_\alpha \leq \| F \|$

From (15) and (16), it follows that

$$\| \hat{F} \| \leq \| F \|$$

This completes the proof.

THEOREM 5.5. $(\lambda(\alpha, n))^{**}$ <u>is isometrically isomorphic to</u> $\Lambda(\alpha, n)$.

PROOF. If F is a functional in $(\lambda(\alpha, n))^{**}$, then the function $\hat{F}$ belongs to $\Lambda(\alpha, n)$ by Proposition 3.8. By Corollary 5.3 and Theorem 5.4, the mapping $F \rightarrow \hat{F}$ of $(\lambda(\alpha, n))^{**}$ into $\Lambda(\alpha, n)$ is linear, one to one, onto and norm preserving and hence is an isometric isomorphism.

CHAPTER 4 : EXTREME POINTS

6. EXTREME POINTS IN $(\lambda(\alpha, 2))^*$

We shall now identify the extreme points of the unit sphere of the dual of $\lambda(\alpha, 2)$, when $0 < \alpha < 1$. φ is an extreme point of a convex set if it is not the mid point of any segment lying in the set. By proposition 4.2 it is enough to consider the corresponding problem for a linear space of continuous functions under the sup. norm.

PROPOSITION 6.1. *Let X be a locally compact topological space and $C_0(X)$ the space of complex valued continuous functions on X that vanish at infinity. Let A be a closed linear subspace of $C_0(X)$. Then A is a Banach space under the sup. norm. For each $x \in X$ let φ_x in A^* be defined by*

$$\varphi_x(f) = f(x) \qquad f \in A$$

Then every extreme point of the unit sphere of A^* is of the form

$$\eta \varphi_x$$

for some x and complex number η with $|\eta| = 1$.

PROOF. This is lemma V.8.6 of Dunford and Schwartz [4]. A converse of this result was proved by De Leeuw [7]. First we introduce the notion of peaking functions.

DEFINITION 6.2. Let x be a point of X. We say that a function h in A *peaks at x relative to A* if $h(x) = 1$ and

$$|h(y)| \leq 1 \qquad y \in X, \; y \neq x$$

with equality holding only for those y in X that satisfy either

$$g(y) = g(x) \qquad \text{all } g \in A$$

or

$$g(y) = -g(x) \qquad \text{all } g \in A .$$

PROPOSITION 6.3. [De Leeuw. Lemma 3.2 p.61 [7]] . Let x be a point of X. Suppose that there is a function in A that peaks at x relative to A. Then the functional φ in A^* defined by

$$\varphi(g) = g(x) \qquad g \in A$$

is an extreme point of the unit sphere of A^*. If $|\eta| = 1$, then $\eta\varphi_x$ is also an extreme point.

We shall now establish the existence of peaking functions for our space.

PROPOSITION 6.4. Let $0 < \alpha < 1$. Given $x_0 \in (0, 1)$, there exists a function $g \in \lambda(\alpha, 1)$ such that $g(x) \geq 0$ for all x and $g(x) = 0$ only at the points $x_0 + k$, where k is an integer.

PROOF. Define a continuous function g on $[0,1]$ by

$$g(x) = \frac{x_0 - x}{x_0} \qquad 0 \leq x \leq x_0$$

$$= \frac{x - x_0}{1 - x_0} \qquad x_0 \leq x \leq 1$$

In the interval $[0,1]$, $g(x) = 0$ only at x_0 and $g(0) = g(1) = 1$. Moreover it is clear that $g(x) \geq 0$. Extend g by periodicity to the whole of R, that is define

$$g(x + 1) = g(x) ,$$

for all x. It is clear $g(x) \geq 0$ for all x and $g(x) = 0$ only

at x_0+k. We shall now show that $g \in \lambda(\alpha, 1)$. It is enough to prove that $g \in \Lambda(1, 1)$.

Let $x, x+h$ be any points of R. Assume $h > 0$.

1) If $x, x+h$ both lie in the same line segment, then

$$|g(x+h) - g(x)| = \frac{h}{x_0} \text{ or } \frac{h}{1-x_0}$$

2) Let $0 \leq x \leq x_0 \leq x+h \leq 1$. Then

$$|g(x+h) - g(x)| = \left| \frac{h x_0 + x - x_0}{x_0(1-x_0)} \right|$$

$$\leq \frac{h x_0 + |x - x_0|}{x_0(1-x_0)} \leq \frac{h x_0 + h}{x_0(1-x_0)}$$

$$= \frac{1+x_0}{x_0(1-x_0)} \cdot h$$

3) Let $x_0 \leq x \leq 1 \leq x+h \leq 1+x_0$. Here

$$g(x+h) - g(x) = g(x+h-1) - g(x)$$

$$= \frac{x_0 - (x+h-1)}{x_0} - \frac{x - x_0}{1-x_0}$$

$$= \frac{(1-x) + h(x_0 - 1)}{x_0(1-x_0)}$$

so that

$$|g(x+h) - g(x)| \leq \frac{(1-x) + h(1-x_0)}{x_0(1-x_0)}$$

$$\leq \frac{h + h(1-x_0)}{x_0(1-x_0)} = \frac{2-x_0}{x_0(1-x_0)} h$$

4) Let $0 \leq x \leq x_0$ and $1 \leq x+h \leq 1 + x_0$. Choose any x_1 in $[x_0 , 1]$. Using (2) and (3) above we can find constants $C_1 , C_2 > 0$ such that

$$|g(x + h) - g(x_1)| \leq C_1 |x + h - x_1|$$

and

$$|g(x_1) - g(x)| \leq C_2 |x_1 - x|$$

But $|x_1 - x| \leq h$ and $|x + h - x_1| \leq h$. These give

$$\begin{aligned} |g(x + h) - g(x)| &\leq |g(x + h) - g(x_1)| + |g(x_1) - g(x)| \\ &\leq C_1 |x + h - x_1| + C_2 |x_1 - x| \\ &\leq C_1 h + C_2 h = (C_1 + C_2) h \end{aligned}$$

5) $0 \leq x \leq x_0$ and $1 + x_0 \leq x + h \leq 2$.

In this case, we choose real numbers x_1 and x_2 such that $x_1 \in [x_0 , 1]$ and $x_2 \in [1, 1 + x_0]$. By (1), (2), (3) proved above, we can find constants C_1 , C_2 , C_3 such that

$$|g(x + h) - g(x_2)| \leq C_1 |x + h - x_2|$$

$$|g(x_2) - g(x_1)| \leq C_2 |x_2 - x_1|$$

$$|g(x_1) - g(x)| \leq C_3 |x_1 - x|$$

using the inequalities

$$|x + h - x_2| \leq h, \quad |x_2 - x_1| \leq h, \quad |x_1 - x| \leq h$$

we have

$$|g(x+h) - g(x)| \leq |g(x+h) - g(x_2)| + |g(x_2) - g(x_1)| + |g(x_1) - g(x)|$$
$$\leq (C_1 + C_2 + C_3)\, h$$

6) Let $x_0 \leq x \leq 1$, $1 + x_0 \leq x+h \leq 2$.

Choose any x_1 in $[1, 1 + x_0]$. Then

$$|g(x+h) - g(x)| \leq |g(x+h) - g(x_1)| + |g(x_1) - g(x)|$$
$$\leq C_1 |x + h - x_1| + C_2 |x_1 - x|$$
$$\leq C_1 h + C_2 h = (C_1 + C_2)\, h$$

Finally let $x \in [k, k+1]$ and $x+h \in [k+m, k+m+1]$ where $m \geq 2$. Then

$$g(x+h) - g(x) = g(x + h - k - m) - g(x - k)$$

since $x+h-k-m$ and $x-k$ both belong to $[0, 1]$ by virtue of (1) and (2) we can find a constant c such that

$$|g(x + h - k - m) - g(x - k)| \leq C\, |h - m| \leq C$$

since $h - m \leq 1$. But

$$h \geq k + m - x \geq k + m - k - 1 = m - 1 \geq 1$$

Hence

$$|g(x+h) - g(x)| \leq C. \leq C.h$$

we have thus proved that

$$|g(x+h) - g(x)| \leq C.\, |h|$$

for all $x, h \in R$, which shows that g belongs to $\Lambda(1,1)$. This completes the proof.

PROPOSITION 6.5. <u>Given any</u> x_0 <u>such that</u> $0 < x_0 < 1$, <u>there exists a function</u> $g \in \Lambda(\infty, n)$ <u>such that</u> $g(x) \geq 0$ <u>for all</u> x <u>and</u> $g(x) = 0$ <u>only at the points</u> $x_0 + k$, <u>where</u> k <u>is an integer</u>.

PROOF. The function F defined by

$$F(x) = 1 + \sin 2\pi x$$

is periodic with period 1 and infinitely differentiable and hence belongs to $\Lambda(n, n)$ for each positive integer n. Moreover $F(x) \geq 0$ for all x and $F(x) = 0$ only at the points $k + \frac{3}{4}$, where k is an integer. Set

$$g(x) = F(x - x_0 + \tfrac{3}{4})$$

Then g satisfies our requirements.

COROLLARY 6.6. <u>Let</u> $0 < x_0 < 1$. <u>There exists a function</u> $g \in \lambda(\alpha, n)$ $(\alpha < n)$ <u>such that</u> $g(x) \geq 0$ <u>for all</u> x <u>and</u> $g(x) = 0$ <u>only at the points</u> $x_0 + k$, <u>where</u> k <u>is an integer</u>.

