AN ALTERNATE VAUGHAN'S IDENTITY IN THE TERNARY GOLDBACH PROBLEM

By

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SUMMARY

This thesis deals with the study of three different problems in analytic number theory. It is divided into five chapters. The first three chapters study the ternary Goldbach problem through an alternative approach suggested by Helfgott in his recent breakthrough on the problem. Chapter 4 deals with the study of correlations of arithmetic functions of a certain type and in Chapter 5, we study the distribution of values of the Oppenheim factorization function.

Chapter 1 consists of an introduction, where we describe the Hardy-Littlewood circle method and give a brief overview of Helfgott's approach and the alternative route suggested by him.

In the next two chapters, we prove explicit results for the type-I sums and type-II sums occurring in this new approach. In Chapter 2, we prove a general result with good constants for type-I sums. In Chapter 3, we prove different versions of the large sieve inequality to handle the type-II sums, which are bilinear exponential sums. Our main results lie in the case when both sequences are supported on primes.

In Chapter 4, we study correlations of a certain class of arithmetic functions and improve the error terms in their asymptotic formulas. We apply this method to study similar shifted sums over primes and improve upon an earlier result. The method generalises to study similar shifted sums of more than two functions.

In Chapter 5, we study a problem of a combinatorial nature, concerning Oppenheim's factorization function. It counts the number of nontrivial unordered factorizations of a positive integer. We obtain an upper bound for the number of its distinct values upto a given parameter. It improves the earlier known bounds on this quantity and we also give heuristic arguments to indicate that our bound is essentially the best possible.

Notation

Symbol	**Description**
$\mathbb{R}$	The set of real numbers
$\mathbb{R}_{\geq 0}$	Set of non-negative real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{Z}$	The set of integers
$\mathbb{N}$	Set of positive integers
$\mathbb{Z}_{\geq 0}$	Set of non-negative integers
$\mathbb{Z}^+(r)$	$\mathbb{Z}^r \setminus \{(0,\ldots,0)\}$.
x/yz	Denotes $\frac{x}{yz}$ for nonzero reals x, y and z
$\|x\|$	Distance of x from the nearest integer
$e(x)$	$e^{2\pi i x}$
$\lfloor x \rfloor$, $\lceil x \rceil$	Floor and Ceil functions, respectively
O, $\ll$	Big O notation
O^*	Big O notation with implied constant 1.
o	Little o notation
α	An element of $\mathbb{R}/\mathbb{Z}$ with an approximation $2\alpha = a/q + \delta/x$ in Chapters 1, 2, 3
δ_0	$\max\{2, \|\delta\|/5\}$
η	Non-negative function supported on $[0,1]$, twice differentiable on $(0,1)$ with L^1-norm 1 and $\eta(0) = \eta(1) = \eta'(1) = 0$

$n \sim x$	$x/2 < n \leq x$
p	A prime number
$\|\boldsymbol{a}\|$, $\|\boldsymbol{b}\|$	L^2-norms of the sequences $\{a_n\}$ and $\{b_m\}$ in Chapter 3.
$\boldsymbol{\alpha}$	Denotes $(\alpha_1, \ldots, \alpha_r)$, with $\alpha_i \in \mathbb{Z}_{\geq 0}$ in Chapter 5
$f_{\leq U}$, $f_{>U}$, $f_{(U,V)}$	The restriction of a function f to the intervals $[1, U]$, (U, ∞) and (U, V), respectively
$F_0(x)$	An upper bound for $q/\varphi(q)$ when $x \geq \max\{3, q\}$
$f * g$	Dirichlet convolution of arithmetic functions f and g.
f', f'', $f^{(k)}$	Denote the first, second and k-th derivatives of f, respectively
$Supp(f)$	Support of a function f
$C^k(I)$	The class of functions k-times differentiable on I with a continuous k-th derivative
$1_{[a,b]}$	Characteristic function of the interval $[a, b]$
$\hat{f}$	Fourier transform of f normalized by $\hat{f}(t) = \int_{\mathbb{R}} f(x)e(-xt)\,dx$
$\lvert f\rvert_1$, $\lvert f'\rvert_1$	Denotes of the L^1-norm of f and f'. If f is differential outside finitely many points, $\lvert f'\rvert_1$ denotes the total variation of f
$(T^l F)(x)$	Defined by $\int_0^\infty e^{-(l+1)t} F(x+t)\,dt$
$\lvert\mathcal{I}\rvert$	Length of an interval $\mathcal{I}$
$\lvert S\rvert$, $\#S$	Cardinality of a set S
$\left(\frac{\cdot}{p}\right)$	Legendre symbol for a prime p
$v_p(n)$	Largest power of a prime p that divides n
(d_1, d_2), $[d_1, d_2]$	GCD and LCM, respectively of positive integers d_1 and d_2
$[d_1, \ldots, d_k]$	LCM of positive integers d_1, d_2, $\ldots$, d_k
μ	Möbius function
Λ	Von Mangoldt function
τ	Divisor function
φ	Euler totient function
σ	Sum of divisors function

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Chapter 1

Introduction

In this chapter, we give a brief description of the Hardy-Littlewood circle method and outline Helfgott's approach to the ternary Goldbach problem.

1.1 The Circle method

Let $\eta : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be compactly supported and twice differentiable, except for finitely many points. Define

$$S_\eta(\alpha, x) := \sum_n \Lambda(n)\eta\left(\frac{n}{x}\right) e(n\alpha). \tag{1.1}$$

The version of Hardy and Littlewood [HL23] and subsequent versions are without a smoothing, i.e., with the brutal truncation $n \leq x$. The use of a smoothing is a major ingredient in Helfgott's proof. He works with two different smoothing functions.

To show that an odd positive integer N is expressible as a sum of three primes, one considers the quantity

$$\int_{\mathbb{R}/\mathbb{Z}} S_{\eta_*}(\alpha, x)^2 S_\eta(\alpha, x) e(-N\alpha)\, d\alpha = \sum_{\sum n_i = N} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\eta_*\left(\frac{n_1}{x}\right)\eta_*\left(\frac{n_2}{x}\right)\eta\left(\frac{n_3}{x}\right). \tag{1.2}$$

When (1.2) is positive, it implies the existence of prime powers n_1, n_2 and n_3 such that $n_1 + n_2 + n_3 = n$. Since the contribution to the above sum when at least one of the n_i's is a proper prime power (not a prime) is negligible, it enough to show that the integral in (1.2) is positive.

1.1.1 The Major and Minor arcs

The integral over $\mathbb{R}/\mathbb{Z}$ in (1.2) is divided into two parts, namely the *Major arcs* (denoted by $\mathfrak{M}$) and the *Minor arcs* (denoted by $\mathfrak{m}$). The major arcs are small neighbourhoods around rationals having small denominators. The complimentary set forms the minor arcs. The major arcs are normally defined as follows:

$$\mathfrak{M} = \bigsqcup_{q \le R} \bigsqcup_{\substack{1 \le a \le q \\ (a,q)=1}} \mathfrak{M}_{a,q}, \qquad \text{where} \quad \mathfrak{M}_{a,q} = \left\{ \alpha \in \mathbb{R}/\mathbb{Z} : \|\alpha - a/q\| \le \frac{R}{qx} \right\}. \tag{1.3}$$

Here $\|.\|$ denotes the distance from the nearest integer and $R > 1$ is a parameter, which is normally taken be a power of $\log x$. The arcs $\mathfrak{M}_{a,q}$ can be made disjoint provided x is large enough.

Helfgott's choice of the major arcs was slightly different. It was as follows:

Definition 1.1. Let

$$\mathfrak{M}_{C_0,r} = \bigsqcup_{q \le (q,2)r} \bigsqcup_{\substack{1 \le a \le q \\ (a,q)=1}} \left\{ \alpha \in \mathbb{R}/\mathbb{Z} : \|\alpha - a/q\| \le \frac{C_0(q,2)r}{qx} \right\}. \tag{1.4}$$

Helfgott chooses $C_0 = 8$ and $r = r_0 = 1.5 \cdot 10^5$. This finite choice of r comes from a verification of the Generalized Riemann hypothesis upto a certain height for all L-functions with modulus less than $3 \cdot 10^5$ which was carried out by Platt [Pla16].

1.1.2 Estimation of the integral

Splitting the integral in (1.2) into major and minor arcs, we obtain

$$\int_{\mathfrak{M}} S_{\eta_*}(\alpha,x)^2 S_\eta(\alpha,x) e(-N\alpha)\, d\alpha + \int_{\mathfrak{m}} S_{\eta_*}(\alpha,x)^2 S_\eta(\alpha,x) e(-N\alpha)\, d\alpha$$

To estimate the integral over $\mathfrak{M}$, an asymptotic formula for $S_{\eta*}(\alpha,x)$ and $S_\eta(\alpha,x)$, for $\alpha \in \mathfrak{M}_{a,q}$ is obtained, using which one can integrate over each $\mathfrak{M}_{a,q}$ and then sum over all $(a,q) = 1$ and all $q \leq R$. This will lead to an asymptotic formula

$$\int_{\mathfrak{M}} \sim C_{\eta_*,\eta} \cdot x^2.$$

To bound the minor arc contributions, the following method is used:

$$\left| \int_{\mathfrak{m}} S_{\eta_*}(\alpha,x)^2 S_\eta(\alpha,x) e(-N\alpha)\, d\alpha \right| \leq \max_{\alpha\in\mathfrak{m}} |S_\eta(\alpha,x)| \cdot \int_{\mathfrak{m}} |S_{\eta_*}(\alpha,x)|^2\, d\alpha.$$

The integral over $\mathfrak{m}$ above can now be extended to $\mathbb{R}/\mathbb{Z}$. An application of the Parseval's identity gives a bound of the order $x \log x$ for $\int_{\mathfrak{m}} |S_{\eta_*}(\alpha,x)|^2\, d\alpha$. Helfgott used a version of the large sieve inequality due to Ramaré [Ram09] in order to get rid of the additional $\log x$ factor above. The idea is to divide the integral over $\mathfrak{m}$ into disjoint annulus of arcs in (1.4) and apply Ramaré's version of the large sieve.

For $\alpha \in \mathfrak{m}$, he obtained a bound of the form [HH13, Theorem 3.1.1]

$$|S_\eta(\alpha,x)| \leq \frac{C_1 x \log r}{\sqrt{r}} + C_2 x^{5/6}.$$

Note that the trivial bound is x. Since $r \geq 1.5 \cdot 10^5$ and $\log r/\sqrt{r}$ is decreasing, a constant saving over the trivial bound is obtained.

1.2 An alternate Vaughan's identity

To give an upper bound for $S_\eta(\alpha, x)$ for $\alpha \in \mathfrak{m}$, one needs to deal with sums over primes where a decomposition of the Von-Mangoldt function, called the *Vaughan*'s identity is commonly used. The standard version of Vaughan's identity is as follows:

$$\Lambda = \mu_{\leq U} * \log - \Lambda_{\leq V} * \mu_{\leq U} * 1 \, + \, 1 * \mu_{>U} * \Lambda_{>V} \, + \, \Lambda_{\leq V},$$

where $U, V > 1$ are any parameters and $*$ denotes the Dirichlet convolution. Though there are free parameters U and V, the identity is not *log-free*, i.e., summing over the RHS and using trivial bounds, one obtains two additional factors of $\log x$ compared to the LHS. Hence, one needs extra work to get rid of these logarithmic factors.

In [HH13, Pg 49, Eq (3.17)], Helfgott mentions an alternate version of the Vaughan's identity which is essentially *log-free*. It originates from the work of Bombieri [Bom76]:

$$\Lambda \cdot \log^2 = \mu * \log^3 - 3(\mu * \log^2 * \Lambda_{\leq V}) \, - \, 3(\Lambda \cdot \log) * \Lambda_{>V} + F_{3,V}, \qquad (1.5)$$

where $F_{3,V} = -\Lambda_{>V} * \Lambda * \Lambda + 2(\Lambda_{\leq V} * \Lambda * \Lambda)$. For $n = n_1 n_2 n_3$, with $n_1 < n_2 < n_3$ and $V^3 < n$, we have

$$F_{3,V}(n) = \Lambda(n_1)\Lambda(n_2)\Lambda(n_3) \cdot \begin{cases} -6, & \text{if all } \; n_i > V, \\ 6, & n_1 < n_2 < V < n_3, \\ 0, & n_1 < V < n_2 < n_3 \\ 12, & \text{if all } \; n_i \leq V. \end{cases}$$

We have two more small parameters that occur as we further split the second and third terms in (1.5) to separately deal with the contribution from the tail.

Let

$$S_{\eta,2}(\alpha,x) = \sum_{n \text{ odd}} \Lambda(n)(\log n)^2\, \eta(n/x)e(n\alpha). \tag{1.6}$$

Using the identity (1.5), this decomposes into

$$S_{\eta,2}(\alpha,x) = S_{I,1,\eta} - 3S_{I,2,\eta} - 3S_{II,\eta} + S_{3,\eta}, \tag{1.7}$$

where

$$\begin{aligned}
S_{I,1,\eta} &= \sum_{m \text{ odd}} \mu(m) \sum_{n \text{ odd}} \log^3 n\, e(mn\alpha)\eta(mn/x),\\
S_{I,2,\eta} &= \sum_{\substack{l \leq V \\ l \text{ odd}}} \Lambda(l) \sum_{m \text{ odd}} \mu(m) \sum_{n \text{ odd}} \log^2 n\, e(lmn\alpha)\eta(lmn/x),\\
S_{II,\eta} &= \sum_{m \text{ odd}} (\Lambda \cdot \log)(m) \sum_{\substack{n > V \\ n \text{ odd}}} \Lambda(n)e(mn\alpha)\eta(mn/x)\\
S_{3,\eta} &= \sum_{n \text{ odd}} F_{3,V}(n)e(n\alpha)\eta(n/x),
\end{aligned} \tag{1.8}$$

The trivial bound for $S_{\eta,2}(\alpha,x)$ is of the order $x\log^2 x$. Using (1.5), we are likely to obtain a bound of the form

$$|S_{\eta,2}(\alpha,x)| \leq C_1 x + C_2 x \log V + \frac{C_3(r)x\log^2 x}{\sqrt{r}} + C_4 x^{\theta},$$

where $\theta < 1$. In the above bound, there are terms proportional to x and $x\log V$. These terms are quite large, and just better than the trivial bound. Therefore, it becomes important to get the smallest possible constants here. The source of such terms is $S_{I,1,\eta}$, $S_{I,2,\eta}$ and the tail of $S_{II,\eta}$ and $S_{3,\eta}$.

Terms in such a decompositions can be classified into two types, namely the *type-I* and *type-II* sums. In type-I sums, the sum over one of the variables m and n consists of a nice function (like logarithm), which allows some cancellation in the exponential sum. For instance, the sums $S_{I,1,\eta}$ and $S_{I,2,\eta}$ are type-I sums. In type-II sums, one cannot obtain a cancellation directly by summing over one variable. We may need to apply the Cauchy-Schwarz inequality followed by an application of the large sieve inequality.

1.3 Choice of smoothing functions

Now, we mention the type of smoothing functions we will work with. We deal with separate smoothing functions in the type-I and type-II sums. They will be normalized so that their L^1-norm is 1.

In the type-I sums, we work with a general smoothing η, but our aim is to apply the bounds with the following function:

Definition 1.2. Let $\eta_0 : \mathbb{R} \to [0, \infty)$ be defined by:

$$\eta_0(t) = \begin{cases} 70t(1-t)^5(1-3t+4t^2+t^3/2), & t \in [0,1], \\ 0, & \text{otherwise.} \end{cases} . \tag{1.9}$$

Then η_0 is non-negative, differentiable on $[0, 1]$ and twice differentiable in $(0, 1]$. It also satisfies $|\eta_0|_1 = 1$ and $\eta_0(0) = \eta_0(1) = \eta_0'(1) = 0$.

This choice of the above smoothing is purely for numerical reasons, as calculations done on a program have suggested that certain important quantities stay small under this choice. This is certainly not an optimal choice.

In the type-II sums, we work with the same smoothing η_2 as chosen by Helfgott and Tao. It is defined as follows:

Definition 1.3. Let $\eta_2 : \mathbb{R} \to [0, \infty)$ be defined by

$$\eta_2(x) = 4 \int_0^\infty 1_{[1/2,1]}(t) 1_{[1/2,1]}(x/t) \frac{dt}{t} = 4 \begin{cases} \log 4x, & x \in (1/4, 1/2), \\ \log 1/x, & x \in (1/2, 1), \\ 0, & \text{otherwise.} \end{cases} \tag{1.10}$$

One of the main advantage is that it allows the bilinear sum to be decomposed dyadically in the two variables, which will be evident when we discuss these sums in Chapter 3.

Chapter 2

Type-I sums

In this chapter, we prove results for the type-I sums. We work with a smoothing function η satisfying some general conditions, but our aim is to apply it to the function η_0 defined in (1.9).

The main result of this chapter broadly follows the approach of Helfgott's, but with certain modifications and differences to adapt the arguments in the current setting. We have done some of the calculations in the appendix.

2.1 Smoothing functions and hypothesis

We begin with some general hypothesis on the smoothing function η.

Definition 2.1 (General conditions on η)**.** Let $\eta : \mathbb{R} \to \mathbb{R}$ be a function satisfying the following conditions:

$$\begin{aligned} &\eta \geq 0, \qquad \mathrm{Supp}(\eta) \subseteq [0,1], \qquad \eta \in C^2(0,1), \\ &|\eta|_1 = 1 \quad \text{and} \quad \eta(0) = \eta(1) = \eta'(1) = 0, \end{aligned} \tag{C1}$$

where $\mathrm{Supp}(f)$ denotes the support of f and $C^k(I)$ denotes functions k times differentiable in I with a continuous k^{th} derivative.

Henceforth, η is a function satisfying (C1). The advantage of taking η over η_2 in (1.10) is that it is twice differentiable and there the L^1-norms are comparitively smaller to that of η_2.

Definition 2.2. Let $\eta : \mathbb{R} \to \mathbb{R}$ satisfy (C1). Then for $u_0 \geq 3$ and $y > u_0$, define

$$\eta_{(y),k,u_0}(t) = \begin{cases} \eta(t)(\log yt)^k, & t > u_0/y, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

It can be seen that

$$\widehat{\eta_{(y),k,u_0}}(t) = \sum_{l=0}^{k} \binom{k}{l} (\log y)^l \, (\widehat{\eta_{(y),0,u_0} \cdot \log^{k-l}})(t), \tag{2.2}$$

where $\hat{f}$ denotes the Fourier transform of f defined by

$$\hat{f}(t) = \int_{\mathbb{R}} f(x) e(-xt)\, dx. \tag{2.3}$$

For functions that are C^1 except for a finite number of points, we define (following Helfgott and Tao) the L^1 norms in terms of their total variation.

Definition 2.3 (L^1-norm as a total variation)**.** Let $f : [a,b] \to \mathbb{R}$ be C^1 except for the set of points $\{x_1, \ldots, x_n\}$. We define $|f'|_1$ to be the total variation of f. In particular,

$$|f'|_1 = \int_a^b |f'(t)|\, dt \; + \sum_{j=1}^{n} \left| f(x_j^+) - f(x_j^-) \right|. \tag{2.4}$$

This coincides with the usual definition if f were C^1 in all of $[a,b]$. We can similarly define $|f^{(k)}|_1$ for $k \geq 2$.

Hypothesis 2.4. We assume that there are non-negative functions which act as upper bounds for the the L^1-norms of $\eta_{(y),k,u_0}$ and its derivatives. In particular,

$$
\begin{aligned}
|\eta_{(y),k,u_0}|_1 &\le P_{0,k,u_0}(\log y),\\
|\eta'_{(y),k,u_0}|_1 &\le P_{1,k,u_0}(\log y),\\
|\eta''_{(y),k,u_0}|_1 &\le P_{2,k,u_0}(\log y),
\end{aligned} \tag{H1}
$$

for all $y > u_0$. We also define $P^{(j)}_{k,u_0}$, $j = 0, 2$ as:

$$
P^{(0)}_{k,u_0} := \sqrt{P_{0,k,u_0} \cdot P_{2,k,u_0}} \quad \text{and} \quad P^{(2)}_{k,u_0} := P_{1,k,u_0}\sqrt{\frac{P_{2,k,u_0}}{P_{0,k,u_0}}}. \tag{H2}
$$

and assume that

$$
P_{j,k,u_0}(\log y), \quad P^{(0)}_{k,u_0}(\log y) \quad \text{and} \quad P^{(2)}_{k,u_0}(\log y) \quad \text{are increasing for all } y > u_0. \tag{H3}
$$

Further, assume there are positive constants $C_{j,k,\eta}$, $j = 0, 1, 2$, such that

$$
\begin{aligned}
&P_{j,k,u_0}(\log y) = C_{j,k,\eta} \cdot (\log y)^k, \quad \text{for all } y > u_0\\
&\text{and} \quad C_{2,k,\eta} \le 1000 \cdot C_{1,k,\eta}.
\end{aligned} \tag{H4}
$$

Consequently, we have

$$
P^{(0)}_{k,u_0}(\log y) = \sqrt{C_{0,k,\eta}\, C_{2,k,\eta}} \cdot (\log y)^k \quad \text{and} \quad P^{(2)}_{k,u_0}(\log y) = C_{1,k,\eta}\sqrt{\frac{C_{2,k,\eta}}{C_{0,k,\eta}}} \cdot (\log y)^k. \tag{2.5}
$$

Remark 2.5. From (H4), it follows that one can assume $P_{j,k,u_0}(\log y)/y$ to be decreasing for $y > e^k$.

Explicit values for the constants $C_{j,k,\eta}$ (in terms of certain norms involving η) have been computed in Proposition A.7 in the appendix.

Remark 2.6. Note that $\eta_{(y),k,u_0}$ and $\eta'_{(y),k,u_0}$ are not continuous (hence are not differentiable) at $t = u_0/y$. So, to evaluate $|\eta'_{(y),k,u_0}|_1$ and $|\eta''_{(y),k,u_0}|_1$ w.r.t. Definition

2.3, we have to add additional contributions (which are jumps of discontinuity at u_0/y)

$$|\eta(u_0/y)(\log u_0)^k| \quad \text{and} \quad \left|\eta'(u_0/y)(\log u_0)^k + k(\log u_0)^{k-1}\frac{\eta(u_0/y)}{u_0/y}\right|,$$

respectively to the integral from u_0/y to 1.

Definition 2.7. For all integers $l \geq 0$ and η satisfying (C1), we define

$$c_{\eta,l} = |\eta \cdot \log^l|_1, \quad c'_{\eta,l} = |(\eta \cdot \log^l)'|_1 \quad \text{and} \quad b_{\eta,l} = \max\left\{2c_{\eta,l}, \frac{c'_{\eta,l}}{5\pi}\right\}. \tag{2.6}$$

Then for any $\delta \in \mathbb{R}$ and $\delta_0 = \max\{2, |\delta|/5\}$, we have

$$\min\left\{c_{\eta,l}, \frac{c'_{\eta,l}}{\pi|\delta|}\right\} \leq \frac{b_{\eta,l}}{\delta_0}. \tag{2.7}$$

Definition 2.8. Let $l > -1$ be a real number and suppose $F : \mathbb{R} \to \mathbb{R}$ satisfies

$$F \geq 0, \quad F \in C^1(x_0, \infty) \quad \text{and} \quad \int_{x_0}^{\infty} e^{-(l+1)t} F(t) < \infty, \quad \text{where } x_0 > 0. \tag{C2}$$

Then for $x > x_0$, define

$$(T^l F)(x) = \int_0^{\infty} e^{-t(l+1)} F(x+t)\, dt. \tag{2.8}$$

2.2 The main result

Let $\alpha \in \mathbb{R}/\mathbb{Z}$ and $Q_0 > 1$ be given. By Dirichlet approximation, we have

$$2\alpha = a/q + \delta/x, \quad |\delta/x| \leq 1/qQ_0, \quad (a,q) = 1 \quad \text{and} \quad q \leq Q_0, \tag{AP1}$$

and assume q to be the *smallest* possible. Define

$$\delta_0 = \delta_0(\alpha, Q_0) = \max\{2, |\delta|/5\} \quad \text{and} \quad s = s(\alpha, Q_0) = \delta_0 q. \tag{2.9}$$

Let

$$s_0 = \min\left\{s, \frac{x}{5Q_0}\right\}. \tag{2.10}$$

Our aim is to bound the following sum:

Definition 2.9. Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function and let $x \geq 1$. Suppose $3 \leq u_0 \leq s_0$ and η satisfies (C1). For $k = 1, 2, 3$, we consider the sum

$$S_{\eta,k,f}(\alpha, x) = \sum_{\substack{m < x/u_0 \\ m \text{ odd}}} f(m) \sum_{\substack{n > u_0 \\ n \text{ odd}}} (\log n)^k e(mn\alpha)\eta(mn/x). \tag{2.11}$$

Remark 2.10. The condition $m < x/u_0$ above is forced upon by the fact that η is supported in $[0, 1]$ and that $n > u_0$.

In the next theorem, we prove the main type-I bound for the sum $S_{\eta,k,f}$ with the assumption of the certain reasonable hypothesis we have given earlier. We prove the following bound for $S_{\eta,k,f}(\alpha, x)$.

Theorem 2.11 (Type I bound)**.** *Let $x \geq 10^{18}$ and Q_0 be a given parameter. Let α be as in (AP1) and s, s_0 be as in (2.9), (2.10) with $s \geq s_0 \geq 1.5 \cdot 10^5$. Let η be as in (C1) and $S_{\eta,k,f}(\alpha, x)$ be as in (2.11) with $k \in \{1, 2, 3\}$. Suppose that (H1), (H2), (H3) and (H4) hold. Also, assume $q \leq \sqrt{x/5}$ and that*

$$\begin{gathered} 3 \leq u_0 \leq s_0, \\ \sqrt{x/5} \leq Q_0 \leq x/10^6, \\ |f(m)| \leq \kappa, \quad \textit{for all } m \leq x. \end{gathered} \tag{H5}$$

Then with P_{j,k,u_0}, $P^{(0)}_{k,u_0}$ and $P^{(2)}_{k,u_0}$ in (H1) and (H2), T^l as in (2.8), we have

$$|S_{\eta,k,f}(\alpha,x)| \le x\kappa \left(\frac{T^0 P^{(0)}_{k,u_0}(\log u_0)}{\pi u_0} + \frac{T^1 P^{(2)}_{k,u_0}(\log u_0)}{2\pi u_0^2} + L_k(s) \right) + R_{k,q}(s,x,f), \tag{2.12}$$

where

$$L_k(s) = A_k + \frac{B_k (\log 10s)^{k+1}}{s}, \tag{2.13}$$

with

$$A_k = 0.002\, C_{0,k,\eta} + 0.00003\, C_{1,k,\eta} + 10^{-6}\, C_{2,k,\eta},$$
$$B_k = 0.108 \sqrt{C_{0,k,\eta}\, C_{2,k,\eta}} + 0.002\, C_{2,k,\eta} + 10^{-7} C_{1,k,\eta} \sqrt{\frac{C_{2,k,\eta}}{C_{0,k,\eta}}}.$$

Here, $R_{k,q}(s,x,f)$ is a decreasing function of s for $s \ge 1.5 \cdot 10^5$ and satisfies

$$R_{\eta,k,q}(s,x,f) \ge \frac{x}{2s} \sum_{i=0}^{k} \sum_{j=0}^{i} \binom{k}{i} \binom{i}{j} b_{\eta,k-i} (\log 10s)^{i-j} \left| m_{2q,j}\left(\frac{x}{10\delta_0 q^2}, f \right) \right|, \tag{2.14}$$

where

$$m_{q,k}(x,f) = \sum_{\substack{m \le x \\ (m,q)=1}} \frac{f(m)}{m} \left(\log \frac{x}{m} \right)^k$$

and $b_{\eta,l}$'s are given in (2.6). Furthermore, for $k = 1,2,3$, it can be seen that the RHS of (2.12) is decreasing for $s \ge 1.5 \cdot 10^5$. Explicit expressions for $R_{k,q}$ in the case $\eta = \eta_0$ (with η_0 given in (1.2)) are given in Proposition A.11.

In the next corollary, we show that essentially the same result holds even if we relax the condition $q \le \sqrt{x/5}$ in Theorem 2.11.

Corollary 2.12. *Let $x \ge 10^{18}$ and η be as in (C1). Let Q_0 be a given parameter, α be as in (AP1) and $S_{\eta,k,f}$ be as in (2.11) with $k \in \{1,2,3\}$. Suppose that (H1), (H2), (H3), (H4) and (H5) hold and $s \ge s_0 \ge 1.5 \cdot 10^5$ be as in (2.9) and (2.10).*

Then, we have

$$|S_{\eta,k,f}(\alpha,x)| \le x\kappa\left(\frac{T^0 P^{(0)}_{k,u_0}(\log u_0)}{\pi u_0} + \frac{T^1 P^{(2)}_{k,u_0}(\log u_0)}{2\pi u_0^2} + L_k(s_0)\right) + R_{k,q}(s_0,x,f). \tag{2.15}$$

Proof. We will prove Corollary 2.12 assuming Theorem 2.11. The only difference in the hypothesis from Theorem 2.11 is the condition $q \le \sqrt{x/5}$. Consider two cases:

Case (i): $q \le \sqrt{x/5}$. In this case, we can directly apply Theorem 2.11 and obtain the bound (2.12) for $S_{\eta,k,f}(\alpha,x)$ in terms of s. Since $s \ge s_0$ and the RHS of (2.12) is decreasing, we prove (2.15) in this case.

Case (ii): $q > \sqrt{x/5}$. We let $Q_0' = \sqrt{x/5}$. Using the parameter Q_0', we seek another Dirichlet approximation for 2α, i.e., $2\alpha = a_1/q_1 + \delta_1/x$ with $(a_1,q_1) = 1$, $q_1 \le Q_0'$ and $|\delta_1|/x \le 1/q_1Q_0'$. Then $|\delta_1|/x$ cannot be $O^*(1/q_1Q_0)$ (as q was the smallest possible satisfying (AP1) and $q_1 \le Q_0' < q$) and therefore

$$|\delta_1|q_1 \ge x/Q_0.$$

Now, $q_1 \le \sqrt{x/5}$ and $\sqrt{x/5} = Q_0' \le x/10^6$, since $x \ge 10^{18}$. We can now apply Theorem 2.11 with q_1 in place of q and Q_0' in place of Q_0. Letting

$$s_1 = s_1(\alpha, Q_0') = \max\left\{2, \frac{|\delta_1|}{5}\right\} q_1 \ge \frac{|\delta_1|q_1}{5} > \frac{x}{5Q_0} > 1.5\cdot 10^5,$$

we find that $3 \le u_0 \le s_0 \le s_1$. This leads us to the bound (2.12) with s replaced by s_1. Since $s_1 > s_0$ and the RHS of (2.12) is decreasing, we prove (2.15). □

2.3 Preliminary lemmas

In this section, we give some preliminaries for the proof of Theorem 2.11. First, we provide bounds on trigonometric sums.

2.3.1 Trigonometric sums

Let $f : \mathbb{R} \to \mathbb{R}$ be compactly supported and piecewise C^k except for a finite number of points. From [HH13, Eq (2.1), Pg 32], we have

$$\widehat{f}(t) = O^*\left(\frac{|\widehat{f^{(k)}}|_\infty}{(2\pi t)^k}\right) = O^*\left(\frac{|f^{(k)}|_1}{(2\pi t)^k}\right), \quad \text{for } k \geq 0, \tag{2.16}$$

where $f^{(k)}$ denotes the k-th derivative of f and $|f^{(k)}|_1$ is w.r.t. Definition 2.3.

