Correlation of multiplicative functions

By

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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Pranendu Darbar

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I would like to dedicate my thesis
to my beloved parents

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Contents

Summary

Motivated by Halász's theorem, one would like to find the asymptotic behaviour of the following so called k-point correlation function.

$$M_k(x) := \sum_{n \le x} g_1(F_1(x)) \dots g_k(F_k(x)) \tag{1}$$

where g_j's are multiplicative functions with modulus less than or equal to 1 and $F_j(x)$'s are polynomials with integer coefficients. We divide this thesis into two parts.

In the first part of this thesis, by using a work of Warlimont and using a variant of the Turán-Kubilius inequality, we study the asymptotic behaviour of the k-point correlation function (1), where g_j's are multiplicative functions with values in the unit disc and F_j's are square-free and relatively co-prime polynomials.

The estimation of (1) is used to get information on the behaviour of the distribution of the sum of additive functions

$$f_1(F_1(x)) + \dots + f_k(F_k(x)),$$

where $f_1, f_2, \dots, f_k$ are real-valued additive functions.

In the final part of this thesis, we study the asymptotic behaviour of the correlation functions over polynomial ring $\mathbb{F}_q[x]$.

Consider the polynomial ring $\mathbb{F}_q[x]$ over a field with q elements. Let $\mathcal{M}_{n,q}$ denote the set of monic polynomials of degree n over $\mathbb{F}_q$, so that $|\mathcal{M}_{n,q}| = q^n$. Let $\mathcal{P}_{n,q}$ be the set of monic irreducible polynomials of degree n over $\mathbb{F}_q$. Our arithmetical functions are complex valued functions ψ on the monic polynomials $\mathcal{M}_q = \cup_{n=1}^{\infty}\mathcal{M}_{n,q}$.

From a general point of view, Our goal is to find an asymptotic formula for the following correlations of arithmetical functions $\psi_1, \ldots, \psi_k$ on $\mathcal{M}_q$ at $(h_1, \ldots, h_k) \in \mathbb{F}_q[x]^k$,

$$S_k(n,q) := \sum_{f \in \mathcal{M}_{n,q}} \psi_1(f+h_1)\ldots\psi_k(f+h_k)$$

and

$$R_k(n,q) := \sum_{P \in \mathcal{P}_{n,q}} \psi_1(P+h_1)\ldots\psi_k(P+h_k)$$

when the parameter $q^n = |\mathcal{M}_{n,q}|$ is large (and $n > \deg(h_i)$ for all i to avoid technical difficulties). This parameter can be large, in particular, either when $n \to \infty$, which we call the large degree limit, or when $q \to \infty$, which we call the large finite field limit.

In this thesis, by using Selberg sieve and Turán-Kubilius inequality over function fields, we study the asymptotic behaviour of the correlation functions $S_2(n,q)$ and $R_2(n,q)$ in the large degree limit. As an application, we prove a truncated variant of Chowla's conjecture over function field in large degree limit.

Notations

- $\mathbb{N}$ Set of all natural numbers.
- $\mathbb{Z}$ Set of all integers.
- $\mathbb{R}$ Set of all real numbers.
- $\mathbb{C}$ Set of all complex numbers.
- $\wp$ Set of all prime numbers.
- $\pi(x) = \#\{p \in \wp : p \leq x\}$.
- $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$.
- For $x \in \mathbb{R}$, we write $e(x) := e^{2\pi i x}$.
- $\Re(s)$ Real part of the complex number s.
- $\in$ Belongs to.
- $\ni$ Such that.
- $\notin$ Does not belong to.
- $\exists$ There exists.
- $[,]$ Least common multiple.
- $(,)$ Greatest common divisor.

- $f|g$ $\quad f$ divides g.
- $f \nmid g$ $\quad f$ does not divide g.
- $p^m \| n$ $\quad p^m | n$ but $p^{m+1} \nmid n$.
- $\lceil x \rceil$ The least integer greater than or equal to x.
- $\implies$ Implies.
- **card**(A) or $\#A$ or $|A|$ Cardinality of a set A.
- $\mathbb{F}_q$ Field with q elements.
- $\mathbb{F}_q^*$ Group of units in $\mathbb{F}_q$.
- $\mathbb{F}_q[x]$ Set of all polynomials over $\mathbb{F}_q$.
- $f * g$ Convolution of functions f and g.
- $\deg(f)$ Degree of the polynomial f.
- $\mathcal{M}_{n,q}$ Set of all monic polynomials of degree n over $\mathbb{F}_q$.
- $\mathcal{M}_{\leq n,q}$ Set of all monic polynomials of degree $\leq n$ over $\mathbb{F}_q$.
- $\mathcal{P}_{n,q}$ Set of all monic irreducible polynomials of degree n over $\mathbb{F}_q$.
- $\mathcal{M}_q := \bigcup_{n=1}^{\infty} \mathcal{M}_{n,q}$ and $\mathcal{P}_q := \bigcup_{n=1}^{\infty} \mathcal{P}_{n,q}$.
- $|f|$ Norm of the polynomial f, where $f \in \mathbb{F}_q[x]$.
- $\pi_A(n) = \#\mathcal{P}_{n,q}$, where $A := \mathbb{F}_q[x]$.
- $f(x) = o(g(x))$ if and only if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.
- $f(x) \sim g(x)$ if $\frac{f(x)}{g(x)} - 1 = o(1)$.
- $f(x) = O(g(x))$ or $f \ll g$ Means that there exists an absolute constant $C > 0$ and some large $x_o \in \mathbb{R}$ such that $|f(x)| \leq C|g(x)|$ for all $x \geq x_o$.

Chapter 1

Introduction

1.1 Background of multiplicative number theory

The set of positive integers $\mathbb{N}$ is a semigroup under both addition and multiplication. The interaction between these two operations create many difficult problems in analytic number theory.

A function $f : \mathbb{N} \to \mathbb{C}$ is said to be multiplicative if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. Such functions are completely determined by their values at prime powers. One of the classical objective of analytic number theory is understanding mean value of multiplicative functions. Determining the mean values of multiplicative functions is of considerable importance due to several applications to fundamental problems.

1.1.1 The distribution of primes

Many of the oldest problems in number theory concern the distribution of prime numbers. The prime number theorem (PNT) counts the number of primes less than or equal to a given positive integer. This was first conjectured in the late 18th century by Gauss and Legendre, and proven independently by Hadamard and de la Vallée-Poussin in 1896.

Theorem 1.1.1 (Prime Number Theorem)**.** *Let $x \geq 2$ and $\pi(x)$ denote the number of primes $p \leq x$. Then there is a constant $c > 0$ such that*

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(xe^{-c\sqrt{\log x}}\right).$$

Riemann's approach to prime number theorem (PNT) was based upon considering what is now called Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \Re(s) > 1.$$

One can extend ζ analytically to the whole complex plane except for a simple pole at $s = 1$. The prime number theorem is a consequence of zero free region to the left of line $\Re(s) = 1$.

It was believed that any proof of the prime number theorem must use the theory of complex variables until Erdös [12] and Selberg [30] independently discovered an elementary proof in 1949.

One way to proceed to prove the prime number theorem is via the identity

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n\geq 1} \frac{\mu(n)}{n^s}, \quad \Re(s) > 1,$$

where μ is the Möbius function. It turns out that absence of zeros of $\zeta(s)$ on the line $\Re(s) = 1$ and thus the prime number theorem in the form $\pi(x) \sim \frac{x}{\log x}, \quad x \to \infty$ is easily seen to be equivalent to

$$\sum_{n\leq x} \mu(n) = o(x), \quad x \to \infty.$$

Landau proved the following in his thesis which demonstrates a concrete link between the theory of multiplicative functions and the theory of primes.

Theorem 1.1.2 (Landau)**.** *The PNT is equivalent to the statement that*

$$\sum_{n\le x}\mu(n)=o(x).$$

The Möbius function is a particular example of multiplicative function. So given a multiplicative function f such that $|f(n)|\le 1$ for all $n\in\mathbb{N}$, it is natural to ask that under which condition the mean value

$$M_f(x):=\frac{1}{x}\sum_{n\le x}f(n)$$

is large, namely $M_f(x)\gg 1$ for all x. There are some obvious examples, such as $f(n)=1$. For $f(n)=n^{it}$,

$$M_f(x):=\frac{1}{x}\sum_{n\le x}n^{it}\sim\frac{x^{it}}{1+it}.$$

Thus $\lim_{x\to\infty}|M_f(x)|$ exists but $\lim_{x\to\infty}M_f(x)$ does not. Motivated by this, we will discuss about the mean value of multiplicative functions in the following section.

1.1.2 Mean value of multiplicative functions

Definition 1.1.1. *Let $f:\mathbb{N}\to\mathbb{U}$ be a multiplicative function, and let $x>1$ be a real number. Recall that*

$$M_f(x):=\frac{1}{x}\sum_{n\le x}f(n).$$

The mean value of f is defined as $M_f:=\lim_{x\to\infty}M_f(x)$, should this limit exists.

The basic heuristic suggests that when $x \to \infty$, we have

$$M_f(x) \to M_f$$

where

$$M_f := \prod_{p\geq 2}\left(1-\frac{1}{p}\right)\left(\sum_{k\geq 0}\frac{f(p^k)}{p^k}\right).$$

Erdös and Wintner conjectured this to be true when $f : \mathbb{N} \to [-1,1]$. In 1967, Wirsing [42] settled this conjecture. As a consequence, Wirsing's theorem implies prime number theorem in a non quantitative form.

Definition 1.1.2. *An arithmetic function* $f : \mathbb{N} \to \mathbb{U}$ *is said to be close to* 1 *if*

$$\sum_p \frac{1-f(p)}{p} < \infty. \tag{1.1}$$

Following Granville and Soundararajan [14], we define the "distance" between two multiplicative functions $f, g : \mathbb{N} \to \mathbb{U}$

$$\mathbb{D}(f,g;y;x) := \left(\sum_{y<p\leq x}\frac{1-\Re(f(p)\overline{g(p)})}{p}\right)^{1/2}$$

where $\mathbb{U} = \{z \in \mathbb{C} : |z| \leq 1\}$. We would also use $\mathbb{D}(f,g;x) := \mathbb{D}(f,g;1;x)$. We remark that, $\mathbb{D}(f,f;\infty) = 0$ if and only if $|f(p)| = 1$ for all prime $p \geq 2$. The importance of this "distance" is that it satisfies the triangle inequality

$$\mathbb{D}(f,g;y;x) + \mathbb{D}(g,h;y;x) \geq \mathbb{D}(f,h;y;x)$$

for any multiplicative functions $f, g, h : \mathbb{N} \to \mathbb{U}$.

Definition 1.1.3. *Let* $f, g : \mathbb{N} \to \mathbb{U}$ *be multiplicative functions. The function* f *pretends to be* g *if* $\mathbb{D}(f,g;\infty) < \infty$.

Note that, if f is close to 1 then f pretends to be 1. In this way, Wirsing's theorem says that the mean value of real valued multiplicative function f is zero unless f "pretends" to be 1. Wirsing's theorem was generalized by Halász [16] in the following way.

Theorem 1.1.3 (Halász, 1971). *Let $f : \mathbb{N} \to \mathbb{U}$ be multiplicative. Then*

$$\sum_{n \leq x} f(n) = o(x)$$

unless there exist $t \in \mathbb{R}$ such that $\mathbb{D}(f, n^{it}, \infty) < \infty$ in which case, as $x \to \infty$ we have

$$M_f(x) = \frac{x^{it}}{1+it} \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(\sum_{k \geq 0} \frac{f(p^k) p^{-kit}}{p^k} \right) + o(1).$$

The quantitative improvements of Halász's and Wirsing's theorem have been obtained by several authors (for example [14]).

1.2 Correlation of multiplicative functions

The theme of the above discussion in Section 1.1 was the estimation of mean value of a multiplicative function f satisfies various mild constraints. A further natural question in the same vein concerns simultaneous values of f along an interval $[1, x]$. Motivated by Halász's theorem, one would like to find the asymptotic behaviour of the following so called k-point correlation function:

$$M_k(x) := \frac{1}{x} \sum_{n \leq x} g_1(F_1(x)) \ldots g_k(F_k(x)) \tag{1.2}$$

where g_j's are multiplicative functions with modulus less than or equal to 1 and $F_j(x)$'s are polynomials with integer coefficients.

The k-point correlation problem is natural and significant. Let λ be the Liouville function and h_j's are distinct non-negative integers. In particular, if $g_j = \lambda$ and $F_j(x) = x + h_j, j = 1, 2, \ldots, k \geq 2$ then we have the following famous conjecture of Chowla.

Conjecture 1.2.1 (Chowla, [4]). *For any distinct natural number $h_1, \ldots, h_k$, one has*

$$\sum_{n \leq x} \lambda(n + h_1) \ldots \lambda(n + h_k) = o(x) \quad \text{as } x \to \infty.$$

As a particular case of Chowla conjecture one expects that

$$\sum_{n \leq x} \lambda(n)\lambda(n + 1) = o(x) \quad \text{as } x \to \infty. \tag{1.3}$$

Twin prime conjecture states that there are infinitely many primes p such that $p+2$ is also prime. Hildebrand [17] writes that " It is possible that (1.3) lies as deep as the the twin prime conjecture, for it amounts to resolving, the 'parity problem' in sieve theory, which constitutes the main obstacle to proving the twin prime conjecture by sieve methods ([13], [19]) ".

In the same paper [17], Hildebrand also writes that " One would naturally expect the above conjecture (1.3), but even the much weaker relation

$$\liminf_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \lambda(n)\lambda(n + 1) < 1$$

is not known and seems to be beyond reach of the present methods ". Recently, Matomäki and Radziwill [24] settle this conjecture in a stronger form.

Theorem 1.2.1 (Matomäki and Radziwill, 2015). *For every integer $h \geq 1$, there exists $\delta(h) > 0$ such that*

$$\frac{1}{x} \left| \sum_{n \leq x} \lambda(n)\lambda(n + h) \right| \leq 1 - \delta(h)$$

for large enough $x > 1$. In fact the same results holds for any completely multiplicative function $f : \mathbb{N} \to [-1, 1]$ such that $f(n) < 0$ for some $n > 0$.

On the basis of this work of Matomäki and Radziwill [24], Tao [38] established the following logarithmically averaged version of Chowla conjecture.

Theorem 1.2.2 (Tao, 2016)**.** *Let a_1, a_2 be natural numbers, and let b_1, b_2 be integers such that $a_1b_2 - a_2b_1 \neq 0$. Let $1 \leq w(x) \leq x$ be a quantity depending on x that goes to infinity as $x \to \infty$. Then one has*

$$\sum_{x/w(x)<n\leq x} \frac{\lambda(a_1n + b_1)\lambda(a_2n + b_2)}{n} = o(\log w(x))$$

as $n \to \infty$.

Chowlas conjecture remains open for any $h_1, \ldots, h_k$ with $k \geq 2$, although there are a number of partial results available. See [24], [26], [25], [37], [38] for some recent results in this direction.

In [20], Kátai studied the asymptotic behaviour of the sum (1.2) when $F_j(x)$'s are special polynomials and some assumptions on g_j's but did not provide any error term. In [36], Stepanauskas studied the asymptotic formula for sum (1.2) with explicit error term when $F_j(x)$'s are linear polynomials and g_j's are close to 1 (see (1.1)). Recently, Klurman [21] studied the 2-point correlation function. In the following section we study 3-point correlation function.

1.2.1 Triple correlation of multiplicative functions

From now onwards, let $F(n); F_1(n), F_2(n), F_3(n)$ be positive integer-valued polynomials with integer coefficients and these are not divisible by the square of any irreducible polynomial. Also suppose that $F_j(n), F_k(n)$ are relatively prime for $j \neq k$ and for all n. Let v and v_j denote the degree of the polynomials $F(n)$ and $F_j(n)$

respectively.

Let $\varrho(d_1, d_2, d_3)$ be the number of solutions of the congruence system

$$F_j(n) \equiv 0 \pmod{d_j} \quad j = 1, 2, 3.$$

Let $\varrho(d)$ and $\varrho_j(d)$ denote the number of solutions of the congruences

$$F(n) \equiv 0 \pmod{d} \quad \text{and} \quad F_j(n) \equiv 0 \pmod{d}$$

respectively.

Suppose $g_j : \mathbb{N} \to \mathbb{U}$ and $h_j : \mathbb{N} \to \mathbb{C}$ be multiplication functions such that $h_j = \mu * g_j$, $j = 1, 2, 3$. For $x \geq r \geq 2$, We also define

$$P(x) := \prod_{p \leq x} w_p \quad \text{and} \quad P(r, x) := \prod_{r < p \leq x} w_p \tag{1.4}$$

where

$$w_p := \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \frac{h_1(p^{m_1}) h_2(p^{m_2}) h_3(p^{m_3})}{[p^{m_1}, p^{m_2}, p^{m_3}]} \varrho(p^{m_1}, p^{m_2}, p^{m_3})$$

In [6], we find an asymptotic formula for $M_3(x)$ with explicit error term which is stated as follows.

Theorem 1.2.3. *Let $F_j(x), j = 1, 2, 3$ be polynomials as above of degree $v_j \geq 2$. Let g_1, g_2 and g_3 be multiplicative functions close to 1 and whose modulus does not exceed 1. Then there exists a positive absolute constant c and a natural number γ depending on polynomials $F_1(x), F_2(x)$ and $F_3(x)$ such that for all $x \geq r \geq \gamma$ and for all $1 - \frac{1}{v_1+v_2+v_3} < \alpha < 1$, we have*

$$\begin{aligned} x^{-1} M_3(x) - P(x) \ll \frac{1}{x} \left(F_1(x) F_2(x) F_3(x)\right)^{1-\alpha} \exp\left(\frac{c r^{\alpha}}{\log r}\right) + (r \log r)^{-\frac{1}{2}} \\ + \sum_{j=1}^{3} (\mathbb{D}(g_j, 1; r; x) + \mathbb{D}(g_j, 1; x; F_j(x))) + \frac{1}{x} C(r, x) + \frac{1}{\log x} \end{aligned}$$

where $P(x)$ is defined by (1.4) and

$$C(r,x)=\sum_{j=1}^{3}\sum_{m=1}^{v_j-1}\sum_{\substack{p^m\leq F_j(x)\\ p>r}}|g_j(p^m)-1|\varrho_j(p^m).$$

Remark 1.2.1. *Theorem 1.2.3 can be extended for $M_k(x), k\geq 4$. We have replaced the notations $S_1(r,x)$ and $T(x)$ in the article [6] by "distance" functions.*

Remark 1.2.2. *For any $\gamma\geq 2$, let D_γ denote the set of those tuples $\{d_1,d_2,d_3\}$ of natural numbers for which all the prime factors of d_i do not exceed γ. Since the congruence system*

$$F_1(n)\equiv 0 \pmod a, F_2(n)\equiv 0 \pmod a, F_3(n)\equiv 0 \pmod a$$

have common solution for finitely many values of a (See [39], Lemma 2.1) then we can choose γ so that $\varrho(d_1,d_2,d_3)=0$ if $\{d_1,d_2,d_3\}\notin D_\gamma$ and $\left(\prod_{p>\gamma}p, \prod_{i\neq j}(d_i,d_j)\right)>1$. Therefore we have

$$P(x)=P_1(\gamma)P_2(\gamma,x)$$

where

$$(1.5)\qquad P_1(\gamma)=\prod_{p\leq\gamma}w_p \quad \text{and} \quad P_2(\gamma,x)=\prod_{\gamma<p\leq x}\left(1+\sum_{j=1}^{3}\sum_{m=1}^{\infty}\frac{h_j(p^m)}{p^m}\varrho_j(p^m)\right).$$

For example, consider the linear polynomials $F_1(x)=x, F_2(x)=x+2, F_3(x)=x+4$. We see that the above congruence system have a common solution $\pmod 2$ only. In this case $\gamma=2$. We observe that γ should depend on the polynomials.

Remark 1.2.3. *The Theorem 1.2.3 is true for all $x\geq r\geq\gamma$ but to get a good error term we will chose $r=(\log x)^{\frac{1}{\alpha}}$, where α is defined as in Theorem 1.2.3.*

Corollary 1.2.1 is a polynomial version with the degree of the polynomial greater than or equal to 2, of a theorem of Kátai ([20], Theorem 5).

Corollary 1.2.1. *Let $F_j(n)$ and $g_j (j = 1, 2, 3)$ be as in Theorem 1.2.3.*

$$\sum_p \frac{(g_j(p) - 1)\varrho_j(p)}{p} < \infty.$$

Suppose that

$$(g_j(p^\alpha) - 1)\, \varrho_j(p^\alpha) \to 0, \qquad as\ p \to \infty$$

for $\alpha = 1$, when $v_j \geq 2$ and for $\alpha = 1, 2, \cdots, v_j - 2$, when $v_j \geq 3$, then there exist a natural number γ depending on polynomials F_1, F_2 and F_3 such that we have

$$M_x(g_1, g_2, g_3) \to P_1(\gamma)P_2(\gamma), \qquad as\ x \to \infty$$

where $P_1(\gamma)$ is defined by (1.5) *and*

$$(1.6) \qquad P_2(\gamma) := \prod_{p>\gamma} \left(1 + \sum_{m=1}^{\infty} \sum_{j=1}^{3} \frac{h_j(p^m)\varrho_j(p^m)}{p^m}\right), \quad h_j = \mu * g_j.$$

As an application of the Theorem 1.2.3, we get the following corollary.

Corollary 1.2.2. *Let $\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$, be Euler's totient function and $\sigma(n) = \sum_{d|n} d$. Let $F_1(x) = x^2 + a_1, F_2(x) = x^2 + a_2, F_3(x) = x^2 + a_3, 0 < t < 1$, where a_1, a_2, a_3 are taken such that $F_j(x), j = 1, 2, 3$ satisfies the assumption of Theorem 1.2.3. Then there exist a natural number γ depending on a_1, a_2 and a_3 such that for all $x \geq \gamma$,*

$$\frac{1}{x} \sum_{n \leq x} \frac{\phi(n^2 + a_1)\phi(n^2 + a_2)\phi(n^2 + a_3)}{\sigma(n^2 + a_1)\sigma(n^2 + a_2)\sigma(n^2 + a_3)} = P_1'(\gamma) \prod_{p>\gamma} w_p' + O\left(\frac{1}{(\log x)^t}\right),$$

$$\frac{1}{x} \sum_{n \leq x} \frac{\phi(n^2 + a_1)\phi(n^2 + a_2)\phi(n^2 + a_3)}{(n^2 + a_1)(n^2 + a_2)(n^2 + a_3)} = P_1''(\gamma) \prod_{p>\gamma} w_p'' + O\left(\frac{1}{(\log x)^t}\right),$$

where

$$w'_p = \left(1 - \sum_{j=1}^{3} \frac{\left(\frac{a_j}{p}\right) \varrho_j(p)}{p^2} + \left(1 - \frac{1}{p}\right)^2 \sum_{j=1}^{3} \sum_{m=1}^{\infty} \frac{\left(\frac{a_j}{p}\right) \varrho_j(p)}{1 + p + \cdots + p^m}\right),$$

$$w''_p = \left(1 - \sum_{j=1}^{3} \frac{\left(\frac{a_j}{p}\right) \varrho_j(p)}{p^2}\right), \quad \left(\frac{a_j}{p}\right) \text{ is the Legendre symbol of } a_j \text{ and } p,$$

$P'_1(\gamma)$ *and* $P''_2(\gamma)$ *are defined by* (1.5) *in which* $g_j(n), j = 1, 2, 3$ *are replaced by* $\phi(n)/\sigma(n)$ *and* $\phi(n)/n$ *respectively.*

In [6], we also studied the mean value of the following triple correlation functions with various assumption on g_j's:

$$M_x(g_1, g_2, g_3) := \frac{1}{x} \sum_{n \leq x} g_1(n+2) g_2(n+1) g_3(n). \tag{1.7}$$

Definition 1.2.1. *A multiplicative function* g *is called good function if there exists* $\kappa \in \mathbb{C}$ *such that for each* $u > 0$

$$\sum_{p \leq x} |g(p) - \kappa| \ll \frac{x}{(\log x)^u}.$$

We also define for $\tau \in \mathbb{R}$ and $r \geq 1$

$$\theta_\tau(n) = \prod_{p|n} \left(1 + \sum_{m=1}^{\infty} \frac{g_3(p^m)}{p^{m(1+i\tau)}}\right)^{-1}, \quad M_x(g_3) := \frac{1}{x} \sum_{n \leq x} g_3(n)$$

$$Q_\tau(r) = \prod_{p \leq r} \left(1 - \frac{2\theta_\tau(p)}{p-1} + \theta_\tau(p) \sum_{m=1}^{\infty} \frac{g_1(p^m) + g_2(p^m)}{p^m}\right),$$

$$P_3(r, x) = \prod_{r < p \leq x} \left(1 - \frac{2}{p} + \left(1 - \frac{1}{p}\right) \sum_{m=1}^{\infty} \frac{g_1(p^m) + g_2(p^m)}{p^m}\right).$$

The following theorem gives an asymptotic formula with explicit error term of the sum (1.7) when g_1, g_2 are close to 1 (see definition 1.1) and g_3 is a good function (see definition 1.2.1).

Theorem 1.2.4. *Let g_1, g_2 and g_3 be multiplicative functions whose modulus does not exceed 1 and g_3 be a good function. Assume further that there exist a positive constant c_1 such that*

$$\left|1 + \sum_{k=1}^{\infty} \frac{g_3(2^k)}{2^{k(1+i\xi)}}\right| \geq c_1$$

for $\xi = 0$, if g_3 is real valued and for all $\xi \in \mathbb{R}$, if g_3 is not real valued. Then there exist positive absolute constants c, c' and a real $\tau, |\tau| \leq (\log x)^{1/19}$, such that for all $x \geq r \geq 2$ and for all $\frac{1}{2} < \alpha < \frac{5}{9}$, we have

$$M_x(g_1, g_2, g_3) - M_x(g_3)P_3(r, x)Q_\tau(r) \ll x^{1-2\alpha} \exp\left(c\frac{r^\alpha}{\log r}\right) + \frac{(\log r)^c}{(\log x)^{c'}}$$
$$+ \frac{\exp\left(c(\log\log r)^2\right)}{(\log x)^{1/19}} + \sum_{j=1}^{2} \mathbb{D}(g_j, 1; r; x - 4 + j) + (r \log r)^{-\frac{1}{2}}.$$

For real-valued g_3, we may set $\tau = 0$ in the expression of $Q_\tau(r)$.

The following theorem tells us that if g_3 is Möbius function then under certain hypothesis on g_1, g_2, the mean value of triple correlation function (1.7) is zero.

Theorem 1.2.5. *Let g_1 and g_2 be multiplicative functions which does not exceed 1 and*

$$\sum_p \sum_{j=1}^{2} \frac{|g_j(p) - 1|^2}{p} < \infty.$$

Then as $x \to \infty$,

$$M_x(g_1, g_2, \mu) = \frac{1}{x} \sum_{n \leq x} g_1(n+2)g_2(n+1)\mu(n) = o(1).$$

Assumption 1.2.6. *For every given $A > 0$,*

$$\sum_{n \leq x} \mu(n+1)\mu(n) \exp(2\pi i n\alpha) = O\left(\frac{x}{(\log x)^A}\right)$$

holds uniformly for all real α and implied constant depends only on A.

Theorem 1.2.7. *Let g_1 be a multiplicative function such that $|g_1(n)| \leq 1$ for all n and*

$$\sum_{p} \frac{|g_1(p) - 1|^2}{p} < \infty.$$

Suppose that Assumption 1.2.6 holds. Then as $x \to \infty$,

$$M_x(g_1, \mu, \mu) = \frac{1}{x} \sum_{n \leq x} g_1(n+2)\mu(n+1)\mu(n) = o(1).$$

The following corollary is a direct application of the Theorem 1.2.5.