PROOF. This follows from the fact that if $\alpha < n$, then by virtue of Theorem 1.2, we have $\Lambda(n, n) \subset \lambda(\alpha, n))$.

THEOREM 6.7. <u>For each point</u> x_0 <u>in</u> U, <u>there is a function</u> f <u>in</u> $\lambda(\alpha, n)$ <u>such that</u> f <u>peaks at</u> x_0 <u>relative to</u> $\mathcal{J} \cdot \lambda(\alpha, n)$.

PROOF. Since $x_0 \in U$, we have $-1 \leq x_0 \leq 0$. By the invariance of $\lambda(\alpha, n)$ and $\|\cdot\|$ under translation, we may suppose that $-1 < x_0 < 0$. Let g be a function in $\lambda(\alpha, n)$ such that $0 \leq g(x) \leq 1$, for all $x \in U$, $g(x_0) = 0$ and $g(x) \neq 0$

for $x \neq x_0$. Then $\| g \|_{\alpha} \leq \| g \|$ and $\| g \|_{\alpha} \leq \| g \|$

Set

$$f = 1 - \frac{g}{\| g \|_{\alpha} + \| g \|_{\alpha}}$$

Then (i) $0 \leq f(x) \leq 1$ for all $x \in U$ (ii) $f(x) = 1$ if and only if $x = x_0$ and (iii) $\| f \|_{\alpha} < 1$. Thus $f \in \lambda(\alpha, n)$ and $\tilde{f}$ peaks at x_0 relative to $\mathcal{J}$. $\lambda(\alpha, n)$.

PROPOSITION 6.8. <u>Let</u> $0 < x_0 < \frac{1}{2}$. <u>Suppose</u> $0 < \alpha < 1$. <u>Then there exists a function</u> F <u>belonging to</u> $\lambda(\alpha, 2)$ <u>satisfying the conditions</u>

$$|F(x)| < x_0^{\alpha}$$

$$\Delta^2_{x_0} F(0) = x_0^{\alpha}$$

<u>and</u>

$$|\Delta^2_{x} F(0)| < x^{\alpha}$$

<u>for all</u> $x \neq x_0$

PROOF. Define a continuous function F on $[0,1]$ by

$$F(x) = \begin{cases} -\dfrac{x_0^{\alpha-1}}{2} \cdot x & 0 \leq x \leq x_0 \\ \dfrac{x_0^{\alpha-1}}{2}(x - 2x_0) & x_0 \leq x \leq 2x_0 \\ 0 & 2x_0 \leq x \leq 1 \end{cases}$$

Then

$$F(x_0) = -\frac{x_0^{\alpha}}{2}$$

and F is linear in each of the intervals $[0, x_0]$ and $[x_0, 2x_0]$ with $F(0) = F(2x_0) = 0$. Extend F to R by periodicity so that

$$F(x+1) = F(x)$$

for all $x \in R$. It is clear that

$$|F(x)| \leq \frac{x_0^{\alpha}}{2} < x_0^{\alpha}$$

and that $F \in \Lambda(1, 1)$ and hence $\in \lambda(\alpha, 2)$ since

$$\Lambda(1, 1) \subset \lambda(\alpha, 1) \subset \lambda(\alpha, 2)$$

Now

$$\Delta_{x_0}^2 F(0) = F(2x_0) - 2F(x_0) + F(0)$$
$$= x_0^{\alpha}$$

We now assert that if $x \neq x_0$, then

$$|\Delta_x^2 F(0)| < x^{\alpha}$$

CASE I. Let $0 \leq x \leq \frac{x_0}{2}$, then $\Delta_x^2 F(0) = 0$ and the required inequality is obvious.

CASE II. $\frac{x_0}{2} \leq x \leq x_0$. Let $x = \lambda x_0$. Then $\frac{1}{2} \leq \lambda \leq 1$ and

$$\Delta_x^2 F(0) = F(2x) - 2F(x) + F(0)$$

$$= - \frac{x_0^{\alpha-1}}{2} \left[2x - 2x_0 + 2x \right]$$

$$= - x_0^{\alpha} \quad (2\lambda - 1)$$

Thus

$$\frac{\left| \Delta_x^2 F(0) \right|}{x^{\alpha}} = \frac{x_0^{\alpha}(2\lambda-1)}{x_0^{\alpha} \cdot \lambda^{\alpha}} = \frac{2\lambda-1}{\lambda^{\alpha}}$$

$$\leq \frac{2\lambda-1}{\lambda} = 2 - \frac{1}{\lambda} \leq 2-1 = 1$$

CASE III. If $x > x_0$, then $x^{\alpha} > x_0^{\alpha}$ so that

$$\left| \Delta_x^2 F(0) \right| = \left| F(2x) - 2F(x) \right|$$

$$\leq \left| F(2x) - F(x) \right| + \left| F(x) \right|$$

$$\leq 2. \| F \|_{\alpha} \leq x_0^{\alpha} < x^{\alpha}$$

PROPOSITION 6.9. <u>Suppose</u> $0 < \alpha < 1$. <u>Then there exists a function</u> $F \in \lambda(\alpha, 2)$ <u>satisfying the conditions</u>

$$\left| F(x) \right| < \left(\tfrac{1}{2}\right)^{\alpha}$$

$$\Delta_{\frac{1}{2}}^2 F(\tfrac{1}{4}) = \left(\tfrac{1}{2}\right)^{\alpha}$$

<u>and</u>

$$\left| \Delta_x^2 F(\tfrac{1}{4}) \right| < x^{\alpha}$$

<u>for</u> $x \neq \frac{1}{2}$.

PROOF. The required function is defined by

$$F(x) = \begin{cases} -(\frac{1}{2})^{\alpha} & (x - \frac{1}{4}) \qquad \frac{1}{4} < x \leq \frac{3}{4} \\ (\frac{1}{2})^{\alpha} & (x - \frac{5}{4}) \qquad \frac{3}{4} \leq x \leq \frac{5}{4} \end{cases}$$

and

$$F(x) = F(x+1)$$

for all $x \in R$. The rest of the proof is as in Proposition 6.8 and hence omitted.

THEOREM 6.10. <u>Let</u> $0 < \alpha < 1$. <u>For each point</u> $(x_0, h_0) \in V$ <u>there is a function</u> f <u>in</u> $\lambda(\alpha, 2)$ <u>such that</u> $\tilde{f}$ <u>peaks at</u> (x_0, h_0) <u>relative to</u> $\int \cdot \lambda(\alpha, 2)$.

PROOF. (i) Let $y_0 = (x_0, h_0) \in V$ with $0 \leq x_0 \leq 1$ $0 < h_0 < \frac{1}{2}$. By the invariance of $\lambda(\alpha, 2)$ and the norm under translation we may suppose that $x_0 = 0$. Let f be the function given by Proposition 6.8. Then f has the following properties

$$|f(x)| < x^{\alpha}$$

$$|\Delta^2_{h_0} f(0)| = h_0^{\alpha}$$

$$|\Delta^2_{x} f(0)| < x^{\alpha} \qquad \text{for } x \neq h_0$$

and f is periodic with period 1, which is linear in each of the intervals $[0, h_0]$, $[h_0, 2h_0]$ and zero on $[2h_0, 1]$. Let y be the point $(1, 1 + h_0) \in V$. Then $\tilde{f}(y_0) = \tilde{f}(y) = 1$, $|\tilde{f}(z)| < 1$ if $z \in W$ $z \neq y$, $z \neq y_0$ and $\tilde{g}(y_0) = \tilde{g}(y)$ for all $g \in \lambda(\alpha, 2)$ so $\tilde{f}$ peaks at y_0 relative to $\int \cdot \lambda(\alpha, 2)$.

(ii) Let $y_0 = (x_0, h_0)$, $0 \leq x_0 \leq 1$, $h_0 = \frac{1}{2}$. By the invariance of $\lambda(\alpha, 2)$ and norm under translation, we may suppose that $(x_0, h_0) = (\frac{1}{4}, \frac{1}{2})$. Let f be the function in $\lambda(\alpha, 2)$ given by Proposition 6.9. Then $\tilde{f}$ satisfies our requirements and it peaks at y_0 relative to $j.\lambda(\alpha, 2)$.

Our result on extreme points can be stated as follows.

THEOREM 6.11. <u>A functional</u> φ <u>in</u> $(\lambda(\alpha, 2))^*$ <u>where</u> $0 < \alpha < 1$ <u>is an extreme point of the unit sphere of</u> $(\lambda(\alpha, 2))^*$ <u>if and only if it is either of the form</u>

(1) $\varphi(f) = \mu f(\rho) \qquad f \in \lambda(\alpha, 2)$

<u>for</u> ρ <u>in</u> R <u>and</u> μ <u>a complex number with</u> $|\mu| = 1$ <u>or of the form</u>

(2) $\varphi(f) = \mu \cdot \dfrac{\Delta_t^2 f(s)}{|t|^\alpha} \qquad f \in \lambda(\alpha, 2)$

<u>for</u> $s \in R$ <u>and</u> $0 < t \leq \frac{1}{2}$ <u>and</u> μ <u>is a complex number with</u> $|\mu| = 1$

PROOF. Setting $X = W$ and $A = \{\tilde{f} : f \in \lambda(\alpha, 2)\}$ we see that by Proposition 6.1, every extreme point of the unit sphere of $j.\lambda(\alpha, 2)^*$ is given by

(3) $\varphi(f) = \eta \tilde{f}(x) \qquad \tilde{f} = j.(\lambda(\alpha, 2))^*$

for a point x in W and η a complex number with $|\eta| = 1$. This gives the representation (1) and (2) according as x belongs to U or x belongs to V as W is the disjoint union of U and V.

