Phenomenological and foundational aspects of non-Markovianity

By

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List of Publications arising from the thesis

Journal

1. "Generalized formalism for information backflow in assessing Markovianity and its equivalence to divisibility", Sagnik Chakraborty, *Phys. Rev. A* **97** 032130 (2018).

2. "Non-Markovianity of qubit evolution under the action of spin environment", Sagnik Chakraborty, Arindam Mallick, Dipanjan Mandal, Sandeep K. Goyal, Sibasish Ghosh, *Scientific Reports* **9** (1) 2987 (2019)

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1. "Degree of non-Markovianity and equivalence between information backflow and divisibility for non-invertible dynamical maps", Sagnik Chakraborty and Dariusz Chruscinski.

Further Publications of candidate not substantially used in the thesis

Journal

1. "Quantum ratchet in disordered quantum walk", Sagnik Chakraborty, Arpan Das, Arindam Mallick, CM Chandrashekar, *Annalen der Physik* **529** (8), 1600346 (2017).

2. "Universal detection of entanglement in two-qubit states using only two copies", Suchetana Goswami, Sagnik Chakraborty, Sibasish Ghosh, and A. S. Majumdar, *Phys. Rev. A*, **99**, 012327 (2019).

3. "Information flow versus divisibility for qubit evolution", Sagnik Chakraborty, Dariusz Chruscinski, *Phys. Rev. A* **99** 042105 (2019).

Pre-print

1. "On thermalization of two-level quantum systems", Prathik Cherian J, Sagnik Chakraborty, Sibasish Ghosh, *arXiv:1604.04998v2*.
2. "Quantum precision thermometry with weak measurement", Arun K Pati, Chiranjib Mukhopadhyay, Sagnik Chakraborty, Sibasish Ghosh, *arXiv:1901.07415*.

Presented Talk/Posters in School/ Conferences:

1. Talk on “On non-Markovianity of qubit evolution under action of spin environment”, at Advanced School and Workshop on Quantum Science and Quantum Technologies, 4 - 15 September 2017, ICTP, Trieste, Italy.

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3. Poster on “Generalized Formalism for Information Backflow in assessing Markovianity and its equivalence to Divisibility”, at International Symposium on New Frontiers in Quantum Correlations (ISNFQC18), 29th January - 2nd February 2018, S. N. Bose National Centre for Basic Sciences, Kolkata, India.

4. Poster on “Generalized Formalism for Information Backflow in assessing Markovianity and its equivalence to Divisibility”, at Meeting on Quantum Information Processing and Applications (QIPA-2018), 2nd - 8th December 2018, Harish-Chandra Research Institute, Allahabad, India.

Dedicated to my family, friends and well wishers

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Synopsis

Introduction and Motivation

With the advent of new experimental techniques, Quantum Information is no longer a subject of solely theoretical interest. Numerous quantum protocols, principles and results discovered theoretically, are now being tested, verified and reinforced upon, by experiments. Along with these new developments, comes a serious challenge of controlling and understanding real life quantum systems, which are inherently open to the environment. This has been the motivation of my PhD work, which is mainly concentrated on phenomenological and foundational understanding of the open quantum systems.

An open quantum system is essentially a system kept in contact with an environment, where both the system and the environment are evolving jointly through a global unitary operation. The resulting evolution on the system side, which is generally expressed as a solution of a *master equation*, is the focus of our study. During this evolution, the system interacts with the environment and gives rise to complex patterns of information flow between them. Often this information flow gives rise to scenarios where the system evolution retains memory of earlier times. This distinct property of remembrance of the earlier dynamics is used to classify open quantum dynamics into two broad categories: *Markovian* or memoryless and *non-Markovian*. Although the classical analog of this classification is well defined [1], the definition of Markovianity in quantum regime is debated. There are numerous prescriptions which capture different aspects of this complex behavior, but a single unified description is yet to be found. Our work is, firstly to study Markovianity from an operational point of view, investigating challenges faced during analyzing real life open quantum systems, and secondly to study the fundamental aspects of it, progressing towards a unified definition of Markovianity in quantum regime.

In the first part of our work we considered a phenomenological model, where a system qubit is interacting with an environment consisting of a finite number of qubits, and by tuning the coupling between the system and individual environment qubits, we have analyzed the Markovianity, non-Markovianity and transition between the two, of the dynamics. In the second part of our work, we attempted to tackle of problem of having a number of non-equivalent definitions of Markovianity in quantum regime. Notably all the different definitions of quantum Markovianity can be classified into two broad classes, namely, Information Backflow (IB) and completely positive divisibility. We have put forward a general framework for the IB approach of Markovianity that not only includes a large number, if not all, of IB prescriptions proposed so far, but also is equivalent to completely positive divisibility for a large class of quantum evolutions including all invertible ones. Following the common approach of IB, where monotonic decay of some physical property or some information quantifier is seen as the definition of Markovianity, we proposed in our framework a general description of what should be called a proper "physicality quantifier" to define Markovianity. We elucidated different properties of our framework and used them to propose an idea of degree of non-Markovianity, which would capture varied strengths of non-Markovianity present in the dynamics.

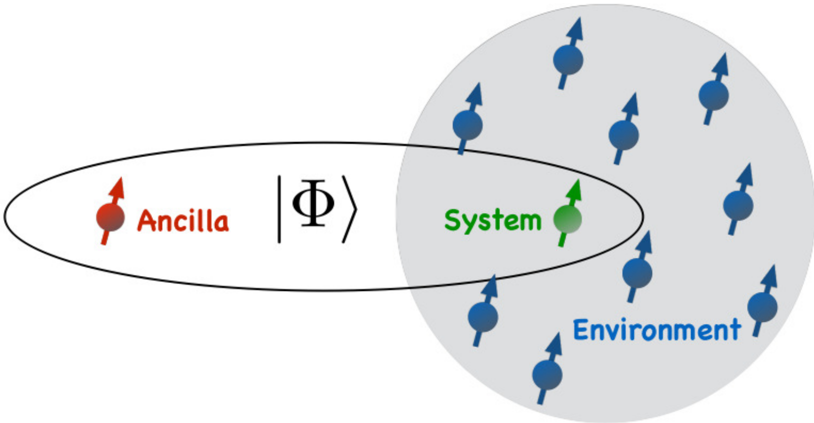

Figure 1: The model.

Non-Markovianity of a spin-environment

In our paper [2], we considered a system qubit interacting with an environment consisting of N non-interacting qubits through an interaction Hamiltonian

$$\begin{aligned} H_{se}(t) &= \hbar\alpha \sum_{n=1}^{N} \Big\{ g_n^*(t)\sigma_+^{(s)} \otimes [\, |0\rangle\langle 0| \otimes .. \otimes \sigma_-^{(n)} \otimes .. \otimes |0\rangle\langle 0|\,]_e \\ &+ g_n(t)\sigma_-^{(s)} \otimes [\, |0\rangle\langle 0| \otimes .. \otimes \sigma_+^{(n)} \otimes .. \otimes |0\rangle\langle 0|\,]_e \Big\} \end{aligned} \tag{1}$$

where $\sigma_+^{(n)} = |0\rangle\langle 1|, \sigma_-^{(n)} = |1\rangle\langle 0|$ and $g_n(t)$'s are the coupling factors between the system and the n^{th} environment qubit. Note here that we have worked in the interaction picture, and as a result, we have not considered the self-Hamiltonians of the system and the environment. When we compare Eq. (1) with the usual Hamiltonian of a spin bath model [3,4], given by

$$H_{spin-bath} = \hbar\alpha \sum_{n=1}^{N} (\sigma_x^{(s)}\sigma_x^{(n)} + \sigma_y^{(s)}\sigma_y^{(n)}), \tag{2}$$

in the interaction picture, we find that the only difference comes from the $|0\rangle\langle 0|$ factors arising in Eq. (1), which are replaced by $\mathbb{1}$ for the spin bath Hamiltonian. As a result of this difference, the dynamics of the spin bath model is not entirely the same as our model.

In the former, an exchange of one quanta of energy takes place between the system and a particular environment qubit, when the rest of the environment qubits are allowed to be in any state, whereas in the latter, the exchange will only take place when the rest of the environment qubits are in their ground state. This difference, although significant in general, will not play a major role when the state of the environment is close to the ground state; or in other words, temperature of the environment is low. Thus we see for low temperatures our model serves as a close approximation to the spin bath model. Moreover the Hamiltonian of our model can be analytically solved for a large number of cases for arbitrary number of environment qubits.

In our paper [2], we considered different functional forms and strengths of system-environment coupling i e. $g_n(t)$ that gives rise to non-Markovianity in the system dynamics of the above model. We chose the following forms of coupling: (i) time-independent and homogeneous over environment qubits, (ii) time-independent but inhomogeneous over environment qubits, (iii) homogeneous over the environment but is time-dependent, and (iv) both time-dependent and inhomogeneous. Note here, in order to ascertain non-Markovianity, we used the Rivas-Huelga-Plenio (RHP) measure of non-Markovianity, which requires attaching an ancilla to the system and setting the initial system-ancilla state to the maximally entangled state, as can be seen in Fig. 1. The RHP measure then detects non-Markovianity whenever the entanglement between the system and ancilla shows a non-monotonic decrease over time.

In cases (i) and (ii), where $g_n(t) = g$ and $g_n(t) = g_n$, we found the dynamics is always non-Markovian. We also found the Kraus operators of the system dynamics for these cases. In case (iii) we have $g_n(t) = g(t)$ and we found the dynamics is always non-Markovian if the the coupling is a polynomial function of time i e, $g(t) = \sum_{k=0}^{n} c_k t^k$. We also found that if coupling is exponetial i e. $g(t) = exp(-\gamma t)$, then the dynamics is non-Markovian if the real part γ_r of γ fails to be positive or violates the inequality $\alpha\sqrt{N} \geq \gamma_r \pi$. Also for this case we found the Kraus operators of the system dynamics.

In case (iv), we studied the most general form of coupling in our model, which is both time-dependent and site-dependent. We employed numerical simulations to study non-Markovianity in this case and we also presented a step by step algorithm describing the numerical technique. We considered the form of coupling $g_n(t) = e^{-\gamma_1 nt}$ and found that, for this case, there is a crossover from non-Markovianity to Markovianity as the coupling strengths is decreased i e. γ_1 is increased. We also found the extreme values of γ_1 which shows this transition. These values act as critical values for the transition from non-Markovianity to Markovianity, and on plotting them as a function of N (no. of environment qubits) they appeared to saturate to some fixed values. Moreover, we analyzed another form of coupling $g_n(t) = t^{-n\gamma}$ and showed that it is non-Markovian for the parameter value $\gamma = 0.3$.

Generalized formalism for Information backflow

In this part, we considered the problem of having a number of non-equivalent definitions of Markovianity in quantum regime. All prescriptions suggested so far can be broadly classified into two main categories: completely positive divisibility (CPD) and information backflow (IB). The CPD approach comes from a mathematical point of view, where a dynamical process is called Markovian if evolution up to a particular time t, can be broken down into two valid quantum evolutions: one up to an intermediate time s (for any $s < t$), followed by another from s to t. The IB approach, on the other hand, describes a dynamical process to be Markovian if some quantifier, i.e., some physical property or some quantifier of information, decreases in a monotonic fashion under the action of the process. Examples of quantifiers include distinguishability, generalized trace-distance, quantum mutual information, interferometric power, and local quantum uncertainty. The IB approach can be further subdivided into two classes based on the type of quantifier used: one which uses quantifiers based on the system only, and the other that uses an ancilla to define the quantifier. With so many notions of Markovianity present, even within the

IB category, one is compelled to look for inter-relations, hierarchies, or equivalence that might be present within them.

In our paper [5], we attempted to tackle this problem by constructing a formalism that is independent of any particular form of quantifier. We proposed a generalized form of quantifier, called the physicality quantifier (PQ), for the whole of IB category. We set a minimal requirement criteria for any quantity to qualify as a PQ, i.e., it should be nonincreasing under any physical process. In doing so, we found that a large class, if not all, of quantifiers considered so far in the literature come as special cases of our generalized form. We also showed that our generalized formalism for IB is also equivalent to CPD for invertible dynamical maps. Note here, by invertible dynamical map Λ_t, we mean Λ_t is invertible for all t.

Preliminaries. If $\mathcal{H}$ is a Hilbert space, let $\mathcal{L}(\mathcal{H})$ be the space of all linear operators and $\mathcal{P}_+(\mathcal{H})$ the set of all density matrices on $\mathcal{H}$. Let $\mathcal{T}(\mathcal{H}, \mathcal{H})$ denote the space of all linear maps from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})$. Now, consider a $d-$dimensional system and a $d-$dimensional ancilla with Hilbert spaces $\mathcal{H}_\mathcal{S}$ and $\mathcal{H}_\mathcal{A}$, respectively. A dynamical map $\Lambda_t \in \mathcal{T}(\mathcal{H}_\mathcal{S}, \mathcal{H}_\mathcal{S})$ is a completely positive (CP) trace preserving (TP) map describing evolution up to a time t. The full dynamics is described by a family of time-parametrized CPTP maps $\Lambda := \{\Lambda_t\}_t$.

Definition 1. A dynamical map Λ_t is said to be divisible if it can be expressed as,

$$\Lambda_t = V_{t,s}\Lambda_s, \tag{3}$$

for any $t > s$, where $V_{t,s} \in \mathcal{T}(\mathcal{H}_\mathcal{S}, \mathcal{H}_\mathcal{S})$. If $V_{t,s}$ is (completely) positive and trace preserving for any $t > s$, the dynamics is called (completely) positive divisible, and abbreviated as (C)PD.

An ensemble of states on the system $\mathcal{E}_S := \{p_i; \rho_i\}_{i=1}^n$ is defined as a finite collection of states $\rho_i \in \mathcal{P}_+(\mathcal{H}_\mathcal{S})$ with *a priori* probabilities p_i. Similarly, we define $\mathcal{E}_{SA} := \{p_i; \xi_i\}_{i=1}^n$ on system-ancilla with $\xi_i \in \mathcal{P}_+(\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{A})$. Let $\mathcal{F}_S^n := \{\mathcal{E}_S \mid \mathcal{E}_S = \{p_i; \rho_i\}_{i=1}^n\}$ and

$\mathcal{F}^n_{SA} := \{\mathcal{E}_{SA} \mid \mathcal{E}_{SA} = \{p_i;\xi_i\}^n_{i=1}\}$ be the collections of all ensembles with n elements. We define the sets of all possible ensembles of any size by,

$$\mathcal{F}_S := \bigcup_{n=1}^{\infty} \mathcal{F}^n_S \quad ; \quad \mathcal{F}_{SA} := \bigcup_{n=1}^{\infty} \mathcal{F}^n_{SA}. \tag{4}$$

We proposed to define two types of PQ, $\mathcal{I}_S : \mathcal{F}_S \mapsto \mathbb{R}$ and $\mathcal{I}_{SA} : \mathcal{F}_{SA} \mapsto \mathbb{R}$, as real bounded functions on ensembles of quantum states, which follow the following condition. For convenience, functions $f_S(\mathcal{E}_S)$ and $f_{SA}(\mathcal{E}_{SA})$, are also represented by symbols $f_S\{p_i;\rho_i\}$ and $f_{SA}\{p_i;\xi_i\}$, respectively.

Condition. Let $T \in \mathcal{T}(\mathcal{H}_\mathcal{S}, \mathcal{H}_\mathcal{S})$ be any CPTP map, acting on the system. For a given form of $\mathcal{I}_S$ or $\mathcal{I}_{SA}$, the following is true:

$$\begin{aligned} \mathcal{I}_S\{p_i;T[\rho_i]\} &\leq \mathcal{I}_S\{p_i;\rho_i\}, \\ \mathcal{I}_{SA}\{p_i;(T\otimes I)[\xi_i]\} &\leq \mathcal{I}_{SA}\{p_i;\xi_i\} \end{aligned}$$

where I denotes the identity map in $\mathcal{T}(\mathcal{H}_\mathcal{A}, \mathcal{H}_\mathcal{A})$.

For a dynamical map Λ_t we defined *dynamic physicality quantifiers* $\Phi^{\mathcal{I}_S}_t$ and $\Phi^{\mathcal{I}_{SA}}_t$ based on $\mathcal{I}_S$ and $\mathcal{I}_{SA}$, in the following way,

$$\Phi^{\mathcal{I}_S}_t\{p_i;\rho_i\} := \mathcal{I}_S\{p_i;\Lambda_t[\rho_i]\}, \tag{5}$$

$$\Phi^{\mathcal{I}_{SA}}_t\{p_i;\xi_i\} := \mathcal{I}_{SA}\{p_i;(\Lambda_t\otimes I)[\xi_i]\}. \tag{6}$$

We first defined Markovianity in terms of each valid form of PQ, and then we generalized the notion to include subsets of PQ having fixed size or type (system or system-ancilla). Note here that special PQ's focused on only n element ensembles are denoted as $\mathcal{I}^n_S$ or $\mathcal{I}^n_{SA}$, and defined as $\mathcal{I}^n_S(\mathcal{E}_S) = 0$ for $\mathcal{E}_S \notin \mathcal{F}^n_S$, and $\mathcal{I}^n_{SA}(\mathcal{E}_{SA}) = 0$ for $\mathcal{E}_{SA} \notin \mathcal{F}^n_{SA}$.

Definition 2. A dynamical map Λ_t, is called $\mathcal{I}_S$*-Markovian* ($\mathcal{I}_{SA}$*-Markovian*) for some form of $\mathcal{I}_S$ ($\mathcal{I}_{SA}$), if $\Phi^{\mathcal{I}_S}_t(\mathcal{E}_S)$ ($\Phi^{\mathcal{I}_{SA}}_t(\mathcal{E}_{SA})$) decreases monotonically with time t, for any

$\mathcal{E}_S \in \mathcal{F}_S$ ($\mathcal{E}_{SA} \in \mathcal{F}_{SA}$).

Definition 3. A dynamical map Λ_t is called *n-S-Markovian* (*n-SA-Markovian*) if $\Phi_t^{\mathcal{I}^n_S}(\mathcal{E}_S)$ ($\Phi_t^{\mathcal{I}^n_{SA}}(\mathcal{E}_{SA})$) decreases in a monotonic fashion with time t, for any form of $\mathcal{I}^n_S$ ($\mathcal{I}^n_{SA}$) and any choice of ensemble $\mathcal{E}_S \in \mathcal{F}^n_S$ ($\mathcal{E}_{SA} \in \mathcal{F}^n_{SA}$).

Definition 4. A dynamical map Λ_t is called *S-Markovian* (*SA-Markovian*) if it is n-S-Markovian (n-SA-Markovian) for any value of n.

Finally, we gave our generalized definition of Markovianity for backflow of information: any dynamics which is both S-Markovian and SA-Markovian, is called *IB-Markovian*.

General properties of the formalism. We first note a hierarchy within our Markovianity classes, which is apparent from their definition: any n-S-Markovian (n-SA-Markovian) class is a subset of $(n+1)$-S-Markovian ($(n+1)$-SA-Markovian) class. We then presented a result, that makes SA-Markovianity an equivalent criteria to IB-Markovianity.

Theorem 1. If any dynamical maps Λ_t is n-SA-Markovian, then it is n-S-Markovian.

We then showed how IB-Markovianity is related to CPD and in what way generalized trace-distance (GTD) on extended space plays a vital role in relating these two quantities. GTD is a quantity that gives the best possible distinguishing probability of a pair of quantum states occurring with different probabilities p_1 and p_2. The corresponding PQ $\mathcal{I}_S^{GTD}$ is given by,

$$\mathcal{I}_S^{GTD}\{p_1, p_2, \rho_1, \rho_2\} := ||p_1\rho_1 - p_2\rho_2||_1, \tag{7}$$

where $||A||_1 = Tr\,\sqrt{A^\dagger A}$. The definition of GTD can also be easily extended to system-ancilla space. We call it *generalized trace-distance measure extended* (GTDE) and define it in the following way,

$$\mathcal{I}_{SA}^{GTDE}\{p_1, p_2, \xi_1, \xi_2\} := ||p_1\xi_1 - p_2\xi_2||_1, \tag{8}$$

where $\xi_i \in \mathcal{P}_+(\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{A})$.

Definition 5. A dynamical map Λ_t is called *GTD-Markovian* or *GTDE-Markovian* in the sense of definition 2, if the respective PQ are $\mathcal{I}_S^{GTD}$ and $\mathcal{I}_{SA}^{GTDE}$.

Next we present the two main theorems of our work in this context.

Theorem 2. For an invertible dynamical map Λ_t, the following are equivalent: (i) Λ_t is GTDE-Markovian, (ii) Λ_t is CPD, (iii) Λ_t is SA-Markovian or IB-Markovian, (iv) Λ_t is 2-SA-Markovian.

Theorem 3. A qubit dynamical map Λ_t is GTD-Markovian if and only if it is 2-S-Markovian.

Extending the generalized formalism

In our ongoing paper [6] we extended our formalism by introducing a new system based quantifier

$$\mathcal{I}_n^{\mathcal{A}}(\mathcal{E}_S) = \begin{cases} \left\| \sum_{i=0}^{n-1} A_i \otimes \rho_i \right\|_1 & \mathcal{E}_S \in \mathcal{F}_S^n \\ 0 & \mathcal{E}_S \notin \mathcal{F}_S^n \end{cases} \tag{9}$$

where $\mathcal{A} = \{A_i\}_{i=0}^{n-1}$ for $A_i \in \mathcal{L}(\mathcal{H}_S)$ and $\rho_i \in \mathcal{P}_+(\mathcal{H}_S)$. Using this quantifier we showed the following results.

Theorem 4. S-Markovianity, SA-Markovianity and CPD are equivalent for *image-nonincreasing* dynamical maps.

Here image-nonincreasing dynamical maps is a large class of dynamics called defined by the following condition

$$\mathrm{Im}(\Lambda_t) \subset \mathrm{Im}(\Lambda_s) \tag{10}$$

for any $s < t$. Note that all invertible dynamical maps fall in this class. Moreover for qubit dynamical maps we showed that

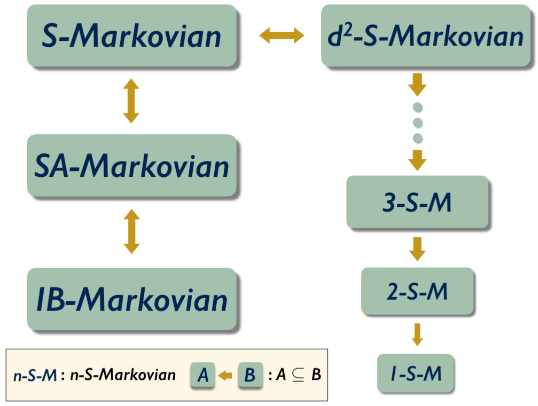

Figure 2: Hierarchy of Markovianity classes for image-nonincreasing or qubit dynamics.

Theorem 5. For any qubit dynamical map S-Markovianity, SA-Markovianity and CPD are equivalent.

