Topology of generalized Dold manifolds

By

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MATH10201404004

The Institute of Mathematical Sciences, Chennai

A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

October, 2019

Homi Bhabha National Institute

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LIST OF PUBLICATIONS ARISING FROM THE THESIS

Journal

1. **Nath, Avijit; Sankaran, Parameswaran** *On generalized Dold manifolds* Osaka J. Math. **56** (2019), no. 1, 75–90.

2.

Chapters in books and lecture notes

1.

2.

Conferences

1.

2.

Others

1.

2.

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Dedicated to my beloved parents
and my brother

ACKNOWLEDGEMENTS

First and foremost, I would like to deeply thank Prof. Parameswaran Sankaran for being an excellent advisor. His constant encouragement, generosity, consistent guidance and care throughout the doctoral studies paved the way for my thesis. I owe a debt of sincere gratitude for the wisdom he shared, and for the time and energy he provided to start me down the road to Algebraic and Differential Topology.

Besides my advisor, I would like to immensely thank to the rest of my doctoral committee members Prof. K.N. Raghavan, Prof. D.S. Nagaraj, Prof. S. Viswanath and Prof. Sanoli Gun for their necessary suggestions during my doctoral studies. I wish to thank my brother Arijit Nath for his presence during my doctoral studies at Chennai, for many illuminating discussions in mathematics I had with him and for being constant source of inspiration and motivation. I am thankful to Pranendu Darbar for being a good friend of mine. My sincere thanks also goes to all of my teachers.

Last but not least, I offer my heartiest thanks to my parents for their unconditional love, unflagging moral support and sacrifices without which this journey would never have been possible.

Contents

Summary

Classical Dold manifolds were defined as the orbit space of $\mathbb{Z}_2$ action on the product of a sphere and a complex projective space where $\mathbb{Z}_2$ acts on the sphere by antipodal involution and the complex projective space by complex conjugation. Dold has given the description of $\mathbb{Z}_2$- cohomology ring of Dold manifolds. He also obtained the formula for Stiefel- Whitney polynomial of Dold manifolds. He used these manifolds to obtain generators for unoriened cobordism ring in odd dimesions. Ucci obtained the formula for stable tangent bundle of Dold manifolds. Korbaš obtained the criterion for parallelizability and stable parallelizability of Dold manifolds.

In this thesis, we obtain a generalization of the Dold manifolds where we replace the complex projective space by an almost complex manifold admitting a complex conjugation. We call them as generalized Dold manifolds. We obtain a description of the tangent bundle, under a mild hypothesis a formula for Stiefel-Whitney polynomial of generalized Dold manifolds, a criteria for orientability and spin structures as an applications of simple compuations of first and second Stiefel-Whitney classes. Using the description of tangent bundle, we also obtain estimates for span and stable span of generalized Dold manifolds. We obtain a very general criterion for (non) vanishing of cobordism classes of generalized dold manifolds. We applied our results by taking almost complex manifolds as complex Grassmann manifolds, more generally as complex flag manifolds.

Our proof to determine estimates for span and stable span of generalized Dold

manifolds involves Bredon-Kosiński's theorem, certain functor introduced by Lam to study the immersions of flag manifolds.

We apply Stefiel-Whitney numbers argument to determine the (non) vanishing of cobordism classes of generalized Dold manifolds. We also use the theory of Clifford algebras, a result of Conner and Floyd concerning cobordism of manifolds admitting stationary point free action of elementary abelian 2-group actions to obtain the results of (non) vanishing of cobordism classes of generalized Dold manifolds corresponding to Grassmann manifolds.

Chapter 1

Preleminaries

The aim of this thesis is to study the topology of generalized Dold manifolds $P(m, X)$. Specifically we shall describe the tangent bundle of the generalized Dold manifolds, obtain (i) a formula for their Stiefel-Whitney classes, (ii) results on their (stable) parallelizability and estimates for span and stable span when X is a complex flag manifold, and, (iii) results on non(vanishing) of the unoriented cobordism class of $P(m, X)$ when X is a complex Grassmann manifold.
In this section, we will briefly recall various well-known definitions, results which will be used later in this thesis.

1.1 Span and stable span

Next, we begin by recalling the notions of span and stable span of a smooth connected manifold and general results yielding estimates for span and stable span.
Let M be a connected smooth manifold. A vector field on M is a section $s : M \to TM$ of its tangent bundle τM. A set of vector fields $s_1, \ldots, s_r$ is said to be *everywhere linearly independent* if the tangent vectors $s_1(x), s_2(x), \ldots, s_r(x) \in T_xM$ are linearly independent $\forall x \in M$. The maximum number r of eveywhere linearly independent

vector fields that exist on M is called the *span* of M and is denoted $Span(M)$.

The notion of span can be extended to an arbitary vector bundle ξ over a finite dimensional CW-complex B as follows: Span of ξ is defined to be the largest non-negative integer r such that ξ admits r cross-sections $s_1, \dots, s_r : B \to E(\xi)$ such that $s_1(x), s_2(x), \dots, s_r(x)$ are linearly independent $\forall x \in B$: is denoted by $Span(\xi)$. The stable span of ξ, denoted $Span^0(\xi)$ is defined as $Span(\xi \oplus k\epsilon) - k$ where $k \geq 1$ is any integer such that $rank(\xi) + k > dimB$. It turns out that the definition of $Span^0(\xi)$ is independent of k so long as $rank(\xi) + k > dimB$. This follows from the fact that if the rank of a real vector bundle η be n and B is a CW complex of dimension $d \leq n$, then $\mathrm{span}(\eta) \geq n - d$. See [11, Theorem 1.1, Ch. 9]. It follows that if $n > d$, then $\mathrm{Span}(\eta) = \mathrm{Span}^0(\eta)$.

We define the *stable span* of M to be the integer $Span(TM^n \oplus \epsilon) - 1$ and it is denoted by $Span^0(M)$. Note that $0 \leq Span(M) \leq Span^0(M) \leq dim(M)$.
A manifold M^n is called *parallelizable* if its tangent bundle τM is trivial and *stably parallelizable* if its tangent bundle is stably trivial, that is, $\tau M \oplus s\epsilon \cong (n+s)\epsilon_{\mathbb{R}}$. In fact, we can choose $s = 1$.

It is easily seen that all spheres are stably parallelizable. However, Bott, Milnor [4] and Kervaire [12] showed that the only spheres which are *parallelizable* are $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$.

From the work of Radon [24] and Hurwitz [10] one has the lower bound $Span(\mathbb{S}^n) \geq \rho(n+1) - 1$ where ρ is the Radon-Hurwitz function defined as follows. Write $n = 2^{4a+b} \times (2c+1)$ then $\rho(n) := 8a + 2^b$ where $a, c \geq 0$ and $0 \leq b \leq 3$. Using K-theory and Adams operations Adams [1] showed that $Span(\mathbb{S}^n) \leq \rho(n+1) - 1$, thereby determining the span of spheres. We will need to use the Adams φ-function in the thesis. It is defined as follows $\varphi(n) := \#\{j | j \equiv 0, 1, 2, 4 \mod 8; 1 \leq j \leq n\}$. The significance of $\varphi(n)$ is that $2^{\varphi(n)}([\xi] - 1) = 0$ in $KO(\mathbb{R}P^m)$ where ξ is the Hopf line bundle over $\mathbb{R}P^m$.

We will now recall results concerning estimates for the span of a smooth compact connected manifolds.

The following is a useful observation. Let $\pi : E \to B$ be a smooth fibre bundle, then $\tau E = \pi^*(\tau B) \oplus \eta$, where η is a bundle tangential along the fibres. Hence $Span^0(E) \geq Span^0(B)$ and $Span(E) \geq Span(B)$.
If the fibre of $\pi : E \to B$ is connected and $i : F \to E$ denotes the inclusion, then $i^*(\tau E) = \epsilon^{dimB} \oplus \tau F$; hence

$$Span^0(B) \leq Span^0(E) \leq Span^0(F) + dimB.$$

In particular, *stable parallelizability* of total space E implies *stable parallelizability* of the fibre F.

The following result in full generality is due to Hopf.

Theorem 1.1.1. (H. Hopf [9]) *Let M be a compact connected smooth manifold. Then $Span(M) \geq 1$ if and only if the Euler-Poincaré characteristic $\chi(M)$ of M is zero.*

If G be a compact connected Lie group, T be a maximal torus, $N(T)$ denotes the normalizer of T in G, $W = N(T)/T$ be the Weyl group of G. Then $\chi(G/T) = |W|$. So $Span(G/T) = 0$. It can be shown that G/T is stably parallelizable.
Bredon and Kosiński gave a criterion for determining the span of stably parallelizable manifolds as follows:

Theorem 1.1.2. (G. Bredon and A. Kosiński [5]) *Let M^n be a smooth compact manifold of dimension n. Suppose M^n is stably parallelizable. Then*
(i) M^n is parallelizable or $Span(M^n) = Span(\mathbb{S}^n) = \rho(n+1) - 1$.
(ii) If n is even, M^n is parallelizable if and only if Euler characteristic $\chi(M) = 0$.
(iii)If $n = 2m+1$ is odd and $n \neq 1, 3, 7$; M^n is parallelizable if and only if $\hat{\chi}_2(M) = 0$ where $\hat{\chi}_2(M) := \sum_{0 \leq j \leq m} \dim_{\mathbb{Z}_2} H^{2j}(M; \mathbb{Z}_2) \mod 2$ is Kervaire mod 2 semi-characteristic of M^n.

Atiyah and Dupont [3] defined the twisted Kervaire semi-characteristic, denoted $R_L(M)$ as follows:

Let M be an n-dimensional closed connected manifold and $\tilde{M}$ be an orientation double cover. Assume that ${w_1}^2(M) = 0$ or equivalently $\beta_2(w_1(M)) = 0$ where $\beta_2 = Sq^1$ is the Bockstein homomorphism coming from the exact sequence of the cofficient $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$. Existence of such an element determines a covering projection $\Gamma \to M$ with deck transformation group $\mathbb{Z}_4$. Let L be the line bundle associated to this covering projection whose total space is $\Gamma \times_{\mathbb{Z}_4} \mathbb{C}$ where $\mathbb{Z}_4$ acts on $\mathbb{C}$ just multiplication by i. The cohomology $H^*(M, L)$ denotes the de Rham cohomology with coefficients in L. It admits a non-degenerate Poincaré-duality paring $H^{n-p}(M; L) \times H^p(M; L) \to H^n(M; \Omega^n \otimes \mathbb{C}) \cong \mathbb{C}$ in view of the isomorphism $L \otimes L \cong \Omega^n \otimes \mathbb{C}$. Here Ω^n is the determinant of the cotangent bundle of M. Then the twisted semi-characteristic is defined as $R_L(M) = (1/2)(\sum_{0 \le k \le n} \dim_{\mathbb{C}}(H^k(M; L)))$ mod 2. When $w_1(M) = 0$, that is, when M is orientable, then L and Ω^n are trivial and we have $R_L(M) = \kappa(M)$.

Theorem 1.1.3. (U. Koschorke [15, §20]) *Let M be a smooth compact connected manifold of dimension d.*

(a) If $d \equiv 0 \mod 2$, and $\chi(M) = 0$, then $Span^0(M) = Span(M)$.

(b) If $d \equiv 1 \mod 4$ and if $w_1(M)^2 = 0$, then $Span^0(M) = Span(M)$ if the twisted Kervaire semi-characteristic $R_L(M)$ vanishes; if $R_L(M) \neq 0$, then $Span(M) = 1$.

(c) If $d \equiv 3 \mod 8$ and $w_1(M) = w_2(M) = 0$, then $Span^0(M) = Span(M)$ if $\hat{\chi}_2(M) = 0$; if $\hat{\chi}_2(M) \neq 0$, then $Span(M) = 3$.

1.2 Tangent bundle of complex flag manifolds

Consider $\mathbb{C}^n$ with its standard Hermitian inner product. Suppose that $n_1, n_2, \ldots n_r$ are sequence of positive integers such that $\sum_{1 \le j \le r} n_j = n$. By an $(n_1, n_2, \ldots, n_r)$-complex flag we mean a sequence $(V_1, V_2, \dots, V_r)$ mutually orthogonal $\mathbb{C}$-vector subspaces of $\mathbb{C}^n$ such that $dim V_i = n_i$. The space of all such flags can be identified as the homogenous space $U(n)/(U(n_1) \times \cdots \times U(n_r))$, denoted by $\mathbb{C}G(n_1, \dots, n_r)$. It is a complex manifold of complex dimension $\frac{1}{2}(n^2 - \sum_{1 \leq j \leq r} n_j^2) = \sum_{1 \leq i < j \leq r} n_i n_j$. Clearly $\mathbb{C}G(n_1, n_2)$ is the complex Grassmann manifold $\mathbb{C}G_{n,n_1}$. Lam [16] has given the following description of tangent bundle of flag manifolds.

