Holographic Conformal Partial Waves

By

ATANU BHATTA

PHYS10201204005

The Institute of Mathematical Sciences, Chennai

A thesis submitted to the

Board of Studies in Physical Sciences

In partial fulfilment of requirements

For the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

July, 2018

Holographic Conformal Partial Waves

By
ATANU BHATTA
PHYS10201204005

The Institute of Mathematical Sciences, Chennai

A thesis submitted to the

Board of Studies in Physical Sciences

In partial fulfilment of requirements

For the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

July, 2018

Homi Bhabha National Institute

Recommendations of the Viva Voce Board

As members of the Viva Voce Board, we certify that we have read the dissertation prepared by Mr. Atanu Bhatta entitled “Holographic conformal partial waves" and recommend that it maybe accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

____ Date:

Chair - Balachandran Sathiapalan

____ Date:

Guide/Convener - Nemani Venkata Suryanarayana

____ Date:

Member 1 - Partha Mukhopadhyay

____ Date:

Member 2 - Satyavani Vemparala

____ Date:

Member 3 - Sanatan Digal

____ Date:

External Examiner - Abhijit Gadde

Final approval and acceptance of this dissertation is contingent upon the candidate’s submission of the final copies of the dissertation to HBNI.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it may be accepted as fulfilling the dissertation requirement.

Date:

Place: IMSc, Chennai Guide - Nemani Venkata Suryanarayana

STATEMENT BY AUTHOR

Atanu Bhatta

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Atanu Bhatta

List of Publications arising from the thesis[1]

1. "Holographic Conformal Partial Waves as Gravitational Open Wilson Networks"

   Atanu Bhatta, *Prashanth Raman and Nemani V. Suryanarayana*;

   JHEP **1606** (2016) 119

2. "Scalar Blocks as Gravitational Wilson Networks"

   Atanu Bhatta, *Prashanth Raman and Nemani V. Suryanarayana*;

   JHEP **1812** (2018) 125

List of other Publications, Not included in the thesis

1. "Sine-Gordon theory: Entanglement entropy and holography"

   Pinaki Banerjee, **Atanu Bhatta** *and B. Sathiapalan*;

   Phys. Rev. D **96**, 126014 (2017)

Atanu Bhatta

[1]As it is standard in the High Energy Physics Theory (hep-th) community the names of the authors on any paper appear in their alphabetical order.

Conferences and workshops attended

1. “National String Meeting" at NISER, Bhubaneswar, India (2017).
2. “Student Talks on Trending Topics in Theory” at CMI, Chennai, India (2017).
3. “Spring School on Superstring Theory and Related Topics " organized by ICTP at ICTP, Italy (2017).
4. “Advanced String School" organized by IOP, Bhubaneswar at Puri, India (2017).
5. “Indian String Meeting" at IISER Pune, Pune, India (2016).
6. “Indian String Meeting" at Puri, India (2014).
7. “Advanced String School" organized by IOP, Bhubaneswar at Puri, India (2014).
8. “Asymptotia : Workshop on Asymptotic Symmetries in Classical and Quantum Gravity" held at Chennai Mathematical Institute, Chennai, India (2013).

Seminars presented

1. Talks

   - May, 2017 at **Studets Talks in Trending Topics in Theory**, Chennai Mathematical Institute, Chennai, India.
     Talk title: “Holographic Conformal Partial Waves as Gravitational Open Wilson Networks."
   - December, 2017 at **National String Meeting**, National Institute of Science Education and Research, Bhubaneswar, India.
     Talk title: “Holographic Conformal Partial Waves as Gravitational Open Wilson Networks."

- February, 2018 at Institute of Physcis, Bhubaneswar, India.
  Talk title: “Holographic Conformal Partial Waves as Gravitational Open Wilson Networks."

2. <u>Poster presentation</u>

   - December, 2016 at **Indian String Meeting**, Indian Institute of Science Education and Research, Pune, India.
     Title: “Sine-Gordon Theory : Entanglement entropy and holography."

To My Parents

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my advisor Nemani Suryanarayana for his exceptional guidance, support and encouragement during my graduate studies. I have benefited immensely from my interactions with him over these years. His insightful approach to research has been awe-inspiring and deeply influential.

I would like to thank Balachandran Sathiapalan for his guidance and support over the past six years. It was a privilege to discuss physics with him.

I would also like to thank my other doctoral committee members Ghanashyam Date, Partha Mukhopadhyay, Satyavani Vemparala and Sanatan Digal for their support and guidance.

I have also benefited substantially from discussions with Alok Laddha who encouraged and inspired me immensely. I am very thankful to him for being so approachable.

I am very thankful to IMSc string theory group for interactions at various levels. I want to thank the faculty members who taught me some wonderful courses in IMSc. I am deeply indebted to the team behind Wolfram Mathematica.

I wish to thank my collaborators Prashanth Raman and Pinaki Banerjee for intense collaborations and discussions. I am obliged to my colleagues Ankit, Arnab, Debangshu, Jahanur, Madhu, Nirmalya, Prafulla, Renjan, Rohan, Sourav, Sudipta and Swastik. It would be remiss if I did not mention Shankhadeep, someone who is always a phone call away for making me feel at ease.

It is right time to thank everyone else with whom I had the pleasure of hanging out at IMSc and share memorable moments. Thanks also to all members of the “IMSc football group", who have accepted me in the field despite my complete lack of skills. I enjoyed every bit of it over the past one year.

I am extremely thankful to IMSc office staff. I am also thankful to the canteen staff, cleaning staff and security guards for being so helpful and friendly.

No words can do justice to how much I am grateful to my parents and all other members of my family. This thesis would not have been possible without their constant love, support, encouragement and patience. I thank Pritha for always being on my side whenever she felt I needed help – be it academic or non-academic.

ABSTRACT

We propose a method to holographically compute the conformal partial waves (CPW) in any decomposition of correlation functions of primary operators in conformal field theories (CFT) using open Wilson network (OWN) operators in the holographic gravitational dual. The Wilson operators are the gravitational ones where gravity is written as a gauge theory in the first order Hilbert-Palatini formalism. We show that the OWNs solve the corresponding differential equations expected by the CPW and this proves our proposal. We apply this method to compute the global conformal blocks and partial waves in 2d CFTs reproducing many of the known results. We also demonstrate the computation of four-point scalar partial waves in general dimension using our method. In the process we introduce the concept of OPE modules, that helps us simplify the computations. Our result for scalar partial waves is naturally given in terms of the Gegenbauer polynomials.

Contents

Synopsis

Overview

The gauge/gravity correspondence (holography) has emerged as one of the most useful tools in exploring quantum properties of conformal field theories and gravitational/string theories over the last two decades. It frames an equivalence between a d-dimensional conformal field theory (CFT_d) and a string theory in a $(d+1)$-dimensional Anti-de Sitter (AdS_{d+1}) background geometry. The conjecture and a duly formed dictionary of AdS/CFT correspondence [1–3] equip us with a prescription to compute CFT correlation functions in terms of so-called Witten diagrams in the bulk AdS gravity.

The correlation functions of a set of primary operators in a given CFT admit an expansion in terms of conformal partial waves (CPW); in a sense these are the basic building blocks of CFT correlators. In general (for $d > 2$, and for $d \leq 2$ only the global part) the CPWs satisfy two types of differential equations, namely the ward identity and the conformal casimir equations [4–6]. In principle one can solve these partial differential equations with appropriate boundary conditions to get explicit expressions for the CPWs. Assuming the AdS/CFT holds one may wonder if the CPWs are computable using bulk theory. This question is answered recently with a interesting method [7] which involves computing the so-called geodesic Witten diagrams (GWD) in the bulk.

The GWD methods to compute CFT correlation functions and the partial waves are well suited when the bulk gravitational theory is formulated in terms of metric, namely the Einstein-Hilbert formulation. However, sometimes it is convenient/essential to write the gravitational theory as extensions of tetrad formulation (Hilbert-Palatini formalism, also known as first order formalism). For example the higher spin gauge theories in three dimensions are described as Chern-Simons theories in which the gravity sector is written in the first order formalism [8–10]. It is also essential when one deals with spinors in a gravitational background. In dimensions greater than three, the theory of gravity with negative cosmological constant can be written in the Hilbert-Palatini formalism, a BF-type gauge theory [11–13]. Therefore it is important to ask how to compute CPWs in this formalism.

In this thesis we answer this question providing a new prescription to compute CPWs in Euclidean CFT$_d$ holographically in the Hilbert-Palatini formulation of Euclidean AdS$_{d+1}$ gravity. In the first order formalism, the gravitational theory is seen as a gauge theory where the diffeomorphisms and the local Lorentz transformations (LLT) are corresponding gauge symmetries of the theory. The basic fields, vielbeins (e^a) and the spin connections (ω^{ab}) of AdS$_{d+1}$ gravity can be packaged into one $so(1, d+1)$ algebra valued gauge connection.

$$A = \frac{1}{2}\omega^{ab} M_{ab} + \frac{1}{\ell} e^a \, M_{0a}$$

where M_{0a} and M_{ab} are the generators of $so(1, d+1)$ with $a, b = 1, \cdots d+1$ and ℓ determines the radius of the AdS space. The action for this connection, A is in general an appropriate BF theory. The flat connections satisfying

$$F = dA + A \wedge A = 0$$

describe locally AdS$_{d+1}$ spaces. In $d \leq 2$ this is also the equation of motion (all solutions are locally AdS). Given two points in the space-time, in a gauge theory one can define a bi-local operator, the Wilson line which depends on the representation of the gauge algebra (R) and the co-ordinates (X_i) of the given bulk points. The co-ordinates of the bulk points are denoted by $X^\alpha = \{\rho, x^a\}$ where x^a are the boundary co-ordinates. We glue three Wilson lines joining at a point by contracting the representation indices with the Clebsch-Gordan coefficients (CGC) of the $so(1, d+1)$ gauge algebra. This results in a trivalent vertex which we can treat as a basic building block and construct a spin-network with open Wilson lines (OWN). Such an OWN operator with N end-points, $W_N(X_1, R_1; \ldots X_N, R_N)$ transforms covariantly as a tensor operator under gauge transformations. It turns out that the leading order contribution of the expectation value of an OWN operator computed in a specific state, namely the cap-state, evaluates CPW of primary operators in boundary CFT when the end-points of the corresponding OWN (X_i) are taken to the boundary.

When the external points of an OWN are taken to boundary our computation reduces to simple Feynman-like rules that require the notion of what we describe as “legs" (also known as conformal wave functions) and the CG coefficients. Furthermore, our results introduce a simplification in the computation of OWN using OPE modules which are close analogs of the well studied OPE blocks.

The thesis contains the following results:

- **Differential equations satisfied:** The expectation value of a generic OWN satisfies the conformal ward identity and conformal Casimir equation when the external points are taken to boundary [14].

- **Computing blocks in $d \leq 2$:** Using our prescription explicit evaluations of OWNs with two, three, four and five points in AdS_3 reproduces corresponding $2d$ CPWs which are already know in the literature [14]. The most general cap-state in $1d$ has been provided and four scalar conformal partial waves have been calculated in $1d$ [15].

- ***d*-dimensional scalar block:** Our prescription is implemented explicitly to compute the CPWs for scalar primary operators in any dimensional CFT_d. Taking various limits of our answer for 4-point scalar CPWs we reproduce the known results in $d = 2, 3, 4$ and 1 [15].

In what follows we present a brief summary of these results.

Basic ingredients

As alluded earlier the basic ingredients needed to compute an OWN explicitly are the Wilson line, CGCs and the cap-states.

Wilson line

The non-local Wilson line operators are defined as

$$W_Y^X(R, C) = P\exp\left[\int_Y^X A\right]$$

where C is a curve connecting X and Y, R is a representation of the gauge algebra. A is the pull back of the gauge connection onto the curve C and as usual P means path ordering. Under a gauge transformation $A \to hAh^{-1} + hdh^{-1}$ the Wilson line operator transforms covariantly as

$$W_Y^X \to h(X)W_Y^X h^{-1}(Y)$$

For any locally AdS space one can write $A = g\,dg^{-1}$ locally where g is an element of $SO(1, d+1)$ and satisfies the following equation

$$dg + \frac{1}{2}\omega^{ab}M_{ab}\,g + \frac{1}{\ell}e^a M_{0a}\,g = 0$$

for flat connections. The above equation is invariant under LLT and this makes $g(X)$ an element of the coset $so(1, d+1)/so(d+1)$ which is the Euclidean AdS$_{d+1}$. Thus the Wilson line becomes

$$\mathrm{P}e^{\int_Y^X A} = g(X)g^{-1}(Y)$$

Finally one finds the $g(X)$ for AdS_{d+1} geometry described by the metric

$$\ell^{-2}\,ds^2 = d\rho^2 + e^{2\rho}\sum_{a=1}^{d} dx^a\,dx^a$$

to be

$$g(X) = e^{-\rho M_{0(d+1)}}\,e^{-x^a(M_{0a}+M_{a(d+1)})}g_0\,.$$

The CFT lives on the boundary of the AdS space at $\rho \to \infty$.

CG coefficients

The second ingredient in the definition of the OWN expectation values is the Clebsch-Gordan (CG) coefficients of the gauge algebra $so(1, d+1)$. One glues the three OWLs joining at a point by contracting the representation indices into the CG coefficients. These are invariant tensors in $R_1 \otimes R_2 \otimes R_3$

$$R_1[g(X)]_{\mathbf{m}_1\mathbf{m}'_1} R_2[g(X)]_{\mathbf{m}_2\mathbf{m}'_2} C^{R_1,R_2;R_3}_{\mathbf{m}'_1,\mathbf{m}'_2,\mathbf{m}'_3} R_3[g(X)^{-1}]_{\mathbf{m}'_3\mathbf{m}_3} = C^{R_1,R_2;R_3}_{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3} \tag{1}$$

where $R_i[g(X)]_{\mathbf{m}_i\mathbf{m}'_i}$ is used to denote the matrix elements of the group element $g(x)$ in the representation R_i with basis elements labeled collectively by $\mathbf{m}_i$. In terms of the algebra elements this reads:

$$R_1[M_{\alpha\beta}]_{\mathbf{m}_1\mathbf{m}'_1} C^{R_1,R_2;R_3}_{\mathbf{m}'_1,\mathbf{m}_2,\mathbf{m}_3} + R_2[M_{\alpha\beta}]_{\mathbf{m}_2\mathbf{m}'_2} C^{R_1,R_2;R_3}_{\mathbf{m}_1,\mathbf{m}'_2,\mathbf{m}_3} = C^{R_1,R_2;R_3}_{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}'_3} R_3[M_{\alpha\beta}]_{\mathbf{m}'_3\mathbf{m}_3} \tag{2}$$

which is the recursion relation that determines the CGC.

The equation (1) eliminates the coordinate dependence of the junction where the Wilson lines meet when the OWN is evaluated in flat connections. This can be done at every trivalent vertex in our spin-network eliminating the dependence of the locations of the vertices. Therefore OWNs are topological.

The cap-states

We are interested in computing the partial waves of the correlation function of a set of primary operators in the dual CFT. According to AdS/CFT the dual of a boundary primary operator is a bulk field. The fields in the bulk transform in finite dimensional non-unitary representations of LLT. Therefore, we would like to restrict each open end of W_N down to the corresponding finite dimensional representation of the LLT algebra $so(d+1)$. Then in the boundary limit each end point will transform as the corresponding dual primary.

This step can be achieved by capping the i^{th} external leg of the OWN in the representation R_i with appropriate cap-states. Then the OWN transforms in the finite dimensional representation of $so(d+1)$. We can construct such special states which we call the cap-states, $|R\rangle\!\rangle$. For scalar such cap-states are constructed in [16] (see also [17, 18] for $d = 2$ case).

Once projected on to the cap-state, it turns out that one particular component of such a tensor has the leading fall-off behavior, as the points X_i approach the boundary, compared to the other components. That specific component will be related to a partial wave in the correlator of the corresponding primary operators of the CFT.

Legs and OPE module

As mentioned earlier to proceed further we need to compute the **in-going legs** and the **out-going legs**. These are obtained by computing the matrix elements of $g(X)$ and $g^{-1}(X)$ between the cap-state $|R\rangle\!\rangle$ and the normalized basis elements $|R, \mathbf{m}\rangle$ of the corresponding module in the boundary limit. These are appropriately called conformal wave functions in the literature. Conformal waves provide appropriate representations and the conjugate representations of conformal algebra when the generators are represented as differential operators.

The standard way to find the CGCs is to solve the recursion relations they satisfy with the appropriate unitary conditions. But our prescription inherently leads one to calculate the specific CGCs required to compute corresponding CPWs in an easier method. One can amputate three legs from the three-point function of the primary operators in the boundary CFT to find the CGCs.

To compute the CPWs (such as four-point one) we actually need CGCs with two legs attached. It suggests that this object can be easily obtained from three-point function amputating only one leg. This quantity depends on the co-ordinates and the conformal data of the two boundary operator insertions and carries the labels of the module of the third primary. We refer to this as **OPE module** which is a close cousin of the so-called OPE blocks [19, 20].

Having all the basic ingredients in hand we analyze the properties of the OWNs. We show that a generic OWN satisfy global Ward identity (a first order PDE) and conformal Casimir equation (a second order PDE).

Differential equations satisfied

In the first order formalism the Killing vectors of AdS_{d+1} are given by

$$(\xi_{[\alpha\beta]})^{\mu} = -\ell\, E^{\mu}_{a}(R[g^{-1}])_{\alpha\beta}{}^{0a} = -\ell\, E^{\mu}_{a}(R[g])^{0a}{}_{\alpha\beta}$$

where E^μ_a is the inverse vielbein and $(R[g^{-1}])_{\alpha\beta}{}^{0a}$ are matrix elements of g in its adjoint representation. Using the following identities

$$
\begin{aligned}
g(X)\, M_{\alpha\beta} &= \xi^\mu_{\alpha\beta}(X)\partial_\mu g(X) + \frac{1}{2} M_{bc} g(X) \left[\omega^{bc}_\mu(X)\, \xi^\mu_{\alpha\beta}(X) + (R[g(X)])^{bc}{}_{\alpha\beta}\right] \\
M_{\alpha\beta} g^{-1}(X) &= -\xi^\mu_{\alpha\beta}(X)\partial_\mu g^{-1}(X) + \frac{1}{2}\left[\omega^{bc}_\mu(X)\, \xi^\mu_{\alpha\beta}(X) \right. \\
&\qquad \left. + (R[g(X)])^{bc}{}_{\alpha\beta}\right] g^{-1}(X) M_{bc}
\end{aligned}
$$

it can be shown that these matrix elements simply turn out to be those obtained by the action of the boundary conformal transformation of a primary operator on the matrix element without the insertion in the boundary limit. Finally we make use the recursion relations that CGCs satisfy (2.16) to show that the OWN satisfy the global Ward identity as well as the conformal Casimir equations.

Everything has been stated so far can be implemented in full generality in $d \leq 2$ and for scalar conformal partial waves in general dimensions.

CPWs in $2d$ CFT

We start with Euclidean AdS$_3$ with boundary $\mathbb{R}^2$. The Lie algebra of the gauge group in this case is $so(1,3)$ which can be decomposed as $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$. The coset element $g(x)$ takes the following form

$$
\begin{aligned}
g(x) &= e^{\rho(L_0+\bar{L}_0)} e^{-L_{-1}(x_1+ix_2)} e^{-\bar{L}_{-1}(x_1-ix_2)} \\
&= e^{\rho(L_0+\bar{L}_0)} e^{-zL_{-1}} e^{-\bar{z}\bar{L}_{-1}}
\end{aligned} \tag{3}
$$

where L_{-1}, L_0, L_1 $(\bar{L}_{-1}, \bar{L}_0, \bar{L}_1)$ are generators of $sl(2,\mathbb{R})$ algebra (another copy) and $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$.

The cap-state

A (quasi) primary operator $O_{h,\bar{h}}(z,\bar{z})$ in $2d$-CFT carries the representation label $(h,\bar{h})$. Using state operator correspondence we define the primary state as

$$O_{h,\bar{h}}(0,0)|0\rangle = |h,\bar{h}\rangle \tag{4}$$

where $|0\rangle$ conformally invariant vacuum state. The state $|h,\bar{h}\rangle$ becomes the lowest weight state in the discrete series representation of the conformal group $so(1,3)$

$$L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle; \qquad \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle \tag{5}$$

$$L_1|h,\bar{h}\rangle = 0; \qquad \bar{L}_1|h,\bar{h}\rangle = 0 \tag{6}$$

The rest of the states in the module are

$$|h,\bar{h};\, k,\bar{k}\rangle \equiv |h,k\rangle \otimes |\bar{h},\bar{k}\rangle = (L_{-1})^k \left(\bar{L}_{-1}\right)^{\bar{k}} |h,\bar{h}\rangle \tag{7}$$

where $k,\bar{k} \in \mathbb{Z}_+$. Working in the basis we find the appropriate cap-state for the primary operator to be

$$\begin{aligned} |h,\bar{h};\, j,p\rangle\!\rangle = \lambda(h,\bar{h}) \sum_{n=0}^{\infty} (-1)^{\frac{p}{2}+n} \sqrt{\frac{(n+p)!}{n!p!}} \\ \times \sqrt{\frac{\Gamma(2\bar{h}+n)}{\Gamma(2h+n+p)\Gamma(2\bar{h}-2h+1-p)}}\; |h,n+p\rangle \otimes |\bar{h},n\rangle \end{aligned} \tag{8}$$

with

$$\lambda(h,\bar{h})^2 = (-1)^{\bar{h}-h} \frac{(2\bar{h}-2h)!\Gamma(2h)}{\Gamma(2\bar{h})}$$

where $p \in \{0, 2j\}$ and $\bar{h} - h = j \geq 0$. For scalar primaries $(h = \bar{h})$ the above expression agrees with the known results [16].

Legs

Using (3) and (8) we obtain the **in-going leg** to be

$$\lim_{\rho\to\infty} e^{\rho(h+\bar{h})}\,\langle\!\langle h,\bar{h};-j,p|g(X)|h,\bar{h};k,\bar{k}\rangle \;=\; \frac{\lambda(-1)^{-p/2}\frac{\Gamma(2\bar{h}+\bar{k})\Gamma(2\bar{h}-p+k)}{\Gamma(2\bar{h}-p)}}{\sqrt{k!\bar{k}!\Gamma(2h+k)\Gamma(2\bar{h}+\bar{k})\Gamma(2j+1-p)p!}} \times e^{-\rho(2j-p)}z^{-2\bar{h}}\bar{z}^{-2\bar{h}}z^{p-k}\bar{z}^{-\bar{k}} \tag{9}$$

and the **out-going leg**

$$\lim_{\rho\to\infty} e^{\rho(h+\bar{h})}\,\langle h,\bar{h};k,\bar{k}|g^{-1}(X)|h,\bar{h};j,p\rangle\!\rangle \;=\; \frac{\lambda(-1)^{p/2}}{\Gamma(2h+p)}\sqrt{\frac{k!\bar{k}!\Gamma(2h+k)\Gamma(2\bar{h}+\bar{k})}{p!\Gamma(2\bar{h}-2h+1-p)}} \times e^{-\rho p}\frac{z^{k-p}}{(k-p)!}\frac{\bar{z}^{\bar{k}}}{\bar{k}!} \tag{10}$$

In the above expressions for the legs we have neglected the sub-leading terms in $e^{-\rho}$. In the boundary limit $\rho\to\infty$ one can see that the most leading term will come from $p = 2j$ in (9) and $p = 0$ in (10). The thesis shows that these legs reproduce the two-point function for primary operators.

CG coefficients

Stripping the legs we just found from three point function of (quasi) primary operators with conformal weights $(h_1,\bar{h}_1)$, $(h_2,\bar{h}_2)$ and $(h,\bar{h})$ we find the CGCs explicitly. Thus the $sl(2,\mathbb{R})$ CGCs take the form

$$C^{h_1,h_2;h}_{k_1,k_2;k} = \frac{1}{\prod_{i=1}^{3}\sqrt{k_i!(\Gamma(2h_i+k_i)}}f(k_1,k_2;k_3)$$

with

$$f(k_1,k_2;k_3) \sim \Gamma(2h_2+k_1+k_2)\Gamma(k_3-k_2)k_2!\,\delta_{h_1+k_1+h_2+k_2-h_3-k_3} \times {}_3F_2\left(\begin{matrix} -k_1,-k_3,h_3+h_1-h_2 \\ 1+k_2-k_3,1-2h_2-k_1-k_2 \end{matrix};1\right)$$

CPWs

Having got all the ingredients ready we evaluate an OWN with four external points. In the boundary limit this results in

$$|z|^{\Delta-l}\left[z^l\,{}_2F_1[\tfrac{\Delta-l}{2}-h_{12},\tfrac{\Delta-l}{2}-h_{34},\Delta-l,z]{}_2F_1[\tfrac{\Delta+l}{2}-h_{12},\tfrac{\Delta+l}{2}-h_{34},\Delta+l,\bar{z}] +(z\to\bar{z},h_{ij}\to\bar{h}_{ij})\right] \qquad (11)$$

where $h_{ij}=h_i-h_j$ etc. and $\Delta=h+\bar{h}$, $l=h-\bar{h}$. The above expression exactly matches with the well known result for (global) four-point conformal partial waves for primary operators with conformal dimensions $(h_i,\bar{h}_i)$ and $(h,\bar{h})$ being that of the exchanged primary in the intermediate channel [6].

In a similar way we compute five-point conformal block with perfect agreement with known results [21].

Similar general analysis is done in the thesis to compute the OWNs in $d=1$.

Scalar CPWs in arbitrary dimension

The next part of the thesis will contain the implementation of our prescription to compute four-point CPWs for scalar primary operators in CFT$_d$. We also show that our methodology reproduces the known answers.

Legs

The cap-states for a scalar primary operator in a d-dimensional CFT are found in [16]. A generic basis state $|\varDelta; l, \mathbf{m}, s\rangle$ in the module for scalar representation $(\varDelta, l = 0)$ of $so(1, d+1)$ is defined as

$$
\begin{aligned}
(P^2)^s\, M^l_{\mathbf{m}}(\mathbf{P})|\varDelta\rangle &= A_{l,s}|\varDelta; \{l, \mathbf{m}, s\}\rangle \\
\langle\varDelta|(K^2)^s\, M^l_{\mathbf{m}}(\mathbf{K}) &= A^*_{l,s}\langle\varDelta; \{l, \mathbf{m}, s\}|
\end{aligned}
$$

where $M^l_{\mathbf{m}}(\mathbf{x}) = x^l\, Y_{l;\mathbf{m}}(\varOmega_x)$; $Y_{l;\mathbf{m}}(\varOmega_x)$ being the hyperspherical harmonics and $\mathbf{m}$ denotes $(m_1, \ldots, m_{d-2})$ [22, 23]. Assuming the orthonormality, $\langle\varDelta; \{l', \mathbf{m}', s'\}|\varDelta; \{l, \mathbf{m}, s\}\rangle = \delta_{ss'}\delta_{ll'}\delta_{\mathbf{m}\mathbf{m}'}$ we find the normalization constant to be

$$
|A_{l,s}|^2 = \frac{2^{2l+4s}\varGamma\,[l+s+d/2]\,\varGamma[\varDelta+l+s]\varGamma\left[\varDelta+s-\frac{(d-2)}{2}\right]s!}{4\,a^2\,\pi^{\frac{d}{2}}\varGamma[d/2]\varGamma[\varDelta]\varGamma\left[\varDelta-\frac{(d-2)}{2}\right]}
$$

where a is a d dependent constant. Then the **in-going leg** becomes

$$
\lim_{\rho\to\infty} e^{\rho\varDelta}\langle\!\langle\varDelta|g(x)|\varDelta; \{l, \mathbf{m}, s\}\rangle = 4a\pi^{d/2}\frac{2^{l+2s}}{A_{l,s}} \times (x^2)^{-\varDelta-l-s}\, M^l_{\mathbf{m}}(\mathbf{x})\,(\varDelta)_{l+s}\left(\varDelta-\frac{d-2}{2}\right)_s \tag{12}
$$

while the **out-going leg** takes the following form

$$
\lim_{\rho\to\infty} e^{\rho\varDelta}\langle\varDelta; \{l, \mathbf{m}, s\}|g^{-1}(y)|\varDelta\rangle\!\rangle = \frac{4a\pi^{d/2}}{2^{l+2s}}\, A_{l,s}\,\frac{(y^2)^s}{(s)!\,\varGamma(l+s+d/2)}\, M^{l*}_{\mathbf{m}}(\mathbf{y}) \tag{13}
$$

where $(a)_n$ is the Pochhammer symbol. In the thesis we checked that these legs used in our prescription reproduces the right two-point function for the scalar primaries.

CPWs

Having found the legs (12) and (13) we can compute the **OPE modules** required for computing an OWN. To compute an OWN with end points $(\infty, \mathbf{u}, \mathbf{x}, \mathbf{0})$ at the boundary with $\mathbf{u}\cdot\mathbf{u} = 1$ we need two types of OPE modules. One can be obtained starting from scalar three-point function with operator insertions at $(\infty, \mathbf{u}, \mathbf{y})$ by amputating the out-going leg that ends at $\mathbf{y}$ when the other one is obtained by amputating the in-going leg starting from $\mathbf{y}$ from the three-point function with the operator insertions $(\mathbf{y}, \mathbf{0}, \mathbf{x})$.

