General Linear Group and Symmetric Group: Commuting Actions and Combinatorics

By

Arghya Sadhukhan

Institute of Mathematical Sciences, Chennai.

A thesis submitted to the

Board of Studies in Mathematical Sciences
In partial fulfillment of requirements

for the Degree of

Master of Science

of

HOMI BHABHA NATIONAL INSTITUTE

July, 2017

ACKNOWLEDGEMENTS

This Master's thesis is the conclusion of my studies in IMSc, Chennai and I would like to mention and thank warmly all the people and friends that contributed to make my time here enjoyable and enriching. My deepest gratitude goes to my thesis advisor, Prof. Amritanshu Prasad, whose door was always wide open (*literally*) for stimulating discussion sessions. His flexible approach and insightful remarks, coupled with extreme patience in hearing out my confused ramblings have always been a huge help; in particular, he has guided me in many instances to chalk out more insightful proofs of classically known results. I have been tremendously benefited from his initiative of organizing the courses by Prof. Xavier Viennot and running the weekly 'Algebraic Combinatorics' seminars, they have been instrumental in shaping my outlook towards mathematics research as a whole. Thanks goes to him also for his course on SAGE, for initiating me to his contagious fervor for combinatorial experimentations, and at the same time letting me pursue other interesting flavors of representation theory and providing gracefully suggestions of beautiful references in many instances.

A great many thanks to Prof. S Viswanath, for initiating us into representation theory and Prof. K N Raghavan, for his further exposition of the theory of Lie algebras; I owe to them for such marvelous exposition, deepening my algebraic inclinations and also for introducing us to the likes of Bourbaki's exposition. The courses that I took with them were experiences to cherish.

I must mention the fellow doctoral students and friends, from whom I have learned so many things and got help whenever in need: Digjoy, Arun, Karthick, Oorna, Ujjal, Prosenjit. In particular, I would like to thank Mrigendra, for his ever willingness to discuss my confusions in Lie algebra stuffs and Mita, for her constant encouragement to discuss mathematics at large and for her help in making me wiser in using SAGE, as much as possible. I am also indebted to the ever increasing group of algebraic combinatorics enthusiasts at IMSc, we have shared so many fruitful interactive sessions in the seminars.

My special thanks goes to our administrative staff members for all the help I have had from them to make my studies here hassle free, especially our registrar Mr. S V Prasad and Admin. officer Indra, for their resolute support in a moment of personal crisis. I wish to thank IMSc from my heart for inculcating an excellent environment for research to thrive in. I am especially indebted to the amazing library of our institute.

Finally, I want to thank my parents for their selfless and unending love and support, without which I would have never made it to this.

ABSTRACT

In this thesis, we have explored the representation theories of two prototypical examples of finite and infinite groups, the symmetric group and the general linear group, over the base field of complex numbers. More specifically, we are interested in understanding the connection between these two groups' representations and seeing the ramifications they have on each other, while trying to make the exposition combinatorial in nature all the while. Robinson-Schensted-Knuth correspondence and its dual have been employed to deduce many character identities throughout, which in turn yield nontrivial facts about representations. After discussing concrete realizations of irreducible representations of these two groups and establishing the bridge between these worlds, we use this machinery to go back and forth, which in turn shed new lights on Gelfand models of symmetric groups. Finally, we use SAGE computations to work out concrete answer to a naturally motivated question we raised in this thesis.

Contents

1 Weyl's Unitary Trick **6**
1.1 Matrix Coefficients and the Peter-Weyl theorem 6
1.2 The Trick 7
1.3 Lifting K-representations to G-representations 9
1.4 Characters of Polynomial Representations 12
1.5 Branching Rule and Gelfand-Tsetlin Basis 17

2 Explicit Constructions **21**
2.1 Using Tensor Space 21
2.2 Via Young's Symmetrizer 24
2.3 Realizing $\mathbb{S}_\lambda$ via Matrix Minors 27
2.4 Kernel of φ 32
2.5 Multiplicity-Free Sum of GL_n Polynomial Representations-I 34

3 Relation to the Symmetric Group **36**
3.1 Specht Modules 36
3.2 Examples 38
3.3 Schur-Weyl Duality 40
3.4 Consequences of Schur-Weyl Duality 44
3.4.1 Character Theoretic Considerations 44
3.4.2 Invariant Theoretic Considerations 47
3.4.3 An Interesting Map 49

4 Perspectives on Gelfand Model and Some Computations **52**
4.1 Multiplicity-Free Sum of GL_n Polyreps-II 52
4.2 Gelfand Model for Symmetric Group 54
4.3 An Old Question Revisited 58

Introduction

In this thesis, we want to explore representation theory of the general linear group GL_n and see how that blends with the representation theory of the symmetric group S_n inside it, over the base field of complex numbers. Now, a representation of $GL_n(\mathbb{C})$ of course means just a group homomorphism $\rho : GL_n(\mathbb{C}) \to GL(V)$ for some finite dimensional $\mathbb{C}$-vector space V. Writing $U = \mathbb{C}^n$ to be the standard n-dimensional representation of GL_n, we can jot down some of the examples that come under this purview:

$(i)V = U, (ii)V = U^*, (iii)V = \mathbb{C}, \rho(g) = (\det g)^k$, for some $k \in \mathbb{Z}, (iv)V = \mathrm{Sym}^k W, or \ \bigwedge^k W$ where W is any of the previous examples, $(v)V = (\mathrm{Sym}^2 U \otimes U) \bigcap (U \otimes \bigwedge^2 U)$, where the intersection is happening inside $U^{\otimes 3}$; this is the 'easiest' nontrivial example of a Schur functor that we discuss in this thesis.

But there is a technical point that we have to take care of: *there are some representations that do not fit into our framework.* For example, complex conjugation gives rise to representations like $g \mapsto \overline{g}$. Moreover, we can use other field automorphisms of $\mathbb{C}$ (of which there are uncountably many to consider, see https://math.stackexchange.com/questions/412010/wild-automorphisms-of-the-complex-numbers), to get highly discontinuous maps. In addition to all these, since as a group, $\mathbb{C}^*$ has a lot of automorphisms (relying on the axiom of choice), we could compose the det representation with any of these. To make things worse, we could take any of these bizarre examples and tensor them (or take direct sum) with the 'normal' examples to produce even weirder ones! Thus we see that if all we ask for is a group homomrphism, then there are too many, and it will be intractable to classify all of them; so we have to impose some further conditions on the nature of the homomorphism so as to get relieved from this analytic mess and restrict ourselves to more algebraic examples as described above.
From the viewpoint of algebraic geometry, a proposed solution is to require $\rho : GL_n \to GL_m$ (taking $V = \mathbb{C}^m$)to be an algebraic map, i.e. that the matrix coefficients of $\rho(g)$ are polynomial in the matrix coefficients g_{ij} of $g = (g_{ij})$ and of $(det\ g)^{-1}$. This means that after choosing a basis of V, so that $\rho(g) = (\rho(g)_{kl}) \in GL_m$, we require that for each $(k, l) \in [m] \times [m]$, there be polynomials P_{kl} in $n^2 + 1$ entries such that

$$\rho(g)_{kl} = P_{kl}(g_{11}, g_{12}, \ldots, g_{nn}, det(g)^{-1})$$

We call this kind of ρ to be an algebraic, or more commonly **rational representation** of $GL_n(\mathbb{C})$, and if there are no occurences of det^{-1} in the matrix coefficients of $\rho(g)$, then we call ρ to be a **polynomial representation**. It can be seen that this notion is independent of the basis chosen for the representation space. Of our examples above, all but $(ii), (iii)$ are polynomial representation; (ii) is always non-polynomial rational, and (iii) is non-polynomial rational precisely when k is a negative integer. These are the type of examples we want to concetrate on and classify in this thesis. Part of the justification behind this comes from an analytic point of view as well: if we instead require the map $\rho : GL_n(\mathbb{C}) \to GL_m(\mathbb{C})$ to be given by holomorphic functions, we will end up in obtaining the same set of representations of GL_n (this is proved in Chapter 1), so we can study analytic representations with fewer analytic prerequisites, by simply looking at rational representations. But a result in Chapter 1 states that all rational representations arise from tensoring a polynomial representation with negative power of the det representation, so it suffices to concentrate on the polynomial representation theory of $GL_n(\mathbb{C})$.

We now proceed to give an outline of the exposition contained in this thesis. Chapter 1 deals with the interplay of continuous representations of the unitary group $U_n(\mathbb{C})$ and analytic represenations of $GL_n(\mathbb{C})$, a general theme which is commonly referred to as **Weyl's unitary trick**, originating from Hermann Weyl's classic book [17] published in 1938. In particular, assuming the Peter-Weyl theorem and certain other results about continuous representations of the unitary group, we prove that the characters of irreducible GL_n polynomial representations are the well known Schur polynomials, a well known class (in fact, a basis) of symmetric polynomials. Thus the irrducible polynomial representations of GL_n are indexed by partitions of any number with at most n parts. A crucial step in the proof is provided by the famous **Robinson-Schensted-Knuth (RSK) correspondence**, a cornerstone result published in 1970 [2]. We have employed the RSK correspondence as an unifying theme in many proofs in this thesis; it is, so to speak, the main bridge between combinatorics and representation theory here, and all this becomes possible because of the relation of polynomial representations with semistandard tableaux via their characters - the Schur polynomials. Since characters determine representations, this also let us get hold of the branching problem, and gives rise to the notion of a **Gelfand-Tsetlin basis** in an irreducible polynomial representation of GL_n.

In Chapter 2, we construct the irreducible polynomial representations concretely: by exploiting upon a clue provided by the dictionary we have set up between representations and symmetric polynomials. We give three different realizations of $V_\lambda(n)$, the irreducible polynomial representation of $GL_n(\mathbb{C})$ corresponding to the partition λ: using the tensor space $V^{\otimes|\lambda|}$, another employing **Schur functor** and a third one in terms of certain matrix minors. The third realization comes, as anticipated at the end of Chapter 1, with a basis indexed by semistandard Young tableaux of shape λ, abbreviated hereafter as $SSYT(\lambda)$, with the entries in $[n] := \{1, 2, \cdots, n\}$; but we show that in spite of its similarity with the Gelfand-Tsetlin basis, it is not the Gelfand-Testlin basis. It remains as an intriging problem to compute the basis change matrix in this scenario, which we hope to solve in future.

In Chapter 3, we turn our attention to the symmetric group S_n embedded in GL_n as permutation matrices. We deduce from the (GL_d, GL_d) duality (which was proved in the first chapter using RSK correspondence) that the (1^d)-weight space of $V_\lambda(d)$ (where λ is a partition of d) is the Specht module Sp_λ, the irreducible representation of S_d corresponding to λ. This, coupled with again the (GL_d, GL_n) duality in turn yields the ubiquitous **Schur-Weyl duality**, a result that allows us to relate the irreducible representations of S_d with the degree d irreducible polynomial representations of GL_n, for any positive integer d and n. This was proved in Schur's celebrated paper [6] using double commutant theorem, but here we resort to a totally different method; in fact we prove that the (GL_d, GL_n) duality is equivalent to Schur-Weyl duality. In general, when two groups or two algebras have commuting actions on the same space, their representation theories and combinatorics become intimately connected. It is important to note that this 'bridge' between the world of representations of two groups (here GL_n and S_d) having commuting actions on the tensor space $(\mathbb{C}^n)^{\otimes d}$, exist in other cases as well; see [20] for an introduction to **partition algebras** and [27] for other instances of commuting actions and their various applications to the theory of symmetric functions and knot theory. We next discuss some direct consequences of Schur-Weyl duality, most notably the *Frobenius character formula for* S_n (which is the character theoretic incarnation of Schur-Weyl duality), first fundamental theorem for GL_n and the Frobenius characterstic map.

In the final chapter, we explore a particular theme: **Gelfand models** (defined in section 4.2) for the symmetric groups. First we describe a representation, necessarily infinite dimensional, in which every $V_\lambda(n)$ occurs exactly once; we prove this employing the RSK correspondence and Schutzenberger's lemma and later sketch out a known proof of this fact using a symmetric function identity. We show how this fact gives rise to an involution model of S_n, and in particular proves the main result in the article [32] in a completely different way. On the way, we also derive another realization of the model, in the sense of Bernstein-Gelfand-Gelfand (see section 4.2) and thus see that how it sheds new light on the work of Klyachko[31] and Inglis et all[30]. These proofs are new to the best of our knowledge and they rely crucially on combinatorics. In the final section of this thesis, we sketch some computations using the **Sage mathematical software** to the following natural question: what happens if we pick up in $V_\lambda(n)$ some weight space other than (1^n)? *It is amusing to notice that to get hold of these weight spaces, we use Schur-Weyl duality, which is itself proved using the nature of the* (1^n) *weight space, and everything starts from*

the (GL_d, GL_n) duality, which the RSK correspondence proves so effortlessly!
We have tried to make this exposition combinatorial in nature. In many places, we have closely followed the treatment of a course given by David Speyer (see [33]) at the University of Michigan, Ann Arbor. The only prerequisite to go through this thesis is a working knowldege of the theory of symmetric functions, the reader is referred to [21], Chapter 4 for a lightning introduction. One can consult this book also for other applications of RSK correspondense to representation theory. Our exposition here touches on different works of Frobenius and Schur, [22] is a masterly chronicle to their life and work.

Chapter 1

Weyl's Unitary Trick

Our route to exploring representation theory of $GL_n(\mathbb{C})$ will be via that of the unitary group $U_n(\mathbb{C})$ sitting inside it. An important reason for such an approach is that $U_n(\mathbb{C})$ is a compact group, and for such classes of groups there is a strong machinery, called the ***Peter-Weyl theorem*** which makes them amenable to representation theoretic study. The fact that $U_n(\mathbb{C})$ is the maximal compact subgroup of $GL_n(\mathbb{C})$ implies that their representation theories are intricately linked, this is Weyl's general folklore and we will see this principle in action.

1.1 Matrix Coefficients and the Peter-Weyl theorem

Let G be a topological group, i.e. G is equipped with a topology in which the group operations, multiplication and inversion, are continuous. A continuous representation (ρ, V) of such groups mean that $\rho : G \to GL(V)$ is a continuous homomorphism of topological groups. Let $\mathcal{C}^0(G)$ be the set of continuous function $G \to \mathbb{C}$. We want to understand such functions in terms of representations of G. In fact, an important subclass of such functions arises as follows. Let V be any finite dimensional continuous representation and $\lambda \in (End\ V)^*$, then $\lambda \circ \rho_V \in \mathcal{C}^0(G)$. These are called *matrix coefficients*. Matrix coefficients form a ring (because direct sum and tensor of representations is again a representation), and we denote this ring by $\mathcal{O}(G)$. In fact, there is a nice criterion for detecting when a continuous function $f : G \to \mathbb{C}$ is a matrix coefficient: *precisely when* $Span\{(g_1, g_2) \cdot f : g_1, g_2 \in G\}$ *is finite dimensional*, where we note that $G \times G$ acts on $C^0(G)$ by $((g_1, g_2) \cdot f)(g) = f(g_1^{-1} g g_2)$. But an important question is: how much in abundance matrix coefficients are, or in other words why should there be lots of finite dimensional continuous representation of a topological group. The answer varies with groups; for instance, the basic statement of Fourier analysis tells us that under left regular action of S^1 on $L^2(S^1)$, every irreducible continuous representation of S^1 occurs in the decomoposition, and all of them are 1 dimensional, whereas if we replace S^1 by $\mathbb{R}$, $L^2(\mathbb{R})$

has no finite dimensional $\mathbb{R}$ subrepresentation. Peter-Weyl theorem asserts the following.

Theorem 1.1.1. For a compact group G, all of its irreducible representations (or, irreps, in abbreviated form) are finite dimensional and $\mathcal{O}(G) \cong \bigoplus(\operatorname{End} V)^* \cong \bigoplus V^* \otimes V$, where the direct sum is over all the isomorphism classes of G; the summands are pairwise orthogonal and this is a decomposition of $G \times G$ representations.

Here the embedding is as follows: given $\lambda \in (\operatorname{End} V)^*$, the isomorphism takes it to the function $g \mapsto \lambda(\rho_V(g))$ on G. This is an ubiquitous result for compact groups, therefore we do not prove it here, see [16]. In particular, since for finite group G, $\mathcal{O}(G) = \mathbb{C}[G]$ (due to our assertion about matrix coefficients), this theorem implies Fourier decomposition for finite groups.
Many of the results from finite group representations carry over, verbatim or with appropriate modification, to the setup of compact groups; this is mainly beacause of the existence of an unimodular *Haar measure* on them, which in turn ensures that the useful technique of 'averaging over the group' in the case of finite groups is available for compact groups as well. In particular,
(i) every continuous representation is a direct sum of simple ones, and
(ii) *character determines representation*, just as in the case of finite group.
Character is defined in the usual sense, for a representation (ρ, V) of G, its character is $\chi_V(g) = \operatorname{Trace}(\rho(g))$. Let us record here a corollary of the Peter-Weyl theorem and (ii) above, which we will need in a later section.

Corollary 1.1.2. Let G be a compact group. If W is any finite dimensional $G \times G$ subrepresentation of $C^0(G)$, then $W \cong \bigoplus_{V \in S} V^* \otimes V$, where S is a subset of the set of isomorphism classes of G.

1.2 The Trick

We will use the following notation throughout in this section:

$$\begin{aligned} G &= GL_n(\mathbb{C}), \\ K &= U_n(\mathbb{C}), \\ T &= \{\text{diagonal matrices in } \operatorname{GL}_n(\mathbb{C})\}, \\ S &= K \cap T. \end{aligned}$$

Note that K, the unitary group, is compact, as is S. Our goal is to go from understanding K to understanding G. The following lemma is a starting point.

Lemma 1.2.1. Let f be an analytic function defined on an open neighborhood U of 0 in $\mathbb{C}^n$. If $f \equiv 0$ on $\mathbb{R}^n \cap U$, then $f = 0$.

Proof. By induction: the base case is clear (since $\mathbb{C}^0 = \mathbb{R}^0 =$ a point). Now if $f \neq 0$, write its power series as:

$$f(z_1, \ldots, z_n) = z_n^N g(z_1, \ldots, z_{n-1}) + z_n^{N+1} h(z_1, \ldots, z_n),$$

with h and g analytic and $g \neq 0$. Now divide by z_n^N and observe that

$$\frac{f}{z_n^N}\bigg|_{U\cap(\mathbb{R}^n \setminus \mathbb{R}^{n-1}\times\{0\})} = 0.$$

So by continuity,

$$g + z_n h\big|_{U\cap(\mathbb{R}^{n-1}\times\{0\})} = 0$$

as well. In particular, since $z_n = 0$ on this part, we just get $g = 0$ on $U \cap (\mathbb{R}^{n-1} \times \{0\})$. By induction we conclude $g = 0$ everywhere, a contradiction. □

Therefore we have the following observation: *Let V be a finite-dimensional $\mathbb{C}$-vector space and W an $\mathbb{R}$-subspace with $V = W \oplus iW$. If $f : V \to \mathbb{C}$ is analytic and $f|_W = 0$, then $f = 0$.* This gives us what we want:

Lemma 1.2.2. If $f : G \to \mathbb{C}$ is analytic and $f|_K = 0$, then $f = 0$.

Proof. Define $g(X) = f(\exp(i \cdot X))$ from $Mat_{n\times n}(\mathbb{C}) \to \mathbb{C}$. Then g is analytic, being a composition of analytic maps and $g = 0$ on the set of Hermitian matrices (because if X is hermitian, $i \cdot X$ is skew hermitian, and therefore $exp(i \cdot X)$ is unitary). Now apply our observation with $V = Mat_{n\times n}(\mathbb{C})$ and W= the subspace of hermitian matrices. □

This is a useful trick: restricting to a compact subgroup in order to conclude something about the whole group which is not compact. Let us apply this to obtain some representation theoretic conclusions.

(i) If V, W are analytic G-representations, then $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_K(V|_K, W|_K)$. Reason: Given a linear map $A \in \mathrm{Hom}(V, W)$, saying that A commutes with the G-action i.e. $A \in \mathrm{Hom}_G(V, W)$, is just the statement $A \cdot \rho_V(g) = \rho_W(g) \cdot A$. This is an equality of analytic functions of g; so, by our lemma, equality holds on G if and only if it holds on K. In other words, the Hom-spaces do not change under restricting to a compact subgroup.

(ii) If V, W are analytic G-representations, then

$$V \cong W \text{ (as } G\text{-reps)} \qquad \Longleftrightarrow \qquad V|_K \cong W|_K \text{ (as } K\text{-reps)}.$$

Reason: The left-hand statement is equivalent to the existence of a square matrix of full rank in $\mathrm{Hom}_G(V, W)$. The right-hand statement is analogous, but with $\mathrm{Hom}_K(V|_K, W|_K)$. Now, apply the previous application.

(iii) Let V be an analytic G-representation. If W is a K-stable subspace, then W is also a G stable subspace, i.e. a G subrepresentation of V. Reason: We can find a set of linear functionals $\lambda_1, \cdots, \lambda_l \in V^*$ such that $W = \bigcap ker \lambda_i$. Therefore, showing that $\rho_V(g) \cdot w \in W \forall g in G$ is equivalent to showing that $\lambda_i(\rho_V(g) \cdot w) = 0 \forall i \in [r], g \in G$. But then $\lambda_i(\rho_V(-) \cdot w) : G \to \mathbb{C}$ are analytic functions which are already zero on K.

The last result has two immediate corollaries:

Corollary 1.2.3. An analytic G representation V is G-simple if and only if $V|_K$ is K-simple.

Corollary 1.2.4. Every analytic G representation is a direct sum of simple G representations.

The last corollary uses, besides the previous corollary, the fact from the Peter-Weyl theorem that the same statement is true for continuous representations of the compact group K. In particular, the last statement is *hard* to prove without passing to a compact subgroup!

An abstract summary of this section is that if one starts with something from G, one can just study it on K. It is not a priori obvious that we can go the other way, i.e., that K-representations extend to G-representations.

1.3 Lifting K-representations to G-representations

Our goal now is to prove:

Theorem 1.3.1. Let V be a continuous K-representation. Then V lifts to a rational G-representation.

We are asserting the existence of a f, given a ϕ, such that the following diagram is commutative, where the horizontal map is the inclusion of K in G.

$$\begin{array}{ccc} K & \xrightarrow{i} & G \\ & \overset{\phi}{\searrow} & \downarrow f \\ & & GL(V) \end{array}$$

We begin by analyzing the characters of representations of the unitary group K. These will eventually give us a 'hint' as to how to find the appropriate rational representations of G. We first analyze χ_V on the compact torus S, bearing in mind that every unitary matrix is diagonalizable and χ_V is a class function.

Lemma 1.3.2. Let V be a continuous K-representation. Then $\chi_V|_S : S \to \mathbb{C}$ is a symmetric Laurent polynomial in the eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_n}$.