Let ξ_j denote the canonical n_j-plane bundle over $\mathbb{C}G(\mu)$ whose fibre over a flag $\underline{V} = (V_1, \dots, V_r) \in \mathbb{C}G(\mu)$ is the vector space V_j. Then obviously we have the $\mathbb{C}$-bundle isomorphism $\xi_1 \oplus \cdots \oplus \xi_r \cong n\epsilon_{\mathbb{C}}$. The tangent bundle of the complex flag manifolds has the following description:

$$\tau \mathbb{C}G(\mu) \cong \oplus_{1 \leq i < j \leq r} Hom_{\mathbb{C}}(\xi_i, \xi_j) \cong \oplus_{1 \leq i < j \leq r} \overline{\xi}_i \otimes_{\mathbb{C}} \xi_j.$$

In particular, $\tau \mathbb{C}G_{n,k} = Hom_{\mathbb{C}}(\gamma_{n,k}, \gamma_{n,k}^{\perp})$ where $\gamma_{n,k}$ denotes the canonical k-plane bundle over $\mathbb{C}G_{n,k}$.

The only complex Grassmann manifold that is stably parallelizable as real manifolds is $G_{2,1} \cong \mathbb{C}P^1 \cong \mathbb{S}^2$. This was settled by Trew and Zvengrowski [31]. The case of stable parallelizability of complex flag manifolds was completely determined by Sankaran and Zvengrowski [28] and they obtained the following:

Theorem 1.2.1. (Sankaran and Zvengrowski [28]) *Let $\mu = (n_1, \dots, n_r)$ where $n_1 \geq \ldots \geq n_r \geq 1$, $r \geq 3$, and let $n := \sum_{1 \leq j \leq r} n_j$. Then $\mathbb{C}G(\mu)$ is stably parallelizable if and only if $n_j = 1$ for all j.*

We recall a certain functor μ^2 introduced by Lam [16, §4-5]. The functor $\mu^2 = \mu^2_{\mathbb{C}}$ associates a real vector bundle to a complex vector bundle. We assume the base space to be paracompact so that every complex vector bundle over it admits a Hermitian metric. If V is any complex vector space $\mu^2(V)$ is defined as $\mu^2(V) = \overline{V} \otimes_{\mathbb{C}} V/Fix(\theta)$ where $\theta : \overline{V} \otimes V \to \overline{V} \otimes V$ is the conjugate complex linear automorphism defined as

$\theta(u \otimes v) = -v \otimes u$. As with any *continuous* functor ([18, §3(f)]), μ^2 is determined by its restriction to the category of finite dimensional complex vector spaces.

The functor μ^2 has the following properties where $\omega, \omega_1, \omega_2$ are all complex vector bundles over a base space X. Lam proved these properties and used them to obtain upper bounds on immersion codimensions of complex flag manifolds.
(i) $\text{rank}(\mu^2(\omega)) = n^2$ where n is the rank of ω as a complex vector bundle.
(ii) $\mu^2(\omega) \cong \epsilon_{\mathbb{R}}$ if ω is a complex line bundle. Indeed, choosing a positive Hermitian metric on ω, the map $E(\mu^2(\omega)) \ni [u \otimes zu] \mapsto (p_\omega(u), Re(z).||u||^2) \in X \times \mathbb{R}, z \in \mathbb{C}$ is a well-defined, $\mathbb{R}$-linear non-zero homomorphism. Since the ranks agree, it is a bundle isomorphism.
(iii) $\mu^2(\omega_1 \oplus \omega_2) = \mu^2(\omega_1) \oplus (\overline{\omega}_1 \otimes_{\mathbb{C}} \omega_2) \oplus \mu^2(\omega_2)$.

1.3 Cobordism

Let M and N be two closed n-dimensional smooth manifold. A cobordism from M to N is a smooth compact $(n+1)$ manifold W with boundary such that ∂W is diffeomorphic to the disjoint union $M \sqcup N$. Two manifolds are said to be cobordant if there exists a cobordism between them. Note that cobordism is an equivalence relation on the set of all diffeomorphism classes of compact smooth manifolds. The set of equivalence classes-called the cobordism classes forms a ring where addition corresponds to disjoint union and multiplication corresponds to cartesian product of two manifolds.
In fact, it is a $\mathbb{Z}_2$-polynomial algebra with one generator x_n in each dimension n not of the form $2^k - 1$ for $k \geq 1$. A smooth compact manifold M is said to be indecomposable if x_n as taken to be $[M]$. Equivalently, M is indecomposable if it is not cobordant to disjoint union of products of strictly lower dimensional manifolds.

Let X be a smooth compact manifold of dimension $2n$ with an almost complex structure J. Let σ be a complex conjugation, that is, $T\sigma : TX \to TX$ is a conjugate linear isomorphism on each fibre, $T\sigma \circ J = -J \circ T\sigma$. The following theorem is due to Conner and Floyd [7].

Theorem 1.3.1. *Let σ be a conjugation on a almost complex manifold X^{2n} and $F = Fix(\sigma) \subset X$. Then F is an n-dimensional manifold if it is nonempty and* $[X^{2n}]_2 = [F \times F]_2$

For example, $\mathbb{C}G_{n,k}$ is unoriented cobordant to $\mathbb{R}G_{n,k} \times \mathbb{R}G_{n,k}$.

Stiefel-Whitney numbers: Let M be a closed n-dimensional manifold and $\mu_M \in H_n(M; \mathbb{Z}_2)$ denote *mod* 2 fundamental class. Let $I = i_1, i_2, \cdots, i_n$ denote a partition of n, that is, $n = i_1 + i_2 + \cdots + i_n$ with $i_1 \geq \cdots \geq i_n \geq 0$. We denote by $w_I(M)$ the class $w_{i_1}(M) \smile \cdots \smile w_{i_n}(M)$ and by $w_I[M]$ the *mod* 2 integer $\langle w_{i_1}(M) \smile \cdots \smile w_{i_n}(M), \mu_M \rangle \in \mathbb{Z}_2$. The element $w_I[M] \in \mathbb{Z}_2$ is called the I-th Stiefel-Whitney number of M.

The following theorem gives a necessary and sufficient condition for a manifold to bound.

Theorem 1.3.2. (Thom-Pontrjagin [18]) *A smooth closed n-dimensional manifold is a boundary of an $(n+1)$-dimensional compact smooth manifold with boundary if and only if all Stiefel-Whitney numbers of M are zero.*

Here the necessity part is due to Pontrjagin and sufficient part is a very deep result of Thom [30]. From this theorem, we can conclude that two manifolds are unoriented cobordant if and only if they have the same Stiefel-Whitney numbers.

$(\mathbb{Z}_2)^r$ action and unoriented cobordism: We begin by recalling the following result due to Conner and Floyd.

Theorem 1.3.3. (Conner and Floyd [7, Theorem 30.1]) *If $(\mathbb{Z}_2)^r$ acts on a smooth compact manifold M without stationary points, then $[M]_2 = 0$ in $\mathfrak{N}_*$.*

In his paper, Sankaran [25] considered certain $(\mathbb{Z}_2)^r$ action on $\mathbb{R}G_{n,k}$ without stationary points and using the fact that $[\mathbb{C}G_{n,k}] = [\mathbb{R}G_{n,k}]^2$, he obtained the following results:

Theorem 1.3.4. *$\mathbb{C}G_{n,k}$ bounds if and only if $\nu_2(k) < \nu_2(n)$.*

As for cobordism of complex flag manifolds, we have the following results

Theorem 1.3.5. (Sankaran and Varadaranjan [27]) *Let $n = n_1 + \cdots + n_r$, $r \geq 3$, The flag manifolds $\mathbb{C}G(n_1, \ldots, n_r)$ is an unoriented boundary in the following cases:*
(i) $n_i = n_j$, for some $i \neq j$, $1 \leq i, j \leq r$,
(ii) for some i, $\nu_2(n_i) < \nu_2(n)$, where $\nu_2(n)$ denotes the highest exponent of 2 that divides n.

Equivariant cobordism: Let $G = (\mathbb{Z}_2)^k$. Let $R_n(G)$ denote the vector space over $\mathbb{Z}_2$ with basis the set of isomorphism classes of G- real representation of dimension n. Then $R_*(G) = \oplus_{n \geq 0} R_n(G)$ is a graded commutative algebra over $\mathbb{Z}_2$ with a unit where multiplication in $R_*(G)$ is given by $[V_1].[V_2] = [V_1 \oplus V_2]$. In fact, $R_*(G)$ is a graded polynomial algebra over $\mathbb{Z}_2$ where the algebra generators are elements of $\hat{G} = Hom_{\mathbb{Z}_2}(G, \mathbb{Z}_2)$.
Consider a smooth G-action on φ on M with finite stationary point set S. For each $x \in S$, we have a real linear representation of G on T_xM, the tangent space at $x \in M$. We denote the tangential representation class by $[T_xM] \in R_n(G)$. Then the map $[M^d, \varphi] \mapsto \sum_{x \in S} [T_xM]$ an algebra homomorrphism $\eta_* : Z_*(G) \to R_*(G)$. By a result of Stong, η_* is in fact a monomorphism.

$(\mathbb{Z}_2)^n$-Action on Grassmann manifolds: Let $D_n \subset O(n)$ denote the diagonal subgroup of the orthogonal group. Then $D_n \cong (\mathbb{Z}_2)^n$ acts on $\mathbb{R}^k$ and on $\mathbb{C}^k$ in the obvious manner. Therefore we have an action of $(\mathbb{Z}_2)^n$ on $\mathbb{C}G_{n,k}$. and the stationary points of this action are $\langle e_{i_1}, e_{i_2}, \ldots, e_{i_k} \rangle := E_\lambda$ where $\lambda := 1 \leq i_1 < i_2 < \cdots i_k \leq n$. So there are $\binom{n}{k}$ stationary points of this action. We will denote the above action by φ.

The following theorem may be obtained from the result of Mukherjee [19] who considered equivariant cobordism of $\mathbb{R}G_{n,k}$.

Theorem 1.3.6. *(Mukherjee [19]) (i)* $(\mathbb{C}G_{n,k}, \varphi)$ *bounds equivariantly if* $n = 2k$
(ii) $(\mathbb{C}G_{n,k}, \varphi)$ *does not bound equivariantly if* $n \neq 2k$.

Chapter 2

Vector bundles over $P(m, X, \sigma)$

Let $\sigma : X \to X$ be an involution of a path connected paracompact Hausdorff topological space and let ω be a complex vector bundle over X. Denote by $\omega^\vee$ the dual vector bundle $Hom_{\mathbb{C}}(\omega, \epsilon_{\mathbb{C}})$. Here $\epsilon_{\mathbb{F}}$ (or more briefly ϵ when there is no danger of confusion) denotes the trivial $\mathbb{F}$-line bundle over X where $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Note that, since X is paracompact, ω admits a Hermitian metric and so $\omega^\vee$ is isomorphic to the conjugate bundle $\overline{\omega}$. Here $\overline{\omega}$ is the complex conjugate of ω. In this chapter we introduce the notion of a σ-conjugate vector bundle, we associate to a σ-conjugate vector bundle ω over X a real vector bundle $\hat{\omega}$ over $P(m, X, \sigma)$. We obtain a splitting principle, which will be used in §2.4 to obtain a formula for the Stiefel-Whitney classes of $\hat{\omega}$ under some mild restrictions on X.

2.1 Notion of σ-conjugate vector bundles

The following definition generalises simultaneously the notion of a complex vector bundle being isomorphic to its conjugate and that of an involution of an almost complex manifold being a 'conjugation', in the sense of Conner and Floyd [7, §24].

Definition 2.1.1. *Let $\sigma : X \to X$ be an involution and let ω be a complex vector bundle* *over X. We say that ω is a σ-conjugate bundle (or more briefly σ-conjugate) if there exists an involutive bundle map $\hat{\sigma} : E(\omega) \to E(\omega)$ that covers σ and is conjugate complex linear on the fibres of $E(\omega) \to X$.*

In view of the requirement that $\hat{\sigma}$ be an involution, σ-conjugacy is stronger than merely requiring that $\overline{\omega} \cong \sigma^*(\omega)$.

Example 2.1.2. *(i) Let σ be any involution on X. When $\omega = n\epsilon_{\mathbb{C}}$, the trivial complex vector bundle of rank n, we have $E(\omega) = X \times \mathbb{C}^n$. Then $(x, \sum z_j e_j) \mapsto (\sigma(x), \sum \overline{z}_j e_j)$ defines an involutive bundle map $\hat{\sigma}$ which is conjugate complex linear on each fibre and so $n\epsilon_{\mathbb{C}}$ is σ-complex conjugate to itself.*

(ii) Let $X = \mathbb{C}G_{n,k}$ and let $\sigma : X \to X$ be the involution $L \mapsto \overline{L}$. Then the canonical k-plane bundle $\gamma_{n,k}$ over X is σ-conjugate to itself. Indeed $v \mapsto \overline{v}$, $v \in L \in \mathbb{C}G_{n,k}$, is the required involutive bundle map $E(\gamma_{n,k}) \to E(\gamma_{n,k})$ that covers σ. Similarly the orthogonal complement $\beta_{n,k} := \gamma_{n,k}^{\perp}$ is also σ-conjugate to itself.

(iii) If $X \subset \mathbb{C}P^N$ is a complex projective manifold defined over $\mathbb{R}$ and $\sigma : X \to X$ is the restriction of complex conjugation $[z] \mapsto [\overline{z}]$, then the tangent bundle τX of X is a σ-conjugate bundle. Indeed the differential of σ, namely $T\sigma : TX \to TX$, is the required bundle map $\hat{\sigma}$ of τX that covers σ. As mentioned above, this classical case was generalized by Conner and Floyd [7, §24] to the case when X is an almost complex manifold.