The following lemma provides cancellations in the trigonometric sums. It is [Tao14, Corollary 3.2] and is implied by a change of variable in [HH13, Eq (2.2), Eq (2.3)].

Theorem 2.13. *Let $\alpha \in \mathbb{R}/\mathbb{Z}$ and $f : \mathbb{R} \to \mathbb{R}$ be compactly supported and piecewise C^2. Then*

$$\left|\sum_{n \text{ odd}} f(n)e(n\alpha)\right| \leq \frac{1}{2}\min\left\{|f|_1 + |f'|_1, \frac{|f'|_1}{|\sin 2\pi\alpha|}, \frac{|f''|_1}{(\sin 2\pi\alpha)^2}\right\}.$$

Remark 2.14. If in (2.16) and Theorem 2.13, f is C^2 except for a finite number of points, one can consider $|f'|_1$ and $|f''|_1$ as total variations of f and f', respectively following Definition 2.3.

Remark 2.15. Unlike Helfgott, we do not consider $|\widehat{f''}|_\infty$ in the final bound above, but instead use the weaker bound $|f''|_1$. This is done in order because of the complications that arise when estimating the Fourier transforms of $\eta'_{(y),k,u_0}$ and $\eta''_{(y),k,u_0}$.

Next, we state a lemma from [HH13].

Lemma 2.16.

(a) Let α be as in (AP1) and $1 \le y_1 < y_2 \le \frac{x}{2|\delta|q}$ with $y_2 - y_1 \le q$. Then

$$\sum_{\substack{y_1 < n \le y_2 \\ q \nmid n}} \min\left\{A, \frac{C}{|\sin 2\pi n\alpha|^2}\right\} \le \frac{20q^2}{3\pi^2} C.$$

(b) Let α be as in (AP1) and $1 \le y_1 < y_2 \le \frac{x}{2|\delta|q}$ with $y_2 - y_1 \le q$. Suppose further that $\pi B/e \ge Cq$. Then

$$\sum_{\substack{y_1 < n \le y_2 \\ q \nmid n}} \min\left\{\frac{B}{|\sin 2\pi n\alpha|}, \frac{C}{|\sin 2\pi n\alpha|^2}\right\} \le \frac{4Bq}{\pi}.$$

Proof. Here, (a) is the first bound of [HH13, Lemma 4.1.2], but with $Q_0/2$ replaced by $\frac{x}{2|\delta|q}$ ($\ge Q_0/2$), since in the proof, we only need $n|\delta|/x \le 1/2q$, which is ensured by $y_2 \le \frac{x}{2|\delta|q}$.

Part (b) is the first bound of [HH13, Lemma 4.1.3]. □

Now, we save the factor 2 when the sum runs over odd numbers. Although the saving may seem modest, it is going to play a crucial role when estimating the final sum. The proof of this lemma follows closely to that of [HH13, Lemma 4.1.1].

Lemma 2.17. *Let $y \ge 1$ and let $2\alpha = a/q + O^*(1/q^2)$. Then, we have*

$$\sum_{\substack{y < n \le y+2q \\ n \text{ odd}}} \min\left\{A, \frac{C}{|\sin 2\pi n\alpha|^2}\right\} \le 6A + \frac{4q}{\pi}\sqrt{AC}.$$

Proof. We can assume that $C \le A$ since otherwise the trivial bound Aq is better than the given bound.

Case (i): q *odd.* Write $n = m_0 + 2j$, where $j \in (-q/2, q/2]$. Then

$$2n\alpha = (m_0 + 2j)\left(\frac{a}{q} + O^*(1/q^2)\right) = \frac{2aj + c}{q} + O^*(3/2q).$$

Let $r = 2aj + c \,(\mathrm{mod}\ q)$, so that as j varies in $(-q/2, q/2]$, r also varies through $(-q/2, q/2]$. We bound the terms with $r = 0, \pm 1, \pm 2$ by A. For terms with $|r| \geq 3$, it follows that $\|2n\alpha\| = \|r/q + O^*(3/2q)\| \geq |r|/q - 3/2q > (|r| - 2)/q$. Letting $r' = |r| - 2$, the given sum is at most

$$5A + 2 \sum_{1 \leq r' \leq q/4} \min\left\{A, \frac{C}{\sin^2 \frac{\pi r'}{q}}\right\}.$$

Now, we use the first bound above when $r' \leq \frac{q}{\pi} \sin^{-1} \sqrt{C/A}$ and the second bound otherwise. The number of such values of r' is at most $\frac{q}{\pi} \sin^{-1} \sqrt{C/A}$. For the terms satisfying $r' > \frac{q}{\pi} \sin^{-1} \sqrt{C/A}$, we can replace the sum by an integral (owing to the convexity of $\sin^2$ in $(0, \pi/2)$). Therefore, this is at most

$$5A + 2A\left(\frac{q}{\pi} \sin^{-1} \sqrt{C/A}\right) + 2C \int_{\frac{q}{\pi}\sin^{-1}\sqrt{C/A}}^{q/4} \frac{1}{\sin^2 \frac{\pi t}{q}}\, dt$$

$$\leq 5A + \frac{2Aq}{\pi} \sin^{-1} \sqrt{C/A} + \frac{2Cq}{\pi}\sqrt{\frac{A}{C} - 1} \leq 5A + \frac{4q}{\pi}\sqrt{AC},$$

where we used the inequality $\sin^{-1} x + x\sqrt{1 - x^2} \leq 2x$, for $0 \leq x \leq 1$.

Case (ii): q *even.* We consider the sum in an interval of length q, i.e., we have

$$\sum_{\substack{y < n \leq y + q \\ n \text{ odd}}} \min\left\{A, \frac{C}{|\sin 2\pi n\alpha|^2}\right\}.$$

As before, write $n = m_0 + 2j$, with j in $(-q/4, q/4]$, so that $2n\alpha = r/q + O^*(1/q)$, where $r = 2aj + c \,(\mathrm{mod}\ q)$. Let $\rho = c \,(\mathrm{mod}\ 2) \in \{0, 1\}$. Then, we can replace r by $2r - \rho$, with r ranging in $(-q/4, q/4]$. Again, bound the terms corresponding to

$r = 0, \pm 1$ by A. For $|r| \geq 2$, we have $\|2n\alpha\| = \|(2r-\rho)/q + O^*(1/q)\| \geq 2(|r|-1)/q$. Letting $r' = |r| - 1$, this sum is at most

$$3A + 2 \sum_{1 \leq r' \leq q/4} \min \left\{ A, \frac{C}{\sin^2 \frac{2\pi r'}{q}} \right\}.$$

As before, the first bound is used when $2r' \leq \frac{q}{\pi} \sin^{-1} \sqrt{C/A}$, the number of which is at most $\frac{q}{2\pi} \sin^{-1} \sqrt{C/A}$. For terms with $2r' > \frac{q}{\pi} \sin^{-1} \sqrt{C/A}$, we replace the sum by an integral, to get

$$3A + 2A \left(\frac{q}{2\pi} \sin^{-1} \sqrt{C/A} \right) + 2C \int_{\frac{q}{2\pi} \sin^{-1} \sqrt{C/A}}^{q/4} \frac{1}{\sin^2 \frac{2\pi t}{q}} \, dt \leq 3A + \frac{2q}{\pi} \sqrt{AC},$$

as before. Since the result is established for an interval of length q, twice this bound holds for an interval of length $2q$. This completes the proof. □

2.3.2 Alternate approximation for α

We may sometimes want the q obtained in (AP1) to be large. If our q happens to be small, an alternate approximation for α (with a parameter other than Q_0) is sought.

The following lemma can be extracted from the proof of [HH13, Lemma 4.2.1]

Lemma 2.18. *Let $2\alpha = a/q + \delta/x$ in (AP1) with $\delta \neq 0$. Then we can always find an approximation a'/q', different from a/q, such that*

$$2\alpha = a'/q' + \delta'/x, \quad (a', q') = 1, \quad |\delta'|/x \leq 1/(q')^2 \quad \textit{and} \quad \frac{x}{2|\delta|q} < q' \leq \frac{2x}{|\delta|q}.$$

Proof. Let $Q_1 = x/|\delta|q$. Then, we have $2\alpha = a/q + O^*(1/qQ_1)$ and $q \leq Q_1$ (since $|\delta|/x \leq 1/q^2$). Letting $Q_2 = 2Q_1$, there is an approximation a'/q', different from a/q such that $2\alpha = a'/q' + \delta'/x$ with $q' \leq Q_2$ and $|\delta'|/x \leq 1/q'Q_2 \leq 1/(q')^2$. The approximation is different from a/q because δ/x cannot be $O^*(1/qQ_2)$, because of the choice of Q_1. By the triangle inequality, we have

$$\frac{1}{qq'} \le \left|\frac{a}{q} - \frac{a'}{q'}\right| \le \frac{1}{qQ_1} + \frac{1}{2q'Q_1}.$$

It then follows that $q' \ge Q_1 - \frac{q}{2} > \frac{Q_1}{2} = \frac{x}{2|\delta|q}$. The other bound follows from $q' \le Q_2 = 2Q_1 = 2x/|\delta|q$, proving the lemma. □

This leads us to the following:

Lemma 2.19. *Let $\alpha \in \mathbb{R}/\mathbb{Z}$ and q be as obtained in (AP1). Let δ_0 be as in (2.9). If $q \le \sqrt{x/5}$, there is an approximation*

$$2\alpha = a'/q' + \delta'/x, \quad (a', q') = 1, \quad |\delta'|/x \le 1/(q')^2 \quad \textit{and} \quad \frac{\delta_0 q}{2} \le q' \le \frac{2x}{5\delta_0 q}, \tag{AP2}$$

where it is possible that a'/q' equals a/q obtained in (AP1).

Proof. We consider two cases:

Case (i): $|\delta| \le 10$. In this case, one has $\delta_0 = 2$ and so we take $a'/q' = a/q$. Then $q' = \delta_0 q/2$ and also $q' = q \le x/5q = 2x/5\delta_0 q$, since $q \le \sqrt{x/5}$ (a hypothesis of Theorem 2.11). We also have $|\delta'|/x = |\delta|/x \le 1/q^2 = 1/(q')^2$.

Case (ii): $|\delta| > 10$. In this case, we have $\delta_0 = |\delta|/5$. By Lemma 2.18, there is an approximation a'/q' such that $2\alpha = a'/q' + \delta'/x$ with $|\delta'|/x \le 1/(q')^2$ and

$$\frac{x}{2|\delta|q} \le q' \le \frac{2x}{|\delta|q} = \frac{2x}{5\delta_0 q}.$$

We now show that $\frac{x}{2|\delta|q} \ge \frac{\delta_0 q}{2} = \frac{|\delta|q}{10}$. This is equivalent to $|\delta|q \le \sqrt{5x}$, which is true because $|\delta|q \le x/Q_0$ and $Q_0 \ge \sqrt{x/5}$ from (H5). This proves the lemma. □

2.3.3 Other important lemmas

Lemma 2.20. *Let $1 \leq Y < X$, $\rho > 0$ and $l > -1$ be real numbers. Suppose that $F : \mathbb{R} \to \mathbb{R}$ satisfies (C2) with $x_0 = \log \frac{X}{Y}$ and that*

$$F'(t) \geq 0 \quad \text{for } t > \log \frac{X}{Y}. \tag{C3}$$

Then

$$\sum_{0 \leq m \leq Y-\rho} (m+\rho)^l F\left(\log \frac{X}{m+\rho}\right) \leq Y^{l+1} T^l F\left(\log \frac{X}{Y}\right) \;+\; l^+ Y^l T^{l-1} F\left(\log \frac{X}{Y}\right) + \rho^l F\left(\log \frac{X}{\rho}\right),$$

where $l^+ = \max\{l, 0\}$ and T^l is as in (2.8), i.e., $(T^l F)(x) = \int_0^\infty e^{-t(l+1)} F(x+t)\, dt$.

Moreover, for $l = -1$, we have

$$\sum_{0 \leq m \leq Y-\rho} \frac{1}{m+\rho} F\left(\log \frac{X}{m+\rho}\right) \leq \int_{\log \frac{X}{Y}}^{\log \frac{X}{\rho}} F(t)\, dt + \rho^{-1} F\left(\log \frac{X}{\rho}\right).$$

If $\rho = 0$ and the range of sum is $1 \leq m \leq Y$, we obtain the same bound by looking at the sum $0 \leq m \leq Y - \rho$ with $\rho = 1$ by a change of variable.

Proof. Suppose that $l > -1$. By the Euler summation formula, we have

$$\begin{aligned}\sum_{0 \leq m \leq Y-\rho} (m+\rho)^l F\left(\log \frac{X}{m+\rho}\right) &= \int_{0^-}^{Y-\rho} (t+\rho)^l F\left(\log \frac{X}{t+\rho}\right) dt \\ &\quad + \int_{0^-}^{Y-\rho} \{t\}\, (t+\rho)^{l-1} (lF - F')\left(\log \frac{X}{t+\rho}\right) dt \\ &\quad + \rho^l F\left(\log \frac{X}{\rho}\right) - \{Y-\rho\}\, Y^l F\left(\log \frac{X}{Y}\right).\end{aligned}$$

Bound $\{t\}$ by 1 and ignore F' in the second term as $F' \geq 0$ (also ignore lF if $l < 0$) above. Moreover, the negative term on the third line can be ignored, which gives

$$\sum_{0 \leq m \leq Y-\rho} (m+\rho)^l F\left(\log \frac{X}{m+\rho}\right) \leq \int_\rho^Y t^l F\left(\log \frac{X}{t}\right) dt + l^+ \int_\rho^Y t^{l-1} F\left(\log \frac{X}{t}\right) dt + \rho^l F\left(\log \frac{X}{\rho}\right),$$

where $l^+ = \max\{l, 0\}$. A change of variable $\lambda = \log \frac{Y}{t}$ gives the desired bound.

Now, we consider the case $l = -1$. We have

$$\sum_{0 \leq m \leq Y-\rho} (m+\rho)^{-1} F\left(\log \frac{X}{m+\rho}\right) = \int_\rho^Y \frac{F(\log \frac{X}{t})}{t} dt - \int_\rho^Y \{t-\rho\} \frac{(F+F')(\log \frac{X}{t})}{t^2} dt + \rho^{-1} F\left(\log \frac{X}{\rho}\right) - \{Y-\rho\} Y^{-1} F\left(\log \frac{X}{Y}\right).$$

Ignoring the negative terms and letting $\lambda = \log \frac{X}{t}$, we prove the lemma. □

2.4 Proof of Theorem 2.11

We have

$$S_{\eta,k,f}(\alpha, x) = \sum_{\substack{m < x/u_0 \\ m \text{ odd}}} f(m) \sum_{\substack{n > u_0 \\ n \text{ odd}}} (\log n)^k e(mn\alpha) \eta(mn/x),$$

with α as in (AP1). Let

$$M = \frac{x}{10\delta_0 q}. \tag{2.17}$$

We split the sum $S_{\eta,k,f}$ into three parts S_1, S_2 and S_3, i.e.,

$$S_{\eta,k,f} = \sum_{\substack{m \leq M \\ q \mid m \\ m \text{ odd}}} + \sum_{\substack{m \leq M \\ q \nmid m \\ m \text{ odd}}} + \sum_{\substack{M < m < x/u_0 \\ m \text{ odd}}} = S_1 + S_2 + S_3. \tag{2.18}$$

We write

$$g_m(t) = \eta_{(x/m),k,u_0}(mt/x) = \begin{cases} \eta(mt/x)(\log t)^k, & \text{if } t > u_0, \\ 0, & \text{otherwise,} \end{cases} \tag{2.19}$$

by abuse of notation. Then from (H1) (with a change of variable), we have

$$|g_m|_1 \le \frac{x}{m} P_{0,k,u_0}\left(\log \frac{x}{m}\right), \quad |g_m'| \le P_{1,k,u_0}\left(\log \frac{x}{m}\right), \quad |g_m''|_1 \le \frac{m}{x} P_{2,k,u_0}\left(\log \frac{x}{m}\right).$$

First consider S_1. We note that $\alpha = a/2q + \delta/2x + \gamma$, with $\gamma = 0$ or $1/2$. For the sum over n, we note that $q \mid m$, and therefore

$$\begin{aligned} \sum_{n \text{ odd}} g_m(n) e(mn\alpha) &= \sum_{n \text{ odd}} g_m(n) e\left(mn\left(\frac{a}{2q} + \frac{\delta}{2x} + \gamma\right)\right) \\ &= u \sum_{n \text{ odd}} g_m(n) e\left(\frac{mn\delta}{2x}\right), \end{aligned} \tag{2.20}$$

with $u = e(a/2 + \gamma)$ as both m, n are odd. Also since $(\widehat{\Phi(t)e(t\alpha)}) = \widehat{\Phi}(t - \alpha)$ and

$$\sum_{n \text{ odd}} \Phi(n) = \frac{1}{2} \sum_n \left(\widehat{\Phi}(n) - \widehat{\Phi}(n + 1/2)\right),$$

(2.20) equals

$$\begin{aligned} &\frac{u}{2} \sum_n \left(\widehat{g_m}\left(n - \frac{m\delta}{2x}\right) - \widehat{g_m}\left(n - \frac{m\delta}{2x} + \frac{1}{2}\right)\right) \\ &= u \frac{x}{2m} \sum_n \left(\widehat{\eta_{(x/m),k,u_0}}\left(\frac{xn}{m} - \frac{\delta}{2}\right) - \widehat{\eta_{(x/m),k,u_0}}\left(\frac{xn}{m} - \frac{\delta}{2} + \frac{x}{2m}\right)\right). \end{aligned} \tag{2.21}$$

Using (2.16), the second term in (2.21) can be bounded as:

$$\begin{aligned}
&\le \frac{x}{2m}\sum_n \left|\widehat{\eta_{(x/m),k,u_0}}\left(\frac{xn}{m}-\frac{\delta}{2}+\frac{x}{2m}\right)\right| \le \frac{x}{2m}\sum_n \frac{|\eta''_{(x/m),k,u_0}|_1}{(2\pi)^2\left(\frac{xn}{m}-\frac{\delta}{2}+\frac{x}{2m}\right)^2}\\
&\le \frac{x}{8\pi^2 m}P_{2,k,u_0}\left(\log\frac{x}{m}\right)\frac{m^2}{x^2}\sum_n \frac{1}{\left(n-\frac{m\delta}{2x}+\frac{1}{2}\right)^2}\\
&\le \frac{m}{8\pi^2 x}P_{2,k,u_0}\left(\log\frac{x}{m}\right)\left(16+\sum_{n=1}^{\infty}\left(\frac{1}{\left(n+\frac{1}{4}\right)^2}+\frac{1}{\left(n-\frac{1}{2}\right)^2}\right)\right)\\
&\le 0.307\frac{m}{x}P_{2,k,u_0}\left(\log\frac{x}{m}\right),
\end{aligned}$$

where we use, in the second line

$$\left|\frac{m\delta}{2x}\right| \le M\cdot\frac{|\delta|}{2x} \le \frac{x}{10\delta_0 q}\cdot\frac{|\delta|}{2x} \le 1/4q \le 1/4,$$

as $M = x/10\delta_0 q$ and $|\delta| \le 5\delta_0$.

Next, consider the contribution to the first term of (2.21) from $n \neq 0$, which is

$$\begin{aligned}
&\le \frac{x}{2m}\sum_{n\neq 0}\frac{|\eta''_{(x/m),k,u_0}|_1}{(2\pi)^2\left(\frac{xn}{m}-\frac{\delta}{2}\right)^2} \le \frac{x}{8\pi^2 m}P_{2,k,u_0}\left(\log\frac{x}{m}\right)\frac{m^2}{x^2}\sum_{n\neq 0}\frac{1}{\left(n-\frac{1}{4}\right)^2}\\
&\le 0.048\frac{m}{x}P_{2,k,u_0}\left(\log\frac{x}{m}\right).
\end{aligned}$$

Hence, (2.21) equals

$$\frac{x}{2m}u\cdot\widehat{\eta_{(x/m),k,u_0}}(-\delta/2) + O^*\left(0.355\frac{m}{x}P_{2,k,u_0}\left(\log\frac{x}{m}\right)\right).$$

Summing over $m \le M$, $q \mid m$ and m odd and using the bound $|f(m)| \le \kappa$ in the error term, we get

$$|S_1| \le \frac{x}{2q}\left|\sum_{\substack{m\le M/q\\(m,2q)=1}}\frac{f(m)}{m}\widehat{\eta_{(x/mq),k,u_0}}(-\delta/2)\right| + O^*\left(\frac{0.355\kappa}{x}\sum_{\substack{m\le M\\ q|m\\ m\text{ odd}}} m\,P_{2,k,u_0}\left(\log\frac{x}{m}\right)\right). \tag{2.22}$$

By (2.2), (2.6), (2.7) and (2.16), the main term of (2.22) is

$$
\begin{aligned}
&\leq \frac{x}{2q}\sum_{l=0}^{k}\binom{k}{l}|\widehat{\eta\cdot\log^{k-l}}(-\delta/2)|\left|\sum_{\substack{m\leq M/q\\(m,2q)=1}}\frac{f(m)}{m}\left(\log\frac{x}{mq}\right)^{l}\right|\\
&\leq \frac{x}{2q}\sum_{l=0}^{k}\binom{k}{l}\min\left\{c_{\eta,k-l},\frac{c'_{\eta,k-l}}{\pi|\delta|}\right\}\left|\sum_{\substack{m\leq M/q\\(m,2q)=1}}\frac{f(m)}{m}\left(\log\frac{x}{mq}\right)^{l}\right|\\
&\leq \frac{x}{2q}\sum_{l=0}^{k}\binom{k}{l}\min\left\{c_{\eta,k-l},\frac{c'_{\eta,k-l}}{\pi|\delta|}\right\}\sum_{l'=0}^{l}\binom{l}{l'}\left(\log\frac{x}{M}\right)^{l-l'}\left|m_{2q,l'}\left(\frac{M}{q},f\right)\right|\\
&\leq \frac{x}{2\delta_0 q}\sum_{l=0}^{k}\sum_{l'=0}^{l}\binom{k}{l}\binom{l}{l'}b_{\eta,k-l}\,(\log 10\delta_0 q)^{l-l'}\left|m_{2q,l'}\left(\frac{x}{10\delta_0 q^2},f\right)\right|.
\end{aligned}
\tag{2.23}
$$

It remains to bound the error term of (2.22). This is at most

$$
\begin{aligned}
&\frac{0.355\kappa M}{x}\sum_{\substack{m\leq M\\ q|m}}P_{2,k,u_0}\left(\log\frac{x}{m}\right)\leq\frac{0.355\kappa M}{x}\sum_{1\leq m\leq M/q}P_{2,k,u_0}\left(\log\frac{x}{mq}\right)\\
&\leq\frac{0.355\kappa M}{x}\left(\frac{M}{q}T^0P_{2,k,u_0}\left(\log\frac{x}{M}\right)+P_{2,k,u_0}\left(\log\frac{x}{q}\right)\right)\\
&\leq\frac{0.355x\kappa}{10^2\delta_0^2q^3}T^0P_{2,k,u_0}(\log 10\delta_0 q)+\frac{0.355\kappa}{10\delta_0 q}P_{2,k,u_0}(\log x),
\end{aligned}
\tag{2.24}
$$

where we apply Lemma 2.20 with $F=P_{2,k,u_0}$, $l=0$, $X=x/q$, $Y=M/q$ and $\rho=1$. The condition (C3) of Lemma 2.20 holds from (H3) and the fact that $X/Y=x/M=10\delta_0 q>u_0$ (since $u_0\leq\delta_0 q$ holds from (H5)).

For S_2, we apply Theorem 2.13 to the n-sum with g_m as in (2.19). Let

$$
K=\min\left\{\frac{q}{2},\frac{\rho_0 x}{q}\right\},\quad\text{where}\quad\rho_0=\frac{\pi}{e}\cdot\frac{C_{1,k,\eta}}{C_{2,k,\eta}}.
\tag{2.25}
$$

Then clearly $K\geq 1/2$ (since $\rho_0\geq 1000\pi/e$ from (H4) and $q\leq Q_0\leq x/10^6$ from (H5)). We now split the m-sum into two parts, namely $m\leq K$ and $K<m\leq M$,

and then use $|f(m)| \leq \kappa$ from (H5), to obtain

$$\frac{|S_2|}{\kappa} \leq \frac{1}{2} \sum_{\substack{m \leq K \\ q \nmid m}} \min \left\{ \frac{|g'_m|_1}{|\sin 2\pi m\alpha|}, \frac{|g''_m|_1}{(\sin 2\pi m\alpha)^2} \right\}$$
$$+ \frac{1}{2} \sum_{\substack{K < m \leq M \\ q \nmid m}} \min \left\{ |g_m|_1 + |g'_m|_1, \frac{|g''_m|_1}{(\sin 2\pi m\alpha)^2} \right\} = S_{21} + S_{22}.$$

We first consider S_{21}. We use Lemma 2.16 (b) with $|g'_m|_1 \leq P_{1,k,u_0}(\log \frac{x}{m}) \leq P_{1,k,u_0}(\log x) = B$ and $|g''_m|_1 \leq (x/m)^{-1} P_{2,k,u_0}(\log \frac{x}{m}) \leq (x/K)^{-1} P_{2,k,u_0}(\log x/K) = C$, since P_{j,k,u_0} is increasing from (H3) and $P_{j,k,u_0}(\log y)/y$ is decreasing from Remark 2.5 (because $x/K \geq 2x/q \geq 2 \cdot 10^6 > e^k$, for $k = 1, 2, 3$). We need to verify $K \leq x/2|\delta|q$ (to ensure $y_2 \leq x/2|\delta|q$), which holds since $K \leq q/2$ and $q/2 \leq x/2|\delta|q$ (as $|\delta|/x \leq 1/q^2$). We also need to verify the condition $\pi B/e \geq Cq$, i.e.,

$$\frac{\pi P_{1,k,u_0}(\log x)}{e} \geq \frac{q\, P_{2,k,u_0}(\log \frac{x}{K})}{x/K}.$$

By (H4), the above is true if $K \leq x/q \cdot \pi/e \cdot C_{1,k,\eta}/C_{2,k,\eta} = \rho_0 x/q$, which holds by the definition of K. Therefore, from Lemma 2.16 (b), we have

$$S_{21} \leq \frac{2q}{\pi} P_{1,k,u_0}(\log x). \tag{2.26}$$

For S_{22}, it follows from Remark 2.5, that $(m/x)P_{2,k,u_0}\left(\log \frac{x}{m}\right)$ is increasing in m

for $m \le M$ (as $x/m \ge x/M = 10\delta_0 q \ge 20 > e^k$, for $k = 1,2,3$) and therefore

$$
\begin{aligned}
S_{22} &\le \frac{1}{2} \sum_{\substack{K<m\le M \\ q\nmid m}} \min\left\{ \frac{x}{m} P_{0,k,u_0}\left(\log\frac{x}{m}\right) + P_{1,k,u_0}\left(\log\frac{x}{m}\right), \frac{\frac{m}{x} P_{2,k,u_0}\left(\log\frac{x}{m}\right)}{(\sin 2\pi m\alpha)^2} \right\} \\
&\le \frac{1}{2} \sum_{j=0}^{M/q-\rho} \sum_{\substack{jq<m-q/2\le(j+1)q \\ q\nmid m}} \min\left\{ \frac{x/q}{j+\rho} P_{0,k,u_0}\left(\log\frac{x/q}{j+\rho}\right) + P_{1,k,u_0}\left(\log\frac{x/q}{j+\rho}\right), \right. \\
&\qquad\qquad \left. \frac{\frac{(j+\rho)q}{x} P_{2,k,u_0}\left(\log\frac{x/q}{j+\rho}\right)}{(\sin 2\pi m\alpha)^2} \right\},
\end{aligned}
$$

where $\rho = K/q$. We will apply Lemma 2.16(a) to the above sum. The condition $y_2 \le x/2|\delta|q$ is true because $M = \frac{x}{10\delta_0 q} \le \frac{x}{2|\delta|q}$. Therefore,

$$
S_{22} \le \frac{10q^3}{3\pi^2 x} \sum_{j=0}^{M/q-\rho} (j+\rho) P_{2,k,u_0}\left(\log\frac{x/q}{j+\rho}\right)
$$

We bound $j+\rho$ by M/q and apply Lemma 2.20 with $F = P_{2,k,u_0}$, $l = 0$, $X = x/q$, $Y = M/q$ and $\rho = K/q$, to obtain

$$
\begin{aligned}
S_{22} &\le \frac{10q^2 M}{3\pi^2 x}\left(\frac{M}{q} T^0 P_{2,k,u_0}\left(\log\frac{x}{M}\right) + P_{2,k,u_0}\left(\log\frac{x}{K}\right)\right) \\
&\le \frac{x}{60\pi^2\,\delta_0 q} T^0 P_{2,k,u_0}(\log 10\delta_0 q) + \frac{q}{6\pi^2} P_{2,k,u_0}(\log 2x).
\end{aligned} \tag{2.27}
$$

For S_3, we use the alternate approximation (AP2) for 2α, i.e., $2\alpha = a'/q' + \delta'/x$, where $\delta_0 q/2 \le q' \le 2x/5\delta_0 q$. Splitting the sum into intervals of length $2q'$, we write

$$
\begin{aligned}
\frac{S_3}{\kappa} &\le \frac{1}{2} \sum_{j=0}^{\frac{x}{2u_0 q'} - \frac{M}{2q'}} \sum_{\substack{2jq'<m-M\le 2(j+1)q' \\ m \text{ odd}}} \min\left\{ \frac{x}{2q'} \frac{P_{0,k,u_0}\left(\log\frac{x/2q'}{j+\frac{M}{2q'}}\right)}{j+\frac{M}{2q'}} + P_{1,k,u_0}\left(\log\frac{x/2q'}{j+\frac{M}{2q'}}\right), \right. \\
&\qquad\qquad \left. \frac{\frac{(j+1)+\frac{M}{2q'}}{x/2q'} P_{2,k,u_0}\left(\log\frac{x/2q'}{j+\frac{M}{2q'}}\right)}{(\sin 2\pi m\alpha)^2} \right\}
\end{aligned} \tag{2.28}
$$

We use Lemma 2.17 to the inner sum and multiply by $1/2$. Then it is at most

$$\begin{aligned}
&\frac{3\,x/2q'}{j+\frac{M}{2q'}} P_{0,k,u_0}\left(\log \frac{x/2q'}{j+\frac{M}{2q'}}\right) + 3P_{1,k,u_0}\left(\log \frac{x/2q'}{j+\frac{M}{2q'}}\right)\\
&\quad + \frac{2q'}{\pi}\sqrt{\begin{aligned}&\frac{j+1+\frac{M}{2q'}}{j+\frac{M}{2q'}}(P_{0,k,u_0}\cdot P_{2,k,u_0})\left(\log\frac{x/2q'}{j+\frac{M}{2q'}}\right)\\ &\quad+\frac{2q'}{x}\left(j+1+\frac{M}{2q'}\right)(P_{1,k,u_0}\cdot P_{2,k,u_0})\left(\log\frac{x/2q'}{j+\frac{M}{2q'}}\right)\end{aligned}}\\
&= S_{31} \;+\; S_{32}.
\end{aligned} \tag{2.29}$$

First we deal with S_{31} (first line of (2.29)). To sum S_{31} over j, apply Lemma 2.20 to both the terms with $F = P_{0,k,u_0}$, $l=-1$ and $F = P_{1,k,u_0}$, $l=0$, respectively and with $X = x/2q'$, $Y = x/2u_0q'$ and $\rho = M/2q'$, to obtain

$$\begin{aligned}
S_{31} &\le \frac{3x}{2q'}\left(\int_{\log u_0}^{\log\frac{x}{M}} P_{0,k,u_0}(t)\,dt + \frac{2q'}{M}P_{0,k,u_0}\left(\log\frac{x}{M}\right)\right)\\
&\quad + 3\left(\frac{x}{2u_0q'}T^0P_{1,k,u_0}(\log u_0) + P_{1,k,u_0}\left(\log\frac{x}{q'}\right)\right)\\
&\le \frac{3x}{\delta_0 q}\left(\int_{\log u_0}^{\log 10\delta_0 q} P_{0,k,u_0}(t)\,dt + \frac{1}{u_0}T^0P_{1,k,u_0}(\log u_0)\right)\\
&\quad + 30\delta_0 q\, P_{0,k,u_0}(\log 10\delta_0 q) + 3P_{1,k,u_0}(\log x),
\end{aligned} \tag{2.30}$$

where we substitute $M = x/10\delta_0 q$ and use $q' \ge \delta_0 q/2$.