Proof. We know $V|_S$ breaks up as a direct sum of S-simple representations. Since S is abelian, every simple representation of it is one-dimensional: it can be easily shown that they are of the form

$$e^{i\theta_1}, \ldots, e^{i\theta_n} \mapsto e^{i(k_1\theta_1+\cdots+k_n\theta_n)}$$

for some $k_1, \ldots, k_n \in \mathbb{Z}$, $i^2+1=0$. This shows $\chi_V|_S$ is a Laurent polynomial in the $e^{i\theta_j}$'s. To see that it is symmetric, take a permutation $w \in S_n \subset U(n)$. Then we have

$$w \cdot \begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_n} \end{pmatrix} w^{-1} = \begin{pmatrix} e^{i\theta_{w(1)}} & & & \\ & e^{i\theta_{w(2)}} & & \\ & & \ddots & \\ & & & e^{i\theta_{w(n)}} \end{pmatrix}$$

We know χ_V is a class function, so we conclude that

$$\chi_V(e^{i\theta_1}, \ldots, e^{i\theta_n}) = \chi_V(e^{i\theta_{w(1)}}, \ldots, e^{i\theta_{w(n)}}).$$

Thus $\chi_V|_S$ is symmetric. □

Let us revert to our original focus: polynomial and rational representations of G. A word of clarification. Although we have defined the character of a G-representation in the usual way, we assert the following: if χ_V is the character of a polynomial representation V of G, then χ_V is a symmetric polynomial in the eigenvalues of g, meaning that there is a symmetric polynomial s_V of n variables such that $\chi_V(g) = s_V(t_1, t_2, \cdots, t_n)$, where t_i's are the eigenvalues of g counted with multiplicity. Reason: this is obviously true for diagonal matrices (as seen from this lemma), and therefore for diagonalizable matrices g (since character is class function); since character is a continuous function on G and diagonalizable matrices are dense in G, its values are determined by its restriction to the diagonalizable ones. Therefore henceforth we will treat characters of polynomial representations of G to be members of $\bigwedge_n$, and if the polynomial representation is homogeneous of degree d then $\chi \in \bigwedge_n^{(d)}$: by saying that $\rho : G \to GL(V)$ is a homogenous polynomial representation, we mean that the all matrix coefficients of $\rho(g)$ are homogenous polynomials of same degree (if this common degree is d, then we call that ρ is a polynomial representation of degree d). With this remark out of our way, we note the following.

Lemma 1.3.3. Every irreducible polynomial representation of G is a homogenous one.

For the proof of the last lemma, see [21], Chapter 6, where it is proved that any polynomial representation is a sum of homogeneous ones, which immediately implies this. Now we begin the lifting process: we find some representations of G whose characters resemble what we are looking for. We now know to look for symmetric Laurent polynomials in the eigenvalues $x_1, \ldots, x_n$ of our matrices.

Lemma 1.3.4. For any $f \in \Lambda_n^{\pm}$, the ring of symmetric Laurent polynomials in n variables, there are rational representations W^+ and W^- of G such that

$$\chi_{W^+}\big|_S - \chi_{W^-}\big|_S = f.$$

Proof. Clearing denominators, we know that for some N, $(x_1 \cdots x_n)^N f \in \Lambda_n$. We write this in the basis of monomial symmetric function and separate the terms with positive and negative coefficients, as in

$$(x_1 \cdots x_n)^N f = \sum_\lambda c_\lambda e_\lambda - \sum_\lambda d_\lambda e_\lambda,$$

with $c_\lambda, d_\lambda \in \mathbb{N}$. Note that if $\mathbb{C}^n$ is the defining representation of G, then $\chi_{\wedge^k \mathbb{C}^n} = e_k(x_1, \ldots, x_n) := \sum_{1 \le i_1 < \cdots < \imath_k \le n} x_{i_1} \ldots x_{i_k}$ So, set

$$U^+ = \bigoplus_\lambda (\bigwedge^{\lambda_1} \mathbb{C}^n \otimes \cdots \otimes \bigwedge^{\lambda_n} \mathbb{C}^n)^{\oplus c_\lambda}, \qquad W^+ = (\det)^{-N} \otimes U^+.$$

Observe that the character of W^+ is precisely $(x_1 \cdots x_n)^{-N} \sum_\lambda c_\lambda e_\lambda$. Similarly, set

$$U^- = \bigoplus_\lambda (\bigwedge^{\lambda_1} \mathbb{C}^n \otimes \cdots \otimes \bigwedge^{\lambda_n} \mathbb{C}^n)^{\oplus d_\lambda}, \qquad W^+ = (\det)^{-N} \otimes U^-.$$

Then W^+ and W^- are the desired rational representations of G. □

Having found the matching characters and representations, we get hold on K by diagonalization, then on G by analyticity.

Proof of lifting theorem. The restriction of χ_V to S is in $\Lambda_n^{\pm}$. So, by the previous lemma, we can find rational G representations W^+ and W^- such that

$$\chi_V|_S = \chi_{W^+}|_S - \chi_{W^-}|_S.$$

We know every unitary matrix is unitarily diagonalizable, so we can decompose K as

$$K = \bigcup_{k \in K} kSk^{-1}.$$

This shows that, in fact, equality holds on all of K: $\chi_V|_K = \chi_{W^+}|_K - \chi_{W^-}|_K$. Since we know that representations of compact groups are determined by their characters, we have.

$$V \oplus W^- \cong W^+$$

as K representations. In particular, V is a K subrepresentation of W^+, hence also a G

subrepresentation of W^+. $\square$

The same approach yields a similar lifting property for polynomial representations.

Lemma 1.3.5. If $f \in \Lambda_n$, the ring of symmetric polynomials in n variables, then there exist polynomial G representations W^+ and W^- such that $\chi_{W^+} - \chi_{W^-} = f$. If V is a K representation such that χ_V is in Λ_n, then V lifts to a polynomial representation of G.

Thus we make the following conclusion: characters of polynomial GL_n-irreducible polynomial representations span Λ_n, hence (by linear independence) are a basis for it. Similarly, characters of rational GL_n-irreducible representations are a basis for $\Lambda_n^{\pm}$. Since χ_V's are homogenous for irreducible V, our basis works in each degree separately. So we deduce an useful numerical consequence: *the number of nonequivalent isomorphism class of polynomial irreducible representations of GL_n such that the character has degree d = number of partitions of d with at most n parts.*
We will see in Chapter 3 how this numerical equality gains more concrete representation theoretic significance, in the light of *Schur-Weyl duality.*
As a direct corollary of the last theorem, we assert that one can study analytic representations of GL_n without much analytic intervention!

Corollary 1.3.6. The irreducible analytic representations of G are precisely the irreducible rational ones.

Proof. Take an analytic irreducible representation V of G, so it is a continuous irreducible G-representation as well; restricting to K gives us a continuous K-irreducible representation, but then by the lifting theorem we know that $V|_K$ lifts to a rational representation of G which is also irreducible. $\square$

Therefore any analytic representation of G is a direct sum of rational representations, justifying our previous remark.

1.4 Characters of Polynomial Representations

The goal for this section is to prove:

Theorem 1.4.1. The characters of GL_n-irreducible polynomial representations are the Schur polynomials in n variables.

Recall that, for a partition λ (of any integer) with at most n parts, the Schur polynomial s_λ in n variables $x^1, \ldots, x_n$ is defined as follows: for each semistandard Young tableaux T of shape λ with entries in $[n]$, define weight of the tableaux $wt_x(T) := x^T = \prod_{i=1}^n x_i^{t_i}$, where t_i denotes the number of occurences of i in T. Then $s_\lambda(x_1, \ldots, x_n) := \Sigma_{T \in SSYT(\lambda,\ entry \in [n])} wt_x(T)$. We will deduce the theorem from the following Peter-Weyl-like theorem.

Theorem 1.4.2. As a $GL_n \times GL_n$ representation, we have

$$\mathbb{C}[z_{ij}] \cong \bigoplus_{V \text{nonisomorphic polynomial irreps}} V^* \otimes V$$

Proof. Note that $\mathbb{C}[z_{ij}]$ is the algebra of polynomial functions in the n^2 matrix entries of GL_n. We have a map $\mathbb{C}[z_{ij}] \to C^0(K)$ by restricting functions to the unitary group. Since polynomials in the z_{ij} are analytic functions, this map is injective by our lemma. We claim that it lands in $\mathcal{O}(K)$. Reason: $\mathbb{C}[z_{ij}] = \bigoplus_d \mathbb{C}[z_{ij}]_d$, where $\mathbb{C}[z_{ij}]_d$ is homogenous polynomials of degree d. Now, $\mathbb{C}[z_{ij}]_d$ is clearly a finite dimensional $K \times K$ subrepresentation of $C^0(K)$. So, by characterization of matrix coefficients, it is in $\mathcal{O}(K)$.

Therefore, $\mathbb{C}[z_{ij}] \cong \bigoplus_{V \in S} V^* \otimes V$ for some set S of simple representations of K. We now have to determine what S is.

Let V occur as a tensor factor in one of the direct summands. Looking at the $1 \times G$ action on V, it is clear that V is a polynomial G representation, so every representation $V \in S$ is the restriction of a polynomial representation of G to K.

On the other hand, if V is a polynomial representation of G, then the embedding $\mathrm{End}(V)^* \to C^0(G)$ clearly lands in $\mathbb{C}[z_{ij}]$. Explicitly, we are asserting that $\lambda(\rho_V(g))$ is a polynomial in the z's, given that the entries of $\rho_V(g)$ are such a polynomial; this is obvious.

So we conclude that S is the set of polynomial representations of G restricted to K, so we have the decomposition as $G \times G$ represenation as desired. □

Let us note a combinatorial consequence of the last theorem. Take the character of the two sides on an element $diag(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}) \times diag(y_1, y_2, \ldots, y_n)$; the inverses in the first term are precisely there to cancel the inverses defining the action of $G \times G$ on $C^0(G)$.

On the left hand side, one calculates that z_{ij} transforms by multiplication with $x_i y_j$. So the character of the left hand side is

$$\prod_{1 \leq i,j \leq n} \frac{1}{1 - x_i y_j}.$$

On the right hand side, $diag(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}) \times diag(y_1, y_2, \ldots, y_n)$ acts on $V^* \otimes V$ by

$$\chi_{V^*}(x_1^{-1}, \ldots, x_n^{-1})\chi_V(y_1, \ldots, y_n) = \chi_V(x_1, \ldots, x_n)\chi_V(y_1, \ldots, y_n).$$

So we deduce the following.

Corollary 1.4.3.

$$\prod_{1 \leq i,j \leq n} \frac{1}{1 - x_i y_j} = \sum_{V \text{a polynomial irrep}} \chi_V(x_1, \ldots, x_n)\chi_V(y_1, \ldots, y_n).$$

We want to prove that the χ_V are the Schur polynomials. First we show that similar equation holds for the Schur polynomials and then leverage this to get what we want.

Lemma 1.4.4.

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-x_iy_j} = \sum\nolimits_{\lambda, l(\lambda)\le n} s_\lambda(x_1,\ldots,x_m)s_\lambda(y_1,\ldots,y_n)$$

Proof. A typical term in the LHS is $\prod_{i,j}(x_iy_j)^{m_{ij}}$ and a typical term in the RHS looks like $x^Ty^U := x_1^{t_1}x_2^{t_2}\cdots x_m^{t_m}y_1^{u_1}y_2^{u_2}\cdots y_n^{u_n}$, where t_i and u_j denotes the number of occurences of i and j in two SSYTs T and U of shape λ. *Robinson-Schensted-Knuth correspondence* associates with a $m \times n$ matrix $M = (m_{ij})$ two SSYT, T and U, of same shape in a bijective way such that $\Sigma_{j=1}^n m_{ij} = u_i, \Sigma_{i=1}^m m_{ij} = t_j$. This is precisely what we need to conclude that each term in the LHS appears in the RHS and vice versa. □

Proof of main theorem. We assert that if f_α is any family of symmetric polynomials with obeying $\prod 1/(1-x_iy_j) = \sum f_\alpha(x)f_\alpha(y)$, then the list of f_α contains each $\pm s_\lambda$ exactly once, plus possibly some occurrences of the 0 function. By the condition, we have

$$\sum_\alpha f_\alpha(x)f_\alpha(y) = \sum_\lambda s_\lambda(x)s_\lambda(y).$$

Let $f_\alpha = \sum_\lambda a_{\alpha\lambda}s_\lambda$. Comparing coefficients of $s_\lambda(x)s_\lambda(y)$, we see that $\sum_\alpha a_{\alpha\lambda}^2 = 1$. So, for fixed λ, exactly one $a_{\alpha\lambda}$ is ± 1 and the rest are zero. Comparing coefficients of $s_\lambda(x)s_\mu(y)$, we see that, for fixed α, at most one $a_{\alpha,\lambda}$ is nonzero. So the χ_V are $\pm$ the s_λ's, and maybe some zero functions. But it is clear that the χ_V are nonzero and have nonnegative coefficients, so we conclude that the characters of GL_n polyreps are the Schur polynomials $s_\lambda(x_,\cdots,x_n)$, where $l(\lambda)\le n$. □

Henceforth, we call $V_\lambda(n)$(or just $V_\lambda(n)$, if omitting n does not beget ambiguity) to be the irreducible representation of GL_n with character $s_\lambda(x_1,\ldots,x_n)$.

Remark 1.4.5. • Let us introduce here the *Hall inner product* on $\bigwedge$: different degree components are declared orthogonal ($\langle\bigwedge^m,\bigwedge^n\rangle = 0$) and s_λ's, for $\lambda \vdash k, l(\lambda)\le n$ are declared orthonormal basis of $\bigwedge^k$ and then this product is bilinearly extended. It follows immediately from the above that $\langle\chi_V,\chi_W\rangle_{Hall} = \dim \mathrm{Hom}_G(V,W)$, since the Schurs polynomials are orthonormal. In fact, under the correspondence $V \mapsto \psi_V$, which takes a virtual polyrep to its generalized character, we have for a partition $\lambda = (\lambda_1,\cdots,\lambda_l), l\le n$,

$$\otimes_{i=1}^{l} Sym^{\lambda_i}(\mathbb{C}^n) \leftrightarrow h_\lambda(x_1,\cdots,x_n)$$
$$\otimes_{i=1}^{l} {\textstyle\bigwedge}^{\lambda_i}(\mathbb{C}^n) \leftrightarrow e_\lambda(x_1,\cdots,x_n)$$

tensor product of reps $\leftrightarrow$ multiplication of symmetric polynomials and so on.

• If we look at the coordinate ring of GL_n, namely $\mathbb{C}[z_{ij}][\det^{-1}]$, we get $\bigoplus V^* \otimes V$ where the sum is over rational representations.

• The characters of the rational irreps are of the form

$$(x_1 x_2 \dots x_n)^{-N} s_\lambda(x_1, \dots, x_n).$$

Proof: Given a symmeric Laurent polynomial $f = \chi_X$, clearing denominator one has $(x_1 \dots x_n)^N f \in \bigwedge_n$ (for some $N \in \mathbb{N}$), and thus $\exists \lambda$ with $(x_1 \dots x_n)^N f = \Sigma_\lambda s_\lambda(x_1, \cdots, x_n)$, whence $X \cong (det)^{-N} \otimes (\oplus V_\lambda)$ as rational representation of $GL(V)$; but irreducibility of X ensures that at most one (and therefore exactly one) s_λ occurs in the expression. Hence, *irreducible rational representations are tensor product of some negative power of the determinant representation with an irreducible polynomial representation.* We also have

$$s_{(\lambda_1+1, \lambda_2+1, \dots, \lambda_n+1)}(x_1, x_2, \dots, x_n) = (x_1 x_2 \dots x_n) s_{(\lambda_1, \lambda_2, \dots, \lambda_n)}(x_1, x_2, \dots, x_n).$$

As a result, the same symmetric Laurent polynomial can be expressed using more than one pair (λ, N) as above. A nonredundant indexing set is the set of integer sequences $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, where we do **not** impose that $\mu_n \geq 0$. The correspondence is that $\mu_i = \lambda_i - N$.

• What is the character of V_λ^*, the contragradient of the representation V_λ of $GL(V)$? Since in terms of matrix, the representation is just $g \mapsto (\rho(g)^t)^{-1}$, we are essentially asking: what is $s_\lambda(x_1^{-1}, \cdots, x_n^{-1})$? Take $m \geq \lambda_1$, then [1], Exercise 7.41 tells that

$$(x_1 x_2 \dots x_n)^m s_\lambda(x_1^{-1}, \cdots, x_n^{-1}) = s_{\bar{\lambda}}(x_1, \cdots, x_n)$$

where $\bar{\lambda} = (m - \lambda_n, \cdots, m - \lambda_1)$. In particular it shows that

$$(V_{(\lambda_1, \dots, \lambda_n)})^* \cong V_{`(-\lambda_n, \dots, -\lambda_1)'}$$

Of course, $(-\lambda_n, \dots, -\lambda_1)$ is not a partition, but we can make the last statement precise (and therefore drop the ‘’) in terms of weight of the highest weight vector in both the representation, but we will not go into that; see [25] for related concepts.

• We can look at $\mathbb{C}[z_{ij}]$ where $1 \leq i \leq m$ and $1 \leq j \leq n$ as a $GL_m \times GL_n$ representation. The last lemma tells us the characters of the following representation matches, so it validates the isomorphism of representations as ‘characters determine representations’ (reason for the last statement: if V, W are GL_n polynomial representations and $\chi_V = \chi_W$, then this equality of characters hold at the level of U_n representation also, so $V|_{U_n} \cong W|_{U_n}$ as U_n continuous representations, whereas the isomorphism holds when we drop the restriction symbol!)

$$\mathbb{C}[z_{ij}] \cong \bigoplus_{\lambda} V_\lambda(m)^* \otimes V_\lambda(n)$$

. The summands with $\ell(\lambda) > \min(m,n)$ are zero, so we can equivalently write

$$\mathbb{C}[z_{ij}] \cong \bigoplus_{\ell(\lambda)<\min(m,n)} V_\lambda(m)^* \otimes V_\lambda(n).$$

• Let us note another consequence of the last lemma: $GL_m \times GL_n$ acts on $\mathbb{C}^m \otimes \mathbb{C}^n$ by their defining representation on each tensor factor respectively (since these actions *commute*, it *is* a joint representation), so it acts on $Sym(\mathbb{C}^m \otimes \mathbb{C}^n) = \bigoplus_{k\in\mathbb{N}} Sym^k(\mathbb{C}^m \otimes \mathbb{C}^n)$. Suppose $\{e_i : i \in [m]\}$ and $\{\hat{e_j} : j \in [n]\}$ are respectively the basis of $\mathbb{C}^m$ and $\mathbb{C}^n$. Since under the action of $diag(x_1, \cdots, x_m) \times diag(y_1, \cdots, y_n)$, $e_i \otimes \hat{e_j}$ transforms by multiplication by x_iy_j, the lemma serves as the character theoretic validation of the following decomposition of this representation into irreducibles

$$Sym(\mathbb{C}^m \otimes \mathbb{C}^n) \cong \bigoplus_{\ell(\lambda)\le\min(m,n)} V_\lambda(m) \otimes V_\lambda(n).$$

Considering each graded piece separately

$$Sym^k(\mathbb{C}^m \otimes \mathbb{C}^n) \cong \bigoplus_{\lambda\vdash k,\ell(\lambda)\le\min(m,n)} V_\lambda(m) \otimes V_\lambda(n).$$

This is known as **Howe duality** for the pair (GL_m, GL_n), see [8] for a proof which comes under the general theme of 'multiplicity-free action'.

• Similarly $GL_m \times GL_n$ acts on $\bigwedge(\mathbb{C}^m \otimes \mathbb{C}^n)$. We assert that it breaks up as follows

$$\bigwedge(\mathbb{C}^m \otimes \mathbb{C}^n) \cong \bigoplus_{l(\lambda)\le m,\lambda_1\le n} V_\lambda(m) \otimes V_{\bar{\lambda}}(n).$$

It suffices to show the following identity of characters

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_iy_j) = \sum\nolimits_{\lambda,l(\lambda)\le m,\lambda_1\le n} s_\lambda(x_1, \ldots, x_m)s_{\bar{\lambda}}(y_1, \ldots, y_n)$$

This is where the dual RSK correspondence ([21], Chapter 4) comes into play! It asserts that there is a bijecive correspondence between matrices A whose entries are in $\{0,1\}$, with pair (P,Q) of same shape such that P', Q are SSYTs having $col(A) = type(P)$ and $row(A) = type(Q)$. Now notice that the coefficient of $x^\alpha y^\beta$ on left hand side counts the number of 0-1 matrices with row sum α and column sum β: for every term appearing in the product, create a matrix which has 1 in the (i,j)th place if x_iy_j is a factor of this term and 0 otherwise. On the right hand side, the required coefficient counts SSYTs whose shapes are transposes of each other with type α for one and β for the other, so we are done. □

This is the **skew Howe duality** for the pair (GL_m, GL_n), see [8]. Note that these Young diagrams fit inside $m \times n$ rectangle, so there are only finitely many summands on the right, which matches with the fact that the exterior algebra on the left is finite dimensional: $\bigwedge^k(\mathbb{C}^m \otimes \mathbb{C}^n) = 0, \forall k > mn$.

As a corollary of this result we get

$$\mathrm{Hom}_{GL_m}(\bigwedge^{|\lambda|}(\mathbb{C}^m \otimes \mathbb{C}^n), V_\lambda) \cong V_{\lambda'}$$

as GL_n representations.

Let us point to the most common instances of these dualities: $Sym(\mathbb{C}^m) = \oplus_i\, Sym^i(\mathbb{C}^m)$ and $\bigwedge \mathbb{C}^m = \oplus_i \bigwedge^i \mathbb{C}^m$, these are the dualities for the pair (GL_m, GL_1); therefore the Howe dualities generalize the special cases of known decompositions of the symmetric algebra and exterior algebra (see next chapter for the identification of $Sym^i(\mathbb{C}^n)$ and $\bigwedge^i(\mathbb{C}^n)$ as irreducible representations).

1.5 Branching Rule and Gelfand-Tsetlin Basis

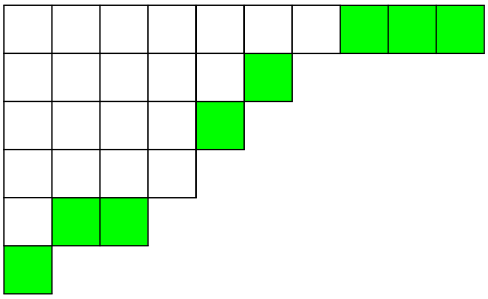
Now that we have proved that Schur polynomials are irreducible characters, and characters determine representations, we can get hold on the branching problem for GL_n; in fact, **any symmetric function identity can be leveraged to deduce some representation theoretic consequence!**

For a given group G and a subgroup H of G, the branching problem asks the following: which irreducible representations of H occur in the restriction of a particular G-irreducible representation? For our case, note that the family of GL_n's (for $n \in \mathbb{N}$) constitute an infinite tower of groups, where each $GL_n \subset GL_{n+1}$: embed a $n \times n$ matrix g into GL_{n+1} as g', where $g'_{ij} = g_{ij} \forall i, j \in [n], g'_{i,n+1} = g'_{n+1,j} = 0, g'_{n+1,n+1} = 1$. Therefore we ask: what are the possible μ's in the decomposition $V_\lambda(n)|_{GL_{n-1}} \cong \bigoplus_{l(\mu)\leq n-1} V_\mu(n-1)^{\oplus c_\mu}$?

Theorem 1.5.1. $V_\mu(n-1)$ occurs in the restriction of $V_\lambda(n)$ iff $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-2} \geq \mu_{n-1} \geq \lambda_n$, and for each such μ, $V_\mu(n-1)$ occurs once the decomposition, i.e. the decomposition is multiplicity-free.

If two partitions λ and μ satisfies such inequalities then we say that μ interlaces λ, and write $\lambda \succ \mu$(or $\mu \prec \lambda$). Note that this is equivalent to saying that the Young diagram of μ is obtained from the Young diagram of λ by removing a *horizontal strip*, i.e. a subset of cells of the λ diagram which does not contain more than two successive cells in a column(in other words, a 2×1 domino).

For instance, the colored cells in this picture constitute a horizontal strip in the diagram of $(10, 6, 5, 4, 3)$.



Proof. Let us prove the equality of desired characters: evaluating both side's character on $diag(x_1, \cdots, x_{n-1})$ and keeping in mind how GL_{n-1} sits inside GL_n, it boils down to showing

$$s_\lambda(x_1, x_2, \cdots, x_{n-1}, 1) = \Sigma_{\lambda \succ \mu} s_\mu(x_1, \cdots, x_{n-1})$$

This is easy, and in fact a direct bijection is evident between the terms of the two sides: a typical term $x^T = x_1^{t_1} \cdots x_n^{t_n}$ of $s_\lambda(x_1, \cdots, x_n)$ comes from $T \in SSYT(\lambda)$ with i occuring t_i times, therefore setting $x_n = 1$ has the effect on this term as deleting from T all the cells labeled with n(call that U) and taking x^U, i.e. $x^T|_{x_n=1} = x^U$. But notice that the configuration of cells in the diagram of T which can be filled up by n is precisely a horizontal strip. Therefore our claim follows. □

See [13] for a completely algebraic proof of the branching rule for all the classical groups.