Finally gluing the OPE modules thus obtained we have evaluated the OWN to be

$$\begin{aligned} W^{(d)}_{\Delta,0}(\Delta_i, \mathbf{x}) &= (x^2)^{\frac{(\Delta-\Delta_3-\Delta_4)}{2}} \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{d/2}\,\Gamma(d/2)} \sum_{l,s} \frac{(2l+d-2)\left(\frac{\Delta-\Delta_{12}}{2}\right)_{l+s}\left(\frac{\Delta+\Delta_{34}}{2}\right)_{l+s}}{s!\,(d/2)_{l+s}\,(\Delta)_{l+s}\left(\Delta-\frac{d-2}{2}\right)_s} \\ &\quad \times \left(\frac{\Delta-\Delta_{12}}{2}-\frac{d-2}{2}\right)_s \left(\frac{\Delta+\Delta_{34}}{2}-\frac{d-2}{2}\right)_s x^{l+2s}\, C_l^{\frac{d-2}{2}}\left(\frac{\mathbf{x}\cdot\mathbf{u}}{x}\right) \end{aligned} \tag{14}$$

where $C^{\mu}_l(x)$ are the Gegenbauer polynomials. The above expression matches with the known results for four-point partial wave for scalar primaries with operator dimensions (Δ_i) with $i = 1, 2, 3, 4$ located at $(\infty, \mathbf{u}, \mathbf{x}, \mathbf{0})$ at the boundary respectively [6].

We take limits of our answer in d to match with the known results in the special cases of interest, namely $d \leq 4$ [6, 24].

Conclusion

This synopsis contains a brief summary of our prescription [14, 15] and its successful implementation in some specific cases. We have shown that a generic gravitational open Wilson network satisfies the defining differential equations for conformal partial waves. We reproduced many known results which include CPWs for primary operators in $d \leq 2$ as well as scalar CPWs in an arbitrary dimensions. This prescription has been found applicable in different contexts.

Plan of the thesis:

1. The first chapter will contain a brief introduction to AdS/CFT correspondence in-

cluding the first order formalism of AdS gravity and various methods for computing conformal partial waves.

2. The second chapter will contain specification of our prescription and its properties.

3. The third chapter will contain implementation of the prescription in $d \leq 2$ to compute conformal blocks from OWNs.

4. The fourth chapter will contain a detail calculation of scalar blocks in arbitrary dimensions using OWNs and recovering some special cases.

5. The fifth chapter will contain a discussion of the results, open problems and general outlooks.

List of Figures

Chapter 1

Introduction

Conformal transformations of a space are those coordinate transformations that leave the metric invariant up to a (local) scale factor. Such coordinate transformations form a group called the conformal group of that space. The group of conformal transformations of the d-dimensional flat spacetime includes, in addition to the Poincaré transformations, (global) scaling and coordinate inversion. A quantum field theory that is invariant under such transformations is called a Conformal Field theory (CFT).

Conformal field theories have been studied for a long time because of their relevance to many physical systems. Theories with scale invariance are used to describe many lattice systems such as the Ising models, near their phase transitions, as the characteristic length scale of the systems becomes infinity. And systems with scale invariance, more often than not, also admit the full conformal invariance. Also, a generic QFT at the fixed point of its renormalization group flow is expected to be described by a CFT. At a more formal level CFTs have been playing a pivotal role in the famous AdS/CFT correspondence. The fact that the conformal symmetry algebra is infinite dimensional in $d = 2$ has lead to enormous control over them. Even in higher dimensions the conformal symmetry lends itself helpful towards classification of such field theories via the conformal bootstrap program.

Some of the best known and well studied examples of CFTs include massless free field theories in any dimension, (infinitely large) classes of exact CFTs, such as Minimal models, WZW models etc., in two dimensions, the Wilson-Fisher fixed point of the $O(N)$ vector models in $d = 4$, and a host of supersymmetric CFTs that have relevance to holography, such as the $\mathcal{N} = 4$ SU(N) SYM theory in $d = 4$, the ABJM theory in $d = 3$, $D1 - D5$ CFT in $d = 2$ etc.

We will now review some relevant aspects of CFTs in general dimensions that will be useful for us later in this thesis.

Conformal group

Let us consider a d-dimensional Riemann manifold with a (Euclidean) metric $g_{ab}(x)$. Then its conformal transformations are defined as the coordinate transformations, $x \to x'(x)$ which leave the metric g_{ab} invariant up to a (local) scale factor, i.e.,

$$g'_{ab}(x) = \Omega^2(x) g_{ab}(x). \tag{1.1}$$

where $a, b = 1, \dots d$. We will consider the subgroup of conformal transformations connected to identity. When the manifold in question is simply the flat space $\mathbb{R}^d$ with Cartesian coordinates, and considering infinitesimal coordinate transformations $x^a \to x'^a = x^a + \xi^a(x)$, the condition (1.1) becomes the conformal Killing vector equation

$$\partial_a \xi_b + \partial_b \xi_a = \frac{2}{d} \partial_c \xi^c \, \delta_{ab} \tag{1.2}$$

For $d \geq 3$, the general solution to this equation is

$$\xi^a = a^a + \omega^a{}_b \, x^b + \lambda \, x^a + b^c (x^2 \delta^a_c - 2 x^a x_c) \tag{1.3}$$

with $\omega_{ab} + \omega_{ba} = 0$. The number of independent parameters of these conformal transformations is $\frac{1}{2}(d+2)(d+1)$ and the commutator algebra of the vector fields $\xi^a \partial_a$ is a representation of the corresponding algebra. When $d = 1$ the conformal transformations is $Diff(\mathbb{R}^1)$, and for $d = 2$ it is generated by two commuting copies of Witt algebras. The global parts of the conformal group even in $d = 1$ and $d = 2$ cases are isomorphic to $so(1,2)$ and $so(1,3)$ respectively.

The conformal group (in dimension $d \geq 3$ and the global subgroup in $d \leq 2$) contains as a subgroup the (Euclidean version of the) Poincare group $ISO(d)$ generated by $\frac{1}{2}d(d-1)$ rotation (Lorentz) generators L_{ab}, d momenta P_a. The rest of the generators of the conformal group are the d special conformal generators K_a and one dilatation D. These generators obey the following commutation relations:

$$\begin{aligned} [P_a, K_b] &= -2(\delta_{ab}D + L_{ab}), \qquad [L_{ab}, V_c] = -\delta_{ac}V_b + \delta_{bc}V_a, \\ [D, P_a] &= P_a, \qquad\qquad\qquad\quad [D, K_a] = -K_a, \\ [L_{ab}, L_{cd}] &= \delta_{ad}L_{bc} - \delta_{ac}L_{bd} + \delta_{bc}L_{ad} - \delta_{bd}L_{ac} \end{aligned} \tag{1.4}$$

where $V_a = P_a,\ K_a$. Using the standard identifications

$$D = -M_{0,d+1},\quad P_a = M_{0a} + M_{a,d+1},\quad K_a = -M_{0a} + M_{a,d+1},\quad \text{and} \quad M_{ab} = L_{ab} \tag{1.5}$$

it is seen that the Lie algebra of the Euclidean conformal group is isomorphic to $so(1, d+1)$ algebra

$$[M_{\alpha\beta}, M_{\gamma\delta}] = \eta_{\alpha\delta}M_{\beta\gamma} - \eta_{\alpha\gamma}M_{\beta\delta} + \eta_{\beta\gamma}M_{\alpha\delta} - \eta_{\beta\delta}M_{\alpha\gamma} \tag{1.6}$$

where $\alpha, \beta = 0, 1, \ldots, d+1$ and $\eta_{\alpha\beta} = \mathrm{diag}\{-1, 1, \ldots 1\}$. These can be more naturally thought of as Lorentz rotations of $\mathbb{R}^{1,d+1}$ spacetime with co-ordinates x^α. The corresponding Minkowski version is obtained by Wick rotating the x^{d+1} coordinate of this $(d+2)$-dimensional space into a time-like one, and the resulting algebra is $so(2, d)$.

In $d = 2$ when the co-ordinates (x^1, x^2) on the Euclidean plane $\mathbb{R}^2$ are written as complex variables as

$$z = x^1 + ix^2; \qquad \bar{z} = x^1 - ix^2 \tag{1.7}$$

the conformal transformations becomes

$$z \to f(z); \qquad \bar{z} \to \bar{f}(\bar{z}) \tag{1.8}$$

where $f(z)$ and $\bar{f}(\bar{z})$ are any holomorphic and anti-holomorphic functions respectively. As an infinite number of parameters is needed to specify any analytic functions the two-dimensional conformal group is infinite dimensional. The corresponding Lie algebra reads,

$$\begin{aligned} [l_n, l_m] &= (n-m)l_{n+m} \\ [\bar{l}_n, \bar{l}_m] &= (n-m)\bar{l}_{n+m} \\ [l_n, \bar{l}_m] &= 0\,, \end{aligned} \tag{1.9}$$

and the generators are represented by

$$l_n = -z^{n+1}\partial_z; \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{1.10}$$

This algebra (1.9) is called Witt algebra. As realised by the 2d CFTs at quantum level, a quantity called the "central charge", c (which is measure of the number degrees of freedom) appears in the algebra of the corresponding charges, and each copy of the Witt algebras (1.9) is modified to a Virasoro algebra,

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12}n(n^2-1)\delta_{n+m,0} \\ [L_n, \bar{L}_m] &= 0\,, \end{aligned} \tag{1.11}$$

where L_n are the quantum generators. The set comprising $\{L_{-1},\ L_0,\ L_1\}$ is called the global part of the conformal algebra and forms an $sl(2,\mathbb{R})$ subalgebra of Virasoro algebra.

In $d = 1$ the conformal group becomes even larger as the CKV equation (1.2) is identically satisfied for any $\xi(x)$. It contains all diffeomorphisms of the line $\mathbb{R}^1$ – whose algebra can be taken to be one copy of Witt algebra, generated by the vector fields $T_n = -x^{n+1}\partial_x$. The maximal finite dimensional subalgebra is again $sl(2,\mathbb{R})$ generated by $\{T_0, T_{\pm 1}\}$.

Having described the conformal symmetries as coordinate transformations, we now want to discuss some basic facts about field theories which are invariant under conformal transformations, namely the conformal field theories.

Conformal field theory

A conformal field theory can be defined with the spectrum of local operators and associated conformal data, namely, the conformal weights and spins, the OPE coefficients etc., without referring to any microscopic theory (e.g. Lagrangian description etc.).

Local operators

An operator which transforms under finite conformal transformation, $x \xrightarrow{g} x'$ as

$$\mathcal{O}^i(x) \xrightarrow{g} \mathcal{O}'^i(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} R^i{}_j(g)\,\mathcal{O}^j(x) = \Omega^{\Delta}\, R^i{}_j(g)\,\mathcal{O}^j(x) \tag{1.12}$$

is called (quasi) primary operator with conformal dimension Δ. Under infinitesimal transformations the operator transforms as

$$\begin{aligned} P_a\,\mathcal{O}^i(x) &= \partial_a \mathcal{O}^i(x) \\ L_{ab}\,\mathcal{O}^i(x) &= (x_a\partial_b - x_b\partial_a)\,\mathcal{O}^i(x) + (\Sigma_{ab})^i{}_j \mathcal{O}^j(x) \\ D\,\mathcal{O}^i(x) &= x^\nu \partial_\nu\,\mathcal{O}^i + \Delta\,\mathcal{O}^i(x) \end{aligned}$$

$$K_a\,\mathcal{O}^i(x) \;=\; \left(2x_a x^b\partial_b - x^2\partial_a + 2x_a\Delta\right)\mathcal{O}^i(x) + 2x^b(\Sigma_{ab})^i{}_j\,\mathcal{O}^j(x) \qquad (1.13)$$

where i, j are the irreducible representation indices of $so(d)$. One finds the finite transformations by exponentiating the above relations.

All the operators present in a CFT can be classified into conformal families each corresponding to a specific primary operator $\mathcal{O}$ and its descendants. The conformal families are denoted as $[\mathcal{O}]$. Under a conformal transformation the elements of a given conformal family transform under themselves. Thus each conformal family provides an irreducible representation of conformal group. The vacuum state $|0\rangle$ in a CFT is invariant under conformal transformations (under only global part in $d \leq 2$). The primary operator with conformal dimension Δ acts on the vacuum to create an state $|\Delta\rangle = \mathcal{O}(0)\,|0\rangle$ where we have suppressed the internal indices of the primary operator $\mathcal{O}$ for simplicity. The fact that the operator $\mathcal{O}(x)$ is primary reflects from

$$K_a|\Delta\rangle = 0; \quad D|\Delta\rangle = \Delta|\Delta\rangle; \quad M_{ab}|\Delta\rangle = \Sigma_{ab}|\Delta\rangle \qquad (1.14)$$

Using the rest of the conformal generators P_a the infinite tower of descendant states can be obtained from $|\Delta\rangle$.

Apart from the kinematic quantities (determined by symmetries of the theory) one needs some dynamical input to specify a CFT. Operator product expansion (OPE) in a CFT defines such dynamical quantities, namely OPE coefficients. The basic idea of OPE is to write the product of two local operators in a CFT as series expansion of the operators present in the CFT

$$\mathcal{O}_n(x)\,\mathcal{O}_m(y) \sim \sum_p C^p_{nm}(x-y;\partial_y)\,\mathcal{O}_p(y). \qquad (1.15)$$

where $C^p_{nm}(x-y;\partial_y)$ are differential operators and (n, m) labels are used to indicate different local operators with internal indices suppressed. The right hand side is constrained by symmetries and associativity. The coordinate dependence of $C^p_{nm}(x-y, \partial_y)$ is fixed by the conformal symmetries up to some constant coefficients, called OPE coefficients, that are the dynamical inputs to the theory. We denote the OPE coefficients as C^{p}_{nm}. If one knows the spectrum of primary operators and all their OPE coefficients of a given CFT in principle one can write down all the correlation functions of the CFT.

Having discussed the local operators and the operator algebra in a CFT we now consider a special local operator, the stress tensor which plays an important role in CFT.

Stress tensor

The Poincare symmetries guarantee the existence of a stress tensor T_{ab} which is symmetric ($T_{ab} = T_{ba}$) and conserved ($\partial^a T_{ab} = 0$) [25]. In addition the scale invariance implies the tracelessness of the stress tensor, $T^a{}_a = 0$. All the generators of the conformal symmetries in the quantum theory can be obtained using the stress tensor by computing the Noether charges

$$Q_\xi = \int d^{d-1}x\, \xi^{\mu}_{[\alpha\beta]} T^0_{\mu} \tag{1.16}$$

where ξ is the Killing vector corresponding to the conformal symmetry.

However in general the vacuum expectation value $\langle T^a{}_a \rangle$ is found to be non-vanishing in even-dimensional curved spacetime. This is called trace anomaly. For example when we quantize a conformal field theory on a two-dimensional curved manifold with curvature $R(x)$, then the expectation value of the trace of the energy-momentum tensor becomes,

$$\langle T^a{}_a(x) \rangle = \frac{c}{24\pi} R(x) \tag{1.17}$$

where c is the central charge. But in the rest of this thesis we only consider the CFT where the trace anomaly is absent. So the trace anomaly does not play any role in our discussion.

The consequence of infinitesimal conformal transformations at quantum level may be expressed via Ward identities which involve the stress tensor and other conserved currents in the theory.

Ward identities

Any infinitesimal transformation can be written as,

$$\mathcal{O}'(x) \sim \mathcal{O}(x) - \frac{1}{2}\omega^{\alpha\beta} M_{\alpha\beta}\mathcal{O}(x) \tag{1.18}$$

where we have suppressed the internal indices of the local operator $\mathcal{O}$ for simplicity and the conformal generators $M_{\alpha\beta}$ are written in appropriate representation with the infinitesimal, constant parameters $\omega^{\alpha\beta}$.

Now if we denote the product of n local operators, at coordinates $x_i, i = 1, ..., n$ by X, then for any $\omega^{\alpha\beta}$ we can write the Ward identity for the conserved current $j^a_{[\alpha\beta]}$ as,

$$\partial_a \langle j^a_{[\alpha\beta]} X\rangle = -\sum_{i=1}^{n} \delta(x - x_i) \langle \mathcal{O}(x_1) \ldots M^{(i)}_{\alpha\beta}\mathcal{O}(x_i) \ldots \mathcal{O}(x_n)\rangle \tag{1.19}$$

Now considering the action of generators (1.13) upon a generic local operator we can easily deduce the Ward identities for conformal transformations from (1.19). For translational invariance,

$$\partial_a\langle T^a{}_b X\rangle = -\sum_i \delta(x - x_i) \frac{\partial}{\partial x_i^b} \langle X\rangle \tag{1.20}$$

Using (1.20) we find the Ward identity associated with rotation to be

$$\langle (T^{ab} - T^{ba}) X\rangle = -\sum_i \delta(x - x_i) \Sigma_i^{ab} \langle X\rangle \tag{1.21}$$

and finally for scale invariance,

$$\langle T^a{}_a X\rangle = -\sum_i \delta(x - x_i) \Delta_i \langle X\rangle. \tag{1.22}$$

Equations (1.20-1.22) are the three Ward identities associated with conformal transformation. We see that the stress tensor becomes conserved, symmetric and traceless within the correlation functions, except at the points of operator insertions in the correlator. One can integrate (1.19) over the space that includes all the points x_i to show

$$\sum_i M^{(i)}_{\alpha\beta}\langle X\rangle = 0 \tag{1.23}$$

considering the fact that the correlator $\langle j^a_{[\alpha\beta]} X\rangle$ vanishes sufficiently fast at the boundary of the space considered.

The correlations functions solve the Ward identities. For a given quantum field theory, if the correlation functions are known then one can extract all other information from these. One can constrain the correlation functions to some extent if the QFT has a large symmetry algebra. In particular if the QFT is conformally invariant we may be able to say a little more about the correlation functions than otherwise. In the next section we shall address the constraints imposed on correlation functions by conformal invariance in a generic CFT.

Conformal correlators and partial waves

In a CFT two- and three-point functions are fixed by the conformal symmetry apart from some constant coefficients. For example, two-point correlation function for the scalar quasi-primary operators, O_1 and O_2 with conformal dimensions Δ_1 and Δ_2 respectively takes the form,

$$\langle O_1(x_1)O_2(x_2)\rangle = \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta}} & \text{if } \Delta_1 = \Delta_2 = \Delta \\ 0 & \text{if } \Delta_1 \neq \Delta_2 \end{cases} \tag{1.24}$$

where C_{12} is a constant coefficient. The three-point function for scalar primaries becomes

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{13}|^{\Delta_3+\Delta_1-\Delta_2}} \tag{1.25}$$

where $x_{ij} \equiv x_i - x_j$ and C_{123} are the OPE coefficients that depend on the theory.

However unlike two- and three- point functions, the n-point functions cannot be fixed by conformal invariance for $n \geq 4$. There exist conformally invariant cross ratios – given four points one can construct two independent cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad \text{and} \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{1.26}$$

The n-point functions may have an arbitrary dependence on such cross ratios.

Nevertheless we can use (1.15) in n-point function to separate it out into kinematic and dynamic parts. This procedure is called partial waves expansions. For example the four-point function of four scalar primary operators can be decomposed as:

$$\langle O_1(x_1) O_2(x_2) O_3(x_3) O_4(x_4) \rangle = \sum_O C_{12O} C^O{}_{34} W_{\Delta,l}(\Delta_i;\ x_i) \tag{1.27}$$

where C_{12O} are the OPE coefficients and the partial wave $W_{\Delta,l}(\Delta_i;\ 1x_i)$ is

$$W_{\Delta,l}(\Delta_i;\ x_i) = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}(\Delta_1-\Delta_2)} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}(\Delta_3-\Delta_4)} \frac{G_{\Delta,l}(u,v)}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)} (x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}} \tag{1.28}$$

The pre-factor is determined by the conformal invariance and the conformal block $G_{\Delta,l}(u,v)$ depends only on the conformally invariant cross ratios u, v. Thus the basic building blocks of the correlation functions of primary operators in a CFT are the partial waves which in-turn are proportional to conformal blocks. The conformal blocks are important inputs into the Bootstrap approach [26–55] to constrain the dynamical data of any CFT. There are different methods to compute conformal partial waves.

1. **Solving differential equations:** In general (for $d \geq 2$, and for $d \leq 2$ restricting to the contribution of global conformal blocks alone) the partial wave $W_{\Delta,l}(x_i)$ is

expected to satisfy two types of differential identities:

$$\begin{aligned}(M^{(1)}_{AB} + M^{(2)}_{AB} + M^{(3)}_{AB} + M^{(2)}_{AB})W_{\Delta,l}(x_i) &= 0 \\ (M^{(1)}_{AB} + M^{(2)}_{AB})(M^{AB}_{(1)} + M^{AB}_{(2)})W_{\Delta,l}(x_i) &= C_2(\Delta, l)W_{\Delta,l}(x_i) \\ &= (M^{(3)}_{AB} + M^{(4)}_{AB})(M^{AB}_{(3)} + M^{AB}_{(4)})W_{\Delta,l}(x_i)\end{aligned} \tag{1.29}$$

where $M^{(i)}_{AB}$ is the operator representing the global conformal transformation generator acting on the primary operator $O_i(x_i)$ and C_2 is the quadratic Casimir of the representation of the operator O being exchanged in the intermediate channel. The first one is the reflection of the fact that the partial wave is covariant under the global conformal transformations. The second is the conformal Casimir equation [4–6].[1] These can be solved with appropriate boundary conditions to obtain explicit expressions [4–6] for the partial waves.

2. **Using projectors:** The projector onto the conformal family containing a primary O (internal indices are suppressed) is defining as

$$|O| = \sum_{\alpha,\beta} |\,\alpha\,\rangle\, \mathcal{N}^{-1}_{\alpha\beta}\, \langle\,\beta\,|; \qquad \mathcal{N}_{\alpha\beta} \equiv \langle \alpha\,|\beta\rangle \tag{1.30}$$

where α, β represents the primary and its descendents in the conformal family. The projector satisfies the relation

$$I = \sum_{O} |O| \tag{1.31}$$

where is the sum is carried over all the primaries present in the theory.

One can insert the projector inside the correlator, e.g. for four-point function one

[1] If there are non-trivial higher order Casimir operators of the conformal algebra then one has to impose the corresponding differential equations as well.

can write

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \sum_{O} \langle 0|\mathcal{R}\{O_3(x_3)\,O_4(x_4)\}\,|O|\;\mathcal{R}\{O_3(x_3)\,O_4(x_4)\}\,|0\rangle \tag{1.32}$$

where $\mathcal{R}$ stands for radial ordering and one can choose the origin such that $|x_{3,4}| \geq |x_{1,2}|$. Each term in the above sum associates with the square of OPE coefficients and a conformal block with the primary operator O and its descendents exchanged in the intermediate channel.

3. **Series expansion:** We discuss this method with a simple case which corresponds to the four-point function of identical scalar operators O with the operator dimension Δ_O. The conformal transformations can be used to put all four operators on a plane [56, 57] (see also [58]). The operator insertions can be mapped to $\mathbf{x}_1 = -\mathbf{r}$, $\mathbf{x}_2 = \mathbf{r}$, $\mathbf{x}_3 = -\mathbf{u}$ and $\mathbf{x}_4 = \mathbf{u}$ with $\mathbf{u}\cdot\mathbf{u} = 1$ and $\mathbf{r} = \rho\,\mathbf{n}$. In radial quantization this mapping takes the operators to diametrically opposite points $\pm\mathbf{n}$ and $\pm\mathbf{u}$ respectively on S^{d-1} with $\mathbf{n}\cdot\mathbf{u} = \cos\theta$. The two pairs $(1,2)$ and $(3,4)$ are separated by time $\tau = -\log\rho$.

   The set of descendants of a primary operator O_Δ with conformal weight Δ and spin j are denoted as $|\Delta, j;\, m\rangle^{\mu_1 \ldots \mu_j}$ with energy $\Delta + m$. Rotational invariance implies

$$\langle 0|O(-\mathbf{r})\,O(\mathbf{r})|\Delta, j;\, m\rangle^{\mu_1\ldots\mu_j} = C\,\frac{\rho^{\Delta+m}}{\rho^{2\Delta_O}}\left(n_{\mu_1}\ldots n_{\mu_j} - \text{traces}\right) \tag{1.33}$$

   where C is a constant. Finally using the above relation and inserting the projector into the four-point function one can write the conformal block in terms of Gegenbauer polynomials $C_j^{\mu}(\cos\theta)$ as

$$G_{\Delta,l}(u,v) = \sum_{m,j} B_{m,j}\,\rho^{\Delta+m}\,C_j^{\frac{d-2}{2}}(\cos\theta) \tag{1.34}$$

   where $j \in \{|l-m|,\ |l-m|+2, \ldots (l+m)\}$ and m sums over all positive even integers.

Several different techniques, e.g. embedding-space formalism [59–66], use of shadow operators [62] etc. have been used along with the methods described above to deal with the spinning operators as well as some other difficult problems.

4. **Recursion relations:** There exist some powerful recursion relations that enable one to compute the conformal partial waves in a given dimension in terms of those in lower dimensions [62,67]. For instance, one such recursion relation among the even dimensional conformal partial waves was given in [62]. Recently a generalization of such a relation among the odd dimensional conformal partial waves has been provided [15].

5. **Mellin space technique:** The Mellin transform is defined as

$$f(x) = \int_C ds\, x^{-s}\, \tilde{f}(s) \tag{1.35}$$

Assuming $\tilde{f}(s)$ falls off at infinity sufficiently fast the power law behavior of $f(x)$ is captured by finding the poles of $\tilde{f}(s)$. For example, if $f(x) = 1/x^{\Delta}$ with $\Delta > 0$, then $\tilde{f}(s) = 1/(s - \Delta)$.

The Mellin amplitude $\mathcal{M}_n(s_{ij})$ of a general n-point CFT correlation functions of primaries O_i with conformal weight Δ_i is given by [68, 69]

$$\langle O_1(x_1) \ldots O_n(x_n) \rangle = \int_{-i\infty}^{i\infty} [ds_{ij}] \prod_{i<j=1}^{n} x_{ij}^{-2s_{ij}}\, \Gamma(s_{ij})\, \mathcal{M}_n(s_{ij}) \tag{1.36}$$

with the constraints on the Mellin variables, $\sum_{j \neq i} s_{ij} = \Delta_i$. For example let us consider the Mellin transform of the four-point function (1.27). In this case there are two independent Mellin variables which can be written as

$$s_{12} = \frac{\Delta_1 + \Delta_2}{2} - s \qquad \text{and} \qquad s_{14} = -t \tag{1.37}$$

Then the Mellin amplitude $B^{(s)}_{\Delta,l}(s,t)$ of the s-channel conformal block $G^{(s)}_{\Delta,l}(u,v)$ is given by [54, 70, 71]

$$B^{(s)}_{\Delta,l}(s,t) = e^{i\frac{\pi}{2}(2s-\Delta+l)} \frac{(-2\pi i)\,\Gamma(\frac{\Delta-l}{2}-s)}{\Gamma(s+1-n+\frac{\Delta-l}{2})\Gamma(\frac{\Delta_1+\Delta_2}{2}-s)\Gamma(\frac{\Delta_3+\Delta_4}{2}-s)} P^{(s)}_{\Delta-n,l}(s,t) \tag{1.38}$$

where $P^{(s)}_{\Delta-n,l}(s,t)$ are the Mack polynomials.

The Mellin space approach to compute conformal blocks has following advantages. The Mellin amplitudes are meromorphic function having only simple poles. The poles in different channel (s,t) correspond to twist $\tau = \Delta - l$ of the operators in that channel. The residues at the poles are related to three-point functions (factorization property). This leads to the recursion relations which can be solved to compute Mellin amplitudes in terms of lower point amplitudes [72–74].

6. **Holographic approach:** The gauge/gravity correspondence has emerged as one of the most useful tools in exploring quantum properties of conformal field theories and gravitational/string theories over the last two decades. It frames an equivalence between a d-dimensional conformal field theory (CFT$_d$) and a string theory in a $(d+1)$-dimensional Anti-de Sitter (AdS$_{d+1}$) background geometry. The conjecture and a duly formed dictionary of AdS/CFT correspondence [1–3] equip us with a prescription to compute CFT correlation functions in terms of so-called Witten diagrams in the bulk AdS gravity. Given this it is a natural question to ask if the partial waves can be computed using holographic techniques. This thesis addresses this issue and provides a novel prescription to compute conformal partial waves using holography.

Before proceeding further let us review some essential aspects of the AdS/CFT correspondence.