Corollary 1.2.3. *If ϕ, μ and σ are as above then as $x \to \infty$, we have*

$$\sum_{n \leq x} \frac{\phi(n+2)}{(n+2)} \frac{\phi(n+1)}{(n+1)} \mu(n) = o(x),$$
$$\sum_{n \leq x} \frac{\phi(n+2)}{\sigma(n+2)} \frac{\phi(n+1)}{\sigma(n+1)} \mu(n) = o(x).$$

1.3 Probabilistic number theory

Let $\Omega_N := \{n : 1 \leq n \leq N\}$ be equipped with the probability measure ν_N on Ω_N obtained by assigning the uniform probability $1/N$ to each element. An arithmetic function may be viewed as a sequence of random variable

$$f_N = (f, \nu_N) \quad (N = 1, 2, \ldots)$$

taking the values $f(n), 1 \leq n \leq N$, with probability $1/N$.

From classical probability theory we recall that a distribution function is a non-decreasing function $F : \mathbb{R} \to [0, 1]$, which is right-continuous and satisfies $F(-\infty) =$

$0, F(+\infty) = 1$. Therefore F has only countably many jump discontinuities.

The set $\mathcal{D}(F)$ of points of discontinuities of F is thus at most countable and only contains jump discontinuities. We denote by $\mathcal{C}(F)$ the complement of $\mathcal{D}(F)$.

A sequence $\{F_n\}_{n=1}^{\infty}$ of distribution functions is said to converge weakly to a function F if we have

$$\lim_{n\to\infty} F_n(z) = F(z) \quad (z \in \mathcal{C}(F)).$$

1.3.1 Limiting distributions of arithmetic functions

Let us consider a real valued arithmetic function f. For each $N \geq 1$ the function

$$F_N(z) := \nu_N\{n : f(n) \leq z\} = \frac{1}{N}|\{n \leq N : f(n) \leq z\}| \tag{1.8}$$

is a distribution function.

Definition 1.3.1. *A real arithmetical function f is said to possess a (limiting) distribution function F (or is said to have a limit law with distribution function F) if the sequences F_N defined by* (1.8) *converges weakly to a distribution function F.*

In the probabilistic study of an arithmetic function, a natural normalization is obtained by introducing the expectation and variance of f relative to ν_N, namely

$$\mathbb{E}_N(f) := \int_{-\infty}^{+\infty} z dF_N(z) = \frac{1}{N}\sum_{n\leq N} f(n),$$

and

$$\begin{aligned}\mathbb{V}_N(f) = \mathbb{D}_N(f)^2 &:= \int_{-\infty}^{+\infty} \{z - \mathbb{E}_N(f)\}^2 dF_N(z) \\ &= \frac{1}{N}\sum_{n\leq N}\{f(n) - \mathbb{E}_N(f)\}^2.\end{aligned}$$

This suggests a different approach to the problem of the distribution of values of an arithmetic function. Instead of studying the asymptotic behaviour of $F_N(z)$, we consider that of

$$G_N(z) := \nu_N\{n : f(n) \leq \mathbb{E}_N(f) + z\mathbb{D}_N(f)\}.$$

This perspective may be considered as central in the probabilistic theory of numbers. The sequence of distribution functions $F_N(z)$ or $G_N(z)$ contains all the information concerning the arithmetic function f.

1.3.2 Distribution of the sum of additive functions

We will start with the following definitions of additive function and characteristic function of a distribution function.

Definition 1.3.2. *A function $f : \mathbb{N} \to \mathbb{C}$ is said to be additive if $f(mn) = f(m) + f(n)$ whenever $(m, n) = 1$.*

Definition 1.3.3. *The characteristic function of a distribution function F is the Fourier transform of the Stieltjes measure $dF(z)$, defined as*

$$\phi(\tau) := \int_{-\infty}^{+\infty} e^{i\tau z} dF(z) \quad (\tau \in \mathbb{R}).$$

It is a uniformly continuous function on the real line, satisfying

$$|\phi(\tau)| \leq 1 = \phi(0) \quad (\tau \in \mathbb{R}).$$

In this section, we will discuss the behaviour of the distribution of the sum

$$f_1(F_1(n)) + f_2(F_2(n)) + f_3(F_3(n)), \tag{1.9}$$

where f_1, f_2 and f_3 are real-valued additive functions and $F_1(x), F_2(x), F_3(x)$ are polynomials with integer coefficients.

The following theorem gives the behaviour of the distribution of the sum (1.9) when F_j's are polynomial of degree greater than or equal to 2. This is an application of the Theorem 1.2.3.

Theorem 1.3.1. *Let $t, z \in \mathbb{R}$. Suppose that f_1, f_2 and f_3 be real-valued additive functions and $F_j(n)$ be as in Theroem 1.2.3 of degree $v_j \geq 2$ for all $j = 1, 2, 3$. Assume that*

$$\sum_{|f_j(p)|\leq 1} \frac{f_j^2(p)}{p} \varrho_j(p) < \infty, j = 1, 2, 3,$$
$$\sum_{|f_j(p)|>1} \frac{\varrho_j(p)}{p} < \infty, j = 1, 2, 3,$$
$$\sum_{j=1}^{3} \sum_{|f_j(p)|\leq 1} \frac{f_j(p)\varrho_j(p)}{p} < \infty,$$
$$f_j(p^m)\varrho_j(p^m) \to 0,$$

for $m = 1$, when $v_j \geq 2$ and for $m = 1, 2, \cdots, v_j - 2$, when $v_j \geq 3$. Then the distribution function

$$G_x(z) := \frac{1}{x} \# \{n | n \leq x, f_1(F_1(n)) + f_2(F_2(n)) + f_3(F_3(n)) \leq z\}$$

converges weakly to a limit distribution as $x \to \infty$, and there exist a natural number γ depending on polynomials F_1, F_2 and F_3 such that the characteristic function say $\phi(t)$ of this limit distribution is equal to $P_1(\gamma)P_2(\gamma)$, where $P_1(\gamma)$ and $P_2(\gamma)$ are defined by (1.5) and (1.6) respectively with g_j is replaced by $\exp(itf_j), j = 1, 2, 3$.

1.4 Multiplicative number theory over $\mathbb{F}_q[x]$

We begin this section with some preliminaries on Number Theory over Function Fields. We will use [28] as a general reference.

1.4.1 Polynomials over finite field

Let $\mathbb{F}_q$ denote a finite field with q elements. We will denote by $A = \mathbb{F}_q[x]$ the polynomial ring over $\mathbb{F}_q$. For a detailed discussion of the similarities between A and $\mathbb{Z}$, see Rosen's book ([28], Chapter 1). Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in \mathbb{F}_q[x].$$

Definition 1.4.1. *If $a_n \neq 0$ we say f has degree n, i.e., $\deg f = n$ and in this case we define the sign of f to be $a_n \in \mathbb{F}_q^*$, $(\mathrm{sgn}(f) = a_n)$. We have that $\mathrm{sgn}(0) = 0$ and $\deg(0) = -\infty$.*

We now present some properties of degree and sign.

Proposition 1.4.1. *Let, $f, g \in A$ be non-zero polynomials. Then*

(i) $\deg(fg) = \deg(f) + \deg(g)$,

(ii) $\mathrm{sgn}(fg) = \mathrm{sgn}(f)\,\mathrm{sgn}(g)$,

(iii) $\deg(f + g) \leq \max(\deg(f), \deg(g))$ *and equality holds if* $\deg(f) \neq \deg(g)$.

(iv) A/fA is a finite ring with $q^{\deg(f)}$ elements.

A polynomial $f \in A$ is called monic if $\mathrm{sgn}(f) = 1$. Let $\mathcal{M}_{n,q}$ be the set of all monic polynomials of degree n over $\mathbb{F}_q$. Let $\mathcal{M}_{\leq n,q}$ be the set of all monic polynomials of degree $\leq n$ over $\mathbb{F}_q$. Also let $\mathcal{M}_q = \bigcup_{n=1}^{\infty} \mathcal{M}_{n,q}$. A polynomial $f \in A$ is reducible if we can write $f(x) = g(x)h(x)$ with $\deg(f) > 0$ and $\deg(g) > 0$, otherwise it is called irreducible. A monic irreducible polynomial is called a prime polynomial. Let $\mathcal{P}_{n,q}$ denote the set of all prime polynomials of degree n over $\mathbb{F}_q$. Also let $\mathcal{P}_q = \bigcup_{n=1}^{\infty} \mathcal{P}_{n,q}$. Now we have the following important definition associated with $f \in A$.

Definition 1.4.2. *For $f \in A$, we define its norm by*

$$|f| = \begin{cases} q^{\deg(f)} & \text{if } f \neq 0 \\ 0 & \text{if } f = 0. \end{cases}$$

The letters P and Q will be used for a prime polynomial in A. The next proposition is the Chinese remainder Theorem for $\mathbb{F}_q[x]$.

Proposition 1.4.2 ([28])**.** *Let $m_1, \ldots, m_t$ be elements of A which are pairwise coprime. Let $m = m_1 \ldots m_t$ and Ψ_i be the natural homomorphism from A/mA to A/m_iA. Then the map $\Psi : A/mA \to A/m_1A \oplus \ldots \oplus A/m_tA$ is given by*

$$\Psi(a) = (\Psi_1(a), \ldots, \Psi_t(a))$$

is a ring isomorphism.

We can also define the analogue of the Möbius function and Euler totient function for $\mathbb{F}_q[x]$ as follows

$$\mu(f) = \begin{cases} (-1)^t & \text{if } f = \alpha P_1 \ldots P_t, \quad \alpha \in \mathbb{F}_q^*, \quad P_i \neq P_j, \quad \forall i \neq j \\ 0 & \text{otherwise }, \end{cases}$$

where each P_j is a distinct monic irreducible polynomial and

$$\Phi(f) = \sum_{\substack{g \text{ monic} \\ \deg g < \deg f \\ (f,g)=1}} 1 = |\,(A/fA)^*\,|. \tag{1.10}$$

Definition 1.4.3. *A function Ψ from $\mathcal{M}_q$ to $\mathbb{C}$ is called an arithmetic function. An arithmetic function Ψ is called even if*

$$\Psi(cf) = \Psi(f), \quad \forall f \in \mathcal{M}_q \text{ and } \forall c \in \mathbb{F}_q^*.$$

and an arithmetic function Ψ is called multiplicative if

$$\Psi(fg) = \Psi(f)\Psi(g), \qquad \text{whenever } f \text{ and } g \text{ are coprime.}$$

Definition 1.4.4. *The zeta function of $A = \mathbb{F}_q[x]$, denoted by $\zeta_A(s)$, is defined by the infinite series*

$$\zeta_A(s) := \sum_{f \in \mathcal{M}_q} \frac{1}{|f|^s} = \prod_{P \in \mathcal{P}_q} (1 - |P|^{-s})^{-1}, \quad \Re(s) > 1.$$

And it is easy to show that

$$\zeta_A(s) = \frac{1}{1 - q^{1-s}}.$$

1.4.2 Prime Number Theorem in $\mathbb{F}_q[x]$

Let x be a real number and $\pi(x)$ be the number of positive prime numbers less than or equal to x. The classical prime number theorem states that $\pi(x)$ is asymptotic to $x/\log(x)$. We now present the analogue of this theorem for polynomials over finite fields.

Theorem 1.4.1 (Prime Polynomial Theorem)**.** *Let $\pi_A(n)$ denote the number of monic irreducible polynomials in A of degree n. Then we have*

$$\pi_A(n) = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right). \tag{1.11}$$

Remark 1.4.1. *If we denote $x = q^n$, then we have*

$$\pi_A(n) = \frac{x}{\log_q x} + O\left(\frac{\sqrt{x}}{\log_q x}\right), \tag{1.12}$$

which analogous to the conjectured form of classical prime number theorem.

1.4.3 Mean value of multiplicative functions over $\mathbb{F}_q[x]$

One of the fruitful analogies in number theory is the one between the integers $\mathbb{Z}$ and the polynomial ring $\mathbb{F}_q[x]$. Thus, for instance prime numbers correspond to the monic irreducible polynomials over $\mathbb{F}_q[x]$ and the fundamental theorem of arithmetic applies. In the recent paper [15], Granville, Harper, and Soundararajan initiated the study of mean values of multiplicative functions over the function field $\mathbb{F}_q[x]$ by proving a quantitative analog of the celebrated theorem of Halasz. We begin by introducing the objects of study, borrowing the notations from [15].

Let $\mathbb{U}$ denote the unit disc. For a multiplicative function $\psi : \mathcal{M}_q \to \mathbb{U}$, we write

$$\sigma(n;\psi) := \frac{1}{q^n} \sum_{f \in \mathcal{M}_{n,q}} \psi(f).$$

The mean value of ψ is defined as $\sigma_\psi := \lim_{n\to\infty} \sigma(n;\psi)$, should this limit exist. As pointed out in [15], a direct analog of Wirsings theorem is however false in the function field setting. Indeed, consider the function $\psi(f) = (-1)^{\deg(f)}$ for which $\sigma(n;\psi) = (-1)^n$ clearly oscillates and hence σ_ψ does not exist. In [22], Klurman proved the following Wirsing's theorem over function field.

Theorem 1.4.2. *Let $\psi : \mathcal{M}_q \to \mathbb{U}$ be a multiplicative function. Then either $\psi(f)$ or $(-1)^{\deg(f)}\psi(f)$ has a mean value.*

Let $e(\alpha) = \exp(2\pi i\alpha)$. A natural function field analog of the function $h_t(n) = n^{it}, t \in \mathbb{R}$ is the function $h_\theta(M) = \exp(\theta \deg(M)), M \in \mathcal{M}_q$ and $\theta \in [0, 1)$.

Definition 1.4.5. *A multiplicative function $\psi : \mathcal{M}_q \to \mathbb{U}$ is said to be close to* 1 *if*

$$\sum_{P \in \mathcal{P}_q} \frac{1 - \psi(P)}{q^{\deg P}} < \infty.$$

Following Klurman [22], we define the "distance" between two multiplicative function $\psi_1, \psi_2 : \mathcal{M}_q \to \mathbb{U}$ by

$$\mathbb{D}^2(\psi_1, \psi_2; m, n) = \sum_{\substack{m \le \deg P \le n \\ P \in \mathcal{P}_q}} \frac{1 - \Re(\psi_1(P)\overline{\psi_2(P)})}{q^{\deg P}}.$$

We also write $\mathbb{D}(\psi_1, \psi_2; n) := \mathbb{D}(\psi_1, \psi_2; 1, n)$. Usually, the distance $\mathbb{D}(\psi_1, \psi_2; \infty)$ is infinite. However, in the case $\mathbb{D}(\psi_1, \psi_2; \infty) < \infty$ we say that ψ_1 "pretends" to be ψ_2. Also observe that, if ψ is close to 1 then ψ pretends to be 1. For any multiplicative function $\psi : \mathcal{M}_q \to \mathbb{U}$ we define

$$\mathcal{P}(\psi, n) := \prod_{\substack{\deg(P) \le n \\ P \in \mathcal{P}_q}} \left(1 - \frac{1}{q^{\deg P}}\right)\left(\sum_{k \ge 0} \frac{\psi(P^k)}{q^{k \deg P}}\right).$$

In [22], Klurman established the following explicit version of Halász's theorem.

Theorem 1.4.3. *For a multiplicative function $\psi : \mathcal{M}_q \to \mathbb{U}$ one of the following holds:*

- *If $\mathbb{D}(\psi(P), e(\theta \deg(P)); \infty) = \infty$ for all $\theta \in [0, 1)$, then*

$$\lim_{n \to \infty} \frac{1}{q^n} \sum_{f \in \mathcal{M}_{n,q}} \psi(f) = 0.$$

- *If* $\mathbb{D}(\psi(P), e(\theta_0 \deg(P); \infty)) < \infty$ *for some* $\theta_0 \in [0, 1)$, *then for any given* $\epsilon > 0$

$$\frac{1}{q^n} \sum_{f \in \mathcal{M}_{n,q}} \psi(f) = e(n\theta_0)\mathcal{P}(\psi(P)e(-\theta_0 \deg(P)), n) + O_\epsilon\left(\mathbb{D}(\psi(P), e(-\theta_0 \deg P); m; n) + \frac{1}{n^{1-\epsilon}}\right)$$

where $m = \lceil (1-\epsilon)\frac{\log n}{\log q} \rceil$.

1.5 Selberg sieve over $\mathbb{F}_q[x]$

In 1947 Selberg [29] introduced a new approach to sieving which is based on global optimization. In this chapter, we will discuss Selberg sieve for polynomial ring over finite fields and an application of it which will be used in Chapter 5.

The Selberg sieve is a technique for estimating the size of "sifted sets" of positive integers which satisfy a set of conditions expressed by congruences. In this section, we will extend Selberg sieve ([9], Lemma 2.1) to polynomials over finite fields and give an application of it which also appears in [8]. The main theorem of this section is as follows.

Theorem 1.5.1. *Let us consider the following set of polynomials*

$$\mathcal{A} = \{a_M \in \mathcal{M}_q : M \in \mathcal{M}_{n,q}\}.$$

Also let r *and* z *be positive integers such that*

$$\tilde{Q} = \prod_{\deg P \leq r} P \quad \textit{and} \quad \mathcal{D} = \{D : D | \tilde{Q}, \deg(D) \leq z\}.$$

Let Ψ *be a real-valued non-negative arithmetic function on* $\mathcal{M}_q$. *Suppose that there exist a multiplicative function* η *supported on square-free polynomials with irreducible* *factors of degree atmost r such that*

$$0 \leq \eta(P) < 1, \quad P \in \mathcal{P}_q$$

and for $D \in \mathcal{D}$,

$$\sum_{\substack{M \in \mathcal{M}_{n,q} \\ a_M \equiv 0(D)}} \Psi(M) = \eta(D)X + R_D(n),$$

where $X, R_D(n)$ are real numbers, $X \geq 0$. Now consider the following sum

$$S(n, \tilde{Q}) = \sum_{\substack{M \in \mathcal{M}_{n,q} \\ a_M \in \mathcal{A} \\ (a_M, \tilde{Q}) = 1}} \Psi(M).$$

If $h(D) = \frac{1}{\eta(D)} \prod_{P|D} (1 - \eta(P))$, for $D \in \mathcal{D}$, then we have

$$S(n, \tilde{Q}) \leq X.L^{-1} + \sum_{\substack{D|\tilde{Q} \\ \deg(D) \leq 2z}} 3^{\omega(D)} |R_D(n)|,$$

where $L = \sum_{M \in \mathcal{D}} \frac{1}{h(M)}$.

In [8], we prove the following application of the Theorem 1.5.1, which is useful to prove a variant of Turán-Kubilius inequality over function field.

Theorem 1.5.2 ([8], Lemma 6)**.** *Given two coprime polynomials $B, M \in \mathbb{F}_q[x]$, let $\pi_A(n; M, B)$ denotes the number of primes $P \in \mathcal{P}_{n,q}$ which satisfies $P \equiv B \pmod{M}$. Then for any $n > \deg B$ we have*

$$\Theta(n) := \sum_{\substack{\frac{n}{2} < \deg Q \leq n \\ Q \in \mathcal{P}_q}} \Phi(Q) \pi_A^2(n; Q, B) \ll |\mathcal{P}_{n,q}|^2$$

where Φ is defined by (1.10).

1.6 Correlation of multiplicative functions over $\mathbb{F}_q[x]$

Our goal here is to study asymptotic behaviour of the correlations of arithmetic functions $\psi_1, \dots, \psi_k$ on $\mathcal{M}_q$ at $(h_1, \dots, h_k) \in \mathbb{F}_q[x]^k$. Let

$$S_k(n,q) := \sum_{f \in \mathcal{M}_{n,q}} \psi_1(f+h_1)\dots\psi_k(f+h_k) \tag{1.13}$$

and

$$R_k(n,q) := \sum_{P \in \mathcal{P}_{n,q}} \psi_1(P+h_1)\dots\psi_k(P+h_k) \tag{1.14}$$

This parameter can be large, in particular, either when n is much larger than q, which we call the large degree limit, or when q is much larger than n, which we call the large finite field limit.

In the large degree limit, one knows no more than what is known in number fields assuming the Generalized Riemann Hypothesis (which is a theorem in function fields). In the large finite field limit one can often go much further than what can be done in the number field setting or in the large degree limit. An extensive study by several authors ([1], [3], [5]) has led to a complete understanding of (1.13) in this limit for the family of arithmetic functions depending on cycle structure.

1.6.1 Correlation of multiplicative functions in the large finite field limit

In [5], Carmon and Rudnick prove the following function field analog of Chowlas conjecture on the correlation of the Möbius function in the large finite field limit.

To formulate it, we recall that the Möbius function of a non-zero polynomial $\mathbb{F}_q[x]$ is defined by $\mu(f) = (-1)^r$ if $f = cP_1 \dots P_r$ with $0 \neq c \in \mathbb{F}_q$ and $P_1, \dots, P_r$ are distinct monic irreducible polynomials, and $\mu(f) = 0$ otherwise.

Theorem 1.6.1. *Fix $r > 1$ and assume that $n > 1$ and q is odd prime power. Then for any choice of distinct polynomials $\alpha_1, \dots, \alpha_r \in \mathbb{F}_q[x]$, with $\max \deg \alpha_j < n$, and $\epsilon_i \in \{1, 2\}$, not all even,*

$$\sum_{f \in \mathcal{M}_{n,q}} \mu(f + \alpha_1)^{\epsilon_1} \dots \mu(f + \alpha_r)^{\epsilon_r} \ll_{r,n} q^{n - \frac{1}{2}}.$$

Thus for fixed $r, n > 1$,

$$\lim_{q \to \infty} \frac{1}{|\mathcal{M}_{n,q}|} \sum_{f \in \mathcal{M}_{n,q}} \mu(f + \alpha_1)^{\epsilon_1} \dots \mu(f + \alpha_r)^{\epsilon_r} = 0.$$

It has been conjectured that there are infinitely many twin primes, and a more general quantitative form, due to Hardy and Littlehood, asserts that given distinct integers $a_1, \dots, a_r$, the number $\pi(x; a_1, \dots, a_r)$ of integers $n \leq x$ for which $n + a_1, \dots, n + a_r$ are simultaneously prime is asymptotically

$$\pi(x; a_1, \dots, a_r) \sim C_{a_1, \dots, a_r} \frac{x}{(\log x)^r}, \quad x \to \infty,$$

for a certain constant $C_{a_1, \dots, q_r}$, which is positive whenever there are no local congruence obstructions.

Bary-Soroker [32] proved that for given n, r, any sequence of finite fields $\mathbb{F}_q$ of odd cardinality q, and distinct polynomials $h_1, \dots, h_r \in \mathbb{F}_q[x]$ of degree less than n, the number $\pi_q(n; h_1, \dots, h_r)$ of monic polynomials $f \in \mathcal{M}_{n,q}$ such that $f + h_1, \dots, f + h_r$ are simultaneously irreducible satisfies

$$\pi_q(n; h_1, \dots, h_r) \sim \frac{q^n}{n^r}, \quad q \to \infty.$$

1.6.2 Correlation of multiplicative functions in the large degree limit

Let $\psi_j : \mathcal{M}_q \to \mathbb{U}$ and $\alpha_j : \mathcal{M}_q \to \mathbb{C}$ be multiplicative functions such that $\alpha_j = \mu * \psi_j$ for all $j = 1, 2$. For fixed polynomials $h_j \in \mathbb{F}_q[x]$ with $\deg(h_j) < n$ for all $j = 1, 2$ and $n \geq r$, we define

$$Q(n) := \prod_{\deg P \leq n} \upsilon_P \quad \text{and} \quad Q(r,n) = \prod_{r<\deg P \leq n} \upsilon_P, \tag{1.15}$$

$$Q^{'}(n) := \prod_{\deg P \leq n} \upsilon'_P \quad \text{and} \quad Q'(r,n) = \prod_{r<\deg P \leq n} \upsilon'_P \tag{1.16}$$

where

$$\upsilon_P := \sum_{\substack{m_1=0 \\ (P^{m_1}, P^{m_2})|(h_2-h_1)}}^{\infty} \sum_{m_2=0}^{\infty} \frac{\alpha_1(P^{m_1})\alpha_2(P^{m_2})}{q^{\deg([p^{m_1}, P^{m_2}])}}, \quad \upsilon'_P := \sum_{\substack{m_1=0 \\ (P^{m_1}, P^{m_2})|(h_2-h_1)}}^{\infty} \sum_{m_2=0}^{\infty} \frac{\alpha_1(P^{m_1})\alpha_2(P^{m_2})}{\Phi[P^{m_1}, P^{m_2}]}.$$

In [8], we investigate the asymptotic behaviour of the above sums (1.13) and (1.14) for $k = 2$, i.e. $S_2(n,q)$ and $R_2(n,q)$ in large degree limit. The following theorem gives the asymptotic behaviour of $S_2(n,q)$ with explicit error term in large degree limit.

Theorem 1.6.2. *Let ψ_1 and ψ_2 be multiplicative functions on $\mathcal{M}_q$ with modulus less than or equal to 1. Suppose that ψ_1 and ψ_2 are close to 1 and $\gamma := \deg(h_2 - h_1) \geq \left\lceil \frac{\log 9}{\log q} \right\rceil$. Then there exists a positive absolute constant c such that for all $n \geq r \geq \gamma$ and for all $\frac{1}{2} < \alpha < 1$, we have*

$$\frac{S_2(n,q)}{q^n} - Q(n) \ll \mathbb{D}(\psi_1, 1; r, n) + \mathbb{D}(\psi_2, 1; r, n) + q^{(1-2\alpha)n} \exp\left(\frac{cq^{\alpha r}}{r}\right) + (rq^r)^{-\frac{1}{2}}$$

where $Q(n)$ is defined by (1.15).

The following theorem gives the asymptotic behaviour of $R_2(n,q)$ with explicit error term in large degree limit.

Theorem 1.6.3. *Let ψ_1 and ψ_2 be multiplicative functions on $\mathcal{M}_q$ with modulus less than or equal to* 1*. Suppose that both ψ_1 and ψ_2 are close to* 1 *and $\gamma := \deg(h_2 - h_1) \geq \left\lceil \frac{\log 17}{\log q} \right\rceil$. Then there exists a positive absolute constant c such that for all $n \geq r \geq \gamma$ and for all $\frac{1}{2} < \alpha < 1$, we have*

$$\frac{R_2(n,q)}{|\mathcal{P}_{n,q}|} - Q'(n) \ll \mathbb{D}(\psi_1, 1; r, n) + \mathbb{D}(\psi_2, 1; r, n) + n^{-A} \exp\left(\frac{cq^{\alpha r}}{r}\right) + (rq^r)^{-\frac{1}{2}}$$

where $A > 0$ is arbitrary constant and $Q'(n)$ is as defined in (1.16)*.*

Remark 1.6.1. *Note that γ is fixed here since the polynomials h_1 and h_2 are fixed. Also we write*

$$Q(n) = Q_1(\gamma)Q_2(\gamma, n) \quad \text{and} \quad Q'(n) = Q'_1(\gamma)Q'_2(\gamma, n)$$

where

$$Q_1(\gamma) = \prod_{\deg P \leq \gamma} \upsilon_P, \quad Q'_1(\gamma) = \prod_{\deg P \leq \gamma} \upsilon'_P, \tag{1.17}$$

$$Q_2(\gamma, n) = \prod_{\gamma < \deg P \leq n} \left(1 + \sum_{j=1}^{2} \sum_{m=1}^{\infty} \frac{\psi_j(P^m) - \psi_j(P^{m-1})}{q^{m \deg P}}\right), \tag{1.18}$$

$$Q_2(\gamma) := Q_2(\gamma, \infty),$$

$$Q'_2(\gamma, n) = \prod_{\gamma < \deg P \leq n} \left(1 + \sum_{j=1}^{2} \sum_{m=1}^{\infty} \frac{\psi_j(P^m) - \psi_j(P^{m-1})}{\Phi(P^m)}\right), \tag{1.19}$$

$$Q'_2(\gamma) := Q'_2(\gamma, \infty).$$

The following corollary is a direct application of Theorem 1.6.2 and Theorem 1.6.3.