Now that we know how $V_\lambda(n)$ restricts from GL_n to GL_{n-1}, we can further restrict it from GL_{n-1} to GL_{n-2}, and obtain similar decomposition. In particular, doing this all the way down to $GL_1 = \mathbb{C}^*$, we have

$$V_\lambda(n) = \bigoplus_{\tau_\lambda \equiv (\lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \lambda^{(n)} = \lambda)} V_{\tau_\lambda}$$

where by V_{τ_λ} we denote the 1 dimensional GL_1 irrep(since GL_1 is abelian, all of its irreps are 1 dimensional) $V_{\lambda^{(0)}}(1)$, where $\lambda^{(0)}$ is the partition which arises from successively removing horizontal strips along the collection τ_λ, i.e each $\lambda^{(i)} \setminus \lambda^{(i-1)}$ is a horizontal strip. We now resort to a specific example to avoid notational complexity.

Example 1.5.2. Take $n = 3, \lambda = (2, 1, 0)$. Then there are 8 sets, each consisiting of three partitions, where each partition is interlaced by the next one in its set, that enumerate the one dimensional GL_1 irreps occuring in the decomposition of $V_{(2,1,0)}(3)$:

$$\{(2,1,0),(2,1),(2)\}, \{(2,1,0),(2,1),(1)\}, \{(2,1,0),(1,1),(1)\}, \{(2,1,0),(1,0),(1)\},$$
$$\{(2,1,0),(1,0),(0)\}, \{(2,1,0),(2,0),(2)\}, \{(2,1,0),(2,0),(1)\}, \{(2,1,0),(2,0),(0)\}$$

Each of these sets of three partitions can be reassembled in an obvious way to triangular patterns, for instance the first set corresponds to

$$\begin{matrix} & & 2 & & \\ & 2 & & 1 & \\ 2 & & 1 & & 0 \end{matrix}$$

These are called ***Gelfand-Tsetlin patterns*** or GT patterns. A GT pattern with partiton λ as its bottom row can be associated bijectively to a semistandard Young tableaux of shape λ with entries in $[n]$: in the digram of λ, fill in the cells of the skew diagram $\lambda^{(i)} \setminus \lambda^{(i-1)}$ with i, for all $i \in [n]$. Under this correspondence, our above written GT pattern goes to

1	1
2	

Therefore in our example we see that $dim\ V_{(2,1,0)}(3) = 8$, and we even have a basis comprising of elements indexed by $SSYT(\lambda)$, if we choose a nonzero vector in each of the 1 dimensional V_{τ_λ}. Of course, this basis is defined upto multiplication by scalar, but it has the property of 'well adaptedness' with respect to representations which is described in the following definition.

Definition 1.5.3. A basis $\{v_1, v_2, \cdots, v_N\}$ of $V_\lambda(n)$, where $N = dim\ V_\lambda(n)$ is called a *Gelfand Tsetlin Basis*, if $\forall k \leq n, \exists$ a decomposition $[N] = \coprod_{1 \leq i \leq M_k} S_i^{(k)}$ into disjoint union, such that for each $i \in [M_k]$, $Span\ \{v_\alpha : \alpha \in S_i^{(k)}\}$ is a irreducible constituent of $V_\lambda(n)|_{GL_k}$(therefore M_k denotes the number of irreducible summands in $V_\lambda(n)|_{GL_k}$).

It can be shown that a GT basis is essentially unique: that is, if $\{v_i\}, \{w_i\}$ are two GT bases, then after reordering one must have $v_i = c_i w_i$ for some scalars $c_i \in \mathbb{C}$. In our example, if we name our basis vectors $\{v_1, \cdots, v_8\}$ of $V_{(2,1,0)}(3)$ in the same order as we wrote the sets of partitions associated to them, then for $k = 2$ the decomposition described in the definition is $\{1, \cdots, 8\} = \{1,2\} \coprod \{3\} \coprod \{4,5\} \coprod \{6,7,8\}$, whereas for $k = 1$ each element is itself a disjoint member of the decomposition. In fact, from our discussions it is evident that *the basis vectors for which the first k partitions are same, or equivalently the successive k rows in their GT pattern starting from the bottom one are same, or equivalently the cells containing $n, n-1, \cdots, n-k+1$ are same in the associated Young diagram, lies in the same GL_k irreducible summand upon restriction.*

If we normalize the GT basis vectors with respect to the GL_n invariant inner product, then it is possible to give explicit formulas for the action of GL_n on this basis of $V_\lambda(n)$, see [19] for these formulas or [12] for similar discussions for all Cartan types of Lie algebras. Therefore we have seen that each poly irrep $V_\lambda(n)$ of GL_n has GT basis, and these are indexed by GT patterns with fixed bottom row having parts of λ or equivalently semistandard Young tableaux of shape λ. In next chapter we will construct the $V_\lambda(n)$'s concretely and we will see that they automatically posess a basis whose elements are naturally indexed by semistandard tableaux, but unfortunately that is not GT basis!

Chapter 2

Explicit Constructions

Let $V = \mathbb{C}^n$, λ be a partition of N with at most n parts. We want to construct V_λ, the $GL(n)$ irreducible representation with character s_λ. We have the two $GL(n)$-representations

$$H = \bigotimes_k \operatorname{Sym}^{\lambda_k} V, \qquad \text{and} \qquad E = \bigotimes_k \Lambda^{(\lambda')_k} V$$

which have characters $\chi_H = h_\lambda$ and $\chi_E = e_{\lambda'}$, respectively. Here λ' denotes the conjugate partition of λ. Recall that

$$h_\lambda = s_\lambda + \sum_{\mu \prec \lambda} \kappa_{\lambda\mu} s_\mu, \qquad \text{and} \qquad e_{\lambda'} = s_\lambda + \sum_{\mu \succ \lambda} \kappa_{\lambda\mu'} s_\mu$$

so the equality $\langle h_\lambda, e_{\lambda'} \rangle = 1$ comes from the s_λ term. Using the dictionary we have set up in the last chapter, it means that in terms of representations ,

$$H = V_\lambda \oplus \bigoplus_{\mu \prec \lambda} V_\mu^{\oplus \kappa_{\lambda\mu}},$$
$$E = V_\lambda \oplus \bigoplus_{\mu \succ \lambda} V_\mu^{\oplus \kappa_{\lambda'\mu'}}.$$

and also $\operatorname{Hom}_{\mathrm{GL}(V)}(E, H) \cong \mathbb{C}$. It follows that the only $GL(n)$ irrep that H and E have in common is a single copy of V_λ and any non-zero $GL(n)$-equivariant map $E \to H$ or $H \to E$ is actually an isomorphism from one copy of V_λ to the other copy of V_λ, so if φ is a nonzero $\mathrm{GL}(V)$-equivariant homomorphism $E \to H$, then $\operatorname{Im}(\varphi) \cong V_\lambda$. Our next goal will be to describe such a map φ explicitly.

2.1 Using Tensor Space

We use embeddings (and projections) of E (and H) into $V^{\otimes N}$. Note that $\operatorname{Sym}^k V$ is can be thought of as either a subspace or a quotient of $V^{\otimes k}$. Viewing $\operatorname{Sym}^k V$ as the subspace of $V^{\otimes k}$ of S_k-invariant tensors, there is the standard inclusion

$$\begin{aligned} \operatorname{Sym}^k V &\to V^{\otimes k} \\ v_1 \cdots v_k &\mapsto \frac{1}{k!} \sum_{w \in S_k} v_{w(1)} \otimes \cdots \otimes v_{w(k)}. \end{aligned}$$

Viewing $\operatorname{Sym}^k V$ as a quotient of $V^{\otimes n}$, we have the projection map

$$\begin{aligned} V^{\otimes k} &\to \operatorname{Sym}^k V \\ v_1 \otimes \cdots \otimes v_k &\mapsto v_1 \cdots v_k \end{aligned}$$

which equates different permutations of a tensor. Similarly for the exterior powers, there are maps $\bigwedge^k V \to V^{\otimes k}$ and $V^{\otimes k} \to \bigwedge^k V$ defined by

$$\begin{aligned} v_1 \wedge \cdots \wedge v_k &\mapsto \frac{1}{k!} \sum_{w \in S_k} (-1)^w v_{w(1)} \otimes \cdots \otimes v_{w(k)} \\ v_1 \otimes \cdots \otimes v_k &\mapsto v_1 \wedge \cdots \wedge v_k \end{aligned}$$

so that $\bigwedge^k V$ can also be viewed as either a subspace or a quotient of $V^{\otimes k}$. (Note: $(-1)^w$ is the parity of the permutation.)

The map $E \to H$ is constructed out of the two parts $E \to V^{\otimes N} \to H$, inclusion and projection. Let the cells of a Young tableau of shape λ index the components of $V^{\otimes N}$ (recall that $N = |\lambda|$), and let the columns index the components of E, and the rows index the components of H. For the map $E \to V^{\otimes N} \to H$, "include by column, and project by row."

Example 2.1.1. Consider the following partition:

$$\lambda = (4, 2, 1) \qquad \lambda' = (3, 2, 1, 1) \qquad\qquad T = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 2 & 5 \\ \hline 7 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

The leftmost column of the tableau corresponds with $\bigwedge^3 V$, the first component of E. It maps to the first, seventh and third components of $V^{\otimes 7}$, which in turn project to $\operatorname{Sym}^4 V$, $\operatorname{Sym}^2 V$, and V, respectively (the first, second, and third rows). In effect, under the inclusion,

$$v_1 \wedge v_2 \wedge v_3 \otimes v_4 \wedge v_5 \otimes v_6 \otimes v_7 \mapsto$$
$$\frac{1}{3!}\frac{1}{2!} \sum_{\pi \in Perm\{1,2,3\}, \sigma \in Perm\{4,5\}} (-1)^\pi (-1)^\sigma v_{\pi(1)} \otimes v_6 \otimes v_{\pi(3)} \otimes v_{\sigma(4)} \otimes v_7 \otimes v_{\sigma(5)} \otimes v_{\pi(2)}$$

and under the projection, this gets mapped to

$$\frac{1}{3!}\frac{1}{2!}\sum_{\pi\in Perm\{1,2,3\},\sigma\in Perm\{4,5\}}(-1)^{\pi}(-1)^{\sigma}v_{\pi(1)}v_6v_{\sigma(4)}v_7\otimes v_{\pi(2)}v_{\sigma(5)}\otimes v_{\pi(3)}$$

Here $PermX$ denotes the group of permutations on the set X. The last expression is neatly written as

$$\tfrac{1}{|S_{\lambda'}|}\sum_{\alpha\in S_{\lambda'}}(-1)^{\alpha}v_{\alpha(1)}v_{\alpha(4)}v_{\alpha(6)}v_{\alpha(7)}\otimes v_{\alpha(2)}v_{\alpha(5)}\otimes v_{\alpha(3)}$$

where $S_{\lambda'} = S_3\times S_2\times S_1\times S_1$ is the Young subgroup associated to the partition λ'. The claim is that this map is equivariant (obvious, as each factor is) and nonzero; check the image of any basis vector of E: the nonzero summands appearing in the image are themselves basis vectors of H. In particular, take wedge of e_i's in accordance with the appearence of i's in the columns of T from top to bottom and then take tensor of these, varying columns from left to right and call this basis vector e_T (in our case, $e_T = e_1\wedge e_7\wedge e_3\otimes e_4\wedge e_6\otimes e_2\otimes e_5$). Then

$$\varphi(e_T) = \frac{1}{|S_{\lambda'}|}\hat{e_{T'}} + \textit{linear combination of } \textbf{other} \textit{ basis vectors} \tag{2.1}$$

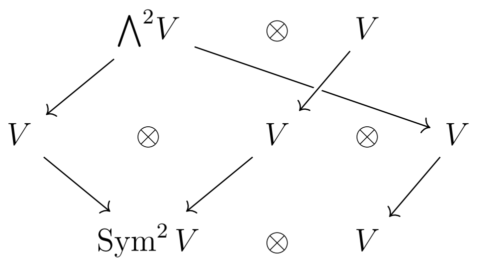
where $\hat{e_{T'}}$ denotes the basis vector (as the notation suggests) of H obtained from taking symmetric product of e_i's in accordance with the appearence of i's in the rows of T from left to right and then take tensor of these, varying rows from top to bottom (in our case $\hat{e_{T'}} = e_1e_4e_2e_5\otimes e_7e_6\otimes e_3$); a moment's thought would reveal that the equation 2.1 holds true essentially due to the fact that we are using a filling T of the shape λ where each entry occurs *only once*. Hence the image of φ is nonzero, as $\varphi(e_T)\neq 0$.

For a smaller example, consider

$\lambda = (2,1)$ $\lambda^T = (2,1)$

1	2
3	

and the picture



which is simple enough that we will write the map explicitly. The component of the map from $\bigwedge^2 V$ to $V\otimes V$ is $u\wedge v\mapsto \frac{1}{2}(u\otimes v - v\otimes u)$. On an arbitrary pure tensor in $\bigwedge^2 V\otimes V$, the whole map is

$$\begin{array}{ccccc} \bigwedge^2 V \otimes V & \to & V \otimes V \otimes V & \to & \mathrm{Sym}^2 V \otimes V \\ (u \wedge v) \otimes w & \mapsto & \frac{1}{2}(u \otimes w \otimes v - v \otimes w \otimes u) & \mapsto & \frac{1}{2}((uw) \otimes v - (vw) \otimes u). \end{array}$$

One special case is

$$(u \wedge v) \otimes w + (v \wedge w) \otimes u + (w \wedge u) \otimes v \mapsto 0$$

(which is suggestive of the Jacobi identity).

Since $E \to H$ is a $GL(V)$-equivariant map, it commutes with torus action. In weight $x_i^2 x_j$, E has one eigenvector $(e_i \wedge e_j) \otimes e_i$, and its image is non-zero. In weight $x_i x_j x_k$, E has 3 eigenvectors, $(e_i \wedge e_j) \otimes e_k$, $(e_j \wedge e_k) \otimes e_i$, $(e_k \wedge e_i) \otimes e_j$ and their images span a 2 dimensional subspace of H. The corresponding Schur function is

$$s_{21}(x) = \sum x_i^2 x_j + 2 \sum x_i x_j x_k.$$

Remark 2.1.2. 1. One naive strategy to construct V_λ would be to map both E and H inside $V^{\otimes N}$ and intersect the images. But this might not work. This is because even though both E and H have a copy of V_λ, their images in $V^{\otimes N}$ might be isomorphic, but not the same, in which case their images would not intersect.
2. We could think of H and E as subspaces of $V^{\otimes N}$. Let a_λ be projection onto $H \subset V^{\otimes N}$, and b_λ be projection onto $E \subset V^{\otimes N}$; they are defined concretely later. Then we need to look at the image of $a_\lambda b_\lambda$. Note the image of $b_\lambda a_\lambda$ will be isomorphic to $a_\lambda b_\lambda$, but not equal, unless E meets H.
3. One natural question that arises: what is the kernel of φ? We investigate this in a later section.

2.2 Via Young's Symmetrizer

We will explore the second option for now. What is a_λ? It is

$$a_\lambda : V^{\otimes N} \to H \to V^{\otimes N},$$

the composition of projection and inclusion. It projects from $V^{\otimes N}$ to a copy of H inside $V^{\otimes N}$. From the previous section, the map is

$$a_\lambda(v_1 \otimes \cdots \otimes v_N) = \frac{1}{\lambda_1! \cdots \lambda_k!} \sum_{w \in S_{\lambda_1} \times \cdots \times S_{\lambda_k}} V_{w(1)} \otimes \cdots \otimes V_{w(N)}$$

with a sum over permutations $w \in S_N$ that preserve the rows of the λ-tableau.

Similarly, b_λ is

$$b_\lambda : V^{\otimes N} \to E \to V^{\otimes N}$$
$$b_\lambda(v_1 \otimes \cdots \otimes v_N) = \frac{1}{(\lambda^T)_1! \cdots (\lambda^T)_\ell!} \sum_w (-1)^w \, V_{w(1)} \otimes \cdots \otimes V_{w(N)}$$

with a sum over permutations $w \in S_N$ that preserve the columns of the λ-tableau, so that b_λ projects from $V^{\otimes N}$ to a copy of E inside $V^{\otimes N}$.

Definition 2.2.1. The composition $a_\lambda b_\lambda$ of both projections is called the ***Young symmetrizer***, and is written c_λ. We can think of V_λ as the image of c_λ in $V^{\otimes N}$.

Theorem 2.2.2. For any partition λ of any integer N with at most n parts,

$$V \mapsto V_\lambda$$

is a functor from the category of finite dimensional vector spaces to the category of polynomial representations of $GL(V)$ (classically it is called the ***Schur Functor*** $\mathbb{S}_\lambda$).

Proof. We first want to show that

$$(c_\lambda)^2 = k_\lambda c_\lambda$$

for some nonzero constant k_λ, so that c_λ is almost an idempotent.

Think of V_λ as a subset of $V^{\otimes N}$ and write c_λ as a composition of projection and inclusion

$$c_\lambda : V^{\otimes N} \xrightarrow{\pi} V_\lambda \xrightarrow{i} V^{\otimes N}.$$

The map

$$V_\lambda \xrightarrow{i} V^{\otimes N} \xrightarrow{\pi} V_\lambda$$

is between irreducible representations, and by Schur's lemma

$$\pi \circ i = k_\lambda \, \mathrm{Id}$$

for some nonzero constant k_λ. Then

$$(c_\lambda)^2 = (i \circ \pi) \circ (i \circ \pi) = i \circ (\pi \circ i) \circ \pi = k_\lambda (i \circ \pi) = k_\lambda c_\lambda$$

as desired.

This constant k_λ does not depend on V. Indeed, we can think about a_λ, b_λ, and c_λ as elements of $\mathbb{C}[S_n]$, e.g.

$$a_\lambda = \frac{1}{\lambda_1! \cdots \lambda_k!} \left(\sum_{w \in S_{\lambda_1} \times \cdots \times S_{\lambda_k}} w \right) \in \mathbb{C}[S_n]$$

keeping in mind that S_n acts on $V^{\otimes N}$ by permuting the factors of simple tensors, whence $V^{\otimes N}$ is a $\mathbb{C}[S_n]$ module and a_λ, b_λ are therefore operators on $V^{\otimes N}$, so that the computation $(c_\lambda)^2 = (a_\lambda b_\lambda)^2$ is independent of the choice of V, in fact one can compute this entirely within the group algebra of the symmetric group. Here we are using the fact that S_N, and thus its group algebra acts on $V^{\otimes N}$ and c_λ is an idempotent operator on $V^{\otimes N}$. With some more work, we can show that $k_\lambda = N!/dimSp_\lambda$, where Sp_λ is the irreducible ***Specht module*** of S_N associated to the partition λ of N, but that is not relevant here.
Now we show functoriality. Any linear map $\alpha : U \to V$ lifts easily to a map $U_\lambda \to V_\lambda$.

$$\begin{array}{ccc} U_\lambda & \xrightarrow{i} & U^{\otimes N} \\ & & \downarrow{\scriptstyle \alpha^{\otimes N}} \\ V_\lambda & \xleftarrow[\frac{1}{k_\lambda}\pi]{} & V^{\otimes N} \end{array}$$

Let $\beta : V \to W$ be another map, and consider the composition $U_\lambda \to V_\lambda \to W_\lambda$.

$$\begin{array}{ccc} U_\lambda & \xrightarrow{i} & U^{\otimes N} \\ & & \downarrow{\scriptstyle \alpha^{\otimes N}} \\ V_\lambda & \xleftarrow{\frac{1}{k_\lambda}\pi} & V^{\otimes N} \\ & \searrow{\scriptstyle i} & \\ & & V^{\otimes N} \\ & & \downarrow{\scriptstyle \beta^{\otimes N}} \\ W_\lambda & \xleftarrow[\frac{1}{k_\lambda}\pi]{} & W^{\otimes N} \end{array} \implies \begin{array}{ccc} U_\lambda & \xrightarrow{i} & U^{\otimes N} \\ & & \downarrow{\scriptstyle \alpha^{\otimes N}} \\ & & V^{\otimes N} \\ & & \downarrow{\scriptstyle \frac{1}{k_\lambda}c_\lambda} \\ & & V^{\otimes N} \\ & & \downarrow{\scriptstyle \beta^{\otimes N}} \\ W_\lambda & \xleftarrow[\frac{1}{k_\lambda}\pi]{} & W^{\otimes N} \end{array} \implies \begin{array}{ccc} U_\lambda & \xrightarrow{i} & U^{\otimes N} \\ & & \downarrow{\scriptstyle \alpha^{\otimes N}} \\ & & V^{\otimes N} \\ & & \downarrow{\scriptstyle \beta^{\otimes N}} \\ & & W^{\otimes N} \\ & & \downarrow{\scriptstyle \frac{1}{k_\lambda}c_\lambda} \\ W_\lambda & \xleftarrow[\frac{1}{k_\lambda}\pi]{} & W^{\otimes N} \end{array}$$

Note that 'c_λ commutes with β'. Also $\frac{1}{k_\lambda}\pi \circ \frac{1}{k_\lambda}c_\lambda = \frac{1}{k_\lambda}\pi$, so

$$\implies \begin{array}{ccc} U_\lambda & \xrightarrow{i} & U^{\otimes N} \\ & & \downarrow{\scriptstyle \alpha^{\otimes N}} \\ & & V^{\otimes N} \\ & & \downarrow{\scriptstyle \beta^{\otimes N}} \\ W_\lambda & \xleftarrow[\frac{1}{k_\lambda}\pi]{} & W^{\otimes N} \end{array} \implies \begin{array}{ccc} U_\lambda & \xrightarrow{i} & U^{\otimes N} \\ & & \downarrow{\scriptstyle (\beta \circ \alpha)^{\otimes N}} \\ W_\lambda & \xleftarrow[\frac{1}{k_\lambda}\pi]{} & W^{\otimes N} \end{array}$$

This proves the functoriality. $\square$

2.3 Realizing $\mathbb{S}_\lambda$ via Matrix Minors

The other strategy is to modify the approach laid out in Remark 1 from Section 1. We embed both E and H into $\mathbb{C}[z_{ij}]$, the polynomial ring in n^2 variables in such a way that their images meet, and then we intersect them to get V_λ or $\mathbb{S}_\lambda(V)$.
Embed H into $\mathbb{C}[z_{ij}]$ as polynomials which have deg λ_i in $z_{i1}, z_{i2}, \ldots, z_{in}$ for $1 \le i \le l(\lambda)$, that is, embed a factor $\mathrm{Sym}^{\lambda_k} V$ into $\mathbb{C}[z_{kj}]^{(\lambda_k)}$ as $e_{i_1}e_{i_2}\ldots\ e_{i_{\lambda_k}} \mapsto\ z_{ki_1}z_{ki_2}\ldots\ z_{ki_{\lambda_k}}$ and extend in an obvious manner. For E, send $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_l}$ to the determinant of the $l \times l$ submatrix of the matrix $(z_{ij})_{n\times n}$, using top l rows and column i_j for $1 \le j \le l$, then send tensor of such wedge, which we called e_T in section 1(filling in a λ-tableau in such manner), to the product of corresponding determinants and extend by linearity to get a map from E to $\mathbb{C}[z_{ij}]$. We call such product of determinants D_T, it is the image of e_T for a filling T of the diagram of λ . Note that the image of E in $\mathbb{C}[z_{ij}]$ lands in the image of H in $\mathbb{C}[z_{ij}]$.