We now bound S_{32}. By using $\sqrt{A+B} \le \sqrt{A} + \frac{B}{2\sqrt{A}}$, S_{32} without the j-sum is

$$\frac{2q'}{\pi}\sqrt{\frac{j+1+\frac{M}{2q'}}{j+\frac{M}{2q'}}P^{(0)}_{k,u_0}\left(\log\frac{x/2q'}{j+\frac{M}{2q'}}\right)} + \frac{2(q')^2}{\pi x}\left(j+\frac{M}{2q'}+1\right)P^{(2)}_{k,u_0}\left(\log\frac{x/2q'}{j+\frac{M}{2q'}}\right), \tag{2.31}$$

where $\sqrt{P_{0,k,u_0}\,P_{1,k,u_0}} = P^{(0)}_{k,u_0}$ and $P_{1,k,u_0}\sqrt{P_{2,k,u_0}/P_{0,k,u_0}} = P^{(2)}_{k,u_0}$ from (H2).

To sum the first term of (2.31) over j, we use the inequality $\sqrt{\frac{j+1+\frac{M}{2q'}}{j+\frac{M}{2q'}}} \leq 1+\frac{1/2}{j+\frac{M}{2q'}}$ and apply Lemma 2.20 with $F = P^{(0)}_{k,u_0}$ and $l = 0$ and $l = -1$, respectively to the two terms with $X = x/2q'$, $Y = x/2u_0q'$ and $\rho = M/2q'$. Then it is at most

$$\begin{aligned} &\frac{2q'}{\pi}\left(\frac{x}{2u_0q'}T^0 P^{(0)}_{k,u_0}(\log u_0) + P^{(0)}_{k,u_0}(\log 10\delta_0 q)\right) \\ &\quad + \frac{q'}{\pi}\left(\int_{\log u_0}^{\log 10\delta_0 q} P^{(0)}_{k,u_0}(t)\,dt + \frac{2q'}{M}P^{(0)}_{k,u_0}(\log 10\delta_0 q)\right) \\ &\leq \frac{x}{\pi u_0}T^0 P^{(0)}_{k,u_0}(\log u_0) + \frac{x}{10\pi\delta_0 q}\left(40P^{(0)}_{k,u_0}(\log 10\delta_0 q) + \int_{\log u_0}^{\log 10\delta_0 q} P^{(0)}_{k,u_0}(t)\,dt\right) \end{aligned} \tag{2.32}$$

where we used $q' \leq 4M$ and $M = x/10\delta_0 q$.

To sum the second term of (2.31), apply Lemma 2.20 with $F = P^{(2)}_{k,u_0}$ and $l = 1$ and $l = 0$, respectively, with $X = x/2q'$, $Y = x/2u_0q'$ and $\rho = M/2q'$. Then, this is

$$\begin{aligned} &\frac{2(q')^2}{\pi x}\left(\left(\frac{x}{2u_0q'}\right)^2 T^1 P^{(2)}_{k,u_0}(\log u_0) + 2\left(\frac{x}{2u_0q'}\right)T^0 P^{(2)}_{k,u_0}(\log u_0)\right. \\ &\quad \left. + \left(1+\frac{M}{2q'}\right)P^{(2)}_{k,u_0}\left(\log\frac{x}{M}\right)\right) \\ &\leq \frac{x}{2\pi u_0^2}T^1 P^{(2)}_{k,u_0}(\log u_0) + \frac{4x}{5\pi u_0\delta_0 q}T^0 P^{(2)}_{k,u_0}(\log u_0) + \frac{9x}{25\pi(\delta_0 q)^2}P^{(2)}_{k,u_0}(\log 10\delta_0 q), \end{aligned} \tag{2.33}$$

From (2.18), (2.22), (2.23), (2.24), (2.26), (2.27), (2.29), (2.30), (2.32), (2.33) and using $q \leq \sqrt{x/5} \leq Q_0$ and observing

$$1.5\cdot 10^5 \leq s = \delta_0 q \leq \max\{2\sqrt{x/5}, x/5Q_0\} = 2\sqrt{x/5}, \tag{2.34}$$

gives

$$|S_{\eta,k,f}(\alpha,x)| \leq x\kappa\left(\frac{T^0 P^{(0)}_{k,u_0}(\log u_0)}{\pi u_0} + \frac{T^1 P^{(2)}_{k,u_0}(\log u_0)}{2\pi u_0^2} + L_{k,u_0}(s,x)\right) + R_{k,q}(s,x,f),$$

where

$$L_{k,u_0}(s,x) = \frac{1}{x}\left(3P_{1,k,u_0}(\log x) + \frac{0.355}{10^6}P_{2,k,u_0}(\log x)\right) + \frac{1}{\sqrt{5x}}\left(\frac{2}{\pi}P_{1,k,u_0}(\log x) + \frac{1}{6\pi^2}P_{2,k,u_0}(\log 2x) + 60P_{0,k,u_0}(\log x)\right) + \frac{1}{s}\Bigg(\frac{4}{\pi}P^{(0)}_{k,u_0}(\log 10s) + \frac{1}{10\pi}\int\limits_{\log u_0}^{\log 10s} P^{(0)}_{k,u_0}(t)\,dt + \frac{4}{5\pi u_0}T^0 P^{(0)}_{k,u_0}(\log u_0) + \frac{1}{60\pi^2}T^0 P_{2,k,u_0}(\log 10s)\Bigg) + \frac{1}{s^2}\left(0.00355T^0 P_{2,k,u_0}(\log 10s) + \frac{9}{25\pi}P^{(2)}_{k,u_0}(\log 10s)\right). \tag{2.35}$$

and $R_{\eta,k,q}$ is as in (2.14), i.e.,

$$R_{\eta,k,q}(s,x,f) \geq \frac{x}{2s}\sum_{i=0}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j} b_{k-i}(\log 10s)^{i-j}\left|m_{2q,j}\left(\frac{x}{10qs}, f\right)\right|.$$

The simplified bound (2.13) for $L_{k,u_0}(s,x)$ is now obtained from Proposition A.9 (we remove the dependence on x), with the constants A_k and B_k being explicitly determined therein.

This completes the proof of Theorem 2.11.

Chapter 3

Type-II sums

In this chapter, we discuss different versions of the large sieve inequality to bound the type-II sums, which take the form of bilinear exponential sums. We use some of the standard results on the large sieve inequality, which includes the version for prime support. We also make use of certain combinatorial results, where the Brun-Titchmarsh theorem plays a central role.

3.1 The bilinear exponential sum

Definition 3.1. Let $\{a_n\}_{n\geq 1}$, $\{b_m\}_{m\geq 1}$ be sequences of complex numbers and let I and J be intervals. A typical sum we consider takes the form

$$S_{\eta_2}(I, J, \alpha) = \sum_{m\in I}\sum_{n\in J} a_n\, b_m e(mn\alpha)\,\eta_2(mn/x), \tag{3.1}$$

where $\eta_2 : [0,1] \to \mathbb{R}$ is defined by (1.10), i.e.,

$$\eta_2(x) = 4\int_0^\infty 1_{[1/2,1]}(t)1_{[1/2,1]}(x/t)\frac{dt}{t} = 4\begin{cases} \log 4x, & x \in (1/4,1/2), \\ \log \frac{1}{x}, & x \in (1/2,1), \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

This is the same smoothing as chosen by Helfgott [HH13] and Tao [Tao14]. One of the main advantages is that it allows us to break the bilinear sum dyadically in the variables m and n, making it easier to apply the large sieve inequalities. In particular,

$$\begin{aligned} S_{\eta_2}(I,J,\alpha) &= 4\sum_{m\in I}\sum_{n\in J} a_n\, b_m e(mn\alpha)\int_0^\infty 1_{[1/2,1]}(nW/x)1_{[1/2,1]}(n/W)\,\frac{dW}{W} \\ &= 4\int_0^1 \left(\sum_{\substack{m\in I\\ m\sim x/W}}\ \sum_{\substack{n\in J\\ n\sim W}} a_n b_m e(mn\alpha)\right)\frac{dW}{W}, \end{aligned}$$

where $m \sim x$ means $x/2 < m \le x$. It is therefore enough to consider sums

$$S(\mathcal{M},\mathcal{N},\alpha) := \sum_{n\in\mathcal{N}}\sum_{m\in\mathcal{M}} a_n b_m e(mn\alpha), \tag{3.3}$$

where $\mathcal{M}$ and $\mathcal{N}$ are intervals satisfying:

$$\mathcal{M} \subseteq [M,2M], \quad \mathcal{N} \subseteq [N,2N] \quad \text{and} \quad MN = x/4, \tag{H6}$$

for some $M, N \ge 1$. We denote by

$$\|\boldsymbol{a}\| := \left(\sum_{n\in\mathcal{N}} |a_n|^2\right)^{1/2} \quad \text{and} \quad \|\boldsymbol{b}\| := \left(\sum_{m\in\mathcal{M}} |b_m|^2\right)^{1/2}. \tag{3.4}$$

We consider (3.3) under different cases depending upon the support of the sequences $\{a_n\}$ and $\{b_m\}$, namely (i) when both sequences are supported on odd numbers, (ii) when one is supported on primes and the other on odd numbers and (iii) when both sequences are supported on the primes.

Various results for (i) and (ii) are present in [HH13, Proposition 5.2.4]. We give bounds of similar nature and will include their proofs. In addition, we prove a few variations in Theorem 3.4, allowing us to obtain better constants in the tail of $S(\mathcal{M}, \mathcal{N}, \alpha)$, i.e., when one of M and N is large and the other small. For (iii), we prove two versions in Proposition 3.16, which lead to Theorem 3.3.

Let $Q_0 > 0$ be a given parameter. By Dirichlet's theorem we have an approximation

$$2\alpha = a/q + \delta/x, \quad |\delta|/x \leq 1/qQ_0, \quad (a,q) = 1, \quad q \leq Q_0, \tag{AP3}$$

and let q be the smallest possible. As in Chapter 2, we define

$$\delta_0 = \delta_0(\alpha, Q_0) = \max\{2, |\delta|/5\}. \tag{3.5}$$

By the Cauchy-Schwarz inequality, we have

$$|S(\mathcal{M}, \mathcal{N}, \alpha)| \leq \|\boldsymbol{b}\| \left(\sum_{m \in \mathcal{M}} \left| \sum_{n \in \mathcal{N}} a_n e(mn\alpha) \right|^2 \right)^{1/2}. \tag{3.6}$$

The second quantity in the above product is estimated by application of a large sieve inequality, which provides a bound of the form

$$\sum_{m \in \mathcal{M}} \left| \sum_{n \in \mathcal{N}} a_n e(mn\alpha) \right|^2 \leq \Delta(\mathcal{M}, \mathcal{N}) \cdot \|\boldsymbol{a}\|^2,$$

for some constant $\Delta(\mathcal{M}, \mathcal{N})$ depending only on $\mathcal{M}$ and $\mathcal{N}$. Therefore,

$$|S(\mathcal{M}, \mathcal{N}, \alpha)| \leq \Delta(\mathcal{M}, \mathcal{N})^{1/2} \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\|. \tag{3.7}$$

3.2 Main results

We now list the main results of this chapter. As was the case in [HH13], our bounds become better as δ becomes larger.

First, we state the version when one of $\{a_n\}$ and $\{b_m\}$ supported on odd numbers.

Theorem 3.2. *Let α be as in (AP3) and $\mathcal{M}$, $\mathcal{N}$ be as in (H6). Let $\{a_n\}$, $\{b_m\}$ be sequences of complex numbers with $\{a_n\}$ supported on the odd numbers and assume one of the following holds:*

(i) $|\delta| \le 10$ *and* $|\delta|/x \le 1/2q^2$, *or*

(ii) $|\delta| \ge 10$ *and* $M + q \le x/|\delta|q$.

Then

$$|S(\mathcal{M},\mathcal{N},\alpha)| \le \left(\frac{N}{2} + 2M + \delta_0 q + \frac{x}{4\delta_0 q}\right)^{1/2} \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\|.$$

Let F_0 be an increasing function that satisfies

$$\frac{q}{\varphi(q)} \le F_0(x), \quad \text{for all } x \ge \max\{3, q\}. \tag{3.8}$$

An explicit choice for F_0 is given in Lemma A.5.

Next, we consider the case when both $\{a_n\}$ and $\{b_m\}$ are supported on primes. In this case, we save two logarithmic factors.

Theorem 3.3. *Let α be as in (AP3) and $\delta_0 = \max\{2, |\delta|/5\}$ be as in (3.5). Let $\{a_n\}$ and $\{b_m\}$ be sequences supported on the primes and $\mathcal{M}$, $\mathcal{N}$ satisfy (H6). If*

$$7\delta_0 q < N \le x/440\delta_0 q \quad (\textit{so that } 110\delta_0 q \le M \le x/28\delta_0 q),$$

then

$$|S(\mathcal{M},\mathcal{N},\alpha)| \le \left(\frac{x\,F_0(\delta_0 q)^2}{\delta_0 q\,\log\frac{M}{10\delta_0 q}\,\log\frac{N}{\delta_0 q}}\right)^{1/2} \|\boldsymbol{a}\|\cdot\|\boldsymbol{b}\|. \tag{3.9}$$

where F_0 is as in (3.8).

The next result is a variant of the case when one variable is supported on the primes. It is useful in saving a constant in the tail of $S(\mathcal{M},\mathcal{N},\alpha)$. We are able to save a factor close to 2 over the standard large sieve for primes.

Theorem 3.4. *Let $75\sqrt{x} \le Q_0 < x/1000$, α be as in (AP3) and δ_0 be as in (3.5). Let $\{a_n\}$ be supported on the odd numbers and $\{b_m\}$ be supported on the primes. Let $\mathcal{M}$, $\mathcal{N}$ be as in (H6) and F_0 be as in (3.8). Then*

(a) If $q \le x/10Q_0$ and $M \ge x/28\delta_0 q$ (so that $N \le 7\delta_0 q$), then

$$|S(\mathcal{M},\mathcal{N},\alpha)| \le \left(\frac{5M}{9} + \frac{8.2x}{\delta_0 q}\right)^{1/2} \left(\frac{F_0(15\delta_0 q)}{\log\frac{x}{448(\delta_0 q)^2}}\right)^{1/2} \|\boldsymbol{a}\|\cdot\|\boldsymbol{b}\|.$$

(b) If $x/10Q_0 < q \le Q_0/100$, we have

$$|S(\mathcal{M},\mathcal{N},\alpha)| \le \left(\frac{5M}{9} + \frac{4.06x}{q}\right)^{1/2} \left(\frac{F_0(30q)}{\log\frac{Q_0}{25q}}\right)^{1/2} \|\boldsymbol{a}\|\cdot\|\boldsymbol{b}\|.$$

(c) If $M \ge 200Q_0$, we have

$$|S(\mathcal{M},\mathcal{N},\alpha)| \le e^{\pi|\delta|} \left(\frac{8M}{15} + \frac{2x}{q}\right)^{1/2} \left(\frac{F_0(30q)}{\log\frac{M}{q}}\right)^{1/2} \|\boldsymbol{a}\|\cdot\|\boldsymbol{b}\|.$$

Remark 3.5. The standard large sieve for primes on the m-variable would have given the factor $(2+\epsilon)M$ instead of $5M/9$ or $8M/15$ in the above scenario, although with $F_0(q)$ instead of $F_0(30q)$ (which does not make much difference since $F_0(x)$ is of the order $\log\log x$).

3.3 Preliminaries

We now give a background and list the known results on the large sieve inequality, including the large sieve for primes. We will provide the proofs in some cases. In a later section, we will discuss some combinatorial results which will aid the proof of Theorems 3.3 and 3.4.

3.3.1 Large sieve inequality

Definition 3.6. For any $x \in \mathbb{R}$, define $\|x\|$ to be the distance of x to the nearest integer, or the norm in $\mathbb{R}/\mathbb{Z}$. More precisely, let

$$\|x\| := \min\{|x-n| : n \in \mathbb{Z}\}.$$

Definition 3.7. A set of points $\{\alpha_r\}_{r\in\mathcal{R}}$ in $\mathbb{R}/\mathbb{Z}$ is said to be *well-spaced* if there is a $\delta > 0$, such that

$$\|\alpha_r - \alpha_s\| \geq \delta, \quad \text{for all } r \neq s.$$

They are alternatively called a set of δ-*spaced* points.

Given a set of δ-spaced points $\{\alpha_r\}_{r\in\mathcal{R}}$, the large sieve problem asks for a bound of the form

$$\sum_{r\in\mathcal{R}} \left| \sum_{n\in\mathcal{N}} a_n e(n\alpha_r) \right|^2 \leq \Delta(\mathcal{R},\mathcal{N}) \sum_{n\in\mathcal{N}} |a_n|^2,$$

for a suitable quantity $\Delta(\mathcal{R},\mathcal{N})$ depending only on the sets $\mathcal{R}$ and $\mathcal{N}$. The large sieve inequality answers this question with $\Delta(\mathcal{R},\mathcal{N}) = N + \delta^{-1}$.

Theorem 3.8 (Large-sieve inequality)**.** *Let $\{\alpha_r\}_{r\in\mathcal{R}}$ be a set of δ-spaced points in $\mathbb{R}/\mathbb{Z}$ and $\{a_n\}$ be a sequence of complex numbers. Let $\mathcal{N}$ be an interval length at* *most N. Then*

$$\sum_{r\in\mathcal{R}}\left|\sum_{n\in\mathcal{N}}a_n e(n\alpha_r)\right|^2 \le (N+\delta^{-1})\sum_{n\in\mathcal{N}}|a_n|^2.$$

Proof. For integer N, see Iwaniec-Kowalski [IK04, Theorem 7.7] (with $N+\delta^{-1}-1$ instead of $N+\delta^{-1}$), Montgomery-Vaughan [MV73, Theorem 1] or Richert [RS76, Theorem 2.3]. When N is not an integer, N is replaced by $N+1$ (as there are most $N+1$ integers in $\mathcal{N}$), which gives the factor $(N+1+\delta^{-1}-1)$, giving the required bound. □

Theorem 3.9 (Weighted large sieve inequality)**.** *Let $\{\alpha_r\}_{r\in\mathcal{R}}$ be a set of points in $\mathbb{R}/\mathbb{Z}$ and let*

$$\delta_r = \min_{\substack{s\in\mathcal{R}\\ s\neq r}}\|\alpha_r-\alpha_s\|.$$

Let $\{a_n\}$ be a sequence of complex numbers and $\mathcal{N}$ be an interval of length at most N. Then

$$\sum_{r\in\mathcal{R}}\left(N+1+3/2\cdot\delta_r^{-1}\right)^{-1}\left|\sum_{n\in\mathcal{N}}a_n e(n\alpha_r)\right|^2 \le \sum_{n\in\mathcal{N}}|a_n|^2.$$

Proof. When N is an integer, this holds (without the 1) due to Montgomery-Vaughan [MV73, Theorem 1] and Richert [RS76, Theorem 2.4]. Again, when N is not a integer, N is replaced by $N+1$, which gives the required version. □

3.3.2 Large sieve inequality for primes

In this section, we discuss the large sieve inequalities for primes. We begin with Montgomery's inequality from [Mon68].

Lemma 3.10 (Montgomery's inequality)**.** *Let r be a squarefree positive integer and let $\{a_n\}$ be supported on integers coprime to r. If $S(\alpha)=\sum_{n\le N}a_n e(n\alpha)$, we have*

$$\frac{1}{\varphi(r)}|S(0)|^2 \le \sideset{}{^*}\sum_{a\,(mod\ r)}\left|S\left(\frac{a}{q}\right)\right|^2.$$

Proof. See [Mon68] or [IK04, Lemma 7.15]. □

Now, we obtain large sieve inequalities for the primes by using Montgomery's inequality. Parts (a) and (c) of the next lemma are similar to Lemma [HH13, Lemma 5.2.1] with change of notation. Part (b) is the large sieve for primes and similar to [HH13, Eq (5.4.3)].

Lemma 3.11. *Let $\{a_n\}$ be a sequence supported on the primes and $\mathcal{N}$ be an interval of length at most N. Let $\{\alpha_m\}_{m\in\mathcal{R}}$ be a set of points in $\mathbb{R}/\mathbb{Z}$. Let $b_m \in \mathbb{Z}$, β_m, $\gamma \in \mathbb{R}$, and $0 < \theta < 1/2$ be such that*

$$\alpha_m = b_m/q + \beta_m + \gamma, \quad \textit{and} \quad |\beta_m - \beta_{m'}| \leq \theta, \ \textit{for all } m, m' \in \mathcal{R}.$$

(a) Suppose $b_m \equiv b_{m'} \pmod{q}$, $m \neq m'$ implies $|\beta_m - \beta_{m'}| \geq \rho$ (set $\rho = \infty$ if b_m's are distinct modulo q) and let $\phi = \min\{1/q - \theta, \rho\}$. Then

$$\sum_{m\in\mathcal{R}} \left| \sum_{n\in\mathcal{N}} a_n e(n\alpha_m) \right|^2 \leq \left(N + \phi^{-1}\right) \sum_{n\in\mathcal{N}} |a_n|^2.$$

Here we do not require $\{a_n\}$ to be supported in primes.

(b) Suppose $\{b_m\}_{m\in\mathcal{R}}$ are all distinct $\pmod{q}$ and $14q < N \leq 5/(4\theta)$. Then

$$\sum_{m\in\mathcal{R}} \left| \sum_{n\in\mathcal{N}} a_n e(n\alpha_m) \right|^2 \leq \frac{q}{\varphi(q)} \frac{2N}{\log \frac{N}{2q}} \sum_{n\in\mathcal{N}} |a_n|^2.$$

(c) Suppose $b_m \equiv b_{m'} \pmod{q}$, $m \neq m'$ implies $|\beta_m - \beta_{m'}| \geq \rho$ and that $1/Nq \leq \theta + \rho < 1/q$. Then

$$\sum_{m\in\mathcal{R}} \left| \sum_{n\in\mathcal{N}} a_n e(n\alpha_m) \right|^2 \leq \frac{2q}{\varphi(q)} \frac{1}{\log \frac{1}{q(\theta+\rho)}} \left(N + \rho^{-1}\right) \sum_{n\in\mathcal{N}} |a_n|^2.$$

Proof. For (a), it is seen that for distinct $m, m' \in \mathcal{R}$

$$\|\alpha_m - \alpha_{m'}\| = \|(b_m - b_{m'})/q + (\beta_m - \beta_{m'})\| \geq \begin{cases} 1/q - \theta, & b_m \not\equiv b_{m'} \pmod q \\ \rho, & b_m \equiv b_{m'} \pmod q. \end{cases}$$

Letting $\phi = \min\{1/q - \rho, \rho\}$ and applying the standard large sieve inequality (Theorem 3.8) gives the required bound.

We now prove (b). Since $\{a_n\}$ are supported on the primes, Montgomery's inequality gives (with $S(x) = \sum_n a_n e(n(\alpha_m)) \cdot e(nx)$)

$$\frac{\mu^2(r)}{\varphi(r)} \left| \sum_{n \in \mathcal{N}} a_n e(n\alpha_m) \right|^2 \leq \sideset{}{^*}\sum_{a' \pmod r} \left| \sum_{n \in \mathcal{N}} a_n e(n(\alpha_m + a'/r)) \right|^2, \tag{3.10}$$

for all $r \leq R \leq \sqrt{N}$ and $(r, q) = 1$. Here R is a parameter to be chosen later. Let

$$\psi(m, a'/r) = \alpha_m + a'/r.$$

This is a double-indexed set over m and the fractions a'/r, with $r \leq R$ and $(r, q) = 1$ of elements in $\mathbb{R}/\mathbb{Z}$. Their separation is at least

$$\begin{aligned} \|\psi(m, a'/r) - \psi(m', a''/r')\| &= \|(b_m - b_{m'})/q + (\beta_m - \beta_{m'}) + (a'/r - a''/r')\| \\ &\geq \begin{cases} 1/qrR - \theta, & a'/r \neq a''/r',\ m \neq m' \\ 1/q - \theta, & a'/r = a''/r'. \end{cases} \end{aligned}$$

Multiply both sides of (3.10) by $(N + 1 + 3/2(1/qrR - \theta)^{-1})^{-1}$ and sum over $m \in \mathcal{R}$

and $r \le R$, $(r,q)=1$ and use the weighted large sieve (Theorem 3.9), to get

$$\begin{aligned}&\left(\sum_{\substack{r\le R\\(r,q)=1}} \frac{\mu^2(r)}{\varphi(r)}\left(N+1+\frac{3}{2}\left(1/qrR-\theta\right)^{-1}\right)^{-1}\right)\sum_{m\in\mathcal{R}}\left|\sum_{n\in\mathcal{N}} a_n e(n\alpha_m)\right|^2\\ &\le \sum_{\substack{r\le R\\(r,q)=1}}\ \sideset{}{^*}\sum_{a' \pmod r}\ \sum_{m\in\mathcal{R}}\left(N+1+\frac{3}{2}\left(1/qrR-\theta\right)^{-1}\right)^{-1}\left|\sum_{n\in\mathcal{N}} a_n e(n(\alpha_m+a'/r))\right|^2\\ &\le \sum_{n\in N}|a_n|^2.\end{aligned} \tag{3.11}$$

Choose

$$R=\left(\frac{N}{3q}\right)^{1/2}>2, \tag{3.12}$$

since $N>14q$. Now, as $\theta\le 5/(4N)$, we have

$$\frac{1}{qrR}-\theta\ge\frac{1}{qrR}-\frac{5}{4N}=\frac{R}{Nr}\left(\frac{N}{qR^2}-\frac{5r}{4R}\right)\ge\frac{R}{Nr}(3-5/4)=\frac{7R}{4Nr}.$$

and therefore

$$N+1+\frac{3}{2}(1/qrR-\theta)^{-1}\le N+1+\frac{6Nr}{7R}\le N+\frac{Nr}{R}+\left(1-\frac{Nr}{7R}\right)<N\left(1+\frac{r}{R}\right).$$

This is because $\frac{Nr}{7R}\ge\frac{N}{7R}=\frac{2\sqrt{Nq}}{7}>\frac{2\sqrt{14}q}{7}>1$, since $N>14q$. Therefore,

$$\begin{aligned}&\sum_{\substack{r\le R\\(r,q)=1}} \frac{\mu^2(r)}{\varphi(r)}\left(N+1+\frac{3}{2}\left(1/qrR-\theta\right)^{-1}\right)^{-1}\\ &\ge\frac{1}{N}\sum_{\substack{r\le R\\(r,q)=1}}\frac{\mu^2(r)}{\varphi(r)}\left(1+\frac{r}{R}\right)^{-1}\ge\frac{1}{N}\frac{\varphi(q)}{q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\left(1+\frac{r}{R}\right)^{-1}\\ &\ge\frac{1}{N}\frac{\varphi(q)}{q}\left(\frac{\log\frac{N}{3q}}{2}+0.25068\right)>\frac{1}{N}\frac{\varphi(q)}{q}\frac{\log\frac{N}{2q}}{2}.\end{aligned}$$

where the inequality $\sum_{r\le R}\mu^2(r)/\varphi(r)\cdot(1+r/R)^{-1}\ge\log R+0.25068$ holds for all

$R \geq 2$ by a lemma of Montgomery-Vaughan [MV73, Lemma 8], for $R \geq 100$ and a verification by Helfgott for $2 \leq R < 100$. Using this in (3.11), we prove (b).

For (c), Montgomery's inequality gives

$$\frac{\mu^2(r)}{\varphi(r)} \left| \sum_{n \in \mathcal{N}} a_n e(n\alpha_m) \right|^2 \leq \sum_{a' \pmod r}^{*} \left| \sum_{n \in \mathcal{N}} a_n e(n(\alpha_m + a'/r)) \right|^2, \tag{3.13}$$

for all $r \leq R < N$, $(r, q) = 1$. Let $\psi(m, a'/r) = \alpha_m + a'/r$. Similar to the earlier cases, we find that when $(m, a'/r) \neq (m', a''/r')$, we have

$$\begin{aligned} \|\psi(m, a'/r) - \psi(m', a''/r')\| &= \left\| \frac{b_m - b_{m'}}{q} + (\beta_m - \beta_{m'}) + a'/r - a''/r' \right\| \\ &\geq \begin{cases} 1/qR^2 - \theta, & b_m \not\equiv b_{m'} \pmod q, \ a'/r \neq a''/r' \\ 1/R^2 - \theta, & b_m \equiv b_{m'} \pmod q, \ a'/r \neq a''/r' \\ \rho, & b_m \equiv b_{m'} \pmod q, \ a'/r = a''/r'. \end{cases} \end{aligned} \tag{3.14}$$

Let

$$R^2 = \frac{1}{q(\theta + \rho)}.$$

The conditions $1/(Nq) \leq \theta + \rho < 1/q$ ensure that $1 < R \leq \sqrt{N}$. We find that the first and third quantities in (3.14) have the smallest value ($= \rho$). Summing (3.13) over r and applying the large sieve inequality along with the bound

$$\sum_{\substack{r \leq R \\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \geq \frac{\varphi(q)}{q} \log R,$$

we prove (c). □

3.3.3 Combinatorial lemmas

Now, we discuss some combinatorial results which will be useful in the proof Theorems 3.3 and 3.4.

We first recall the Brun-Titchmarsh theorem from [MV73, Theorem 2].

Theorem 3.12 (Brun-Titchmarsh)**.** *Let q be a positive integer and $\mathcal{I}$ be an interval with $|\mathcal{I}| > q$. For any residue class $(a, q) = 1$, let $\pi(\mathcal{I}, q, a)$ denote the number of primes in $\mathcal{I}$ congruent to $a \,(\text{mod } q)$. Then, we have*

$$\pi(\mathcal{I}, q, a) \leq \frac{2|\mathcal{I}|}{\varphi(q) \log \frac{|\mathcal{I}|}{q}}.$$

The next lemma provides an upper bound for the maximum number of subsets, the the primes in an interval $\mathcal{I}$ can be partitioned so as to ensure that they satisfy some given properties.

Lemma 3.13. *Let q be a positive integer and $\mathcal{I} = (x, x + y)$ be an interval with $x > q$ and $|\mathcal{I}| = y > q$. Then, we have the following:*

(a) *Let*

$$B_1 = B_1(\mathcal{I}, q) = \frac{2|\mathcal{I}|}{\varphi(q) \log \frac{|\mathcal{I}|}{q}}. \tag{3.15}$$

Then the primes of $\mathcal{I}$ can be partitioned into at most $\lfloor B_1 \rfloor$ subsets S_j, $1 \leq j \leq \lfloor B_1 \rfloor$, such that no two primes in S_j occupy the same residue class mod q.

(b) *Let $3 < L \leq |\mathcal{I}|/q$ be a given parameter and*

$$B_2 = B_2(L, q) = \frac{q}{\varphi(q)} \frac{2L}{\log L}. \tag{3.16}$$

Then one can partition the primes in $\mathcal{I}$ into at most $\lceil B_2 \rceil$ subsets S_j, $1 \leq j \leq \lceil B_2 \rceil$, such that primes congruent mod q in any S_j are separated by at least

Lq. More precisely, if p, p' are distinct primes in S_j satisfying $p \equiv p' \pmod{q}$, then $|p - p'| \geq Lq$.