As a result of this theorem, we found a simplified hierarchical structure of Markovianity classes as given in Fig 2. This description also provided a useful insight, that our formalism can be used to define a *degree* of non-Markovianity, which would capture varied intensities of memory effects present in image-nonincreasing or qubit dynamics. The higher the least value of n, for which a dynamics fails to be n-S-Markovian, the weaker is the effect of memory in the dynamics. We then tested our new quantifier on the following dynamics known as *eternal non-Markovian* dynamics, given by

$$\Lambda_t[\rho] = \sum_{i=0}^{3} p_i(t)\sigma_i\rho\sigma_i, \tag{11}$$

where $p_0(t) = \frac{1+e^{-\gamma t}}{2}$, $p_1(t) = p_2(t) = \frac{1-e^{-\gamma t}}{4}$, $p_3(t) = 0$, and σ_i's are Pauli matrices with $\sigma_0 = \mathbb{1}$. We found a PQ of the form $\Phi_t^{\mathcal{I}_4^{\mathcal{A}}}$ is non-monotonic over time under the action of the dynamics. Hence, we conclude the above dynamics is non-Markovian, at least with respect to 4−S Markovianity class.

Next we considered the qubit random unitary dynamics again. In Eq. (11), we chose

$$p_i(t) = \alpha_i[1 - p_0(t)] \quad ; \quad i = 1, 2, 3, \tag{12}$$

where $\alpha_i \geq 1$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Also note we must have $p_0(0) = 1$. We therefore considered two choices of dynamics with the following forms: (a) $p_0(t) = 1/(1 + t)$, (b) $p_0(t) = (1 + \cos t)/2$. Using the PQ in Eq. (9), in both cases we found the dynamics is non-Markovian, at least with respect to 2−S Markovianity class.

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Chapter 1

Introduction

The discovery of quantum theory, in the early twentieth century, fundamentally changed the way Physics was perceived at that time. It brought about a change in the core principles that formed the basis of contemporary Physics. Since then there has been numerous occasions when quantum theory have jolted our "common sense" and startled our understanding about the surrounding world. Perhaps, its most recent contribution is to the area of Information Theory. *Quantum Information Theory* is an area, which brings in elements of Information Theory and Quantum Theory together to provide a machinery, that has been able to produce promising results like quantum teleportation [1], superdense coding [2], and Shor's algorithm [3, 4]. These developments and many more such results have led to coinage of the term, which is also the the newest promise of quantum theory: *Quantum Technology*.

1.1 The onset of Quantum Technology

In the last decade, quantum technology has emerged to be a field with not only theoretical predictions but also real life implementations. In particular, the quantum random number generator which derives its working principle from the Born rule of quantum mechanics [5], is already a developed and commercially available technology. Technologies are now being developed which actually deal with quantum states and employ quantum properties like superposition and entanglement [6]. It is being expected that in the decade to come quantum technology will make significant contribution in providing security to communication systems [7]. There are also other areas like quantum simulation, quantum computation and quantum metrology which are expected to develop significantly in the coming years. These developments are being made possible due to the significant achievements made in experimental techniques, recently.

It should also be stressed that although quantum technology is very promising, it still faces major impediments on its way to becoming a viable technological alternative. One of the main problems in this direction is due to dissipation of useful quantum properties like coherence and entanglement, on exposure to the environment. As a result, in recent years the study of quantum systems interacting with environment has drawn a lot of interest.

1.2 Motivation of the thesis

Real life quantum systems are inherently interacting with the environment. This type of systems are called open quantum systems, where, although the system and the environment evolves jointly through a unitary process, the evolution on the system side is not necessarily unitary. To understand this type of evolutions, more general rules for quantum evolution beyond the standard unitary evolution via Schrödinger dynamics have been considered.

Interaction of the system with its environment gives rise to complex patterns of information flow between them and often this results in scenarios where the system evolution retains memory of earlier times. This distinct property has been used to classify open quantum evolution into two broad categories: Markovian or memoryless and non-Markovian. Although the classical analog of this classification is well defined [8], the definition of

Markovianity in a quantum regime is debated. There are numerous prescriptions which capture different aspects of this complex behavior, but a single unified description is yet to be found. Therefore, a deeper understanding of Markovianity is required to develop a consistent theory in the quantum regime. The motivation of the thesis is to understand quantum Markovianity from a phenomenological as well as a foundational perspective.

On the phenomenological side, we choose a model mimicking a real physical scenario and study aspects of Markovianity on it, by using the understanding about Markovianity that is currently prevalent. On the foundational side, we look for more general definitions of Markovianity, which will help proceed towards a consistent and complete theory of Markovianity in the quantum regime.

1.3 Outline of the thesis

In chapter 2, we discuss the basics of open quantum system and some standard techniques used in this area of research.

In chapter 3, we introduce the concept of Markovianity in quantum regime and discuss different approaches, proposed so far, to describe quantum Markovianity.

In chapter 4, we discuss the contents of the first paper[1], on which the thesis is based. We deal with a phenomenological open quantum system, where a qubit system is interacting with an environment consisting of finite number of qubits. The interaction is chosen in such a way that for low temperatures, it is expected to produce similar results as that of the spin bath model. The spin bath model is a commonly used model in several physical scenarios like quantum theory of magnetism [9], quantum spin glasses [10] and theory of conductors and superconductors [11]. In our model, we choose different forms of coupling between the system and individual environment qubits and study aspects of Markovianity

[1]"Non-Markovianity of qubit evolution under the action of spin environment", Sagnik Chakraborty, Arindam Mallick, Dipanjan Mandal, Sandeep K. Goyal, Sibasish Ghosh, Scientific Reports 9 (1), 2987 (2019).

of it. We show that, for a number of forms of coupling, the model could be analytically solved for arbitrary number of environment qubits. Moreover, a transition from non-Markovian to Markovian regime can be witnessed for certain forms of coupling. In those cases, we find the critical values of parameters which show the transition and study their dependence on number of environment qubits and initial environment temperature.

In chapter 5, we discuss the contents of the second paper[2], on which the thesis is based. We propose a general framework for quantum Markovianity which takes into account a large number, if not all, of the prescriptions for Markovianity proposed earlier in literature. Following a common approach of Markovianity, where monotonic decay of some physical property or some information quantifier is seen as the definition of Markovianity, we propose in our framework a general description of what should be called a proper "physicality quantifier" to define Markovianity. We show for invertible evolutions our prescription provides a unified understanding of Markovianity. We also show that our framework allows for a hierarchy of Markovianity classes arranged in order of strength of Markovianity. In particular, for qubit evolutions we find the necessary and sufficient condition for any dynamical map to belong to a particular class.

In chapter 6, we discuss the contents of the third paper[3], on which the thesis is based. We introduce a new quantifier for defining Markovianity and extend our formalism to non-invertible evolutions. We show that monotonicity over time of only system based quantifiers are sufficient to establish Markovianity for qubit dynamical maps and so called image non-increasing dynamical maps of higher dimension. Moreover, we demonstrate that, there is a simple hierarchical structure of Markovianity classes for any qubit or image non-increasing dynamical map of dimension d, which provides a degree of non-Markovianity to the dynamics with $d^2 - 1$ ($d = 2$ for qubits) being the most non-Markovian and zero being Markovian. We also used our results to estimate the degree of non-

[2]"Generalized formalism for information backflow in assessing Markovianity and its equivalence to divisibility", Sagnik Chakraborty, Phys. Rev. A 97 032130 (2018).

[3]"Degree of non-Markovianity and equivalence between Information Backflow and Divisibility for Non-invertible Dynamical maps", Sagnik Chakraborty and Dariusz Chruscinski, (manuscript in preparation).

Markovianity of a number of dynamical maps.

Finally, in chapter 7, we summarize the results obtained in the thesis and discuss plausible future directions to the works discussed in the thesis.

Chapter 2

Open Quantum Evolutions

In this chapter, we present the basic understanding of open quantum systems, that will be used in the thesis. Moreover, we present short discussions on some standard techniques used in this field.

2.1 Preliminaries

If $\mathcal{H}$ is a Hilbert space, let $\mathcal{L}(\mathcal{H})$ be the space of all linear operators and $\mathcal{P}_+(\mathcal{H})$ the set of all density matrices on $\mathcal{H}$. Let $\mathcal{T}(\mathcal{H},\mathcal{H})$ denote the space of all linear maps from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})$. Note that, elements of $\mathcal{T}(\mathcal{H},\mathcal{H})$ are also called *super-operators* or *maps*. We now introduce the *operator-vector correspondence*, which is a one-to-one correspondence between the operator space $\mathcal{L}(\mathcal{H})$ and the state space $\mathcal{H}\otimes\mathcal{H}$.

The *vector correspondence* of an operator $A \in \mathcal{L}(\mathcal{H})$, given by $A = \sum_{i,j} a_{ij} |i\rangle\langle j|$, is defined as $vec(A) = \sum_{i,j} a_{ij} |i\rangle |j\rangle$, where $\{ |i\rangle \}$ forms an orthonormal basis in $\mathcal{H}$. Note that $vec(A) \in \mathcal{H}\otimes\mathcal{H}$.

2.2 The most general quantum evolution

Although, there is a lot of debate [12] about what should be the most general description of quantum evolution, we present here our perception on the matter, and the definition we use in the thesis.

Let us first discuss *trace preserving* (TP) and *completely positive* (CP) maps. Consider a system S having a $d-$dimensional Hilbert space $\mathcal{H}_d = \mathbb{C}^d$. A linear map $\Lambda \in \mathcal{T}(\mathcal{H}_d, \mathcal{H}_d)$ is called TP if

$$Tr(\Lambda[\rho]) = Tr(\rho), \tag{2.1}$$

for any $\rho \in \mathcal{P}_+(\mathcal{H}_d)$. This condition is equivalent to preserving normalization condition of probability distributions before and after the action of the map. The TP property is recognized as a fundamental rule that the most general quantum evolution must obey.

Next we present another major criteria, which is often regarded as another basic requirement for the most general quantum evolution. The map Λ is called CP if

$$(\Lambda \otimes \mathbb{1}_n)[\xi] \geq 0 \;\; ; \quad n \in \{1, 2, \ldots\}, \tag{2.2}$$

where $\xi \in \mathcal{P}_+(\mathcal{H}_d \otimes \mathcal{H}_n)$ is any joint state of the system S and an $n-$dimensional ancilla with Hilbert space $\mathcal{H}_n$. Eq. (2.2) reflects that if S is actually a sub-system of larger composite consisting of S and the ancilla, the action of Λ on S does not map a physical state (density matrix) of the system-ancilla to an unphysical state (non-positive operator). The CP and the TP conditions are often recognized as the defining conditions for the most general quantum evolution.

There is also an alternate view that the most general quantum evolution should include scenarios where the input system S is not able to share arbitrary entanglement with an arbitrary ancilla (which is the basic presumption behind the CP condition). This type of scenarios occur in practice. For example, if the initial joint state of S and the environment

E is chosen to be a maximally entangled state, there can be no entanglement between S and any ancilla system. This fact is guaranteed by the monogamy of entanglement [13]. In this scenario, if the composite $S + E$ is evolves through a joint unitary evolution, the evolution on the system side will not necessarily be CP. In order to deal this type of scenarios, there are prescriptions which further generalize the notion of most general quantum evolution [12, 14, 15].

Although, this view is of significant importance, for the purpose of the thesis we will stick to situations where the input system is able to share arbitrary entanglement with arbitrary ancillae. As a result the system evolution must be a CPTP map.

2.3 Choi-Jamiolkowski representation and the Kraus representation

The *Choi-Jamiolkowski* representation or the *CJ matrix* $J(\Lambda)$ is a representation of a map $\Lambda \in \mathcal{T}(\mathcal{H}_d, \mathcal{H}_d)$ in form of a $d^2 \times d^2$ matrix, as given below

$$J(\Lambda) = \sum_{i,j=1}^{d} \Lambda[\,|i\rangle\langle j|\,] \otimes |i\rangle\langle j|, \tag{2.3}$$

where $\{\,|i\rangle\,\}_{i=1}^{d}$ forms an orthonormal basis in $\mathcal{H}_d$.

The *Kraus* representation of Λ is given by

$$\Lambda[\rho] = \sum_{i=1}^{n} A_i \rho B_i^{\dagger}, \tag{2.4}$$

where $A_i, B_i \in \mathcal{L}(\mathcal{H}_d)$ and n is some positive integer [16]. As Λ is linear, we can always find a set $\{A_i, B_i\}$ which obey Eq. (2.4). It can be also be seen that the CJ matrix and the

Kraus representation are related through a simple identity [16] :

$$J(\Lambda) = \sum_{i=1}^{n} vec(A_i)\, vec(B_i)^\dagger \tag{2.5}$$

This identity leads to the remarkable result that it is sufficient to consider only $n = d$ in Eq. (2.2) for checking the CP condition. We do not present the complete proof here, as it can be found in [16]. Nevertheless, note that Λ is CP if and only if $J(\Lambda)$ is a positive operator i e. $J(\Lambda) \geq 0$ or Λ has a Kraus representation of the following form

$$\Lambda[\rho] = \sum_{i=1}^{n} A_i \rho A_i^\dagger. \tag{2.6}$$

Moreover, if Λ is TP the following condition should hold:

$$\sum_{i=1}^{n} A_i^\dagger A_i = \mathbb{1}_d \tag{2.7}$$

2.4 Description of open quantum systems

Consider a system S with a finite dimensional Hilbert space $\mathcal{H}_S$, interacting with an environment E with a finite dimensional Hilbert space $\mathcal{H}_E$. Let S and E evolve jointly through a global unitary operator $U_{SE}(t, 0)$, where t represents time. We consider the initial state of the system-environment $\rho_{SE} \in \mathcal{P}_+(\mathcal{H}_S \otimes \mathcal{H}_E)$ to be

$$\rho_{SE} = \rho \otimes \rho_E, \tag{2.8}$$

where $\rho \in \mathcal{P}_+(\mathcal{H}_S)$ and $\rho_E \in \mathcal{P}_+(\mathcal{H}_E)$. The time evolved state of the system is then given by

$$\rho(t) = Tr_E\Big(U_{SE}(t, 0)\, \rho_{SE}\, U_{SE}(t, 0)^\dagger\Big). \tag{2.9}$$

The time evolution map $\Lambda_t \in \mathcal{T}(\mathcal{H}_S, \mathcal{H}_S)$, defined as $\Lambda_t[\rho] = \rho(t)$, is called the *dynamical* *map* representing the evolution of the open quantum system S. Note that, our consideration of choosing ρ_{SE} to be in a product form in Eq. (2.8) ensures the CP property of the dynamical map [8]. This is the basic model of open quantum system that we will be using in the thesis.

Although this description of dynamical maps is easy to understand, but often it becomes very difficult to know the full evolution operator U_{SE} or the initial environment state ρ_E. This is due to the fact that the environment is not always in control of the experimenter. In those situations it becomes more convenient to have a definition of dynamical maps which is solely defined on the system Hilbert space. As a result, we often define a dynamical map to be just a time-parametrized family of CPTP map $\{\Lambda_t\}_t$, where each Λ_t represents evolution up to time t.

2.5 Master Equations

In this section, we present a description of the open quantum evolution in terms of a differential equation. As the *von-Neumann equation* governs the evolution of isolated systems, for open quantum systems there is *master equation*. Master equations often provide more physical insight about the dynamics than the dynamical map. As for example, the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) form of master equations, which we will discuss in the next chapter, not only describes dynamical maps obeying the *semigroup* property but also provides insight about the self-Hamiltonian of the system and the factors governing the dissipation in the dynamics.

A typical time local master equation has the following form

$$\frac{d\rho}{dt} = \mathcal{L}_t[\rho] \tag{2.10}$$

where $\mathcal{L}_t \in \mathcal{T}(\mathcal{H}_S, \mathcal{H}_S)$ is called the time local generator of the dynamical map Λ_t. The time local generator is related to the dynamical map in the following way [17],

$$\Lambda_t = \mathcal{T}exp\left(\int_0^t \mathcal{L}_u du\right), \tag{2.11}$$

where $\mathcal{T}$ denotes time-ordering [5]. There is also another equivalent approach to deal with dynamical maps called the Nakajima–Zwanzig projection operator technique [18,19], which shows under sufficiently general conditions, the master equation of an open quantum system has the following form

$$\frac{d\rho}{dt} = \int_0^t \mathcal{K}_{t-\tau}\rho_\tau d\tau, \tag{2.12}$$

where the function $\mathcal{K}_t$ is called the memory kernel.

2.6 Chapter summary

In this chapter, we have presented a short discussion on the most general form of quantum evolution. We explored the two main schools of thought about the basic requirements of such evolutions. We also discussed completely positive trace preserving maps and their representation as Choi-Jamiolkowski matrix and Kraus representation. We then provided a description of open quantum system which we will be using in the thesis, and finally we introduced master equations. The idea of master equations will be used in the next chapter to discuss ideas of Markovianity.

Chapter 3

Markovianity in open quantum systems

In this chapter we introduce the idea of *Markovian* or memoryless dynamics in open quantum systems. Markovianity is already a well defined concept in classical probability theory [8, 20, 21]. If $\mathcal{X}$ denotes a discrete set of events and the probability of occurrence of $x_n \in \mathcal{X}$ at time t_n is given by $P\{x_n; t_n\}$, the stochastic (probabilistic) process is called Markovian if

$$P(x_3; t_3 | x_2, t_2; x_2, t_1) = P(x_3; t_3 | x_2, t_2) \tag{3.1}$$

for any $t_3 > t_2 > t_1$. The idea behind Markovian processes is that it does not retain memory of earlier times.

In open quantum systems, information exchange between system and environment is an essential feature. Information that has been previously transferred to the environment may come back and affect the system, and this may appear as a memory-effect on the system. When this information backflow from the environment is negligible we have a situation analogous to the classical Markovian process and the system dynamics is called memory-less or Markovian. On the other hand, when this information backflow affects the system significantly i.e. when some long past history of the system influences its present state, the system dynamics becomes retentive, and is called non-Markovian.

3.1 Dynamical semigroup

The *semigroup* property of dynamical maps was initially recognized as the defining property for Markovianity in quantum regime. A dynamical map Λ_t is called a semigroup if

$$\Lambda_{t+s} = \Lambda_t \Lambda_s, \tag{3.2}$$

for all $t, s \geq 0$ [8]. In the pioneering works by Gorini-Kossakowski-Sudarshan (GKS) [22] and Lindblad [23], the master equation governing semi-group dynamical maps on a $d-$dimensional systems was found to be[1]

$$\frac{d\rho}{dt} = -i[H, \rho] + \frac{1}{2} \sum_{k,l=1}^{d^2-1} c_{kl} \Big([F_k, \rho F_l^\dagger] + [F_k \rho, F_l^\dagger] \Big), \tag{3.3}$$

where $H^\dagger = H$, $Tr(H) = 0, Tr(F_k) = 0$, $Tr[F_k^\dagger F_l] = 0$ for $k, l = 1, 2, \ldots, d^2 - 1$, and $[c_{kl}]$ is a positive matrix. This form was actually obtained by GKS [22, 24]. Around the same time Lindblad [23, 24] came up with an equivalent form valid for both finite and infinite dimensional systems:

$$\frac{d\rho}{dt} = -i[H, \rho] + \frac{1}{2} \sum_j \Big([V_j, \rho V_j^\dagger] + [V_j \rho, V_j^\dagger] \Big) \tag{3.4}$$

Eqs. (3.3) or (3.4) is often called the GKLS form of master equation.

The GKLS form can also be represented as

$$\frac{d\rho}{dt} = -i[H, \rho] + \Phi[\rho] - \frac{1}{2} \{\Phi^*[\mathbb{1}], \rho\}, \tag{3.5}$$

where Φ is a CP map given by $\Phi[\rho] = \sum_j V_j \rho V_j^\dagger$, while the conjugate map Φ^* is given by $\Phi^*[\rho] = \sum_j V_j^\dagger \rho V_j$. A brief history of the events that led to the GKLS form can found in a recent article [24].

[1] taking $\hbar = 1$

3.2 Completely positive divisibility

A generalization of the idea of semigroup to define Markovianity was proposed by Rivas Huelga and Plenio (RHP) in [25]. They proposed the idea of *completely positive divisibility* or CP-divisibility, which is a generalization of divisible processes in classical probability theory.

A dynamical map Λ_t is called divisible if it can be expressed as,

$$\Lambda_t = V_{t,s}\,\Lambda_s, \tag{3.6}$$

for any $t > s$, where $V_{t,s} \in \mathcal{T}(\mathcal{H}_S, \mathcal{H}_S)$. If $V_{t,s}$ is (completely) positive and trace preserving for any $t > s$, the dynamics is called (completely) positive divisible, and abbreviated as (C)PD. $V_{t,s}$ can be seen as the intermediate evolution from s to t, and it is uniquely defined only when Λ_t is invertible i e. $V_{t,s} = \Lambda_t \Lambda_s^{-1}$.

Recently, Chruscinski et. al. [26] showed that the necessary and sufficient condition for divisibility is

$$Ker(\Lambda_s) \subseteq Ker(\Lambda_t), \tag{3.7}$$

for any $t > s$, where $Ker(\Lambda)$ represents kernel or null space of Λ. It can also be shown that the master equation for a CPD dynamical map is of the GKLS form with time dependent coefficients i e. H and Φ in Eq. (3.5) are time dependent [17, 27]. For a detailed mathematical characterization of divisibility and (C)PD, refer to [26].

Consider the following dynamical map

$$\Lambda_t[\rho] = (1 - p(t))\rho + \frac{p(t)}{2}(\rho + \sigma_3 \rho \sigma_3) \tag{3.8}$$

where σ_3 is the Pauli z-matrix and $p(t)$ is a time-dependent probability with $p(0) = 0$. For retaining invertibility of Λ_t we choose $p(t) < 1$ for all finite t. Now from Eq. (2.10) we know $\mathcal{L}_t = \dot{\Lambda}_t \Lambda_t^{-1}$ where

$$\dot{\Lambda}_t[\rho] = -\frac{\dot{p}(t)}{2}(\rho - \sigma_3 \rho \sigma_3),$$

and

$$\Lambda_t^{-1}[\rho] = \frac{1}{2 - 2p(t)}\left[2\rho - p(t)(\rho + \sigma_3 \rho \sigma_3)\right]$$

Hence,

$$\mathcal{L}_t[\rho] = \frac{\dot{p}(t)}{2 - 2p(t)}(\sigma_3 \rho \sigma_3 - \rho) \tag{3.9}$$

Therefore, comparing with Eq. (3.5) we get $H = 0$ and $\Phi[\rho] = \frac{\dot{p}(t)}{2-2p(t)}\sigma_3 \rho \sigma_3$. Hence, Λ_t is CPD if and only if $\dot{p}(t) \geq 0$ for all t.

Interestingly, for a master equation of the form Eq. (2.10), a criteria for checking if it is in GKLS form, is given by [27]

(a) $\mathcal{L}[X^\dagger] = \mathcal{L}[X]$

(b) $Tr(\mathcal{L}[X]) = Tr(X)$

(c) $\omega_\perp J(\mathcal{L}) \omega_\perp \geq 0$, where $J(\mathcal{L})$ is the CJ matrix of $\mathcal{L}$, $\omega_\perp = \mathbb{1} - \omega$ and $\omega = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\, i\rangle$.