Example 2.3.1. Take $n \ge 7, \lambda = (4,2,1)$. Take a filling T of λ,

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 2 & 5 \\ \hline 7 & 6 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

Then the second map above sends $e_T = e_1 \wedge e_7 \wedge e_3 \otimes e_4 \wedge e_6 \otimes e_2 \otimes e_5$ to

$$D_T = \begin{vmatrix} z_{11} & z_{17} & z_{13} \\ z_{21} & z_{27} & z_{23} \\ z_{31} & z_{37} & z_{33} \end{vmatrix} \cdot \begin{vmatrix} z_{14} & z_{16} \\ z_{24} & z_{26} \end{vmatrix} \cdot z_{12} \cdot z_{15}$$

One can see this product has degree 4 in $z_{11}, \ldots z_{1n}$, degree 2 in $z_{21}, \ldots, z_{2n}$, degree 1 in $z_{31}, \ldots, z_{3n}$, thus lie in the image of $Sym^4V \otimes Sym^2V \otimes V$ inside $\mathbb{C}[z_{ij}]$ via the first map. Thus we see $V_{(4,2,1)}(n)$ is spanned by all products of the form

$$\begin{vmatrix} z_{1i} & z_{1j} & z_{1k} \\ z_{2i} & z_{2j} & z_{2k} \\ z_{3i} & z_{3j} & z_{3k} \end{vmatrix} \cdot \begin{vmatrix} z_{1l} & z_{1m} \\ z_{2l} & z_{2m} \end{vmatrix} \cdot z_{1p} \cdot z_{1q}$$

Note that it is now easy to see that we have a nonzero map $E \to H$: These products of minors are clearly nonzero (as long as $\ell(\lambda) \le n$, so we can form large enough determinants inside an $n \times n$ matrix). The next calculation shows that this is equivariant too, if we let $g \in GL_n$ act on $\mathbb{C}[z_{ij}]$ via: $g = (g_{ij}) : z_{i,j} \mapsto \sum_k z_{i,k}g_{k,j}$.

Lemma 2.3.2. The map $E \to H$ is GL_n equivariant.

Proof. The element $g = (g_{ij})$ takes $e_i \mapsto \sum_j g_{j,i} e_j$ by taking the column vectors of g; so $e_T \mapsto \sum_{j_1,\dots,j_d} g_{j_1,i_1} g_{j_2,i_2} \dots g_{j_d,i_d} e_{T'}$ where T is the filling obtained from T by replacing its entries $i_1, \dots, i_d$ with $j_1, \dots, j_d$ correspondingly.

On the other hand, the determinant $D_{i_1,\dots,i_p}$ gets mapped to:

$$\det \begin{pmatrix} z_{1,i_1} & z_{1,i_2} & \dots & z_{1,i_p} \\ z_{2,i_1} & z_{2,i_2} & \dots & z_{2,i_p} \\ \vdots & \vdots & \ddots & \vdots \\ z_{p,i_1} & z_{p,i_2} & \dots & z_{p,i_p} \end{pmatrix} \mapsto \det \begin{pmatrix} \sum_{j_1} z_{1,j_1} g_{j_1,i_1} & \dots & \sum_{j_p} z_{1,j_p} g_{j_p i_p} \\ \sum_{j_1} z_{2,j_1} g_{j_1,i_1} & \dots & \sum_{j_p} z_{2,j_p} g_{j_p,i_p} \\ \vdots & \ddots & \vdots \\ \sum_{j_1} z_{p,j_1} g_{j_1,i_1} & \dots & \sum_{j_p} z_{p,j_p} g_{j_p,i_p} \end{pmatrix}$$

which is $\sum_{j_1,\dots,j_d} g_{j_1,i_1} \dots g_{j_d,i_d} D_{T'}$. □

Example 2.3.3. (i) $V_{(2,1)}(n)$ is spanned by

$$\begin{vmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{vmatrix} \cdot z_{1i}, \begin{vmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{vmatrix} \cdot z_{1k}$$

where i, j, k are distinct, with the relation

$$\begin{vmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{vmatrix} \cdot z_{1k} - \begin{vmatrix} z_{1i} & z_{1k} \\ z_{2i} & z_{2k} \end{vmatrix} \cdot z_{1j} + \begin{vmatrix} z_{1j} & z_{1k} \\ z_{2j} & z_{2k} \end{vmatrix} \cdot z_{1i} = 0$$

The last equation exactly corresponds to the fact $(e_i \wedge e_j) \otimes e_k + (e_j \wedge e_k) \otimes e_i + (e_k \wedge e_i) \otimes e_l$ will be mapped to 0 under the map from E to H that we talked about in section 1, and it is easier to see now because this equation is the expansion of $\begin{vmatrix} z_{1i} & z_{1j} & z_{1k} \\ z_{1i} & z_{1j} & z_{1k} \\ z_{2i} & z_{2j} & z_{2k} \end{vmatrix}$ which is 0.

(ii)$V_{1^k}(n)$ as span of $k \times k$ top justified minors. This is isomorphic to $\wedge^k V$.

(iii)$V_k(n)$ is realized as polynomials of degree k in $z_{11}, \dots, z_{1n}$.

(iv)If $\lambda = \overbrace{(d, \dots, d)}^{k}$, the image V_λ in $\mathbb{C}[z_{ij}]$ is spanned by the k-fold products of the $d \times d$ top justified minors. Since the image of V_λ will not use any z_{ij} for $i > d$, $V_\lambda \subset \mathbb{C}[z_{ij}]_{1 \le i \le d, 1 \le j \le n}$. Note that V_λ here is actually $det^{\otimes k}$.

As anticipated in Chapter 1, we now prove the 'semistandard basis theorem'.

Theorem 2.3.4. $\{D_T : T \in SSYT(\lambda)$, T has entry at most n $\}$ is a basis for $V_\lambda(n)$.

Proof. First, we put an order on the monomials in $\mathbb{C}[z_{ij}]$. Represent each monomial in the z_{ij} as an $n \times n$ matrix whose (i, j)th entry is the exponent of z_{ij} in the monomial. Order the monomials using the the lexicographical order reading left to right then top to bottom on these matrices.

For instance, $z_{11}z_{23}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} > \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = z_{12}z_{21}^2$

Note that this assignment of polynomials to largest monomial in it ('initial monomial') $p \mapsto in(p)$ has the following key property:

- If $p \geq p'$ and $q \geq q'$, where p, q, p', q' are monomials, then $pq \geq p'q'$. This implies that $in(fg) = in(f)in(g)$, therefore $in(D_T) = \prod$ (product of diagonal terms in each matrix minor), where the product varies over all columns of T.

Example:

$$\begin{aligned} D_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} &= D_{13}D_2 = (z_{11}z_{23} - z_{13}z_{21})z_{12} \\ &= z_{11}z_{23}z_{12} - z_{13}z_{21}z_{12} \end{aligned}$$

We have

$$\begin{pmatrix} 1 & 1 & \\ & & 1 \end{pmatrix} > \begin{pmatrix} & 1 & 1 \\ 1 & & \end{pmatrix}$$

so the initial monomial for $D_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}$ is $z_{11}z_{23}z_{12}$.

We will first prove that the D_T for $T \in \mathrm{SSYT}(\lambda)$ are linearly independent in $V_\lambda(n)$ and then show that they span $V_\lambda(n)$.

Claim 1: The correspondence $SSYT(\lambda) \ni T \mapsto in(D_T)$ is injective.

Proof. We show that it is possible to construct T entirely from $in(D_T)$; call this polynomial p. We are going to construct from p a tableau T column by column, starting with the first one. Note that if a column of T consists of $a_1, \dots, a_k$ (from top to bottom), then this column contributes to $in(D_T)$ the factor $z_{1a_1} \cdots z_{ka_k}$; in fact $in(D_T)$ is the product of such terms. Therefore going backword, find the smallest integers $a_1, \dots, a_k$, where $k = l(\lambda)$, such that z_{ia_i} is a factor of p for every $i \in [k]$. Then it is clear that the first column of T consists of $a_1, \dots, a_k$, ordered from top to bottom. Now just remove the factor $z_{1a_1} \cdots z_{ka_k}$ from p, name this new polynomial to be p and repeat the same thing until the rest of the columns are constructed, i.e. p becomes 1.

□

The linear independence of the D_T over $T \in \mathrm{SSYT}(\lambda)$ follows from this claim. Suppose that there is a nontrivial relation: $\Sigma_{T \in \mathrm{SSYT}(\lambda)} a_T D_T = 0$, where a_T's are nonzero real numbers. Among such T's, pick T^*, the one with maximum (in the said order) initial monomial, note that by the claim above, there is an unique maximum one; but then there is no other monomial in this equation to cancel off $in_{D_{T^*}}$ (obviously) and we run into a contradiction.

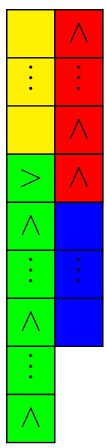
We still need to show that the D_T, $T \in \text{SSYT}(\lambda)$, span $V_\lambda(n)$. Note that this follows from the fact that the D_T are linearly independent and there are $\dim V_\lambda(n)$ of them, but we provide a more constructive proof below.

We already know that D_T, T any tableaux of λ, span $V_\lambda(n)$. So we just need a consistent method to express D_T, when T is not a SSYT, as a linear combination of D_U, $U \in \text{SSYT}(\lambda)$.

For this, we order the tableaux lexicographically, reading down the columns in order from left to right, i.e. we read off the entries in the filling T in the order shown below

1	4	6	7
2	5		
3			

It suffices to show that *if T is not semistandard, then $D_T \in Span_{U<T}(D_U)$*, because then repeatedly applying this to the D_U's for non SSYT U's would eventually produce the desired linear combination in terms of SSYT terms. We therefore prove this claim in the rest of this section.
If any column of T is non increasing, then sorting it produces a smaller T' and $D_T = \pm D_{T'}$. So we may assume that the columns are increasing. If T is not semistandard, then we have two adjacent columns like this:



Break these columns up into $I_1 \sqcup I_2$, $J_1 \sqcup J_2$, where I_1, I_2, J_1, J_2 is yellow, green, red and blue in the above diagram.

Claim 2: Let $s \geq t > 0$. Let $I = I_1 \sqcup I_2$ and $J = J_1 \sqcup J_2$ with $|I| = s$, $|J| = t$, and $|I_2| + |J_1| = s + 1$. Let $I_2 \cup J_1 = \{r_1 r_2 \cdots r_{s+2}\}$ where elements can be repeated. Then

$$\sum_{\omega \in S_{s+1}} (-1)^\omega D_{I_1 r_{\omega(1)} r_{\omega(2)} \cdots r_{\omega(|I_2|)}} D_{r_{\omega(|I_2|+1)} \cdots r_{\omega(s+1)} J_2} = 0$$

Proof. This expression is an antisymmetric multilinear function of the $s+1$ vectors of the columns of $\begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix}$ indexed by $I_2 \cup J_1$, and only it uses their top s entries. So it is an element of $s+1$ of a dimension s vector space, so it is 0. $\square$

Example 2.3.5. Take $I = \{1\} \sqcup \{2, 3\} = I_1 \sqcup I_2$, $s = 3$, and $J = \{4, 5\} \sqcup \emptyset = J_1 \sqcup J_2$, $t = 2$. Observe that *the terms in the sum are constant on the cosets of* $S_{|I_2|} \times S_{s+1-|I_2|}$ *in* S_{s+1}, so we can restrict to just summing over these cosets (reason: need to show that any $\pi \in S_{|I_2|} \times S_{s+1-|I_2|}$, $(-1)^\pi D_{I_1 r_{\pi(1)} r_{\pi(2)} \cdots r_{\pi(|I_2|)}} D_{r_{\pi(|I_2|+1)} \cdots r_{\pi(s+1)} J_2} = D_I D_J$, and suffices to show this for any transposition in this subgroup, but there the result is obvious).
So the equation in the claim boils down to really the following one, consisting of $|S_4/S_2 \times S_2| = 6$ terms instead of 24 terms.

$$\begin{matrix} D_{123}D_{45} & -D_{124}D_{35} & +D_{125}D_{34} \\ +D_{134}D_{25} & -D_{135}D_{24} & +D_{145}D_{23} \end{matrix} = 0$$

$$D_{\begin{smallmatrix}1&4\\2&5\\3&\end{smallmatrix}} - D_{\begin{smallmatrix}1&3\\2&5\\4&\end{smallmatrix}} + D_{\begin{smallmatrix}1&3\\2&4\\5&\end{smallmatrix}} + D_{\begin{smallmatrix}1&2\\3&5\\4&\end{smallmatrix}} - D_{\begin{smallmatrix}1&2\\3&4\\5&\end{smallmatrix}} + D_{\begin{smallmatrix}1&2\\4&3\\5&\end{smallmatrix}} = 0$$

We can use this relation *to express a non SSYT,* $D_{\begin{smallmatrix}1&2\\4&3\\5&\end{smallmatrix}}$*, by a linear combination of SSYT terms.*

This also resolves the general case. Claim 2 coupled with our last observation shows that

$$D_{I_1 \sqcup I_2} D_{J_1 \sqcup J_2} = \sum \pm D_{I_1,()} D_{(),J_2}.$$

Start working from left to right of the given non SSYT T and sort the entries so that the columns are strictly increasing. Suppose we have rectified columns $1, \cdots, i$ and $(i-1, i)$ is the first trouble-making pair, in the sense that some cell of column $i-1$ has larger entry (say u) than its right neighbour(say v) in column i, and this is the first such instance. Pick out these two column and treating them as a tableau $T_{(i-1,i)}$, apply the lemma. If the entries between $u = u_1$ and $v = v_j$(in the reading order said above) are $u_2, \cdots, u_i, v_1, \cdots, v_{j-1}$, with the u's in column $i-1$ and v's in column i, then by our assumption $u_i > u_{i-1} > \cdots > u_2 > u > v > v_{j-1} > \cdots > v_1$. Now the tableaux S_α associated to the summands of RHS of the equation above (meaning D_{S_α}'s are the summands) have some of the entries among $\{u_1, \cdots, u_i\}$ of the first column interchanged with entries from $\{v_1, \cdots, v_j\}$ of the second column, therefore $T_{(i-1,i)} > S_\alpha$ in our order. Therefore repeated application of Plücker relations will help expanding $D_{T_{(i-1,i)}}$ in terms of D_{SSYT}s. After having rectified fully the tableau upto column i this way, carry on similar process. Multiplying by the polynomials corresponding to the columns that remain unchanged, the conclusion follows. □

Now we show that although we have found a basis of $V_\lambda(n)$ indexed by SSYT of shape λ with entries in $[n]$, this is not a Gelfand-Tsetlin basis, which can be seen in the following way. Consider the representation $V_{(2,1,0)}(3)$ of GL_3. By our discussion in the last section of Chapter 1, if the claim were true, then the subspace of $V_{(2,1,0)}(3)$ spanned by all semistandard tableaux with 3-s in a given set of boxes would be GL_2 invariant subspace(i.e. an irrep GL_2). But calculation reveals that, if $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, (so that A represents $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in GL_2 \subset GL_3$)then

$$\begin{aligned} A \cdot D_{\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}} &= D_{\begin{smallmatrix} 2 & 1 \\ 3 & \end{smallmatrix}} + D_{\begin{smallmatrix} 2 & 2 \\ 3 & \end{smallmatrix}} \\ &= D_{\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}} + D_{\begin{smallmatrix} 2 & 3 \\ 1 & \end{smallmatrix}} + D_{\begin{smallmatrix} 2 & 2 \\ 3 & \end{smallmatrix}} \\ &= D_{\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}} - D_{\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}} + D_{\begin{smallmatrix} 2 & 2 \\ 3 & \end{smallmatrix}} \end{aligned}$$

Although our basis is not the GT basis, it is not so bad either; this is a weight basis, much like the GT basis, meaning that $diag(t_1, \cdots, t_n) \cdot D_T = \prod_i t_i^{\text{number of i's in T}} D_T$, evident from the definition of D_T. Similar equations hold for GT basis elements v_T, see [12]. We make the following remark: although the semistandard basis is not the GT basis (even though it resembles the GT basis, being indexed by SSYT's), the highest weight vectors in these bases matches, as predicted via the bijection between GT pattern and SSYT.

Proposition 2.3.6. D_{T_0} is a highest weight vector of $V_\lambda(n)$, where T_0 is the SSYT having all i's in the i-th row; also v_{ζ_0} (the GT basis vector indexed by ζ_0) is another highest weight vector of $V_\lambda(n)$, where ζ_0 is the GT pattern associated to the SSYT T_0. Therefore these two vectors are just a scalar multiple of each other.

The first claim follows from a straightforward calculation; for the second, see [12].

2.4 Kernel of φ

We go back to answering the question raised in Remark 1 in Section 1, and on the way we derive another realization of V_λ using the results in the last section. We want to find

out kernel of the map $E = \bigotimes_k \Lambda^{\lambda_k^{\mathrm{T}}} V \mapsto H = \bigotimes_k \mathrm{Sym}^{\lambda_k} V$, given by $e_T \mapsto D_T$, and therefore conclude that $V/Kernel \cong V_\lambda$. We will need the notion of an exchange between the columns of a tableau. This depends on a choice of two columns of a Young diagram λ, and a choice of a set of the same number of boxes in each column. For any filling T of λ, the corresponding exchange is the filling S obtained from T by interchanging the entries n the two chosen set of boxes, maintaining the vertical order in each;the entries outside these chosen boxes are fixed. We write $V^{\times\lambda}$ for the Cartesian product of $n = |\lambda|$ copies of V, which is labelled by the n boxes of the diagram of λ: an element $\mathbf{v}$ of $V^{\times\lambda}$ is given by specifying an element of V for each box in λ. Following [18], define V^λ to be the universal target module for the following type of maps ρ:
(i) ρ is multilinear.
(ii) ρ is alternating in the entries of any column of λ.
(iii) For any $\mathbf{v}$ in $V^{\times\lambda}$, $\rho(\mathbf{v}) = \Sigma\rho(\mathbf{w})$,where the sum is over all $\mathbf{w}$ obtained from $\mathbf{v}$ by an exchange between two given columns,with a given subset of boxes chosen from the top in the right chosen column.
For example, for $\lambda = (2, 2, 2)$, the third condition says that the following equations hold:

$$\rho\left(\begin{array}{|c|c|}\hline x & u \\ \hline y & v \\ \hline z & w \\ \hline\end{array}\right) = \rho\left(\begin{array}{|c|c|}\hline u & x \\ \hline y & v \\ \hline z & w \\ \hline\end{array}\right) + \left(\begin{array}{|c|c|}\hline x & y \\ \hline u & v \\ \hline z & w \\ \hline\end{array}\right) + \left(\begin{array}{|c|c|}\hline x & z \\ \hline y & v \\ \hline u & w \\ \hline\end{array}\right),$$

$$\rho\left(\begin{array}{|c|c|}\hline x & u \\ \hline y & v \\ \hline z & w \\ \hline\end{array}\right) = \rho\left(\begin{array}{|c|c|}\hline u & x \\ \hline v & y \\ \hline z & w \\ \hline\end{array}\right) + \rho\left(\begin{array}{|c|c|}\hline u & x \\ \hline y & z \\ \hline v & w \\ \hline\end{array}\right) + \rho\left(\begin{array}{|c|c|}\hline x & y \\ \hline u & z \\ \hline v & w \\ \hline\end{array}\right), \rho\left(\begin{array}{|c|c|}\hline x & u \\ \hline y & v \\ \hline z & w \\ \hline\end{array}\right) = \rho\left(\begin{array}{|c|c|}\hline u & x \\ \hline v & y \\ \hline w & z \\ \hline\end{array}\right)$$

This means that we have a linear map $V^{\times\lambda} \to V^\lambda$, denoted by $\mathbf{v} \mapsto \mathbf{v}^\lambda$, satisfying these three conditions and for any other $\phi : V^{\times\lambda} \to F$ satisfying these conditions, there is an unique linear map $\phi^* : V^\lambda \to F$ such that $\phi(\mathbf{v}) = \phi^*(\mathbf{v}^\lambda)$.
Now, the universal object satisfying $(i), (ii)$ is simply $\bigotimes_k \Lambda^{(\lambda')_k} V$, if we number λ down the column from left to right and the alleged map from $V^{\times\lambda} \to \bigotimes_k \Lambda^{(\lambda')_k} V$, which we write $\mathbf{v} \mapsto \wedge\mathbf{v}$, is also the obvious one, e.g.

$$\rho\left(\begin{array}{|c|c|}\hline x & u \\ \hline y & v \\ \hline z & w \\ \hline\end{array}\right) \mapsto (x \wedge y \wedge z) \otimes (u \wedge v \wedge w) \in \Lambda^3 V \otimes \Lambda^3 V$$

Then $V^\lambda = \bigotimes_k \Lambda^{(\lambda')_k} V/Q_\lambda$ (so it exists), where Q_λ is the subspace generated by all element of the form $\wedge\mathbf{v} - \Sigma \wedge \mathbf{w}$, the sum over all $\mathbf{w}$ obtained from $\mathbf{v}$ by the exchange procedure in (iii) for all possible choices of columns and boxes. We claim that $V^\lambda \cong V_\lambda$: suffices to show that (a) the D_T's satisfy similar relation as in (iii), so that φ does factor through the quotient and kernel is inside Q_λ, whence $V^\lambda \subset V_\lambda$, and that (b) the dimension matches (thus equating the kernel with Q_λ), or even $dim\ V^\lambda \geq dim\ V_\lambda$ would do. Once we show (a), it is straightforward to see that the images of e_T in the quotient V^λ, where $T \in SSYT(\lambda)$, are independent, as their images D_T for $T \in SSYT(\lambda)$ are. In fact, they also span V^λ: *condition (iii) is really a statement about 'straightening out the non SSYT's' which we did in the last section for the D_T's, these are Plücker relations in disguise.* Therefore all we need is the following.

Lemma 2.4.1. Property (iii) for the D_T's follows from Claim 2 in previous section, applied to suitable matrices.

Proof. Notice that the following (which goes by the name Sylvester's lemma) is really a (weaker) restatement of the claim: *for any $M, N \in Mat_{p\times p}(\mathbb{C})$ and $k \in [p]$, $\det(M)\cdot\det(N) = \Sigma \det(M')\cdot\det(N')$, where the sum is over all pairs (M', N') obtained from (M, N) by interchanging a fixed set of k columns of N with any k columns of M, preserving the ordering of columns.* Now for our purpose, suppose the two columns of T in which exchange takes place have entries $i_1, \cdots, i_p$ in the first and $j_1, \cdots, j_q$ in the second column.

Set $M = \begin{pmatrix} z_{1i_1} & \cdots & z_{1i_p} \\ \vdots & \ddots & \vdots \\ z_{pi_1} & \cdots & z_{pi_p} \end{pmatrix}$ $N = \begin{pmatrix} z_{1j_1} & \cdots & z_{1j_q} & 0_{q\times p-q} \\ \vdots & \ddots & & \vdots \\ z_{pj_1} & \cdots & z_{pj_q} & I_{(p-q)\times(p-q)} \end{pmatrix}$

Sylvester's lemma, applied to this situation, precisely translates to the required equation. □

2.5 Multiplicity-Free Sum of GL_n Polynomial Representations-I

There is a simple way to construct all the polynomial representations $\mathbb{S}_\lambda(V)$ of GL_n at once, and their direct sum over all partitions λ can be made, in the approach of Deryuts, into commutative graded ring, which we denote by $\mathbb{S}(V)$. This is similar to the fact that the algebras $SymV = \bigoplus Sym^k V$ and $AltV = \bigoplus \wedge^k V$ are easier to describe than the individual graded pieces.

First, observe that the map $e_T \mapsto D_T$ is symmetric in the entries of columns of same length: if two columns of T are of same length, and if T_0 is the tableau obtained from T interchanging those two columns and leaving everything else same, then $D_T = D_{T_0}$; thus it factors through $A^{\mathbf{a}}(V) = Sym^{a_n}(\wedge^n V) \otimes Sym^{a_{n-1}}(\wedge^{n-1} V) \otimes \cdots \otimes Sym^{a_1}(V)$ where a_i = number of columns in λ of length $i = \lambda_i - \lambda_{i-1}$. It shows that V_λ sits inside $A^{\mathbf{a}}(V)$.

