MULTIPLIERS OF SEGAL ALGEBRAS

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Kasturi Nagarajan.

(Kasturi Nagarajan)
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If \( G \) is a locally compact abelian group, then there exists a nonnegative regular measure \( \mu \) on \( G \), the so called Haar measure of \( G \) which is not identically zero and which is translation invariant. That is to say

\[
\mu(B + x) = \mu(B)
\]

for every \( x \in G \) and every Borel set \( B \) in \( G \). If \( \mu \) and \( \mu' \) are two Haar measures on \( G \), then \( \mu = \lambda \mu' \) where \( \lambda \) measure is a positive constant. If \( G \) is compact it is customary to normalize \( \mu \) so that \( \mu(G) = 1 \). If \( G \) is discrete any set consisting of a single point is assigned the measure 1.

Let \( G \) be a locally compact abelian group with character group \( \Gamma \). Let \( dx \) and \( d\gamma \) denote the elements of the normalized Haar measures on \( G \) and \( \Gamma \) respectively. For \( 1 \leq p < \infty \), \( L^p(G) \) is the Lebesgue space of equivalence classes of complex valued functions \( f \) on \( G \) such that

\[
\| f \|_p = \left[ \int_G |f(x)|^p \, dx \right]^{1/p} < \infty.
\]

When \( p = \infty \), \( \| f \|_\infty \) denotes the essential supremum of \( |f| \).

If \( f, g \in L^1(G) \), the convolution is given by

\[
f \ast g(t) = \int_G f(t-x) g(x) \, dx, \quad t \in G.
\]

\( M_b(G) \) is the Banach space of all bounded regular complex-valued measures \( \mu \) on \( G \) normed by
A linear translation invariant operator \( \tau_y \) is defined on a space of functions \( X \) on \( G \) by the formula

\[
\tau_y f(x) = f(x+y), \quad x \in G, \quad f \in X.
\]

A space \( A \) of functions on \( G \) is defined to be translation-invariant if \( f \in A \) implies \( \tau_y f \in A \) for every \( y \in G \). Let \( A \) and \( B \) be two translation invariant spaces of functions on \( G \). A linear operator \( T : A \rightarrow B \) is translation invariant if it commutes with translations, that is,

\[
T \tau_x = \tau_x T \quad \text{for all } x \in G.
\]

A bounded linear translation invariant operator from \( A \rightarrow B \) is termed a multiplier from \( A \) to \( B \). In the case of \( L^1(G) \rightarrow L^1(G) \) there are various equivalent definitions. A multiplier on \( L^1(G) \) is either a continuous linear operator \( T \) which commutes with translation operators or which commutes with convolutions. Notice that in some spaces translation operators may be defined even though convolutions are not. Another definition is the following: A function \( \phi \) defined on the character group \( \Gamma \) is called a multiplier for \( L^1(G) \) if \( \hat{\phi} \hat{f} \in [L^1(G)]^\wedge \) whenever \( f \in L^1(G) \), where \( \wedge \) denotes the Fourier transform.
A linear subspace $S(G)$ of $L^2(G)$ is called a Segal Algebra if the following four conditions are satisfied:

(a) $S(G)$ is dense in $L^2(G)$
(b) $S(G)$ is a Banach space under some norm $\| \cdot \|_g$ and
\[
\| f \|_g \geq \| f \|_1, \quad f \in S(G)
\]
(c) Let $y \in G$. Then $\tau_y f \in S(G)$ for every $f \in S(G)$ and the mapping $y \mapsto \tau_y f$ is continuous from $G$ into $S(G)$.
(d) $\| \tau_y f \|_g = \| f \|_g$ for all $f \in S(G)$ and all $y \in G$.

Various properties of Segal algebras are collected below in the form of lemmas.

**Lemma 0.1.** For every $f \in S(G)$ and arbitrary $h \in L^2(G)$, the vector-valued integral $\int h(y) \, \tau_y f \, dy$ exists as an element of $S(G)$ and
\[
\int h(y) \, \tau_y f \, dy = h \ast f.
\]

Moreover
\[
\| h \ast f \|_g \leq \| h \|_1 \| f \|_g.
\]

It follows immediately that if $h \in S(G)$, then
\[
\| h \ast f \|_g \leq \| h \|_1 \| f \|_g \leq \| h \|_g \| f \|_g,
\]
which shows that $S(G)$ is actually a Banach algebra and it is an ideal in $L^2(G)$.

**Lemma 0.2.** Let $\mu$ be a bounded complex-valued measure on $G$.
Then for any $f \in S(G)$ the vector-valued integral $\int \tau_y f \, du(y)$
exists as an element of $s(0)$ and

$$\int_{\mathbb{R}} e^{-iy} d\nu(y) = \mu \ast f$$

Further

$$\| \mu \ast f \|_g \leq \| \mu \|_\infty \| f \|_1.$$ 

Thus $s(0)$ is an ideal in $L^1_{\text{loc}}(\mathbb{R}).$

**Lemma 0.3.** $s(0)$ contains all $f \in L^1(\mathbb{R})$ such that its Fourier transform $\hat{f}$ has compact support.

**Lemma 0.4.** In every compact set $K \subset \mathbb{R}$, there is a constant $C_K > 0$ such that every $f \in s(0)$ whose Fourier transform vanishes outside $K$ satisfies the inequality

$$\| f \|_g \leq C_K \| f \|_1.$$

**Lemma 0.5.** Given any $f \in s(0)$, there is for every $\epsilon > 0$ a $\gamma \in s(0)$ such that the Fourier transform $\hat{\gamma}$ has compact support and

$$\| \gamma \ast f - f \|_g < \epsilon.$$

**Lemma 0.6.** Every Segal algebra has approximate units of $L^1_{\text{loc}}(\mathbb{R}).$

The proof of these lemmas can be found in Reiter [20, pp.126-199] and [21, pp.15-39, p.97].

If $s(0)$ is a Segal algebra on $G$, a multiplier on $s(0)$ is a bounded linear operator on $s(0)$ which commutes with translations. We denote by $M(s)$ the set of all multipliers on $s(0)$. 
**Theorem 0.7.** Let \( T \in M(\mathbb{C}) \). If \( f, g \in s(\mathbb{C}) \), then

\[
T(f \ast g) = T \ast g = f \ast Tg
\]

This is proved by Umd in [1].

The following representation theorem for a multiplier on a Segal algebra has been proved by Umd [2].

**Theorem 0.8.** If \( T \in M(\mathbb{C}) \), then there is a unique measure \( \sigma \) such that

\[
Tf = \sigma \ast f
\]

for all \( f \in s(\mathbb{C}) \).

We shall give now give some examples of Segal algebras.

**Example.** Let \( 1 \leq k < \infty \) and let \( T \) be the circle group. The Banach algebra \( C^k(\mathbb{T}) \) of all functions \( f \) with \( k \) continuous derivatives on \( T \) under the norm

\[
\| f \|_{C^k} = \sum_{j=0}^{k} \| f^{(j)} \|_{\infty}
\]

where \( f^{(j)} \) denotes the \( j \)th derivative of \( f \).

**Example 2.** Let \( T \) be the circle group. Let

\[
\mathcal{B} = \left\{ f \in L^1(\mathbb{T}) : \| f - D_N \ast f \|_2 \to 0 \text{ as } N \to \infty \right\}
\]

where \( D_N \) is the Dirichlet kernel of order \( N \) defined by

\[
D_N(t) = \sum_{j=-N}^{N} e^{ijt}.
\]

Then \( \mathcal{B} \) is a Segal algebra with norm

\[
\| f \|_{\mathcal{B}} = \sup_{N \geq 1} \| D_N \ast f \|_1
\]
EXAMPLE 3. The Banach algebra $L^A(R)$ of all functions $f \in L^2(R)$ which are absolutely continuous on the real line $R$ with the derivative $f' \in L^2(R)$ under the norm

$$\|f\|_{L^A} = \|f\|_1 + \|f'\|_1.$$ 

EXAMPLE 4. The space $W(R)$ of all continuous functions $f$ on $R$ for which

$$\int_{-\infty}^{\infty} \max_{k-\frac{1}{2} \leq x \leq k + \frac{1}{2}} |f(x)| < \infty,$$

under the norm

$$\|f\|_W = \sup_{\alpha \in \mathbb{R}} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} |f(x+\alpha)|.$$

EXAMPLE 5. The algebra $A^p(\mathbb{C})$ for $1 \leq p < \infty$ consisting of all those $f$ in $L^2(\mathbb{C})$ such that its Fourier transform $\hat{f} \in L^p(\mathbb{R})$ with the norm

$$\|f\|_{A^p} = \|f\|_1 + \|\hat{f}\|_p.$$ 

EXAMPLE 6. Let $\omega$ be a real valued even continuous function on $\mathbb{R}$ such that

$$\omega(\gamma + \gamma') \leq \omega(\gamma) \omega(\gamma')$$

for all $\gamma, \gamma' \in \mathbb{R}$. For $1 \leq p < \infty$ we define $A^p_\omega(\mathbb{C})$ to be the set of all functions $f$ in $L^2(\mathbb{C})$ such that $f \circ \omega \in L^p(\mathbb{R})$. We introduce a norm by

$$\|f\| = \|f\|_1 + \|\hat{f} \circ \omega\|_p, f \in A^p_\omega(\mathbb{C}).$$

Then $A^p_\omega(\mathbb{C})$ is a Segal algebra (see Neiter, pp. 25).

EXAMPLE 7. Consider a nested family $\mathcal{Q}$ of strictly positive functions $\omega$ on $\mathbb{R}$ which are measurable, summable with respect
to $dV$, together with the norm $M(a)$, satisfying the following conditions

1) For each $a \in \Omega$, $M(a)$ is finite and

$$0 < \int a dV \leq M(a).$$

2) If $a \in \Omega$, then $1/a$ is locally $L^\infty$ on $\Gamma$.

3) For each positive number $\lambda$ and each $a \in \Omega$, we have $\lambda a \in \Omega$ and

$$M(\lambda a) = \lambda M(a).$$

4) If $a_1, a_2 \in \Omega$, then $a_1 + a_2 \in \Omega$ and

$$M(a_1 + a_2) \leq M(a_1) + M(a_2).$$

5) $\Omega$ is complete under the norm $M$. That is, for any sequence $\{a_n\}^\infty_{n=1} \subset \Omega$ such that

$$\sum_{n=1}^\infty M(a_n) < \infty,$$

is in $\Omega$ and

$$M(a) \leq \sum_{n=1}^\infty M(a_n).$$

Let $\Omega_0$ be the subset of $\Omega$ consisting of those $a$ such that

$$M(a) = 1.$$

For a fixed $p$ satisfying $1 < p < \infty$, we set

$$\omega = \frac{1}{p-1}.$$

For each $a \in \Omega_0$, we consider the Banach spaces $L^p_{\omega}(\Gamma)$ and

$$L^p_{\omega}(\Gamma)$$

of functions measurable on $\Gamma$ having the respective norms

$$\|f\|_{L^p_{\omega}} = \left\{ \int_{\Gamma} |f|^p \omega \, dV \right\}^{1/p}.$$
and
\[ \| \phi \|_{L^q} = \left\{ \int \phi \cdot \phi \cdot dV \right\}^{1/q} \]
and set
\[ \Lambda^p(\Gamma) = \bigcup_{\alpha \in \mathbb{Z}_0} L^p(\Gamma) \]
and set
\[ \Omega^q(\Gamma) = \bigcap_{\alpha \in \mathbb{Z}_0} L^q(\Gamma) \]
where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Then we have the following theorem of Bourling [11].

**Theorem 0.11.** Let \( 1 < p < \infty \) and \( q = p/p-1 \), then both
\[ \Lambda^p(\Gamma) \] and \( \Omega^q(\Gamma) \) are Banach spaces if they are supplied with the norm given by
\[ \| f \|_{\Lambda^p} = \inf_{\alpha \in \mathbb{Z}_0} \| f \|_{L^p(\Gamma)} \]
and
\[ \| \phi \|_{\Omega^q} = \sup_{\alpha \in \mathbb{Z}_0} \| \phi \|_{L^q(\Gamma)} \]
respectively.

Let \( s^p(G) = \left\{ f \in L^1(G) : \hat{f} \in \Lambda^p(\Gamma) \right\} \). We introduce a norm on \( s^p(G) \) by setting
\[ \| f \|_G = \| f \|_1 + \| \hat{f} \|_{\Lambda^p(\Gamma)} \quad f \in s^p(G). \]

Then if \( G \) is a locally compact non-discrete abelian group and \( 1 \leq p < \infty \), \( s^p(G) \) is a Segal algebra (see Unni [6]).
EXAMPLE 2. Let \( G \) be a locally compact abelian group with character group \( \Gamma \). Let \( \alpha \) be a locally bounded function on \( G \) with \( \alpha(\gamma) \geq 1 \) for all \( \gamma \in \Gamma \). Let \( S(\alpha) = \{ f \in L^1(G) : \lim \hat{f}(\gamma) \alpha(\gamma) = 0 \} \) i.e., for every \( \varepsilon > 0 \) there exists a compact subset \( K \) of \( \Gamma \) such that
\[
\left\{ \gamma : \hat{f}(\gamma) \alpha(\gamma) > \varepsilon \right\} \subseteq K.
\]
Then \( S(\alpha) \) is a Segal algebra with norm
\[
||f||_g = \sup \left\{ \hat{f}(\gamma) \alpha(\gamma) : ||f||_1 \right\}.
\]

In this thesis we study multipliers on some of the Segal algebras included in the examples above, particularly the algebras \( A^p(G) \), \( A^p_0(G) \), \( B^p(G) \), \( C^b(G) \) and \( W(G) \), and characterise various spaces of multipliers. This introduction is followed by the characterisation of the space of multipliers on \( A^p(G) \), when \( p > 2 \), answering the questions raised by Larson \( [1] \) and Lai \( [2] \).

The principal results are contained in Theorems 1.10, 1.14 and 1.19.

Chapter two is concerned with the Segal algebras \( A^p_0(G) \) and \( B^p(G) \), when \( G \) is a compact abelian group. We prove that there is a continuous algebra isomorphism of \( \mathcal{M}(A^p_0(G)) \) onto \( L^Q(\Gamma) \), when \( 1 \leq p \leq 2 \) and a continuous linear isomorphism of \( \mathcal{M}(A^p_0(G)) \) onto continuous linear isomorphism the dual of a certain Banach space of continuous function if \( p > 2 \). Analogous results are also obtained for \( B^p(G) \). In Chapter three we characterise the multiplier spaces on the segal algebras \( W(G) \), \( C^b(G) \) and \( B(\alpha) \).

Chapter four discusses bijective and isometric isomorphisms of multiplier algebras. In particular we prove that for two groups
$a_1$ and $a_2$, a bijective isomorphism of the multiplier spaces of the segal algebras induces a topological isomorphism between $a_1$ and $a_2$, thereby generalizing the results of Cauchy and Temari. In the case of $A^p_0(S)$, an isometric isomorphism of the multiplier algebras also induces a topological isomorphism of the underlying groups. In Chapter five we discuss the characterizations of the multipliers from $S(S)$ into $L^1(S)$. The sixth chapter is concerned with the problem of the restriction of a multiplier to a subset of the dual space $S'$. We have extended the result of Pigno concerning the restriction of multipliers from $L^1(S)$ to $L^{p_1} \cap L^{p_2}(S)$ to the case when both the exponents $p_1$ and $p_2$ are greater than two. In the last chapter, we have answered the question raised by Larson as to whether there exist non-zero closed translation invariant subspaces $X$ of $L^p(S)$ satisfying $X \cap L^1(S) = \{0\}$. 
CHAPTER II

MULTIPLIERS OF AP(G) ALGEBRAS

In this chapter we shall study the multipliers of $A^p(G)$ as also the multipliers of $A^p(G)$.

Throughout this chapter $G$ will be a locally compact abelian group. The space $A^p(G)$ is the subset of $L^2(G)$ consisting of those functions $f$ whose Fourier transform $\hat{f}$ belongs to $L^p(\Gamma)$. Given the norm

$$\| f \|_s = \| f \|_2 + \| \hat{f} \|_p, \quad f \in A^p(G)$$

it turns out that $A^p(G)$, $1 \leq p < \infty$, is a dense ideal in $L^2(G)$ and forms a semisimple commutative Banach algebra under convolution and in fact is a Segal algebra.

We shall now introduce some notation and terminology. Let $C_c(G)$ denote the space of all continuous functions on $G$ with compact support and let $C_0(G)$ be the space of continuous functions on $G$ vanishing at infinity. For a fixed $p$ satisfying $1 \leq p < \infty$, let $q$ denote the conjugate index of $p$ given by $1/p + 1/q = 1$.

Let $B = C_0(G) \times L^q(\Gamma)$. Then the linear space $B$ can be made into a normed linear space by introducing either of the equivalent norms:

$$\| (f, g) \| = \max \left( \| f \|_\infty, \| g \|_q \right)$$
(2) \[ \| (f, g) \|_{\text{sum}} = \| f \|_p + \| g \|_q \]

where \( f \in C_0(\Omega) \) and \( g \in L^q(\Gamma) \).

Let \( H_q \) be the closure in \( H \) of \( \left\{ (f, -\hat{f}) : f \in \Lambda^2(\Omega) \right\} \)

where \( \hat{f} \) is the reflexive function given by \( \hat{f}(x) = f(-x) \), for all \( x \in \Omega \) and \( \hat{f} \) is the Fourier transform of \( f \).

The space \( C_0(\Omega) \bigvee_{H_q} L^q(\Gamma) \) is the quotient space \( E_q = H/H_q \) with the quotient norm, more explicitly the norm is given by

\[ \| (g, h) \|_q = \inf \left\{ \| g' \|_p + \| h' \|_q : (g', h') = (g, h) \mod H_q \right\} \]

The space \( C_0(\Omega) \cap L^q(\Gamma) \) is the space \( E_q \), the norm being restriction to \( E_q \) of the maximum norm (1) on \( C_0(\Omega) \times L^q(\Gamma) \).

Let \( A_p(\Omega) \) the set of all functions \( u \) which can be expressed as

\[ u = \sum_{i=1}^n f_i \times g_i \]

where \( f_i \in A^p(\Omega) \), \( g_i \in C_0 = \left\{ g \in C_0(\Gamma) : \hat{g}, \hat{g} \in K_q \right\} \)

and \( \sum_{i=1}^n \| f_i \|_p \| g_i \|_q < \infty \), where \( \| g_i \|_q = \| \hat{g_i}, \hat{g_i} \|_q \).

Define a norm on \( A_p(\Omega) \) by

\[ \| u \|_p = \inf \left\{ \sum_{i=1}^n \| f_i \|_p \| g_i \|_q : (f_i, g_i) = (f, g) \mod E_q \right\} \]
where the infimum is taken over all functions $f_i \in A^p(G_r)$, $g_i \in C_q$, for the representation of $u = \sum_{i=1}^{\infty} f_i \times g_i$ as an element of $A^p(G_r)$.

The following result is then valid.

**Theorem 1.1.** [ ] For $1 < p \leq 2$, the space $A^p(G)$ is a dense linear subspace of $C_0(G)$ and is a Banach space with respect to the norm $\| \cdot \|_p$ and the topology thus defined is stronger than both the top uniform topology and the topology induced from $A^p(G)$.

Let $M(L^1, A^p)$ denote the space of multipliers from $L^1(G)$ to $A^p(G)$ and $M(A^p)$ the space of multipliers of $A^p(G)$.

Concerning multipliers, the following results were proved by Lal.

**Theorem 1.2.** [ ] For $1 < p \leq 2$, the multiplier space $M(A^p)$ is isometrically isomorphic to the topological dual $[A^p(G)]^\ast$ of $A^p(G)$.

**Theorem 1.3.** [ ] For $1 < p \leq 2$, $M(L^1, A^p) \subseteq A^p(G)$ where $\subseteq$ denotes the isometric isomorphism between the two spaces.

The proof of Theorem 1.3 makes use of the following result of Liu and Rosij.
Theorem 1.4. For $1 \leq p \leq 2$,
\[ C_0(G, \mathbb{C}) \vee L^q(G, \mathbb{R})^* \cong A^p(G, \mathbb{R}). \]

Theorems 1.2, 1.3 and 1.4 were left open when $p > 2$. The question whether $A^p(G, \mathbb{R})$ is actually a dual space for $p > 2$ was raised by Larson \[ \text{[16, p.301]} \]. He also asked for a description of $M(A^p)$ similar to Theorem 1.2 when $p > 3$ \[ \text{[16, p.301]} \].

Theorem 1.3 for $p > 3$ was left open by Lai \[ \text{[12, p.668]} \]. We shall prove below the results analogous to Theorems 1.3, 1.3 and 1.4 when $p > 2$.

We will often make use of the following well-known results.

**Theorem 1.5.** Let $G$ be a locally compact abelian group. Assume $\mathfrak{F}$ is a multiplier from $L^2(G)$ into itself. Then there exists a unique measure $\mu \in \mathbb{M}(G)$ such that
\[ \mathfrak{F} = \mu \hat{f} \]
for each $f \in L^2(G)$.

**Theorem 1.6.** If $\{ e_\alpha \}$ is an approximate identity for $L^2(G)$ with $\| e_\alpha \| = 1$ for all $\alpha$ and $e_\alpha$ has compact support for each $\alpha$, then $e_\alpha$ converges uniformly to one on compact subsets of $G$.

Let $A$ be a Banach algebra. A Banach $A$-module is a Banach space $V$ which is an $A$-module in the algebraic sense and in which the following norm inequality is satisfied:
\[ \| a v \| \leq \| a \| \| v \|, \quad a \in A, \quad v \in V. \]
Here $\|a\|$ denotes the norm in $A$ and $\|u\|$ and $\|v\|$ are norms in $V$.

If $A$ is a Banach algebra and $V$ an $A$-module, then $V$ is said to be an essential $A$-module if $AV$, the linear manifold spanned by $\{av : a \in A, v \in V\}$ is dense in $V$.

We shall now state some results that are needed in the course of our proofs.

**Theorem 1.7** [ ] If $A$ is a Banach algebra with bounded approximate identity $\{I_j\}$ and $V$ is an $A$-module, then $V$ is essential if and only if $I_jv \to v$ for every $v \in V$.

**Theorem 1.8** [ ] Let $A$ be a Banach algebra with bounded approximate identity and $W$ is an $A$-module. If the $A$-module $W^*$ is essential, then $W$ is also essential.

It follows from Theorem 1.7 that every Segal algebra is an essential $L^2(G)$ module.

**Lemma 1.9** [ ] Let $A$ be a normed algebra with bounded approximate identity $\{e_\alpha\}$ such that $\|e_\alpha\|_A \leq 1$ and $B$ a normed $A$-module such that $\alpha e_\alpha \to \alpha$ for all $\alpha \in B$, limit being taken over $\alpha$. Then there is a natural isometry $M(A, B^*) \to B^*$

where $B^*$ denotes the topological dual of $B$. 


THEOREM 1.10. \[
\left[ C_0(G_t) \cap L^q(M) \right]^{\ast} = M_{bd}(G_t) \bigcap L^p(M)
\]
for \(1 < q < \infty, \frac{1}{p} + \frac{1}{q} = 1\), where
\[
J = \{ (\mu, h) \in M_{bd}(G_t) \times L^p(M) : \int_{G_t} \hat{f} \, d\mu = \int_{\Gamma} \hat{f} \, h \, d\gamma \}
\]
for all \(f \in A^1(G_t)\).

LEMMA 1.11. \[\text{If } \mu \in M_{bd}(G_t) \text{ then } \mu \in L^1(G_t)
\]
if and only if the mapping \(G_t \to M_{bd}(G_t) \text{ given by } s \to T_s \mu\)
is continuous.

We now define a linear subspace \(B_p(G_t)\) of \(M_{bd}(G_t)\) by
\[
B_p(G_t) = \{ \mu \in M_{bd}(G_t) : \hat{\mu} \in L^p(M) \}
\]
If we set
\[
\|\mu\|_{B_p} = \|\mu\| + \|\hat{\mu}\|_p
\]
then \(B_p(G_t)\) becomes a Banach space under the norm \(\|\cdot\|_{B_p}\). Our
characterization of the space of multipliers from \(L^1\) to \(A^p\) is
given by

THEOREM 1.12. \[\text{If } p > 2, \text{ then } M(C^1, A^p) \text{ is}
\]
isometrically isomorphic to \(B_p(G_t)\).

REMARK. When \(p > 2\), there are measures \(\mu\) whose Fourier
Stieltjes transform belongs to \(L^p(\Gamma)\) but \(\mu\) is not absolutely
continuous.
PROOF OF THEOREM 1.12. Let \( \{ \hat{c}_\alpha \} \) be an approximate identity for \( L^2(\mathbb{C}) \) with \( \| \hat{c}_\alpha \|_1 = 1 \) for all \( \alpha \). \( \hat{c}_\alpha \) has compact support for all \( \alpha \). Suppose \( \mu \in B_p(\mathbb{C}) \). If \( f \in L^2(\mathbb{C}) \), we know that \( \mu \hat{f} \in L^2(\mathbb{C}) \) and \( \hat{f} \in L^p(\mathbb{C}) \) since \( \hat{\mu} \in L^p(\mathbb{C}) \) and \( \hat{\mu} \) is bounded. Define \( T_\mu : L^2(\mathbb{C}) \to L^2(\mathbb{C}) \) by

\[
T_\mu(f) = \mu \hat{f}, \quad f \in L^2(\mathbb{C}).
\]

Then \( T_\mu \) is a well defined linear map which is moreover bounded since

\[
\| T_\mu f \|_2 = \| \mu \hat{f} \|_2 = \| \mu \|_1 \| \hat{f} \|_1 + \| \hat{\mu} \|_p \| \hat{f} \|_p \leq \| \mu \|_1 \| \hat{f} \|_1 + \| \hat{\mu} \|_p \| \hat{f} \|_p \leq \| \hat{f} \|_1 [\| \mu \|_1 + \| \hat{\mu} \|_p] = \| \hat{f} \|_1 \| \mu \|_B_p.
\]

Hence

\[
(4) \quad \| T_\mu \|_2 \leq \| \mu \|_B_p.
\]

Since \( T_\mu \) is also translation invariant, it follows that \( T_\mu \in \mathcal{M}(L^1_B(\mathbb{C})) \).

The mapping

\[
\Lambda : B_p(\mathbb{C}) \to \mathcal{M}(L^1_B(\mathbb{C}))
\]

given by

\[
\Lambda(\mu) = T_\mu
\]

is a well defined linear map.
First we show that $\wedge$ is onto. For this purpose, let $T \in \mathcal{M}(L^1_p, A^p)$. The inequalities

$$
\| T \|_1 \leq \| T \|_S \leq \| T \|_1 \| \| \phi \|_1
$$

imply that $T$ can be extended to a bounded linear translation invariant map of $L^1_p(\mathbb{C})$ into itself. Then by Theorem 1.4 there exists a $\mu \in \mathcal{M}(\mathbb{C})$ such that

$$
T \phi = \mu \cdot \phi \text{ for all } \phi \in L^1_p(\mathbb{C}).
$$

Now since $e^s \in A^p(\mathbb{C})$, using (6) and (3) we have

$$
\| T e^s \phi \|_p = \| T e^s \phi \|_S \leq \| T \|_1 \| e^s \phi \|_1 = \| T \|_1
$$

This implies that

$$
\| T e^s \phi \|_p \leq \| T \|_1 \text{ for all } \phi.
$$

Then by Alaoglu's theorem [1, p. 324], there exists a subnet $\{ e^s \phi_p \}$ and an element $g \in L^p(\mathbb{C})$ such that $\hat{e}^s \phi_p \to g$ weakly in $L^p(\mathbb{C})$. This means that for each $\phi \in L^q(\mathbb{C})$ we have

$$
\int_{\mathbb{C}} e^s \phi_p \chi \; d\sigma \to \int_{\mathbb{C}} g \chi \; d\sigma.
$$

Now consider $\chi \in C_c(\mathbb{C})$. Since $e^s \phi_p$ converges uniformly to one on compact subsets of $\mathbb{C}$, by Theorem 1.6, we obtain

$$
\lim_{p \to \infty} \int_{\mathbb{C}} e^s \phi_p \chi \; d\sigma = \int_{\mathbb{C}} g \chi \; d\sigma.
$$

From this and (6) we get

$$
g \chi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{C}} e^s \chi \; d\sigma.$$
\[ \sum_{\Gamma} g \chi_{\Gamma} \, dx = \sum_{\Gamma} f \chi_{\Gamma} \, dx. \]

Since (9) is valid for every continuous function \( \chi \) with compact support on \( \Gamma \), we conclude that \( \hat{\mu} = g \) a.e. on \( \Gamma \). Thus \( \mu \in L^p(\Gamma) \) and so \( \mu \in E_p \). Hence \( T = \mu \) and \( \Lambda \) is onto.

Next we show that \( \| T \mu \| = \| \mu \|_{E_p} \) which will prove that \( \Lambda \) is an isometry. Since \( \{ e_\alpha \} \) is an approximate identity for \( L^1(\alpha) \), we have

\[ \| \mu \ast f \ast e_\alpha - \mu \ast f \|_1 \to 0 \quad \text{as} \quad \alpha \to 0 \quad f \in L^1(G) \]

Then if \( f \in C_c(\alpha) \), there exists a subnet \( \{ e_{\alpha_p} \} \) of \( \{ e_\alpha \} \) such that

\[ \mu \ast f \ast e_{\alpha_p} \to \mu \ast f \text{ pointwise on } G \] so that

\[ \text{which implies} \]

\[ \lim_{p} \mu \ast f \ast e_{\alpha_p}(\alpha) = \mu \ast f(\alpha) \]

which is the same as

\[ \lim_{p} \int_{G} \left( \mu \ast e_{\alpha_p} \right)(x) f(x) \, dx = \int_{G} f(x) \, d\mu(x). \]

Moreover

\[ \| \mu \ast e_{\alpha_p} \|_1 \leq \| \mu \ast e_{\alpha_p} \|_1 \leq \| T \mu \|_1 \| e_{\alpha_p} \|_1 \leq \| T \mu \|_1. \]

We can then find a subnet \( \{ e_{\alpha_{p_q}} \} \) of \( \{ e_{\alpha_p} \} \) and a measure \( \nu \).
such that \( \mu * \delta_{x_0} \) converges weakly to \( \nu \). Hence if \( f \in \mathcal{C}_c(\mathbb{R}) \), we have

\[
\lim_{\mu \rightharpoonup \nu} \int_{\mathbb{R}} f(x) \, d\mu(x) = \int_{\mathbb{R}} f(x) \, d\nu(x)
\]

(12) \[
\lim_{\mu \rightharpoonup \nu} \int_{\mathbb{R}} f(x) \, d\mu(x) = \int_{\mathbb{R}} f(x) \, d\nu(x)
\]

From (11) and (12) we get

\[
\int_{\mathbb{R}} f(x) \, d\mu(x) = \int_{\mathbb{R}} f(x) \, d\nu(x), \quad f \in \mathcal{C}_c(\mathbb{R})
\]

Since \( \mathcal{C}_c(\mathbb{R}) \) is dense in \( \mathcal{C}_0(\mathbb{R}) \), we have \( \nu = \mu \). We have thus proved that there exists a subset \( \mathcal{C}_c(\mathbb{R}) \) of \( \mathcal{C}_0(\mathbb{R}) \) which converges weakly to \( \mu \) in \( \mathcal{M}(\mathbb{R}) \).