3.3 Information Backflow

The information backflow approach is inspired from the fact that a Markovian dynamics is characterized by unidirectional flow of information from the system to the environment. As for example, in the Lindblad master equation [8] the non-negativity of the entropy production rate signifies unidirectional information flow from the system to the environment, and thereby is a signature of Markovianity. A dynamics is called Markovian from the information backflow approach, if some information quantifier decays over time in a monotonic way. Any departure from monotonicity of such quantifier is seen as a backflow of information from the environment back to the system. As for example, Breuer, Laine and Piilo (BLP) [28] used distinguishability of (any) two time evolved states [28],

$$\mathcal{I}(\Lambda_t, \rho_1, \rho_2) := ||\Lambda_t[\rho_1] - \Lambda_t[\rho_2]||_1, \tag{3.10}$$

where $||A||_1 = Tr\sqrt{A^\dagger A}$, as the quantifier of information, and therefore any departure from monotonicity of $\mathcal{I}(\Lambda_t, \rho_1, \rho_2)$ is a signature of non-Markovianity.

Consider the following example [29]

$$\frac{d\rho}{dt} = \sum_{i=1}^{3} \gamma_i(t)(\sigma_i\rho\sigma_i - \rho), \tag{3.11}$$

with $\gamma_i(t)$ are real functions of time, and $\{\sigma_i\}$ are the Pauli matrices. The corresponding dynamical map has the form

$$\Lambda_t[\rho] = \sum_{i=0}^{3} p_i(t)\sigma_i\rho\sigma_i, \tag{3.12}$$

where $\sigma_0 = \mathbb{1}$, and $p_i(t)$'s are probabilities with $p_0(0) = 1$.

We can find,

$$\begin{aligned}
p_0(t) &= \frac{1}{4}[1 + \lambda_1(t) + \lambda_2(t) + \lambda_3(t)] \\
p_1(t) &= \frac{1}{4}[1 + \lambda_1(t) - \lambda_2(t) - \lambda_3(t)] \\
p_2(t) &= \frac{1}{4}[1 - \lambda_1(t) + \lambda_2(t) - \lambda_3(t)] \\
p_3(t) &= \frac{1}{4}[1 - \lambda_1(t) - \lambda_2(t) + \lambda_3(t)]
\end{aligned}$$

where $\lambda_\alpha(t)$'s are the eigenvalues for eigenvector σ_α i e. $\Lambda_t[\sigma_\alpha] = \lambda_\alpha(t)\sigma_\alpha$ for $\alpha = 0, 1, 2, 3$ and

$$\lambda_i(t) = \exp(-2\Gamma_j(t) - 2\Gamma_k(t)),$$

where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$ and $\Gamma_k(t) = \int_0^t \gamma_k(\tau) d\tau$.

Any traceless hermitian operator $\Delta = \sum_{k=1}^{3} x_k \sigma_k$ can be written as $\Delta = \alpha(\rho_1 - \rho_2)$, where $\alpha > 0$. Hence $\Delta_t = \Lambda_t[\Delta] = \sum_{k=1}^{3} x_k \lambda_k(t) \sigma_k$ is also traceless and hermitian. Therefore,

$$||\Delta_t||_1 = Tr\sqrt{\Delta_t^2} = 2\xi(t),$$

where $\xi(t) = \sqrt{\sum_{k=1}^{3} x_k^2 \lambda_k(t)^2}$. As a result

$$\frac{d}{dt}||\Delta_t||_1 = \frac{1}{\xi(t)} \sum_k x_k^2 \frac{d}{dt} \lambda_k(t)^2$$

Hence $\frac{d}{dt}||\Delta_t||_1 \leq 0$ for all possible x_k only when $\frac{d}{dt}\lambda_k(t)^2 \leq 0$, which is equivalent to

$$\gamma_i(t) + \gamma_j(t) \geq 0 \ ; \ i, j = 1, 2, 3 \tag{3.13}$$

Hence the dynamical map Λ_t is BLP-Markovian if the above condition is satisfied.

Different other quantifiers for measuring non-Markovianity were also suggested, like measure of entanglement [25], quantum mutual information [30], etc. These quantifiers describe different non-equivalent aspects of Markovianity. Only recently, there has been attempts to unify all these different definitions [26, 31, 32] to provide a unified approach to information backflow.

3.4 Chapter summary

In this chapter, we gave a brief motivation to quantum Markovianity starting from classical Markovian processes to quantum semi-group. We then discussed two major prescriptions to quantum Markovianity: CP-divisibility and Information backflow. These prescriptions have been extensively used in recent years to describe Markovianity. We also presented examples to discusses these two concepts. In the next chapter, we will use these concepts to discuss Markovianity in a phenomenological model of spin systems.

Chapter 4

Non-Markovianity in a spin bath

In recent years, non-Markovianity has been used as a resource in a number of information theoretic protocols, namely, channel discrimination [33], preserving coherence and correlation [34–51] and retrieving quantum correlations in both quantum and classical environments [52–56]. Non-Markovian effects also play important roles in areas ranging from fundamental physics of strong fields [57, 58] to energy transfer process of photosynthetic complexes [59].

Owing to its diverse applications, various aspects of non-Markovianity are now being studied. Lately, researchers have been focusing on transition from non-Markovian to Markovian dynamics [60–65]. Some of them have dealt with bosonic baths of finite and infinite degrees of freedom, while some have considered a qudit system as the environment. But in all of these studies, system-environment interaction has been considered to be homogeneous in space, and the issue of non-Markovian to Markovian transition in terms of system-environment coupling strength has not been addressed. Note that, non-Markovian to Markovian transition is, in general, not a trivial issue, as in most cases finite dimensional environments give rise to non-Markovianity.

In our study, we attempt to analyze the problem of whether the transition can be engineered for the spin bath model [66]. We particularly choose the spin bath model since it has wide ranging applications in simulating real physical scenarios [66–68]. In our attempt, we face a serious difficulty in diagonalizing the spin bath Hamiltonian, either analytically or numerically, for larger number of spins in the environment. Although, analytic solutions do exist for constant coupling [69] and some special forms of time dependent coupling [70], general solution for arbitrary forms of system-environment coupling of the spin bath Hamiltonian are hard to find. We therefore, try to circumvent the problem by choosing a simple model, which we argue, is a close approximation to the spin bath model for low temperatures. We choose an exchange type of interaction between a system qubit and individual environment qubits, where for each environment qubit the coupling can be chosen to be of different time dependent forms. But unlike the spin bath case, in our model, when the exchange interaction takes place between the system and a particular environment qubit, the rest of the environment qubits remain in a ground state; which also closely resembles the state of environment for low temperatures. As we will see in the following, this approximation helps us to calculate and analyze non-Markovian to Markovian transition for different types of system-environment coupling.

We present four scenarios here, for different forms of system-environment coupling: (i) the coupling is time-independent and homogeneous over environment qubits, (ii) the coupling is time-independent but inhomogeneous over environment qubits, (iii) the coupling is homogeneous over the environment but is time-dependent, and (iv) the coupling is both time-dependent and inhomogeneous. We find that cases (i) and (ii) always give rise to non-Markovian system dynamics. For cases (iii) and (iv), we find that some functional forms of coupling for certain ranges of coupling strengths gives rise to non-Markovianity. For example in case (iii), polynomial forms of coupling always give rise to non-Markovian system dynamics, while exponential coupling give rise to non-Markovian system dynamics only for certain ranges of parameter values. In case (iv) we find that a cross-over from non-Markovianity to Markovianity can be achieved by varying the strength of coupling. We also calculate, the extremal values of coupling parameter beyond which non-Markovianity can no longer be detected. Thus we see, these extremal values act as critical values for transition from non-Markovian to Markovian regime. It is worth mentioning here that, for the purpose of detecting non-Markovianity we use Rivas-Huelga-Plenio (RHP) measure of non-Markovianity as proposed in [25]. Although there are different approaches of defining Markovianity and each approach represent different aspects of Markovianity, for the purpose of the present problem we choose, detection by the RHP measure as the definition of Markovianity.

Similar works on this line were done in [71–73]. But in the first approach [71], the system qubit directly interacts with a single environment qubit and the rest of the environment qubits, only have an indirect effect on the system via the environment qubit directly attached. Also, the coupling parameters involved do not have any time dependence. In the second approach [72], the transition from Markovianity to non-Markovianity was shown with a two tier environment; the first one being a multiple-spin system, while the second one was a bosonic bath. Also in [73], the coupling between the system and individual environment qubits were constant in space and time. We take into account all these factors and present a detailed study of a spin environment and cover *all* the relevant cases.

4.1 Detecting non-Markovianity through Entanglement

As mentioned in the previous chapter, there are a number of non-equivalent definitions of Markovianity, each representing different aspects of this property, for our purpose here we consider information backflow, in terms of measure of entanglement i e. the RHP measure as the description of Markovianity.

Let us first discuss entanglement measure of two-qubit states. The entanglement between two two-level systems (two qubits) can be characterized by the Peres-Horodecki criterion [74, 75] which states that a two-qubit state ρ_{as}, shared between a system qubit s and an ancilla qubit a, is entangled if and only if the partial transpose of this state, i.e. $(\rho_{\mathrm{as}})^{T_s}$, is not a positive-semidefinite operator i.e. $(\rho_{as})^{T_s} \ngeq 0$.

Notably, for a two-qubit entangled state, the operator $(\rho_{\text{as}})^{T_s}$ has exactly one negative eigenvalue λ [76, 77]. Thus $|\lambda|$ may be used as a measure of entanglement for the state ρ_{as}. Formally, the entanglement measure can be defined as follows

$$E(\rho_{as}) = |\lambda| = \frac{||(\rho_{as})^{T_s}||_1 - 1}{2} \tag{4.1}$$

where, $||A||_1 = Tr\,\sqrt{A^\dagger A}$ is the trace norm of a matrix A. Note that, $E(\rho_{as})$ is nothing but the negativity of the bipartite state ρ_{as} [78].

We will use this measure of entanglement as the quantifier for ascertaining Markovianity of the dynamics from the information backflow approach. Using entanglement to detect non-Markovianity was first done by Rivas, Huelga and Plenio in [25], and this measure has been so called the RHP measure of non-Markovianity. Following their technique we attach an ancilla to the system, on which a dynamical map Λ_t is acting. Following the information backflow approach, the dynamical map Λ_t is called Markovian if $E((\mathbb{1}\otimes\Lambda_t)[\,|\Phi^+\rangle\langle\Phi^+|\,])$ is a monotonically decreasing function of time t, where $|\Phi^+\rangle\langle\Phi^+|$ is the maximally entangled state, given by

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(\,|00\rangle + |11\rangle\,). \tag{4.2}$$

4.2 The model

In this section, we present our model and discuss the motivation behind choosing it. We also describe the technique in detail, by which non-Markovianity is detected in the system dynamics .

We consider two qubits, one of which is called the system (s) and the other, the ancilla (a). The system qubit is placed in an environment consisting of N non-interacting qubits (see Fig. 4.1). We take the interaction between the system qubit and the environment in

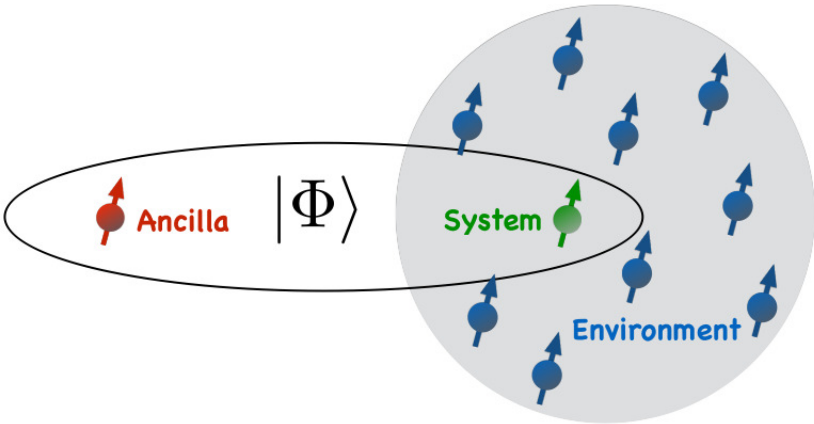

Figure 4.1: Schematic diagram of system qubit and ancilla qubit sharing a maximally entangled state $|\Phi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$. The system is interacting with an environment consisting of finite number of non-interacting qubits.

the following form,

$$
\begin{aligned}
\tilde{H}_{se}(t) &= \hbar\alpha\Big[\, |1\rangle_s \langle 0| \otimes \sum_{n=1}^{N} \tilde{g}_n^*(t)\, |0..0_n..0\rangle_e \langle 0..1_n..0| \\
&+ \ |0\rangle_s \langle 1| \otimes \sum_{n=1}^{N} \tilde{g}_n(t)\, |0..1_n..0\rangle_e \langle 0..0_n..0| \,\Big],
\end{aligned} \tag{4.3}
$$

where $|0\rangle$ and $|1\rangle$, respectively represent the ground and excited state of each qubit.

The coupling strength $\tilde{g}_n(t)$ is in general, a complex number and can also be time-dependent as well as site-dependent, and α is a real parameter with the dimension of frequency. The extra factor α is introduced to make the coupling strengths $\tilde{g}_n(t)$ dimensionless. For all practical purposes α can be assumed to be 1.

The free Hamiltonians of the system and the environment are respectively given by,

$$
H_s = \frac{\hbar\omega_s}{2}\sigma_z, \tag{4.4}
$$

$$
H_e = \sum_n \frac{\hbar\omega_n}{2}\sigma_z^{(n)}. \tag{4.5}
$$

It is convenient to work in the interaction picture where we replace the total Hamiltonian $H = H_s + H_e + \tilde{H}_{\text{se}} \equiv H_0 + \tilde{H}_{\text{se}}$ by the interaction picture Hamiltonian $H_{\text{se}}(t) =$

$\exp(iH_0t/\hbar)\tilde{H}_{\rm se}\exp(-iH_0t/\hbar)$ which reads,

$$\begin{aligned}
H_{\rm se}(t) &= \hbar\alpha\Big[\,|1\rangle_s\langle 0|\otimes\sum_{n=1}^{N}\tilde{g}_n^*(t)e^{i\delta\omega_n t}\,|0..0_n..0\rangle_e\langle 0..1_n..0| \\
&\quad+|0\rangle_s\langle 1|\otimes\sum_{n=1}^{N}\tilde{g}_n(t)e^{-i\delta\omega_n t}\,|0..1_n..0\rangle_e\langle 0..0_n..0|\,\Big] \\
&= \hbar\alpha\Big[\,|1\rangle_s\langle 0|\otimes\sum_{n=1}^{N}g_n^*(t)\,|0..0_n..0\rangle_e\langle 0..1_n..0| \\
&\quad+|0\rangle_s\langle 1|\otimes\sum_{n=1}^{N}g_n(t)\,|0..1_n..0\rangle_e\langle 0..0_n..0|\,\Big]
\end{aligned} \tag{4.6}$$

where $\delta\omega_n=\omega_s-\omega_n$ and $g_n(t)=\tilde{g}_n(t)e^{-i\delta\omega_n}$. Henceforth, our discussion will be based on the Hamiltonian $H_{se}(t)$.

We also consider the initial state of the environment to be in the thermal state,

$$\rho_e(0)=\Big[p\,|0\rangle\langle 0|+(1-p)\,|1\rangle\langle 1|\Big]^{\otimes N}, \tag{4.7}$$

where $p=(1+e^{-\beta})^{-1}$ and β is a positive real parameter which can be identified as the inverse of the temperature T of the environment.

4.2.1 Motivation behind the model

Here we argue that, our model is a close approximation to the 'spin bath' model [67, 68] for low temperatures. Note that the Hamiltonian in Eq. (4.6) can also be written as,

$$\begin{aligned}
H_{\rm se}(t) &= \hbar\alpha\sum_{n=1}^{N}\Big\{g_n^*(t)\sigma_+^{(s)}\otimes[\,|0\rangle\langle 0|\otimes..\sigma_-^{(n)}..\otimes|0\rangle\langle 0|\,]_e \\
&\quad+g_n(t)\sigma_-^{(s)}\otimes[\,|0\rangle\langle 0|\otimes..\sigma_+^{(n)}..\otimes|0\rangle\langle 0|\,]_e\Big\}
\end{aligned} \tag{4.8}$$

where $\sigma_+=|0\rangle\langle 1|$ and $\sigma_-=|1\rangle\langle 0|$.

When we compare Eq. (4.8) with the usual Hamiltonian of a spin bath model [67, 68] in the interaction picture :

$$H_{spin-bath} = \hbar\alpha \sum_{n=1}^{N} (\sigma_x^{(s)}\sigma_x^{(n)} + \sigma_y^{(s)}\sigma_y^{(n)}), \tag{4.9}$$

we find that the only difference comes from the $|0\rangle\langle 0|$ factors arising in Eq. (4.8), which are replaced by $\mathbb{1}$ for the spin bath Hamiltonian. As a result of this difference, the dynamics of the spin bath model is not entirely the same as our model. In the former, an exchange of one quanta of energy takes place between the system and individual environment qubit, when the rest of the environment qubits are allowed to be in any state, whereas in the latter, the exchange will only take place when the rest of the environment qubits are in their ground state. This difference, although significant in general, will not play a major role when the state of the environment is close to the ground states, or in other words, temperature of the environment is low. Note that low temperature of environment correspond to values of p in Eq. (4.7), which are very close to 1, and this also confirms the fact that for low temperatures ρ_e is close to the ground state.

Thus we see for low temperatures our model serves as a close approximation to the spin bath model. The main advantage of our model is the fact that our Hamiltonian is easily diagonalizable, and for certain types of couplings, as we discuss latter in detail, allows for exact determination of the system dynamics in terms of Kraus operators, for *any* number of environment qubits.

We also stress that, although our model shows similarity to the spin bath model for low temperatures, we find solutions and analyze the dynamics of our model for any temperature whatsoever. The reason behind this is that our model being analytically solvable for certain types of couplings, allows for an opportunity to exactly solve the dynamics for any number of environment qubits, which is not often the case for systems with large number of spins. Note that, even for the spin-bath Hamiltonian, it is not easy to find the exact solutions for non-zero temperature.

4.2.2 Diagonalizing the Hamiltonian of our model

There are only two non-zero eigenvalues of the Hamiltonian $H_{se}(t)$ and they are,

$$\mathcal{E}_{\pm}(t) = \pm\hbar\alpha\sqrt{\sum_{n=1}^{N}|g_n(t)|^2} = \pm\mathcal{E}(t), \tag{4.10}$$

corresponding to the eigenvectors,

$$|\chi_{\pm}(t)\rangle_{se} = \frac{1}{\sqrt{2}}[\,|1\rangle_s \otimes |0\rangle_e^{\otimes N} \pm |\xi(t)\rangle_{se}\,], \tag{4.11}$$

where $|\xi(t)\rangle_{se} = |0\rangle_s \otimes |\beta_0\rangle_e$ and,

$$|\beta_0\rangle_e = \frac{\hbar\alpha}{\mathcal{E}(t)}\sum_{n=1}^{N} g_n(t)\,|0..1_n..0\rangle_e\,. \tag{4.12}$$

Thus, the time evolution operator $U(t,0)$ corresponding to the Hamiltonian H_{se} is,

$$U(t,0) = \mathcal{T}\exp\left[-\frac{i}{\hbar}\int_0^t H_{se}(\tau')\,d\tau'\right], \tag{4.13}$$

where $\mathcal{T}$ represents time ordering.

We prepare the system and ancilla qubits in a maximally entangled state $|\Phi^+\rangle$, as given in Eq. (4.2). Due to the interaction of the system qubit with the environment, the entanglement between the system and the ancilla qubit will evolve with time. The deviation of this time evolution of the entanglement, from monotonic decay is used to establish the non-Markovian character of the dynamics. Note here, that this idea was used by Rivas et al [25] to devise a measure of non-Markovianity. In the present case, we follow this technique to consider the system dynamics to be non-Markovian whenever the entanglement between system and ancilla, as described above, shows non-monotonic behaviour, otherwise we consider the dynamics to be Markovian.

The joint initial state of the system, ancilla and the environment is of the form,

$$\rho_{\text{ase}}(0) = |\Phi^+\rangle\langle\Phi^+| \otimes \rho_e(0), \tag{4.14}$$

which evolves to,

$$\rho_{ase}(t) = [\mathbb{1}_a \otimes U(t,0)]\,\rho_{\text{ase}}(0)\,[\mathbb{1}_a \otimes U^\dagger(t,0)]. \tag{4.15}$$

Therefore, reduced time-evolved system-ancilla state can be calculated by tracing out the environment part,

$$\rho_{as}(t) = \text{Tr}_e\, \rho_{\text{ase}}(t). \tag{4.16}$$

4.3 System - Environment couplings

In this section, we introduce various classes of system-environment coupling, and in each case, we study their effect on the evolution of the system-ancilla joint state. We classify all the couplings into four major classes : (A) when the coupling parameter $g_n(t)$ is independent of the site index n (homogeneous) and time-independent; (B) when $g_n(t)$ is inhomogeneous but time-independent; (C) when $g_n(t)$ is homogeneous but time-dependent, and (D) when $g_n(t)$ is inhomogeneous and time-dependent. For each class, we calculate the entanglement of the time evolved state of system-ancilla, and thereby try to characterize the non-Markovian behaviour of the system dynamics. Henceforth, we assume α to be 1.

4.3.1 Case A : Homogeneous and time-independent coupling

We have here the simplest situation, where the coupling of the system with all the environment qubits are uniform and time-independent i.e. $g_n(t) = g$, a constant. As a result, the non-zero eigenvalues of the Hamiltonian, as given in Eq. (4.10), takes the form $\mathcal{E}_{\pm}(t) = \pm\mathcal{E} = \pm\hbar\sqrt{N}\,|g| \equiv \pm\hbar\omega_0$, where $\omega_0 = \sqrt{N}\,|g|$ is a constant with the dimension of frequency. The time-evolution operator $U(t,0)$ is of the form,

$$U(t,0) = \left(e^{-i\omega_0 t} - 1\right)|\chi_+\rangle\langle\chi_+| + \left(e^{i\omega_0 t} - 1\right)|\chi_-\rangle\langle\chi_-| + \mathbb{1}. \tag{4.17}$$

Using the above form and the form of ρ_e given in Eq. (4.7), we find the Kraus operators $K_{mn}(t)$ of system dynamics, which are defined in the following way,

$$\rho_s(0) \to \rho_s(t) = \sum_{m,n=1}^{N} K_{mn}(t)\,\rho_s(0)\,K^{\dagger}_{mn}(t), \tag{4.18}$$

where the N^2 Kraus operators are given by,

$$K_{mn}(t) = \sqrt{p^{N-s_n}(1-p)^{s_n}}
\begin{bmatrix} \frac{\cos\omega_0 t - 1}{\omega_0^2}\, g_{N-\log m}(t)\, g^{*}_{N-\log n}(t) + \delta_{mn} & -\frac{i}{\omega_0}\sin\omega_0 t\; g_{N-\log m}(t)\;\delta_{0n} \\ -\frac{i}{\omega_0}\sin\omega_0 t\; g^{*}_{N-\log n}(t)\;\delta_{0m} & (\cos\omega_0 t - 1)\;\delta_{0m}\delta_{0n} + \delta_{mn} \end{bmatrix} \tag{4.19}$$

where $m, n = 1, \ldots, N$, $\log x$ refers to $\log_2 x$, and s_n is the number of 1's in the binary equivalent of n. For example, if $n = 6$, then the binary equivalent of n is 110. Therefore $s_n = 2$. Moreover, $g_{N-\log n}(t)$ is to be interpreted as a function $g_x(t)$ defined in the following way :

$$g_x(t) = \begin{cases} g_x(t) & x \in \{0, .., N-1\} \\ 0 & x \notin \{0, .., N-1\} \end{cases} \tag{4.20}$$

A detailed calculation is given in Appendix A.