The AdS/CFT correspondence

The $(d+1)$-dimensional Anti-de-Sitter (AdS) spacetime which solves the Einstein equation in vacuum with negative cosmological constant, Λ is given by the hyper-surface in $\mathbb{R}^{2,d}$,

$$-X_{-1}^2 - X_0^2 + X_1^2 + \cdots\cdot + X_d^2 = -l^2. \tag{1.39}$$

with the curvature $R = -d(d+1)/l^2$. The Euclidean version of AdS_{d+1} which we will work on can can be obtained starting from $\mathbb{R}^{1,d+1}$. In Poincaré patch parameterized by $\{\rho, x^i\}$ the metric of AdS_{d+1} becomes,

$$l^{-2}\, ds^2 = d\rho^2 + e^{2\rho} \sum_{i=1}^{d} dx_i dx^i. \tag{1.40}$$

The conformal boundary of AdS_{d+1} $(\rho \to \infty)$ is identified with $\mathbb{R}^d$ [3].

The AdS/CFT conjecture states the theory of quantum gravity (often called "bulk theory") in AdS_{d+1} is dual to d-dimensional conformal field theory (often called "boundary theory") on the Minkowski space. Further each field in the bulk is associated with an operator on the boundary and vice versa. For instance, a scalar field of mass m in the bulk is dual to an operator of scaling dimension $\Delta = d/2 + \sqrt{(d/2)^2 + m^2}$ on the boundary. Similarly the graviton in the bulk is dual to the energy-momentum tensor on the boundary and a gauge field is dual to conserved currents. The bulk action generally takes the form

$$S_{bulk} = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x\, \sqrt{g}\, (R - 2\Lambda + \mathcal{L}_{matter}) \tag{1.41}$$

where G_{d+1} is the Newton constant in $(d+1)$-dimension and $\Lambda = -\frac{d(d-1)}{2l^2}$.

The AdS/CFT conjecture for computation of correlation functions relates the on-shell (evaluated in the space of solutions with appropriate boundary conditions) bulk partition function

$$\mathcal{Z}_{bulk} = \int \mathcal{D}\phi \, e^{-S_{bulk}} \tag{1.42}$$

to the generating functional of the CFT correlators in the boundary theory

$$\mathcal{Z}_{bulk}[\Phi_0(\mathbf{x})] = \langle \, e^{\int d^d\mathbf{x} \;\Phi_0(\mathbf{x})\, O(\mathbf{x})} \, \rangle_{CFT} \tag{1.43}$$

where the boundary value of a generic bulk field Φ_0 becomes the source of the dual operator O in the generating functional. We have suppressed the internal indices of the bulk field as well as the boundary operators for simplicity. Differentiating (1.43) with respect to Φ_0 one can evaluate the CFT correlation functions

$$\frac{\delta}{\delta\Phi_0(\mathbf{x_1})} \cdots \frac{\delta}{\delta\Phi_0(\mathbf{x_n})} \mathcal{Z}_{bulk}[\Phi_0(\mathbf{x})] \Big|_{\Phi_0=0} = \langle \, O(\mathbf{x_1}) \cdots O(\mathbf{x_n}) \, \rangle \tag{1.44}$$

For illustration let us consider a massive scalar field ϕ in the matter sector in (1.41)

$$\mathcal{L}_{matter} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + V(\phi) \tag{1.45}$$

where $V(\phi)$ is the interaction term in the action. When the theory is free i.e. $V(\phi) = 0$ the equation of motion is given by

$$\left(\Box_g + m^2\right)\phi \equiv \frac{1}{\sqrt{g}}\partial_\mu \left(\sqrt{g} g^{\mu\nu}\partial_\nu\phi\right) + m^2\phi = 0 \tag{1.46}$$

Using the Green function $K_{d,\Delta}(\rho, \mathbf{x};\, \mathbf{x}')$ the solution of (1.46) can be found with the boundary condition

$$\lim_{\rho\to\infty} e^{\rho\,(d-\Delta)}\, \phi(\rho, \mathbf{x}) = \phi_0(\mathbf{x}) \tag{1.47}$$

and regularity requirements at $(\rho \to \infty)$ as

$$\phi(\rho, \mathbf{x}) = \int d^d\mathbf{x}'\, K_{d,\Delta}(\rho, \mathbf{x};\, \mathbf{x}')\, \phi_0(\mathbf{x}') \tag{1.48}$$

where the “bulk-to-boundary propagator" $K_{d,\Delta}(\rho, \mathbf{x};\, \mathbf{x}')$ satisfies the following conditions

$$\begin{aligned}
\left(\Box_g + m^2\right) K_{d,\Delta}(\rho, \mathbf{x};\, \mathbf{x}') &= 0 \\
\lim_{\rho\to\infty} e^{\rho\,(d-\Delta)}\, K_{d,\Delta}(\rho, \mathbf{x};\, \mathbf{x}') &= \delta^{(d)}(\mathbf{x} - \mathbf{x}') \\
\lim_{\rho\to-\infty} K_{d,\Delta}(\rho, \mathbf{x};\, \mathbf{x}') &= 0
\end{aligned} \tag{1.49}$$

One finds the bulk-to-boundary propagators for scalar fields to take the form [75]

$$K_{d,\Delta}(\rho, \mathbf{x};\, \mathbf{x}') = \left(\frac{e^{-\rho}}{e^{-2\rho} + |\mathbf{x} - \mathbf{x}'|^2}\right)^{\Delta} \tag{1.50}$$

If we add a cubic interaction $V(\phi) = -\frac{\lambda}{3}\phi^3$ to the action, the equation of motion becomes

$$(\Box_g + m^2)\phi = \lambda\phi^2 \tag{1.51}$$

which can be solved perturbatively in λ. Substituting $\phi = \phi_{(0)} + \lambda\,\phi_{(1)} + O(\lambda^2)$ in (1.51) we get

$$(\Box_g + m^2)\,\phi_{(0)} = 0; \qquad (\Box_g + m^2)\,\phi_{(1)} = \phi_{(0)}^2 \tag{1.52}$$

To solve these equations we introduce the Green function $G_{d,\Delta}(\rho, \mathbf{x};\, \rho', \mathbf{x}')$, also known as “bulk-to-bulk propagator" which are defined as

$$\begin{aligned}
(\Box_g + m^2)\, G_{d,\Delta}(\rho, \mathbf{x};\, \rho', \mathbf{x}') &= \frac{1}{\sqrt{g}}\, \delta(\rho - \rho')\, \delta^{(d)}(\mathbf{x} - \mathbf{x}') \\
\lim_{\rho\to\infty} e^{\rho(d-\Delta)} G_{d,\Delta}(\rho, \mathbf{x};\, \rho', \mathbf{x}') &= 0 \\
\lim_{\rho\to-\infty} G_{d,\Delta}(\rho, \mathbf{x};\, \rho', \mathbf{x}') &= 0
\end{aligned} \tag{1.53}$$

The bulk-to-bulk propagator for bulk scalar field is given by [75]

$$G_{d,\Delta}(\rho, \mathbf{x};\, \rho', \mathbf{x}') = e^{-\Delta\sigma(\rho,\mathbf{x};\rho',\mathbf{x}')}\, {}_2F_1\left(\Delta,\, \frac{d}{2};\, \Delta+1-\frac{d}{2};\, e^{-2\sigma(\rho,\mathbf{x};\rho',\mathbf{x}')}\right) \tag{1.54}$$

where $\sigma(\rho, \mathbf{x};\, \rho', \mathbf{x}')$ is the geodesic distance between points $(\rho, \mathbf{x})$ and $(\rho', \mathbf{x}')$

$$\sigma(\rho, \mathbf{x};\, \rho', \mathbf{x}') = \log\left(\frac{1+\sqrt{1-\xi^2}}{\xi}\right); \qquad \xi = \frac{2e^{-(\rho+\rho')}}{e^{-2\rho}+e^{-2\rho'}+|\mathbf{x}-\mathbf{x}'|^2} \tag{1.55}$$

The solutions of equations of motion (1.52) are

$$\begin{aligned} \phi_{(0)}(\rho, \mathbf{x}) &= \int d^d\mathbf{x}'\, K_{d,\Delta}(\rho, \mathbf{x};\, \mathbf{x}')\, \phi_0(\mathbf{x}') \\ \phi_{(1)}(\rho, \mathbf{x}) &= \int d\rho' \int d^d\mathbf{x}' \int d^d\mathbf{x}'' \int d^d\mathbf{x}''' \;\sqrt{g}\, \phi_0(\mathbf{x}'')\, K_{d,\Delta}(\rho', \mathbf{x}';\, \mathbf{x}'') G_{d,\Delta}(\rho, \mathbf{x};\, \rho', \mathbf{x}') \\ &\qquad \times K_{d,\Delta}(\rho', \mathbf{x}';\, \mathbf{x}''')\, \phi_0(\mathbf{x}''') \end{aligned} \tag{1.56}$$

Diagrammatically this can be shown as

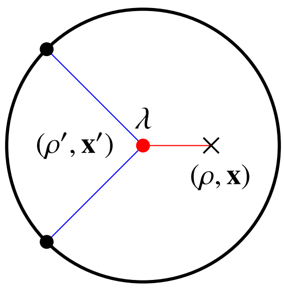


Figure 1.1: Witten diagram for solving ϕ.

where the blue lines represent the bulk-to-boundary propagators, red lines represent the bulk-to-bulk propagators and the cubic vertex is denoted as red dot. One substitutes the solution for ϕ in the bulk partition function and then differentiating with respect to the boundary value ϕ_0 computes three-point function in a perturbative expansion in λ. The cubic vertex contributes to the four-point function at $O(\lambda^2)$ as shown in the following diagram (s-channel)

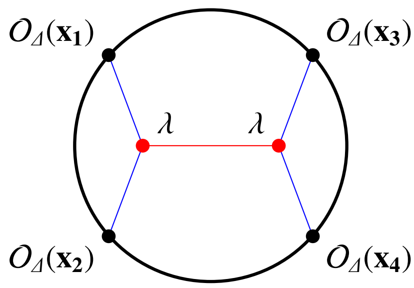


Figure 1.2: 4-point function using cubic vertex

along with the similar diagrams with the end-points permuted (t-channel and u-channel).

In a similar manner one can consider the quartic interaction $-\frac{g}{4}\phi^4$ in addition to the cubic interaction in the bulk action and compute the four-point function as

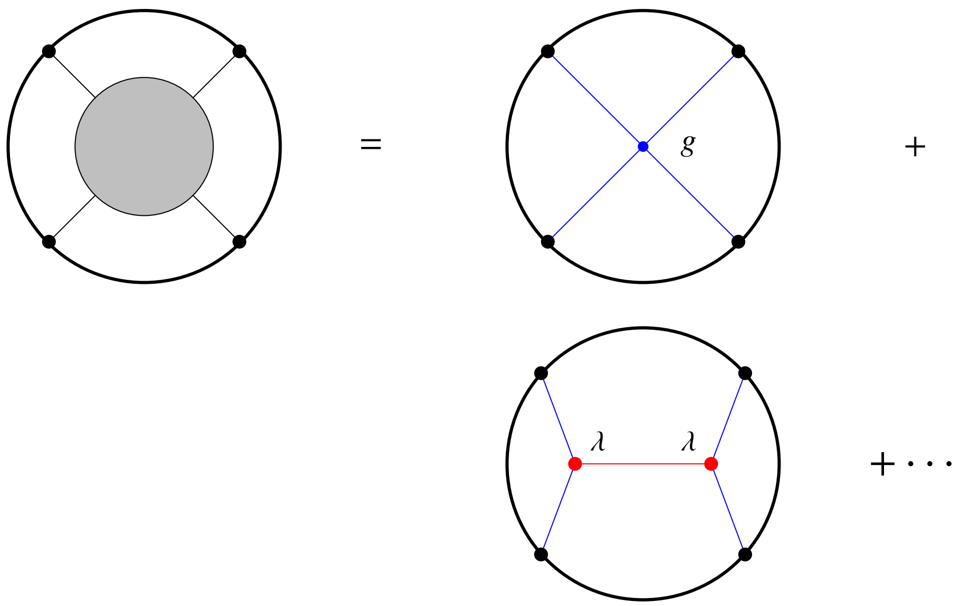


Figure 1.3: Witten diagrams for 4-point function.

The rules to evaluate such diagrams are fairly simple. The vertex points are connected to the locations of boundary operators via bulk-to-boundary propagators. Two vertices in the bulk can be connected through bulk-to-bulk propagators. Finally one has to integrate the locations of vertices over the bulk geometry.

Similarly the correlators of non-scalar boundary operators can be computed by considering the appropriate fields and the interaction terms in the bulk theory [76].

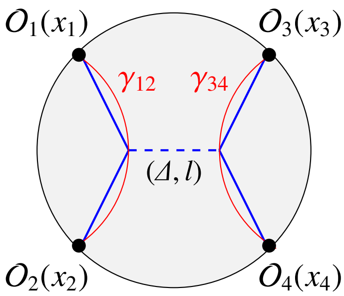
Since AdS/CFT provides a natural avenue to answer questions in CFT_d in terms of AdS_{d+1} gravity (and vice versa) it is natural to ask how to compute the conformal partial waves of a given correlation function of primary operators in a CFT holographically.

Holographic conformal partial waves

Recently a novel prescription has been provided to compute conformal partial waves holographically by evaluating so-called geodesic-Witten diagrams in the bulk AdS_{d+1} gravity [7].

Geodesic Witten diagrams

A geodesic Witten diagram is a similar object to an ordinary Witten diagram. The only difference is that instead of integrating the locations of vertices over the bulk one has to integrate the vertices over the geodesics connecting each pair of the boundary operator insertions. For example the four-point semi-classical scalar conformal partial waves can be obtained by evaluating the diagram



$$\propto \quad W_{\Delta,l}\,(\Delta_i;\ \mathbf{x}_i)$$

Figure 1.4: Geodesic Witten diagram

In Fig. 1.4 the solid blue lines represent the bulk-to-boundary propagators, the dashed blue lines are bulk-to-bulk propagators for traceless symmetric tensor fields and the geodesics connecting the pairs of boundary points γ_{12} and γ_{34} are denoted by red lines. The diagram becomes

$$W_{\Delta,l}\,(\Delta_i;\,\mathbf{x}_i) \;=\; \int_{\gamma_{12}} d\lambda \int_{\gamma_{34}} d\lambda'\, G_{b\partial}(X(\lambda),\mathbf{x}_1;\,\Delta_1)\, G_{b\partial}(X(\lambda),\mathbf{x}_2;\,\Delta_2)$$

$$\times\, G_{bb}(X(\lambda),X(\lambda');\,\Delta,l) \times G_{b\partial}(X(\lambda'),\mathbf{x}_3;\,\Delta_3)\, G_{b\partial}(X(\lambda'),\mathbf{x}_4;\,\Delta_4) \tag{1.57}$$

when the geodesics γ_{12} and γ_{34} are parameterized by λ and λ' respectively. Carrying out the integration in the above expression one can obtain the conformal partial waves. It is shown in [7] that a generic geodesic Witten diagram satisfies the global Ward identity and the conformal Casimir equation (1.29).

This method has been applied in locally AdS_3 geometries. Considering the backreaction of heavy operator insertions the semi-classical Virasoro block in $2d$ CFT has been computed [77, 78]. Using the prescription the global conformal partial waves are obtained when the boundary CFT lives on the torus. It is shown that the one-point toroidal block can be written as geodesic Witten diagrams in thermal AdS_3 [79]. This method is also generalized further to compute spinning blocks [80, 81], conformal blocks involving anti-symmetric tensor primaries [82] and fermions [83] etc.

The geodesic Witten diagrams (GWD) methods to compute CFT the partial waves are well suited when the bulk gravitational theory is formulated in terms of metric, namely the Einstein-Hilbert formulation. However, sometimes it is convenient/essential to write the gravitational theory as extensions of tetrad formulation (Hilbert-Palatini formalism, also known as first order formalism). For example, the higher spin gauge theories in three dimensions are described as Chern-Simons theories in which the gravity sector is written in the first order formalism [8–10]. It is also essential when one deals with spinors in a gravitational background. In dimensions greater than three, the theory of gravity with negative cosmological constant can be written in the Hilbert-Palatini formalism, a BF-type gauge theory [11, 13]. Therefore, it is important to ask how to compute CPWs in this formalism.

In this thesis this question has been answered with a new prescription to compute CPWs in Euclidean CFT_d holographically in terms of gravitational open Wilson line networks in the Hilbert-Palatini formulation of Euclidean AdS_{d+1} gravity.

Gravitational open Wilson network

In the Hilbert-Palatini formulation of AdS_{d+1} gravity the gravitational fields, namely the vielbein 1-forms e^a and the spin-connection 1-forms ω^{ab} can be packaged into one $so(1, d+1)$ adjoint valued gauge connection

$$A = \frac{1}{2}\omega^{ab} M_{ab} + \frac{1}{l} e^a\, M_{0a} \tag{1.58}$$

where M_{0a} and M_{ab} are the generators of $so(1, d+1)$ with $a, b = 1, \cdots d+1$ as in (1.6). The parameter l with dimensions of length sets the radius of AdS_{d+1} vacuum. The action for the connection A is in general an appropriate BF type theory [11–13].

$$S[A,\ B,\ \Phi] = \int_M \text{Tr}\ (B \wedge F) + \frac{1}{2}\text{Tr}\ (B \wedge \Phi(B)) \tag{1.59}$$

where F is the field strength $F = dA + A \wedge A$, B is a Lie algebra valued $(d-1)$-form field and Φ is the Lagrange multiplier which couples with B in a specific way to form a Lie algebra valued 2-form, denoted by $\Phi(B)$. For example, in $d = 1$ the action becomes [84–86]

$$S = \int_{M_2} \text{Tr}\ (\Phi\, F) \tag{1.60}$$

where Φ is a zero-form field and the gauge algebra is $sl(2, \mathbb{R})$. This action is dynamically equivalent to that of Jackiw-Teitelboim model [87, 88]. In $d = 2$ the corresponding action is

$$S = \int_{M_3} \text{Tr}\ (B \wedge F) \tag{1.61}$$

where B is a one-form field. This action can be recast as Chern-Simmons action [8–10]

$$S_{CS} = \frac{k}{4\pi} \int_{M_3} \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{1.62}$$

where the connection takes values in the adjoint representation of $so(1,3)$. In $d = 3$ the action contains a fixed $SO(5)$ vector v^A [13]

$$S = \int_{M_4} \left(B^{IJ} \wedge F_{IJ} - \frac{1}{2} B^{IJ} \wedge B^{KL} \epsilon_{IJKLM} v^M \right) \tag{1.63}$$

where $I, J = 1, \ldots, 5$. The flat connection, $F = 0$ leads to the equations

$$de^a + \omega^a{}_b \wedge e^b = 0 \tag{1.64}$$

$$d\omega^{ab} + \omega^c{}_d \wedge \omega^{db} + \frac{1}{l^2} e^a \wedge e^b = 0. \tag{1.65}$$

The eq. (1.64) is the torsionless condition and the eq. (1.65) describes the locally AdS_{d+1} spaces. In $d \leq 2$, (1.64) and (1.65) becomes the equations of motion for the gauge fields. This implies that all solutions of the gravity theory in $d \leq 2$ are locally AdS space where as there are more general solutions in $d \geq 3$. One can couple matter particles as external sources to the AdS_{d+1} gravity considering the Wilson line operators for the gauge connection (1.58) in an appropriate representation of the gauge algebra along the curve given by the trajectory of the particle [89, 90].

Our aim is to provide a prescription to compute the conformal partial waves of the boundary CFT in this first order formalism of bulk gravity. Recall that these (global) partial waves constitute the non-dynamical building blocks of any CFT correlation function. Therefore one expects that they should be computable holographically without putting too much dynamical information about the bulk theory and its interactions. Indeed, our proposal on the gravity side uses certain “tree-level" (no loop) open Wilson network (OWN) operators for the connection (1.58). The operators of our interests will be associated to directed trivalent open spin networks (every line in the graphs carries representation labels, i.e. “spins" of the conformal algebra $so(1, d+1)$) such as the graphs in Fig. 1.5 with their end points on the boundary of AdS_{d+1}. We propose that (a subset of) the expectation values of such OWNs compute the (global) partial waves of the dual CFT [14].

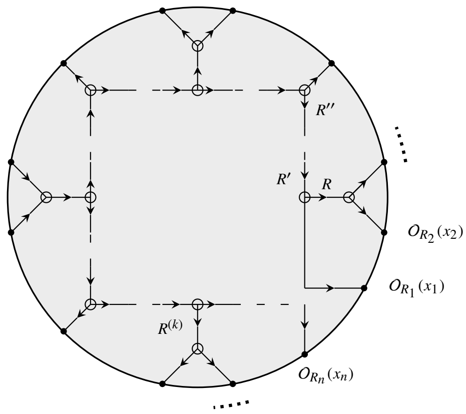


Figure 1.5: A typical directed trivalent Open Wilson Network

The representations of the external legs are determined by the data of the primary operators in a correlation function of the CFT whose partial waves we want to compute. We glue together the lines joining at a vertex with an appropriate Clebsch-Gordan coefficients (invariant tensors of the three representations involved) of the gauge algebra. We further project the external Wilson lines on to those states in the representation of that external leg that transform in a finite dimensional representation of local Lorentz/rotation algebra $so(d+1)$. We call these states as “cap-states". Such states that transform in the scalar representation of the local Lorentz algebra have appeared in the recent literature in the context of construction of local bulk operators [16–18].

It will be shown that the open Wilson network operators evaluated the background of pure AdS geometry satisfies the global conformal Ward identities such as (1.29) when the external points are taken to the boundary. We will also establish that a certain subset of the diagrams satisfies the appropriate set of conformal Casimir equations such as the one in (1.29). We will also evaluate such diagrams explicitly in the limit of the ends going to the boundary of the AdS space and reproduce the known answers in several cases; general cases in $d \leq 2$ and scalar conformal waves in general d.

The rest of the thesis is organised as follows. In chapter 2 we elaborate on the construction of the open Wilson networks of interest and show that they satisfy the right identities such as the global conformal Ward identities and the conformal Casimir equations. In chapter 3 we initiate explicit evaluation of these diagrams in $d \leq 2$ and compare our results with the known answers in the literature. The chapter 4 contains the construction of the modules and the conformal wave functions required for the computation of scalar blocks. We also introduce the concept of OPE module here and use it to carry out the computation of the 4-point scalar blocks in arbitrary dimensions. We also show here how our answers match with several known results in $d \leq 4$. We provide a discussion of our results and open questions in chapter 5. The appendices contain some relevant mathematical results used in the thesis.

Chapter 2

CPW as OWNs: generalities

We are interested in providing a prescription to compute partial waves of correlation functions of the dual CFT_d in terms of the first order action of AdS_{d+1} gravity. As alluded to in the introduction the basic ingredients are the gauge covariant and non-local Wilson line operators:

$$W_y^x(R, C) = P \exp\left[\int_y^x A\right] \tag{2.1}$$

where x and y are two points in the space (with a boundary $\mathbb{R}^d$), C is a curve connecting those, R is a representation of the gauge algebra $so(1, d + 1)$ and A is the pull back of the gauge connection onto the curve C. As usual the symbol P denotes the standard path ordering prescription. Under a gauge transformation $A \to hAh^{-1} + hdh^{-1}$ the Wilson line operator transforms covariantly as

$$W_y^x \to h(x) W_y^x h^{-1}(y) . \tag{2.2}$$

One can consider open Wilson network operators such as the one alluded to in the introduction where we take several directed open Wilson lines in different representations of the gauge algebra and glue the ends of any three lines ending at the same point by contracting the representation indices into the Clebsch-Gordan coefficients relating those representations. Such an open Wilson network operator will depend on the coordinates of the end points, the representations of each open Wilson line and, in general, the spin network $\mathcal{N}$ that went into its construction. Let us call such an operator with n end points $W_{\mathcal{N}}(x_1, R_1; x_2, R_2; \cdots; x_n, R_n)$. Then under gauge transformations it will transform covariantly as a tensor in the tensor product of all the representations $(R_1, \cdots, R_n)$. We will be interested in only those representations of $so(1, d+1)$ which are related to the unitary (infinite dimensional) irreducible representation of the corresponding $so(2, d)$ relevant to the Lorentzian CFT.

In a general gauge theory we cannot hope that the expectation value of such an open Wilson network operator represents any physical quantity as it will not be gauge invariant. However the gauge transformations of (1.58) with gauge group $SO(1, d+1)$ can be split into two subclasses representing both Local Lorentz transformations and the diffeomorphisms in the Euclidean AdS_{d+1} gravity in the Hilbert-Palatini formulation. If we call the generators of $SO(1, d+1)$ as $\{M_{ab}, M_{0a}\}$ with $a, b = 1, \cdots, d+1$ where M_{ab}'s generate the maximal compact subalgebra $so(d+1)$ and M_{0a}'s are like the boost operators of the Lorentz group - then the gauge transformation with parameter in the subalgebra $so(d+1)$ correspond to (the Euclidean analogs of) the Local Lorentz transformation of the vielbein e^a and the spin-connection ω^{ab}.

Furthermore, since the bulk theory is supposed to describe geometries that are asymptotically AdS_{d+1} which have a boundary the observables do not necessarily have to be invariant under all the gauge transformations but only under *small* gauge transformations, namely, those which do not have a non-trivial action on the boundary [91].

According to AdS/CFT the dual of a boundary primary operator is a bulk field. The fields in the bulk transform in *finite* dimensional representations of the group of local Lorentz transformations. For instance if one considers a bulk field $\phi^{\nu_1\nu_2\cdots}_{\mu_1\mu_2\cdots}$ with spacetime indices then the (inverse) vielbeins can be used to convert all the spacetime indices into the tangent space indices so that the field transforms in a finite dimensional irreducible representation of the tangent space rotation algebra $so(d+1)$. Therefore, we would first like to project the quantity $W_{\mathcal{N}}(x_1, R_1; x_2, R_2; \cdots ; x_n, R_n)$ which is an element of the tensor product of the infinite dimensional representations R_i of $so(1, d+1)$ down to that of the finite dimensional representations of the local Lorentz algebra $so(d+1)$. This step can be achieved by projecting the i^{th} external leg of the Wilson network operator in the representation R_i of $so(1, d+1)$ onto vectors in this representation which provide the appropriate finite dimensional representation of the sub-algebra $so(d+1)$. As we will see (explicitly later in chapter 3 and 4) such special states do exist and their construction is closely related to those in [16, 18]. It will turn out that one particular component of such a tensor has the leading fall off behaviour, as the points x_i approach the boundary, compared to the other components. This component will be related to the partial wave of the corresponding primary operators of the CFT.

Having defined the open Wilson network (OWN) operators of interest classically, the next issue is how to define the expectation value of these operators in the quantum gauge theory. One can use a path integral definition [92] for this. However we will not attempt to do this in this thesis (see [93] for an extension beyond the semi-class analysis in $d = 2$). Instead we will restrict ourselves to computing the values of these operators in the background of pure AdS space - which corresponds to evaluating the expectation values of these operators in the (semi-) classical limit.

For any locally AdS space the corresponding gauge field strength of A in (1.58) vanishes. For such pure gauge configurations one can take $A = g\, dg^{-1}$ locally where g is an element of the group $SO(1, d+1)$. Then it follows from the definition (1.58) that such a configuration describes a given space with the corresponding e^a and ω^{ab} satisfying the equation:

$$dg + \frac{1}{2}\omega^{ab} M_{ab}\, g + \frac{1}{l} e^a M_{0a}\, g = 0. \tag{2.3}$$

If we are interested in finding the gauge field A for a given geometry with given e^a and ω^{ab} we just have to solve this equation for g and then use $A = -dg\, g^{-1}$. The integrability condition of the equation (2.3) reads:

$$[\partial_\mu + \frac{1}{2}\omega_\mu^{ab} M_{ab} + \frac{1}{l} e_\mu^a M_{0a}, \partial_\nu + \frac{1}{2}\omega_\nu^{cd} M_{cd} + \frac{1}{l} e_\nu^c M_{0c}]\, g(x) = 0 \tag{2.4}$$

which may be written as:

$$\frac{1}{2}[R_{\mu\nu}{}^{ab} + \frac{1}{l^2}(e_\mu^a e_\nu^b - e_\nu^a e_\mu^b)] M_{ab}\, g(x) + \frac{1}{l}(\partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega_\mu^{ab} e_\nu^b - \omega_\nu^{ab} e_\mu^b) M_{0a}\, g(x) \;=\; 0 \tag{2.5}$$

Thus any configuration that satisfies the equations $F = 0$ will lead to a g and the integrability does not impose any further conditions. In higher dimensions integrability will impose non-trivial constraints as $F = 0$ is not the equation on motion.