On the other hand, Theorems 6.7 and 6.10 show that for each point $x \in W$, there is a function $f \in \lambda(\alpha, 2)$ such that $\tilde{f}$

peaks at x relative to $j.\lambda(\alpha, 2)$. Then by Proposition 6.3 every φ given by (3) is an extreme point of the unit sphere of $(\lambda(\alpha, 2))^*$. This completes the proof of the theorem.

7. ISOMETRIES OF $\lambda(\alpha, 2)$

We shall now obtain some properties of the isometries on the space $\lambda(\alpha, 2)$ when $0 < \alpha < 1$. The most obvious isometries of the space $\lambda(\alpha, 2)$ have the following form.

Let β be a real number and η a complex number with absolute value one. Consider the mapping

$$U, V : \lambda(\alpha, 2) \longrightarrow \lambda(\alpha, 2)$$

given by

$$(1) \quad U f(x) = \eta f(\beta + x) \qquad x \in R$$

and

$$(2) \quad V f(x) = \eta f(\beta - x) \qquad x \in R$$

It is easy to verify that U, V are linear mappings of $\lambda(\alpha, 2)$ onto itself and also satisfy the relations

$$(3) \quad \| U f \| = \| f \| \qquad f \in \lambda(\alpha, 2)$$

and

$$(4) \quad \| V f \| = \| f \| \qquad f \in \lambda(\alpha, 2)$$

Thus U and V are linear isometries of $\lambda(\alpha, 2)$ onto itself.

We shall now establish some results connected with isometries. Let Ext.S* denote the set of all extreme points of the unit sphere of $(\lambda(\alpha, 2))^*$. Since T is a linear isometry of $\lambda(\alpha, 2)$ onto itself, it follows that the adjoint operator T^* is also a linear isometry of $(\lambda(\alpha, 2))^*$ leaving Ext.S* invariant. That is

(5) $T^* \operatorname{Ext}(S^*) = \operatorname{Ext}. S^*$.

PROPOSITION 7.1. <u>Let</u> f <u>be a function in</u> $\lambda(\alpha, 2)$. <u>Then</u> f <u>is a constant function</u> f <u>if and only if</u>

(6) $$\left\{ |\varphi(f)| \; ; \; \varphi \in \operatorname{Ext}. S^* \right\}$$

<u>consists of atmost two numbers.</u>

PROOF. If f is a constant, it is evident from the representation of extreme points given by Theorem 6.11 that the set (6) consists of atmost two elements. To prove the converse, let us suppose that (6) consists of atmost two elements. Because f belongs to $\lambda(\alpha, 2)$, it follows that 0 is in the closure of

$$\left\{ \frac{\Delta_h^2 f(x)}{|h|^\alpha} \; ; \; x, h \in R, \; h \neq 0 \right\}$$

and these are elements of Ext.S^* by Theorem 6.11. Hence 0 must belong to the set (6). If there is no other element in (6), then by Theorem 6.11 $f(x) = 0$ for all $x \in R$ which implies that f is the zero function. On the other hand suppose that the set (6) is given by $(0, \beta)$, where $\beta > 0$. Since $f \in \lambda(\alpha, 2)$ by definition, there exists $\epsilon > 0$ such that

$$\left| \frac{\Delta_h^2 f(s)}{h^\alpha} \right| < \beta$$

if $|h| < \epsilon$ and for all $s \in R$. Because the set (6) is $(0, \beta)$, by virtue of Theorem 6.11, we conclude that each number

$$\frac{|\Delta_h^2 f(s)|}{|h|^\alpha}$$

is equal to either 0 or ρ . Thus

$$\Delta_h^2 f(s) = 0$$

for all s if $|h| < \rho$. This implies that the second difference of f is identically zero. Hence f must be a linear function. By periodicity f must be a constant. This completes the proof.

PROPOSITION 7.2. <u>Let</u> T <u>be a linear isometry of</u> $\lambda(\alpha, 2)$. <u>Then there is a complex number</u> η <u>with</u> $|\eta| = 1$ <u>such that</u>

(7) $$\{T^* \varphi_x : x \in R\} = \{\eta \varphi_x : x \in R\}$$

PROOF. Let g be a constant function, then by Proposition 7.1

$$\{|\varphi(g)| : \varphi \in \text{Ext.}S^*\}$$

consists of two numbers. Now

$$\begin{aligned}\{|\varphi(T g)| &: \varphi \in \text{Ext.}S^*\} \\ &= \{|T^* \varphi(g)| : \varphi \in \text{Ext.}S^*\} \\ &= \{|\varphi(g)| : \varphi \in T^* \text{Ext.}S^*\} \\ &= \{|\varphi(g)| : \varphi \in \text{Ext.}S^*\}\end{aligned}$$

by virtue of (5). Hence

$$\{|\varphi(T g)| : \varphi \in \text{Ext.}S^*\}$$

consists of atmost two numbers and by Proposition 7.1, Tg must be a constant function. Let

$$T g(x) = \eta \qquad x \in R$$

Then

$$
\begin{aligned}
T^*\{\varphi_x : x \in R\} &= T^*\{\varphi : \varphi \in \mathrm{Ext.}S^*,\ \varphi(T\,g) = \eta\} \\
&= \{T^*\varphi : \varphi \in \mathrm{Ext.}S^*,\ \varphi(T\,g) = \eta\} \\
&= \{T^*\varphi : T^*\varphi \in \mathrm{Ext.}S^*,\ T^*\varphi(g) = \eta\} \\
&= \{\psi : \psi \in \mathrm{Ext.}S^*,\ \psi(g) = \eta\} \\
&= \{\eta\,\varphi_x : x \in R\}
\end{aligned}
$$

which is (7). Since T is an isometry, $|\eta| = 1$.

PROPOSITION 7.3. <u>If</u> $s, t \in R$ <u>and</u> $|t| \leq \frac{1}{2}$, <u>then</u>

$$\| \varphi_{s+2t} - 2\,\varphi_{s+t} + \varphi_s \| = |t|^{\alpha}$$

PROOF. We have already proved that

$$\| \varphi_{s+2t} - 2\,\varphi_{s+t} + \varphi_s \| \leq |t|^{\alpha}$$

It is therefore enough to prove the opposite inequality. Suppose that $|h| < \frac{1}{2}$. By the invariance of $\lambda(\alpha, 2)$ and the norm under translation, we may suppose that $s = 0$ and $0 < h < \frac{1}{2}$. Let f be the function given by Proposition 6.8. Then

$$\| f \| = 1$$

and

$$
\begin{aligned}
|\varphi_{s+2t}(f) - 2\,\varphi_{s+t}(f) + \varphi_s(f)| &= |\Delta_t^2 f(s)| \\
&= |t|^{\alpha}
\end{aligned}
$$

Therefore

$$(8) \quad \| \varphi_{s+2t} - 2\varphi_{s+t} + \varphi_s \| \geq |t|^{\alpha}$$

if $|t| < \frac{1}{2}$. The Proposition 6.9 establishes the inequality (8) when $t = \frac{1}{2}$. This completes the proof.

CHAPTER 5 : APPLICATIONS

8. A MULTIPLIER PROBLEM

Let $L^1 = L^1[0,1]$ be the space of periodic complex valued Lebesgue measurable functions on $[0, 1]$ and $L^\infty = L^\infty[0,1]$ consists of those L^1 - functions which are essentially bounded. $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denotes the norms in L^1 and L^∞ respectively.

Let $U(\alpha, n)$ denote the sub class of L^1 consisting of those functions f whose nth difference satisfy the integral Lipschits condition

$$(1) \qquad \int_0^1 |\Delta_\delta^n f(t)|\, dt \leq B.\ \delta^\alpha$$

where B is a constant depending only on the function f, while $V(\alpha, n)$ is the sub class satisfying the condition

$$(2) \qquad \int_0^1 |\Delta_\delta^n f(t)|\, dt = o(\delta^\alpha)$$

We shall now identify $U(\alpha, n)$ and $V(\alpha, n)$ with certain multiplier classes. Let us recall that if f and g are any two L^1 - functions, then their convolution product $f * g$ is defined by

$$(f * g)(x) = \int_0^1 f(t).g(x-t).dt$$

and $f * g$ is an L^1-function. The multiplier class $(L^\infty, \Lambda(\alpha, n))$ (respy. $(L^\infty, \lambda(\alpha, n))$) is the collection of all those f in L^1 such that $f * g \in \Lambda(\alpha, n)$ (respy. $\lambda(\alpha, n)$) for each $g \in L^\infty$.

THEOREM 8.1.

(i) $(L^\infty, \Lambda(\alpha, n)) = U(\alpha, n)$

(ii) $(L^\infty, \lambda(\alpha, n)) = V(\alpha, n)$

PROOF. We shall prove only (i). The proof of (ii) is analogous.

Suppose first that f is in $U(\alpha, n)$ so that (i) holds and let g be any L^∞ function. Then

$$\begin{aligned} |\Delta_\delta^n f * g(x)| &= \left|\int_0^1 \Delta_\delta^n f(x-t).\, g(t).\, dt\right| \\ &\leq \|g\|_\infty . \int_0^1 |\Delta_\delta^n f(x-t)| .dt \\ &\leq \|g\|_\infty . B.\, \delta^\alpha \end{aligned}$$

Thus $f * g \in \Lambda(\alpha, n)$ and $U(\alpha, n) \subset (L^\infty, \Lambda(\alpha, n))$.