Proof. First, we prove (a). We assume that S_j's are empty to begin with and partition the primes of $\mathcal{I}$ in the following manner:

- For any residue class $(a, q) = 1$, consider the primes of $\mathcal{I}$ congruent to $a \pmod{q}$, (at most B_1 in number by the Brun-Titchmarsh theorem). Write them in increasing order as $\{p_1(a), \ldots, p_{\lfloor B_1 \rfloor}(a)\}$ and place them in different subsets, i.e., put $p_j(a)$ in S_j for all j. It is possible there may not be these many primes.
- Repeat the above step for all coprime residue classes $a \pmod{q}$.

Having done this process, it is clear that in any given S_j, no two primes can be congruent to the same residue class modulo q.

Now, we prove (b). Let S_j's be empty sets to begin with. We do the following:

- Let $(a, q) = 1$ be a residue class $\pmod{q}$ and enumerate the primes in $\mathcal{I}$, congruent to $a \pmod{q}$ in the increasing order as $\{p_1(a), p_2(a), \ldots, p_k(a)\}$. To place $p_j(a)$, let $j_0 = j \pmod{\lceil B_2 \rceil} \in \{1, 2, \ldots, \lceil B_2 \rceil\}$ and put $p_j(a)$ in S_{j_0}. In other words, put $p_1(a)$ in S_1, $p_2(a)$ in S_2, etc., and put $p_{\lceil B_2 \rceil + 1}$ again in S_1 and continue this cyclically.
- Repeat the above step for all coprime residue classes $a \pmod{q}$.

Claim: If p and p' are distinct primes in S_j with $p \equiv p' \pmod{q}$, then $|p - p'| \geq Lq$.

To prove the claim, first note that since $L > 3$, we have $B_2 > 5$, or $\lceil B_2 \rceil \geq 6$. Let p, p' be distinct primes in S_j satisfying $p \equiv p' \equiv a \pmod{q}$, where $(a, q) = 1$. It then follows by construction that $p = p_i(a)$ and $p' = p_{i'}(a)$, with $i \equiv i' \pmod{\lceil B_2 \rceil}$. As $i \neq i'$, we have $|i - i'| \geq \lceil B_2 \rceil$ and therefore there are at least $\lceil B_2 \rceil \geq 6$ primes

($\equiv a \pmod{q}$) in $J = (p, p']$ (or $(p', p]$). Write $|p - p'| = hq$, with $h \geq 6$. By the Brun-Titchmarsh theorem, there are at most $\frac{2hq}{\varphi(q)\log h}$ primes in J, which are congruent to $a \pmod{q}$. Therefore

$$\frac{2hq}{\varphi(q)\log h} \geq B_2 = \frac{2Lq}{\varphi(q)\log L},$$

from which it follows that $h \geq L$ (as $x/\log x$ is increasing for $x \geq e$) and hence $|p - p'| = hq \geq Lq$. This proves the claim and completes the proof of (b). $\square$

This leads to the following extension:

Lemma 3.14. *Let q be a positive integer and $\mathcal{I} = (x, x+y)$ be an interval with $x > q$ and $|\mathcal{I}| = y > q$. Let $B_1 = B_1(\mathcal{I}, q)$ and $B_2 = B_2(L, q)$ be as in (3.15) and (3.16), respectively. Then for any $d \mid q$, we have the following:*

(a) The primes of $\mathcal{I}$ can be partitioned into at most $\varphi(d)\lfloor B_1 \rfloor$ subsets $S_{j,i}$, $1 \leq j \leq \lfloor B_1 \rfloor$, $1 \leq i \leq \varphi(d)$, such that in any $S_{j,i}$, the primes occupy distinct residue classes (mod q) and any two primes are congruent mod d. When $d = q$, each $S_{j,i}$ has at most one element and the condition holds trivially.

(b) Let $3 < L \leq |\mathcal{I}|/q$ be a given parameter. Then the primes of $\mathcal{I}$ can be partitioned into at most $\varphi(d)\lceil B_2 \rceil$ subsets $S_{j,i}$, $1 \leq j \leq \lceil B_2 \rceil$, $1 \leq i \leq \varphi(d)$ such that in any $S_{j,i}$, distinct primes congruent mod q are separated by at least Lq and any two primes in $S_{j,i}$ are congruent mod d. When $d = q$, any two primes of $S_{j,i}$ are separated by at least Lq (as any two are congruent mod q).

Proof. To prove (a), we apply Lemma 3.13 (a) to partition the primes of $\mathcal{I}$ into at most $\lfloor B_1 \rfloor$ subsets to ensure all primes in any subset S_j occupy different residue classes mod q. Then, we further partition each S_j into $\varphi(d)$ subsets $S_{j,i}$, $1 \leq i \leq \varphi(d)$ depending on the residue class mod d occupied i.e., the primes congruent to the same class mod d are placed in the same subset. This ensures that for any p, p' in $S_{j,i}$, one has $p \equiv p' \pmod{d}$. If $d = q$, then $S_{j,i}$ can have at most one element since primes of S_j were incongruent mod q.

For (b), use Lemma 3.13 (b) to divide $\mathcal{I}$ into at most $\lceil B_2 \rceil$ subsets so that in every S_j, primes congruent mod q are separated by at least Lq. Again, further split every S_j into $\varphi(d)$ more parts depending on the residue class mod d occupied. This ensures that the difference of any p, p' in $S_{j,i}$ is divisible by d. Again, when $d = q$, it means that any two primes of $S_{j,i}$ are separated by at least Lq. □

3.4 Proof of the main results

Now, we give the proof of Theorems 3.2, 3.3 and 3.4

3.4.1 Proof of Theorem 3.2

We prove the following proposition, which yields Theorem 3.2 as a corollary.

Proposition 3.15. *Let α be as in (AP3) and $\{a_n\}$ be a sequence of supported on the odd numbers. Let $\mathcal{M}$ and $\mathcal{N}$ be as in (H6). Then*

(a) Suppose that $|\delta|/x \leq 1/2q^2$ (this holds whenever $q \leq Q_0/2$). Then

$$\sum_{m \in \mathcal{M}} \left| \sum_{n \in \mathcal{N}} a_n e(mn\alpha) \right|^2 \leq \left\lceil \frac{M}{q} \right\rceil \left(\frac{N}{2} + 2q \right) \sum_{n \in \mathcal{N}} |a_n|^2.$$

(b) Suppose $\delta \neq 0$ and $M + q \leq x/|\delta|q$. Then

$$\sum_{m \in \mathcal{M}} \left| \sum_{n \in \mathcal{N}} a_n e(mn\alpha) \right|^2 \leq \left(\frac{N}{2} + \frac{x}{|\delta|q} \right) \sum_{n \in \mathcal{N}} |a_n|^2$$

Proof of Proposition 3.15. Part (a) follows from the standard large sieve. As $\{a_n\}$

is supported on odd numbers, a change of variable in the n-sum makes it

$$\sum_{m\in\mathcal{M}}\left|\sum_{n\in\mathcal{N}'}a_{2n+1}e(mn(2\alpha))\right|^2, \tag{3.17}$$

where $\mathcal{N}'$ denotes the set $\{(n-1)/2 : n\in\mathcal{N}\}$, which is contained in an interval of length at most $N/2$. We apply Lemma 3.11 (b) with

$$\alpha_m = m(2\alpha) = b_m/q + \beta_m + \gamma, \quad \text{where } b_m = ma,\ \beta_m = m\delta/x,\ \gamma = 0. \tag{3.18}$$

To ensure that b_m's are distinct mod q, $\mathcal{M}$ is divided into intervals of length q (at most $\lceil M/q\rceil$ in number), which ensures that any $m\neq m'$ in such an interval are distinct mod q. Then $|\beta_m-\beta_{m'}| = |(m-m')\delta|/x \le q|\delta|/x \le 1/2q = \theta$, since $|\delta|/x \le 1/2q^2$. Applying Lemma 3.11 (b) with $\phi = 1/q-\theta = 1/2q$ for all such intervals, we get

$$\sum_{m\in\mathcal{M}}\left|\sum_{\substack{n\in\mathcal{N}\\ n\text{ odd}}}a_n e(mn\alpha)\right|^2 \le \left\lceil\frac{M}{q}\right\rceil\left(\frac{N}{2}+2q\right)\sum_{n\in\mathcal{N}'}|a_{2n+1}|^2 = \left\lceil\frac{M}{q}\right\rceil\left(\frac{N}{2}+2q\right)\sum_{n\in\mathcal{N}}|a_n|^2.$$

For (b), again a change of variable reduces this to (3.17) and the n-sum runs over $\mathcal{N}'$ of length at most $N/2$. We use Lemma 3.11 (a) with α_m from (3.18). For $m,m'\in\mathcal{M}$, we have $|\beta_m-\beta_{m'}| = |(m-m')\delta/x| \le M|\delta|/x = \theta$. Moreover, if $b_m\equiv b_{m'} \pmod q$, $m\neq m'$, then $m\equiv m' \pmod q$ which means $|\beta_m-\beta_{m'}|\ge q|\delta|/x = \rho$. We also have $\theta+\rho = M|\delta|/x + q|\delta|/x \le 1/q$ by hypothesis. Therefore, we get

$$\begin{aligned}\sum_{m\in\mathcal{M}}\left|\sum_{\substack{n\in\mathcal{N}\\ n\text{ odd}}}a_n e(mn\alpha)\right|^2 &= \sum_{m\in\mathcal{M}}\left|\sum_{n\in\mathcal{N}'}a_{2n+1}e(mn(2\alpha))\right|^2 \le \left(\frac{N}{2}+\frac{x}{|\delta|q}\right)\sum_{n\in\mathcal{N}'}|a_{2n+1}|^2\\ &\le \left(\frac{N}{2}+\frac{x}{|\delta|q}\right)\sum_{n\in\mathcal{N}}|a_n|^2.\end{aligned}$$

This completes the proof. □

Proof of Theorem 3.2. By the Cauchy-Schwarz inequality, we reduce to (3.6) and apply the large sieve to the second quantity. This gives the bound (3.7) with $\Delta(\mathcal{M},\mathcal{N})$ as obtained from Proposition 3.15.

If (i) holds, then $\delta_0 = 2$ (as $|\delta| \le 10$) and Proposition 3.15 (a) gives

$$\Delta(\mathcal{M},\mathcal{N}) = \left\lceil \frac{M}{q} \right\rceil \left(\frac{N}{2} + 2q\right) = \left(1 + \frac{M}{q}\right)\left(\frac{N}{2} + 2q\right) = \frac{N}{2} + 2M + \delta_0 q + \frac{x}{4\delta_0 q}.$$

If (ii) holds, then $\delta_0 = |\delta|/5$ and Proposition 3.15 (b) gives

$$\Delta(\mathcal{M},\mathcal{N}) = \frac{N}{2} + \frac{x}{|\delta| q} = \frac{N}{2} + \frac{x}{5\delta_0 q}.$$

Comparing the above bounds, we see that the bound in (i) is larger. This completes the proof. □

Next, we prove Theorem 3.3.

3.4.2 Proof of Theorem 3.3

We prove the following proposition, which immediately yields Theorem 3.3.

Proposition 3.16. *Let α be as in (AP3), $\{a_n\}$ be a sequence supported on the primes and $\mathcal{M}$ and $\mathcal{N}$ be as in (H6). Then*

(a) If $|\delta| \le 10$ and $14q < N \le x/56q$ (so that $14q \le M < x/56q$), then

$$\sum_{\substack{m\in\mathcal{M} \\ m \text{ prime}}} \left| \sum_{n\in\mathcal{N}} a_n e(mn\alpha) \right|^2 \le \frac{q}{\varphi(q)} \frac{2q}{\varphi(2q)} \frac{x}{2q \log \frac{M}{2q} \log \frac{N}{2q}} \sum_{n\in\mathcal{N}} |a_n|^2.$$

(b) If $|\delta| \geq 10$ and $2|\delta|q/5 \leq N \leq x/88|\delta|q$ (so that $22|\delta|q \leq M \leq 5x/8|\delta|q$), then

$$\sum_{\substack{m\in\mathcal{M}\\ m\ prime}} \left|\sum_{n\in\mathcal{N}} a_n e(mn\alpha)\right|^2 \leq \frac{q}{\varphi(q)} \frac{2q}{\varphi(2q)} \frac{5x}{|\delta|q \log\frac{M}{2|\delta|q} \log\frac{7N}{|\delta|q}} \sum_{n\in\mathcal{N}} |a_n|^2.$$

Proof of Proposition 3.16. We need to save logarithmic factors over both the variables m and n. For the n-sum, the large sieve for primes gives the saving and for the m-sum, saving comes from the Brun-Titchmarsh Theorem.

Let us prove (a). We make use of Lemma 3.11 (a). As $2\alpha = a/q + \delta/x$, we have

$$\alpha_m = m\alpha = b_m/q + \beta_m + \gamma, \quad \text{where } b_m = \frac{(m-1)a}{2}, \ \ \beta_m = \frac{(m-1)\delta}{2x}, \ \ \gamma = \alpha. \tag{3.19}$$

We have to ensure that b_m's are all distinct $\pmod q$, which is the same as $2q \nmid m - m'$. We use Lemma 3.13 for this purpose. Let

$$B_1 = \frac{2M}{\varphi(2q)\log\frac{M}{2q}}.$$

From Lemma 3.13 (a) with $\mathcal{I} = \mathcal{M}$ and $2q$ in place of q, we can partition the primes of $\mathcal{M}$ into subsets S_j, $1 \leq j \leq \lfloor B_1 \rfloor$ such that primes in every S_j occupy distinct residue classes mod $2q$, i.e., b_m, $m \in S_j$ are distinct $\pmod q$. Moreover, the errors β_m and $\beta_{m'}$ are separated by at most $|(m-m')\delta|/2x \leq M|\delta|/2x = \theta$. We apply Lemma 3.11 (a) for each S_j with α_m as in (3.19) and $\theta = M|\delta|/2x = |\delta|/(8N)$. The conditions $14q < N \leq 5/(4\theta) = 10N/|\delta|$ hold since $|\delta| \leq 10$. Therefore,

$$\sum_{\substack{m\in\mathcal{M}\\ m \text{ prime}}} \left|\sum_{n\in\mathcal{N}} a_n e(mn\alpha)\right|^2 \leq B_1 \cdot \frac{q}{\varphi(q)} \frac{2N}{\log\frac{N}{2q}} \sum_{n\in\mathcal{N}} |a_n|^2 = \frac{q}{\varphi(q)} \frac{2M}{\varphi(2q)\log\frac{M}{2q}} \frac{2N}{\log\frac{N}{2q}} \sum_{n\in\mathcal{N}} |a_n|^2$$
$$\leq \frac{q}{\varphi(q)} \frac{2q}{\varphi(2q)} \frac{x}{2q\log\frac{M}{2q}\log\frac{N}{2q}} \sum_{n\in\mathcal{N}} |a_n|^2,$$

where we use $MN = x/4$ from (H6).

To prove (b), we use Lemma 3.11 (c). Recall that $\alpha_m = m\alpha$ as in (3.19). Again, $b_m \equiv b_{m'} \pmod q$ is the same as $m \equiv m' \pmod{2q}$. Let

$$L = \frac{M}{2|\delta|q} \geq 11 \quad \text{and} \quad B_2 = \frac{2q}{\varphi(2q)} \frac{2L}{\log L} \geq 9.$$

Using Lemma 3.13 (b) with $\mathcal{I} = \mathcal{M}$ and with $2q$ in place of q, we partition the primes of $\mathcal{M}$ into at most $\lceil B_2 \rceil \leq 10B_2/9$ subsets S_j, $1 \leq j \leq \lceil B_2 \rceil$ such that for distinct primes m, m' in S_j satisfying $m \equiv m' \pmod{2q}$ (or $b_m \equiv b_{m'} \pmod q$), we have $|m - m'| \geq 2Lq$. This means that

$$|\beta_m - \beta_{m'}| = \left| \frac{(m-m')\delta}{2x} \right| \geq \frac{Lq|\delta|}{x} = \rho,$$

whenever $b_m \equiv b_{m'} \pmod q$ with $m \neq m'$. We already know that $|\beta_m - \beta_{m'}| \leq \theta = M|\delta|/2x$ from the proof of (a). Also

$$\theta + \rho = \frac{Lq|\delta|}{x} + \frac{M|\delta|}{2x} = \frac{M(1+|\delta|)}{2x} \leq \frac{11M|\delta|}{20x} < 1/q,$$

since $1 + |\delta| \leq 11|\delta|/10$, for $|\delta| \geq 10$ and $M \leq 5x/8|\delta|q$. Similarly, we have $\theta + \rho = M(1+|\delta|)/2x > 1/Nq$ since $|\delta| \geq 10$ and $MN = x/4$. Therefore, the conditions of Lemma 3.11 (c) hold and applying it for each subset S_j (at most $\lceil B_2 \rceil \leq 10B_2/9$ in number), we get

$$\begin{aligned}
\sum_{\substack{m \in \mathcal{M} \\ m \text{ prime}}} \left| \sum_{n \in \mathcal{N}} a_n e(mn\alpha) \right|^2 &\leq \frac{10B_2}{9} \cdot \frac{2q}{\varphi(q)} \frac{\left(N + \frac{x}{L|\delta|q}\right)}{\log \frac{20x}{11M|\delta|q}} \sum_{n \in \mathcal{N}} |a_n|^2 \\
&= \frac{20}{9} \frac{q}{\varphi(q)} \frac{2q}{\varphi(2q)} \frac{2L}{\log L} \frac{\left(N + \frac{x}{L|\delta|q}\right)}{\log \frac{20x}{11M|\delta|q}} \sum_{n \in \mathcal{N}} |a_n|^2 \\
&\leq \frac{q}{\varphi(q)} \frac{2q}{\varphi(2q)} \frac{5x}{|\delta|q \log \frac{M}{2|\delta|q} \log \frac{7N}{|\delta|q}} \sum_{n \in \mathcal{N}} |a_n|^2,
\end{aligned}$$

where we use

$$N + \frac{x}{L|\delta|q} = N + \frac{2x}{M} = 9N \quad \text{and} \quad \log\frac{20x}{11M|\delta|q} = \log\frac{80N}{11|\delta|q} > \log\frac{7N}{|\delta|q}.$$

This completes the proof. □

Proof of Theorem 3.3. From Proposition 3.16, the bound (3.7) holds with

$$\Delta(\mathcal{M},\mathcal{N}) = \begin{cases} \frac{q}{\varphi(q)}\frac{2q}{\varphi(2q)}\frac{x}{2q\log\frac{M}{2q}\log\frac{N}{2q}}, & |\delta| \le 10 \\ \frac{q}{\varphi(q)}\frac{2q}{\varphi(2q)}\frac{5x}{|\delta|q\log\frac{M}{2|\delta|q}\log\frac{7N}{|\delta|q}}, & |\delta| > 10. \end{cases}$$

Substituting $\delta_0 = 2$ when $|\delta| \le 10$ and $\delta_0 = |\delta|/5$ when $|\delta| > 10$, and comparing the log factors, we get the required bound. □

3.4.3 Proof of Theorem 3.4

For Theorem 3.4, we will need the following proposition:

Proposition 3.17. *Let $75\sqrt{x} \le Q_0 < x/1000$ and α be as in (AP3). Let $l_0 \ge 1$ be a square-free positive integer and $\mathcal{M}$, $\mathcal{N}$ be as in (H6). Let $\{a_n\}$ be a sequence supported on odd numbers and F_0 be as in (3.8). Then we have the following:*

(a) Suppose $q \le Q_0/100$ and let

$$M_0 = \frac{1}{25}\max\left\{\frac{x}{10q}, Q_0\right\} = \begin{cases} \frac{x}{250q}, & q \le x/10Q_0, \\ \frac{Q_0}{25}, & x/10Q_0 < q \le Q_0/100. \end{cases} \tag{3.20}$$

If $|\delta| \le 10$ and $M \ge M_0$ (so that $N \le x/4M_0$), we have

$$\sum_{\substack{m\in\mathcal{M} \\ m \text{ prime}}} \left|\sum_{n\in\mathcal{N}} a_n e(mn\alpha)\right|^2 \le \left(\frac{25M}{12}\frac{\varphi(l_0)}{l_0} + 128M_0\varphi(l_0)\right)\frac{F_0(l_0q)}{\log\frac{M_0}{q}}\sum_{n\in\mathcal{N}}|a_n|^2.$$

(b) Suppose $|\delta| \geq 10$ (which implies $q \leq x/10Q_0$) and that $M \geq 5x/28|\delta|q$ (so that $N \leq 7|\delta|q/5$). Then

$$\sum_{\substack{m\in\mathcal{M}\\ m \text{ prime}}} \left|\sum_{n\in\mathcal{N}} a_n e(mn\alpha)\right|^2 \leq \left(\frac{25M}{12}\frac{\varphi(l_0)}{l_0} + \frac{101\,x}{24|\delta|q}\varphi(l_0)\right)\frac{F_0(l_0q)}{\log\frac{32x}{99(|\delta|q)^2}}\sum_{n\in\mathcal{N}}|a_n|^2.$$

(c) Suppose $\delta = 0$, i.e., $2\alpha = a/q$ and that $M \geq 200Q_0$. Then

$$\sum_{\substack{m\in\mathcal{M}\\ m \text{ prime}}} \left|\sum_{n\in\mathcal{N}} a_n e\left(mn\alpha\right)\right|^2 \leq \left(2M\frac{\varphi(l_0)}{l_0} + \frac{x}{4q}\varphi(l_0)\right)\frac{F_0(l_0q)}{\log\frac{M}{q}}\sum_{n\in\mathcal{N}}|a_n|^2.$$

Proof of Proposition 3.17. The proceeds in a similar manner to Theorem 3.3.

We first prove (a). As $\{a_n\}$ are supported on odd numbers, the sum reduces to (3.17). So, we apply Lemma 3.11 (b) with α_m given in (3.18). With M_0 in (3.20), we have $M_0 > q$ in both cases because $q \leq Q_0/100$ and $Q_0 \geq 75\sqrt{x}$. We split $\mathcal{M}$ into at at most $\lceil M/M_0\rceil$ intervals of length M_0. For any such interval $\mathcal{M}_0$, let

$$B_1 = \frac{2M_0}{\varphi(q)\log\frac{M_0}{q}}.$$

Let $g = (l_0, q)$. Then by Lemma 3.14 (a) with $\mathcal{I} = \mathcal{M}_0$ and $d = g = (l_0, q)$, we can partition the primes of $\mathcal{M}_0$ into at most $\varphi(g)\lfloor B_1\rfloor$ subsets $S_{j,i}$ such that primes in $S_{j,i}$ occupy distinct residue classes mod q and are congruent mod g (when $g = q$, $S_{j,i}$ has at most one element and the separation between α_m is ∞). Therefore, for $m \neq m'$ in $S_{j,i}$, we have

$$\|\alpha_m - \alpha_{m'}\| = \left\|\frac{a(m-m')}{q} + \frac{(m-m')\delta}{x}\right\| \geq \frac{g}{q} - \frac{M_0|\delta|}{x} \geq \frac{24g}{25q},$$

since $g \mid m - m'$ and

$$\frac{M_0|\delta|}{x} \leq \begin{cases} \frac{|\delta|}{250q} \leq \frac{10}{250q} \leq \frac{g}{25q}, & q < x/10Q_0, \\ \frac{Q_0}{25qQ_0} \leq \frac{1}{25q} \leq \frac{g}{25q}, & x/10Q_0 < q \leq Q_0/100. \end{cases}$$

Therefore,

$$\sum_{m \in S_{j,i}} \left| \sum_{n \in \mathcal{N}} a_n e(mn\alpha) \right|^2 = \sum_{m \in S_{j,i}} \left| \sum_{n \in \frac{\mathcal{N}-1}{2}} a_{2n+1} e(2mn\alpha) \right|^2 \leq \left(\frac{N}{2} + \frac{25}{24} \frac{q}{g} \right) \sum_{n \in \mathcal{N}} |a_n|^2.$$

Summing the above over all $1 \leq i \leq \varphi(g)$, $1 \leq j \leq \lfloor B_1 \rfloor$ and over all intervals of length M_0 (at most $\lceil M/M_0 \rceil$ in number), we have

$$\begin{aligned} \sum_{\substack{m \in \mathcal{M} \\ m \text{ prime}}} \left| \sum_{n \in \mathcal{N}} a_n e(mn\alpha) \right|^2 &\leq \left\lceil \frac{M}{M_0} \right\rceil \varphi(g) B_1 \left(\frac{N}{2} + \frac{25}{24} \frac{q}{g} \right) \sum_{n \in \mathcal{N}} |a_n|^2 \\ &\leq \varphi(g) \frac{2M_0 \lceil M/M_0 \rceil}{\varphi(q) \log \frac{M_0}{q}} \left(\frac{N}{2} + \frac{25}{24} \frac{q}{g} \right) \sum_{n \in \mathcal{N}} |a_n|^2 \\ &\leq \left(\varphi(g) \frac{N(M + M_0)}{q \log \frac{M_0}{q}} + \frac{25}{12} \frac{\varphi(g)}{g} \frac{M + M_0}{\log \frac{M_0}{q}} \right) \frac{q}{\varphi(q)} \sum_{n \in \mathcal{N}} |a_n|^2, \end{aligned} \tag{3.21}$$

where we are using $M_0 \lceil M/M_0 \rceil \leq M + M_0$. Since $(q, l_0/g) = 1$, one has

$$\frac{q}{\varphi(q)} = \frac{\varphi(l_0/g)}{l_0/g} \frac{ql_0/g}{\varphi(ql_0/g)} \leq \frac{\varphi(l_0/g)}{l_0/g} F_0(l_0 q). \tag{3.22}$$

Therefore, from (3.21) (and using $MN = x/4$ from (H6)), we obtain

$$\begin{aligned} \sum_{\substack{m \in \mathcal{M} \\ m \text{ prime}}} \left| \sum_{n \in \mathcal{N}} a_n e(mn\alpha) \right|^2 \leq &\frac{25M}{12} \frac{F_0(l_0 q)}{\log \frac{M_0}{q}} \frac{\varphi(l_0)}{l_0} \sum_{n \in \mathcal{N}} |a_n|^2 \\ &+ \left(\frac{x}{4q} + M_0 \left(\frac{N}{q} + \frac{25}{12 l_0} \right) \right) \frac{\varphi(l_0) F_0(l_0 q)}{\log \frac{M_0}{q}} \sum_{n \in \mathcal{N}} |a_n|^2 \end{aligned}$$

Using the bounds $M_0N/q \le x/4q$ and $x/4q \le 250M_0/4$ with $l_0 \ge 1$, we prove (a).

The proof of (b) is similar to that of (a). Again, as $\{a_n\}$ are supported on odd numbers, the given expression reduces to (3.17). We use Lemma 3.11 (b) with α_m's in (3.18). Let $g = (l_0, q)$ and let

$$M_0' = \min\left\{\frac{32gx}{33|\delta|q}, M\right\} \quad \text{and} \quad L = \frac{5M_0'}{16|\delta|q} = \min\left\{\frac{10gx}{33(|\delta|q)^2}, \frac{5M}{16|\delta|q}\right\}. \tag{3.23}$$

Now split $\mathcal{M}$ into at most $\lceil M/M_0' \rceil$ intervals of length M_0'. For any such interval $\mathcal{M}_0'$, we use Lemma 3.14 (b). It is seen that $L < M_0'/q$, since $|\delta| \ge 10$. We also have

$$L = \begin{cases} \dfrac{10gx}{33(|\delta|q)^2} \ge \dfrac{10gx}{33(x/Q_0)^2} \ge \dfrac{10 \cdot (75)^2 g}{33} > 300, & M_0' < M, \\ \dfrac{5M}{16|\delta|q} \ge \dfrac{25x}{448(|\delta|q)^2} \ge \dfrac{25x}{448(x/Q_0)^2} > 300, & M_0' = M, \end{cases}$$

as $M \ge 5x/28|\delta|q$, $|\delta|q \le x/Q_0$ and $Q_0 \ge 75\sqrt{x}$. Let

$$B_2 = \frac{q}{\varphi(q)} \frac{2L}{\log L} \ge 100,$$

since $L > 300$. Hence, $\lceil B_2 \rceil \le 1 + B_2 \le 1.01B_2$. We apply Lemma 3.14 (b) with $\mathcal{I} = \mathcal{M}_0'$, L from (3.23) and $d = g = (l_0, q)$ to partition $\mathcal{M}_0'$ into at most $\varphi(g)\lceil B_2 \rceil$ subsets $S_{j,i}$, such that any two primes in $S_{j,i}$ are congruent mod g and distinct primes $m \equiv m' \pmod q$ satisfy $|m - m'| \ge Lq$. This implies

$$|\beta_m - \beta_{m'}| = \left|\frac{(m-m')\delta}{x}\right| \ge \frac{Lq|\delta|}{x} = \rho, \quad \text{say},$$

whenever $b_m \equiv b_m' \pmod q$, $m \ne m'$. Also, in any $S_{j,i}$, we have $|\beta_m - \beta_{m'}| \le M_0'|\delta|/x \le 32g/33q = \theta$, say. Therefore, the α_m's in $S_{j,i}$ are separated by at least

$$\|\alpha_m - \alpha_m'\| = \left\|\frac{(m-m')a}{q} + \frac{(m-m')\delta}{x}\right\| \ge \begin{cases} g/q - \theta = \frac{g}{33q}, & b_m \not\equiv b_{m'} \pmod q \\ \frac{L|\delta|q}{x}, & b_m \equiv b_{m'} \pmod q. \end{cases}$$

It is easily seen that $\frac{g}{33q} \geq \frac{10g}{33|\delta|q} \geq \frac{L|\delta|q}{x} = \rho$ since $|\delta| \geq 10$ and $L \leq \frac{10gx}{33(|\delta|q)^2}$ by definition of L. Therefore, the large sieve applied to each $S_{j,i}$ gives

$$\sum_{m\in S_{j,i}} \left|\sum_{n\in\mathcal{N}} a_n e(mn\alpha)\right|^2 \leq \left(\frac{N}{2} + \frac{x}{L|\delta|q}\right) \sum_{n\in\frac{\mathcal{N}-1}{2}} |a_{2n+1}|^2 = \left(\frac{N}{2} + \frac{x}{L|\delta|q}\right) \sum_{n\in\mathcal{N}} |a_n|^2.$$

Summing over all $S_{j,i}$ and all intervals $\mathcal{M}_0'$, we get

$$\begin{aligned}
\sum_{\substack{m\in\mathcal{M}\\ m \text{ prime}}} \left|\sum_{n\in\mathcal{N}} a_n e(mn\alpha)\right|^2 &\leq 1.01 B_2 \left\lceil \frac{M}{M_0'} \right\rceil \varphi(g) \left(\frac{N}{2} + \frac{x}{L|\delta|q}\right) \sum_{n\in\mathcal{N}} |a_n|^2 \\
&= 1.01 \varphi(g) \frac{q}{\varphi(q)} \frac{2L}{\log L} \left\lceil \frac{M}{M_0'} \right\rceil \left(\frac{N}{2} + \frac{x}{L|\delta|q}\right) \sum_{n\in\mathcal{N}} |a_n|^2 \\
&= 1.01 \frac{\varphi(l_0)}{l_0/g} \left(1 + \frac{M}{M_0'}\right) \left(NL + \frac{2x}{|\delta|q}\right) \frac{F_0(l_0 q)}{\log L} \sum_{n\in\mathcal{N}} |a_n|^2
\end{aligned} \tag{3.24}$$

where we use (3.22) in the last line. Now, we note that

$$\begin{aligned}
\left(1 + \frac{M}{M_0'}\right)\left(NL + \frac{2x}{|\delta|q}\right) &\leq NL + \frac{LMN}{M_0'} + \frac{2x}{|\delta|q} + \frac{2xM}{M_0'|\delta|q} \\
&\leq \frac{Lx}{4M} + \frac{Lx}{4M_0'} + \frac{2x}{|\delta|q} + \frac{2x}{|\delta|q} \max\left\{\frac{33M|\delta|q}{32xg}, 1\right\} \\
&\leq \frac{5x}{64|\delta|q} + \frac{5x}{64|\delta|q} + \frac{2x}{|\delta|q} + \max\left\{\frac{33M}{16g}, \frac{2x}{|\delta|q}\right\} \\
&\leq \frac{133x}{32|\delta|q} + \frac{33M}{16g}.
\end{aligned}$$

where we use $L/M \leq L/M_0' = 5/16|\delta|q$, substitute the value of M_0' and then finally use $\max\{a, b\} \leq a + b$. Substituting the above in (3.24), we obtain

$$\begin{aligned}
\sum_{\substack{m\in\mathcal{M}\\ m \text{ prime}}} \left|\sum_{n\in\mathcal{N}} a_n e(mn\alpha)\right|^2 &\leq 1.01 \frac{\varphi(l_0)}{l_0/g} \left(\frac{133x}{32|\delta|q} + \frac{33M}{16g}\right) \frac{F_0(l_0 q)}{\log \frac{5M_0'}{16|\delta|q}} \sum_{n\in\mathcal{N}} |a_n|^2 \\
&\leq \left(\frac{21x}{5|\delta|q}\varphi(l_0) + \frac{25M}{12}\frac{\varphi(l_0)}{l_0}\right) \frac{F_0(l_0 q)}{\log \frac{25x}{448(|\delta|q)^2}} \sum_{n\in\mathcal{N}} |a_n|^2.
\end{aligned} \tag{3.25}$$

where we use $M_0' = \min\{32xg/33|\delta|q, M\} \geq 5x/28|\delta|q$ (since $M \geq 5x/28|\delta|q$) in the log factor. This proves (b).