We then find time evolved state of the system-ancilla, using Eqs. (4.14), (4.15) and (4.16),

$$
\begin{aligned}
\rho_{\mathrm{as}}(t) = |\Phi^+\rangle\langle\Phi^+| - \frac{1}{2}\Big[& p\kappa_0(\,|11\rangle\langle 11| - |10\rangle\langle 10|\,) \\
&+ (1-p)\kappa_0(\,|00\rangle\langle 00| - |01\rangle\langle 01|\,) \\
&+ \delta_0(|00\rangle\langle 11| + |11\rangle\langle 00|)\Big],
\end{aligned} \tag{4.21}
$$

where $\kappa_0 = p^{N-1}\sin^2(\omega_0 t)$ and $\delta_0 = 2p^{N-1}\sin^2\left(\frac{\omega_0 t}{2}\right)$. The only possible negative eigenvalue of $[\rho_{\mathrm{as}}(t)]^{T_s}$, if any, is of the form,

$$
\lambda(t) = \frac{1}{4}\left[\kappa_0 - \sqrt{\left(1-2p\right)^2\kappa_0^2 + 4(1-\delta_0)^2}\right]. \tag{4.22}
$$

We present the plot of $E(\rho_{as}(t)) = \lambda(t)$ versus time, latter in the Result section.

4.3.2 Case B : Inhomogeneous and time-independent coupling

Consider a system, where a single two-level system (perhaps an ion as an impurity) is placed in a spin lattice. The lattice sites, closest to the impurity interacts very strongly with the system, while, as we go away from the impurity site, the strength of interaction becomes weaker and weaker. In such cases the interaction parameter $g_n(t)$ is in general inhomogeneous, but there is no explicit time dependence. To mimic this situation let us consider $g_n(t) = g_n$ in our model. Hence, $\mathcal{E}_\pm(t) = \pm\mathcal{E} = \hbar\omega$ in Eq. (4.10) are also time-independent. Note, in this case also $\omega = \sqrt{\sum_{n=1}^{N} |g_n|^2}$ is a constant with dimensions of frequency. Following this, the analysis is same as in the last subsection. As a result, the evolution operator $U(t,0)$, the Kraus operators $K_{mn}(t)$, the time evolved state $\rho_{\mathrm{as}}(t)$ and the only possible negative eigenvalue, if any, of $[\rho_{\mathrm{as}}(t)]^{T_s}$ for this case are of the same forms as in Eqs (4.17), (4.19), (4.21) and (4.22) respectively, except for ω_0, in appropriate places, replaced by ω.

4.3.3 Case C : Homogeneous and time-dependent coupling

So far we have considered only couplings which are independent of time. In this section, we consider time-dependent and homogeneous couplings. We take an *arbitrary* real function of time, which is independent of site index n i.e. $g_n(t) = g(t)$. Note that our coupling operator between system and individual environment qubit, as given in Eq. (4.6), is of the form $\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+$ which can also be expressed as $\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y$. Thus, our system-environment coupling is a special case of the XY coupled Hamiltonian. Such coupling with time-dependent coefficients have been used to show non-trivial entanglement dynamics [79, 80].

Fortunately, the Hamiltonians $H_{se}(t)$ in this case commutes at different times, which makes the analysis similar to the one in section 4.3.2. The only difference being the non-zero eigenvalues in Eq. (4.10) to be of the form $\mathcal{E}_\pm(t) = \pm\hbar\sqrt{N}|g(t)| = \pm\mathcal{E}(t)$, which is no longer constant in time. The whole treatment of the dynamics of the system and the ancilla remains the same if we replace ω_0 and $\omega_0 t$, in Eqs. (4.17), (4.19), (4.21) and (4.22), by $\sqrt{N}\,|g(t)|$ and $\Omega(t)$, respectively, where

$$\begin{aligned}\Omega(t) &= \frac{1}{\hbar}\int_0^t \mathcal{E}(\tau)d\tau = \int_0^t \left(\sum_{n=1}^{N} |g_n(\tau)|^2\right)^{\frac{1}{2}} d\tau \\ &= \sqrt{N}\int_0^t |g(\tau)|d\tau. \end{aligned} \tag{4.23}$$

4.3.4 Case D : Inhomogeneous and time-dependent coupling

The most general class of coupling $g_n(t)$ is when it depends both on the site n and time t. The interaction Hamiltonian in such a situation does not commute at different times and this makes the calculation for solving the dynamics difficult. However, we can use numerical methods to simulate the time-evolution and get the solution for $\rho_{\rm as}(t)$. One can obtain the following results analytically before starting the simulation part.

Analytical Part: Only two of the eigenvalues of the Hamiltonian in Eq. (4.6) are non-zero, as given in Eq. (4.10). The remaining $(2^{N+1} - 2)$ of the eigenvalues are zero. A possible choice for these null space eigenvectors are found in the following way:

Step I: We first feed the eigenvectors (corresponding to non-zero eigenvalues) given in Eq. (4.11) as rows of a matrix A. Note, A is a $2 \times 2^{(N+1)}$ matrix.

Step II: By row reduction method [81] we find out a basis $\mathcal{B}$ for the Null space of A. Note that $\mathcal{B}$ is not necessarily ortho-normal.

Simulation Part: Obtaining an orthonormal basis $\mathcal{B}'$ from $\mathcal{B}$ analytically, is a challenging job. We therefore resort to numerical techniques for this case.

Step I: From $\mathcal{B}$, using Gram-Schmidt Orthonormalization procedure [81], we find an orthonormal basis $\mathcal{B}'$. Note, $\mathcal{B}'$ forms the set of eigenvectors of the Hamiltonian corresponding to zero eigenvalues.

Step II: As, the eigenvectors are time-dependent, the Hamiltonian is not different-time commuting. Hence, the evolution operator may be found numerically from the following expression,

$$\begin{aligned} U_{se}(t,0) &= \mathcal{T}exp\Big[-\frac{i}{\hbar}\int_0^t H_{se}(\tau')\ d\tau'\Big] \\ &= \lim_{m\to\infty}\Bigg[exp\Big[-\frac{i}{\hbar}\int_{(m-1)\tau}^{m\tau} H_{se}(\tau')\ d\tau'\Big] \\ &\times\ exp\Big[-\frac{i}{\hbar}\int_{(m-2)\tau}^{(m-1)\tau} H_{se}(\tau')\ d\tau'\Big]\times\ldots \\ &\times\ exp\Big[-\frac{i}{\hbar}\int_0^{\tau} H_{se}(\tau')\ d\tau'\Big]\Bigg], \end{aligned} \tag{4.24}$$

where $\mathcal{T}$ represents time ordering.

Step III: We evolve the initial ancilla -system-environment state $\rho_{ase}(0)$ by the unitary operator $U_{ase}(t,0) = \mathbb{1}_a \otimes U_{se}(t,0)$ and get the time evolved state $\rho_{ase}(t)$.

Step IV: We trace out the environment from $\rho_{ase}(t)$ and get $\rho_{as}(t) = Tr_e[\rho_{ase}(t)]$. We then evaluate our entanglement measure $E(t)$ given in Eq. (4.1), on $\rho_{as}(t)$ and plot it as a function of time.

Table 4.1: Nature of dynamics for different forms of coupling

Coupling	**Value of p**	**Value of N**	**Parameter values**	**Nature**
Homogeneous and time independent	any value	any value	–	non-Markovian
Inhomogeneous and time independent	any value	any value	–	non-Markovian
$g(t) = \sum_m c_m t^m$	any value	any value	–	non-Markovian
$g(t) = e^{-\gamma t}$	any value	any value	$Re(\gamma) \leq 0$, $Re(\gamma) > \alpha\sqrt{N}/\pi$	non-Markovian
$g(t) = \frac{1}{1+\gamma t}$	0.6	4	$\gamma = 0.4, 0.8$	non-Markovian
	0.5	4	$\gamma = 0.4, 0.8$	
	0.6	8	$\gamma = 0.4, 0.8$	
	0.5	8	$\gamma = 0.4, 0.8$	
$g_n(t) = e^{-\gamma_1 n t}$	1.0	3	$\gamma_1 = 0.45, .., 0.95$	transition from non-Markovianity to Markovianity
	0.6	3	$\gamma_1 = 0.15, .., 0.65$	
	0.6	6	$\gamma_1 = 0.15, .., 0.35$	
	1.0	8	$\gamma_1 = 0.45, .., 0.95$	
$g_n(t) = \frac{1}{1+t^{n\gamma}}$	1.0	3	$\gamma = 0.35, 0.55, 0.75$	non-Markovian
	0.6	3	$\gamma = 0.35, 0.55, 0.75$	
	0.6	6	$\gamma = 0.35, 0.55, 0.75$	
	1.0	6	$\gamma = 0.35, 0.55, 0.75$	

4.4 Results

In this section, we show that some of the classes of the couplings that we have considered in the previous section always results in non-Markovian dynamics. However, there are also some classes for which we can tune the parameters to find a transition from non-Markovian

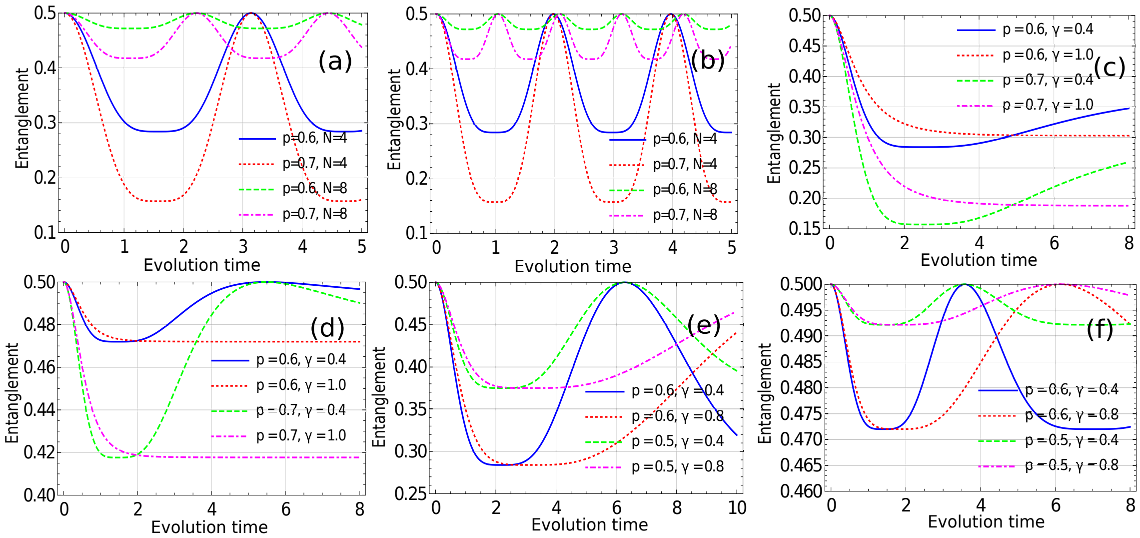

Figure 4.2: Plots showing the system-ancilla entanglement dynamics in different scenarios. For simplicity we have considered $\alpha = 1$. (a) When the coupling is homogeneous and time-independent i.e. $g = 1$. (b) When the coupling is inhomogeneous and time-independent i.e. $g_n = \sqrt{n}$. (c) When the coupling is homogeneous and time-dependent, i.e. $g_n(t) = g(t) = \exp(-\gamma t)$ and $N = 4$. (d) Coupling is $g(t) = \exp(-\gamma t)$ and $N = 8$. (e) Coupling is of the form, $g(t) = \frac{1}{1+\gamma t}$ and $N = 4$. (f) Coupling is $g(t) = \frac{1}{1+\gamma t}$ and $N = 8$.

to Markovian dynamics. In order to do so, we plot the entanglement dynamics between the system and ancilla for each class as a function of time, and observe if there is any departure from monotonicity in the plot. As mentioned in Sec 4.2, this technique helps in characterizing non-Markovianity present in the system dynamics. In Figs. 4.2 and 4.3, we present entanglement dynamics for different classes of system-environment coupling considered in the previous section. Also in Table 5.1, we provide a concise summary of all the resuts obtained in this section. We now present our findings for each class of system-environment coupling.

4.4.1 For couplings in Cases A and B

In Figs. 4.2(a) and 4.2(b), we plot the entanglement as a function of time for the homogeneous time-independent, and the inhomogeneous time-independent couplings, respectively, i.e, $g_n(t) = g$ and $g_n(t) = g_n$, respectively. Note here, that g and g_n, for all values

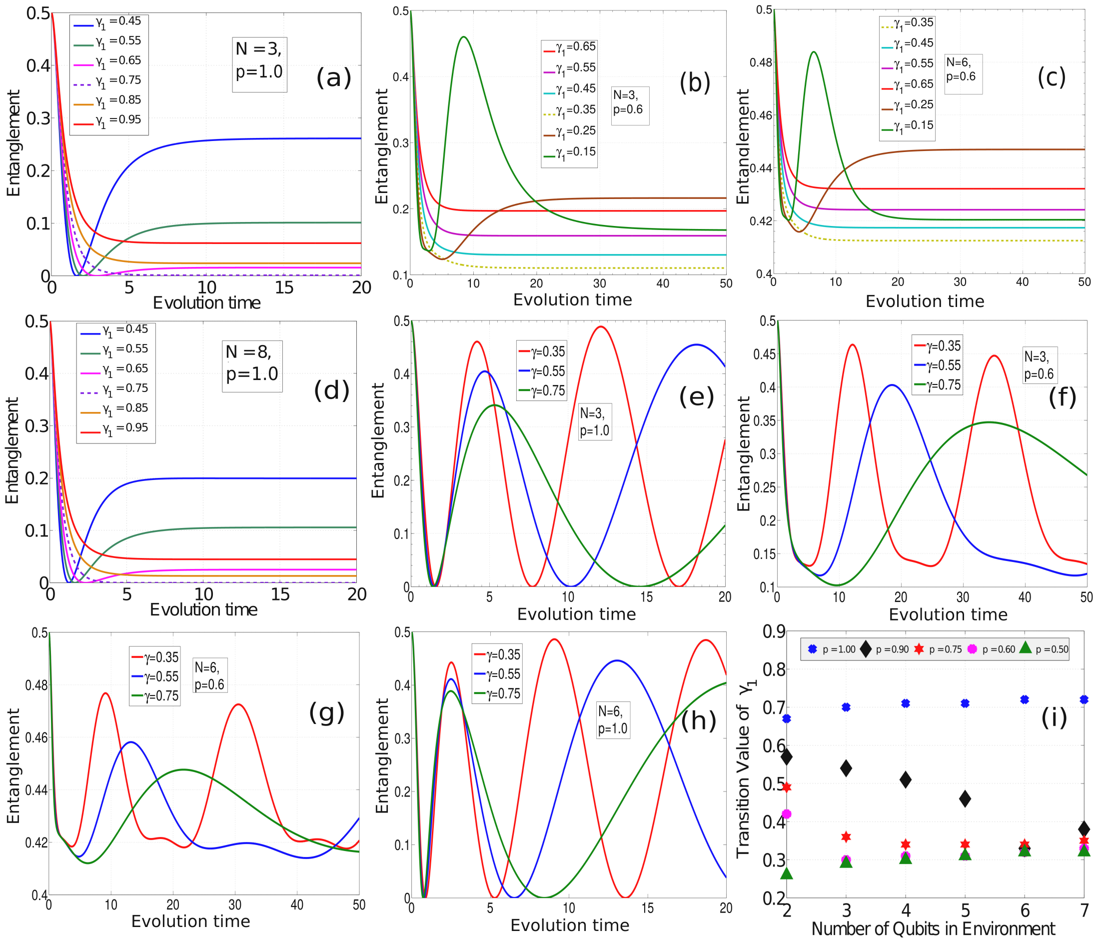

Figure 4.3: Plots showing the system-ancilla entanglement dynamics in different scenarios. For simplicity we have considered $\alpha = 1$. (a),(b),(c),(d) When the coupling is inhomogeneous and time-dependent i.e. $g_n(t) = e^{-\gamma_1 n t}$. (e),(f),(g),(h) Coupling is $g_n(t) = \frac{1}{1+t^{n\gamma}}$. (i) Transition values of γ_1 for coupling $g_n(t) = e^{-\gamma_1 n t}$ as a function of N for different values of p.

of n, are arbitrary complex functions. The forms of the entanglement measure, as given in Eq. (4.22) suggests a periodic behaviour for both the classes, which can also be seen in Figs. 4.2(a) and 4.2(b). As a result, we conclude for both of these classes of couplings, the dynamics is always non-Markovian.

4.4.2 For couplings in Case C

We consider homogeneous and time-dependent couplings and find that if $g(t)$ is some polynomial function of t, we will get $\Omega(t)$ as a polynomial function of t. This gives rise to a periodic function $\lambda(t)$. As a result, the dynamics is non-Markovian in general.

If $g(t) = \exp(-\gamma t)$ then non-Markovianity can be witnessed if the real part γ_r of γ fails to be positive or violates the inequality $\alpha\sqrt{N} \geq \gamma_r \pi$. Figs. 4.2(c) and 4.2(d) show the entanglement vs time plot for two values of γ_r; one of which violates the above mentioned inequality. We also consider the case $g(t) = \frac{1}{1+\gamma t}$, and show in Figs 4.2(e) and 4.2(f) that the dynamics is non-Markovian for various values of γ. N and p.

4.4.3 For couplings in Case D

For inhomogeneous time-dependent couplings, the dynamics can be made both non-Markovian and Markovian by choosing the strength of the coupling appropriately. We consider two special cases of inhomogeneous time-dependent coupling: (i) $g_n(t) = e^{-\gamma_1 nt}$, and (ii) $g_n(t) = \frac{1}{1+t^{n\gamma}}$.

For analyzing coupling (i), we plot the system-ancilla entanglement measure as a function of time in Figs. 4.3(a), 4.3(b), 4.3(c) and 4.3(d), for different values of the coupling parameter γ_1, at fixed values of N and p. In Figs 4.3(a), 4.3(b), 4.3(c) and 4.3(d), monotonically decreasing entanglement values show signs of Markovianity and non-monotonic decay are evidence of non-Markovianity. As expected, increasing the coupling parameter γ_1 i.e., decreasing coupling strength, leads to the transition from non-Markovian to Markovian dynamics. The figures also show an interesting feature that, after sufficient time, the entanglement in the system-ancilla state saturates to fixed values irrespective of their Markovian or non-Markovian nature. This feature can be signs of possible equilibration of the system ancilla state.

Next, we find the extremal values of γ_1 for which non-Markovianity is witnessed. These extremal values serve as transition parameters from non-Markovianity to Markovianity. On plotting these transition values as a function of N (see Fig. 4.3(i)), it appears that a saturation is reached as N is increased for values $p = 0.5$ and $p = 1.0$. We perceive, this is the result of the fact that for this type of coupling i.e. $g_n(t) = e^{-\gamma_1 nt}$, the larger is the value of N, the smaller is its effect on the system dynamics. Although a definitive conclusion about whether the saturation persists over large N can only be made after computing the transition values for larger values of N, it is a computationally demanding process with the computational facilities available at our disposal.

For coupling (ii), the dynamics shows non-Markovianity for various values of γ, N and p, as shown in Figs 4.3(e), 4.3(f), 4.3(g) and 4.3(h).

4.5 Chapter summary

In this chapter, we have addressed the question of how non-Markovianity of a dynamics changes with the interaction between the system and the environment and also with size of the environment. We have taken a simple model constituting of a few qubits, which can also be seen as a close approximation to the spin bath model for low temperatures. Even in this minimalistic scenario, we were able to find a transition from non-Markovian to Markovian dynamics by tuning the system-environment interaction. This is somewhat counterintuitive as it is generally conceived that for having Markovian dynamics the bath/environment should have infinite degrees of freedom, although there are exceptions [82]. We also found, in our model, that if the interaction Hamiltonian is time-independent, the dynamics is always non-Markovian, irrespective of the size of environment. In the case of site-independent interaction, polynomial forms and certain cases of exponential forms of interaction show non-Markovianity. Lastly, we study time-dependent and site-dependent interaction for certain forms of system-environment coupling. In this last case, we also saw a transition from non-Markovian to Markovian regime. Interestingly, the transition values appear to saturate to a certain value depending on the initial temperature of the environment, as the number of environment qubits increases.

The above observations point out some interesting features about the nature of non-Markovianity that might be seen in the spin-bath model for low temperatures. These features can be useful in further theoretical understanding and real life implementation of the spin-bath model. Firstly, we found the dynamics to be non-Markovian for time-independent coupling for any number of qubits in the environment. We also saw in the time-dependent and site-dependent case, fixed number of environment qubits can give rise to both Markovian or non-Markovian dynamics depending on the strength of coupling. These observations show that, at least for the spin-bath model, non-Markovianity is not an effect of size of the environment. Secondly, the rapidly oscillatory feature of the entanglement dynamics for the couplings in cases A and B suggests that a strongly non-Markovian dynamics can be engineered in the spin-bath model by choosing these forms of couplings. Thirdly, the Kraus operators of the dynamics found in cases A,B and C could be used as the solution of the spin bath model for low temperatures.

It is noteworthy here, that although we draw analogies between our model and the spin-bath model, in case D we performed our simulations with only a few number of qubits in the environment. Therefore, it might not be justified to call the environment in our model or the spin-bath model itself, a “bath" as the term usually refers to large dimension or large number of degrees of freedom. Nevertheless, for the sake of convention and ease of understanding we retain this nomenclature in our study.

Examining this type of spin environment is recently drawing some amount of interest [66]. Studies on similar lines was also done recently in [69], where an analysis of a qubit system interacting with a sea of spins was given.

Chapter 5

Generalized formalism for Information backflow

Studying aspects of Markovianity in open quantum system has recently received renewed focus from researchers working in various disciplines of physics. As mentioned in chapter 3, the definition of Markovianity in quantum regime is debated. Although a number of prescriptions have been proposed, they are, in general, non-equivalent and capture different aspects of Markovianity. A single unified description which take into account all these different aspects is yet to be found.

All the prescriptions of Markovianity suggested so far can be broadly classified into two main categories: *completely positive divisibility* (CPD) [17,25], and *information backflow* (IB) [28,30,83–88]. The CPD approach comes from a mathematical point of view, where a dynamical process is called Markovian if evolution up to a time t, can be broken down into two valid quantum evolutions: one up to an intermediate time s (for any $s < t$), followed by another from s to t. The IB approach, on the other hand describes a dynamical process to be Markovian if some *quantifier*, i.e. some physical property or some quantifier of information, decreases in a monotonic fashion under the action of the process. The IB category can be further divided into two classes based on the type of quantifier used.