Corollary 1.6.1. *Let $\gamma = \deg(h_2 - h_1)$ for fixed polynomials h_1, h_2 in $\mathbb{F}_q[x]$. Assume that ψ_1 and ψ_2 be multiplicative functions on $\mathcal{M}_q$ with modulus less than or equal to*

1. *Suppose that* ψ_1 *and* ψ_2 *are close to* 1. *Then we have, as* $n \to \infty$

$$\frac{S_2(n,q)}{q^n} \to Q_1(\gamma)Q_2(\gamma) \quad \text{and} \quad \frac{R_2(n,q)}{|\mathcal{P}_{n,q}|} \to Q_1'(\gamma)Q_2'(\gamma)$$

where $Q_1(\gamma), Q_1'(\gamma)$ *are defined by* (1.17) *and* $Q_2(\gamma), Q_2'(\gamma)$ *are defined by* (1.18) *and* (1.19) *respectively.*

Remark 1.6.2. *Theorem 1.6.2 and Theorem 1.6.3 can be extended for* $S_k(n,q)$ *and* $R_k(n,q)$ *for* $k \geq 3$.

We define the truncated Liouville function over function field by

$$\lambda_y(P^\alpha) = \begin{cases} (-1)^\alpha \quad (= \lambda(P^\alpha)) & \text{if } \deg P \leq y \\ 1 & \text{if } \deg P > y. \end{cases}$$

It is very interesting to establish

$$\sum_{f \in \mathcal{M}_{n,q}} \lambda_y(f)\lambda_y(f+h) = o(q^n), \qquad \text{as } n \to \infty.$$

Note that, if $y = n$ then the above problem is an analog of Chowla's conjecture over function fields in large degree limit. The following is an weaker result for smaller value of y.

Theorem 1.6.4. *There is a positive absolute constant* C *such that if* $n \geq 2$, $2 \leq y \leq \log n$ *and a fixed* $h \in \mathbb{F}_q[x]$ *with* $\deg h \leq y$, *then*

$$\Bigg| \sum_{f \in \mathcal{M}_{n,q}} \lambda_y(f)\lambda_y(f+h) \Bigg| < C \frac{\log^4 y}{y^4} q^n.$$

As a direct application of Theorem 1.6.2, we get an asymptotic formula for simultaneously k-free monic polynomials.

Corollary 1.6.2. *For a fixed $a \in \mathbb{F}_q^*$ and natural number $k \geq 2$, let us consider*

$$\mathcal{F}_k = \Big\{ f \in \mathcal{M}_{n,q} : f \text{ and } f+a \text{ are both } k\text{- free polynomial of degree } n \Big\}.$$

Then we have

$$\frac{1}{q^n} \sum_{\substack{f \in \mathcal{M}_{n,q} \\ f \in \mathcal{F}_k}} 1 = \prod_P \left(1 - \frac{2}{q^{k \deg P}}\right) + O\left(\frac{1}{n^B}\right)$$

for any $B < 1$.

The following corollary is a direct application of Theorem 1.6.2 and Theorem 1.6.3.

Corollary 1.6.3. *For a fixed $a \in \mathbb{F}_q^*$, we define Euler Phi function over function field by*

$$\Phi(f) = |f| \prod_{P|f} \left(1 - \frac{1}{|P|}\right).$$

Then we have

$$\frac{1}{q^n} \sum_{f \in \mathcal{M}_{n,q}} \frac{\Phi(f)\Phi(f+a)}{|f||f+a|} = \prod_P \left(1 - \frac{2}{q^{2 \deg P}}\right) + O\left(\frac{1}{n^B}\right)$$

and

$$\frac{1}{|\mathcal{P}_{n,q}|} \sum_{P \in \mathcal{P}_{n,q}} \frac{\Phi(P)\Phi(P+a)}{|P||P+a|} = \prod_P \left(1 - \frac{2}{q^{\deg P}(q^{\deg P} - 1)}\right) + \frac{1}{(\log n)^B}$$

for any $B < 1$.

1.7 Probabilistic number theory over $\mathbb{F}_q[x]$

Definition 1.7.1. *A function $\psi : \mathcal{M}_q \to \mathbb{C}$ is called additive if*

$$\psi(fg) = \psi(f) + \psi(g), \quad \text{whenever } f \text{ and } g \text{ are coprime.}$$

Let $\psi : \mathcal{M}_q \to \mathbb{R}$ be a real valued additive function. Define $\Omega := \mathcal{M}_{n,q}$, which is a finite set of q^n elements. Let $\psi_n(f) := \{x_1, \ldots, x_l\}$. The subset $A_i := \{f \in \Omega : \psi_n(f) = x_i\}, \quad i = 1, \ldots, t$, of Ω are pairwise disjoint and form a partition of Ω. The σ-field $\mathfrak{F}$ generated by this partition consists of union of a finite number of subsets A_i.

Consider a real-valued additive function ψ on $\mathcal{M}_q$. For a positive integer n and a real number x, write

$$\nu_n(\psi, x) = \sum_{\substack{f \in \mathcal{M}_{n,q} \\ \psi(f) \leq x}} 1.$$

Definition 1.7.2. *If there exists a distribution function* Ψ *such that* $\frac{1}{q^n}\nu_n(\psi, x)$ *converges point-wise to* $\Psi(x)$ *as* $n \to \infty$ *, then we say that* ψ *has the limit distribution function* $\Psi(x)$*.*

For $A \in \mathfrak{F}$, let $\nu(A) = \frac{|A|}{q^n}$, where $|A|$ is the cardinality of A. Then ν is a probability measure on $\mathfrak{F}$ and $(\Omega, \mathfrak{F}, \nu)$ is a finite probability space. Now ψ_n is a random variable on $(\Omega, \mathfrak{F}, \nu)$ and measurable on $\mathfrak{F}$. The distribution function of ψ_n is

$$\nu[\psi_n \leq x] = \frac{1}{q^n} \left|\{f \in \mathcal{M}_{n,q} : \psi_n(f) \leq x\}\right| = \frac{\nu_n(\psi, x)}{q^n}.$$

In the above setup, one can ask the following question.

Question 1.7.1. *For any two real-valued additive function* ψ *and* $\tilde{\psi}$ *on* $\mathcal{M}_q$ *does there exist a distribution function* $\Psi(x)$ *such that as* $n \to \infty$

$$\frac{1}{q^n}\nu_n\left(f \in \mathcal{M}_{n,q} : \psi(f + h_1) + \tilde{\psi}(f + h_2) \leq x\right) \to \Psi(x) \quad \forall x \qquad \text{as } n \to \infty$$

for any two fixed $h_1, h_2 \in \mathbb{F}_q[x]$ *with* $\deg(h_i) < n$ *for all* $i = 1, 2$*.*

As an application of Theorem 1.6.2, the following theorem gives an answer of the

Question 1.7.1.

Theorem 1.7.1. *Let $t, x \in \mathbb{R}$ and h_1, h_2 be fixed polynomials in $\mathbb{F}_q[x]$. We define $\gamma := \deg(h_2 - h_1)$. Assume that $\tilde{\psi_1}$ and $\tilde{\psi_2}$ be real-valued additive functions on $\mathcal{M}_q$ and the following series converges:*

$$\sum_{|\tilde{\psi_i}(P)|\leq 1} \frac{\tilde{\psi_i}(P)}{q^{\deg P}}, \qquad \sum_{|\tilde{\psi_i}(P)|\leq 1 \forall i} \frac{\tilde{\psi_1}(P)+\tilde{\psi_2}(P)}{q^{\deg P}}, \qquad \sum_{|\tilde{\psi_i}(P)|>1} q^{-\deg P} \quad \forall i = 1, 2.$$

Then the distribution function

$$F_n(x) := \frac{1}{|\mathcal{M}_{n,q}|}\left|\left\{f \in \mathcal{M}_{n,q} : \tilde{\psi_1}(f+h_1) + \tilde{\psi_2}(f+h_2) \leq x\right\}\right|$$

converges weekly towards a limit distribution as n $\to \infty$ whose characteristic function say $G(t)$ is equal to $Q_1(\gamma)Q_2(\gamma)$, where $Q_1(\gamma)$ and $Q_2(\gamma)$ are defined (1.17) *and* (1.18) *respectively with ψ_j is replaced by* $\exp(it\tilde{\psi_j}), \forall j = 1, 2$.

Theorem 1.7.2. *Let $t, x \in \mathbb{R}$ and h_1, h_2 be fixed polynomials in $\mathbb{F}_q[x]$. We define $\gamma := \deg(h_2 - h_1)$Assume that $\tilde{\psi_1}$ and $\tilde{\psi_2}$ be real-valued additive functions on $\mathcal{M}_q$ and series in the Hypothesis of Theorem 1.7.1 converges. Then the distribution function*

$$F'_n(x) := \frac{1}{|\mathcal{P}_{n,q}|}\left|\left\{P \in \mathcal{P}_{n,q} : \tilde{\psi_1}(P+h_1) + \tilde{\psi_2}(P+h_2) \leq x\right\}\right|$$

converges weekly towards a limit distribution as n $\to \infty$ whose characteristic function say $H(t)$ is equal to $Q'_1(\gamma)Q'_2(\gamma)$, where $Q'_1(\gamma)$ and $Q'_2(\gamma)$ are defined (1.17) *and* (1.19) *respectively with ψ_j is replaced by* $\exp(it\tilde{\psi_j}), \forall j = 1, 2$.

As a direct consequence of Theorem 1.7.1 and Theorem 1.7.2, we get the following corollary.

Corollary 1.7.1. *Let $z, t \in \mathbb{R}$ and $a \in \mathbb{F}_q^*$. The distribution functions*

$$\frac{1}{q^n}\left|\{f \in \mathcal{M}_{n,q} : \frac{\Phi(f)\Phi(f+a)}{|f||f+a|} \leq e^z\right|$$

and

$$\frac{1}{|\mathcal{P}_{n,q}|}\Big|\{P \in \mathcal{P}_{n,q} : \frac{\Phi(P)\Phi(P+a)}{|P||P+a|} \le e^z\Big|$$

converge weakly towards limit distribution , as $n \to \infty$*. The characteristic functions of these limit distributions are*

$$\prod_{\deg P}\left(1+\frac{2\left(\left(1-q^{-\deg P}\right)^{it}-1\right)}{q^{\deg P}}\right) \quad \textit{and} \quad \prod_{\deg P}\left(1+\frac{2}{\Phi(P)}\left(\left(1-q^{-\deg P}\right)^{it}-1\right)\right)$$

respectively.

Chapter 2

Correlation of multiplicative functions

In this chapter, we prove results regarding mean values of correlation functions over integers. These results of this chapter are contained in [6].

Let $g_j : \mathbb{N} \to \mathbb{C}$ denotes multiplicative functions such that $|g_j(n)| \leq 1$ for all n. Let $F(n); F_1(n), F_2(n), F_3(n)$ be positive integer-valued polynomials with integer coefficients and these are not divisible by the square of any irreducible polynomial. Also suppose that $F_j(n), F_k(n)$ are relatively prime for $j \neq k$ and for all n. Let v and v_j denote the degree of the polynomials $F(n)$ and $F_j(n)$ respectively. Let $\varrho(d_1, d_2, d_3)$ be the number of solutions of the system of congruence

$$F_j(n) \equiv 0 \pmod{d_j} \quad j = 1, 2, 3.$$

Let $\varrho(d)$ and $\varrho_j(d)$ denote the number of solutions of the congruences

$$F(n) \equiv 0 \pmod{d} \quad \text{and} \quad F_j(n) \equiv 0 \pmod{d}$$

respectively.

Suppose $g_j : \mathbb{N} \to \mathbb{U}$ and $h_j : \mathbb{N} \to \mathbb{C}$ be multiplication functions such that $h_j = \mu * g_j$, $j = 1, 2, 3$. For $x \geq r \geq 2$, We also define

$$P(x) := \prod_{p \leq x} w_p \quad \text{and} \quad P(r, x) := \prod_{r < p \leq x} w_p \tag{2.1}$$

where

$$w_p := \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \frac{h_1(p^{m_1}) h_2(p^{m_2}) h_3(p^{m_3})}{[p^{m_1}, p^{m_2}, p^{m_3}]} \varrho(p^{m_1}, p^{m_2}, p^{m_3}).$$

Let us consider the following triple correlation function:

$$M_x(g_1, g_2, g_3) := \frac{1}{x} \sum_{n \leq x} g_1(F_1(n)) g_2(F_2(n)) g_3(F_3(n)). \tag{2.2}$$

In [6], we find an asymptotic formula for $M_x(g_1, g_2, g_3)$ with explicit error term which is stated as follows.

Theorem 2.0.1. *Let $F_j(x), j = 1, 2, 3$ be polynomials as above of degree $v_j \geq 2$. Let g_1, g_2 and g_3 be multiplicative functions close to 1 and whose modulus does not exceed 1. Then there exists a positive absolute constant c and a natural number γ depending on polynomials $F_1(x), F_2(x)$ and $F_3(x)$ such that for all $x \geq r \geq \gamma$ and for all $1 - \frac{1}{v_1+v_2+v_3} < \alpha < 1$, we have*

$$\begin{aligned} x^{-1} M_3(x) - P(x) \ll & \frac{1}{x} \left(F_1(x) F_2(x) F_3(x)\right)^{1-\alpha} \exp\left(\frac{c r^{\alpha}}{\log r}\right) + (r \log r)^{-\frac{1}{2}} \\ & + \sum_{j=1}^{3} (\mathbb{D}(g_j, 1; r; x) + \mathbb{D}(g_j, 1; x; F_j(x))) + \frac{1}{x} C(r, x) + \frac{1}{\log x} \end{aligned}$$

where $P(x)$ is defined by (2.1) and

$$C(r, x) = \sum_{j=1}^{3} \sum_{m=1}^{v_j - 1} \sum_{\substack{p^m \leq F_j(x) \\ p > r}} |g_j(p^m) - 1| \varrho_j(p^m).$$

Remark 2.0.1. *For any $\gamma \geq 2$, let D_γ denote the set of those tuples $\{d_1, d_2, d_3\}$*

of natural numbers for which all the prime factors of d_i *do not exceed* γ. *Since the congruence system*

$$F_1(n) \equiv 0 \pmod{a}, F_2(n) \equiv 0 \pmod{a}, F_3(n) \equiv 0 \pmod{a}$$

have common solution for finitely many values of a (See [39]) then we can choose γ *so that* $\varrho(d_1, d_2, d_3) = 0$ *if* $\{d_1, d_2, d_3\} \notin D_\gamma$ *and* $\left(\prod_{p>\gamma} p, \prod_{i\neq j}(d_i, d_j)\right) > 1$. *Therefore we have*

$$P(x) = P_1(\gamma)P_2(\gamma, x)$$

where

$$(2.3) \qquad P_1(\gamma) = \prod_{p\leq\gamma} w_p \quad \textit{and} \quad P_2(\gamma, x) = \prod_{\gamma<p\leq x} \left(1 + \sum_{j=1}^{3}\sum_{m=1}^{\infty} \frac{h_j(p^m)}{p^m}\varrho_j(p^m)\right).$$

Remark 2.0.2. *The Theorem 2.0.1 is true for all* $x \geq r \geq \gamma$ *but to get an good error term we will chose* $r = (\log x)^{\frac{1}{\alpha}}$, *where* α *is defined as in Theorem 2.0.1.*

2.1 Proof of Theorem 2.0.1

We begin with some lemmas. The following lemma gives that the number of solution of a congruence is bounded over any power of primes.

Lemma 2.1.1 ([11], Lemma 3). *Let* $F(m)$ *be arbitrary primitive polynomial of degree* v *with integer coefficients and with discriminant* D. *Let* $D \neq 0$. *Then the number of solution of the congruence* $F(m) \equiv 0 \pmod{p^\alpha}$ *is* $\varrho(p)$ *when* $p \nmid D$, *and smaller than* vD^2 *when* $p|D$. *Further,* ϱ *is a multiplicative function and* $\varrho(p^\alpha) \leq c$, c *depends only on* F.

The following lemma ensures the existence of γ in Theorem 2.0.1.

Lemma 2.1.2 ([39], Lemma 2.1)**.** *If $F_1(m)$ and $F_2(m)$ are relatively prime polynomials with integer coefficients, then the congruences*

$$F_1(m) \equiv 0 \pmod a, F_2(m) \equiv 0 \pmod a$$

have common roots for atmost finitely many values of a.

Now we prove a polynomial version of classical Turán-Kubilius inequality which is one of the main tool to prove Theorem 2.0.1.

Lemma 2.1.3. *Let $f(p^m)$ be the sequence of complex numbers for all primes p, $m \geq 1$ and let $F(n)$ is a polynomial as above of degree v. Then we have*

$$\sum_{n \leq x} |K(F(n)) - A(x)| \ll xB(F(x)) + \sum_{m=1}^{v-1} \sum_{p^m \leq F(x)} |f(p^m)| \varrho(p^m) + \frac{x}{\log x},$$

where

$$K(n) := \sum_{p^m \| n} f(p^m), \quad A(x) := \sum_{p^m \leq x} \frac{f(p^m)\varrho(p^m)}{p^m}, \quad B^2(x) := \sum_{p^m \leq x} \frac{|f(p^m)|^2 \varrho(p^m)}{p^m}.$$

Proof. We write $\quad K(F(n)) = \sum_{p^m \| F(n)} f(p^m) = g_x(F(n)) + h_x(F(n)),$

where

$$g_x(F(n)) = \sum_{\substack{p^m \| F(n) \\ p^m \leq x^{\frac{1}{2}}}} f(p^m) \quad \text{and} \quad h_x(F(n)) = \sum_{\substack{p^m \| F(n) \\ p^m > x^{\frac{1}{2}}}} f(p^m).$$

Now

$$\sum_{n \leq x} |K(F(n)) - A(x)| \leq \sum_{n \leq x} \left| g_x(F(n)) - A(x^{1/2}) \right| + \sum_{n \leq x} |h_x(F(n))| + \sum_{n \leq x} \left| A(x^{1/2}) - A(x) \right|.$$

From Turán-Kubilius inequality ([9], Lemma 4.11), we have

$$\sum_{n\le x}\left|g_x(F(n)) - A(x^{1/2})\right| \ll xB(x^{1/2}).$$

From Lemma 2.1.1 and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\left|A(x) - A(x^{1/2})\right| \le \sum_{x^{1/2}<p^m\le x}\frac{|f(p^m)|\,\varrho(p^m)}{p^m}\\ &\le \Bigg(\sum_{x^{1/2}<p^m\le x}\frac{|f(p^m)|^2\,\varrho(p^m)}{p^m}\Bigg)^{1/2}\Bigg(\sum_{x^{1/2}<p^m\le x}\frac{\varrho(p^m)}{p^m}\Bigg)^{1/2} = O(B(x)).\end{aligned}$$

Again by Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\sum_{n\le x}|h_x(F(n))| = \sum_{n\le x}\Bigg|\sum_{\substack{p^m\|F(n)\\ p^m>x^{1/2}}} f(p^m)\Bigg|\\ &\ll x\sum_{x^{1/2}<p^m\le F(x)}\frac{|f(p^m)|\varrho(p^m)}{p^m} + \sum_{x^{1/2}<p^m\le F(x)}|f(p^m)|\,\varrho(p^m)\\ &\ll x\Bigg(\sum_{x^{1/2}<p^m\le F(x)}\frac{|f(p^m)|^2\varrho(p^m)}{p^m}\Bigg)^{1/2}\Bigg(\sum_{x^{1/2}<p^m\le F(x)}\frac{\varrho(p^m)}{p^m}\Bigg)^{1/2}\\ &+ \sum_{x^{1/2}<p^m\le F(x)}|f(p^m)|\,\varrho(p^m)\\ &\ll xB(F(x)) + \sum_{m=1}^{v-1}\sum_{p^m\le F(x)}|f(p^m)|\,\varrho(p^m) + \frac{x}{\log x},\end{aligned}$$

which proves the lemma. □

Proof of Theorem 2.0.1. For an integer $r \ge 3$, we define multiplicative functions $g_{j,r}$ and $g^*_{j,r}$, $j = 1, 2, 3$ by

$$g_{j,r}(p^m) = \begin{cases} g_j(p^m) & \text{if } p \le r \\ 1 & \text{if } p > r, \end{cases} \qquad g^*_{j,r}(p^m) = \begin{cases} 1 & \text{if } p \le r \\ g_j(p^m) & \text{if } p > r \end{cases}$$

and multiplicative function $h_{j,r}$, $j=1,2,3$ by

$$h_{j,r}(p^m) = \begin{cases} g_j(p^m) - g_j(p^{m-1}) & \text{if } p \le r \\ 0 & \text{if } p > r \end{cases}$$

so that, $g_{j,r} = 1 * h_{j,r}$, $j = 1,2,3$.

We can write

$$M_x(g_1,g_2,g_3) - P'(x) = P'(r,x)\Big(\frac{1}{x}\sum_{n\le x} g_{1,r}(F_1(n))g_{2,r}(F_2(n))g_{3,r}(F_3(n)) - P'(r)\Big)$$
$$+\frac{1}{x}\sum_{n\le x}\prod_{j=1}^{3} g_{j,r}(F_j(n))\Big(g^*_{1,r}(F_1(n))g^*_{2,r}(F_3(n))g^*_{3,r}(F_3(n)) - P'(r,x)\Big).$$

Set

$$\eta_j(p) := \sum_{m=1}^{\infty} \frac{(g_j(p^m)) - g_j(p^{m-1}))\,\varrho_j(p^m)}{p^m}, \quad j=1,2,3.$$

From Lemma 2.1.1, we have

$$|\eta_j(p)| \le 2c_{F_j}\frac{1}{p-1} \le \frac{1}{6} \text{ if } p \ge 1 + 12c_{F_j} =: p_j.$$

Let $p_4 := \max(p_1,p_2,p_3)$. If $r \ge p_4$, then using hypothesis of the theorem we obtain that for any $x \ge r \ge p_4$

$$P'(r,x) = \prod_{r<p\le x}\left(1 + \sum_{j=1}^{3}\eta_j(p)\right) = \exp\Big(\sum_{r<p\le x}\sum_{j=1}^{3}\left(\eta_j(p) + O\left(|\eta_j(p)|^2\right)\right)\Big)$$
$$= \exp\Big(\sum_{r<p\le x}\sum_{j=1}^{3}\frac{(g_j(p)-1)\varrho_j(p)}{p} + O\Big(\sum_{r<p\le x}\frac{1}{p^2}\Big)\Big) \ll 1.$$

So

$$|M_x(g_1,g_2,g_3) - P'(x)| \ll \Big|\frac{1}{x}\sum_{n\le x} g_{1,r}(F_1(n))g_{2,r}(F_2(n))g_{3,r}(F_3(n)) - P'(r)\Big|$$

$$+\frac{1}{x}\sum_{n\le x}\left|g_{1,r}^*(F_1(n))g_{2,r}^*(F_2(n))g_{3,r}^*(F_3(n))-P'(r,x)\right| =: E_1+E_2.$$

Estimation of E_1. We have

$$\frac{1}{x}\sum_{n\le x}\prod_{j=1}^{3} g_{j,r}(F_j(n)) = \frac{1}{x}\sum_{\substack{d_j\le F_j(x)\\ j=1,2,3}}\prod_{j=1}^{3} h_{j,r}(d_j)\sum_{\substack{n\le x\\ d_j\mid F_j(n)\\ j=1,2,3}} 1$$

$$= \frac{1}{x}\sum_{d_1\le F_1(x)}\sum_{d_2\le F_2(x)}\sum_{d_3\le F_3(x)} h_{1,r}(d_1)h_{2,r}(d_2)h_{3,r}(d_3)\frac{x}{[d_1,d_2,d_3]}\varrho(d_1,d_2,d_3)$$

$$+O\left(\frac{1}{x}\sum_{\substack{d_j\le F_j(x)\\ j=1,2,3}}\prod_{j=1}^{3} h_{j,r}(d_j)\varrho(d_1,d_2,d_3)\right) =: P_1'+E_3.$$

Now we observe that

$$\sum_{d_j=1}^{\infty}\frac{|h_{j,r}(d_j)|\varrho_j(d_j)}{d_j}\le \exp\left(c_{F_j}\sum_{p\le r}\frac{1}{p}\right)\ll(\log r)^{c_{F_j}}$$

and for $0<\alpha<1$,

$$\sum_{d_j=1}^{\infty}\frac{|h_{j,r}(d_j)|\varrho_j(d_j)}{d_j^{1-\alpha}}\le\prod_{p\le r}\left(1+\sum_{m=1}^{\infty}\frac{|h_{j,r}(p^m)|\varrho_j(p^m)}{p^{m(1-\alpha)}}\right)\le \exp\left(c_{F_j}\frac{r^\alpha}{\log r}\right).$$

So we can say that

$$E_3\ll\frac{1}{x}\sum_{\substack{d_j\le F_j(x)\\ j=1,2,3}}\prod_{j=1}^{3}|h_{j,r}(d_j)|\varrho_j(d_j)\ll\frac{1}{x}\left(\prod_{j=1}^{3}F_j(x)\right)^{1-\alpha}\sum_{\substack{d_j=1\\ j=1,2,3}}^{\infty}\prod_{j=1}^{3}\frac{|h_{j,r}(d_j)|}{d_j^{1-\alpha}}\varrho_j(d_j)$$

$$\ll\frac{1}{x}\left(F_1(x)F_2(x)F_3(x)\right)^{1-\alpha}\exp(c_F\frac{r^\alpha}{\log r}).$$

Now

$$P_1'=P'(r)+O\left(\sum_{k=1}^{3}\sum_{\substack{d_j=1\\ j=1,2,3\\ d_k>F_k(x)}}^{\infty}\frac{|h_{1,r}(d_1)h_{2,r}(d_2)h_{3,r}(d_3)|}{[d_1,d_2,d_3]}\varrho(d_1,d_2,d_3)\right) =: P'(r)+E_4.$$

Again from the above observations, we have

$$E_4 \ll \sum_{k=1}^{3} \sum_{\substack{d_j=1\\ j=1,2,3\\ d_k>F_k(x)}}^{\infty} \frac{|h_{1,r}(d_1)h_{2,r}(d_2)h_{3,r}(d_3)|}{d_1d_2d_3}\varrho_1(d_1)\varrho_2(d_2)\varrho_3(d_3)$$

$$\ll \left(F_1(x)^{-\alpha}+F_2(x)^{-\alpha}+F_3(x)^{-\alpha}\right)\exp\left(c_F\frac{r^{\alpha}}{\log r}\right).$$

Estimation of E_2. We will use a technique of R.Warlimont [41]. Let

$$N_r' = \left\{ n\le x| \quad \exists k\in\{1,2,3\} \text{ and } \exists p>r \text{ such that } p^m\|F_k(n), |1-g_k(p^m)|>\frac{1}{2}\right\}.$$

Decompose E_2 into

$$E_2 = \frac{1}{x}\sum_{n\in N_r'}\left|g_{1,r}^*(F_1(n))g_{2,r}^*(F_2(n))g_{3,r}^*(F_3(n))-P'(r,x)\right|$$
$$+\frac{1}{x}\sum_{n\notin N_r'}\left|g_{1,r}^*(F_1(n))g_{2,r}^*(F_2(n))g_{3,r}^*(F_3(n))-P'(r,x)\right| =: E_5+E_6.$$