So, if we define,

$$\begin{aligned}\mathbb{A}(V) &= Sym(V \bigoplus \wedge^2 V \bigoplus \cdots \bigoplus \wedge^n V) \\ &= \bigoplus Sym^{a_n}(\wedge^n V) \otimes Sym^{a_{n-1}}(\wedge^{n-1} V) \otimes \cdots \otimes Sym^{a_1}(V)\end{aligned}$$

then it is the direct sum of all $A^{\mathbf{a}}(V)$ just considered, over all n-tuples $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{N}^n$. Thus it contains a multiplicity-free direct sum for the irreducible polynomial representations as a subspace; we go modulo the correct ideal to get the explicit V_λ as the precise summands.
Define $\mathbb{S}(V) = \mathbb{A}(V)/I$, where I is the graded, two-sided ideal generated by all elements (Plücker relations) of the form

$$(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_q)$$
$$- \sum (v_1 \wedge \cdots \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_r \cdots \wedge v_p) \cdot (v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_r} \wedge w_{r+1} \wedge \cdots \wedge w_q)$$

for all $n \geq p \geq q \geq r \geq 1$ and all $v_i, w_j \in V$, where the sum is over all $1 \leq i_1 < i_2 < \cdots < i_r \leq p$, and the elements $w_1, w_2, \cdots, w_r$ are inserted at the corresponding places in $v_1 \wedge \cdots \wedge v_p$. Observe the generators of $I^{\mathbf{a}} = I \cap A^{\mathbf{a}}$ precisely matches with those of the kerφ as dictated in property (iii) from the last section, where $\mathbf{a}$ and λ are related as said earlier. Thus $\mathbb{S}(V) = \mathbb{A}(V)/I = \bigoplus A^{\mathbf{a}}(V)/I^{\mathbf{a}} = \bigoplus V_\lambda$, the sum of being over all partitions λ with at most n parts. We shall find another realization of this instance in a later chapter.

Chapter 3

Relation to the Symmetric Group

In this chapter we begin translating some of the results on GL_n representations into results about S_n representations. We will also eventually establish ***Schur-Weyl Duality***, a cornerstone result with substantial subsequent generalizations, which allows us to go back and forth between GL_n and S_d for any n, d.

3.1 Specht Modules

Let us note that if we restrict to T_n, the torus inside GL_n, then $V_\lambda(n)$ breaks up into T_n irreducibles: $V_\lambda = \oplus(V_\lambda)_\mu$, where $(V_\lambda(n))_\mu := \{v \in V_\lambda : diag(t_1, \cdots, t_n) \cdot v = t_1^{\mu_1} \cdots t_n^{\mu_n} v\}$ for a composition $\mu = (\mu_1, \cdots, \mu_n)$ of n (meaning that $\Sigma_i \mu_i = n$). Our first point of departure is to observe that $(V_\lambda(n))_\mu$, the μ *weight space* (and μ is called a *weight* of the vectors in this weight space) in $V_\lambda(n)$ is the subspace spanned by all the D_T's, where T has shape λ and content μ: for such a tableau T, D_T is inside the claimed weight space(note that GL_n acts from the right) and both the dimension of μ-weight spaces and number of SYT's of shape λ, content μ has to add up to dim V_λ. *In fact, the Freudenthal formula in this case becomes* $dim(V_\lambda)_\mu = K_{\lambda\mu}$. Note that in particular, $(V_\lambda)_\mu = 0$ unless $|\lambda| = |\mu|$. Next thing to notice is that S_n sits inside GL_n as permutation matrices, and they permute the weight spaces in a given representtion of GL_n: *if $\sigma \in S_n$ is a permutation, then it maps the $(a_1, \ldots, a_n)$ weight space to the $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ weight space.* Take a diagonal matrix $d = diag(t_1, \ldots, t_n)$ and consider its action on σu, where u is in the $(a_1, \ldots, a_n)$ weight space. We have

$$\begin{aligned} d\sigma u = \sigma\left(\sigma^{-1}d\sigma\right)u = \sigma diag(t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(n)})u = \sigma\left(t_{\sigma^{-1}(1)}^{a_1} t_{\sigma^{-1}(2)}^{a_2} \cdots t_{\sigma^{-1}(n)}^{a_n} \cdot u\right) \\ = \left(t_{\sigma^{-1}(1)}^{a_1} t_{\sigma^{-1}(2)}^{a_2} \cdots t_{\sigma^{-1}(n)}^{a_n}\right) \cdot \sigma u = \left(t_1^{a_{\sigma(1)}} t_2^{a_{\sigma(2)}} \cdots t_n^{a_{\sigma(n)}}\right) \cdot \sigma u \end{aligned}$$

So σu is in the desired weight space.

In particular, we see that S_n acts on the $(1, \ldots, 1) = (1^n)$ weight space. Let us separate this instance.

Definition 3.1.1. Let $\lambda \vdash n$. The ***Specht module*** Sp_λ is the (1^n) weight space of $V_\lambda(n)$. Note that it is a module for the group algebra of the symmetric group and a basis for the Specht module is given by the SSYTs of shape λ and entries $1, \ldots, n$, each occurring once. These are precisely the ***standard Young tableaux***. In particular,

$$\dim Sp_\lambda = \#\{\text{standard Young tableaux of shape } \lambda\}.$$

Our main theorem is the following:

Theorem 3.1.2. As λ varies over the partitions of n, Sp_λ varies over the irreducible representations of S_n, each occurring once.

Proof. Let $V = \mathbb{C}^n$. We know that, as a $\mathrm{GL}_n \times \mathrm{GL}_n$ representation,

$$Sym^n(V \otimes V) = \bigoplus_{\lambda \vdash n} V_\lambda(n) \otimes V_\lambda(n).$$

where the action is given by $(g, \hat{g}) \cdot v_1 \otimes w_1 \ldots v_n \otimes w_n = gv_1 \otimes \hat{g}w_1 \ldots gv_n \otimes \hat{g}w_n$.
Motivated by the definition of Sp_λ, consider the subspace of $Sym^n(V \otimes V)$ which is of weight (1^n) for the both the left GL_n action and right GL_n action; in other words, we consider the subspace with basis vectors $\{e_{\alpha(1)} \otimes \hat{e}_{\beta(1)} \cdots e_{\alpha_n} \otimes \hat{e}_{\beta(n)} : \alpha, \beta \in S_n\}$., where $\{e_i : i \in [n]\}$ and $\{\hat{e}_j : j \in [n]\}$ are the usual bases of the left and right copies of $\mathbb{C}^n$. Notice that $e_{\alpha(1)} \otimes \hat{e}_{\beta(1)} \cdots e_{\alpha_n} \otimes \hat{e}_{\beta(n)} = e_1 \otimes \hat{e}_{\beta\alpha^{-1}(1)} \cdots e_n \otimes \hat{e}_{\beta\alpha^{-1}(n)}$, so we might as well take the basis vectors to be $\{e_\gamma := e_1 \otimes \hat{e}_{\gamma(1)} \cdots e_n \otimes \hat{e}_{\gamma(n)} : \gamma \in S_n\}$.
But then, the next calculation shows $e_\gamma \mapsto \gamma$ is an $S_n \times S_n$isomorphism of this weight space with the regular representation $\mathbb{C}[S_n]$ of $S_n \times S_n$:

$$\begin{aligned}(\pi, \zeta) \cdot e_\gamma &= \pi e_1 \otimes \zeta e_{\gamma(1)} \cdots \pi e_n \otimes \zeta e_{\gamma(n)} \\ &= e_{\pi(1)} \otimes e\zeta\gamma(1) \cdots e_{\pi(n)} \otimes e\zeta\gamma(n) \\ &= e_1 \otimes e_{\zeta\gamma\pi^{-1}(1)} \cdots e_n \otimes e_{\zeta\gamma\pi^{-1}(n)} \\ &\mapsto \zeta\gamma\pi^{-1} \\ &= (\pi, \zeta) \cdot \gamma\end{aligned}$$

Now in order to get hold of the right hand side, taking into account that $(V_\lambda)_{(1^n)} = 0$ unless $|\lambda| = n$, the right hand side finally boils down to $\bigoplus_{\lambda \vdash n} Sp_\lambda \otimes Sp_\lambda$. Therefore, we obtain

$$\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} Sp_\lambda \otimes Sp_\lambda$$

But we already know from Fourier decomposition of finite groups that

$$\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} F_\lambda^* \otimes F_\lambda$$

if $\{F_\lambda : \lambda \vdash n\}$ is the complete set of S_n irreducibles; we know that the irreducible representations of S_n are self dual, so we can replace the F_λ^* (first tensor factor of each summands) from the usual decomposition by F_λ here. Therefore if $Sp_\lambda = \Sigma_{\mu \vdash n} F_\mu^{\bigoplus c_{\lambda\mu}}$, then from these two equations one sees that $c_{\lambda\mu} = \delta_{\lambda\mu}$ whence $Sp_\lambda = F_\lambda$. Thus the Sp_λ's are precisely the irreducibles. $\square$

In a nutshell, this gives us the following:

Corollary 3.1.3. Restriction to the (1^n) weight space gives an equivalence of categories

$$\left\{ \mathrm{GL}_n \text{ polynomial irreps where } t{\cdot}\mathrm{Id} \text{ acts by } t^n, i.e. \text{ of degree n} \right\} \longrightarrow \left\{ S_n \text{ representations} \right\}.$$

3.2 Examples

Here are four basic examples of Specht modules.

Example 1. Consider Sp_λ with $\lambda = (n)$. This is (1^n) weight space of $V_{(n)}(n) = Sym^n(\mathbb{C}^n) \cong \mathbb{C}[x_1, \cdots, x_n]^{(n)}$. so $Sp_{(n)} = \mathbb{C}[x_1 x_2 \cdots x_n]$, and S_n acts trivially. So, this is the trivial representation.

Example 2. $Sp_{(1^n)}$ is the subspace of $V_{(1^n)}(n)$ of degree $(1, \ldots, 1)$ and $V_\lambda(n) \cong \wedge^n \mathbb{C}^n$, which is the 1 dimensional determinant representation, so the Specht module is one dimensional, $Sp_\lambda = \mathbb{C} \cdot det(z_{ij})_{n \times n}$, and $\sigma \in S_n$ acts by permuting the columns in the determinant, which introduces a sign of $(-1)^\sigma$. So, this is the sign representation of S_n.

Example 3. We consider $Sp_{(n-1,1)}$. This is the $\mathbb{C}$-span of the products

$$D_{ij} \cdot \frac{z_{11} \cdots z_{1n}}{z_{1i} z_{1j}} = \det \begin{vmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{vmatrix} \cdot z_{11} \cdots \widehat{z_{1i}} \cdots \widehat{z_{1j}} \cdots z_{1n}.$$

where $1 \leq i, j \leq n, i \neq j$. The dimension is the number of SYTs of shape λ, which is $n-1$ (corresponding to the choices of the box on the second row), so there are various relations between the above generators. In particular, letting $p = z_{11} \cdots z_{1n}$ and $w_k = \frac{z_{2k}}{z_{1k}}$, we see that our generators above are given by $p(w_j - w_i)$, which leads to lots of relations. A nice way of expressing it is:

$$Sp_{(n-1,1)} \cong \{a_1 w_1 + \cdots a_n w_n : \sum a_i = 0\} \subset \mathbb{C}^n,$$

which identifies it as the "standard representation" (the subrep of the "permutation representation" $\mathbb{C}^n$ that is orthogonal to the trivial subrep).

Example 4. Take the transpose of our last partition, so $\lambda = (2, 1^{n-1})$ Similar to the above, we have

$$Sp_{(2,1^{n-1})} = Span\{z_{1k} \cdot D_{1\cdots\widehat{k}\cdots n} : k = 1, \cdots n\}$$

This gives n generators, but there are only $n-1$ standard Young tableaux of this shape, so there is one relation. The relation is just the alternating sum:

$$\sum (-1)^k z_{1k} \cdot D_{1\cdots\widehat{k}\cdots n} = 0.$$

In particular, we can write

$$Sp_{(2,1^{n-1})} \cong \mathbb{C}^n/(e_1 + \cdots + e_n),$$

where the S_n action is given by (the obvious action) $\otimes$ (the sign action).

These example provide evidence for the following equality (which is true):

$$Sp(\lambda') = Sp(\lambda) \otimes (\text{sign})$$

A natural question to ask at this point is that what happens if we pick up some other weight space of $V_\lambda(n)$: take the weight $\mu = (\mu_1, \cdots, \mu_n)$ and form $V_{n,\lambda,\mu} = \bigoplus_{\sigma \in S_n/S_\mu} (V_\lambda)_{\sigma\cdot\mu}$, the direct sum running over all distinct weight spaces obtained from permuting the weight coordinates; this is by construction a representation of S_n, in fact one immediately observes that $V_{n,\lambda,\mu} := Ind_{S_\mu}^{S_n}(V_\lambda)_\mu$, and we want an explicit decomposition, that is to find the $c_\nu^{\lambda\mu}$'s in $V_{n,\lambda,\mu} = \bigoplus_{\nu \in S_n} Sp_\nu^{c_\nu^{\lambda\mu}}$. A more specific question would be to ask what are the irreducible constituents of $Res_{S_n}^{GL_n} V_\lambda(n)$. We will see in next chapter that there does exist a general answer of the later question in closed form using the concept of ***Plethysm***, but it is not much amenable for explicit calculation.

3.3 Schur-Weyl Duality

In this section, we draw an ubiquitous connection between the representation theories of S_d and GL_n. Observe that both GL_n and S_d acts on $V^{\otimes d}$, where $V = \mathbb{C}^n$, in the following way:

$$g \cdot (v_1 \otimes \cdots \otimes v_d) := gv_1 \otimes \cdots \otimes gv_d$$
$$\pi \cdot (v_1 \otimes \cdots \otimes v_d) := v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(d)}, \forall g \in GL_n, \pi \in S_n$$

Since this two action commutes with each other, we have a joint representation space of these two groups:

$$(\pi, g) \cdot (v_1 \otimes \cdots \otimes v_d) = gv_{\pi^{-1}(1)} \otimes \cdots \otimes gv_{\pi^{-1}(d)}$$

Also, the action of GL_n as specified above give rise to a polynomial representation. Now recall, over algebraically closed field such as $\mathbb{C}$, if for groups G and H, every representation is a direct sum of simples, then $G \times H$ also has this property, and $G \times H$ simples are of the form $\sigma \otimes \rho$, where σ and ρ are respectively G and H simple (Reason: Let π be $G \times H$ simple. Let $\sigma \subset \pi$ be a $G \cong G \times 1$-simple subrepresentation. Consider $\rho = \mathrm{Hom}_G(\pi, \sigma)$, which is a finite dimensional representation of $H \cong 1 \times H$ and take a subrepresentation ρ. The natural evaluation map $\sigma \otimes \rho \to \pi$ as a $G \times H$ representation is nonzero, and therefore it is both surjective and injective because the source and target modules are irreducible.) So while working over $\mathbb{C}$, our expectation would be that the tensor space breaks up under the joint action into various $V_\lambda \otimes Sp_\mu$, where μ is a partition of d and λ has at most n parts. Schur Weyl duality precisely determine the nature of these decomposition. Viewed from another perspective, one can motivate this result in the following way as well (afterall this result does not depend upon the field being algebraically closed, we will later outline the general proof scheme found as in Schur's celebrated paper): if G is a finite group and X a finite-dimensional representation of G and we are in the semisimple case, then X breaks up into a direct sum $X = \bigoplus n_i V_i$ of irreducible representations V_i with some multiplicities n_i. However, this direct sum decomposition is not canonical if the multiplicities $n_i \geq 1$. In the worst case, G may act trivially on X, and then X is a direct sum of $\dim X$ copies of the trivial representation. Actually choosing such a direct sum decomposition is equivalent to choosing a basis of X.

However, there is an alternate and completely canonical way of describing a representation in terms of its irreducible subrepresentations without choosing a direct sum decomposition as above. As a first hint, note that $n_i = \dim \mathrm{Hom}_G(V_i, X)$.This suggests that it might be useful to *replace* n_i *with the vector space* $Hom(V_i, X)$, and in fact, this turns out to be a great idea: there is a canonical evaluation map $V_i \otimes \mathrm{Hom}_G(V_i, X) \to X$, whose image is precisely the V_i-isotypic component of X, and this gives an alternate canonical decomposition of X as

$$X = \bigoplus V_i \otimes \mathrm{Hom}_G(V_i, X)$$

which does not require making any choices. One can think of $\mathrm{Hom}_G(V_i, X)$ as the multiplicity space associated to V_i, the *correct canonical replacement* for the multiplicity n_i. The idea of the ***Double Commutant Theorem*** is to think about what kind of structure these multiplicity spaces have. So far we have been using them only as vector spaces, but in fact they are $End_G\ X$ modules with the obvious action being given by post-composition of linear maps. The double commutant theorem asserts the following:

Theorem 3.3.1. Let X be a finite dimensional vector space and A be a semisimple subalgebra of $End\ X$, and $B = End_A X$. Then
(i)B is semisimple. (ii)$A = End_B X$ (hence the name, 'double' commutant) (iii)As an $A \times B$ module, we have the decomposition $X \cong \bigoplus_i U_i \otimes W_i$, where U_i and W_i are all the simple modules for A and B respectively. Therefore via this theorem we get a bijective correspondence between simple modules for A and those of its commutant, via $U_i \mapsto W_i$.

This is all general nonsence, and Schur in his celebrated paper [7] applied this to the following case, where $G = S_d$ and $X = V^{\otimes d}$; if $char\ k \geq d$ then $A = kS_d$ is semisimple algebra, therefore if we identify it with its image inside End$V$ (and call that A), then that is also a semisimple subalgebra, being a quotient of semisimple algebra. Schur's crucial observation was that $End_{S_d} V^{\otimes d}$ is the associative algebra of transformations on $V^{\otimes d}$ generated by GL_n, that is, what is termed as the ***Schur Algebra*** $S(n, d)$. In his doctoral discertation [6], Schur already showed that modules for the Schur algebra $S(n, d)$ are nothing but polynomial representation for GL_n of degree d, and this correspondence takes simple modules to irreducible representations. Therefore the double commutant theorem yields in our case a bijective correspondence between S_d irreps and GL_n poly irreps of degree d and one gets the decomposition of the tenosor space in terms of them. Notice that by our knowledge of explicit constructions from last chapter, GL_n poly irreps of degree d is indexed by partitions of d with at most n parts (for other partitions of d, they become zero vector space), so the said correspondence is really bijective when $n \geq d$, i.e. we are in the *stable range*. To finish up the proof one needs to assert that the irrep of S_d associated to the partition λ pairs up with the poly irrep of GL_n indexed by the same partition λ under this correspondence, see standard references, e.g. [21] and [5]. Notice that part of this result claims that the map $kS_d \to End_{GL_n} V^{\otimes d}$, which originates from the commutativity of the actions of S_d and GL_n on $V^{\otimes d}$, is surjective. What we do here is completely in the opposite direcion and actually we derive it from our previous identification of Specht module and $GL_d - GL_n$ duality. Another such approach of deriving Schur Weyl duality from $GL_d - GL_n$ duality is laid well in [8]; in fact, these two are equivalent, as we will see later. We will also deduce the surjectivity of the map discussed above, as a corollory.

Theorem 3.3.2. (**Schur-Weyl Duality, or abbreviated, SWD**)
As an $S_d \times GL_n$ representation, $(\mathbb{C}^n)^{\otimes d} \cong \bigoplus_{|\lambda|=d, l(\lambda)\leq n} Sp_\lambda \otimes V_\lambda(n)$.

Proof. Take our old acquaitance, the (GL_d, GL_n) duality

$$Sym^d(\mathbb{C}^d \otimes \mathbb{C}^n) \cong \bigoplus_{l(\lambda)\leq min\{d,n\}} V_\lambda(d) \otimes V_\lambda(n)$$

and pick up the (1^d) weight space of both side under left GL_d action.
As before, inside $Sym^d(\mathbb{C}^d \otimes \mathbb{C}^n)$, this weight space has the basis $\{e_{\alpha(1)} \otimes \hat{e}_{\beta(1)} \cdots e_{\alpha_n} \otimes \hat{e}_{\beta(n)} | \alpha \in S_n, \beta : [d] \to [n]\}$, or equivalently $\{e_\gamma := e_1 \otimes \hat{e}_{\gamma(1)} \cdots e_n \otimes \hat{e}_{\gamma(n)} | \gamma : [d] \to [n]\}$. All we need to note is that the map $e_\gamma \mapsto \otimes_{i=1}^d e_{\gamma(i)}$ gives rise to an $S_d \times GL_n$-intertwiner isomorphism between the sought weight space and $V^{\otimes d}$; one just needs to calculate, as before,

$$\begin{aligned}(\pi, g) \cdot e_\gamma &= \pi e_1 \otimes g\hat{e}_{\gamma(1)} \cdots \pi e_d \otimes g\hat{e}_{\gamma(d)} \\ &= e_1 \otimes g\hat{e}_{\gamma\pi^{-1}(1)} \cdots e_d \otimes g\hat{e}_{\gamma\pi^{-1}(d)} \\ &\mapsto \otimes_{i=1}^d g\hat{e}_{\gamma\pi^{-1}(i)} \\ &= (\pi, g) \cdot \otimes_{i=1}^d e_{\gamma(i)}\end{aligned}$$

The last equality follows becuase if we write $\otimes_{i=1}^d e_{\gamma(i)} =: \otimes v_i$, then $\otimes v_{\pi^{-1}(i)} = \otimes e_{\gamma\pi^{-1}(i)}$. On the right hand side, picking up (1^d) weight space of under left GL_d action we finally have,

$$\begin{aligned}V^{\otimes d} &\cong \bigoplus_{l(\lambda)\leq min\{d,n\}} (V_\lambda(d))_{(1^d)} \otimes V_\lambda(n) \\ &\cong \bigoplus_{\lambda\vdash d, l(\lambda)\leq n} Sp_\lambda \otimes V_\lambda(n) \\ &\cong \bigoplus_{\lambda\vdash d, l(\lambda)\leq n} Sp_\lambda \otimes V_\lambda(n)\end{aligned}$$

□

As pointed out earlier, we can now rederive the (GL_d, GL_n) duality from SWD.

Corollary 3.3.3. $Sym^k(V \otimes W) \cong \bigoplus_{|\lambda|=k, l(\lambda)\leq min\{d,n\}} V_\lambda \otimes W_\lambda$ as $GL(V) \times GL(W)$ representation, where $V = \mathbb{C}^d, W = \mathbb{C}^n$.