Notice that the equation (2.3) for g has a gauge invariance. It is covariant under an arbitrary local Lorentz transformation: $e^a \to (\Lambda)^{ac} e^c$, $\omega^{ab} \to (\Lambda)^{ac}\omega^{cd}(\Lambda^{-1})^{db} + (\Lambda)^{ac} d(\Lambda^{-1})^{cb}$ and $g \to \Lambda g$ where Λ is any element of the subgroup $SO(d+1)$ and $(\Lambda)^{ab}$ are the matrix element of Λ in the vector representation thus defining an equivalence relation between g and Λg. This makes the physical solution (representative of the gauge equivalence class) g an element of the coset $so(1, d+1)/so(d+1)$. Notice also that the equation satisfied by g is equivalent to

$$dg^{-1} - \frac{1}{2}\omega^{ab} g^{-1} M_{ab} - \frac{1}{l} e^a g^{-1} M_{0a} = 0 \tag{2.6}$$

This coset element g turns out to be one of the ingredients in our prescription to compute boundary partial waves. Before turning to the other ingredients let us point out a relation between Killing vectors of the AdS_{d+1} geometry and matrix elements of g in the adjoint representation. We claim that the components of the Killing vectors $(l_{[\alpha\beta]})^\mu$ are given by

$$(l_{[\alpha\beta]})^\mu = -l\, E_a^\mu (R[g^{-1}])_{\alpha\beta}{}^{0a} = -l\, E_a^\mu (R[g])^{0a}{}_{\alpha\beta} \tag{2.7}$$

where E^μ_a is the inverse vielbein and $(R[g^{-1}])_{\alpha\beta}{}^{0a}$ are matrix elements of g in its adjoint representation. One can recognize the similarity of Killing spinor equation of AdS_{d+1} with the equations (2.3, 2.6). To prove the relation (2.7) let us start with the definition of the vector operator (an element of the Lie algebra):

$$\xi_\mu = e^a_\mu \, g^{-1} M_{0a} \, g. \tag{2.8}$$

and calculate $D_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\lambda_{\mu\nu} \xi_\lambda$. One can show using the equations satisfied by g and g^{-1} (2.3, 2.6) that

$$\partial_\mu \xi_\nu = (\partial_\mu e^a_\nu + \omega_\mu{}^{ac} e^c_\nu) \, g^{-1} M_{0a} g + \frac{1}{l} e^a_\mu e^b_\nu \, g^{-1} M_{ab} g = \Gamma^\lambda_{\mu\nu} e^a_\lambda \, g^{-1} M_{0a} g + \frac{1}{l} e^a_\mu e^b_\nu \, g^{-1} M_{ab} g \tag{2.9}$$

where we have used $\partial_\mu e^a_\nu + \omega_\mu{}^{ac} e^c_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda = 0$. This implies

$$D_\mu \xi_\nu = \frac{1}{l} e^a_\mu e^b_\nu \, g^{-1} M_{ab} g \, . \tag{2.10}$$

Taking the symmetric part clearly shows that the vector operators ξ_μ satisfy the Killing vector equations $D_\mu \xi_\nu + D_\nu \xi_\mu = 0$. The relation between the Killing vector components (2.7) found here is a generalisation of a similar relation in the Killing spinor context to a more general representation of the local Lorentz algebra. This Killing vector operator can be expanded into a linear combination of the generators

$$\xi_\mu = e^a_\mu \left[(R[g^{-1}])^{0b}{}_{0a} M_{0b} + \frac{1}{2} (R[g^{-1}])^{bc}{}_{0a} M_{bc} \right] \tag{2.11}$$

where $R[g^{-1}]$ is the representation of g^{-1} in the adjoint representation. One can raise the index on the Killing operator as $\xi^\mu = E^\mu_a g^{-1} M_{0a} g$:

$$\xi^\mu = E^\mu_a \left[(R[g^{-1}])^{0b}{}_{0a} M_{0b} + \frac{1}{2} (R[g^{-1}])^{bc}{}_{0a} M_{bc} \right]. \tag{2.12}$$

This enable one to immediately read out the components of the Killing vectors

$$(l^{[\alpha\beta]})^{\mu} = l\, E^{\mu}_{a}(R[g^{-1}])^{\alpha\beta}{}_{0a} \tag{2.13}$$

or equivalently

$$(l_{[\alpha\beta]})^{\mu} = -l\, E^{\mu}_{a}(R[g^{-1}])_{\alpha\beta}{}^{0a} = -l\, E^{\mu}_{a}(R[g])^{0a}{}_{\alpha\beta} \tag{2.14}$$

It can be verified that these Killing vector fields $l^{\mu}_{\alpha\beta}(x)\partial_{\mu}$ satisfy the same algebra as their corresponding algebra generators $M_{\alpha\beta}$. We will make use of (2.7) to establish some differential equations satisfied by our OWN operators shortly.

The second ingredient is the set of states in the representation space which a given external Wilson line is in that transform in a (finite dimensional) irreducible representation of the subalgebra $so(d+1)$.[1] We will construct several examples of such states later in this thesis and make use of them to compute the OWN operators.

The last ingredient in the computation of the OWN expectation values is the Clebsch-Gordan coefficients (CGC) of the gauge algebra $so(1, d+1)$. Here we propose a method to derive them using the 3-point functions.

For this first recall that the CGC*s* are defined as the invariant tensors in the product of three representations. That is, the CGC*s* that appear in the tensor product decomposition $R_1 \otimes R_2 \to R_3$ satisfy:

$$R_1[g(x)]_{\mathbf{m}_1\mathbf{m}'_1} R_2[g(x)]_{\mathbf{m}_2\mathbf{m}'_2} C^{R_1,R_2;R_3}_{\mathbf{m}'_1,\mathbf{m}'_2;\mathbf{m}'_3} R_3[g(x)^{-1}]_{\mathbf{m}'_3\mathbf{m}_3} = C^{R_1,R_2;R_3}_{\mathbf{m}_1,\mathbf{m}_2;\mathbf{m}_3} \tag{2.15}$$

where $R_i[g(x)]_{\mathbf{m}_i\mathbf{m}'_i}$ is used to denote the matrix elements of $g(x)$ in the representation R_i, whose basis elements are collectively labelled by $\mathbf{m}_i$. In terms of the algebra elements

[1] An identical problem has appeared in [16] in a closely related context.

M_{AB} with $A, B = 0, 1, \cdots, d+1$, this eq. (2.15) reads:

$$R_1[M_{AB}]_{\mathbf{m}_1\mathbf{m}_1'} C^{R_1,R_2;R_3}_{\mathbf{m}_1',\mathbf{m}_2;\mathbf{m}_3} + R_2[M_{AB}]_{\mathbf{m}_2\mathbf{m}_2'} C^{R_1,R_2;R_3}_{\mathbf{m}_1,\mathbf{m}_2';\mathbf{m}_3} = C^{R_1,R_2;R_3}_{\mathbf{m}_1,\mathbf{m}_2;\mathbf{m}_3'} R_3[M_{AB}]_{\mathbf{m}_3'\mathbf{m}_3} \tag{2.16}$$

which is the recursion relation that determines the CGC. Now we argue that this is equivalent to the conformal Ward identity of the 3-point function of primary operators corresponding to the irreps (R_1, R_2, R_3). The prescription of [14] for the 3-point function of scalar primaries is to extract the leading term, *i.e,* the coefficient of $e^{-\rho(\Delta_1+\Delta_2+\Delta_3)}$ term – in the boundary limit of

$$\langle\langle R_1|g(x_1)|R_1,\mathbf{m}_1\rangle \, \langle\langle R_2|g(x_2)|R_2,\mathbf{m}_2\rangle \, C^{R_1,R_2;R_3}_{\mathbf{m}_1,\mathbf{m}_2;\mathbf{m}_3} \langle R_3,\mathbf{m}_3|g^{-1}(x_3)|R_3\rangle\rangle \tag{2.17}$$

It will be shown shortly that this type of quantities by construction satisfy the conformal Ward identity. It is of course true that the Ward identity completely determines the coordinate dependence of the 3-point function. Therefore, the question of finding the CGC is translated into finding expressions for the quantities $\langle\langle R|g(x)|\Delta,\mathbf{m}\rangle$ and $\langle R,\mathbf{m}|g^{-1}(x)|R\rangle\rangle$ in the large radius limit, and then amputating them from the corresponding 3-point function (Fig. 2).

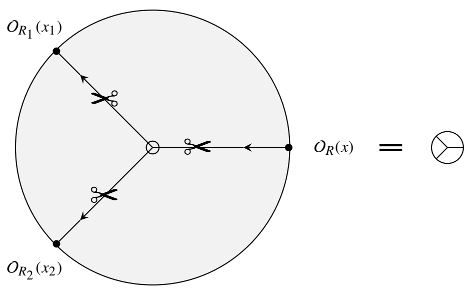


Figure 2.1: Clebsch-Gordan coefficients.

The expressions for these for general representations in $d \le 2$ and scalars in general d are derived in the appendix A and B respectively.

CPW as OWN*s*

Now we are ready to provide the prescription to compute various boundary partial waves. This can be obtained by simply evaluating the OWN in the flat connection corresponding to the AdS_{d+1} geometry. The Wilson line evaluated in a flat connection (corresponding to a locally AdS geometry) is

$$\mathrm{P}e^{\int_y^x A} = g(x)g^{-1}(y) \tag{2.18}$$

where the coset elements of $g(x)$ and $g^{-1}(x)$ are the solutions of (2.3) and (2.6) respectively. This is taken in an irreducible (infinite dimensional) representation of $so(1, d+1)$ algebra- particularly, the one which would correspond to unitary representation of $so(2, d)$ which is the relevant gauge algebra of the Lorentzian case – as mentioned earlier. Such a representation is labeled by the appropriate Casimirs $(\Delta, l_1, \cdots, l_{[d/2]})$ of $so(1, d+1)$. On the other hand the representations of the subalgebra $so(d+1)$ are labeled by the "angular momenta" $(j_1, \cdots j_{[d/2]})$. We will label a state in the finite dimensional irrep of $so(d+1)$ found as a linear combination of states in the infinite dimensional irrep of $so(1, d+1)$ by $|\{\Delta, l_i\} : \{j_i\}, \{m_i\}\rangle\rangle$.

We are now ready to form OWN operators that transform nicely under the local Lorentz rotation (LLR) algebra. We start with a spin network of the type given in the introduction in Fig. 1.5. Associate to it a Wilson network operator as prescribed above. Then project each external leg with an outgoing arrow with a ket-type state in a representation of LLR algebra, and each incoming external leg of the operator onto a bra-type dual state. This results in an object with n floating indices (for an OWN with n external legs) each of which transforms either by $R[h]$ or $R[h^{-1}]$ (depending on the index carried by the outgoing leg or the incoming leg of the OWN). It turns out that the quantity that satisfies the global conformal ward identities of a partial wave of the correlation function of primary operators corresponds to one particular component of this tensor.

Locations of vertices do not matter

Since we are restricting ourselves to computing the expectation values of the OWN operators classically we simply evaluate them in the flat connection corresponding to the AdS_{d+1} background. Now we will show that for this computation the positions of the vertices do not matter. For this we first note that at each vertex we have the following combination depending on the position of that vertex:

$$R_1[g(x)]_{\mathbf{m}_1\mathbf{m}'_1} R_2[g(x)]_{\mathbf{m}_2\mathbf{m}'_2} C^{R_1,R_2;R_3}_{\mathbf{m}'_1,\mathbf{m}'_2;\mathbf{m}'_3} R_3[g(x)^{-1}]_{\mathbf{m}'_3\mathbf{m}_3} = C^{R_1,R_2;R_3}_{\mathbf{m}_1,\mathbf{m}_2;\mathbf{m}_3} \tag{2.19}$$

where $R_i[g(x)]_{\mathbf{m}_i\mathbf{m}'_i}$ is used to denote the matrix elements of $g(x)$ in the representation R_i, whose basis elements are collectively labelled by $\mathbf{m}_i$. and $C^{R_1,R_2;R_3}_{\mathbf{m}_1,\mathbf{m}_2;\mathbf{m}_3}$ is the relevant CGC. This is the relevant object when the trivalent vertex has two legs in representations R_1 and R_2 going out of it and the one in representation R_3 going into it. If the arrows are reversed on all legs then we simply have to replace the corresponding g by g^{-1}. Now we use the identity (A.37) in appendix A to replace this by one CGC eliminating the coordinate dependence of the junction. This can be done at every trivalent vertex in our spin network thus eliminating the dependence of the locations of all the vertices as claimed.

Differential equations satisfied

The global blocks/partial waves are expected to satisfy some differential relations as stated in the introduction. Now we want to show that an OWN such as the one in Fig.1.5 will also satisfy the same set of differential identities expected of the corresponding partial wave. To proceed further we note the following identities:

$$g(x)\, M_{\alpha\beta} \;=\; l^{\mu}_{\alpha\beta}(x)\partial_\mu g(x) + \frac{1}{2} M_{bc}\, g(x)\left[\omega^{bc}_{\mu}(x) l^{\mu}_{\alpha\beta}(x) + (R[g(x)])^{bc}{}_{\alpha\beta}\right] \tag{2.20}$$

$$M_{\alpha\beta}g^{-1}(x) \;=\; -l^{\mu}_{\alpha\beta}(x)\partial_{\mu}g^{-1}(x) + \frac{1}{2}\left[\omega^{bc}_{\mu}(x)l^{\mu}_{\alpha\beta}(x) + (R[g(x)])^{bc}{}_{\alpha\beta}\right]g^{-1}(x)M_{bc} \quad (2.21)$$

where $g(x)$ ($g^{-1}(x)$) is a solution to (2.3) ((2.6)), $l^{\mu}_{\alpha\beta}(x)$ are the components of the Killing vector of the background geometry carrying the indices of the corresponding algebra generator $M_{\alpha\beta}$ of the left hand side.

To prove the identity (2.20) we start with the left hand side

$$\begin{aligned} g(x)\,M_{\alpha\beta} &= g(x)\,M_{\alpha\beta}\,g^{-1}(x)g(x) \\ &= (R[g(x)])^{0a}{}_{\alpha\beta}\,M_{0a}\,g(x) + \frac{1}{2}(R[g(x)])^{bc}{}_{\alpha\beta}\,M_{bc}\,g(x) \end{aligned} \quad (2.22)$$

From (2.3) we obtain

$$M_{0a}\,g(x) = -lE^{\mu}_{a}\,\partial_{\mu}g(x) - \frac{l}{2}E^{\mu}_{a}\,\omega^{bc}_{\mu}\,M_{bc}\,g(x) \quad (2.23)$$

Now substituting this in (2.22) and using the relation (2.14) we recover right hand side of (2.20). Similarly one can use the differential equation (2.6) and (2.14) to prove the identity (2.21).

The ingredients in our OWN are the matrix elements of $g(x)$ or $g^{-1}(x)$ between a generic state $|\Delta; \mathbf{l_i},\, \mathbf{m_i}\rangle$ in the representation of the particular external leg and the state $|\{\Delta, l_i\} : \{j_i\}, \{m_i\}\rangle\rangle$ in the finite dimensional representation of the subalgebra $so(d+1)$. It turns out that these quantities in the limit of bulk point x approaching the boundary of AdS_{d+1} can be computed (which will be done in the later chapters). We can also compute these with either the additional insertions of $M_{\alpha\beta}$ to the right of $g(x)$ or $-M_{\alpha\beta}$ to the left of $g^{-1}(x)$ depending on the direction of the external leg. It can be shown (again will be illustrated explicitly later on) further that these matrix elements simply turn out to be those obtained by the action of the boundary conformal transformation of a primary (or descendent) operator on the matrix element without the insertion in the boundary limit. Now the left hand side of the global conformal Ward identity is simply given by the boundary limit of sum of the OWN operators with the insertion of the corresponding generator ($M_{\alpha\beta}$ after $g(x)$ if it is an ingoing leg and $-M_{\alpha\beta}$ before g^{-1} for the outgoing one). Using the recursion relations that the CGCs are expected to satisfy it can be seen that this sum will vanish. This establishes the identity that under simultaneous transformation of the primary operators under the global conformal transformations the OWN expectation value is left invariant.

Now we turn to the Casimir equations that the global conformal blocks are expected to satisfy. Because the partial wave decomposition of a correlation function involves taking the contribution of one primary (and its global descendants) they are expected to satisfy the conformal Casimir equation with eigenvalue given by the Casimir invariant of the primary in question. One expects one Casimir equation for each channel of decomposition of the correlator. In our context this translates to the expectation that our OWN operator (associated to a spin network such as the one in Fig. 1.5) satisfies a Casimir equation for each ("1-particle reducible") edge of the spin network graph that when cut the diagram falls apart into two disjoint pieces (which is the case of any intermediate leg of a *tree-level* network, i.e., without closed loops). We now want to argue that this is indeed the case.

We will use the 4-point partial wave (in the s-channel decomposition) in $d = 2$ as the illustrative example. In this case there are two independent quadratic Casimir operators (one for each of the two commuting $sl(2,\mathbb{R})$ algebras in $so(1,3)$). The partial wave is expected to satisfy one Casimir equation corresponding to the quadratic Casimir of the full algebra $so(1,3)$. However, our OWNs satisfy two equations - one for each of the two quadratic Casimirs of $so(1,3)$. It will turn out that there are two OWNs for each intermediate ("1-particle reducible") edge (connecting two trivalent vertices) with the same eigenvalue of the quadratic Casimir of the full algebra $so(1,3)$ related by the interchange $h \leftrightarrow \bar{h}$ in that edge. Therefore any linear combination of these two OWNs will satisfy the Casimir equation. Then one should be guided by the boundary conditions expected (from the OPEs in the CFT as one takes the coincidence limits of various vertices). We will comment on this aspect again later on.

The value of a 4-point OWN (see Fig. 3.3) with external legs in representations $(h_i, \bar{h}_i)$ with $i = 1, 2, 3, 4$ becomes[2]

$$
\begin{aligned}
&\langle\!\langle h_1, \bar{h}_1; j_1, m_1 | g(x_1) | h_1, \bar{h}_1; k_1, \bar{k}_1 \rangle \langle\!\langle h_2, \bar{h}_2; j_2, m_2 | g(x_2) | h_2, \bar{h}_2; k_2, \bar{k}_2 \rangle \\
&\langle h_3, \bar{h}_3; k_3, \bar{k}_3 | g^{-1}(x_3) | h_3, \bar{h}_3; j_3, m_3 \rangle\!\rangle \langle h_4, \bar{h}_4; k_4, \bar{k}_4 | g^{-1}(x_4) | h_4, \bar{h}_4; j_4, m_4 \rangle\!\rangle \\
&\times C^{h_1 h_2 h}_{k_1, k_2, k} \times C^{\bar{h}_1 \bar{h}_2 \bar{h}}_{\bar{k}_1 \bar{k}_2 \bar{k}} \times C^{h_3 h_4 h}_{k_3 k_4 k} \times C^{\bar{h}_3 \bar{h}_4 \bar{h}}_{\bar{k}_3 \bar{k}_4 \bar{k}}
\end{aligned}
\tag{2.24}
$$

where the sum over repeated indices is assumed. Then the action of the Casimir differential operator on the partial wave in eq.(1.29)

$$
(M^{(1)}_{AB} + M^{(2)}_{AB})(M^{AB}_{(1)} + M^{AB}_{(2)}) W_{h,\bar{h}}(x_i) \tag{2.25}
$$

is obtained by summing over three diagrams with the first one with an insertion of the Casimir operator $M_{\alpha\beta} M^{\alpha\beta}$ after $g(x_1)$, the second one with an insertion of $M_{\alpha\beta} M^{\alpha\beta}$ after $g(x_2)$ and the third one with one insertion of $M_{\alpha\beta}$ after $g(x_1)$ and one insertion of $M^{\alpha\beta}$ after $g(x_2)$ with a factor of two and summing over the α and β indices. Let us consider the Casimir made of $\{L_0, L_{\pm 1}\}$ first (see appendix A for our conventions on $so(1,3)$ generators). Then the answer of this sum contains the following terms:

$$
\begin{aligned}
&(L^{(1)}_a L^a_{(1)})_{k_1 k'_1} C^{h_1 h_2 h}_{k'_1 k_2 k} + (L^{(2)}_a L^a_{(2)})_{k_2 k'_2} C^{h_1 h_2 h}_{k_1 k'_2 k} + (L^{(1)}_a)_{k_1 k'_1} (L^a_{(2)})_{k_2 k'_2} C^{h_1 h_2 h}_{k'_1 k'_2 k} + (L^a_{(1)})_{k_1 k'_1} (L^{(2)}_a)_{k_2 k'_2} C^{h_1 h_2 h}_{k'_1 k'_2 k} \\
&= (L^a_{(1)})_{k_1 k''_1} \left[(L^{(1)}_a)_{k''_1 k'_1} C^{h_1 h_2 h}_{k'_1 k_2 k} + (L^{(2)}_a)_{k_2 k'_2} C^{h_1 h_2 h}_{k''_1 k'_2 k} \right] + (L^a_{(2)})_{k_2 k''_2} \left[(L^{(1)}_a)_{k_1 k'_1} C^{h_1 h_2 h}_{k'_1 k''_2 k} + (L^{(2)}_a)_{k''_2 k'_2} C^{h_1 h_2 h}_{k_1 k'_2 k} \right] \\
&= (L^a_{(1)})_{k_1 k'_1} C^{h_1 h_2 h}_{k'_1 k_2 k'} (L^{(0)}_a)_{k' k} + (L^{(2)}_a)_{k_2 k'_2} C^{h_1 h_2 h}_{k_1 k'_2 k'} (L^{(0)}_a)_{k' k} \\
&= \left[(L^a_{(1)})_{k_1 k'_1} C^{h_1 h_2 h}_{k'_1 k_2 k'} + (L^{(2)}_a)_{k_2 k'_2} C^{h_1 h_2 h}_{k_1 k'_2 k'} \right] (L^{(0)}_a)_{k' k} = C^{h_1 h_2 h}_{k_1 k_2 k''} (L^a_{(0)})_{k'' k'} (L^{(0)}_a)_{k' k}
\end{aligned}
\tag{2.26}
$$

where we denoted the matrix elements of the generator L_a in the representation $(h_1, \bar{h}_1)$ by $L^{(1)}_a$ etc. and in representation $(h, \bar{h})$ by $L^{(0)}_a$. Substituting this result back into the sum of diagrams we started with we see that the result is simply given by the value of the

[2]The leading term of this particular OWN corresponds to the 4-point conformal partial waves of correlators of primary operators with conformal weight $(h_i, \bar{h}_i)$ when the end points of the external legs of the OWN are taken to boundary. This will be shown in the next chapter.

Casimir operators $L_a L^a$ in the representation $(h, \bar{h})$ times the original diagram. It can be easily checked that our digram satisfies the corresponding Casimir equation for the second quadratic Casimir operator made of $\{\bar{L}_0, \bar{L}_{\pm 1}\}$ as well. This proof generalises to any spin network straightforwardly. The identity can also be generalised to higher dimensions as well. Finally one just needs to ensure that the right boundary conditions are imposed to show that our OWN operators indeed compute the partial waves- this will be discussed case by case.

Chapter 3

Conformal blocks in $d \leq 2$

Having defined and elaborated on the OWN operators we now turn to computing them explicitly in the $d \leq 2$ cases in this chapter. In the course of this various properties argued in the previous chapter will be demonstrated. We are starting with $d = 2$ case.

Euclidean AdS_3 with boundary $\mathbb{R}^2$

We consider the Eucidean AdS_3 geometry with boundary $\mathbb{R}^2$. The metric is

$$l^{-2}\, ds^2 = d\rho^2 + e^{2\rho}(dx_1^2 + dx_2^2) \tag{3.1}$$

where l is the radius of AdS_3 and the ranges of the coordinates are $-\infty < \rho, x_1, x_2 < \infty$. In these coordinates $\rho \to \infty$ is the conformal boundary. The Killing vectors of this geometry are:

$$\begin{aligned} L_{-1} &= -\partial_z, \;\; L_0 = \tfrac{1}{2}\partial_\rho - z\partial_z, \;\; L_1 = z\partial_\rho - z^2\partial_z + e^{-2\rho}\partial_{\bar{z}} \\ \bar{L}_{-1} &= -\partial_{\bar{z}}, \;\; \bar{L}_0 = \tfrac{1}{2}\partial_\rho - \bar{z}\partial_{\bar{z}}, \;\; \bar{L}_1 = \bar{z}\partial_\rho - \bar{z}^2\partial_{\bar{z}} + e^{-2\rho}\partial_z \end{aligned} \tag{3.2}$$

where $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$ and the Killing vectors satisfy the commutator algebra $[L_m, L_n] = (m-n)L_{m+n}, [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}, [L_m, \bar{L}_n] = 0$. Let us choose the frame: $e^1 = l\, e^\rho\, dx_1,\ e^2 = l\, e^\rho\, dx_2,\ e^3 = l\, d\rho$. Then the non-vanishing spin-connections are: $\omega^{a3} = \frac{1}{l} e^a$ for $a = 1, 2$. The equation $dg + \frac{1}{2}\omega^{ab} M_{ab} g + \frac{1}{l} e^a M_{0a} g = 0$ satisfied by the coset element g in this case reads

$$dg + d\rho\, M_{03} g + e^\rho dx_1\, (M_{13} + M_{01})g + e^\rho dx_2\, (M_{23} + M_{02})g = 0. \tag{3.3}$$

Its solution may be written as

$$g = e^{-\rho M_{03}} e^{-x_1(M_{13}+M_{01})} e^{-x_2(M_{23}+M_{02})} g_0 \tag{3.4}$$

up to a multiplication by a constant group element on the right. Written in terms of the generators of the two $sl(2, \mathbb{R})$ factors in $so(1, 3)$ as in appendix A

$$\begin{aligned} L_0 &= -J_0^{(+)},\ \ L_1 = i(J_1^{(+)} + i\, J_2^{(+)}),\ \ ,\ \ L_{-1} = -i(J_1^{(+)} - i\, J_2^{(+)}) \\ \bar{L}_0 &= J_0^{(-)},\ \ \bar{L}_1 = -i(J_1^{(-)} - i\, J_2^{(-)}),\ \ \bar{L}_{-1} = i(J_1^{(-)} + i\, J_2^{(-)}) \end{aligned} \tag{3.5}$$

where

$$J_1^{(\pm)} = \frac{i}{2}(-iM_{23} \pm M_{01}),\ \ J_2^{(\pm)} = \frac{i}{2}(-iM_{31} \pm M_{02}),\ \ J_0^{(\pm)} = \frac{1}{2}(-iM_{12} \pm M_{03}) \tag{3.6}$$

the coset element g reads:

$$g = e^{\rho(L_0+\bar{L}_0)} e^{-L_{-1}(x_1+ix_2)} e^{-\bar{L}_{-1}(x_1-ix_2)} g_0\,. \tag{3.7}$$

Since the Wilson line for locally AdS takes the form $W_x^y(R, C) = g(x)g^{-1}(y)$ we can choose g_0 to be identity I without loss of generality. The basic building blocks of the open Wilson networks are again $\langle h, \bar{h}; k, \bar{k}|g^{-1}(x)|h, \bar{h}; j, m\rangle\!\rangle$ and $\langle\!\langle h, \bar{h}; j, m|g(x)|h, \bar{h}; k, \bar{k}\rangle$. To evaluate these we need the states $|h, \bar{h}; j, m\rangle\!\rangle$ and $\langle\!\langle h, \bar{h}; j, m|$.