For (c), we have $2\alpha = a/q$, since $\delta = 0$ and we no longer have to bother about the size of M (earlier we required $M_0|\delta|/x \leq (1-\epsilon)/q$). Again, as $\{a_n\}$ are supported on odd numbers, the expression reduces to (3.17) with $2\alpha = a/q$. We use Lemma 3.11 (a) with α_m from (3.18) and in addition, $\beta_m = 0$. Let $g = (l_0, q)$ and set

$$B_1 = \frac{2M}{\varphi(q)\log\frac{M}{q}}.$$

Again, split $\mathcal{M}$ into at most $\varphi(g)\lfloor B_1 \rfloor$ subsets $S_{j,i}$ so that primes in $S_{j,i}$ are incongruent mod q and congruent mod g. Then for distinct primes $m, m' \in S_{j,i}$, we have

$$\|\alpha_m - \alpha_{m'}\| = \left\| \frac{(m-m')a}{q} \right\| \geq \frac{g}{q}.$$

Therefore

$$\sum_{\substack{m \in S_{j,i} \\ m \text{ prime}}} \left| \sum_{n \in \mathcal{N}} a_n e(mn\alpha) \right|^2 \leq \left(\frac{N}{2} + \frac{q}{g} \right) \sum_{n \in \mathcal{N}} |a_n|^2.$$

Summing over all $1 \leq i \leq \varphi(g)$, $1 \leq j \leq \lfloor B_1 \rfloor$, we obtain

$$\begin{aligned} \sum_{\substack{m \in \mathcal{M} \\ m \text{ prime}}} \left| \sum_{n \in \mathcal{N}} a_n e(mn\alpha) \right|^2 &\leq \varphi(g) B_1 \left(\frac{N}{2} + \frac{q}{g} \right) \sum_{n \in \mathcal{N}} |a_n|^2 \\ &\leq \varphi(g) \frac{2M}{\varphi(q)\log\frac{M}{q}} \left(\frac{N}{2} + \frac{q}{g} \right) \sum_{n \in \mathcal{N}} |a_n|^2. \end{aligned}$$

Proceeding in the same manner as the proof of (a) from (3.21) onwards with $M = M_0$ and using (3.22), we will obtain the required bound. This completes the proof. □

Proof of Theorem 3.4. By the Cauchy-Schwarz inequality, the given expression for $S(\mathcal{M}, \mathcal{N}, \alpha)$ reduces to (3.7), i.e., $|S(\mathcal{M}, \mathcal{N}, \alpha)| \leq \Delta(\mathcal{M}, \mathcal{N})^{1/2} \|\boldsymbol{a}\| \|\boldsymbol{b}\|$, with $\Delta(\mathcal{M}, \mathcal{N})$ obtained from Proposition 3.17 according to the case under consideration.

Let us first prove (a). We apply Proposition 3.17 (a) and (b) with $l_0 = 2{\cdot}3{\cdot}5 = 30$. We then have $\varphi(l_0) = 8$ and $\frac{\varphi(l_0)}{l_0} = \frac{4}{15}$.

If $|\delta| \le 10$ (so that $\delta_0 = 2$), Proposition 3.17 (a) (with $M_0 = x/250q$) gives

$$\Delta(\mathcal{M},\mathcal{N}) = \left(\frac{25M}{12}\cdot\frac{4}{15} + \frac{128x}{250q}\cdot 8\right)\cdot\frac{F_0(30q)}{\log\frac{x}{250q^2}} \le \left(\frac{5M}{9} + \frac{8.192x}{\delta_0 q}\right)\cdot\frac{F_0(15\delta_0 q)}{\log\frac{x}{63(\delta_0 q)^2}}.$$

If $|\delta| \ge 10$ (so that $\delta_0 = |\delta|/5$), Proposition 3.17 (b) gives

$$\Delta(\mathcal{M},\mathcal{N}) = \left(\frac{25M}{12}\cdot\frac{4}{15} + \frac{21x}{5|\delta|q}\cdot 8\right)\cdot\frac{F_0(30q)}{\log\frac{25x}{448(|\delta|q)^2}} \le \left(\frac{5M}{9} + \frac{6.72x}{\delta_0 q}\right)\cdot\frac{F_0(15\delta_0 q)}{\log\frac{x}{448(\delta_0 q)^2}}.$$

Comparing the estimates in the above two cases and choosing the weakest amongst them, we get the desired bound by using $F_0(30q) \le F_0(15\delta_0 q)$.

The proof of (b) follows in the same manner as (a) above with Proposition 3.17 (a) (as $q > x/10Q_0$ implies $|\delta| \le 10$) applied with $M_0 = Q_0/25$.

We now prove (c). Since $2\alpha = a/q + \delta/x$, it follows that $\alpha = a'/q' + \delta/2x$, where either $q' = q$ or $q' = 2q$. So we write

$$e(mn\alpha) = e\left(\frac{mna'}{q'}\right)e\left(\frac{mn\delta}{2x}\right) = e\left(\frac{mna'}{q'}\right)\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{2\pi i mn\delta}{2x}\right)^k.$$

Therefore, we obtain

$$S(\mathcal{M},\mathcal{N},\alpha) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\pi i\delta}{x}\right)^k\sum_{n\in\mathcal{N}} n^k a_n \sum_{m\in\mathcal{M}} m^k b_m e\left(\frac{mna'}{q'}\right) \tag{3.26}$$

From (3.7) with the $\{a_n\}$, $\{b_m\}$ replaced by $\{n^k a_n\}$, $\{m^k b_m\}$ and apply Proposition

3.17 (c) (as $\delta = 0$ above) with $l_0 = 30$ to get

$$\begin{aligned}
&\left|\sum_{n\in\mathcal{N}} n^k a_n \sum_{m\in\mathcal{M}} m^k b_m e\left(\frac{mna'}{q'}\right)\right| \\
&\leq \left(2M\cdot\frac{4}{15}+\frac{x}{4q}\cdot 8\right)^{1/2}\left(\frac{F_0(30q)}{\log\frac{M}{q}}\right)^{1/2}\|\{n^k a_n\}\|\cdot\|\{m^k b_m\}\| \qquad (3.27)\\
&\leq \left(\frac{8M}{15}+\frac{2x}{q}\right)^{1/2}\left(\frac{F_0(30q)}{\log\frac{M}{q}}\right)^{1/2}(2N)^k(2M)^k\|\boldsymbol{a}\|\cdot\|\boldsymbol{b}\|,
\end{aligned}$$

since $\|\{n^k a_n\}\| = \left(\sum_{n\in\mathcal{N}} n^{2k}|a_n|^2\right)^{1/2} \leq (2N)^k\|\boldsymbol{a}\|$ and similarly we have $\|\{m^k b_m\}\| \leq (2M)^k\|\boldsymbol{b}\|$. Implementing this in the above and substituting in (3.26), we find that

$$\begin{aligned}
|S(\mathcal{M},\mathcal{N},\alpha)| &\leq \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{4\pi|\delta|MN}{x}\right)^k\left(\frac{8M}{15}+\frac{2x}{q}\right)^{1/2}\left(\frac{F_0(30q)}{\log\frac{M}{q}}\right)^{1/2}\|\boldsymbol{a}\|\cdot\|\boldsymbol{b}\|\\
&\leq e^{\pi|\delta|}\left(\frac{8M}{15}+\frac{2x}{q}\right)^{1/2}\left(\frac{F_0(30q)}{\log\frac{M}{q}}\right)^{1/2}\|\boldsymbol{a}\|\cdot\|\boldsymbol{b}\|,
\end{aligned}$$

where we use $MN = x/4$ from (H6). This completes the proof. □

Chapter 4

Correlations of certain arithmetic functions

4.1 Introduction

In this chapter we study partial sums of products of shifted arithmetic functions of a certain type. We make use of the convolution method to establish these results.

Definition 4.1. A function $F : \mathbb{N} \to \mathbb{C}$ is called an *arithmetic* function. It is called *multiplicative* if

$$F(mn) = F(m)F(n) \quad \text{for all} \;\; m, n \in \mathbb{N} \;\text{ with }\; (m, n) = 1.$$

Definition 4.2. Let F and G be arithmetic functions. Then the *Dirichlet convolution* of F and G, denoted by $F * G$, is defined by

$$(F * G)(n) = \sum_{d|n} F(d)G\left(\frac{n}{d}\right), \quad \text{for all } n \in \mathbb{N}.$$

Let F and G are arithmetic functions and $h \in \mathbb{Z}$. In [BG15] the authors obtain an asymptotic formula for the sum

$$\sum_{n\le x} F(n)G(n-h), \tag{4.1}$$

where $F = f * 1$ and $G = g * 1$ and $F(p)$ and $G(p)$ are close to 1 for primes p. We give an improved asymptotic formula in Theorem 4.4.

One of the methods of estimating the sum (4.1) is the *Convolution method*. We write $F(n) = \sum_{d_1|n} f(d_1)$ and $G(n-h) = \sum_{d_2|n-h} g(d_2)$, so that the sum becomes

$$\begin{aligned}\sum_{n\le x} \sum_{\substack{d_1|n \\ d_2|n-h}} f(d_1)g(d_2) &= \sum_{(d_1,d_2)|h} f(d_1)g(d_2) \sum_{\substack{n\le x \\ n\equiv 0 \pmod{d_1} \\ n\equiv h \pmod{d_2}}} 1 \\ &= \sum_{\substack{d_1,d_2\le x \\ (d_1,d_2)|h}} f(d_1)g(d_2) \left(\frac{x}{[d_1,d_2]} + O(1)\right).\end{aligned}$$

Then one proves that the main term is $x\sum_{(d_1,d_2)|h} \frac{f(d_1)g(d_2)}{[d_1,d_2]}$ and estimates the error term. The same method can be applied to obtain an asymptotic formula for

$$\sum_{n\le x} \mu^2(n)G(n-h), \tag{4.2}$$

where μ is the Möbius function defined by

$$\mu(n) = \begin{cases} 1, & n = 1, \\ (-1)^r, & n = p_1 \dots p_r, \quad p_i \text{ distinct}. \\ 0, & \text{otherwise} \end{cases}$$

Much work has been done on this and related problems. In [Mir49b], Mirsky considers the sum $\sum_{n\le x} F_1(n+k_1)\dots F_s(n+k_s)$, where $F_j = 1 * f_j$ and $f_j(p) = O(p^{-\sigma+\epsilon})$ for all j. In [Ste97], Stepanauskas considers (4.1) under a weaker hypothesis that $\sum_p (f(p) + g(p) - 2)/p < \infty$. In [ŠS07], Stepanauskas and Siaulys also consider the sum $\sum_{p\le x} F(p+1)G(p+2)$, where the sum runs over the primes. In [CMS16], Coppola, Murty, Saha consider the sum (4.1) under a general condition that F and G admit a Ramanujan expansion. More results of this kind are found in papers of Carlitz [Car66], Choi and Schwarz [CS02], Katai [Kát69] and Rearick [Rea66].

Since the aforementioned results are proved under different hypothesis (plural), it is difficult to compare the strength of our results directly with earlier results. However the functions like $\varphi_s(n)/n^s$ and $\sigma_s(n)/n^s$ serve as a common thread between them and our Theorem 4.4.

4.2 Main results

In [BG15], Balasubramanian and Giri proved the following asymptotic formula for the sum (4.1). Their main result is the following:

Theorem A. *For arithmetic functions f and g, let $E_f(x) = \sum\limits_{n\le x} |f(n)|$ and $E_g(x) = \sum\limits_{n\le x} |g(n)|$. Then if $F = 1 * f$ and $G = 1 * g$ we have*

$$\sum_{n\le x} F(n)G(n-h) = x\,C(h) + O\left(hE_f(x)E_g(x)\right).$$

where

$$C(h) = \sum_{\substack{d_1,d_2\ge 1\\ (d_1,d_2)|h}} \frac{f(d_1)g(d_2)}{[d_1,d_2]}.$$

Now we define a class of arithmetic functions.

Definition 4.3. For any $\alpha > 0$, denote by $\mathcal{A}_\alpha$ the family of arithmetic functions g for which there is a positive real number C such that satisfying $|g(n)| \le \frac{C}{n^\alpha}$ for all $n \in \mathbb{N}$.

Henceforth, assume $0 < \alpha \leq \beta$ and define:

$$E(x;\alpha,\beta) = \begin{cases} x^{1-\alpha}, & \alpha < \min\{1,\beta\}, \\ x^{1-\alpha}\log x, & \alpha = \beta < 1 \text{ or } 1 = \alpha < \beta, \\ \log^2 x, & \alpha = \beta = 1, \\ 1, & 1 < \alpha < \beta. \end{cases} \tag{4.3}$$

We prove the following:

Theorem 4.4. *Let $F = f * 1$, $G = g * 1$, with $f \in \mathcal{A}_\alpha$ and $g \in \mathcal{A}_\beta$, with $0 < \alpha \leq \beta$. Then, uniformly for all $h \in \mathbb{Z}$ with $|h| \leq \frac{x}{2}$, we have*

$$\sum_{H < n \leq x} F(n)G(n-h) = (x-H)C(h) + O\left(E(x;\alpha,\beta)\right),$$

where $H = \max\{h, 0\}$ and

$$C(h) = \sum_{\substack{d_1,d_2 \geq 1 \\ (d_1,d_2)|h}} \frac{f(d_1)g(d_2)}{[d_1,d_2]}.$$

Moreover, the O-constant in the error term depends only on α and β.

Remark 4.5. Theorem 4.4 also covers $h = 0$. In this case, there are no restrictions on d_1, d_2 in the expression for $C(h)$. Also, since $f(a) \ll d_1^{-\alpha}$ and $g(b) \ll d_2^{-\beta}$, the series for $C(h)$ is well-defined and admits a product expansion whenever f and g are multiplicative, i.e.,

$$C(h) = \prod_p \left(\sum_{\substack{e_1,e_2 \geq 0 \\ \min(e_1,e_2) \leq v_p(h)}} \frac{f(p^{e_1})g(p^{e_2})}{p^{\max(e_1,e_2)}} \right),$$

where $v_p(h) = \begin{cases} \infty, & h = 0, \\ m, & h \neq 0 \text{ and } p^m \mid\mid |h|. \end{cases}$

This method also applies to study the sum (4.2). Let

$$E_1(x;\alpha) = \begin{cases} x^{1-\alpha}, & 0 < \alpha \leq 1/2, \\ x^{1/2}, & \alpha > 1/2. \end{cases} \tag{4.4}$$

In [BGS17], we prove the following:

Theorem 4.6. *Let $G = g * 1$, with $g \in \mathcal{A}_\alpha$ for some $\alpha > 0$. Then, uniformly for all $|h| \leq \frac{x}{2}$ and $\epsilon > 0$ we have*

$$\sum_{H < n \leq x} \mu^2(n) G(n-h) = (x - H) K(h) + O_\epsilon \left(x^\epsilon E_1(x;\alpha) \right), \tag{4.5}$$

where $H = \max\{h, 0\}$ and

$$K(h) = \sum_{\substack{a,b \geq 1 \\ (a^2,b) \mid h}} \frac{\mu(a) g(b)}{[a^2, b]}.$$

Remark 4.7. Later in Section 4.4.5, we shall indicate how the x^ϵ in the error term of Theorem 4.6 may be replaced with a power of $\log x$ provided α is not close to $1/2$.

Remark 4.8. Theorem 4.6 covers the case $h = 0$. Also, $K(h)$ is well-defined since $g \in A_\alpha$. Again, for g multiplicative, $K(h)$ admits a product expansion

$$K(h) = \prod_p \left(\sum_{\max(2e_1, e_2) \leq v_p(h)} \frac{\mu(p^{e_1}) g(p^{e_2})}{p^{\max(2e_1, e_2)}} \right).$$

We also prove asymptotic formula for the shifted sum of product of k arithmetic functions $F_1, \ldots, F_k$, with $F_j = 1 * f_j$ and $f_j \in \mathcal{A}_\alpha$. We have the following result:

Theorem 4.9. *Let $0 < \alpha < 1$ and k be a positive integer satisfying $k = o(\log\log\log x)$. Let $F_1, \ldots, F_k$ be arithmetic functions satisfying $F_j = 1 * f_j$, with $f_j \in \mathcal{A}_\alpha$ for all j. Let $a_1, \ldots, a_k$ be integers satisfying $|a_j| \leq x/2$. Then, for any $\epsilon > 0$ and x sufficiently large (depending upon ϵ and k)*

(a)

$$\sum_{n \leq x} F_1(n + a_1) \ldots F_k(n + a_k) = C_1 x + O_\epsilon\left(x^{1-\alpha+\epsilon}\right),$$

where

$$C_1 = \sum_{\substack{d_1,\ldots,d_k \geq 1 \\ (d_i,d_j)|a_i - a_j}} \frac{\prod_{j=1}^k f_j(d_j)}{[d_1, \ldots, d_k]}.$$

(b)

$$\sum_{n \leq x} F_1(n^2 + a_1) \ldots F_k(n^2 + a_k) = C_2 x + O\left(x^{1-\alpha+\epsilon}\right),$$

where

$$C_2 = \sum_{\substack{d_1,\ldots,d_k \geq 1 \\ (d_i,d_j)|a_i - a_j}} \lambda(d_1, \ldots, d_k) \frac{\prod_{j=1}^k f_j(d_j)}{[d_1, \ldots, d_k]}.$$

Here $\lambda(d_1, \ldots, d_k)$ denotes the number of solutions modulo $[d_1, \ldots, d_k]$ to the system of congruences $n^2 \equiv -a_j \pmod{d_j}$, for all $1 \leq j \leq k$.

Remark 4.10. The method can be extended to study $\sum_{n \leq x} F_1(P_1(n)) \ldots F_k(P_k(n))$, where P_j's are polynomials with integer coefficients for each j.

Remark 4.11. The condition $f_j \in \mathcal{A}_\alpha$ and the bound for λ from Lemma 4.25 ensures that C_1 and C_2 are well defined. If the functions f_j are multiplicative, the sum C_1 admits the Euler product

$$C_1 = \prod_p \left(\sum_{\substack{e_1,\ldots,e_k \geq 0 \\ \min(e_i,e_j) \leq v_p(a_i - a_j)}} \frac{\prod_{j=1}^k f_j(p^{e_j})}{p^{\max(e_1,\ldots,e_k)}} \right).$$

The constant C_2 can be computed in the following manner: suppose for simplicity that $a_j = j$ and that $f_j = f$, where f is multiplicative and supported on square-free numbers. This means the d_j are square-free. First, consider those d_j's free of primes $< k$. Then they would be pairwise coprime and therefore system of congruences $n^2 \equiv -j \pmod{d_j}$, $1 \leq j \leq k$ has exactly $\lambda(d_1, \dots, d_k) = \prod_{j=1}^{k} \prod_{p|d_j} \left(1 + \left(\frac{-j}{p}\right)\right)$ solutions modulo $[d_1, \dots, d_k] = \prod d_j$. The contribution for the d_j composed of primes less than k has to evaluated separately according to local constraints. This gives

$$\begin{aligned} C_2 &= A_2 \prod_{p \geq k} \left(\sum_{\substack{e_1, \dots, e_k \in \{0,1\} \\ \min\{e_i, e_j\} = 0}} \frac{\prod_{j=1}^{k} \left(1 + \left(\frac{-j}{p}\right)\right) f(p^{e_j})}{p^{\max\{e_1, \dots, e_k\}}} \right) \\ &= A_2 \prod_{p \geq k} \left(1 + \frac{f(p)}{p} \left(k + \sum_{j=1}^{k} \left(\frac{-j}{p}\right)\right)\right), \end{aligned}$$

where A_2 corresponds to the finite Euler product for primes $< k$.

We now consider the sum

$$\sum_{p \leq x} F(p+h) G(p+k). \tag{4.6}$$

We prove an asymptotic formula for (4.6) in the particular case $F(n) = G(n) = \frac{\varphi(n)}{n}$ and $h = 1$, $k = 2$. The same method applies for F, G satisfying $F = f * 1$, $G = g * 1$ and f, g in $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$, respectively for all values of h, k. We prove:

Theorem 4.12. *Fix $A > 0$. Then*

$$\sum_{p \leq x} \frac{\varphi(p+2)}{p+2} \frac{\varphi(p+1)}{p+1} = \frac{li(x)}{2} \prod_{p > 2} \left(1 - \frac{2}{p(p-1)}\right) + O\left(\frac{x}{(\log x)^{A-1}}\right),$$

where the O-constant depends only on A. Here $li(x) = \int_2^x \frac{dt}{\log t}$.

Remarks and comparison to previous results

Now, we compare the main results of this chapter with earlier results of a similar type.

Let $f \in \mathcal{A}_\alpha$ and $g \in A_\beta$ with $0 < \alpha \le \beta < 1$. Then Theorem 4.4 gives:

$$\sum_{n \le x} F(n)G(n-h) = xC(h) + O(E(x; \alpha, \beta)),$$

for all h with $|h| \le \frac{x}{2}$. Note that Theorem A of [BG15] gives the error term $O(hx^{2-\alpha-\beta})$, so our result improves this in terms of h, α and β.

Next, we take $F(n) = n/\varphi(n)$ and $G(n) = \sigma(n)/n$ in Theorem 4.4, so that $f(p) = \frac{1}{p-1}$, $f(p^k) = 0$ for $k \ge 2$ and $g(n) = 1/n$. Thus, we can take $\alpha = 1 - \epsilon$ and $\beta = 1$ in Theorem 4.4, to get

Corollary 4.13.

$$\sum_{n \le x} \frac{\sigma(n+1)}{n+1} \frac{n}{\varphi(n)} = x \prod_p \left(1 + \frac{2p+1}{p(p^2-1)}\right) + O(x^\epsilon). \tag{4.7}$$

$$\sum_{n \le x} \frac{\sigma(n+1)}{\varphi(n)} = x \prod_p \left(1 + \frac{2p+1}{p(p^2-1)}\right) + O(x^\epsilon). \tag{4.8}$$

We remark that Stepanauskas [Ste97] has proved (4.8) with an error term $O\left(\frac{x}{(\log x)^2}\right)$, which is much larger than $O(x^\epsilon)$.

Taking $F(n) = \sigma_s(n)/n^s$, $G(n) = \sigma_t(n)/n^t$ in Theorem 4.4, where $s \le t$ and $\sigma_s(n) = \sum_{d|n} d^s$, we have $f(n) = 1/n^s$ and $g(n) = 1/n^t$. This gives

Corollary 4.14. *Uniformly for $|h| \le N/2$, we have*

$$\sum_{n \le N} \frac{\sigma_s(n)}{n^s} \frac{\sigma_t(n+h)}{(n+h)^t} = (N-H) \frac{\zeta(s+1)\zeta(t+1)}{\zeta(s+t+2)} \sigma_{-(s+t+1)}(h) + E(N; s, t), \tag{4.9}$$

where the O-term depends only on s, t and is independent of h. The error term is $E(N; s, t)$ defined in (4.3).

We compare (4.9) above with Corollary 1 of [CMS16], where the error term is dependent on h, and is given by

$$\begin{cases} O\left(N^{1-s}(\log N)^{4-2s}\right), & s < 1, \\ O(\log^3 N), & s = 1, \\ O(1), & s > 1. \end{cases}$$

Similar remarks apply for Corollary 2 of [CMS16].

Remark 4.15. Letting $G(n) = \varphi(n)/n$ in Theorem 4.6 with $h = 0$, we have

$$\sum_{n \leq x} \mu^2(n) \frac{\varphi(n)}{n} = x \prod_p \left(1 - \frac{2}{p^2}\right)\left(1 + \frac{1}{p^3 - 2p}\right) + O\left(x^{1/2}\right). \tag{4.10}$$

Now, observe that the Dirichlet series of $\mu^2(n)\varphi(n)/n$ is

$$\sum_{n=1}^{\infty} \frac{\mu^2(n)\varphi(n)}{n^{1+s}} = \frac{\zeta(s)K(s)}{\zeta(2s)},$$

where $K(s)$ is absolutely convergent in $\Re(s) > 0$. Due to Landau's theorem, the error term of (4.10) is $\Omega(x^{1/2-\epsilon})$, if the zeta function were to have a zero close to $Re(s) = 1$ and hence cannot be improved other than terms of the type $\exp\left(-c(\log x)^{2/5}(\log\log x)^{3/5}\right)$, unless one assumes a good zero-free region for $\zeta(s)$.

For $0 < \alpha \leq 1$, let

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha \quad \text{and} \quad \varphi_\alpha(n) = \sum_{d|n} \frac{\mu(d)}{d^\alpha} = n^\alpha \prod_{p|n}\left(1 - \frac{1}{p^\alpha}\right).$$

Theorem 4.9 leads to the following when $k = 3$. The constants are computed by the method mentioned in remark 4.11.

Corollary 4.16.

$$\sum_{n\le x}\frac{\sigma_\alpha(n+1)}{(n+1)^\alpha}\frac{\sigma_\alpha(n+2)}{(n+2)^\alpha}\frac{\sigma_\alpha(n+3)}{(n+3)^\alpha}=Ax\prod_{p>2}\left(1+\frac{3}{p^{\alpha+1}-1}\right)+O(x^{1-\alpha+\epsilon}),$$

$$\sum_{n\le x}\frac{\varphi_\alpha(n^2+1)}{(n^2+1)^\alpha}\frac{\varphi_\alpha(n^2+2)}{(n^2+2)^\alpha}\frac{\varphi_\alpha(n^2+3)}{(n^2+3)^\alpha}=Bx\prod_{p>2}\left(1-\frac{3+\left(\frac{-1}{p}\right)+\left(\frac{-2}{p}\right)+\left(\frac{-3}{p}\right)}{p^{\alpha+1}}\right)$$
$$+O(x^{1-\alpha+\epsilon}),$$

where A, B are Euler factors for $p=2$ and $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. The Euler product above is $\prod_{p\equiv 1\,(mod\ 24)}\left(1-\frac{6}{p^2}\right)\prod_{p\equiv 13,17,19\,(mod\ 24)}\left(1-\frac{4}{p^2}\right)\prod_{p\equiv 5,7,11\,(mod\ 24)}\left(1-\frac{2}{p^2}\right)$.

Also, Theorem 4.12 improves upon Corollary 1 of [ŠS07], where the authors estimate the error term by $O\left(\frac{\mathrm{li}(x)}{(\log\log x)^B}\right)$, which is much larger.

4.3 Preliminary lemmas

In this section we give some preliminary lemmas for the proof of the main results. We assume throughout that $0<\alpha\le\beta$. Recall that

$$E(x)=E(x;\alpha,\beta)=\begin{cases}x^{1-\alpha}, & \alpha<\min(1,\beta)\\ x^{1-\alpha}\log x, & \alpha=\beta<1 \text{ or } 1=\alpha<\beta,\\ \log^2 x, & \alpha=\beta=1,\\ 1, & 1<\alpha<\beta.\end{cases}$$

The statements of the lemmas in this section stand true for all $0<\alpha\le\beta$. However, we give the proofs of these lemmas only in the case $\alpha<\min\{1,\beta\}$ for ease of exposition. The proof follows with minor changes in the other cases. When $\alpha<\min\{1,\beta\}$ we find that

$$E(x)=O(x^{1-\alpha}).$$

We begin with the following:

Lemma 4.17.

(a) If $y \geq 1$, then

$$\sum_{mn \geq y} \frac{1}{m^{1+\alpha} n^{1+\beta}} = O\left(\frac{E(y)}{y}\right).$$

(b) If $x \geq 1$, then

$$S = \sum_{[a,b] \geq x} \frac{1}{a^{\alpha} b^{\beta} [a,b]} = O\left(\frac{E(x)}{x}\right).$$

Proof. Let us first prove (a). Since $\beta > \alpha$ the sum equals

$$\begin{aligned}\sum_{n \geq 1} \frac{1}{n^{1+\beta}} \sum_{m \geq y/n} \frac{1}{m^{1+\alpha}} &= \sum_{n \leq y} \frac{1}{n^{1+\beta}} \sum_{m \geq y/n} \frac{1}{m^{1+\alpha}} + \sum_{n > y} \frac{1}{n^{1+\beta}} \sum_{m \geq 1} \frac{1}{m^{1+\alpha}} \\ &= O\left(\frac{1}{y^{\alpha}} \sum_{n \leq y} \frac{1}{n^{1+\beta-\alpha}}\right) + O\left(\sum_{n > y} \frac{1}{n^{1+\beta}}\right) = O(y^{-\alpha}),\end{aligned}$$

as required.

For (b), we split the sum depending on $l = \gcd(a, b)$. Write $a = ml$ and $b = nl$, so that

$$S \leq \sum_{l \geq 1} \frac{1}{l^{1+\alpha+\beta}} \sum_{mn \geq x/l} \frac{1}{m^{1+\alpha} n^{1+\beta}}.$$

Thus, using (a)

$$\begin{aligned}S &\ll \sum_{l \leq x} \frac{1}{l^{\alpha+\beta}} \frac{E(x/l)}{x} + \sum_{l > x} \frac{1}{l^{1+\alpha+\beta}} \sum_{m,n \geq 1} \frac{1}{m^{1+\alpha} n^{1+\beta}} \\ &\ll x^{-\alpha} \sum_{l \leq x} \frac{1}{l^{1+\beta}} + \sum_{l > x} \frac{1}{l^{1+\alpha+\beta}} \ll x^{-\alpha}.\end{aligned}$$

This completes the proof of (b). $\square$

Next, we have the following lemma:

Lemma 4.18.