Proof. Note that as a $GL_d \times GL_n$ module,

$$\begin{aligned}Sym^k(V \otimes W) &\cong ((V \otimes W)^{\otimes k})^{S_k} \\ &\cong (V^{\otimes k} \otimes W^{\otimes k})^{\triangle S_k} \\ &\cong (\oplus_{|\lambda|=k, l(\lambda)\leq d} V_\lambda \otimes Sp_\lambda \otimes \oplus_{|\mu|=k, l(\mu)\leq n} V_\mu \otimes Sp_\mu)^{\triangle S_k} \\ &\cong (\oplus_{\lambda,\mu} V_\lambda \otimes W_\mu \otimes (Sp_\lambda \otimes Sp_\mu))^{\triangle S_k} \\ &\cong \oplus_{\lambda,\mu} V_\lambda \otimes W_\mu \otimes (Sp_\lambda \otimes Sp_\mu)^{\triangle S_k} \\ &\cong \oplus_{\lambda,\mu} V_\lambda \otimes W_\mu \otimes Hom_{S_k}(Sp_\lambda, Sp_\mu) \\ &\cong \oplus_{\lambda, |\lambda|=k, l(\lambda)\leq d,n} V_\lambda \otimes W_\lambda\end{aligned}$$

Here we have used SWD in the third isomorphism, self duality of Sp_λ's in the penultimate isomorphism and Schur's lemma in the last isomorphism; also, when we factor $(V \otimes W)^{\otimes k}$ into the tensor product of $V^{\otimes k}$ and $W^{\otimes k}$, we see that the product $S_k \times S_k$ acts on this space. Our original copy of S_k is identified with the diagonal subgroup $\triangle S_k$ of $S_k \times S_k$; this is the meaning of the notation $\triangle S_k$. □

Remark 3.3.4. Counting the dimension of both side of the decomposition yield for us, when $n \geq d$

$$n^d = \bigoplus_{|\lambda|=d} f_\lambda dim V_\lambda = \bigoplus_{|\lambda|=d} |SYT(\lambda)||SSYT(\lambda,\ entry\ \in [n])|$$

This numerical identity, a priori, is a hint towards the duality and can alternatively be proved as a consequence of our favorite RSK correspondence: LHS is cardinality of the set of all $d \times n$ integer matrices all having row sum 1, and each of them corresponds to a pair of tableaux of same shape, the first one of which is standard and the second one semistandard and is filled with entries in $[n]$. In fact RSK directly yields the so-called Young's rule for decomposing permutation representation of symmetric group originating from its action on set partitions, which in turn proves Frobenius' formula (see later) and hence the SWD, see [21], Chapter 3.

Remark 3.3.5. Note that SWD generalizes the usual decomposition $V \otimes V \cong Sym^2 V \oplus Alt^2 V$; it is no longer true that third (and higher) tensor power of V admits such simple decomposition into two pieces: for example, just a dimension check on both the sides would reveal that. Schur-Weyl duality supplies the missing pieces to make the decomposition correct.

Remark 3.3.6. SWD can often be used to make constructions "natural". For example, we know from symmetric function theory that there is an algebra isomorphism $\omega : \Lambda \to \Lambda$, which takes s_λ to $s_{\lambda'}$. Is there a functor on the category of GL_n-representations which realizes it?
Fix $d \leq n$, V is standard representation. Let $\mathcal{C}$ be the category of polynomial GL_n representations, where $t \cdot Id$ acts by t^d. Then we define a functor $\mathcal{C} \to \mathcal{C}$ by:

$$W \longrightarrow Hom_{S_d}(Hom_{GL_n}(W, V^{\otimes d}) \otimes Sgn, V^{\otimes d}).$$

Thus functor takes representations with character f to representations with character $\omega(f)$. Note that it is really a contrived way to go from V_λ to $V_{\lambda'}$, we have already seen a more direct way to do so at the end of Section 1.4.

In general, when $d \leq n$, Schur-Weyl duality is an equivalence of categories between

$$\{\text{polynomial representations of } GL_n \text{ on which } t \cdot Id \text{ acts by } t^d\}$$

and

$$\{S_d \text{ representations}\}$$

Note that this generalizes the 'restriction to all one weight space' functor; the correspondence here are $W \mapsto \mathrm{Hom}_{GL(V)}(V^{\otimes d}, W)$ and $W \mapsto \mathrm{Hom}_{S_d}(V^{\otimes d}, W)$.

3.4 Consequences of Schur-Weyl Duality

3.4.1 Character Theoretic Considerations

Let us compute the trace of a generic element $diag(x_1, \cdots, x_n) \times w_\mu$ on both side of this decomposition, where for w_μ we henceforth take the standard element $(1, \cdots, \mu_1)(\mu_1 + 1, \cdots, \mu_2) \cdots (\mu_{t-1} + 1, \cdots, \mu_t)$ with cycle decomposition type $\mu = (\mu_1, \cdots, \mu_t)$. On the left hand side take the usual basis of simple tensors and consider the action on a simple tensor $e_{i_1} \otimes \cdots \otimes e_{i_d}$. The effect of applying $x = diag(x_1, \cdots, x_n)$ is simply to multiply the tensor by $x_{i_1} \cdots x_{i_d}$; the action of $w = w_\mu$ transforms this x-eigenvector to $e_{i_{w(1)}} \otimes \cdots \otimes e_{i_{w(d)}}$. Therefore the only basis vectors that contribute to the trace are those for which the first μ_1 tensor factors are the same, the next μ_2 tensor factors are the same, and so forth, and in such case the contribution is a term of the form $x_{j_1}^{\mu_1} x_{j_2}^{\mu_2} \cdots x_{j_t}^{\mu_t}$, where $1 \leq j_1, \cdots, j_t \leq n$. Thus taking sum of all such terms and recalling the definition of *power sum symmetric function*, e.g. $p_m(x_1, \cdots, x_n) = x_1^m + \cdots + x_n^m$ and $p_\mu(x_1, \cdots, x_n) = p_{\mu_1}(x_1, \cdots, x_n) \cdots p_{\mu_t}(x_1, \cdots, x_n)$, we get that $trace(diag(x_1, \cdots, x_n) \times w_\mu; V^{\otimes d} = p_\mu(x_1, \cdots, x_n)$. Computing the trace of $diag(x_1, \cdots, x_n) \times w_\mu$ on the right hand side we conclude that

$$p_\mu(x_1, \cdots, x_n) = \bigoplus\nolimits_{\lambda \vdash d, l(\lambda) \leq n} \chi_\lambda(w_\mu) s_\lambda(x_1, \cdots, x_n)$$

This is the classical *Frobenius Character Formula*, which states that *the transition matrix between power sum symmetric function p_μ's and Schur function s_λ's(with $|\lambda| = d = |\mu|$), the two bases of the ring of symmetric function is the character table of the symmetric group S_d.* Thus **Frobenius formula is the character theoretic incarnation of Schur-Weyl duality** and therefore, is equivalent to it due to the slogan *"charater determines representation".*

Frobenius formula can further be used to prove the famous *hook length formula* for $f_\lambda := dim Sp_\lambda$, which says that

$$f_\lambda = \frac{n!}{\prod_{x \in \lambda} h_x}$$

where h_x is the hook length of the cell x in the diagram for the partition λ of n; hook length of the cell $x =$ nummber of cells strictly to the right of $x+$ number of cells strictly below $x+1$. For example, each of the cells in the following tableau is filled with its hook length.

6	4	2	1
3	1		
1			

See [5] or [21] for related discussions, also the book [11] contains some interesting history of the discovery of this beautiful formula. In fact, it is well known and can be easily proved that this is equivalent to all the other description of the character of the symmetric group, e.g. the recursive Murghnahan-Nakayama rule; see, for example, [3], Chapter 1, Ex. 3.11. We would like to note another consequence, relatively less well known, of the Frobenius formula: the hook length formula for dim V_λ due to Frobenius, see [9]. In [4], Diaconis and Greene showed that this can be proved using the hook length formula for the symmetric group and a property of certain very important (in studying representation theory) elements in the group algebra of the symmetric group, ubiquitously known as the *Young Jucys Murphy* elements. These are defined as follows:

$$X_1 := 0, X_i := (1\ i) + (2\ i) + \cdots + (i-1\ i), \forall\ i = 2, \ldots, n$$

These elements inside $\mathbb{C}[S_n]$ generate the algebra $GZ(n)$ of operators diagonal in the Gelfand Tsetlin basis of all the irreps of S_n(yes, each of Sp_λ possess GT basis! And, they are indexed, naturally enough, by $SYT(\lambda)$: all these follow from the the sections 1.5 and 3.1), where we keep in mind that $\mathbb{C}[S_n] \cong \oplus_{\lambda \vdash n} End\ Sp_\lambda$; therefore, they act on each Sp_λ by scalar, and on the GT basis for Sp_λ (which is, by previous assertion, an eigenbasis of the X_i's in Sp_λ) their action is given by $X_i \cdot v_T = c_i v_T$. Here c_i = column number of the cell in T containing $i-$ row number of the cell in T containing i. Also, for each cell x in the diagram of a partition λ, we define the content c_x of cell x to be = column number of $x-$row number of cell x For instance, each of the cells in the next tableau is filled with its content.

0	1	2	3
a	0		
b			

Here $a = -1, b = -2$. For proofs of these assertion about YJM elements and related further discussions, see [10].

Assume $n = d$. Let us start by noting that the orthogonality of the irreducible characters of symmetric group allows us to transform the Frobenius formula to the following form; see [23], page 48.

$$s_\lambda(x_1, \cdots, x_n) = \Sigma_{\mu \vdash n} \frac{1}{z_\mu} \chi_\lambda(w_\mu) p_\mu(x_1, \cdots, x_n)$$

where z_μ=size of the centralizer of w_μ in S_n; actually it is this form of Frobenius formula that comes to use while proving Murnaghan-Nakayama rule, see [23], page $79-83$. From this, we know that

$$dim\ V_\lambda = s_\lambda(1,\cdots,1) = \Sigma_{\mu\vdash n}\frac{1}{z_\mu}\chi_\lambda(w_\mu)n^{l(\mu)} = \frac{1}{n!}\Sigma_{\pi\in S_n}\chi_\lambda(w_\pi)n^{l(\pi)}$$

where $l(\mu)$ denotes as before the number of parts of the partition μ, and $l(\pi)$ denotes the similar thing for the partition of cycle type of π. Frobenius' hook length formula claims the following.

Theorem 3.4.1.

$$dim\ V_\lambda = \prod_{x\in\lambda}\frac{n+c_x}{h_x}$$

The authors in [4] derives this from the following proposition, which is the central ingredient of the paper for deriving different character formulas.

Lemma 3.4.2. Suppose q is indeterminate. We have the following equality in the group algebra of the symmetric group

$$\prod_{2\leq i\leq n}(I+qY_i) = \Sigma_{\pi\in S_n}q^{n-l(\pi)}\pi$$

The proof of this lemma proceeds straightforwardly by induction on n, into which we will not delve; substituting $q=1/n$ yields the for us the useful formula

$$\Sigma_{\pi\in S_n}n^{l(\pi)}\pi = n\prod_{2\leq i\leq n}(nI+Y_i)$$

Take traces corresponding to the λth representation(i.e. Sp_λ) on both side of the equation. We obtain on the left $\Sigma_{\pi\in S_n}\chi_\lambda(w_\pi)n^{l(\pi)} = n!s_\lambda(1,\cdots,1)$. Since the YJM elements act by scalars on any representation space, in the λth representation the right hand side expression acts as a scalar matrix, whence the trace is f_λ times the $(1,1)$th entry of the matrix, which equals $f_\lambda n\prod_{2\leq i\leq n}(n+c_i)$, c_i here being the content of the cell containing i in the first(or any other) SYT on shape λ. Using the formula for f_λ, it finally boils down to $\prod_{x\in\lambda}\frac{n!}{h_x}\prod_{x\in\lambda}(m+c_x)$. Comparing the two sides, we get the result.□
Historically speaking, this result was discovered much before its counterpart for the symmetric group!

Remark 3.4.3. Take Frobenius' formula and use the expression for Schur polynomials $s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$, where $\delta = (n, n-1,\cdots,1,0)$ and $a_\mu = det(x_i^{\mu_j})$. Then with little observation one can conclude that *the character value* $\chi_\lambda(w_\mu)$*=coefficient of* $x^{\lambda+\delta}$ *in* $a_\delta p_\mu$. It is important to note, e.g. found in [14], Lecture 4, that appropriate generalization of this result holds for any connected complex reductive algebraic groups, in the form of *Generalized Frobenius Formula*; the result, quite expectedly, uses the Weyl's formula for the group's character, much like what we did here for the GL_n character s_λ.

Remark 3.4.4. A special case of the Frobenius formula for $\mu = (1^d)$ i.e. $w_\mu = 1$ yield another well known symmetric function identity, which can be proved using RSK correspondence in the same line as remark 3.3.3.

$$(x_1 + \cdots + x_n)^d = \bigoplus_{|\lambda|=d, l(\lambda)\leq n} f_\lambda s_\lambda(x_1, \cdots, x_n)$$

3.4.2 Invariant Theoretic Considerations

Schur-Weyl duality can be used to prove interesting results in invariant theory, which we do now. Let $V = \mathbb{C}^n$, and consider the space of invariants $(V^{\otimes m} \otimes V^{*\otimes k})^{GL(V)}$, where V has the defining representation. Since for any scalar $\xi \in \mathbb{C}^*$, $\xi I \in GL(V)$ acts by ξ^{m-k}, there are no nonzero invariant tensors unless $m = k$. Hence we assume $k = m$, and then $V^{\otimes m} \otimes V^{*\otimes m} \cong V^{\otimes m} \otimes (V^{\otimes m})^* \cong End(V^{\otimes m})$ as $GL(V)$ representation. The first isomorphism results from $V^{*\otimes m} \cong (V^{\otimes m})^*$, being given by $f_1 \otimes \cdots \otimes f_m \mapsto (v_1 \otimes \cdots \otimes v_m \mapsto f_1(v_1)f_2(v_2)\cdots f_m(v_m))$ and the second isomorphism arises from the map given by

$$\otimes_{i=1}^m v_i \otimes_{i=1}^m f_i \mapsto (\otimes_{i=1}^m w_i \mapsto \prod_{i=1}^m f_i(w_i) \otimes_{i=1}^m v_i).$$

Therefore we are actually asking for the description of the centralizer algebra $End_{GL(V)}V^{\otimes m}$. As we pointed out earlier, we will shortly prove Schur's result that $End_{GL(V)}V^{\otimes m} \cong kS_m$, when $n \geq k$, by using Schur-Weyl duality (which is exactly opposite to what Schur did), and in general the previously descibed map $kS_m \to End_{GL(V)}V^{\otimes m}$ is surjective. In particular, we can produce an explicit basis of invariants as a corollary of this. Let $e_1, \cdots, e_n$ be the standard basis for V and $e_1^*, \cdots, e_n^*$ be the dual basis for V^*. For a tuple $I = (i_1, \cdots, i_m) \in [n]^m$, set $e_I = e_{i_1} \otimes \cdots \otimes e_{i_m}$ and $e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_m}^*$. Recall that the action of S_m on m-tensors is given by $\sigma_m(\pi) \cdot e_I = e_{\pi \cdot I}$. Thus if we define $C_\pi = \Sigma_{|I|=m} e_{\pi\cdot I} \otimes e_I^*$, under the isomorphism of the invariants with the centralizer algebra, C_π corresponds to $\sigma_m(\pi)$:

$$C_\pi(e_J) = \Sigma_{|I|=m} e_I^*(e_J) e_{\pi\cdot I} = e_{\pi\cdot J} = \sigma_m(\pi) e_J$$

where $e_I^*(e_J) := \prod_{\alpha=1}^m e_{i_\alpha}^*(e_{j_\alpha})$.
Therefore by our previous assertion of the surjectivity, the following classical result follows.

Theorem 3.4.5. (**First Fundamental Theorem for GL_n, tensor invariants version**) For $m \geq 1, (V^{\otimes m} \otimes V^{*\otimes m})^{GL(V)} = span\{C_\pi : \pi \in S_m\}$, and the latter collection is a basis, when we are in the stable range $n \geq m$.

Since the vector space $V^{\otimes m} \otimes V^{*\otimes m}$ is self dual as $GL(V)$ representation, each of the mixed tensors C_π can also be viewed as a linear functional on $V^{\otimes m)} \otimes V^{*\otimes m}$ which are $GL(V)$-fixed. Since $End_{GL(V)}V^{\otimes m} \cong Hom_{GL(V)}(V^{\otimes m}, Hom(V^{*\otimes m}, \mathbb{C})) \cong Hom_{GL(V)}(V^{\otimes m} \otimes V^{*\otimes m}, \mathbb{C})$, keeping track of identification at each stage we have an alternate version of tensor FFT in terms of what are called *total contractions*.

Corollary 3.4.6. (**First Fundamental Theorem for GL_n, invariant forms version**) The space of $GL(V)$ invariant linear forms on $V^{\otimes m} \otimes V^{*\otimes m}$ is spanned by the contractions of vectors with covectors

$$v_1 \otimes v_2 \otimes \cdots \otimes v_m \otimes v_1^* \otimes v_2^* \otimes \cdots \otimes v_m^* \mapsto \prod_{1 \leq i \leq m} v_{\pi(i)}^*(v_i).$$

Before going anywhere further, we should prove our standing assumption, from which we deduced these results.

Theorem 3.4.7. The map $\mathbb{C}[S_d] \to End_{GL(V)} V^{\otimes d}$ is always surjective, and is an isomorphism when $d \leq n = dim\ V$.

Proof. We know from SWD that, as vector spaces we have

$$\begin{aligned} End_{GL(V)} V^{\otimes d} &\cong End_{GL(V)}(\oplus_{\lambda \vdash d, l(\lambda) \leq n} Sp_\lambda \otimes V_\lambda(n)) \\ &\cong \oplus_{\lambda \vdash d, l(\lambda) \leq n} End_{GL(V)}(Sp_\lambda \otimes V_\lambda(n)) \\ &\cong \oplus_{\lambda \vdash d, l(\lambda) \leq n} End\ Sp_\lambda. \end{aligned}$$

But the we already know that $\mathbb{C}[S_d]$ spans $\oplus_{\lambda \vdash n} End\ Sp_\lambda$, so our map is surjective: some of the Sp_λ might not appear on the RHS(the ones with $l(\lambda) \geq n$), but the ones that appear do so once. Also, when $d \leq n$, everyone shows up in the RHS, so their dimension matches and hence surjectivity implies isomorphism. □

Next we discuss about GL_n invariant polynomials. First, some generalities, that applies to any reductive linear algebraic group; take G any group and V be any finite dimensional representation, although the arguments are valid for regular representation, see [14]. Then G acts as an automorphism group of the commutative algebra $\mathcal{P}(V)$ of complex valued polynomial functions on V: $g \cdot f(v) = f(g^{-1} \cdot v)$. Since G acts by automorphisms, the space $\mathcal{J} = \mathcal{P}(V)^G$ of G-invariants is a subalgebra of $\mathcal{P}(V)$. The basic result in this regard is due to Hilbert, it asserts that $\mathcal{J}$ *is finitely generated as an algebra over* $\mathbb{C}$. We say that $\{\phi_1, \cdots, \phi_n\}$ is a set of *basic invariants* if
(i) $\{\phi_1, \cdots, \phi_n\}$ *generates* $\mathcal{J}$ *as an algebra over* $\mathbb{C}$.
(ii) *each* ϕ_i *is homogenous of some degree* d_i, with n as small as possible, subject to $(i), (ii)$.
By this assertion, there always exists a basic set of invariants, the polynomials are not unique but their degrees are uniquely determined.
Now fix $G = GL_n$ and take V to be its defining representation. Let G act on $V^{\oplus m}$ and $V^{*\oplus k}$ by its natural action on each summand. Then $\mathcal{P}(V^{*\oplus k} \oplus V^{\oplus m})^G$ is the algebra of GL_n invariant polynomial functions of k covectors and m vectors. Involving other general linear groups, we can furnish an obvious algebra of G invariant polynomials together with a set of quadratic generators. Notice that, if we arrange the m copies of V in a column

of matrices and similarly arrange the k copies of V^* in rows of matrices, then we have G isomorphisms $V^{\oplus m} \cong M_{n\times m}, V^{*\oplus k} \cong M_{k\times n}$, where the action on these matrices are left multiplication by g and right multiplication by g^{-1} respectively, for $g \in G$. In this picture we see that $GL_k \times GL_m$ acts on $M_{k\times n} \oplus M_{n\times m}$ by

$$(a, b) \cdot (x \oplus y) := ax \oplus yb^{-1}$$

Since this action commutes with the G action, the induced action on functions make $\mathcal{P}(M_{k\times n} \oplus M_{n\times m})^G$ into a representation of $GL_k \times GL_m$.
Define the multiplication map

$$\mu : M_{k\times n} \oplus M_{n\times m} \to M_{k\times m}, x \oplus y \mapsto xy$$

Obviously $\mu(xg^{-1} \oplus gy) = \mu(x \oplus y)\ \forall g \in G$, so we have the pullback as an algebra homomorphism

$$\mu^* : \mathcal{P}(M_{k\times m}) \to \mathcal{P}(V^{*\oplus k} \oplus V^{\oplus m})^G, \mu^*(f)(x \oplus y) = f(xy)$$

In particular taking $f = x_{ij}$, i.e. the function on matrices in $M_{k\times m}$ which picks out the (i, j)th matrix entry we get

$$\mu^*(x_{ij})(v_1^*, \cdots, v_k^*, v_1, \cdots, v_m) = v_i^*(v_j)$$

The polynomial FFT is the assertion that the method just described to construct invariants furnishes the full algebra of polynomial invariants.

Theorem 3.4.8. (**First Fundamental Theorem for GL_n, polynomial invariants version**) μ^* is surjective, whence the km quadratic polynomials $\phi_{ij} = \mu^*(x_{ij})$ produces a set of basic invariants for $\mathcal{P}(V^{*\oplus k} \oplus V^{\oplus m})^{GL(V)}$

The proof follows from the invariant forms version of FFT from the corollary, see [14], section 7 for the detailed proof and answer to similar questions in case of orthogonal and symplectic groups.

3.4.3 An Interesting Map

Recall that when $d \leq n$, Schur-Weyl duality sets up an equivalence of the category of S_d representation and that of GL_n polynomial representation of degree d. Every element in the first category has a character, which is a class function on S_d; every element in the later category also has the notion of character which is a symmetric function of degree d in n variables. Therefore taking characters of both the sides (and linearly extending) furnishes a linear map

$$\mathcal{F} = \mathcal{F}_n^{(d)} \colon \{\text{Class functions on } S_d\} \to \bigwedge\nolimits_n^{(d)}$$

This map is called the ***Frobenius Characteristic Map***. By construction, it sends χ_λ to $s_\lambda(x_1, \cdots, x_n)$, therefore automatically we get the general formula:

$$\mathcal{F}(\xi) = \frac{1}{d!}\Sigma_{\sigma\in S_d}\xi(\sigma)p_\sigma(x_1, \cdots, x_n)$$

Reason: in order to show that this is the correct form, it suffices to check this for a basis of class functions on S_d, e.g. on the irreducible characters χ_λ's, and the alternative form of Frobenius' character formula validates this.
Since $\mathcal{F}$ sends a collection of basis vectors of class functions on S_d to a collection of basis vectors of $\bigwedge_n^{(d)}$, this is clearly an isomorphism of vector spaces. Moreover, since both the domain and the target is equipped with inner product and the basis vectors noted earlier are orthogonal, i.e. $\langle\chi_\lambda, \chi_\mu\rangle_{CF_d} = \delta_{\lambda\mu} = \langle s_\lambda, s_\mu\rangle_{Hall}$, $\mathcal{F}$ **is an isometric isomorphism**: $\langle\xi, \eta\rangle_{CF_d} = \langle\mathcal{F}(\xi), \mathcal{F}(\eta)\rangle_{Hall}$, for any class functions ξ, η. For a closed form of the inverse map involving supersymmetric function, the reader is referred to [26].
All this is valid in the stable range $d \le n$; however, when $d > n$, the Schur modules V_λ for $\lambda \vdash d, l(\lambda) > n$ are zero (as we saw in our explicit constructions as well); equivalently, for such λ, $s_\lambda = 0$. Therefore although $\mathcal{F}$ is still surjective in this case, it has a nontrivial kernel $Span\{\chi_\lambda : \lambda \vdash d, l(\lambda) > n\}$.
Geissinger gave a representation theoretic interpretation of the bialgera $\bigwedge$ of symmetric functions, which we discuss now. Naturally, we have to combine our previous discussions $for\ all\ d\ and\ n$, in the following way. For a general finite group G, let $X(G) = \bigoplus_{\chi\in\hat{G}}\mathbb{Z}\chi$, the additive group of generalized characters, which is also isomorphic to the free Abelian group on the irreducible isomorphism classes of G. $X(G)$ comes equipped with a ring structure and is called the ring of generalized characters, but we will not be using that at all. Instead, for $G = S_d$, denote $\mathcal{R}_d = X(S_d)$ and set $\mathcal{R} = \bigoplus \mathcal{R}_d$; construct $\mathcal{F}_n : \mathcal{R} \to \bigwedge_n$ by letting $\mathcal{F}_n = \mathcal{F}_n^{(d)}$ on $\mathcal{R}_d$. Note that $\bigwedge_n$ is a graded ring (graded by $\mathbb{N}$) under usual multiplication of polynomials, meaning that $\bigwedge_n^{(k)} \times \bigwedge_n^{(l)} \subset \bigwedge_n^{(k+l)}$. A natural question at this point is: can we endow $\mathcal{R}$ with a "multiplication" so that it becomes a graded ring with this multiplication and $\mathcal{F}_n$ is furthermore a homomorphism of graded ring? The answer is in the affirmative, and is provided in the next theorem.