Now given \( \varepsilon > 0 \), since \( f \in L^p(\mathbb{R}) \), there exists a compact subset \( K \) of \( \mathbb{R} \) such that

\[
\int_{\mathbb{R} \setminus K} |f(x)|^p \, dx < \frac{\varepsilon^p}{2^{2p+1}}
\]

Since \( \delta_{x_0} \) converges to one uniformly on \( K \),

\[
\int_{\mathbb{R}} f(x) \, d\delta_{x_0}(x) - \int_{\mathbb{R}} f(x) \, d\mu(x) \text{ converges to zero.}
\]

Hence there exists a finite subset \( B \) of indices such that for all \( \alpha \not\in B \)
\[ \int_{\mathbb{R}} \hat{\mu}(x) e^x - \mu(x) e^x \, dx < \frac{\varepsilon}{2} \quad \text{for all } x \in \mathbb{R}. \]

Thus
\[ \| \hat{\mu} - \mu \|_p < \frac{\varepsilon}{2} \quad \text{for all } \mu \in \mathcal{B}. \]

(15) \[ \| \hat{\mu} \|_p \leq \| \hat{\mu} - \mu \|_p + \frac{\varepsilon}{2} \quad \text{for all } \mu \in \mathcal{B}. \]

Given \( \varepsilon > 0 \), there exists \( \mu \in C_c(\mathbb{R}) \) such that \( \| \mu \|_{L}^2 \leq 1 \) and
\[ \| \mu \|_p \leq \| \mu - \hat{\mu} \|_p + \frac{\varepsilon}{4}. \]

Since
\[ \int_{\mathbb{R}} f(x) \, dx = \lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x) \mu(x) e^x \, dx, \]
we can find a finite subset \( \mathcal{C} \) of indices such that
\[ \left| \int_{\mathbb{R}} f(x) \, dx - \int_{\mathbb{R}} f(x) \mu(x) e^x \, dx \right| < \frac{\varepsilon}{4} \quad \text{for all } \mu \in \mathcal{C}. \]
Hence
\[ |f|_{L^p} < |f|_{L^p(\mu \ast e^\varphi)} + \frac{\varepsilon}{4}, \quad \forall \varphi \in \mathcal{C}, \]
\[ < \|f\|_{L^p} \|\mu \ast e^\varphi\|_1 + \frac{\varepsilon}{4}, \quad \forall \varphi \in \mathcal{C}, \]
\[ < \|\mu \ast e^\varphi\|_1 + \frac{\varepsilon}{4}, \quad \forall \varphi \in \mathcal{C}. \]

We then have
\[ \|\mu\| \leq |\int f \, d\mu| + \frac{\varepsilon}{4} \leq \|\mu \ast e^\varphi\|_1 + \frac{\varepsilon}{2}, \quad \forall \varphi \in \mathcal{C}. \]

Let \( D = B \cup C \). Choose \( \varphi_0 \in D \). Then
\[ \|\mu\|_p + \|\mu\|_p \leq \|\mu \ast e^{\varphi_0}\|_p + \|\mu \ast e^{\varphi_1}\|_1 + \varepsilon \]
\[ = \|\mu \ast e^{\varphi_1}\|_1 + \varepsilon \leq \|\mu\|_p \|\varphi_0\|_1 + \varepsilon = \|\mu\|_p + \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, we get
\[ \|\mu\|_p + \|\mu\|_p \leq \|\mu\|_p. \]

This inequality along with the opposite inequality (4) gives
\[ \|\mu\|_p + \|\mu\|_p = \|\mu\|_p \]
as desired.

In the case of the Segal algebra \( A^p_\infty(\mathcal{G}) \) for any weight function \( \omega \) on \( \mathcal{G} \) given as in Example 5, if \( M(A^p_\infty(\mathcal{G})) \) denotes the multiplier algebra of this Segal algebra, then we have

**Theorem 1.19.** If \( M(L^1, A^p_\infty(\mathcal{G})) \subseteq A^p_\infty(\mathcal{G}) \) for
\[ 1 \leq p < 2 \]
and
\[ M(L^1, A^p_\infty(\mathcal{G})) \subseteq B^p_\infty(\mathcal{G}) \] if \( p > 2 \).
where
\[ B_\omega^p(\mathcal{M}) = \{ \mu \in M_{bd}(\mathcal{M}) : \int \mu \omega \in L^p(\mathcal{M}) \} \]

with
\[ \| \mu \|_{B_\omega^p} = \| \mu \| + \| \int \mu \omega \|_p \]

**Proof.** The proof is similar to that of Theorem 1.12.

The question whether \( A^p(\mathcal{M}) \) is a dual for \( p > 2 \) is answered by the following result.

**Theorem 1.14** \( \text{For } p > 2, A^p(\mathcal{M}) \) is not a dual space.

**Proof.** Suppose for some \( p > 2 \), \( A^p(\mathcal{M}) \) is the dual of a normed linear space \( B \). Since \( A^p(\mathcal{M}) \) is an \( L^1(\mathcal{M}) \) module, \( B^{\text{op}} \) is an \( L^1(\mathcal{M}) \) module. Since \( B \subset B^{\text{op}} \), \( B \) is an \( L^1(\mathcal{M}) \) module.

Now \( A^p(\mathcal{M}) = B^* \) is an essential \( L^1(\mathcal{M}) \) module. Hence by Theorem 1.8, \( B \) is an essential \( L^1(\mathcal{M}) \) module. By Lemma 1.9, we then conclude that
\[ M(C^1, A^p) = M(C^1, B^*) \supsetneq B^* = A^p(\mathcal{M}). \]

This is a contradiction to the fact that
\[ M(C^1, A^p) \subseteq B^p(\mathcal{M}) \supsetneq A^p(\mathcal{M}) \text{ for } p > 2, \]

This completes the proof.

We shall now obtain a representation of \( M(A^p) \) analogous to that of Theorem 1.2. First we prove

**Lemma 1.15** \( \text{For } p > 2, \frac{1}{p} + \frac{1}{q} = 1 \)

**Proof.** By Theorem 1.10,
\[ k_q^* \supseteq M_{bd}(\mathcal{M}) \wedge L^p(\mathcal{M}) \]
Let \( \{ \mu, \lambda \} \in \mathcal{F} \) and let \( \mu \in \mathcal{L}^2(G) \). Since
\[
\hat{\mu} \in L^2(G)
\]
by Parseval's relation we have
\[
\int_G \hat{f} \hat{\mu} \, d\mu = \int_G \hat{f} \hat{\lambda} \, d\lambda.
\]
But by the definition of \( \mathcal{F} \),
\[
\int_G \hat{f} \, d\mu = \int_G \hat{f} \, d\lambda.
\]
Thus
\[
\int_G \hat{f} \hat{\mu} \, d\lambda = \int_G \hat{f} \hat{\lambda} \, d\lambda.
\]
(15)
\[
\int_G \hat{f} \hat{\mu} \, d\lambda = \int_G \hat{f} \hat{\mu} \, d\lambda.
\]
Now (15) holds for every \( \hat{f} \in C_c(G) \). Hence \( \hat{\mu} = \hat{\lambda} \) a.e. on \( G \) and we conclude that \( \mu \in L^p(G) \) and so \( \mu \in B_p(G) \).
Hence
\[
\mathcal{F} = \left\{ \mu \mid \hat{\mu} \in C_c(G) : \mu \in B_p(G) \right\}.
\]
If we equip \( B_p(G) \) with the norm
\[
\|\mu\| = \sup \left\{ \int_G |\mu| \mu^* M_{bd}(G) \right\} \left\| \hat{\mu} \right\|_p \lesssim \left\| \hat{\mu} \right\|_p
\]
equivalent to the one defined earlier, by definition of
\[
M_{bd}(G) \triangleq L^p(G)
\]
we have
\[
B_p(G) = M_{bd}(G) \triangleq L^p(G) = K_0
\]
This completes the proof of Lemma 1.15.
Lemma 1.16 \[ \text{For } \mu \in B_p(G), \ p > 2 \text{ and } g \in [C(I^0)]^n \text{ we have} \]

\[
\| \mu \|_{B_p} = \sup \left\{ \left| \mu \times g(0) + \int \frac{\partial g}{\partial x} \, dx \right| : \| g \| \leq 1 \right\}
\]

where \[ \{g, h \in K_q, g \in C(G_0), h \in L^2(M) \} \]

Proof. By Parseval's relation, we have

\[
\begin{align*}
\int \frac{\partial g}{\partial x} \, dx &= \int \frac{\partial g}{\partial x} \left( \int \left< \alpha, y \right> \, d\mu(\alpha) \right) \, dx \\
&= \int \frac{\partial g}{\partial x}(x) \, d\mu(x) = \mu \times g(0)
\end{align*}
\]

Consider a linear functional of the form

\[ L_\mu(g_1, h_1) = \int g_1(x) \, d\mu(x) + \int h_1(x) \, d\mu(x) \]

for \( g, h \in [C(I^0)]^n \) and \( (g_1, h_1) \in C(G_0) \times L^2(M) \).

Since \((g, h) = (h, g) \mod H_q\), we have

\[ \| (g, h) \| = \| (h, g) \| = \inf \left( \| g \|_w + \| h \|_q \right) \]

and so

\[ L_\mu(g_1, h_1) = L_\mu(h_1, g_1) = \frac{1}{2} \left( L_\mu(g_1, g_1) + L_\mu(h_1, h_1) \right). \]
By Lemma 1.16

\[ \| \mu \|_{B_p} = \sup \left\{ | \mu (\varphi, \varphi) | : \| \varphi \| \leq 1 \right\} \]

\[ = \sup \left\{ | \mu (\varphi_1, \varphi_1) | : \| \varphi_1 \| = 1 \right\} \]

\[ = \sup \left\{ | \frac{1}{a} (\mu (\varphi_1 + h, \varphi_1 + h) + \mu (\varphi_1, \varphi_1)) | : \| \varphi_1 \| = 1 \right\} \]

\[ \leq \sup \left\{ | \mu (\varphi_1, \varphi_1) | : \| \varphi_1 \| \leq 1 \right\} \]

But

\[ \sup \left\{ | \mu (\varphi_1, \varphi_1) | : \| \varphi_1 \| \leq 1 \right\} \]

\[ \leq \sup \left\{ | \mu (\varphi_1, \varphi_1) | : \| \varphi_1 \| \leq 1 \right\} \]

\[ = \| \mu \|_{B_p} \]

This proves Lemma 1.16.

Let \( p > 2 \). We define the space \( C_P (G) \) to be the set of all functions \( \mu \) such that

\[ \mu = \sum_{i=1}^{\infty} f_i \times g_i \]
where $f_i \in A^p(G_G) \quad g_i \in C_q = \left\{ g \in L^2(C_G(G_G))^n : \sum_{j=1}^n g_j, g_j \in K_q \right\}$

and

$$\sum_{i=1}^n \| f_i \|_G \| g_i \| < \infty$$

where $\| g \| = \left\| \left( g_1, g_2, \ldots \right) \right\|_G$.

Define $\| u \|_p$ by

$$\| u \|_p = \inf \left\{ \| u \| : \sum_{i=1}^n \| f_i \|_G \| g_i \| \right\}$$

where the infimum is taken over all the representations of $u$ as an element of $C_p(G_G)$.

It is easy to verify that $\| u \|_p$ is actually a norm on $C_p(G_G)$. We now assert that $C_p(G_G)$ is a Banach space under the norm. To this end, let $\{ u_n \} \subset C_p(G_G)$ be a Cauchy sequence. It suffices to show that a subsequence of $\{ u_n \}$ converges to an element of $C_p(G_G)$. We may assume, without loss of generality, that our sequence is such that

$$\| u_{n+1} - u_n \|_p < \frac{1}{2^n} \quad n = 1, 2, \ldots$$

Let $\| u_1 \|_p = C$. By the definition of the norm in $C_p(G_G)$, we can always find elements $f_{1k} \in A^p(G_G)$ and $g_{1k} \in C_q$ such that

$$u_1 = \sum_{k=1}^n f_{1k} * g_{1k}$$

and

$$u_{n+1} - u_n = \sum_{k=1}^n f_{n+1, k} * g_{n+1, k}$$
with $u_1 + (u_n - u_1) + (u_{n-1} - u_1) + ... + (u_1 - u_1) = \sum_{k=1}^{n} \frac{u_{n+1} - u_n}{n}$.

(16) $\sum_{k=1}^{n} \| f_k \|_S \| g_k \|_S \leq C + 1$

and

(17) $\sum_{k=1}^{n} \| f_{n+1} \|_S \| g_{n+1} \|_S < \frac{1}{2^{n-1}}, n = 1, 2, ...$

Now define

$u = f_1 \star g_1 + f_2 \star g_2 + f_3 \star g_3 + ...$

Then

$\| f_1 \|_S \| g_1 \|_S + \| f_2 \|_S \| g_2 \|_S + \| f_3 \|_S \| g_3 \|_S + ... < C + 3$

and thus $u \in C_p(G')$. We now show that $u_n$ converges to $u$ in $C_p(G')$. Given $\varepsilon > 0$, choose a natural number $n_0$ such that

$\sum_{\gamma=n_0}^{\infty} \frac{1}{2^{\gamma-1}} < \varepsilon$. If $n > n_0$ then

(18) $u - u_{n+1} = u - \left[ (u_{n+1} - u_n) + (u_n - u_{n-1}) + (u_{n-1} - u_{n-2}) + ... + (u_2 - u_1) + u_1 \right]$. 
\[ u_1 + (u_2 - u_1) + (u_3 - u_2) + \cdots + (u_{n+1} - u_n) \]
\[ = b_{11} \times g_{11} + b_{12} \times g_{12} + b_{13} \times g_{13} + b_{14} \times g_{14} + \cdots + b_{21} \times g_{21} + b_{22} \times g_{22} + b_{23} \times g_{23} + \cdots + b_{n+1,1} \times g_{n+1,1} + b_{n+1,2} \times g_{n+1,2} + \cdots \]

\[ u = b_{11} \times g_{11} + b_{12} \times g_{12} + b_{21} \times g_{21} + b_{31} \times g_{31} + b_{22} \times g_{22} + b_{32} \times g_{32} + b_{33} \times g_{33} + b_{23} \times g_{23} + b_{14} \times g_{14} + \cdots + b_{n+1,1} \times g_{n+1,1} + b_{n+1,2} \times g_{n+1,2} + b_{n+1,3} \times g_{n+1,3} + \cdots + b_{1,n+1} \times g_{1,n+1} + \cdots \]

Thus from (16)
\[ u - u_{n+1} = b_{n+2,1} \times g_{n+2,1} + b_{n+2,2} \times g_{n+2,2} + \cdots + b_{n+3,1} \times g_{n+3,1} + b_{n+3,2} \times g_{n+3,2} + \cdots \]

Therefore
\[ \| u - u_{n+1} \|_{C_p} \leq \sum_{j=1}^{\infty} \| f_{n+2, j} \|_{L} \| g_{n+2, j} \|_{L} + \sum_{j=1}^{\infty} \| f_{n+3, j} \|_{L} \| g_{n+2, j} \|_{L} + \ldots. \]

\[ \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots = \frac{1}{2^n} \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}}. \]

Hence \( u_n \) converges to \( u \) in \( C_p(G) \).

**Theorem 1.17.** \( C_p(G) \) is a dense linear subspace of \( C_0(G) \) and is a Banach space with respect to the norm \( \| \| \cdot \| \|_p \) and the topology so defined is stronger than the uniform topology.

**Proof.** We need only to verify only the last part of the theorem. If \( u \in C_p(G) \), then \( u \) has the representation

\[ u = \sum_{i} f_i \times g_i \]

where \( f_i \in A_p(G) \), \( g_i \in C_0 \), and \( \sum_{i} \| f_i \|_{L} \| g_i \|_{L} < \infty \).

From the inequalities

\[ \| \sum_{i=m}^{n} f_i \times g_i \|_{W} \leq \sum_{i=m}^{n} \| f_i \|_{L} \| g_i \|_{L} \]

where the right hand side tends to zero as \( m, n \to \infty \), we conclude that \( u = \sum_{i} f_i \times g_i \) is therefore a uniformly continuous function on \( G \) and the norm \( \| \| \cdot \| \|_p \) is stronger than the uniform norm. To complete the proof of the theorem, we have to show that \( C_p(G) \) is dense in \( C_0(G) \). This is because of the fact that the
algebra of continuous functions on $G$ generated by
\[ \{ f \ast g : f \in A^p(G), g \in C_c(G) \} \]
is a self adjoint sub-algebra of $C^p_0(G)$ and separates points on $G$, thus it is uniformly dense in $C^p_0(G)$ by the Stone-Weierstrass theorem. This completes the proof of the theorem.

**Theorem 1.13**

If $p > a$, then the multiplier space $M(A^p)$ is isometrically isomorphic to the topological dual $C^*_p(G)$ of $C^p_0(G)$.

**Proof.** Suppose $T \in M(A^p)$. Define a linear functional on $C^p_0(G)$ by
\[
\mu(u) = \sum_{i=1}^{\infty} \left( \int T f_i(x) g_i(x) \, dx + \int T f_i(x) \hat{g_i}(x) \, dx \right)
\]

\[
= \sum_{i=1}^{\infty} \left( \int T f_i \ast g_i(x) \, dx \right) + \sum_{i=1}^{\infty} \int T f_i(x) \hat{g_i}(x) \, dx
\]

for $u = \sum_{i=1}^{\infty} f_i \ast g_i$ in $C^p_0(G)$ with $f_i \in A^p(G)$, $g_i \in C_c(G)$

and
\[
\sum_{i=1}^{\infty} \| f_i \|_s \| g_i \|_l < \infty
\]

To show that $\mu$ is well defined, it suffices to show that if
\[
u = \sum_{i=1}^{\infty} f_i \ast g_i = 0 \quad \text{and} \quad \sum_{i=1}^{\infty} \| f_i \|_s \| g_i \|_l < \infty
\]
than $\mu(u) = 0$. Let $f_\alpha \to 1$ be an approximate identity for $A^p(G)$. Let $h_\alpha = T^\alpha$. Then

$$\|h_\alpha \ast f - T^\alpha f\|_{A^p} \to 0$$

(19)

For $f \in A^p(G)$, $g \in L^p(G \times G)$, we have

$$\|h_\alpha \ast f \ast g\|_X \leq \int_0^\infty \|h_\alpha \ast T^\alpha f\|_X \|g\|_X \, d\alpha$$

(20)

By assumption, the series $u = \sum_{i=1}^\infty f_i \ast g_i$ is uniformly convergent to zero on $G$. We therefore have

$$h_\alpha \ast u = \sum_{i=1}^\infty h_\alpha \ast f_i \ast g_i = 0$$

(22)

By (19) we have

$$T^\alpha f_i \ast g_i(0) = \lim_{\alpha} h_\alpha \ast f_i \ast g_i(0)$$

Therefore

$$\sum_{i=1}^\infty T^\alpha f_i \ast g_i(0) = \lim_{\alpha} \sum_{i=1}^\infty h_\alpha \ast f_i \ast g_i(0)$$

$$\sum_{i=1}^\infty \lim_{\alpha} h_\alpha \ast f_i \ast g_i(0) = 0 \text{ from (21)}$$
\[ \int T_{\hat{b}_i}(x) \hat{g}_i(x) \, dx = \int \hat{g}_i(x) \int_{\mathcal{G}_i} \langle x, y \rangle \, T_{\hat{b}_i}(x) \, dy \, dx \]
\[ = \int T_{\hat{b}_i}(x) \left[ \int_{\mathcal{G}_i} \hat{g}_i(x) \langle x, y \rangle \, dy \right] \, dx \]
\[ = \int T_{\hat{b}_i}(x) \, \hat{g}_i(x) \, dx \]
\[ = T_{\hat{b}_i} \ast \hat{g}_i(0) \]

Therefore
\[ \mu(w) = \sum_{i=1}^{\infty} \left[ \int_{\mathcal{G}_i} T_{\hat{b}_i}(x) \hat{g}_i(x) \, dx + T_{\hat{b}_i} \ast \hat{g}_i(0) \right] \]
\[ = 2 \sum_{i=1}^{\infty} T_{\hat{b}_i} \ast \hat{g}_i(0) = 0 \quad \text{from (22)} \]

thus proving that \( \mu \) is well defined.

Since \( A^\infty(G) \subset B_p(G) \), by Lemma 1.16,

\[ |\mu(w)| \leq \sum_{i=1}^{\infty} \| T_{\hat{b}_i} \|_s \| \hat{g}_i \| \leq \| T \|_s \sum_{i=1}^{\infty} \| \hat{b}_i \|_s \| g_i \| \]

It follows that
\[ |\mu(w)| \leq \| T \| \| w \|, \]

that is

\[ \| \mu \| \leq \| T \| \]

\[ (23) \]
On the other hand from Lemma 1.16 we have

\[ \| T \| = \sup_{\| f \|_S \leq 1} \| T f \|_S = \sup_{\| f \|_S \leq 1} \mu \left( \frac{f \times g}{\| f \|_S, \| g \|_S} \right) \]

\[ \leq \sup_{\| f \times g \|_C \leq 1} \| f \|_S \| g \|_S \leq 1 \]

From (23) and (24) we have

\[ \| T \| = \| \mu \| \]

It only remains to show that the mapping from \( \mathcal{L}(\mathcal{P}) \) into \( [C^r_\mu(G)]^* \) is surjective. Let \( \mu \in [C^r_\mu(G)]^* \). For a fixed \( f \in \mathcal{P}(G) \) define the linear functional

\[ L(g) = \mu \left( \frac{f \times g}{\| f \|_S, \| g \|_S} \right) \quad \text{for} \quad g \in \left[ C^r_\mu(G) \right]^* \]

Now

\[ \| L(g) \| \leq \| \mu \| \| f \|_S \| g \|_C \leq \| \mu \| (\| f \|_S \| g \|_S) \]

It may be extended to an element of \( K^*_G \) and hence defines a unique \( T f \in B_p(G) \), the dual space of \( K^*_G \) by Lemma 1.15. Since

\[ \| T f \|_{B_p} = \| T \| \leq \| \mu \| \| f \|_S \]

\( T \) is a bounded linear operator from \( \mathcal{P}(G) \) into \( B_p(G) \). Also
\[ \mu (f \circ g) = T f \circ g (0) + \int \mu (T f (\varphi (\sigma)) d\sigma), \quad f \in \mathcal{A} (\mathbf{C}) , \]

\[ g \in [C_0 (\mathbf{C})]^N . \]

Therefore if \( y \in G \), we have

\[ T (c_y f) \times g (0) + \int T c_y f \circ g (\sigma) d\sigma = \mu (c_y f \times g) \]

\[ = \mu (f \circ c_y g) = \int T f \circ c_y g (\sigma) d\sigma + T f \times c_y g (0) \]

\[ = \int c_y T f (\varphi (\sigma)) g (\varphi (\sigma)) d\sigma + c_y T f \times g (0) \]

that is

\[ c_y (T f) = T (c_y f) , \]

that is \( T \) commutes with translations. Also the map from

\[ G \rightarrow M (\mathbf{C}) \text{ given by } y \rightarrow c_y (T f) \text{ is continuous for every } f \in \mathcal{A} (\mathbf{C}) . \]

Hence by Lemma 1.11, \( T f \) must be absolutely continuous. \( T f \in B_p (\mathbf{C}) \) and \( T f \) absolutely continuous implies \( T f \in \mathcal{A} (\mathbf{C}) \) for every \( f \in \mathcal{A} (\mathbf{C}) \). Thus \( T \) defines a bounded linear translation-invariant map from \( \mathcal{A} (\mathbf{C}) \) into itself, that is \( T \in M (\mathcal{A}) \). This proves that the mapping of \( M (\mathcal{A}) \) into \([C_0 (\mathbf{C})]^*\) is surjective which completes the proof of the theorem.

**Remark 1.19.** Theorems 1.12 and 1.14 have been obtained independently by Burknell, Krogstad and Larson in [J].
CHAPTER II

MUTLIPLICATIVE ON THE ALGEBRAS $A_p^ω(G)$ AND $S^p(G)$

This chapter deals with characterizations of the spaces $M(A_p^ω(G))$ and $M(S^p(G))$.

In the case when $G$ is a noncompact locally compact abelian group, a characterization of the multipliers on the algebras $A_p^ω(G)$ and $S^p(G)$ have been given by Hochava Murthy and Unni in the following theorem.

THEOREM 3.1. Let $G$ be a non-discrete, noncompact locally compact abelian group and $1 ≤ p < ∞$. If $T ∈ M(A_p^ω(G))$ then there exists a unique measure $μ ∈ M_{bd}(G)$ such that

$$Tf = μ * f$$

for all $f ∈ A_p^ω(G)$. Further $M(A_p^ω(G))$ is isometrically isomorphic to $M_{bd}(G)$.

THEOREM 3.2. Let $G$ be a non-discrete, noncompact locally compact abelian group and $1 ≤ p < ∞$. If $T ∈ M(S^p(G))$ then there exists a unique measure $μ ∈ M_{bd}(G)$ such that

$$Tf = μ * f$$

for all $f ∈ S^p(G)$. Further $M(S^p(G))$ is isometrically isomorphic to $M_{bd}(G)$.

In the case when $G$ is compact abelian, we shall give
have characterizations of $M(A^p(G))$ and $M(S^p(G'))$ similar to the one given by Larson for the algebra $A^p(G)$ as the dual of Banach spaces. Throughout this section we shall assume that $G$ is a compact abelian group.

For a semisimple commutative Banach algebra the following multiplier theorem is valid.

**Theorem 2.3.** If $A$ is a semisimple commutative Banach algebra and $T$ a multiplier on $A$, then there exists a continuous function $\varphi$ defined on the maximal ideal space $\Delta$ of $A$ satisfying $\hat{T}_x = \varphi \hat{x}$ for all $x \in A$, where $\hat{x}$ denotes the self-adjoint transform of $x$, with $\|\varphi\|_{\infty} \leq \|T\|$.

We remark that every Segal algebra is a semisimple commutative Banach algebra with maximal ideal space $N$. Thus Theorem 2.3 is valid for all Segal algebras.

We shall now prove

**Theorem 2.4.** When $1 \leq p \leq 2$, there exists a continuous algebra isomorphism of $M(A^p(G))$ onto $L^0(M)$.

**Proof.** If $T \in M(A^p(G))$ there exists a function $\hat{T} \in L^0(M)$ satisfying

$$\hat{T}_f = \hat{\varphi} \hat{f}, \quad f \in A^p(G),$$

with $\|\varphi\|_{\infty} \leq \|T\|$.

The mapping $\hat{T} : \tilde{T} \mapsto \hat{T}$ is a well defined continuous linear map of $M(A^p(G))$ into $L^0(M)$. Also if $\hat{T}_1 = \hat{T}_2.$
\[ T_1, \ T_2 \in MCA_\omega^p(G) \] then
\[ \hat{T}_1 f = \hat{T}_2 f, \ \forall f \in A_\omega^p(G) \]

which implies that
\[ T_1 f = T_2 f \]
that is
\[ T_1 = T_2 \]

Thus the mapping \( \beta \) is one to one. We claim \( \beta \) is an onto map. To this end, let \( \varphi \in L^\omega(C) \). Now \( f \in A_\omega^p(G) \) implies
\[ \varphi \hat{f} \omega \in L^p(C) \]. Since
\[ \omega(\gamma) > 1, \ \forall \gamma \in \Gamma, \ \varphi \hat{f} \omega \in L^p(C) \]

Thus \( \varphi \hat{f} \in L^p(C) \cap L^\omega(C) \). Since \( 1 < p \leq 2 \), we see that
\[ \varphi \hat{f} \in L^2(C) \]. Then there exists \( g \in L^2(G) \) such that
\[ \hat{g} = \varphi \hat{f} \]. Since \( G \) is compact, \( g \in L^1(G) \). Moreover
\[ \hat{g} \omega = \varphi \hat{f} \omega \in L^p(C) \] which implies that \( g \in A_\omega^p(G) \). Now
\[ \|g\|_S = \|g\|_1 + \|\hat{g} \omega\|_p \leq \|g\|_q + \|\hat{g} \omega\|_p \leq \|\hat{g}\|_p + \|\hat{g} \omega\|_p = 2\|g\|_p + 2\|\hat{f} \omega\|_p \leq 2\|g\|_p + \|\hat{f} \omega\|_p \leq 2\|g\|_p + \|\hat{f} \omega\|_1 \leq 2\|g\|_S \|f\|_S \]
The mapping \( T: A_0^p(G) \rightarrow A_0^p(G) \) given by \( T\frac{f}{\|f\|_p} = g \)

is a bounded linear map which can be easily verified to be translation invariant. Therefore \( T \in M(A_0^p(G)) \) with \( \hat{T} = \varphi \).

Hence \( \beta \) is onto. This completes the proof.

We thus see that in the case \( 1 \leq p \leq 2 \), there are multipliers in \( M(A_0^p(G)) \) which do not correspond to measures \( \mu \in M_{bd}(G) \) since not every function in \( L^p(G) \) is the Fourier-Stieltjes transform of a measure in \( M_{bd}(G) \). In the case of \( p > 2 \), we now give an example of a multiplier in \( M(A_0^p(G)) \) which does not correspond to any measure in \( M_{bd}(G) \).