(a) If $y \geq 1$, then

$$\sum_{mn \leq y} \frac{1}{m^{\alpha} n^{\beta}} = O\left(E(y)\right).$$

(b) If $x \geq 1$, then

$$\sum_{\substack{[a,b] \leq x \\ (a,b)=l}} \frac{1}{a^{\alpha} b^{\beta}} = O\left(\frac{E(x)}{l^{1+\beta}}\right).$$

(c) If $x \geq 1$, then

$$\sum_{[a,b] \leq x} \frac{1}{a^{\alpha} b^{\beta}} = O\left(E(x)\right).$$

Proof. For (a), we write the given sum as

$$\sum_{mn \leq y} \frac{1}{m^{\alpha} n^{\beta}} = \sum_{n \leq y} \frac{1}{n^{\beta}} \sum_{m \leq y/n} \frac{1}{m^{\alpha}} = O\left(\sum_{n \leq y} \frac{1}{n^{\beta}} \left(\frac{y}{n}\right)^{1-\alpha}\right) = O\left(y^{1-\alpha}\right),$$

In order to prove (b), we again split the sum depending on the value of $l = \gcd(a, b)$ and use (a). Writing $a = ml$ and $b = nl$ as before, we find that the sum equals

$$\frac{1}{l^{\alpha+\beta}} \sum_{\substack{mn \leq x/l \\ (m,n)=1}} \frac{1}{m^{\alpha} n^{\beta}} \ll \frac{1}{l^{\alpha+\beta}} \left(\frac{x}{l}\right)^{1-\alpha},$$

which proves (b).

Now (c) follows directly from (b). □

The preceding lemmas lead us to:

Lemma 4.19.

(a) Let $y \geq 1$ and $|k| \leq \frac{y}{2}$. Then

$$S_1 = \sum_{K < m \leq y} \sum_{\substack{a \mid m \\ b \mid m-k \\ ab \geq y}} a^{-\alpha} b^{-\beta} = O\left(E(y)\right)$$

where $K = \max\{k, 0\}$ *and the* O*-constant is dependent only on* α *and* β.

(b) Let $x \geq 1$ *and* $|h| \leq \frac{x}{2}$. *Then*

$$S_2 = \sum_{H<n\leq x} \sum_{\substack{c|n \\ d|n-h \\ [c,d]\geq x}} c^{-\alpha} d^{-\beta} = O\left(E(x)\right)$$

where $H = \max\{h, 0\}$ *and the* O*-constant depends only on* α, β.

Proof. We first prove (a). Set $m = ac$, $m - k = bd$ and write the sum in terms of c and d to get

$$S_1 = \sum_{K<m\leq y} \sum_{\substack{c|m \\ d|m-k \\ cd\leq \frac{m(m-k)}{y}}} \left(\frac{m}{c}\right)^{-\alpha} \left(\frac{m-k}{d}\right)^{-\beta}.$$

Since $cd \leq \frac{m(m-k)}{y} \leq m - k$, it follows that $m - k \geq cd$. Therefore

$$S_1 \ll \sum_{cd\leq y} c^{\alpha} d^{\beta} \sum_{\substack{m\equiv 0 \,(\mathrm{mod}\ c) \\ m\equiv k \,(\mathrm{mod}\ d) \\ cd+k\leq m\leq y}} m^{-\alpha}(m-k)^{-\beta}.$$

The congruences on m reduce to $m \equiv r \pmod{[c, d]}$. We now replace m by $m + K$, so that the sum over m is at most

$$\ll \sum_{\substack{m\equiv r' \,(\mathrm{mod}\ [c,d]) \\ cd\leq m\leq y-K}} m^{-\alpha-\beta} \ll \sum_{\substack{m\equiv 0 \,(\mathrm{mod}\ [c,d]) \\ cd\leq m\leq 2y}} m^{-\alpha-\beta}.$$

Let $l = \gcd(c, d)$ and write $m = j[c, d]$, where $l \leq j \leq \frac{2y}{[c,d]}$. The sum over m is then

$$\ll [c, d]^{-\alpha-\beta} \sum_{l\leq j\leq \frac{2y}{[c,d]}} j^{-\alpha-\beta},$$

which means that

$$S_1 \ll \sum_{\substack{l,j \\ l\le j}} j^{-\alpha-\beta} \sum_{\substack{[c,d]\le 2y/j \\ (c,d)=l}} \frac{c^\alpha d^\beta}{[c,d]^{\alpha+\beta}} \ll \sum_{\substack{l,j \\ l\le j}} j^{-\alpha-\beta} l^{\alpha+\beta} \sum_{\substack{[c,d]\le 2y/j \\ (c,d)=l}} c^{-\beta} d^{-\alpha}.$$

Applying Lemma 4.18 (b) to the inner sum above, we get

$$S_1 \ll y^{1-\alpha} \sum_j \frac{1}{j^{1+\beta}} \sum_{l\le j} \frac{1}{l^{1-\alpha}} \ll y^{1-\alpha} \sum_{j\le y} \frac{1}{j^{1+\beta-\alpha}} \ll E(y),$$

which proves (a).

For (b), we split the given sum depending on $(c,d)=l$ to get

$$S_2 = \sum_{H<n\le x} \sum_{\substack{l|n \\ l|n-h}} \sum_{\substack{c|n \\ d|n-h \\ cd\ge lx \\ (c,d)=l}} c^{-\alpha} d^{-\beta} = \sum_{H<n\le x} \sum_{l|h} \sum_{\substack{lc|n \\ ld|n-h \\ cd\ge x/l \\ (c,d)=1}} (lc)^{-\alpha} (ld)^{-\beta}$$

$$\ll \sum_{l|h} l^{-\alpha-\beta} \sum_{\substack{H<n\le x \\ n\equiv 0 \ (\mathrm{mod}\ l)}} \sum_{\substack{c|\frac{n}{l} \\ d|\frac{n-h}{l} \\ cd\ge x/l}} c^{-\alpha} d^{-\beta}.$$

Let $n/l = n'$ and $h/l = h'$. Then the given sum becomes

$$S_2 \ll \sum_{l|h} l^{-\alpha-\beta} \sum_{H/l<n'\le x/l} \sum_{\substack{c|n' \\ d|n'-h' \\ cd\ge x/l}} c^{-\alpha} d^{-\beta}.$$

Hence by (a), it follows that

$$S_2 \ll \sum_{l|h} l^{-\alpha-\beta} E(x/l) \ll x^{1-\alpha} \sum_{l|h} \frac{1}{l^{1+\beta}} \ll E(x).$$

This proves (b). $\square$

Now we give the preliminary lemmas for the proof of Theorem 4.6. Recall that

$$E_1(x) = E_1(x;\alpha) = \begin{cases} x^{1-\alpha}, & 0 < \alpha \le 1/2, \\ x^{1/2}, & \alpha > 1/2. \end{cases}$$

We have the following:

Lemma 4.20.

(a) Let c be a positive integer. Then

$$S = \sum_{H<n\le y} \sum_{\substack{ca^2|n \\ b|n-h \\ a^2b>z}} b^{-\alpha} = O\left(\frac{y^\epsilon}{c} \left(\frac{y}{z}\right)^\alpha E_1(y) \right).$$

(b)

$$\sum_{H<n\le x} \sum_{\substack{a^2|n \\ b|n-h \\ [a^2,b]>x}} b^{-\alpha} = O(x^\epsilon E_1(x)).$$

Proof. For (a), observe that as $ca^2 \mid n$, we have $ca^2 \le y$. Split the sum over a, b dyadically i.e., let $a \sim A$ and $b \sim B$, where $n \sim x$ denotes $x < n \le 2x$. Then

$$\begin{aligned} S_{A,B} &= \sum_{H<n\le y} \left(\sum_{\substack{ca^2|n \\ a\sim A}} 1 \right) \left(\sum_{\substack{b|n-h \\ b\sim B}} b^{-\alpha} \right) \ll B^{-\alpha} y^\epsilon \sum_{H<n\le y} \sum_{\substack{ca^2|n \\ a\sim A}} 1 \\ &\ll y^\epsilon B^{-\alpha} \sum_{a\sim A} \sum_{\substack{H<n\le y \\ n\equiv 0 \, (\mathrm{mod}\ a^2c)}} 1 \ll y^\epsilon B^{-\alpha} \sum_{a\sim A} \left(\frac{y}{ca^2} + O(1) \right) \ll \frac{y^{1+\epsilon}}{cAB^\alpha}. \end{aligned}$$

Summing the above over $A = 2^Q$ and $B = 2^R$ over powers of 2 with $A \le y^{1/2}$, $B \le y$ and $A^2B > z$, we get

$$S = \frac{y^{1+\epsilon}}{c} \sum_{\substack{A=2^Q\le y^{1/2} \\ B=2^R\le y \\ A^2B>z}} \frac{1}{AB^\alpha} \ll \frac{y^{1+\epsilon}}{cz^\alpha} \sum_{\substack{A=2^Q \\ A\le y^{1/2}}} A^{2\alpha-1} \ll \frac{y^{1+\epsilon}}{cz^\alpha} \cdot y^{\alpha-1/2} \ll \frac{y^\epsilon}{c} \left(\frac{y}{z}\right)^\alpha E_1(y).$$

For (b), let $(a^2,b) = l_1^2 l_2$, with l_2 square-free. This means that $a = kl_1l_2$ and $b = ml_1^2l_2$, where $[a^2,b] = k^2m(l_1l_2)^2$. Also, for fixed l_1, l_2, the given sum is

$$\sum_{H<n\leq x} \sum_{\substack{k^2l_1^2l_2^2|n \\ ml_1^2l_2|n-h \\ k^2m(l_1l_2)^2>x \\ l_1^2l_2|h}} b^{-\alpha}.$$

Write $h = h'l_1^2l_2$ and $n = n'l_1^2l_2$, so that the sum becomes

$$\ll \sum_{\substack{H<n\leq x \\ n\equiv 0(l_1^2l_2)}} \sum_{\substack{k^2l_1^2l_2|n \\ ml_1^2l_2|n-h \\ k^2m>x/(l_1l_2)^2}} b^{-\alpha} \ll (l_1^2l_2)^{-\alpha} \sum_{H/l_1^2l_2<n'\leq x/(l_1^2l_2)} \sum_{\substack{l_2k^2|n' \\ m|n'-h' \\ k^2m>x/(l_1l_2)^2}} m^{-\alpha}.$$

Applying (a) to the above sum with $y = \frac{x}{l_1^2l_2}$, $z = \frac{x}{(l_1l_2)^2}$ and $c = l_2$, we find that for a fixed l_1 and l_2 the sum becomes

$$\ll (l_1^2l_2)^{-\alpha}\left(\frac{x}{l_1^2l_2}\right)^{\epsilon} l_2^{\alpha-1}E_1\left(\frac{x}{l_1^2l_2}\right).$$

Summing over $l_1^2l_2 \leq x$, we obtain the required bound. □

Remark 4.21. In the last step of the above proof, we sum over all $l_1^2l_2 \leq x$ instead of just $l_1^2l_2 \mid h$. This means that the O-constant is indeed independent of h.

Lemma 4.22. *With the notation as before, we have*

(a)

$$\sum_{a^2b\leq y} b^{-\alpha} = O(E_1(y)).$$

(b)

$$\sum_{[a^2,b]\leq x} b^{-\alpha} = O(E_1(x)).$$

Proof. For (a), the proof follows in the same way as that of Lemma 4.18 (a).

To prove (b), let $(a^2, b) = l_1^2 l_2$, where l_2 is square-free. Write $a = kl_1 l_2$ and $b = ml_1^2 l_2$ like the proof of Lemma 4.20 (b). The given sum then reduces to that of (a). Summing over $l_1^2 l_2 \le x$ gives us the desired result. □

Lemma 4.23.

(a)

$$\sum_{a^2 b > y} \frac{1}{a^2 b^{1+\alpha}} = O\left(\frac{E_1(y)}{y}\right).$$

(b)

$$\sum_{[a^2, b] > x} \frac{b^{-\alpha}}{[a^2, b]} = O\left(\frac{E_1(x)}{x}\right).$$

Proof. For (a), we follow the proof of Lemma 4.17 (a). For (b), let $(a^2, b) = l_1^2 l_2$ with l_2 square-free. Then $a = kl_1 l_2$ and $b = ml_1^2 l_2$. The sum then reduces to a sum of the kind in part (a). Summing over l_1, l_2 then gives the desired result. □

Definition 4.24. Define a multiplicative function $H(n)$ by

$$H(n) = \prod_{p \mid n} p^{\left\lfloor \frac{v_p(n)}{2} \right\rfloor}. \tag{4.11}$$

In particular, for s square-free, we have $H(r^2 s) = r$.

Lemma 4.25. *Let a, m be positive integers and $h \neq 0$. Let $\lambda(m, a, h)$ denote the number of solutions modulo m to the congruence $ax^2 \equiv h \pmod{m}$. Then*

$$\lambda(m, a, h) \le H(m)\tau(m),$$

where τ stands for the divisor function.

Proof. If $(a, m) > 1$, then $(a, m) \mid h$. Canceling that factor, the congruence becomes

$$ax_1^2 \equiv h_1 \pmod{m_1}, \tag{4.12}$$

where $m_1 = m/(a,m)$ and $(m_1, a_1) = 1$. Note that any given solution to (4.12) lifts to a unique solution of the congruence $ax^2 \equiv h \pmod{m}$. As $(m_1, a_1) = 1$, (4.12) is the same as $x^2 \equiv k \pmod{m_1}$. Writing $m_1 = q_1 q_2$, with q_1 being the product of prime powers p^l with $v_p(m_1) \leq v_p(k)$ and q_2 being the product of those prime powers p^l with $v_p(m_1) > v_p(k)$.

The equation $x^2 \equiv k \pmod{q_1}$ is same as $x^2 \equiv 0 \pmod{q_1}$ having at the most $H(q_1)$ solutions. Also, $x^2 \equiv k \pmod{q_2}$ has at most $\tau(q_2)$ solutions. Combining the two, we find the total number of solutions to be at most $H(q_1)\tau(q_2)$. As $q_1 \mid m$, we get $H(q_1) \leq H(m)$ and since $\tau(q_2) \leq \tau(m)$, the proof is complete. ☐

Lemma 4.26. *For $H(n)$ as in (4.11), we have*

$$\sum_{n \leq x} \frac{H(n)\tau(n)}{n^\beta} = \begin{cases} O(1), & \beta > 1, \\ O(\log^5 x), & \beta = 1, \\ O(x^{1-\beta}(\log x)^4), & 0 < \beta < 1. \end{cases} \tag{4.13}$$

Proof. For any $n \leq x$, we can write it uniquely as $n = r^2 s$, with s squarefree. Moreover, we then have $H(n) = H(r^2 s) = r$. We have

$$\begin{aligned}\sum_{n \leq x} H(n)\tau(n) &= \sum_{r^2 s \leq x} r \cdot \tau(r^2 s) \leq \sum_{r \leq \sqrt{x}} r \cdot \tau(r^2) \sum_{s \leq x/r^2} \tau(s) \leq \sum_{r \leq \sqrt{x}} r \cdot \tau(r^2) \frac{x}{r^2} \log \frac{x}{r^2} \\ &\leq x \log x \sum_{r \leq \sqrt{x}} \frac{\tau(r^2)}{r} = x \log x \sum_{r \leq \sqrt{x}} \frac{1}{r} \sum_{k \mid r^2} 1 = x \log x \sum_{k \leq x} \sum_{\substack{r^2 \equiv 0 \pmod{k} \\ r \leq \sqrt{x}}} \frac{1}{r}\end{aligned}$$

Now, write $k = a^2 b$, with b square-free, so that $k \mid r^2$ implies $ab \mid r$. So, the above is

$$x \log x \sum_{a^2 b \leq x} \sum_{\substack{r \leq \sqrt{x} \\ r \equiv 0 \pmod{ab}}} \frac{1}{r} \leq x \log^2 x \sum_{a^2 b \leq x} \frac{1}{ab} \ll x(\log x)^4.$$

The result now follows from partial summation. ☐

Lemma 4.27. *Let k and L be positive integers. The number of tuples $(d_1, \ldots, d_k)$*

of positive integers satisfying $[d_1, \ldots, d_k] = L$ *is at most* $\tau(L)^k$.

Proof. Let $J(L)$ denote the number of solutions to $[d_1, \ldots, d_k] = L$. Since J is multiplicative, it is enough to look at prime powers. For $L = p^e$, the number of solutions to $[p^{e_1}, \ldots, p^{e_k}] = p^e$, or $\max\{e_1, \ldots, e_k\} = e$ is clearly bounded by $(e+1)^k = \tau(p^e)^k$. The proof now follows from the multiplicativity of J. $\square$

4.4 Proof of the main results

4.4.1 Proof of Theorem 4.4

We have

$$S = \sum_{H<n\le x} \sum_{\substack{a|n \\ b|n-h}} f(a)g(b) = \sum_{H<n\le x} \sum_{[a,b]\le x} f(a)g(b) + \sum_{H<n\le x} \sum_{[a,b]>x} f(a)g(b).$$

The second term on the right is $O(E(x))$ by Lemma 4.19 (b). The first term is

$$\sum_{[a,b]\le x} f(a)g(b) \sum_{\substack{H<n\le x \\ n\equiv 0 \pmod{a} \\ n\equiv h \pmod{b}}} 1 = \sum_{\substack{[a,b]\le x \\ (a,b)|h}} f(a)g(b) \left(\frac{x-H}{[a,b]} + O(1) \right)$$

Also, the O-term above is $O\left(E(x)\right)$ by Lemma 4.18 (c). The main term is then

$$(x-H) \sum_{(a,b)|h} \frac{f(a)g(b)}{[a,b]} - (x-H) \sum_{\substack{(a,b)|h \\ [a,b]>x}} \frac{f(a)g(b)}{[a,b]}.$$

Clearly, the first term is $(x-H)C(h)$ and the second term is $O(E(x))$ by Lemma 4.17 (b). This completes the proof of Theorem 4.4.

4.4.2 Proof of Theorem 4.6

The given sum can be written as

$$S = \sum_{H<n\le x} \mu^2(n)G(n-h) = \sum_{H<n\le x} \sum_{\substack{a^2|n \\ b|n-h}} \mu(a)g(b) = T_1 + T_2,$$

where T_1 corresponds to $[a^2, b] \le x$ and T_2 corresponds to $[a^2, b] > x$. We note that $T_2 = O(x^\epsilon E_1(x))$ by Lemma 4.20 (b). Now

$$T_1 = \sum_{a,b} \mu(a)g(b) \sum_{\substack{n\equiv 0 \pmod{a^2} \\ n\equiv h \pmod{b} \\ H<n\le x}} 1 = \sum_{\substack{[a^2,b]\le x \\ (a^2,b)|h}} \mu(a)g(b) \left(\frac{x-H}{[a^2,b]} + O(1) \right) = T_3 \ + \ T_4.$$

We first estimate T_3, which is

$$T_3 = (x-H) \sum_{(a^2,b)|h} \frac{\mu(a)g(b)}{[a^2,b]} + O\left(x \sum_{[a^2,b]\ge x} \frac{|g(b)|}{[a^2,b]} \right).$$

The main term is $(x-H)K(h)$ and the error is $O\left(E_1(x)\right)$ by Lemma 4.23 (b). Also,

$$T_4 = O\left(\sum_{[a^2,b]\le x} |g(b)| \right) = O\left(E_1(x)\right),$$

by Lemma 4.22 (b). This completes the proof of Theorem 4.6.

4.4.3 Proof of Theorem 4.9

For (a), denote the sum by S_1 and let L be the LCM of $d_1, \ldots, d_k$. Then

$$S_1 = \sum_{n\le x} \sum_{d_j|n+a_j} \prod_{j=1}^{k} f_j(d_j) = \sum_{n\le x} \sum_{\substack{d_j|n+a_j \\ L\le x}} \prod_{j=1}^{k} f_j(d_j) \ + \sum_{n\le x} \sum_{\substack{d_j|n+a_j \\ L>x}} \prod_{j=1}^{k} f_j(d_j)$$

$$= S_{11} \ + \ S_{12}.$$

For the second term, we have

$$
\begin{aligned}
S_{12} &\ll \sum_{n\le x}\sum_{\substack{L|\prod(n+a_j)\\ L>x}}\sum_{[d_1,\dots,d_k]=L}(d_1\dots d_k)^{-\alpha} \ll \sum_{n\le x}\sum_{\substack{L|\prod(n+a_j)\\ L>x}}\frac{\tau(L)^k}{L^\alpha}\\
&\ll x^{-\alpha+\epsilon}\sum_{n\le x}\tau\left(\prod(n+a_j)\right) \ll x^{1-\alpha+\epsilon},
\end{aligned}
$$

since by Lemma 4.27, the number of $d_1,\dots,d_k$ satisfying $[d_1,\dots,d_k]=L$ is at most $\tau(L)^k = O(x^{\frac{2k^2}{\log\log x}}) = O(x^\epsilon)$. This is because $L\le x^k$, $\tau(n+a_j)\ll x^{\frac{2}{\log\log x}}$ and $k=o(\log\log\log x)$. The first term is

$$
S_{11} = \sum_{\substack{d_1,\dots,d_k\\ L\le x}}\prod_{j=1}^k f_j(d_j)\sum_{\substack{n\equiv -a_j \pmod{d_j}\\ n\le x}} 1.
$$

Note that the n-sum is nonempty $\iff$ $(d_i,d_j)\mid a_i-a_j$ for all i, j. We write $\sum'$ to denote this condition. In this case, the solution is unique modulo L and hence

$$
\begin{aligned}
S_{11} &= \sideset{}{'}\sum_{\substack{d_1,\dots,d_k\\ L\le x}}\prod_{j=1}^k f_j(d_j)\sum_{\substack{n\equiv -a_j \pmod{d_j}\\ n\le x}} 1 = \sideset{}{'}\sum_{\substack{d_1,\dots,d_k\\ L\le x}}\prod_{j=1}^k f_j(d_j)\left(\frac{x}{[d_1,\dots,d_k]}+O(1)\right)\\
&= x\sideset{}{'}\sum_{d_1,\dots,d_k}\frac{\prod_{j=1}^k f_j(d_j)}{[d_1,\dots,d_k]} + O\left(x\sum_{\substack{d_1,\dots,d_k\\ L>x}}\frac{\prod_{j=1}^k f_j(d_j)}{[d_1,\dots,d_k]}\right) + O\left(\sum_{\substack{d_1,\dots,d_k\\ L\le x}}\prod_{j=1}^k f_j(d_j)\right)
\end{aligned}
\tag{4.14}
$$

The first term in (4.14) gives the desired main term C_1x. The series for C_1 is convergent owing to the fact that $f_j(d)\ll d^{-\alpha}$. Using $f_j(d_j)\ll d_j^{-\alpha}$ and the fact that number of d_j's satisfying $[d_1,\dots,d_k]=L$ is at most $\tau(L)^k \ll x^{\frac{2k^2}{\log\log x}} \ll x^\epsilon$, we find that the second O-term above is at most

$$
\ll x^{1+\epsilon}\sum_{L>x}\frac{1}{L^{1+\alpha}} \ll x^{1-\alpha+\epsilon}.
$$

Similarly, the third error term in (4.14) is

$$x^{\epsilon}\sum_{L\le x}L^{-\alpha}\ll x^{1-\alpha+\epsilon}.$$

Combining the estimates S_{11} and S_{12}, we prove (a).

Now, we prove (b). Denoting the given sum by S_2, we have

$$S_2=\sum_{n\le x}\sum_{\substack{d_1,\ldots,d_k\\ d_j|n^2+a_j}}\prod_{j=1}^{k}f_j(d_j)=\sum_{n\le x}\sum_{\substack{d_1,\ldots,d_k\\ d_j|n^2+a_j\\ L\le x}}\prod_{j=1}^{k}f_j(d_j)+\sum_{n\le x}\sum_{\substack{d_1,\ldots,d_k\\ d_j|n^2+a_j\\ L>x}}\prod_{j=1}^{k}f_j(d_j)$$

$$=S_{21}\ +\ S_{22}.$$

As in (a), the second term S_{22} is at most

$$S_{22}\ll\sum_{n\le x}\sum_{\substack{L|\prod(n^2+a_j)\\ L>x}}\sum_{[d_1,\ldots,d_k]=L}\left(\prod d_j\right)^{-\alpha}\ll\sum_{n\le x}\sum_{\substack{L|\prod(n^2+a_j)\\ L>x}}\frac{\tau(L)^k}{L^{\alpha}}$$
$$\ll x^{-\alpha+\epsilon}\sum_{n\le x}\tau\left(\prod(n^2+a_j)\right)\ll x^{1-\alpha+\epsilon},$$

where we again use Lemma 4.27, $\tau(L)^k\ll x^{\epsilon}$ and that $\tau\left(\prod(n+a_j)\right)\ll x^{\epsilon}$ from the proof of (a). The first term is

$$S_{11}=\sum_{\substack{d_1,\ldots,d_k\\ L\le x}}\prod_{j=1}^{k}f_j(d_j)\sum_{\substack{n^2\equiv -a_j\ (\mathrm{mod}\ d_j)\\ n\le x}}1$$

To have a solution to the congruence $n^2\equiv -a_j\,(\mathrm{mod}\ d_j)$, first we need to have $(d_i,d_j)\mid a_i-a_j$ for all i, j. Again, we write $\sum'$ to denote this condition. Let $\lambda(d_1,\ldots,d_k)$ be the number of solutions modulo $L=[d_1,\ldots,d_k]$ to the system of congruences $n^2 \equiv -a_j \pmod{d_j}$. Therefore,

$$
\begin{aligned}
S_{21} &= \sideset{}{'}\sum_{\substack{d_1,\dots,d_k \\ L \le x}} \lambda(d_1,\dots,d_k) \prod_{j=1}^{k} f_j(d_j) \left(\frac{x}{L} + O(1)\right) \\
&= x \sideset{}{'}\sum_{d_1,\dots,d_k} \frac{\lambda(d_1,\dots,d_k)\prod_{j=1}^{k} f_j(d_j)}{L} + x \sideset{}{'}\sum_{\substack{d_1,\dots,d_k \\ L > x}} \frac{\lambda(d_1,\dots,d_k)\prod_{j=1}^{k} f_j(d_j)}{L} \qquad (4.15) \\
&\quad + \sideset{}{'}\sum_{\substack{d_1,\dots,d_k \\ L \le x}} \lambda(d_1,\dots,d_k) \prod_{j=1}^{k} f_j(d_j)
\end{aligned}
$$

The first term of (4.15) gives the main term $C_2 x$. We shall estimate the second and third terms of (4.15). We note that the system of congruences $n^2 \equiv -a_j \pmod{d_j}$ reduces to $n^2 \equiv b \pmod{L}$ and this has at most $H(L)\tau(L)$ solutions modulo L by Lemma 4.25. Hence, the second term of (4.15) is at most

$$
\begin{aligned}
x \sum_{x < L \le x^k} \frac{H(L)\tau(L)}{L^{1+\alpha}} &\ll x^{1-\alpha} \sum_{L \le x^k} \frac{H(L)\tau(L)}{L} \ll x^{1-\alpha}(\log x^k)^6 = x^{1-\alpha}(k \log x)^6 \\
&\ll x^{1-\alpha+\epsilon},
\end{aligned}
$$

from Lemma 4.26 and that $k = o(\log\log\log x)$. The third term of (4.15) is at most

$$
\sum_{L \le x} \frac{H(L)\tau(L)}{L^{\alpha}} \ll x^{1-\alpha+\epsilon},
$$

by Lemma 4.26. Combining the estimates for S_{21} and S_{22}, we complete the proof.

4.4.4 Proof of Theorem 4.12

Write the given sum as

$$
\begin{aligned}
S = \sum_{p \le x} \frac{\varphi(p+2)}{p+2} \frac{\varphi(p+1)}{p+1} &= \sum_{p \le x} \sum_{\substack{a \mid p+2 \\ b \mid p+1}} \frac{\mu(a)\mu(b)}{ab} \\
&= T_1 \ + \ T_2 \ + \ T_3,
\end{aligned}
$$

where T_1 corresponds to $[a,b] \le (\log x)^A$, T_2 for $(\log x)^A < [a,b] \le x$ and T_3 for $[a,b] > x$. Now

$$T_3 \le \sum_{n\le x} \sum_{\substack{a|n+2 \\ b|n+1 \\ [a,b]\ge x}} \frac{1}{ab} = O\left(\log^2 x\right), \tag{4.16}$$

by Lemma 4.19 (b). Moreover

$$T_2 \le \sum_{n\le x} \sum_{\substack{a|n+2 \\ b|n+1 \\ (\log x)^A<[a,b]\le x}} \frac{1}{ab} = \sum_{\substack{(a,b)=1 \\ (\log x)^A<[a,b]\le x}} \frac{1}{ab}\left(\frac{x}{ab}+O(1)\right) = O\left(\frac{x}{(\log x)^{A-1}}\right). \tag{4.17}$$

Next, we have

$$T_1 = \sum_{p\le x} \sum_{\substack{[a,b]\le(\log x)^A \\ a|p+2 \\ b|p+1}} \frac{\mu(a)\mu(b)}{ab} = \sum_{[a,b]\le(\log x)^A} \frac{\mu(a)\mu(b)}{ab} \sum_{\substack{p\le x \\ p\equiv -2 \pmod a \\ p\equiv -1 \pmod b}} 1. \tag{4.18}$$

For $p \ne 2$, the p-sum survives only if $(a,b)=1$ and a is odd. Thus

$$T_1 = \sum_{\substack{a \text{ odd}\ge 1 \\ (a,b)=1 \\ ab\le(\log x)^A}} \frac{\mu(a)\mu(b)}{ab}\left(\frac{\mathrm{li}(x)}{\varphi(ab)} + O\left(\frac{x}{(\log x)^A}\right)\right),$$

by Siegel's theorem on primes in arithmetic progressions. Clearly, the O-term is $O\left(\frac{x}{(\log x)^{A-1}}\right)$ and the main term is

$$\mathrm{li}(x) \sum_{\substack{a \text{ odd} \\ (a,b)=1}} \frac{\mu(a)\mu(b)}{ab\varphi(ab)} - \mathrm{li}(x) \sum_{\substack{a \text{ odd} \\ (a,b)=1 \\ ab>(\log x)^A}} \frac{\mu(a)\mu(b)}{ab\varphi(ab)}.$$

The second term is $O\left(\frac{x}{(\log x)^{A-1}}\right)$ and the first term is clearly $\frac{\mathrm{li}(x)}{2}\prod_{p>2}\left(1-\frac{2}{p(p-1)}\right)$.

4.4.5 Replacing x^ϵ by a power of $\log x$ in Theorem 4.6

Now, we sketch how x^ϵ can be replaced by a power of $\log x$ in the error term of Theorem 4.6, provided that α is not close to $1/2$. We recall that x^ϵ comes from Lemma 4.20, and therefore we restrict our attention to this lemma. Recall that

$$S = \sum_{\substack{A=2^k \le x^{1/2} \\ B=2^l \le x \\ A^2B > x}} S_{A,B}, \quad \text{where } S_{A,B} = \sum_{H<n\le x} \sum_{\substack{a^2|n \\ b|n-h \\ a\sim A \\ b\sim B}} \mu(a) b^{-\alpha}.$$

Here A and B run over powers of 2 and satisfy $A \le x^{1/2}$, $B \le x$ as well as $A^2B > x$.

Case I: $x^{0.05} \le A \le x^{0.45}$. In this case, Lemma 4.20 (a) tells us that $S_{A,B} \ll x^{1+\epsilon}/(AB^\alpha)$. Summing A, B over powers of 2, we have

$$S \ll x^{1+\epsilon} \sum_{\substack{x^{0.05}\le A\le x^{0.45} \\ x/A^2 < B \le x}} \frac{1}{AB^\alpha} \ll x^{1-\alpha+\epsilon} \sum_{x^{0.05}\le A\le x^{0.45}} A^{2\alpha-1} \ll \begin{cases} x^{0.55-0.1\alpha+\epsilon}, & \alpha > 1/2 \\ x^{0.95-0.9\alpha+\epsilon}, & \alpha < 1/2. \end{cases},$$

and the above is $\ll E_1(x)$ whenever $\epsilon < 0.1|\alpha - 1/2|$.