One, which uses quantifiers based on the system only, which we call the *information backflow system only* (IBS) class, and the other that uses an ancilla to define the quantifier, which we call the *information backflow system-ancilla* (IBSA) class. The IBS class involves system based quantifiers like *distinguishability* [28, 83] and *generalized trace-distance* [84]. A number of measures to quantify the IBS class has also been suggested, namely, *fidelity* [85], *temporal steering weight* [86], etc. Any such measure can be used as a quantifier to define Markovianity. Similarly, the IBSA class involves joint system-ancilla quantifiers like *quantum mutual information* [30], *interferometric power* [87], *local quantum uncertainty* [88], etc.

With so many notions of Markovianity present, even within the IB category, one is compelled to look for inter-relations, hierarchies, or equivalence, that might be present within them. A number of studies [89–92] have already shown that these notions are not in general equivalent. To our knowledge, all the prescriptions under the IB category can be shown to be CPD but the converse is not true. Attempts have been made to find hierarchies [29, 93] or equivalence [26, 31, 94] within these approaches, either by concentrating on specific models or by using modified forms of some quantifier. But a general universal description that applies to any generic dynamics and any meaningful quantifier is not yet found.

In the work reported in this chapter, we attempt to tackle this problem by constructing a formalism that is independent of any particular form of quantifier. We give a generalized form of quantifier, called the *physicality quantifier* (PQ), for the whole of IB category. We set a minimal requirement criteria for any quantity to qualify as a PQ i.e. it should be non-increasing under any physical process. In doing so, we found that a large class, if not all, of quantifiers considered so far in the literature come as special cases of our generalized form. Basically we consider an ensemble of quantum states, and define a PQ as a real bounded function on the ensemble, that is non-increasing under any CPTP map. We call a given dynamics, *information backflow Markovian* or *IB-Markovian* if all possible PQ, defined on any ensemble, decreases in a monotonic fashion with time. IB-Markovianity is then sub-divided into *system only* (S) and *system-ancilla* (SA) class, where in the latter case the ancilla considered, is of same dimension as the system. A dynamics is said to belong to S-Markovian class, if the choice of ensemble for PQ is restricted to include system states only, whereas it belongs to SA-Markovian class, if we allow arbitrary system-ancilla joint states in the ensemble. Latter, we show that the S-Markovian class is a subset of the SA-Markovian class. Thus, SA-Markovianity comes out as an equivalent criteria to IB-Markovianity. Also, it can be easily inferred from the definition of PQ, that our formalism is automatically CPD.

The motivation behind choosing this specific definition of PQ are two fold. On one hand, as we demonstrate here, it allows for a lucid description of information flow between system and environment, which clearly shows backflow, whenever there is a departure from monotonicity in decay of a PQ. On the other hand, it results in filtering out of those quantities which are decreasing, even for isolated systems, under unitary evolution. Thus our definition serves as a minimal criteria, that captures only those quantities which reflect information exchange between system and environment.

We then examine different properties of our formalism and prove that, for invertible dynamics, the following are equivalent: (i) SA-Markovianity or IB-Markovianity, (ii) Markovianity with respect to generalized trace-distance measure (GTD) on an extended system-ancilla space, and (iii) CPD. We also show that our formalism can be used to construct an infinite family of non-Markovianity measures, which would capture varied strengths of memory effects present in the dynamics. Moreover, we prove that for qubit dynamics, GTD (defined only on the system) serves as a sufficient criteria for IB-Markovianity, for quantifiers defined on one or two system states; like fidelity, distinguishability, etc. Finally, we present some applications of our formalism and discuss the context of our formalism with respect to incoherent and unital dynamics.

5.1 Preliminaries

Consider a $d-$dimensional system and a $d-$dimensional ancilla with Hilbert spaces $\mathcal{H}_{\mathcal{S}}$ and $\mathcal{H}_{\mathcal{A}}$, respectively. An invertible dynamics is a dynamical map Λ_t which is invertible for all t. Note that, most of the physical dynamical maps are invertible. Even the thermalization process, where any initial state evolves asymptotically towards a fixed thermal state, is invertible for finite times.

An ensemble on the system $\mathcal{E}_S := \{p_i; \rho_i\}_{i=1}^n$ is defined as a finite collection of states $\rho_i \in \mathcal{P}_+(\mathcal{H}_{\mathcal{S}})$ with *a priori* probabilities p_i. Similarly, we define $\mathcal{E}_{SA} := \{p_i; \xi_i\}_{i=1}^n$ on system-ancilla with $\xi_i \in \mathcal{P}_+(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{A}})$. Let $\mathcal{F}_S^n := \{\mathcal{E}_S \mid \mathcal{E}_S = \{p_i; \rho_i\}_{i=1}^n\}$ and $\mathcal{F}_{SA}^n := \{\mathcal{E}_{SA} \mid \mathcal{E}_{SA} = \{p_i; \xi_i\}_{i=1}^n\}$ be the collection of all ensembles with n elements. We define the set of all possible ensembles of any size by,

$$\mathcal{F}_S := \bigcup_{n=1}^{\infty} \mathcal{F}_S^n \quad ; \quad \mathcal{F}_{SA} := \bigcup_{n=1}^{\infty} \mathcal{F}_{SA}^n. \tag{5.1}$$

5.2 Generalized formalism for information backflow

We now present a generalized formalism for the IB category in such a way that a large class of IB prescriptions proposed so far [28, 30, 83–88] falls into it. We identify an essential feature which is common to all quantifiers considered in the literature i.e. they are unitarily invariant and non-increasing under CPTP maps. This suggests a ready generalization of the definition of quantifier. Therefore, we propose to define two types of PQ, $\mathcal{I}_S : \mathcal{F}_S \mapsto \mathbb{R}$ and $\mathcal{I}_{SA} : \mathcal{F}_{SA} \mapsto \mathbb{R}$, as real bounded functions on ensembles of quantum states, which follow condition 1 (given below). For convenience, any function $f_S(\mathcal{E}_S)$ or $f_{SA}(\mathcal{E}_{SA})$, is also represented by symbols $f_S\{p_i; \rho_i\}$ or $f_{SA}\{p_i; \xi_i\}$, respectively.

Condition 1. *Let $T \in \mathcal{T}(\mathcal{H}_{\mathcal{S}}, \mathcal{H}_{\mathcal{S}})$ be any CPTP map, acting on the system. For a given* *form of $\mathcal{I}_S$ or $\mathcal{I}_{SA}$, the following are true:*

$$\mathcal{I}_S\{p_i; T[\rho_i]\} \leq \mathcal{I}_S\{p_i; \rho_i\},$$
$$\mathcal{I}_{SA}\{p_i; (T \otimes I)[\xi_i]\} \leq \mathcal{I}_{SA}\{p_i; \xi_i\}$$

where I denotes the identity map in $\mathcal{T}(\mathcal{H}_\mathcal{A}, \mathcal{H}_\mathcal{A})$.

Note that, this readily implies invariance of PQ under any unitary evolution [1], i e.

$$\mathcal{I}_S\{p_i; U\rho_i U^\dagger\} = \mathcal{I}_S\{p_i; \rho_i\}, \tag{5.2}$$
$$\mathcal{I}_{SA}\{p_i; (U \otimes \mathbb{1})\xi_i(U^\dagger \otimes \mathbb{1})\} = \mathcal{I}_{SA}\{p_i; \xi_i\}, \tag{5.3}$$

where $U \in \mathcal{L}(\mathcal{H}_\mathcal{S})$ and $\mathbb{1} \in \mathcal{L}(\mathcal{H}_\mathcal{A})$ are unitary and identity operators, respectively.

As any PQ is real and bounded, for any given form we can always choose an equivalent form, by adding the lower bound of the PQ, which would have the same monotonic or non monotonic nature as the PQ and would also be positive. We would therefore restrict ourselves to only positive PQ. Note that, distance measures like *p-norms* [95] on qubit space, and general measures of information like *order-α* Renyi divergences in any dimension for certain ranges of α [96], obey condition 1.

We further sub-divide PQ according to n, i.e. number of elements present in the ensemble. We define special types of PQ, $\mathcal{I}_S^n$ and $\mathcal{I}_{SA}^n$, which are focused on ensembles of size n, in the following way: $\mathcal{I}_S^n(\mathcal{E}_S) = 0$ and $\mathcal{I}_{SA}^n(\mathcal{E}_{SA}) = 0$, for any $\mathcal{E}_S \notin \mathcal{F}_S^n$ and $\mathcal{E}_{SA} \notin \mathcal{F}_{SA}^n$. Note that as $\mathcal{I}_S^n$ and $\mathcal{I}_{SA}^n$ are valid PQ, they obey condition 1. Observe that, different forms of quantifiers used in the literature, are defined on different subsets of $\mathcal{F}_S^n$ or $\mathcal{F}_{SA}^n$, for various values of n. For example, GTD [84] is defined on all elements of $\mathcal{F}_S^2$, whereas distinguishability [28] is defined only on those elements of $\mathcal{F}_S^2$ for which $p_1 = p_2 = 1/2$. Likewise, quantum mutual information [30] is defined on all elements of $\mathcal{F}_{SA}^1$. To fit these

[1]Consider the CPTP maps $\rho \to U\rho U^\dagger$ and $\rho \to U^\dagger \rho U$ for any unitary U.

Table 5.1: Physicality Quantifiers and their Class and Type

Type	Class	Physicality quantifier
$\mathcal{I}_S^2$	2-S-Markovian	Fidelity [85] State distinguishability [20] Generalized trace-distance [84]
$\mathcal{I}_S^m$	m-S-Markovian[2]	Temporal steering weight [86]
$\mathcal{I}_{SA}^1$	1-SA-Markovian	Quantum mutual information [30] Interferometric power [87] Local quantum uncertainty [88]
$\mathcal{I}_{SA}^2$	2-SA-Markovian	Generalized trace-distance extended

quantifiers in our formalism, we define compatible PQ in each case, which are of the form $\mathcal{I}_S^n$ or $\mathcal{I}_{SA}^n$. For example for GTD, we define a PQ, $\mathcal{I}_S^{GTD}$, which takes the same value as GTD for elements in $\mathcal{F}_S^2$, and zero otherwise. Similarly for distinguishability, we define $\mathcal{I}_S^{BLP}$ such that, $\mathcal{I}_S^{BLP}\{p_1 = 1/2, p_2 = 1/2, \rho_1, \rho_2\} = ||\rho_1 - \rho_2||_1$ and $\mathcal{I}_S^{BLP}(\mathcal{E}_S) = 0$, for $\mathcal{E}_S \notin \mathcal{F}_S^2$ or $p_i \neq 1/2$. Note that BLP stands for Breuer, Laine, and Piilo, who were the first to use distinguishability as a measure of non-Markovianity [28]. In a similar way, we find that a large class of system-ancilla quantifiers considered so far [30, 87, 88] also correspond to PQ of the form $\mathcal{I}_{SA}^n$, for different values of n. In particular, $\mathcal{I}_S^2$ and $\mathcal{I}_{SA}^1$ corresponds to a large number of cases in the literature (see Table 5.1). Refer to Appendix B.1, for a detailed disposition of how each quantifier corresponds to PQ.

For a dynamical map Λ_t we define *dynamic physicality quantifiers* $\Phi_t^{\mathcal{I}_S}$ and $\Phi_t^{\mathcal{I}_{SA}}$ based on $\mathcal{I}_S$ and $\mathcal{I}_{SA}$, in the following way,

$$\Phi_t^{\mathcal{I}_S}\{p_i; \rho_i\} := \mathcal{I}_S\{p_i; \Lambda_t[\rho_i]\}, \tag{5.4}$$

$$\Phi_t^{\mathcal{I}_{SA}}\{p_i; \xi_i\} := \mathcal{I}_{SA}\{p_i; (\Lambda_t \otimes I)[\xi_i]\}. \tag{5.5}$$

We now verify the perception, that non-monotonic decay of a PQ, rightly represents

[2] If Alice makes measurement $M_{a|x}$, where $a = 1, \ldots, m_1$ and $x = 1, \ldots, m_2$, then $m = m_1 m_2$ (see Appendix B.1).

backflow of information from environment to the system. For any form of $\mathcal{I}_S$ or $\mathcal{I}_{SA}$, consider $\mathcal{I'}_S$ or $\mathcal{I'}_{SA}$ to be the same quantity defined on environment-system ensemble $\{p_i;\zeta^i_{ES}\}_i$ or environment-system-ancilla ensemble $\{p_i;\zeta^i_{ESA}\}_i$, respectively. As any open system dynamics is a result of unitary evolution $U_{ES}(t)$ of the system and environment considered jointly [8], we conclude from Eqs. (5.2) and (5.3), that $\Phi_t^{\mathcal{I'}_S}$ and $\Phi_t^{\mathcal{I'}_{SA}}$, defined in the sense of Eqs. (5.4) and (5.5), are constant in time. Now, consider $I_{env}^{\mathcal{I'}_S}(t) = \Phi_t^{\mathcal{I'}_S} - \Phi_t^{\mathcal{I}_S}$ or $I_{env}^{\mathcal{I'}_{SA}}(t) = \Phi_t^{\mathcal{I'}_{SA}} - \Phi_t^{\mathcal{I}_{SA}}$ to represent the information content of the environment and system (ancilla) combined, that cannot be obtained by knowing the system (ancilla) ensemble alone. Hence, we get $I_{env}^{\mathcal{I'}_S}(t) + \Phi_t^{\mathcal{I}_S} = constant$ and $I_{env}^{\mathcal{I'}_{SA}}(t) + \Phi_t^{\mathcal{I}_{SA}} = constant$, which means the net information content remains unchanged. Note, as partial tracing is always isomorphic to a CPTP map[3], from condition 1, we find $I_{env}^{\mathcal{I'}_S}(t)$ and $I_{env}^{\mathcal{I'}_{SA}}(t)$ are both positive quantities. Therefore, we conclude non-monotonic decay of $\Phi_t^{\mathcal{I}_S}$ and $\Phi_t^{\mathcal{I}_{SA}}$ rightly signifies information backflow from environment to the system. Now we define Markovianity in terms of each valid form of PQ.

Definition 2. A dynamical map Λ_t, is called *$\mathcal{I}_S$-Markovian* (*$\mathcal{I}_{SA}$-Markovian*) for some form of $\mathcal{I}_S$ ($\mathcal{I}_{SA}$), if $\Phi_t^{\mathcal{I}_S}(\mathcal{E}_S)$ ($\Phi_t^{\mathcal{I}_{SA}}(\mathcal{E}_{SA})$) decreases monotonically with time t, for any $\mathcal{E}_S \in \mathcal{F}_S$ ($\mathcal{E}_{SA} \in \mathcal{F}_{SA}$).

There are numerous examples in the literature for the above definition [28, 30, 83–88]. We generalize the above notion in the following way,

Definition 3. A dynamical map Λ_t is called *n-S-Markovian* (*n-SA-Markovian*) if $\Phi_t^{\mathcal{I}^n_S}(\mathcal{E}_S)$ ($\Phi_t^{\mathcal{I}^n_{SA}}(\mathcal{E}_{SA})$) decreases in a monotonic fashion with time t, for any form of $\mathcal{I}^n_S$ ($\mathcal{I}^n_{SA}$) and any choice of ensemble $\mathcal{E}_S \in \mathcal{F}^n_S$ ($\mathcal{E}_{SA} \in \mathcal{F}^n_{SA}$).

We now give generalized definition of Markovianity for all PQ defined on system and system-ancilla.

Definition 4. A dynamical map Λ_t is called *S-Markovian* (*SA-Markovian*) if it is n-S-

[3]Consider the map $\Omega[X] = \sum_i K_i X K_i^\dagger$, where $K_i = |\psi\rangle\langle i|_E \otimes \mathbb{I}_S$ and $\{|i\rangle\}_i$ forms an orthonormal basis in environment space.This CPTP map is isomorphic to partial trace operation $Tr_E(\bullet)$.

Markovian (n-SA-Markovian) for all values of n.

Finally, we give our generalized definition of Markovianity for backflow of information: any dynamics which is both S-Markovian and SA-Markovian, is called *IB-Markovian*.

5.3 General properties of the formalism

We first note a hierarchy within our Markovianity classes, which is apparent from their definition: any n-S-Markovian (n-SA-Markovian) class is a subset of $(n+1)$-S-Markovian ($(n+1)$-SA-Markovian) class. This observation provides a useful insight, that our formalism can be used to construct an infinite family of non-Markovianity measures, which would capture varied intensities of memory effects present in the dynamics. The higher the least value of n, for which a dynamics fails to be n-S-Markovian or n-SA-Markovian, the weaker is the effect of memory in the dynamics. Also note, as any PQ obeys condition 1, all IB-Markovian dynamics are automatically CPD. We now present a result, that makes SA-Markovianity an equivalent criteria to IB-Markovianity.

Theorem 1. *If any dynamical maps Λ_t is n-SA-Markovian, then it is n-S-Markovian.*

This result is expected, as any PQ on the system can be seen as a PQ on system-ancilla by choosing the system ensemble states to be reduced density matrices of the system-ancilla ensemble states. See Appendix B.2 for detailed proof. Figure 6.1 gives a concise representation of all the hierarchies present in our formalism.

We now show how IB-Markovianity is related to CPD and in what way GTD on extended space plays a vital role in relating these two quantities. GTD is a quantity that gives the best possible distinguishing probability of a pair of quantum states, occurring with different probabilities [97]. Suppose, Alice prepares one of two states ρ_1 and ρ_2 with probabilities p_1 and p_2 and sends the ensemble to Bob. The best possible probability for him to distinguish between these two states with a single-shot experiment is given by the

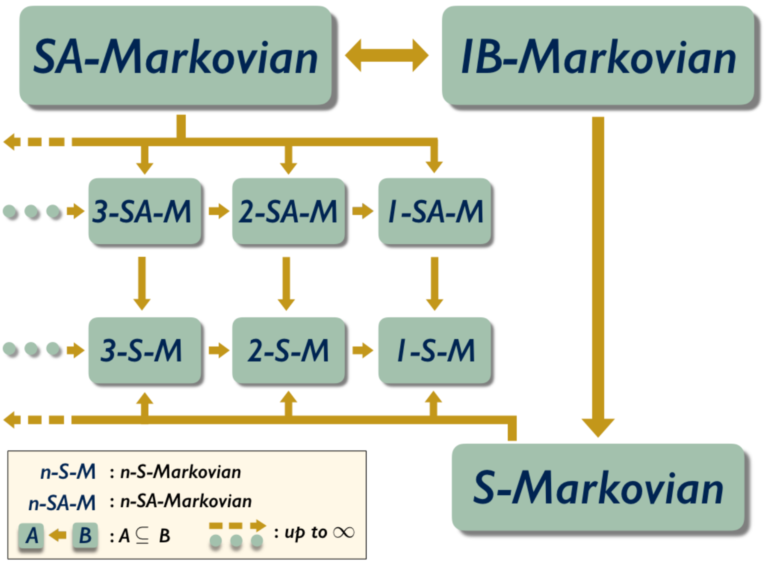

Figure 5.1: Hierarchy of different classes of IB-Markovianity.

GTD of the ensemble. The corresponding PQ, $\mathcal{I}_S^{GTD}$ is given by,

$$\mathcal{I}_S^{GTD}\{p_1, p_2, \rho_1, \rho_2\} := ||p_1\rho_1 - p_2\rho_2||_1, \tag{5.6}$$

where $||A||_1 = Tr\sqrt{A^\dagger A}$. It was first proposed in [92] and latter in [84] that GTD can be used as a quantifier to define Markovianity. The definition of GTD can also be easily extended to system-ancilla space. We call it *generalized trace-distance measure extended* (GTDE) and define it in the following way,

$$\mathcal{I}_{SA}^{GTDE}\{p_1, p_2, \xi_1, \xi_2\} := ||p_1\xi_1 - p_2\xi_2||_1, \tag{5.7}$$

where $\xi_i \in \mathcal{P}_+(\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{A})$. Note that $\mathcal{I}_S^{GTD}$ and $\mathcal{I}_{SA}^{GTDE}$ are special forms of $\mathcal{I}_S^2$ and $\mathcal{I}_{SA}^2$, respectively.

Definition 5. A dynamical map Λ_t is called *GTD-Markovian* or *GTDE-Markovian* in the sense of definition 2, if the respective PQ are $\mathcal{I}_S^{GTD}$ and $\mathcal{I}_{SA}^{GTDE}$.

Now we present one of the main theorems of this chapter.

Theorem 2. *For an invertible dynamical map Λ_t, the following are equivalent: (i) Λ_t is*

GTDE-Markovian, (ii) Λ_t *is CPD,* (iii) Λ_t *is SA-Markovian or IB-Markovian,* (iv) Λ_t *is* 2*-SA-Markovian.*

The non-intuitive part of the theorem is $(i) \implies (ii)$, which can be easily deduced by using a result by Kossakowski [98], where TP contraction of trace-norm was shown to be the necessary and sufficient criteria for positivity of maps (see Appendix B.3). Thus we see the GTDE criteria is not only necessary but also sufficient for the whole of IB-Markovianity class, for invertible dynamics.

5.4 The qubit case

We now show that for qubit dynamics, the GTD criteria is an equivalent criteria for 2-S-Markovianity. From theorem 1, this implies that GTD is also a sufficient criteria for 1-S-Markovianity. Therefore, for qubit dynamics GTD serves as a sufficient criteria of Markovianity for quantifiers defined on one or two states of the system, namely, distinguishability [28], fidelity [85], etc. The existence of this result is due to Alberti and Uhlmann [99] and latter also by Chefles et al. [100] and Huang et al. [101], who showed that TP contractivity of trace-norm is necessary and sufficient condition for existence of physical transformations between two pairs of qubit states. The following lemma, and consequently the theorem can be easily deduced by applying this result (see Appendix B.4). For this section, we assume $\mathcal{H}_\mathcal{S}$ and $\mathcal{H}_\mathcal{A}$ to be the Hilbert spaces for a qubit system and qubit ancilla, respectively.