Suppose that f belongs to $(L^\infty, \Lambda(\alpha, n))$. We have seen that $\Lambda(\alpha, n)$ is a Banach space when equipped with the norm

$$\|h\| = \sup_{x,\delta} \left\{ |h(x)| \;,\; \frac{|\Delta_\delta^n f(x)|}{|\delta|^\alpha} \right\}$$

We associate with f a continuous linear operator T from a Banach space L^∞ to the Banach space $\Lambda(\alpha, n)$ by putting

$$Tg = f * g \qquad g \text{ in } L^\infty .$$

The fact that T is continuous is an immediate consequence of the closed graph theorem. In fact if $g_n \to g$ in L^∞ and $Tg_n \to h$ in $\Lambda(\alpha, n)$ then for each x in $[0,1]$, we have

$$\begin{aligned} |Tg(x) - h(x)| &\leq |f * g(x) - f * g_n(x)| \\ &\quad + |Tg_n(x) - h(x)| \\ &\leq \|f\|_1 . \|g - g_n\|_\infty \\ &\quad + \|Tg_n - h\| \\ &\to 0 \quad \text{as } n \to \infty \end{aligned}$$

Thus $Tg = h$ so that T is continuous. Hence there is a constant $\| T \|$ such that

$$\| Tg \| \leq \| T \| \cdot \| g \|_{\infty}$$

The definition of Tg in $\Lambda(\alpha, n)$ implies that for any pair x, δ of points in $[0,1]$ we have

$$\left| \int_0^1 \Delta_\delta^n f(x-t) \cdot g(t) \cdot dt \right| \leq \| T \| \cdot \| g \|_{\infty} \cdot \delta^{\alpha}$$

we take

$$g(t) = \exp. \left\{ -i \cdot \arg \Delta_\delta^n f(x-t) \right\}$$

to obtain

$$\int_0^1 \left| \Delta_\delta^n f(x-t) \right| dt \leq \| T \| \cdot \delta^{\alpha}$$

This proves that $f \in U(\alpha, n)$. This completes the proof of (i).

9. LACUNARY INTERPOLATION OF ENTIRE FUNCTIONS

While the problem of lacunary interpolation for polynomials and trigonometric polynomials have drawn the attention of many a mathematician for the past few decades, see for e.g. [1,2,6,12-14,16] in the case of entire function of exponential type, being the natural counterpart for approximants in the case of infinite interval, only the problem of interpolation, where the values of the function and the consecutive derivatives are given at prescribed points, is so far known. We shall now initiate the study of lacunary interpolation for entire functions. The most elementary case of (0,2) interpolation is being persued at the moment, the remaining ones being retained for future study. For properties of entire functions of exponential type one refers [3,5] .

We first notice that a trigonometric polynomial of order n is an entire function of exponential type n. Our (0,2) lacunary interpolation problem can be stated as follows.

Given the distinct points

$$(1) \qquad \frac{k\pi}{\sigma} \quad (k = 0, \pm 1, \pm 2, \ldots)$$

where σ is a given positive number and arbitrary sequence of numbers

$$\left\{ a_k \right\}_{k=-\infty}^{k=+\infty} \quad \text{and} \quad \left\{ b_k \right\}_{k=-\infty}^{k=+\infty}$$

it is to be decided whether or not there exists an entire function $f_\sigma(x)$ of exponential type 2σ, satisfying the conditions

$$f_\sigma\left(\frac{k\pi}{\sigma}\right) = a_k \quad \text{and} \quad f_\sigma''\left(\frac{k\pi}{\sigma}\right) = b_k$$

The natural questions that would arise are

(i) Does there exist such an f_σ at all?

(ii) If there is such an f_σ, is it uniquely determined?

(iii) If f_σ exists and is unique, can it be represented in a suitable form?

and (iv) assuming that a_k's are the values of a given function $f(x)$ defined on the real line R at the set of points x_k what are the conditions under which $f_\sigma(x)$ converges to $f(x)$ uniformly as $\sigma \to \infty$?

Since the given points are in arithmetic progression, it is easily seen that if $U(x)$ is an entire function of exponential type 2σ taking the value 1 at $x = 0$ and $U(\frac{k\pi}{\sigma})$ is equal to 0 for k different from 0 and $U''(\frac{k\pi}{\sigma})$ is equal to 0 for all k, then $U(x-\frac{k\pi}{\sigma})$ will take the value 1 at $\frac{k\pi}{\sigma}$ and will be zero at other points. Moreover the second derivative vanishes at all points of the sequence (1). Similarly if $V(x)$ is an entire function, 0 at all points and $V''(0) = 1$ while $V''(\frac{k\pi}{\sigma}) = 0$ for $k \neq 0$, then $V(x-\frac{k\pi}{\sigma})$ has the same properties except that it takes the value 1 at $\frac{k\pi}{\sigma}$ instead of 0. If we said that

$$(2) \quad F(x) = \sum_{-\infty}^{\infty} a_k\, U\left(x - \frac{k\pi}{\sigma}\right) + \sum_{-\infty}^{\infty} b_k\, V\left(x - \frac{k\pi}{\sigma}\right)$$

then $F(x)$ satisfies the conditions

$$(3) \quad F\left(\frac{k\pi}{\sigma}\right) = a_k \;;\; F''\left(\frac{k\pi}{\sigma}\right) = b_k$$

So the required function will be of the form (2) provided it is an

entire function of exponential type 2σ. This automatically implies that the series on the R.H.S of (2) will have to be convergent and suitable conditions on a_k's and b_k's are necessary. Our problem first reduces to the problem of finding the entire functions U and V as mentioned above.

THEOREM 9.1. Let $\sigma > 0$ be given. If

$$x_k = \frac{k\pi}{\sigma} \qquad k = 0, \pm 1, \pm 2, \ldots\ldots$$

then the even entire functions U_σ and V_σ of exponential type 2σ satisfying the conditions

$$(4) \quad U_\sigma(0) = 1, \quad U_\sigma\left(\frac{k\pi}{\sigma}\right) = 0 \qquad k = \pm 1, \pm 2, \ldots\ldots$$

$$(5) \quad U''_\sigma\left(\frac{k\pi}{\sigma}\right) = 0 \quad \text{for } k = 0, \pm 1, \pm 2, \ldots\ldots$$

and

$$(6) \quad V_\sigma\left(\frac{k\pi}{\sigma}\right) = 0 \qquad k = 0, \pm 1, \pm 2, \ldots\ldots$$

$$(7) \quad V''_\sigma(0) = 1, \quad V''_\sigma\left(\frac{k\pi}{\sigma}\right) = 0 \quad \text{for } k = \pm 1, \pm 2, \ldots\ldots$$

are given by

$$(8) \quad U_\sigma(x) = \frac{\sin \sigma x}{\sigma x}\left\{1 - x\int_0^x \frac{\sin \sigma t - \sigma t}{\sigma t^3}\, dt\right\}$$

and

$$(9) \quad V_\sigma(x) = \frac{\sin \sigma x}{2\sigma}\int_0^x \frac{\sin \sigma t}{\sigma t}\, dt$$

U_σ and V_σ are also unique.

PROOF. Let us first obtain the explicit form of the function U(x). Because of condition (4) imposed on U, U has the form

$$U(x) = \frac{\sin \sigma x}{\sigma x} \varphi(x)$$

where $\varphi(x)$ is an even entire function of exponential type σ with $\varphi(0) = 1$. Differentiating twice we get

$$U''(x) = \left(\frac{\sin \sigma x}{\sigma x}\right)'' \varphi(x) + 2\left(\frac{\sin \sigma x}{\sigma x}\right)' \varphi'(x) + \frac{\sin \sigma x}{\sigma x} \varphi''(x)$$

Now

$$\left(\frac{\sin \sigma x}{\sigma x}\right)' = \frac{\sigma x \cos \sigma x - \sin \sigma x}{\sigma x^2}$$

and

$$\left(\frac{\sin \sigma x}{\sigma x}\right)'' = \frac{-\sigma^2 x^2 \sin \sigma x - 2\sigma x \cos \sigma x + 2 \sin \sigma x}{\sigma x^3}$$

The values of $\frac{\sin \sigma x}{\sigma x}$, $\left(\frac{\sin \sigma x}{\sigma x}\right)'$ and $\left(\frac{\sin \sigma x}{\sigma x}\right)''$ at $x = 0$ and $x = \frac{k\pi}{\sigma}$ for $k = \pm 1, \pm 2, \ldots$ are obtained below.