**Example 2.5.** Let \( E \subset \Gamma \) be an infinite Sidon set. Let \( m = p/2 \), \( n = m^{-1} \). Choose \( \gamma \) such that \( 0 < \gamma < 2 \) and \( \gamma n > 2 \). Let \( \varphi \) be a bounded function on \( \Gamma \) satisfying \( \varphi(\gamma) = 0 \), \( \gamma \notin E \), \( \frac{1}{\gamma} \varphi(\gamma) < \infty \), \( \frac{1}{\gamma} \varphi(\gamma) \gamma_n < \infty \)

\[ \|f\|_p = \frac{1}{\gamma} \varphi(\gamma) \gamma_n \]

If \( \frac{f}{\|f\|_p} \in A_0^p(G) \), then

\[ \frac{1}{\gamma} \varphi(\gamma) \left( \int |\frac{f}{\|f\|_p} \varphi(\gamma) \gamma |^2 \right)^{\frac{1}{2}} \leq \left( \int |\varphi(\gamma) \gamma_n |^2 \right)^{\frac{1}{2}} \gamma \left( \frac{1}{\gamma} \varphi(\gamma) \right)^{\frac{1}{2}} \left( \int \left( \frac{1}{\gamma} \varphi(\gamma) \gamma_n \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \]

\[ \leq \left( \int \varphi(\gamma) \gamma_n \right)^{\frac{1}{2}} \left( \int \frac{1}{\gamma} \varphi(\gamma) \gamma_n \right)^{\frac{1}{2}} \gamma \left( \frac{1}{\gamma} \varphi(\gamma) \right)^{\frac{1}{2}} \left( \int \left( \frac{1}{\gamma} \varphi(\gamma) \gamma_n \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \]

\[ \leq \|f\|_p^{\frac{1}{2}} \left( \int \varphi(\gamma) \gamma_n \right)^{\frac{1}{2}} \left( \int \frac{1}{\gamma} \varphi(\gamma) \gamma_n \right)^{\frac{1}{2}} \gamma \left( \frac{1}{\gamma} \varphi(\gamma) \right)^{\frac{1}{2}} \left( \int \left( \frac{1}{\gamma} \varphi(\gamma) \gamma_n \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \]

\[ \leq \|f\|_p^{\frac{1}{2}} \left( \int \varphi(\gamma) \gamma_n \right)^{\frac{1}{2}} \left( \int \frac{1}{\gamma} \varphi(\gamma) \gamma_n \right)^{\frac{1}{2}} \gamma \left( \frac{1}{\gamma} \varphi(\gamma) \right)^{\frac{1}{2}} \left( \int \left( \frac{1}{\gamma} \varphi(\gamma) \gamma_n \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \]
so that
\( \begin{align*}
&\sum_{y} |\varphi(y) \hat{f}(y)|^2 < K \|f\|_p^2
\end{align*} \)
for some constant \( K \) which depends only on \( \varphi \).

Since Fourier transformation is an isometry on \( L^2 \), we can find a \( g \in L^2(G) \subset L^1(G) \) such that
\( \hat{g} = \varphi \hat{f} \omega \)

Further
\( \|g\|_2 \leq \|g\|_q \leq \|g\|_2 = \|\hat{g}\|_2 = \|\varphi \hat{f} \omega\|_2 < K^{1/2} \|\hat{f} \omega\|_p. \)

Since \( g \in L^2(G) \) and \( 1 < q < 2 \), by Hausdorff-Young theorem we have \( \hat{g} \in L^p(G) \) and
\( \|\hat{g}\|_p \leq \|g\|_2 < K^{1/2} \|\hat{f} \omega\|_p. \)

Analogously we prove that
\( \begin{align*}
&\sum_{y} |\varphi(y) \hat{f}(y)|^2 \leq K \|\hat{f}\|_p^2 \leq K \|\hat{f} \omega\|_p^2
\end{align*} \)

Then there exists \( g' \in L^2(G) \subset L^1(G) \) such that
\( \hat{g'} = \varphi \hat{f} \)

with
\( \|g'\|_1 \leq \|g'\|_2 = \|\hat{g'}\|_2 = \|\varphi \hat{f}\|_2 < K^{1/2} \|\hat{f} \omega\|_p. \)
From (8) and (6) we see that $\hat{g} = \hat{A} \omega$. The inequalities (4) and (7) show that $g^1 \in A_p^p(G)$ and

$$\|g^1\|_s = \|g^1\|_1 + \|g^1\|_p = \|g^1\|_1 + \|\hat{g}\|_p \leq 2k^{\frac{1}{2}} \|\hat{f}\|_p \leq 4k^{\frac{1}{2}} \|\hat{f}\|_s.$$ 

The mapping $T: A_p^p(G) \to A_p^p(G)$ given by $Tf = g^1$ is a multiplier on $A_p^p(G)$ with $f = \varphi$. Since $\varphi(x) = \omega$, $\varphi = \hat{f}$ for any bounded regular Radon measure $\mu \in M_{\text{ord}}(G)$.

**Theorem 8.6.** Then $p > 2$, there exists a continuous linear isomorphism of $M(A_p^p(G))$ onto the dual space of a Banach space of continuous functions.

**Proof.** We prove our theorem by showing that there exists a continuous linear isomorphism of $M(A_p^p(G))$ onto a dual $[R(G)]^*$ and then proving that the completion of $R(G)$ can be embedded into a space of continuous functions.

Let $A(G) = \{ \hat{f} : f \in L^1(G) \}$. For each $T \in M(A_p^p(G))$ set

$$\beta_T(f) = \int \hat{T}(x) \hat{f}(x) \, dx, \quad f \in A(G).$$

Then $\beta_T$ is a linear form on $A(G)$ and

$$|\beta_T(f)| \leq \|T\|_s.$$
Introduce a norm on \( A(G) \) by

\[
\| f \| = \sup \left\{ |f_T(b)| : T \in M(A_\omega^p(G)), \| T \| \leq 1 \right\}
\]

It is easy to verify that (9) defines a seminorm on \( A(G) \). We shall now show that \( \| \cdot \| \) is actually a norm on \( A(G) \).

To this end, let \( f \in A(G) \) such that \( \| f \| = 0 \). Then

\[
b_T(b) = 0 \quad \text{for all } T \in M(A_\omega^p(G)).
\]

Now let \( y \in G \). Since \( T_y \in M(A_\omega^p(G)) \), taking \( T = T_y \) we have

\[
b_f(-y) = b_T(b) = 0,
\]

since \( T_y(y) = \left\langle y, y \right\rangle \). This being true for every \( y \in G \), we see that \( f = 0 \).

Our space \( R(G) \) is the space \( A(G) \) with the norm given by (9). The norm itself will be denoted by \( \| \cdot \|_R \).

Now for each \( T \in M(A_\omega^p(G)) \), \( b_T \) is a well defined linear form on \( R(G) \) and

\[
\| b_T(b) \| = \left| \frac{b_T(b)}{\| T \|} \right| \| T \| \leq \| T \| \| f \|_R
\]

by the definition of \( \| f \|_R \). Thus \( b_T \in [R(G)]^* \), the dual of \( R(G) \). Consider the mapping

\[
\beta : M(A_\omega^p(G)) \to [R(G)]^*
\]

given by

\[
\beta(T) = b_T
\]

Then \( \beta \) is well defined and linear. The inequalities

\[
\| \beta(T) \| = \| b_T \| \leq \| T \|
\]
for all $T \in MC_{\omega}^{0}(G_{t})$ show that $\beta$ is continuous. We now claim that $\beta$ is both one to one and onto. If $\beta_{T_{1}} = \beta_{T_{2}}$ then

$$
\int_{\Gamma} T_{1}(\tau) \hat{b}(\tau) \, d\tau = \int_{\Gamma} T_{2}(\tau) \hat{b}(\tau) \, d\tau, \quad \hat{b} \in L^{1}(\Gamma).
$$

Therefore $\frac{\Delta_{T_{1}}}{\Delta_{T_{1}}} = \frac{\Delta_{T_{2}}}{\Delta_{T_{2}}}$ as functions of $L^{0}(\Gamma)$ which implies that $\beta$ is one to one.

To prove that the mapping $\beta$ is onto. Let $\lambda \in [R_{G_{t}}]^{*}$, we denote by $B(G_{t})$ the set of all functions in $L^{1}(G_{t})$ whose Fourier transform has compact support. If $f, g \in B(G_{t})$ then $f \ast g \in A(G_{t})$. Then

$$
|\beta_{T}(f \ast g)| = |\int_{\Gamma} T_{1}(\tau) \hat{f}(\tau) \hat{g}(\tau) \, d\tau| = |T_{1}f \ast g(0)|
$$

$$
\leq \|T_{1}f\|_{1} \|g\|_{\infty} \leq \|T_{1}\| \|f\|_{L^{1}} \|g\|_{\infty}
$$

so that

$$
(10) \quad \|f \ast g\|_{R} \leq \|f\|_{L^{1}} \|g\|_{\infty}
$$

Also

$$
\beta_{T}(f \ast g) = \int_{\Gamma} T_{1}(\tau) \hat{f}(\tau) \hat{g}(\tau) \omega(\tau) \hat{\omega}(\tau) \, d\tau.
$$

The inequality

$$
|\beta_{T}(f \ast g)| \leq \|T_{1}\|_{\infty} \|f\|_{L^{p}} \|g\|_{\infty} \|\hat{\omega} \|_{p'}
$$
then implies

\[(11) \quad \| f \times g \|_{p} \leq \| f \|_{\text{sup} p} \| \frac{\phi}{\omega} \|_{p} \| g \|_{q} \]

We have already taken an element \( \lambda \in [R(G_{0})]^{\times} \). Let \( f \in B(G_{0}) \) be fixed. Define a linear form on \([B(G_{0})]^{\wedge}\) by

\[ F_{f} (\hat{g}) = \lambda (f \times g), \quad \hat{g} \in [B(G_{0})]^{\wedge}. \]

Then

\[ |F_{f} (\hat{g})| = |\lambda (f \times g)| \leq \lambda \| f \|_{\text{sup} p} \| g \|_{q} \]

which by virtue of (11) gives

\[(12) \quad |F_{f} (\hat{g})| \leq \lambda \| f \|_{\text{sup} p} \| g \|_{q} \| \frac{\phi}{\omega} \|_{p} \]

Now \( L^{q,1/\omega} (\Gamma) \) is the Lebesgue space of measurable functions \( h \) on \( \Gamma \) satisfying

\[ \| h \|_{L^{q,1/\omega} (\Gamma)} = \left[ \int_{\Gamma} \max \left(0, |h(x)|^q \right) \frac{dx}{\omega(x)} \right]^{1/q} < \infty. \]

(13) implies then that \( F_{f} \) is a linear form on \([B(G_{0})]^{\wedge}\) bounded in the norm of \( L^{q,1/\omega} (\Gamma) \) and hence can be extended as a continuous linear functional to the whole space \( L^{q,1/\omega} (\Gamma) \). Now \( q < \infty \) and \( L^{q,1/\omega} (\Gamma) \) is the dual space of \( L^{q,1/\omega} (\Gamma) \). Hence
there exists an $h \in L^p_c(\Gamma)$ such that

$$(13) \quad F_b^f (g) = \lambda(f \times g) = \int G(x) \, h(x) \, dx$$

for all $\hat{g} \in \mathcal{B}(C(\Gamma))$ and

$$(14) \quad \|h\|_{L^p_c} = \|h \circ \lambda\|_{L^p} = \|F_b^f \| \leq \|\lambda\| \|f\|_{L^p}.$$ 

We now define another linear form $G_b^f$ on $B(\mathcal{G})$ by setting

$$G_b^f (g) = \lambda(f \times g), \quad g \in B(\mathcal{G}).$$

Then by virtue of (13) we have

$$|G_b^f (g)| \leq \|\lambda\| \|f \times g\|_{L^p_c} \leq \|\lambda\| \|f\|_{L^p} \|g\|_{L^p}.$$ 

Hence $G_b^f$ can be extended to a continuous linear functional on $C_0(\Gamma)$ since $B(\mathcal{G})$ is dense in $C_0(\Gamma)$). Therefore there exists $\mu \in M_{bd}(\Gamma)$ such that

$$(15) \quad G_b^f (g) = \lambda(f \times g) = \int g(x) \, d\mu(x), \quad g \in B(\mathcal{G}),$$

with

$$(16) \quad \|\mu\| \leq \|\lambda\| \|f\|_{L^p} \|g\|_{L^p}.$$ 

Define $T_b^f = \mu$ for every $f \in B(\mathcal{G})$. Then

$$T_b^f \times g(\cdot) = \lambda(f \times g), \quad f, g \in B(\mathcal{G}).$$
\[ T(T_y f) \ast g(o) = \lambda(T_y f \ast g) = \lambda(f \ast T_y g) = T_y(T_y f) \ast g(o). \]

Thus
\[ T(T_y f) = T_y(T_y f), \quad y \in G. \]

Since the mapping \( G \to M_{\text{ba}}(G) \) given by \( y \mapsto T_y(T_y f) \)

is continuous. Therefore by Lemma 11, \( T_y \) is absolutely continuous, that is \( T_y \in L^1(G) \). Now suppose \( f, g \in B(G) \).

Then
\[ F_f(g) = \lambda(f \ast g) = G_{T_y f}(g) \]

so that, by (13) and (15) we have
\[ \int \hat{g}(\gamma) \overline{h(\gamma)} \, d\gamma = \int g(\gamma) \, d\mu(\gamma). \]

The equality
\[ \int \hat{g}(\gamma) \overline{h(\gamma)} \, d\gamma = \int \hat{g}(\gamma) \overline{\hat{f}(\gamma)} \, d\gamma \]

holds for every \( \hat{g} \in [B(G)]^\cap \). This implies that \( h(\gamma) = \hat{f}(\gamma) \)

a.e. on \( M \). Therefore \( \hat{\mu}_o \in L^p(M) \) and

\[ \|
\hat{\mu}_o\|_p = \|h_{\omega}\|_p < \|\lambda\|_2 \|f\|_2. \]
by (14). On the other hand, by virtue of (16) 

\[ \| T_f \|_1 = \| \mu \| \leq \| \lambda \|_{R^*} \| f \|_{L^1} \]

Since \( T_f = \mu \), (17) and (18) imply that \( T_f \in A_p^p(G) \) and 

\[ \| T_f \|_{L^1} = \| T_f \|_1 + \| T(\phi) \|_p \leq \| T \|_{L^1} \| \phi \|_{L_p} + \| \phi \|_{L^p} \| \lambda \|_{R^*} = 2 \| \lambda \|_{R^*} \| f \|_{L^1} \]

The mapping \( T \) which maps \( B(G) \) into \( A_p^p(G) \) can therefore be extended to a map from \( A_p^p(G) \) into \( A_p^p(G) \) which is bounded linear and also translation invariant, i.e., \( T \) can be extended to an element of \( M C A_p^p(G) \), with norm 

\[ \| T \| \leq 2 \| \mu \| \]

We now claim that for this \( T \), \( \beta_T = \lambda \).

We actually have 

\[ \beta_T(f * g) = \int f(\sigma) \hat{\phi}(\sigma) \sigma \sigma d\sigma = \lambda(f * g), \]

\[ f \in B(G). \]

Since \( B(G) \) is dense in \( A(G) \), we see that 

\[ \beta_T(f) = \lambda(f), \quad f \in A(G) \]

Hence

\[ \beta_T = \lambda \]

and the mapping \( \beta \) is an isomap. To complete the proof of our
theorem, we propose to show that the completion of $R(G_c)$ can be embedded into space of continuous functions.

Let $\mathcal{F}$ be a fixed but arbitrary element of $R(G_c)$. Consider $T = T_x$ for some $x \in G_c$. Since $\|T\| = 1$ and

$$\tilde{T}(\gamma) = \langle -x, \gamma \rangle$$

for all $\gamma \in G$, we have

$$\|\tilde{T}(\mathcal{F})\| = \int_{G} \langle -x, \gamma \rangle \mathcal{F}(\gamma) d\gamma = |\mathcal{F}(x)|$$

so that

$$(20)$$

$$\|\tilde{T}(\mathcal{F})\|_{\infty} \leq \|\tilde{T}(\mathcal{F})\|_R$$

Since $\mathcal{F}$ is arbitrary, (19) holds for every $\mathcal{F} \in R(G_c)$.

If $\overline{R(G_c)}$ is the completion of $R(G_c)$ then $\overline{R(G_c)}$ can be thought of as the Cauchy sequences $\{\mathcal{F}_n\}$ of elements of $R(G_c)$. From (19) it follows that every Cauchy sequence in $R(G_c)$ norm is also a Cauchy sequence in the essential supremum norm. Hence to each Cauchy sequence in $R(G_c)$ there exists a continuous function $f$ on $G_c$ such that $\mathcal{F}_n \to f$ in the essential supremum norm. Setting $i(\{\mathcal{F}_n\}) = f$ we have a well defined linear map from $\overline{R(G_c)}$ into $C(G_c)$ the space of continuous functions on $G_c$.

The inequality (19) shows that this mapping is continuous.

To prove $i$ is injective, it suffices to prove that if $\{\mathcal{F}_n\} \in R(G_c)$ is Cauchy and

$$\lim_m \|\mathcal{F}_n\|_{\infty} = 0$$

then
\[
\lim_{n} \|f_{n}\|_{R} = 0. \quad \text{For } g \in L^{1}(G), \text{ let } T_{g} \text{ be the element of } \text{MCA}_{\infty}(G) \text{ given by }
\]
\[
T_{g}(f) = g \ast f, \quad f \in \text{AP}_{\infty}(G).
\]

Then
\[
\|T_{g}\| \leq \|g\|_{1} \quad \text{and} \quad T_{g}^{\vee}(\gamma) = g(\gamma), \quad \gamma \in \Gamma.
\]

If \( g \in \text{B}(G) \), then
\[
\lim_{n} \|f_{n}\|_{1} = \lim_{n} \left| \int f_{n}(\gamma) T_{g}^{\vee}(\gamma) d\gamma \right| = \lim_{n} \left| g \ast f_{n}(0) \right| 
\]
\[
\leq \lim_{n} \|g\|_{1} \|f_{n}\|_{\alpha} = 0.
\]

Hence
\[
\lim_{n} \|T_{g}^{\vee} f_{n}\| = 0, \quad f \in \text{B}(G).
\]

Let \( \{\varphi_{\alpha}\} \subset \text{B}(G) \) be an approximate identity for \( \text{AP}_{\infty}(G) \) satisfying \( \|\varphi_{\alpha}\|_{1} \leq 1 \). Then \( T_{\varphi_{\alpha}} = h_{\alpha} \in \text{B}(G) \) and
\[
\|T_{f} - T_{h_{\alpha}} f\|_{S} = \|T_{f} - h_{\alpha} \ast f\|_{S}
\]
\[
= \|T_{f} - T_{\varphi_{\alpha}} \ast f\|_{S} \leq \|T\| \|f - \varphi_{\alpha} \ast f\|_{S}
\]
\[
\rightarrow 0, \quad f \in \text{AP}_{\infty}(G).
\]

Therefore \( T_{f} = \lim_{\alpha} T_{\varphi_{\alpha}}(f) \), \( f \in \text{AP}_{\infty}(G) \) in the \( \text{AP}_{\infty}(G) \) norm. Also
\[ \| T_{\alpha}(f) \|_S = \| T_{\alpha}x \|_S = \| \Phi \|_S \]

\[ \leq \| \Phi \|_1 \| T_f \|_S \]

\[ \leq \| T \| \| f \|_S. \]

Hence

\[ \| T_{\alpha} \| \leq \| T \| \text{ for all } \alpha. \]

We now claim that

\[ P_{T_{\alpha}}(u) \rightarrow P_T(u), \quad u \in R(G_\varepsilon). \]

If \( f, g \in B(G_\varepsilon) \) then \( f \times g \in R(G_\varepsilon) \) and

\[ \beta_{T_{\alpha}}(f \times g) = T_{\alpha}f \times g(0) \rightarrow T_f \times g(0) = \beta_T(f \times g), \]

since \( T_{\alpha}f \rightarrow T_f \) in the \( L^1(G_\varepsilon) \) norm. Further if \( u \in R(G_\varepsilon) \) and \( \varepsilon > 0 \) is given we can find \( f, g \in B(G_\varepsilon) \) such that

\[ \| f \times g - u \|_R < \varepsilon/3 \| T \|. \]

Now

\[ \| f \times g - u \|_R < \varepsilon/3 \| T \|. \]
\[ |P_{Th_a}(u) - \beta_T(u)| \leq |P_{Th_a}(u) - P_{Th_a}(b \times g)| + |P_{Th_a}(b \times g) - \beta_T(b \times g)| + |\beta_T(b \times g) - \beta_T(u)| \]

\[ \leq \|T_{Th_a}\| \|u - b \times g\|_R + |P_{Th_a}(b \times g) - \beta_T(b \times g)| + \|T\| \|b \times g - u\|_R \]

\[ \leq 2\|T\| \|b \times g - u\|_R + |P_{Th_a}(b \times g) - \beta_T(b \times g)|. \]

The first term on the right-hand side can be made small by choosing \( f \) and \( g \) and then the second term is made small because of (92). We thus conclude that

\[ \lim_{n \to \infty} \beta_{Th_a}(f_n) = \beta_T(u), \quad u \in R(G_t) \]

Since \( \beta_{Th_a}(f_n) \) converges to \( \beta_T(f_n) \) for every \( n \).

Therefore \( \lim_{n \to \infty} \beta_{Th_a}(f_n) \) converges to \( \lim_{n \to \infty} \beta_T(f_n) \).

Since \( \lim_{n \to \infty} \beta_{Th_a}(f_n) = 0 \) for all \( \alpha \), \( \lim_{n \to \infty} \beta_T(f_n) = 0 \) by (92).

This is true for all \( T \in M\mathcal{A}_\infty^P(G_t) \).

Corresponding to every \( \epsilon > 0 \), and every integer \( n \) there exists \( T_n \in M\mathcal{A}_\infty^P(G_t) \) satisfying

\[ \|T_n\| \leq 1, \quad \|f_n\|_R \leq |\beta_T(f_n)| + \epsilon/3. \]

Since \( \{f_n\} \) is Cauchy in \( R(G_t) \), there exists a \( N \) such that

\[ \|f_m - f_n\|_R < \epsilon/3, \quad m, n \geq N. \]
Hence for $m > N$ we have

$$\|f_n\|_R \leq \| \beta_{TN}^* (f_n - f_m) \| + \| \beta_{TN}^* (f_m) \| + \epsilon / 3$$

$$\leq \| T_N \| \| f_n - f_m \| + \| \beta_{TN}^* (f_m) \| + \epsilon / 3.$$

Since

$$\lim_{m \to \infty} \| \beta_{TN}^* (f_m) \| = 0,$$

we have

$$\| f_n \|_R \leq 2 \epsilon / 3.$$

Hence

$$\| f_n \|_R \leq \| f_n - f_n \|_R + \| f_n \|_R < \epsilon, \quad n > N,$$

that is

$$\lim_{n \to \infty} \| f_n \|_R = 0$$

as desired. This completes the proof of the theorem.

In the case of the algebra $SP(G_0)$, for $1 \leq p < \infty$, we have the following characterizations of the multiplier space $M_{SP(G_0)}$.

**Theorem 2.7.** (a) If $1 \leq p < 2$, there exists a continuous isomorphism of $M_{SP(G_0)}$ onto $L^p(G_0)$. (b) If $p > 2$, there exists a continuous isomorphism of $M_{SP(G_0)}$ onto the dual space of a Banach space of continuous functions.
The proof of (a) is similar to that of Theorem 2.4. In the case of (b) the construction of the Banach space of continuous functions is similar to that given in Theorem 2.6 except that in the construction of $R(G_t)$, elements of $M_{\beta}^p(G_t)$ are used instead of elements of $M(C_{\beta}^p(G_t))$.

We give below an example similar to Example 2.5 of an element in $M(C_{\beta}^p(G_t))$ when $p > 2$ which is not given by convolution with a bounded Radon measure in $M_{\beta}^p(G_t)$.

**Example 2.5.** Let $E \subseteq \mathbb{R}^n$ be an infinite Sidon set. Let $m = \frac{p}{2}, n = \frac{m}{m-1}$. Choose $\gamma$ such that $0 < \gamma < 2$ and $\gamma > 2$. Choose $\varphi$ as in Example 2.5. Since $\varphi \in L^\infty(\mathbb{R}^n)$, if $f \in C(\mathbb{R}^n)$, then we have $\varphi * f \in \Lambda^p C(\mathbb{R}^n)$ with

$$||\varphi * f||_A \leq ||\varphi||_A ||f||_A \leq ||\varphi||_{\infty} ||f||_A$$

Since $\varphi \in \Lambda^p C(\mathbb{R}^n)$, consider $\omega \in \mathbb{N}$ such that

$$\sum_{\omega \in \mathbb{N}} 1^{p-1} \frac{w^p}{\omega^n} < \infty$$

Since

$$\sum_{\omega \in \mathbb{N}} \omega^p \leq N(\omega) = 1,$$

and $\mathbb{M}$ is discrete,

$$\omega(\gamma) \leq 1, \quad \gamma \in \mathbb{M}$$

Therefore
\[
\frac{1}{4} \int \phi(x) \, \mathrm{d}x \leq \frac{1}{4} \int \phi(x) \, \mathrm{d}x < \infty
\]
that is \( \hat{f} \in L^p(M) \) with
\[
\| \hat{f} \|_p \leq \| \hat{f} \|_{p,\omega}.
\]
This being true for all \( \omega \in \mathcal{F} \), for which \( \hat{f} \in L^p,\omega(M), \)
\[
\| \hat{f} \|_p \leq \| \hat{f} \|_\Lambda.
\]
As in Example 2.5, we can prove that
\[
\| \phi \hat{f} \|_2 \leq K \| \hat{f} \|_p
\]
where \( K \) is a constant depending only on \( \phi \). Then
\[
\| \phi \hat{f} \|_2 \leq K \| \hat{f} \|_\Lambda < \infty.
\]
Hence there exists \( g \in L^2(G_\varepsilon) = L^1(G_\varepsilon) \) satisfying \( \hat{g} = \phi \hat{f} \) with
\[
\tag{24}
\| g \|_1 \leq \| g \|_2 \leq \| g \|_2 \leq K \| \hat{f} \|_{p,\omega} \leq K \| \hat{f} \|_\Lambda.
\]
From (23) and (24), the mapping \( T : \mathcal{S}^p(G_\varepsilon) \rightarrow \mathcal{S}^p(G_\varepsilon) \) given by
\[
Tf = g
\]
is a multiplier on \( \mathcal{S}^p(G_\varepsilon) \) with \( \hat{T} = \phi \). Since
\[
\| \phi \|_1 = \infty, \phi \neq \hat{\mu} \text{ for any bounded measure } \mu \in M_{\text{b,comp}}(G_\varepsilon).
\]
CHAPTER III

MULTIPLIERS ON THE SEGAL ALGEBRAS $\mathcal{C}(\mathbb{R}), \mathcal{C}^k(\mathbb{R}), S(\mathbb{R})$ AND $W(\mathbb{R})$

In this chapter we shall discuss the multipliers on the Wiener algebra and the algebras $\mathcal{C}^k(\mathbb{R}), S(\mathbb{R})$ and $W(\mathbb{R})$. First we consider the Wiener algebras and give a necessary and sufficient condition for a function $F$ defined on the real line to him to be the Fourier transform of a multiplier. Next we consider the algebra $S(\mathbb{R})$ and give a characterization of the multipliers from $L^1(\mathbb{G})$ into $S(\mathbb{R})$. If $G$ is a compact abelian group, the space of multipliers from $L^1(\mathbb{G})$ to the Segal algebra $S(\mathbb{G})$ is characterized and particular cases are then considered. These include $\mathcal{C}^k(\mathbb{R})$ and $V(\mathbb{R})$.

Let $W(\mathbb{R})$ denote the class of all continuous functions $f$ defined on the real line $\mathbb{R}$ satisfying

$$\sum_{k=-\infty}^{\infty} \max_{|k| \leq K} |f(x)| < \infty.$$ 

$W(\mathbb{R})$ is then a Segal algebra if we define the norm on $W(\mathbb{R})$ by

$$||f||_S = \sup_{x \in \mathbb{R}} \sum_{k=-\infty}^{\infty} \max_{|k| \leq K} |f(x)|.$$ 

We observe that if
\[ \|f\|_W = \sum_{k=-\infty}^{\infty} \max_{k \leq \alpha \leq k+1} |f(\alpha)| \]

then

\[ \|f\|_W \leq \|f\|_S \leq 2\|f\|_W \]

so that the norms \( \| \cdot \|_S \) and \( \| \cdot \|_W \) are equivalent.

The following theorem gives a necessary and sufficient condition for a bounded continuous function defined on the real line to be the Fourier transform of a multiplier.

**Theorem 3.3.** A bounded and continuous function \( F \) defined on the real line \( \mathbb{R} \) is the \( \mathcal{F} \)-Fourier transform of an element in \( \text{M}[\text{W}^0(\mathbb{R})] \) if and only if there exist \( f_{b_n} \in \text{C}[\text{W}^0(\mathbb{R})] \) such that

\[ f_{b_n} \to F \]

uniformly on compact subsets of \( \mathbb{R} \) and

\[ \|T_{f_{b_n}}\| \leq K \]

for some constant \( K \) for all \( n \) where \( T_{g} \) is the multiplier corresponding to \( g \in \text{W}[\text{W}^0(\mathbb{R})] \) defined by
\[ T_g(f) = g \ast f, \quad f \in W(C). \]

**Proof.** Let \( T \in M(W(C)) \) be such that \( \hat{T} = F \), that is
\[ \hat{T}(x) = F(x) \hat{f}(x), \quad f \in W(C), x \in R. \]

Let \( \epsilon_n \) be an approximate identity for \( W(C) \) satisfying the conditions that each \( \hat{\epsilon}_n \) has compact support and \( \| \epsilon_n \|_1 = 1 \) for all \( n \). Set \( f_n = T \epsilon_n \). Then
\[ \hat{f}_n = F \hat{\epsilon}_n. \]

Since \( \hat{\epsilon}_n \to 1 \) uniformly on compact subsets of \( R \), it follows that
\[ \hat{f}_n \to F \]
uniformly on compact subsets of \( R \). Also
\[ \| T_{f_n}(g) \|_S = \| f_n \ast g \|_S = \| T_{\epsilon_n} \ast g \|_S = \| \epsilon_n \ast T_{f_n} \|_S \leq \| \epsilon_n \|_1 \| T_{f_n} \|_S = \| T_{f_n} \|_S, \]
where \( f_n \to f \in W(C) \).