(iv) If ω, η are σ-conjugate complex vector bundles over X, then so are $\Lambda^r(\omega)$, $Hom_{\mathbb{C}}(\omega, \eta)$, $\omega \otimes \eta$, and $\omega \oplus \eta$. For example, if $\hat{\sigma}$ and $\tilde{\sigma}$ are complex conjugate bundle involutions of $E(\omega)$ and $E(\eta)$ respectively, both covering σ, then $Hom(\omega, \eta) \ni f \mapsto \tilde{\sigma} \circ f \circ \hat{\sigma} \in Hom(\omega, \eta)$ is verified to be a conjugate complex linear bundle involution that covers σ.

(v) Any subbundle η of a σ-conjugate complex vector bundle ω over X is also σ-conjugate provided $\hat{\sigma} : E(\omega) \to E(\omega)$ satisfies $\hat{\sigma}(E(\eta)) = E(\eta)$.

2.2 Vector bundle associated to $(\eta, \hat{\sigma})$

Let η be a *real* vector bundle over X with projection $p_\eta : E(\eta) \to X$ and let $\hat{\sigma} : E(\eta) \to E(\eta)$ be an involutive bundle isomorphism that covers σ. We obtain a real vector bundle, denoted $\hat{\eta}$, over $P(m, X, \sigma)$. Indeed $(v, e) \mapsto (-v, \hat{\sigma}(e))$ defines a fixed point free involution of $\mathbb{S}^m \times E(\eta)$ with orbit space $P(m, E(\eta), \hat{\sigma})$. The map $p_{\hat{\eta}} : P(m, E(\eta), \hat{\sigma}) \to P(m, X, \sigma)$ defined as $[v, e] \mapsto [v, p_\eta(e)]$ is the projection of the bundle $\hat{\eta}$.

This construction is applicable when η is the underlying real vector bundle, denoted $\rho(\omega)$, of a σ-conjugate complex vector bundle ω with $\hat{\sigma}$ as in Definition 2.1.1.

If β is a subbundle of η such that $\hat{\sigma}(E(\beta)) = E(\beta)$, then the restriction of $\hat{\sigma}$ to $E(\beta)$ defines a bundle $\hat{\beta}$ which is evidently a subbundle of $\hat{\eta}$.

We shall denote by ξ the real line bundle over $P(m, X, \sigma)$, often referred to as the Hopf bundle, associated to the double cover $\mathbb{S}^m \times X \to P(m, X, \sigma)$. Its total space has the description $\mathbb{S}^m \times X \times_{\mathbb{Z}_2} \mathbb{R}$ consisting of elements $[v, x, t] = \{(v, x, t), (-v, \sigma(x), -t) \mid v \in \mathbb{S}^m, x \in X, t \in \mathbb{R}\}$. Denote by $\pi : P(m, X, \sigma) \to \mathbb{R}P^m$ the map $[v, x] \mapsto [v]$. Then π is the projection of a fibre bundle with fibre X. The map $E(\xi) \to E(\zeta)$ defined as $[v, x, t] \mapsto [v, t]$ is a bundle map that covers the projection $\pi : P(m, X, \sigma) \to \mathbb{R}P^m$ and so $\xi \cong \pi^*(\zeta)$ where ζ is the Hopf line bundle over $\mathbb{R}P^m$.

If $\sigma(x_0) = x_0 \in X$, then we have a cross-section $s : \mathbb{R}P^m \to P(m, X)$ defined as $[v] \mapsto [v, x_0]$. Note that $s^*(\xi) = \zeta$.

2.3 Dependence of $\hat{\omega}$ on $\hat{\sigma}$

It should be noted that the definition of $\hat{\eta}$ depends not only on the real vector bundle η but also on the bundle map $\hat{\sigma}$ that covers σ. For example, let $k, l \geq 0$ be integers and let $n = k + l \geq 1$. Denote by $\hat{\varepsilon}_{k,l} : X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ the involutive bundle map of $n\epsilon_{\mathbb{R}}$ covering σ defined as $(x, t_1, \ldots, t_n) = (\sigma(x), -t_1, \ldots, -t_k, t_{k+1}, \ldots, t_n)$. Then the bundle over $P(m, X, \sigma)$ associated to $(n\epsilon_{\mathbb{R}}, \hat{\varepsilon_{k,l}})$ is isomorphic to $k\xi \oplus l\epsilon_{\mathbb{R}}$.

When $\omega = \tau X$ is the tangent bundle over an almost complex manifold X and $\sigma : X \to X$ is a smooth involution such that the differential $T\sigma : TX \to TX$ is a conjugate complex morphism (i.e. satisfies $J_{\sigma(x)} \circ T_x\sigma = -T_x\sigma \circ J_x \; \forall x \in X$ where J is the almost complex structure on X), we always take $\hat{\sigma}$ to be $T\sigma$. The bundle $\hat{\tau}X$ is thus defined with respect to $T\sigma$. Also in the case of a trivial complex vector bundle $d\epsilon_{\mathbb{C}}$, it is understood that $\hat{\sigma} : X \times \mathbb{C}^d \to X \times \mathbb{C}^d$ is the conjugation $(x, u) \mapsto (\sigma(x), \overline{u})$. We shall refer to this $\hat{\sigma}$ as the *standard* conjugation on $d\epsilon_{\mathbb{C}}$. Note that $\rho(d\epsilon_{\mathbb{C}}) = 2d\epsilon_{\mathbb{R}}$ and $\hat{\sigma}$ may be identified with $\varepsilon_{d,d}$.

Let ω be a σ-conjugate complex vector bundle and let $\hat{\sigma} : E(\omega) \to E(\omega)$ be a bundle involution that cover σ. Let η be a real vector bundle which is isomorphic to the real vector bundle $\rho(\omega)$ underlying ω. Suppose that $f : \rho(\omega) \to \eta$ is a bundle isomorphism that covers the identity map of X. Set $\tilde{\sigma} := f \circ \hat{\sigma} \circ f^{-1}$. Then $\widetilde{\sigma}$ is an involution of η that covers σ and hence defines a vector bundle $\hat{\eta}$ over $P(m, X, \sigma)$.

Lemma 2.3.1. *We keep the above notations.*
(i) The real vector bundles $\hat{\omega}$ and $\hat{\eta}$ over $P(m, X, \sigma)$ associated to the pairs $(\omega, \hat{\sigma})$ and $(\eta, \tilde{\sigma})$ are isomorphic. In particular $\hat{\omega} \cong \hat{\bar{\omega}}$.
(ii) Suppose that $\rho(\omega) = \eta_0 \oplus \eta_1$ where $\eta_j, j = 0, 1$ are real vector bundles. Suppose that $\hat{\sigma}(E(\eta_j)) = E(\eta_j)$, then $\hat{\omega}$ is isomorphic to $\hat{\eta}_0 \oplus \hat{\eta}_1$ where $\hat{\eta}_j$ is defined with respect to the pair $(\eta_j, \hat{\sigma}|_{E(\eta_j)})$, $j = 0, 1$.
(iii) Let $n = k + l \geq 1$. Suppose that $\rho(\omega) \oplus n\epsilon_{\mathbb{R}} \cong N\epsilon_{\mathbb{R}}$, where $N := 2d + n$, and that

the involutive bundle map $(\hat{\sigma}, \varepsilon_{k,l})$ on $\rho(\omega) \oplus n\epsilon_{\mathbb{R}}$ covering σ equals $\varepsilon_{d+k,d+l}$, then $\hat{\omega} \oplus k\xi \oplus l\epsilon_{\mathbb{R}} \cong (d+k)\xi \oplus (d+l)\epsilon_l$.

Proof. We will only prove (i). Consider the map $\varphi : \mathbb{S}^m \times E(\omega) \to \mathbb{S}^m \times E(\eta)$ defined as $\varphi(v, e) = (v, f(e))\ \forall v \in \mathbb{S}^m, e \in E(\omega)$. The $\varphi((-v, \sigma(e))) = (-v, f(\hat{\sigma}(e))) = (-v, \tilde{\sigma}(f(e)))$. Thus φ is $\mathbb{Z}_2$-equivariant and so induces a vector bundle homomorphism $\overline{\varphi} : P(m, E(\omega), \hat{\sigma}) \to P(m, E(\eta), \tilde{\sigma})$ that covers the identity map of $P(m, X, \sigma)$. Restricted to each fibre, the map $\overline{\varphi}$ is an $\mathbb{R}$-linear isomorphism since this is true for f. Thus $\overline{\varphi}$ is a bundle map that covers the identity map of $P(m, X, \sigma)$. Therefore $\hat{\omega}$ and $\hat{\eta}$ are isomorphic vector bundles. Finally, let $\eta = \overline{\omega}$. Taking $f : E(\omega) \to E(\eta)$ to be $e \mapsto \overline{e}$, we see that $\hat{\omega} \cong \hat{\overline{\omega}}$. □

Example 2.3.2. *(i) Since $\epsilon_{\mathbb{C}} \cong 2\epsilon_{\mathbb{R}}$ and the complex conjugation on $\epsilon_{\mathbb{C}}$ corresponds to $[v, x; s, t] \mapsto [v, x, s, -t]$ on $\epsilon_{\mathbb{R}} \oplus \epsilon_{\mathbb{R}}$, the above lemma yields $\hat{\epsilon}_{\mathbb{C}} \cong \epsilon_{\mathbb{R}} \oplus \xi$.*
(ii) Consider the Riemann sphere $\mathbb{S}^2 = \mathbb{C}P^1$. Let $\gamma \subset 2\epsilon_{\mathbb{C}}$ be the tautological (complex) line bundle over $\mathbb{C}P^1$ and let β be its orthogonal complement. As complex line bundles one has the isomorphism $\beta \cong \overline{\gamma}$. It follows that from the above lemma that $\hat{\gamma} \cong \hat{\beta}$.
(iii) Suppose that $X = \mathbb{C}G_{n,k}$ and let $\sigma : X \to X$ be the conjugation $L \to \overline{L}$. As seen in Example 2.1.2(ii), $v \mapsto \overline{v}$ define conjugations of $\gamma_{n,k}, \beta_{n,k}$ that cover σ. Note that $\gamma_{n,k} \oplus \beta_{n,k} = n\epsilon_{\mathbb{C}}$. By the above lemma we obtain that $\hat{\gamma}_{n,k} \oplus \hat{\beta}_{n,k} \cong d\hat{\epsilon}_{\mathbb{C}} \cong d\epsilon_{\mathbb{R}} \oplus d\xi$. Also, the conjugations on $\gamma_{n,k}, \beta_{n,k}$ induce an involution, denoted $\hat{\sigma}$, on $Hom(\gamma_{n,k}, \beta_{n,k})$ as in Example 2.1.2(iv). But this latter bundle is isomorphic to the tangent bundle $\tau\mathbb{C}G_{n,k}$. The bundle map $\hat{\sigma}$ corresponds to the bundle involution $T\sigma : T\mathbb{C}G_{n,k} \to T\mathbb{C}G_{n,k}$ under the identification of $Hom(\gamma_{n,k}, \beta_{n,k})$ with $\tau\mathbb{C}G_{n,k}$.

2.4 Splitting principle

Denote by $Flag(\mathbb{C}^r)$ the complete flag manifold $\mathbb{C}G(1, \ldots, 1)$. Let ω be a complex vector bundle over X of rank $r \geq 1$ endowed with a Hermitian metric and let $q : Flag(\omega) \to X$

be the $Flag(\mathbb{C}^r)$-bundle associated to ω. Thus the fibre over an $x \in X$ is the space $\{(L_1, \ldots, L_r) \mid L_1 + \cdots + L_r = p_\omega^{-1}(x), L_j \perp L_k, 1 \leq j < k \leq r, \dim_{\mathbb{C}} L_j = 1\} \cong Flag(\mathbb{C}^r)$ of complete flags in $p_\omega^{-1}(x) \subset E(\omega)$. The vector bundle $q^*(\omega)$ splits as a Whitney sum $q^*(\omega) = \oplus_{1 \leq j \leq r} \omega_j$ of complex line bundles ω_j over $Flag(\omega)$ with projection $p_j : E(\omega_j) \to Flag(\omega)$. The fibre over a point $\mathbf{L} = (L_1, \ldots, L_r) \in Flag(\omega)$ of the bundle ω_j is the vector space $L_j \subset p_\omega^{-1}(q(\mathbf{L}))$.

Suppose that $\sigma : X \to X$ is an involution and that ω is a σ-conjugate bundle. Let $\hat{\sigma} : E(\omega) \to E(\omega)$ be the σ-conjugation on ω. We shall write $\overline{e}$ for $\hat{\sigma}(e), e \in E(\omega)$. One has the involution $\theta : Flag(\omega) \to Flag(\omega)$ defined as $\mathbf{L} = (L_1, \ldots, L_r) \mapsto (\overline{L}_1, \ldots, \overline{L}_r) =: \overline{\mathbf{L}}$. Here $\overline{V}$ denotes the subspace $\hat{\sigma}(V) \subset p_\omega^{-1}(\sigma(x))$ when $V \subset p_\omega^{-1}(x)$. The bundle $q^*(\omega)$ is a θ-conjugate bundle where bundle involution $\hat{\theta}$ is defined by $\hat{\sigma}$: that is, $\hat{\theta}(\mathbf{L}, e) = (\overline{\mathbf{L}}, \overline{e})$.