Now

$$E_5 \ll \frac{1}{x}\sum_{n\in N_r'}1 \ll \frac{1}{x}\sum_{\substack{p^m\le F_j(x)\\ |1-g_j(p^m)|>1/2\\ p>r}}\left(\frac{x\varrho_j(p^m)}{p^m}+\varrho_j(p^m)\right)$$

$$\ll \sum_{\substack{p^m\le F_j(x)\\ |1-g_j(x)|>1/2\\ p>r}}\frac{\varrho_j(p^m)}{p^m}+\frac{1}{x}\sum_{\substack{p^m\le F_j(x)\\ |1-g_j(p^m)|>1/2\\ p>r}}\varrho_j(p^m)$$
$$\ll \sum_{r<p\le F_j(x)}\frac{|1-g_j(p)|^2\varrho_j(p)}{p}+\sum_{p>r}\frac{1}{p^2}+\frac{1}{x}\sum_{\substack{p^m\le F_j(x)\\ p>r, m<v_j}}|1-g_j(p^m)|\varrho_j(p^m)$$
$$+\frac{1}{\log x}\ll \sum_{j=1}^{3}(\mathbb{D}^2(g_j,1;r;x)+\mathbb{D}^2(g_j,1;x;F_j(x)))+(r\log r)^{-1}+\frac{1}{x}C(r,x)+\frac{1}{\log x}.$$

Since we know that if $\Re(u) \leq 0, \Re(v) \leq 0$, then

$$|\exp(u) - \exp(v)| \leq |u - v| \quad \text{and} \tag{2.4}$$

$$\log(1+z) = z + O(|z|^2), \quad \text{if } |z| \leq 1, |\arg(z)| \leq \frac{\pi}{2} \tag{2.5}$$

We obtain

$$\begin{aligned}
E_6 \ll & \frac{1}{x} \sum_{n \leq x} \sum_{j=1}^{3} \left| \sum_{\substack{p^m \| F_j(n) \\ p > r}} (g_j(p^m) - 1) - \sum_{\substack{p^m \leq x \\ p > r}} \frac{g_j(p^m) - 1}{p^m} \varrho_j(p^m) \right| \\
& + \frac{1}{x} \sum_{n \leq x} \left| \sum_{\substack{p^m \leq x \\ p > r}} \sum_{j=1}^{3} \frac{(g_j(p^m) - 1)\varrho_j(p^m)}{p^m} - \log P'(r, x) \right| \\
& + O\left(\frac{1}{x} \sum_{n \leq x} \sum_{j=1}^{3} \sum_{\substack{p^m \| F_j(n) \\ p > r}} |g_j(p^m) - 1|^2 \right) =: E_{61} + E_{62} + E_{63}.
\end{aligned}$$

From Lemma 2.1.3, we have

$$\begin{aligned}
E_{61} & \ll \sum_{j=1}^{3} \left(\sum_{\substack{p^m \leq F_j(x) \\ p > r}} \frac{|g_j(p^m) - 1|^2 \varrho_j(p^m)}{p^m} \right)^{1/2} + \frac{1}{x} C(r, x) + \frac{1}{\log x} \\
& \ll \sum_{j=1}^{3} (\mathbb{D}(g_j, 1; r; x) + \mathbb{D}(g_j, 1; x; F_j(x))) + (r \log r)^{-1/2} + \frac{1}{x} C(r, x) + \frac{1}{\log x},
\end{aligned}$$

$$\begin{aligned}
E_{62} & = \left| \sum_{j=1}^{3} \sum_{r < p \leq x} \frac{(g_j(p) - 1)\varrho_j(p)}{p} + O\left(\sum_{p > r} \frac{1}{p^2} \right) - \sum_{r < p \leq x} \sum_{j=1}^{3} \frac{(g_j(p) - 1)\, \varrho_j(p)}{p} \right| \\
& \ll \sum_{p > r} \frac{1}{p^2} \ll (r \log r)^{-1}
\end{aligned}$$

and

$$E_{63} \ll \sum_{j=1}^{3} \sum_{\substack{p^m \leq F_j(x) \\ p > r}} \frac{|g_j(p^m) - 1|^2 \varrho_j(p^m)}{p^m} + \frac{1}{x} C(r, x) + \frac{1}{\log x}$$

$$\ll \sum_{j=1}^{3}(\mathbb{D}^2(g_j,1;r;x)+\mathbb{D}^2(g_j,1;x;F_j(x)))+(r\log r)^{-1}+\frac{1}{x}C(r,x)+\frac{1}{\log x}.$$

Combining all these estimates for all $1-\frac{1}{v_1+v_2+v_3}<\alpha<1$, we have

$$M_x(g_1,g_2,g_3)-P'(x)\ll \frac{1}{x}\left(F_1(x)F_2(x)F_3(x)\right)^{1-\alpha}\exp\left(c\frac{r^\alpha}{\log r}\right)$$
$$+\sum_{j=1}^{3}(\mathbb{D}(g_j,1;r;x)+\mathbb{D}(g_j,1;x;F_j(x)))+(r\log r)^{-1/2}+\frac{1}{x}C(r,x)+\frac{1}{\log x},$$

which proves the theorem.

2.1.1 Application of Theorem 2.0.1

As an application of the Theorem 2.0.1, we get the following corollary.

Corollary 2.1.1. *Let* $\phi(n)=n\prod_{p|n}\left(1-\frac{1}{p}\right)$, *be Euler's totient function and* $\sigma(n)=\sum_{d|n}d$. *Let* $F_1(x)=x^2+a_1, F_2(x)=x^2+a_2, F_3(x)=x^2+a_3, 0<t<1$, *where* a_1,a_2,a_3 *are taken such that* $F_j(x), j=1,2,3$ *satisfies the assumption of Theorem 2.0.1. Then there exist a natural number* γ *depending on* a_1,a_2 *and* a_3 *such that for all* $x\geq\gamma$,

$$\frac{1}{x}\sum_{n\leq x}\frac{\phi(n^2+a_1)\phi(n^2+a_2)\phi(n^2+a_3)}{\sigma(n^2+a_1)\sigma(n^2+a_2)\sigma(n^2+a_3)}=P_1'(\gamma)\prod_{p>\gamma}w_p'+O\left(\frac{1}{(\log x)^t}\right),$$
$$\frac{1}{x}\sum_{n\leq x}\frac{\phi(n^2+a_1)\phi(n^2+a_2)\phi(n^2+a_3)}{(n^2+a_1)(n^2+a_2)(n^2+a_3)}=P_1''(\gamma)\prod_{p>\gamma}w_p''+O\left(\frac{1}{(\log x)^t}\right),$$

where

$$w_p'=\left(1-\sum_{j=1}^{3}\frac{\left(\frac{a_j}{p}\right)\varrho_j(p)}{p^2}+\left(1-\frac{1}{p}\right)^2\sum_{j=1}^{3}\sum_{m=1}^{\infty}\frac{\left(\frac{a_j}{p}\right)\varrho_j(p)}{1+p+\cdots+p^m}\right),$$
$$w_p''=\left(1-\sum_{j=1}^{3}\frac{\left(\frac{a_j}{p}\right)\varrho_j(p)}{p^2}\right),\quad \left(\frac{a_j}{p}\right)$$ *is the Legendre symbol of* a_j *and* p,

$P_1'(\gamma)$ and $P_2''(\gamma)$ are defined by (2.3) in which $g_j(n), j = 1,2,3$ are replaced by $\phi(n)/\sigma(n)$ and $\phi(n)/n$ respectively.

Proof. To prove Corollary 2.1.1, we will use the following standard congruence lemma.

Lemma 2.1.4. *Let p be an odd prime and $(a,p) = 1$, then $x^2 \equiv a \pmod{p^k}$ has exactly two solutions if a is a quadratic residue of p, and no solution if a is quadratic non-residue of p. Further, if a is odd, then the congruence $x^2 \equiv a \pmod 2$ is always solvable and has exactly one solution.*

Proof of Corollary 2.1.1. We see that by Lemma 2.1.4, as $p \to \infty$

$$\sum_p \frac{1-(g_j(p))\varrho_j(p)}{p} \leq \sum_p \frac{2}{p(p+1)} < \infty, \quad (1-g_j(p))\varrho_j(p) \leq \frac{2}{p+1} \to 0,$$

where $g_j(n) = \phi(n)/\sigma(n), j = 1,2,3$, and as $p \to \infty$

$$\sum_p \frac{1-(g_j(p))\varrho_j(p)}{p} \leq \sum_p \frac{2}{p^2} < \infty, \quad (1-g_j(p))\varrho_j(p) \leq \frac{2}{p} \to 0,$$

where $g_j(n) = \phi(n)/n, j = 1,2,3$.

The remainder term for both sum are estimated from the remainder term of Theorem 2.0.1 by choosing

$$\alpha = \frac{5+c_{21}}{6} \quad \text{and} \quad r = c_{22}(\log x \log\log x)^{1/\alpha},$$

for sufficiently small $c_{21}, c_{22} > 0$.

Hence by Theorem 2.0.1, the corollary is proved. □

Corollary 2.1.2 is a polynomial version with the degree of the polynomial greater than or equal to 2, of a theorem of Kátai ([20], Theorem 5).

Corollary 2.1.2. *Let $F_j(n)$ and $g_j (j = 1, 2, 3)$ be as in Theorem 2.0.1.*

$$\sum_p \frac{(g_j(p) - 1)\varrho_j(p)}{p} < \infty. \tag{2.6}$$

Suppose that

$$(g_j(p^\alpha) - 1)\,\varrho_j(p^\alpha) \to 0, \qquad \text{as } p \to \infty \tag{2.7}$$

for $\alpha = 1$, when $v_j \geq 2$ and for $\alpha = 1, 2, \cdots, v_j - 2$, when $v_j \geq 3$, then we have

$$M_x(g_1, g_2, g_3) \to P_1(\gamma)P_2(\gamma), \qquad \text{as } x \to \infty$$

where $P_1(\gamma)$ is defined by (2.3) *and*

$$P_2(\gamma) := \prod_{p>\gamma}\left(1 + \sum_{m=1}^{\infty}\sum_{j=1}^{3} \frac{h_j(p^m)\varrho_j(p^m)}{p^m}\right), \quad h_j = \mu * g_j. \tag{2.8}$$

Proof. We need the following lemmas.

Lemma 2.1.5 ([40])**.** *Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be two complex sequences such that*

$$\sum_{n=1}^{\infty} \left(|u_n|^2 + |v_n|\right) < \infty.$$

Then we have

$$\prod_{n=1}^{\infty}(1 + u_n + v_n) < \infty \text{ if and only if } \sum_{n=1}^{\infty} u_n < \infty.$$

Lemma 2.1.6 ([20], Lemma 6)**.** *Let $F(n)$ be a polynomial as above of degree $v \geq 2$. we have the relation:*

$$\textbf{card}\left\{n \leq x : F(n) \equiv 0 \pmod{p^{v-1}}, y_1(x) < p\right\} = o(x),$$

when $y_1 = y_1(x)$ tends to infinity as $x \to \infty$.

Proof of Corollary 2.1.2. Since $|g_j(p) - 1|^2 \leq 2(1 - \Re(g_j(p)))$, from (2.6) and Lemma 2.1.1 we have

$$\mathbb{D}(g_j, 1; \infty) < \infty, \quad j = 1, 2, 3. \tag{2.9}$$

From (2.6) and Lemma 2.1.5, we have $P_1(\gamma)$ and $P_2(\gamma)$ are convergent.
From Lemma 2.1.6, it is easy to see that as $p \to \infty$,

$$\left(1 - g_j(p^{v_j - 1})\right) \varrho_j(p^{v_j - 1}) \to 0, \quad j = 1, 2, 3. \tag{2.10}$$

So by setting $r = (\log x)^{\frac{1}{\alpha}}$ and from (2.7), (2.9), (2.10), the error term in Theorem 2.0.1 is $o(1)$ as $x \to \infty$. Hence by Theorem 2.0.1, the corollary is proved. □

In [6], we also studied the mean value of the following triple correlation functions with various assumption on g_j's.

$$M'_x(g_1, g_2, g_3) := \frac{1}{x} \sum_{n \leq x} g_1(n+2) g_2(n+1) g_3(n). \tag{2.11}$$

The following theorem tells us that if g_3 is Möbius function then under certain hypothesis on g_1, g_2, the mean value of triple correlation function (2.11) is zero.

Theorem 2.1.1. *Let $g_1, g_2 : \mathbb{N} \to \mathbb{U}$ be multiplicative functions and*

$$\sum_p \sum_{j=1}^{2} \frac{|g_j(p) - 1|^2}{p} < \infty. \tag{2.12}$$

Then as $x \to \infty$,

$$M'_x(g_1, g_2, \mu) = \frac{1}{x} \sum_{n \leq x} g_1(n+2) g_2(n+1) \mu(n) = o(1).$$

2.2 Proof of Theorem 2.1.1

Proof. We begin with the following lemma.

Lemma 2.2.1 ([7], Theorem 1). *Define $e(y) := e^{2\pi iy}$. For any given $K > 0$,*

$$\sum_{n\le x}\mu(n)e(n\theta) = O\left(\frac{x}{(\log x)^K}\right)$$

uniformly in θ, where the implied constant depends on K.

Proof of Theorem 2.1.1. We set

$$R(r,x) = \prod_{r<p\le x}\left\{1-\frac{2}{p}+\left(1-\frac{1}{p}\right)\sum_{m=1}^{\infty}\frac{g_1(p^m)+g_2(p^m)}{p^m}\right\},$$

where r will be chosen later. It is easy to see that $|R(r,x)| \le 1$. Therefore

$$M'_x(g_1,g_2,\mu) = R(r,x)\frac{1}{x}\sum_{n\le x}g_{1,r}(n+2)g_{2,r}(n+1)\mu(n)+$$
$$+\frac{1}{x}\sum_{n\le x}g_{1,r}(n+2)g_{2,r}(n+1)\mu(n)\left(g^*_{1,r}(n+2)g^*_{2,r}(n+1)-R(r,x)\right).$$

So

$$|M'_x(g_1,g_2,\mu)| \le |M'_x(g_{1,r},g_{2,r},\mu)| + \frac{1}{x}\sum_{n\le x}\left|\prod_{j=1}^{2}g^*_{j,r}(n+3-j)-R(r,x)\right|$$
$$=: T_1+T_2.$$

Estimation of T_1. Recall that $h_{j,r} = \mu * g_{j,r}$. So we have

$$\sum_{n\le x}g_{1,r}(n+2)g_{2,r}(n+1)\mu(n) = \sum_{n\le x}\sum_{d_1|n+2}\sum_{d_2|n+1}h_{1,r}(d_1)h_{2,r}(d_2)\mu(n)$$
$$= \sum_{\substack{d_1\le x+2\\ d_2\le x+1}}h_{1,r}(d_1)h_{2,r}(d_2)\sum_{\substack{n\le x\\ d_1|n+2\\ d_2|n+1}}\mu(n) = \sum_{\substack{d_1\le x+2\\ d_2\le x+1\\ (d_1,d_2)=1}}h_{1,r}(d_2)h_{2,r}(d_2)\sum_{\substack{n\le x\\ n\equiv v(d_1d_2)}}\mu(n)$$

$$= \sum_{\substack{d_1 \le y \\ d_2 \le y \\ (d_1,d_2)=1}} \prod_{j=1}^{2} h_{j,r}(d_j) \sum_{\substack{n \le x \\ n \equiv v(d_1 d_2)}} \mu(n) + \sum_{k=1}^{2} \sum_{\substack{d_j \le x+3-j \\ j=1,2 \\ d_k > y \\ (d_1,d_2)=1}} \prod_{j=1}^{2} h_{j,r}(d_j) \sum_{\substack{n \le x \\ n \equiv v(d_1 d_2)}} \mu(n),$$

where v is the unique solution of the system of linear congruence $n \equiv -2(d_1)$, $n \equiv -1(d_2), 0 \le v \le d_1 d_2 - 1$ and $y := \log x$.

So

$$\begin{aligned} xT_1 &\ll \sum_{\substack{d_j \le y \\ j=1,2}} |h_{1,r}(d_1) h_{2,r}(d_2)| \Bigg| \sum_{\substack{n \le x \\ n \equiv v(d_1 d_2)}} \mu(n) \Bigg| + \sum_{j,k=1}^{2} \sum_{\substack{d_j \le x+3-j \\ d_k > y}} |h_{j,r}(d_j)| \left(\frac{x}{d_1 d_2} + 1 \right) \\ &=: T_{11} + T_{12}. \end{aligned}$$

From Lemma 2.2.1, we have

$$\begin{aligned} \sum_{\substack{n \le x \\ n \equiv v(d_1 d_2)}} \mu(n) &= \sum_{n \le x} \mu(n) \frac{1}{d_1 d_2} \sum_{l=1}^{d_1 d_2} e\left(\frac{(n-v)l}{d_1 d_2} \right) \\ &= \frac{1}{d_1 d_2} \sum_{l=1}^{d_1 d_2} e\left(\frac{-vl}{d_1 d_2} \right) \sum_{n \le x} \mu(n) e\left(\frac{nl}{d_1 d_2} \right) \ll \frac{x}{(\log x)^K}. \end{aligned}$$

So

$$\begin{aligned} T_{11} &\ll \frac{x}{(\log x)^K} \sum_{d_1, d_2 \le y} |h_{1,r}(d_1) h_{2,r}(d_2)| \\ &\ll \frac{x}{(\log x)^K} y^4 \sum_{d_1,d_2=1}^{\infty} \frac{|h_{1,r}(d_1) h_{2,r}(d_2)|}{d_1^2 d_2^2} \ll \frac{x}{\log x}, \quad \text{if } K \ge 5. \end{aligned}$$

Let $0 < \alpha < 1$. Now from the following two estimates

$$\begin{aligned} \sum_{d_j=1}^{\infty} \frac{|h_{j,r}(d_j)|}{d_j^{\alpha}} &= \prod_{p \le r} \left(1 + \sum_{m=1}^{\infty} \frac{|h_{j,r}(p^m)|}{p^{m\alpha}} \right) \le \prod_{p \le r} \left(1 + \frac{2}{p^{\alpha} - 1} \right) \\ &\le \exp\left(c_1 \sum_{p \le r} \frac{1}{p^{\alpha}} \right) \le \exp\left(c_2 \frac{r^{1-\alpha}}{\log r} \right) \end{aligned} \tag{2.13}$$

and

$$\sum_{d_j=1}^{\infty}\frac{|h_{j,r}(d_j)|}{d_j}\le \exp\Big(c_3\sum_{p\le r}\frac{1}{p}\Big)\ll (\log r)^{c_4}. \tag{2.14}$$

Also for $0<\gamma<1$ we obtain

$$\begin{aligned}T_{12} &\ll x\sum_{\substack{d_j\le x+3-j\\ j=1,2\\ d_k>y}}\frac{|h_{1,r}(d_1)h_{2,r}(d_2)|}{d_1d_2}+\sum_{\substack{d_j\le x+3-j\\ j=1,2\\ d_k>y}}|h_{1,r}(d_1)h_{2,r}(d_2)|\\ &\ll xy^{-\gamma}\exp\left(c_2\frac{r^{\gamma}}{\log r}\right)(\log r)^{c_4}+x^{2\alpha}\exp\left(2c_2\frac{r^{1-\alpha}}{\log r}\right)\\ &\ll xy^{-\gamma}\exp\left(c_5\frac{r^{\gamma}}{\log r}\right)+x^{2\alpha}\exp\left(2c_2\frac{r^{1-\alpha}}{\log r}\right).\end{aligned}$$

Taking $1-\alpha=\gamma=\frac{2}{3}$ we have,

$$T_{12}\ll xy^{-\frac{2}{3}}\exp\left(c_5\frac{r^{2/3}}{\log r}\right)+x^{2/3}\exp\left(2c_2\frac{r^{2/3}}{\log r}\right).$$

Setting $r=(\log\log x)^{3/2}$ we have

$$T_{12}\ll\frac{x}{y^{2/3}}(\log x)^{1/6}+xx^{-1/3}(\log x)^{1/6}\ll\frac{x}{(\log x)^{1/2}}.$$

So $T_1=o(1)$ as $x\to\infty$.

Estimation of T_2. We closely follow the method of R.Warlimont [41]. Let

$$N_r=\left\{n\le x|\quad \exists j\in\{1,2\}\ \text{ and }\exists p>r\text{ such that }p^m\|n+3-j,|1-g_j(p^m)|>\frac{1}{2}\right\}$$

Decompose T_2 as

$$\begin{aligned}T_2&=\frac{1}{x}\sum_{n\in N_r}\left|g_{1,r}^*(n+2)g_{2,r}^*(n+1)-R(r,x)\right|\\ &+\frac{1}{x}\sum_{n\notin N_r}\left|g_{1,r}^*(n+2)g_{2,r}^*(n+1)-R(r,x)\right| =: T_5+T_6.\end{aligned}$$

Now

$$T_5 \ll \frac{1}{x}\sum_{j=1}^{2}\sum_{r<p\le x+3-j}\frac{x+3-j}{p}|1-g_j(p)|^2+\sum_{j=1}^{2}\sum_{p>r}\sum_{m\ge 2}\frac{1}{p^m}$$
$$\ll \sum_{j=1}^{2}\mathbb{D}^2(g_j,1;r;x+3-j)+(r\log r)^{-1}.$$

From (2.4) and (2.5), we have

$$T_6 \le \frac{1}{x}\sum_{j=1}^{2}\sum_{n\le x}\Bigg|\sum_{\substack{p^m\|n+3-j\\ p>r}}(g_j(p^m)-1)-\sum_{\substack{p^m\le x\\ p>r}}\frac{g_j(p^m)-1}{p^m}\Bigg|$$
$$+\frac{1}{x}\Bigg|\sum_{\substack{p^m\le x\\ p>r}}\sum_{j=1}^{2}\frac{g_j(p^m)-1}{p^m}-\log R(r,x)\Bigg|+O\Bigg(\frac{1}{x}\sum_{n\le x}\sum_{\substack{p^m\|n+3-j\\ j=1,2,p>r}}|g_j(p^m)-1|^2\Bigg)$$
$$=: T_7+T_8+T_9.$$

Now by Cauchy-Schwarz inequality and Turán-Kubilius inequality ([9], Lemma 4.4), we have

$$T_7 \ll \Bigg(\sum_{j=1}^{2}\sum_{\substack{p^m\le x+3-j\\ p>r}}|g_j(p^m)-1|^2\Bigg)^{\frac{1}{2}}+\frac{1}{x}\ll\Bigg(\sum_{j=1}^{2}\sum_{r<p\le x+3-j}|g_j(p)-1|^2\Bigg)^{\frac{1}{2}}$$
$$+\Bigg(\sum_{p>r}\frac{1}{p^2}\Bigg)^{\frac{1}{2}}+\frac{1}{x}\ll\sum_{j=1}^{2}\mathbb{D}(g_j,1;r;x+3-j)+(r\log r)^{-\frac{1}{2}}+x^{-1}.$$

Now similar to estimation of E_{62}, we have

$$T_8 \ll \Bigg|\sum_{\substack{p^m\le x\\ p>r}}\sum_{j=1}^{2}\frac{g_j(p^m)-1}{p^m}-\log R(r,x)\Bigg|\ll\sum_{p>r}\frac{1}{p^2}\ll(r\log r)^{-1}$$

and

$$T_9 \ll \frac{1}{x}\Bigg\{\sum_{j=1}^{2}\sum_{\substack{p^m\le x+3-j\\ p>r}}\frac{|g_j(p^m)-1|^2}{p^m}\Bigg\}\ll\sum_{j=1}^{2}\mathbb{D}^2(g_j,1;r;x+3-j)+(r\log r)^{-1}.$$

Combining above calculations, we have

$$T_2 \ll (r\log r)^{-1/2} + \sum_{j=1}^{2} \mathbb{D}(g_j, 1; r; x+3-j).$$

By the above choice of r and from (2.12), we have $T_2 = o(1)$ as $x \to \infty$. Which proves the required theorem. $\square$

The following theorem gives an asymptotic formula with explicit error term for $M'_x(g_1, g_2, g_3)$ as in (2.11) when g_1, g_2 are close to 1 (see definition 1.1) and g_3 is a good function (see definition 1.2.1). Recall that

$$\theta_\tau(n) = \prod_{p|n}\left(1 + \sum_{m=1}^{\infty} \frac{g_3(p^m)}{p^{m(1+i\tau)}}\right)^{-1}, \quad \tau \in \mathbb{R}, \qquad M_x(g_3) = \frac{1}{x}\sum_{n\le x} g_3(n),$$

$$Q_\tau(r) = \prod_{p\le r}\left(1 - \frac{2\theta_\tau(p)}{p-1} + \theta_\tau(p)\sum_{m=1}^{\infty}\frac{g_1(p^m)+g_2(p^m)}{p^m}\right),$$

$$P_3(r,x) = \prod_{r<p\le x}\left(1 - \frac{2}{p} + \left(1-\frac{1}{p}\right)\sum_{m=1}^{\infty}\frac{g_1(p^m)+g_2(p^m)}{p^m}\right).$$

Theorem 2.2.1. *Let g_1, g_2 and g_3 be multiplicative functions whose modulus does not exceed 1 and g_3 be a good function. Assume further that there exist a positive constant c_1 such that*

$$\left|1 + \sum_{k=1}^{\infty}\frac{g_3(2^k)}{2^{k(1+i\xi)}}\right| \ge c_1 \tag{2.15}$$

for $\xi = 0$ if g_3 is real valued, and for all $\xi \in \mathbb{R}$, if g_3 is not real valued. Then there exist positive absolute constants c, c' and a real $\tau, |\tau| \le (\log x)^{1/19}$, such that for all $x \ge r \ge 2$ and for all $\frac{1}{2} < \alpha < \frac{5}{9}$, we have

$$M'_x(g_1,g_2,g_3) - M_x(g_3)P_3(r,x)Q_\tau(r) \ll x^{1-2\alpha}\exp\left(c\frac{r^\alpha}{\log r}\right) + \frac{(\log r)^c}{(\log x)^{c'}}$$
$$+ \frac{\exp\left(c(\log\log r)^2\right)}{(\log x)^{1/19}} + \sum_{j=1}^{2}\mathbb{D}(g_j,1;r;x-4+j) + (r\log r)^{-\frac{1}{2}}.$$

For real-valued g_3 we may set $\tau = 0$ in the expression of $Q_\tau(r)$.

2.3 Proof of Theorem 2.2.1

Lemma 2.3.1 ([10], Theorem 2)**.** *Let g be a multiplicative function whose modulus does not exceed* 1*. Then there is a real τ with $|\tau| \leq (\log x)^{1/19}$ such that*

$$\sum_{\substack{n\leq x\\(n,D)=1}} g(n) = \theta_\tau(D)\sum_{n\leq x} g(n) + O\left(\frac{x(\log\log 3D)^2}{(\log x)^{1/19}}\right) \tag{2.16}$$

holds uniformly for $x \geq 2$ and odd integers D. If in addition, the condition (2.15) *is satisfied then* (2.16) *holds for even integers as well. For real-valued g we may set $\tau = 0$.*

The following lemma is a special case of a theorem of Wolke [43].

Lemma 2.3.2 ([43], Theorem 1)**.** *Let g be a good function which is multiplicative with modulus ≤ 1. Then for given any $A > 0$ there is a corresponding $A_1 > 0$, possibly depending on g, such that for $x \geq 2$ and $Q = x^{1/2}(\log x)^{-A_1}$, we have*

$$\sum_{d\leq Q}\max_{(l,d)=1}\max_{u\leq x}\left|\sum_{\substack{n\leq u\\ n\equiv l(d)}} g(n) - \frac{1}{\phi(d)}\sum_{\substack{n\leq u\\(n,d)=1}} g(n)\right| \ll \frac{x}{(\log x)^A}.$$

In case $-\tau \in \mathbb{N}$ or $\tau = 0$ then

$$\sum_{d\leq Q}\max_{l}\max_{u\leq x}\left|\sum_{\substack{n\leq u\\ n\equiv l(d)}} g(n)\right| \ll \frac{x}{(\log x)^A}.$$

The following lemma is a two dimensional version of standard Cauchy-Schwarz inequality.