This implies that
\[ (1) \quad \| T_{f_n} \| \leq \| T \| \]
for all \( n \), thus completing the necessary part of the theorem.

To prove the sufficiency, we proceed as follows. Let \( f_n \)
be a sequence of functions in \( W(C) \) satisfying the condition that
\[ f_n \to F \quad \text{uniformly on compact subsets of } R \quad \text{and} \quad \| T_{f_n} \| \leq K. \]
$\|T_{f_n}\| \leq K$ for some constant $K$. Let $f \in L^1(\mathbb{R})$ be such that $\hat{f}$ has compact support. Then by the definition of the sequence $\{\hat{f}_n\}$,

$$\hat{f}_n \to \hat{f},$$

uniformly. Since all the functions $\hat{f}_n$ and $\hat{f}$ are supported by a fixed compact set,

$$\|\hat{f}_n - \hat{f}\|_1 \to 0 \quad \text{as} \quad n \to \infty.$$

Now by considering the inverse Fourier transform we have

$$\|f_n \ast f - g\|_2 \to 0 \quad \text{as} \quad n \to \infty,$$

where $g$ is the Fourier transform of $f$. Then

$$\max_{k \leq 2sk+1} |g(x+k)| = \max_{k \leq 2sk+1} \lim_{n \to \infty} |f_n \ast f(x+k)|$$

$$= \lim_{n \to \infty} \max_{k \leq 2sk+1} |f_n \ast f(x+k)|$$

so that

$$\sum_{k=-\infty}^{\infty} \max_{k \leq 2sk+1} |g(x+k)| \leq \sum_{k=-\infty}^{\infty} \lim_{n \to \infty} \max_{k \leq 2sk+1} |f_n \ast f(x+k)|$$

$$\leq \liminf_{n \to \infty} \sum_{k=-\infty}^{\infty} \max_{k \leq 2sk+1} |f_n \ast f(x+k)|$$

(by Fatou's Lemma).
\[
\leq \liminf_{n \to \infty} \| T f_n \|_s = \liminf_{n \to \infty} \| f_n \|_s \leq \liminf_{n \to \infty} \| T f_n \|_s \leq K \| f \|_s
\]

The above inequality is true for all \( \alpha \in \mathbb{R} \) which therefore implies that

(9) \[ \| g \|_s \leq K \| f \|_s \]

Thus \( g \in W(CR) \). If we define \( T f = g \), we have a mapping \( T \) from \( B(CR) \) into \( W(CR) \), where \( B(CR) = \{ f \in W(CR) : f \} \) has compact support. This mapping is linear and continuous, since \( B(CR) \) is dense in \( W(CR) \), \( T \) can be extended to \( W(CR) \) as a multiplier on \( W(CR) \). Moreover, the relations

\[ \frac{T f}{\| f \|_s} = f = F f \]

imply that \( \frac{T}{\| f \|_s} = F \). This completes the proof of the theorem.

Remark 3.3. Let \( N(CR) = \{ F : F \) is bounded and continuous on \( R \) with the property that there exists a sequence \( f_n^3 \subset W(CR) \) satisfying \( \| T f_n \|_s \leq K \) for all \( n \) and \( f_n^3 \to F \) uniformly on compact subsets of \( R \) \}. Then \( N(CR) \) is a Banach space with the following norm
\[ \left\| f_n \right\|_{L^1(K)} \leq \inf_{p \in P} \left\| f_n \right\|_{L^1(p)} \leq K \] for all \( n \) and \( f_n \to F \) uniformly on compact subsets of \( \mathbb{R}^d \).

Equations (1) and (2) then imply that \( M[L^1(K)] \) is isometrically isomorphic to \( N(K) \).

**Theorem 3.2.** can be generalised to a general Segal algebra on a compact abelian group as follows.

**Theorem 3.2.** Let \( G \) be a compact abelian group with character group \( \Gamma \). Let \( S(G) \) be a Segal algebra on \( G \).

Let

\[ N(\Gamma) = \{ \phi \in L^1(\Gamma) : \exists f \in S(G) \text{ such that } \left\| f \right\|_{L^1(\Gamma)} \leq K \text{ for some constant } K \text{ and } f_n \to \phi \text{ uniformly on compact subsets of } \mathbb{R}^d \} \]

\( N(\Gamma) \) is then a Banach space with the norm given by:

\[ \left\| \phi \right\|_{N(\Gamma)} = \inf \{ K : \exists f \in S(G) \text{ such that } \left\| f \right\|_{L^1(\Gamma)} \leq K \text{ and } f_n \to \phi \text{ uniformly on compact subsets of } \mathbb{R}^d \} \]

The space of multipliers \( M[S(G)] \) is then isometrically isomorphic to \( N(\Gamma) \).

**Proof.** The proof follows along the same lines as in Theorem 3.2 and is hence omitted.
Let \( G \) be a locally compact non-discrete abelian group with character group \( \hat{G} \). Let \( \alpha \) be a locally bounded function on \( G \) with \( \alpha(y) \geq 1 \) for all \( y \in G \). Let \( S(\alpha) \) be defined by

\[
S(\alpha) = \left\{ f \in L^1(G) : \lim_{y \to \infty} f(y) \alpha(y) = 0 \right\},
\]

that is, for every \( \varepsilon > 0 \) there exists a compact subset \( K \) of \( \hat{G} \) such that \( |\hat{f}(\xi)\alpha(\xi)| < \varepsilon \), \( \forall \xi \in K \). Then Hirschman [1] has proved that \( S(\alpha) \) is a Segal algebra with norm

\[
\|f\|_\infty = \sup_{y \in G} |\hat{f}(y)\alpha(y)| L + \|f\|_1.
\]

We have obtained a characterization of the multipliers from \( L^1(G) \) into \( S(\alpha) \) as follows.

**Theorem 3.4.** Let \( B = \{ \mu \in M_b(G), \sup_{y \in G} |\mu(y)\alpha(y)| < \infty \} \).

Then \( B \) is a Banach space with norm given by

\[
\|\mu\|_B = \|\mu\|_1 + \sup_{y \in G} |\mu(y)\alpha(y)|.
\]

The space \( M[L^1, S(\alpha)] \) of multipliers from \( L^1(G) \) into \( S(\alpha) \) is then isometrically isomorphic to \( B \).

**Proof.** The fact that \( B \) is a Banach space is easily verified. To prove \( M[L^1, S(\alpha)] \cong B \), let \( T \in M[L^1, S(\alpha)] \). Then

\[
Tf = T \times f, \quad f \in L^1(G).
\]

Therefore by Theorem 1.5, there exists a bounded Borel measure \( \mu \) such that
Let \( \{ e_\beta \} \) be an approximate identity for \( S(\alpha) \) satisfying
\[
\| e_\beta \|_1 \leq 1, \quad e_\beta \text{ has compact support for all } \beta.
\]
Then \( \mu \in B \) with
\[
\| \mu \|_{\infty} = \| \mu \|_1 + \sup_{\beta} | \tilde{e}_\beta(\alpha) \mu(\alpha, x) |.
\]

Now
\[
| f(\alpha) \alpha(x) | = \lim_{\beta} | f(\alpha) \tilde{e}_\beta(\alpha) \alpha(x) |
\leq \lim_{\beta} \sup_{\gamma} | f(\alpha) \tilde{e}_\beta(\alpha) \alpha(x) |
\leq \lim_{\beta} \| \mu \|_{\infty} \| e_\beta \|_1 \leq \lim_{\beta} \| \mu \|_{11} = \| T \|.
\]
Therefore \( \mu \in B \) with
\[
\sup_{\gamma} | f(\alpha) \alpha(x) | \leq \| T \|.
\]

Conversely let \( \mu \in B \). Then since
\[
| f(\alpha) \tilde{e}_\beta(\alpha) \alpha(x) | \leq | f(\alpha) \alpha(x) | \| e_\beta \|_1 = | f(\alpha) \alpha(x) | \leq \sup_{\gamma} | f(\alpha) \alpha(x) |,
\]
we have
\[
\| \mu \|_{\infty} = \| \mu \|_1 + \sup_{\beta} | \tilde{e}_\beta(\alpha) \mu(\alpha, x) |.
\]
\[
\leq \| \mu \|_1 + \sup_{\beta} | \tilde{e}_\beta(\alpha) \alpha(x) | = \| \mu \|_{11}.
\]
(d) being true for all \( \{e_p^3 \} \) by a result in \([\] \), \( \mu \) corresponds to an element \( T_\mu \in M[n^{-1}, s(\alpha)] \) given by
\[
T_\mu f = \mu \times f, \quad f \in L^1(\sigma).
\]

By lemma (d):
\[
\|T_\mu\| \leq \|\mu\|_B.
\]

We have thus established a mapping from \( M[n^{-1}, s(\alpha)] \) into \( B_\beta \) which is linear and onto. It only remains to prove that this mapping is an isometry. If \( \mu \in B \), then \( \mu \in C_0(M) \).

This is because, if we take an infinite sequence \( \{b_{n_k}\} \) of elements of \( M \), we can find an infinite subsequence \( \{b_{n_k}\} \) such that
\[
|\alpha(b_{n_k})| > \frac{1}{k}
\]

This result is true because of a result in \([\] \) which says that \( \alpha \) is bounded on \( M \) if and only if \( M \) is compact. Now \( \mu \in B \) implies \( \mu \) satisfies for some constant \( K \), the following inequality
\[
|\alpha(x)\alpha(y)| \leq K, \quad x, y \in M
\]

Therefore
\[
|\alpha(b_{n_k})| |\alpha(b_{n_k})| \leq K
\]

that is
\[
|\alpha(b_{n_k})| \leq \frac{K}{|\alpha(b_{n_k})|} \leq \frac{K}{K}
\]
that is \( \mu (B_{\varepsilon n}) \to 0 \) as \( n \to \infty \). Therefore

\[ \mu \in C_0(G) \] which implies that given \( \varepsilon > 0 \), there exists a compact subset \( C \) of \( G \) satisfying

\[ |\mu (x)| = \varepsilon, \quad x \in C. \]

Since \( \hat{\phi}_\beta \to 1 \) uniformly on \( C \), given \( \varepsilon > 0 \), there exists a finite subset \( J \) of the index set such that

\[ |\hat{\phi}_\beta (x) \mu (x) - \hat{\phi}_\beta (x) \mu (x)| < \varepsilon/2, \quad x \in C, \beta \in J, \]

that is

\[ |\mu (x) - \mu(x)| < |\hat{\phi}_\beta (x) \mu (x)| + \varepsilon/2, \quad x \in C, \beta \in J. \]

Choosing \( \varepsilon < \frac{\| \mu \| \_\infty}{2} \), we have

\[ \sup \| \mu (x) - \mu(x) \| \leq \sup |\hat{\phi}_\beta (x) \mu (x)| + \varepsilon/2 \]

Now \( \mu \times \phi \) converges weakly to \( \mu \) in \( M_0(G) \). Therefore for a given \( \varepsilon > 0 \), take \( f \in C_0(G) \) such that

\[ \| f \| \_\infty \leq 1, \quad |\int f \mu| + \varepsilon/4 \geq \| \mu \| \_1. \]

Then there exists a finite subset \( J \) of indices such that

\[ |\int f \mu - \int f (\mu \times \phi) \, dx| < \varepsilon/2 \]
for all \( p \neq P \).

Take \( \overline{P} = L \cup P \). Then from (4), and (7) and (8) we have

\[
\| \mu \|_{B} + \sup_{\gamma \in \gamma} | \alpha(\gamma) \| \leq \sup_{\gamma \in \gamma} | \alpha(\gamma) \| + 1 + \| \mu \|_{B} \leq \| \tilde{T} \mu \|_{B}
\]

that is

\[
(9) \quad \| \mu \|_{B} \leq \| \tilde{T} \mu \|_{B} + 2
\]

(9) being true for every \( \varepsilon > 0 \), we have

\[
(10) \quad \| \mu \|_{B} \leq \| \tilde{T} \mu \|_{B}
\]

(4) and (10) combine to show that

\[
\| \mu \|_{B} = \| \tilde{T} \mu \|_{B}
\]

This proves that the mapping from \( M_{d}, S_{G} \) into \( B \) is an isometric isomorphism.

Let \( G \) be a compact abelian group with character group \( \Gamma \). Let \( S(G) \) be a Segal algebra on \( G \). Define \( \tilde{B} \) in the following way:

\[
\tilde{B} = \{ \mu \in \text{Rad}(G) : \text{there exists } \tilde{f}_{\alpha, x} \in S(G) \text{ satisfying } \| \tilde{f}_{\alpha, x} \|_{S} \leq K, \alpha \in A \text{ and } \tilde{f}_{\alpha} \to \tilde{f} \text{ pointwise on } \Gamma \}.
\]

Then \( \tilde{B} \) becomes a Banach space with norm given by
there exists \( \{ f_a \}_{a \in A} \subset S(G) \) satisfying \( \| f_a \|_S \leq K \) \( a \in A \) and \( f_a^* \to \mu \) pointwise.

Then we have the following characterization of the multipliers from \( L^1(G) \) into \( S(G) \).

**Theorem 3.5.** The space of multipliers \( \mathcal{M}(L^1, S) \) is isometrically isomorphic to \( \mathbb{B} \).

**Proof.** Let \( \{ e_\alpha \} \) be an approximate identity for \( L^1(G) \) consisting of functions \( \{ e_\alpha \} \) satisfying \( \| e_\alpha \|_2 = 1 \) for all \( \alpha \), has compact support for all \( \alpha \). Let \( T \in \mathcal{M}(L^1, S) \). Then \( T \in M(L^0, L^1) \). Therefore by Theorem 1.5, there exists a bounded Radon measure \( \mu \) such that

\[
Tf = \mu * f, \quad f \in L^1(G).
\]

Then we prove that \( \mu \in \mathbb{B} \). Define \( h_\alpha = \mu * e_\alpha \). \( h_\alpha = f e_\alpha \to f \) pointwise on \( G \). Also

\[
\| h_\alpha \|_S = \| \mu * e_\alpha \|_S \leq \| T \| (\| e_\alpha \|_1) = \| T \|
\]

for all \( \alpha \). This proves that \( \mu \in \mathbb{B} \) with

\[
(11) \quad \| \mu \| \leq \| T \|
\]

by the definition of the norm in \( \mathbb{B} \). Now let \( \mu \in \mathbb{B} \). Then
there exists a net \( f_n \) of functions in \( S(G) \) for every \( \varepsilon > 0 \) satisfying

\[
(12) \quad f_n(x) \to f(x), \quad x \in G, \quad \|f_n\|_S \leq \|f\|_S + \varepsilon.
\]

If \( f \in B(G) \), then since \( f \) has compact support

\[
\|f_n - f\|_\infty \to 0,
\]

that is

\[
\|f_n f - f^\wedge f\|_1 \to 0,
\]

which implies that

\[
\|f_n x f - x f\|_\infty \to 0,
\]

that is, since \( G \) is compact we have

\[
(13) \quad \|f_n x f - x f\|_1 \to 0.
\]

Since the net of functions \( f_n x f \) and \( x f \) have compact support inside a fixed compact subset \( K \) of \( G \), by Lemma 0.4, there exists a constant \( M \) such that

\[
\|f_n x f - x f\|_L \leq M \|f_n - x\|_L
\]

for all \( \alpha \), that is

\[
\lim_{\alpha} \|f_n x f - x f\|_L = 0.
\]

Therefore
\[ \| \mu * f \|_s = \lim_{\alpha} \| \mu \alpha * f \|_s \leq \lim_{\alpha} \| \mu \alpha \|_s \| f \|_1 \]
\[ \leq (\| \mu \|_1 + \varepsilon) \| f \|_1 . \]

This proves that the mapping \( T_\mu \) defined on \( L^1(G) \) by
\[ T_\mu(f) = \mu * f \]
defines an element of \( H(L^1, s) \) with
\[ (14) \quad \| T_\mu \| \leq \| \mu \|_1 + \varepsilon . \]
The inequality (14) is true for every \( \varepsilon > 0 \). Thus
\[ (15) \quad \| T_\mu \| \leq \| \mu \|_1 . \]

From (11) and (15) we then have
\[ \| T_\mu \| = \| \mu \|_1 . \]

The mapping \( \mu \rightarrow T_\mu \) from \( \tilde{B} \) into \( (L^1, s) \) is a linear map which is moreover an isometry. Therefore \( \tilde{B} \) is isometrically isomorphic to \( (L^1, s) \).

Durham and Goldberg in [ ] have defined \( \tilde{S} = \{ f \in L^1(G) : \) there exists \( \{ f_n \} \subset C(G) \) satisfying \( \| f_n \|_s \leq K \) for some \( K \) and \( \| f_n - f \|_1 \rightarrow 0 \) as \( n \rightarrow \infty \) \}. They have normed it as
\[ \| f \|_s = \| f \|_1 , \]

where for the norm to be truly by the definition of a thingy...
\[ \|f\|_\mathcal{S} = \inf \{ K : \text{there exists } \{f_n\} \subset \mathcal{S}(G) \text{ satisfying } \|f_n\|_\mathcal{S} \leq K \text{ for all } n \text{ and } \lim_{n \to \infty} \|f_n - f\|_1 = 0 \} \]

It is easily seen that \( \mathcal{S} \) is a closed ideal in \( \mathcal{B} \) and that it is isometrically embedded in \( \mathcal{B} \). We also have

**Corollary 3.6.** \( \mathcal{B} \cap \mathcal{L}^1(G) = \mathcal{S} \).

**Proof.** By Corollary 2 in [1], we have

\[ (L^1, \mathcal{S}) \cap \mathcal{L}^1(G) = \mathcal{S} \]

This implies from Theorem 3.5 that

\[ \mathcal{B} \cap \mathcal{L}^1(G) = \mathcal{S} \]

**Corollary 3.7.** \( \mathcal{S}(G) \) is a closed ideal in \( \mathcal{B} \) with

\[ \|f\|_\mathcal{B} = \|f\|_\mathcal{S} \]

**Proof.** Since \( \mathcal{S}(G) \) is a closed ideal in \( \mathcal{B} \) and \( \mathcal{S} \) is isometrically embedded in \( \mathcal{B} \), we therefore have the required result.

**Theorem 3.8.** If \( \mu \in \mathcal{B} \), \( \|\mathcal{I} \mu - \mu\|_1 \to 0 \) as \( y \to e \) the identity of \( G \) if and only if \( \mu \in \mathcal{S}(G) \).

**Proof.** If \( f \in \mathcal{S}(G) \), since by Corollary 3.7

\[ \|f\|_\mathcal{B} = \|f\|_\mathcal{S} \]

the assertion of the theorem is true by the definition of a Segal algebra. Conversely if there exists a \( \mu \in \mathcal{B} \) with

\[ \|\mathcal{I} \mu - \mu\|_1 \to 0 \text{ as } y \to e. \]
we have to prove that $\mu \in SC(G_0)$. Since $\mu \in \mathbb{F}$, there exists

$$\{f_{\alpha}^{\gamma} \in SC(G_0) \text{ satisfying}$$

$$f_{\alpha}^{\gamma}(x) \to \mu(\alpha), \ x \in \Gamma, \ \|f_{\alpha}^{\gamma}\| \leq \|\mu\|_1 + \varepsilon.$$ 

It is easily proved that there exists a subnet of $\{f_{\alpha}^{\gamma}\}$ say $\{f_{\alpha_{\rho}}^{\gamma}\}$ which converges weakly in $Mb_a(G_0)$ to $\mu$. This implies that

$$\|\mu\|_1 \leq \|\mu\|_1 + \varepsilon,$$

and since this is true for every $\varepsilon > 0$, we have

$$\|\mu\|_1 \leq \|\mu\|_1.$$ 

Therefore

$$\|\mu\|_1 \leq \|\mu\|_1$$

implies

$$\|\mu\|_1 \to 0 \text{ as } \varepsilon \to 0.$$ 

From Lemma 1.11, $\mu$ is absolutely continuous, that is

$$\mu \in B \cup \mathcal{L}(G) = \mathcal{B}.$$ 

By corollary 1.6 in [1], we therefore see that $\mu \in SC(G_0)$.

Similarly, using Theorem 1.4 in [1], we can prove the following theorem

**Theorem 3.9.** If $\mu \in \mathbb{F}$, $\mu \in SC(G_0)$ if and only if.
given \( \varepsilon > 0 \) there exists \( \delta \) such that

\[ |I(x) - \mu(x)| < \varepsilon \]

for all \( x \in I \).

We now give two applications of Theorem 3.6 to special cases of Segal algebras. Let \( \mathcal{S} \) denote the circle group. Let \( \mathcal{V}(T) \) be defined by

\[ \mathcal{V}(T) = \{ f \in L^1(T) : \| f - D_N \|_1 \to 0 \text{ as } N \to \infty \} \]

where \( D_N \) is the Dirichlet kernel of order \( N \). Then \( \mathcal{V}(T) \) is a Segal algebra with norm given by

\[ \| f \|_\mathcal{V} = \sup_{N \geq 1} \| f - D_N \|_1. \]

In order to investigate \( \mathcal{M}(L^1(T), \mathcal{V}) \) it is enough to determine the space \( \mathcal{B} \) for this algebra. If \( \mathcal{V}(T) = \{ \mu \in M_{bd}(T) : \sup_{N \geq 1} \| D_N \times \mu \|_1 < \infty \} \), then \( \mathcal{V}(T) \) is a Banach space with norm given by

\[ \| \mu \|_\mathcal{V} = \sup_{N \geq 1} \| D_N \times \mu \|_1. \]

We now prove the following theorem:

**Theorem 3.10.** \( \mathcal{B} \) is isometrically isomorphic to \( \mathcal{V}(T) \).

**Proof.** Let \( \mu \in \mathcal{B} \). Since \( D_N \in L^1(T), \mu \times D_N \in L^1(T) \), let \( f_\omega \) be the net of functions in \( \mathcal{V}(T) \) satisfying

\[ \| f_\omega \|_\mathcal{V} \leq \| \mu \|_\mathcal{V} + \varepsilon, \quad f_\omega(n) \to \hat{\mu}(n), \quad n \in \mathbb{Z} \]
for some \( \varepsilon > 0 \) where \( \mathbb{Z} \) denotes the group of all integers. Then since \( D_n \) has finite support, by a proof analogous to that of Theorem 3,6, we obtain

\[
\lim_{\alpha} \| f_{\alpha} \ast D_n - \mu \ast D_n \|_1 = 0
\]

which implies that

\[
\| \mu \ast D_n \|_1 = \lim_{\alpha} \| f_{\alpha} \ast D_n \|_1 \leq \lim_{\alpha} \sup_n \| f_{\alpha} \ast D_n \|_1 \leq \lim_{\alpha} \| f_{\alpha} \|_1 \leq \| \mu \|_1 + \varepsilon.
\]

Therefore

\[
\sup_n \| \mu \ast D_n \|_1 \leq \| \mu \|_1 + \varepsilon.
\]

This being true for all \( \varepsilon > 0 \), we have

\[
\sup_n \| \mu \ast D_n \|_1 \leq \| \mu \|_1,
\]

that is \( \mu \in \mathbb{V}(T) \) with

\[
(16) \quad \| \mu \|_1 \leq \| \mu \|_1
\]

Conversely if \( \mu \in \mathbb{V}(T) \), let \( \varepsilon_0 \) be an approximate identity for \( L^1(T) \) consisting of functions satisfying \( \| \varepsilon_0 \|_1 = 1 \) and \( \varepsilon_0 \) has compact support for all \( n \). Then if \( f_n = (x \ast e_n) \) satisfies

\[
f_n(m) = e_n(m) \widehat{\alpha}(m) \rightarrow \widehat{\alpha}(m), \quad m \in \mathbb{Z},
\]
Also

\[ \| f \|_s = \| \mu \|_{en} \|_s = \sup_N \| \mu \|_{en} \| D_N \|_1 \]

\[ \leq \| \mu \|_{en} \|_s \sup N \| \mu \|_{D_N} \|_1 \]

\[ = \| \mu \|_{en} \|_s \]

Therefore by the definition of \( B \), \( \mu \in B \) with

(17)

\[ \| \mu \|_s \leq \| \mu \|_N \]

(16) and (17) then combine to show that

\[ \| \mu \|_N = \| \mu \|_s \]

Thus there exists a linear, one to one, onto correspondence between \( B \) and \( V(C^1) \) thus proving the required result.

Regarding the Segal algebra \( C^k(\Omega) \), which is defined for an integer \( k \geq 0 \) as

\[ C^k(\Omega) = \{ f \in L^1(\Omega) \text{ has } k \text{ continuous derivatives on } \Omega \} \]

with norm

\[ \| f \|_s = \sum_{j=0}^{k} \frac{1}{j!} \| f^{(j)} \|_{L^1} \]

\( f^j \) denoting the \( j \)th differential of \( f \), we have the following characterization of \( B \).
THEOREM 3.11. \( \mathbb{R} \) is isometrically isomorphic to the space 
\[ C^k(T) = \{ f : f \text{ is } k \text{ times differentiable on } T \} \], the \( k \)th derivative is a bounded function, and the norm 
\[ \| f \|_{C^k} = \frac{1}{k!} \| f^{(k)} \| \].

PROOF. Let \( \mu \in \mathbb{R} \). Then there exists \( f_{\mu} \in C^k(T) \) with

\[ f_{\mu}^{(n)} \to \mu^n, \quad n \in \mathbb{Z}, \quad \| f_{\mu} \|_{C^k} \leq \| \mu \|_1 + 2 \]

for some \( \varepsilon > 0 \). Therefore

\[ \| f_{\mu} \|_{C^k} \leq \| f_{\mu} \|_{C^k} \leq \| \mu \|_1 + 2 \]

for all \( \mu \), which implies that there exists a function \( g \in L^\infty(T) \) and a subset of \( f_{\mu} \) say \( f_{\mu_3} \) such that

\[ f_{\mu_3} \to g \]

weakly in \( L^\infty(T) \). Since \( e^{im\lambda} \in L^1(T) \) for all \( m \in \mathbb{Z} \) we therefore have

\[ f_{\mu_3}^{(n)} \to \mu^n, \quad m \in \mathbb{Z} \]

(18) and (19) then show that

\[ \hat{g}(m) = \mu(m), \quad m \in \mathbb{Z} \]

From the uniqueness theorem for Fourier Stieltjes transforms, we
then have

\[ \int g(x) \, dx = d\mu(x), \quad x \in \mathbb{R}. \]

Since

\[ \| \frac{\delta}{\delta x} \|_\infty \leq \| \frac{\delta}{\delta x} \|_S \leq \| \mu \|_S + \epsilon \]

for all \( \alpha \beta \), there exists a subset of \( \{ \frac{\delta}{\delta x} \beta \} \) say \( \{ \frac{\delta}{\delta x} \beta \} \)

and an element \( h \in L^\infty(T) \) such that

\[ \frac{\delta}{\delta x} \beta \rightarrow h \]

weakly in \( L^\infty(T) \). This again implies that

\[ \frac{\delta}{\delta x} \beta \left( m \right) \rightarrow h \left( m \right), \quad m \in \mathbb{Z} \]

which from (13) then shows that

\[ h(m) = \frac{1}{\imath} \, g(x)(m), \quad m \in \mathbb{Z} \]

which implies that the distributional derivative of \( g \) is \( h \). Since

\[ h \in L^\infty(T) \subset L^1(T) \]

this implies by a result in [3] that \( g \) is absolutely continuous, and that its derivative exists almost everywhere. Similarly proceeding, we see that \( h \) is absolutely continuous with a derivative existing almost everywhere in \( L^\infty(T) \) and so on.

Therefore \( g \) will have \( k \) derivatives, the \( k \)th derivative being a function in \( L^\infty(T) \) which proves that \( g \in C^k(T) \). Also

at each stage

\[ \| g \|_S \leq \lim_{\alpha \beta} \| \frac{\delta}{\delta x} \beta \|_S \], \quad \| g \|_\infty \leq \lim_{\alpha \beta} \| \frac{\delta}{\delta x} \beta \|_\infty \]
Therefore there exists a subnet of \( \{ \bar{x}_3 \} \) say \( \{ \bar{x}_s \} \) satisfying

\[
\| \bar{x}_s \|_{\infty} \leq \lim_{\alpha \to \delta} \| \tilde{x}_s \|_{\infty}, \quad \| \tilde{x}_s \|_{\infty} \leq \lim_{\alpha \to \delta} \| \tilde{x}_s \|_{\infty}.
\]

This implies that

\[
\| \bar{x}_s \|_{\infty} = \sum_{j=0}^{k} \frac{1}{2^j} \| \tilde{x}_s \|_{\infty} \leq \lim_{\alpha \to \delta} \sum_{j=0}^{k} \frac{1}{2^j} \| \tilde{x}_s \|_{\infty} = \lim_{\alpha \to \delta} \| \tilde{x}_s \|_{\infty} \leq \| \mu \|_{\infty} + \varepsilon.
\]

(23) being true for all \( \varepsilon > 0 \) we then have

\[
\| \bar{x}_s \|_{\infty} \leq \| \mu \|_{\infty} + \varepsilon,
\]

that is from (23) we then have \( \mu \in C^k \) with

(23)

\[
\| \mu \|_{\infty} \leq \| \mu \|_{\infty}.
\]

Conversely if \( \mu \in C^k (T) \) then if \( \bar{x}_s \) is the approximate identity considered in Theorem 3.10, set \( \bar{x}_n = \mu \times \bar{x}_n \). Then

\[
\mu \times \bar{x}_n \in C^k (T) \text{ with}
\]

(23)

\[
\| \bar{x}_n \|_{\infty} = \| \mu \times \bar{x}_n \|_{\infty} = \sum_{j=0}^{k} \frac{1}{2^j} \| \mu \times \bar{x}_n \|_{\infty} = \sum_{j=0}^{k} \frac{1}{2^j} \| \mu \|_{\infty} \| \bar{x}_n \|_{\infty}
\]
\[ f_n^\wedge (m) = \mu^\wedge (m) e_n^\wedge (m) \to \hat{\mu}^\wedge (m), \ m \in \mathbb{Z}. \]

Therefore by the definition of \( \tilde{B} \) with \( \mu \in \tilde{B} \):

\[ \| \mu \| \leq \lim_{n} \| f_n \| \leq \| \mu \| \leq \| \mu \| \]

from (24). From (24) and (23) we then have

\[ \| \mu \|_{\tilde{B}} = \| \mu \|, \ \mu \in \tilde{B}. \]

Therefore we have established a one to one onto correspondence between \( \tilde{F} \) and \( \mathcal{C}^\wedge (T) \) which is easily seen to be linear. By (25) we see that it is also an isometry. This completes the proof of the theorem.