We define $\hat{\theta}_j : E(\omega_j) \to E(\omega_j)$ as $\hat{\theta}_j(\mathbf{L}, e) = (\overline{\mathbf{L}}, \overline{e})$. Evidently $\hat{\theta}_j$ is an involution, covers θ, and is conjugate linear when restricted to any fibre. Thus ω_j is a θ-conjugate bundle.

Recall from §2.2 that $\hat{\omega}$ is the real vector bundle with projection $p_{\hat{\omega}} : P(m, E(\omega), \hat{\omega}) \to P(m, X, \sigma)$. Likewise, we have the real 2-plane bundle $\hat{\omega}_j$ over $P(m, Flag(\omega), \theta)$ with projection $p_{\hat{\omega}_j} : P(m, E(\omega_j), \hat{\theta}_j) \to P(m, Flag(\omega), \theta)$. Since $q \circ \theta = \sigma \circ q$, we have the induced map $\hat{q} : P(m, Flag(\omega), \theta) \to P(m, X, \sigma)$ defined as $[v, \mathbf{L}] \mapsto [v, q(\mathbf{L})]$. The map $\hat{q}$ is in fact the projection of a fibre bundle with fibre the flag manifold $Flag(\mathbb{C}^r)$. Since $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_r)$, applying Lemma we see that $\hat{q}^*(\hat{\omega}) \cong \oplus_{1 \leq j \leq r} \hat{\omega}_j$.

Recall that the first Chern classes mod 2 of the canonical complex line bundles ξ_j over $Flag(\mathbb{C}^r)$, $1 \leq j \leq r$, generate the $\mathbb{Z}_2$-cohomology algebra $H^*(Flag(\mathbb{C}^r); \mathbb{Z}_2)$. In fact $H^*(Flag(\mathbb{C}^r); \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_r]/I$ where I is the ideal generated by the elementary symmetric polynomials in $c_1, \ldots, c_r$. Here the generators $c_j + I$ may be identified with the (integral) Chern class $c_1(\xi_j)$. In particular $H^*(Flag(\mathbb{C}^r); \mathbb{Z})^{S_r} = H^0(Flag(\mathbb{C}^r); \mathbb{Z}) \cong \mathbb{Z}$.

Since $\hat{\omega}_j$ restricts to the (real) 2-plane bundle $\rho(\xi_j)$, we have $c_1(\xi_j) = i^*(w_2(\omega_j))$ where $i : Flag(\mathbb{C}^r) \cong \hat{q}^{-1}([v, \mathbf{L}]) \to P(m, Flag(\omega), \theta)$ is fibre inclusion, we see that the $Flag(\mathbb{C}^r)$-bundle is $\mathbb{Z}_2$-totally non-cohomologous to zero. By Leray-Hirsch theorem, we have $H^*(P(m, Flag(\omega), \theta); \mathbb{Z}_2) \cong H^*(P(m, X, \sigma); \mathbb{Z}_2) \otimes H^*(Flag(\mathbb{C}^r); \mathbb{Z}_2)$. In particular, $H^*(P(m, Flag(\omega), \theta); \mathbb{Z}_2)$ is a free module over the algebra $H^*(P(m, X, \sigma); \mathbb{Z}_2)$ of rank $\dim_{\mathbb{Z}_2} H^*(Flag(\mathbb{C}^r); \mathbb{Z}_2) = r!$. It follows that $\hat{q}$ induces a monomorphism in mod 2 cohomology.

The symmetric group S_r operates on $Flag(\omega)$ by permuting the components of each flag $\mathbf{L} = (L_1, \ldots, L_r)$ and the projection $q : Flag(\omega) \to X$ is constant on the S_r-orbits. Moreover, $\theta \circ \lambda = \lambda \circ \theta$ for each $\lambda \in S_r$. This implies that the S_r action on $Flag(\omega)$ extends to an action on $P(m, Flag(\omega), \theta)$ where $\lambda([v, \mathbf{L}]) = [v, \lambda(\mathbf{L})]$. The projection $\hat{q} : P(m, Flag(\omega), \theta) \to P(m, X, \sigma)$ is constant on S_r-orbits. It follows that the image of the ring homomorphism $\hat{q}^* : H^*(P(m, X, \sigma); \mathbb{Z}_2) \to H^*(P(m, Flag(\omega), \theta); \mathbb{Z}_2)$ is contained in the subring $H^*(P(m, Flag(\omega), \theta); \mathbb{Z}_2)^{S_r}$ of elements fixed by the induced action of S_r on $H^*(P(m, Flag(\omega), \theta); \mathbb{Z}_2)$. As the S_r-action induces the identity map of $P(m, X, \sigma)$, we see that it acts as $H^*(P(m, X, \sigma); \mathbb{Z}_2)$-module automorphisms on $H^*(P(m, Flag(\omega), \theta); \mathbb{Z}_2)$. We summarise the above discussion in the proposition below.

Proposition 2.4.1. *Let ω be a σ-conjugate complex vector bundle of rank r and let $q^* : Flag(\omega) \to X$ be the associated $Flag(\mathbb{C}^r)$-bundle over X. Then:*
(i) the ω_j are θ-conjugate line bundles for $1 \leq j \leq r$, and, $\hat{q}^*(\hat{\omega}) = \oplus_{1 \leq j \leq r} \hat{\omega}_j$.
(ii) $\hat{q} : P(m, Flag(\omega), \theta) \to P(m, X, \sigma)$ induces monomorphism in cohomology, moreover, $H^*(P(m, Flag(\omega), \theta); \mathbb{Z}_2)$ is isomorphic, as an $H^*(P(m, X, \sigma); \mathbb{Z}_2)$-module, to a free module with basis a $\mathbb{Z}_2$-basis of $H^*(Flag(\mathbb{C}^r); \mathbb{Z}_2)$.
(iii) The image of $\hat{q}^*$ is contained in the subalgebra invariant under the action of the symmetric group S_r on $H^*(P(m, Flag(\omega), \theta); \mathbb{Z}_2)$.

We end this section with the following lemma which will be used in the sequel.

Lemma 2.4.2. *We keep the above notations. Let ω be a σ-conjugate complex vector* *bundle over X. Suppose that $Fix(\sigma) \neq \emptyset$ and that $H^1(X;\mathbb{Z}_2) = 0$. Then $Fix(\theta) \neq \emptyset$ and $H^1(P(m, Flag(\omega), \theta);\mathbb{Z}_2) \cong H^1(P(m, X, \sigma);\mathbb{Z}_2) \cong \mathbb{Z}_2$.*

Proof. Let $\sigma(x) = x \in X$ and set $V := p_\omega^{-1}(x)$. Then $\hat{\sigma}$ restricts to a conjugate complex isomorphism $\hat{\sigma}_x$ of V onto itself. Thus $V \cong \overline{V}$. Then, setting $Fix(\hat{\sigma}_x) =: U \subset V$, we see that V is the $\mathbb{C}$-linear extension of U, that is, $V = U \otimes_{\mathbb{R}} \mathbb{C}$. The Hermitian product on V restricts to a (real) inner product on U. Let $(K_1, \ldots, K_r)$ be a complete real flag in U and define $L_j := K_j \otimes_{\mathbb{R}} \mathbb{C} \subset V$. Then it is readily seen that $\mathbf{L} = (L_1, \ldots, L_r)$ belongs to $Flag(\omega)$ and is fixed by θ.

Since $H^1(X;\mathbb{Z}_2) = 0$, we have $H^1(P(m, X, \sigma);\mathbb{Z}_2) \cong H^1(\mathbb{R}P^m;\mathbb{Z}_2) \cong \mathbb{Z}_2$, using the Serre spectral sequence of the X-bundle with projection $\pi : P(m, X, \sigma) \to \mathbb{R}P^m$. The same argument applied to the $Flag(\mathbb{C}^r)$-bundle with projection $q : Flag(\omega) \to X$ yields that $H^1(Flag(\omega);\mathbb{Z}_2) \cong H^1(X;\mathbb{Z}_2) = 0$. Now using the $Flag(\omega)$-bundle with projection $\hat{q} : P(m, Flag(\omega), \theta) \to P(m, X, \sigma)$, we obtain that $H^1(P(m, Flag(\omega), \theta);\mathbb{Z}_2) \cong H^1(P(m, X, \sigma);\mathbb{Z}_2) \cong \mathbb{Z}_2$. □

Chapter 3

A formula for Stiefel-Whitney classes of $\hat{\omega}$

Denote the Stiefel-Whitney polynomial $\sum_{0\leq i\leq q} w_i(\eta)t^i$ of a rank q real vector bundle η by $w(\eta;t)$ and similarly the Chern polynomial $\sum_{0\leq i\leq q} c_j(\alpha)t^j$ of a complex vector bundle α of rank q by $c(\alpha;t)$. Recall that when α is regarded as a real vector bundle, we have $w(\alpha;t) = c(\alpha;t^2) \mod 2$. (See [18].) In this chapter we obtain a formula for the Stiefel-Whitney classes of the vector bundle $E(\hat{\omega}) \to P(m, X, \sigma)$ associated to a σ-conjugate vector bundle over X under a mild hypothesis on X (See Proposition 3.2.4). In Lemma 3.2.2, we first consider the case where ω is a line bundle.

We shall make no notational distinction between $c_j(\alpha) \in H^{2j}(X;\mathbb{Z})$ and its reduction mod 2 in $H^{2j}(X;\mathbb{Z}_2)$. We will mostly be working with the coefficient ring $\mathbb{Z}_2$.

Since $\hat{\omega}$ restricted to any fibre of $\pi : P(m, X, \sigma) \to \mathbb{R}P^m$ is isomorphic to ω (regarded as a real vector bundle), we obtain that, the total Stiefel-Whitney polynomial $j^*(w(\hat{\omega};t)) = w(\omega;t) = c(\omega, t^2)$ where $j : X \to P(m, X, \sigma)$ is the typical fibre inclusion. Suppose that $z_1, \ldots, z_q$ are the Chern roots of α, that is, $c_j(\alpha)$ is formally the j-th elementary symmetric polynomial $e_j(z_1, \ldots, z_q)$ in the indeterminates $z_1, \ldots, z_q$. Thus we have a formal factorization $c(\alpha;t) = \prod_{1\leq j\leq q}(1+z_jt)$. Similarly, we have a formal factorization $w(\eta;t) = \prod_{1\leq j\leq q}(1+y_jt)$ where $y_j, 1\leq j\leq q$, are the 'Stiefel-Whitney roots' of the real vector bundle η. Lemma 3.1.1 yields the Stiefel-Whitney classes of $\hat{\omega}$ when ω is a complex line bundle. Using this and the splitting principle, we will obtain a formula for the Stiefel-Whitney classes when ω is of arbitrary rank. The lemma was obtained in the special case of Dold manifolds in [32, Prop. 1.4].

3.1 The vector bundles $\hat{\omega}$ and $\xi \otimes \hat{\omega}$

Recall that ξ is the line bundle associated to the double cover $\mathbb{S}^m \times X \to P(m, X, \sigma)$ and is isomorphic to $\pi^*(\zeta)$ where ζ is the Hopf line bundle over $\mathbb{R}P^m$.

Lemma 3.1.1. *Let $\sigma : X \to X$ be an involution with non-empty fixed point set and let ω be a complex vector bundle of rank r over X. With the above notations, we have $\hat{\omega} \cong \xi \otimes \hat{\omega}$.*

Proof. The total space of the bundle $\xi \otimes \hat{\omega}$ has the description $E(\xi\otimes\hat{\omega}) = \{[v, x; t\otimes e] \mid [v, x] \in P(m, X; \sigma), t \in \mathbb{R}, e \in p_\omega^{-1}(x)\}$ where $[v, x; t\otimes e] = \{(v, x; t\otimes e), (-v, \sigma(x); -t\otimes\hat{\sigma}(e))\}$; here $\hat{\sigma} : E(\omega) \to E(\omega)$ is an *involutive* bundle map that covers σ and is conjugate linear isomorphism on each fibre. Thus we have the equality $\hat{\sigma}(\sqrt{-1}te) = -\sqrt{-1}t\hat{\sigma}(e)$. Observe that $[v, x; \sqrt{-1}te] = [-v, \sigma(x); \hat{\sigma}(\sqrt{-1}te)] = [-v, \sigma(x), -\sqrt{-1}t\hat{\sigma}(e)]$ and so the map $h : E(\xi\otimes\hat{\omega}) \to E(\hat{\omega})$, $[v, x; t\otimes e] \mapsto [v, x; \sqrt{-1}te] = [-v, \sigma(x); -\sqrt{-1}t\hat{\sigma}(e)]$ is a well-defined isomorphism of real vector bundles. $\square$

Remark 3.1.2. *Recall from §1.2 the functor μ^2 defined by Lam. The following properties of μ^2 will be used in the Chapter 4.*

(i) If $\hat{\sigma} : E(\omega) \to E(\omega)$ is a complex conjugation of ω covering an involution $\sigma : X \to X$, then $\mu^2(\hat{\sigma}) : E(\mu^2(\omega)) \to E(\mu^2(\omega))$ is a bundle map covering σ. In particular $\mu^2(\overline{\omega}) \cong \mu^2(\omega)$.