At $x = 0$

$$\left(\frac{\sin \sigma x}{\sigma x}\right)_{x=0} = 1$$

$$\left(\frac{\sin \sigma x}{\sigma x}\right)'_{x=0} = \lim_{x \to 0} \frac{\sigma x \cos \sigma x - \sin \sigma x}{\sigma x^2}$$

$$= \lim_{x \to 0} \frac{1}{\sigma x^2} \left[\sigma x\left(1 - \frac{\sigma^2 x^2}{2!} + \cdots\right) - \left(\sigma x - \frac{\sigma^3 x^3}{3!} + \cdots\right)\right]$$

$$= 0$$

$$\left(\frac{\sin \sigma x}{\sigma x}\right)''_{x=0} = \lim_{x \to 0} \frac{1}{\sigma x^3}\left[-\sigma^2 x^2\left(\sigma x - \frac{\sigma^3 x^3}{3!} + \cdots\right) - 2\sigma x\left(1 - \frac{\sigma^2 x^2}{2!} + \cdots\right) + 2\left(\sigma x - \frac{\sigma^3 x^3}{3!} + \cdots\right)\right]$$

$$= \frac{1}{\sigma}\left[-\sigma^3 + 2\cdot\frac{\sigma^3}{2!} - 2\frac{\sigma^3}{3!}\right]$$

$$= -\sigma^2/3$$

At $x = \frac{k\pi}{\sigma}$

$$\left(\frac{\sin \sigma x}{\sigma x}\right)_{\frac{k\pi}{\sigma}} = \frac{\sin k\pi}{k\pi} = 0$$

$$\left(\frac{\sin \sigma x}{\sigma x}\right)'_{\frac{k\pi}{\sigma}} = \frac{k\pi \cos k\pi - \sin k\pi}{\sigma \cdot \frac{k^2\pi^2}{\sigma^2}}$$

$$= \frac{\sigma}{k\pi}(-1)^k$$

$$\left(\frac{\sin \sigma x}{\sigma x}\right)''_{\frac{k\pi}{\sigma}} = \frac{-\sigma^2 \frac{k^2\pi^2}{\sigma^2}\sin k\pi - 2\sigma\frac{k\pi}{\sigma}\cos k\pi + 2\sin k\pi}{\sigma \cdot k^3 \cdot \pi^3/\sigma^3}$$

$$= -2(-1)^k \cdot \left(\frac{\sigma}{k\pi}\right)^2$$

Since $U''_\sigma(0) = 0$, we have

$$0 = -\frac{\sigma^2}{3}\varphi(0) + 1 \cdot \varphi''(0) = -\frac{\sigma^2}{3} + \varphi''(0)$$

so that

$$\varphi''(0) = \frac{\sigma^2}{3}$$

On the other hand, by virtue of condition (5) we have $U''_\sigma\left(\frac{k\pi}{\sigma}\right) = 0$

so that

$$0 = -2(-1)^k \cdot \left(\frac{\sigma}{k\pi}\right)^2 \varphi\left(\frac{k\pi}{\sigma}\right)$$
$$+ 2(-1)^k \cdot \left(\frac{\sigma}{k\pi}\right) \varphi'\left(\frac{k\pi}{\sigma}\right)$$

From this we get

$$\frac{k\pi}{\sigma} \varphi'\left(\frac{k\pi}{\sigma}\right) - \varphi\left(\frac{k\pi}{\sigma}\right) = 0$$

If we set

$$F(x) = x\varphi'(x) - \varphi(x)$$

we see that

$$F(0) = -1, \quad F\left(\frac{k\pi}{\sigma}\right) = 0$$

Since $F(x)$ is an even entire function of exponential type σ we may take $F(x) = -\frac{\sin \sigma x}{\sigma x}$. Hence φ satisfies the differential equation

$$(10) \quad x\varphi'(x) - \varphi(x) = -\frac{\sin \sigma x}{\sigma x}$$

To solve theequation (10), we make the substitution

$$g(x) = \varphi(x) - 1$$

so that

$$g(0) = 0 \text{ and } g'(x) = \varphi'(x)$$

Then (10) becomes

$$x g'(x) - g(x) = -\frac{\sin \sigma x - \sigma x}{\sigma x}$$

Integrating,

$$\frac{g(x)}{x} = -\int_0^x \frac{\sin \sigma t - \sigma t}{\sigma t^3} \cdot dt + c$$

Since φ is even, so is g and hence the constant of integration C becomes zero. Then

$$g(x) = -x \int_0^x \frac{\sin \sigma t - \sigma t}{\sigma t^3} \cdot dt$$

so that

$$\varphi(x) = 1 - x \int_0^x \frac{\sin \sigma t - \sigma t}{\sigma t^3} \cdot dt$$

Therefore

$$U_\sigma(x) = \frac{\sin \sigma x}{\sigma x} \left\{ 1 - x \int_0^x \frac{\sin \sigma t - \sigma t}{\sigma t^3} \cdot dt \right\}$$

This gives (8).

We shall proceed to find V. We recall the conditions (6) and (7) as

$$V\left(\frac{k\pi}{\sigma}\right) = 0 \qquad k = 0, \pm 1, \pm 2, \ldots$$

$$V''(0) = 1, \; V''\left(\frac{k\pi}{\sigma}\right) = 0 \qquad k = \pm 1, \pm 2, \ldots$$

Because of condition (6), we can assume

(11) $V(x) = \sin \sigma x \; \psi(x)$

where $\psi(x)$ is an odd entire function of exponential type σ.

Differentiating twice, we obtain

$$V''(x) = -\sigma^2 \sin\sigma x\, \psi(x) + 2\sigma\cos\sigma x\, \psi'(x) + \sin\sigma x . \psi''(x)$$

$$1 = V''(0)$$

$$= 2\sigma\psi'(0)$$

and

$$0 = V''\left(\frac{k\pi}{\sigma}\right) \quad \text{for } k = \pm 1, \pm 2, \ldots$$

$$= 2\sigma\cos k\pi \,.\, \psi'\left(\frac{k\pi}{\sigma}\right)$$

That is

$$\psi'(0) = \frac{1}{2\sigma} \quad \text{and} \quad \psi'\left(\frac{k\pi}{\sigma}\right) = 0$$

Because of the above conditions, we can take

$$(12) \qquad \psi'(x) = \frac{1}{2\sigma} \cdot \frac{\sin\sigma x}{\sigma x}$$

Integrating

$$\psi(x) = \frac{1}{2\sigma} \int_0^x \frac{\sin\sigma t}{\sigma t}\, dt$$

the constant of integration being zero because $\psi(x)$ is odd. Hence

$$V_\sigma(x) = \frac{\sin\sigma x}{2\sigma} \int_0^x \frac{\sin\sigma t}{\sigma t} \cdot dt$$

This gives (9).

It is clear that the integrands given in (8) and (9) are entire functions and so the functions U_σ, V_σ given in the formulae (8) and (9) are actually entire. We shall now prove

that U_σ and V_σ are unique. We shall prove the uniqueness of U_σ only, the proof being similar in the case of V_σ.

Suppose U_1 is another entire function of exponential type 2σ, satisfying the conditions

$$U_1(0) = 1, \quad U_1\left(\frac{k\pi}{\sigma}\right) = 0 \quad (k = \pm 1, \pm 2, \ldots)$$

$$U_1''\left(\frac{k\pi}{\sigma}\right) = 0 \quad (k = 0, \pm 1, \pm 2, \ldots)$$

Let

$$F(z) = U_\sigma(z) - U_1(z)$$

Then

$$F\left(\frac{k\pi}{\sigma}\right) = 0; \quad F''\left(\frac{k\pi}{\sigma}\right) = 0$$

for all k. And F is an entire function of type 2σ. Since an entire function of exponential type σ, vanishing at the points

$$\frac{k\pi}{\sigma} \quad (k = 0, \pm 1, \pm 2, \ldots)$$

is given by

$$C. \sin \sigma z$$

[See 17, P.180], by the condition on $F(z)$, we have

(A) $\quad F(z) = C. \sin \sigma z. g(z)$

where $g(z)$ is an odd entire function of exponential type σ. The condition

$$F''\left(\frac{k\pi}{\sigma}\right) = 0 \quad \text{for all } k$$

implies

$$g'\left(\frac{k\pi}{\sigma}\right) = 0 \quad \text{for all } k$$

and hence

$$g'(z) = C_1 \sin \sigma z$$

which is an odd function. This contradicts the fact that $g(z)$ is odd. Hence C of the equation (A) must be zero. So

$$F(z) = 0$$

which means

$$U_\sigma(z) = U_1(z)$$

This completes the proof of Theorem 9.1.

Let us now proceed to find the bounds of $|U_\sigma(x)|$ and $|V_\sigma(x)|$.