We also have the following characterization of the multipliers on \( \mathcal{C}^\wedge (T) \).

**Theorem 2.12.** The space of multipliers \( M[\mathcal{C}^\wedge (T)] \) is isometrically isomorphic to \( H_{\text{bd}}(T) \), the space of bounded Radon measures on \( T \).

**Proof.** If \( \mu \in H_{\text{bd}}(T) \), since \( \mathcal{C}^\wedge (T) \) is a Segal algebra from Lemma 0.2, \( \mu * f \in \mathcal{C}^\wedge (T) \) with

\[ \| \mu * f \| \leq \| f \| \| \mu \|. \]
Therefore the mapping \( T \) given by \( T_{\mu} f = \mu \ast f, f \in C^r(T) \) belongs to \( M[\mathcal{C}^r(T)] \) with

\[ ||T|| \leq ||\mu||. \]

Conversely let \( T \in M[\mathcal{C}^r(T)] \). From Theorem 0.3, we see that there exists a unique pseudomeasure \( \sigma \) satisfying

\[ T \phi = \sigma \ast \phi, \phi \in C^r(T) \]

For \( \phi \in C^r(T) \), \( \phi \) is a continuous function on \( T \) and hence belongs to \( L^2(T) \). Therefore \( \sigma \ast \phi \) is a well-defined function belonging to \( L^2(T) \) with

\[ (\sigma \ast \phi)^*(m) = \sigma^* (m) (\phi^*)^*(m), m \in \mathbb{Z}. \]

Also

\[ T_{\phi} \phi = (\sigma \ast \phi)^* \neq \sigma \ast f^*. \]

Therefore

\[ (\sigma \ast \phi)^*(m) = \sigma^* (m) (\phi^*)^*(m), m \in \mathbb{Z}, f \in C^r(T). \]

Let \( f \in C(T) \). Consider \( h(x) = f(x) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt \). Then

\[ F(x) = \int_{-\pi}^{\pi} h(t) dt \]

defines a continuous periodic function on \( T \) with

\[ F^1(x) = h(x) = f(x) - f(0). \]
Repeating the procedure \( k \) times, we obtain a function \( g \in C^k(T) \) with
\[
g^k(x) = f(x) - \hat{f}(0), \quad x \in T.
\]
Define a mapping \( T' \) on \( C^k(T) \) as follows
\[
T'f = (Tg)^k + \hat{f}(0)A(0).
\]
Then
\[
\hat{T'}f(m) = (Tg)^k(m), \quad m \in 2, m \neq 0
\]
\[
= (Tg)^k(0) + \hat{f}(0)A(0), \quad m = 0,
\]
that is from (27) we have
\[
(28) \quad \hat{T'}f(m) = \hat{f}^k(m) = \hat{A}(m)g^k(m), \quad m \neq 0
\]
\[
= \hat{A}(0)g^k(0) + \hat{A}(0)f(0) = \hat{A}(0)f(0), \quad m = 0.
\]
\( T' \) is then a well defined linear mapping of \( C^k(T) \) into itself. To prove that it is continuous, we apply the closed graph theorem. Let \( \{f_n\} \) converge to \( f \) in \( C^k(T) \), and \( T'f_n \) converge to \( g \) in \( C(T) \). Then
\[
(29) \quad \lim_n f_n^k(m) = \hat{f}(m), \quad m \in 2
\]
and
\[
(30) \quad \lim_n (Tf_n)^k(m) = \hat{A}(m), \quad m \in 2.
\]
But from (29) we have
\[(31) \quad \lim n (T^1 f_n)^\wedge (m) = \lim n A(m) h_n^\wedge (m) = A(m) \hat{f}^\wedge (m) \]

from (2). Thus (30) and (31) together give
\[g^\wedge (m) = A(m) \hat{f}^\wedge (m) = T^1 \hat{f}^\wedge (m), \quad m \in \mathbb{Z}^+
\]
This proves that \( g = T^1 f \) and that the mapping \( T^1 \) is continuous.
Also since
\[(T^1 f)^\wedge (m) = \hat{f}^\wedge (m) A(m), \quad m \in \mathbb{Z}^+
\]
\( T^1 \) defines a multiplier on \( \text{CCT} \). Since the multipliers on \( \text{CCT} \)
correspond to measures in \( H_{bd}(T)\), there exists a measure \( \mu \in \text{M}_{bd}(T) \) such that
\[(32) \quad T^1 f = \mu * f, \quad f \in \text{CCT}
\]
with
\[(33) \quad \|T^1\| = \|\mu\|.
\]
Now if \( u \in \text{C}^r(T) \), there exists a \( v \in \text{C}^r(T) \) such that
\( v - k = u - \hat{u}(0) \). Therefore
\[T^1 u = \mu * u = (T^1 \mu)^\wedge + A(0) \hat{u}(0)
\]
\[= o * v - k + A(0) \hat{u}(0)
\]
\[= o * [u - \hat{u}(0)] + A(0) \hat{u}(0)
\]
\[= o * u, \quad u \in \text{C}^r(T).
\]
Therefore
\[ \sigma \ast u = \mu \ast u, \quad u \in C^r(\mathbb{T}). \]

This implies therefore that \( \sigma = \mu \). Also
\[
\|T\|_1 = \|\mu\|_1 = \sup_{\omega \in CR(T)} \|T\omega\|_1 = \sup_{\omega \in CR(T)} \|T\omega\|_1, \quad \|\omega\|_\infty \leq 1
\]
\[
\leq \sup_{\omega \in CR(T)} \|T\omega\|_1 \leq \sup_{\omega \in CR(T)} \|T\omega\|_1, \quad \|\omega\|_\infty \leq 1
\]
\[
\leq \|T\|_1.
\]

From (34) and (36) we then have
\[ \|\mu\|_1 = \|T\|_1. \]

This proves the required result.

We now give an example of an element of \( M[WCR] \) which
does not correspond to convolution with a bounded measure. First
we prove a lemma.

**Lemma 3.13.** Every element in \( WCR \) is the sum \( \sum f_i \)
of
translates of elements with support in \((-1,1)\).

Let \( f \in WCR \). For each integer \( n \) define a continuous func-
tion \( h_n \) with support contained in \([2n-1, 2n+1]\) and satisfying
\[
h_n(x) = f(x), \quad x \in [2n-1/2, 2n+1/2]
\]
then
\[
h_n(x) = \gamma (n) = n \int_{n}^{n+1} f(x) dx.
\]
Define
\[ g_n(x) = f(x) - h_n(x) - h_{n+1}(x) \]
for \( x \in [2n+\frac{1}{2}, 2n+\frac{3}{2}] \)
and 0 outside \([2n+1/2, 2n+3/2] \).

Since \( f, h_n, h_{n+1} \) are continuous so is \( g_n \). Moreover
\[ g_n(2n+1/2) = g_n(2n+3/2) = 0 \]
so that the support of \( g_n \) is \([2n+1/2, 2n+3/2] \subset [2n, 2n+2] \).

Thus \( h_n \) and \( g_n \) are continuous functions with support contained in open intervals of length 2 and hence can be thought of as translates of continuous functions with support contained in \((-1, 1) \).

Further, it is clear from the construction that
\[ f(x) = \sum_{n=-d}^{d} (h_n(x) + g_n(x)) \]

If
\[ N_k = \max \{ |f(x)| : k \leq x \leq k+1 \} \]
then
\[ \| h_n \|_\infty \leq N_{2n-1} + N_{2n} \]
and
\[ \| g_n \|_\infty \leq 2 [N_{2n} + N_{2n+1}] \]

If we set
\[ f_{2n-1} = h_n, \quad n=0,1,2,\ldots \]
\[ f_{2n} = g_n \]
then
\[ f = \sum_{j=-d}^{d} f_j = \sum_{n=-d}^{d} (h_n + g_n) \]
and
\[ \frac{\sum_{j=-d}^{d} \| f_j \|_{d}}{\sum_{n=-d}^{d} \| g_n \|_{d}} = \frac{\sum_{k=0}^{N_k} \| h_k \|_{d}}{\sum_{k=0}^{N_k} \| g_k \|_{d}} = 3 \frac{\sum_{k=0}^{N_k} \| h_k \|_{d}}{\sum_{k=0}^{N_k} \| g_k \|_{d}} = 3 \frac{\| f \|_{1}}{\| g \|_{1}}. \]

**Theorem 3.14.** There exists a function \( F \), which is locally integrable, on the real line such that for every \( f \in W(\mathbb{R}) \) with \( F \cdot f \in W(\mathbb{R}) \) and
\[ \| F \cdot f \|_{1} \leq K \| f \|_{1} \]
for some constant \( K \) independent of \( f \in W(\mathbb{R}) \). Also, the multipliers so defined on \( W(\mathbb{R}) \) do not correspond to convolution with a bounded linear measure on \( \mathbb{R} \).

**Proof.** Define
\[ F(t) = \frac{1}{2\pi} \int_{n^2}^{(n+1)^2} e^{2\pi int} \, dt, \quad n^2 < t < (n+1)^2, \quad n = 0, 1, 2, \ldots \]
and
\[ F(t) = 0 \quad \text{for} \quad t \leq 0. \]

Then \( F(t) \) is a measurable function on the real line \( \mathbb{R} \). However, \( F(t) \) is not absolutely integrable since
\[ \sum_{n=1}^{\infty} \int_{n^2}^{(n+1)^2} |F(t)| \, dt = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{n^2}^{(n+1)^2} 1 \, dt = \sum_{n=1}^{\infty} \frac{(n+1)^2 - n^2}{n^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2} = \infty. \]

Therefore if we define
\[ m(E) = \int_{E} |F(t)| \, dt \]
for every \( E \).
for all Borel subsets of $\mathbb{R}$ then $\mu$ defines a measure on the real line, but $\mu$ is not a bounded since

$$\int_{-\infty}^{\infty} |F(t)| \, dt = \int_{-\infty}^{\infty} |F'(t)| \, dt = \infty$$

Let $f \in WCR$ be a continuous function with compact support in $[-1, 1]$. Then

$$F \ast f(x) = \int_{-1}^{1} F(x-t) f(t) \, dt$$

For $n \geq 2$, when $n^2 - 1 \leq x \leq n^2 + 1$ and $-1 \leq t \leq 1$, we have

$$n^2 - 2 \leq x - t \leq n^2 + 2$$

that is

$$(x-1)^2 < n^2 - x \leq x - t \leq n^2 + 2 \leq (n+1)^2.$$  

Therefore

$$|F \ast f(x)| = \left| \int_{-1}^{1} F(x-t) f(t) \, dt \right| \leq \|f\|_\infty \int_{-1}^{1} |F(x-t)| \, dt$$

$$\leq \|f\|_\infty \left[ \frac{4}{n^2} + \frac{1}{(n+1)^2} \right] x^2 \leq \frac{4 \|f\|_\infty}{n^2}$$

that is for $n \geq 2$ we have

\begin{equation}
(36) \quad \|F \ast f\|_\infty \leq \frac{4 \|f\|_\infty}{n^2} \quad \left[ n^2 - 1, n^2 + 1 \right]
\end{equation}
For \( \eta / 2 \leq x \leq (n+1)^2 - 1 \) and \(-1 \leq t \leq +1\),
\[
\eta^2 < x - t < (n+1)^2,
\]
which implies that
\[
F(x-t) = \frac{1}{(n+1)^2} e^{2\pi i n t}
\]
Thus
\[
F \ast f(x) = \frac{1}{(n+1)^2} \int_{-1}^{+1} e^{2\pi i n t} f(t) dt = \frac{\hat{f}(n)}{(n+1)^2},
\]
that is
\[
|F \ast f(x)| = \frac{1}{n^2} |\hat{f}(n)|.
\]
Since \( F \) has its support in \([0, \eta]\) and \( f \) in \([-1, 1]\), \( F \ast f \) is supported by \([-1, \eta]\). Therefore we are only left to consider the case \(-1 \leq x \leq 3\) and \(-1 \leq t \leq +1\). For
- \( -2 \leq x-t \leq 0 \), \( F(x-t) = 0 \)
- \( 0 \leq x-t \leq 1 \), \( F(x-t) = 1 \)
- \( 1 \leq x-t \leq 4 \), \( F(x-t) = \frac{1}{4} e^{2\pi i (x-t)} \)
Therefore for \(-1 \leq x \leq 3, -1 \leq t \leq +1\), we have
\[
|F \ast f(x)| \leq \frac{1}{3} \left\| f_{\omega_2} \right\|_2^2 = \frac{4}{3} \left\| f_{\omega} \right\|_2^2
\]
From (3.4, 3.7, 3.8) we then have

\[ \| f \ast b \| W = \sum_{k=-\infty}^{\infty} \sup_{x \leq k} | f \ast b(x) | \leq 4 \| f \| \omega + \sum_{n=2}^{\infty} \frac{8 \| f \| \omega}{n^2} + \sum_{n=2}^{\infty} \frac{2 (Qn-1)^2}{(n+1)^4} \]

\[ \leq 4 \| f \| \omega + \sum_{n=2}^{\infty} \frac{8 \| f \| \omega}{n^2} + [\sum_{n=2}^{\infty} \frac{2 (Qn-1)^2}{(n+1)^4}]^{1/2} \]

\[ \leq 4 \| f \| \omega + 8 \| f \| \omega \times \left( \sum_{n=2}^{\infty} \frac{1}{n^2} + 2 \| f \| \omega \left[ \sum_{n=2}^{\infty} \frac{(Qn-1)^2}{(n+1)^4} \right]^{1/2} \right) \]

\[ = 4 \| f \| \omega \left[ 4 + 8 \sum_{n=2}^{\infty} \frac{1}{n^2} + 2 \left[ \sum_{n=2}^{\infty} \frac{(Qn-1)^2}{(n+1)^4} \right]^{1/2} \right] \]

Let \[ C = 4 + 8 \sum_{n=2}^{\infty} \frac{1}{n^2} + 2 \left[ \sum_{n=2}^{\infty} \frac{(Qn-1)^2}{(n+1)^4} \right]^{1/2} \]

a constant independent of \( f \). Then we have the following

(3.9) \[ \| f \ast b \| W \leq C \| f \| \omega . \]

If \( g \in W(\mathbb{R}) \), then by Lemma 3.13 there exist continuous functions \( g_k \) with compact support in \([-1, 1]\) and elements \( b_k \in \mathbb{R} \) such that
\[ g = \sum_{k=-\infty}^{\infty} b_k g_k, \]
and
\[ \sum_{k=-\infty}^{\infty} \| g_k \|_\infty \leq 3 \| g \|_W \text{ from (35)}. \]

Then
\[ F \ast g(x) = \sum_{k=-\infty}^{\infty} F \ast b_k g_k(x). \]

Now
\[ \| F \ast b_k g_k \|_W \leq \| F \ast b_k g_k \|_s = \| F \ast g_k \|_s \leq 2 \| F \ast g_k \|_W \text{ from (9)}. \]

Therefore
\[ \sum_{k=-\infty}^{\infty} \| F \ast b_k g_k \|_W \leq 2c \sum_{k=-\infty}^{\infty} \| g_k \|_\infty \leq 2c \times 3 \| g \|_W. \]

which implies that
\[ \| F \ast g \|_W \leq \sum_{k=-\infty}^{\infty} \| F \ast b_k g_k \|_W \leq 6c \| g \|_W. \]

This proves that the function \( F \) defines an element of \( M(\mathbb{R}) \) which does not correspond to convolution with a bounded Radon measure on \( \mathbb{R} \).
CHAPTER IV

BIPOSITIVE AND ISOMETRIC ISOMORPHISMS OF MULTIPLIER ALGEBRAS

Biopositive and isometric isomorphisms of multiplier algebras were considered by Gundy [1]. When \( 1 \leq p < \infty \), he showed that if \( G_1 \) and \( G_2 \) are locally compact groups and \( m_p(G_1), m_p(G_2) \) are the multiplier algebras of \( L^p(G_1) \) and \( L^p(G_2) \) respectively, then a biopositive or isometric isomorphism between \( m_p(G_1) \) and \( m_p(G_2) \) induces a topological isomorphisms of the groups \( G_1 \) and \( G_2 \). In the case of abelian groups, the analogous results for the algebras \( A^p(G) \cap L^p(G) \) and \( L^1 \cap C_0(G) \) were given by Tewari [2]. We shall prove here that the results are true for Segal algebras in general. We prove the following result.

**Theorem 4.1.** [1] If \( S(G_1) \) and \( S(G_2) \) are the Segal algebras on the locally compact abelian groups \( G_1 \) and \( G_2 \), and \( M[S(G_1)] \) and \( M[S(G_2)] \) are their multiplier algebras, then a biopositive isomorphism \( \Lambda \) of \( M[S(G_1)] \) onto \( M[S(G_2)] \) induces a topological isomorphism of the group \( G_1 \) onto \( G_2 \).

Our Theorem includes the results of Tewari in the case of biopositive isomorphism of the multiplier algebras. On the other hand when \( G_t \) is compact, \( L^2(G_t) \) is a Segal algebra and the following example given by Gundy [2] shows that in the case of Segal algebras an isometric algebra isomorphism between the
multiplier algebras fail to induce a topological isomorphism between the groups.

**EXAMPLE 4.2.** Take $G_1 = T$, the circle group and $G_2 = T \times T$ the two dimensional torus. Then $m(L^2(G_1)) \cong l^\infty(Z)$ where $Z$ is the additive group of integers and

$$m[L^2(G_2)] \cong l^\infty(\mathbb{Z} \times \mathbb{Z})$$

the algebras $l^\infty$ being taken with pointwise operations and the usual sup norm. Each of $Z$ and $\mathbb{Z} \times \mathbb{Z}$ is complete. Let $\varphi$ be any one-to-one correspondence between $\mathbb{Z} \times \mathbb{Z}$ and $Z$. The mapping $T_\varphi$ of $l^\infty(Z)$ onto $l^\infty(\mathbb{Z} \times \mathbb{Z})$ defined by

$$T_\varphi \psi(m,n) = \psi(\varphi(m,n))$$

is an isometric isomorphism of $l^\infty(Z)$ onto $l^\infty(\mathbb{Z} \times \mathbb{Z})$. However $Z$ and $\mathbb{Z} \times \mathbb{Z}$ are not algebraically isomorphic, therefore $G_1$ and $G_2$ are not isomorphic.

However in the case of the special fgal algebras $A^p_\infty(G_1)$ and $S^p(G_1)$, for $1 \leq p < \infty$, we have the following results.

**Theorem 4.3.** Let $G_1$ and $G_2$ be two locally compact abelian groups and $\omega_1, \omega_2$ two weight functions on $\Gamma_1$ and $\Gamma_2$ respectively. If there exists an isometric algebra isomorphism $\Lambda$ between the multiplier algebras $M[A^p_\omega(G_1)]$ and
Then \( \mathcal{M}[A_{0,2}^{p} C_{0,2}] \) and \( \mathcal{G}_{1} \) are topologically isomorphic.

In the case of the algebras \( \mathcal{S}^{p}(C) \) defined in Example 7 of the introductory chapter, we have the following.

**Theorem 4.4. [1]** If \( \mathcal{G}_{1} \) and \( \mathcal{G}_{2} \) are two locally compact abelian groups, then an isometric algebra isomorphism between the multiplier algebras \( \mathcal{M}[\mathcal{S}^{p}(C_{1})] \) and \( \mathcal{M}[\mathcal{S}^{p}(C_{2})] \) for \( 1 \leq p < \infty \) induces a topological isomorphism between the groups \( \mathcal{G}_{1} \) and \( \mathcal{G}_{2} \).

We shall now recall a few definitions and preliminary results.

By a positive multiplier on a space \( A \) of functions on \( G \) we mean a multiplier \( T \) satisfying: \( f \in A \), \( f \geq 0 \) a.e. on \( G \) implies \( Tf \geq 0 \) a.e. on \( G \). If \( A \) and \( B \) are two spaces of functions on \( G \) then a homotopy isomorphism \( \Lambda \) between the multiplier spaces \( \mathcal{M}(A) \) and \( \mathcal{M}(B) \) is an algebraic isomorphism which satisfies the condition that \( \Lambda T \) is a positive multiplier in \( \mathcal{M}(B) \) if \( T \) is a positive multiplier in \( \mathcal{M}(A) \).

An isometric isomorphism \( \Lambda \) between \( \mathcal{M}(A) \) and \( \mathcal{M}(B) \) is an algebraic isomorphism of \( \mathcal{M}(A) \) onto \( \mathcal{M}(B) \) for which holds the equality.
\[ \|\lambda \| T \| = \| \| T \| \|, \quad T \in M(A) \]

Similar definitions hold if \( A \) and \( B \) are spaces of functions on the groups \( G_1 \) and \( G_2 \), respectively.

**Theorem 4.6.** \( L \) \( \Rightarrow \)

If \( A \) is a semisimple commutative Banach algebra and \( M(A) \) denotes the space of multipliers on \( A \), then \( M(A) \) is also a semisimple commutative Banach algebra.

**Theorem 4.7.** \( L \) \( \Rightarrow \)

If \( G \) is a locally compact abelian group, \( T \) is a positive multiplier on \( L^2(G) \) and \( \sigma \) is the pseudomeasure corresponding to \( T \), that is,

\[ T \sigma f = \sigma (f \sigma f), \quad f \in L^2(G), \]

then \( \sigma \) reduces to a positive bounded measure in \( M_{bd}(G) \).

**Theorem 4.8.** \( L \) \( \Rightarrow \)

If \( T \) is a norm-preserving multiplier of \( L^1(G) \), then there exists \( a \in G \) and a complex number \( \lambda \) of absolute value one such that

\[ T = \lambda \delta_a. \]

**Theorem 4.9.** \( L \) \( \Rightarrow \)

Let \( F(G_1) \) and \( F(G_2) \) be ideals of \( L^1(G_1) \) and \( L^1(G_2) \), respectively, which are Banach algebras in their own norm and let \( M[F(G_1)] \) denote the multiplier algebra of \( F(G_1) \).

If \( \Lambda \) is a bijective or isometric algebra isomorphism of \( F(G_1) \) onto \( F(G_2) \), then \( \Lambda \) induces a bijective or isometric algebra isomorphism of \( M[F(G_1)] \) onto \( M[F(G_2)] \).
THEOREM 4.9. \( \hat{\phi} \) is a homomorphism if
\[ L^1(G_1) \rightarrow M_{b_\delta}(G_2), \]
then \( \hat{\phi} = \phi \circ \alpha \)
where \( \alpha \) is a continuously differentiable map of \( G_1 \) into \( M_1 \) and \( Y \) belongs to the compact ring of \( G_2 \).

THEOREM 4.10. \( \parallel \) is the locally
connected Hausdorff. The spaces \( M_{b_\delta}(G) \) are isometrically
isomorphic if and only if \( G_1 \) and \( G_2 \) are topologically isomorphic.

Let
\[ c_\infty^+(G) = \{ f \in C_c(G) : f \geq 0 \text{ on } G \} \]
\[ B_\infty^+(G) = \{ f \in B(G) : f \geq 0 \text{ on } G \} \]
For \( \alpha \in G \), let \( \delta_\alpha \) denote the Dirac measure at \( \alpha \). We then have

LEMMA 4.11. If \( T \) is a positive multiplier on a locally connected
space \( S(G) \) on a locally compact abelian group \( G \) there exists \( \mu \in M_{b_\delta}(G) \)
such that
\[ T \hat{f} = \mu \hat{f} \]

PROOF. To each multiplier \( T \) on \( S(G) \) there corresponds
by theorem 0. a unique pseudomeasure \( \mu \) such that
\[ T \hat{f} = \mu \hat{f}, \quad f \in S(G). \]
Now suppose that $T$ is a positive multiplier. Then

$$\sigma \ast f \geq 0 \quad \text{a.e. on } G \quad \text{or } f \in B^+(G).$$

Let $f \in L^2(G)$ such that $f \geq 0$ a.e. on $G$. To each positive integer $n$, we can find a compact subset $K_n$ of $G$ with the property that

$$\int_{G \setminus K_n} f(x) \, dx = \frac{1}{2n} \text{ for all } n.$$

Let

$$g_n(x) = f(x) \cdot \chi_{K_n}(x)$$

where $\chi_{K_n}$ is the characteristic function of $K_n$. Let $\{e_\alpha\}$ be an approximate identity for $L^1(G)$ satisfying $e_\alpha \geq 0$ on $G$ and $e_\alpha$ has compact support for all $\alpha$. Then $\{e_\alpha f\}$ is also an approximate identity for $L^2(G)$. Hence there exists $e_n \in \{e_\alpha\}$ satisfying

$$\|e_n \ast f\|_2 \leq \frac{1}{2n}. $$

Let

$$h_n = g_n \ast e_n.$$

Then

$$\|e_n \ast h_n - \sigma \ast f\|_2 \leq \|e_n \ast g_n - \sigma \ast f\|_2$$

$$+ \|e_n \ast f - \sigma \ast f\|_2$$

$$\leq \|e_n\|_1 \|\chi_{K_n} - f\|_2 + \frac{1}{2n} \quad \text{by (2)}$$

$$\leq \|e_n\|_1 \cdot \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$
Hence $\sigma \times h_n$ converges to $\sigma \times f$ in the $L^2(G)$ norm. We can find a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ such that

$$\sigma \times h_{n_k} \rightarrow \sigma \times f \text{ a.e. on } G.$$ 

Now set $h_{n_k} = e^{h_{n_k}} \times g_{n_k}$. Then $h_{n_k} \in L^1(G)$ and

$$\sigma h_{n_k} \text{ has compact support. Moreover } h_{n_k} \geq 0 \text{ on } G \text{ so that } \sigma h_{n_k} \in L^1(G).$$

Then $\sigma \times h_{n_k} \geq 0$ on $G$ for all $n_k$ and hence $\sigma \times f \geq 0$ a.e. on $G$.

We have thus proved that if $f \in L^2(G)$ such that $f \geq 0$ a.e. on $G$, then $\sigma \times f \geq 0$ a.e. on $G$ also. This implies that $\sigma$ defines a positive multiplier on $L^2(G)$. By Theorem 4.6, therefore there exists a unique positive measure $\mu \in \mathcal{M}_b(G)$ such that

$$\sigma \times f = \mu \times f, \quad f \in L^2(G).$$

But $B(G) \subset L^2(G)$. Therefore

$$Tf = \sigma \times f = \mu \times f, \quad f \in B(G).$$

Since $B(G)$ is dense in $S(G)$, we conclude that

$$Tf = \mu \times f, \quad f \in S(G).$$

This completes the proof.
PROOF OF THEOREM 4.1. Let $S(G_1)$ and $S(G_2)$ be Segal algebras on the locally compact abelian groups $G_1$ and $G_2$ respectively. Suppose $H[S(G_1)]$ and $H[S(G_2)]$ denote their multiplier algebras. Suppose that $\Lambda$ is a bijective isomorphism of $H[S(G_1)]$ onto $H[S(G_2)]$. For each element $a \in G_1$ the translation operators $T_a$ and $T_{-a}$ are positive multipliers on $S(G_1)$. Since $\Lambda$ is a bijective isomorphism of $H[S(G_1)]$ onto $H[S(G_2)]$ it follows that $\Lambda T_a$ and $\Lambda T_{-a}$ are positive multipliers on $S(G_2)$. By Lemma 4.11 there exist positive bounded measures $\mu$ and $\nu$ in $M_b(G_2)$ such that
\[
\Lambda t_a(f) = \mu * f, \quad \Lambda t_{-a}(f) = \nu * f, \quad f \in S(G_2).
\]
Since $\Lambda$ is an algebraic isomorphism
\[
\Lambda(t_a o t_{-a}) = \Lambda(t_{e_1}) = T_{e_2} = \Lambda T_a o \Lambda T_{-a}
\]
where $e_1$ and $e_2$ denote the identities of $G_1$ and $G_2$, respectively. Therefore we have
\[
\mu * \nu = \delta_{e_2}
\]
We now claim that both $\mu$ and $\nu$ are measures with one point support. If possible, let $b_1$ and $b_2$ be two points in the support of $\mu$ and $c$ a point in the support of $\nu$. Let $\gamma \in C_0^+(G_2)$
be such that $0 \leq \gamma \leq 1$. Define measures $\mu_1$ and $\nu_1$ by

$$
\mu_1 = \langle \tau_{b_1}, x \rangle \mu + \langle \tau_{b_2}, y \rangle \mu, \quad \nu_1 = \langle \tau_c, z \rangle \nu
$$

Then $\mu_1$ and $\nu_1$ are bounded positive measures. Moreover $\mu_1 \leq \mu$ and $\nu_1 \leq \nu$ so that

$$
\mu_1 \times \nu_1 \leq \mu \times \nu
$$

But $\mu_1 \times \nu_1$ has at least two points in its support which is a contradiction. This proves our assertion that both $\mu$ and $\nu$ have one point support. Therefore there exist $a_1, b_1 \in G_{\mathbb{R}}$ and positive real numbers $\lambda_1, \lambda_2$ such that

$$
\mu = \lambda_1 \delta_{a_1}, \quad \nu = \lambda_2 \delta_{b_1}
$$

Since

$$
\mu \times \nu = \delta_{a_1}, \quad b_1 = -a_1, \quad \lambda_1 = \frac{\lambda_2}{\lambda_2}
$$

Therefore to every $a \in G_{\mathbb{R}}$ we can associate a $a \in G_{\mathbb{R}}$ and a positive real number $\lambda(a)$ such that

$$
\lambda(a) (f) = \lambda(a) \sum f(c) \tau_{\mu}(c) \tau_{\nu}(c) \quad f \in SC(G_{\mathbb{R}})
$$

Since $\lambda$ is an algebraic isomorphism, it follows that $\phi$ is an algebraic isomorphism of $G_{\mathbb{R}}$ onto $G_{\mathbb{R}}$ and that $\lambda$ is an algebraic isomorphism of $G_{\mathbb{R}}$ onto the set of positive real numbers. Now if $\lambda(a) > 1$ for some $a \in G_{\mathbb{R}}$, there exists a sequence of elements $\{a_n\} \subset G_{\mathbb{R}}$ such that

$$
\lambda(a_n) > n^3 \quad \text{for all } n.
$$
Consider the positive multiplier on $S(G_1)$ defined by

$$\lambda = \sum_{n=1}^{\infty} \frac{1}{n^2} \tau_n$$

Then $\lambda - \frac{1}{n^2} \tau_n$ is a positive multiplier on $S(G_1)$ for all $n$. Since $\wedge$ is bipositive, $\lambda(\bigcap_{n} \tau_n)$ is positive for every $n$. If $\mu$ denotes the positive measure in $M_{bd}(G_2)$ corresponding to $\lambda$, then we have

$$\mu \geq \frac{1}{n^2} \lambda(a_n) \sum_{k} \varphi(k_n) \quad \text{for each } n$$

that is

$$\mu \geq n^3 \cdot \frac{1}{n^2} \sum_{k} \varphi(a_n) = n \sum_{k} \varphi(a_n) \quad \text{for each } n,$$

which is a contradiction to the fact that $\mu$ is a bounded measure.