Lemma 1. *If a qubit dynamical map* Λ_t *is GTD-Markovian, then for any* $t > s$ *and any collection of states* $\rho_1, \rho_2, \sigma_1, \sigma_2 \in \mathcal{P}_+(\mathcal{H}_\mathcal{S})$ *such that* $\rho_i = \Lambda_s[\sigma_i]\ ;\ i = 1, 2$*, there exists a CPTP map* $T_{12} \in \mathcal{T}(\mathcal{H}_\mathcal{S}, \mathcal{H}_\mathcal{S})$ *such that,*

$$V_{t,s}[\rho_i] = T_{12}[\rho_i]\ ;\ i = 1, 2. \tag{5.8}$$

Theorem 3. *A qubit dynamical map Λ_t is GTD-Markovian if and only if it is 2-S-Markovian.*

Proof. As Λ_t is GTD-Markovian, it can be expressed as Eq. (3.6) (see Proposition 2 of [26]). For any form of physicality quantifier $\mathcal{I}_S^2$, let us choose any $\rho_1, \rho_2, \sigma_1, \sigma_2 \in \mathcal{P}_+(\mathcal{H}_S)$ and $t > s$, such that $\rho_i = \Lambda_s[\sigma_i]$ for $i = 1, 2$. As Λ_t is GTD-Markovian, using lemma 1 we get $\mathcal{I}_S^2\{p_1, p_2, V_{t,s}[\rho_1], V_{t,s}[\rho_2]\} = \mathcal{I}_S^2\{p_1, p_2, T_{12}[\rho_1], T_{12}[\rho_2]\}$. Note, as T_{12} is CPTP and $\mathcal{I}_S^2$ obeys Condition 1, this implies $\Phi_t^{\mathcal{I}_S^2} \leq \Phi_s^{\mathcal{I}_S^2}$ for any initial ensemble $\{p_1, p_2, \sigma_1, \sigma_2\}$ and $t > s$. Hence, we conclude that if Λ_t is GTD-Markovian then it is 2-S-Markovian. The converse statement is easy to prove, as $\mathcal{I}_S^{GTD}$ is a physicality quantifier of the form $\mathcal{I}_S^2$. □

It was shown in Ref. [84], that GTD is an equivalent criteria to P-divisibility for invertible dynamics. Therefore, the above theorem shows that P-divisibility is equivalent to 2-S-Markovianity for invertible qubit dynamics. A detailed algorithm for constructing T_{12} of lemma 1 is given in theorems 2.1 and 2.2 of [101].

5.5 Applications of the formalism

5.5.1 Minimum strength of Non-Markovianity required to be used as a resource: Case studies

A number of protocols have been suggested, where backflow of information in a non-Markovian process has been used as a resource to enhance the efficiency of the protocol. All these protocols require different minimum strengths of non-Markovianity to enable the enhancement of efficiency.

We now present two such scenarios and in each case we identify the minimum strength of non-Markovianity required.

Preserving channel capacities :

Non-Markovianity was used by Bylicka et al [21, 33] to preserve *channel capacity* over long channels. They used the fact that *classical* and *quantum channel capacities*, given by C_c and C_q, show a non-monotonic decay over time, whenever the dynamical map is non-Markovian.

$$C_c[\Lambda_t] = \sup_{\rho} I(\rho, \Lambda_t), \tag{5.9}$$

$$C_q[\Lambda_t] = \sup_{\rho} I_c(\rho, \Lambda_t), \tag{5.10}$$

where $I(\rho, \Lambda_t)$ is the quantum mutual information between the initial and the time evolved state of the system, defined in the following way,

$$I(\rho, \Lambda_t) := S(\rho) + I_c(\rho, \Lambda_t). \tag{5.11}$$

Here $S(\rho) = -\rho \log \rho$ is the von Neumann entropy and I_c is the quantum coherent information, given by,

$$I_c(\rho, \Lambda_t) = S(\Lambda_t[\rho]) - S((\Lambda_t \otimes I)[\,|\Psi\rangle\langle\Psi|\,]), \tag{5.12}$$

where $\rho = Tr_A(\,|\Psi\rangle\langle\Psi|\,)$ and $|\Psi\rangle$ is a purification of ρ in a higher dimensional system-ancilla space with ancilla dimension same as the system. It can be easily seen that $I(\rho, \Lambda_t)$ is a PQ of the form $\mathcal{I}^1_{SA}$.

Therefore, in order to have a revival of channel capacities, the dynamics should be non-Markovian at least with respect to 1-SA-Markovianity class.

As a thermodynamic resource :

Recently, Bylicka et al. [102] used a non-Markovian dynamics to obtain revival of extractable work from an $n-$qubit system. The main ingredient behind this result is non-increasing nature of quantum mutual information between system and ancilla, under action of arbitrary CPTP map on one side of system or ancilla. *Hence, we conclude that in order to obtain a revival of extractable work, the dynamical process must be non-Markovian at least with respect to 1-SA-Markovianity class.*

5.5.2 Relation to the problem of existence of physical transformations between states

The problem of whether there exists a physical transformation between two sets of quantum states, is a well researched topic with various partial and complete results available [99–101, 103, 104]. We will follow the notation in [100] and denote the existence of physical transformation between two sets of quantum states, each containing n elements, by $\{\rho_1, \rho_2, \ldots, \rho_n\} \implies \{\sigma_1, \sigma_2 \ldots, \sigma_n\}$. Formally speaking, this means there exists a CPTP map T connecting them i.e. $T[\rho_i] = \sigma_i$ for all $i = 1, \ldots, n$. In [103], it was shown that this problem can be reformulated as a semidefinite programming problem and thus using convex optimization techniques [105], it can be checked algorithmically. For any given dynamics Λ_t, it can be easily shown (in a similar way as in Theorem 3) that, if $\{\rho_1(s), \rho_2(s), \ldots, \rho_n(s)\} \implies \{\rho_1(t), \rho_2(t), \ldots, \rho_n(t)\}$ for any $t > s$ and $\{\rho_i\}_{i=1}^n$, then Λ_t is n-S-Markovian. Note, here $\rho_i(t) = \Lambda_t[\rho_i]$ represents the time evolved states.

5.5.3 A family of new non-Markovianity measures

As this formalism provides a general structure for constructing PQ, it is expected that a number of new PQ will emerge, that was previously unknown to the literature. Any such

PQ can be used to device a non-Markovianity measure in the following way,

$$\mathcal{N}_{\mathcal{I}_S}(\Lambda_t) := \int_{t\in\left\{t':\frac{d\left(\Phi_t^{\mathcal{I}_S}\right)}{dt}>0\right\}} \frac{d(\Phi_t^{\mathcal{I}_S})}{dt}dt. \tag{5.13}$$

Here $\Phi_t^{\mathcal{I}_S}$ is, as defined in Eq. (5.4). Note, positive time derivative of $\Phi_t^{\mathcal{I}_S}$ implies, departure from monotonic decay of the quantity over time. Similarly for PQ's of the form $\mathcal{I}_{SA}$, we can define non-Markovianity measures in the same way. Thus our formalism provides a platform for an infinite family of non-Markovianity measures. For example, in [106] a family of new metrics $g_D(A,B)$ on the space of linear operators of finite dimension were suggested, which are monotonic (decreasing) under stochastic (CPTP) maps i.e. $g_D(T[A],T[A]) \leq g_D(A,A)$, for any CPTP map T, any operator A and positive operator D. See [106] for more details about $g_D(A,B)$. Any such metric can used to device a new PQ of the form,

$$\mathcal{I}_S^{g_D}\{\rho\} = g_D(\rho,\rho). \tag{5.14}$$

Note that this is a PQ of the form $\mathcal{I}_S^1$. Also, [95] presents a collection of norms, the $p-$norms, that can used as PQ on qubit space in the following way,

$$\mathcal{I}_S^{p-norm}\{q_1,q_2,\rho_1,\rho_2\} = \begin{cases} ||\rho_1-\rho_2||_p & q_1=q_2=1/2 \\ 0 & q_i\neq 1/2 \end{cases} \tag{5.15}$$

where $||A||_p = [Tr(A^\dagger A)^{p/2}]^{1/p}$, $p\geq 1$, $\rho_1,\rho_2 \in \mathcal{P}_+(\mathcal{H}_S)$ with $\mathcal{H}_S=\mathbb{C}^2$ and q_1,q_2 are probabilities. Note, $p=1$ gives our usual trace-norm and we have, $\mathcal{I}_S^{1-norm} = \mathcal{I}_S^{BLP}$. We now combine two forms of $p-$norms to define a new PQ,

$$\mathcal{I}_S^{p_1-p_2}(\mathcal{E}_S) := q_1\mathcal{I}_S^{p_1-norm}(\mathcal{E}'_S) + q_2\mathcal{I}_S^{p_2-norm}(\mathcal{E}'_S) \tag{5.16}$$

where $\mathcal{E}_S = \{q_1,q_2,\rho_1,\rho_2\}$ and $\mathcal{E}'_S = \{1/2,1/2,\rho_1,\rho_2\}$. Note that $p-$norms defined on qubit space obey condition 1 and hence $\mathcal{I}_S^{p-norm}$ and $\mathcal{I}_S^{p_1-p_2}$ qualify as PQ of the form $\mathcal{I}_S^2$.

To present an example, we consider the the random unitary dynamics Λ_t on a qubit system, which has been extensively studied in the literature. We follow the notations and results presented in [29] to test our new PQ. The random unitary dynamics is given by,

$$\Lambda_t[\rho] = \sum_{\alpha=0}^{3} r_\alpha(t)\sigma_\alpha \rho \sigma_\alpha, \tag{5.17}$$

where $\{r_0(t), r_1(t), r_2(t), r_3(t)\}$ is a time dependent probability distribution with $r_0(0) = 1$. Also note, the Pauli matrices σ_α happen to be the eigenvectors of Λ_t with time-dependent eigenvalues $\lambda_i(t)$, i.e. $\Lambda_t[\sigma_i] = \lambda_i(t)\sigma_i$ for $i = 0, \ldots, 3$. We find, $\lambda_i = \sum_{j=1}^{4} H_{ij} r_j$, where $H = [H_{ij}]$ is the Hadamard matrix given by,

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Note that $\lambda_0(t) = 1$ for all t. The master equation of this dynamics is given by,

$$\frac{d}{dt}\rho_t = \sum_{i=0}^{3} \gamma_i(t)\sigma_i \rho_t \sigma_i, \tag{5.18}$$

where $\gamma_i = \frac{1}{4}\sum_{j=0}^{3} H_{ij}\frac{\dot{\lambda}_j(t)}{\lambda_j(t)}$ for all $i = 0, \ldots, 3$. This readily implies,

$$\sum_{i=0}^{3} \gamma_i(t) = 0. \tag{5.19}$$

Note that, this identity simplifies the above master equation to the following form,

$$\frac{d}{dt}\rho_t = \sum_{i=1}^{3} \gamma_i(t)(\sigma_i \rho_t \sigma_i - \rho_t). \tag{5.20}$$

Also, we find the following,

$$\lambda_1(t) = e^{-2[\Gamma_2(t)+\Gamma_3(t)]}, \tag{5.21}$$

$$\lambda_2(t) = e^{-2[\Gamma_1(t)+\Gamma_3(t)]}, \tag{5.22}$$

$$\lambda_3(t) = e^{-2[\Gamma_1(t)+\Gamma_2(t)]}, \tag{5.23}$$

where $\Gamma_k(t) = \int_0^t \gamma_k(\tau)d\tau$, for $k = 1, 2, 3$. Now, consider the ensembles $\mathcal{E}_S$ and $\mathcal{E}'_S$, used in Eq. (5.16). As $\rho_1 - \rho_2$ is a traceless hermitian operator, we get $\rho_1 - \rho_2 = \sum_{k=1}^{3} x_k \sigma_k$, where x_1, x_2 and x_3 are real numbers. Thus we have,

$$||\Lambda_t[\rho_1 - \rho_2]||_p = 2^{1/p}\eta(t), \tag{5.24}$$

where $\eta(t) = \sqrt{\sum_{k=1}^{3} \lambda_k(t)^2 x_k^2}$. This implies,

$$\frac{d}{dt}||\Lambda_t[\rho_1 - \rho_2]||_p = \frac{2^{1/p-1}}{\eta(t)} \sum_{k=1}^{3} x_k^2 \frac{d}{dt}|\lambda_k(t)|^2. \tag{5.25}$$

Thus we have,

$$\frac{d\left(\Phi_t^{\mathcal{I}_S^{p_1-p_2}}(\mathcal{E}_S)\right)}{dt} = \left[q_1 2^{1/p_1-1} + q_2 2^{1/p_2-1}\right] \frac{d\left(\Phi_t^{\mathcal{I}_S^{BLP}}(\mathcal{E}'_S)\right)}{dt}, \tag{5.26}$$

where,

$$\frac{d\left(\Phi_t^{\mathcal{I}_S^{BLP}}(\mathcal{E}'_S)\right)}{dt} = \frac{1}{\eta(t)} \sum_{k=1}^{3} x_k^2 \frac{d}{dt}|\lambda_k(t)|^2. \tag{5.27}$$

So we see $\mathcal{I}_S^{p_1-p_2}$ witnesses non-Markovianity whenever $\mathcal{I}_S^{BLP}$ witnesses the same, and vice-versa.

Example 1. On choosing $\gamma_1(t) = \gamma_2(t) = 1$ and $\gamma_3(t) = \sin t$, we get $\Gamma_1(t) = \Gamma_2(t) = t$

and $\Gamma_3(t) = 1 - \cos t$. So we have,

$$\lambda_1(t) = e^{-2(1+t-\cos t)}, \tag{5.28}$$

$$\lambda_2(t) = e^{-2(1+t-\cos t)}, \tag{5.29}$$

$$\lambda_3(t) = e^{-4t}. \tag{5.30}$$

Thus we get,

$$\begin{aligned}\Phi_t^{\mathcal{I}_S^{p_1-p_2}}(\mathcal{E}_S) = &\,(q_1 2^{1/p_1-1} + q_2 2^{1/p_2-1}) \\ &\times \left[(x_1^2 + x_2^2)\, e^{-4(1+t-\cos t)} + x_3^2\, e^{-8t}\right]^{\frac{1}{2}}\end{aligned} \tag{5.31}$$

Since q_1, q_2, p_1 and p_2 are all positive and x_1, x_2 and x_3 are real, it can be easily seen that the above function is monotonically decreasing with t. Hence, we conclude that the above dynamics is $\mathcal{I}_S^{p_1-p_2}$–Markovian for any $p_1, p_2 \geq 1$. Also, note this dynamics is not CPD in general, as $\gamma_3(t)$ can take negative values [29].

Example 2. Choose $r_1(t) = r_2(t) = \frac{1-r_0(t)}{4}$ and $r_3(t) = \frac{1-r_0(t)}{2}$. Therefore $\lambda_1(t) = \lambda_2(t) = \frac{3r_0(t)-1}{2}$ and $\lambda_3(t) = r_0(t)$. Also, $\gamma_1(t) = \gamma_2(t) = -\frac{\dot{r}_0(t)}{4r_0(t)}$ and $\gamma_3(t) = -\frac{(3r_0(t)+1)}{4r_0(t)}\frac{\dot{r}_0(t)}{(3r_0(t)-1)}$. So, we have,

$$\Phi_t^{\mathcal{I}_S^{p_1-p_2}}(\mathcal{E}_S) = (q_1 2^{1/p_1-1} + q_2 2^{1/p_2-1})\Big[\frac{1}{4}(x_1^2 + x_2^2)\,(3r_0(t) - 1)^2 + x_3^2\, r_0(t)^2\Big]. \tag{5.32}$$

Figure 5.2 shows time evolution of $\Phi_t^{\mathcal{I}_S^{2-3}}$ (i e. when $p_1 = 2$ and $p_2 = 3$) for different initial ensembles and two different forms of $r_0(t)$: e^{-t} and $\frac{1+\cos t}{2}$. We find in both cases the dynamics is $\mathcal{I}_S^{2-3}$– non-Markovian.

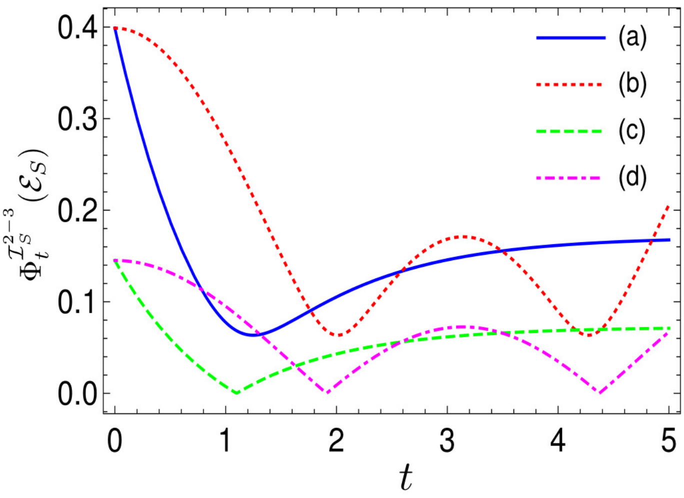

Figure 5.2: Plot of dynamic PQ vs time for random unitary dynamics with $r_1(t) = r_2(t) = \frac{1-r_0(t)}{4}$ and $r_3(t) = \frac{1-r_0(t)}{2}$, for different initial ensembles $\mathcal{E}_S$ and different functional forms of $r_0(t)$. We consider $\mathcal{E}_S = \{q_1, q_2, \rho_1, \rho_2\}$, where $\rho_i = \frac{1}{2}(\sigma_0 + \sum_{k=1}^{3} n_k^i \sigma_k)$ and $n^i = (s_i \sin\theta_i \cos\phi_i\ ,\ s_i \sin\theta_i \sin\phi_i\ ,\ s_i \cos\theta_i)$. The different cases considered here are: (a) $r_0(t) = e^{-t}$, $q_1 = 0.7, q_2 = 0.3,\ s_1 = 1, \theta_1 = \phi_1 = \pi/2,\ s_2 = 0.6, \theta_1 = \pi, \phi_1 = 0$; (b) $r_0(t) = \frac{1+\cos t}{2}$, $q_1 = 0.7, q_2 = 0.3,\ s_1 = 1, \theta_1 = \phi_1 = \pi/2,\ s_2 = 0.6, \theta_1 = \pi, \phi_1 = 0$; (c) $r_0(t) = e^{-t}$, $q_1 = 0.3, q_2 = 0.7,\ s_1 = 0.7, \theta_1 = 2\pi/3, \phi_1 = \pi/6,\ s_2 = 0.4, \theta_1 = 5\pi/6, \phi_1 = \pi/3$; (d) $r_0(t) = \frac{1+\cos t}{2}$, $q_1 = 0.3, q_2 = 0.7,\ s_1 = 0.7, \theta_1 = 2\pi/3, \phi_1 = \pi/6$, $s_2 = 0.4, \theta_1 = 5\pi/6, \phi_1 = \pi/3$.

5.6 Discussions

A number of measures of non-Markovianity were suggested for incoherent and unital dynamics [107–109], which are not non-increasing under arbitrary CPTP maps. Therefore, they are not guaranteed to be monotonically decreasing under arbitrary CPD dynamics, which are well accepted Markovian dynamics. Hence, in spite of them being useful non-Markovianity measures for certain types of dynamics, we do not consider them as appropriate quantifiers for describing Markovianity, in general.

5.7 Chapter summary

In this chapter, we have provided a generalized formalism for describing the IB approach of Markovianity. We provided a general form of a quantifier, called the physicality quantifier, whose monotonic decay with time was seen as the defining criteria for Markovianity. We defined the physicality quantifier, to be any real bounded function on the ensemble space, that is non-increasing under CPTP maps. In doing so, we found that a large number of prescriptions for IB-Markovianity in the literature, come as special cases of our formalism. Also, by using our formalism we showed that for invertible dynamics, IB-Markovianity is equivalent to CP-divisibility, as well as to Markovianity with respect to generalized trace-distance measure in extended system-ancilla space.

We showed hierarchies of different subclasses of our formalism and argued that it can be used to construct an infinite family of non-Markovianity measures, which would capture varied strengths of memory effects present in the dynamics. We also used the formalism to show that generalized trace-distance measure for qubit dynamics, serve as sufficient criteria of IB-Markovianity for a number of prescriptions suggested earlier. Finally, we presented certain applications of our formalism. We expect our formalism will shed light into further understanding of physical and mathematical structure of quantum Markovianity and enhance its applicability to more varied scenarios.

Chapter 6

Extending the generalized formalism to non-invertible dynamical maps

In this chapter, we extend the *Generalized formalism for Information Backflow* (GIB) to include non-invertible dynamical maps. We introduce a new physicality quantifier (PQ) and show that monotonic decrease of only system based PQ's are sufficient to establish completely positive divisibility (CPD) as well as Information backflow Markovianity (IB-Markovianity) for all image non-increasing dynamical maps.

For qubit systems, we further extend the result to include all dynamical maps – invertible as well as non-invertible. Moreover, using the new PQ, we show that a simplified hierarchical structure of GIB can be found for image non-increasing and qubit dynamical maps, which have only d^2 Markovianity classes for any $d-$dimensional ($d = 2$ for qubit case) dynamical map.

This classification provides a *degree* to the non-Markovianity of dynamical maps, an idea first introduced in Ref [110]. In Ref [110], degree was determined by $k-$divisibility, a representation of digression of sections of dynamics from complete positivity. In our case, degree is determined by subsets of PQ's which decay monotonically under the action of the dynamics. Moreover, in most cases where non-Markovianity is used as a resource, revival over time of different PQ's serve as the essential factor responsible for the usefulness of the non-Markovianity as a resource. Therefore, we expect our description of degree will be more useful in applications of non-Markovianity as a resource. We also use this description to determine the degree of non-Markovianity of some commonly used dynamical maps.

6.1 Preliminaries

Since, in this chapter we will use only system based PQ's, we slightly modify the notations to make it easier to follow.

We denote the system Hilbert space by $\mathcal{H} = \mathbb{C}^d$. An ensemble of states of the system is denoted by $\mathcal{E} = \{p_i; \rho_i\}_{i=1}^n$, where $\rho_i \in \mathcal{P}_+(\mathcal{H})$ and p_i's are the *apriori* probabilities. We denote the set of all ensembles with fixed size n as $\mathcal{F}_n$, and the set of all ensemble of any size as

$$\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n. \tag{6.1}$$

A PQ on the system is denoted by $\mathcal{I}$. PQ's focused on ensembles with fixed size is denoted as $\mathcal{I}^n(\mathcal{E}) = 0$ if $\mathcal{E} \notin \mathcal{F}_n$. For a dynamical maps Λ_t, a dynamical PQ is denoted by $\Phi_t^{\mathcal{I}^n}$.

Definition 1. A dynamical maps Λ_t is called image-nonincreasing if $\mathrm{Im}(\Lambda_t) \subset \mathrm{Im}(\Lambda_s)$ for any $t > s$, where $\mathrm{Im}(\Lambda)$ represents the image of a map $\Lambda \in \mathcal{T}(\mathcal{H}, \mathcal{H})$.

Note that, for invertible dynamical maps, $\mathrm{Im}(\Lambda_t) = \mathcal{L}(\mathcal{H})$ for all t. Hence, all invertible dynamical maps are image-nonincreasing. We call the map $\Pi_{\mathcal{M}} \in \mathcal{T}(\mathcal{H}_S, \mathcal{H}_S)$, a *projector* onto a subspace $\mathcal{M} \subset \mathcal{L}(\mathcal{H}_S)$ if $\mathrm{Im}(\Pi_{\mathcal{M}}) \subset \mathcal{M}$ and $\Pi_{\mathcal{M}}[X] = X$ for all $X \in \mathcal{M}$.

6.2 Extending the generalized formalism

In this section, we extend the GIB to image non-increasing dynamical maps by introducing a new system based PQ, $\mathcal{I}^n_{\mathcal{A}}$, of the form :

$$\mathcal{I}^n_{\mathcal{A}}(\mathcal{E}) = \begin{cases} ||\sum_{i=1}^n A_i \otimes \rho_i||_1 & \mathcal{E} \in \mathcal{F}_n \\ 0 & \mathcal{E} \notin \mathcal{F}_n \end{cases}, \tag{6.2}$$

where $\mathcal{E} = \{p_i; \rho_i\}_{i=1}^n$, $\mathcal{A} = \{A_i\}_{i=1}^n$ with $A_i \in \mathcal{L}(\mathcal{H})$, and $||A||_1 = \sqrt{A^\dagger A}$ is the trace-norm. Let us first recall a theorem on image non-increasing dynamical maps from [26].