THEOREM 9.2. <u>$U_\sigma(x)$ and $V_\sigma(x)$ of Theorem 9.1 have the following bounds</u>

$$|U_\sigma(x)| \leq \frac{e^2 + 4e - 1}{2e}$$

$$|V_\sigma(x)| \leq \frac{8 + \pi^2}{4\pi\sigma^2}$$

PROOF. From (8)

$$U_\sigma(x) = \frac{\sin \sigma x}{\sigma x} - \frac{\sin \sigma x}{\sigma x} \int_0^x \frac{\sin \sigma t - \sigma t}{\sigma t^3}\, dt$$

Now

$$\left| \int_0^1 \frac{\sin t - t}{t^3}\, dt \right| \leq \int_0^1 \left| \frac{\sin t - t}{t^3} \right| dt$$

$$\leq \int_0^1 [\sin h.1 - 1]\, dt$$

$$= [\sin 1 - 1].1$$

$$= \frac{e^2 - 2e - 1}{2e}$$

If $t \geqslant 1$

$$\left| \int_1^x \frac{\sin t - t}{t^3}\, dt \right| \leq \int_1^x \frac{t - \sin t}{t^3}\, dt$$

$$\leq 2 \int_1^x \frac{1}{t^2}\, dt$$

$$= 2\left(1 - \frac{1}{x}\right) \leq 2$$

Hence

$$\left| \int_0^x \frac{\sin t - t}{t^3}\, dt \right| \leq \frac{e^2 - 2e - 1}{2e} + 2$$

$$= \frac{e^2 + 2e - 1}{2e}$$

so we find

$$\left| \int_0^x \frac{\sin \sigma t - \sigma t}{\sigma t^3}\, dt \right| = \left| \sigma \int_0^{x\sigma} \frac{\sin t' - t'}{t'^3}\, dt' \right|$$

$$\leq \sigma . \frac{e^2 + 2e - 1}{2e}$$

using (8)

$$|U_\sigma(x)| \leq \left| \frac{\sin \sigma x}{\sigma x} \right| + \frac{|\sin \sigma x|}{|\sigma|} \left| \int_0^x \frac{\sin \sigma t - \sigma t}{\sigma t^3}\, dt \right|$$

(13)

$$\leq 1 + \frac{e^2 + 2e - 1}{2e} = \frac{e^2 + 4e - 1}{2e}$$

Having obtained the bounds of $|U_\sigma(x)|$, let us now get the bounds of $|V_\sigma(x)|$. From (12) we see that

$$\psi'(x) = \frac{1}{2\sigma}\,\frac{\sin\sigma x}{\sigma x}$$

so that, because $\psi(0)$ is odd, we have

$$\psi(x) = \frac{1}{2\sigma}\int_0^x \frac{\sin\sigma t}{\sigma t}\,.\,dt$$

$$= \frac{1}{2\sigma^2}\int_0^{x\sigma} \frac{\sin t}{t}\,.\,dt$$

If $0 \leq x \leq \pi/2$, then

$$\int_0^x \frac{\sin t}{t}\,dt \leq \int_0^x dt \leq \int_0^{\pi/2} dt = \pi/2$$

so that

$$\left|\int_0^x \frac{\sin t}{t}\,dt\right| \leq \pi/2$$

If $x > \pi/2$

$$\int_0^x \frac{\sin t}{t}\,dt = \int_0^{\pi/2} \frac{\sin t}{t}\,dt + \int_{\pi/2}^x \frac{\sin t}{t}\,dt$$

we have already seen that

(i) $$\left|\int_0^{\pi/2} \frac{\sin t}{t}\,dt\right| \leq \pi/2$$

Taking up the second integral, we see that

$$\int_{\pi/2}^x \frac{\sin t}{t}\,dt = -\frac{\cos t}{t}\Big|_{\pi/2}^x + \int_{\pi/2}^x \frac{\cos t}{t^2}\,dt$$

$$= -\frac{\cos x}{x} + \int_{\pi/2}^x \frac{\cos t}{t^2}\,dt$$

It is easy to observe that

(ii) $$\left| -\frac{\cos x}{x} \right| \leq \frac{2}{\pi}$$

But

(iii) $$\left| \int_{\pi/2}^{x} \frac{\cos t}{t^2}\, dt \right| \leq \int_{\pi/2}^{x} \frac{1}{t^2}\, dt = \frac{2}{\pi} - \frac{1}{x}$$
$$\leq \frac{2}{\pi}$$

Combining (i), (ii) and (iii), we obtain

$$\left| \int_0^x \frac{\sin t}{t}\, dt \right| \leq \left| \int_0^{\pi/2} \frac{\sin t}{t}\, dt \right| + \left| \int_{\pi/2}^{x} \frac{\sin t}{t}\, dt \right|$$
$$\leq \frac{\pi}{2} + \frac{2}{\pi} + \frac{2}{\pi}$$
$$= \frac{8 + \pi^2}{2\pi}$$

So

$$|\psi(x)| \leq \frac{1}{2\sigma^2} \left[\frac{8 + \pi^2}{2\pi} \right] = \frac{8 + \pi^2}{4\pi\sigma^2}$$

As we know from (11)

$$V_\sigma(x) = \sin \sigma x \; \psi(x)$$

we get

(14) $$|V_\sigma(x)| \leq 1 \cdot \frac{8 + \pi^2}{4\pi\sigma^2} = \frac{8 + \pi^2}{4\pi\sigma^2}$$

Thus the proof of the theorem is complete.

THEOREM 9.3. <u>If</u> $\{a_k\}$; $\{b_k\}$ <u>are two sequences of complex numbers such that</u>

(15) $$\sum_{k=-\infty}^{\infty}{}' \left| \frac{a_k}{k} \right| < \infty$$

and

(16) $$\sum_{k=-\infty}^{\infty} |b_k| < \infty$$

then

(17) $$R_\sigma(x) = \sum_{k=-\infty}^{\infty} a_k \, U\left(x - \frac{k\pi}{\sigma}\right) + \sum_{k=-\infty}^{\infty} b_k \, V\left(x - \frac{k\pi}{\sigma}\right)$$

<u>represents an entire function of exponential type</u> 2σ <u>satisfying the conditions</u>

(18) $R_\sigma(x_k) = a_k$; $R''_\sigma(x_k) = b_k$

PROOF. That the conditions (18) are satisfied from the representation (17) is obvious and it is enough to prove that $R_\sigma(x)$ does represent an entire function of exponential type 2σ. Setting

$$F_1(z) = \sum_{k=-\infty}^{\infty} a_k \, U\left(z - \frac{k\pi}{\sigma}\right)$$

and

$$F_2(z) = \sum_{k=-\infty}^{\infty} b_k \, V\left(z - \frac{k\pi}{\sigma}\right)$$

we shall show that F_1 and F_2 are entire functions of exponential type 2σ. Now

$$\varphi(z) = 1 - z \int_0^z \frac{\sin \sigma t - \sigma t}{\sigma t^3} \, dt$$

Since

$$\frac{\sin \sigma t - \sigma t}{\sigma t^3} = O\left(\frac{1}{t^2}\right)$$

for large t, it follows that $\varphi(z)$ is bounded for real z. Since $\varphi(z)$ has the same order and type as $\frac{\sin \sigma z - \sigma z}{\sigma z^3}$, we conclude that $\varphi(z)$ is an entire function of exponential type σ bounded on the real axis. Hence there exists a constant M such that

$$(19) \qquad |\varphi(x+iy)| \leq M e^{\sigma|y|}$$

Now

$$\begin{aligned} F_1(z) &= \sum_{-\infty}^{\infty} a_k \; U\left(z - \frac{k\pi}{\sigma}\right) \\ &= \sum_{-\infty}^{\infty} a_k \frac{\sin \sigma\left(z - \frac{k\pi}{\sigma}\right)}{\sigma\left(z - \frac{k\pi}{\sigma}\right)} \cdot \varphi\left(z - \frac{k\pi}{\sigma}\right) \\ &= \sum_{-\infty}^{\infty} a_k \cdot (-1)^k \cdot \frac{\sin \sigma z}{\sigma z - k\pi} \varphi\left(z - \frac{k\pi}{\sigma}\right) \\ &= a_0 \frac{\sin \sigma z}{\sigma z} \varphi(z) + \sum_{-\infty}^{\infty}{}' \frac{(-1)^k \cdot a_k \cdot \sin \sigma z}{\sigma z - k\pi} \varphi\left(z - \frac{k\pi}{\sigma}\right) \end{aligned}$$

Now given z, we can choose k_0 such that $|\sigma z| \leq \frac{k_0 \pi}{2}$.

Then for $|k| \geq |k_0|$, we have $|\sigma z| \leq \frac{k_0 \pi}{2} \leq |k| \cdot \frac{\pi}{2}$

so that

$$(20) \qquad \begin{aligned} \frac{1}{|\sigma z - k\pi|} &= \frac{1}{|k| \cdot \pi \left|1 - \frac{\sigma z}{k\pi}\right|} \leq \frac{1}{|k| \cdot \pi \left[1 - \left|\frac{\sigma z}{k\pi}\right|\right]} \\ &\leq \frac{1}{|k| \cdot \pi \left[1 - \frac{1}{2}\right]} = \frac{2}{|k| \cdot \pi} \end{aligned}$$

Since $|\sin \sigma z| \leq e^{\sigma |y|}$, using (20), (19) and (15) we conclude that $F_1(z)$ does represent an entire function of exponential type 2σ, because of condition (15).

Since $V(x)$ is bounded on the real axis and is an entire function of exponential type 2σ, condition (16) implies that $F_2(z)$ is also an entire function of exponential type 2σ.

Thus $R_\sigma(z)$ is an entire function of exponential type 2σ satisfying (18). This completes the proof of Theorem 9.3.

Having established the bounds for $|U_\sigma(x)|$ and $|V_\sigma(x)|$, we will try to establish the conditions for the convergence of $R_\sigma(x)$. Before we proceed to establish the convergence let us introduce the class W introduced by Wiener [18] of continuous functions on $(-\infty, \infty)$ satisfying the condition

$$\sum_{k=-\infty}^{\infty} \max_{0 \leq x \leq 1} |f(k+x)| < \infty$$

Set

$$\|f\|_W = \sum_{k=-\infty}^{\infty} \max_{0 \leq x \leq 1} |f(k+x)|$$

The space of Wiener functions has been studied in detail by Goldberg [11].