Therefore

$$\lambda(a) = 1, \quad a \in G_1$$

It remains to show that $\varphi$ is a topological isomorphism.

It is enough to show that $\varphi$ is continuous. Then the same argument with $\varphi^{-1}$ will prove the continuity of $\varphi^{-1}$. In order to prove $\varphi$ is continuous, it suffices to prove that if $a_i \to a$ in $G_1$, then $\varphi(a_i) \to \varphi(a)$ in $G_2$. Suppose this not. There exists an open neighbourhood $V$ of $e_2$ and an infinite subnet of $\{ \varphi(a_n) \}_{n \geq 1}$ such that all the elements of that subnet lie outside $V$ for all sufficiently large indices $n$. We shall assume
without loss of generality that \( \{ \phi(a_i) \} \) belongs to the complement of \( V \) for all \( i \).

Now \( \{ \phi(a_i) \} \) is a norm bounded net of measures in \( M_{bd}(G_{a_2}) \). Therefore by Alaoglu's theorem there exists a subnet of \( \{ \phi(a_i) \} \) which, without loss of generality, we assume to be itself, such that \( \hat{\phi}(a_i) \) converges to \( \mu \) weakly in \( M_{bd}(G_{a_2}) \) for some positive measure \( \mu \in M_{bd}(G_{a_2}) \).

Take \( h \in C_c^+(G_{a_1}) \). Denote the multipliers generated by \( h \) and \( S_{a_i} \times h \) by \( W_h \) and \( W_{S_{a_i} \times h} \) respectively. Since \( a_i \to a \) in \( G_{a_1} \), we have

\[
S_{a_i} \times h \to h
\]

in \( M_{bd}(G_{a_2}) \). Now \( M_{bd}(G_{a_2}) \) norm is stronger than the \( M_{ESC}(G_{a_2}) \) norm. Therefore we have

\[
W_{S_{a_i} \times h} \to W_h \quad \text{in} \quad M_{ESC}(G_{a_2})
\]

In other words

\[
T_{a_i} \circ W_h \to W_h \quad \text{in} \quad M_{ESC}(G_{a_2})
\]

Since \( M_{ESC}(G_{a_2}) \) and \( M_{ESC}(G_{a_2}) \) are two semi-simple commutative Banach algebras and \( \Lambda \) is a bijective isomorphism between them, \( \Lambda \) is continuous. Hence

\[
\Lambda t_a \circ \Lambda W_h = \Lambda (t_{a_i} \circ W_h) \to \Lambda W_h
\]
in $M(S(G_{12}))$, $\Lambda_{W_h}$ being a positive multiplier on $S(G_{12})$, we can find a corresponding positive measure $\mu_1 \in M_{bd}(G_{12})$. Then

$$\sum \varphi(ai) \ast \mu_1 \ast f \to \mu_1 \ast f$$

in the $S(G_{12})$ norm for all $f \in S(G_{12})$. But the $L^1(G_{12})$ norm is weaker than the $S(G_{12})$ norm so that

$$\sum \varphi(ai) \ast \mu_1 \ast f \to \mu_1 \ast f, \quad f \in S(G_{12})$$

in the $L^1(G_{12})$ norm also. Since $S(G_{12})$ is dense in $L^1(G_{12})$ we conclude that

$$(a) \quad \sum \varphi(ai) \ast \mu_1 \ast f \to \mu_1 \ast f, \quad f \in L^1(G_{12})$$

in the $L^1(G_{12})$ norm. We have also proved that

$$(b) \quad \sum \varphi(ai) \to \mu \text{ weakly in } M_{bd}(G_{12})$$

From (a) it follows that

$$\sum \varphi(ai) \ast f(x) \to \mu \ast f(x), \quad f \in C_c(G_{12}).$$

Thus if $f \in C_c(G_{12})$, then $\mu_1 \ast f \in C_c(G_{12})$ so that

$$\sum \varphi(ai) \ast \mu_1 \ast f(x) \to \mu_1 \ast \mu \ast f(x), \quad x \in G_{12}$$

On the other hand (a) implies

$$\sum \varphi(ai) \ast \mu_1 \ast f \to \mu_1 \ast f$$

in $L^1(G_{12})$ so that there exists a subnet $\{\varphi(ai_k)\}_k$ of
\{ \phi(\alpha) \} \text{ such that}

\[ \mathbb{E}_\phi(\alpha \in H) \times \mu_1 \times f(x) \to \mu_1 \times f(x) \]

for almost all \( x \in \mathbb{R}_2 \). Thus we have

\[ \mu \times \mu \times f = \mu_1 \times f \text{ a.e. on } \mathbb{R}_2. \]

Since both the functions are continuous, we have

\[ \mu \times \mu_1 \times f = \mu_1 \times f, \quad f \in C_c(\mathbb{R}_2). \]

But \( C_c(\mathbb{R}_2) \) is dense in \( C_0(\mathbb{R}_2) \). Thus we must have

\[ \mu \times \mu_1 = \mu_1 \]

and so

\[ W_\mu \circ \Lambda W_h = \Lambda W_h \]

where \( W_\mu \) is the multiplier on \( S(\mathbb{R}_2) \) corresponding to \( \mu \).

This gives

\[ \Lambda^{(-1)}(W_\mu) \circ W_h = W_h \]

and so

\[ \Lambda^{(-1)}(W_\mu) \circ W_h(f) = W_h(f), \quad f \in S(\mathbb{R}_2). \]

Thus

\[ \Lambda^{(-1)}(W_\mu)(h \times f) = h \times f, \quad f \in S(\mathbb{R}_2). \]

If \( \mu_1 \) is the positive measure corresponding to \( \Lambda^{(-1)} W_\mu \).
we have
\[ D_1 \ast \eta \ast f = \eta \ast f, \quad \eta \in C_c^+ (G \mathbb{R}), \quad f \in S (G \mathbb{R}), \]
so that
\[ D_1 = \mathcal{E}_{\eta_1} \]
and therefore
\[ \Lambda^{-1} (W_\mu) = \mathcal{E}_{\eta_2} \]
which gives
\[ W_\mu = \Lambda (\mathcal{E}_{\eta_1}) = \mathcal{E}_{\eta_2} . \]
This implies
\[ \mu = \mathcal{E}_{\eta_2} . \]
Hence
\[(a) \quad \mathcal{E}_{\eta (ai)} \to \mathcal{E}_{\eta_2} \text{ weakly in } H_{bd} (G \mathbb{R}) . \]
Consider a neighbourhood \( W \) of \( \eta_2 \) whose closure is compact. There exists a neighbourhood \( U \) of \( \eta_2 \) with closure \( \overline{W} \subset U \). There exists a function \( f \in L^1 (G \mathbb{R}) \) satisfying \( \hat{f} = 0 \) outside \( U \) and \( \hat{f} = 1 \) on \( \overline{U} \). Since \( f \in C_0 (G \mathbb{R}) \) by (a) we have
\[ \hat{f} \mathcal{E}_{\eta (ai)} \to \hat{f} \mathcal{E}_{\eta_2} = 1 . \]
Hence there exists an infinite subset of \( \{ \eta (ai) \} \) say \( \{ \eta (ai_k) \} \) inside \( W \subset \overline{W} \). Since \( \overline{W} \) is compact, there exists an infinite subset of \( \{ \eta (ai_k) \} \) say \( \{ \eta (ai_k) \} \) and an \( a \in G \mathbb{R} \) such
that

\[ \phi(\alpha_{k,e}) \rightarrow a_1. \]

Hence \( a_1 \notin V \) since \( \phi(\alpha_{k,e}) \) belongs to complement of \( V \) for all \( i,k,e \). Since \( \phi(\alpha_{k,e}) \rightarrow a_1 \)

\[ f(\phi(\alpha_{k,e})) \rightarrow f(a_1) \quad f \in C_0(C(G_2)) \]

that is

\[ \Xi_{1,e} \rightarrow \Xi_{1,a_1} \quad \text{weakly} \]

Combining this with (3) we have \( \Xi_{1,e} = \Xi_{1,a_1} \) which is impossible since \( a_1 + e_2 \). Hence \( \phi \) is continuous, thus completing the proof of the theorem.

**Corollary 4.10.** A bijective isomorphism of \( S(C G_1) \) onto \( S(C G_2) \) induces a topological isomorphism of \( G_1 \) onto \( G_2 \).

**Proof.** From Theorem 4.8, a bijective isomorphism of \( S(C G_1) \) onto \( S(C G_2) \) induces a bijective isomorphism of \( M[S(C G_1)] \) onto \( M[S(C G_2)] \). The conclusion then follows from Theorem 4.1.

Before we prove Theorem 4.9, we need a few lemmas.

**Lemma 4.12.** Let \( G \) be a locally compact abelian group and \( \Gamma \) a norm-preserving multiplier of \( A_0^p(G) \) for a suitable weight function \( \omega \). Then there exist a \( \alpha \in \Gamma \) and a complex number \( \Lambda \) of absolute value 1 such that \( T = \Lambda T_\alpha \).
PROOF. Case 1. \( G \) is compact. Let \( y \in G \). Then
\[
y \times y = y
\]
and hence
\[
T(y \times y) = T(y) \times y = T(y)
\]
that is
\[
T(y) = \phi(y)y
\]
where \( \phi(y) \) is a complex number. Since \( T \) is norm preserving it follows that \( |\phi(y)| = 1 \), \( y \in G \). For any \( f = \sum_{i=1}^{n} a_i \phi_i \) in \( B(G_i) \) we then have
\[

\|T \left( \sum_{i=1}^{n} a_i \phi_i \right) \|_s = \|T \left( \sum_{i=1}^{n} a_i \phi_i \right) \|_1 + \left[ \sum_{i=1}^{n} a_i \phi_i \phi_i \| \phi_i \|^2 \right]^{1/p}.
\]

Since \( T \) is norm preserving
\[
\|Tf\|_s = \|f\|_s = \|f\|_1 + \left[ \sum_{i=1}^{n} a_i \phi_i \phi_i \| \phi_i \|^2 \right]^{1/p}.
\]

Therefore
\[
\|Tf\|_1 = \|f\|_1, \quad f \in B(G_i).
\]

Since \( B(G_i) \) is dense in \( L^1(G_i) \), there exists a unique norm preserving multiplier \( T' \) of \( L^1(G_i) \) into itself such that
\[
T'f = Tf, \quad f \in A_p^0(G_i).
\]
Hence by Theorem 4.7, there exists \( \lambda \) and \( \sigma \) as desired such that
\[
Tf = \lambda T \sigma f, \quad f \in A_p^0(G_i).
\]
Case 2. When $G$ is noncompact locally compact abelian

we have

$$M[A^0_0(G)] \sim M_{bd}(G).$$

by Theorem 2.1. Therefore there exists a measure $\mu \in M_{bd}(G)$ such

that

$$T_f = \mu * f, \quad f \in A^0_0(G),$$

with

$$\| \mu \| = 1.$$

Now

$$\| \mu * f \|_p = \| \mu * f \|_1 + \| f \|_p \| \omega \|_p = \| f \|_1 + \| f \|_p \| \omega \|_p$$

Therefore from (8) we have

$$\| \mu * f \|_1 = \| f \|_1, \quad f \in A^0_0(G).$$

Since $A^0_0(G)$ is dense in $L^1(G)$ an application of Theorem
4.7 yields the desired result.

Lemma 4.14. Let $G$ be a compact abelian group. Let $\omega$ be a net of multipliers in $N[A^0_0(G)]$ such that $\omega$ is norm bounded. Then there exists a subnet $\omega$ of $\omega$.
Note that
\[ \lim \int\int \mu_k(x) \hat{f}(x) \, dx = \int \int \mu(x) \hat{f}(x) \, dx, \quad f \in L^1_\infty(G). \]

for some multiplier \( m \in M_{\text{pl}}(G) \).

**Proof.** Let \( 1 \leq p \leq 2 \). By Theorem 2.4 there exists a net \( \{\delta_k\} \) of pseudomeasures corresponding to \( m \).

Then \( \{\delta_k\} \) is a norm bounded net of pseudomeasures. By Almgren's theorem, there exists a pseudomeasure \( \sigma \) and a subnet of \( \{\delta_k\} \) say \( \{\delta_{k_i}\} \) satisfying
\[ \int \int \mu_k(x) \hat{f}(x) \, dx \rightarrow \int \int \mu(x) \hat{f}(x) \, dx, \quad f \in L^1_\infty(G). \]

Let \( m \in M_{\text{pl}}(G) \) correspond to \( \sigma \). Then we have
\[ \lim \int\int \mu_k(x) \hat{f}(x) \, dx = \int \int \mu(x) \hat{f}(x) \, dx, \quad f \in L^1_\infty(G). \]

**Case 1.** \( 2 \leq p < \infty \). By Theorem 2.6, \( m \) corresponds to a norm bounded net of elements in \( [L^p(G)]^* \). Hence there exists an element \( x \) of \( [L^p(G)]^* \) and a subnet of \( \{\delta_{k_i}\} \) converging to \( x \) in the weak topology of \( [L^p(G)]^* \) that is
\[ \int f(x) \mu_k(x) \, dx \rightarrow \int f(x) \mu(x) \, dx, \quad f \in L^1_\infty(G). \]

This completes the proof of the lemma.
Lemma 4.15. Let $G_1$ and $G_2$ be locally compact abelian groups with character groups $\Gamma_1$ and $\Gamma_2$ respectively. For $1 \leq p < \infty$, if there exists an algebra isomorphism $\Psi$ of $M[A_{\alpha_1}^p(G_1)]$ onto $M[A_{\alpha_2}^p(G_2)]$, either both the groups are connected or both of them are noncompact.

Proof. We prove that if one of the groups say $G_1$ is compact, the other is also compact. Suppose $G_1$ is noncompact. By Theorem 2.1

$$M[A_{\alpha_2}^p(G_2)] \cong M_{\text{bd}}(G_2).$$

Thus $\Psi$ can be considered as an algebra isomorphism of $M[A_{\alpha_2}^p(G_2)]$ onto $M_{\text{bd}}(G_2)$. Since every element of $L^1(G_1)$ gives an element of $M[A_{\alpha_1}^p(G_1)]$, the restriction of $\Psi$ to $L^1(G_1)$ is an algebra isomorphism of $L^1(G_1)$ into $M_{\text{bd}}(G_2)$. By theorem 4.9 it follows that there exists a subset $Y$ of $\Gamma_2$ and a piecewise affine map $\alpha$ of $Y$ into $\Gamma_1$ such that for every $f \in L^1(G_1)$

$$\Psi f(Y) = \int f(\alpha(y)) \text{ if } y \in Y$$

$$= 0 \text{ if } y \not\in Y$$

Since by Theorem 0.19 every element of $M[A_{\alpha_1}^p(G_1)]$ gives rise to a pseudomeasure, we identify each element of $M[A_{\alpha_1}^p(G_1)]$ with the corresponding pseudomeasure on $G_1$. Thus if $\sigma \in M[A_{\alpha_1}^p(G_1)]$ and $f \in \mathcal{BC}(G_1)$, we have $\sigma \times f \in L^1(G_1)$ with
(6) \([\chi (\sigma * f)]^*(x) = (\sigma * f)^*(\chi (x)) = \sigma^*(\chi (x))^*\]

On the other hand

\[\chi (\sigma * f) = \chi (\sigma) * \chi (f)\]

Therefore

(7) \[\left[ \chi (\sigma * f) \right]^*(x) = \left[ \chi (\sigma) \right]^*(x) \cdot \left[ \chi (f) \right]^*(x)\]

\[= \left[ \chi (\sigma) \right]^*(x) \cdot \chi (x) \cdot \sigma^*(\chi (x)), \forall \gamma \in Y\]

From (6) and (7) we therefore have

(8) \[\sigma^*(\chi (x)) \cdot \chi (x) = \left[ \chi (\sigma) \right]^*(x) \cdot \chi (x) \cdot \sigma^*(\chi (x)), \forall \gamma \in Y\]

Since (8) holds for every \(f \in B (G_1)\), we have

(9) \[\left[ \chi (\sigma) \right]^*(x) = \delta^* (\chi (x)), \forall \gamma \in Y\]

Now we prove \(\phi\) is one to one on \(Y\). Let \(\gamma_1, \gamma_2 \in Y\) be such that \(\gamma_1 \neq \gamma_2\). Choose \(\mu \in M_{ba} (G_1)\) such that

\[\mu (\gamma_1) \neq \mu (\gamma_2)\]

Let \(\sigma \in M_{\mu} (G_1)\) be such that \(\chi (\sigma) = \mu\)

Theorem (3): Then from (9) we have

\[\mu (\gamma_1) = \delta^* (\chi (\gamma_1)) = \mu (\gamma_1) = \delta^* (\chi (\gamma_2)) = \delta (\chi (\gamma_2)).\]
Therefore

\[ \alpha(\gamma_1) \neq \alpha(\gamma_2) \]

that is, \( \alpha \) is one to one.

Next we show that \( \alpha(\gamma) = \Gamma_1 \). Since \( \Gamma_1 \) is discrete, \( \alpha(\gamma) \) is closed in \( \Gamma_1 \). If \( \alpha(\gamma) \neq \Gamma_1 \), there exists \( \beta \in B(\Gamma_1) \) such that \( \beta = 0 \) on \( \alpha(\gamma) \) but \( \beta \) is not identically zero. Since \( \beta \circ \alpha = 0 \), we have \( \gamma(\beta) = 0 \), which contradicts the one to oneness of \( \gamma \). Finally we prove that \( \gamma = \Gamma_2 \).

If \( \gamma = \Gamma_2 \), since \( \gamma \) is a closed subset of \( \Gamma_2 \), there exists \( \mu \in \mathcal{M}_{b,d}(\Gamma_2) \) such that \( \mu = 0 \to \mu = 0 \) on \( \gamma \). Choose \( \sigma \in \mathcal{M}_{b,d}(\Gamma_1) \) such that \( \gamma(\sigma) = \mu \). By (9) we therefore have

\[ \gamma(\sigma) = \gamma(\alpha(\gamma)) = \sigma(\alpha(\gamma)) = 0, \quad \forall \gamma \in \gamma. \]

Since \( \alpha(\gamma) = \Gamma_1 \), \( \sigma = 0 \) on \( \Gamma_1 \) which implies that \( \sigma = 0 \) and hence \( \mu = 0 \) which is a contradiction. This proves that \( \gamma = \Gamma_2 \).

Thus \( \alpha \) is a piecewise affine homeomorphism of \( \Gamma_2 \) onto \( \Gamma_1 \). Since \( \Gamma_1 \) is discrete, \( \Gamma_2 \) is also discrete and hence \( \Gamma_1 \) is compact. This completes the proof of the lemma.

**Proof of Theorem 4.3.** Since \( \wedge \) is an algebraic isomorphism of \( \mathcal{M}[A_{d_1}^{p}(\Gamma_1)] \) onto \( \mathcal{M}[A_{d_2}^{p}(\Gamma_2)] \), either both \( \Gamma_1 \) and \( \Gamma_2 \) are compact or both of them are noncompact. We shall consider two cases separately.
Case 1. Suppose \( G_{1} \) and \( G_{2} \) are both noncompact. By Theorem 9.1,
\[
M \left[ \mathcal{A}_{0,1}^{p}(G_{1}) \right] \cong M_{\text{bd}}(G_{1})
\]
\[
M \left[ \mathcal{A}_{0,2}^{p}(G_{2}) \right] \cong M_{\text{bd}}(G_{2})
\]
Since \( \Lambda \) is an isometric algebra isomorphism of \( M \left[ \mathcal{A}_{0,1}^{p}(G_{1}) \right] \) onto \( M \left[ \mathcal{A}_{0,2}^{p}(G_{2}) \right] \), it follows that
\[
M_{\text{bd}}(G_{1}) \cong M_{\text{bd}}(G_{2})
\]
From Theorem 4.10, this implies that \( G_{1} \) and \( G_{2} \) are topologically isomorphic.

Case 2. Suppose \( G_{1} \) and \( G_{2} \) are both compact. For each \( a \in G_{1} \), \( \Lambda \tau_{a} \) and \( \Lambda \tau_{-a} \) are norm preserving multipliers on \( \mathcal{A}_{0,2}^{p}(G_{2}) \) since \( \Lambda \) is an isometric isomorphism of \( M \left[ \mathcal{A}_{0,2}^{p}(G_{1}) \right] \) onto \( M \left[ \mathcal{A}_{0,2}^{p}(G_{2}) \right] \). Therefore by Lemma 4.13, there exist \( a_{1}, b_{1} \in G_{2} \) and complex numbers \( \lambda_{1}, \lambda_{2} \) of absolute value 1 such that
\[
\Lambda \tau_{a} = \lambda_{1} \tau_{a_{1}}, \quad \Lambda \tau_{-a} = \lambda_{2} \tau_{b_{1}}.
\]
Since \( \Lambda \) is an algebraic isomorphism, \( b_{1} = -a_{1} \) and \( \lambda_{1} = \overline{\lambda_{2}} \).
Also it can be easily verified that the mapping \( \phi : G_{1} \to G_{2} \) given by \( \phi(a) = a_{1} \) and the mapping \( \Lambda \) which maps \( G_{1} \) into the set of all complex numbers of modulus one are algebraic isomorphisms.
Since $\varphi$ is an algebraic isomorphism of $G_{1}$ onto $G_{2}$, all that remains to complete the proof is to show that $\varphi$ is continuous.

To show that $\varphi$ is continuous, we proceed as in the proof of Theorem 4.1 to show that if $a_{i} \to e_{1}$ in $G_{1}$, then $\varphi(a_{i}) \to e_{2}$ in $G_{2}$. Suppose not. Then there exists an open neighbourhood $V$ of $e_{2}$ such that $\{\varphi(a_{i})\}$ belongs to complement of $V$ for all $i$.

The net of multipliers $\{\Lambda L_{ai}\}$ is a norm bounded net in $M[A_{o1}^{p}(G_{1})]$. Therefore by Lemma 4.16, there exists a multiplier $m$ and a subnet of $\Lambda L_{ai}$ which, without loss of generality we assume to be $\Lambda L_{ai}$ itself, satisfying

\begin{equation}
\int_{G_{1}} f(x) \Delta L_{ai}(x) d\lambda \to \int_{G_{1}} f(x) m(x) d\lambda, \quad f \in A(G_{1})
\end{equation}

For $h \in L^{1}(G_{1})$, let $W_{h}$ be the multiplier defined by convolution with $h$ on $A_{o1}^{p}(G_{1})$. Since $a_{i} \to e_{1}$ in $G_{1}$, $\Lambda L_{ai} h \to h$ in $M_{b\sigma}(G_{1})$. Since the topology of $M_{b\sigma}(G_{1})$ is stronger than that induced by $M[A_{o1}^{p}(G_{1})]$ we have $W_{\Lambda L_{ai} h} \to W_{h}$.
in $\text{M} \left[ A_{\omega_1} \right]$. Thus

$\Lambda_{\omega_1} \circ \Lambda W_h \rightarrow \Lambda W_h$

in $\text{M} \left[ A_{\omega_1} \right]$. $\Lambda$ is an isometric isomorphism of

$\text{M} \left[ A_{\omega_1} \right]$ into $\text{M} \left[ A_{\omega_2} \right]$. Therefore $\Lambda$ is continuous. We then have

$\Lambda_{\omega_1} \circ \Lambda W_h \rightarrow \Lambda W_h$

in $\text{M} \left[ A_{\omega_2} \right]$, so that

$\Lambda_{\omega_1} \circ \Lambda W_h \rightarrow \Lambda W_h$

in $\text{L}^\infty \left( \mathbb{R}_2 \right)$.

This gives

(11) \[ \lim \int_{\Gamma_2} \Lambda_{\omega_1} \circ \Lambda W_h \cdot f(x) \, dx = \int_{\Gamma_2} \Lambda W_h \cdot f(x) \, dx \quad \text{for} \quad f \in \text{AC} \]

From (10) we have

(12) \[ \lim \int_{\Gamma_2} \Lambda_{\omega_1} \circ \Lambda W_h \cdot f(x) \, dx = \int_{\Gamma_2} \Lambda W_h \cdot f(x) \, dx \quad \text{for} \quad f \in \text{AC} \]

From (10) and (12) we therefore have

\[ \int_{\Gamma_2} \Lambda W_h \cdot f(x) \, dx = \int_{\Gamma_2} \Lambda W_h \cdot f(x) \, dx \]

Since $B \left( \mathbb{R}_2 \right)$ is dense in $A \left( \mathbb{R}_2 \right)$, this implies that

(13) \[ \Lambda W_h \left( x \right) \cdot m \left( \sigma \right) = \Lambda W_h \left( x \right), \quad \text{a.e. on} \ \Gamma_2 \]
\[ \hat{W}_h(\alpha) A(\alpha) = m \circ \overrightarrow{W}_h(\alpha) \]

This (10) gives
\[ m \circ \overrightarrow{W}_h = \overrightarrow{W}_h \]

so that
\[ \overrightarrow{W}_h = \overrightarrow{W}_h \]

that is
\[ \overrightarrow{W}_h(\alpha) = \alpha \overrightarrow{f} \quad \text{for } f \in A^{p}_{\ell}(G_1) \]

Since \( A^{p}_{\ell}(G_1) \) is a Segal algebra, it is an essential \( L^1(G_1) \) module and hence
\[ \overrightarrow{W}_h = T e_1 \]

which implies
\[ m = T e_2 \]

(10) then gives

(14)
\[ \lim_{n \to \infty} \sum_{k=1}^{N} \overrightarrow{W}_{\alpha_k}(\alpha) f(\alpha) \, d\alpha = \sum_{k=1}^{N} \overrightarrow{C}(\alpha_k) \, d\alpha, \quad f \in A^{p}_{\ell}(G_2) \]

Now \( \{\lambda(a_1)\} \) is a bounded sequence of complex numbers.

Therefore there exists a subset of \( \{\lambda(a_1)\} \) which without loss of generality we denote by \( \{\hat{\lambda}(a_1)\} \) itself and a complex number \( \lambda \) of modulus 1 such that \( \{\hat{\lambda}(a_1)\} \) converges to \( \lambda \).

Now \( \{\hat{\lambda}(a_1)\} \subseteq \Gamma^2_2 \). Since \( \Gamma^2_2 \) is compact, there exists a subset of \( \{\hat{\lambda}(a_1)\} \) say \( \{\lambda(a_1)\} \) and an element \( a \in \Gamma^2_2 \) such that \( \{\lambda(a_1)\} \) converges to \( a \) in \( \Gamma^2_2 \).
belongs to complement of $V$ for all $k$, and therefore $a^l$ belongs to complement of $V$, that is $a^l = e_2$. Let $f \in B(G_2)$. Then

$$\int_{\Gamma_2} \chi(x, y) \phi(y) f_1(y) f(y) \, dy = \int_{\Gamma_2} (\phi(y) - \phi(y)) f(y) \, dy$$

$$= \chi(x, y) f(\phi(y)) f(y) \, dy$$

Is continuous and therefore we have from (15)

$$\int_{\Gamma_2} \chi(x, y) \phi(y) f_1(y) f(y) \, dy \rightarrow \int f \, dy$$

Now

$$\Delta f(-a^l) = \int_{\Gamma_2} \chi(x, y) f(y) f(y) \, dy = \int_{\Gamma_2} \chi(x, y) f(y) f(y) \, dy$$

(16) then implies that

(16) then implies that

that is

(27) gives rise to an isomorphism between $A_0^0(G_2)$ and $A_0^0(G_2)$.

(24) and (27) then combine to show that
\[ \int_{\Gamma} \varphi(n) \, d\gamma = \int_{\Gamma_2} \varphi_1(x) \varphi_2(x) \, d\gamma, \quad f \in B(C_{\alpha_2}). \]

The densest of \( B(C_{\alpha_2}) \) in \( A(C_{\alpha_2}) \) then shows that

\[ \varphi_1(x) = 1 \quad \text{a.e. on} \ \Gamma_2, \]

that is

\[ \lambda = 1 \quad \text{and} \quad \alpha_1 = \alpha_2 \]

which is a contradiction. Therefore \( \varphi \) is continuous. This completes the proof of the theorem.

**Corollary 4.36.** If \( G_1 \) and \( G_2 \) are two locally compact

closed groups with character spaces \( \Gamma_1 \) and \( \Gamma_2 \) and \( \omega_1, \omega_2 \)

are the weight functions on \( \Gamma_1 \) and \( \Gamma_2 \) respectively, then \( G_1 \)

and \( G_2 \) are isomorphic as topological groups if and only if there

exists an isometric or topological isomorphism between \( A_{\omega_1}^p(G_1) \)

and \( A_{\omega_2}^p(G_2) \) for some \( p \) satisfying \( 1 \leq p < \infty \).

**Proof.** Any isometric algebra isomorphism between \( A_{\omega_1}^p(G_1) \)

and \( A_{\omega_2}^p(G_2) \) gives rise to an isomorphism between \( M[CA_{\omega_1}^p(G_1)] \)

and \( M[CA_{\omega_2}^p(G_2)] \) by Theorem 4.3. An application of Theorem 4.3

then completes the proof of the corollary.
The proof of Theorem 4.4 is exactly similar to that of Theorem 4.3 and is hence omitted. From Theorem 4.4 and in known we can derive the following corollary similar to corollary 4.16.