Theorem 4. *([26]) If an image non-increasing dynamical map Λ_t obeys the condition*

$$\frac{d}{dt}||(\Lambda_t \otimes I)H||_1 \leq 0, \tag{6.3}$$

for any hermitian $H \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$, then Λ_t is CPD.

We now present the principal result of this chapter.

Theorem 5. *Any image non-increasing dynamical map Λ_t is CPD if and only if it is d^2-S Markovian.*

Proof. The *only if* part can be trivially shown by comparing the definitions of CPD and d^2-S Markovianity. For proving the *if* part, we choose a set of density matrices $\{\rho_i\}_{i=1}^{d^2}$ which forms a basis in $\mathcal{L}(\mathcal{H})$. Therefore, for any hermitian operator $H \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$, there exists a set $\mathcal{A} = \{A_i\}_{i=1}^{d^2} \subset \mathcal{L}(\mathcal{H})$ such that $H = \sum_{i=1}^n A_i \otimes \rho_i$. Using the set $\mathcal{A}$ we can construct a PQ of the form $\mathcal{I}^{d^2}_{\mathcal{A}}$, as given in Eq. (6.2). Therefore, using d^2-S Markovianity of Λ_t, we get $\frac{d}{dt}[\Phi_t^{\mathcal{I}^{d^2}_{\mathcal{A}}}(\mathcal{E})] = \frac{d}{dt}||(\Lambda_t \otimes I)[H]||_1 \leq 0$, where $\mathcal{E} = \{p_i; \rho_i\}_{i=1}^n$. Hence, using theorem 4, we conclude Λ_t is CPD. □

6.3 A complete formalism for qubit dynamics

We first show some examples of dynamical maps which are not image non-increasing.

Example 1. Consider the following qubit dynamics.

$$\Lambda_t[\rho] = (1 - q_0(t) - q_1(t))\,\rho + q_0(t)\,|0\rangle\langle 0| + q_1(t)\,|1\rangle\langle 1| \tag{6.4}$$

where $q_0(t), q_1(t) \geq 0$, $q_0(0) = q_1(0) = 0$ and $0 \leq q_0(t) + q_1(t) \leq 1$ for all $t \geq 0$. Let us choose the $q_i(t)$'s in the following form

$$q_0(t) = \theta(t)e^{-t} \;\;;\;\; q_1(t) = \theta(t)(1 - e^{-t}), \tag{6.5}$$

where

$$\theta(t) = \begin{cases} \sqrt{1 - (t-1)^2} & 0 \leq t \leq 1 \\ 1 & t > 1 \end{cases} \tag{6.6}$$

Note that $\theta(t)$ is a continuous and differentiable function of t. As a result, Λ_t is also continuous and differentiable. Using this form of $\theta(t)$, it can be easily seen that $\mathrm{Im}(\Lambda_t)$, for $t \geq 1$, is spanned by a single density matrix σ_t of the form

$$\sigma_t = e^{-t}\,|0\rangle\langle 0| + (1 - e^{-t})\,|1\rangle\langle 1| \;\;;\;\; t > 1. \tag{6.7}$$

Therefore it can be easily seen that $\mathrm{Im}(\Lambda_{t_1}) \cap \mathrm{Im}(\Lambda_{t_2})$ is an empty set for any $t_2 > t_1 > 1$. As a result Λ_t is not image non-increasing.

Example 2. Consider another qubit dynamics

$$\begin{aligned}\Lambda_t[\rho] = \Big(1 - \theta(t)\Big)\,\rho + \theta(t)\Big(&e^{-it\sigma_x}\,|0\rangle\langle 0|\,\rho\,|0\rangle\langle 0|\,e^{it\sigma_x} \\ &+ e^{-it\sigma_x}\,|1\rangle\langle 1|\,\rho\,|1\rangle\langle 1|\,e^{it\sigma_x}\Big),\end{aligned} \tag{6.8}$$

where $\theta(t)$ is as defined in Eq. (6.6). Note here that for any $t > 1$, $\mathrm{Im}(\Lambda_t)$ is spanned by density matrices $e^{-it\sigma_x} |0\rangle \langle 0| e^{it\sigma_x}$ and $e^{-it\sigma_x} |1\rangle \langle 1| e^{it\sigma_x}$. As a result, for $t > 1$ all density matrices in $\mathrm{Im}(\Lambda_t)$ has block vectors along the z-axis, rotated by an angle t about the x-axis. Hence it can be easily seen that the dynamics is not image non-increasing.

Thus we see, in order to take into account dynamics of the above kind, a complete description of IB is required which applies to even those dynamical maps which are not image non-increasing. We show below that our description of GIB is complete for qubits.

Consider the following results from [111],

Lemma 2. *([111]) Let $\mathcal{H} = \mathbb{C}^2$ be the Hilbert space of a qubit system and let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a subspace spanned of a set of density matrices. If $\mathcal{M}$ has dimension 3 then there exists no CPTP projector $\Pi_{\mathcal{M}}$ onto $\mathcal{M}$.*

Using this lemma and a result from [26], we show the following result.

Theorem 6. *If Λ_t represents a divisible dynamical map on a qubit system i e. $\mathcal{H} = \mathbb{C}^2$ then the dimension of Im(Λ_t) must not be 3 for any t.*

Proof. As Λ_t is divisible, it admits a decomposition $\Lambda_t = V_{t,s}\Lambda_s$ for all $t > s$. We prove this theorem by contradiction. Let us assume there exists a time t, such that $\mathrm{Im}(\Lambda_t)$ has dimension 3. It was shown in [26] that if t_1 is the smallest time for which Λ_{t_1} is non-invertible, that is, Λ_s is invertible for $0 \leq s < t_1$, then

$$\Pi_{t_1} = \lim_{\epsilon \to 0^+} V_{t_1, t_1 - \epsilon}. \tag{6.9}$$

is a projector onto $\mathrm{Im}(\Lambda_{t_1})$. Evidently $t_1 < t$. $\mathrm{Im}(\Lambda_t)$ is the image of $\mathrm{Im}(\Lambda_{t_1})$ under the action of a linear map V_{t,t_1}. As a result, dimension of $\mathrm{Im}(\Lambda_{t_1})$ must be greater or equal to that of $\mathrm{Im}(\Lambda_t)$. Using the fact that dimension of $\mathrm{Im}(\Lambda_{t_1})$ must be less than 4, we conclude that the dimension of $\mathrm{Im}(\Lambda_{t_1})$ is 3. Also note that if we choose a set of linearly independent states $\{\rho_1, \rho_2, \rho_3, \rho_4\}$, then the set $\{\rho_1(t_1), \rho_2(t_1), \rho_3(t_1), \rho_4(t_1)\}$ spans $\mathrm{Im}(\Lambda_{t_1})$, where

$\rho_i(t) = \Lambda_t[\rho_i]$. As a result Π_{t_1} is a CPTP projector on a 3-dimensional subspace spanned by density matrices. Using lemma 2, we conclude that this is impossible. □

Before proving the next theorem, let us recollect an old result proved by Alberti and Ulhmann [99], where it was shown that if $\{\sigma_1, \sigma_2\}$ and $\{\sigma_1', \sigma_2'\}$ are sets of input and output qubit states of a linear map Γ then there exists a CPTP map $T \in \mathcal{T}(\mathcal{H}, \mathcal{H})$ which matches the action of Γ on σ_1 and σ_2 i e. $T[\sigma_i] = \Gamma[\sigma_i] = \sigma_i'$ for $i = 1, 2$, if and only if

$$||\sigma_1 - \delta\sigma_2||_1 \geq ||\sigma_1' - \delta\sigma_2'||_1 \tag{6.10}$$

for all $\delta > 0$.

Theorem 7. *Any dynamical map Λ_t on a qubit system i e. $\mathcal{H} = \mathbb{C}^2$ is 4-S Markovian if and only if Λ_t is CPD.*

Proof. Proving the *if* part is trivial. For proving the *only if* part, we first choose

$$\{\rho_1, \rho_2, \rho_3, \rho_4\} \subset \mathcal{P}_+(\mathcal{H})$$

to be a set of four linearly independent density matrices. Therefore, any operator $X \in \mathcal{L}(\mathcal{H})$, can be written as $X = \sum_{i=1}^4 c_i\,\rho_i$, for a set of complex numbers $\{c_i\}$. Let us choose a PQ of the form $\mathcal{I}_{\mathcal{A}}^4$, as in Eq. (6.2), where $\mathcal{A} = \{\ A_1 = c_1 A,\ A_2 = c_2 A,\ A_3 = c_3 A, A_4 = c_4 A\}$ for some $A \in \mathcal{L}(\mathcal{H})$. Now, using 4-S Markovianity of Λ_t, we get $\frac{d}{dt}[\Phi_t^{\mathcal{I}_{\mathcal{A}}^4}(\mathcal{E})] = ||A||_1 \times \frac{d}{dt}[||\Lambda_t[X]||_1] \leq 0$ for any $X \in \mathcal{L}(\mathcal{H})$ and any $t \geq 0$, where $\mathcal{E} = \{p_i; \rho_i\}_{i=1}^4$. As a result, using proposition 2 of [26], we conclude that Λ_t is divisible i e. $\Lambda_t = V_{t,s} \circ \Lambda_s$ for any $t > s$. Thus, from Theorem 6 we conclude, the dimension of $\mathrm{Im}(\Lambda_s)$ must be 4, 2 or 1. We consider each case separately.

(a) If dimension of $\mathrm{Im}(\Lambda_s)$ is 4 for any s, the dynamics is invertible and hence image-nonincreasing. As a result, using Theorem 5 we conclude, the dynamics is CPD.

(b) If dimension of $\mathrm{Im}(\Lambda_s)$ is 2, then two of $\{\Lambda_s[\rho_1], \Lambda_s[\rho_2], \Lambda_s[\rho_3], \Lambda_s[\rho_4]\}$ must be linearly independent and also span the subspace $\text{Im}(\Lambda_s)$. Without loss of generality, we choose $\sigma_1 = \Lambda_s[\rho_1]$ and $\sigma_2 = \Lambda_s[\rho_2]$ to be independent. Note that, the action of $V_{t,s}$ is fixed only on the subspace $\text{Im}(\Lambda_s)$. Let us choose a PQ of the form $\mathcal{I}^2_{\mathcal{A}}$, where $\mathcal{A} = \{A_1 = A,\ A_2 = -\delta A\}$ for some $A \in \mathcal{L}(\mathcal{H})$ and $\delta \geq 0$. As Λ_t is 4−S Markovian, it must also be 2−S Markovian. As a result for $\mathcal{E} = \{p_1, \rho_1; p_2, \rho_2\}$, we get $\Phi_t^{\mathcal{I}^2_{\mathcal{A}}}(\mathcal{E}) = ||A||_1 \times ||V_{t,s}[\sigma_1] - \delta\ V_{t,s}[\sigma_2]||_1 \leq ||A||_1 \times ||\sigma_1 - \delta\sigma_2||_1$ for any $\delta \geq 0$. Now, using Eq. (6.10), we conclude that there exists a CPTP map $T \in \mathcal{T}(\mathcal{H}, \mathcal{H})$ such that $T[\sigma_i] = V_{t,s}[\sigma_i]$ for $i = 1, 2$. Note here that T is defined on the full space $\mathcal{L}(\mathcal{H})$. Thus we see there is a CPTP extension of $V_{t,s}$ defined on the full space $\mathcal{L}(\mathcal{H})$. Hence, we conclude that the dynamics is CPD.

(c) If dimension of $\text{Im}(\Lambda_s)$ is 1, then there must exist a $\sigma \in \mathcal{P}_+(\mathcal{H})$ such that $\Lambda_s[\rho] = \sigma$ for all $\rho \in \mathcal{P}_+(\mathcal{H})$. In this case, the action of $V_{t,s}$ is specified only on the 1 dimensional subspace spanned by σ. Let us choose the CPTP projector $\Pi_\sigma \in \mathcal{T}(\mathcal{H}, \mathcal{H})$ of the form $\Pi_\sigma[X] = Tr[X]\sigma$. It can be easily seen that $V_{t,s}\Pi_\sigma$ serves as a CPTP extension of $V_{t,s}$ on the full space. Hence, we conclude the dynamics is CPD. □

6.3.1 Degree of non-Markovianity

From theorem 5 and 7, we conclude that only system based quantifiers are sufficient to establish CPD of image non-increasing and qubit dynamical maps. Moreover recall from chapter 5, that if a dynamical map is CPD it must also be S-Markovian and SA-Markovian. Hence, we conclude that d^2-S Markovianity (4-S Markovianity in the case of qubits), S Markovianity, SA Markovianity and CPD are all equivalent for image non-increasing (qubit) dynamical maps.

As a result the hierarchical structure of Markovianity classes, proposed in chapter 5, can be significantly simplified to the form given in Fig. 6.1. Note that this simplification readily suggests existence of a *degree of non-Markovianity* (DONM) for image non-ncreasing and

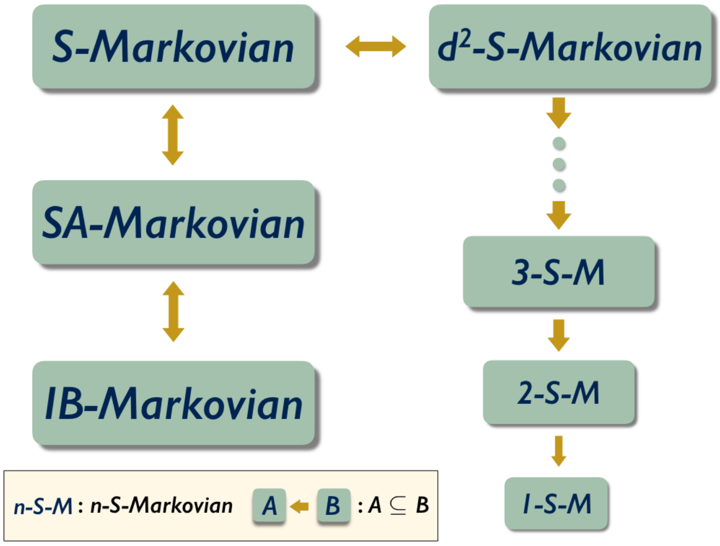

Figure 6.1: Hierarchy of Markovianity classes of qubit ($d = 2$) and image-nonincreasing dynamical maps.

qubit dynamical maps :

$$\mathrm{DONM}(\Lambda_t) := d^2 - \max\{n \mid \Lambda_t \text{ is } n\text{-S Markovian}\}. \tag{6.11}$$

Note that any dynamical map can always be shown to be 1-S Markovian[1]. As a result, DONM of any image non-increasing and qubit ($d = 2$) dynamics can take a minimum value of zero, when it is equivalent to CPD and a maximum value of $d^2 - 1$, when it is maximally non-Markovian. Thus, higher the value of DONM, stronger is the effect of non-Markovianity.

6.4 Applications of the formalism

In this section, using numerical simulations, we estimate the minimum value of DONM of a number of commonly used dynamics.

[1]For any PQ of the form $\mathcal{I}^1$, if $\rho_t = \Lambda_t[\rho]$ is the time evolved state of ρ after time t, consider the CPTP map T_{ρ_t} to be $T_{\rho_t}[X] = Tr[X]\rho_t$. Therefore, for any $t > s$, $\Phi_t^{\mathcal{I}^1}\{\rho\} = \mathcal{I}^1\{\rho_t\} = \mathcal{I}^1\{T[\rho_s]\} \leq \mathcal{I}^1\{\rho(s)\} = \Phi_s^{\mathcal{I}^1}\{\rho\}$. Hence Λ_t is 1-S Markovian.

6.4.1 Eternal non-Markovianity in qubits

Consider the qubit dynamics known as *eternal non-Markovian* [112] dynamics, given by

$$\Lambda_t[\rho] = \sum_{i=0}^{3} q_i(t)\sigma_i\rho\sigma_i, \tag{6.12}$$

where $q_0(t) = \frac{1+e^{-\gamma t}}{2}$, $q_1(t) = q_2(t) = \frac{1-e^{-\gamma t}}{4}$, $q_3(t) = 0$, and σ_i's are Pauli matrices with $\sigma_0 = \mathbb{1}$. Consider a PQ of the form $\mathcal{I}^4_{\mathcal{A}}$, as given in Eq. (6.2), with $\mathcal{A} = \{A_i\}_{i=0}^{3}$ to be of the following form

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \;\; ; \;\; A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{6.13}$$

$$A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;\; ; \;\; A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.14}$$

We then consider an ensemble

$$\mathcal{E} = \{\frac{1}{4}; \rho_0; \frac{1}{4}; \rho_1; \frac{1}{4}; \rho_2; \frac{1}{4}; \rho_3\}, \tag{6.15}$$

where $\rho_i = \frac{1}{2}(\mathbb{1} + x_i\sigma_x + y_i\sigma_y + z_i\sigma_z)$ and $x_i = r_i \sin\theta_i \cos\phi_i$, $y_i = r_i \sin\theta_i \sin\phi_i$ and $z_i = r_i \cos\theta_i$. Here we consider

$$r_0 = 0.8\,,\ \theta_0 = 0.9\pi\,,\ \phi_0 = 3.1\pi; \tag{6.16}$$

$$r_1 = 0.5\,,\ \theta_1 = 0.2\pi\,,\ \phi_1 = 1.8\pi; \tag{6.17}$$

$$r_2 = 0.3\,,\ \theta_2 = 1.9\pi\,,\ \phi_2 = 0.3\pi; \tag{6.18}$$

$$r_3 = 0.9\,,\ \theta_3 = 1.3\pi\,,\ \phi_3 = 2.1\pi; \tag{6.19}$$

We now plot the dynamic PQ $\Phi_t^{\mathcal{I}^4_{\mathcal{A}}}(\mathcal{E})$ as a function of time, as given in Fig 6.2(d), and find that the dynamics fails to be 4-S Markovian. As a result, we conclude DONM(Λ_t)

is at least 1. Also note that, as Λ_t is invertible and P-divisible [113], it must also be 2-S Markovian [32]. Thus DONM of Λ_t is at most 2.

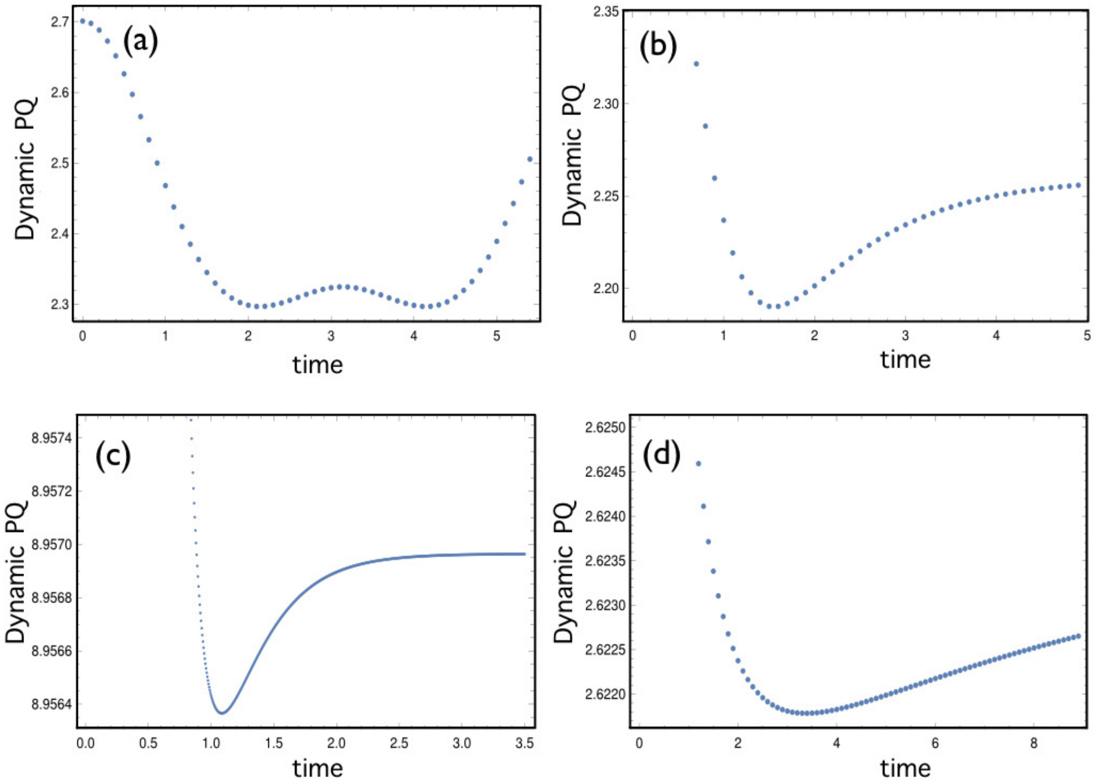

Figure 6.2: Violation of: (a) 2-S Markovianity class, as discussed in subsection 6.4.3(b), (b) 2-S Markovianity class, as discussed in subsection 6.4.3(a), (c) 5-S Markovianity class, as discussed in subsection 6.4.2, (d) 4-S Markovianity class, as discussed in subsection 6.4.1.