Constructing the states $|h, \bar{h}; j, m\rangle\rangle$

We consider the discrete sereis representation of $so(1,3)$. The basis is spanned by the states constructed over the lowest weight state $|h, \bar{h}\rangle \equiv |h\rangle \otimes |\bar{h}\rangle$ such that

$$L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle; \qquad \bar{L}_0|h, \bar{h}\rangle = \bar{h}|h, \bar{h}\rangle \tag{3.8}$$

$$L_1|h, \bar{h}\rangle = 0; \qquad \bar{L}_1|h, \bar{h}\rangle = 0 \tag{3.9}$$

The rest of the states are obtained by applying L_{-1} and $\bar{L}_{-1}$ on the highest weight state as

$$(L_{-1})^k \left(\bar{L}_{-1}\right)^{\bar{k}} |h, \bar{h}\rangle \tag{3.10}$$

We denote such states as $|h, \bar{h};\ k, \bar{k}\rangle \equiv |h, k\rangle \otimes |\bar{h}, \bar{k}\rangle$. The sub-algebra $so(3)$ in $so(1,3)$ is generated by:

$$L_0 - \bar{L}_0 = iM_{12}, \;\; L_1 + \bar{L}_{-1} = iM_{23} + M_{13}, \;\; L_{-1} + \bar{L}_1 = -iM_{23} + M_{13} \tag{3.11}$$

We define $|h, \bar{h}; j, m\rangle\rangle$ as[1]

$$|h, \bar{h}; j, m\rangle\rangle = \sum_{n=0}^{\infty} C_{n,p}(h,\ \bar{h}) \;\; |h, n+p\rangle \otimes |\bar{h}, n\rangle \tag{3.12}$$

such that it satisfies the following conditions (corresponding to $h^D = -j$ as expected from [94])

$$\begin{aligned}
(L_0 - \bar{L}_0)|h, \bar{h}; -j, p\rangle\rangle &= (-j + p)|h, \bar{h}; -j, p\rangle\rangle \\
(L_{-1} + \bar{L}_1)|h, \bar{h}; -j, p\rangle\rangle &= \sqrt{(p+1)(-2j+p)}\, |h, \bar{h}; -j, p+1\rangle\rangle \\
(L_1 + \bar{L}_{-1})|h, \bar{h}; -j, p\rangle\rangle &= \sqrt{p(-2j+p-1)}\, |h, \bar{h}; -j, p-1\rangle\rangle
\end{aligned} \tag{3.13}$$

[1] When the bulk field dual to the primary operator at hand is a gauge field one expects classes of the cap states reflecting the gauge symmetry [16] – we do not consider this case here.

where $p = j + m$ and $\bar{h} - h = j \geq 0$. (When $h - \bar{h} \geq 0$ one can write a similar set of states obtained by interchanging h with $\bar{h}$ and the order of the states in the tensor product.) We assume that $j \in \mathbb{Z}/2$ and thus $p = 0, 1, \cdots 2j$. The above conditions (3.13) result in the recursion relations for $C_{n,p}$

$$\begin{aligned} C_{n+1,p}\sqrt{(n+1)(2\bar{h}+n)} + C_{n,p}\sqrt{(2h+n+p)(n+p+1)} &= C_{n,p+1}\sqrt{(p+1)(2\bar{h}-2h+p)} \\ C_{n,p}\sqrt{(n+p)(2h+n+p-1)} + C_{n-1,p}\sqrt{n(2\bar{h}+n-1)} &= C_{n,p-1}\sqrt{p(2\bar{h}-2h+p-1)} \end{aligned} \tag{3.14}$$

Solving the above recursion relations for $C_{n,p}$ we find

$$C_{n,p}(h,\bar{h}) = (-1)^{\frac{p}{2}+n}\lambda(h,\bar{h})\sqrt{\frac{(n+p)!}{n!p!}}\sqrt{\frac{\Gamma(2\bar{h}+n)}{\Gamma(2h+n+p)\Gamma(2\bar{h}-2h+1-p)}} \tag{3.15}$$

Therefore the cap-states become

$$\begin{aligned} |h,\bar{h};j,m\rangle\!\rangle = \lambda(h,\bar{h})\sum_{n=0}^{\infty}(-1)^{\frac{p}{2}+n}&\sqrt{\frac{\Gamma(2\bar{h}+n)}{\Gamma(2h+n+p)\Gamma(2\bar{h}-2h+1-p)}} \\ &\times\sqrt{\frac{(n+p)!}{n!p!}}\;|h,n+p\rangle\otimes|\bar{h},n\rangle \end{aligned} \tag{3.16}$$

The factor $\lambda(h,\bar{h})$ is arbitrary constant at this stage - will be chosen to be

$$\lambda(h,\bar{h})^2 = (-1)^{\bar{h}-h}\frac{(2\bar{h}-2h)!\Gamma(2h)}{\Gamma(2\bar{h})} \tag{3.17}$$

for convenience. Notice that when $j = 0$ this state is closely related to the one written down in [18] and for other j advocated for in [16] (see also [17]).

These cap-states form a non-unitary finite dimensional representation of the twisted diagonal $sl(2,\mathbb{R})$ generated by $\{L_0^D := L_0 - \bar{L}_0, L_1^D := L_1 + \bar{L}_{-1}, L_{-1}^D := L_{-1} + \bar{L}_1\}$. The local Lorentz group $SO(3)$ is generated by $\{J_3 = L_0^D, J_+ = \pm iL_{-1}^D, J_- = \pm iL_1^D\}$. Then these states can be seen to provide the unitary representation labeled by the angular momentum j (with the identification $|h,\bar{h};-j,p\rangle\!\rangle \to |h,\bar{h};j,m\rangle\!\rangle$ which we will use interchangeably) of the $su(2)$ algebra generated by these $\{J_3, J_\pm\}$.

Computing $\langle\!\langle h,\bar{h};j,m|g(x)|h,\bar{h};k,\bar{k}\rangle$ in $\rho\to\infty$ limit

We start with eq.(3.7) with $z = x_1 + ix_2$

$$g(x) = e^{\rho(L_0+\bar{L}_0)}e^{-zL_{-1}}e^{-\bar{z}\bar{L}_{-1}} \tag{3.18}$$

Then

$$\begin{aligned}
&\langle\!\langle h,\bar{h};-j,p|g(x)|h,\bar{h};k,\bar{k}\rangle \\
=\;& \lambda\sum_{n=0}^{\infty}(-1)^{\frac{p}{2}+n}\sqrt{\frac{(n+p)!}{n!p!}}\sqrt{\frac{\Gamma(2\bar{h}+n)}{\Gamma(2h+n+p)\Gamma(2\bar{h}-2h+1-p)}} \\
&\times\langle h,n+p|e^{\rho L_0}e^{-zL_{-1}}|h,k\rangle\langle\bar{h},n|e^{\rho\bar{L}_0}e^{-\bar{z}\bar{L}_{-1}}|\bar{h},\bar{k}\rangle \\
=\;& \lambda\frac{(-z)^{p-k}(-\bar{z})^{-k}e^{\rho(h+\bar{h}+p)}(-1)^{n+\frac{p}{2}}}{\sqrt{\Gamma(2h+k)\Gamma(2\bar{h}+\bar{k})k!\bar{k}!p!\Gamma(2j+1-p)}}\sum_{n=\max(k-p,\bar{k})}^{\infty}\frac{(n+p)!\Gamma(2\bar{h}+n)}{(n+p-k)!(n-\bar{k})!}\left(-e^{2\rho}|z|^2\right)^n \\
\overset{k-p\geq\bar{k}}{=}\;& \lambda\frac{e^{\rho(h+\bar{h}+p)}(-z)^{p-k}(-\bar{z})^{-\bar{k}}k!\Gamma(2\bar{h}+k-p)(-1)^{p/2}}{\sqrt{k!\bar{k}!p!\Gamma(2h+k)\Gamma(2\bar{h}+\bar{k})\Gamma(2\bar{h}-2h+1-p)}}\frac{(-e^{2\rho}|z|^2)^{k-p}}{(k-\bar{k}-p)!} \\
&\times{}_2F_1(k+1,2\bar{h}+k-p,k-p-\bar{k}+1,-e^{2\rho}|z|^2) \\
\overset{k-p\leq\bar{k}}{=}\;& \lambda\frac{e^{\rho(h+\bar{h}+p)}(-z)^{p-k}(-\bar{z})^{-\bar{k}}\Gamma(2\bar{h}+\bar{k})(\bar{k}+p)!(-1)^{p/2}}{\sqrt{p!k!\bar{k}!\Gamma(2h+k)\Gamma(2\bar{h}+\bar{k})\Gamma(2\bar{h}-2h+1-p)}}\frac{(-e^{2\rho}|z|^2)^{\bar{k}}}{(\bar{k}-k+p)!} \\
&\times{}_2F_1(\bar{k}+1+p,2\bar{h}+\bar{k},p+\bar{k}-k+1,-e^{2\rho}|z|^2)
\end{aligned}$$

We would like to take the $\rho \to \infty$ of these expressions. For this we use the well-known Euler's identity (see, for instance, [95])

$$ {}_2F_1(\alpha,\beta,\gamma,x) = (1-x)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta,\gamma,x) \tag{3.19} $$

Using this we have

$$ \begin{aligned} &{}_2F_1(k+1, 2\bar{h}+k-p, k-p-\bar{k}+1, -e^{2\rho}|z|^2) \\ &\overset{\rho\to\infty}{\longrightarrow} (-1)^{p+\bar{k}}(e^{2\rho}|z|^2)^{p-k-2\bar{h}}\frac{(k-\bar{k}-p)!\Gamma(2\bar{h}+\bar{k})}{\Gamma(2\bar{h}-p)k!} \end{aligned} \tag{3.20} $$

$$ \begin{aligned} &{}_2F_1(\bar{k}+1+p, 2\bar{h}+\bar{k}, p+\bar{k}-k+1, -e^{2\rho}|z|^2) \\ &\overset{\rho\to\infty}{\longrightarrow} (-1)^{k}(e^{2\rho}|z|^2)^{-\bar{k}-2\bar{h}}\frac{(\bar{k}-k+p)!\Gamma(2\bar{h}+k-p)}{\Gamma(2\bar{h}-p)(p+\bar{k})!} \end{aligned} \tag{3.21} $$

When we take the $\rho \to \infty$ limit both the cases reduce to the same expression given by

$$ \begin{aligned} \langle\langle h,\bar{h};-j,p|g(x)|h,\bar{h};k,\bar{k}\rangle \to \lambda(-1)^{-p/2}&\frac{e^{\rho(h-3\bar{h}+p)}z^{-2\bar{h}}\bar{z}^{-2\bar{h}}z^{p-k}\bar{z}^{-\bar{k}}}{\sqrt{k!\bar{k}!\Gamma(2h+k)\Gamma(2\bar{h}+\bar{k})\Gamma(2j+1-p)p!}} \\ &\times\frac{\Gamma(2\bar{h}+\bar{k})\Gamma(2\bar{h}-p+k)}{\Gamma(2\bar{h}-p)}+\cdots \end{aligned} $$

At this point let us note that since p runs from 0 to $2j$ the leading terms in the $\rho \to \infty$ limit comes by setting $p = 2j$ which goes as $e^{-\rho(h+\bar{h})}$ and lower values of p lead to sub-leading terms in this limit. Therefore, the special case of $p = 2j$ should correspond to insertion of a primary operator at the corresponding boundary point. To substantiate this we now show that the matrix element $\langle\langle h,\bar{h};j,m|g(x)M_{\alpha\beta}|h,\bar{h};k,\bar{k}\rangle$ is the conformal transformation of the answer without the insertion of $M_{\alpha\beta}$. By explicit computation of eqs.(3) we find

$$\begin{aligned}
g(x)L_{-1} &= -\partial_z g(x) \\
g(x)L_0 &= -z\partial_z g(x) + \frac{1}{2}(\partial_\rho + (L_0 - \bar{L}_0))g(x) = (-z\partial_z + L_0)g(x) \\
g(x)L_1 &= -z^2\partial_z g(x) + z(\partial_\rho + (L_0 - \bar{L}_0))g(x) + e^{-\rho}(L_1 + \bar{L}_{-1})g(x) + e^{-2\rho}\partial_{\bar{z}} g(x) \\
g(x)\bar{L}_{-1} &= -\partial_{\bar{z}} g(x) \\
g(x)\bar{L}_0 &= -\bar{z}\partial_{\bar{z}} g(x) + \frac{1}{2}(\partial_\rho - (L_0 - \bar{L}_0))g(x) \\
g(x)\bar{L}_1 &= -\bar{z}^2\partial_{\bar{z}} g(x) + \bar{z}(\partial_\rho - (L_0 - \bar{L}_0))g(x) + e^{-\rho}(L_{-1} + \bar{L}_1)g(x) + e^{-2\rho}\partial_z g(x)
\end{aligned} \tag{3.22}$$

Notice that the terms leading in $\rho \to \infty$ limit do not mix different states in the irrep of the twisted diagonal $sl(2,\mathbb{R})$ whereas the sub-leading ones do. Using matrix elements just computed we can write down the effect of insertion of L_n's and $\bar{L}_n$'s. We find:

$$\begin{aligned}
\langle\langle h,\bar{h}; j, j|g(x)L_{-1}|h,\bar{h}; k,\bar{k}\rangle &= -\partial_z \langle\langle h,\bar{h}; j, j|g(x)|h,\bar{h}; k,\bar{k}\rangle \\
\langle\langle h,\bar{h}; j, j|g(x)L_0|h,\bar{h}; k,\bar{k}\rangle &= -(z\partial_z + h)\langle\langle h,\bar{h}; j, j|g(x)|h,\bar{h}; k,\bar{k}\rangle \\
\langle\langle h,\bar{h}; j, j|g(x)L_1|h,\bar{h}; k,\bar{k}\rangle &= -(z^2\partial_z + 2zh)\langle\langle h,\bar{h}; j, j|g(x)|h,\bar{h}; k,\bar{k}\rangle \\
\langle\langle h,\bar{h}; j, j|g(x)\bar{L}_{-1}|h,\bar{h}; k,\bar{k}\rangle &= -\partial_{\bar{z}} \langle\langle h,\bar{h}; j, j|g(x)|h,\bar{h}; k,\bar{k}\rangle \\
\langle\langle h,\bar{h}; j, j|g(x)\bar{L}_0|h,\bar{h}; k,\bar{k}\rangle &= -(\bar{z}\partial_{\bar{z}} + \bar{h})\langle\langle h,\bar{h}; j, j|g(x)|h,\bar{h}; k,\bar{k}\rangle \\
\langle\langle h,\bar{h}; j, j|g(x)\bar{L}_1|h,\bar{h}; k,\bar{k}\rangle &= -(\bar{z}^2\partial_{\bar{z}} + 2\bar{z}\bar{h})\langle\langle h,\bar{h}; j, j|g(x)|h,\bar{h}; k,\bar{k}\rangle
\end{aligned} \tag{3.23}$$

in the $\rho \to \infty$ limit. These results (3.23) establish that the functions

$$\Phi^{(h,\bar{h})}_{k,\bar{k}}(x) := \langle\langle h,\bar{h}; j, j|g(x)|h,\bar{h}; k,\bar{k}\rangle \tag{3.24}$$

in the $\rho \to \infty$ provide a representation of the (global) conformal algebra $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$ with the action of its generators given by the differential operator representation appropriate for its action on primaries under the corresponding conformal transformation (see for instance [96]). Remarkably the representation (for either the holomorphic or the anti-holomorphic part) that we find here is the same as the one used in [97] (see also [94]) in the study of unitary irreps of $sl(2,\mathbb{R})$ algebra around $z \to \infty$ in the complex plane.

Computing $\langle h,\bar{h};k,\bar{k}|g^{-1}(x)|h,\bar{h};j,m\rangle\!\rangle$ in $\rho\to\infty$ limit

This can also be computed and the calculation is simpler than the previous one. Starting with

$$g^{-1}(x) = e^{\bar{z}\bar{L}_{-1}}e^{zL_{-1}}e^{-\rho(L_0+\bar{L}_0)} \tag{3.25}$$

we obtain

$$\begin{aligned}
&\langle h,\bar{h};k,\bar{k}|g^{-1}(x)|h,\bar{h};j,m\rangle\!\rangle \\
&= \lambda\sum_{n=0}^{\infty}(-1)^{n+\frac{p}{2}}\sqrt{\frac{(n+p)!}{n!p!}}\sqrt{\frac{\Gamma(2\bar{h}+n)}{\Gamma(2h+n+p)\Gamma(2\bar{h}-2h+1-p)}} \\
&\qquad\times\langle h,k|e^{zL_{-1}}e^{-\rho L_0}|h,n+p\rangle\langle\bar{h},\bar{k}|e^{\bar{z}\bar{L}_{-1}}e^{-\rho\bar{L}_0}|\bar{h},n\rangle \\
&= \lambda(-1)^{p/2}e^{-\rho(h+\bar{h}+p)}\sqrt{\frac{\Gamma(2h+k)\Gamma(2\bar{h}+\bar{k})k!\bar{k}!}{p!\Gamma(2\bar{h}-2h+1-p)}} \\
&\qquad\times\sum_{n=0}^{\min(k-p,\bar{k})}\frac{(-e^{-2\rho}|z|^{-2})^n}{n!(\bar{k}-n)!\Gamma(2h+n+p)(k-n-p)!} \\
&= \lambda(-1)^{p/2}\frac{e^{-\rho(h+\bar{h}+p)}}{\Gamma(2h+p)}\sqrt{\frac{k!\bar{k}!\Gamma(2h+k)\Gamma(2\bar{h}+\bar{k})}{p!\Gamma(2\bar{h}-2h+1-p)}}\frac{z^{k-p}}{(k-p)!}\frac{\bar{z}^{\bar{k}}}{\bar{k}!} \\
&\qquad\times{}_2F_1[-\bar{k},-k+p,2h+p;-e^{-2\rho}|z|^{-2}] \\
&\overset{\rho\to\infty}{=} \lambda(-1)^{p/2}\frac{e^{-\rho(h+\bar{h}+p)}}{\Gamma(2h+p)}\frac{z^{k-p}}{(k-p)!}\frac{\bar{z}^{\bar{k}}}{\bar{k}!}\sqrt{\frac{k!\bar{k}!\Gamma(2h+k)\Gamma(2\bar{h}+\bar{k})}{p!\Gamma(2\bar{h}-2h+1-p)}}+O(e^{-\rho(h+\bar{h}+p+1)})
\end{aligned} \tag{3.26}$$

Let us note that the leading term from this leg comes from $p = 0$ ($m = -j$) which again goes as $e^{-\rho(h+\bar{h})}$ and higher values of p give sub-leading terms in the $\rho \to \infty$ limit. In this case the $p = 0$ answer corresponds to insertion of a primary with dimensions $(h, \bar{h})$ at the boundary point. This can again be seen on similar lines as before by first observing the identities (3):

$$\begin{aligned}
L_{-1}g^{-1}(x) &= \partial_z g^{-1}(x) \\
L_0 g^{-1}(x) &= z\partial_z g^{-1}(x) - \frac{1}{2}(\partial_\rho g^{-1}(x) - g^{-1}(x)(L_0 - \bar{L}_0)) \\
L_1 g^{-1}(x) &= z^2\partial_z g^{-1}(x) + z(-\partial_\rho g^{-1}(x) + g^{-1}(x)(L_0 - \bar{L}_0)) + e^{-\rho}g^{-1}(x)(L_1 + \bar{L}_{-1}) - e^{-2\rho}\partial_{\bar{z}}g^{-1}(x) \\
\bar{L}_{-1}g^{-1}(x) &= \partial_{\bar{z}} g^{-1}(x) \\
\bar{L}_0 g^{-1}(x) &= \bar{z}\partial_{\bar{z}} g^{-1}(x) - \frac{1}{2}(\partial_\rho g^{-1}(x) + g^{-1}(x)(L_0 - \bar{L}_0)) \\
\bar{L}_1 g^{-1}(x) &= \bar{z}^2\partial_{\bar{z}} g^{-1}(x) - \bar{z}(\partial_\rho g^{-1}(x) + g^{-1}(x)(L_0 - \bar{L}_0)) + e^{-\rho}g^{-1}(x)(L_{-1} + \bar{L}_1) - e^{-2\rho}\partial_z g^{-1}(x)
\end{aligned}$$

Using these we can show as $\rho \to \infty$ that:

$$\begin{aligned}
\langle h,\bar{h};k,\bar{k}|(-L_{-1})g^{-1}(x)|h,\bar{h};j,-j\rangle\!\rangle &= -\partial_z\langle h,\bar{h};k,\bar{k}|g^{-1}(x)|h,\bar{h};j,-j\rangle\!\rangle \\
\langle h,\bar{h};k,\bar{k}|(-L_0)g^{-1}(x)|h,\bar{h};j,-j\rangle\!\rangle &= -(z\partial_z + h)\langle h,\bar{h};k,\bar{k}|g^{-1}(x)|h,\bar{h};j,-j\rangle\!\rangle \\
\langle h,\bar{h};k,\bar{k}|(-L_1)g^{-1}(x)|h,\bar{h};j,-j\rangle\!\rangle &= -(z^2\partial_z + 2zh)\langle h,\bar{h};k,\bar{k}|g^{-1}(x)|h,\bar{h};j,-j\rangle\!\rangle \\
\langle h,\bar{h};k,\bar{k}|(-\bar{L}_{-1})g^{-1}(x)|h,\bar{h};j,-j\rangle\!\rangle &= -\partial_{\bar{z}}\langle h,\bar{h};k,\bar{k}|g^{-1}(x)|h,\bar{h};j,-j\rangle\!\rangle \\
\langle h,\bar{h};k,\bar{k}|(-\bar{L}_0)g^{-1}(x)|h,\bar{h};j,-j\rangle\!\rangle &= -(\bar{z}\partial_{\bar{z}} + \bar{h})\langle h,\bar{h};k,\bar{k}|g^{-1}(x)|h,\bar{h};j,-j\rangle\!\rangle \\
\langle h,\bar{h};k,\bar{k}|(-\bar{L}_1)g^{-1}(x)|h,\bar{h};j,-j\rangle\!\rangle &= -(\bar{z}^2\partial_{\bar{z}} + 2\bar{z}\bar{h})\langle h,\bar{h};k,\bar{k}|g^{-1}(x)|h,\bar{h};j,-j\rangle\!\rangle
\end{aligned} \tag{3.27}$$

This again means that the functions

$$\Psi^{(h,\bar{h})}_{k,\bar{k}}(x) := \langle h,\bar{h};k,\bar{k}|g^{-1}(x)|h,\bar{h};j,-j\rangle\!\rangle \tag{3.28}$$

in the $\rho \to \infty$ limit also provide a representation of the generators of algebra $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$ after the implementation of the automorphism $M \to -M^T$ where M is any generator, thereby providing a dual representation to that of $\Phi^{(h,\bar{h})}_{k,\bar{k}}$s. Again the (anti-) holomorphic part has appeared in [94, 97].

The last ingredient we want is the CGC*s* of unitary irreducible positive discrete series representations [97] of $sl(2,\mathbb{R})$. These have been known for a long time [98] which we rework in the Appendix A using our conventions. These are given as

$$C^{h_1 h_2;h_3}_{k_1 k_2;k_3} = \langle h_1, h_2; k_1, k_2 | h_1, h_2; h_3, k_3 \rangle = \frac{1}{\prod_{i=1}^{3} \sqrt{k_i!(\Gamma(2h_i + k_i)}} f(k_1, k_2; k_3) \tag{3.29}$$

with $f(k_1, k_2; k_3)$ are as given by (A.31) in the Appendix A. We will not fix the normalisation as we do not need it.

Finally we are ready to put together various components of our OWN diagrams with n external legs with the corresponding representations $(h_i, \bar{h}_i)$ and compute them explicitly. The final answer will be proportion to $e^{-\rho\ \sum_{i=1}^{n} h_i} \times e^{-\rho\ \sum_{i=1}^{n} \bar{h}_i}$ times a function that is a product of a holomorphic part and an anti-holomorphic part. Let us now summarize the rules to compute the holomorphic part:

Feynman rules for OWN :

- For each in-going external leg in representation $(h_i, \bar{h}_i)$ we associate the factor:

$$\left[i^{h_i} z_i^{-2h_i - k_i} \sqrt{\frac{\Gamma(2h_i + k_i)}{k_i! \Gamma(2h_i)}} \right]$$

- For each out-going external leg in representation $(h_i, \bar{h}_i)$ associate the factor

$$\left[(-i)^{h_i} z_i^{k_i} \sqrt{\frac{\Gamma(2h_i + k_i)}{k_i! \Gamma(2h_i)}} \right]$$

- For each trivalent vertex with two in-going (out-going) edges in representations $(h_m, \bar{h}_m)$, $(h_n, \bar{h}_n)$ and one out-going (in-going) edge in the representation $(h_l, \bar{h}_l)$ we associate a CGC $C^{h_m h_n;h_l}_{k_m k_n;k_l}$.

- Finally sum over all repeated k_is.

The rules to compute the anti-holomorphic factor in the OWN are simply obtained from the above ones by replacing $h_i \to \bar{h}_i$, $k_i \to \bar{k}_i$ and then complex conjugating the rest. The boundary CFT answers are the same as the OWN answers but for the ρ-dependent pre-factors.

The 2-point function recovered

The 2-point function of two primary operators in a CFT is completely determined by the symmetries and therefore it is our simplest partial wave. We should be able to derive it from the simplest OWN which is just a line with the end points approaching the boundary.

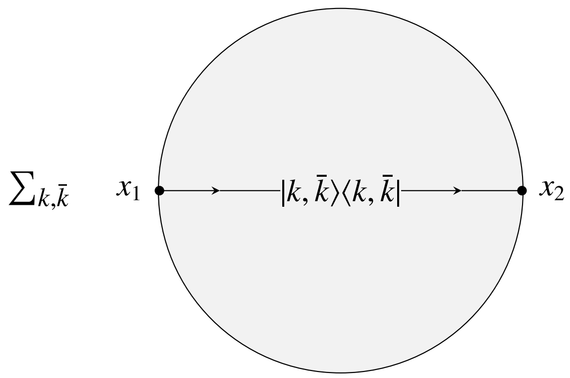


Figure 3.1: Spin network for 2-point function

According to our prescription it is given by[2]

$$\langle\langle h,\bar{h};j,j|g(x_1)g^{-1}(x_2)|h,\bar{h};j,-j\rangle\rangle$$

[2]Our prescription for the 2-point function appears similar to the one used in [99, 100] – as it is also the matrix element of the Wilson line operator between the highest and the lowest weight states of a non-unitary (finite dimensional) irrep of an $sl(2,\mathbb{R})$ algebra. Note, however, that [99, 100] use a non-unitary irrep (taken to be the same) for either $sl(2,\mathbb{R})$ component of the gauge algebra $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$ – where as we use a non-unitary irrep of the twisted diagonal $sl(2,\mathbb{R})$ sub-algebra constructed from two (generically distinct) unitary (infinite dimensional) irreps of each $sl(2,\mathbb{R})$ component of the gauge algebra.

$$
\begin{aligned}
&= \sum_{k,\bar{k}=0}^{\infty} \langle\!\langle h,\bar{h};j,j|g(x_1)|h,\bar{h};k,\bar{k}\rangle\langle h,\bar{h};k,\bar{k}|g^{-1}(x_2)|h,\bar{h};j,-j\rangle\!\rangle \\
&\overset{\rho_1=\rho_2=\rho\to\infty}{=} \lambda^2 \frac{(-1)^{-j}\Gamma(2\bar{h})}{\Gamma(2h)(2j)!} z_1^{-2h}\bar{z}_1^{-2\bar{h}} \sum_{k,\bar{k}=0}^{\infty} \frac{\Gamma(2h+k)}{k!\Gamma(2h)}\left(\frac{z_2}{z_1}\right)^k \frac{\Gamma(2\bar{h}+\bar{k})}{\bar{k}!\Gamma(2\bar{h})}\left(\frac{\bar{z}_2}{\bar{z}_1}\right)^{\bar{k}} \\
&= \frac{e^{-2\rho(h+\bar{h})}}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}} \qquad (3.30)
\end{aligned}
$$

where we have used the value of λ as in (3.17). Therefore the correctly normalised 2-point function is obtained by taking

$$
\begin{aligned}
\langle O_{(h,\bar{h})}(z_1,\bar{z}_1)O_{(h,\bar{h})}(z_2,\bar{z}_2)\rangle &= e^{2\rho(h+\bar{h})}\langle\!\langle h,\bar{h};j,j|g(z_1,\bar{z}_1,\rho)g^{-1}(z_2,\bar{z}_2,\rho)|h,\bar{h};j,-j\rangle\!\rangle\Big|_{\rho\to\infty} \\
&= \frac{1}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}} \qquad (3.31)
\end{aligned}
$$

It should be clear that when we consider the 2-point function of primaries with different conformal dimensions the corresponding Wilson line vanishes as the Wilson line operator does not change the representation of the state $\langle\!\langle h\bar{h};j,j|$ ($|h,\bar{h};j,-j\rangle\!\rangle$) when it acts to the left (right) and the resultant overlap simply vanishes as the representations $(h_1,\bar{h}_1)$ and $(h_2,\bar{h}_2)$ will be orthogonal when $h_1 \neq h_2$ or $\bar{h}_1 \neq \bar{h}_2$.

Note that this computation suggests that the conjugate to the state $\langle\!\langle h,\bar{h};j,m|$ should be taken to be $|h,\bar{h};j,-m\rangle\!\rangle$. This is not an unreasonable choice as the conformal transformation that takes the representation provided by the functions $\langle h,\bar{h};k,\bar{k}|g^{-1}(x)|h,\bar{h};j,m\rangle\!\rangle$ to the representation provided by $\langle\!\langle h,\bar{h};j,m|g(x)|h,\bar{h};k,\bar{k}\rangle$ is $z\to -1/z$, $\bar{z}\to -1/\bar{z}$. In polar coordinates on the complex plane this is $r\to 1/r$ which corresponds to the time-reversal operation on the cylinder under the state-operator correspondence. It is well known that the time-reversal operation acts on angular momentum eigenstates in this fashion.

The 3-point function recovered

We can now turn to computing the 3-point function which is also a partial wave on its own. For this we consider a three-pronged Open Wilson Network as given below:

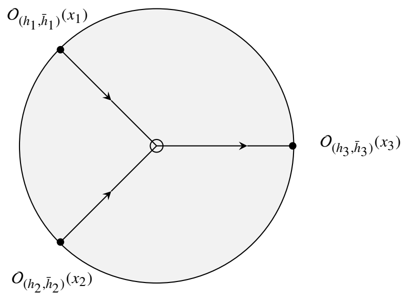


Figure 3.2: Spin network for CFT 3-point function.