Define the translation T_y by

$$T_y f(x) = f(x+y)$$

and the one sided second difference Δ_h^2 by

$$\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x).$$

It is easy to see that

$$\Delta_h^2 f(x) = T_{2h} f(x) - 2 T_h f(x) + f(x)$$

$$= (T_{2h} f - 2T_h f + f)\ (x)$$

If $x, y \in R$ and $f \in W$, then both T_y and $\Delta_h^2 f$ are in W and

(21) $\| T_y f \|_W \leq 2 \| f \|_W$

It is also well known that the mapping $y \to T_y f$ is continuous from R into W. That is

(22) $\| T_y f - f \|_W \to 0$ as $y \to 0$

DEFINITION 9.4. If a function $f(x)$ is bounded on R the <u>modulus of smoothness of order</u> $k \geq 1$ is defined by

$$\omega_k(f\,;\,t) = \omega_k(t) = \sup_{|h| \leq t} \| \Delta_h^k f \|_W .$$

where

$$\Delta_h^k f(x) = \sum_{r=0}^{k} (-1)^{k-r} \cdot \binom{k}{r} f(x + r h)$$

The following properties of the modulus of smoothness are used in this section.

(1) If the function $f(x)$ has a bounded derivative of the rth order [r an integer] on R then for any integer $k \geq 0$

$$\omega_{k+r}(f\,;\,t) \leq 2t^r \ \omega_k(f^{(r)}\,;\,t)$$

(2) If $n \geq 0$ is an integer, then

$$\omega_k(n\,t) \leq 2\,n^k\,\omega_k(t)$$

The proof is exactly similar to the ones given by Timan [17] . The constant 2 appears in the R.H.S. of (1) and (2) because the translation operator on W has a norm bounded by 2 by (21).

THEOREM 9.5. Let f be a function of the class W, which has a continuous derivative on R. Suppose

$$(23)\quad g_\sigma(x) = \left(\frac{\sin \sigma x/2}{\sigma x}\right)^4$$

$$(24)\quad m_\sigma = \int_{-\infty}^{\infty} \left(\frac{\sin \sigma t/2}{\sigma t}\right)^4 . dt$$

If

$$(25)\quad F_\sigma(x) = \frac{1}{m_\sigma} \int_{-\infty}^{\infty} g_\sigma(t)\,\left[2\,f(x+t) - f(x+2t)\right] dt$$

$$= \frac{1}{m_\sigma} \int_{-\infty}^{\infty} G_\sigma(x-t).\ f(t).dt$$

where

$$(26)\quad G_\sigma(h) = 2\,g_\sigma(h) - \tfrac{1}{2}\,g_\sigma\left(\tfrac{h}{2}\right)$$

then

(1) $\| f - F_\sigma \|_W \leq c_1\,\omega_2(f\,;\tfrac{1}{\sigma})$

(2) $\| F'' \|_W \leq c_2 \cdot \sigma \cdot \omega_1(f', \tfrac{1}{\sigma})$

where C_1 , C_2 are constants.

PROOF. If $f \in L'$ and $g \in W$, then $f * g$ belongs to W. Thus $F_\sigma \in W$. Then

$$f(x) - F_\sigma(x) = \frac{1}{m_\sigma} \int_{-\infty}^{\infty} g_\sigma(t).\, f(x).dt$$

$$= \frac{1}{m_\sigma} \int_{-\infty}^{\infty} g_\sigma(t) \left[2f(x+t) - f(x+2t) \right] dt$$

$$= \frac{1}{m_\sigma} \int_{-\infty}^{\infty} g_\sigma(t).\, \Delta_t^2 f(x).\, dt$$

which gives

$$(27) \quad \| f - F_\sigma \|_W \leq \frac{1}{m_\sigma} \int_{-\infty}^{\infty} \| g_\sigma(t).\, \Delta_t^2 f \|_W \, dt$$

$$= \frac{1}{m_\sigma} \int_{-\infty}^{\infty} g_\sigma(t) \| \Delta_t^2 f \|_W \, dt$$

$$\leq \frac{1}{m_\sigma} \int_{-\infty}^{\infty} g_\sigma(t).\, \omega_2(f;\, t)\, dt$$

Now using the property that

$$(28) \quad \omega_2(f;\, \lambda t) \leq 2(\lambda + 1)^2 \,.\, \omega_2(f\,;\, t)$$

where $\lambda > 0$, we see that

$$\omega_2(f\,;\, t) \leq 2(\sigma t + 1)^2 \omega_2(f\,;\, \tfrac{1}{\sigma})$$

By virtue of (28), (27) becomes

$$\| f - F_\sigma \|_W \leq \frac{2}{m_\sigma} \int_{-\infty}^{\infty} g_\sigma(t)\, \omega_2(f;\, \tfrac{1}{\sigma})\, (\sigma t + 1)^2 dt$$

$$= \frac{2\sigma^2}{m_\sigma} \omega_2(f;\, \tfrac{1}{\sigma}) \left[\int_{-\frac{1}{\sigma}}^{\frac{1}{\sigma}} g_\sigma(t).\, (t + \tfrac{1}{\sigma})^2 \, dt \right.$$

$$\left. + \int_{|t| \geq \frac{1}{\sigma}} g_\sigma(t).\, (t + \tfrac{1}{\sigma})^2 \, dt \right]$$

Now

$$\int_{-\frac{1}{\sigma}}^{\frac{1}{\sigma}} g_\sigma(t).(t+\tfrac{1}{\sigma})^2\,dt \le \frac{4}{\sigma^2}\int_{-\frac{1}{\sigma}}^{\frac{1}{\sigma}} g_\sigma(t)\,dt$$

$$\le \frac{4}{\sigma^2}\int_{-\infty}^{\infty} g_\sigma(t)\,dt = \frac{4}{\sigma^2}.m_\sigma$$

and

$$\frac{\sigma^2}{m_\sigma}\int_{|t|\geq\frac{1}{\sigma}} g_\sigma(t)(t+\tfrac{1}{\sigma})^2.dt \le \frac{4\sigma^2}{m_\sigma}\int_{|t|\geq\frac{1}{\sigma}} t^2.g_\sigma(t).dt$$

$$= \frac{4\sigma^2}{m_\sigma}\int_{|t|\geq\frac{1}{\sigma}} t^2.\left(\frac{\sin\sigma t/2}{\sigma t}\right)^4 dt$$

$$= 4\,\frac{\int_{|x|\geq 1} x^2\left(\frac{\sin x/2}{x}\right)^4 dx}{\int_{-\infty}^{\infty}\left(\frac{\sin x/2}{x}\right)^4 dx}$$

$$\le 4\,\frac{\int_{-\infty}^{\infty}\left(\frac{\sin x/2}{x}\right)^2 dx}{\int_{-\infty}^{\infty}\left(\frac{\sin x/2}{x}\right)^4 dx}$$

Hence

$$(29)\quad \|f - F_\sigma\|_W \le 8\left[1+\frac{\int_{-\infty}^{\infty}\left(\frac{\sin x/2}{x}\right)^2 dx}{\int_{-\infty}^{\infty}\left(\frac{\sin x/2}{x}\right)^4 dx}\right]\omega_2(f;\tfrac{1}{\sigma})$$

$$\le C.\,\omega_2(f;\tfrac{1}{\sigma})$$

Then from (25)

$$F_\sigma(x) = \frac{1}{m_\sigma}\int_{-\infty}^{\infty} G_\sigma(x-t).\,f(t).dt$$

Hence

$$F''_\sigma(x) = \frac{1}{m\sigma}\int_{-\infty}^{\infty} G'_\sigma(x-t).\, f'(t).dt$$

$$= \frac{1}{m\sigma}\int_{-\infty}^{\infty} 2\, g'_\sigma(x-t)\, f'(t)dt$$

$$= \frac{1}{m\sigma}\int_{-\infty}^{\infty} \frac{1}{4} g'_\sigma\left(\frac{x-t}{2}\right) f'(t)dt$$

$$= \frac{2}{m\sigma}\int_{-\infty}^{\infty} g'_\sigma(t).f'(x-t)dt$$

$$= \frac{1}{2m\sigma}\int_{-\infty}^{\infty} g'_\sigma(t).f'(x-2t).dt$$

$$= \frac{2}{m\sigma}\int_{0}^{\infty} g'_\sigma(t)\left[f'(x-t) - f'(x+t)\right] dt$$

$$= \frac{1}{2m\sigma}\int_{0}^{\infty} g'_\sigma(t)\left[f'(x-2t) - f'(x+2t)\right] dt$$

So

$$\|F''_\sigma\|_w \le \frac{2}{m\sigma}\int_0^\infty |g'_\sigma(t)| \,.\, \|\Delta_{2t} f'\|_w\, dt$$

$$+ \frac{1}{2m\sigma}\int_0^\infty |g'_\sigma(t)| \,.\, \|\Delta_{4t} f'\|_w\, dt$$

$$\le C_1 \frac{1}{m\sigma}\int_0^\infty |g'_\sigma(t)|\, \omega_1(f'; t)\, dt$$

$$\le C_1 \frac{1}{m\sigma}\int_0^\infty |g'_\sigma(t)|\, \omega_1(f'; \tfrac{1}{\sigma})\, \sigma\, (t+\tfrac{1}{\sigma})\, dt$$

$$= \frac{C_1.\sigma.\omega_1(f'; \frac{1}{\sigma})}{m\sigma}\int_0^\infty |g'_\sigma(t)|\, (t+\tfrac{1}{\sigma})\, dt$$

$$= \frac{C_1.\sigma.\omega_1(f'; \frac{1}{\sigma})}{m\sigma}\int_0^{\frac{1}{\sigma}} |g'_\sigma(t)|\, (t+\tfrac{1}{\sigma})\, dt$$

$$+ \frac{C_1.\sigma.\omega_1(f'; \frac{1}{\sigma})}{m\sigma}\int_{\frac{1}{\sigma}}^\infty |g'_\sigma(t)|\, (t+\tfrac{1}{\sigma})\, dt$$

$$\leq \frac{2\,C_1 \cdot \omega_1(f';\tfrac{1}{\sigma})}{m_\sigma} \int_0^{\frac{1}{\sigma}} |g_\sigma'(t)|\,dt$$

$$+ \frac{2\,C_1 \cdot \sigma \cdot \omega_1(f';\tfrac{1}{\sigma})}{m_\sigma} \int_{1/\sigma}^{\infty} |g_\sigma'(t)| \cdot t\,dt$$