**Corollary 4.17.** If $G_1$ and $G_2$ are two locally compact abelian groups with duals $\Gamma_1$ and $\Gamma_2$ and $Sp(G_1)$, $Sp(G_2)$ are two spectral algebras of locally type for $p$ satisfying $1 < p < \infty$. Then $G_1$ and $G_2$ are isometric as topological groups if and only if there exists an isometric algebra isomorphism between $Sp(G_1)$ and $Sp(G_2)$.

**Remark 4.12.** If $G_1$ and $G_2$ are noncompact, isometric can be replaced by norm decreasing in Theorems 4.3 and 4.4 since the multiplier algebras involved are isometrically isomorphic to $M_{ba}(G_i)$ as Banach algebras.
CHAPTER V

MULTIPLIERS FROM A SEGAL ALGEBRA INTO $L^p(G)$

Figa-Talamanca [12] has given a characterization of the multipliers on $L^p(G)$ for a locally compact abelian group $G$ for $1 \leq p < \infty$, as the dual space of a Banach space of continuous functions. Larson [14] has given a similar characterization for the multiplier space of $A^p(G)$ when $G$ is compact abelian and $p > 2$. We give here an analogous characterization of the space of multipliers from a Segal algebra $S(G)$ on a locally compact abelian group into $L^1(G)$.

Let $M(S, L^1)$ denote the space of all multipliers from $S(G)$ into $L^1(G)$. Durnham in [5] has proved the following result.

THEOREM 5.1. If $S_1(G)$ and $S_2(G)$ are two Segal algebras on the locally compact abelian group $G$ and $T$ is a multiplier from $S_1(G)$ into $S_2(G)$ then there exists a bounded continuous function $\phi$ defined on $\mathbb{R}$ satisfying

$$\hat{Tf}(\alpha) = \phi(\alpha) \hat{f}(\alpha), \quad \alpha \in \mathbb{R}, \quad f \in S_1(G)$$

with

$$||\phi||_{\infty} \leq ||T||.$$
The following theorem of Rudin [ ] will be needed in the sequel.

**Theorem 5.2.** $A(C)$ consists precisely of the convolutions $F_1 \ast F_2$ with $F_1$ and $F_2$ in $L^2(C)$.

If $\varphi \in L^1(C)$ and $L\varphi$ is defined by

$$L\varphi(f) = \varphi \ast f, \quad f \in S(C),$$

then $L\varphi$ is a multiplier from $S(C)$ into $L^1(C)$. Let $T \in M(S, L^1)$.

Consider the linear form on $A(C)$ defined as follows

$$\beta_T(f) = \int \hat{f}(\xi) \hat{T}(\xi) d\alpha,$$

where $\hat{\cdot}$ denotes the unique bounded continuous function corresponding to $f$ satisfying

$$\hat{\varphi}(\xi) = \hat{T}(\xi) \hat{\varphi}(\xi), \quad f \in S(C),$$

with

$$\|T\|_{L^1} \leq \|f\|_{L^1},$$

the existence of which is given by Theorem 5.1. Hence since

$$|\beta_T(f)| \leq \int |\hat{f}(\xi)| |\hat{T}(\xi)| d\alpha \leq \|T\|_{L^1} \|f\|_{L^1},$$

$\beta_T$ is a well-defined linear form on $A(C)$. Remark $A(C)$ as
follows:

(2) \[ \| f \| = \sup \{ |\beta_T(f)| : T \in \mathcal{M}(S, L^1), \| T \| \leq 1 \} \]

That the above defines a seminorm on \( A(G) \) can be easily verified. To prove that it defines a norm, we have to prove \( \| f \| = 0 \) implies \( f = 0 \) on \( G \). \( \| f \| = 0 \) implies \( \beta_T(f) = 0 \), \( T \in \mathcal{M}(S, L^1) \). Taking \( T = c_y \) for some \( y \in G \), we have

\[ \bar{A}(y) = \langle -y, x \rangle, \quad x \in X, \quad \| y \| \leq 1; \]

\[ \beta_T(f) = \int X \langle -y, x \rangle f(x) \, dx = f(-y) = 0, \quad y \in G. \]

Hence \( f = 0 \) on \( G \). With this new norm let us denote \( A(G) \) by \( B_\Sigma(G) \). Denoting the norm on \( B_\Sigma(G) \) by \( \| \cdot \|_B \), we have

**Theorem 5.3.** \( \mathcal{M}(S, L^1) \) is isometrically isometric to the dual of the convolution of \( B_\Sigma(G) \). The weak operator topology on \( \mathcal{M}(S, L^1) \) is stronger than the weak topology on norm bounded subsets of \( \mathcal{M}(S, L^1) \).

We first prove two lemmas.

**Lemma 5.4.** If \( H = \{ f \times g : f \in B(G), g \in C_0(G) \} \) then \( \| \cdot \| \) is dense in \( B_\Sigma(G) \).
Proof. If \( f \in B_s(G_t) \),

\[
\| f \|_B = \sup \| T f \|_{L^1} : T \in M_s(G_t), \| T \|_1 \leq 1
\]

\[
= \sup \| S \hat{T}(x) \hat{f}(y) \|_{L^1} : T \in M_s(G_t), \| T \|_1 \leq 1
\]

\[
\leq \sup \| T \|_1 \| \hat{f} \|_1, T \in M_s(G_t), \| T \|_1 \leq 1
\]

\[
\leq \| \hat{f} \|_1.
\]

Therefore if we give another norm to \( A(G) \) as

\[
\| f \|' = \| \hat{f} \|_1
\]

then

\[
\| f \|_B \leq \| f \|' \quad \forall f \in B_s(G_t).
\]

If we therefore prove that \( H \) is dense in \( A(G) \) in \( \| \cdot \|' \), we have our required result. If \( f \in A(G_t) \), there exists \( g, h \in L^2(G_t) \) such that \( f = g * h \) by Theorem 5.2. Let \( \varepsilon > 0 \) be given. Now \( C_c(G_t) \) is dense in \( L^2(G_t) \). Therefore exists \( g' \in C_c(G_t) \) such that

\[
\| g - g' \|_2 < \frac{\varepsilon}{2 \| h \|_2}
\]

Now

\[
\| g * h - g' * h \|_2 = \| \hat{g} \hat{h} - \hat{g'} \hat{h} \|_1 = \| (\hat{g} - \hat{g'}) \hat{h} \|_1
\]

\[
\leq \| \hat{g} - \hat{g'} \|_2 \| h \|_2.
\]
That is
\[ \| g \ast h - g' \ast h' \|_1 \leq \| g - g' \|_2 \| h - h' \|_2 < \frac{\varepsilon}{2} \]
from (2). Similarly there exists \( h' \in B(G) \) such that
\[ \| h' - \hat{h} \|_2 < \frac{\varepsilon}{2\| g \|_2} . \]
Therefore we have
\[ \| g' \ast h - g' \ast h' \|_1 = \| \hat{g}' \ast \hat{h} - \hat{g}' \ast \hat{h}' \|_1 \]
\[ = \| \hat{g}' \ast (\hat{h} - \hat{h}') \|_1 \leq \| g' \|_2 \| \hat{h} - \hat{h}' \|_2 < \frac{\varepsilon}{2} \]
from (4). Combining (3) and (5) we then have
\[ \| g \ast h - g' \ast h' \|_1 \leq \| g \ast h - g' \ast h' \|_1 + \| g' \ast h - g' \ast h' \|_1 \]
\[ \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]
Therefore \( \Pi \) is dense in \( B_S(G) \).

LEMMA 5, 5. If \( T \in M(S, L) \), then \( T = \lim \lambda_q \) in the weak operator topology where \( \{ \lambda_q \} \) is a net of functions in \( S(G) \) with \( \| \lambda_q \ast f \|_1 \leq K \| f \|_1 \) for some constant \( K \in \text{compact set} \) and \( \text{invariant of} \ f \in S(G) \) and \( \text{invariant of} \ \ast \).

PROOF. Let \( \varepsilon < \frac{1}{2} \) be an approximate identity satisfying
\[ \| \varepsilon - \lambda_q \|_1 = 1 \text{ and } \lambda_q \text{ has compact support for all } q \]. Set \( \lambda_q = T \lambda_q \).
Then \( \phi_\alpha \in B(G) \). Now

\[
\lim \| \phi_\alpha \ast f - T_f \|_1 = \lim \| T h_\alpha \ast f - T_f \|_1 = \lim \| T (h_\alpha \ast f) - f \|_1 \leq \lim \| T \| \| h_\alpha \ast f - f \|_1
\]

Thus for \( f \in S(G) \) we have

\[
\lim \phi_\alpha \ast f = T_f
\]

in the \( L^1(G) \) norm which shows that

\[
T = \lim \phi_\alpha
\]

in the strong operator topology which in turn implies

\[
T = \lim \phi_\alpha
\]

in the weak operator topology. Also

\[
\| \phi_\alpha \ast f \|_1 = \| T (h_\alpha \ast f) \|_1 \leq \| T \| \| h_\alpha \ast f \|_1 \leq \| T \| \| f \|_1,
\]

for all \( \alpha \) and for all \( f \in S(G) \). This proves the required result.

**Proof of Theorem 5.2.** Given \( T \in M(C(G, L^1)) \), we have to define an element of \( [B_s(G)]^* \) corresponding to it. Consider the linear form \( p_T \) defined on \( [B_s(G)] \) as in \((1)\). Since
\[ |\beta_T(f)| = \|T\| |\beta_T(f)| \leq \|T\| \|f\|_B \]

by the definition of the norm of a function in \( \mathcal{B}_S(\mathcal{C}_0) \), \( \beta_T \) is a continuous linear functional on \( \mathcal{B}_S(\mathcal{C}_0) \) with

\[ \|\beta_T\|_{\mathcal{B}^*} \leq \|T\|. \]

If \( f \in \mathcal{B}(\mathcal{C}_0) \), \( g \in \mathcal{C}_c(\mathcal{C}_0) \), then \( f \ast g \in \mathcal{B}_S(\mathcal{C}_0) \) and

\[ \|f \ast g\|_B = \sup \left\{ \|\beta_T(f \ast g)\| : T \in \mathcal{M}(\mathcal{C}_1, \ell^1), \|T\| \leq 1 \right\} = \sup \left\{ |T_f \ast g(0)| : T \in \mathcal{N}(\mathcal{C}_1, \ell^1), \|T\| \leq 1 \right\} \leq \sup \left\{ \|T_f\|_1 \|g\|_{\ell^\infty} : T \in \mathcal{N}(\mathcal{C}_1, \ell^1), \|T\| \leq 1 \right\} \leq \sup \left\{ \|T\| \|f\|_S \|g\|_{\ell^\infty} : T \in \mathcal{M}(\mathcal{C}_1, \ell^1), \|T\| \leq 1 \right\} \leq \|f\|_S \|g\|_{\ell^\infty} \]

Therefore

\[ \|f \ast g\|_B \leq \|f\|_S \|g\|_{\ell^\infty}. \]
Now
\[ \| T \| = \sup \{ \| Tg \| : g \in C_0(G), \ 1 \leq \| g \|_\infty \leq 1 \} \]
\[ = \sup \{ \| \langle f, g \rangle \| : f \in C_0(G), \ 1 \leq \| f \|_\infty \leq 1 \} \]
\[ \leq \sup \{ \| f \|_\infty \| g \|_1 : f \in C_0(G), \ 1 \leq \| f \|_\infty \leq 1 \} \]
\[ \leq \sup \{ \| f \|_\infty \| g \|_1 \| : f \in C_0(G), \ 1 \leq \| f \|_\infty \leq 1 \} \]

from (7). This implies that
\[ (c) \quad \| T \|_\infty \leq \| \beta_T \|_{B^*}. \]
\[ (c) \quad \] and (c) together give
\[ \| T \| = \| \beta_T \|_{B^*}. \]

We have therefore defined a map \( \Lambda \) from \( M(G, L^1) \) into \( [B_\sigma(G)]^* \). This can be easily verified to be a linear map.

Also \( \Lambda \) is an isometry. It only remains to show that \( \Lambda \) is a linear map. Given \( f \in B_\sigma(G) \) for a given \( f \in B(G) \).
consider the linear form $F_f$ defined on $C_c(G)$ as follows.

$$F_f(g) = \beta(f \ast g)$$

Then

$$|F_f(g)| = |\beta(f \ast g)| \leq \|\beta\|_{B^*} \|f \ast g\|_B \leq \|\beta\|_{B^*} \|f\|_B \|g\|_G$$

from (7). Therefore $F_f$ defines a continuous linear functional on $C_c(G)$ endowed with the supremum norm. Since $C_c(G)$ is dense in $C_0(G)$ there corresponds a measure $Tf \in M_{bd}(G)$ satisfying

(9) $$T_f \ast g(0) = F_f(g) = \beta(f \ast g), \quad g \in C_c(G)$$

and

(10) $$\|T_f\| \leq \|\beta\|_{B^*} \|f\|_B \|g\|_G$$

Given $y \in G$, we have

$$T_y(T_f \ast g(0)) = T_f \ast T_y g(0) = \beta(f \ast T_y g)$$

$$= \beta(T_y f \ast g) = T(T_y f) \ast g(0), \quad g \in C_c(G),$$

from which we deduce that

$$T_y(T_f) = T(T_y f), \quad y \in G, \quad f \in B(G).$$

The mapping $G \rightarrow M_{bd}(G)$ given by $y \rightarrow T_y(T_f)$ is therefore
continuous which implies by Theorem that $Tf_\cdot$ is absolutely continuous, that is $Tf \in L^1(G)$. From (10) we see that the mapping $T$ can be extended to the whole of $S(G)$, since $T$ is a translation bounded linear map from $S(G)$ into $L^1(G)$, $T \in M(S, L^1)$. Also

$$\beta_T(f \ast g) = T(f \ast g) = \beta(f \ast g), f \in B(G), g \in L^1(G)$$

Thus $\beta_T$ and $\beta$ coincide on $H$. Since $H$ is dense in $B_s(G)$ by Lemma 5.4, $\beta_T$ and $\beta$ coincide on $B_s(G)$ that is $\beta_T = \beta$. Therefore the mapping $\Lambda$ is onto. $\Lambda$ then defines an isometric isomorphism between $M(G, L^1)$ and the dual of $B_s(G)$. The dual of $B_s(G)$ is the same as that of its completion. We have therefore proved the first half of the theorem.

To prove the second half of the theorem, suppose $\\{T_\alpha\} \subset M(G, L^1)$ satisfies $\|T_\alpha\| \leq 1$ and $T_\alpha$ converges to $T$ in the norm weak operator topology in the limit of $\alpha$. Let $\\{h_n\}$ be a Cauchy sequence in $B_s(G)$ and let $\varepsilon > 0$ be given. We have to prove that there exists an infinite subset of the $\\{T_\alpha\}$, say $\\{T_\alpha_\beta\}$ such that

$$\lim_{n \to \infty} |\beta_{T_\alpha_\beta}(h_n) - \beta_T(h_n)| < \varepsilon.$$

Let $n_0$ be chosen such that
Let \( k = f \ast g \) with \( f \in B(G_n), g \in C_c(G) \) be given such that

\[
\| h_{n_0} - k \|_{p_0} < \frac{\varepsilon}{5}, \quad n > n_0
\]

Since \( T_\lambda \) converges to \( T \) in the weak operator topology, there exists an infinite subset of \( \{ T_\lambda \} \) say \( \{ T_\lambda \} \) satisfying

\[
\| T_\lambda f \ast g (0) - T f \ast g (0) \| < \frac{\varepsilon}{5}.
\]

Then for \( n > n_0 \) from (11), (12) and (13) we have

\[
\| \beta_{T_\lambda} (h_n) - \beta_T (h_n) \| = \| \beta_{T_\lambda} (h_n) - \beta_{T_\lambda} (h_{n_0}) + \beta_{T_\lambda} (h_{n_0}) - \beta_T (h_{n_0}) \| + \| \beta_T (h_{n_0}) - \beta_T (h_n) \|
\]

\[
\leq \| T_\lambda \| \| h_n - h_{n_0} \|_{p_0} + \| h_{n_0} - k \| \| T_\lambda \| + \| \beta_{T_\lambda} f \ast g (0) - T f \ast g (0) \|
\]

\[
\leq \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon.
\]

This proves that the weak operator topology on \( M C(S, L^1) \) is stronger than the weak* topology on \( B_0 (G_n) \), thus completing the theorem.
Theorem 5.6. The completion \( B_\infty(G) \) of \( B_\infty(G) \) can be identified with a space of continuous functions.

Proof. For each \( x \in G \), \( T = c_x \in M(G, L^1) \) with \( \|c_x\| \leq 1 \). Therefore \( \|B_{c_x}(h)\| \leq \|h\|_B \), \( h \in B_\infty(G) \).

But
\[
B_{c_x}(h) = \int c_x(\gamma) h(\gamma) d\gamma = \int <c_x, \gamma> h(\gamma) d\gamma
\]
\[
= h(-x) .
\]

This implies that
\[
\|h(x)\| \leq \|h\|_B , \quad h \in B_\infty(G) , \quad x \in G
\]
that is
\[
(24) \quad \|h\|_\infty \leq \|h\|_B , \quad h \in B_\infty(G) .
\]

Let \( \{h_n\} \) be a Cauchy sequence in \( B_\infty(G) \). The inequality
\[
\|h_n - h_m\|_\infty \leq \|h_m - h_n\|_B
\]
shows that \( \{h_n\} \) is a Cauchy sequence in the supremum norm also.

There exists then a continuous function \( h \) such that
\[
\lim_{n \to \infty} \|h - h_n\|_\infty = 0
\]

To each Cauchy sequence in \( B_\infty(G) \), we associate the continuous function so obtained which gives us a linear map assigning a continuous function on \( G \) to each element of the completion of \( B_\infty(G) \). To complete the proof of our theorem, it is enough if we show that
This map is one to one. To this end let \( \{ h_n \} \) be a Cauchy sequence in \( B_2(\mathbb{R}) \) such that

\[
\lim_{n \to \infty} \| h_n \|_\alpha = 0
\]

We need to prove that

\[
\lim_{n \to \infty} \| h_n \|_B = 0
\]

Let us now consider the expression

\[
\beta_{L_2}(h_n) = \int \frac{f}{x} h_n(x) \, dx, \quad f \in B(\mathbb{R})
\]

Then

\[
\beta_{L_2}(h_n) = f \ast h_n(0)
\]

and

(16) \[
\lim_{n \to \infty} | \beta_{L_2}(h_n) | = \lim_{n \to \infty} | f \ast h_n(0) | \leq \lim_{n \to \infty} \| f \|_{L_1} \| h_n \|_B = 0
\]

If \( F \) is the linear form on \( M(\mathbb{R}, L^1) \) defined by

\[
F(T) = \lim_{n \to \infty} \beta_T(h_n)
\]

Then \( F \) is continuous with respect to the weak* topology on \( M(\mathbb{R}, L^1) \)

Therefore there exists \( h \in B_2(\mathbb{R}) \) such that

(16) \[
F(T) = \lim_{n \to \infty} \beta_T(h_n) = \beta_T(h), \quad T \in M(\mathbb{R}, L^1).
\]
Now \( \lim_{\alpha} \mathcal{L}_{\alpha} \) in the weak operator topology where \( \mathcal{L}_{\alpha} \) is a net of functions in \( \mathcal{B}(G) \) by Lemma 5.6. Hence \( \lim_{\alpha} \mathcal{L}_{\alpha} \) in the weak* topology by Theorem 5.3. Therefore using (16) we have

\[
F(T) = \lim_{\alpha} F(\mathcal{L}_{\alpha}) = \lim_{\alpha} \lim_{\beta} \beta_{\alpha}(h) = \lim_{\alpha} \lim_{\beta} \beta_{\alpha}(h) = 0.
\]

This being true for all \( T \in M(G, L^1) \), it follows from (16) that \( \beta_T(h) = 0 \) for all \( h \in G \).

That is, \( h = 0 \).

Since \( \mathcal{E}_{\eta} \) is Cauchy, we have \( \lim_{n} \|h_n\|_B = 0 \) and our assertion is proved.

**Theorem 5.7.** \( \{T_x : x \in G \} \) in dense in \( M(G, L^1) \) in the weak* topology.

**Proof.** If \( h \in B(G) \) satisfies

\[
\mathcal{R}_{\alpha}(h) = 0, \quad x \in G,
\]

then

\[
\beta_{T_x}(h) = h(-x) = 0, \quad x \in G.
\]

This shows that \( h(x) = 0 \) for all \( x \in G \) so that \( h = 0 \) on \( G \). Now by an application of the Hahn-Banach theorem, we see that \( \mathcal{L}_{\alpha} \) is
weak* dense in the dual of the completion of $B_s(G)$. Therefore we have the required result.

We now give an application of Theorem 5.3 to character Segal algebras. A character Segal algebra is a Segal algebra $S(G)$ satisfying $f \in S(G)$, $\gamma \in \Gamma$ implies $\gamma f \in S(G)$ and

$$\|\gamma f\|_S = \|f\|_S, \quad f \in S(G), \quad \gamma \in \Gamma.$$ 

**Theorem 5.3.** If $S(G)$ is a character Segal algebra on a locally compact abelian group $G$ with dual group $\Gamma$ and $T_0 \in M(L^1(G))$, $\mu \in M_{\text{b}}(\Gamma)$, then there exists a $T_1 \in M(L^1(G))$ with $T_1 \mu = T_0 \mu$ and

$$\|T_1\| \leq \|\mu\| \|T_0\|.$$

**Proof.** If $h \in B_s(G)$, $\mu \in M_{\text{b}}(\Gamma)$, then $h \mu \in B_s(G)$ and

$$\|\partial_{\gamma} (h \mu)\| = \|S^{+}(\gamma \mu) \hat{h} \| \leq \|S^{+}(\gamma \mu)\| \|\hat{h}\| = \|\gamma \mu\| \|\hat{h}\|,$$

so that

$$\|\partial_{\gamma} (h \mu)\| \leq \|\gamma \mu\| \|\hat{h}\|.$$

Hence, as $h \mu \in B_s(G)$, we have

$$\|\partial_{\gamma} (h \mu)\| \leq \|\gamma \mu\| \|\hat{h}\|.$$

Therefore, by Theorem 5.3, there exists a $T_1 \in M(L^1(G))$ with $T_1 \mu = T_0 \mu$ and

$$\|T_1\| \leq \|\mu\| \|T_0\|.$$
\[ \sup_{\omega \in \Gamma} \left| \int \mathcal{A}(x + \omega) \, o(x) \, \alpha(x) \, d\mu(x) \right| \]
\[ = \sup_{\omega \in \Gamma} \left| \int \mathcal{A}(x + \omega) \, o(x) \, \alpha(x) \, d\mu(x) \right| \]

Since \( \mathcal{A}(\cdot) \) is a character Segal algebra, it is easy to verify that \( T_{\omega} \) defined by
\[ T_{\omega} f(x) = \omega(-x) \, T(\omega f)(x), \quad f \in \mathcal{A}(\mathcal{G}), \quad x \in \mathcal{G} \]
belongs to \( \mathcal{M}(\mathcal{S}, \mathcal{L}) \) and
\[ \mathcal{A}(\cdot) = \mathcal{A}(\cdot + \omega) \]

Also
\[ \| T_{\omega} \| \leq \| T \|. \]

Therefore from (17) we have
\[ |\beta_{T}(\cdot) | \leq \sup_{\omega \in \Gamma} |\beta_{T_{\omega}}(\cdot) | \| f \| \leq \sup_{\omega \in \Gamma} \| T_{\omega} \| \| f \| \| \mu \|
\]
that is
\[ (\exists) \quad |\beta_{T}(\cdot) | \leq \| f \| \| \mu \| \| T \|. \]
(12) being true for all \( T \in \mathcal{M}(S, \mathbb{L}^1) \), by the definition of the norm in \( B_\delta(\mathcal{G}) \) we see that

\[
\| h \|_{B_\delta} \leq \| h \|_{\mathcal{G}} \| \mu \|.
\]

Given \( T_0 \in \mathcal{M}(S, \mathbb{L}^1), \mu \in \mathcal{M}_{\text{ba}}(\mathcal{H}) \), define the linear form \( \varphi \) on \( B_\delta(\mathcal{G}) \) as follows

\[
\varphi(h) = \beta_{T_0}(\mu h), \quad h \in B_\delta(\mathcal{G}).
\]

Then

\[
|\varphi(h)| = |\beta_{T_0}(\mu h)| \leq \| \beta_{T_0} \| \| \mu h \|_{B_\delta}, \quad h \in B_\delta(\mathcal{G}).
\]

From (12) we then have

\[
|\varphi(h)| \leq \| T_0 \| \| h \|_{B_\delta} \| \mu \|, \quad h \in B_\delta(\mathcal{G}).
\]

Therefore \( \varphi \) defines a bounded linear functional on \( B_\delta(\mathcal{G}) \) with

\[
\| \varphi \| \leq \| T_0 \| \| \mu \|.
\]

By Theorem 5.3, there exists an element \( \varphi_{\text{Top}} \in \mathcal{M}(S, \mathbb{L}^1) \) satisfying

\[
\beta_{\varphi_{\text{Top}}} = \varphi, \quad \| \varphi_{\text{Top}} \| = \| \varphi \|.
\]

Then from (12) we have

\[
\| \varphi_{\text{Top}} \| \leq \| T_0 \| \| \mu \|.
\]

Moreover
\[ \hat{\alpha}_{\nu} (h) = \int_{\Gamma} T_{\nu} (x) \hat{\alpha} (x) \, dx = \varphi (h) = \beta_{\nu} (h^\mu) \]

\[ = \int_{\Gamma} T_{\nu} (x) \hat{\mu} (x) \, dx, \quad h \in B_\nu (G_{\nu}) \]

Hence

\[ T_{\nu} (x) \hat{\mu} (x) = T_{\nu} \times \mu (x), \quad \nu \in \Gamma. \]

thus proving the required result.
CHAPTER VI

ON A THEOREM OF PIGNON

Let $G$ be a locally compact abelian group with character group $\Gamma$. Let $B$ be a subset of $\Gamma$. In this chapter we study the restrictions to $B$ of the multipliers on various spaces.

The space $\left( L^1, L^{p_1} \cap L^{p_2}, E \right)$ for $1 \leq p_1, p_2 \leq \infty$ is defined to be the set of all functions $\varphi$ on $B$ satisfying the condition that for every $f \in L^1(G)$, there exists $g \in L^{p_1} \cap L^{p_2}(G)$ such that

$$\hat{g} = \varphi \hat{f}$$

a.e. on $B$.

Pigne \cite{Pigne} has proved the following

THEOREM 6.1. For $1 \leq p_1 < 2$, $1 \leq p_2 \leq \infty$,

$$\left( L^1, L^{p_1} \cap L^{p_2}, E \right) = \left( L^{p_1} \cap L^{p_2} \right)^{1/E}$$

where $\left( L^{p_1} \cap L^{p_2} \right)^{1/E}$ denotes the space of restrictions to $B$ of the Fourier transforms of functions in $L^{p_1} \cap L^{p_2}(G)$.

Here we generalize the theorem to the case when both $p_1$ and $p_2$ are greater than two. For this we need the concept of the Fourier transform of a function in $L^p(G)$ as a quasi-measure as given by Gaudry in \cite{Gaudry}. 

Let $K$ be a compact subset of $G$. $D_K(G_t)$ is the vector space of all those continuous functions $u$ which can be represented as

$$u = \sum_{i=1}^{\infty} f_i \cdot g_i$$

where $f_i, g_i$ are continuous functions on $G$ with support contained in $K$ and $\sum_{i=1}^{\infty} \|f_i\|_\infty \cdot \|g_i\|_\infty < \infty$. If $u \in D_K(G_t)$ we define

$$\|u\|_{D_K} = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_\infty \cdot \|g_i\|_\infty \right\}$$

where the infimum is taken over all such representations of $u$. Then $D_K(G_t)$ is a Banach space. $\mathcal{D}(G)$ is then defined as the internal inductive limit of the Banach spaces $D_K(G_t)$. This means that $\mathcal{D}(G)$ is the vector space $\bigcup_K D_K(G_t)$ and the neighbourhood base at the origin for the topology on $\mathcal{D}(G)$ is given by open sets of the form

$$U_r = \bigcup_K \left\{ u \in D_K(G_t) : \|u\|_{D_K} < r \right\}.$$

$\mathcal{D}(G)$ is then a locally convex topological vector space and $D(G_t) \subset \mathcal{C}(G_t)$. The space of continuous linear functionals on $\mathcal{D}(G)$ is denoted by $\mathcal{Q}(G_t)$ and the elements of $\mathcal{Q}(G_t)$ are
called quasi-measures. Then $S$ is a quasi-measure if and only if $S$ is linear and the restriction of $S$ to $D_K(G)$ is continuous in the topology of $D_K(G)$ for each compact set $K$. Then Gauzky has proved the following results on quasi-measures.