Following our prescription we associate the following answer to this diagram:

$$\sum_{k_i=0}^{\infty}\left[(-1)^{-\frac{h_1}{2}}e^{-\rho h_1}z_1^{k_1}\sqrt{\frac{\Gamma(2h_1+k_1)}{k_1!\Gamma(2h_1)}}\right]\times\left[(-1)^{-\frac{h_2}{2}}e^{-\rho h_2}z_2^{k_2}\sqrt{\frac{\Gamma(2h_2+k_2)}{k_2!\Gamma(2h_2)}}\right]$$
$$\times\left[(-1)^{\frac{h_3}{2}}e^{-\rho h_3}z_3^{-2h_3-k_3}\sqrt{\frac{\Gamma(2h_3+k_3)}{k_3!\Gamma(2h_3)}}\right]\times C^{h_1h_2h_3}_{k_1k_2k_3}$$
$$\times\sum_{\bar{k}_i=0}^{\infty}\left[(-1)^{\frac{\bar{h}_1}{2}}e^{-\rho\bar{h}_1}\bar{z}_1^{\bar{k}_1}\sqrt{\frac{\Gamma(2\bar{h}_1+\bar{k}_1)}{\bar{k}_1!\Gamma(2\bar{h}_1)}}\right]\times\left[(-1)^{\frac{\bar{h}_2}{2}}e^{-\rho\bar{h}_2}\bar{z}_2^{\bar{k}_2}\sqrt{\frac{\Gamma(2\bar{h}_2+\bar{k}_2)}{\bar{k}_2!\Gamma(2\bar{h}_2)}}\right]$$
$$\times\left[(-1)^{-\frac{\bar{h}_3}{2}}e^{-\rho\bar{h}_3}\bar{z}_3^{-2\bar{h}_3-\bar{k}_3}\sqrt{\frac{\Gamma(2\bar{h}_3+\bar{k}_3)}{\bar{k}_3!\Gamma(2\bar{h}_3)}}\right]\times C^{\bar{h}_1\bar{h}_2\bar{h}_3}_{\bar{k}_1\bar{k}_2\bar{k}_3} \tag{3.32}$$

Clearly the answer is a product of holomorphic and anti-holomorphic pieces each of which can be computed separately. The holomorphic part becomes:

$$\sum_{k_i=0}^{\infty}\left[(-1)^{-\frac{h_1}{2}}e^{-\rho h_1}z_1^{k_1}\sqrt{\frac{\Gamma(2h_1+k_1)}{k_1!\Gamma(2h_1)}}\right]\times\left[(-1)^{-\frac{h_2}{2}}e^{-\rho h_2}z_2^{k_2}\sqrt{\frac{\Gamma(2h_2+k_2)}{k_2!\Gamma(2h_2)}}\right]$$
$$\times\left[(-1)^{\frac{h_3}{2}}e^{-\rho h_3}z_3^{-2h_3-k_3}\sqrt{\frac{\Gamma(2h_3+k_3)}{k_3!\Gamma(2h_3)}}\right]\times\delta_{k_1+k_2-k_3+h_1+h_2-h_3}$$
$$\times\frac{\Gamma(k_3-k_2)k_2!\Gamma(2h_2+k_1+k_2)\Gamma(h_3+h_1-h_2)}{\sqrt{k_1!k_2!k_3!\Gamma(2h_1+k_1)\Gamma(2h_2+k_2)\Gamma(2h_3+k_3)}}{}_3F_2\left(\begin{matrix}-k_1,-k_3,2h_1+k_1+k_2-k_3\\1+k_2-k_3,1-2h_2-k_1-k_2\end{matrix};1\right)$$
$$\sim\quad z_3^{-2h_3}\sum_{k_i=0}^{\infty}\frac{1}{k_1!k_3!}\frac{\Gamma(2h_2+k_1+k_2)\Gamma(k_3-k_2)}{\Gamma(h_1+h_2-h_3)\Gamma(2h_2+k_1+k_2-k_3)}$$
$$\times z_1^{k_1}{}_3F_2\left(\begin{matrix}-k_1,-k_3,2h_1+k_1+k_2-k_3\\1+k_2-k_3,1-2h_2-k_1-k_2\end{matrix};1\right)z_2^{k_2}z_3^{-k_3}\delta_{k_1+k_2-k_3+h_1+h_2-h_3}$$

$$\sim \quad z_3^{-2h_3} \sum_{k_i=0}^{\infty} \sum_{n=0}^{\min(k_1,k_3)} \frac{\Gamma(k_3-k_2-n)\Gamma(2h_2+k_1+k_2-n)\Gamma(2h_1+k_1+k_2-k_3+n)}{\Gamma(h_1+h_2-h_3)\Gamma(2h_2+k_1+k_2-k_3)\Gamma(2h_1+k_1+k_2-k_3)} \times \frac{z_1^{k_1} z_2^{k_2} z_3^{-k_3}}{n!(k_1-n)!(k_3-n)!} \delta_{k_1+k_2-k_3+h_1+h_2-h_3} \tag{3.33}$$

where we have used the explicit series representation of ${}_3F_2$. Next we write $k_1 = n_2 + n_3$, $k_3 = n_1 + n_2$ $n = n_2$ and $k_2 = n_1 - n_3 + h_3 - h_1 - h_2$. Then the four sums can be reduced to three sums over n_1, n_2, n_3 at the expense of removing the Kronecker delta to obtain an expression exactly proportional to

$$\sim \quad \frac{1}{(z_1-z_2)^{h_1+h_2-h_3}(z_2-z_3)^{h_2+h_3-h_1}(z_3-z_1)^{h_3+h_1-h_2}} \tag{3.34}$$

where (and henceforth) “~" indicates that we have dropped some non-zero constant factors. Similarly the the anti-holomorphic part will give an answer proportional to

$$\frac{1}{(\bar{z}_1-\bar{z}_2)^{\bar{h}_1+\bar{h}_2-\bar{h}_3}(\bar{z}_2-\bar{z}_3)^{\bar{h}_2+\bar{h}_3-\bar{h}_1}(\bar{z}_3-\bar{z}_1)^{\bar{h}_3+\bar{h}_1-\bar{h}_2}} \tag{3.35}$$

Multiplying both these factors together one recovers the precise coordinate behaviour of the 3-point function of primaries in the CFT.

The 4-point partial wave recovered

For this we consider the OWN in Fig. 3.3 below:

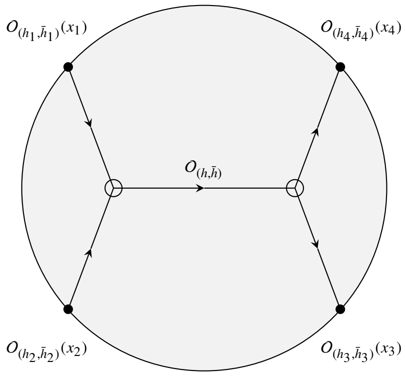


Figure 3.3: Partial Wave of 4-point function.

whose answer is

$$\sum_{k_i,\bar{k}_i} \langle\langle h_1,\bar{h}_1; j_1, j_1|g(x_1)|h_1,\bar{h}_1;k_1,\bar{k}_1\rangle \times \langle\langle h_2,\bar{h}_2; j_2, j_2|g(x_2)|h_2,\bar{h}_2;k_2,\bar{k}_2\rangle$$
$$\times\langle h_3,\bar{h}_3;k_3,\bar{k}_3|g^{-1}(x_3)|h_3,\bar{h}_3; j_3,-j_3\rangle\rangle \times \langle h_4,\bar{h}_4;k_4,\bar{k}_4|g^{-1}(x_4)|h_4,\bar{h}_4; j_4,-j_4\rangle\rangle$$

$$\times \sum_{k,\bar{k}} C^{h_1h_2h}_{k_1k_2k} \times C^{hh_3h_4}_{kk_3k_4} \times C^{\bar{h}_1\bar{h}_2\bar{h}}_{\bar{k}_1\bar{k}_2\bar{k}} \times C^{\bar{h}\bar{h}_3\bar{h}_4}_{\bar{k}\bar{k}_3\bar{k}_4} \tag{3.36}$$

where $x_i = (z_i, \bar{z}_i, \rho \to \infty)$. One can in principle compute this quantity and recover the full coordinate dependence of this 4-point partial wave (as guaranteed by the differential relations we had established earlier). However to simplify the presentation and as it is standard we take $z_1 \to \infty$, $z_2 \to 1$, $z_3 \to z$ and $z_4 \to 0$. Then the partial wave in the decomposition of the 4-point function of four primaries of dimensions $(h_i, \bar{h}_i)$ for $i = 1, 2, 3, 4$ takes the form:

$$W_{(h,\bar{h})}(x) = z_1^{-2h_1}\bar{z}_1^{-2\bar{h}_1} G_{(h,\bar{h})}(z) \tag{3.37}$$

From the CFT it is known that G takes the following form:

$$G_{(h,\bar{h})}(z,\bar{z}) \sim z^{-h_3-h_4}\bar{z}^{-\bar{h}_3-\bar{h}_4}\mathcal{F}_h(z)\bar{\mathcal{F}}_{\bar{h}}(\bar{z}) \tag{3.38}$$

where $\mathcal{F}$ and $\bar{\mathcal{F}}$ are supposed to be the corresponding conformal blocks. We can now compute using our prescription this partial wave and hence the blocks. When we set $z_1 \to \infty$, $z_2 \to 1$, $z_3 \to z$ and $z_4 \to 0$ then each component in the above expression simplifies as follows:

$$\begin{aligned}
\langle\!\langle h_1, \bar{h}_1; j_1, j_1 | g(x_1) | h_1, \bar{h}_1; k_1, \bar{k}_1 \rangle &\to (-1)^{\frac{h_1}{2}} e^{-\rho h_1} z_1^{-2h_1} \delta_{k_1,0} \times (-1)^{-\frac{\bar{h}_1}{2}} e^{-\rho \bar{h}_1} \bar{z}_1^{-2\bar{h}_1} \delta_{\bar{k}_1,0} \\
\langle\!\langle h_2, \bar{h}_2; j_2, j_2 | g(x_2) | h_2, \bar{h}_2; k_2, \bar{k}_2 \rangle &\to (-1)^{\frac{h_2}{2}} e^{-\rho h_2} \sqrt{\frac{\Gamma(2h_2 + k_2)}{k_2! \Gamma(2h_2)}} \\
&\qquad \times (-1)^{-\frac{\bar{h}_2}{2}} e^{-\rho \bar{h}_2} \sqrt{\frac{\Gamma(2\bar{h}_2 + \bar{k}_2)}{\bar{k}_2! \Gamma(2\bar{h}_2)}} \\
\langle h_3, \bar{h}_3; k_3, \bar{k}_3 | g^{-1}(x_3) | h_3, \bar{h}_3; j_3, -j_3 \rangle\!\rangle &\to (-1)^{-\frac{h_3}{2}} e^{-\rho h_3} z^{k_3} \sqrt{\frac{\Gamma(2h_3 + k_3)}{k_3! \Gamma(2h_3)}} \\
&\qquad \times (-1)^{\frac{\bar{h}_3}{2}} e^{-\rho \bar{h}_3} \bar{z}^{\bar{k}_3} \sqrt{\frac{\Gamma(2\bar{h}_3 + \bar{k}_3)}{\bar{k}_3! \Gamma(2\bar{h}_3)}} \\
\langle h_4, \bar{h}_4; k_4, \bar{k}_4 | g^{-1}(x_4) | h_4, \bar{h}_4; j_4, -j_4 \rangle\!\rangle &\to (-1)^{-\frac{h_4}{2}} e^{-\rho h_4} \delta_{k_4,0} \times (-1)^{\frac{\bar{h}_4}{2}} e^{-\rho \bar{h}_4} \delta_{\bar{k}_4,0} \qquad (3.39)
\end{aligned}$$

We now need the CGC*s* which again fall into different cases. Let us first assume that we have $h_1 + h_2 \geq h$ and $h_3 + h_4 \geq h$. In this case the relevant CGC*s* are

$$C^{h_1,h_2,h}_{0,k_2,k} \sim \delta_{h_1+h_2+k_2-h-k} \times \frac{\Gamma(k-k_2)}{\sqrt{\Gamma(2h_1)}} \times \sqrt{\frac{k_2! \Gamma(2h_2 + k_2)}{k! \Gamma(2h + k)}} \qquad (3.40)$$

$$C^{h,h_3,h_4}_{k,k_3,0} \sim \delta_{h_3+h_4+k_3-h-k} \times \frac{\Gamma(k-k_3)}{\sqrt{\Gamma(2h_4)}} \times \sqrt{\frac{k_3! \Gamma(2h_3 + k_3)}{k! \Gamma(2h + k)}} \qquad (3.41)$$

The holomorphic part of (3.36) becomes

$$\begin{aligned}
\sim \; & (-1)^{\frac{1}{2}(h_1+h_2-h_3-h_4)} e^{-\rho(h_1+h_2+h_3+h_4)} z_1^{-2h_1} \sum_{k_i,k} \sqrt{\frac{\Gamma(2h_2 + k_2)}{k_2! \Gamma(2h_2)}} \sqrt{\frac{\Gamma(2h_3 + k_3)}{k_3! \Gamma(2h_3)}} z^{k_3} \\
& \times \delta_{h_1+h_2-h+k_2-k} \frac{\Gamma(k-k_2)}{\sqrt{\Gamma(2h_1)}} \sqrt{\frac{k_2! \Gamma(2h_2 + k_2)}{k! \Gamma(2h + k)}} \times \delta_{h_3+h_4+k_3-h-k} \frac{\Gamma(k-k_3)}{\sqrt{\Gamma(2h_4)}} \sqrt{\frac{k_3! \Gamma(2h_3 + k_3)}{k! \Gamma(2h + k)}}
\end{aligned}$$

$$\sim \quad e^{-\rho(h_1+h_2+h_3+h_4)} z_1^{-2h_1} \sum_{k=0}^{\infty} \frac{\Gamma(h-h_1+h_2+k)\Gamma(h+h_3-h_4+k)}{\Gamma(2h+k)k!} z^k$$

$$\sim \quad e^{-\rho(h_1+h_2+h_3+h_4)} z_1^{-2h_1} z^{h-h_3-h_4} {}_2F_1[h-h_1+h_2, h+h_3-h_4, 2h, z] \tag{3.42}$$

where in the second step we carried out the sums over k_2 and k_3 using the Kronecker deltas. One gets for the anti-holomorphic part

$$\sim e^{-\rho(\bar{h}_1+\bar{h}_2+\bar{h}_3+\bar{h}_4)} \bar{z}_1^{-2\bar{h}_1} \bar{z}^{\bar{h}-\bar{h}_3-\bar{h}_4} {}_2F_1[\bar{h}-\bar{h}_1+\bar{h}_2, \bar{h}+\bar{h}_3-\bar{h}_4, 2\bar{h}, \bar{z}]. \tag{3.43}$$

Comparing our answer with (3.37, 3.38) we recover the well-known answer [101] for the 4-point spinning global conformal block

$$\begin{aligned} \mathcal{F}_h(z) &\sim z^h \, {}_2F_1[h-h_1+h_2, h+h_3-h_4, 2h, z] \\ \bar{\mathcal{F}}_{\bar{h}}(\bar{z}) &\sim \bar{z}^{\bar{h}} \, {}_2F_1[\bar{h}-\bar{h}_1+\bar{h}_2, \bar{h}+\bar{h}_3-\bar{h}_4, 2\bar{h}, \bar{z}] \end{aligned} \tag{3.44}$$

As mentioned above in this calculation we have assumed $h \le h_3 + h_4$ and $h \le h_1 + h_2$. There are three other possibilities which can also be computed easily using the appropriate expressions of the CGC*s* (A.31) to give answers exactly of the same form.

Note that what we have computed satisfies two independent conformal Casimir equations – one for each of the two $sl(2,\mathbb{R})$ factors in the 2d global conformal algebra with eigenvalues $2h(h-1)$ and $2\bar{h}(\bar{h}-1)$ respectively. The global partial wave however is supposed to satisfy one conformal Casimir equation with the Casimir operator given by the sum of these two Casimirs with eigenvalue $2h(h-1) + 2\bar{h}(\bar{h}-1)$. This eigenvalue is invariant under $h \leftrightarrow \bar{h}$. The OWN considered above continues to be a solution to this one Casimir equation. But there is a second independent solution with the same eigenvalue obtained from the above OWN by interchanging h with $\bar{h}$.[3] Therefore any linear combination of these two OWNs would provide a solution to the conformal Casimir equation. A basis

[3]The two states $|h, \bar{h}\rangle$ and $|\bar{h}, h\rangle$, when $h - \bar{h}$ is an integer, could be thought of as the duals of the two physical polarisations of an appropriate higher spin field in the bulk with spin $|h - \bar{h}|$ as in, say [102].

in this space of solutions can be taken to be the symmetric and the antisymmetric combinations under $h \leftrightarrow \bar{h}$. As advocated, say, in [6] the symmetric combination is the one satisfying the appropriate boundary conditions. This in our context gives us the $G_{\Delta,l}(z,\bar{z})$ given by $z^{-h_3-h_4}\bar{z}^{-\bar{h}_3-\bar{h}_4}$ times :

$$|z|^{\Delta-l}\Big(z^l\,{}_2F_1\left[\frac{\Delta-l}{2}-h_{12},\frac{\Delta-l}{2}-h_{34},\Delta-l,z\right]{}_2F_1\left[\frac{\Delta+l}{2}-h_{12},\frac{\Delta+l}{2}-h_{34},\Delta+l,\bar{z}\right]$$
$$+\,(z\to\bar{z},h_{ij}\to\bar{h}_{ij})\Big) \tag{3.45}$$

where $h_{ij}=h_i-h_j$ etc. and $\Delta=h+\bar{h}$, $l=h-\bar{h}$. This is our final answer for the 4-point partial wave of primaries and clearly matches with the result for scalar conformal partial waves given in [6]

$$G_{\Delta,l}=|z|^{\Delta-l}\Big[z^l\,{}_2F_1\left(\frac{\Delta-\Delta_{12}+l}{2},\frac{\Delta+\Delta_{34}+l}{2},\Delta+l;z\right)$$
$$\times\,{}_2F_1\left(\frac{\Delta-\Delta_{12}-l}{2},\frac{\Delta+\Delta_{34}-l}{2},\Delta-l;\bar{z}\right)+(z\leftrightarrow\bar{z})\Big]$$

when we take $h_i=\bar{h}_i$ as it is appropriate for scalar operators in the external legs and satisfies the same boundary conditions as $z,\bar{z}\to 0$. We have considered in here the case of the boundary conditions imposed when cross ratios $(z,\bar{z})$ approach zero. One can similarly consider diagrams that compute blocks with boundary conditions imposed as $(z,\bar{z})$ approaches $(1,1)$ or (∞,∞).

The 5-point conformal block recovered

The last example we consider here is the conformal partial wave that appears in the pants decomposition of the 5-point function of primaries. For this we consider the following

Open Wilson Network (Fig. 3.4).

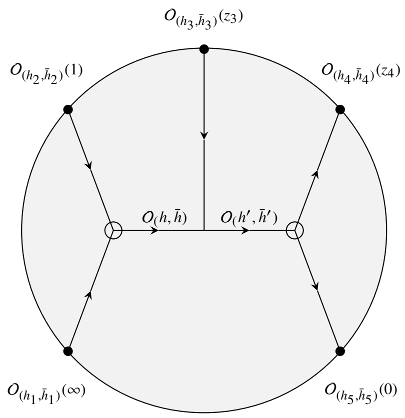


Figure 3.4: A Partial Wave of 5-point function.

The value of the holomorphic part of this diagram is up to a factor $e^{-\rho(h_1+h_2+h_3+h_4+h_5)}$:

$$\sim z_3^{-2h_3} \sum_{k_2,k_3,k_4,k,k'} C^{h_1 h_2;h}_{0k_2;k} \sqrt{\frac{\Gamma(2h_2+k_2)}{k_2!\Gamma(2h_2)}} C^{hh_3;h'}_{kk_3;k'} \sqrt{\frac{\Gamma(2h_3+k_3)}{k_3!\Gamma(2h_3)}} C^{h_4 h_5;h'}_{k_4 0;k'} \sqrt{\frac{\Gamma(2h_4+k_4)}{k_4!\Gamma(2h_4)}} z_3^{-k_3} z_4^{k_4} \tag{3.46}$$

Let us further assume that $h_1 + h_2 \geq h$, $h + h_3 \geq h'$ and $h_4 + h_5 \geq h'$.[4] The CGC*s* are given by

$$C^{h_1,h_2,h}_{0,k_2,k} \sim \delta_{h_1+h_2+k_2-h-k} \frac{\Gamma(k-k_2)}{\sqrt{\Gamma(2h_1)}} \sqrt{\frac{k_2!\Gamma(2h_2+k_2)}{k!\Gamma(2h+k)}}$$

$$C^{h_4 h_5;h'}_{k_4 0;k'} \sim \delta_{h_4+h_5+k_4-h'-k'} \frac{\Gamma(k'-k_4)}{\sqrt{\Gamma(2h_5)}} \sqrt{\frac{k_4!\Gamma(2h_4+k_4)}{k'!\Gamma(2h'+k')}} \tag{3.47}$$

and

$$C^{hh_3;h'}_{kk_3;k'} = \delta_{h+h_3-h'+k+k_3-k'} \frac{\Gamma(k'-k_3)k_3!\Gamma(2h_3+k+k_3)\Gamma(h+h'-h_3)}{\sqrt{k!k_3!k'!\Gamma(2h+k)\Gamma(2h_3+k_3)\Gamma(2h'+k')}}$$

[4]All the other possibilities can also be considered and computed with the suitable CGC*s* resulting in expressions with the same coordinate dependence.

$$\times\, {}_3F_2\left(\begin{matrix} -k, -k', 2h+k+k_3-k' \\ 1+k_3-k', 1-2h_3-k-k_3 \end{matrix}; 1\right) \tag{3.48}$$

Then the value of 5-point block thus becomes

$$z_3^{-2h_3} \sum_{k,k'} \sum_{k_2,k_3,k_4} z_3^{-k_3} z_4^{k_4} \;{}_3F_2\left(\begin{matrix} -k, -k', 2h+k+k_3-k' \\ 1+k_3-k', 1-2h_3-k-k_3 \end{matrix}; 1\right)$$
$$\times \frac{\Gamma(k-k_2)\Gamma(k'-k_4)\Gamma(k'-k_3)\Gamma(2h_3+k+k_3)\Gamma(2h_2+k_2)\Gamma(2h_4+k_4)\Gamma(h+h'-h_3)}{k!k'!\Gamma(2h+k)\Gamma(2h'+k')}$$
$$\times \delta_{h_1+h_2+k_2-h-k}\;\delta_{h_4+h_5+k_4-h'-k'}\;\delta_{h+h_3-h'+k+k_3-k'} \tag{3.49}$$

The 5-point global block has been computed recently using the CFT methods in [21] and to compare with their answer we write above expression as

$$z_3^{-2h_3} z_3^{h-h_3-h'} z_4^{h'-h_4-h_5} \sum_{k,k'} \sum_{k_2,k_3,k_4} q_1^{-h_3-k_3+h_4+k_4+h_5-h} q_2^{h_4+k_4+h_5-h'}$$
$$\times {}_3F_2\left(\begin{matrix} -k, -k', 2h+k+k_3-k' \\ 1+k_3-k', 1-2h_3-k-k_3 \end{matrix}; 1\right)$$
$$\times \frac{\Gamma(k-k_2)\Gamma(k'-k_4)\Gamma(k'-k_3)\Gamma(2h_3+k+k_3)\Gamma(2h_2+k_2)\Gamma(2h_4+k_4)\Gamma(h+h'-h_3)}{k!k'!\Gamma(2h+k)\Gamma(2h'+k')}$$
$$\times \delta_{h_1+h_2+k_2-h-k}\;\delta_{h+h_3-h'+k+k_3-k'}\;\delta_{h_4+h_5+k_4-h'-k'} \tag{3.50}$$

where $q_1 = z_3$ and $q_2 = z_4/z_3$. Then we do k_2, k_3, k_4 sums using 1st, 2nd and 3rd Kronecker deltas respectively in the above expression. The result becomes

$$\sim z_3^{-2h_3} z_3^{h-h_3-h'} z_4^{h'-h_4-h_5} \sum_{k,k'} \frac{q_1^k q_2{}^{k'}}{k!k'!}$$
$$\times \frac{\Gamma(-h_1+h+h_2+k)\Gamma(h-h'+h_3+k)\Gamma(-h+h'+h_3+k')\Gamma(h'+h_4-h_5+k')}{\Gamma(2h+k)\Gamma(2h'+k')}$$
$$\times {}_3F_2\left(\begin{matrix} -k, -k', h+h'-h_3 \\ h-h'-h_3-k'+1, -h+h'-h_3-k+1 \end{matrix}; 1\right) \tag{3.51}$$

The hypergeometric function here can be rewritten using the (Shepperd's) identity [95]

$$ {}_3F_2\left(\begin{matrix}-n, a, b\\ d, e\end{matrix};1\right) = \frac{(e-a)_n}{(e)_n}\, {}_3F_2\left(\begin{matrix}-n, a, d-b\\ d, a+1-n-e\end{matrix};1\right) \tag{3.52} $$

as

$$ {}_3F_2\left(\begin{matrix}-k, -k', h+h'-h_3\\ h-h'-h_3-k'+1, -h+h'-h_3-k+1\end{matrix};1\right) $$
$$ = \frac{\Gamma(h-h'+h_3+k-k')\Gamma(h-h'+h_3)}{\Gamma(h-h'+h_3-k')\Gamma(h-h'+h_3+k)}\; {}_3F_2\left(\begin{matrix}-k, -k', -2h'-k'+1\\ h-h'-h_3-k'+1, h_3+h-h'-k'\end{matrix};1\right) \tag{3.53} $$

Finally we get the holomorphic part of the 5-point block in the form

$$ \sim z_3^{-2h_3} z_3^{h-h_3-h'} z_4^{h'-h_4-h_5} \sum_{k,k'=0}^{\infty} F_{k,k'} q_1^k q_2{}^{k'} \tag{3.54} $$

where

$$ F_{k,k'} = \frac{1}{k!k'!} \frac{\Gamma(h+h_2-h_1+k)\Gamma(h'+h_4-h_5+k')}{\Gamma(2h+k)\Gamma(2h'+k')} \tau_{k,k'} \tag{3.55} $$

with

$$ \begin{aligned} \tau_{k,k'} \quad &= \frac{\Gamma(h_3-h+h'+k')\Gamma(h_3+h-h'+k-k')}{\Gamma(h_3+h-h'-k')} \\ &\times {}_3F_2\left(\begin{matrix}-k, -k', -2h'-k'+1\\ h-h'-h_3-k'+1, h_3+h-h'-k'\end{matrix};1\right) \end{aligned} \tag{3.56} $$

which apart from a purely h_i-dependent pre-factor is exactly identical to the one obtained in [21]. The anti-holomorphic part can also be computed on similar lines and put together with the holomorphic part to find the contribution of the OWN in Fig.(5) to the 5-point partial wave.

The 5-point block is a solution to two pairs of Casimir equations (two each for each of the two intermediate edges) as discussed earlier. However just as in the case of the 4-point partial wave we need impose only two Casimir equations. Then we have four OWN diagrams related to the one in Fig.(5) under $(h \leftrightarrow \bar{h})$ or $(h' \leftrightarrow \bar{h}')$ all of which solve these two equations. Again the generic solution would be a linear combination of all four solutions and one would have to pick appropriate combinations depending on the boundary conditions one imposes.

Now that we have demonstrated our method at work successfully one can in principle compute straightforwardly the higher point (global) blocks as well as the partial waves for a given decomposition of that higher point function of primaries. We next turn to $d = 1$ analysis.