(ii) If $\hat{\sigma}$ is a conjugation of a complex line bundle ω with a Hermitian metric $\langle .,. \rangle$ covering an involution σ such that $\langle u, v\rangle_x = \overline{\langle \hat{\sigma}(u), \hat{\sigma}(v)\rangle}_{\sigma(x)}, u, v \in p_\omega^{-1}(x), x \in X$, then $\mu^2(\hat{\sigma}) : \mu^2(\omega) \to \mu^2(\omega)$ is the identity on each fibre since $||\hat{\sigma}(u)|| = ||u||$. Note that $\mu^2(\omega) \cong \epsilon_{\mathbb{R}}$ since ω is a line bundle.

3.2 Stiefel-Whitney classes of $\hat{\omega}$

We shall make the following simplifying assumptions.
(a) $\sigma : X \to X$ has a fixed point. As observed already, the X-bundle $\pi : P(m, X, \sigma) \to \mathbb{R}P^m$ admits a cross-section $s : \mathbb{R}P^m \to P(m, X, \sigma)$. It follows that $\pi^* : H^*(\mathbb{R}P^m; \mathbb{Z}_2) \to H^*(P(m, X, \sigma); \mathbb{Z}_2)$ is a monomorphism. We shall identify $H^*(\mathbb{R}P^m; \mathbb{Z}_2)$ with its image under π^*.
(b) $H^1(X; \mathbb{Z}_2) = 0$. This implies that $H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}_2)$ induced by the homomorphism $\mathbb{Z} \to \mathbb{Z}_2$ of the coefficient rings is surjective.

Example 3.2.1. *(i) Let X be the complex flag manifold $\mathbb{C}G(n_1, \ldots, n_r)$ and let $\sigma : X \to X$ be defined by the complex conjugation on $\mathbb{C}^n$, $n = \sum n_j$. Then $Fix(\sigma)$ is the real flag manifold $\mathbb{R}G(n_1, \ldots, n_r) = O(n)/(O(n_1) \times \cdots \times O(n_r))$ so assumption (a) holds. Since X is simply connected,(b) also holds. .*

(ii) Let ω be a σ-conjugate complex vector bundle of rank r. Suppose that $Fix(\sigma) \neq$ and that $H^1(X; \mathbb{Z}_2) = 0$. Let $\theta : Flag(\omega) \to Flag(\omega)$ be the associated involution of the $Flag(\mathbb{C}^r)$ manifold bundle over X. (See §2.4.) Then $Fix(\theta) \neq \emptyset$ and $H^1(Flag(\omega); \mathbb{Z}_2) = 0$.

In the Serre spectral sequence of the bundle $(P(m, X), \mathbb{R}P^m, X, \pi)$, we have $E_2^{0,k} = H^0(\mathbb{R}P^m; \mathcal{H}^k(X; \mathbb{Z}_2))$ where $\mathcal{H}^k(X; \mathbb{Z}_2)$ denotes the local coefficient system on $\mathbb{R}P^m$. The action of the fundamental group $\mathbb{Z}_2$ (respectively $\mathbb{Z}$ when $m = 1$) of $\mathbb{R}P^m$ on $H^*(X; \mathbb{Z}_2)$ is generated by the involution $\sigma^* : H^*(X; \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2)$. Hence

$E_2^{0,2} = H^2(X;\mathbb{Z}_2)^{\mathbb{Z}_2} = Fix(\sigma^*)$. In order to emphasise the dimension, we shall write $H^2(\sigma;\mathbb{Z}_2)$ instead of σ^*. Also (b) implies that $E_3^{0,2} = E_2^{0,2}$ and (a) implies that the transgression $E_3^{0,2} = H^2(X;\mathbb{Z}_2) \to E_3^{3,0} = H^3(\mathbb{R}P^m;\mathbb{Z}_2)$ is zero. It follows that $E_3^{0,2} = E_\infty^{0,2}$ and that the image $j^* : H^2(P(m,X);\mathbb{Z}_2) \to H^2(X;\mathbb{Z}_2)$ equals $Fix(H^2(\sigma;\mathbb{Z}_2))$, where $j : X \hookrightarrow P(m,X)$ is the fibre inclusion.

We have the exact sequence for all $m \geq 1$:

$$0 \to H^2(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{\pi^*} H^2(P(m,X,\sigma);\mathbb{Z}_2) \xrightarrow{j^*} Fix(H^2(\sigma;\mathbb{Z}_2)) \to 0.$$

The homomorphism s^* yields a splitting and allows us to identify $Fix(H^2(\sigma;\mathbb{Z}_2))$ as a subspace of $H^2(P(m,X,\sigma);\mathbb{Z}_2)$, namely the kernel of s^*. We shall denote the image of an element $u \in Fix(H^2(\sigma;\mathbb{Z}_2))$ by $\tilde{u}$.

Part (iii) of the lemma was obtained in the special case of Dold manifolds in [32, Prop. 1.4]. Recall that ζ is the Hopf bundle over $\mathbb{R}P^m$.

Lemma 3.2.2. *Suppose that $\sigma(x_0) = x_0$ and $H^1(X;\mathbb{Z}_2) = 0$. Let $s : \mathbb{R}P^m \to P(m,X,\sigma)$ be defined as $v \mapsto [v, x_0]$ and let ω be a σ-conjugate complex vector bundle over X of rank r. Then (i) $s^*(\hat{\omega}) = r\epsilon \oplus r\zeta$, (ii) $c_k(\omega) \in Fix(H^{2k}(\sigma;\mathbb{Z}_2))$, $k \leq r$, and, (iii) if $r = 1$, then $w(\hat{\omega}) = 1 + w_1(\xi) + \tilde{c}_1(\omega)$.*

Proof. (i) Since $\sigma(x_0) = x_0$, $\hat{\sigma}$ restricts to a conjugate complex linear automorphism $\hat{\sigma}_0$ of $V := p_\omega^{-1}(x_0)$. Let $U \subset V$ is the eigenspace of $\hat{\sigma}_0$ corresponding to eigenvalue 1 of $\hat{\sigma}_0$. Then $\sqrt{-1}U$ is the -1 eigenspace. The vector bundle $s^*(\hat{\omega})$ is isomorphic to the Whitney sum of the bundles $\mathbb{S}^m \times_{\mathbb{Z}_2} U \to \mathbb{R}P^m$ and $\mathbb{S}^m \times_{\mathbb{Z}_2} \sqrt{-1}U \to \mathbb{R}P^m$. Evidently these bundles are isomorphic to $r\epsilon_\mathbb{R}$ and $r\xi$ respectively.

(ii) Since $\hat{\sigma} : E(\omega) \to E(\omega)$ is a *conjugate* complex linear bundle map covering σ, we have $\sigma^*(\omega) \cong \overline{\omega}$. So $\sigma^*(c_k(\omega)) = c_k(\sigma^*(\omega)) = (c_k(\overline{\omega})) = (-1)^k c_k(\omega) \in H^{2k}(X;\mathbb{Z})$. Therefore $c_k(\omega) \in Fix(H^{2k}(\sigma;\mathbb{Z}_2))$, $k \leq r$.

(iii) Since $c_1(\omega) \in Fix(H^2(\sigma;\mathbb{Z}_2))$, the element $\tilde{c}_1(\omega)$ is meaningful. It remains to show that $w_2(\hat{\omega}) = \tilde{c}_1(\omega)$. Since $j^*(\hat{\omega}) = \omega$, we see that
$j^*(w_2(\hat{\omega})) = w_2(\omega) = c_1(\omega) \in Fix(H^2(\sigma;\mathbb{Z}_2))$. On the other hand, $w_2(s^*(\hat{\omega})) = 0$. So, under our identification of $Fix(H^2(\sigma;\mathbb{Z}_2))$ with the kernel of s^*, we have
$w_2(\hat{\omega}) = \tilde{c}_1(\omega)$. □

Remark 3.2.3. *The above lemma shows that the element of $\tilde{c}_1(\omega) \in H^2(P(m,X);\mathbb{Z}_2)$ is independent of the choice of the fixed point $x_0 \in X$ (used in the definition s^*) since it equals $w_2(\hat{\omega})$.*

Suppose that ω is a σ-conjugate complex vector bundle of rank r over X. Since $q^*(\omega)$ splits as a Whitney sum $q^*(\omega) = \oplus_{1\le j\le r}\omega_j$, where $q : Flag(\omega) \to X$ is the $Flag(\mathbb{C}^r)$-bundle, in view of Example 3.2.1, we have $c_1(\omega_j) \in Fix(H^2(\theta;\mathbb{Z}_2))$. Therefore we obtain their 'lifts' $\tilde{c}_1(\omega_j) \in H^2(P(m, Flag(\omega);\theta);\mathbb{Z}_2)$. The bundle $\hat{q}^*(\hat{\omega}))$ splits as $\hat{q}^*(\hat{\omega}) = \oplus_{1\le j\le r}\hat{\omega}_j$, where $\hat{q} : P(m, Flag(\omega), \theta) \to P(m, X, \sigma)$ is the projection of the $Flag(\mathbb{C}^r)$-bundle.

Therefore $e_j(\tilde{c}_1(\omega_1), \ldots, \tilde{c}_1(\omega_r)) = e_j(w_2(\hat{\omega}_1), \ldots, w_2(\hat{\omega}_r)) = w_j(\hat{\omega})$ is in $H^{2j}(P(m,X,\sigma);\mathbb{Z}_2)$. Here e_j stands for the j-th elementary symmetric polynomial.

Notation: *Set $\tilde{c}_j(\omega) := e_j(w_2(\hat{\omega}_1), \ldots, w_2(\hat{\omega}_r)) \in H^{2j}(P(m,X,\sigma);\mathbb{Z}_2),\ 1 \le j \le r$.*

When $j > r$, $\tilde{c}_j = 0$. Observe that $\tilde{c}_j(\omega)$ restricts to $c_j(\omega) \in H^{2j}(X;\mathbb{Z}_2)$ on any fibre of $\pi : P(m,X,\sigma);\mathbb{Z}_2) \to \mathbb{R}P^m$.

We have the following formula for the Stiefel-Whitney classes of $\hat{\omega}$.

Proposition 3.2.4. *We keep the above notations. Let ω be a σ-conjugate complex vector bundle over X. Suppose that $H^1(X;\mathbb{Z}_2) = 0$ and that $Fix(\sigma) \neq \emptyset$. Then*

$$w(\hat{\omega};t) = \sum_{0\le j\le r} (1+xt)^{r-j}\tilde{c}_j(\omega)t^{2j},$$

where $x = w_1(\xi)$.

Proof. The case when ω is a line bundle was settled in Lemma 3.2.2. In the more general case, it follows from the case of line bundles by the splitting principle, namely Proposition 2.4.1. We omit the routine details. $\square$

It is more convenient to rewrite the above formula using the bundle isomorphism $\hat{q}^*(\hat{\omega}) = \hat{\omega}_1 \oplus \cdots \hat{\omega}_r$. This leads to the formula

$$w(\hat{\omega}; t) = \prod_{1 \leq j \leq r} (1 + xt + \tilde{c}_1(\omega_j)t^2).$$

Chapter 4

Tangent bundle of generalized Dold manifolds

In this chapter we describe Stiefel-Whitney classes of $P(m, X)$ and obtain bounds for span and stable span of $P(m, X)$. When X is a complex flag manifolds and we obtain results on (stable) parallelizability of $P(m, X)$.

4.1 Stiefel-Whitney classes of $P(m, X)$

Let X be an almost complex manifold and let $\sigma : X \to X$ be a complex conjugation. the complex vector bundle τX is σ-conjugate. The manifold $P(m, X, \sigma)$ will be more briefly denoted $P(m, X)$. We assume that $Fix(\sigma)$ is non-empty and hence a smooth manifold of dimension $d = (1/2)dimX$. Also assume that $H^1(X; \mathbb{Z}_2) = 0$. The tangent bundle of $P(m, X)$ admits a Whitney sum decomposition

$$\tau P(m, X) = \pi^*(\tau \mathbb{R}P^m) \oplus \hat{\tau} X.$$

Using the fact that $w(\mathbb{R}P^m) = (1+x)^{m+1}$ where $x = w_1(\xi)$, and applying Proposition 3.2.4, we have the following expression for the Stiefel-Whitney polynomial of $P(m, X)$. Note that $H^1(P(m,X); \mathbb{Z}_2) \cong H^1(\mathbb{R}P^m; \mathbb{Z}_2) \cong \mathbb{Z}_2$, since $H^1(X; \mathbb{Z}_2) = 0$. For the definition of $\tilde{c}_j(X)$, see §3.2.