Lemma 2.3.3. *Let $1 \leq i, j, k \leq y$. If x_j, x_k and c_{jk} are non-negative real numbers,* *then*

$$\sum_{j\le y}\sum_{k\le y} x_j x_k c_{jk} \le \Big(\sum_{j\le y}\sum_{k\le y} x_j^2 x_k^2 c_{jk}\Big)^{1/2}\Big(\sum_{j\le y}\sum_{k\le y} c_{jk}\Big)^{1/2}.$$

Proof. By applying Cauchy-Schwarz inequality, we have

$$\sum_{j\le y}\sum_{k\le y} x_j x_k c_{jk} \le \sum_{j\le y}\Big(\sum_{k\le y} x_j^2 x_k^2 c_{jk}\Big)^{1/2}\Big(\sum_{k\le y} c_{jk}\Big)^{1/2} =: \sum_{j\le y} a_j b_j.$$

Applying Cauchy-Schwarz inequality again, we have

$$\sum_{j\le y} a_j b_j \le \Big(\sum_{j\le y} a_j^2\Big)^{1/2}\Big(\sum_{j\le y} b_j^2\Big)^{1/2} = \Big(\sum_{j,k\le y} x_j^2 x_k^2 c_{jk}\Big)^{1/2}\Big(\sum_{j,k\le y} c_{jk}\Big)^{1/2}$$

which completes the proof. □

Proof of Theorem 2.2.1. We set

$$\begin{aligned} R &:= M'_x(g_1,g_2,g_3) - M_x(g_3)P_3(r,x)Q_\tau(r) \\ &= P_3(r,x)\left(M'_x(g_{1,r},g_{2,r},g_3) - M_x(g_3)Q_\tau(r)\right) \\ &+ \frac{1}{x}\sum_{n\le x} g_{1,r}(n+2)g_{2,r}(n+1)g_3(n)\left(g^*_{1,r}(n+2)g^*_{2,r}(n+1) - P_3(r,x)\right). \end{aligned}$$

It is easy to see that $|P_3(r,x)| \le 1$. Therefore

$$\begin{aligned} R &\ll |M'_x(g_{1,r},g_{2,r},g_3) - M_x(g_3)Q_\tau(r)| \\ &\quad + \frac{1}{x}\sum_{n\le x}\left|g^*_{1,r}(n+2)g^*_{2,r}(n+1) - P_3(r,x)\right| =: U_1 + U_2. \end{aligned}$$

Estimation of U_1. We have

$$M'_x(g_{1,r},g_{2,r},g_3) = \frac{1}{x}\sum_{n\le x}\sum_{d_1|n+2}\sum_{d_2|n+1} h_{1,r}(d_1)h_{2,r}(d_2)g_3(n)$$

$$
\begin{aligned}
&= \frac{1}{x} \sum_{d_1 \le x+2} h_{1,r}(d_1) \sum_{\substack{d_2 \le x+1 \\ (d_1,d_2)=1}} h_{2,r}(d_2) \sum_{\substack{n \le x \\ n \equiv v(d_1 d_2)}} g_3(n) \\
&= \frac{1}{x} \sum_{\substack{d_j \le y \\ j=1,2 \\ (d_1,d_2)=1}} h_{1,r}(d_1) h_{2,r}(d_2) \sum_{\substack{n \le x \\ n \equiv v(d_1 d_2)}} g_3(n) \\
&+ \frac{1}{x} \sum_{k=1}^{2} \sum_{\substack{d_j \le x+3-j \\ j=1,2 \\ d_k > y}} h_{1,r}(d_1) h_{2,r}(d_2) \left(\frac{x}{d_1 d_2} + 1 \right) =: P_2 + U_{11},
\end{aligned}
$$

where v is the unique solution of the system $n \equiv -2(d_1), n \equiv -1(d_2)$ and $y := x^{1/4}(\log x)^{-\frac{\beta}{2}}, \quad \beta > 0$.

From (2.13), (2.14) for $0 < \alpha, \gamma < 1$, we have

$$
\begin{aligned}
U_{11} &\ll y^{-\gamma} \exp\left(c_6 \frac{r^\gamma}{\log r} \right) + x^{1-2\alpha} \exp\left(2c_2 \frac{r^\alpha}{\log r} \right) \\
&\ll x^{-\gamma/4} (\log x)^{\frac{\gamma\beta}{2}} \exp\left(c_6 \frac{r^\gamma}{\log r} \right) + x^{1-2\alpha} \exp\left(2c_2 \frac{r^\alpha}{\log r} \right).
\end{aligned}
$$

We also obtain

$$
\begin{aligned}
P_2 &= \frac{1}{x} \sum_{d_1 \le y} \sum_{\substack{d_2 \le y \\ (d_1,d_2)=1}} \frac{h_{1,r}(d_1) h_{2,r}(d_2)}{\phi(d_1 d_2)} \sum_{\substack{n \le x \\ (n, d_1 d_2)=1}} g_3(n) + O\left(\frac{1}{x} \sum_{\substack{d_j \le y \\ j=1,2}} |h_{jr}(d_j)| \right) \\
&+ O\left(\frac{1}{x} \sum_{\substack{d_j \le y \\ j=1,2}} |h_{1,r}(d_1) h_{2,r}(d_2)| \left| \sum_{\substack{n \le x \\ n \equiv v(d_1 d_2)}} g_3(n) - \frac{1}{\phi(d_1 d_2)} \sum_{\substack{n \le x \\ (n, d_1 d_2)=1}} g_3(n) \right| \right) \\
&=: P_3 + U_{12} + U_{13}.
\end{aligned}
$$

By Lemma 2.3.2 and Lemma 2.3.3, we have

$$
\begin{aligned}
U_{13} \ll \frac{1}{x} &\left(\sum_{l \le y^2} \left| \sum_{\substack{n \le x \\ n \equiv v(l)}} g_3(n) - \frac{1}{\phi(l)} \sum_{\substack{n \le x \\ (n,l)=1}} g_3(n) \right| \right)^{1/2} \times \\
&\left(\sum_{\substack{d_j \le y \\ j=1,2}} |h_{1,r}(d_1)|^2 |h_{2,r}(d_2)|^2 \left| \sum_{\substack{n \le x \\ n \equiv v(d_1 d_2)}} g_3(n) - \frac{1}{\phi(d_1 d_2)} \sum_{\substack{n \le x \\ (n, d_1 d_2)=1}} g_3(n) \right| \right)^{1/2}
\end{aligned}
$$

$$\ll \frac{1}{(\log x)^{A/2}}\left(\sum_{\substack{d_j\le y\\ j=1,2}}\frac{|h_{1,r}(d_1)|^2|h_{2,r}(d_2)|^2}{\phi(d_1)\phi(d_2)}\right)^{1/2}.$$

Observe that

$$\sum_{d\le y}\frac{|h(d)|^2}{\phi(d)}\le \exp\left(c_7\sum_{p\le r}\frac{1}{p}\right)\le (\log r)^{c_8}.$$

Hence we have

$$U_{13}\ll \frac{(\log r)^{c_8}}{(\log x)^{A/2}}.$$

Now from (2.13), we get

$$U_{12}\ll \frac{y^{2(1-\alpha)}}{x}\sum_{\substack{d_j=1\\ j=1,2}}^{\infty}\frac{|h_{1,r}(d_1)h_{2,r}(d_2)|}{d_1^{1-\alpha}d_2^{1-\alpha}}\ll x^{-\frac{1}{2}(1+\alpha)}(\log x)^{-\beta(1-\alpha)}\exp\left(\frac{2c_2r^\alpha}{\log r}\right).$$

Using Lemma 2.3.1, there exist a real τ with $|\tau|\le(\log x)^{\frac{1}{19}}$ such that

$$\begin{aligned}P_3=&\frac{1}{x}\sum_{d_1\le y}\sum_{\substack{d_2\le y\\ (d_1,d_2)=1}}\frac{h_{1,r}(d_1)h_{2,r}(d_2)}{\phi(d_1d_2)}\theta_\tau(d_1d_2)\sum_{n\le x}g_3(n)\\ &+O\left(\sum_{\substack{d_j\le y\\ j=1,2\\ (d_1,d_2)=1}}\frac{|h_{1,r}(d_1)h_{2,r}(d_2)|}{\phi(d_1)\phi(d_2)}\frac{(\log\log 3d_1d_2)^2}{(\log x)^{1/19}}\right)=:P_{4\tau}+U_{14}.\end{aligned}$$

Now,

$$\begin{aligned}\sum_{\substack{d_1,d_2\le y\\ (d_1,d_2)=1}}\frac{h_{1,r}(d_1)h_{2,r}(d_2)}{\phi(d_1)\phi(d_2)}\theta_\tau(d_1)\theta_\tau(d_2)=&\sum_{\substack{d_1,d_2=1\\ (d_1,d_2)=1}}^{\infty}\frac{h_{1,r}(d_1)h_{2,r}(d_2)}{\phi(d_1)\phi(d_2)}\theta_\tau(d_1)\theta_\tau(d_2)\\ +O\left(\sum_{k=1}^{2}\sum_{\substack{d_1,d_2\le y\\ d_k>y}}\frac{h_{1,r}(d_1)h_{2,r}(d_2)}{\phi(d_1)\phi(d_2)}\theta_\tau(d_1)\theta_\tau(d_2)\right)&=:P_{5\tau}+U_{15}.\end{aligned}$$

Now from the following two estimates

$$\sum_{d>y}\frac{|h_{j,r}(d)\theta_\tau(d)|}{\phi(d)}\le y^{-\alpha}\exp\left(c_9\sum_{p\le r}\frac{1}{p^{1-\alpha}}\right)\ll x^{-\frac{1-\alpha}{4}}(\log x)^{\frac{\beta(1-\alpha)}{2}}\exp\left(c_{10}\frac{r^\alpha}{\log r}\right)$$

and

$$\sum_{d=1}^{\infty}\frac{|h_{j,r}(d)\theta_\tau(d)|}{\phi(d)}\le\exp\biggl(c_{11}\sum_{p\le r}\frac{1}{p}\biggr)\le(\log r)^{c_{12}}.$$

We deduce that

$$U_{15}\ll x^{-\frac{1-\alpha}{4}}(\log x)^{\frac{\beta(1-\alpha)}{2}}\exp\left(c_{13}\frac{r^\alpha}{\log r}\right)$$

,

$$\begin{aligned}P_{5\tau}&=\prod_{p\le r}\biggl(1+\sum_{m=1}^{\infty}\frac{(h_1(p^m)+h_2(p^m))\,\theta_\tau(p^m)}{\phi(p^m)}\biggr)\\&=\prod_{p\le r}\biggl(1-\frac{2\theta_\tau(p)}{p-1}+\theta_\tau(p)\sum_{m=1}^{\infty}\frac{g_1(p^m)+g_2(p^m)}{p^m}\biggr)=Q_\tau(r).\end{aligned}$$

Observe that

$$\begin{aligned}&\sum_{d=1}^{\infty}\frac{|h_{j,r}(d)|\log\log d}{\phi(d)}=\prod_{p\le r}\biggl(1+\sum_{\alpha=1}^{\infty}\frac{|h_{j,r}(p^\alpha)|\alpha\log\log p}{\phi(p^\alpha)}\biggr)\\&\ll\prod_{p\le r}\left(1+\frac{2p\log\log p}{(p-1)^2}\right)\ll\exp\biggl(c_{14}\sum_{p\le r}\frac{\log\log p}{p}\biggr)\ll\exp\left(c_{15}(\log\log r)^2\right).\end{aligned}$$

Hence

$$U_{14}\ll\frac{\exp\left(2c_{15}(\log\log r)^2\right)}{(\log x)^{1/19}}.$$

Using the same method as in estimation of T_2, we have

$$U_2\ll\sum_{j=1}^{2}\mathbb{D}(g_j,1;r;x+3-j)+(r\log r)^{-1/2}.$$

Combining these results, we get

$$\begin{aligned}R&\ll\left(x^{-\frac{\gamma}{4}}(\log x)^{\gamma\beta}+x^{\frac{\alpha-1}{4}}(\log x)^{\frac{\beta(1-\alpha)}{2}}\right)\exp\left(c_{16}\frac{r^\gamma}{\log r}\right)\\&+x^{1-2\alpha}\exp\left(c_{17}\frac{r^\alpha}{\log r}\right)+\frac{(\log r)^{c_8}}{(\log x)^{\frac{A}{2}}}+x^{-\frac{(1+\alpha)}{2}}(\log x)^{\beta(1-\alpha)}\exp\left(c_{18}\frac{r^\alpha}{\log r}\right)\\&+\frac{\exp\left(2c_{15}(\log\log r)^2\right)}{(\log x)^{1/19}}+\sum_{j=1}^{2}\mathbb{D}(g_j,1;r;x+3-j)+(r\log r)^{-\frac{1}{2}}.\end{aligned}$$

By choosing $\alpha = \gamma$, we get the required theorem.

We recall the following assumption related to 2-point Chowla type conjecture.

Assumption 2.3.1. *For every given $A > 0$,*

$$\sum_{n\le x} \mu(n+1)\mu(n)e(n\alpha) = O\left(\frac{x}{(\log x)^A}\right)$$

holds uniformly for all real α and implied constant depends on A.

Theorem 2.3.2. *Let g_1 be a multiplicative function such that $|g_1(n)| \le 1$ for all n and*

$$\sum_{p} \frac{|g_1(p)-1|^2}{p} < \infty. \tag{2.17}$$

Suppose that Assumption 2.3.1 holds. Then as $x \to \infty$,

$$M'_x(g_1,\mu,\mu) = \frac{1}{x}\sum_{n\le x} g_1(n+2)\mu(n+1)\mu(n) = o(1).$$

Proof. Set

$$T(r,x) = \prod_{r<p\le x}\left\{1-\frac{1}{p}+\left(1-\frac{1}{p}\right)\sum_{m=1}^{\infty}\frac{g_1(p^m)}{p^m}\right\},$$

where r will be chosen later. Now

$$\sum_{n\le x} g_1(n+2)\mu(n+1)\mu(n) = T(r,x)\sum_{n\le x} g_{1,r}(n+2)\mu(n+1)\mu(n)$$
$$+\sum_{n\le x} g_{1,r}(n+2)\mu(n+1)\mu(n)\left(g^*_{1,r}(n+2)-T(r,x)\right).$$

It is easy to see that $|T(r,x)| \le 1$. Therefore

$$\Big|\sum_{n\le x} g_1(n+2)\mu(n+1)\mu(n)\Big| \le \Big|\sum_{n\le x} g_{1,r}(n+2)\mu(n+1)\mu(n)\Big|$$
$$+\sum_{n\le x}\big|g_{1,r}^*(n+2)-T(r,x)\big| =: V_1+V_2.$$

Estimation of V_1. We have

$$\begin{aligned}V_1 &= \Big|\sum_{n\le x}\sum_{d|n+2} h_{1,r}(d)\mu(n+1)\mu(n)\Big|\\ &\le \Big|\sum_{d\le y} h_{1,r}(d)\sum_{\substack{n\le x\\ n\equiv -2(d)}}\mu(n+1)\mu(n)\Big|\\ &+\Big|\sum_{y<d\le x+2} h_{1,r}(d)\sum_{\substack{n\le x\\ n\equiv -2(d)}}\mu(n+1)\mu(n)\Big| =: V_{11}+V_{12},\end{aligned}$$

where $y := \log x$. Under Assumption 2.3.1, we have

$$\sum_{\substack{n\le x\\ n\equiv -2(d)}}\mu(n+1)\mu(n) = \sum_{n\le x}\mu(n+1)\mu(n)\frac{1}{d}\sum_{l=1}^{d} e\left(\frac{(n-2)l}{d}\right)$$
$$=\frac{1}{d}\sum_{l=1}^{d} e\left(\frac{-2l}{d}\right)\sum_{n\le x}\mu(n+1)\mu(n)e\left(\frac{nl}{d}\right) \ll \frac{x}{(\log x)^A}.$$

By choosing $A=3$, we get

$$V_{11} \ll \frac{x}{(\log x)^3}\sum_{d\le y}|h_{1,r}(d)| \ll \frac{xy^2}{(\log x)^3}\sum_{d=1}^{\infty}\frac{|h_{1,r}(d)|}{d^2} \ll \frac{x}{\log x}.$$

Now using (2.13), for $0<\alpha<1$, we have

$$\begin{aligned}V_{12} &\ll \sum_{y<d\le x+2}|h_{1,r}(d)|\left(\frac{x}{d}+1\right) \ll (x+2)\sum_{d>y}\frac{|h_{1,r}(d)|}{d}\\ &\ll \frac{(x+2)}{y^\alpha}\sum_{d=1}^{\infty}\frac{|h_{1,r}(d)|}{d^{1-\alpha}} \ll \frac{x}{y^\alpha}\exp\left(c_2\frac{r^\alpha}{\log r}\right).\end{aligned}$$

By taking $r = (\log\log x)^{\frac{1}{\alpha}}$, we have

$$V_{12} \ll xy^{-\alpha}y^{\frac{\alpha}{2}} = \frac{x}{(\log x)^{\frac{\alpha}{2}}}.$$

So as $x \to \infty$ we have, $V_1 = o(x)$. From a similar calculation as in the estimation of T_2, we have

$$V_2 \ll x(r\log r)^{-\frac{1}{2}} + x\left(\sum_{r<p\leq x+2} \frac{|g_1(p) - 1|^2}{p}\right)^{1/2}.$$

From (2.17) and $r = (\log\log x)^{\frac{1}{\alpha}}$ we have as $x \to \infty$, $V_2 = o(x)$ which proves the required theorem. $\square$

Chapter 3

Distribution of the sum of additive functions

In this chapter, we will discuss about the behaviour of distributions of the sum of additive functions. The result of this chapter is contained in [6].

In this chapter, we will discuss the behaviour of the distribution of the sum

$$f_1(F_1(n)) + f_2(F_2(n)) + f_3(F_3(n)), \tag{3.1}$$

where f_1, f_2 and f_3 are real-valued additive functions and $F_1(x), F_2(x), F_3(x)$ are special polynomials with integer coefficients.

The following theorem gives the behaviour of the distribution of the sum (3.1) when F_j's are polynomial of degree greater than or equal to 2, which is an application of Theorem 2.0.1.

Suppose that $g_j : \mathbb{N} \to \mathbb{U}$ and $h_j : \mathbb{N} \to \mathbb{C}$ be multiplication functions such that

$h_j = \mu * g_j$, $j = 1, 2, 3$. We recall that

$$P_1(\gamma) := \prod_{p \le \gamma} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \frac{h_1(p^{m_1}) h_2(p^{m_2}) h_3(p^{m_3})}{[p^{m_1}, p^{m_2}, p^{m_3}]} \varrho(p^{m_1}, p^{m_2}, p^{m_3}), \tag{3.2}$$

$$P_2(\gamma) := \prod_{p > \gamma} \left(1 + \sum_{m=1}^{\infty} \sum_{j=1}^{3} \frac{h_j(p^m) \varrho_j(p^m)}{p^m}\right). \tag{3.3}$$

Theorem 3.0.1. *Let $z, t \in \mathbb{R}$. Let f_1, f_2 and f_3 be real-valued additive functions and $F_j(n)$, $j = 1, 2, 3$ are as above of degree $v_j \ge 2$. Assume that*

$$\sum_{|f_j(p)| \le 1} \frac{f_j^2(p)}{p} \varrho_j(p) < \infty, j = 1, 2, 3, \tag{3.4}$$

$$\sum_{|f_j(p)| > 1} \frac{\varrho_j(p)}{p} < \infty, j = 1, 2, 3, \tag{3.5}$$

$$\sum_{j=1}^{3} \sum_{|f_j(p)| \le 1} \frac{f_j(p) \varrho_j(p)}{p} < \infty, \tag{3.6}$$

$$f_j(p^m) \varrho_j(p^m) \to 0, \tag{3.7}$$

for $m = 1$, when $v_j \ge 2$ and for $m = 1, 2, \cdots, v_j - 2$, when $v_j \ge 3$. Then the distribution function

$$G_x(z) := \frac{1}{x} \# \{n | n \le x, f_1(F_1(n)) + f_2(F_2(n)) + f_3(F_3(n)) \le z\} \tag{3.8}$$

converges weakly towards a limit distribution as $x \to \infty$, and there exist a natural number γ depending on polynomials $F_1(x), F_2(x)$ and $F_3(x)$ such that the characteristic function say $\phi(t)$ of this limit distribution is equal to $P_1(\gamma) P_2(\gamma)$, where $P_1(\gamma)$ and $P_2(\gamma)$ are defined by (3.2) *and* (3.3) *respectively with g_j is replaced by* $\exp(itf_j), j = 1, 2, 3$.

3.1 Proof of Theorem 3.0.1

Proof. We begin with some lemmas which are often used in probabilistic theory. The famous continuity theorem of Paul Lévy connects weak convergence of distribution functions to point-wise convergence of characteristic functions.

Lemma 3.1.1 ([40], Theroem 2.4)**.** *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of distribution functions and $\{\phi_n\}_{n=1}^{\infty}$ the sequence of their characteristic functions. Then F_n converges weakly to a distribution function F if, and only if, ϕ_n converges point-wise on $\mathbb{R}$ to a function ϕ which is continuous at 0. Furthermore, in this case, ϕ is the characteristic function of F, and the convergence of ϕ_n to ϕ is uniform on any compact subset.*

The continuity theorem immediately provides the following criterion.

Lemma 3.1.2 ([40], Theorem 2.6)**.** *Let f be a real arithmetic function. Then f possesses a distribution function F if, and only if, the sequence of functions*

$$\phi_N(\tau) := \frac{1}{N} \sum_{n \leq N} e^{i\tau f(n)}$$

converges point-wise on $\mathbb{R}$ to a function $\phi(\tau)$ which is continuous at 0. In this case, ϕ is the characteristic function of F.

Proof of Theorem 3.0.1. We will use Lemma 3.1.1 and Lemma 2.1.5 to prove this application. The characteristic functions of the distribution (3.8) equal

$$\phi_x(t) := \frac{1}{x} \sum_{n \leq x} \exp\left(it\left(f_1(F_1(n)) + f_2(F_2(n)) + f_3(F_3(n))\right)\right). \tag{3.9}$$

Since

$$\sum_p \sum_{j=1}^{3} \frac{\left(\exp\left(itf_j(p)\right)-1\right)\varrho_j(p)}{p} = it\sum_{j=1}^{3} \sum_{|f_j(p)|\le 1} \frac{f_j(p)\varrho_j(p)}{p}$$
$$+O\Big(t^2\sum_{j=1}^{3} \sum_{|f_j(p)|\le 1} \frac{f_j^2(p)\varrho_j(p)}{p}\Big)+O\Big(\sum_{j=1}^{3} \sum_{|f_j(p)|>1} \frac{\varrho_j(p)}{p}\Big)$$

then from the convergence of the series (3.4), (3.5), (3.6) and from Lemma 2.1.5 we deduce that $P_1(\gamma)$ and $P_2(\gamma)$ are convergent for every real t. Further, the infinite product $P_1(\gamma)P_2(\gamma)$ is continuous at $t=0$ because it converges uniformly for $|t|\le T$ where $T>0$ is arbitrary.

Since for $j=1,2,3$

$$\sum_p \frac{(1-\Re(\exp(itf_j(p))))\varrho_j(p)}{p} \ll t^2 \sum_{|f_j(p)|\le 1} \frac{|f_j(p)|^2\varrho_j(p)}{p} + \sum_{|f_j(p)|>1} \frac{\varrho_j(p)}{p},$$

then from the convergence of (3.4) and (3.5) it follows that $\mathbb{D}(\exp(itf_j),1;r;x)$ and $\mathbb{D}(\exp(itf_j),1;x;F_j(x))$ tends to zero when $r,x\to\infty$.
Now from (3.7) and Lemma 2.1.6, it is easy to see that
$(\exp\left(itf_j(p^m)-1\right)\varrho_j(p^m))\to 0$ when $p\to\infty, m<v_j, j=1,2,3$.
Then $\frac{1}{x}C(r,x)\to\ 0$ as $r,x\to\infty$. Choosing $r=\log x$ in our Theorem 2.0.1 we get that the remainder term disappears when $x\to\infty$.

Thus the characteristic function $\phi_x(t)$ has the limit $\phi(t)=P_1(\gamma)P_2(\gamma)$ for every real t and this limit is continuous at $t=0$.

Therefore by Lemma 3.1.1, the corollary is proved. □

Chapter 4

Selberg sieve over $\mathbb{F}_q[x]$

We begin this chapter with some background on Number Theory over Function Fields. In 1947 Selberg [29] introduced a new approach to sieving which is based on global optimization. In this chapter, we will discuss Selberg sieve for polynomial ring over finite fields and an application of it which will be used in Chapter 5.

4.1 Basic summation estimates over $\mathbb{F}_q[x]$

In this section, we present some useful estimates for polynomials over finite fields, which also appears in [8].

Lemma 4.1.1. *i) Let $q > 1$ and $\gamma > 0$. For $n \geq \frac{2\log(n/2)}{\log q}$, we have*

$$\sum_{m \leq n} q^m m^{-\gamma} = O(q^n n^{-\gamma}).$$

ii) We have

$$\sum_{\substack{P \in \mathcal{P}_q \\ \deg P \leq n}} q^{-\deg P} = \log n + c_1 + O(1/n),$$

where c_1 is an absolute constant.

iii) Also we have

$$\sum_{\substack{m\deg P\le n/2\\ m\ge 1}} q^{m\deg P} = O\left(\frac{q^n}{n}\right) \quad \textit{and} \quad \sum_{\substack{m\deg P\le n\\ m\ge 1}} q^{-(m+1)\deg P} = O(1).$$

Proof. i) $\sum_{m\le n} q^m m^{-\gamma} \ll q^{n/2}\sum_{m\le n/2} m^{-\gamma} + n^{-\gamma}\sum_{n/2<m\le n} q^m \ll q^n n^{-\gamma}.$

ii) Using (1.11), we have

$$\sum_{\deg P\le n} q^{-\deg P} = \sum_{m\le n} q^{-m}|\mathcal{P}_{m,q}| = \sum_{m\le n} q^{-m}\left(\frac{q^m}{m} + O\left(\frac{q^{m/2}}{m}\right)\right)$$
$$= \log n + c_1 + O(1/n).$$

iii) Using i) and (1.11) we have

$$\begin{aligned}\sum_{m\deg P\le n/2} q^{m\deg P} &= \sum_{\deg P\le n/2} q^{\deg P} + \sum_{\substack{m\deg P\le n/2\\ m\ge 2}} q^{m\deg P}\\ &= \sum_{m\le n/2} q^m|\mathcal{P}_{m,q}| + O\left(\frac{q^{3n/4}}{n}\right)\\ &= \sum_{m\le n/2}\frac{q^{2m}}{m} + O\left(\frac{q^{3n/4}}{n}\right) = O\left(\frac{q^n}{n}\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{\substack{m\deg P\le n\\ m\ge 1}} q^{-(m+1)\deg P} &= \sum_{\deg P\le n} q^{-2\deg P} + \sum_{\substack{m\deg P\le n\\ m\ge 2}} q^{-(m+1)\deg P}\\ &= \sum_{m\le n} q^{-2m}|P_{m,q}| + O(1) = O(1).\end{aligned}$$

□

4.2 Selberg sieve over $\mathbb{F}_q[x]$

The Selberg sieve provides majorants for cardinality of certain arithmetic sequences, such as the primes and the twin primes. In other words, it is a technique for estimating the size of "sifted sets" of positive integers which satisfy a set of congruence. In this section, we will extend Selberg sieve ([9], Lemma 2.1) to polynomials over finite fields and give an application of it which also appears in [8]. The main theorem of this section is as follows.