Theorem 3.4.9. The map $\mathcal{F}_n$ is a surjective homomorphism of $\mathbb{N}$-graded rings, where the graded ring structure of the domain comes from the following multiplication: if θ, ρ are representation of S_k and S_l respectively, then define $\theta \circ \rho = Ind_{S_k\times S_l}^{S_{k+l}}\theta\otimes\rho$ and then extend $\mathbb{Z}$-linearly.

Proof. Notice that this binary operation is associative, for induction is transitive, and since the subgroup $S_k \times S_l$ is conjugate to $S_l \times S_k$ in S_{k+l}, this operation is also commutative. Therefore the only thimg to check is that this "induction product" in $\mathcal{R}$ corresponds to multiplication of polynomials. It suffices to check this only on the irreducible characters and see where the Schur functor sends it to. Thus for $\lambda \vdash k, \mu \vdash l, l(\lambda) \leq n, l(\mu) \leq n$,

$$\begin{aligned} Sp_\lambda \circ Sp_\mu &= \mathbb{C}[S_{k+l}] \otimes_{\mathbb{C}[S_k \times S_l]} (Sp_\lambda \otimes Sp_\mu) \\ &\mapsto V^{\otimes k+l} \otimes_{\mathbb{C}[S_{k+l}]} \mathbb{C}[S_{k+l}] \otimes_{\mathbb{C}[S_k \times S_l]} (Sp_\lambda \otimes Sp_\mu) \\ &\cong V^{\otimes k+l} \otimes_{\mathbb{C}[S_k] \otimes \mathbb{C}[S_l]} (Sp_\lambda \otimes Sp_\mu) \\ &\cong V^{\otimes k} \otimes_{\mathbb{C}[S_k]} Sp_\lambda \otimes V^{\otimes l} \otimes_{\mathbb{C}[S_l]} Sp_\mu \\ &\cong V_\lambda \otimes V_\mu \end{aligned}$$

and this last representation of $GL(V)$ has character $s_\lambda s_\mu$. □

Next, the rings $\bigwedge_n$ have to be combined in the following way. We have an evaluation homomorphism for every $n \in \mathbb{N}$

$$r_n : \bigwedge_{n+1} \to \bigwedge_n ; x_{n+1} \mapsto 0, x_i \mapsto x_i \forall\ i \in [n]$$

and each of the elements of this collection are compatible with the homomorphisms $\mathcal{F}_n : \mathcal{R} \to \bigwedge_n$, meaning that $r_n \mathcal{F}_{n+1} = \mathcal{F}_n$; the last claim can be checked on the generators, see [16], Chapter 34. Thus, $((\bigwedge_i)_{i\in\mathbb{N}}, (r_{ij})_{i\leq j\in\mathbb{N}})$, where $r_{ij} := r_{j-1}r_{j-2}\cdots r_i$, is an example of an *inverse system*, so if we take the inverse limit $\bigwedge := \varprojlim \bigwedge_n = \{\hat{f} \in \prod_{i\in\mathbb{N}} \bigwedge_i | f_i = r_{ij}(f_j) \forall i \leq j\}$, there is an induced ring homomorphism $\hat{\mathcal{F}} : \mathcal{R} \to \bigwedge$. But $\bigwedge$ is precisely the ring of symmetric functions, so we obtain the following from the last theorem.

Theorem 3.4.10. $\hat{\mathcal{F}} : \mathcal{R} \to \bigwedge$ is an algebra isomorphism.

Chapter 4

Perspectives on Gelfand Model and Some Computations

In this chapter, we will see how we can exploit Schur-Weyl duality, a bridge connecting the representation theories of symmetric group and general linear group, to transport information between these two worlds, which can in turn shed new lights on the work of Klyachko et al. [31] on Gelfand model of symmetric group; we will also see some explicit computations, answering the question raised in Chapter 3, section 2.

4.1 Multiplicity-Free Sum of GL_n Polyreps-II

Theorem 4.1.1. Let $V = \mathbb{C}^n$. Then as a $GL(V)$ representation,

$$Sym(V \oplus \wedge^2 V) \cong \bigoplus_{\lambda, l(\lambda) \leq n} V_\lambda(n)$$

Also,

$$Sym^k(V \oplus \wedge^2 V) \cong \bigoplus_{\lambda, l(\lambda) \leq n, |\lambda| + o(\lambda) = 2k} V_\lambda(n)$$

where we denote the number of odd length columns of the diagram of λ by $o(\lambda)$.

Proof. Notice that the first claim follows from the second one, since $\lambda \equiv o(\lambda) \pmod 2$. Now $Sym^k(V \oplus \wedge^2 V) = \bigoplus_{i+j=k} Sym^i\ V \otimes Sym^j(\wedge^2 V)$; we want to get hold of the character of LHS of the second equation, so just take the obvious basis $\{e_{l_1} e_{l_2} \dots e_{l_i} \otimes e_{\alpha_1} \wedge e_{\beta_1} \dots e_{\alpha_j} \wedge e_{\beta_j} : i + j = k, \alpha_1 \leq \dots \leq \alpha_m, \beta_1 \leq \dots \leq \beta_m, \alpha_m \neq \beta_m \forall m\}$ and compute the trace of action of a generic element $diag(t_1, \dots, t_n)$ on this. Fix i, j and consider the direct summand $Sym^i\ V \otimes Sym^j(\wedge^2 V)$. On a basis vector $e_{l_1} e_{l_2} \dots e_{l_i} \otimes e_{\alpha_1} \wedge e_{\beta_1} \dots e_{\alpha_j} \wedge e_{\beta_j}$, the action of $diag(t_1, \dots, t_n)$ contributes $t_{l_1} \dots t_{l_i} t_{\alpha_1} t_{\beta_1} \dots t_{\alpha_j} t_{\beta_j}$ to the total trace. Record this data in a symmetric matrix (a_{mn}), where for $m < n$, $a_{mn} = |\{p \in [j] : (\alpha_p, \beta_p) = (m, n)\}|$ and $a_{mm} = |\{q \in [i] : l_q = m\}|$ and notice that such matrices have trace i. Now the coefficient of $t^\mu = t_1^{\mu_1} \dots t_n^{\mu_n}$ in the trace of the left hand side is the number of solutions of the equations, obtained by equating the exponents in $t_{l_1} \dots t_{l_i} t_{\alpha_1} t_{\beta_1} \dots t_{\alpha_j} t_{\beta_j} = t_1^{\mu_1} \dots t_n^{\mu_n}$, or in other words *the number of symmetric ('recording') matrices of with rowsum* μ *(i.e. sum of entries of* i*-th row*$=\lambda_i$*) and trace* i. Now for a fixed i, the coeffcient of t^μ in the trace of $diag(t_1, \dots, t_n)$'s action on $\bigoplus_{o(\lambda)=i,|\lambda|+o(\lambda)=2k} V_\lambda(n)$ is $\sum_{o(\lambda)=i,|\lambda|+o(\lambda)=2k} K_{\lambda\mu}$, (since $s_\lambda = \sum K_{\lambda\mu} m_\mu$) i.e. the number of SSYT with $(shape, type) = (\lambda, \mu)$ such that $o(\lambda) = i, |\lambda| + o(\lambda) = 2k$. We want an bijection of certain number of specific matrices with these tableaux and therefore RSK correspondence comes into play! The symmetry property of RSK correspondence tells that if $A \in IntegerMatrices \mapsto (P, Q) \in SSYT(\lambda) \times SSYT(\lambda)$, then $A^t \mapsto (Q, P)$, therefore RSK induces a bijection between symmetric matrices having specified row sum μ and SSYT's of type μ. Furthermore, by a result of Schùtzenberger, *the matrices with fixed trace* i *corresponds to exactly the tableaux having* i *odd columns in this circumstance.* Combining all this with varying i, j such that $i + j = k$ proves our claim. □

Remark 4.1.2. A direct proof for the first decomposition falls out of, as always, an identity involving symmetric functions:

$$\prod_{i=1}^n \frac{1}{1-x_i} \prod_{1\le i<j\le n} \frac{1}{1-x_i x_j} = \sum_{\lambda, l(\lambda)\le n} s_\lambda(x_1, \dots, x_n)$$

This proves the decomposition, because $Sym(V \oplus \wedge^2 V) = Sym\ V \otimes Sym(\wedge^2 V) = (\oplus_{k\in\mathbb{N}} Sym^k\ V) \otimes (\oplus_{l\in\mathbb{N}} Sym^l(\wedge^2 V))$ has character

$$\sum_{k\in\mathbb{N}} h_k(x_1, \dots, x_n)) \cdot (\sum_{l\in\mathbb{N}} h_l(x_1x_2, \dots, x_1x_n, x_2x_3, \dots\dots, x_{n-1}x_n)) = \prod_{i=1}^n \frac{1}{1-x_i} \prod_{1\le i<j\le n} \frac{1}{1-x_i x_j}$$

To prove the identity one again uses the RSK correspondence; multiplying out terms in the LHS, one sees that a typical term looks like $\prod_{i=1}^n x_i^{m_i} \prod_{1\le i<j\le n} (x_i x_j)^{m_{ij}}$, therefore the coefficient of x^μ is the number of solutions of the system of linear equations $m_i + \sum_{i<j} m_{ij} + \sum_{j<i} m_{ji} = \mu_i, \forall i \in [l(\mu)]$, i.e. the number of symmetric matrices with row sum μ, which is equinumerous with the number of SSYT with type μ and entries filled from $[n]$ i.e. $\sum_\lambda K_{\lambda\mu}$, the coefficient of x^μ in the RHS.
In fact the second claim also follows directly from the following identity, see [1],

$$\prod_{i=1}^n \frac{1}{1-qx_i} \prod_{1\le i<j\le n} \frac{1}{1-x_i x_j} = \sum_{\lambda, l(\lambda)\le n} q^{o(\lambda)} s_\lambda(x_1, \dots, x_n)$$

Remark 4.1.3. $\wedge V \otimes Sym(Sym^2 V)$ is also a multiplicity-free direct sum of the Schur modules for $GL(V)$; note that its character is

$$(\sum_{k=1}^n e_k(x_1, \dots, x_n)) \cdot (\sum_{l\in\mathbb{N}} h_l(x_1^2, x_1x_2, \dots, x_2^2, x_2x_3, \dots\dots, x_{n-1}x_n, x_n^2)) = \prod_{i=1}^n (1+x_i) \prod_{1\le i\le j\le n} \frac{1}{1-x_i x_j} = \prod_{i=1}^n \frac{1}{1-x_i} \prod_{1\le i<j\le n} \frac{1}{1-x_i x_j}$$

We also record here the following fact, which can be proved in exactly analogous way as the theorem.

Corollary 4.1.4. $Sym^k(\wedge^2 V) \cong \bigoplus_{\lambda \vdash 2k, o(\lambda)=0} V_\lambda$

All these results can be proved alternatively, considering highest weight vectors of the representations, see [14].

4.2 Gelfand Model for Symmetric Group

For a finite (in general, compact) group G and a complex G representation (π, V), there is a nice criterion for determining whether V can be endowed with a nondegenerate G-invariant bilinear form B, i.e. $B(g \cdot v, g \cdot w) = B(v, w)$; note that this issue should be contrasted with the unitarizability of π, which says that on V there is always an invariant nondegenerate hermitian form. It can be shown that such B exists if and only if V is self contragradient G representation (i.e. $V \cong V^*$), and in that case if V is an irreducible G representation then B is unique upto a scalar multiple and for some constant $\epsilon = \epsilon(\pi) = \pm 1$, we must have $B(v, w) = \epsilon B(w, v)$. If V is not self contragradient, then we define $\epsilon(\pi) = 0$. Frobenius and Schur proved

$$\Sigma_{g \in G} \chi_V(g^2) = \epsilon(\pi)$$

We remark that this assertion is true for compact groups also, if we replace the sum in the last equation by integration with respect to the unique Haar measure on G. This ***Frobenius-Schur index*** ϵ affords a concrete interpretation, in the following way.
(i) $\epsilon(\pi) = 1$ if and only if the bilinear form on V is symmetric; then $\pi(G)$ is conjugate to a subgroup of the orthogonal group $O(n)$, where $n = dim\ V$.
(ii) $\epsilon(\pi) = -1$, if and only if the bilinear form on V is alternating; then $\pi(G)$ is conjugate to a subgroup of the symplectic group $Sp(n)$ and n is even.
We therefore say that π is *orthogonal* or *symplectic* type if $\epsilon(\pi) = 1$ or -1, respectively, and when $\epsilon(\pi) = 0$, we say π is a *complex* type. Note that this is not a priori true for a representation V of a finite group G to be of not more than one type, however (i) and (ii) asserts that for irreps of compact groups, one and exactly one of these situation arises. The terminology of 'complex type' is justified due to the following reason: $\epsilon(\pi) \neq 0$ if and only if $\chi_V \in \mathbb{R}$ for all $g \in G$, in particular $\epsilon(\pi) = 1$ if and only if there is a basis of V in which all the representation matrices $\pi(g)$ have real entries (so when $\epsilon(\pi) = 0$, things are very 'complex' indeed!) We note a necessary and sufficient condition for $\epsilon(\pi) = \pm 1$ for all irreps π of G: every $g \in G$ is conjugate to g^{-1}.
This is all general story, and the proofs of these assertions can be found in [24], Section 6.2; also one can find out vast generalizaions of Frobenius-Schur index in [16]. Our point of interest revolves around the symmetric groups, and gets mileage from the following

fact: for a finite group G and a fixed element $x \in G$,

$$\Sigma_{\pi \in \hat{G}} \chi_\pi(x)\epsilon(\pi) = |\{g \in G : g^2 = x\}|$$

It is classically known, for example using Young's seminormal or orthogonal basis, that all the symmetric group irreps are of orthogonal type, in fact in the former basis the matrix entries of $\pi(g)$ lie in $\mathbb{Z}$. Hence taking $x = 1$ in the last equation, we get for the symmetric groups

$$\Sigma_{\pi \in \hat{S_n}} dim\ V_\pi = number\ of\ involutions\ in\ S_n$$

This last fact can also be skimmed out of our arguments in the proof of $Theorem$ 4.1: on left hand side we have $\Sigma_{\lambda \vdash n}|SYT(\lambda)|$ and on the right, symmetric matrices with only one 1 in each row and column (and the rest of the entry there is 0), where we recall how $S_n \subset GL_n$, and our discussion regarding the symmetry property of the RSK correspondence shows that these two sets are of same cardinality.
Out of this numerical equality emerges a natural question: can one make the set $\mathcal{I} = \{\sigma \in S_n : \sigma^2 = 1\}$ of involutions in S_n into a representation $\mathcal{M}$ of S_n in such a way that it breaks up into a multiplicity free direct sum of all irreps of S_n? The answer is remarkably an yes, although in general it need not happen for a group with all FS index 1! One can check that the obvious action of S_n by conjugaion on $\mathcal{I}$ will not suffice, a further twist of sign is needed to make it work, as expounded in [32]. We will see later how the result of this article follows immediately from the results in section 1.
In literature, a *model* of a representation π, typically irreducible, is an embedding of π in a multiplicity free induced representation, typically induced from an one dimensional representation of a subgroup of G. The project of Bernstein, Gelfand and Gelfand [29] was to find a collection of subgroups $H_1, \cdots, H_n$ of G and characters, typically one dimensional, ψ_i of H_i such that $\bigoplus_i Ind_{H_i}^G \psi_i \cong \mathcal{M}$; it automatically means that $Ind_{H_i}^G \psi_i$ is multiplicity free, so we obtain a model for every irreducible representation of G. The data comprising of such subgroups and characters are ubiquitously known as ***Gelfand Model*** for G.
Rephrasing our old question in this light, we see that for a start we can take $H_i = C_{S_n}(\sigma_i)$, the centralizer of σ in S_n, where $\sigma_1 = 1, \sigma_2 = (1\ 2), \sigma_3 = (1\ 2)(3\ 4)$ and so on; this is a reasonable start, as $\mathcal{I} = \coprod_i \{involutions\ with\ cycle\ type\ (2^i, 1^{n-2i})\} = \coprod_i \{conjugacy\ class\ of\ \sigma_i\} = \coprod_i S_n/C_{S_n}(\sigma_i)$, so that the dimension matches as a precondition. The real question is whether we can specify the required ψ_i's. Note that $C_{S_n}(\sigma_i) = C_{S_{2i}}(\sigma_i) \times S_{n-2i} \subset S_{2i} \times S_{n-2i}$, where S_{n-2i} is the symmetric group on $\{2i+1, \ldots, n\}$ and $C_{S_{2i}}(\sigma_i)$ can be described as isomorphic to the Weyl group of Cartan type B_i: it has order $2^i i!$, has a normal subgroup of order 2^i generated by the transpo-

sitions $(1\ 2), \ldots, (2i-1\ 2i)$ and the quotient is isomorphic to S_i. We denote this group by B_{2i}. A moment's thought would reveal that $B_{2i} \cong \mathbb{Z}_2^i \rtimes S_i$, where S_i acts on $\mathbb{Z}_2^i$ by permuting the coordinates: one sees that S_i permutes the transpositions in σ and each copy of Z_2 acts by reversing the order of appearences in a transposition i.e. changing $(a\ b)$ to $(b\ a)$. We remark that in such realization, the copy of S_i sits inside $A_{2i} \subset S_{2i} \subset S_n$, each of these embedding being the usual ones; as a consequence, we have for any $\sigma \in B_{2i}$

$$|\{(\alpha,\beta), \alpha < \beta, \alpha, \beta \in [2i] : \sigma(\alpha) > \sigma(\beta)\}| \equiv |\gamma \in [i] : \sigma(2\gamma-1) > \sigma(2\gamma)| \pmod 2$$

Therefore if we denote the alternating character of S_{2i} by $\epsilon = \epsilon_{2i}$, then

$$\epsilon_{2i}(b) = (-1)^{|\gamma \in [i]: \sigma(2\gamma-1) > \sigma(2\gamma)|}, \forall\ b \in B_{2i}.$$

This observation is useful in what we are going to prove.

Theorem 4.2.1. For any natural number k, l with $2k + l = n = dimV$, as a S_{2k+l} representation,

$$Ind_{B_{2k}\times S_l}^{S_{2k+l}} \epsilon \otimes 1 \cong \mathrm{Hom}_{GL(V)}(V^{\otimes 2k+l}, Sym^k(\wedge^2 V) \otimes Sym^l\ V)$$

Here by ϵ, we denote the restriction of the alternating character of S_{2k} to B_{2k}. First notice that it suffices to prove that $(Sym^k(\wedge^2 V) \otimes Sym^l\ V)_{(1^n)} \cong Ind_{B_{2k}\times S_l}^{S_n} \epsilon \otimes 1$, because of the following assertion.

Lemma 4.2.2. For any polynomial representation W of $GL(V)$ of degree n, where $V = \mathbb{C}^n$, we have as a S_n representation

$$Hom_{GL(V)}(V^{\otimes n}, W) \cong (W)_{(1^n)}$$

Proof. If $W \cong \bigoplus_{\alpha \in S} V_\alpha^{\oplus c_\alpha}$ for some $S \subset Par(n)$, then

$$\mathrm{Hom}_{GL(V)}(V^{\otimes n}, W) \cong Hom_{GL(V)}(\textstyle\bigoplus_{\lambda \vdash n} Sp_\lambda \otimes V_\lambda, \bigoplus_{\alpha \in S} V_\alpha^{\oplus c_\alpha}) \cong \bigoplus_{\alpha \in S} Sp_\alpha^{\oplus c_\alpha} \cong \bigoplus_{\alpha \in S} (V_\alpha)_{(1^n)}^{\oplus c_\alpha} \cong (W)_{(1^n)}$$

□

Proof of the theorem. Recall that one of the equivalent definitions of induced representation, e.g. see [16], tells us that

$$Ind_{B_{2k}\times S_l}^{S_n} \epsilon \otimes 1 = \{f : S_n \to \mathbb{C} | f((b\sigma)\pi) = \epsilon(b) f(\pi), \forall\ (b,\sigma) \in B_{2k} \times S_l, \pi \in S_n\}$$

where S_n acts by $\pi \cdot f(w) := f(w\pi), \forall w, \pi \in S_n$.
In $Sym^k(\wedge^2 V) \otimes Sym^l\ V$, a basis vector $(e_{i_1} \wedge e_{i_2}) \cdots (e_{i_{2k-1}} \wedge e_{i_{2k}}) \otimes (e_{i_{2k+1}} \cdots e_{i_{2k+l}})$ lies in the (1^{2k+l}) weight space if and only if $[2k+l] = \{i_1, \ldots, i_{2k+l}\}$, therefore the required weight space in the LHS has a subset of these vectors as a basis. Given $w \in S_n$, define

$$e_w := e_{w^{-1}(1)} \wedge e_{w^{-1}(2)} \cdots e_{w^{-1}(2k-1)} \wedge e_{w^{-1}(2k)} \otimes e_{w^{-1}(2k+1)} \cdots e_{w^{-1}(2k+l)}.$$

It is straightforward to see that e_v and e_w are linearly independent unless v, w lie in the same coset of $B_{2k} \times S_l$ in S_n, in which case if $v = (b, u)w$ then $e_v = \epsilon(b)e_w$, so that a basis of the sought weight space is $\{e_w : w \in (B_{2k} \times S_l)\backslash S_n\}$.
Define a linear map from this weight space to the induced representation space by extending $e_w \mapsto f_w$, where we define

$$\begin{aligned} f_w(v) &:= 1 \textit{ when } v = w \\ &:= \epsilon(b) f_w(w) \textit{ when } v = (b, u)w \\ &:= 0 \textit{ otherwise} \end{aligned}$$

By construction, f_w lies in $Ind_{B_{2k}\times S_l}^{S_n} \epsilon \otimes 1$: in fact it is the 'twisted' characteristic function of the right coset of w in $(B_{2k} \times S_l)\backslash S_n$ and they form a basis of the induced representation space, whence $e_w \mapsto f_w$ gives rise to a linear isomorphism of vector spaces. The fact that this is a S_n equivariant isomorphism follows from the calculation below:

$$\pi \cdot e_w = \pi \cdot e_{w^{-1}(1)} \wedge e_{w^{-1}(2)} \cdots e_{w^{-1}(2k-1)} \wedge e_{w^{-1}(2k)} \otimes e_{w^{-1}(2k+1)} \cdots e_{w^{-1}(2k+l)} = e_{\pi w^{-1}(1)} \wedge e_{\pi w^{-1}(2)} \cdots e_{\pi w^{-1}(2k-1)} \wedge e_{\pi w^{-1}(2k)} \otimes e_{\pi w^{-1}(2k+1)} \cdots e_{\pi w^{-1}(2k+l)} = e_{w\pi^{-1}} \mapsto f_{w\pi^{-1}} = \pi \cdot f_w$$

because $f_{w\pi^{-1}}(\sigma) = f_w(\sigma\pi)$, seen from direct evaluation. □

Now we can finally get grip on the main theorem developed in the work of Inglis et al [30].

Theorem 4.2.3. $\bigoplus_{2k+l=n} Ind_{B_{2k}\times S_l}^{S_n} \epsilon \otimes 1 \cong \mathcal{M}$, the model for S_n.