But

$$\frac{1}{m_\sigma} \int_0^{\frac{1}{\sigma}} |g_\sigma'(t)|\,dt \leq \frac{1}{m_\sigma} \int_0^{\frac{1}{\sigma}} |g_\sigma'(t)|\,dt \leq \frac{\sigma}{m_\sigma} \int_0^{\frac{1}{\sigma}} |g_\sigma(t)|\,dt$$

$$\leq \sigma \quad [3,\ \text{Th.11.3.3, P.211}]$$

Now

$$\frac{1}{m_\sigma} \int_{t \geq \frac{1}{\sigma}} |g_\sigma'(t)|\, t\,dt = \frac{1}{m_\sigma} \int_{t \geq \frac{1}{\sigma}} \left| \left\{ \left(\frac{\sin \sigma t/2}{\sigma t} \right)^4 \right\}' \right| t\,dt$$

$$= \frac{1}{m_\sigma} \int_{t \geq \frac{1}{\sigma}} 4 \cdot \left| \left(\frac{\sin \sigma t/2}{\sigma t} \right)^3 \right| \left| \frac{\sigma t/2 \cos \sigma t/2 - \sin \sigma t/2}{\sigma t^2} \right| \times t \cdot dt$$

$$\leq \frac{4}{\sigma \cdot m_\sigma} \int_0^\infty x \left| \left(\frac{\sin x/2}{x} \right)^3 \right| \left| \frac{x/2 \cos x/2 - \sin x/2}{x^2} \right| dx$$

$$= \frac{4 \cdot \int_0^\infty \left| \left(\frac{\sin x/2}{x} \right)^3 \right| \left| \frac{x/2 \cos x/2 - \sin x/2}{x} \right| dx}{\int_{-\infty}^{\infty} \left(\frac{\sin x/2}{x} \right)^4 \cdot dx}$$

$$= C_2$$

So

$$\| F_\sigma'' \|_W \leq 2\,C_1\, \omega_1(f';\tfrac{1}{\sigma}) \cdot \sigma + 2\,C_1 \cdot \omega_1(f';\tfrac{1}{\sigma})\, C_2\, \sigma$$
$$= C_4 \cdot \sigma \cdot \omega_1(f';\tfrac{1}{\sigma})$$

With these preliminaries, we are in a position to state and prove the theorem on the convergence of the sequence of entire function $\{R_\sigma(x; f)\}$

THEOREM 9.6. <u>Let</u> $f(x)$ <u>be bounded, is uniformly continuous and derivable on the whole of the real axis. Then the sequence of entire functions</u> $\{R_\sigma(x; f)\}$ <u>defined by</u>

$$R_\sigma(x; f) = \sum_{k=-\infty}^{\infty} f(x_k)\, U(x - x_k) + \sum_{k=-\infty}^{\infty} b_k\, V(x - x_k)$$

<u>converge to</u> $f(x)$ <u>if</u>

i) $\omega_1(f'; \frac{1}{\sigma}) \longrightarrow 0$ as $\sigma \to \infty$

ii) $\sum_{k=-\infty}^{\infty} |b_k| = O(\sigma^2)$

PROOF: Introducing $F_\sigma(x)$ from (25), which has been established to be an entire function of exponential type 2σ, we write

$$\begin{aligned} R_\sigma(x) - f(x) &= R_\sigma(x) - F_\sigma(x) + F_\sigma(x) - f(x) \\ &= F_\sigma(x) - f(x) + \sum_{-\infty}^{\infty} \Big[f(x_k)U(x-x_k) \\ &\qquad + b_k\, V(x-x_k)\Big] \\ &\quad - \Big[\sum_{-\infty}^{\infty} F_\sigma(x_k)U(x-x_k) + \sum_{-\infty}^{\infty} F''_\sigma(x_k)V(x-x_k)\Big] \\ &= \big[F_\sigma(x)-f(x)\big] + \sum_{-\infty}^{\infty} \big[f(x_k)-F_\sigma(x_k)\big]\, U(x-x_k) \\ &\quad + \sum_{-\infty}^{\infty} b_k V(x-x_k) - \sum_{-\infty}^{\infty} F''_\sigma(x_k)\, V(x-x_k) \end{aligned}$$

So

$$\begin{aligned} |R_\sigma(x)-f(x)| \le\ & |F_\sigma(x)-f(x)| + \sum_{-\infty}^{\infty} |f(x_k)-F_\sigma(x_k)|\, |U(x-x_k)| \\ & + \sum_{-\infty}^{\infty} |F''_\sigma(x_k)|\, |V(x-x_k)| + \sum_{-\infty}^{\infty} |b_k|\, |V(x-x_k)| \end{aligned}$$

We know from Timan [17] that for any function f(x) bounded and uniformly continuous on the whole of the real axis, there exists a sequence of integral functions $\{g_n(x)\}$ of degree σ_n such that

$$\sup_{-\infty < x < \infty} |f(x) - g_n(x)| \to 0 \quad \text{as} \quad \sigma_n \to \infty$$

using the above theorem, we observe that

$$\text{(A)} \qquad |F_\sigma(x) - f(x)| \to 0 \quad \text{as} \quad \sigma \to \infty$$

since f(x) is bounded and is uniformly continuous.

But if $x_k = \frac{k\pi}{\sigma}$, then

$$\sum_{k=-\infty}^{\infty} |f(x_k) - F(x_k)| \leq \sum_{j=-\infty}^{\infty} \sum_{x_k \in [j, j+1]} |f(x_k) - F_\sigma(x_k)|$$

$$\leq \sum_{j=-\infty}^{\infty} \left[\frac{\sigma}{\pi}\right] \max_{x \in [j, j+1]} |f(x) - F_\sigma(x)|$$

$$= \left[\frac{\sigma}{\pi}\right] \| f - F_\sigma \|_W$$

$$\leq \frac{\sigma}{\pi} \| f - F_\sigma \|_W$$

$$\leq \frac{\sigma}{\pi} C. \omega_2(f; \tfrac{1}{\sigma})$$

using (31).

From the property (1) of the modulus of smoothness, we find

$$\omega_2(f; \tfrac{1}{\sigma}) \leq \tfrac{1}{\sigma} \cdot \omega_1(f'; \tfrac{1}{\sigma})$$

So

$$\sum_{k=-\infty}^{\infty} |f(x_k) - F_\sigma(x_k)| \leq \frac{\sigma}{\pi} \quad C. \frac{1}{\sigma} . \omega_1(f'; \tfrac{1}{\sigma})$$
$$= C_6 . \omega_1(f'; \tfrac{1}{\sigma})$$

Hence

(B) $$\sum_{k=-\infty}^{\infty} |f(x_k) - F_\sigma(x_k)| \; |U(x-x_k)|$$
$$\leq \frac{e^2+4e-1}{2e} . C_6 . \omega_1(f'; \tfrac{1}{\sigma})$$
$$\to 0 \text{ as } \sigma \to \infty \text{ from (1)}$$

From (30) we find

$$\| F_\sigma'' \|_w \leq C_4 . \sigma . \omega_1(f'; \tfrac{1}{\sigma})$$
$$\sum_{k=-\infty}^{\infty} |F_\sigma''(x_k)| \leq \left[\frac{\sigma}{\pi}\right] . \| F_\sigma'' \|_w$$
$$\leq \frac{\sigma}{\pi} . C_4 . \sigma . \omega_1(f'; \tfrac{1}{\sigma})$$
$$= C_5 . \sigma^2 . \omega_1(f'; \tfrac{1}{\sigma})$$

Further from (14)

$$|V_\sigma(x - x_k)| \leq \frac{8 + \pi^2}{4\pi\sigma^2}$$

Hence

(C) $$\sum_{k=-\infty}^{\infty} |F_\sigma''(x_k)| \; |V(x-x_k)| \leq \frac{8+\pi^2}{4\pi\sigma^2} . C_5 . \sigma^2 . \omega_1(f'; \tfrac{1}{\sigma})$$
$$= C_7 . \omega_1(f'; \tfrac{1}{\sigma})$$
$$\to 0 \text{ as } \sigma \to \infty \text{ from (1) again}$$

From (14) again

(D) $$\sum_{k=-\infty}^{\infty} |b_k| \; |V(x-x_k)| \leq \frac{8+\pi^2}{4\pi\sigma^2} \sum_{k=-\infty}^{\infty} |b_k|$$
$$\to 0 \text{ as } \sigma \to \infty$$

since we have assumed

$$\sum_{k=-\infty}^{\infty} |b_k| = 0(\sigma^2) \quad \text{in (ii)}$$

combining (A), (B), (C) and (D), we find that

$$|R_\sigma(x; f) - f(x)| \to 0 \quad \text{as } \sigma \to \infty$$

This completes the proof of the Theorem 9.6.

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