Theorem 4.2.1. *Let us consider the following set of polynomials*

$$\mathcal{A} = \{a_M \in \mathcal{M}_q : M \in \mathcal{M}_{n,q}\}.$$

Also let r and z be positive integers such that

$$\tilde{Q} = \prod_{\deg P \leq r} P \quad \textit{and} \quad \mathcal{D} = \{D : D|\tilde{Q}, \deg(D) \leq z\}.$$

Let Ψ be a real-valued non-negative arithmetic function on $\mathcal{M}_q$. Suppose that there exist a multiplicative function η supported on square-free polynomials with irreducible factors of degree atmost r satisfying

$$0 \leq \eta(P) < 1, \quad \textit{for all } P \in \mathcal{P}_q$$

such that for all $D \in \mathcal{D}$,

$$\sum_{\substack{M \in \mathcal{M}_{n,q} \\ a_M \equiv 0 (D)}} \Psi(M) = \eta(D)X + R_D(n), \tag{4.1}$$

where $X, R_D(n)$ are real numbers and $X \geq 0$. Now consider the following sum

$$S(n,\tilde{Q}) = \sum_{\substack{M\in\mathcal{M}_{n,q}\\ a_M\in\mathcal{A}\\ (a_M,\tilde{Q})=1}} \Psi(M).$$

If $h(D) = \frac{1}{\eta(D)}\prod_{P|D}(1-\eta(P))$, for $D \in \mathcal{D}$, then we have

$$S(n,\tilde{Q}) \leq X.L^{-1} + \sum_{\substack{D|\tilde{Q}\\ \deg(D)\leq 2z}} 3^{\omega(D)}|R_D(n)|,$$

where $L = \sum_{M\in\mathcal{D}} \frac{1}{h(M)}$.

Proof. Let λ_D be real numbers supported on monic polynomials D with $D \in \mathcal{D}$ such that $\lambda_1 = 1$. Since Ψ is non-negative arithmetic function, so we have

$$S(n,\tilde{Q}) \leq \sum_{\substack{M\in\mathcal{M}_{n,q}\\ a_M\in\mathcal{A}}} \Psi(M)\Big(\sum_{D|(a_M,\tilde{Q})} \lambda_D\Big)^2.$$

Using hypothesis (4.1), expanding the square and interchanging the order of summation, we have

$$\begin{aligned} S(n,\tilde{Q}) &\leq \sum_{D_1\in\mathcal{D}}\sum_{D_2\in\mathcal{D}} \lambda_{D_1}\lambda_{D_2} \sum_{\substack{M\in\mathcal{M}_{n,q}\\ a_M\equiv 0([D_1,D_2])}} \Psi(M) \\ &= X\sum_{D_1\in\mathcal{D}}\sum_{D_2\in\mathcal{D}} \lambda_{D_1}\lambda_{D_2}\eta([D_1,D_2]) + \sum_{D_1\in\mathcal{D}}\sum_{D_2\in\mathcal{D}} \lambda_{D_1}\lambda_{D_2}R_{[D_1,D_2]}(n) \\ &\leq X\sum_{D_1\in\mathcal{D}}\sum_{D_2\in\mathcal{D}} \lambda_{D_1}\lambda_{D_2}\eta([D_1,D_2]) + \sum_{D|\tilde{Q}} |\mu^{+}_{\mathbf{D}}(D)R_D(n)| \\ &=: X\Gamma + E, \end{aligned}$$

where

$$\mu^+_{\mathcal{D}}(D) = \sum_{\substack{D_1, D_2 \in \mathcal{D} \\ [D_1, D_2] = D}} \lambda_{D_1}\lambda_{D_2}, \quad \Gamma = \sum_{D_1 \in \mathcal{D}} \sum_{D_2 \in \mathcal{D}} \lambda_{D_1}\lambda_{D_2}\eta([D_1, D_2]),$$

$$E = \sum_{D | \tilde{Q}} |\mu^+_{\mathcal{D}}(D) R_D(n)|.$$

We need to choose the parameters λ_D to minimize Γ, consistent with the requirement that $\lambda_1 = 1$. We note that $h(D)$ is multiplicative on the divisors D of $\tilde{Q}$ and satisfies

$$\sum_{M|D} h(M) = \frac{1}{\eta(D)}.$$

Hence

$$\eta([D_1, D_2]) = \frac{\eta(D_1)\eta(D_2)}{\eta((D_1, D_2))} = \eta(D_1)\eta(D_2) \sum_{M|(D_1,D_2)} h(M)$$

and

$$\begin{aligned}\Gamma &= \sum_{D_1 \in \mathcal{D}} \sum_{D_2 \in \mathcal{D}} \lambda_{D_1}\lambda_{D_2}\eta(D_1)\eta(D_2) \sum_{M|(D_1,D_2)} h(M) \\ &= \sum_{M \in \mathcal{D}} h(M) \Big\{ \sum_{\substack{D \in \mathcal{D} \\ D \equiv 0 \pmod{M}}} \eta(D)\lambda_D \Big\}^2.\end{aligned}$$

We make the change of variable

$$y_M = \sum_{\substack{D \in \mathcal{D} \\ D \equiv 0 (M)}} \eta(D)\lambda_D \quad (M \in \mathcal{D})$$

in order to diagonalize Γ. By Möbius inversion formula [[23], Prop. 5.2], we have

$$\lambda_D \eta(D) = \sum_{\substack{M \in \mathcal{D} \\ M \equiv 0 \pmod{D}}} y_M \mu(M/D).$$

Also observe that, if $\lambda_1 = 1$ then

$$\sum_{M \in \mathcal{D}} \mu(M) y_M = 1. \tag{4.2}$$

So, we have

$$\Gamma = \sum_{M \in \mathcal{D}} h(M) \left(y_M - \frac{\mu(M)}{h(M)} L^{-1} \right)^2 + L^{-1}$$

where

$$L = \sum_{M \in \mathcal{D}} \frac{1}{h(M)}.$$

It is now clear that to minimize Γ we should choose the y_M as

$$y_M = \frac{\mu(M)}{h(M)} L^{-1} \tag{4.3}$$

which also satisfies (4.2). Therefore we obtain

$$S(n, \tilde{Q}) \leq XL^{-1} + E.$$

Using (4.3), we see that for any $D \in \mathcal{D}$,

$$\lambda_D = \frac{1}{\eta(D)} \sum_{\substack{M \in \mathcal{D} \\ M \equiv 0(D)}} y_M \mu(M/D) = \frac{1}{\eta(D)} \sum_{\substack{W \in \mathcal{D} \\ \deg(W) \leq z - \deg D \\ (W,D)=1}} \mu(W) y_{DW}$$

$$= \frac{\mu(D)}{Lh(D)\eta(D)} \sum_{\substack{W \in \mathcal{D} \\ \deg(W) \leq z - \deg D \\ (W,D)=1}} \frac{\mu(W)}{h(W)}.$$

Since $h(D)$ is multiplicative function $\tilde{Q}$, then we have

$$\sum_{M|D} \frac{1}{h(M)} = \frac{1}{h(D)\eta(D)}, \quad D \in \mathcal{D}$$

and $|\lambda_D| \leq 1$ for any $D \in \mathcal{D}$. This bound implies that for any $D|\tilde{Q}$,

$$|\mu_{\mathcal{D}}^{+}(D)| \leq \mathbb{1}_{\{\deg P \leq 2z\}} 3^{\omega(D)}.$$

Combining all of the above, we conclude the proof of the theorem. □

4.2.1 Application of Theorem 4.2.1

Now we give an application of the Theorem 4.2.1, which is useful to prove a variant of Turán-Kubilius inequality over function field in chapter 5.

Theorem 4.2.2 ([8], Lemma 6). *Given a modulus $M \in \mathbb{F}_q[x]$ of positive degree and a polynomial h co-prime to M, let $\pi_A(n; M, h)$ denotes the number of primes $P \equiv h \pmod{M}$, where $P \in \mathcal{P}_{n,q}$. Then we have*

$$\Theta(n) := \sum_{\substack{\frac{n}{2} < \deg Q \leq n \\ Q \in \mathcal{P}_q}} \Phi(Q) \pi_A^2(n; Q, h) \ll |\mathcal{P}_{n,q}|^2 \tag{4.4}$$

where h is a fixed polynomial with $\deg(h) < n$.

We start with the following lemma.

Lemma 4.2.1. *Let Δ be a polynomial in $\mathbb{F}_q[x]$. Then we have*

$$\sum_{\substack{M \in \mathcal{M}_{\leq n,q} \\ (M,\Delta)=1}} \frac{\mu^2(M) 3^{\omega(M)}}{|M|} \geq cn^3 \prod_{P|\Delta} \left(1 + \frac{3}{|P|}\right)^{-1},$$

where c is an absolute constant.

Proof. Let us consider

$$F(s) = \sum_{M} \frac{\mu^2(M) 3^{\omega(M)}}{|M|^{s+1}}, \quad \Re(s) > 0.$$

We can write

$$F(s)=\sum_{n=1}^{\infty}\frac{H(n)}{q^{ns}},\qquad \text{where}\qquad H(n)=\sum_{M\in\mathcal{M}_{n,q}}\frac{\mu^2(M)3^{\omega(M)}}{|M|}.$$

Putting $u=q^{-s}$, we define

$$\tilde{F}(u)=\sum_{n=1}^{\infty}H(n)u^n,\quad |u|<1.$$

On the other hand, from Euler product we get

$$\tilde{F}(u)=\frac{\tilde{G}(u)}{(1-u)^3},\qquad \text{where}\qquad \tilde{G}(u)=G(s)=\prod_{P}\left(1+\frac{3}{|P|^{s+1}}\right)\left(1-\frac{1}{|P|^{s+1}}\right)^3.$$

It is easy to see that $G(u)$ is bounded and therefore converges for $|u|<1$. Comparing the coefficient of $\tilde{F}(u)$ we have $H(n)\geq c_1n^2$, where $c_1>0$ is an absolute constant. Using this, we observe that

$$\prod_{P|\Delta}\left(1+\frac{3}{|P|}\right)\sum_{\substack{M\in\mathcal{M}_{\leq n,q}\\(M,\Delta)=1}}\frac{\mu^2(M)3^{\omega(M)}}{|M|}\geq\sum_{M\in\mathcal{M}_{\leq n,q}}\frac{\mu^2(M)3^{\omega(M)}}{|M|}=\sum_{m\leq n}H(m)\geq cn^3$$

which completes the proof of the lemma. □

Proof of Theorem 4.2.2. Expanding square of L.H.S of (4.4), we obtain

$$\Theta(n)=\sum_{\frac{n}{2}<\deg Q\leq n}\Phi(Q)\Bigg(\sum_{\substack{P\in\mathcal{P}_{n,q}\\P\equiv h(Q)}}1\Bigg)\times\Bigg(\sum_{\substack{P'\in\mathcal{P}_{n,q}\\P'\equiv h(Q)}}1\Bigg)=\sum_{\substack{A,B\in\mathcal{M}_{\leq\frac{n}{2},q}\\\deg A=\deg B}}S(A,B),$$

where

$$S(A,B):=\sum_{\substack{\frac{n}{2}<\deg Q\leq n\\AQ+h\in\mathcal{P}_{n,q}\\BQ+h\in\mathcal{P}_{n,q}}}\Phi(Q).$$

Now we have to find upper bound of the set $S(A,B)$. We define the following sets.

$$\mathcal{A} = \{a_M := M(AM+h)(BM+h) : \deg(M) = n - \deg(A)\}$$

and $\quad \wp_{\triangle} = \{P \in \mathcal{P}_q : \deg(P) < [n/2], P \nmid \triangle\}$,
where $\triangle := AB(Ah - Bh)$.

Let us define

$$\varrho(D) = \#\{M \pmod{D} : a_M \equiv 0 \pmod{D}\}$$

for monic polynomial $D \in \mathbb{F}_q[x]$. Also let

$$\tilde{Q} = \prod_{\substack{P \in \wp_{\triangle} \\ \deg P \le \frac{n}{2}}} P \quad \text{and} \quad \mathcal{D} = \{D : D|\tilde{Q}, \deg(D) \le \frac{n}{4}\}.$$

Observe that

$$S(A,B) \le \sum_{\substack{M \in \mathcal{M}_{n-\deg(A),q} \\ a_M \in \mathcal{A} \\ (a_M, \tilde{Q}) = 1}} |M|. \tag{4.5}$$

Now we are ready to apply Theorem 4.2.1 to the R.H.S of the inequality (4.5).

We see that $|M|$ is a real valued non-negative arithmetic function and

$$\sum_{\substack{M \in \mathcal{M}_{n-\deg(A),q} \\ a_M \equiv 0 (D)}} |M| = q^{n-\deg(A)} \sum_{\substack{M \in \mathcal{M}_{n-\deg(A),q} \\ a_M \equiv 0 (D)}} 1 = q^{2(n-\deg(A))} \eta(D)$$

where $\eta(D) = \frac{\varrho(D)}{|D|}$.

Therefore using Theorem 4.2.1, we have

$$S(A,B) \le XL^{-1},$$

where

$$X = q^{2n-2\deg(A)} \quad \text{and} \quad L = \sum_{M\in\mathcal{D}} \frac{1}{h(M)}.$$

Note that, for $D \in \mathcal{D}$

$$h(D) = \frac{1}{\eta(D)} \prod_{P|D} (1-\eta(P)).$$

Note that $\varrho(D) = 3^{\omega(D)}$ for all $D \in \mathcal{D}$. Therfore using Lemma 4.2.1, we obtain

$$L = \sum_{M\in\mathcal{D}} \frac{1}{h(M)} \geq \sum_{M\in\mathcal{D}} \eta(M) = \sum_{\substack{M\in\mathcal{M}_{\leq \frac{n}{4},q} \\ (M,\Delta)=1}} \frac{\mu^2(M)3^{\omega(M)}}{|M|} \geq cn^3 \prod_{P|\Delta} \left(1+\frac{3}{|P|}\right)^{-1}$$

where $c > 0$ is an absolute constant. Combining above results we get

$$S(A,B) \ll \frac{q^{2n-2\deg(A)}}{n^3} \prod_{P|\Delta} \left(1+\frac{3}{|P|}\right).$$

Therefore we have

$$\Theta(n) \ll \frac{q^{2n}}{n^3} \sum_{\substack{A,B\in\mathcal{M}_{\leq \frac{n}{2},q} \\ \deg A = \deg B}} q^{-2\deg A} \prod_{P|AB(A-B)h} \left(1+\frac{3}{|P|}\right).$$

Now we write

$$\begin{aligned}
&\sum_{\substack{A,B\in\mathcal{M}_{\leq \frac{n}{2},q} \\ \deg A = \deg B}} q^{-2\deg A} \prod_{P|AB(A-B)h} \left(1+\frac{3}{|P|}\right) \\
&= \sum_{\substack{A,B\in\mathcal{M}_{\leq \frac{n}{2},q} \\ \deg A = \deg B}} q^{-2\deg A} \prod_{P|A} \left(1+\frac{3}{|P|}\right) \prod_{P|B(A-B)} \left(1+\frac{3}{|P|}\right) \prod_{P|h} \left(1+\frac{3}{|P|}\right).
\end{aligned}$$

Since h is a fixed polynomial then we have

$$\prod_{P|h} \left(1+\frac{3}{|P|}\right) \ll 1$$

with constant depending on q and h. So it is enough to consider the following sum:

$$\sum_{\substack{A,B\in\mathcal{M}_{\le\frac{n}{2},q}\\ \deg A=\deg B}} q^{-2\deg A}\prod_{P|A}\left(1+\frac{3}{|P|}\right)\prod_{P|B(A-B)}\left(1+\frac{3}{|P|}\right)$$
$$=\sum_{A\in\mathcal{M}_{\le\frac{n}{2},q}} q^{-2\deg A}\prod_{P|A}\left(1+\frac{3}{|P|}\right)\sum_{\substack{B\\ \deg B=\deg A}}\prod_{P|B(A-B)}\left(1+\frac{3}{|P|}\right).$$

Now inner sum becomes

$$\sum_{\substack{B\\ \deg B=\deg A}}\prod_{P|B(A-B)}\left(1+\frac{3}{|P|}\right)\le\sum_{\substack{B\\ \deg B=\deg A}}\prod_{P|B}\left(1+\frac{3}{|P|}\right)\prod_{P|(A-B)}\left(1+\frac{3}{|P|}\right)$$
$$=\sum_{\substack{B\\ \deg B=\deg A}}\sum_{D_1|B}\frac{\mu^2(D_1)3^{\omega(D_1)}}{|D_1|}\sum_{D_2|A-B}\frac{\mu^2(D_2)3^{\omega(D_2)}}{|D_2|}$$
$$=\sum_{D_1,D_2}\frac{\mu^2(D_1)\mu^2(D_2)3^{\omega(D_1)}3^{\omega(D_2)}}{|D_1D_2|}\sum_{\substack{B\in\mathcal{M}_{\deg A,q}\\ B\equiv 0(D_1)\\ B\equiv A(D_2)}}1.$$

We observe that $(D_1,D_2)|A$. Let $D=(D_1,D_2)$ and $D_i=DF_i$. Here $(F_i,D)=1,(F_1,F_2)=1$ and $\omega(D_i)=\omega(D)+\omega(F_i)$ for all $i=1,2$.

So we have

$$\sum_{\substack{B\\ \deg B=\deg A}}\prod_{P|B(A-B)}\left(1+\frac{3}{|P|}\right)$$
$$\le\sum_{D|A}\frac{\mu^2(D)3^{\omega(D)}}{|D|^2}\sum_{\substack{F_i\in\mathcal{M}_{\le\deg A-\deg D,q}\\ (F_1,F_2)=1\\ (F_i,D)=1}}\frac{\mu^2(F_1)\mu^2(F_2)3^{\omega(F_1)+\omega(F_2)}}{|F_1F_2|}\sum_{\substack{B'\in\mathcal{M}_{\deg A-\deg D,q}\\ B'\equiv 0(F_1)\\ B'\equiv\frac{A}{D}(F_2)}}1$$
$$=\sum_{D|A}\frac{\mu^2(D)3^{\omega(2D)}}{|D|^2}\sum_{\substack{F_i\in\mathcal{M}_{\le\deg A-\deg D,q}\\ (F_1,F_2)=1\\ (F_i,D)=1}}\frac{\mu^2(F_1)\mu^2(F_2)3^{\omega(F_1)+\omega(F_2)}}{|F_1F_2|}\left(\frac{q^{\deg A-\deg D}}{|F_1F_2|}+O(1)\right)$$

$$= q^{\deg A} \sum_{D|A} \frac{\mu^2(D) 3^{\omega(2D)}}{|D|^3} \sum_{\substack{F_i \in \mathcal{M}_{\leq \deg A - \deg D, q} \\ (F_1, F_2) = 1 \\ (F_i, D) = 1}} \frac{\mu^2(F_1)\mu^2(F_2) 3^{\omega(F_1)+\omega(F_2)}}{|F_1 F_2|^2}$$

$$+ O\left(\sum_{D|A} \frac{\mu^2(D) 3^{\omega(2D)}}{|D|^2} \sum_{\substack{F_i \in \mathcal{M}_{\leq \deg A - \deg D, q} \\ (F_1, F_2) = 1 \\ (F_i, D) = 1}} \frac{\mu^2(F_1)\mu^2(F_2) 3^{\omega(F_1)+\omega(F_2)}}{|F_1 F_2|} \right)$$

$$\ll q^{\deg A}.$$

Hence we obtain

$$\Theta(n) \ll \frac{q^{2n}}{n^3} \sum_{A \in \mathcal{M}_{\frac{n}{2}, q}} q^{-degA} \prod_{P|A} \left(1 + \frac{3}{|P|}\right)$$

$$= \frac{q^{2n}}{n^3} \sum_{A \in \mathcal{M}_{\frac{n}{2}, q}} \sum_{D|A} \frac{\mu^2(D) 3^{\omega(D)}}{|D|} \ll \frac{q^{2n}}{n^2}$$

which completes proof of the lemma.

Chapter 5

Correlation of multiplicative functions over $\mathbb{F}_q[x]$

In this chapter, we will study the mean value of correlation of multiplicative functions over function field in large degree limit. More precisely, for multiplicative functions $\psi_1, \psi_2 : \mathcal{M}_q \to \mathbb{U}$, our aim is to study the asymptotic behaviour of the following correlation functions for a fixed q and $n \to \infty$:

$$S_2(n,q) := \sum_{f \in \mathcal{M}_{n,q}} \psi_1(f+h_1)\psi_2(f+h_2) \tag{5.1}$$

and

$$R_2(n,q) := \sum_{P \in \mathcal{P}_{n,q}} \psi_1(P+h_1)\psi_2(P+h_2), \tag{5.2}$$

where h_1, h_2 are fixed polynomials of degree $< n$ over $\mathbb{F}_q$. The results of this chapter are contained in [8].

Let $\psi_j : \mathcal{M}_q \to \mathbb{U}$ and $\alpha_j : \mathcal{M}_q \to \mathbb{C}$ be multiplicative functions such that $\alpha_j = \mu * \psi_j$ for all $j = 1, 2$. For fixed polynomials $h_j \in \mathbb{F}_q[x]$ with $\deg(h_j) < n$ for all $j = 1, 2$ and $n \geq r$, we define

$$(5.3) \qquad Q(n) := \prod_{\deg P \leq n} \upsilon_P \quad \text{and} \quad Q(r,n) = \prod_{r < \deg P \leq n} \upsilon_P.$$

where

$$\upsilon_P := \sum_{\substack{m_1=0 \\ (P^{m_1}, P^{m_2}) | (h_2 - h_1)}}^{\infty} \sum_{m_2=0}^{\infty} \frac{\alpha_1(P^{m_1}) \alpha_2(P^{m_2})}{q^{\deg([p^{m_1}, P^{m_2}])}}.$$

In [8], we investigate the asymptotic behaviour of the above sums (5.1) and (5.2) for $k = 2$, i.e. $S_2(n,q)$ and $R_2(n,q)$ in large degree limit. The following theorem gives the asymptotic behaviour of $S_2(n,q)$ with explicit error term in large degree limit.

Theorem 5.0.1. *Let ψ_1 and ψ_2 be multiplicative functions on $\mathcal{M}_q$ with modulus less than or equal to 1. Suppose that ψ_1 and ψ_2 are close to 1 and $\gamma := \deg(h_2 - h_1) \geq \left\lceil \frac{\log 9}{\log q} \right\rceil$. Then there exists a positive absolute constant c such that for all $n \geq r \geq \gamma$ and for all $\frac{1}{2} < \alpha < 1$, we have*

$$\frac{S_2(n,q)}{q^n} - Q(n) \ll \mathbb{D}(\psi_1, 1; r, n) + \mathbb{D}(\psi_2, 1; r, n) + q^{(1-2\alpha)n} \exp\left(\frac{cq^{\alpha r}}{r}\right) + (rq^r)^{-\frac{1}{2}}$$

where $Q(n)$ is defined by (5.3).

Remark 5.0.1. *Note that γ is fixed here since the polynomials h_1 and h_2 are fixed. Also we write*

$$Q(n) = Q_1(\gamma) Q_2(\gamma, n)$$

where

$$(5.4) \qquad Q_1(\gamma) = \prod_{\deg P \leq \gamma} \upsilon_P,$$

$$(5.5) \qquad Q_2(\gamma, n) = \prod_{\gamma < \deg P \leq n} \left(1 + \sum_{j=1}^{2} \sum_{m=1}^{\infty} \frac{\psi_j(P^m) - \psi_j(P^{m-1})}{q^{m \deg P}}\right).$$

5.1 Proof of Theorem 5.0.1

The following lemma is a version of standard Chinese remainder theorem over function fields.

Lemma 5.1.1. *Let, $h_1, h_2, g_1, g_2 \in \mathbb{F}_q[x]$. The congruence system*

$$f + h_j \equiv 0 \pmod{g_j} \quad j = 1, 2$$

has a solution if and only if $(g_1, g_2)|(h_2 - h_1)$. If the solution exists, it is unique modulo $[g_1, g_2]$.

We recall the following lemma from Chapter 4 which collects some useful estimation over function field.

Lemma 5.1.2. *We have the following:*

a) Let $q > 1$ and $\gamma > 0$. Then we obtain

$$\sum_{m \le n} q^m m^{-\gamma} = O(q^n n^{-\gamma}).$$

b) We have

$$\sum_{\substack{P \in \mathcal{P}_q \\ \deg P \le n}} q^{-\deg P} = \log n + c_1 + O(1/n)$$

where c_1 is a absolute constant.

c) $$\sum_{\substack{m \deg P \le n/2 \\ m \ge 1}} q^{m \deg P} = O\left(\frac{q^n}{n}\right) \quad \text{and} \quad \sum_{\substack{m \deg P \le n \\ m \ge 1}} q^{-(m+1)\deg P} = O(1).$$

The following lemma is a version of Turán-Kubilius inequality over function field in large degree limit.

Lemma 5.1.3. *For a sequence of complex numbers $\psi(P^m)$, supported on powers P^m of irreducible polynomials P and $m \geq 1$, we have*

$$\sum_{f \in \mathcal{M}_{n,q}} \left| \sum_{P^m \| f+h} \psi(P^m) - \sum_{m \deg P \leq n} \frac{\psi(P^m)}{q^{m \deg P}} \left(1 - q^{-\deg P}\right) \right|^2 \ll q^n \sum_{m \deg P \leq n} \frac{|\psi(P^m)|^2}{q^{m \deg P}}$$

where h is some fixed polynomial with $\deg(h) < n$.

Proof. First we assume that $\psi(P^m) = 0$ for all irreducible polynomials P with $m \deg(P) > \frac{n}{2}$. By simplifying square of modulus on left hand side of the above inequality, the coefficient of $\psi(P^m)\overline{\psi(Q^r)}$, where P and Q are distinct irreducible polynomials, is

$$\sum_{f \in \mathcal{M}_{n,q}} \sum_{P^m, Q^r \| f+h} 1 - \sum_{f \in \mathcal{M}_{n,q}} \sum_{\substack{P^m \| f+h \\ r \deg Q \leq n}} \frac{1 - q^{-\deg Q}}{q^{r \deg Q}} - \sum_{f \in \mathcal{M}_{n,q}} \sum_{\substack{Q^r \| f+h \\ m \deg P \leq n}} \frac{1 - q^{-\deg P}}{q^{m \deg P}}$$
$$+ \sum_{f \in \mathcal{M}_{n,q}} \sum_{\substack{m \deg P \leq n \\ r \deg Q \leq n}} \frac{\left(1 - q^{-\deg P}\right)\left(1 - q^{-\deg Q}\right)}{q^{m \deg P + r \deg Q}}.$$

Observe that

$$\sum_{f \in \mathcal{M}_{n,q}} \sum_{P^m, Q^r \| f+h} 1 = q^n \sum_{\substack{m \deg P \leq n \\ r \deg Q \leq n}} \frac{\left(1 - q^{-\deg P}\right)\left(1 - q^{-\deg Q}\right)}{q^{m \deg P + r \deg Q}}.$$

By treating all three other sums analogously, we find that the coefficient of $\psi(P^m)\overline{\psi(Q^r)}$ is zero if $P \neq Q$. Therefore the coefficients of only diagonal terms will have non-zero coefficients. Using Lemma 5.1.2 and Cauchy-Schwarz inequality the rest of the sum gives

$$\sum_{f \in \mathcal{M}_{n,q}} \sum_{\deg P \leq n/2} \left| \psi\left(P^{v_P(f+h)}\right) - \sum_{m \deg P \leq n} \frac{\psi(P^m)}{q^{m \deg P}} \left(1 - q^{-\deg P}\right) \right|^2$$

$$
\ll \sum_{f\in\mathcal{M}_{n,q}} \sum_{\deg P\le n/2} \left|\psi\left(P^{v_P(f+h)}\right)\right|^2 + \sum_{f\in\mathcal{M}_{n,q}} \sum_{\substack{m;m\deg P\le n\\ \deg P\le n/2}} \left|\frac{\psi(P^m)}{q^{m\deg P}}\left(1-q^{-\deg P}\right)\right|^2
$$

$$
\ll q^n \sum_{m\deg P\le n} \frac{|\psi(P^m)|^2}{q^{m\deg P}} + q^n\left(\sum_{\substack{m;m\deg P\le n\\ \deg P\le n/2}} \frac{|\psi(P^m)|^2}{q^{m\deg P}} \sum_{\substack{m;m\deg P\le n\\ \deg P\le n/2}} \frac{1}{q^{m\deg P}}\left(1-q^{-\deg P}\right)^2\right)
$$

$$
\ll q^n \sum_{m\deg P\le n} \frac{|\psi(P^m)|^2}{q^{m\deg P}},
$$

where $v_P(f)$ is the highest power of P dividing f.