Case II: $A \le x^{0.05}$. In this case, we claim that

$$S_{A,B} \ll \frac{x(\log A)^{10}}{AB^\alpha},$$

To prove it, write $n = a^2c$ (since $a^2 \mid n$) and let

$$T = \left\{(a,b,c,d) : a^2c - bd = h,\ a \sim A, b \sim B\right\} \tag{4.19}$$

This means that $S_{A,B} \ll B^{-\alpha}|T|$. Since $bd = a^2c - h \le 2x$ and $a^2b > x$, we have $d \le 2a^2 \ll x^{0.1}$. To bound the number of elements in T, first fix a, d, so that the congruence $a^2c - h \equiv 0 \pmod{d}$ has at most (a^2, d) solutions in $c \pmod{d}$.

As $c \le x/a^2$, the number of choices for c is at most $\left(\frac{x}{a^2 d} + O(1)\right)(a^2, d)$ and since $a \ll x^{0.05}$, $d \ll x^{0.1}$, the O-term can be absorbed into the main term and therefore

$$|T| \ll \sum_{\substack{a \sim A \\ d \le 2a^2}} \frac{x(a^2, d)}{a^2 d} \ll \frac{x(\log x)^{10}}{A},$$

which proves the claim. Summing $S_{A,B}$ over $A \le x^{0.05}$, $A^2 B > x$ over powers of 2 now gives

$$S \ll x(\log x)^{10} \sum_{\substack{A \le x^{0.05} \\ B \le x}} \frac{1}{AB^\alpha} \ll (\log x)^{10} \begin{cases} x^{0.95-0.9\alpha}, & \alpha > 1/2 \\ x^{1-\alpha}, & \alpha < 1/2. \end{cases} \ll E_1(x)(\log x)^{10}$$

Case III: $A \ge x^{0.45}$, $B > x^{0.2}$. Here again, Lemma 4.20 (a) gives $S_{A,B} \ll x^{1+\epsilon}/(AB^\alpha)$ and summing A and B over powers of 2, we get $S \ll E_1(x)$.

Case IV: $A \ge x^{0.45}$, $B \le x^{0.2}$. In this case, we again claim that

$$S_{A,B} \ll \frac{x(\log B)^{10}}{AB^\alpha}.$$

Just as in Case II, we need an upper bound for $|T|$, with T as given in (4.19). Since $a^2 c \le x$ and $a > x^{0.45}$, one has $c < x^{0.1}$. Fixing c and b, Lemma 4.25, tells us that $a^2 c - h \equiv 0 \pmod{b}$ has at most $L(b)\tau(b)$ solutions for $a \pmod{b}$. Since $a \sim A$, the number of choices for a is at most $\left(\frac{A}{b} + O(1)\right) L(b)\tau(b)$. The O-term can be ignored again as $b \sim B < A$. Also, since $a^2 c \le x$, we have $c \ll x/A^2$. Summing this over $c \ll x/A^2$ and $b \sim B$ and applying Lemma 4.26, the claim follows. Now, summing A and B over powers of 2 in the relevant range, we find that $S \ll (\log x)^{10} E_1(x)$.

Chapter 5

Number of factorizations of an integer

In this chapter, we study a problem concerning the Oppenheim's factorization function, that counts the number of ways of writing a positive integer as a product of factors larger than 1 without taking the order into consideration. We estimate the number of distinct values of this function not exceeding a given parameter x.

5.1 Oppenheim's factorization function

Definition 5.1. Let $f(n)$ denote the number of unordered factorizations of n into factors larger than 1, i.e., $f(n)$ is the number of tuples $(n_1, \ldots, n_r)$, with $1 < n_1 \leq n_2 \leq \cdots \leq n_r$ and $n = n_1 n_2 \ldots n_r$.

For example, $f(18) = 4$, since 18 has the factorizations

$$18, \quad 2 \cdot 9, \quad 3 \cdot 6, \quad 2 \cdot 3 \cdot 3.$$

This function is a multiplicative analogue of the the partition function.

The properties of this function have been studied before. Oppenheim [Opp26] obtained the asymptotic formula

$$\sum_{n\le x} f(n) \sim \frac{x\exp(2\sqrt{\log x})}{2\sqrt{\pi}(\log x)^{3/4}}.$$

Laterf Canfield, Erdős and Pomerance [CEP83] showed that the maximal order of $f(n)$ is

$$n\exp\left((-1+o(1))\frac{\log n\cdot\log\log\log n}{\log\log n}\right),$$

For any $x\ge 1$, let $\mathscr{F}(x)$ be the the set of values of $f(n)$, not exceeding x, i.e.

$$\mathscr{F}(x) = \{f(n) : f(n)\le x\}. \tag{5.1}$$

In [CEP83], the authors claimed that they could prove $\#\mathscr{F}(x) = x^{o(1)}$, as $x\to\infty$, but did not include a proof. In this connection, Luca, Mukhopadhyay and Srinivas [LMS10] proved that

$$\#\mathscr{F}(x) = x^{O(\log\log\log x/\log\log x)}.$$

Their bound was improved by Balasubramanian and Luca [BL11], who proved that

$$\#\mathscr{F}(x) \le \exp\left(9(\log x)^{2/3}\right), \quad \text{for all } x\ge 1.$$

5.2 The main result

In this chapter and [BS17], we further improve this bound. We prove:

Theorem 5.2. *Let $C = 2\pi\sqrt{2/3}$ and x be sufficiently large. Then*

$$\#\mathscr{F}(x) \le \exp\left(C\sqrt{\frac{\log x}{\log\log x}}\left(1+O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right).$$

We have strong reasons to believe that up to the constant C, the above bound is essentially the best possible. We will give reasons for believing the same in the final section.

5.3 Preliminaries

In this section, we give some preliminary background needed for the proof.

5.3.1 A generalized partition function

In [CEP83], the authors made the following observations:

$$f(q^n) = p(n), \qquad q \text{ prime,} \tag{5.2}$$

$$f(p_1 \dots p_r) = B_r, \qquad p_1, \dots, p_r \text{ distinct primes.} \tag{5.3}$$

Here $p(n)$ is the partition function and B_r is the r^{th} *Bell number*, which also happens to be the number of partitions of a set with r distinct elements.

In view of the observations (5.2), (5.3) as well as the remarks made by the authors of [CEP83], we generalize the partition function to $\mathbb{N}^r$.

Notation 5.3. For any $r \geq 1$, let

$$\mathbb{Z}^+(r) := (\mathbb{Z}_{\geq 0})^r \setminus \{\mathbf{0}\}, \quad \text{where } \mathbf{0} = (0, \dots, 0). \tag{5.4}$$

Definition 5.4. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$. A partition of $\boldsymbol{\alpha}$ is an unordered decomposition

$$\boldsymbol{\alpha} = \boldsymbol{\beta_1} + \cdots + \boldsymbol{\beta_l},$$

where $\boldsymbol{\beta_i} \in \mathbb{Z}^+(r)$, for each $1 \leq i \leq l$ and the addition is component-wise. The number of partitions of $\boldsymbol{\alpha}$ is denoted by $p(\boldsymbol{\alpha})$.

Example 5.5. The partitions of $\boldsymbol{\alpha} = (1,2)$ are

$$(1,2), \quad (1,0)+(0,2), \quad (0,1)+(1,1), \quad (0,1)+(0,1)+(1,0).$$

Remark 5.6. When $r = 1$, the above corresponds to the usual *partition function* in $\mathbb{N}$. Moreover, any such partition π satisfying

$$\boldsymbol{\alpha} = \sum_{\boldsymbol{\beta} \in \mathbb{Z}^+(r)} h(\boldsymbol{\beta}) \cdot \boldsymbol{\beta},$$

can be represented as

$$\pi = \prod_{\boldsymbol{\beta} \in \mathbb{Z}^+(r)} \boldsymbol{\beta}^{h(\boldsymbol{\beta})},$$

as in the case $r = 1$. Here, $h(\boldsymbol{\beta})$ is the number of times $\boldsymbol{\beta} \in \mathbb{Z}^+(r)$ appears in the partition (note that all but finitely many $h(\boldsymbol{\beta})$'s are zero). For example, when $r = 2$, the partition π of $(2,3)$ given by $(2,3) = (0,1)+(0,1)+(1,0)+(1,1)$ can be written as $\pi = (0,1)^2 \cdot (1,0) \cdot (1,1)$.

Remark 5.7. The function $p(\boldsymbol{\alpha})$ can be seen as a partition of the multi-set

$$\{1, 1, \ldots, 1, 2, \ldots, 2, \ldots, r, \ldots, r\},$$

with each i having exactly α_i copies, for $1 \leq i \leq r$.

The following lemma generalizes the observations in (5.2) and (5.3).

Lemma 5.8. *If $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$, then $f(n) = p(\boldsymbol{\alpha})$.*

Proof. Let $n = n_1 n_2 \ldots n_l$ be a nontrivial factorization of n, with $n_i > 1$ for each i.

For each $1 \le i \le l$, let

$$n_i = \prod_{j=1}^{r} p_j^{\beta_{ij}} \quad \text{and} \quad \boldsymbol{\beta_i} = (\beta_{i1}, \dots, \beta_{ir}).$$

Then, clearly $\boldsymbol{\beta_i} \in \mathbb{Z}^+(r)$ and $\sum_{i=1}^{l} \boldsymbol{\beta_i} = \boldsymbol{\alpha}$. Therefore, each unordered factorization gives rise to a partition of $\boldsymbol{\alpha}$. Clearly, the partition obtained in this way is unique. The converse follows analogously. □

Hence, $\#\mathscr{F}(x)$ is bounded above by the number of unordered tuples $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$, which satisfy $p(\boldsymbol{\alpha}) \le x$. We record this as the following corollary:

Corollary 5.9.

$$\#\mathscr{F}(x) \le \#\{1 \le \alpha_1 \le \cdots \le \alpha_r : \ p(\boldsymbol{\alpha}) \le x\}.$$

Our job is therefore reduced to determining the distribution of $p(\boldsymbol{\alpha}) \le x$.

5.3.2 A generating function for $p(\boldsymbol{\alpha})$

We give a generating function for $p(\boldsymbol{\alpha})$, which is later used in order to obtain a lower bound for $p(\boldsymbol{\alpha})$.

Notation 5.10. Let $\boldsymbol{q} = (q_1, \dots, q_r)$, with $|q_i| < 1$ for each $1 \le i \le r$. For $\boldsymbol{\beta} \in \mathbb{Z}^+(r)$, we use the notation

$$\boldsymbol{q}^{\boldsymbol{\beta}} := q_1^{\beta_1} \dots q_r^{\beta_r}.$$

We have the following generating function for $p(\boldsymbol{\alpha})$:

Lemma 5.11. *Let*

$$P(\boldsymbol{q}) = \prod_{\boldsymbol{\beta}\in\mathbb{Z}^+(r)} \left(1-\boldsymbol{q}^{\boldsymbol{\beta}}\right)^{-1}.$$

Then $P(\boldsymbol{q})$ is a generating function for $p(\boldsymbol{\alpha})$; i.e., for any $\boldsymbol{\alpha}\in\mathbb{N}^r$, the coefficient of $\boldsymbol{q}^{\boldsymbol{\alpha}}$ in $P(\boldsymbol{q})$ is $p(\boldsymbol{\alpha})$.

Remark 5.12. When $r = 1$, this corresponds to the generating function of the partition function $p(n)$.

Proof of Lemma 5.11. Since the given product converges locally uniformly, we have

$$P(\boldsymbol{q}) = \prod_{\boldsymbol{\beta}\in\mathbb{Z}^+(r)} \left(\sum_{l=0}^{\infty} \boldsymbol{q}^{l\boldsymbol{\beta}}\right) = \sum_{h:\mathbb{Z}^+(r)\to\mathbb{Z}_{\geq 0}} \boldsymbol{q}^{h(\boldsymbol{\beta})\cdot\boldsymbol{\beta}} \tag{5.5}$$

Therefore, the coefficient of $\boldsymbol{q}^{\boldsymbol{\alpha}}$ above is the number of functions $h:\mathbb{Z}^+(r)\to\mathbb{Z}_{\geq 0}$, for which

$$\sum_{\boldsymbol{\beta}\in\mathbb{Z}^+(r)} h(\boldsymbol{\beta})\cdot\boldsymbol{\beta} = \boldsymbol{\alpha}. \tag{5.6}$$

We show that the number of such h equals $p(\boldsymbol{\alpha})$. For a partition π of $\boldsymbol{\alpha}$, write the decomposition

$$\pi = \prod_{\boldsymbol{\beta}\in\mathbb{Z}^+(r)} \boldsymbol{\beta}^{h(\boldsymbol{\beta})}, \tag{5.7}$$

This defines h uniquely. Conversely, any such function h gives rise to a unique decomposition in (5.6). This completes the proof. □

We also need some bounds on certain binomial coefficients. We prove them in the next section.

5.3.3 Some bounds on factorials and binomial coefficients

We begin with the following.

Lemma 5.13. *Let*

$$h_1(x) = \left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}, \quad h_2(x) = \frac{x+1}{x+2}\left(1+\frac{1}{x}\right)^{x+\frac{3}{2}}.$$

Then, as $x \to \infty$, the functions h_1 and h_2 converge to e decreasingly.

Proof. It is clear that both $h_1(x)$ and $h_2(x)$ converge to e as $x \to \infty$. To show that they are decreasing, we will use the following inequality

$$\log\left(1+\frac{1}{x}\right) = \int_x^{x+1} \frac{dt}{t} \leq \frac{1}{2}\left(\frac{1}{x}+\frac{1}{x+1}\right) = \frac{x+\frac{1}{2}}{x(x+1)}, \quad \text{for all } x \geq 1. \tag{5.8}$$

Taking logarithmic derivative of h_1, we get

$$\frac{h_1'(x)}{h_1(x)} = \log\left(1+\frac{1}{x}\right) - \frac{x+\frac{1}{2}}{x(x+1)} \leq 0,$$

by (5.8) for all $x \geq 1$. Therefore, h_1 is decreasing.

To show h_2 is decreasing, we look at

$$\frac{h_2'(x)}{h_2(x)} = \log\left(1+\frac{1}{x}\right) - \frac{x^2+\frac{5}{2}x+3}{x(x+1)(x+2)} \leq \frac{x+\frac{1}{2}}{x(x+1)} - \frac{x^2+\frac{5}{2}x+3}{x(x+1)(x+2)} < 0,$$

for all $x \geq 1$. This completes the proof. □

This leads to the following:

Lemma 5.14. *Let n and k be positive integers. Then*

(a)

$$(k+1)! \leq \frac{2\,k^{k+\frac{3}{2}}}{e^{k-1}},$$

(b)

$$\binom{k+n}{k} \geq \frac{1}{2\sqrt{2}} \frac{(k+n)^{k+n+\frac{1}{2}}}{k^{k+\frac{1}{2}} n^{n+\frac{1}{2}}}.$$

Proof. Proof is by induction on $k \geq 1$ (for any $n \geq 1$).

We first prove (a). When $k = 1$, (a) is trivially true. So, assume that (a) holds for some $k \geq 1$. Then, by induction

$$(k+2)! = (k+2)(k+1)! \leq \frac{2(k+2)k^{k+\frac{3}{2}}}{e^{k-1}}. \tag{5.9}$$

We need to show that the RHS of (5.9) is at most $\frac{2(k+1)^{k+\frac{5}{2}}}{e^k}$, which is equivalent to

$$\frac{k+1}{k+2}\left(1+\frac{1}{k}\right)^{k+\frac{3}{2}} \geq e,$$

and this is true by Lemma 5.13 for the function h_2.

Next, we prove (b). When $k = 1$, this reduces to

$$\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} \leq 2\sqrt{2}.$$

This is true because h_1 is decreasing implying its maximum occurs at $n = 1$.

Now, suppose that (b) holds true for (k, n). We want to prove it holds for $(k+1, n)$ as well. By induction

$$\binom{k+n+1}{k+1} = \frac{k+n+1}{k+1}\binom{k+n}{k} \geq \frac{1}{2\sqrt{2}}\frac{(k+n+1)}{(k+1)}\frac{(k+n)^{k+n+\frac{1}{2}}}{k^{k+\frac{1}{2}}n^{n+\frac{1}{2}}}. \tag{5.10}$$

We need to show that the RHS of (5.10) is at least

$$\frac{1}{2\sqrt{2}}\frac{(k+n+1)^{k+n+\frac{3}{2}}}{(k+1)^{k+\frac{3}{2}}n^{n+\frac{1}{2}}},$$

and this is equivalent to

$$\left(1+\frac{1}{k}\right)^{k+\frac{1}{2}} \geq \left(1+\frac{1}{k+n}\right)^{k+n+\frac{1}{2}},$$

which is true since h_1 is decreasing. This completes the proof. $\square$

Remark 5.15. It was possible to prove Lemma 5.14 using Stirling's formula. We chose this approach because we wanted to a bound valid for all $k, n \geq 1$ without bothering about the error terms occurring in Stirling's approximation.

We prove the following lemma about the exponential of a power series:

Lemma 5.16. *Suppose that*

$$F(\boldsymbol{q}) = a(\mathbf{0}) + \sum_{\boldsymbol{n} \in \mathbb{Z}^+(r)}^{\infty} a(\boldsymbol{n}) \boldsymbol{q}^{\boldsymbol{n}},$$

is convergent in $\{\boldsymbol{q} : |q_i| < 1\}$*, with real coefficients satisfying* $a(\boldsymbol{n}) \geq 0$*, for* $\boldsymbol{n} \in \mathbb{Z}^+(r) \cup \{\mathbf{0}\}$*. Then the power series of* $G(\boldsymbol{q}) = \exp(F(\boldsymbol{q}))$ *around* $\mathbf{0}$ *also has non-negative coefficients.*

Proof. Note that

$$G(\boldsymbol{q}) = \sum_{k=0}^{\infty} \frac{F(\boldsymbol{q})^k}{k!}.$$

Now, since $a(\boldsymbol{n}) \geq 0$, for each $\boldsymbol{n} \in \mathbb{Z}^+(r)$, it follows that the coefficients of $F(\boldsymbol{q})^k$ are non-negative for each $k \geq 0$. Therefore, $G(\boldsymbol{q})$ has non-negative coefficients. $\square$

The next lemma gives an upper bound to number of tuples of positive integers satisfying $\sum n_i \leq y$.

Lemma 5.17. *The number of unordered tuples* $(n_1, \ldots, n_l)$ *in* $\mathbb{N}$ *satisfying*

$$\sum_{i=1}^{l} n_i \leq y,$$

is at most $y \exp\left(\pi\sqrt{2y/3}\right)$*, for all* $y \geq 1$*.*

Proof of Lemma 5.17. Suppose that $\sum_{i=1}^{l} n_i = n \leq y$. From the proof of Theorem 15.3 in [Nat00, Pg 468], we have the upper bound $p(n) \leq \exp\left(\pi\sqrt{2n/3}\right)$, for all $n \geq 1$.

Therefore, the total number of choices for $n_1, \ldots, n_l$ is at most

$$\sum_{n \le y} \exp\left(\pi\sqrt{2n/3}\right) \le y \exp\left(\pi\sqrt{2y/3}\right).$$

□

5.3.4 A lower bound for $p(\boldsymbol{\alpha})$

Now, we obtain a lower bound for $p(\boldsymbol{\alpha})$ in terms of a generalized hypergeometric series.

Lemma 5.18. *Let* $\boldsymbol{\alpha} \in \mathbb{N}^r$*. Then*

$$p(\boldsymbol{\alpha}) \ge \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\prod_{i=1}^{r}\binom{k+\alpha_i}{k}. \tag{5.11}$$

Remark 5.19. The RHS of (5.11) is a generalized hypergeometric series

$$\frac{1}{e}\,{}_rF_r\left(\begin{matrix}\alpha_1+1 & \ldots & \ldots & \alpha_{r-1}+1 & \alpha_r+1 \\ 1 & \ldots & \ldots & 1 & 2\end{matrix}\,;1\right).$$

When $\boldsymbol{\alpha} = (1,1,\ldots,1)$, equality holds in (5.11) and the RHS of (5.11) becomes the Dobiński's formula for the r^{th} Bell number B_r.

Proof of Lemma 5.18. Taking logarithms in the expression for $P(\boldsymbol{q})$ in Lemma 5.11, we get

$$\begin{aligned}\log P(\boldsymbol{q}) &= \sum_{\boldsymbol{\beta}\in\mathbb{Z}^+(r)} -\log(1-\boldsymbol{q}^{\boldsymbol{\beta}}) = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^+(r)}\sum_{m=1}^{\infty}\frac{\boldsymbol{q}^{m\boldsymbol{\beta}}}{m} = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^+(r)}\boldsymbol{q}^{\boldsymbol{\beta}}\sum_{m\mid\beta_i\forall i}\frac{1}{m} \\ &= \sum_{\boldsymbol{\beta}\in\mathbb{Z}^+(r)}\frac{\sigma(\beta_1,\ldots,\beta_r)}{(\beta_1,\ldots,\beta_r)}\boldsymbol{q}^{\boldsymbol{\beta}} = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^+(r)}\boldsymbol{q}^{\boldsymbol{\beta}} + H(\boldsymbol{q}),\end{aligned} \tag{5.12}$$

where $\sigma(\beta_1, \ldots, \beta_r)$ denotes $\sigma(\gcd(\beta_1, \ldots, \beta_r))$, and

$$H(\boldsymbol{q}) = \sum_{\boldsymbol{\beta} \in \mathbb{Z}^+(r)} \left(\frac{\sigma(\beta_1, \ldots, \beta_r)}{(\beta_1, \ldots, \beta_r)} - 1 \right) \boldsymbol{q}^{\boldsymbol{\beta}}. \tag{5.13}$$

Taking exponential in (5.12), we get $P(\boldsymbol{q}) = \exp\left(\sum_{\boldsymbol{\beta} \in \mathbb{Z}^+(r)} \boldsymbol{q}^{\boldsymbol{\beta}} \right) \cdot \exp(H(\boldsymbol{q}))$. We have

$$\sum_{\boldsymbol{\beta} \in \mathbb{Z}^+(r)} \boldsymbol{q}^{\boldsymbol{\beta}} = \sum_{\substack{\beta_1, \ldots, \beta_r \geq 0 \\ \sum \beta_j \geq 1}} q_1^{\beta_1} \ldots q_r^{\beta_r} = \sum_{\beta_1, \ldots, \beta_r \geq 0} q_1^{\beta_1} \ldots q_r^{\beta_r} - 1 = \frac{1}{(1-q_1) \ldots (1-q_r)} - 1. \tag{5.14}$$

Since $H(\boldsymbol{q})$ has non-negative coefficients with constant term 0, it follows by Lemma 5.16 that $\exp(H(\boldsymbol{q}))$ also has non-negative coefficients with constant term 1. So, the coefficient of $\boldsymbol{q}^{\boldsymbol{\alpha}}$ in $P(\boldsymbol{q})$ is at least $1/e$ times the coefficient of $\boldsymbol{q}^{\boldsymbol{\alpha}}$ in $\exp\left(\prod_{i=1}^r (1-q_i)^{-1} \right)$. Since

$$\exp\left(\prod_{i=1}^r (1-q_i)^{-1} \right) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{i=1}^r (1-q_i)^{-k}, \tag{5.15}$$

and $(1-q)^{-k} = 1 + \sum_{n=1}^{\infty} \binom{k+n-1}{k-1} q^n$, the coefficient of $\boldsymbol{q}^{\boldsymbol{\alpha}}$ in (5.15) equals

$$\sum_{k=1}^{\infty} \frac{1}{k!} \prod_{i=1}^r \binom{k+\alpha_i-1}{k-1} = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \prod_{i=1}^r \binom{k+\alpha_i}{k}.$$

This completes the proof. □

For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ and $z > 0$, define

$$g(\boldsymbol{\alpha}, z) = z \prod_{i=1}^r \left(1 + \frac{\alpha_i}{z}\right)^{-1}. \tag{5.16}$$

Now, $g(\boldsymbol{\alpha}, z)$ is a strictly increasing function of z with $g(\boldsymbol{\alpha}, 1) < 1$ since

$$\frac{g'(\boldsymbol{\alpha}, z)}{g(\boldsymbol{\alpha}, z)} = \frac{r+1}{z} - \sum_{i=1}^r \frac{1}{z+\alpha_i} > 0,$$

for all $z > 0$. Therefore, $g(\boldsymbol{\alpha}, z) = 1$ has a unique positive solution $z(\boldsymbol{\alpha}) > 1$. Let

$$N = N(\boldsymbol{\alpha}) = \lfloor z(\boldsymbol{\alpha}) \rfloor . \tag{5.17}$$

Now, we prove a lower bound for $p(\boldsymbol{\alpha})$.

Proposition 5.20. *Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ *and* $N = N(\boldsymbol{\alpha})$ *be as in* (5.17). *Then*

(a)

$$p(\boldsymbol{\alpha}) \geq \frac{e^{N-2}}{2\, N^{\frac{3}{2}}} \prod_{i=1}^{r} \frac{1}{2\sqrt{2N}} \left(1 + \frac{N}{\alpha_i}\right)^{\alpha_i + \frac{1}{2}} .$$

(b) Further, if $p(\boldsymbol{\alpha}) \leq x$, *then for* x *sufficiently large, we have*

$$r \leq R := \frac{2 \log x}{\log \log x} \left(1 + \frac{2 \log \log \log x}{\log \log x}\right) \quad \text{and} \quad N \leq 3 \log x.$$

Notation 5.21. The quantity $N = N(\boldsymbol{\alpha})$ depends entirely on $\boldsymbol{\alpha}$. From now onwards, we denote this by N for the sake of simplicity.

Proof of Proposition 5.20. With N from (5.17), we have

$$g(\boldsymbol{\alpha}, N) \leq 1 \leq g(\boldsymbol{\alpha}, N + 1). \tag{5.18}$$

In particular,

$$\prod_{i=1}^{r} \left(1 + \frac{\alpha_i}{N}\right) \geq N. \tag{5.19}$$

To prove (a), we use the bound given in Lemma 5.18, i.e.,

$$p(\boldsymbol{\alpha}) \geq \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \prod_{i=1}^{r} \binom{k + \alpha_i}{k} = \frac{1}{e} \sum_{k=0}^{\infty} T(\boldsymbol{\alpha}, k), \tag{5.20}$$

where

$$T(\boldsymbol{\alpha}, k) := \frac{1}{(k+1)!} \prod_{i=1}^{r} \binom{k + \alpha_i}{k}.$$

We do not have an asymptotic formula for the sum in (5.20). Fortunately for us, the series converges very rapidly and therefore an optimally chosen term $T(\boldsymbol{\alpha}, k)$ will be good enough to provide a good lower bound.

Applying Lemma 5.14 to $T(\boldsymbol{\alpha}, k)$, we have for any $k \geq 1$, that

$$T(\boldsymbol{\alpha}, k) \geq \frac{e^{k-1}}{2\, k^{k+\frac{3}{2}}} \prod_{i=1}^{r} \frac{1}{2\sqrt{2}} \frac{(k+\alpha_i)^{k+\alpha_i+\frac{1}{2}}}{\alpha_i^{\alpha_i+\frac{1}{2}} k^{k+\frac{1}{2}}} \tag{5.21}$$

Choosing $k = N$ in (5.21), we obtain

$$T(\boldsymbol{\alpha}, N) \geq \frac{e^{N-1}}{2\, N^{N+\frac{3}{2}}} \prod_{i=1}^{r} \frac{1}{2\sqrt{2N}} \left(1 + \frac{\alpha_i}{N}\right)^{N} \left(1 + \frac{N}{\alpha_i}\right)^{\alpha_i+\frac{1}{2}}. \tag{5.22}$$

Using (5.19) in (5.22), we get

$$p(\boldsymbol{\alpha}) \geq \frac{T(\boldsymbol{\alpha}, N)}{e} \geq \frac{e^{N-2}}{2\, N^{\frac{3}{2}}} \prod_{i=1}^{r} \frac{1}{2\sqrt{2N}} \left(1 + \frac{N}{\alpha_i}\right)^{\alpha_i+\frac{1}{2}},$$

which proves (a).

Now we prove (b). From Lemma 5.18, we have

$$p(\boldsymbol{\alpha}) \geq \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \prod_{i=1}^{r} \binom{k+\alpha_i}{k} \geq \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^r}{k!}. \tag{5.23}$$

Considering the term $k = \lceil r/2 \rceil$, and using the inequality $1/k! \geq 1/k^k$, for all $k \geq 1$, we obtain

$$x \geq p(\boldsymbol{\alpha}) \geq \frac{1}{e} \frac{\lceil r/2 \rceil^r}{\lceil r/2 \rceil !} \geq \frac{1}{e} \lceil r/2 \rceil^{\lfloor r/2 \rfloor}.$$

From this, it follows that $r \leq R$, for all $x \geq 24$.

To show $N \leq 3 \log x$, we take logarithms in (a) of Proposition 5.20, to obtain

$$N - 1.04R - 0.5(R+3)\log N - \log x - 2.7 \leq 0.$$

Substituting R, we find that $N \leq 3\log x$, for all $x \geq e^{9540}$. This completes the proof of Proposition 5.20. $\square$

5.4 Proof of Theorem 5.2

Assume x is sufficiently large. Let $\boldsymbol{\alpha} \in \mathbb{N}^r$ be such that $p(\boldsymbol{\alpha}) \leq x$. Taking logarithm in the inequality in Proposition 5.20 (a), and transferring the negative terms to RHS, we obtain

$$N + \sum_{i=1}^{r} (\alpha_i + 0.5) \log\left(1 + \frac{N}{\alpha_i}\right) \leq \log x + 0.5(r+3)\log N + 1.04\, r + 2.7.$$

Using the bounds for N, r from Proposition 5.20 (b) above and simplifying, we get:

$$\sum_{i=1}^{r} \alpha_i \log\left(1 + \frac{N}{\alpha_i}\right) \leq 2\log x \left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right). \tag{5.24}$$

We split the set $\{\alpha_1, \ldots, \alpha_r\}$ into two parts I and J, where

$$I = \{\alpha_i : \alpha_i \leq A(N+1)\} \quad \text{and} \quad J = \{\alpha_i : \alpha_i > A(N+1)\},$$

and $A > 0$ is a parameter depending only on x. We shall choose

$$A = \frac{(\log\log x)^6}{(\log x)^{1/2}}. \tag{5.25}$$

We separately estimate the number of choices for elements in I and J.

For elements of I, we have $\alpha_i \leq A(N+1)$, which means

$$\log\left(1 + \frac{N}{\alpha_i}\right) \geq \log\left(1 + \frac{N}{A(N+1)}\right) \geq \log\left(1 + \frac{1}{2A}\right)$$
$$\geq \frac{\log\log x}{2}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right).$$

for all $\alpha_i \in I$. With this applied to (5.24), we obtain (ignoring the elements of J)

$$\sum_I \alpha_i \leq \frac{4\log x}{\log\log x}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right). \tag{5.26}$$

By Lemma 5.17 applied to (5.26), the number of choices for α_i's in I, is at most

$$\exp\left(2\pi\sqrt{\frac{2\log x}{3\log\log x}}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right). \tag{5.27}$$

Next, we estimate the total number of choices for elements of J. For any $1 \leq i \leq r$, we have $p(\alpha_i) \leq p(\boldsymbol{\alpha}) \leq x$. Also, from [Mar03, Corollary 3.1], we have the lower bound $p(n) \geq \exp(2\sqrt{n})/14$, for all $n \geq 1$. In particular, for each $\alpha_i \in J$, we have

$$\alpha_i \leq \frac{1}{4}(\log 14x)^2 \leq \log^2 x, \quad \text{for all } x \geq 14. \tag{5.28}$$

In the next lemma, we give an upper bound for the cardinality of J.