Proof. We just need to observe that

$$\begin{gathered} \mathrm{Hom}_{GL(V)}(V^{\otimes n}, Sym^k(\wedge^2 V) \otimes Sym^l\ V) \cong \\ \mathrm{Hom}_{GL(V)}(\textstyle\bigoplus_{\lambda\vdash n} Sp_\lambda \otimes V_\lambda, (\bigoplus_{\mu\vdash 2k, o(\mu)=0} V_\mu) \otimes Sym^l V) \\ \cong \mathrm{Hom}_{GL(V)}(\textstyle\bigoplus_{\lambda\vdash n} Sp_\lambda \otimes V_\lambda, \bigoplus_{\mu\vdash 2k+l, o(\mu)=l} V_\mu) \\ \cong \textstyle\bigoplus_{\mu\vdash 2k+l, o(\mu)=l} Sp_\mu \end{gathered}$$

where we have used the *Pieri's formula* in the second isomorphism, which says that for $\lambda \vdash i, V_\lambda \otimes Sym^j(V) \cong_{GL(V)} \oplus V_\mu$, where the direct sum varies over the partitions of $\mu \vdash i + j$ from which λ can be obtained by removing a horizontal strip. But then, these are precisely the $\mu \vdash i + j$ which have j number of odd columns in their diagram, and we are done. □

Here is a second approach. Take the (1^n) weight space of our old equation

$$Sym(V \oplus \wedge^2 V) \cong_{GL(V)} \bigoplus_{\lambda, l(\lambda)\le n} V_\lambda(n)$$

Since $(V_\lambda(m))_{(1^n)} = \delta_{mn}\ Sp_\lambda$, the right hand side automatically reduces to $\bigoplus_{\lambda\vdash n} Sp_\lambda$. On the left hand side, all but the nth graded part vanishes. Note that the obvious map (defined on the basis elements and linearly extended) $(e_{i_1}e_{i_2}\cdots e_{i_l})\otimes(e_{\alpha_1}\wedge e_{\beta_1})\cdots(e_{\alpha_k}\wedge e_{\beta_k})\mapsto(\alpha_1\ \beta_1)\dots(\alpha_k\ \beta_k)$ between this weight space and the free vector space generated on the involutions in S_n is an isomorphism, if we define the action on the later space in the following way: for $\sigma = (\alpha_1\ \beta_1)\cdots(\alpha_k\ \beta_k)\in\mathcal{I}$, where we take $\alpha_i < \beta_i$, define $\rho : S_n \to GL(\mathbb{C}[\mathcal{I}])$ by

$$\rho(\pi)\sigma = (-1)^{|k:\pi(\alpha_k)>\pi(\beta_k)|}(\pi(\alpha_1)\ \pi(\beta_1))\cdots(\pi(\alpha_k)\ \pi(\beta_k))$$

the sign is explained by the way $S_n \subset GL_n$ permutes the basis vector in $\mathbb{C}^n$ and hence acts on the basis vectors of the weight space. This is the direct way of constructing an involution model for the symmetric group, as explained in [32].
We remark that both the versions, which were discovered after 1990, naturally fall out of the first theorem in this chapter, something that was known to even Schur!

4.3 An Old Question Revisited

Let us get back to the question we asked in Section 3.2: for $|\lambda| = n = |\mu|$, what is the decomposition of $V_{n,\lambda,\mu}$ in terms of irreps of S_n? Notice that an answer to this question for all the weights μ occuring in $V_\lambda(n)$ would yield a solution to the problem of describing $Res_{S_n}^{GL_n}V_\lambda(n)$, because $V_\lambda(n) = \oplus_{\mu\vdash}V_{n,\lambda,\mu}$. Now the last problem does have an answer in closed form, see [1], Ex. 7.74 and its solution.

Theorem 4.3.1. Denoting the character of the restriction by ζ_λ, we have for $\mu \vdash n$,

$$\langle\hat{\mathcal{F}}(\zeta_\lambda), s_\mu\rangle = \langle s_\lambda, s_\mu[h]\rangle$$

where $s_\mu[h]$ denotes the **plethysm** of s_μ with the symmetric function $h = \Sigma_{i\geq 0}h_i$, where $h_i = \Sigma_{1\leq i_1\leq\cdots\leq i_k}x_{i_1}\cdots x_{i_k}$ is the complete homogneous symmetric function(therefore it is taken in infinitely many variables $x_1, x_2, \cdots$, in fact whenever we say symmetric functions, it is assumed that they are in infinitely many variables, as opposed to symmetric polynomials).

See [1], Appendix 2, for discussions on plethysm. In our case, it means the following: take the Schur function s_μ and plug in every monomials, i.e.

$$s_\mu[h] = s_\mu(1, x_1, x_2, \dots, x_1^2, x_1x_2, x_1x_3, \dots, x_1^3, x_1^2x_2, \dots)$$

Although the answer is in closed form, in general, plethysms are quite intractable to compute, and the same statement applies to our particular example as well; see [28] and [3] for some computations of plethysms in general. It is worth mentioning that many of the results that we prove in here can be recasted in plethystic languages, they can found in these references.

We will now concentrate on the decomposition of $V_{n,\lambda,\mu}$, and see how Schur Weyl duality can be used to reformulate the problem into a simpler one, which can be computed using a computer algebra system such as Sage [34]! Let $V = \mathbb{C}^n$ and note that $V^{\otimes n} \cong_{S_n \times GL_n} \oplus_{\lambda\vdash n} Sp_\lambda \otimes V_\lambda(n)$ implies that $(V^{\otimes n})_\mu \cong \oplus_{\lambda\vdash n} Sp_\lambda \otimes (V_\lambda(n))_\mu$ for any weight $\mu \vdash n$ for the GL_n action, but this is is not quite an $S_n \times \hat{S_n}$ isomorphism, it is just on the level of vector spaces, because for $\mu \neq (1^n)$, the weight spaces are not stable under the action of the copy of S_n inside GL_n(which we denote by $\hat{S_n}$). The remedy is immediate and obvious, as we did before: take the direct sum over all the weight spaces obtained by permuting the weight coordinates of μ on both the sides. Then we have

$$\begin{aligned}\bigoplus_{\sigma\in S_n/S_\mu}(V^{\otimes n})_{\sigma\cdot\mu} &\underset{S_n\times\hat{S_n}}{\cong} \bigoplus_{\lambda\vdash n}[Sp_\lambda \otimes \{\bigoplus_{\sigma\in S_n/S_\mu}(V_\lambda(n))_{\sigma\cdot\nu}\}] \\ \Rightarrow V_{n,\mu} &\underset{S_n\times\hat{S_n}}{\cong} \bigoplus_{\lambda\vdash n} Sp_\lambda \otimes V_{n,\lambda,\mu} \\ \Rightarrow V_{n,\lambda,\mu} &\underset{\hat{S_n}}{\cong} Hom_{S_n}(Sp_\lambda, V_{n,\mu})\end{aligned}$$

where we abbreviate $V_{n,\mu} := \bigoplus_{\sigma\in S_n/S_\mu}(V^{\otimes n})_{\sigma\cdot\mu}$, it is an $S_n \times \hat{S_n}$ representation and a basis for this space is $\{e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(n)} |\ \alpha : [n] \to [n]$ is any map such that $|\alpha^{-1}(i)| = \mu_{\sigma(i)}$ for some $\sigma \in S_n/S_\mu\}$. Now this is a permutation representation, $S_n \times \hat{S_n}$ permutes this basis vectors and via character theory, we just need to count the number of fixed points under this action to get hold of the decomposition of $V_{n,\mu}$ as a $S_n \times \hat{S_n}$ representation and then by our earlier observation, we will be done! That is, given $(\pi, \zeta) \in S_n \times \hat{S_n}$, we are asking how many of the basis vectors $e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(n)}$ satisfy the last of the equations below.

$$(\pi,\zeta)\cdot \otimes_{i=1}^n e_{\alpha(i)} = \otimes_{i=1}^n e_{\zeta\alpha\pi^{-1}(i)} = \otimes_{i=1}^n e_{\alpha(i)}$$

So our original problem boils down finally to the following counting problem:

Given a collection $\{(\pi,\zeta)\}$ of distinct conjugacy class representatives of $S_n \times \hat{S_n}$, find the number of maps $\alpha : [n] \to [n]$ with specified fiber size $|\alpha^{-1}(i)| = \mu_{\sigma(i)} \forall\ i \in [n]$ such that $\zeta\alpha\pi^{-1} = \alpha$ on $[n]$.

This can be computed in the mathematics software system **Sagemath**. Having done that, we now proceed to find the coefficients $m_{ij} = m_{ij}^{(n,\mu)}$ in

$$V_{n,\mu} \cong \oplus_{(i,j)\in[p(n)]\times[p(n)]}(X_i \otimes X_j)^{\oplus m_{ij}}$$

where the S_n irreps have been indexed in some specific order and denoted as $\{X_i : i \in [p(n)]\}$. Here $p(n)$ denotes the partition function's value at n; keep in mind that $\{X_i \otimes X_j : (i,j) \in [p(n)] \times [p(n)]\}$ are all the $S_n \times \hat{S_n}$ irreps. Let us also index the conjugacy class representatives of S_n as $\{C_i : i \in [p(n)]\}$, then $S_n \times \hat{S_n}$ conjugacy class representatives are $\{(C_i, C_j) : (i,j) \in [p(n)] \times [p(n)]\}$. We denote the $S_n \times \hat{S_n}$ character on $V_{n,\mu}$ as χ and S_n character on X_i as χ_i; define $v_{kl} = \chi(C_k, C_l)$. Also denote the character table of S_n (written in the fixed order of conjugacy classes and irreducible representations) by $A = (a_{ij})$, i.e. $a_{ij} = \chi_i(C_j)$. Then notice that the matrix $A \otimes A$ is the character table of $S_n \times \hat{S_n}$ (in the specified order as before), where $A \otimes A$ is the matrix representing the tensor of the linear map associated with the matrix A with itself; the matrix looks like

$$A \otimes A = \begin{pmatrix} a_{11}A & \cdots & a_{1n}A \\ \vdots & \ddots & \vdots \\ a_{n1}A & \cdots & a_{nn}A \end{pmatrix}$$

The rows and columns are indexed in an obvious manner by pairs $(a, b) \in [p(n)] \times [p(n)]$, and notice that $(A \otimes A)_{(i,j)(k,l)} = a_{ik}a_{jl}$, reflecting the fact that $(\chi_i \otimes \chi_j)(C_k, C_l) = \chi_i(C_k)\chi_j(C_l)$. With this notation, taking character values in the last isomorphism yield

$$\chi(C_k, C_l) = \Sigma_{(i,j)\in[p(n)]\times[p(n)]} m_{ij}\chi_i(C_k)\chi_j(C_l)$$
$$\text{or, } v_{kl} = \Sigma_{(i,j)\in[p(n)]\times[p(n)]} m_{ij}a_{ik}a_{jl}$$
$$\text{or, } \vec{v} = (A \otimes A)^t \vec{m}$$

where $\vec{v}$ denote the column vector of length $p(n) \times p(n)$ with entries v_{kl} written in the alphabetical order of the indices. Since A is an invertible matrix, $(A \otimes A)^t$ is also so; one can check that $det(M_{m\times m} \otimes N_{n\times n}) = (det\ M)^m(det\ N)^n$. Therefore this system of equations can be solved (again using Sagemath) and we can get the coefficients m_{ij}, finally solving our original problem!

We now list some calculations. We remark that for small values $n = 2, 3, V_{n,\lambda,\mu}$ can be decomposed directly by looking at the action of $\hat{S_n}$, but for larger values it becomes intractable to do it directly without resorting to the modified problem: *mainly because when we try to compute character values of $\hat{S_n}$ action on $V_{n,\lambda,\mu}$, one needs to often 'express non SSYT's in terms of SSYT' (as we did in Chapter 2) using the Plucker relations, which is messy.* For each fixed $n \in N$, we can write a table of size $p(n) \times p(n)$ where the rows and columns are indexed by partitions of n, and the (λ, μ)th entry in the table records the deomposition $V_{n,\lambda,\mu}$, or better yet (for larger n), the vector made out of the coefficients $c_\nu^{\lambda\mu}$ with which Sp_ν occurs in the decomposition. For this, we need to impose an order on the set of partitions of n. We use the *lexicographical ordering*: we write $\alpha \triangleright \beta$ if $\alpha_1 \geq \beta_1, \alpha_1 + \alpha_2 \geq \beta_1 + \beta_2$ and so on. We choose this order because of the following important fact: the set of SSYT with shape $\lambda = (\lambda_1, \ldots, \lambda_n)$ and type $\mu = (\mu_1, \ldots, \mu_n)$ is nonempty, or $K_{\lambda\mu} \neq 0$, precisely when $\lambda \triangleright \mu$, see [21], Chapter 3 for a proof. In our case it means that $(V_\lambda)_\mu = 0$ unless $\lambda \triangleright \mu$, and coupled with the fact that $K_{\lambda\mu} = K_{\lambda\pi\cdot\mu}$ for any $\pi \in S_n$, it shows that if we index the rows and columns of our proposed table by the partitions of n arranged in an increasing order in $\triangleright$ order, it will be a lower triangular table, meaning that everything above the diagonal is 0 vector space (or $(0^{p(n)})$ in our intended notation for larger n). With this setup, we record our findings in the tables below for $n = 2, 3, 4, 5$.

Table 4.1: Decomposition for S_2

$\lambda\backslash\mu$	(1^2)	$(2,0)$
(1^2)	$Sp_{(1^2)}$	0
$(2,0)$	$Sp_{(2,0)}$	$Sp_{(1^2)} \oplus Sp_{(2,0)}$

Table 4.2: Decomposition for S_3

$\lambda\backslash\mu$	(1^3)	$(2,1,0)$	$(3,0^2)$
(1^3)	$Sp_{(1^3)}$	0	0
$(2,1,0)$	$Sp_{(2,1,0)}$	$\mathbb{C}[S_3]$	0
$(3,0^2)$	$Sp_{(3,0^2)}$	$\mathbb{C}[S_3]$	$Sp_{(2,1,0)} \oplus Sp_{(3,0^2)}$

We illustrate the Sagemath computation of the Table 4.3. Take $\mu = (2,1,1,0)$, and arrange the partitions of 4 in increasing order (in the lex ordering) for once and all. Then for this weight, the vector $\vec{v}$ recording the character values in our discussion is $(144,0,0,0,0,24,4,0,0,0,0,8,0,0,0,0,0,0,0,0,0,0,0,0,0)$. Next we use the character table of S_4 and compute the vector $\vec{m}$, which records the required multiplicities in $V_{4,(2,1,1,0)} \cong \oplus_{(\alpha,\beta)\in Par(4)\times Par(4)}(Sp_\alpha \otimes Sp_\beta)^{\oplus m_{\alpha\beta}}$; we have ordered the set $Par(4) \times Par(4)$ by dictionary order that arises from the lex order on $Par(4)$. We get

$$\vec{m} = (0,0,0,0,0,1,2,1,1,0,0,1,1,2,1,1,3,2,3,1,0,1,1,2,1)$$

From this we can write the decomposition of $V_{4,\lambda,(2,1,1,0)}$ for any $\lambda \vdash 4$. This is recorded in the second column of the table, this 25-tuple gets stacked in this column successively as 5 5-tuples. Similar computations in Sage helps us complete the full table.

Table 4.3: Decomposition for S_4

$\lambda\backslash\mu$	(1^4)	$(2,1^2,0)$	$(2^2,0^2)$	$(3,1,0^2)$	$(4,0^3)$
(1^4)	$(1,0,0,0,0)$	(0^5)	(0^5)	(0^5)	(0^5)
$(2,1^2,0)$	$(0,1,0,0,0)$	$(1,2,1,1,0)$	(0^5)	(0^5)	(0^5)
$(2^2,0^2)$	$(0,0,1,0,0)$	$(0,1,1,2,1)$	$(0,0,1,1,1)$	(0^5)	(0^5)
$(3,1,0^2)$	$(0,0,0,1,0)$	$(1,3,2,3,1)$	$(0,1,0,1,0)$	$(0,1,1,2,1)$	(0^5)
$(4,0^3)$	$(0,0,0,0,1)$	$(0,1,1,2,1)$	$(0,0,1,1,1)$	$(0,1,1,2,1)$	$(0,0,0,1,1)$

The table for S_5 is illustrated in the next page. Here e_i denotes the 7-tuple , consisting of 1 in the i-th place and 0 everywhere else.

We end with the following remark: there are lots of pattern evident in these tables, and there must be a neat theorem lurking behind all these computations, and this can also shed new light on the plethysm we saw before. We hope to unravel these in future.

Table 4.4: Decomposition for S_5

$\lambda\backslash\mu$	(1^5)	$(2,1^3,0)$	$(2^2,1,0^2)$	$(3,1^2,0^2)$	$(3,2,0^3)$	$(4,1,0^3)$	$(5,0^4)$
(1^5)	e_1	(0^7)	(0^7)	(0^7)	(0^7)	(0^7)	(0^7)
$(2,1^3,0)$	e_2	$(1,2,1,1,0,0,0)$	(0^7)	(0^7)	(0^7)	(0^7)	(0^7)
$(2^2,1,0^2)$	e_3	$(0,1,2,2,2,1,0)$	$(0,0,1,1,2,2,1)$	(0^7)	(0^7)	(0^7)	(0^7)
$(3,1^2,0^2)$	e_4	$(1,3,3,3,2,1,0)$	$(0,1,1,2,1,1,0)$	$(0,1,1,2,1,1,0)$	(0^7)	(0^7)	(0^7)
$(3,2,0^3)$	e_5	$(0,1,2,3,3,3,1)$	$(0,1,2,3,3,3,1)$	$(0,0,1,1,2,2,1)$	$(0,0,0,1,1,2,1)$	(0^7)	(0^7)
$(4,1,0^3)$	e_6	$(0,1,2,3,3,3,1)$	$(0,1,2,3,3,3,1)$	$(0,1,2,3,3,3,1)$	$(0,0,0,1,1,2,1)$	$(0,0,0,1,1,2,1)$	(0^7)
$(5,0^4)$	e_7	$(0,0,0,1,1,2,1)$	$(0,0,1,1,2,2,1)$	$(0,0,1,1,2,2,1)$	$(0,0,0,1,1,2,1)$	$(0,0,0,1,1,2,1)$	$(0,0,0,0,0,1,1)$

Bibliography

[1] Richard P. Stanley, *Enumerative Combinatorics: Volume 2*, Cambridge Studies in Advanced Mathematics Book 62, Cambridge University Press, 1999.

[2] Donald Ervin Knuth, *Permutations, matrices and generalized Young tableaux*, Pacific Journal of Mathematics 34, 1970. Also to be found in the collection **Selected papers on discrete mathematics**, CSLI, Stanford, 2003.

[3] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Second edition, Oxford Mathematical Monograph, Oxford Science Publications, 1995.

[4] Perci Diaconis and Curtis Greene, *Applications of Murphy's elements*, Technical Report No. 335, http://statweb.stanford.edu/ cgates/PERSI/papers/EFSNSF335.pdf, 1989.

[5] William Fulton and Joe Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics, Springer, 1999.

[6] Issai Schur, *Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen*, Doctoral dissertation, Universitt Berlin, 1901. In I. Schur, Gesammelte Abhandlungen I, 170, Springer, Berlin, 1973.

[7] Issai Schur, *Über die rationalen Darstellungen der allgemeinen linearen Gruppe*, Sitzber. Knigl. Preu. Ak. Wiss., Physikal.-Math. Klasse, pages 5875, 1927. In I. Schur, Gesammelte Abhandlungen III, 6885, Springer, Berlin, 1973.

[8] Roger Howe, *Perspectives on Invariant Theory*, The Schur Lectures(1992), Gelbert Research Intitute for Mathematical Sciences, Emmy Noether Research Institute, Bar-Ilan University, 1995.

[9] G. Frobenius, *Über die charaktere der symmetrischen Gruppe*, Sitzungs-berichte Koniğl. Preuss. Akad. Wissenshaften, 1900.

[10] A. Yu. Okounkov and Anatoly Vershik, *A New Approach to the Representation Thoery of the Symmetric Groups. 2*, Zapiski Seminarod POMI (In Russian) v.307, 2004.

[11] Bruce Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Second Edition, Graduate Texts in Mathematics 203, Springer, 2001.

[12] A. I. Molev, *Gelfand-Tsetlin bases for classical Lie algebras*, Handbook of Algebra, Vol. 4, (M. Hazewinkel, Ed.), Elsevier, 2006.

[13] Roe Goodman and Nolan R Wallach, *Symmetry, Representations, and Invariants*, Graduate Texts in Mathematics 255, Springer.

[14] Roe Goodman, *Multiplicity-free spaces and Schur-Weyl-Howe duality*, Representations of real and p-adic groups, 305415, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., Singapore Univ. Press, 2004.

[15] Daniel Bump and David Ginzburg *Generalized Frobenius-Schur numbers*, Journal of Algebra, August 2004.

[16] Daniel Bump, *Lie Groups*, Second Edition, Graduate Texts in Mathematics 225, Springer, 2013

[17] Hermann Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton Landmarks in Mathematics, Princeton University Press, 1939

[18] William Fulton, *Young Tableaux: With Applications to Representation Theory and Geometry*, London Mathematical Society.

[19] Arun Ram and H. Barcelo, *Combinatorial representation theory*, New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97), 23–90, Math. Sci. Res. Inst. Publ., 38 , Cambridge Univ. Press, Cambridge, 1999.

[20] Arun Ram and Tom Halverson, *Partition Algebras*, European J. Combin. 26 no. 6, 2005.

[21] Amritanshu Prasad, *Representation Theory: A Combinatorial Viewpoint*, Cambridge Studies in Advanced Mathematics 147, Cambridge University Press, 2015.

[22] Charles W Curtis, *Pioneers of Representation Theory: Frobenius, Burnside, Schur and Brauer*, History of Mathematics, Volume 15, American Mathematical Society and London Mathematical Society, 1999.

[23] G D James, *The Representation Theory of the Symmetric Groups*, Lecture Notes in Mathematics, Springer, 1978.

[24] Emmanuel Kowalski, *An Introduction to the Representation Theory of Groups*, Graduate Studies in Mathematics Volume 155, American Mathematical Society, 2010.

[25] Joel Kamnitzer, *Lectures on geometric constructions of the irreducible representations of GL_n*, https://arxiv.org/abs/0912.0569, 2010.

[26] Sergey Fomin and J. Y. Thibon, *Inverting the Frobenius Map*, Algebra i Analiz, 10:3 (1998), 183192; St. Petersburg Math. J., 10:3 (1999), 545552.

[27] Georgia Benkart, *Commuting Actions- A Tale of Two Groups*, Lie Algebras and Their Representations: A Symposium on Lie Algebras and Representation Theory, Seoul National University, Contemporary Mathematics 194, American Mathematical Society, 1995.

[28] Nicholas A. Loehr and Jeffrey B. Remmel, *A computational and combinatorial expos of plethystic calculus*, Journal of Algebraic Combinatorics, March 2011.

[29] J. Bernstein, I. Gelfand, S. Gelfand, *Models of representations of compact Lie groups*, Funct. Anal. Appl. 9, 1976.

[30] N. Inglis, J. Richardson and J. Saxl, *An explicit model for the complex representations of S_n* , Arch. Math.(Basel), 1990.

[31] A. A. Klyachko, *Models for complex representations of the groups $GL(n, q)$ and Weyl groups (Russian)*, Dokl. Akad. Nauk SSSR 261, 1981.

[32] Vijay Kodiyalam and D. N. Verma, *A natural representation model for symmetric groups*, https://arxiv.org/abs/math/0402216.

[33] A class on combinatorial representation theory of GL_n, offered by David E Speyer at the University of Michigan, Ann Arbor, Fall 2012.

[34] The Sage Developers, *Sage Mathematics Software (Version 7.1), 2016*, http://www.sagemath.org