**Theorem 6.2.** $D(0)$ is dense in $\Lambda(G)$

**Theorem 6.3.** If $\mu \in M_{bd}(H)$, the mapping $u \rightarrow \mu u$ is continuous from $D_K(G)$ into $D(0)$.

Theorem 6.3 implies that for $\mu \in M_{bd}(H)$, $u \in D_K(G)$

$$||\mu u||_{D_K} \leq ||u||_{D_K} ||\mu||_1.$$

Therefore if $\mu \in M_{bd}(H)$, $S \in \mathcal{G}(G)$, we can define

$$\hat{S} \in \mathcal{G}(G)$$

by

$$\langle \hat{S}, u \rangle = \langle \hat{\mu u}, S \rangle, \ u \in D(G).$$

Then

$$|\langle u, \hat{S} v \rangle| = |\langle \hat{\mu u}, Sv \rangle| \leq ||S||_{\mathcal{G}(G)} ||\mu u||_{D_K} \leq ||S||_{\mathcal{G}(G)} ||u||_{D_K}$$

which implies that

$$\langle \hat{S}, u \rangle \leq ||S||_{\mathcal{G}(G)} ||u||_{D_K}.$$

We also have

**Theorem 6.4.** Every quasi-measure with compact support is a quasi-measure.
For $f \in L^p(G)$, $1 \leq p \leq \infty$, $\hat{f}$ is defined to be the element of $\ell^q(G)$ satisfying

$$\langle \varphi, \hat{f} \rangle = \langle \varphi, f \rangle, \quad \varphi \in D(G)$$

If $\varphi_1, \varphi_2 \in D(G)$ and $E$ is a subset of $G$, $\varphi_1$ is said to be equal to $\varphi_2$ on $E$ if

$$\langle \varphi, \varphi_1 \rangle = \langle \varphi, \varphi_2 \rangle, \quad \varphi \in D(G)$$

where $\varphi$ has compact support, contained in $E$. For $2 \leq p_1, p_2 \leq \infty$, we define $(L^{1}, L^{p_1} \cap L^{p_2}, E)$ to be the set of all quasimeasure $q \in \mathcal{C}(G)$, satisfying the condition that for every $f \in L^1(G)$ there exists $g \in L^{p_1} \cap L^{p_2}(G)$ such that

$$\hat{g} = \hat{f}q \quad \text{on } E.$$

Then we have the following

**Theorem 6.5.**

$$\bigcap (L^{1}, L^{p_1} \cap L^{p_2}, E) = (L^{p_1} \cap L^{p_2})^{\lambda} \quad \text{on } E.$$

**Proof.** Suppose $f \in L^{p_1} \cap L^{p_2}(G)$. If $g \in L^1(G)$ then $q \cdot f \in L^{p_1} \cap L^{p_2}(G)$. Moreover

$$\hat{g}^\wedge = \hat{f}^\wedge$$

Then

$$\hat{f} \in (L^{1}, L^{p_1} \cap L^{p_2}, E)$$

To prove the converse, let $q \in (L^{1}, L^{p_1} \cap L^{p_2}, E)$. Let

$$A = \{ f \in L^{p_1} \cap L^{p_2}(G) : \hat{f} = 0 \quad \text{on } E \}$$

Then $A$ is a linear subspace of $L^{p_1} \cap L^{p_2}(G)$. If we introduce a norm on $L^{p_1} \cap L^{p_2}(G)$ by
(a) \[ \| f \|_{L^p} = \| f \|_{L^p,1} + \| f \|_{L^p,2} . \]

\( L^p \cap L^2(G) \) become a Banach space. We claim that \( A \) is a closed subspace of this Banach space. To this end, let \( \{ f_n \} \subset A \) and \( f \in L^p \cap L^2(G) \) such that

(3) \[ \| f_n - f \|_{L^p} = \| f_n - f \|_{L^p,1} + \| f_n - f \|_{L^p,2} \to 0 \text{ as } n \to \infty . \]

Then we have

(4) \[ \| f_n - f \|_{L^p,1} \to 0 \text{ as } n \to \infty \]

and

(5) \[ \| f_n - f \|_{L^p,2} \to 0 \text{ as } n \to \infty . \]

(6) implies

(6) \[ \lim_{n \to \infty} \int_G f_n g \, dx = \int_G f g \, dx, \quad g \in L^q(G), \]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Let now \( \varphi \in D(G) \) be such that the support of \( \varphi \) is contained in \( S \). Then
\[\langle \phi, f_n^\perp \rangle = \int_{G} f_n(x) \hat{\phi}(x) \, dx\]

by virtue of (6). By hypothesis \(f_n = 0\) on \(E\), then (7) gives

\[\langle \phi, f_n^\perp \rangle = 0\]

We have thus proved that \(f_n^\perp = 0\) on \(E\) and that \(A\) is a closed linear subspace of \(L^p \cap L^q(G)\).

Now consider the transformation

\[T_q : L^1(G) \rightarrow L^p \cap L^q(G) / A\]

given by

\[T_q f = g + A\]

where \(g\) is an element of \(L^p \cap L^q(G)\) such that \(\hat{g} = \hat{f}_q\) on \(E\). If \(\hat{g}_1 = \hat{f}_q\) on \(E\) also, then \(\hat{g} - \hat{g}_1 = 0\) on \(E\) and so \(g - g_1 \in A\). \(T_q\) is thus well defined. It is clearly linear. We shall now show that \(T_q\) is continuous. For this purpose we appeal to the closed graph theorem. Suppose

\[f_n, g_n^\perp \in L^1(G), f_n^\perp + A_n^\perp \subset L^p \cap L^q(G) / A\]

such that

\[f_n \rightarrow f\] and \(g_n^\perp + A \rightarrow g + A\). We want to prove that \(\hat{g} = \hat{f}_q\) on \(E\). Since

\[\lim_{n} f_n = f\]
in $L^1(G)$,

$$\lim_{n} \hat{f}_n q = \hat{f} q$$

in the space of quasimeasures (from (1)), that is

$$\lim_{n} \langle \varphi, \hat{f}_n q \rangle = \langle \varphi, \hat{f} q \rangle, \quad \varphi \in D(G).$$

Let $\varphi \in D(G)$ be such that support $\varphi \subseteq E$. Then

$$\lim_{n} \langle \varphi, \hat{g}_n \rangle = \lim_{n} \langle \varphi, \hat{f}_n q \rangle = \langle \varphi, \hat{f} q \rangle.$$

Since $g_n + A \rightarrow g + A$ in $L^1 \cap L^\infty$, there exists for each integer $n$, an element $h_n \in A$ such that

$$\|g_n - g + h_n\| \leq \|g_n - g\| + \|h_n\| + \frac{1}{n}$$

which implies

$$\|g_n - g + h_n\|_{p_1} + \|g_n - g + h_n\|_{p_2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then

$$\lim_{n} \langle (g_n - g + h_n)^\vee, \varphi \rangle = 0$$

for each $\varphi \in D(G)$. If support $\varphi \subseteq E$ then

$$\langle \varphi, h_n \rangle = 0 \quad \text{for all } n.$$
so that

\[ \lim_{n \to \infty} \langle \phi, \hat{g}_n \rangle = \langle \phi, \hat{g} \rangle. \]

From (11) and (6) we have

\[ \langle \phi, \hat{g} \rangle = \langle \phi, \hat{f} q \rangle \]

for all \( \phi \in D(\mathcal{M}) \) with support \( \phi \subset E \), that is

\[ \hat{g} = \frac{\hat{f}}{q} \quad \text{on} \quad E. \]

Hence \( T_q \) is continuous. There exists a constant \( K \) such that

\[ \| T_q f \|_1 \leq K \| f \|_2 \quad f \in L^1(\mathcal{M}). \]

Let \( \{ e_\alpha \} \) be an approximate identity in \( L^1(\mathcal{M}) \) such that

\[ \| e_\alpha \|_1 = 1 \quad \text{for all} \ \alpha \quad \text{and} \quad e_\alpha \]

has compact support for all \( \alpha \). For each \( \varepsilon > 0 \), there exists \( h_\alpha \in L^p \cap L^q(\mathcal{M}) \)

such that \( T_q e_\alpha = h_\alpha + A \quad \text{and} \)

\[ \| h_\alpha \|_p + \| h_\alpha \|_q \leq \| T_q e_\alpha \| + \varepsilon \leq K \| e_\alpha \|_1 + \varepsilon \leq K + \varepsilon. \]

Hence there exists a subset of \( \{ h_\alpha \} \) which without loss of

generality we denote by \( \{ h_\alpha \} \) itself and a function \( h \in L^p(\mathcal{M}) \)

such that \( h_\alpha \) converges weakly to \( h \) in \( L^p(\mathcal{M}) \). Now there
exists a subset of \( \mathcal{E}_{x_\beta} \) say \( \mathcal{E}_{x_\beta} \) and a function \( g \) in
\( L^2(G) \) satisfying
\[
\lim_{x_\beta} h_{x_\beta} = g
\]
weakly in \( L^2(G) \).

Therefore if \( \phi \in L^q \cap L^2(G) \) where \( \frac{1}{p_i} + \frac{1}{q_i} = 1 \), \( i = 1, 2 \), considering \( \phi \) as an element of \( L^q_i(G) \) we have

\[
(15) \quad \lim_{x_\beta} \int_{G_i} h_{x_\beta} \phi \, dx = \int_{G_i} h \phi \, dx.
\]

Also considering \( \phi \) as an element of \( L^2(G) \) we have

\[
(16) \quad \lim_{x_\beta} \int_{G} h_{x_\beta} \phi \, dx = \int_{G} g \phi \, dx.
\]

Hence from (15) and (16) we have

\[
(17) \quad \int_{G} h \phi \, dx = \int_{G} g \phi \, dx, \quad \phi \in L^q \cap L^2(G),
\]

that is

\[
h = g \quad \text{a.e. on } G.
\]

Hence \( h \in L^1 \cap L^2(G) \). To complete the proof we have to show that \( h = g \) on \( G \). Let \( \phi \in \mathcal{D}(G) \) be such that support of \( \phi = K \subseteq E \). There exists \( f \in L^1(G) \) such that
\[ \hat{f} = 1 \text{ on } \mathbb{R}, \text{ then} \]
\[ \langle \phi, \hat{e}_2 \rangle = \langle \hat{f} \phi, \hat{e}_2 \rangle = \langle \phi, \hat{f} \hat{e}_2 \rangle = \langle \phi, \hat{f} \cdot \hat{e}_2 \rangle = \langle \phi, \hat{f} \cdot \hat{e}_2 \rangle. \]

Since
\[ \lim_{\alpha} \hat{e}_2 \cdot f = \hat{f} \]
in the \( L^1(\mathbb{R}) \) norm,

\[ \lim_{\alpha} \hat{e}_2 \cdot f_\alpha = g \]
in \( L^1(\mathbb{R}) \) by virtue of inequality (1). We therefore have
\[ \lim_{\alpha} \langle \phi, \hat{e}_2 \rangle = \lim_{\alpha} \langle \phi, \hat{e}_2 \cdot f_\alpha \rangle = \langle \phi, \hat{f} \cdot \hat{e}_2 \rangle = \langle \phi, \hat{f} \cdot \hat{e}_2 \rangle = \langle \phi, \hat{g} \rangle. \]

Since support \( \phi \subset \mathbb{R} \) and \( \hat{h}_\alpha = \hat{e}_2 \cdot f_\alpha \) on \( \mathbb{R} \), we have
\[ \lim_{\alpha} \langle \phi, \hat{h}_\alpha \rangle = \lim_{\alpha} \langle \phi, \hat{e}_2 \rangle = \langle \phi, \hat{g} \rangle. \]

Since \( \hat{h}_\alpha \) converges weakly to \( \hat{h} \) in \( L^p(\mathbb{R}) \),

\[ \lim_{\alpha} \langle \phi, \hat{h}_\alpha \rangle = \langle \phi, \hat{h} \rangle, \quad \phi \in D(\mathbb{R}). \]
From (16) and (17) we get
\[
\langle \varphi, \hat{h} \rangle = \langle \varphi, q \rangle
\]
for all \( \varphi \in D(M) \) with support \( \varphi \subset \mathcal{E} \), that is
\[
\hat{h} = q \text{ on } \mathcal{E}.
\]
This completes the proof of the theorem.

If \( q \) is a quasi-measure on \( M \), we now derive a necessary and sufficient condition for \( q \) to be the Fourier transform of a function in \( L^p(G_t) \) for some \( p > 2 \). If \( \{e_\lambda\} \) is the approximate identity considered in the proof of Theorem 6.5, we see that \( \hat{e_\lambda}q \) has compact support for all \( \lambda \). This implies that \( \hat{e_\lambda}q \) is a pseudo-measure for every \( \lambda \) by Theorem 6.4. Hence there exists \( h_\lambda \in L^\infty(G_t) \) satisfying
\[
\hat{h_\lambda} = \hat{e_\lambda}q
\]
for each \( \lambda \). The following theorem then gives a criterion for a quasi-measure in \( G(M) \) to be the Fourier transform of an element of \( L^p(G_t) \).

**Theorem 6.6.** Given a quasi-measure \( q \) on \( M \), there exists \( h \in L^p(G_t) \) for some \( p > 2 \) satisfying \( \hat{h} = q \) if and only if \( h_\lambda \in L^p(G_t) \) for each \( \lambda \) with
\[
\|h_\lambda\|_p \leq K
\]
for all \( c \).

**Proof.** Suppose there exists \( h \in L^p(\Omega) \) such that \( \hat{h} = q \). Then \( h \ast e_\alpha \in L^p(\Omega) \) with
\[
\hat{e_\alpha \ast h} = \hat{e_\alpha} \cdot \hat{h} = \hat{h},
\]

Therefore
\[
\langle \varphi, \hat{e_\alpha \ast h} \rangle = \langle \varphi, \hat{h} \rangle, \quad \varphi \in D(\Gamma)
\]

that is,
\[
\int_{\Omega} h_\alpha(x) \varphi(x) \, dx = \int_{\Omega} e_\alpha \ast h(x) \varphi(x) \, dx, \quad \varphi \in D(\Gamma)
\]

By Theorem 6.3 since \( D(\Gamma) \) is dense in \( A(\Gamma) \), \( D(\Gamma)^\perp \) is dense in \( L^1(\Omega) \) and so
\[
h_\alpha(x) = e_\alpha \ast h \text{ a.e. on } \Omega.
\]

Therefore \( \hat{e_\alpha \ast h} \in L^p(\Omega) \) for each \( \alpha \) with
\[
\|h_\alpha\|_p = \|e_\alpha \ast h\|_p \leq \|e_\alpha\|_1 \|h\|_p \leq \|h\|_p.
\]

Conversely if \( h_\alpha \in L^p(\Omega) \) with \( \|h_\alpha\|_p \leq K \) for all \( \alpha \) then there exists a subseq, say \( \{h_{\alpha_p}\} \), and a \( h \in L^p(\Omega) \) such that \( h_{\alpha_p} \) converges to \( h \) weakly. Hence \( \langle h_{\alpha_p}, \varphi \rangle \) converges to \( \langle h, \varphi \rangle \) for all \( \varphi \in D(\Gamma) \). By arguments similar to that used in the Proof of Theorem 6.3, we can show that
\[ \lim \langle \varphi, e^{\alpha_0} q \rangle = \langle \varphi, q \rangle, \ \varphi \in D(Q) \]

But
\[ e^{\alpha_0} q = \xi \quad \text{Hence} \]

\[ \lim \langle \xi, q \rangle = \langle \xi, q \rangle, \ \varphi \in D(Q) \]

That is
\[ \xi = q. \]

This completes the proof of the theorem.

For \( 1 \leq p < \infty \), we define \((L^1, A^p, E)\) to be the set of all functions \( \varphi \) defined on the subset \( E \) of \( \Gamma \) satisfying the condition that for every \( f \in L^1(G) \), there exists \( g \in A^p(G) \) such that
\[ g = \varphi \hat{f} \quad \text{a.e on } E, \]

We then have the following:

**Theorem 6.3**: \( (L^1, A^p, E) = (A^p)^{\perp/} E \) for \( 1 \leq p < \infty \) and \( (L^1, A^p, E) = (B_p)^{\perp/} E \) for \( 2 < p < \infty \)

where \( A^p \) and \( B_p \) denote the restrictions to \( E \) of the Fourier transform and the Fourier-Stieltjes transform of functions in \( A^p(G) \) and measures in \( B_p(G) \) respectively.

**Proof**. The proof is similar to that of Theorem 6.5 and is hence omitted.
Similarly if $\omega$ is a weight function defined on $\mathbb{R}$ for $1 \leq p < \infty$, we define \((L^1, A^p_\omega, E)\) to be the set of functions $\varphi$ defined on $\mathbb{R}$ such that for every $f \in L^1(\mathbb{R})$ there exists $g \in A^p_\omega(\mathbb{R})$ such that
$$\hat{\varphi} = \varphi \hat{f} \quad \text{a.e. on } E.$$

Then we have the following

**Theorem 6.7.** \((L^1, A^p_\omega, E) = (A^p_\omega)^\prime|_E\) for $1 \leq p \leq 2$ and \((L^1, B^p_\omega, E) = (B^p_\omega)^\prime|_E\) for $p > 2$.

where \((A^p_\omega)^\prime|_E\) and \((B^p_\omega)^\prime|_E\) denote the restrictions to $E$ of the Fourier transforms and the Fourier-Stieltjes transforms of the functions in $A^p_\omega(\mathbb{R})$ and the measures in $B^p_\omega(\mathbb{R})$ respectively.
CHAPTER VII

ON TRANSLATION INVARIANT SUBSPACES

Larson has asked the following question in [1]. If $G$ is a noncompact group and $1 < p < 2$, do there exist nonzero closed translation invariant subspaces $X$ of $L^p(G)$ such that $X \cap L^1(G) = \{0\}$? In this chapter we give an answer to this question.

Let $G$ be a noncompact locally compact abelian group with dual $\Gamma$. If $E$ is a measurable subset of $\Gamma$ and $1 \leq p < \infty$, $E$ is a set of uniqueness for $L^p(G)$, if there exists no nonzero element $g \in L^1(\Gamma)$ such that $g(\gamma) = 0$ for almost all $\gamma$ in $\Gamma \setminus E$ the complement of $E$ in $\Gamma$ and $\hat{g} \in L^p(\hat{G})$. The existence of such a set has been proved by Figa-Talamanca and Gundy in [2].

THEOREM 7.1. If $E$ is a measurable subset of $\Gamma$ with positive Haar measure, then given $\varepsilon > 0$, there exists a subset $F$ of $E$ such that $m(F) > m(E) - \varepsilon$, where $m$ denotes the Haar measure and $F$ is a set of uniqueness of $L^p(G)$ for all $p$ in $1 \leq p < 2$.

Let $X^p_E = \{ f \in L^p(G) : \frac{f}{\gamma} = 0 \text{ outside } E \}$. 

where
\[ \hat{\phi}(\gamma) = \hat{\phi}(-\gamma), \quad \gamma \in \mathbb{R} \, . \]

Then Larsen has proved the following theorem.

**Theorem 7.2.** Let \( G \) be a noncompact locally compact abelian group and \( E \) a measurable subset of \( \mathbb{R} \). If \( 1 \leq p < \infty \), and \( \chi_{\mathbb{R}}^* \cap L^p(G) = 0 \), then \( E \) is a set of uniqueness for \( L^p(G) \).

**Theorem 7.3.** Let \( G \) be a noncompact locally compact abelian group. Let \( 1 \leq p < 2 \) and \( E \) be a measurable subset of \( \mathbb{R} \) with finite linear measure. If \( E \) is a set of uniqueness of \( L^p(G) \) and \( 1 \leq p < 2 \), then \( \chi_{\mathbb{R}}^* \cap L^p(G) = 0 \).

**Theorem 7.4.** If \( G \) is a noncompact locally compact abelian group and \( 1 \leq p < 2 \), then there exists a subset \( E \) of \( \mathbb{R} \) of finite positive linear measure such that \( E \) is a set of uniqueness for \( L^p(G) \), but not a set of uniqueness for \( L^2(G) \).

We then have

**Theorem 7.5.** If \( G \) is a noncompact locally compact abelian group and \( 1 \leq p < 2 \), there exists a non-zero closed translation invariant subspace \( X \) of \( L^p(G) \) such that \( X \cap L^1(G) = 0 \).

**Proof.** First we prove the existence of a measurable subset \( E \) of \( \mathbb{R} \) with finite positive linear measure such that \( E \) is a set of uniqueness for \( L^p(G) \), but not a set of uniqueness for \( L^2(G) \). Suppose there does not
exists such a set. Choose \( q \) satisfying \( 1 < q < p < 2 \). By Theorem 7.4 there exists a measurable subset \( E' \) of \( \mathbb{M} \) with finite positive linear measure such that \( E' \) is a set of uniqueness for \( L^q(G) \) but not a set of uniqueness for \( L^p(G) \).

From Theorem 7.2, since every set of uniqueness for \( L^q(G) \) is a set of uniqueness for \( L^q(G) \), \( E' \) is a set of uniqueness for \( L^q(G) \). By hypothesis, \( E' \) is a set of uniqueness for \( L^p(G) \), thus contradicting the definition of \( E' \). Hence there exists at least one measurable subset \( E \) of \( \mathbb{M} \) of finite positive linear measure such that \( E \) is a set of uniqueness for \( L^p(G) \) but not a set of uniqueness for \( L^p(G) \).

\[
X = X_E^p
\]

is then a closed translation invariant subspace of \( L^p(G) \) which is nonzero, since if \( X_E^p = \{0\} \) by Theorem 7.2, \( E \) will be a set of uniqueness for \( L^p(G) \). Since \( E \) is a set of uniqueness for \( L^p(G) \), by Theorem 7.3,

\[
X \cap L^1(G) = X_E^p \cap L^1(G) = \{0\}
\]

This proves the required result.

**Theorem 7.6.** If \( G \) is a noncompact locally compact abelian group and \( 1 \leq q < r \leq p \) there exists a non-zero closed translation invariant subspace \( X \) of \( L^p(G) \) satisfying

\[
X \cap L^q(G) = \{0\}
\]
PROOF. From Theorem 7.1, we know that there exists a measurable subset \( B \) of \( \mathbb{R} \), the character group of \( G \), such that \( B \) is of finite positive Haar measure and it is a set of uniqueness for \( L^q(G) \). Since the only sets of uniqueness for \( L^p(G) \), for \( p \leq 2 \), are the sets of Haar measure zero, \( B \) is not a set of uniqueness for \( L^p(G) \).

For \( f \in L^p(G) \), as in chapter six, we can define \( \hat{f} \) to be the quasimeasure on \( \mathbb{R} \) satisfying
\[
\langle \varphi, \hat{f} \rangle = \langle \varphi, f \rangle, \quad \varphi \in D(\mathbb{R})
\]

Define \( X_E^p = \{ f \in L^p(G) : \hat{f} = 0 \text{ on } \mathbb{R} \setminus E \} \).

Suppose \( X_E^p = \{ 0 \} \). Let \( g \in L^{1}(\mathbb{R}) \) be such that \( g(x) = 0 \) for almost all \( x \in \mathbb{R} \setminus E \) and \( \hat{g} \in L^p(G) \). Define \( f(x) = g(x) = \hat{g}(-x), x \in \mathbb{R} \). Then \( f \in L^p(G) \) and
\[
\hat{f} = g = 0 \text{ on } \mathbb{R} \setminus E.
\]

Therefore \( f \in X_E^p \), which implies that \( f = 0 \) and hence that \( g = 0 \). This contradicts the fact that \( B \) is not a set of uniqueness for \( L^p(G) \). Using arguments similar to that used in the proof of Theorem 6.5, we can show that \( X \) is closed. Therefore \( X \) is a closed nonzero linear subspace of \( L^p(G) \) which is moreover translation invariant since if \( f \in X \), and \( a \in G \),
\[ \langle \phi, \mathcal{H} \hat{a} \hat{f} \rangle = \langle \mathcal{H} \hat{a} \hat{f}, \phi \rangle, \quad \phi \in D(\mathcal{H}) \]

which implies that
\[ \langle \phi, \mathcal{H} \hat{a} \hat{f} \rangle = 0, \quad \phi \in D(\mathcal{H}) \]

with support \( \phi \subseteq \mathbb{R} \setminus E \).

To complete the proof of the theorem it only remains to verify that \( X \cap L^q(G) = \{0\} \). Suppose not. Let \( f \in X \cap L^q(G) \). Then \( f \in L^1(G) \cap L^q(G) \). Since \( q < 2 \leq p \), \( f \in L^2(G) \).

Therefore
\[ f \in L^2(G), \quad \hat{f} = 0 \text{ a.e. on } \mathbb{R} \setminus E \]

that is \( f \in X_2 \cap L^q(G) \). Since \( E \) is a set of uniqueness for \( L^q(G) \), from Theorem 7.3,
\[ X_2 \cap L^q(G) = \{0\} \]

This implies that \( \hat{f} = 0 \). Therefore
\[ X \cap L^q(G) = \{0\} \]

The proof of the theorem is now complete.

If \( S(G) \) is a regular algebra on a noncompact locally compact abelian group \( G \) and \( M[S(G)] \) is its multiplier algebra, then to every element of \( M[S(G)] \) there corresponds a bounded continuous function on \( \mathbb{R} \). Let \( \mathcal{M}[S(G)] \) denote the space \( \mathcal{M}[S(G)] \). Then we have the following
THEOREM 7.7. \( \text{on } \mathcal{L}(G) \cap C(R^m) \neq C(R^m) \)

PROOF. Let \( F \) be a compact subset of \( R^m \) with positive Haar measure. By Theorem 7.1 there exists a measurable subset \( E \) of \( F \) with finite positive Haar measure, such that \( E \) is a set of uniqueness for \( \mathcal{L}(G) \). If \( \chi_E \) denotes the characteristic function of \( E \), then \( \chi_E \) does not belong to \( \text{on } \mathcal{L}(G) \). If we assume that it does, then we will arrive at a contradiction in the following way. For every \( f \in \mathcal{L}(G) \) there exists \( g \in \mathcal{S}(G) \) such that \( \hat{f} = \chi_E \hat{g} \). Now choose \( f \in \mathcal{S}(G), \hat{f} = 1 \) on \( E \) and \( \hat{f} = 0 \) outside \( F \). Then there exists \( g \in \mathcal{S}(G) \) such that \( \hat{f} \chi_E = \hat{g} \). Now \( \hat{g} \) has compact support and therefore it belongs to \( \mathcal{L}(R^m) \). Also \( \hat{g} = 0 \) a.e. outside \( E \) and \( \chi_E \hat{g} = g \in \mathcal{L}(G) \).

Since \( E \) is a set of uniqueness for \( \mathcal{L}(G) \), we then have \( g = 0 \) which is impossible since \( \hat{g} = \chi_E \hat{g} = 0 \) on \( R^m \). Therefore \( \chi_E \notin \text{on } \mathcal{L}(G) \).

Suppose \( g \cdot \chi_E \in \mathcal{M}(C(G)) \) for every \( g \in \mathcal{L}(R^m) \). Then we have a mapping \( \Lambda \) from \( \mathcal{L}(R^m) \) into \( \mathcal{M}(C(G)) \) given by \( \Lambda(g) = T_g \) where \( T_g \in \mathcal{M}(C(G)) \) satisfies \( T_g = g \cdot \chi_E \).

To prove \( \Lambda \) is continuous, we apply the closed graph theorem.
Let \( \lim_{n} \| g_n - g \|_1 = 0 \) and \( \lim_{n} \| T g_n - T \| = 0 \) in \( M_C(S(G)) \). To prove \( T = T_g \), we have
\[
\| T - T g_n \|_\infty \leq \| T - T^\wedge g_n \|_\infty + \| T^\wedge g_n - T^\wedge g \|_\infty
\]
\[
\leq \| T - T g_n \| + \| g_n \times \chi_E - g \times \chi_E \|_\infty
\]
\[
\leq \| T - T g_n \| + \| \chi_E \|_1 \| g_n - g \|_\infty
\]

The right hand side tends to zero as \( n \to \infty \). Therefore, we have
\[
T = T^\wedge g \quad \text{a.e. on} \quad \Gamma^c,
\]
that is \( T = T_g \). Therefore there exists a constant \( K \) such that
\[
\| T g \| \leq K \| g \|_1, \quad g \in L^1(\mathbb{R}).
\]

Let \( \epsilon^\alpha \) be an approximate identity in \( L^1(\mathbb{R}) \) with
\[
\| \epsilon^\alpha \|_1 \leq 1, \quad \epsilon^\alpha \quad \text{has compact support for all} \; \alpha.
\]
Then
\[
\| T \epsilon^\alpha \| \leq K \| \epsilon^\alpha \|_2 \quad \text{for all} \; \alpha,
\]
that is
\[
\| T \epsilon^\alpha \|_S \leq K
\]
that is
\[
\| T \epsilon^\alpha \|_S \leq K
\]
for all \( f \in S(G_\varepsilon) \) with
\[ \| f \|_\varepsilon \leq 1. \]
Choose \( f \in S(G_\varepsilon) \) such that \( f = 1 \) on \( E \) and \( f = 0 \) outside \( E \). We then have
\[ \| T_\varepsilon f \|_1 \leq \| T_\varepsilon f \|_\varepsilon \leq K \| f \|_\varepsilon \]
for all \( \varepsilon \). There exists a subset of \( \{ e_\alpha \} \) say \( \{ e_\alpha^p \} \) and a measure \( \mu \in M_{bd}(G_\varepsilon) \) satisfying
\[ T_\varepsilon f \rightarrow \mu \]
weakly in \( M_{bd}(G_\varepsilon) \). If \( g \in B(G_\varepsilon) \) we have
\[ \lim_{\varepsilon \to 0} \int_G T_\varepsilon f(x) g(x) \, dx = \int_G g(x) \, d\mu(x) \]
that is
\[ \lim_{\varepsilon \to 0} \int_M T_\varepsilon f(x) g(x) \, dx = \int_M g(x) \, d\mu(x) \]
that is
\[ \lim_{\varepsilon \to 0} \int_M \hat{f}(x) \hat{g}(x) \, dx = \int_M \hat{g}(x) \, d\mu(x) \]
that is
\[ \lim_{\varepsilon \to 0} \int_M e_\varepsilon \times \chi_E \, dx = \int_M \chi_E \, d\mu \]
Since
\[ \lim_{\varepsilon \to 0} \| e_\varepsilon \chi_E - \chi_E \|_1 = 0 \]
we have

\[
\lim_{\alpha \to 0} \int_{\mathbb{R}} \mathcal{F}_\alpha(x) \mathcal{F}_\alpha(y) \, dy = \int_{\mathbb{R}} \mathcal{F}_\alpha(x) \mathcal{F}_\alpha(y) \, dy
\]

(1) and (2) together tell us that

\[
\int_{\mathbb{R}} \mathcal{F}_\alpha(x) \, dx = \int_{\mathbb{R}} \mathcal{F}_\alpha(y) \, dy
\]

This implies that

\[
\mu(x) = \lambda(x) \quad \text{a.e. on } \mathbb{R}.
\]

Now \( \mu \in M_c(S(G)) \) and therefore \( \lambda \in M_c(S(G)) \) which contradicts that which we have already proved. Therefore there exists at least one \( g \in L^2(M) \) such that \( g \ast \lambda \notin M_c(S(G)) \).

Since \( g \ast \lambda \in C_c(G) \), we therefore see that

\[
M_c(S(G)) \cap C_c(G) = C_c(G).
\]

In the case of compact groups, Theorem 7.7 is not valid since \( L^1(G) \) is a Segal algebra for which \( M_c(S(G)) = L^2(M) \). This will therefore satisfy:

\[
M_c(S(G)) \cap C_c(G) = C_c(G).
\]
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