$d = 1$ analysis

First we would like to compute the cap state for $1d$ case and then the $1d$ global blocks. We begin with the infinite dimensional matrix representations [94] of global conformal algebra $sl(2,\mathbb{R})$ for CFT_1:

$$\begin{aligned} L_1|h,n\rangle &= \sqrt{n(2h+n-1)}\,|h,n-1\rangle \\ L_{-1}|h,n\rangle &= \sqrt{(n+1)(2h+n)}\,|h,n+1\rangle \\ L_0|h,n\rangle &= (h+n)\,|h,n\rangle \end{aligned} \tag{3.57}$$

where $D = L_0$, $P = L_{-1}$ and $K = L_1$. The bulk is the $\mathbb{H}^2$ space whose tangent space rotation group is $SO(2)$. Therefore the cap state $|h,\theta\rangle\rangle$ transform as a 1-dimensional irrep of $SO(2)$:

$$(L_1 - L_{-1})|h,\theta\rangle\rangle = \theta|h,\theta\rangle\rangle \tag{3.58}$$

The parameter θ, a purely imaginary number, is related to the spin of the general bulk field – we will elaborate further on this shortly. This equation can be solved for $|h,\theta\rangle\!\rangle$ as a linear combination of states in the module:

$$|h,\theta\rangle\!\rangle = \sum_{n=0}^{\infty} C_n |h,n\rangle \tag{3.59}$$

The C_n can be given by $C_n = \sqrt{\frac{\Gamma(2h)}{n!\,\Gamma(2h+n)}}\, f_n$ with the f_n satisfying the recursion relation

$$f_{n+1} = \theta\, f_n + \; n\,(2h+n-1) f_{n-1}\,. \tag{3.60}$$

It is not difficult to see that the f_n are generated by $G(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f_n$ where

$$G(x) = (1-x)^{-h-\frac{\theta}{2}}(1+x)^{-h+\frac{\theta}{2}}\,. \tag{3.61}$$

We find that the coefficient of $\frac{x^n}{n!}$ in $G(x)$ to be:

$$f_n = (-1)^n \left(h-\frac{\theta}{2}\right)_n {}_2F_1\left(-n,\, h+\frac{\theta}{2},\, -h+\frac{\theta}{2}-n+1;-1\right) \tag{3.62}$$

Having obtained the expression for the most general cap state in $d=1$, we can repeat the rest of the exercises carried out in section 2 on these caps. Working with the coset element

$$g(x) = e^{\rho L_0}\, e^{-xL_{-1}}$$

we can extract the leading terms in the large-ρ limit of $\langle\!\langle h,\theta|g(x)|h,k\rangle$ and $\langle\!\langle h,\theta|g(x)|h,k\rangle$. With some further analysis we find the following simple answers in the $\rho\to\infty$ limit:

$$\begin{aligned}
\lim_{\rho\to\infty} e^{\rho h}\langle\!\langle h,\theta|g(x)|h,k\rangle &= (-1)^{-h-\frac{\theta}{2}}\sqrt{\frac{\Gamma(2h+k)}{k!\,\Gamma(2h)}}x^{-2h-k} \\
\lim_{\rho\to\infty} e^{\rho h}\langle h,k|g^{-1}(y)|h,\theta\rangle\!\rangle &= y^k\sqrt{\frac{\Gamma(2h+k)}{k!\,\Gamma(2h)}}
\end{aligned} \tag{3.63}$$

Notice that even though the general cap states depend on the spin-parameter θ the final expressions for the legs only have very simple dependence on it. For example, putting the legs together and performing the sum over k gives the following result for two point function

$$\lim_{\rho\to\infty} e^{2\rho h}\langle\langle h,\theta|g(x)g^{-1}(y)|h,\theta\rangle\rangle = (-1)^{-h-\frac{\theta}{2}}x^{-2h}\sum_{k=0}^{\infty}\frac{\Gamma(2h+k)}{k!\ \Gamma(2h)}\left(\frac{x}{y}\right)^{k} = (-1)^{-h-\frac{\theta}{2}}\frac{1}{(x-y)^{2h}} \tag{3.64}$$

A comparison of the $d=1$ legs here with the holomorphic part of the $d=2$ case of the previous section enables us to immediately write down the $1d$ blocks [24, 103] by starting with the holomorphic parts of $d=2$ blocks and replacing $h\to\Delta$ and $z\to|x|$.

The interpretation of θ

We have seen so far that the θ despite being part of the cap state does not enter the legs in the $\rho\to\infty$ limit except as a phase factor. So what is the interpretation of this θ? To better understand the role of θ we must first look at the linearised bulk equations satisfied by the the legs $\langle h,k|g^{-1}(x)|h,\theta\rangle\rangle$. To this end we first list the following identities [14] satisfied by $g^{-1}(x)$

$$\begin{aligned} L_0\, g^{-1}(x) &= (-\partial_\rho + x\partial_x)\, g^{-1}(x) \\ L_{-1}\, g^{-1}(x) &= -\partial_x g^{-1}(x) \\ L_1\, g^{-1}(x) &= (2x\,\partial_\rho - x^2\partial_x + e^{-\rho}\partial_x)\, g^{-1}(x) + g^{-1}(x)\, e^{-\rho}(L_1 - L_{-1}) \end{aligned} \tag{3.65}$$

Using these relations we can easily compute the action of the $sl(2,\mathbb{R})$ Casimir operator C_2 on $g^{-1}(x)$

$$\begin{aligned} C_2 &= 2L_0^2 - L_1L_{-1} - L_{-1}L_1 \\ C_2 g^{-1}(x) &= 2\,(\partial_\rho^2 + \partial_\rho + e^{-2\rho}\partial_x^2)\, g^{-1}(x) + e^{-\rho}\partial_x g^{-1}(x)\,(L_1 - L_{-1}) \end{aligned} \tag{3.66}$$

Thus we see that the legs satisfy the second order PDE:

$$(\partial_\rho^2 + \partial_\rho + e^{-2\rho}\partial_x^2 + \theta\, e^{-\rho}\partial_x)\langle h, k|g^{-1}(x)|h, \theta\rangle\!\rangle = \varDelta(\varDelta - 1)\langle h, k|g^{-1}(x)|h, \theta\rangle\!\rangle \tag{3.67}$$

We would now like to interpret this equation as that of a bulk local field in the background AdS_2 geometry with metric $ds^2_{\mathbb{H}^2} = d\rho^2 + e^{2\rho}\, dx^2$.

Since the boundary isometry group is just $\mathbb{Z}_2$ we would expect the boundary conformal primary operators to be characterised by a scaling dimension $\varDelta$ and a parity ± 1. But any general bulk local field in two dimensions (once one trades off the tangent space indices for spacetime ones) has to have only two parameters: the mass and the spin on which the bulk covariant derivative acts as

$$D_\mu\psi(x) = \partial_\mu\psi(x) + \frac{1}{2}\omega_\mu^{ab} L_{ab}\psi(x) \tag{3.68}$$

where L_{ab} is the tangent space rotation generator in the representation of $\psi(x)$. Redefining the coordinates $z = e^{-\rho} + ix$, $\bar{z} = e^{-\rho} - ix$ the metric of AdS_2 becomes $ds^2 = \frac{4dzd\bar{z}}{(z+\bar{z})^2}$. For this geometry we have the following non-zero vielbeins, spin-connections and Christoffel connections:

$$e^+ = \frac{2dz}{(z+\bar{z})^2}, \quad e^- = \frac{2d\bar{z}}{(z+\bar{z})^2}, \quad \omega^+{}_+ = \frac{dz - d\bar{z}}{(z+\bar{z})} = -\omega^-{}_-, \quad \varGamma^z_{zz} = \frac{-2}{z+\bar{z}} = \varGamma^{\bar{z}}_{\bar{z}\bar{z}} \tag{3.69}$$

Since the tangent space is just $\mathbb{R}^2$, there is only one rotation generator L_{+-}, and we can take the field $\psi(x)$ to be an eigenstate of it with eigenvalue $i\theta$. We can now find the Laplacian operator which acts on ϕ. It is easy to show that such a field satisfies the following equation

$$(\Box - m^2)\,\psi(x) \;=\; (\partial_\rho^2 + \partial_\rho + e^{-2\rho}\partial_x^2 + \theta\, e^{-\rho}\partial_x)\psi(x) - (m^2 + \frac{1}{8}\theta^2)\,\psi(x) = 0 \tag{3.70}$$

Comparing (3.67) and (3.70) we make the following identifications:

$$\Delta(\Delta - 1) = m^2 + \frac{\theta^2}{8}\,. \tag{3.71}$$

Therefore, we conclude that, when it is available, the parameter θ represents the spin of the bulk field.[5]

[5] Amusingly the same equation (3.70) shows up when one considers a complex scalar in AdS_2 minimally coupled to a background electric field whose strength is given by θ.

Chapter 4

Scalar CPW from OWN*s*

Even though the general prescription for computing the partial waves of correlators of any set of primaries (in arbitrary representations of the rotation group of the boundary theory) in general CFT$_d$ using OWNs was laid down in chapter 2, to carry through the explicit computations in higher dimensions one needs to find out the basic ingredients. Most of these ingredients are not known yet. For example the CGC*s* in generic representations of conformal group $so(1, d+1)$ are mostly not known. Even when they have existence in the literature (for example the CGC*s* for traceless symmetric representations are found in [104]), they are not in a form we can use. The cap-states for a generic tensor primary in CFT_d are not known.

In this chapter we would like to report some progress in this direction. In particular, we will demonstrate how to implement our prescription explicitly for the scalar CPW $W^{(d)}_{\Delta,0}(\Delta_i, x_i)$ in any CFT$_d$. This will be shown by computing the OWNs in AdS_{d+1} spaces, with all lines (both external and internal) carrying scalar representations. Our results include a simplification of the computations of OWN using the concept of *OPE modules* - which are close analogues of the OPE blocks that were studied in the literature [19, 20]. With this simplification we compute the scalar 4-point blocks in general dimension and show that our prescription reproduces the known answers [6]. Remarkably, our results are naturally given in terms of Gegenbauer polynomial basis [6, 57].

Collecting the Ingredients

We start with collecting these ingredients: (i) Wilson lines, (ii) the cap states, and (iii) CG coefficient.

Wilson Lines

We will be evaluating the OWN in the background of the Euclidean AdS_{d+1} geometry with $\mathbb{R}^d$ boundary (*i.e*, Poincare AdS_{d+1}) with the metric:

$$l^{-2}\, ds^2_{AdS_{d+1}} = d\rho^2 + e^{2\rho} \sum_{i=1}^{d} dx^i dx^i. \tag{4.1}$$

For this, working with the frame:

$$e^i = l\, e^\rho\, dx^i, \quad e^{d+1} = l\, d\rho \tag{4.2}$$

we find only non-vanishing spin-connections to be $\omega^{i(d+1)} = \frac{1}{l} e^i = -\omega^{(d+1)i}$. Solving the eq. (2.3) for $g(x)$ in this frame we find

$$g(x) = e^{-\rho\, M_{0,d+1}} e^{-x_a\, (M_{0,a} + M_{a,d+1})} g_0\,, \tag{4.3}$$

where the algebra generators are taken in the representation R of $so(1, d+1)$. Using the standard identification of $so(1, d+1)$ generators as the conformal generators of $\mathbb{R}^d$ (1.5) the coset element $g(x)$ reads:

$$g(x) = e^{\rho\, D} e^{-x^a\, P_a} g_0. \tag{4.4}$$

This gives us the Wilson lines

$$W_y^x(R,C) = P\exp\left[\int_y^x A\right] = g(x)\,g^{-1}(y) \tag{4.5}$$

Without loss of generality we can choose g_0 to be identity I.

<u>**The Scalar Caps**</u>

To project the external legs of the OWN operator we seek states, in the representation space R carried by that external leg, that transform in a (finite dimensional) irrep of the subalgebra $so(d+1)$ with generators $\{M_{ab}, M_{a,d+1}\}$ [16]. In particular, for the scalar cap this finite dimensional representation is the trivial one, that is, annihilated by $\{M_{ab}, M_{a,d+1}\}$. Let us now construct these states.

In terms of the generators in (1.5) the $so(1,d+1)$ algebra reads (1.4). We work with irreps R of $so(1,d+1)$ that become UIR of $so(2,d)$ obtained by Wick rotation. This implies the following reality conditions

$$M^\dagger_{0,d+1} = M_{0,d+1}, \quad M^\dagger_{0a} = -M_{0,a}, \quad M^\dagger_{a,d+1} = M_{a,d+1}, \quad M^\dagger_{ab} = -M_{ab}\,. \tag{4.6}$$

In terms of the generators in (1.5) these mean:

$$D^\dagger = D, \quad P^\dagger_a = K_a, \quad M^\dagger_{ab} = -M_{ab}. \tag{4.7}$$

Then the scalar cap state $|\Delta\rangle\rangle$ is defined to be a state in the scalar module $(\Delta, l_i = 0)$ that satisfies the conditions:

$$M_{ab}|\Delta\rangle\rangle = (P_a + K_b)|\Delta\rangle\rangle = 0. \tag{4.8}$$

We can construct it as a linear combination of states in the module over the scalar primary (lowest weight) state $|\varDelta\rangle$ which satisfies

$$D|\varDelta\rangle = \varDelta\,|\varDelta\rangle, \quad M_{ab}\,|\varDelta\rangle = K_a\,|\varDelta\rangle = 0. \tag{4.9}$$

Rest of the basis states of the module take form $|\varDelta, k_i\rangle = \mathcal{N}_{\vec{k}}\, P_1^{k_1}\cdots P_d^{k_d}|\varDelta\rangle$. The solution to the scalar cap state equation (4.9) was provided first in [16]. We re-derive it here for completeness. For this note that the cap state has to be a singlet under $so(d)$ and therefore can only depend on P_aP^a. So we write

$$|\varDelta\rangle\rangle = \sum_{n=0}^{\infty} C_n(\varDelta, d)\,(P_aP^a)^n|\varDelta\rangle\,, \tag{4.10}$$

and impose $(P_a + K_a)|\varDelta\rangle\rangle = 0$ to determine the coefficients C_n. Carrying out this straightforward exercise gives

$$C_n(\varDelta, d) = \frac{(-1)^n}{2^{2n} n!\; (\varDelta - \mu)_n} \tag{4.11}$$

With these (4.10) can be seen to be equivalent to the one in [16]. We will need the dual (conjugate under (4.7)) of this cap state which is given by:

$$\langle\langle\varDelta| = \sum_{n=0}^{\infty} C_n(\varDelta, d)\,\langle\varDelta|\,(K_aK^a)^n \tag{4.12}$$

with the same C_n as in (4.11).[1]

In fact one can obtain more general cap states. For instance, in the previous chapter case of $d = 2$, we provided expressions for cap states in the module over the primary state $|h, \bar{h}\rangle$ that transform under (j, m) representation of $so(3)$ algebra. In other dimensions one should seek caps that transform under arbitrary finite dimensional irreps of $so(d+1)$ – to be used in computing the OWNs with primaries that are not just scalars for the vector cap state – provided for illustration). We however will not pursue this further here (work in

[1]This scalar cap in the $d = 2$ case can be seen to be equivalent to that with $h = \bar{h}$ cap used in chapter 2 (see also [18] and more recently [105] for a different perspective).

progress).

CGCs

The last ingredient in the computation of the OWN expectation values is the Clebsch-Gordan coefficients (CGC) of the gauge algebra $so(1, d+1)$. Some of these are known – see for instance [104]. Those are however not in a form that lends itself readily to our purposes. As elucidated in chapter 2 we can extract the relevant CGC from scalar three-point functions. An explicit expression has beed derived in appendix B. However one can bypass this exercise making use of OPE modules which we will introduce later in this chapter.

Processing the Ingredients

To proceed further we need the explicit expression for the **in-going** legs $\langle\langle\Delta|g(x)|\Delta, \mathbf{m}\rangle$ and the **out-going** legs $\langle\Delta, \mathbf{m}|g^{-1}(x)|\Delta\rangle\rangle$ which are matrix elements of $g(x)$ and $g^{-1}(x)$ between the cap states $|\Delta\rangle\rangle$ and normalised basis elements $|\Delta, \mathbf{m}\rangle$ of the scalar module. So we turn to finding a suitable orthonormal basis for the module over a scalar primary $|\Delta\rangle$ next.

Scalar Module for $d \geq 2$

The descendent states take the form $|\Delta, \{k_1, k_2, \dots, k_d\}\rangle \sim \prod_{i=1}^{d} P_i^{k_i}|\Delta\rangle$. These states are eigenstates of the dilatation operator D with eigenvalue $\Delta + \sum_{i=1}^{d} k_i$. States with different eigenvalues of D are orthogonal. The set of states with a given conformal weight form a reducible representation of the rotation algebra $so(d)$ – which can be decomposed into a sum of irreps of $so(d)$. Then states belonging to different irreps will also be orthogonal. Therefore, a more suitable basis to work with would be in terms of the hyperspherical harmonics of the boundary $so(d)$ rotation algebra, $(P^2)^s M^l_{\mathbf{m}}(\mathbf{P})|\Delta\rangle$ where $\mathbf{m}$ denotes

$(m_{d-2}, \cdots, m_2, m_1)$, whose conformal dimension is $\Delta + l + 2s$. In the rest of the thesis we follow the conventions of [23, 106] for hyperspherical functions. It turns out that this choice is responsible for giving the CPW as a sum over contributions of given spin l, namely the Gegenbauer polynomial basis.

We define orthonormal states in this basis as follows

$$(P^2)^s M^l_{\mathbf{m}}(\mathbf{P})|\Delta\rangle \;:=\; A_{l,s}\,|\Delta;\{l,\mathbf{m},s\}\rangle \tag{4.13}$$

$$\langle\Delta|(K^2)^s\, M^l_{\mathbf{m}}{}^{\star}(\mathbf{K}) \;:=\; A^*_{l,s}\,\langle\Delta;\{l,\mathbf{m},s\}| \tag{4.14}$$

with

$$\langle\Delta;\{l',\mathbf{m}',s'\}|\Delta;\{l,\mathbf{m},s\}\rangle = \delta_{ll'}\delta_{\mathbf{mm}'}\delta_{ss'} \tag{4.15}$$

Note that the $so(d)$ symmetry dictates that the normalisations $A_{l,s}$ of these states do not depend on $\mathbf{m}$. To find the normalisation $A_{l,s}$ let us start with the state $O_\Delta(x)|0\rangle$ which can be rewritten as

$$O_\Delta(x)|0\rangle = e^{\mathbf{x}\cdot\mathbf{P}}\,O_\Delta(0)\,e^{-\mathbf{x}\cdot\mathbf{P}}|0\rangle = e^{\mathbf{x}\cdot\mathbf{P}}\,|\Delta\rangle \tag{4.16}$$

where we have used the fact that the vacuum is conformally invariant. The Hermitian conjugation (BPZ dual) is defined as

$$O^\dagger_\Delta(\mathbf{y}) = (y^2)^{-\Delta}\,O^\dagger_\Delta(\mathbf{y}/y^2) \tag{4.17}$$

On the other hand using the Hermiticity of the conformal generators (4.7) we find

$$\left(e^{\mathbf{y}\cdot\mathbf{P}}\,|\Delta\rangle\right)^\dagger = \langle\Delta|\,e^{\mathbf{y}\cdot\mathbf{K}} \tag{4.18}$$

Finally we compute the following inner product

$$\begin{aligned}\langle\Delta|e^{\mathbf{y}\cdot\mathbf{K}}e^{\mathbf{x}\cdot\mathbf{P}}|\Delta\rangle &= (y^2)^{-\Delta}\,\langle 0|\,O^\dagger_\Delta(\mathbf{y}/y^2)\,O_\Delta(\mathbf{x})\,|0\rangle \\ &= \frac{1}{(1-2\,\mathbf{x}\cdot\mathbf{y}+x^2\,y^2)^\Delta}\end{aligned} \tag{4.19}$$

On the left hand side of the above identity we expand the plane waves $e^{\mathbf{x}\cdot\mathbf{P}}$ in terms of spherical waves:

$$e^{\mathbf{x}\cdot\mathbf{P}} = \sum_{l=0}^{\infty}(2l+d-2)(d-4)!!\;j^d_l(x\,P)\,C_l^{\frac{d-2}{2}}\left(\frac{\mathbf{x}\cdot\mathbf{P}}{x\,P}\right) \tag{4.20}$$

where $j^d_l(x)$ is the spherical Bessel function and $C^\mu_l(z)$ is the Gegenbauer polynomials as defined below

$$j^d_l(x) = \sum_{s=0}^{\infty}\frac{(-1)^s(x)^{l+2s}}{(2s)!!(d+2l+2s-2)!!}\,. \tag{4.21}$$

and

$$C_l^{\frac{d-2}{2}}(x) = \frac{1}{(d-4)!!}\sum_{k=0}^{[l/2]}(-1)^k\frac{(2l-2k+d-4)!!}{(2k)!!(l-2k)!!}x^{l-2k} \tag{4.22}$$

Even though this formal expansion looks odd as it apparently depends not only on P whose square is $\mathbf{P}\cdot\mathbf{P}$, but also in the denominator of the argument of the Gegenbauer polynomial – we will shortly see that this is not really a problem once interpreted correctly. One can also write Gegenbauer polynomials in terms of hyperspherical harmonics using the well known identity

$$\sum_{\mathbf{m}} Y^*_{l;\mathbf{m}}(\Omega_x)Y_{l;\mathbf{m}}(\Omega_y) = \frac{\Gamma[\frac{d-2}{2}](2l+d-2)}{4\pi^{d/2}}C_l^{\frac{d-2}{2}}\left(\frac{\vec{x}\cdot\vec{y}}{x\,y}\right)\,. \tag{4.23}$$

Substituting these into the (4.20) we get:

$$e^{\mathbf{x}\cdot\mathbf{P}} = 4\,a\,\pi^{\frac{d}{2}}\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x^2)^s}{2^{l+2s}s!\Gamma[l+s+\frac{d}{2}]}\sum_{\mathbf{m}} M^{l*}_{\mathbf{m}}(\mathbf{x})M^l_{\mathbf{m}}(\mathbf{P})(P^2)^s \tag{4.24}$$

where $M^l_{\mathbf{m}}(\mathbf{x}) = x^l\, Y_{l;\mathbf{m}}(\Omega_x)$ and

$$a = \begin{cases} \frac{1}{2^{(d-2)/2}\Gamma[\frac{d-2}{2}]}, & \text{if } d \text{ is even} \\ \frac{\sqrt{\pi}}{2^{(d-1)/2}\Gamma[\frac{d-2}{2}]}, & \text{if } d \text{ is odd} \end{cases} \tag{4.25}$$

Similarly

$$e^{\mathbf{y}\cdot\mathbf{K}} = 4\, a\, \pi^{\frac{d}{2}} \sum_{l=0}^{\infty}\sum_{s=0}^{\infty} \frac{(y^2)^s}{2^{l+2s} s! \Gamma[l+s+\frac{d}{2}]} \sum_{\mathbf{m}} M^{l*}_{\mathbf{m}}(\mathbf{y}) M^l_{\mathbf{m}}(\mathbf{K})(K^2)^s \tag{4.26}$$

Therefore the left hand side of (4.19) takes the following form

$$\langle\Delta|e^{\mathbf{y}\cdot\mathbf{K}}e^{\mathbf{x}\cdot\mathbf{P}}|\Delta\rangle = \sum_{l=0}^{\infty}\sum_{s=0}^{\infty}(x^2)^s(y^2)^s \sum_{\mathbf{m}} M^l_{\mathbf{m}}(\mathbf{y})\, M^{l*}_{\mathbf{m}}(\mathbf{x})\, |A_{l,s}|^2 \left(\frac{4\, a\, \pi^{\frac{d}{2}}}{s!\; 2^{l+2s}\, \Gamma(l+s+d/2)}\right)^2 . \tag{4.27}$$

Next we want to expand the right hand side of (4.19) in the same basis. For this we first write

$$\frac{1}{(1-2\,\mathbf{x}\cdot\mathbf{y}+x^2\,y^2)^{\Delta}} = \frac{1}{(1-2\,\xi\,t+t^2)^{\Delta}} \tag{4.28}$$

with $t = x\,y$ and $\xi = t^{-1}\mathbf{x}\cdot\mathbf{y}$. We would now like to expand this quantity in terms of Gegenbauer polynomials $C^{\mu}_n(x)$. Luckily this exercise was done in [107] which reads[2]

$$\frac{1}{(1-2\,\xi\,t+t^2)^{\Delta}} = \frac{\Gamma(\mu)}{\Gamma(\Delta)} \sum_{k=0}^{\infty} C^{\mu}_k(\xi)\, t^k\, \frac{\Gamma(\Delta+k)}{\Gamma(\mu+k)}\; {}_2F_1(\Delta+k, \Delta-\mu; \mu+k+1; t^2) \tag{4.29}$$

However, we are interested in expanding the left hand side in d-dimensional hyperspherical harmonics in $\mathbf{x}$ which requires us to choose $\mu = (d-2)/2$. Using the series represen-

[2]This is a remarkable generalisation of how the Gegenbauer Polynomials $C^{\mu}_k(x)$ are defined through its generating function when $\Delta = \mu$.

tation of the hypergeometric function:

$$ {}_2F_1(\varDelta+k,\varDelta-\mu;\mu+k+1;t^2)=\frac{\Gamma(\mu+k+1)}{\Gamma(\varDelta+k)\Gamma(\varDelta-\mu)}\sum_{n=0}^{\infty}\frac{\Gamma(\varDelta+k+n)\Gamma(\varDelta-\mu+n)}{\Gamma(\mu+k+n+1)}\frac{t^{2n}}{n!} \tag{4.30} $$

and using the identity (4.23) we finally arrive at

$$ \frac{1}{(1-2\,\mathbf{x}\cdot\mathbf{y}+x^2\,y^2)^{\varDelta}}=\frac{4\,\pi^{\frac{d}{2}}}{\Gamma(\varDelta)}\sum_{l,s=0}^{\infty}\frac{\Gamma(\varDelta+l+s)\Gamma(\varDelta+s-\frac{d-2}{2})}{\Gamma(l+s+d/2)\,s!}\,(x^2)^{l+2s}(y^2)^{l+2s}\sum_{\mathbf{m}}M^l_{\mathbf{m}}(\mathbf{y})\,M^{l*}_{\mathbf{m}}(\mathbf{x}) \tag{4.31} $$

Comparing (4.27) with (4.31), we get

$$ |A_{l,s}|^2=\frac{2^{2l+4s}\Gamma\left[l+s+d/2\right]\Gamma[\varDelta+l+s]\Gamma\left[\varDelta+s-\frac{(d-2)}{2}\right]s!}{4\,a^2\,\pi^{\frac{d}{2}}\Gamma[d/2]\Gamma[\varDelta]\Gamma\left[\varDelta-\frac{(d-2)}{2}\right]} \tag{4.32} $$

Having found an orthonormal basis for the scalar module we would like to now compute the legs (conformal wave functions) as described in the beginning of this section.