Now we will assume that $\psi(P^m) = 0$ for all monic irreducible polynomials P with $m\deg P \le n/2$. Note that $f+h \in \mathcal{M}_{n,q}$. Therefore, if $f \in \mathcal{M}_{n,q}$ and $\psi(f+h) \neq 0$ then there exist at most one irreducible monic polynomial power $P^m \| f+h$ and $\psi(P^m) \neq 0$. So by using Cauchy-Schwarz inequality, we have

$$
\sum_{f\in\mathcal{M}_{n,q}} \left|\sum_{P^m\|f+h} \psi(P^m) - \sum_{m\deg P\le n} \frac{\psi(P^m)}{q^{m\deg P}}\left(1-q^{-\deg P}\right)\right|^2 \ll q^n \sum_{m\deg P\le n} \frac{|\psi(P^m)|^2}{q^{m\deg P}}.
$$

Finally we write ψ as $\psi_1+\psi_2$, where $\psi_1(P^m) = 0$ for all monic irreducible polynomials with $m\deg P > n/2$ and $\psi_2(P^m) = 0$ with $m\deg P \le n/2$ and combining above calculation we get the required result. □

As a direct consequence of Lemma 5.1.3, using Lemma 5.1.2 and Cauchy-Schwarz inequality twice, we get the following version of Turán-Kubilius inequality over function field.

Lemma 5.1.4. *For a sequence of complex numbers $\{\psi(P^m) : P \in \mathcal{P}_q, m \ge 1\}$, we have*

$$
\sum_{f\in\mathcal{M}_{n,q}} \left|\sum_{P^m\|f+h} \psi(P^m) - \sum_{m\deg P\le n} \frac{\psi(P^m)}{q^{m\deg P}}\right| \ll q^n\left(\sum_{m\deg P\le n} \frac{|\psi(P^m)|^2}{q^{m\deg P}}\right)^{1/2}
$$

where h is some fixed polynomial with $\deg h < n$.

Proof of Theorem 5.0.1. For $r \geq 1$ and $j = 1, 2$, we define multiplicative functions $\psi_{j,r}$ and $\psi^*_{j,r}$, by

$$\psi_{j,r}(P^m) = \begin{cases} \psi_j(P^m) & \text{if } \deg P \leq r \\ 1 & \text{if } \deg P > r \end{cases} \quad \text{and} \quad \psi^*_{j,r}(P^m) = \begin{cases} 1 & \text{if } \deg P \leq r \\ \psi_j(p^m) & \text{if } \deg P > r \end{cases}$$

and multiplicative function $\alpha_{j,r}$ by

$$\alpha_{j,r}(P^m) = \begin{cases} \psi_j(P^m) - \psi_j(P^{m-1}) & \text{if } \deg P \leq r \\ 0 & \text{if } \deg P > r \end{cases}$$

so that $\psi_{j,r} = 1 * \alpha_{j,r}$, $j = 1, 2$.

We write

$$\frac{S_2(n,q)}{q^n} - Q(n) = Q(r,n)\Big(\frac{1}{q^n} \sum_{f \in \mathcal{M}_{n,q}} \psi_{1,r}(f+h_1)\psi_{2,r}(f+h_2) - Q(r)\Big)$$
$$+ \frac{1}{q^n} \sum_{f \in \mathcal{M}_{n,q}} \psi_{1,r}(f+h_1)\psi_{2,r}(f+h_2)\Big(\psi^*_{1,r}(f+h_1)\psi^*_{2,r}(f+h_2) - Q(r,n)\Big).$$

We observe that

$$\Big|\sum_{m=1}^{\infty} \frac{\psi_j(P^m) - \psi_j(P^{m-1})}{q^{m \deg P}}\Big| \leq \sum_{m=1}^{\infty} \frac{2}{q^{m \deg P}} \leq \frac{1}{4}, \quad \text{if } \deg P \geq \frac{\log 9}{\log q}.$$

Using Hypothesis that both ψ_1 and ψ_2 are close to 1 and using Lemma 5.1.2, if $r \geq \lceil \frac{\log 9}{\log q} \rceil$, it is easy to see that $Q(r,n) \ll 1$. Therefore, we have

$$\frac{S_2(n,q)}{q^n} - Q(n) \ll \Big|\frac{1}{q^n} \sum_{f \in \mathcal{M}_{n,q}} \psi_{1,r}(f+h_1)\psi_{2,r}(f+h_2) - Q(r)\Big|$$
$$+ \frac{1}{q^n} \sum_{f \in \mathcal{M}_{n,q}} \Big|\psi^*_{1,r}(f+h_1)\psi^*_{2,r}(f+h_2) - Q(r,n)\Big|. \tag{5.6}$$

Now we see that

$$\sum_{f\in\mathcal{M}_{n,q}}\psi_{1,r}(f+h_1)\psi_{2,r}(f+h_2)=\sum_{f\in\mathcal{M}_{n,q}}\sum_{g_1|f+h_1}\alpha_{1,r}(g_1)\sum_{g_2|f+h_2}\alpha_{2,r}(g_2)$$
$$=\sum_{\substack{g_1\in\mathcal{M}_{\le n,q}\\ g_2\in\mathcal{M}_{\le n,q}}}\alpha_{1,r}(g_1)\alpha_{2,r}(g_2)\sum_{\substack{f\in\mathcal{M}_{n,q}\\ g_1|f+h_1\\ g_2|f+h_2}}1.$$

By using Lemma 5.1.1, we have

$$\Big|\{f\in\mathcal{M}_{n,q}: f\equiv -h_1 \pmod{g_1}, f\equiv -h_2 \pmod{g_2}\}\Big|=\frac{q^n}{|[g_1,g_2]|}+O(1)$$

whenever $(g_1,g_2)|(h_2-h_1)$.

Therefore we obtain

$$\sum_{f\in\mathcal{M}_{n,q}}\psi_{1,r}(f+h_1)\psi_{2,r}(f+h_2)=q^n\sum_{\substack{g_j\in\mathcal{M}_{\le n,q}\forall j\\ (g_1,g_2)|(h_2-h_1)}}\frac{\alpha_{1,r}(g_1)\alpha_{2,r}(g_2)}{|[g_1,g_2]|}+$$
$$+O\Bigg(\sum_{\substack{g_j\in\mathcal{M}_{\le n,q}\\ \forall j=1,2}}|\alpha_{1,r}(g_1)\alpha_{2,r}(g_2)|\Bigg)=:M_1+\mathcal{E}_1.$$

Now we have

$$M_1=q^n\sum_{\substack{g_j\in\mathbb{F}_q[x]\forall j\\ (g_1,g_2)|(h_2-h_1)}}\frac{\alpha_{1,r}(g_1)\alpha_{2,r}(g_2)}{|[g_1,g_2]|}+O\Bigg(q^n\sum_{\deg(g_1)>n}\sum_{g_2\in\mathbb{F}_q[x]}\frac{|\alpha_{1,r}(g_1)\alpha_{2,r}(g_2)|}{|[g_1,g_2]|}\Bigg)$$
$$=q^nQ(r)+\mathcal{E}_2.$$

Since $(g_1,g_2)|(h_2-h_1)$ and (h_2-h_1) is a fixed polynomial we have $|(g_1,g_2)|\ll 1$ with constant depending on q,h_1 and h_2. By writing $[g_1,g_2]=\frac{g_1g_2}{(g_1,g_2)}$ we get

$$\mathcal{E}_2\ll q^n\sum_{\deg(g_1)>n}\frac{|\alpha_{1r}(g_1)|}{|g_1|}\sum_{g_2\in\mathbb{F}_q[x]}\frac{|\alpha_{2r}(g_2)|}{|g_2|}.$$

Using Lemma 5.1.2 (b), we observe that

$$\sum_{g\in\mathbb{F}_q[x]} \frac{|\alpha_{j,r}(g)|}{|g|} = \prod_{\deg P\le r}\left(1+\sum_{k=1}^{\infty}\frac{|\alpha_{j,r}(P^k)|}{q^{k\deg P}}\right) \le \prod_{\deg P\le r}\left(1+\frac{2}{q^{\deg P}-1}\right)$$
$$\ll \exp\left(c\sum_{\deg P\le r} q^{-\deg P}\right) \ll r^{c_1}$$

for some constant $c, c_1 > 0$. For $0<\alpha<1$, using Lemma 5.1.2 (a), we have

$$\sum_{\deg(g)>n} \frac{|\alpha_{j,r}(g)|}{|g|} \le \frac{1}{q^{n\alpha}}\sum_{g\in\mathbb{F}_q[x]}\frac{|\alpha_{j,r}(g)|}{q^{(1-\alpha)\deg(g)}} \ll \frac{1}{q^{n\alpha}}\exp\left(c_2\sum_{\deg P\le r}\frac{1}{q^{(1-\alpha)\deg P}}\right)$$
$$\ll \frac{1}{q^{n\alpha}}\exp\left(c_3\sum_{m\le r}\frac{q^{m\alpha}}{m}\right) \ll \frac{1}{q^{n\alpha}}\exp\left(c_4\frac{q^{r\alpha}}{r}\right)$$

for absolutely constant $c_3>0$ and $c_4>0$. Using these estimates, we get

$$\mathcal{E}_1 \ll q^{(2-2\alpha)n}\exp\left(c\frac{q^{r\alpha}}{r}\right) \quad \text{and} \quad \mathcal{E}_2 \ll q^{(1-\alpha)n}\exp\left(c\frac{q^{r\alpha}}{r}\right).$$

Therefore finally we have to calculate the following sum

$$\mathcal{E}_3 := \sum_{f\in\mathcal{M}_{n,q}}\Big|\psi^*_{1,r}(f+h_1)\psi^*_{2,r}(f+h_2)-Q(r,n)\Big|.$$

For a fixed $f\in\mathcal{M}_q$ and $k=1,2$, we define

$$\mathcal{P}_f(k) := \Big\{P : P^m\| f+h_k \text{ and } |1-\psi_k(P^m)| > \frac{1}{2}\Big\}.$$

Now we consider the following set

$$\mathcal{N}_r = \Big\{f\in\mathcal{M}_{n,q} : \exists k\in\{1,2\} \text{ and } \exists P\in\mathcal{P}_f(k) \text{ with } \deg P > r\Big\}.$$

On the basis of the set $\mathcal{N}_r$ we decompose $\mathcal{E}_3$ into

$$\mathcal{E}_3 = \sum_{f \in \mathcal{N}_r} \Big| \psi_{1,r}^*(f+h_1)\psi_{2,r}^*(f+h_2) - Q(r,n) \Big| + \sum_{f \notin \mathcal{N}_r} \Big| \psi_{1,r}^*(f+h_1)\psi_{2,r}^*(f+h_2) - Q(r,n) \Big|$$
$$=: \mathcal{E}_4 + \mathcal{E}_5.$$

Using Lemma 5.1.2, we see that

$$\begin{aligned}
\mathcal{E}_4 &\ll \sum_{j=1}^{2} \sum_{\substack{f \in \mathcal{M}_{n,q} \\ P^m \| f+h_j \\ |1-\psi_j(P^m)|>1/2 \\ \deg P > r}} 1 \ll q^n \sum_{j=1}^{2} \sum_{\substack{m \deg P \le n \\ |1-\psi_j(P^m)|>1/2 \\ \deg P > r}} \frac{1}{q^{m \deg P}} \\
&\ll q^n \sum_{j=1}^{2} \sum_{r < \deg P \le n} \frac{|1-\psi_j(P)|}{q^{\deg P}} + q^n \sum_{\deg P > r} q^{-2 \deg P} \\
&\ll q^n (\mathbb{D}(\psi_1, 1; r, n) + \mathbb{D}(\psi_2, 1; r, n)) + \frac{q^{n-r}}{r}.
\end{aligned}$$

We recall that if $\Re(u) \le 0, \Re(v) \le 0$, then

$$|\exp(u) - \exp(v)| \le |u - v| \quad \text{and} \tag{5.7}$$

$$\log(1+z) = z + O(|z|^2), \quad \text{if } |z| \le 1, |\arg(z)| \le \frac{\pi}{2}. \tag{5.8}$$

Note that

$$\log Q(r,n) = \sum_{r < \deg P \le n} \log \left(1 + \sum_{j=1}^{2} \sum_{m=1}^{\infty} \frac{\psi_j(P^m) - \psi_j(P^{m-1})}{q^{m \deg P}} \right).$$

Using (5.8), we get

$$\log \psi_{j,r}^*(f+h_j) = \sum_{\substack{P^m \| f+h_j \\ \deg P > r}} (\psi_j(P^m) - 1) + O\Bigg(\sum_{\substack{P^m \| f+h_j \\ \deg P > r}} |\psi_j(P^m) - 1|^2 \Bigg).$$

Therefore again using (5.7) and (5.8), we have

$$\mathcal{E}_5 \ll \sum_{j=1}^{2} \sum_{f\in\mathcal{M}_{n,q}} \Bigg| \sum_{\substack{P^m \| f+h_j \\ \deg P>r}} (\psi_j(P^m)-1) - \sum_{\substack{m\deg P\le n \\ \deg P>r}} \frac{\psi_j(P^m)-1}{q^{m\deg P}} \Bigg|$$
$$+ \sum_{f\in\mathcal{M}_{n,q}} \Bigg| \sum_{j=1}^{2} \sum_{\substack{m\deg P\le n \\ \deg P>r}} \frac{\psi_j(P^m)-1}{q^{m\deg P}} - \log Q(r,n) \Bigg| + O\Bigg(\sum_{f\in\mathcal{M}_{n,q}} \sum_{\substack{P^m \| f+h_j \\ \deg P>r}} |\psi_j(P^m)-1|^2 \Bigg)$$
$$=: \mathcal{E}_6 + \mathcal{E}_7 + \mathcal{E}_8.$$

Now we obtain

$$\mathcal{E}_8 \ll q^n \sum_{j=1}^{2} \sum_{\substack{m\deg P\le n \\ m\ge 1;\deg P>r}} \frac{|\psi_j(P^m)-1|^2}{q^{m\deg P}} \ll q^n \sum_{j=1}^{2} \sum_{r<\deg P\le n} \frac{|\psi_j(P)-1|^2}{q^{\deg P}} + \frac{q^{n-r}}{r}$$
$$\ll q^n \left(\mathbb{D}^2(\psi_1,1;r,n) + \mathbb{D}^2(\psi_2,1;r,n) \right) + \frac{q^{n-r}}{r},$$
$$\mathcal{E}_7 = q^n \Bigg| \sum_{j=1}^{2} \sum_{r<\deg P\le n} \frac{\psi_j(P)-1}{q^{\deg P}} + O\Bigg(\sum_{\deg P>r} q^{-2\deg P} \Bigg) - \sum_{j=1}^{2} \sum_{r<\deg P\le n} \frac{\psi_j(P)-1}{q^{\deg P}} \Bigg|$$
$$\ll q^n \sum_{\deg P>r} q^{-\deg P} \ll \frac{q^{n-r}}{r}.$$

Using Lemma 5.1.4, we have

$$\mathcal{E}_6 \ll q^n \Bigg(\sum_{j=1}^{2} \sum_{\substack{m\deg P\le n \\ m\ge 1;\deg P>r}} \frac{|\psi_j(P^m)-1|^2}{q^{m\deg P}} \Bigg)^{1/2} \ll q^n \sum_{j=1}^{2} \mathbb{D}(\psi_j,1;r,n) + \frac{q^n}{(rq^r)^{\frac{1}{2}}}.$$

Combining the above estimates, we get the theorem.

The following theorem gives an asymptotic formula of the sum (5.2) in large degree limit (when q is fixed and $n \to \infty$). For fixed polynomials $h_1, h_2 \in \mathbb{F}_q[x]$ with $\deg(h_j) < n$ for all $j = 1, 2$ and $n \ge r$, we define

$$Q^{'}(n) := \prod_{\deg P\le n} \upsilon'_P \quad \text{and} \quad Q'(r,n) = \prod_{r<\deg P\le n} \upsilon'_P \tag{5.9}$$

where

$$\upsilon'_P := \sum_{\substack{m_1=0 \\ (P^{m_1},P^{m_2})|(h_2-h_1)}}^{\infty} \sum_{m_2=0}^{\infty} \frac{\alpha_1(P^{m_1})\alpha_2(P^{m_2})}{\Phi[P^{m_1}, P^{m_2}]}.$$

Theorem 5.1.1. *Let ψ_1 and ψ_2 be multiplicative functions on $\mathcal{M}_q$ with modulus less than or equal to 1. Suppose that ψ_1 and ψ_2 are close to 1 and $\gamma := \deg(h_2 - h_1) \geq \lceil \frac{\log 17}{\log q} \rceil$. Then there exists a positive absolute constant c such that for all $n \geq r \geq \gamma$ and for all $\frac{1}{2} < \alpha < 1$, we have*

$$\frac{R_2(n,q)}{|\mathcal{P}_{n,q}|} - Q'(n) \ll \mathbb{D}(\psi_1, 1; r, n) + \mathbb{D}(\psi_2, 1; r, n) + n^{-A} \exp\left(\frac{cq^{\alpha r}}{r}\right) + (rq^r)^{-\frac{1}{2}}$$

*where $A > 0$ is arbitrary constant and $Q'(n)$ is as defined in (**??**).*

Remark 5.1.1. *Note that γ is fixed here since the polynomials h_1 and h_2 are fixed. Also we write*

$$Q'(n) = Q'_1(\gamma)Q'_2(\gamma, n)$$

where

$$Q'_1(\gamma) = \prod_{\deg P \leq \gamma} \upsilon'_P, \tag{5.10}$$

$$Q'_2(\gamma, n) = \prod_{\gamma < \deg P \leq n} \left(1 + \sum_{j=1}^{2} \sum_{m=1}^{\infty} \frac{\psi_j(P^m) - \psi_j(P^{m-1})}{\Phi(P^m)}\right). \tag{5.11}$$

We use this decomposition of $Q'(n)$ in the applications.

5.2 Proof of Theorem 5.1.1

Recall that for a given modulus $M \in \mathbb{F}_q[x]$ of positive degree and a polynomial B coprime to M, let $\pi_A(n; M, B)$ denotes the number of primes $P \equiv B \pmod{M}$, where $P \in \mathcal{P}_{n,q}$. The prime polynomial theorem for arithmetic progression says that

$$\pi_A(n; M, B) = \frac{q^n}{n\Phi(M)} + O\left(\frac{q^{n/2}}{n}\right). \tag{5.12}$$

As in classical case, we want to allow $\deg(M)$ to grow with n. The interesting range of parameter is $\deg(M) < n$ because if $\deg(M) \geq n$ there is at most one monic prime polynomial in arithmetic progression $h \equiv B \pmod{M}$ of degree n. From (5.12) we see that if $n/2 \leq \deg(M) < n$ then error term becomes larger than main term. Therefore, we must assume that $\deg(M) < n/2$.

The following lemma is analog of Brun-Titchmarsh inequality over function field which is a special case of a theorem of Chin-Nung Hsu.

Lemma 5.2.1 ([18], Theorem 4.3)**.** *Let $\pi_A(n; M, B)$ be defines as above and $\Phi(M)$ denotes the number of coprime residues modulo M. Then for $\deg(M) < n$, we have*

$$\pi_A(n; M, B) \leq \frac{2q^n}{\Phi(M)(n - deg(M) + 1)}.$$

Remark 5.2.1. *The inequality in Lemma 5.2.1 is stronger than* (5.12) *if $n/2 < \deg(M) < n$.*

We recall the following lemma from Chapter 4 which is used in next lemma to prove shifted version of Turán-Kubilius inequality over irreducible polynomials.

Lemma 5.2.2. *Using the above notations, we have*

$$\Theta(n) := \sum_{\substack{\frac{n}{2} < \deg Q \leq n \\ Q \in \mathcal{P}_q}} \Phi(Q)\pi_A^2(n; Q, -h) \ll |\mathcal{P}_{n,q}|^2,$$

where h is a fixed polynomial with $\deg(h) < n$.

Lemma 5.2.3. *Let h be a fixed polynomial with $\deg h < n$. For a sequence of* *complex numbers* $\{\psi(P^m) : P \in \mathcal{P}_q, m \geq 1\}$*, we have*

$$\sum_{P\in\mathcal{P}_{n,q}} \Big| \sum_{Q^k \| P+h} \psi(Q^k) - A(n) \Big| \ll |\mathcal{P}_{n,q}| \times B(n)$$

where $A(n) := \sum_{\substack{Q\in\mathcal{P}_q \\ k\deg Q\leq n}} \frac{\psi(Q^k)}{\Phi(Q^k)}\Big(1 - \frac{1}{q^{\deg(Q)}}\Big)$ *and* $B^2(n) := \sum_{\substack{Q\in\mathcal{P}_q \\ k\deg Q\leq n}} \frac{|\psi(Q^k)|^2}{\Phi(Q^k)}$.

Proof. Using triangle inequality, we have

$$\sum_{P\in\mathcal{P}_{n,q}} \Big| \sum_{Q^k\|P+h} \psi(Q^k) - A(n)\Big| \leq \sum_{P\in\mathcal{P}_{n,q}} \Big| \sum_{\substack{Q^k\|P+h \\ k\deg(Q)\leq m}} \psi(Q^k) - A(m)\Big|$$
$$+ \sum_{P\in\mathcal{P}_{n,q}} \Big| \sum_{\substack{Q^k\|P+h \\ k\deg(Q)\leq m}} \psi(Q^k)\Big| + \sum_{P\in\mathcal{P}_{n,q}} |A(n) - A(m)| =: L_1 + L_2 + L_3,$$

where $m < n$ will be chosen later. Using Cauchy-Schwarz inequality and (1.11), we get

$$L_1 \leq \Big(\sum_{P\in\mathcal{P}_{n,q}} 1\Big)^{\frac{1}{2}} \Big(\sum_{P\in\mathcal{P}_{n,q}} \Big|\sum_{Q^k\|P+h} \psi(Q^k) - A(m)\Big|^2\Big)^{\frac{1}{2}} \leq \frac{q^{\frac{n}{2}}}{n^{\frac{1}{2}}} \times L_4^{\frac{1}{2}},$$

where

$$L_4 := \sum_{P\in\mathcal{P}_{n,q}} \Big|\sum_{Q^k\|P+h} \psi(Q^k) - A(m)\Big|^2.$$

Note that

$$\sum_{\substack{P\in\mathcal{P}_{n,q} \\ Q^k\|P+h}} 1 = \pi_A(n, Q^k, -h) - \pi_A(n, Q^{k+1}, -h)$$

and

$$\sum_{\substack{P\in\mathcal{P}_{n,q} \\ Q_1^{k_1}, Q_2^{k_2}\|P+h}} 1 = \pi_A(n, Q_1^{k_1}Q_2^{k_2}, -h) - \pi_A(n, Q_1^{k_1+1}Q_2^{K_2}, -h)$$
$$- \pi_A(n, Q_1^{k_1}Q_2^{k_2+1}, -h) + \pi_A(n, Q_1^{k_1+1}Q_2^{k_2+1}, -h).$$

Using these estimates, (5.12) and by simplifying square of modulus of L_4, we observe that

$$L_4 = \frac{q^n}{n} \sum_{k \deg Q \le m} \frac{|\psi(Q^k)|^2}{\Phi(Q^k)} \left(1 - \frac{1}{q^{\deg Q}}\right)\left(1 - \frac{1}{q^{k \deg Q}}\right)$$
$$+ O\left(\frac{q^{\frac{n}{2}}}{n} \log m \sum_{k \deg Q \le m} \frac{|\psi(Q^k)|^2}{\Phi(Q^k)} + \frac{q^{\frac{n}{2}+m} (\log m)^{\frac{1}{2}}}{n m^{\frac{1}{2}}} \sum_{k \deg Q \le m} \frac{|\psi(Q^k)|^2}{\Phi(Q^k)}\right).$$

By choosing $m = \frac{n}{2}$, we have $L_4 \ll \frac{q^n}{n} \times B^2(n)$. Using Lemma 5.2.1, Lemma 5.2.2 and Cauchy-Schwarz inequality, we have

$$L_2 = \sum_{P \in \mathcal{P}_{n,q}} \left| \sum_{\substack{Q^k \| P+h \\ \frac{n}{2} < k \deg Q \le n}} \psi(Q^k) \right| \le \sum_{\frac{n}{2} < k \deg Q \le n} |\psi(Q^k)| \pi_A(n; Q^k, -h)$$
$$\ll \left(\sum_{\substack{\frac{n}{2} < k \deg Q \le n \\ k \ge 1}} \frac{|\psi(Q^k)|^2}{\Phi(Q^k)} \right)^{\frac{1}{2}} \times \left(\sum_{\substack{\frac{n}{2} < k \deg Q \le n \\ k \ge 1}} \Phi(Q^k) \pi_A^2(n; Q^k, -h) \right)^{\frac{1}{2}}$$
$$\ll B(n) \times (\Theta(n))^{\frac{1}{2}} + B(n) \times q^n \left(\sum_{\substack{\frac{n}{2} < k \deg Q \le n \\ k \ge 2}} \frac{1}{\Phi(Q^k)} \right)^{\frac{1}{2}}$$
$$\ll B(n) \times (\Theta(n))^{\frac{1}{2}} + B(n) \times |\mathcal{P}_{n,q}| \times \frac{n^{\frac{1}{2}}}{q^{n/8}} \ll |\mathcal{P}_{n,q}| \times B(n)$$

and for $m = \frac{n}{2}$ we get

$$L_3 = \sum_{P \in \mathcal{P}_{n,q}} |A(n) - A(m)| \ll |\mathcal{P}_{n,q}| \sum_{\frac{n}{2} < k \deg Q \le n} \frac{|\psi(Q^k)|}{\Phi(Q^k)} \ll |\mathcal{P}_{n,q}| \times B(n)$$

where $\Theta(n)$ is defined as in Lemma 5.2.2. This completes the proof of the lemma. □

Proof of Theorem 5.1.1. We write

$$\frac{R_2(n,q)}{|\mathcal{P}_{n,q}|} - Q'(n) = Q'(r,n)\left(\frac{1}{|\mathcal{P}_{n,q}|} \sum_{P \in \mathcal{P}_{n,q}} \psi_{1,r}(P+h_1)\psi_{2,r}(P+h_2) - Q'(r)\right)$$
$$+ \frac{1}{|\mathcal{P}_{n,q}|} \sum_{P \in \mathcal{P}_{n,q}} \psi_{1,r}(P+h_1)\psi_{2,r}(P+h_2)\left(\psi_{1,r}^*(P+h_1)\psi_{2,r}^*(P+h_2) - Q'(r,n)\right).$$

where $\psi_{j,r}$ and $\psi^*_{j,r}, j = 1, 2$ are defined as in the proof of the Theorem 5.0.1.