6.4.2 Non-Markovianity of Weyl channels

We consider the generalization of Pauli channels to $d-$dimensional space [113], given by

$$\Lambda_t[\rho] = e^{-dt}\rho + (1 - e^{-dt})\bar{\Phi}[\rho], \tag{6.20}$$

where,

$$\bar{\Phi} = \frac{1}{d}(\Phi_1 + \cdots + \Phi_d), \tag{6.21}$$

where we have,

$$\Phi_\alpha = \sum_{l=0}^{d-1} P_l^{(\alpha)} \rho P_l^{(\alpha)} \quad ; \quad \alpha = 1, \ldots, d. \tag{6.22}$$

Here, $P_l^{(\alpha)} = |\psi_l^{(\alpha)}\rangle\langle\psi_l^{(\alpha)}|$, where $\{|\psi_0^{(\alpha)}\rangle, \ldots, |\psi_{d-1}^{(\alpha)}\rangle\}$ for $\alpha = 1, \ldots, d$ along with the computational basis $\{|0\rangle, \ldots, |d-1\rangle\}$ forms mutually unbiased bases (MUB) in $d-$ dimension. The set of MUB's other than the computational basis for a prime dimension p is given by

$$\langle k|\psi_l^\alpha\rangle = \frac{1}{\sqrt{d}} e^{(2\pi i/p)(\alpha k^2 + lk)}, \tag{6.23}$$

for $\alpha = 1, \ldots, p$ and $k, l = 0, \ldots, p-1$ [114, 115]. Here we consider $d = 3$. Now consider a PQ of the form $\mathcal{I}_\mathcal{A}^5$ where $\mathcal{A} = \{A_i\}_{i=0}^4$ is given by

$$A_0 = \begin{pmatrix} 0.197607 & 0.636578 & 0.360871 \\ 0.501448 & 0.103937 & 0.429824 \\ 0.796486 & 0.149248 & 0.347865 \end{pmatrix} ; A_1 = \begin{pmatrix} 0.650907 & 0.287493 & 0.564978 \\ 0.71768 & 0.698053 & 0.355613 \\ 0.811352 & 0.828574 & 0.465072 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0.102362 & 0.487076 & 0.215023 \\ 0.159078 & 0.0562856 & 0.951697 \\ 0.603284 & 0.382475 & 0.272579 \end{pmatrix} ; A_3 = \begin{pmatrix} 0.777797 & 0.503384 & 0.093342 \\ 0.006149 & 0.27209 & 0.756466 \\ 0.958651 & 0.665264 & 0.285476 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 0.736385 & 0.0201729 & 0.946703 \\ 0.315758 & 0.624188 & 0.724125 \\ 0.372077 & 0.593007 & 0.113422 \end{pmatrix}.$$

Consider the ensemble,

$$\mathcal{E} = \{\frac{1}{5}; \rho_0; \frac{1}{5}; \rho_1; \frac{1}{5}; \rho_2; \frac{1}{5}; \rho_3; \frac{1}{5}; \rho_4\}, \tag{6.24}$$

where,

$$\rho_0 = \begin{pmatrix} 0.539296 & 0.048893 & -0.152769 \\ 0.048893 & 0.126133 & 0.072201 \\ -0.152769 & 0.072201 & 0.334571 \end{pmatrix} ; \ \rho_1 = \begin{pmatrix} 0.336771 & -0.186573 & -0.249316 \\ -0.186573 & 0.253417 & 0.024165 \\ -0.249316 & 0.024165 & 0.409812 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 0.365003 & 0.044078 & -0.0102 \\ 0.044078 & 0.305657 & 0.063467 \\ -0.0102 & 0.063467 & 0.32934 \end{pmatrix} ; \ \rho_3 = \begin{pmatrix} 0.218123 & -0.111708 & -0.120676 \\ -0.111708 & 0.323393 & -0.092277 \\ -0.120676 & -0.092277 & 0.458484 \end{pmatrix}$$

$$\rho_4 = \begin{pmatrix} 0.153128 & 0.185705 & 0.025751 \\ 0.185705 & 0.338135 & -0.103136 \\ 0.025751 & -0.103136 & 0.508737 \end{pmatrix} .$$

We now plot the dynamic PQ of $\Phi^{\mathcal{I}^5\mathcal{A}}$ as a function of time in Fig 6.2(c), and conclude that the dynamics fails to be 5-S Markovian. As a result, DONM(Λ_t) must be at least 3.

6.4.3 Some other qubit dynamics

Consider once again the dynamics given in Eq. (6.12). In this case, we choose

$$q_i(t) = \alpha_i[1 - q_0(t)] \quad ; \quad i = 1, 2, 3, \tag{6.25}$$

where $\alpha_i \geq 1$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Note that, we must have $p_0(0) = 1$. We consider two choices of dynamics with the following forms: (a) $q_0(t) = 1/(1 + t)$, (b) $q_0(t) = (1 + \cos t)/2$.

(a) In the first case, we consider a PQ of the form $\mathcal{I}^2_{\mathcal{A}}$ with $\mathcal{A} = \{A_0, A_1\}$, where

$$A_0 = \begin{pmatrix} 0.676533 & 0.143323 \\ 0.530258 & 0.940163 \end{pmatrix},$$
$$A_1 = \begin{pmatrix} 0.420843 & 0.281129 \\ 0.933353 & 0.368185 \end{pmatrix}.$$

We consider an ensemble,

$$\mathcal{E} = \{\frac{1}{2}; \rho_0; \frac{1}{2}; \rho_1\}, \tag{6.26}$$

where, $\rho_i = \frac{1}{2}(\mathbb{1} + x_i\sigma_x + y_i\sigma_y + z_i\sigma_z)$ and $x_i = r_i \sin\theta_i \cos\phi_i$, $y_i = r_i \sin\theta_i \sin\phi_i$ and $z_i = r_i \cos\theta_i$, and we consider

$$r_0 = 1\,,\ \theta_0 = 0,\ \phi_0 = 0; \tag{6.27}$$
$$r_1 = 1\,,\ \theta_1 = 3.5\pi\,,\ \phi_1 = 0; \tag{6.28}$$
$$\tag{6.29}$$

On plotting the dynamic PQ $\Phi_t^{\mathcal{I}^2_{\mathcal{A}}}$ with time, as given in Fig. 6.2(b), we find the dynamics is not 2-S Markovian.

(b) For this case, we choose the same ensemble as in Eq. (6.26), but the form of PQ is $\mathcal{I}^2_{\mathcal{A}}$ with $\mathcal{A} = \{A_0, A_1\}$, where

$$A_0 = \begin{pmatrix} 0.38592 & 0.922887 \\ 0.0302912 & 0.574226 \end{pmatrix},$$
$$A_1 = \begin{pmatrix} 0.590118 & 0.271617 \\ 0.998843 & 0.728935 \end{pmatrix}.$$

The plot of dynamic PQ $\Phi_t^{\mathcal{I}^2_{\mathcal{A}}}$ versus time in Fig 6.2(a) shows that, for this case also the dynamics fails to be 2-S Markovian.

Hence in both cases (a) and (b), DONM(Λ_t) is 3, which is also the maximum value DONM can take for qubit dynamics.

6.5 Chapter summary

In this chapter, we extended the GIB to image non-increasing dynamical maps. We also showed that for qubit dynamical maps, GIB provides a complete description of Markovianity. By introducing a new system based PQ, we showed that a description of DONM could be given for image non-increasing and qubit dynamical maps, which would take a minimum value of zero and a maximum value of $d^2 - 1$ ($d = 2$ for qubits). We showed that higher the value of DONM, stronger is the effect of memory in the dynamics. We also estimated the DONM of a number of commonly used dynamics. We expect this formalism will shed more light into the structure of dynamical maps and DONM will serve useful in characterizing strength of non-Markovianity of dynamical maps.

Chapter 7

Summary and future directions

In this thesis, we studied phenomenological and foundational aspects on non-Markovianity. We first chose a model of spin systems which closely resembles the spin bath model for low temperatures. In this model, a system qubit interacts with a sea of environment qubits, where the coupling between the system and individual environment qubit is determined by a coupling function. The coupling function can depend on time as well as site (location of individual environment qubit). We studied the model for different forms of coupling function. For a number of forms, exact solution of the system dynamics could be analytically found for arbitrary number of environment qubits. We also studied the non-Markovian nature of the dynamics for different coupling forms. For a specific time and site dependent form of coupling, we found a transition from non-Markovian to Markovian regime. We studied the critical values of coupling strength which showed the transition.

As a future direction, we hope this approach can be extended to incorporate other forms of physically realizable couplings. This would help predicting behavior of the spin-bath model in different physical scenarios. As for example, some of the spin-bath models can be realized, in principle, with the present day technology. For those models, our study will provide directions towards studying the non-Markovian to Markovian transition of the system dynamics. Our study could also be extended to environments consisting of a collection of harmonic oscillators. We also expect our approach of simplifying the form of Hamiltonians to fit low-temperature scenarios can be applied to other models of many body interactions.

We also provided a generalized framework for Information backflow for assessing Markovianity in quantum regime. Our framework acts as an umbrella structure for a large number, if not all, of Information backflow approaches proposed earlier in literature. We showed that this framework is equivalent to CP-divisibility for invertible dynamical maps, which we latter extended to qubit dynamical maps and image non-increasing dynamical maps. As a result, for qubit dynamical maps, our framework provides a complete formalism, taking into account both CP-divisibility and Information backflow. As an outcome of this approach, we found a hierarchy of Markovianity classes that provides a degree of non-Markovianity which captures the strength of non-Markovianity present in the dynamics. A number of extensions of this work seems plausible:

- If the equivalence between the generalized formalism and CP-divisibility can be extended to any non-invertible dynamical maps of higher dimension?
- If a necessary and sufficient criteria for checking if a dynamical maps falls in a particular Markovianity class could be found?
- If master equations for each Markovianity class could be found? This would greatly help understand and apply non-Markovianity as a resource in different information theoretic protocols.

We expect our work would help figuring out better techniques of using Markovianity as a resource in quantum technology. In particular, we expect our study will be useful in developing efficient heat engine/refrigerator, efficient control of decoherence, etc.

Appendices

Appendix A

Calculation of Kraus operators

The initial state of the environment is given by

$$\rho_e(0) = \Big[p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|\Big]^{\otimes N} = \sum_{n=0}^{N-1} p^{N-s_n}(1-p)^{s_n}\,|n_2\rangle\langle n_2|\,, \tag{A.1}$$

where n_2 is the binary equivalent of decimal number n and s_n is the number of 1's in n_2. Therefore, the time evolved system state $\rho_s(t)$ is given by

$$\rho_s(t) = Tr_E\Big(U(t,0)\;\rho_s(0)\otimes\rho_e(0)\;U(t,0)^\dagger\Big) = \sum_{m,n=1}^{N} K_{mn}\rho_s(0)K_{mn}^\dagger, \tag{A.2}$$

where

$$\begin{aligned} K_{mn} &= \sqrt{p^{N-s_n}(1-p)^{s_n}}\,\langle m_2|U(t,0)|n_2\rangle\,, \\ &= \sqrt{p^{N-s_n}(1-p)^{s_n}}\Big\{(e^{-i\omega_0 t}-1)\,\langle m_2|\chi_+\rangle\,\langle\chi_+|n_2\rangle \\ &+ (e^{i\omega_0 t}-1)\,\langle m_2|\chi_-\rangle\,\langle\chi_-|n_2\rangle + \mathbb{1}\delta_{mn}\Big\}, \end{aligned}$$

where we have used Eq. (4.17). Also note that in the above equation, $U(t,0)$ and $|\chi_\pm\rangle$ are respectively system-environment operators and states, whereas $|m_2\rangle$ and $|n_2\rangle$ are environment states. Moreover, the identity operator $\mathbb{1}$ is in system Hilbert space.

Using Eq. (4.11), we have

$$\langle m_2|\chi_\pm\rangle = \frac{1}{\sqrt{2}}\Big(\delta_{m0}\,|1\rangle \pm |0\rangle\,\langle m_2|\beta_0\rangle\Big). \tag{A.3}$$

Using Eq.(4.12) and $\mathcal{E}(t) = \hbar\omega_0$, we get

$$\langle m_2|\beta_0\rangle = \frac{1}{\omega_0} g_{N-\log_2 m}(t), \tag{A.4}$$

In the above equation, $g_{N-\log_2 m}(t)$ is to be interpreted as a function $g_x(t)$ given by

$$g_x(t) = \begin{cases} g_x(t) & x \in \{0, .., N-1\} \\ 0 & x \notin \{0, .., N-1\} \end{cases} \tag{A.5}$$

Note that although in this case the coupling $g_n(t)$ is a constant, we use the time dependent form in the above equation, as it will be later useful in deriving the Kraus operators in other sections of chapter 4. Thus we have

$$\begin{aligned} K_{mn} = {} & \sqrt{p^{N-s_n}(1-p)^{s_n}}\Bigg\{\frac{e^{-i\omega_0 t}-1}{2}\Big(\delta_{m0}\,|1\rangle + \frac{1}{\omega_0} g_{N-\log_2 m}(t)\,|0\rangle\Big)\Big(\delta_{n0}\,\langle 1| \\ & + \frac{1}{\omega_0} g^*_{N-\log_2 n}(t)\,\langle 0|\Big) + \frac{e^{i\omega_0 t}-1}{2}\Big(\delta_{m0}\,|1\rangle - \frac{1}{\omega_0} g_{N-\log_2 m}(t)\,|0\rangle\Big)\Big(\delta_{n0}\,\langle 1| \\ & - \frac{1}{\omega_0} g^*_{N-\log_2 n}(t)\,\langle 0|\Big) + \delta_{mn}\mathbb{1}\Bigg\} \\ = {} & \frac{\cos\omega_0 t - 1}{\omega_0^2}\Big[g_{N-\log_2 m}(t) g^*_{N-\log_2 n}(t) + \delta_{mn}\Big]\,|0\rangle\langle 0| \\ & - i\sin\omega_0 t\frac{g_{N-\log_2 m}(t)}{\omega_0}\delta_{n0}\,|0\rangle\langle 1| - i\sin\omega_0 t\frac{g^*_{N-\log_2 n}(t)}{\omega_0}\delta_{m0}\,|1\rangle\langle 0| \\ & + \Big[(\cos\omega_0 t - 1)\delta_{0m}\delta_{0n} + \delta_{mn}\Big]\,|1\rangle\langle 1| \end{aligned}$$

Appendix B

Details about the Generalized formalism

B.1 Physicality quantifiers considered so far in Literature

Most of the quantifiers, suggested in the literature, are defined on ensembles having fixed number of elements. To fit them as valid physicality quantifiers, which are defined on ensembles of any size, we define special forms of physicality quantifiers $\mathcal{I}_S^n$ or $\mathcal{I}_{SA}^n$, which are focused on ensembles of size n. We define, $\mathcal{I}_S^n(\mathcal{E}_S) = 0$ and $\mathcal{I}_{SA}^n(\mathcal{E}_{SA}) = 0$, for any $\mathcal{E}_S \notin \mathcal{F}_S^n$ and $\mathcal{E}_{SA} \notin \mathcal{F}_{SA}^n$. We now show that a large number of quantifiers considered so far, correspond to physicality quantifiers of the form $\mathcal{I}_S^n$ or $\mathcal{I}_{SA}^n$ (for various values of n) in such a way, that the physicality quantifier takes the same value as the quantifier, for ensembles of size n. We denote $\mathbb{Z}_k$ to be the set of positive integers from 1 to k and $[a, b]$ to be the closed interval of real numbers from a to b.

(i) Breuer et. al. [20] considered an equal mixture of states i.e. $p_1 = p_2 = 1/2$ and defined *distinguishability* of two states as their quantifier. We define the physicality quantifier as,

$$\mathcal{I}_S^{BLP}\{p_1, p_2, \rho_1, \rho_2\} = ||\rho_1 - \rho_2||_1; \; p_1 = p_2 = 1/2$$
$$= 0 \qquad ; \; p_i \neq 1/2, \tag{B.1}$$

where $||A||_1 = Tr\sqrt{A^\dagger A}$. The Markovianity criteria, corresponding to this quantifier, is popularly known as the *BLP*-criteria of Markovianity. It can be easily shown that the above quantity is bounded and non-increasing under CPTP maps [97]. Hence, we see $\mathcal{I}_S^{BLP}$ is a particular form of $\mathcal{I}_S^2$.

(ii) Rajagopal et. al. [85] also considered a form that corresponds to $\mathcal{I}_S^2$. They took equal mixture of two states and used *fidelity* as their measure of non-Markovianity i.e $p_1 = p_2 = 1/2$. We slightly modify their definition and define the physicality quantifier in the following form,

$$\mathcal{I}_S^{Fid}\{p_1, p_2, \rho_1, \rho_2\} = 1 - ||\sqrt{\rho_1}\sqrt{\rho_2}||_1; \; p_1 = p_2 = 1/2$$
$$= 0 \qquad ; \; p_i \neq 1/2, \tag{B.2}$$

It is easy to show that $\mathcal{I}_S^{Fid}$ lies in the interval [0, 1] and non-increasing under CPTP maps [85, 97].

(iii) Chen et. al. used *temporal steering weight* (TSW) [86] to quantify Markovianity. In this setting Alice performs measurement $M_{a|x}$, ($a \in \mathbb{Z}_{m_1}$, $x \in \mathbb{Z}_{m_2}$) on a system state , creating an ensemble $\{p(a|x), \rho_{a|x}\}_{a|x}$. The ensemble is then passed through a dynamical map Λ_t and the TSW of the output ensemble is calculated at each instant t. It was shown in [86] that TSW is non-increasing under CPTP maps. Also, TSW refers to maximum value of μ in Eq. (4) of [86]. Therefore, it is evident from the construction that $0 \leq \mu \leq 1$. Thus we see, TSW corresponds to a physicality quantifier of the form $\mathcal{I}_S^m$, where $m = m_1 m_2$.

(iv) Dhar et. al. [87] used *interferometric power* for their criteria of Markovianity. It can be shown to be non-increasing under CPTP maps [87]. Also, for any system-ancilla state, interferometric power is calculated by optimizing over local unitary operations on the ancilla side [116]. Therefore, it can be inferred that interferometric power is bounded. Hence, interferometric power is a valid physicality quantifier and is of the form $\mathcal{I}_{SA}^{1}$.

(v) He et. al. introduced a measure of non-Markovianity based on *local quantum uncertainty* (LQU) [88]. It can be shown that LQU is non-increasing under CPTP maps. Also note, as LQU is determined though minimization of a bounded function over unitaries, it is evident that LQU is bounded [117]. Hence we conclude, LQU corresponds to a physicality quantifier of the form $\mathcal{I}_{SA}^{1}$.

(vi) Luo et. al. [30] used *quantum mutual information* (QMI) as their quantifier. The corresponding physicality quantifier is

$$\mathcal{I}_{SA}^{QMI}(\xi) = S(\xi_S) + S(\xi_A) - S(\xi), \tag{B.3}$$

where $\xi \in \mathcal{P}_+(\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{A})$, $\xi_{S/A} = Tr_{A/S}(\xi)$ are reduced density matrices and $S(\rho) = -\rho \log \rho$ is the usual von Neumann entropy. QMI is known to be bounded [118] and non-increasing under CPTP maps [119]. Note that $\mathcal{I}_{SA}^{QMI}$ is a special form of $\mathcal{I}_{SA}^{1}$.

B.2 Detailed proof of Theorem 1

Proof. Assume Λ_t is n-SA-Markovian. For any system based physicality quantifier $\mathcal{I}_S^n$, we define a real valued function $\mathcal{I}_{SA}^n$ on $\mathcal{F}_{SA}$, such that $\mathcal{I}_{SA}^n(\mathcal{E}_{SA}) = \mathcal{I}_S^n(\mathcal{E}_S)$, where $\mathcal{E}_S = \{p_i;\rho_i\}$, $\mathcal{E}_{SA} = \{p_i;\xi_i\}$ and $Tr_A(\xi_i) = \rho_i$. This implies $\mathcal{I}_{SA}^n(\mathcal{E}_{SA}) = 0$ for $\mathcal{E}_{SA} \notin \mathcal{F}_{SA}^n$ and for any CPTP map $T \in \mathcal{T}(\mathcal{H}_S, \mathcal{H}_S)$, we get $\mathcal{I}_{SA}^n\{p_i;(T\otimes I)[\xi_i]\} = \mathcal{I}_S^n\{p_i;T[\rho_i]\} \leq \mathcal{I}_S^n\{p_i;\rho_i\} = \mathcal{I}_{SA}^n\{p_i;\xi_i\}$. Also note, as $\mathcal{I}_S^n$ is bounded, $\mathcal{I}_{SA}^n$ must also be bounded. Therefore, we see for any physicality quantifier $\mathcal{I}_S^n$ on the system, there exists a physicality quantifier on system-ancilla, which is of the form $\mathcal{I}_{SA}^n$. These imply $\Phi_t^{\mathcal{I}_{SA}^n}\{p_i;\xi_i\} = \Phi_t^{\mathcal{I}_S^n}\{p_i;\rho_i\}$ (see Eqs. (5.4) and (5.5)). Thus, monotonic decrease of $\Phi_t^{\mathcal{I}_{SA}^n}$ implies monotonic decrease of $\Phi_t^{\mathcal{I}_S^n}$. Thus, Λ_t is n-S-Markovian. □

B.3 Detailed proof of Theorem 2

Proof. Note as Λ_t is invertible, it is also divisible i.e. it can be decomposed in the form of Eq. (3.6). Moreover, the intermediate evolutions $V_{t,s}$ must also be trace preserving for any $t > s$.

$(i) \implies (ii)$. Consider any hermitian operator $H \in \mathcal{L}(\mathcal{H_S} \otimes \mathcal{H_A})$. As Λ_s is an invertible and positive map for any $s > 0$, there exists a hermitian operator $\tilde{H}$ such that $(\Lambda_s \otimes I)[\tilde{H}] = H$. Also from [84], we know that any hermitian operator can be written as a positive number multiple of a Helstrom matrix i.e. $\tilde{H} = \lambda(p_1\xi_1 - p_2\xi_2)$ for $\lambda > 0$, $\xi_1, \xi_2 \in \mathcal{P}_+(\mathcal{H_S} \otimes \mathcal{H_A})$ and probabilities p_1, p_2. As Λ_t is GTDE-Markovianity, for any $t > s$, we get $||(V_{t,s} \otimes I)[H]||_1 = ||(\Lambda_t \otimes I)[\tilde{H}]||_1 \leq ||(\Lambda_s \otimes I)[\tilde{H}]||_1 = ||H||_1$ for any hermitian H. Since $V_{t,s}$ is trace preserving, this implies $V_{t,s} \otimes I$ must be a positive map for any $t > s$ [98]. Therefore, $V_{t,s}$ is CP for any $t > s$. Thus, Λ_t is CPD.

$(ii) \implies (iii)$. As Λ_t is CPD, $V_{t,s}$ is CPTP. Hence from condition 1 and Eq. (5.5), we get $\Phi_t^{\mathcal{I}_{SA}} = \mathcal{I}_{SA}\{p_i; (V_{t,s} \otimes I)(\Lambda_s \otimes I)[\xi_i]\} \leq \Phi_s^{\mathcal{I}_{SA}}$ for any form of $\mathcal{I}_{SA}$ and any ensemble $\mathcal{E}_{SA} = \{p_i; \xi_i\}$. Therefore, Λ_t is SA-Markovian. Hence, from theorem 1, Λ_t is IB-Markovian.

$(iii) \implies (iv)$. This follows from the definition of SA-Markovianity.

$(iv) \implies (i)$. This is trivial as $\mathcal{I}_{SA}^{GTDE}$ in Eq. (5.7), given is of the form $\mathcal{I}_{SA}^2$. □

B.4 Detailed proof of Lemma 1

Proof. Note, as Λ_t is GTD-Markovian it is divisibile i.e. it can be expressed as Eq. (3.6) (see Proposition 2 of [26]). First Alberti et. al. [99] and latter Huang et. al. [101] showed that the necessary and sufficient condition for a collection of qubit states $\rho_1, \rho_2, \rho'_1, \rho'_2 \in \mathcal{P}_+(\mathcal{H}_\mathcal{S})$ to have a CPTP map T_{12} connecting them i.e. $T_{12}[\rho_i] = \rho'_i$; $i = 1, 2$, is $||\rho'_1 - x\rho'_2||_1 \leq ||\rho_1 - x\rho_2||_1$ for any $x \geq 0$. Since, p_1 and p_2 in the GTD quantifier in Eq. (5.6) are probabilities, without loss of generality we can choose $p_1 > 0$. For any $t > s$, let us now choose $\rho'_i = V_{t,s}[\rho_i]$; $i = 1, 2$, and $x = p_2/p_1$. Therefore, if Λ_t is GTD-Markovian, the necessary and sufficient condition for the existence of T_{12} connecting ρ_1, ρ_2, ρ'_1 and ρ'_2 is satisfied for any $t > s$. Hence, we conclude there must exist a CPTP map T_{12} satisfying Eq. (5.8). □

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