Theorem 4.1.1. *Let X be a $2d$-dimensional connected almost complex manifold with complex conjugation σ. Suppose that $Fix(\sigma) \neq \emptyset$ and that $H^1(X; \mathbb{Z}_2) = 0$. Then:*

$$w(P(m,X); t) = (1+xt)^{m+1}. \sum_{0 \leq j \leq d} (1+xt)^{d-j} \tilde{c}_j(X) t^{2j},$$

where $x = w_1(\xi)$. □

Remark 4.1.2. *Proposition 3.2.4 can be used to obtain Dold's formula for $w(P(m,n))$ as follows. Since $\tau P(m,n) = \pi^*(\tau \mathbb{R}P^m) \oplus \hat{\tau}\mathbb{C}P^n$, adding $2\epsilon_{\mathbb{R}}$ on both sides, we get $\tau P(m,n) \oplus 2\epsilon_{\mathbb{R}} = \pi^*(\tau\mathbb{R}P^m \oplus \epsilon_{\mathbb{R}}) \oplus \hat{\tau}\mathbb{C}P^n \oplus \epsilon_{\mathbb{R}} = (m+1)\xi \oplus \hat{\tau}\mathbb{C}P^n \oplus \epsilon_{\mathbb{R}} = m\xi \oplus \hat{\tau}\mathbb{C}P^n \oplus \epsilon_{\mathbb{R}} \oplus \xi$. Note that $\hat{\tau}\mathbb{C}P^n \oplus \epsilon_{\mathbb{R}} \oplus \xi = \hat{\tau}\mathbb{C}P^n \oplus \hat{\epsilon_{\mathbb{C}}} = \tau\widehat{\mathbb{C}P^n \oplus \epsilon_{\mathbb{C}}} = (n+1)\hat{\bar{\gamma}} = (n+1)\hat{\gamma}$ where γ denotes the Hopf line bundle over $\mathbb{C}P^n$. By Proposition 3.2.4, $w(\hat{\gamma}) = (1+x+d)$ where $d = \tilde{c}_1(\gamma)$. Therefore, using the above bundle isomorphism*
$w(P(m,n)) = (1+x)^m(1+x+d)^{n+1}$ which is the formula of Dold.

The same formula can also be obtained from Theorem 4.1.1 using $\tilde{c}_j(\mathbb{C}P^n) = \binom{n+1}{j} d^j$ and the relation $d^{n+1} = 0$.

Corollary 4.1.3. *(i) $P(m,X)$ is orientable if and only if $m+d$ is odd.*
(ii) $m \geq 2$, $P(m,X)$ admits a spin structure if and only if X admits a spin structure and $m+1 \equiv d \mod 4$.

Proof. (i) Since $P(m,X) = \mathbb{S}^m \times X/\mathbb{Z}_2$, it is readily seen that $P(m,X)$ is orientable if and only if the antipodal map of $\mathbb{S}^m$ and the conjugation involution σ on X are simultaneously either orientation preserving or orientation reversing. The last condition is equivalent to $m+1 \equiv d \mod 2$.

Alternatively, from Theorem 4.1.1, we obtain that $w_1(P(m,X)) = (m+1+d)x$, which is zero precisely if $m+d$ is odd.

(ii) Using the same formula, we have $w_2(P(m,X)) = (\binom{m+1}{2} + \binom{d}{2})x^2 + \tilde{c}_1(X)$. The existence of a spin structure being equivalent to vanishing of the first and the second Stiefel-Whitney classes, we see that $P(m,X)$ admits a spin structure if and only if $m+1 \equiv d \mod 2$, $\tilde{c}_1(X) = 0$ and $\binom{m+1}{2} \equiv \binom{d}{2} \mod 2$. Equivalently, X admits a spin structure and $m+1 \equiv d \mod 4$. $\square$

4.2 Results of span and stable span of $P(m,X)$

The notions of stable parallelizability and parallelizability were recalled in §1.1. Recall that $\varepsilon_{k,n-k} : X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ is the involutive bundle map of $n\epsilon_{\mathbb{R}}$ covering σ defined as $(x, t_1, \ldots, t_n) \mapsto (\sigma(x), -t_1, \ldots, -t_k, t_{k+1}, \ldots, t_n)$.

Recall that $\varphi(m)$ is the number of positive integers $j \leq m$ such that $j \equiv 0, 1, 2,$ or $4 \mod 8$. Also $\rho(\omega)$ denotes the underlying real vector bundle of the complex vector bundle ω.

Theorem 4.2.1. *Let σ be a conjugation on a connected almost complex manifold X and let $\dim_{\mathbb{R}} X = 2d$. Suppose that $Fix(\sigma) \neq \emptyset$. Then:*

(i) If $P(m,X)$ is stably parallelizable, then X is stably parallelizable and $(m+1+d)$ is divisible by $2^{\varphi(m)}$.

(ii) Suppose that $\rho(\tau X) \oplus n\epsilon_{\mathbb{R}} \cong (2d+n)\epsilon_{\mathbb{R}}$ as real *vector bundle. Suppose that the bundle map $\varepsilon_{d+k,d+n-k}$ of $(2d+n)\epsilon_{\mathbb{R}}$ covering σ restricts to $\hat{\sigma} = T\sigma$ on TX and to $\varepsilon_{k,n-k}$ on $n\epsilon_{\mathbb{R}}$. If $(m+1+d)$ is divisible by $2^{\varphi(m)}$, then $P(m,X)$ is stably parallelizable.*

(iii) Suppose that m is even and that $P(m,X)$ is stably parallelizable. Then $P(m,X)$ is parallelizable if and only if $\chi(X) = 0$.

Proof. (i) Suppose that $P(m,X)$ is stably parallelizable. As $\pi : P(m,X) \to \mathbb{R}P^m$ is the projection of a smooth fibre bundle with fibre X, the bundle $\tau P(m,X)$ restricts to

$m\epsilon_{\mathbb{R}} \oplus \tau X$ on the fibre X. It follows that, X is stably parallelizable.

Let $x_0 \in Fix(\sigma)$ and let $s : \mathbb{R}P^m \to P(m, X)$ be the corresponding cross-section defined as $[v] \mapsto [v, x_0]$. In view of Proposition 3.1.1, we see that $s^*(\tau P(m, X)) = s^*(\pi^*\tau\mathbb{R}P^m \oplus \hat{\tau}X) = \tau\mathbb{R}P^m \oplus s^*(d\epsilon \oplus d\xi) = (\tau\mathbb{R}P^m \oplus d\epsilon) \oplus d\zeta \cong (m+1+d)\zeta \oplus (d-1)\epsilon$. Thus the stable parallelizability of $P(m, X)$ implies that $(m+1+d)([\zeta]-1) = 0$ in $KO(\mathbb{R}P^m)$. By the result of Adams [1] (recalled in §1) it follows that $2^{\varphi(m)}$ divides $(m+1+d)$.

(ii) Our hypothesis implies, using Lemma 2.3.1, that $\hat{\tau}X \oplus (k\xi + (n-k)\epsilon_{\mathbb{R}}) \cong (d+n-k)\epsilon_{\mathbb{R}} \oplus (d+k)\xi$. Therefore $\tau P(m, X) \oplus k\xi \oplus (n-k+1)\epsilon_{\mathbb{R}} \cong k\xi \oplus (n-k+1)\epsilon_{\mathbb{R}} \oplus \pi^*(\tau\mathbb{R}P^m) \oplus \hat{\tau}X \cong (m+1)\xi \oplus \hat{\tau}X \oplus k\xi + (n-k)\epsilon_{\mathbb{R}} \cong (m+1)\xi + (d+k)\xi + (d+n-k)\epsilon_{\mathbb{R}}$. Since $\dim P(m, X) = 2d + m < 2d + n + 1 + m$, we may cancel the factor $k\xi + (n-k)\epsilon_{\mathbb{R}}$ on both sides, leading to an isomorphism $\tau P(m, X) \oplus \epsilon_{\mathbb{R}} \cong (d+m+1)\xi \oplus d\epsilon_{\mathbb{R}}$. Since $\xi = \pi^*(\zeta)$, again using Adams' result it follows that $P(m, X)$ is stably parallelizable if $2^{\varphi(m)}$ divides $(m+d+1)$.

(iii) Note that $P(m, X)$ is even dimensional. By Bredon-Kosiński's theorem 1.1.2, it follows that $P(m, X)$ is parallelizable if and only if its span is at least 1. By Hopf's theorem, span $P(m, X) \geq 1$ if and only if $\chi(P(m, X))$ vanishes. Since $\chi(P(m, X)) = \chi(\mathbb{R}P^m).\chi(X) = \chi(X)$ as m is even, the assertion follows. □

Remark 4.2.2. *(i) Suppose that $P(m, X)$ is stably parallelizable. If m is odd, then $\chi(P(m, X)) = 0$ as $\chi(\mathbb{R}P^m) = 0$. Consequently we obtain no information about $\chi(X)$ from the equality $\chi(P(m, X)) = \chi(\mathbb{R}P^m)\chi(X)$. Let us suppose that $\chi(X) \neq 0$. Since $Span(\mathbb{R}P^m) = Span(\mathbb{S}^m)$, we obtain the lower bound $Span(P(m, X)) \geq Span(\mathbb{S}^m) = \rho(m+1) - 1$, where $\rho(m+1)$ is the Hurwitz-Radon function (See §1.1). From Bredon-Kosiński's theorem 1.1.2, we obtain that $P(m, X)$ is parallelizable if $\rho(m+1) > \rho(m+2d+1)$. For example if $m = (2c+1)2^r - 1$ and $d = 2^s(2k+1)$ with $s < r-1$ then $m+1+2d = ((2c+1)2^{r-1-s} + 2k+1)2^{s+1}$ and so*

$\rho(m+1) = \rho(2^r) > \rho(2^{s+1}) = \rho(m+2d+1)$; consequently $P(m, X)$ is parallelizable.

(ii) The following bounds for the span and stable span of $P(m, X)$ are easily obtained.

- *$\rho(m+1) \leq Span^0(P(m, X)) \leq \min\{d + Span(m+d+1)\zeta, m + Span^0(X)\}$,*
- *$Span(P(m, X)) \geq Span(\mathbb{R}P^m)$.*

If m is even and $\chi(X) = 0$, then $\chi(P(m, X)) = 0$ and so by Theorem 1.1.3(a), we have $Span(P(m, X)) = Span^0(P(m, X))$.

We shall now illustrate Theorem 4.2.1 in the case when X is the complex flag manifold $\mathbb{C}G(n_1, \ldots, n_r) = U(n)/(U(n_1) \times \cdots \times U(n_r))$, where the $n_j \geq 1$ are positive integers and $n = \sum_{1 \leq j \leq r} n_j$, with its standard $U(n)$-invariant complex structure. We assume, without loss of generality, that $n_1 \geq \cdots \geq n_r$. We denote by $P(m; n_1, \ldots, n_r)$ the space $P(m, \mathbb{C}G(n_1, \ldots, n_r))$. Note that $\mathbb{C}G(1, \ldots, 1)$ is the complete flag manifold $Flag(\mathbb{C}^n)$.

The case of the classical Dold manifold corresponds to $r = 2$ and $n_1 \geq n_2 = 1$. The result is then due to J. Korbaš [13]. (Cf. [32], [17].)

Theorem 4.2.3. *Let $m \geq 1$ and $r \geq 2$.*
(i) The manifold $P(m; n_1, \ldots, n_r)$ is stably parallelizable if and only if $n_j = 1$ for all j and $2^{\varphi(m)}$ divides $(m + 1 + \binom{n}{2})$ where $\varphi(m)$ is the number of positive integers $j \leq m$ such that $j \equiv 0, 1, 2,$ or $4 \mod 8$.
(ii) Suppose that $P := P(m; 1, \ldots, 1)$ is stably parallelizable. Then it is parallelizable if $\rho(m+1) > \rho(m+1+n(n-1))$. In particular, P is not parallelizable if m is even.

Proof: When $n_j > 1$ for some j, the flag manifold $X = \mathbb{C}G(n_1, \ldots, n_r)$ is well-known to be not stably parallelizable; see, for example, [28]. So the non-trivial part of the above theorem concerns the case when the flag manifold is stably parallelizable, namely, $n_j = 1$ for all j. It remains to determine which of the $P(m; 1, \ldots, 1)$ are stably parallelizable. This will be done in Proposition 4.2.4 below.

When $n_j = 1$ for all j, the corresponding complex flag manifold has non-vanishing Euler characteristic; in fact, $\chi(X) = n!$, the order of the Weyl group of $U(n)$. When m is even, it follows that the Euler characteristic of $P = P(m; 1, \dots, 1)$ also equals $n!$ and so its span is zero.

Then $\operatorname{Span}(P) \geq \operatorname{Span}(\mathbb{R}P^m) \geq \rho(m+1) - 1$ whereas span of the sphere of dimension $\dim P = m + 2d = m + n(n-1)$ equals $\rho(m+1+n(n-1)) - 1$. By Bredon-Kosiński theorem, P is parallelizable if it is stably parallelizable and
$\rho(m+1) > \rho(m+1+n(n-1))$. $\square$

The proof of the following proposition will require the functor μ^2 defined by Lam. The definition of functor μ^2 was recalled in §1.2. Specifically we use property that
$\mu^2(\omega_1 \oplus \omega_2) = \mu^2(\omega_1) \oplus (\overline{\omega}_1 \otimes_{\mathbb{C}} \omega_2) \oplus \mu^2(\omega_2)$ as real vector bundles where ω_1 and ω_2 are complex vector bundles. Also we will use the properties of μ^2 given in Remark 3.1.2.