Lemma 5.22. *With J as before, we have*

$$\#J \leq \frac{4\sqrt{\log x}}{(\log\log x)^5}.$$

Proof. Recall that $g(\boldsymbol{\alpha}, N+1) \geq 1$, which implies

$$N + 1 \geq \prod_{i=1}^{r}\left(1 + \frac{\alpha_i}{N+1}\right) \geq \prod_{\alpha_i \in J}\left(1 + \frac{\alpha_i}{N+1}\right) \geq (1+A)^{\#J},$$

since $\alpha_i > A(N+1)$, for all $\alpha_i \in J$. Since $A < 1$, we have $\log(1+A) \geq A/2$ and from Proposition 5.20, we have $\log(N+1) \leq \log(1 + 3\log x) \leq 2\log\log x$, for all $x > e^4$. Hence,

$$\#J \leq \frac{\log(N+1)}{\log(A+1)} \leq \frac{4\sqrt{\log x}}{(\log\log x)^5}.$$

This proves the lemma. $\square$

From (5.28) and Lemma 5.22, the number of choices for elements of J is at most

$$\left(\log^2 x\right)^{\#J} \leq \exp\left(\frac{8\sqrt{\log x}}{(\log\log x)^4}\right), \tag{5.29}$$

Therefore, from (5.27) and (5.29), the total number of choices for $\boldsymbol{\alpha}$ is at most

$$\exp\left(2\pi\sqrt{\frac{2\log x}{3\log\log x}}\left(1+O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right).$$

This completes the proof of Theorem 5.2.

5.5 Final remarks

We believe that the bound obtained in Theorem 5.2 is the best possible apart from the constant C. Our reasons for believing the same are as follows:

$$S = \left\{\boldsymbol{\alpha} : \alpha_i \leq \sqrt{\log x}\ \forall\ i \text{ and } \sum \alpha_i \leq \frac{B\log x}{\log\log x}\right\}.$$

Then, for each $\boldsymbol{\alpha} \in S$, we have $p(\boldsymbol{\alpha}) = O(x)$ and the number of elements in this set is at least $\exp\left(c_1\sqrt{\frac{\log x}{\log\log x}}\right)$. However, we are unable to show that the values of $p(\boldsymbol{\alpha})$, as $\boldsymbol{\alpha}$ runs over S, are largely distinct, i.e., they do not repeat too often. Calculations do seem to suggest that the number of such distinct values of $p(\boldsymbol{\alpha})$, as $\boldsymbol{\alpha} \in S$ have a lower bound of a similar order. We will return to this problem later.

Appendix A

Appendix for Chapter 2

A.1 Partial sums of Möbius and related functions

We prove bounds for the partial sums of the Möbius function. We have the following:

Lemma A.1. *Let $\{b_n\}_{n\geq 1}$ be any sequence of reals and for any $k \geq 0$, define*

$$B_k(x) := \sum_{n\leq x} b_n \left(\log \frac{x}{n}\right)^k .$$

Then for all $0 \leq r \leq k-1$, we have

$$B_k(x) = \frac{k(k-1)\dots(k-r)}{r!} \int_1^x \left(\log \frac{x}{t}\right)^{k-r-1} B_r(t)\, \frac{dt}{t}.$$

Proof. Consider the quantity

$$\begin{aligned}\int_1^x \left(\log \frac{x}{t}\right)^{k-r-1} B_r(t)\, \frac{dt}{t} &= \int_1^x \left(\log \frac{x}{t}\right)^{k-r-1} \left(\sum_{n\leq t} b_n \left(\log \frac{t}{n}\right)^r\right) \frac{dt}{t} \\ &= \sum_{n\leq x} b_n \int_n^x \frac{(\log \frac{x}{t})^{k-r-1}(\log \frac{t}{n})^r}{t}\, dt\end{aligned} \tag{A.1}$$

Making the change of variable $\lambda = \left(\log \frac{t}{n}\right) / \left(\log \frac{x}{n}\right)$, we find that

$$\int_n^x \frac{(\log \frac{x}{t})^{k-r-1}(\log \frac{t}{n})^r}{t}\, dt = \left(\log \frac{x}{n}\right)^k \int_0^1 (1-\lambda)^{k-r-1}\lambda^r \, d\lambda = \left(\log \frac{x}{n}\right)^k \frac{(k-r-1)!\, r!}{k!}. \tag{A.2}$$

From (A.1) and (A.2), the proof is complete. □

We define a function f_0 as follows:

Definition A.2. For an interval (A, B), we define the arithmetic function f_0 as

$$f_0 = \mu * \Lambda_{(A,B)}. \tag{A.3}$$

Although the function depends upon the choice of interval (A, B), we denote it by f_0 as the interval (A, B) will be clear when writing f_0.

Definition A.3. Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. Define

$$m_{q,k}(x, f) = \sum_{\substack{m \leq x \\ (m,q)=1}} \frac{f(m)}{m} \left(\log \frac{x}{m}\right)^k.$$

Proposition A.4. *Let $x \geq 1$ and $q \geq 1$ be a positive integer. Then we have the following bounds:*

$$|m_{q,0}(x, \mu)| \leq 1, \qquad |m_{q,0}(x, f_0)| \leq \log B, \tag{A.4}$$

$$|m_{q,1}(x, \mu)| \leq 1.00303 \frac{q}{\varphi(q)}, \qquad |m_{q,1}(x, f_0)| \leq 1.00303 \frac{q}{\varphi(q)} \log B, \tag{A.5}$$

$$|m_{q,2}(x, \mu)| \leq \frac{2q}{\varphi(q)} \log x, \qquad |m_{q,2}(x, f_0)| \leq \frac{2q}{\varphi(q)} \log x \cdot \log B. \tag{A.6}$$

Also, for all $k \geq 3$, we have

$$|m_{q,k}(x, \mu)| \leq k \frac{q}{\varphi(q)} (\log x)^{k-1}. \tag{A.7}$$

Proof. The proof uses a lemma of Granville-Ramaré [GR96] and the bounds of Ramaré [Ram15]. First, we note that

$$|m_{q,k}(x,f_0)| = \left| \sum_{\substack{ab\le x\\ (ab,q)=1\\ A<b\le B}} \frac{\mu(a)}{a}\frac{\Lambda(b)}{b}\left(\log\frac{x}{ab}\right)^k \right| \le \sum_{\substack{A<b\le B\\ (b,q)=1}} \frac{\Lambda(b)}{b}\left| \sum_{\substack{a\le x/b\\ (a,q)=1}} \frac{\mu(a)}{a}\left(\log\frac{x}{ab}\right)^k \right|$$
$$\le \sum_{A<b\le B} \frac{\Lambda(b)}{b}\left| m_{q,k}\left(\frac{x}{b},\mu\right)\right|. \tag{A.8}$$

The first part of (A.4) is [GR96, Lemma 2.10], although a stronger bound is given in [Ram14, Theorem 1.1]. The first parts of (A.5) and (A.6) are due to Ramaré [Ram15, Corollary 1.10, 1.11]. For the second part of (A.4), we use (A.8) with $k=0$ along with the bound $|m_{q,0}(x,\mu)|\le 1$ from the first part of (A.4) to obtain

$$|m_{q,k}(x,f_0)| \le \sum_{A<b\le B}\frac{\Lambda(b)}{b} \le \log B.$$

The second parts of (A.5) and (A.6) are obtained in the same manner.

For (A.7), we use Lemma A.1 with $r=2$ to obtain

$$m_{q,k}(x) = \frac{k(k-1)(k-2)}{2}\int_1^x \left(\log\frac{x}{t}\right)^{k-3} m_{q,2}(t)\,\frac{dt}{t}.$$

Using the bound $|m_{q,2}(t)|\le 2q/\varphi(q)\log t$ from (A.6) above, we have

$$|m_{q,k}(x)| \le \frac{k(k-1)(k-2)}{2}\frac{q}{\varphi(q)}\int_1^x \frac{(\log\frac{x}{t})^{k-3}\cdot 2\log t}{t}\,dt$$
$$\le k(k-1)(k-2)\cdot(\log x)^{k-1}\frac{q}{\varphi(q)}\int_0^1 t(1-t)^{k-3}\,dt \le k\frac{q}{\varphi(q)}(\log x)^{k-1},$$

since $\int_0^1 t(1-t)^{k-3}\,dt = \frac{1}{(k-1)(k-2)}$. This completes the proof. □

Now, we state the following bound from [HH13, Lemma C.2.2], also proved for $q \geq 27$ in [RS62, Theorem 15].

Lemma A.5. *Let q be a positive integer. Then, for any $s \geq \max\{3, q\}$, we have $q/\varphi(q) \leq F_0(s)$, where*

$$F_0(x) = e^{\gamma} \log\log x + \frac{2.50637}{\log\log x}. \tag{A.9}$$

Lemma A.6. *$F_0(x)/x$ is decreasing for all $x \geq 3$.*

Proof. It is enough to show that $F(x) - xF'(x) > 0$. This equals

$$\begin{aligned} &e^{\gamma} \log\log x + \frac{2.50637}{\log\log x} - x\left(\frac{e^{\gamma}}{x\log x} - \frac{2.50637}{x\log x(\log\log x)^2}\right) \\ &= \frac{e^{\gamma}\left((\log\log x)^3 \log x - (\log\log x)^2\right) + 2.50637(1 + \log x \, \log\log x)}{\log x(\log\log x)^2} > 0, \end{aligned}$$

whenever $x \geq 3$. □

A.2 Explicit values of $C_{j,k,\eta}$ and T^l on monomials

In the next proposition, we give explicit values of constants $C_{j,k,\eta}$ in (H4). They will be useful when y is large. When y is small, we will need to resort to explicit numerical calculations using a program, as we need tight constants. The first two terms in (2.12) will be numerically calculated as they are proportional to x.

Proposition A.7. *Let $k = 1, 2, 3$ and $\eta_{(y),k,u_0}$ and P_{j,k,u_0}'s be as in (2.1) and (H1), respectively. Let $C_{j,k,\eta}$, $j = 0, 1, 2$ be as in (H4). Then for all $y \geq u_0$, we can take*

$$C_{0,k,\eta} = 1, \quad C_{1,k,\eta} = |\eta'|_1, \quad C_{2,k,\eta} = 3|\eta'|_\infty + \int_0^1 |\eta''(t)|\, dt + 2k|\eta(t)/t|_\infty. \tag{A.10}$$

Proof. For $j = 0$, we note that

$$|\eta_{(y),k,u_0}|_1 = \int\limits_{u_0/y}^{1} |\eta_{(y),k,u_0}(t)|\,dt \le \int_0^1 \eta(t)(\log yt)^k\,dt \le (\log y)^k \int\limits_0^1 \eta(t)\,dt = (\log y)^k,$$

which implies we can take $C_{0,k,\eta} = 1$.

In the case $j = 1$, we need to add an additional contribution due to the discontinuity of $\eta'_{(y),k,u_0}$ at u_0/y. Observe that $((\log yt)^k)' \ge 0$ and therefore, we have

$$\begin{aligned}
&|\eta'_{(y),k,u_0}|_1 - \eta(u_0/y)(\log u_0)^k \\
&= \int\limits_{u_0/y}^{1} |\eta'_{(y),k,u_0}(t)|\,dt = \int\limits_{u_0/y}^{1} \left|\eta'(t)(\log yt)^k + \eta(t)((\log yt)^k)'\right|\,dt \\
&\le \int_{u_0/y}^{1} |\eta'(t)|(\log yt)^k\,dt + \left(\eta(t)(\log yt)^k|_{u_0/y}^{1} - \int_{u_0/y}^{1} \eta'(t)(\log yt)^k\,dt\right) \\
&\le (\log y)^k \int\limits_0^1 \left(|\eta'(t)| - \eta'(t)\right)\,dt - \eta(u_0/y)(\log u_0)^k = (\log y)^k|\eta'|_1 - \eta(u_0/y)(\log u_0)^k,
\end{aligned}$$

since $\int_0^1 \eta'(t)\,dt = 0$. This shows we can take $C_{1,k,\eta} = |\eta'|_1$.

We now consider the case $j = 2$. Here again, we have to consider additional contribution arising from the discontinuity at u_0/y. Therefore,

$$\begin{aligned}
&|\eta''_{(y),k,u_0}|_1 - \left|\eta'(u_0/y)(\log u_0)^k + k(\log u_0)^{k-1}\frac{\eta(u_0/y)}{u_0/y}\right| \\
&= \int\limits_{u_0/y}^{1} \left|\eta''(t)(\log yt)^k + 2\eta'(t)((\log yt)^k)' + \eta(t)((\log yt)^k)''\right|\,dt \\
&\le (\log y)^k \int\limits_{u_0/y}^{1} |\eta''(t)|\,dt + 2\int\limits_{u_0/y}^{1} |\eta'(t)|((\log yt)^k)'\,dt + \int\limits_{u_0/y}^{1} \eta(t)\left|((\log yt)^k)''\right|\,dt
\end{aligned} \tag{A.11}$$

where we use $((\log yt)^k)' \ge 0$. We note that $((\log yt)^k)'' \ge 0$ if and only if $t \le e^{k-1}/y$. Therefore, we split the last integral in (A.11) into two parts, namely $I_0 =$

$(u_0/y, e^{k-1}/y)_+$ and $I_1 = (e^{k-1}/y, 1)_+$, where $(a,b)_+$ denotes $(a,b) \cap (u_0/y, 1)$ if $a < b$ and is empty otherwise. Therefore, the last integral of (A.11) is

$$\begin{aligned}
&\int_{I_0} \eta(t) \left((\log yt)^k\right)'' \, dt \; - \; \int_{I_1} \eta(t) \left((\log yt)^k\right)'' \, dt \\
&= \eta(t) \left((\log yt)^k\right)' \big|_{I_0} - \eta(t) \left((\log yt)^k\right)' \big|_{I_1} + \sum_{j=0}^{1} (-1)^{j-1} \int_{I_j} \eta'(t) \left((\log yt)^k\right)' \, dt \\
&\leq \eta(t) \left((\log yt)^k\right)' \big|_{I_0} - \eta(t) \left((\log yt)^k\right)' \big|_{I_1} + \int_{u_0/y}^{1} |\eta'(t)| \left((\log yt)^k\right)' \, dt
\end{aligned} \tag{A.12}$$

For the first two terms in (A.12), we consider three cases, namely (i) $e^{k-1} \leq u_0 \leq y$, (ii) $u_0 \leq e^{k-1} \leq y$ and (iii) $u_0 \leq y \leq e^{k-1}$. In case (i), I_0 is empty and $I_1 = (u_0/y, 1)$ and in case (iii), $I_0 = (u_0/y, 1)$ and I_1 is empty. So, the first two terms of (A.12) contribute in the three cases:

$$\pm\, k \frac{\eta(u_0/y)}{u_0/y} (\log u_0)^{k-1}, \quad 2k \frac{\eta(e^{k-1}/y)}{e^{k-1}/y} (k-1)^{k-1} - \frac{\eta(u_0/y)}{u_0/y} (\log u_0)^{k-1}$$

and therefore all of the above are at most $2k \left|\eta(t)/t\right|_\infty (\log y)^{k-1} - k \frac{\eta(u_0/y)}{u_0/y} (\log u_0)^{k-1}$. This means that (A.11) is at most:

$$\begin{aligned}
&(\log y)^k \int_0^1 |\eta''(t)| \, dt + 3 \int_{u_0/y}^{1} |\eta'(t)| ((\log yt)^k)' \, dt + 2k \left|\eta(t)/t\right|_\infty (\log y)^{k-1} \\
&\quad - k \frac{\eta(u_0/y)(\log u_0)^{k-1}}{u_0/y} \\
&\leq \left(\int_0^1 |\eta''(t)| \, dt + 3|\eta'|_\infty + 2k|\eta(t)/t|_\infty \right) (\log y)^k - 3|\eta'|_\infty (\log u_0)^k \\
&\quad - k \frac{\eta(u_0/y)}{u_0/y} (\log u_0)^{k-1}
\end{aligned}$$

This gives us the desired value for $C_{2,k,\eta}$ and completes the proof. □

Now, we compute the value of the operator T^l (defined in (2.8)) for monomials.

Lemma A.8. *Let k be a positive integer $l > -1$ be a real number. Let T^l be as defined in (2.8). Then for $x \geq 1$, we have*

$$T^l x^k \leq \rho_{l,k} \cdot x^k, \quad \textit{where} \;\; \rho_{l,k} = \sum_{r=0}^{k} \frac{\binom{k}{r} r!}{(l+1)^{r+1}}. \tag{A.13}$$

The values of $\rho_{l,k}$, $l \in \{0,1\}$ and $k \in \{1,2,3\}$ are given as follows:

Table A.1: Values of $\rho_{l,k}$

k	1	2	3
$\rho_{0,k}$	2	5	16
$\rho_{1,k}$	0.75	1.25	2.375

Proof. For $x \geq 1$, we have

$$\begin{aligned} T^l x^k &= \int_0^\infty e^{-t(l+1)}(x+t)^k \, dt = \sum_{r=0}^{k} \binom{k}{r} x^{k-r} \int_0^\infty e^{-t(l+1)} t^r \, dt \\ &= \sum_{r=0}^{k} \frac{\binom{k}{r} x^{k-r}}{(l+1)^{r+1}} \int_0^\infty e^{-t} t^r \, dt = \sum_{r=0}^{k} \frac{\binom{k}{r} r!}{(l+1)^{r+1}} x^{k-r} \leq \rho_{l,k} \, x^k \end{aligned}$$

□

A.3 Explicit values in the case $\eta = \eta_0$

Now, we give some explicit values for constants and for P_{j,k,u_0} when $\eta = \eta_0$ given in (1.2). We note that η_0 satisfies (C1) and from Mathematica, we have

$$|\eta_0'|_1 = 6.194\ldots, \quad |\eta_0'|_\infty = |\eta_0(t)/t|_\infty = 70 \quad \text{and} \quad \int_0^1 |\eta_0''(t)| \, dt = 89.327\ldots, \tag{A.14}$$

The following are the explicit values for $c_{\eta_0,l}$, $c_{\eta_0,l'}$ and $b_{\eta_0,l}$ defined in (2.6).

Table A.2: Values of $c_{\eta,l}$, $c_{\eta,l'}$ and $b_{\eta,l}$ when $\eta = \eta_0$

l	$c_{\eta_0,l}$	$c_{\eta_0,l'}$	$b_{\eta_0,l}$
0	1	$6.1948\ldots$	2
1	$1.70906\ldots$	$14.6946\ldots$	$3.418\ \ldots$
2	$3.55424\ldots$	$42.0314\ldots$	$7.1084\ \ldots$
3	$8.66541\ldots$	$143.6278\ldots$	$17.3308\ \ldots$

We would now like to give expressions for P_{j,k,u_0}, $P^{(0)}_{k,u_0}$ and $P^{(1)}_{k,u_0}$ and also compute $T^l P(x)$, with $l = 0, 1$ for these polynomials P_{j,k,u_0} for $k = 1, 2, 3$ in the case $\eta = \eta_0$. From Proposition A.7 for $k = 1, 2, 3$ and using the values from (A.14), we can take $C_{0,k,\eta_0} = 1$, $C_{1,k,\eta_0} = 6.195$ and $C_{2,k,\eta_0} = 720$, i.e.,

$$P_{0,k,u_0}(x) = x^k, \quad P_{1,k,u_0}(x) = 6.195x^k \quad \text{and} \quad P_{2,k,u_0}(x) = 720x^k. \tag{A.15}$$

and therefore

$$P^{(0)}_{k,u_0}(x) = 26.84x^k \quad \text{and} \quad P^{(2)}_{k,u_0}(x) = 166.23x^k, \tag{A.16}$$

Therefore, from Lemma A.8, it follows that

$$T^0 P_{2,k,u_0}(x) = 720\rho_{0,k}x^k, \qquad T^0 P^{(0)}_{k,u_0}(x) = 26.84\rho_{0,k}x^k. \tag{A.17}$$

A.4 Simplification of $L_{k,u_0}(s,x)$ and $R_{\eta_0,k,q}(s,x,f)$

We now give a simplified upper bound for $L_{k,u_0}(s,x)$ in (2.35) for $k = 1, 2, 3$.

Proposition A.9. *Let $x \geq 10^{18}$ and let $L_{k,u_0}(s,x)$ be as given in (2.35). Assume that $s \geq 1.5 \cdot 10^5$. Then, for $k = 1, 2, 3$, we have*

$$L_{k,u_0}(s,x) \leq A_k + \frac{B_k(\log 10s)^{k+1}}{s}, \tag{A.18}$$

where

$$A_k = 0.002\, C_{0,k,\eta} + 0.00003\, C_{1,k,\eta} + 10^{-6}\, C_{2,k,\eta},$$
$$B_k = 0.108\sqrt{C_{0,k,\eta}\, C_{2,k,\eta}} + 0.002\, C_{2,k,\eta} + 10^{-7} C_{1,k,\eta}\sqrt{\frac{C_{2,k,\eta}}{C_{0,k,\eta}}}. \tag{A.19}$$

Proof. We see from (2.35), that

$$\begin{aligned} L_{k,u_0}(s,x) = {} & \frac{1}{x}\left(3P_{1,k,u_0}(\log x) + \frac{0.355}{10^6}P_{2,k,u_0}(\log x)\right) \\ & + \frac{1}{\sqrt{5x}}\left(\frac{2}{\pi}P_{1,k,u_0}(\log x) + \frac{1}{6\pi^2}P_{2,k,u_0}(\log 2x) + 60P_{0,k,u_0}(\log x)\right) \\ & + \frac{1}{s}\Bigg(\frac{4}{\pi}P^{(0)}_{k,u_0}(\log 10s) + \frac{1}{10\pi}\int_{\log u_0}^{\log 10s} P^{(0)}_{k,u_0}(t)\, dt + \frac{4}{5\pi u_0}T^0 P^{(0)}_{k,u_0}(\log u_0) \\ & \qquad + \frac{1}{60\pi^2}T^0 P_{2,k,u_0}(\log 10s)\Bigg) \\ & + \frac{1}{s^2}\left(0.00355 T^0 P_{2,k,u_0}(\log 10s) + \frac{9}{25\pi}P^{(2)}_{k,u_0}(\log 10s)\right). \end{aligned} \tag{A.20}$$

Consider the first two lines of (A.20). Using $P_{j,k,u_0}(\log x) = C_{j,k,\eta}\cdot(\log x)^k$ and $x \geq 10^{18}$, we find that the they contribute at most (for $k = 1, 2, 3$):

$$\leq 0.002\, C_{0,k,\eta} + 0.00003\, C_{1,k,\eta} + 10^{-6}\, C_{2,k,\eta}. \tag{A.21}$$

Next, we look at the coefficient of $1/s$ in (A.20). Using (H4), (2.5) and the value of T^l from Lemma A.8 ($T^l x^k \leq \rho_{l,k} x^k$), we find that this is at most

$$\begin{aligned} & \left(\frac{4\sqrt{C_{0,k,\eta}\, C_{2,k,\eta}}}{\pi} + \frac{C_{2,k,\eta}\cdot \rho_{0,k}}{60\pi^2}\right)(\log 10s)^k + \frac{\sqrt{C_{0,k,\eta}\, C_{2,k,\eta}}}{10(k+1)\pi}(\log 10s)^{k+1} \\ & + \frac{4\rho_{0,k}\sqrt{C_{0,k,\eta}\, C_{2,k,\eta}}}{5\pi}\cdot\frac{(\log u_0)^k}{u_0} \\ & \leq B'_k(\log 10s)^{k+1}, \end{aligned} \tag{A.22}$$

where

$$B'_k = \sqrt{C_{0,k,\eta}\,C_{2,k,\eta}}\left(\frac{4}{14\pi} + \frac{1}{10(k+1)\pi} + \frac{4}{5\pi}\frac{\rho_{0,k}}{14}\left(\frac{k}{14e}\right)^k\right) + \frac{\rho_{0,k}}{14}\frac{C_{2,k,\eta}}{60\pi^2} \le 0.108\sqrt{C_{0,k,\eta}\,C_{2,k,\eta}} + 0.00193\,C_{2,k,\eta} \tag{A.23}$$

Here we use $\log 10s > 14$ which implies $\frac{\rho_{0,k}}{(\log 10s)^{k+1}}\frac{(\log u_0)^k}{u_0} \le \frac{\rho_{0,k}}{14^{k+1}}\left(\frac{k}{e}\right)^k$ (since the maximum of $(\log t)^k/t$ is $(k/e)^k$) and then use the values of $\rho_{0,k}$ from Table A.1.

Next, the coefficient of $1/s^2$ in (A.20) is at most

$$\le \left(0.0568\,C_{2,k,\eta} + 0.1146\,C_{1,k,\eta}\sqrt{\frac{C_{2,k,\eta}}{C_{0,k,\eta}}}\right)(\log 10s)^k. \tag{A.24}$$

Therefore, from (A.21), (A.22), (A.23) and (A.24) and using $s \ge 1.5 \cdot 10^5$, we find

$$L_{k,u_0}(s,x) \le A_k + \frac{B_k(\log 10s)^{k+1}}{s},$$

where A_k and B_k are as in (A.19). This completes the proof. □

Remark A.10. When $\eta = \eta_0$, we have $A_k = 0.0029\ldots$ and $B_k = 4.3379\ldots$.

Next, we bound $R_{\eta,k,q}(s,y,f)$ in the case $\eta = \eta_0$, $f \in \{\mu, f_0\}$ and $k \in \{1,2,3\}$.

Proposition A.11. *Suppose that $\eta = \eta_0$ and $s \ge 1.5 \cdot 10^5$. Let $B \le \sqrt{x}$ and $f_0 = \mu * \Lambda_{(A,B)}$ be as in (A.3). Then, the following are admissible choices:*

$$R_{\eta_0,1,q}(s,x,\mu) = 0.00020506x, \tag{A.25}$$

$$R_{\eta_0,2,q}(s,x,\mu) = \left(0.001824 + \frac{2F_0(s)\log x}{s}\right)x, \tag{A.26}$$

$$R_{\eta_0,2,q}(s,x,f_0) = \frac{2x\log B}{s}\left(\log^2 10s + F_0(s)\log x\right), \tag{A.27}$$

$$R_{\eta_0,3,q}(s,x,\mu) = \left(0.03494 + \frac{4F_0(s)\log^2 x}{s}\right)x, \tag{A.28}$$

Proof. We show that the $R_{\eta_0,k,q}$ given above satisfy the following inequality:

$$R_{\eta_0,k,q}(s,x,f) \geq \frac{x}{2s} \sum_{i=0}^{k} \sum_{j=0}^{i} \binom{k}{i} \binom{i}{j} b_{k-i} (\log 10s)^{i-j} \left| m_{2q,j} \left(\frac{x}{10qs}, f \right) \right|,$$

where we denote $b_{k-i} = b_{\eta_0,k-i}$. For bounds related to $|m_{2q,k}|$, we will use of Proposition A.4. We also bound $q/\varphi(q)$ by $F_0(q) \leq F_0(s)$ and use the fact that $F_0(s)/s$ is decreasing from Lemma A.6. We also note that $(\log 10s)^k/s$ is decreasing as soon as $10s > e^k$, which will clearly hold for $k = 1, 2, 3$ since $s \geq 1.5 \cdot 10^5$. For the values of the constants b_l, we refer to Table A.2, from which allows us to take

$$b_0 = 2, \quad b_1 = 3.42, \quad b_2 = 7.11 \quad \text{and} \quad b_3 = 17.34 \tag{A.29}$$

For (A.25), we have $k = 1$, $f = \mu$ and the expression simplifies to

$$\begin{aligned}
&\frac{x}{2s} \left((b_0 + b_1 \log 10s) \left| m_{2q,0} \left(\frac{x}{10qs}, \mu \right) \right| + b_0 \left| m_{2q,1} \left(\frac{x}{10qs}, \mu \right) \right| \right) \\
&\leq \frac{x}{2s} \left(b_0 + b_1 \log 10s + 1.00303 b_0 F_0(s) \right) \\
&\leq 0.00020506x,
\end{aligned}$$

for $s \geq 1.5 \cdot 10^5$ (as the above is decreasing in s in that range).

For (A.26), the given expression is

$$\begin{aligned}
&\frac{x}{2s} \Bigg((b_2 + 2b_1 \log 10s + b_0 \log^2 10s) \left| m_{2q,0} \left(\frac{x}{10qs}, \mu \right) \right| \\
&\qquad + (2b_1 + 2b_0 \log 10s) \left| m_{2q,1} \left(\frac{x}{10qs}, \mu \right) \right| + b_0 \left| m_{2q,2} \left(\frac{x}{10qs}, \mu \right) \right| \Bigg) \\
&\leq \frac{x}{2s} \Big(b_2 + 2b_1 \log 10s + b_0 \log^2 10s + (2.00606 b_1 + 0.00606 b_0 \log 10s) F_0(s) \\
&\qquad + 2b_0 F_0(s) \log x \Big) \\
&\leq 0.001824x + \frac{2x F_0(s) \log x}{s}.
\end{aligned}$$

For (A.27), we have $k = 2$ and $f = f_0$, and the given expression is

$$\begin{aligned}
&\frac{x}{2s}\Bigg((b_2 + 2b_1 \log 10s + b_0 \log^2 10s)\left|m_{2q,0}\left(\frac{x}{10qs}, f_0\right)\right| \\
&\qquad + (2b_1 + 2b_0 \log 10s)\left|m_{2q,1}\left(\frac{x}{10qs}, f_0\right)\right| + b_0\left|m_{2q,2}\left(\frac{x}{10qs}, f_0\right)\right|\Bigg) \\
&\le \frac{x \log B}{2s}\Bigg(b_2 + 2.00606 b_1 F_0(s) + (2b_1 + 0.00606 b_0 F_0(s)) \log 10s \\
&\qquad + b_0 \log^2 10s + 2b_0 F_0(s) \log x\Bigg) \\
&\le \frac{x \log B}{2s}\left(4 \log^2 10s + 4F_0(s) \log x\right),
\end{aligned}$$

since for $s \ge 1.5 \cdot 10^5$, we have

$$b_2 + 2.00606 b_1 F_0(s) + (2b_1 + 0.00606 b_0 F_0(s)) \log 10s < b_0 \log^2 10s.$$

For (A.28), we have $k = 3$ and $f = \mu$ and the given sum is

$$\begin{aligned}
&\frac{x}{2s}\Bigg(b_3 + 3b_2 \log 10s + 3b_1 \log^2 10s + b_0 \log^3 10s + 6F_0(s) \log \frac{x}{10s}(b_1 + b_0 \log 10s) \\
&\qquad + 1.00303 F_0(s)(3b_2 + 6b_1 \log 10s + 3b_0 \log^2 10s) + 4b_0 F_0(s) \log^2 \frac{x}{10s}\Bigg) \\
&\le \frac{x}{2s}\Bigg(b_3 + 3.00909 b_2 F_0(s) + (3b_2 + 6(0.00303) b_1 F_0(s)) \log 10s \\
&\qquad + (3b_1 + 1.00909 b_0 F_0(s)) \log^2 10s + b_0 \log^3 10s \\
&\qquad + (6b_1 - 2b_0 \log 10s) F_0(s) \log x + 4b_0 F_0(s) \log^2 x\Bigg) \\
&\le 0.03494x + \frac{4x F_0(s) \log^2 x}{s}.
\end{aligned}$$

This completes the proof. □

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