Observe that $Q'(r, n) \ll 1$. Therefore we have

$$\Big|\frac{R_2(n,q)}{|\mathcal{P}_{n,q}|} - Q'(n)\Big| \ll \Big|\frac{1}{|\mathcal{P}_{n,q}|}\sum_{P\in\mathcal{P}_{n,q}} \psi_{1,r}(P+h_1)\psi_{2,r}(P+h_2) - Q'(r)\Big| + \frac{1}{|\mathcal{P}_{n,q}|}\sum_{P\in\mathcal{P}_{n,q}} \big|\psi^*_{1,r}(P+h_1)\psi^*_{2,r}(P+h_2) - Q'(r,n)\big| =: \mathcal{E}_9 + \mathcal{E}_{10} \tag{5.13}$$

Using Lemma 5.1.1, we have

$$\sum_{P\in\mathcal{P}_{n,q}} \psi_{1,r}(P+h_1)\psi_{2,r}(P+h_2) = \sum_{P\in\mathcal{P}_{n,q}} \sum_{g_1|P+h_1} \alpha_{1,r}(g_1) \sum_{g_2|P+h_2} \alpha_{2,r}(g_2)$$
$$= \sideset{}{'}\sum_{\substack{g_j\in\mathcal{M}_{\le n,q}\\ j=1,2}} \alpha_{1,r}(g_1)\alpha_{2,r}(g_2)\pi_A(n;[g_1,g_2],M)$$
$$= \sideset{}{'}\sum_{\substack{g_j\in\mathcal{M}_{\le z,q}\\ j=1,2}} \alpha_{1,r}(g_1)\alpha_{2,r}(g_{)}\Big(\pi_A(n;[g_1,g_2],M) - \frac{q^n}{n\Phi([g_1,g_2])}\Big) + \frac{q^n}{n} \sideset{}{'}\sum_{\substack{g_j\in\mathcal{M}_{\le z,q}\\ j=1,2}} \frac{\alpha_{1,r}(g_1)\alpha_{2,r}(g_2)}{\Phi([g_1,g_2])}$$
$$+ O\Bigg(\sum_{\substack{g_1\in\mathcal{M}_{\le n,q}\\ z<\deg(g_2)\le n}} |\alpha_{1,r}(g_1)\alpha_{2,r}(g_2)|\pi_A(n;[g_1,g_2],M)\Bigg)$$

where M is the monic polynomial for which $M \equiv -h_j \pmod{g_j}, j = 1, 2$ and $0 \le \deg(M) < \deg([g_1, g_2])$, and $\sum'$ means the summation over g_1, g_2 satisfying $(g_1, g_2)|(h_2 - h_1)$, $\alpha_{j,r}, j = 1, 2$ are defined as in the proof of the Theorem 5.0.1 and $r \le z < n$ will be chosen later.

Therefore we have

$$\mathcal{E}_9 \le \frac{1}{|\mathcal{P}_{n,q}|} \sideset{}{'}\sum_{\substack{g_j\in\mathcal{M}_{\le z,q}\\ j=1,2}} |\alpha_{1,r}(g_1)\alpha_{2,r}(g_2)|\Big|\pi_A(n;[g_1,g_2],M) - \frac{q^n}{n\Phi([g_1,g_2])}\Big|$$
$$+ O\Bigg(\frac{1}{|\mathcal{P}_{n,q}|} \sum_{\substack{g_1\in\mathcal{M}_{\le n,q}\\ z<\deg(g_2)\le n}} |\alpha_{1,r}(g_1)\alpha_{2,r}(g_2)|\pi_A(n;[g_1,g_2],M)\Bigg)$$

$$+ O\Bigg(\sum_{\substack{g_1 \in \mathbb{F}_q[x] \\ \deg(g_2) > z}} \frac{|\alpha_{1,r}(g_1)\alpha_{2,r}(g_2)|}{\Phi([g_1, g_2])} \Bigg) =: \mathcal{E}_{11} + \mathcal{E}_{12} + \mathcal{E}_{13}.$$

Using Lemma 5.1.2, for $0 < \alpha < 1$, we have

$$\sum_{\deg(f) > z} \frac{|\alpha_{j,r}(f)|}{\Phi(f)} \leq \frac{1}{q^{\alpha z}} \sum_{f \in \mathbb{F}_q[x]} \frac{|\alpha_{j,r}(f)| q^{\alpha \deg(f)}}{\Phi(f)}$$
$$\leq \frac{1}{q^{\alpha z}} \prod_{\deg(P) \leq r} \Big(1 + \frac{2q^{\alpha \deg P}}{(q^{\deg P} - 1)(1 - q^{(\alpha - 1)\deg P})}\Big) \leq \frac{1}{q^{\alpha z}} \exp\Big(c_1 \sum_{\deg P \leq r} \frac{1}{q^{(1-\alpha)\deg P}}\Big)$$
$$\leq q^{-\alpha z} \exp\Big(c_2 \frac{q^{\alpha r}}{r}\Big).$$

Similarly for $0 < \alpha < 1$, we get

$$\sum_{f \in \mathbb{F}_q[x]} \frac{|\alpha_{j,r}(f)|}{q^{\alpha \deg(f)}} \leq \prod_{\deg P \leq r} \Big(1 + \sum_{t=1}^{\infty} \frac{|\alpha_{j,r}(P^t)|}{q^{t\alpha \deg P}}\Big) \leq \exp\Big(c_3 \frac{q^{(1-\alpha)r}}{r}\Big).$$

Using these estimates and (5.12), we have

$$\mathcal{E}_{11} \leq \frac{q^{2z\alpha}}{|\mathcal{P}_{n,q}|} \max_{\substack{[g_1,g_2] \in \mathcal{M}_{\leq z,q} \\ ([g_1,g_2],M)=1}} \Big|\pi_A(n; [g_1, g_2], M) - \frac{q^n}{n\Phi([g_1, g_2])}\Big| \times \sum_{\substack{g_j \in \mathbb{F}_q[x] \\ j=1,2}} \frac{|\alpha_{1,r}(g_1)\alpha_{2,r}(g_2)|}{|g_1|^\alpha |g_2|^\alpha}$$
$$\leq \frac{q^{2z\alpha}}{|\mathcal{P}_{n,q}|} \max_{\substack{[g_1,g_2] \in \mathcal{M}_{\leq z,q} \\ ([g_1,g_2],M)=1}} \Big|\pi_A(n; [g_1, g_2], M) - \frac{q^n}{n\Phi([g_1, g_2])}\Big| \times \exp\Big(2c_3 \frac{q^{r(1-\alpha)}}{r}\Big)$$
$$\ll \frac{q^{2z\alpha}}{|\mathcal{P}_{n,q}|} \times \frac{q^{\frac{n}{2}}}{n} \times \exp\Big(2c_3 \frac{q^{r(1-\alpha)}}{r}\Big),$$

and

$$\mathcal{E}_{13} \ll \frac{1}{|\mathcal{P}_{n,q}|} \sum_{\substack{g_1 \in \mathcal{M}_{\leq n,q} \\ z < \deg(g_2) \leq n}} |\alpha_{1,r}(g_1)\alpha_{2,r}(g_2)| \Big(\frac{q^n}{|[g_1, g_2]|} + 1\Big)$$
$$\ll \frac{q^n}{|\mathcal{P}_{n,q}|} q^{-\alpha z} r^{c_1} \exp\Big(c_4 \frac{q^{\alpha r}}{r}\Big) + \frac{q^{2n\alpha}}{|\mathcal{P}_{n,q}|} \times \exp\Big(c_5 \frac{q^{r(1-\alpha)}}{r}\Big)$$

and also we have

$$\mathcal{E}_{12} \ll q^{-\alpha z} r^{c_1} \times \exp\Big(c_6 \frac{q^{\alpha r}}{r}\Big) \ll q^{-\alpha z} \exp\Big(c_7 \frac{q^{\alpha r}}{r}\Big).$$

For a fixed $P \in \mathcal{P}_q$ and $k = 1, 2$, we define

$$\mathcal{P}_P(k) := \Big\{ Q : Q^m \| P + h_k \text{ and } |1 - \psi_k(Q^m)| > \frac{1}{2} \Big\}.$$

Now we consider the following set

$$\mathcal{Q}_r = \Big\{ P \in \mathcal{P}_{n,q} : \exists k \in \{1, 2\} \text{ and } \exists Q \in \mathcal{P}_P(k) \text{ with } \deg Q > r \Big\}.$$

Therefore we write

$$\begin{aligned}\mathcal{E}_{10} &= \frac{1}{|\mathcal{P}_{n,q}|} \sum_{P \in \mathcal{Q}_r} \big| \psi^*_{1,r}(P + h_1)\psi^*_{2,r}(P + h_2) - Q'(r, n) \big| \\ &\quad + \frac{1}{|\mathcal{P}_{n,q}|} \sum_{P \notin \mathcal{Q}_r} \big| \psi^*_{1,r}(P + h_1)\psi^*_{2,r}(P + h_2) - Q'(r, n) \big| \\ &=: \frac{1}{|\mathcal{P}_{n,q}|} (\mathcal{E}_{14} + \mathcal{E}_{15}).\end{aligned}$$

We see that

$$\begin{aligned}\mathcal{E}_{14} &\ll \sum_{j=1}^{2} \sideset{}{^*}\sum_{\substack{P \in \mathcal{Q}_r \\ Q^k \| P + h_j}} 1 = \sum_{j=1}^{2} \sideset{}{^*}\sum_{\substack{k \deg Q \leq n \\ k \geq 1}} \pi_A(n; Q^k, -h_j) = \sum_{j=1}^{2} \sideset{}{^*}\sum_{\deg Q \leq n} \pi_A(n; Q, -h_j) \\ &\quad + \sum_{j=1}^{2} \sideset{}{^*}\sum_{\substack{k \deg Q \leq n \\ k \geq 2}} \pi_A(n; Q^k, -h_j) =: \mathcal{E}_{16} + \mathcal{E}_{17},\end{aligned}$$

where $\sum^*$ means the summation varies over conditions in the definition of $\mathcal{Q}_r$.

Using Lemma 5.2.1 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathcal{E}_{16} &\ll \frac{q^n}{n} \sideset{}{^*}\sum_{r<\deg Q\le \frac{n}{2}} \frac{1}{\Phi(Q)} + \Bigg(\sideset{}{^*}\sum_{\frac{n}{2}<\deg Q\le n} \frac{1}{\Phi(Q)} \Bigg)^{\frac{1}{2}} \Bigg(\sideset{}{^*}\sum_{\frac{n}{2}<\deg Q\le n} \Phi(Q)\pi_A^2(n;Q,-h_j) \Bigg)^{\frac{1}{2}} \\
&\ll \frac{q^n}{n} \sum_{r<\deg Q\le \frac{n}{2}} \frac{|1-\psi_j(Q)|^2}{\Phi(Q)} + \Bigg(\sum_{\frac{n}{2}<\deg Q\le n} \frac{|1-\psi_j(Q)|^2}{\Phi(Q)} \Bigg)^{\frac{1}{2}} \times (\Theta(n))^{\frac{1}{2}} \\
&\ll |\mathcal{P}_{n,q}| \times \sum_{j=1}^{2} \mathbb{D}^2(\psi_j, 1; r, \frac{n}{2}) + (\Theta(n))^{\frac{1}{2}} \times \sum_{j=1}^{2} \mathbb{D}(\psi_j, 1; \frac{n}{2}, n).
\end{aligned}$$

Using Lemma 5.1.2 and Lemma 5.2.1, we get

$$\begin{aligned}
\mathcal{E}_{17} &\ll \frac{q^n}{n} \sum_{\substack{k\deg Q\le \frac{n}{2} \\ \deg Q>r; k\ge 2}} \frac{1}{\Phi(Q^k)} + q^n \sum_{\substack{\frac{n}{2}<k\deg Q\le n \\ k\ge 2}} \frac{1}{\Phi(Q^k)} \\
&\ll |\mathcal{P}_{n,q}| \times \frac{1}{rq^r} + |\mathcal{P}_{n,q}| \times \frac{1}{q^{\frac{n}{4}}}.
\end{aligned}$$

Using Lemma 5.2.2 and combining these estimates, we obtain

$$\mathcal{E}_{14} \ll |\mathcal{P}_{n,q}| \times \sum_{j=1}^{2} \mathbb{D}(\psi_j, 1; r, n) + \frac{|\mathcal{P}_{n,q}|}{rq^r} + |\mathcal{P}_{n,q}| \times \frac{1}{q^{\frac{n}{4}}}.$$

Observe that

$$Q'(r,n) = \prod_{r<\deg P\le n} \Bigg(1 - \frac{2}{\Phi(P)} + \sum_{k=1}^{\infty} \frac{\psi_1(P^k)+\psi_2(P^k)}{q^{k\deg P}} \Bigg). \tag{5.14}$$

Using (5.7) and (5.8), we have

$$\begin{aligned}
\big|\psi_{1,r}^*(P+h_1)\psi_{2,r}^*(P+h_2) - Q'(r,n)\big| \le \Bigg| \sum_{j=1}^{2} \sum_{\substack{Q^k\|P+h_j \\ \deg Q>r}} \big(\psi_j(Q^k)-1\big) - \log Q'(r,n) \Bigg| \\
+ O\Bigg(\sum_{j=1}^{2} \sum_{\substack{Q^k\|P+h_j \\ \deg Q>r}} \big|\psi_j(Q^k)-1\big|^2 \Bigg).
\end{aligned}$$

Therefore we get

$$\begin{aligned}\mathcal{E}_{10} \le & \sum_{P\in\mathcal{Q}_r}\sum_{j=1}^{2}\Bigg|\sum_{\substack{Q^k\| P+h_j\\ \deg Q>r}}\big(\psi_j(Q^k)-1\big)-\sum_{\substack{k\deg Q\le n\\ \deg Q>r}}\frac{\psi_j(Q^k)-1}{\Phi(Q^k)}\Bigg|\\ &+\sum_{P\in\mathcal{Q}_r}\Bigg|\sum_{\substack{k\deg Q\le n\\ \deg Q>r}}\frac{\psi_1(Q^k)+\psi_2(Q^k)-2}{\Phi(Q^k)}-\log Q'(r,n)\Bigg|\\ &+O\Bigg(\sum_{P\in\mathcal{Q}_r}\sum_{\substack{Q^k\| P+h_j\\ \deg Q>r}}\big|\psi_j(Q^k)-1\big|^2\Bigg)=:\mathcal{E}_{18}+\mathcal{E}_{19}+\mathcal{E}_{20}.\end{aligned}$$

From (5.14), we see that

$$\begin{aligned}\mathcal{E}_{19}=|\mathcal{P}_{n,q}|\times&\Bigg|\sum_{j=1}^{2}\sum_{r<\deg Q\le n}\frac{\psi_j(Q)-1}{\Phi(Q)}-\sum_{r<\deg Q\le n}\Big(-\frac{2}{\Phi(Q)}+\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\psi_j(Q^k)}{q^{k\deg Q}}\Big)\Bigg|\\ +O\Bigg(|\mathcal{P}_{n,q}|\sum_{\deg Q>r}q^{-2\deg Q}\Bigg)&\ll O\Bigg(|\mathcal{P}_{n,q}|\sum_{\deg Q>r}q^{-2\deg Q}\Bigg)\ll|\mathcal{P}_{n,q}|\times\frac{1}{rq^r}.\end{aligned}$$

Similar to estimation of $\mathcal{E}_{14}$, we get

$$\mathcal{E}_{20}\ll|\mathcal{P}_{n,q}|\times\sum_{j=1}^{2}\mathbb{D}(\psi_j,1;r,n)+\frac{|\mathcal{P}_{n,q}|}{rq^r}+|\mathcal{P}_{n,q}|\times\frac{1}{q^{\frac{n}{4}}}.$$

Using Lemma 5.2.3, we have

$$\mathcal{E}_{18}\ll|\mathcal{P}_{n,q}|\times\Bigg(\sum_{\substack{k\deg Q\le n\\ k\ge1;\deg Q>r}}\frac{|\psi_j(Q^k)-1|^2}{\Phi(Q^k)}\Bigg)^{\frac12}\ll|\mathcal{P}_{n,q}|\times\sum_{j=1}^{2}\mathbb{D}(\psi_j,1;r,n)+\frac{|\mathcal{P}_{n,q}|}{(rq^r)^{\frac12}}.$$

Choosing $z=A\log_q n$, $A>0$ and combining all these estimates, we get the required theorem.

Now we recall the truncated Liouville function over function field by

$$\lambda_y(P^t)=\begin{cases}(-1)^t\quad(=\lambda(P^t)) & \text{if } \deg P\le y\\ 1 & \text{if } \deg P>y.\end{cases}$$

For very small choice of y the following theorem gives a truncated variant of Chowla's conjecture in large degree limit, which is an application of Theorem 5.0.1.

Theorem 5.2.1. *There is a positive absolute constant C such that if $n \geq 2$, $2 \leq y \leq \log n$ and fixed $h \in \mathbb{F}_q[x]$ with $\deg(h) \leq y$, then*

$$\left| \sum_{f \in \mathcal{M}_{n,q}} \lambda_y(f)\lambda_y(f+h) \right| < C \frac{\log^4 y}{y^4} q^n.$$

5.3 Proof of Theorem 5.2.1

Proof. We start with the following lemma. The following lemma gives an upper bound of a product over those irreducible polynomials which divides a certain fixed polynomial.

Lemma 5.3.1 ([2], Lemma 2.2)**.** *Let $f \in \mathbb{F}_q[x]$. Then we have*

$$\prod_{P|f} \left(1 + \frac{1}{|P|}\right) = O(\log(\deg(f))).$$

Proof of Theorem 5.2.1. Choose $r = y$ and $\psi_j = \lambda_y$, $j = 1, 2$. Let $\alpha_j = \mu * \lambda_y$, $j = 1, 2$.

Observe that $\mathbb{D}\left(\lambda_y(P), 1; r, n\right) = 0$ and

$$\alpha_j(P^t) = \begin{cases} 2(-1)^t & \text{if } \deg P \leq y \\ 0 & \text{if } \deg P > y. \end{cases}$$

Using Theorem 5.0.1, we have

$$\frac{1}{q^n} \left| \sum_{f \in \mathcal{M}_{n,q}} \lambda_y(f)\lambda_y(f+h) \right| \leq |Q(n)| + O\left((yq^y)^{-\frac{1}{2}} + q^{(1-2\alpha)n} \exp\left(\frac{cq^{\alpha y}}{y}\right) \right).$$

where $Q(n) = Q_1(y)Q_2(y,n)$ is defined by (5.4).

Since $\deg(h) \leq y$ then we obtain

$$Q_1(y) = \prod_{\deg P \leq y} \sum_{\substack{m_1=0 \\ (P^{m_1},P^{m_2})|h}}^{\infty} \sum_{m_2=0}^{\infty} \frac{\alpha_1(P^{m_1})\alpha_2(P^{m_2})}{q^{[m_1,m_2]\deg P}} =: \prod_{\deg P \leq y} W_P.$$

Now if $\deg P > y$, then we have $Q_2(y,n) = 1$. Also note that $\alpha_1 = \alpha_2 = \alpha_3$ (say). We define the non-negative integer $k(P)$ such that $P^{k(P)} \| h$. So for $\deg P \leq y$, we get

$$W_P = \sum_{m=0}^{k(P)} \frac{\alpha_3(P^m)^2}{q^{m \deg P}} + 2\sum_{m=0}^{k(P)} \alpha_3(P^m) \sum_{l=m+1}^{\infty} \frac{\alpha_3(P^l)}{q^{l \deg P}} \left(1 + 4\sum_{m=0}^{k(P)} \frac{1}{q^{m \deg P}}\right) + 4\sum_{l=1}^{\infty} \frac{(-1)^l}{q^{l \deg P}}$$
$$+ 8\sum_{m=1}^{k(P)} \frac{(-1)^{2m+1}}{q^{(m+1)\deg P}} \sum_{j=0}^{\infty} \frac{(-1)^j}{q^{j \deg P}} = 1 - \frac{4}{q^{k(P)\deg P}\,(q^{\deg P}+1)}.$$

Finally using Lemma 5.1.2 and Lemma 5.3.1 and the hypothesis that $\deg(h) \leq y$, we have

$$Q(n) = \prod_{\deg P \leq y} \left(1 - \frac{4}{q^{\deg P}+1}\right) \prod_{\substack{\deg P \leq y \\ P|h}} \left(1 - \frac{4}{q^{k(P)\deg P}\,(q^{\deg P}+1)}\right)\left(1 - \frac{4}{q^{\deg P}+1}\right)^{-1}$$
$$\leq C_1 \exp\left(-4\sum_{\deg P \leq y} q^{-\deg P} + 4\sum_{\substack{\deg P \leq y \\ P\|h}} q^{-\deg P}\right) \leq C\frac{(\log y)^4}{y^4}.$$

Using the Hypothesis that $2 \leq y \leq \log n$, we have

$$q^{(1-2\alpha)n} \exp\left(\frac{cq^{\alpha y}}{y}\right) \ll (y \log y)^{-1}$$

Combining the above estimates we conclude the proof. $\square$

Chapter 6

Distribution of the sum of additive functions over $\mathbb{F}_q[x]$

In this chapter, we will discuss about the behaviour of distributions of the sum of additive functions over function field in large degree limit. More precisely, our main goal is to study the asymptotic behaviour of the following sum:

$$\tilde{\psi}_1(f + h_1) + \tilde{\psi}_2(f + h_2), \tag{6.1}$$

where $\tilde{\psi}_1, \tilde{\psi}_2$ are additive functions on $\mathcal{M}_q$ and h_1, h_2 are fixed polynomials with $\deg(h_j) < n$ for all j. The results of this chapter are contained in [8].

The following theorem gives the behaviour of the distribution of the sum (6.1) over monic polynomials, which is an application of Theorem 5.0.1.

Let $\psi_j : \mathcal{M}_q \to \mathbb{U}$ and $\alpha_j : \mathcal{M}_q \to \mathbb{C}$ be multiplicative functions such that $\alpha_j = \mu * \psi_j$ for all $j = 1, 2$. For fixed polynomials $h_j \in \mathbb{F}_q[x]$ with $\deg(h_j) < n$ for all $j = 1, 2$

and $n \geq r$, we define

$$Q_1(\gamma) := \prod_{\deg P \leq \gamma} \sum_{\substack{m_1=0 \\ (P^{m_1}, P^{m_2}) | (h_2-h_1)}}^{\infty} \sum_{m_2=0}^{\infty} \frac{\alpha_1(P^{m_1})\alpha_2(P^{m_2})}{q^{\deg([p^{m_1}, P^{m_2}])}}, \tag{6.2}$$

$$Q_1'(\gamma) := \sum_{\substack{m_1=0 \\ (P^{m_1}, P^{m_2}) | (h_2-h_1)}}^{\infty} \sum_{m_2=0}^{\infty} \frac{\alpha_1(P^{m_1})\alpha_2(P^{m_2})}{\Phi[P^{m_1}, P^{m_2}]}, \tag{6.3}$$

$$Q_2(\gamma) = \prod_{\deg P > \gamma} \left(1 + \sum_{j=1}^{2} \sum_{m=1}^{\infty} \frac{\psi_j(P^m) - \psi_j(P^{m-1})}{q^{m \deg P}}\right), \tag{6.4}$$

$$Q_2'(\gamma) = \prod_{\deg P > \gamma} \left(1 - \frac{2}{\Phi(P)} + \sum_{j=1}^{2} \sum_{k=1}^{\infty} \frac{\psi_j(P^k)}{q^{k \deg P}}\right). \tag{6.5}$$

Theorem 6.0.1. *Let $t, x \in \mathbb{R}$ and $\gamma = \deg(h_2 - h_1)$ for fixed polynomials h_1, h_2 in $\mathbb{F}_q[x]$. Assume that $\tilde{\psi_1}$ and $\tilde{\psi_2}$ be real-valued additive functions on $\mathcal{M}_q$ and the following series converges:*

$$\sum_{|\tilde{\psi_i}(P)| \leq 1} \frac{\tilde{\psi_i}(P)}{q^{\deg P}}, \qquad \sum_{|\tilde{\psi_i}(P)| \leq 1 \forall i} \frac{\tilde{\psi_1}(P) + \tilde{\psi_2}(P)}{q^{\deg P}}, \qquad \sum_{|\tilde{\psi_i}(P)| > 1} q^{-\deg P} \quad \forall i = 1, 2.$$

Then the distribution function

$$F_n(x) := \frac{1}{|\mathcal{M}_{n,q}|} \left| \left\{ f \in \mathcal{M}_{n,q} : \tilde{\psi_1}(f + h_1) + \tilde{\psi_2}(f + h_2) \leq x \right\} \right|$$

converges weekly towards a limit distribution as $n \to \infty$ whose characteristic function say $G(t)$ is equal to $Q_1(\gamma)Q_2(\gamma)$, where $Q_1(\gamma)$ and $Q_2(\gamma)$ are defined (6.2) *and* (6.4) *respectively with ψ_j is replaced by $\exp(it\tilde{\psi_j}), \forall j = 1, 2$.*

Theorem 6.0.2. *Let $t, x \in \mathbb{R}$ and $\gamma = \deg(h_2 - h_1)$ for fixed polynomials h_1, h_2 in $\mathbb{F}_q[x]$. Assume that $\tilde{\psi_1}$ and $\tilde{\psi_2}$ be real-valued additive functions on $\mathcal{M}_q$ and series in the Hypothesis of Theorem* **??** *converges. Then the distribution function*

$$F_n'(x) := \frac{1}{|\mathcal{P}_{n,q}|} \left| \left\{ P \in \mathcal{P}_{n,q} : \tilde{\psi_1}(P + h_1) + \tilde{\psi_2}(P + h_2) \leq x \right\} \right|$$

converges weekly towards a limit distribution as $n \to \infty$ whose characteristic function say $H(t)$ is equal to $Q_1'(\gamma)Q_2'(\gamma)$, where $Q_1'(\gamma)$ and $Q_2'(\gamma)$ are defined (6.3) *and* (6.5) *respectively with ψ_j is replaced by* $\exp(it\tilde{\psi_j}), \forall j = 1, 2$.

6.1 Proof of Theorem 6.0.1 and Theorem 6.0.2

Proof. We recall the following lemma from Chapter 3.

Lemma 6.1.1 ([40], Theroem 2.4)**.** *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of distribution functions and $\{\phi_n\}_{n=1}^{\infty}$ the sequence of their characteristic functions. Then F_n converges weakly to a distribution function F if, and only if, ϕ_n converges point-wise on $\mathbb{R}$ to a function ϕ which is continuous at 0. Furthermore, in this case, ϕ is the characteristic function of F, and the convergence of ϕ_n to ϕ is uniform on any compact subset.*

Proof of Theorem 6.0.1. The distribution function is

$$F_n(x) = \frac{1}{q^n}\nu_n\big\{f; \psi_1(f+h_1) + \psi_2(f+h_2) \leq x\big\}$$

and the corresponding characteristic function is

$$G_n(t) = \frac{1}{q^n}\sum_{f\in\mathcal{M}_{n,q}} \exp\left(it\left(\psi_1(f+h_1) + \psi_2(f+h_2)\right)\right).$$

We observe that

$$\sum_P \frac{\exp(it\psi_j(P)) - 1}{q^{\deg P}} = it \sum_{|\psi_j(P)|\leq 1} \frac{\psi_j(P)}{q^{\deg P}} + O\Bigg(t^2 \sum_{|\psi_j(P)|\leq 1} \frac{\psi_j^2(P)}{q^{\deg P}} + \sum_{\psi_j(P)>1} q^{-\deg P}\Bigg).$$

Therefore, from the hypothesis of the theorem we can say that $G(t)$ is convergent for every real t. Further, the infinite product $G(t)$ is continuous at $t = 0$ because it converges uniformly for $|t| \leq T$ where $T > 0$ is arbitrary.

Also notice that, for $j = 1, 2$ we have

$$\mathbb{D}(\psi_j, 1; \infty) \ll t^2 \sum_{|\psi_j(P)| \leq 1} \frac{\psi_j^2(P)}{q^{\deg P}} + \sum_{|\psi_j(P)| > 1} q^{-\deg P}.$$

So, using the hypothesis of the theorem we see that ψ_j is pretend to 1 and choosing $r = \log n$ in Theorem 5.0.1 we get that the remainder term disappears when $n \to \infty$.

Thus by Theorem 5.0.1, the characteristic function $G_n(t)$ has the limit $G(t)$ for every real t and this limit is continuous at $t = 0$. Therefore, by Lemma 6.1.1 we get the required Theorem 6.0.1.

Proof of Theorem 6.0.2. The distribution function is

$$F_n'(x) = \frac{1}{|\mathcal{P}_{n,q}|} \nu_n \big\{ P; \psi_1(P + h_1) + \psi_2(P + h_2) \leq x \big\}$$

and the corresponding characteristic function is

$$H_n(t) = \frac{1}{|\mathcal{P}_{n,q}|} \sum_{P \in \mathcal{P}_{n,q}} \exp\left(it\left(\psi_1(P + h_1) + \psi_2(P + h_2)\right)\right).$$

By following the similar argument as in the proof of Theorem 6.0.1, we get the required Theorem 6.0.2. □

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