Proposition 4.2.4. *The manifold $P(m; 1, \dots, 1) = P(m, Flag(\mathbb{C}^n))$ is stably parallelizable if and only if $2^{\varphi(m)}$ divides $(m + 1 + \binom{n}{2})$.*

Proof. Recall from §1.2 that $\tau\mathbb{C}G(n_1, \dots, n_r) \cong \oplus_{1 \leq i < j \leq r} \overline{\gamma}_i \otimes \gamma_j$ where γ_j is the j-th canonical bundle of rank n_j whose fibre over $(L_1, \dots, L_r) \in \mathbb{C}G(n_1, \dots, n_r)$ is the complex vector space L_j. We have

$$\gamma_1 \oplus \cdots \oplus \gamma_r \cong n\epsilon_{\mathbb{C}}.$$

Applying μ^2 and using the above description of $\tau\mathbb{C}G(n_1, \dots, n_r)$ we obtain the following isomorphism of *real* vector bundles by repeated use of property (iii) of μ^2 listed above:

$$\bigoplus \mu^2(\gamma_j) \oplus \tau(\mathbb{C}G(n_1, \dots, n_r)) \cong n\epsilon_{\mathbb{R}} \oplus \big(\bigoplus_{1 \leq i < j \leq n} \epsilon_{\mathbb{C}}(\overline{e}_i \otimes e_j)\big) \cong n^2 \epsilon_{\mathbb{R}}. \quad (1)$$

(Cf. [16, Theorem 5.1].) Specialising to the case of $X = Flag(\mathbb{C}^n)$ we have $\mu^2(\gamma_j) \cong \epsilon_{\mathbb{R}}$. The involution $\sigma : X \to X$ defined as $\mathbf{L} \mapsto \overline{\mathbf{L}}$ induces a complex conjugation of $\hat{\sigma} = T\sigma$ on τX which preserves the summands $\omega_{ij} := \overline{\gamma}_i \otimes \gamma_j, i < j$, yielding a conjugation $\hat{\sigma}_{ij}$ on it. The bundle involution $\varepsilon_{d,d}$ (covering σ) on the summand on the right $\oplus_{1\leq i<j\leq n}\rho(\epsilon_{\mathbb{C}})$, defined with respect to the basis $\bar{e}_i \otimes e_j, \bar{e}_i \otimes \sqrt{-1}e_j, 1 \leq i < j \leq n$, and $\varepsilon_{0,n}$ on the summand $\oplus_{1\leq i\leq n}\epsilon_{\mathbb{R}}(\bar{e}_i \otimes e_i)$ defined with respect to $\bar{e}_i \otimes e_i, 1 \leq i \leq n$, together define an involution, denoted ε, that covers σ. Under the isomorphism, ε restricts to $T\sigma$ on τX and to $\varepsilon_{0,n}$ on $\oplus_{1\leq i\leq n}\mu^2(\gamma_i)$ defined with respect to a basis $\bar{u}_i \otimes u_i, 1 \leq i \leq n$, where $u_i \in L_i$ with $||u_i|| = 1$. It follows, by using property (ii) of μ^2 given in Remark 3.1.2 and Lemma 2.3.1, that

$$n\epsilon_{\mathbb{R}} \oplus \hat{\tau}Flag(\mathbb{C}^n) \cong n\epsilon_{\mathbb{R}} \oplus \binom{n}{2}(\epsilon_{\mathbb{R}} \oplus \xi).$$

Therefore $(n+1)\epsilon_{\mathbb{R}} \oplus \tau P \cong (m+1)\xi \oplus \hat{\tau}Flag(\mathbb{C}^n) \oplus n\epsilon_{\mathbb{R}} \cong (m+1+\binom{n}{2})\xi \oplus n\epsilon_{\mathbb{R}}$. Hence τP is stably trivial if and only if $(m+1+\binom{n}{2})\xi$ is stably trivial if and only if $(m+1+\binom{n}{2})\zeta$ on $\mathbb{R}P^m$ is stably trivial if and only if $2^{\varphi(m)}$ divides $(m+1+\binom{n}{2})$. This completes the proof. $\square$

Remark 4.2.5. *It is clear that for a given $n \geq 2$, there are only finitely many values $m \geq 1$ for which $P = P(m, Flag(\mathbb{C}^n))$ is parallelizable. In fact, since $2^{\varphi(m)} \geq 2m$ for $m \geq 8$, we must have $m \leq \max\{8, \binom{n}{2}\}$. However the required values of m are highly restricted. For example when $n = 2^s, s \geq 4$, P is parallelizable only when $m \in \{1, 3, 7\}$ and when $n = 2^s - 2, s \geq 5$, $m \in \{2, 6\}$. When $n = 6$, P is not parallelizable for any m.*

Chapter 5

Unoriented cobordism

In this chapter we will obtain results concerning unoriented cobordism classes of $P(m, X)$.

Recall from the work of Thom and Pontrjagin (See Theorem 1.3.2) that the (unoriented) cobordism class of a smooth closed manifold is determined by its Stiefel-Whitney numbers. Proposition 3.2.4 makes it possible to compute certain Stiefel-Whitney numbers of $P(m, X)$ in terms of those of X where X is an almost complex manifold. We assume that complex conjugation σ has a fixed point and that $H^1(X; \mathbb{Z}_2) = 0$.

5.1 Transversality argument

Let $s : \mathbb{R}P^m \to P(m, X)$ be the cross-section corresponding to an $x_0 \in Fix(\sigma)$. We identify $\mathbb{R}P^m$ with its image under s and X with the fibre over $[e_{m+1}] \in \mathbb{R}P^m$. Then $X \cap \mathbb{R}P^m = \{[e_{m+1}, x_0]\}$ and the intersection is transverse. Denoting the Poincaré dual of a submanifold M by $[M]$, we have $[\mathbb{R}P^m] \smile [X] = [\mathbb{R}P^m \cap X] = [\{[e_{m+1}, x_0]\}]$, which is the generator of the top cohomology group $H^{m+2d}(P(m, X); \mathbb{Z}_2) \cong \mathbb{Z}_2$.

We claim that the class $[X] \in H^m(P(m, X); \mathbb{Z}_2)$ equals x^m. To see this, let

$S_j \subset \mathbb{S}^m, 1 \le j \le m$, be the sphere consisting of unit vectors whose j-th coordinate is zero and let X_j be the submanifold $\{[v, x] \mid v \in S_j, x \in X\} \cong P(m-1, X)$. Let $u_0 = (e_1 + \ldots + e_m)/\sqrt{m}$. Then $C := \{[\cos(t)u_0 + \sin(t)e_{m+1}, x_0] \in P(m, X) \mid 0 \le t \le \pi\}$ meets X_j transversally at $[e_{m+1}, x_0]$. So $[C][X_j] \neq 0$. It follows that $[X_j] = x, 1 \le j \le m$, since $H^1(P(m, X); \mathbb{Z}_2) = \mathbb{Z}_2 x$. Now (i) $\cap_{i<j} X_i$ intersects X_j transversely for each $j \le m$, and (ii) $\cap_{1 \le j \le m} X_j \cong X$ is the fibre of $P(m, X) \to \mathbb{R}P^m$ over $[e_{m+1}]$. It follows that $[X] = [X_1] \smile \cdots \smile [X_m] = x^m$ as claimed.

Denote by $\mu_X, \mu_{P(m,X)}$ the mod 2 fundamental classes of $X, P(m, X)$ respectively. Note that $w_{2j}(P(m, X))$ is of the form $w_{2j}(P(m, X)) = \tilde{c}_j(X) + a_1 x^2 \tilde{c}_{j-1}(X) + \ldots + a_t x^{2t} \tilde{c}_{j-t}(X)$ for suitable $a_i \in \{0, 1\}, 1 \le i \le t$ where $t = \min\{m/2, j\}$. Similarly $w_{2j+1}(P(m, X)) = b_0 x \tilde{c}_j(X) + b_1 x^3 \tilde{c}_{j-1}(X) + \ldots + b_t x^{2t+1} \tilde{c}_{j-t}$, $b_i \in \{0, 1\}, 0 \le i \le t$ with $t = \min\{m/2, j\}$. A straightforward calculation using Theorem 4.1.1 reveals that $b_0 = m + 1 + d - j$. Let $J = j_1, \ldots, j_r$ be a sequence of positive integers with $|J| := j_1 + \cdots + j_r = m + 2d$. Then $w_J(P(m, X)) := w_{j_1}(P(m, X)) \ldots w_{j_r}(P(m, X))$ is a polynomial in x over the subring $\mathbb{Z}_2[\tilde{c}_1(X), \ldots, \tilde{c}_d(X)] \subset H^*(P(m, X); \mathbb{Z}_2)$. Since $x^{m+1} = 0$, we see that $w_J(P(m, X)) = 0$ if the number of odd numbers among $j_k, 1 \le k \le r$, exceeds m.

Suppose that $I = i_1, \ldots, i_k; J = 1^m.2I = 1^m, 2i_1, \ldots, 2i_k$, (i.e., $j_t = 1, 1 \le t \le m$) and $P(m, X)$ is non-orientable, so that $w_1(P(m, X)) = x$, we have $w_J(P(m, X)) = x^m.\tilde{c}_I(X))$. Therefore, using $j^*(\tilde{c}_I(X)) = c_I(X) = w_{2I}(X)$, the Stiefel-Whitney number $w_J[P(m, X)] := \langle w_J(P(m, X)), \mu_{P(m,X)} \rangle = \langle x^m.w_{2I}(P(m, X)), \mu_{P(m,X)} \rangle = \langle w_{2I}(X), \mu_X \rangle = w_{2I}[X] \in \mathbb{Z}_2$.

Theorem 5.1.1. *Suppose that $H^1(X; \mathbb{Z}_2) = 0$ and that $Fix(\sigma) \neq \emptyset$.*
(i) Assume that $m \equiv d \mod 2$. If $[X] \neq 0$ in $\mathfrak{N}$, then $[P(m, X)] \neq 0$.
(ii) If $[P(1, X)] \neq 0$, then $[X] \neq 0$.

Proof. (i) Since $m \equiv d \mod 2$, we have $w_1(P(m,X)) = x$. Since the odd Stiefel-Whitney classes $w_{2i+1}(X)$ vanish (as X is an almost complex manifold), $[X] \neq 0$ implies that we must have that $w_{2I}[X] \neq 0$ for some I with $|I| = d$. Then, by our above discussion $w_J[P(m,X)] \neq 0$ where $J = 1^m.2I$. This proves the first assertion.

(ii) Let $m = 1$. Then $\dim P(m,X) = 1 + 2d$ is odd. Using $x^2 = 0$, we have, from the above discussion, that $w_{2j}(P(m,X)) = \tilde{c}_j(X)$ and $w_{2j+1}(P(m,X)) = (d-j)x\tilde{c}_j(X)$. Suppose that $w_J[P(m,X)] \neq 0$. Then we see that exactly one term, say j_k, in J must be odd. Write $j_k = 2s+1$ where $s \geq 0$. If $d-s$ is even, then $w_J[P(m,X)] = 0$. So $d-s$ is odd and we have $w_J(P(1,X)) = x\tilde{c}_I(X)$ where $2I$ is obtained from J by replacing j_k by $j_k - 1$. Therefore $w_{2I}[X] = w_J[P(1,X)] \neq 0$. This completes the proof. □

We now state our results on the cobordism class of generalized Dold manifolds $P(m,X)$ with X a complex Grassmann manifold.

Theorem 5.1.2. *Let $1 \leq k \leq n/2$ and let $m \geq 1$.*
(i) If $\nu_2(k) < \nu_2(n)$, then $[P(m, \mathbb{C}G_{n,k})] = 0$ in $\mathfrak{N}_*$.
(ii) If $m \equiv 0 \mod 2$ and if $\nu_2(k) \geq \nu_2(n)$, then $[P(m, \mathbb{C}G_{n,k})] \neq 0$.

We now turn to the proof of above theorem. The proof will make use of how certain units in a complex Clifford algebra act on its simple modules. We shall now recall the description and certain properties of real and complex Clifford algebras.

We shall use the structure of real and complex Clifford algebras to obtain an action of $G := (\mathbb{Z}_2)^r$ on $P(m,X)$ with $X := \mathbb{C}G_{n,k}$ such that $P(m,X)$ has no G-fixed points. This implies, by 1.3.3, that $[P(m,X)] = 0$.

5.2 Complex Clifford algebras

Let C_r (resp. C'_r) be the Clifford algebra associated to $(\mathbb{R}^r, -||\cdot||^2)$ (resp. $(\mathbb{R}^r, ||\cdot||^2)$). Thus C_r is generated as an $\mathbb{R}$-algebra by the elements $\varphi_1, \cdots, \varphi_r$ which satisfy the relations $\varphi_i^2 = -id\ \forall i$, and $\varphi_i \circ \varphi_j = -\varphi_j \circ \varphi_i, 1 \leq i < j \leq r$. Similarly C'_r is generated as an $\mathbb{R}$-algebra by $\psi_1, \ldots, \psi_r$ which satisfy the relations $\psi_i^2 = id\ \forall i$, and $\psi_i\psi_j = -\psi_j\psi_i, 1 \leq i < j \leq r$. We shall denote by C_r^c the complex Clifford algebra $C_r \otimes_{\mathbb{R}} \mathbb{C}$. Note that $C_r^c \cong C'_r \otimes_{\mathbb{R}} \mathbb{C}$ under an isomorphism that sends φ_j to $\sqrt{-1}\psi_j$. Following the notation in Husemoller's book [11], we denote the matrix algebra $M_m(A)$ over a division ring A by $A(m)$. It is known that C_r^c is isomorphic to $\mathbb{C}(2^p)$ or $\mathbb{C}(2^p) \times \mathbb{C}(2^p)$ according as $r = 2p$ or $r = 2p + 1$.

It is well known that C_r, C'_r are isomorphic to algebras of the form $A(2^t)$ or $A(2^s) \times A(2^s)$ where $A = \mathbb{R}, \mathbb{C}$, or the quaternions $\mathbb{H}$. The value of t is such that $2^r = 2^{2t} \dim_{\mathbb{R}} A$ by comparing the dimensions. The value of s is determined similarly. Since $A \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}, \mathbb{C} \times \mathbb{C}, \mathbb{C}(2)$ according as $A = \mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively, we see that C_r^c is isomorphic to one of the algebras $\mathbb{C}(2^p) \times \mathbb{C}(2^p)$ or $\mathbb{C}(2^p)$ according as $r = 2p + 1$ or $2p$ respectively. For our purposes we will only need to consider even Clifford algebras and so we shall ignore further discussions on odd Clifford algebras.

We consider $\mathbb{C}^{2^p}$ as a module over C_r^c where $r = 2p$. For our purposes, it is important to know whether the elements $\varphi_i \in C_r^c, 1 \leq i \leq r$, or $\psi_i \in C_r^c, 1 \leq i \leq r$, act on $\mathbb{C}^{2^p}$ as *real* transformations, that is if the elements are matrices with *real* entries in $C_r^c = \mathbb{C}(2^p)$. This is guaranteed to be the case if at least one of the algebras C_r or C'_r is isomorphic to $\mathbb{R}(2^p)$. We have isomorphisms of $\mathbb{R}$-algebras $C'_2 \cong \mathbb{R}(2), C_6 \cong \mathbb{R}(8), C_8 \cong \mathbb{R}(16)$. Also, we have the isomorphisms of algebras $C_{r+8} \cong C_r \otimes \mathbb{R}(16), C'_{r+8} \cong C'_r \otimes \mathbb{R}(16)$. Since $\mathbb{R}(k) \otimes \mathbb{R}(l) = \mathbb{R}(kl)$ and $\mathbb{R}(k) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(k)$, using the isomorphism $C_r \otimes_{\mathbb{R}} \mathbb{C} \cong C_r^c \cong C'_r \otimes_{\mathbb{R}} \mathbb{C}$, we see that when $r \equiv 2 \mod 8$, the elements $\psi_i \in C_r^c, 1 \leq i \leq r$, are represented by real matrices and that when $r \equiv 6, 8 \mod 8$, the same property holds for $\varphi_i \in C_r^c, 1 \leq i \leq r$. Therefore, we see that when p is a positive integer such that $p \equiv 3, 4 \mod 4$ (resp. $p \equiv 1 \mod 4$) $\mathbb{C}^{2^p}$ has the structure of a simple C_{2p}^c-module on which $\varphi_i, 1 \leq i \leq 2p$, (resp. $\psi_i, 1 \leq i \leq 2p$) acts as a real transformation, that is, via matrices with real entries.

Let $p \equiv 2 \mod 4$. One may regard $\mathbb{C}^{2^p}$ as a module over C_r^c-module where $r = 2p$ or $2p+1$. However, the corresponding real Clifford algebras C_r, C'_r are *not* matrix algebras over the reals. In this case we proceed as follows. Write $r = 2p = 8q + 4$. We have the isomorphisms $C'_{8q+2} \cong \mathbb{R}(2^{4q+1})$ with its generators $\psi_i, 1 \leq i \leq r-2$. Consider the $\mathbb{R}$-algebra C generated by the elements $\theta_i \in C_r^c, 1 \leq i \leq r$, expressed below as 2×2 block matrices with block sizes 2^{p-1} as follows:

$$\theta_i = \begin{cases} \begin{pmatrix} 0 & \psi_i \\ -\psi_i & 0 \end{pmatrix}, & 1 \leq i \leq r-2, \\ \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, & i = r-1, \\ \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, & i = r. \end{cases}$$

Then the following relations are readily verified: (i) $\theta_i\theta_j = -\theta_j\theta_i$ if $1 \leq i < j \leq r$, and, (ii) $\theta_i^2 = -1$ if $1 \leq i \leq r-2$ and $\theta_i^2 = 1$ if $i = r-1, r$. Moreover, it is easily verified that $\mathbb{R}$-algebra generated by the θ_i equals $\mathbb{R}(2^p)$. Therefore $C \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}(2^p) \cong C_r^c$. In particular, the elements $\theta_i, 1 \leq i \leq r$, act as real transformations on the simple module $\mathbb{C}^{2^p}$ of C_r^c.

Notation: For $1 \leq i \leq r$, we shall denote by $\theta_i \in C_r^c$ the element ψ_i (resp. φ_i) when $r \equiv 2 \mod 8$ (resp. $r \equiv 6, 8 \mod 8$). When $r \equiv 4 \mod 8$, the $\theta_i \in C_r^c$ are as defined above.

The above discussion establishes the validity of the following lemma.

Lemma 5.2.1. *Let $r = 2p$ be any even positive number. With the above notations, the elements $\theta_i \in C_r^c \cong \mathbb{C}(2^p), 1 \leq i \leq r$, satisfy the following conditions:*

Thus C is the real Clifford algebra associated to the indefinite (non-degenerate) quadratic form with signature $(2, r-2)$. See [23, Chapter 13].

(i) $\theta_i\theta_j = -\theta_j\theta_i, i \neq j$ and $\theta_i^2 = \pm 1$ for $i \leq r$,

(ii) the $\mathbb{R}$-subalgebra of C_r^c generated by $\theta_i, 1 \leq i \leq r$, is isomorphic to $\mathbb{R}(2^p)$,

(iii) the $\theta_i \in C_r^c$ act as a real transformation on the simple C_r^c module $\mathbb{C}^{2^p}$. □

We are now ready to prove the Theorem 5.1.2.

Proof. (i). Write $n = 2^p n_0$ where n_0 is odd and $p \geq 1$. Suppose that 2^p does not divide k.

Now let $r = 2p$. We regard $\mathbb{C}^n$ as a sum of n_0 copies of the simple C_r^c-module $\mathbb{C}^{2^p}$. With notations as in Lemmas 5.2.1, let $t_i, 1 \leq i \leq r$, denote the smooth map of the complex Grassmann manifold $\mathbb{C}G_{n,k}$ defined as $V \mapsto \theta_i(V), 1 \leq i \leq r$. Then $t_i^2 = id$ for $i \leq r$ since $\theta_i^2 = \pm 1$. Also $t_i t_j = t_j t_i$ for $1 \leq i < j \leq r$ since $\theta_i\theta_j = -\theta_j\theta_i$. So, the t_i define a smooth action of the group $(\mathbb{Z}/2\mathbb{Z})^r$. Any stationary point V of this action is a complex vector space of dimension k such that $\theta_i(V) = V\ \forall i \leq r$. This means that V is a module of over the $\mathbb{C}$-algebra generated by the $\theta_i, 1 \leq i \leq r$, that is, V is a C_r^c-module. In particular the $(\mathbb{Z}/2\mathbb{Z})^r$-action on $\mathbb{C}G_{n,k}$ is stationary point free since k is not divisible by 2^p.

The fact that the θ_i are real transformations implies that the t_i commute with complex conjugation σ, defined as $\sigma(V) = \overline{V}$. This means that the t_i define involutions, again denoted t_i, on the generalized Dold manifold $P(m, \mathbb{C}G_{n,k})$. Explicitly, $t_i([u, V]) = [u, t_i(V)]$ is meaningful since $(-u, t_i(\overline{V})) = (-u, \overline{t_i(V)}) \sim (u, t_i(V))$.

We claim that the action of $(\mathbb{Z}/2\mathbb{Z})^r$ has no stationary points. Indeed, $[u, V] = t_i([u, V]) = [u, t_i(V)]$ implies that $t_i(V) = V$ and so if $[u, V] \in P(m, \mathbb{C}G_{n,k})$ is a stationary point, then $V \in \mathbb{C}G_{n,k}$ would be a stationary point, contrary to what was just observed. Now, by 1.3.3, it follows that $[P(m, \mathbb{C}G_{n,k})] = 0$.

(ii) Suppose that $\nu_2(n) = \nu_2(k)$. Then $[\mathbb{C}G_{n,k}] \neq 0$ by the main theorem of [26]. (See also [25].) Note that $\dim_{\mathbb{C}} \mathbb{C}G_{n,k}$ is even in this case. If m is also even, then it follows that $[P(m, \mathbb{C}G_{n,k})] \neq 0$ by Theorem 5.1.1(i). $\square$

Let $D_k \subset O(k)$ denote the diagonal subgroup of the orthogonal group. Then $D_k \cong (\mathbb{Z}_2)^k$ acts on $\mathbb{R}^k$ and on $\mathbb{C}^k$ in the obvious manner. This leads to an action of D_{m+1} on $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ and an action of D_n on $\mathbb{C}G_{n,k}$. The set of antipodal points $\{e_j, -e_j\} \subset \mathbb{S}^m$ is stable under the action of D_{m+1} for all $j \leq m+1$. This induces an action of $D_{m+1} \times D_n \cong (\mathbb{Z}_2)^{m+n+1}$ on $P(m, \mathbb{C}G_{n,k})$ with $(m+1)\binom{n}{k}$ stationary points. We consider the $(\mathbb{Z}_2)^{m+n+1}$-equivariant cobordism classes of these manifolds.

We will now turn to the proof of following theorem .

Theorem 5.2.2. *The manifold $P(m, \mathbb{C}G_{n,k})$ bounds equivariantly under the above action of $(\mathbb{Z}_2)^{m+n+1}$ if $n = 2k$.*

Proof. Let $n = 2k$. Define $\alpha : \mathbb{S}^m \times \mathbb{C}G_{n,k} \to \mathbb{S}^m \times \mathbb{C}G_{n,k}$ as $\alpha(v, L) = (v, L^{\perp})$. Then α is a fixed point free involution. Note that $\alpha(-v, \overline{L}) = (-v, \overline{L}^{\perp}) = (-v, \overline{L^{\perp}})$. So α defines a fixed point free involution, denoted β, on $P(m, \mathbb{C}G_{n,k})$ by passing to the quotient. Our claim is that β is $(\mathbb{Z}_2)^{m+n+1}$-equivariant. It is enough to check this on the generators $T_i \in (\mathbb{Z}_2)^{m+n+1}, 1 \leq i \leq m+n+1$. It is trivial to check that $\beta T_i = T_i \beta$ for $1 \leq i \leq m+1$. For $m+2 \leq i \leq m+n+1$, since T_i is real orthogonal transformation $T_i\beta(V) = T_i(V^{\perp}) = (T_i(V))^{\perp} = \beta(T_i(V))\ \forall V \in \mathbb{C}G_{n,k}$. Using this, it follows that $\beta T_i = T_i \beta$ for $m+2 \leq i \leq m+n+1$. Therefore β is $(\mathbb{Z}_2)^{m+n+1}$-equivariant.

Let $M = P(m, \mathbb{C}G_{n,k})$ and $G = (\mathbb{Z}_2)^{m+n+1}$. Consider the $\mathbb{Z}_2$-action generated by the involution η on $M \times [-1, 1]$, defined as $\eta(z, s) = (\beta(z), -s)$. It is fixed point free and G-equivariant. Let W be the quotient of $M \times [-1, 1]$ by the $\mathbb{Z}_2$-action. Then W is a G-manifold with boundary $\partial W \cong M = P(m, \mathbb{C}G_{n,k})$ where the G-action on ∂W coincides with $P(m, \mathbb{C}G_{n,k})$. Hence $P(m, \mathbb{C}G_{n,k})$ bounds G-equivariantly. $\square$

Remark 5.2.3. *It appears to be unknown precisely which (real or complex) flag manifolds are unoriented boundaries. Let $n_1, \ldots, n_r \geq 1$ be integers and let $n = \sum_{1 \leq j \leq r} n_j$. Proceeding as in the case of the $P(m, \mathbb{C}G_{n,k})$ it is readily seen that*

$[\mathbb{C}G(n_1,\ldots,n_r)]$ *and* $[P(m;n_1,\ldots,n_r)]$ *in* $\mathfrak{N}$ *are zero if* $\nu_2(n) > \nu_2(n_j)$ *for some* j. *Also, if* $n_i = n_j$ *for some* $i \neq j$, *then* $\mathbb{C}G(n_1,\ldots,n_r)$ *admits a fixed point free involution, namely the one that swaps the* i*-th and the* j*-component of each flag in* $\mathbb{C}G(n_1,\ldots,n_r)$. *This involution commutes with complex conjugation and so defines an involution of* $P(m;n_1,\ldots,n_r)$ *which is again fixed point free. It follows that* $P(m;n_1,\ldots,n_r) = 0$ *in this case. If* $m \equiv d \mod 2$ *where* $d = \dim_{\mathbb{C}} \mathbb{C}G(n_1,\ldots,n_r) = \sum_{1\leq i<j\leq r} n_i n_j$ *and if* $[\mathbb{C}G(n_1,\ldots,n_r)] \neq 0$, *then* $[P(m;n_1,\ldots,n_r)] \neq 0$ *by Theorem 5.1.1. For example, it is known that* $\chi(\mathbb{C}G(n_1,\ldots,n_r)) = n!/(n_1!.\ldots.n_r!)$. *So if* m *and* d *are even and if* $n!/(n_1!.\ldots.n_r!)$ *is odd, then* $\chi(P(m;n_1,\ldots,n_r)$ *is also odd and so* $[P(m;n_1,\ldots,n_r)] \neq 0$.

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