In-going legs

For this we start with $g(x)=e^{\rho D}e^{-\mathbf{x}\cdot\mathbf{P}}$. Then

$$ \langle\!\langle \varDelta|g(x)|\varDelta;\{l,\mathbf{m},s\}\rangle $$

$$ =\sum_{n=0}^{\infty}(-1)^n C_n\langle\varDelta|\,(K^2)^n\,e^{\rho D}e^{-\mathbf{x}\cdot\mathbf{P}}|\varDelta;\{l,\mathbf{m},s\}\rangle $$

$$ =\sum_{n=0}^{\infty}\frac{(-1)^n C_n}{A_{l,s}}A^*_{0,n}\,\langle\varDelta,\{0,0,n\}|\,e^{\rho D}e^{-\mathbf{x}\cdot\mathbf{P}}\,(P^2)^s\,M^l_{\mathbf{m}}(\mathbf{P})|\varDelta\rangle $$

$$ =\frac{4a\pi^{\frac{d}{2}}}{A_{l,s}}\sum_{n=0}^{\infty}(-1)^n C_n A^*_{0,n}\sum_{l'=0}^{\infty}\sum_{s'=0}^{\infty}\frac{(x^2)^{s'}}{s'!\,2^{l'+2s'}\,\Gamma(l'+s'+d/2)}\sum_{\mathbf{m}'}M^{l'*}_{\mathbf{m}'}(-\mathbf{x}) $$

$$\times \langle \varDelta, \{0,0,n\}|\, e^{\rho D} (P^2)^{s+s'} M^{l'}_{\mathbf{m}'}(\mathbf{P}) M^{l}_{m}(\mathbf{P}) |\varDelta\rangle$$

Now using the identity for the spherical harmonics

$$M^{l}_{\mathbf{m}}(\mathbf{P}) M^{l'}_{\mathbf{m}'}(\mathbf{P}) = \sum_{L} \sum_{\mathbf{n}} \begin{bmatrix} l & l' & L \\ \mathbf{m} & \mathbf{m}' & \mathbf{n} \end{bmatrix} (P^2)^{\frac{l+l'-L}{2}} M^{L}_{\mathbf{n}}(\mathbf{P}) \tag{4.33}$$

where $\left[\begin{smallmatrix} l & l' & L \\ \mathbf{m} & \mathbf{m}' & \mathbf{n} \end{smallmatrix}\right]$ is $so(d)$ CGCs, we find

$$\begin{aligned}
&\langle\!\langle \varDelta | g(x) | \varDelta; \{l, \mathbf{m}, s\} \rangle \\
&= \frac{4a\pi^{\frac{d}{2}}}{A_{l,s}} \sum_{n=0}^{\infty} (-1)^n C_n A^*_{0,n} \sum_{l'=0}^{\infty} \sum_{s'=0}^{\infty} \frac{(x^2)^{s'}}{s'!\, 2^{l'+2s'}\, \Gamma(l'+s'+d/2)} \sum_{\mathbf{m}'} M^{l'*}_{\mathbf{m}'}(-\mathbf{x})\; e^{\rho(\varDelta+l+l'+2(s+s'))} \\
&\qquad \times \sum_{L} \sum_{\mathbf{n}} \begin{bmatrix} l & l' & L \\ \mathbf{m} & \mathbf{m}' & \mathbf{n} \end{bmatrix} \langle \varDelta; \{0,0,n\} | (P^2)^{s+s'+(l+l'-L)/2} M^{L}_{\mathbf{n}}(\mathbf{P}) | \varDelta \rangle \\
&= \frac{4a\pi^{\frac{d}{2}}}{A_{l,s}} \sum_{n=0}^{\infty} (-1)^n C_n A^*_{0,n} \sum_{l'=0}^{\infty} \sum_{s'=0}^{\infty} \frac{(x^2)^{s'}}{s'!\, 2^{l'+2s'}\, \Gamma(l'+s'+d/2)} \sum_{\mathbf{m}'} M^{l'*}_{\mathbf{m}'}(-\mathbf{x})\; e^{\rho(\varDelta+l+l'+2(s+s'))} \\
&\qquad \times \sum_{L} \sum_{\mathbf{n}} \begin{bmatrix} l & l' & L \\ \mathbf{m} & \mathbf{m}' & \mathbf{n} \end{bmatrix} A_{L, s+s'+\frac{l+l'-L}{2}}\, \delta_{L0}\, \delta_{\mathbf{n}0}\, \delta_{n\left(s+s'+\frac{l+l'-L}{2}\right)}
\end{aligned}$$

Carrying out the summation over L and $\mathbf{n}$ we find

$$\begin{aligned}
&\langle\!\langle \varDelta | g(x) | \varDelta; \{l, \mathbf{m}, s\} \rangle \\
&= \frac{4a\pi^{\frac{d}{2}}}{A_{l,s}} \sum_{n=0}^{\infty} (-1)^n C_n A^*_{0,n} \sum_{l'=0}^{\infty} \sum_{s'=0}^{\infty} \frac{(x^2)^{s'}}{s'!\, 2^{l'+2s'}\, \Gamma(l'+s'+d/2)} M^{l}_{\mathbf{m}}(-\mathbf{x})\; e^{\rho(\varDelta+l+l'+2(s+s'))} \\
&\qquad \times \delta_{ll'}\, A_{0, s+s'+\frac{l+l'}{2}}\, \delta_{n\left(s+s'+\frac{l+l'}{2}\right)}
\end{aligned} \tag{4.34}$$

where we have used

$$\sum_{\mathbf{m}'} \begin{bmatrix} l & l' & 0 \\ \mathbf{m} & \mathbf{m}' & 0 \end{bmatrix} M^{l'*}_{\mathbf{m}'}(\mathbf{x}) = \delta_{ll'}\, M^{l}_{\mathbf{m}}(\mathbf{x})\,. \tag{4.35}$$

Therefore

$$
\begin{aligned}
&\langle\langle \Delta | g(x) | \Delta; \{l, \mathbf{m}, s\}\rangle \\
&= \frac{4a\pi^{\frac{d}{2}}}{A_{l,s}} \sum_{n=0}^{\infty} (-1)^n C_n A_{0,n}^* \sum_{s'=0}^{\infty} \frac{(x^2)^{s'}}{s'!\, 2^{l+2s'}\, \Gamma(l+s'+d/2)} M^l_{\mathbf{m}}(-\mathbf{x})\; e^{\rho(\Delta+2(l+s+s'))} A_{0,s+s'+l}\, \delta_{n(s+s'+l)} \\
&= e^{\rho\Delta} \frac{4a\pi^{\frac{d}{2}}}{A_{l,s}} (x^2)^{-l-s} M^l_{\mathbf{m}}(-\mathbf{x}) \sum_{n=0}^{\infty} (-1)^n C_n |A_{0,n}|^2 \frac{\left(e^{2\rho} x^2\right)^n}{(n-s-l)!\, 2^{2n-2s-l}\, \Gamma(n-s+d/2)} \\
&= e^{-\rho\Delta} \frac{4a\pi^{d/2}\, 2^{l+2s}}{A_{l,s}} \times M^l_{\mathbf{m}}(-\mathbf{x}) \times (-1)^{s+l} \times (e^{2\rho})^{\Delta+l+s} (l+d/2)_s\, (\Delta)_{l+s} \\
&\qquad\qquad \times {}_2F_1\left(\Delta+l+s,\, l+s+d/2;\, l+d/2;\, -e^{2\rho} x^2\right)
\end{aligned}
\tag{4.36}
$$

Now we want to take $\rho \to \infty$ limit. We rewrite the hypergeometric function in the above expression using the identity

$$
{}_2F_1(a,\, b;\, c;\, z) = (1-z)^{-a}\, {}_2F_1\left(a,\, c-b;\, c;\, \frac{z}{z-1}\right) \tag{4.37}
$$

as

$$
\begin{aligned}
&{}_2F_1(\Delta+l+s,\, l+s+d/2;\, l+d/2;\, -e^{2\rho}x^2) \\
&\qquad = (1+e^{2\rho}x^2)^{-\Delta-l-s}\, {}_2F_1\left(\Delta+l+s,\, -s;\, l+\frac{d}{2};\, \frac{e^{2\rho}x^2}{1+e^{2\rho}x^2}\right)
\end{aligned}
\tag{4.38}
$$

In the $\rho \to \infty$ limit the argument of the hypergeometric function tends to unity. As the following identity holds

$$
{}_2F_1(-n,\, b;\, c;\, 1) = \frac{(c-b)_n}{(c)_n} = \frac{\Gamma(c-b+n)\Gamma(c)}{\Gamma(c-b)\Gamma(c+n)}\,, \tag{4.39}
$$

to the leading order in $e^{-\rho}$ the in-going leg becomes

$$
\langle\langle \Delta | g(x) | \Delta; \{l, \mathbf{m}, s\}\rangle \to e^{-\rho\Delta} \frac{4a\pi^{d/2}\, 2^{l+2s}}{A_{l,s}} (-1)^{s+l} M^l_{\mathbf{m}}(-\mathbf{x})\, (x^2)^{-\Delta-l-s} (\Delta)_{l+s}\, (d/2-\Delta-s)_s + \cdots
$$

where dots are subleading terms in $\rho \to \infty$ limit. Finally we use $(-x)_n = (-1)^n (x-n+1)_n$

and $(-1)^l M^l_{\mathbf{m}}(\mathbf{x}) = M^l_{\mathbf{m}}(-\mathbf{x})$ to get

$$\lim_{\rho\to\infty} e^{\rho\Delta}\langle\!\langle\Delta|g(x)|\Delta;\{l,\mathbf{m},s\}\rangle = 4\,a\,\pi^{d/2}\,\frac{2^{l+2s}}{A_{l,s}}\,(\Delta)_{l+s}\left(\Delta-\frac{d-2}{2}\right)_s(x^2)^{-\Delta-l-s}\,M^l_{\mathbf{m}}(\mathbf{x}) \tag{4.40}$$

Out-going legs

For this we start with $g^{-1}(y) = e^{\mathbf{y}\cdot\mathbf{P}}e^{-\rho D}$, and compute

$$\begin{aligned}
&\langle\Delta;\{l,\mathbf{m},s\}|g^{-1}(y)|\Delta\rangle\!\rangle\\
&=\sum_{n=0}^{\infty}(-1)^n C_n\,e^{-\rho(\Delta+2n)}\langle\Delta;\{l,\mathbf{m},s\}|e^{\mathbf{y}\cdot\mathbf{P}}(P^2)^n|\Delta\rangle\\
&=4a\pi^{d/2}\sum_{n=0}^{\infty}(-1)^n C_n\,e^{-\rho(\Delta+2n)}\sum_{l'=0}^{\infty}\sum_{s'=0}^{\infty}\frac{(y^2)^{s'}}{s'!\,2^{l'+2s'}\,\Gamma(l'+s'+d/2)}\sum_{\mathbf{m}'}M^{l'*}_{\mathbf{m}'}(\mathbf{y})\\
&\qquad\qquad\times A_{l',s'+n}\,\delta_{ll'}\,\delta_{\mathbf{m}\mathbf{m}'}\,\delta_{s(s'+n)}\\
&=4a\pi^{d/2}\sum_{n=0}^{\infty}(-1)^n C_n\,A_{l,s}\,e^{-\rho(\Delta+2n)}\,\frac{(y^2)^{s-n}}{(s-n)!\,2^{l+2(s-n)}\,\Gamma(l+s-n+d/2)}M^{l*}_{\mathbf{m}}(\mathbf{y})\\
&=e^{-\rho\Delta}\frac{4a\pi^{d/2}}{2^{l+2s}}\,A_{l,s}\sum_{n=0}^{\infty}(-1)^n C_n\,e^{-2n\rho}\frac{2^{2n}\,(y^2)^{s-n}}{(s-n)!\,\Gamma(l+s-n+d/2)}\,M^{l*}_{\mathbf{m}}(\mathbf{y})
\end{aligned}\tag{4.41}$$

As $\rho\to\infty$, to the leading order only the $n=0$ term contributes, so that we have the result

$$\lim_{\rho\to\infty} e^{\rho\Delta}\,\langle\Delta;\{l,\mathbf{m},s\}|g^{-1}(y)|\Delta\rangle\!\rangle = \frac{4a\pi^{d/2}}{2^{l+2s}}\,A_{l,s}\,\frac{(y^2)^s}{(s)!\,\Gamma(l+s+d/2)}\,M^{l*}_{\mathbf{m}}(\mathbf{y}) \tag{4.42}$$

The results of these rather lengthy exercises are (4.40, 4.42). These two sets of functions (4.40) and (4.42) provide a representation and its conjugate representation respectively of the conformal algebra $so(1,d+1)$, on which the conformal generators $\{D, M_{\alpha\beta}, P_\alpha, K_\alpha\}$ act through their differential operator representations on scalar primaries with dimension Δ. One can use these to derive matrix representations of the conformal generators and therefore, can be more appropriately called the *conformal wave functions*.

Finally let us quickly carry out a check on our conformal wave functions, namely, that when they are used in our OWN prescription they have to reproduce the appropriate two-point function for the scalar primaries. According to our prescription the two-point function can be obtained as

$$\begin{aligned}\langle O_\Delta(\mathbf{x})O_\Delta(\mathbf{y})\rangle &= \lim_{\rho\to\infty} e^{2\Delta\rho}\langle\!\langle\Delta|g(x)g^{-1}(y)|\Delta\rangle\!\rangle \\ &= \lim_{\rho\to\infty} e^{2\Delta\rho}\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\mathbf{m}}\langle\!\langle\Delta|g(x)|\Delta;\{l,\mathbf{m},s\}\rangle\langle\Delta;\{l,\mathbf{m},s\}|g^{-1}(y)|\Delta\rangle\!\rangle\end{aligned} \tag{4.43}$$

$\sum_{l,\mathbf{m},s}$

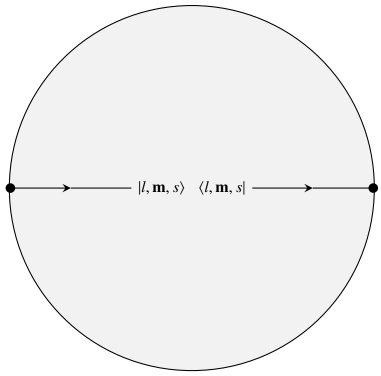


Figure 4.1: 2-point function

As $\rho\to\infty$ the above diagram evaluates to

$$\begin{aligned}&\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\sum_{m=-l}^{l}\langle\!\langle\Delta|g(x)|\Delta;\{l,m,s\}\rangle\langle\Delta;\{l,m,s\}|g^{-1}(y)|\Delta\rangle\!\rangle \\ &= e^{-2\Delta\rho}(x^2)^{-\Delta}(4a\pi^{d/2})^2\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\Delta)_{l+s}\left(\Delta-\frac{d-2}{2}\right)_s}{\Gamma(l+s+d/2)\,s!}\left(\frac{y}{x}\right)^{2s}\times(x^2)^{-l}\sum_{\mathbf{m}}M^l_{\mathbf{m}}(\mathbf{x})M^{l*}_{\mathbf{m}}(\mathbf{y})\end{aligned} \tag{4.44}$$

Finally using (4.23) and comparing with (4.29) we obtain

$$\begin{aligned}\langle O_\Delta(\mathbf{x})O_\Delta(\mathbf{y})\rangle &= 4a^2\pi^{d/2}(x^2)^{-\Delta}\left(1-2\,\frac{\mathbf{x}\cdot\mathbf{y}}{x^2}+\frac{y^2}{x^2}\right)^{-\Delta} \\ &= 4a^2\pi^{d/2}|\mathbf{x}-\mathbf{y}|^{-2\Delta}\end{aligned} \tag{4.45}$$

This is the expected result for two-point function (up to an overall constant factor - which can be gotten rid of by multiplying the cap states by appropriate overall factors).

Introducing OPE module

Finally we need to amputate the legs we have found in the previous section from the correlation function of three scalar primaries to find the CGC we need. The explicit expressions adapted to our method are given in the appendix A. However, to compute, for example, the 4-point conformal partial waves we need CGC*s* that are already connected to two legs at a time – which is obtained easily by starting with an appropriate 3-point function and amputating only one leg. This object depends on the boundary coordinates where two of the primaries are inserted, and carries labels of basis vectors of the module of the third primary. This is a close cousin of the so called OPE block [19, 20] which we call the *OPE module*.

These OPE modules can be characterised by two types of identities. To spell them out let us label the representations of the conformal algebra $so(1, d+1)$ of interest by $(\varDelta, \mathbf{l})$ where $\varDelta$ is the conformal dimension and $\mathbf{l}$ represents all the independent Casimirs of the representation. States in such a representation R can be labelled by $(\varDelta, \mathbf{l}; \mathbf{m}, s)$ where $\mathbf{m}$ is again a collective index of magnetic quantum numbers. It turns out there are two types of these OPE modules which we denote by $\mathcal{B}^{(\varDelta_1,\mathbf{l}_1;\mathbf{x}_1),(\varDelta_2,\mathbf{l}_2;\mathbf{x}_2)}_{(\varDelta_3,\mathbf{l}_3;\mathbf{m}_3,s_3)}$ and $\mathcal{B}^{(\varDelta_3,\mathbf{l}_3;\mathbf{m}_3,s_3)}_{(\varDelta_1,\mathbf{l}_1;\mathbf{x}_1),(\varDelta_2,\mathbf{l}_2;\mathbf{x}_2)}$. Then these OPE modules are supposed to satisfy the Ward identities:

$$
\begin{aligned}
(\mathcal{L}_{x_1}[M_{AB}] + \mathcal{L}_{x_2}[M_{AB}])\,\mathcal{B}^{(\varDelta_1,\mathbf{l}_1;\mathbf{x}_1),(\varDelta_2,\mathbf{l}_2;\mathbf{x}_2)}_{(\varDelta,\mathbf{l};\mathbf{m},s)} &= \mathcal{M}_{(\varDelta,\mathbf{l};\mathbf{m},s)}{}^{(\varDelta,\mathbf{l};\mathbf{m}',s')}[M_{AB}]\mathcal{B}^{(\varDelta_1,\mathbf{l}_1;\mathbf{x}_1),(\varDelta_2,\mathbf{l}_2;\mathbf{x}_2)}_{(\varDelta,\mathbf{l};\mathbf{m}',s')} \\
(\mathcal{L}_{x_1}[M_{AB}] + \mathcal{L}_{x_2}[M_{AB}])\,\mathcal{B}^{(\varDelta,\mathbf{l};\mathbf{m},s)}_{(\varDelta_1,\mathbf{l}_1;\mathbf{x}_1),(\varDelta_2,\mathbf{l}_2;\mathbf{x}_2)} &= -\mathcal{B}^{(\varDelta,\mathbf{l};\mathbf{m}',s')}_{(\varDelta_1,\mathbf{l}_1;\mathbf{x}_1),(\varDelta_2,\mathbf{l}_2;\mathbf{x}_2)}\,\mathcal{M}_{(\varDelta,\mathbf{l};\mathbf{m}',s')}{}^{(\varDelta,\mathbf{l};\mathbf{m},s)}[M_{AB}]
\end{aligned} \tag{4.46}
$$

where we denote the differential operator representation and the matrix representation of the conformal generator M_{AB} by $\mathcal{L}[M_{AB}]$ and $\mathcal{M}[M_{AB}]$ respectively. From these identities it is very easy to see that both types of OPE modules satisfy corresponding conformal Casimir equations. The solutions of these equations can be obtained by starting with the 3-point functions and amputating from them either an in-going leg or an out-going leg. The following picture demonstrates this aspect diagrammatically.

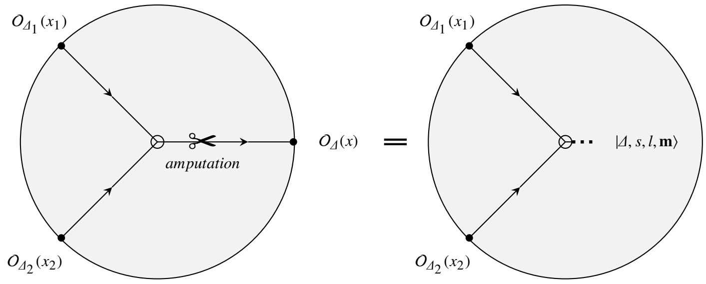


Figure 4.2: OPE module from 3-point function.

Finally the method to obtain the 4-point conformal partial wave using the OWN prescription reduces to taking two types of OPE modules defined above and contracting the module indices. Next we turn to using these to compute the 4-point scalar partial waves.

Computing the 4-point scalar CPW

Having equipped ourselves with all the ingredients needed, we now turn to compute four-point conformal blocks for scalar primaries of conformal weights Δ_i for $i = 1, 2, 3, 4$. For simplicity we take the operator insertion points to be at $\mathbf{x_1} \to \infty$, $\mathbf{x_2} \to \mathbf{u}$, $\mathbf{x_3} \to \mathbf{x}$ and $\mathbf{x_4} \to \mathbf{0}$ with $\mathbf{u} \cdot \mathbf{u} = 1$. As elucidated earlier this four-point conformal block can be computed using two specific OPE modules.

One of the OPE modules we need can be extracted from the three-point function, with the operator insertions at $(\infty, \mathbf{u}, \mathbf{y})$ by amputating the out-going leg anchored at the boundary-point $\mathbf{y}$. The corresponding OPE module is shown in the figure given below.

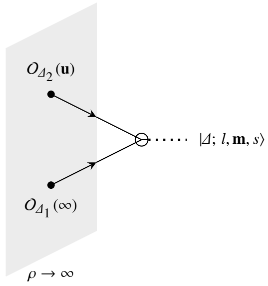


Figure 4.3: An OPE module

The three-point function takes the form

$$\langle O_{\Delta_1}(\infty) O_{\Delta_2}(\mathbf{u}) O_{\Delta}(\mathbf{y})\rangle = \lim_{z\to\infty}(z^2)^{\Delta_1}\langle O_{\Delta_1}(\mathbf{z}) O_{\Delta_2}(\mathbf{u}) O_{\Delta}(\mathbf{y})\rangle = \frac{1}{[(\mathbf{u}-\mathbf{y})^2]^{\frac{\Delta_2+\Delta-\Delta_1}{2}}} \quad (4.47)$$

which can be expanded in terms of hyperspherical harmonics using (4.29) as

$$\begin{aligned}&\langle O_{\Delta_1}(\infty) O_{\Delta_2}(\mathbf{u}) O_{\Delta}(\mathbf{y})\rangle \\ &= (4\pi^{d/2}) \sum_{l=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\frac{\Delta_2+\Delta-\Delta_1}{2}\right)_{l+s}\left(\frac{\Delta_2+\Delta-\Delta_1}{2}-\frac{d-2}{2}\right)_s}{s!\,\Gamma(l+s+d/2)} (y^2)^s \sum_{\mathbf{m}} M^l_{\mathbf{m}}(\mathbf{u})\, M^{l*}_{\mathbf{m}}(\mathbf{y})\end{aligned} \quad (4.48)$$

Amputation of the out-going leg ending at $\mathbf{y}$ from the above expression gives the following answer for the desired OPE module

$$\left[\frac{4\pi^{d/2}}{s!\,(d/2)_{l+s}\,(\Delta)_{l+s}\left(\Delta-\frac{d-2}{2}\right)_s}\right]^{\frac{1}{2}} \left(\frac{\Delta-\Delta_{12}}{2}\right)_{l+s}\left(\frac{\Delta-\Delta_{12}}{2}-\frac{d-2}{2}\right)_s M^l_{\mathbf{m}}(\mathbf{u}) \quad (4.49)$$

where $\Delta_{ij} \equiv \Delta_i - \Delta_j$. Similarly we can find the other OPE module from the three-point function with operator insertions at $(\mathbf{x}, \mathbf{0}, \mathbf{y})$ by amputating in-going leg starting from $\mathbf{y}$. We start with the three-point function

$$\langle O_{\Delta}(\mathbf{y}) O_{\Delta_3}(\mathbf{x}) O_{\Delta_4}(\mathbf{0})\rangle = (y^2)^{\frac{\Delta_3-\Delta_4-\Delta}{2}}(x^2)^{\frac{\Delta-\Delta_3-\Delta_4}{2}} \frac{1}{[(\mathbf{y}-\mathbf{x})^2]^{\frac{\Delta+\Delta_3-\Delta_4}{2}}} \quad (4.50)$$

Expanding in hyperspherical function we get

$$\langle O_{\Delta}(\mathbf{y})O_{\Delta_3}(\mathbf{x})O_{\Delta_4}(\mathbf{0})\rangle$$

$$= (4\pi^{d/2})(x^2)^{\frac{\Delta-\Delta_3-\Delta_4}{2}} \sum_{l,s=0}^{\infty} \frac{\left(\frac{\Delta+\Delta_3-\Delta_4}{2}\right)_{l+s} \left(\frac{\Delta+\Delta_3-\Delta_4}{2} - \frac{d-2}{2}\right)_s}{s!\,\Gamma(l+s+d/2)} (x^2)^s (y^2)^{\Delta-l-s} \sum_{\mathbf{m}} M^l_{\mathbf{m}}(\mathbf{y})\, M^{l*}_{\mathbf{m}}(\mathbf{x}) \tag{4.51}$$

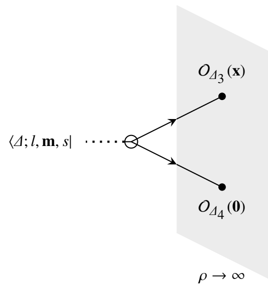


Figure 4.4: Another OPE module.

Now amputating the in-going leg starting from **y** from the above expression for the three-point function, we obtain the othe type of OPE module:

$$(x^2)^{\frac{(\Delta-\Delta_3-\Delta_4)}{2}} \left[\frac{1}{4\pi^{d/2}\, s!\,(d/2)_{l+s}\,(\Delta)_{l+s}\left(\Delta - \frac{d-2}{2}\right)_s}\right]^{\frac{1}{2}} \frac{1}{\Gamma(d/2)}$$
$$\times \left(\frac{\Delta+\Delta_{34}}{2}\right)_{l+s} \left(\frac{\Delta+\Delta_{34}}{2} - \frac{d-2}{2}\right)_s (x^2)^s M^{l*}_{\mathbf{m}}(\mathbf{x}) \tag{4.52}$$

Finally we glue the OPE modules (4.49) and (4.52) to compute the four-point conformal partial waves. Diagrammatically this procedure is shown in the figure 7 given below.

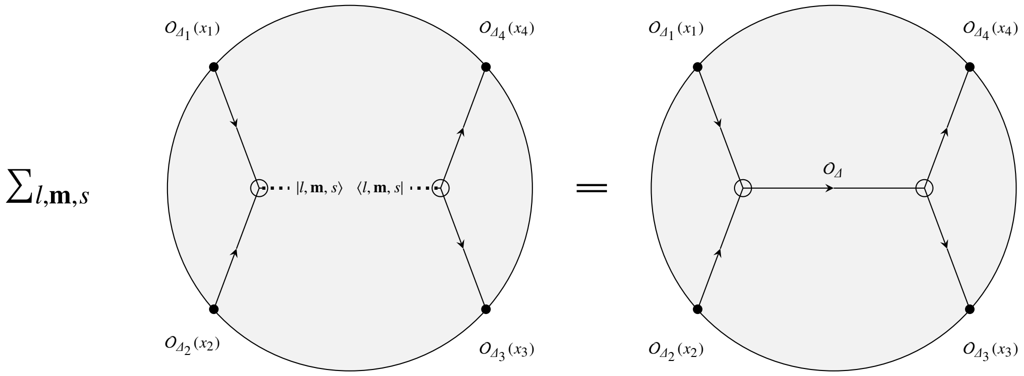


Figure 4.5: 4-point block from OPE modules.

Thus the corresponding four-point conformal partial wave becomes

$$\begin{aligned} W^{(d)}_{\Delta,0}(\Delta_i,\mathbf{x}) &= (x^2)^{\frac{(\Delta-\Delta_3-\Delta_4)}{2}} \frac{1}{\Gamma(d/2)} \sum_{l,s} \frac{\left(\frac{\Delta-\Delta_{12}}{2}\right)_{l+s}\left(\frac{\Delta+\Delta_{34}}{2}\right)_{l+s}}{s!\,(d/2)_{l+s}\,(\Delta)_{l+s}\left(\Delta-\frac{d-2}{2}\right)_s} \\ &\times \left(\frac{\Delta-\Delta_{12}}{2}-\frac{d-2}{2}\right)_s \left(\frac{\Delta+\Delta_{34}}{2}-\frac{d-2}{2}\right)_s (x^2)^s \sum_{\mathbf{m}} M^{l\,\star}_{\mathbf{m}}(\mathbf{x})\, M^{l}_{\mathbf{m}}(\mathbf{u}) \end{aligned} \tag{4.53}$$

Using (4.23) we can also express our result in terms of Gegenbauer polynomials

$$\begin{aligned} W^{(d)}_{\Delta,0}(\Delta_i,\mathbf{x}) &= (x^2)^{\frac{(\Delta-\Delta_3-\Delta_4)}{2}} \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{d/2}\,\Gamma(d/2)} \sum_{l,s} \frac{(2l+d-2)\left(\frac{\Delta-\Delta_{12}}{2}\right)_{l+s}\left(\frac{\Delta+\Delta_{34}}{2}\right)_{l+s}}{s!\,(d/2)_{l+s}\,(\Delta)_{l+s}\left(\Delta-\frac{d-2}{2}\right)_s} \\ &\times \left(\frac{\Delta-\Delta_{12}}{2}-\frac{d-2}{2}\right)_s \left(\frac{\Delta+\Delta_{34}}{2}-\frac{d-2}{2}\right)_s x^{l+2s}\, C_l^{\frac{d-2}{2}}\left(\frac{\mathbf{x}\cdot\mathbf{u}}{x}\right) \end{aligned} \tag{4.54}$$

This is our final result for the scalar conformal partial wave in any $d \geq 2$. As we will see in the next section this result also works for $d = 1$. Notice that as advertised earlier our answer is naturally given in terms of the Gegenbauer polynomial basis. A result for the same quantity already exists in the literature in terms of the cross ratios [6]. We now prove the following identity towards establishing the equivalence between our answers and theirs. We claim

$$\sum_{l,s=0}^{\infty} \frac{\left(\frac{\Delta-\Delta_{12}}{2}\right)_{l+s}\left(\frac{\Delta+\Delta_{34}}{2}\right)_{l+s}}{(\Delta)_{l+s}} \frac{\left(\frac{\Delta-\Delta_{12}}{2}-\mu\right)_s\left(\frac{\Delta+\Delta_{34}}{2}-\mu\right)_s}{(\Delta-\mu)_s} \frac{1+\frac{l}{\mu}}{s!(\mu+1)_{l+s}}(z\bar{z})^{s+\frac{l}{2}} C_l^{\mu}\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right) \tag{4.55}$$

is equal to

$$\sum_{r,q=0}^{\infty} \frac{\left(\frac{\Delta+\Delta_{12}}{2}\right)_r\left(\frac{\Delta-\Delta_{12}}{2}\right)_{r+q}\left(\frac{\Delta-\Delta_{34}}{2}\right)_r\left(\frac{\Delta+\Delta_{34}}{2}\right)_{r+q}}{r!q!\,(\Delta)_{2r+q}\,(\Delta-\mu)_r}(z\bar{z})^r\,(z+\bar{z}-z\bar{z})^q \tag{4.56}$$

To establish this we first note the following identities/definitions:

$$(z\bar{z})^{\frac{l}{2}} C_l^{\mu}\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right) := \sum_{k=0}^{[l/2]}(-1)^k \frac{(\mu)_{l-k}}{k!\,(l-2k)!}(z+\bar{z})^{l-2k}\,(z\bar{z})^k \tag{4.57}$$

$$(z+\bar{z}-z\bar{z})^q = \sum_{p=0}^{q}(-1)^p \binom{q}{p}\,(z+\bar{z})^{q-p}(z\bar{z})^p \tag{4.58}$$

Using the double sum identity:

$$\sum_{q=0}^{\infty}\sum_{p=0}^{q} a_{p,q-p} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{n,m} = \sum_{l=0}^{\infty}\sum_{k=0}^{[l/2]} a_{k,l-2k} \tag{4.59}$$

the first expression can be written as

$$\sum_{s=0}^{\infty} \frac{\left(\frac{\Delta-\Delta_{12}}{2}-\mu\right)_s\left(\frac{\Delta+\Delta_{34}}{2}-\mu\right)_s}{s!\,(\Delta-\mu)_s} \sum_{m,n=0}^{\infty} \frac{\left(\frac{\Delta-\Delta_{12}}{2}\right)_{s+m+2n}\left(\frac{\Delta+\Delta_{34}}{2}\right)_{s+m+2n}}{(\Delta)_{s+m+2n}n!\,m!} \frac{\left(1+\frac{m+2n}{\mu}\right)(\mu)_{m+n}}{(\mu+1)_{s+m+2n}} \times (-1)^n\,(z\bar{z})^{n+s}\,(z+\bar{z})^m \tag{4.60}$$

The second of the expressions can be manipulated to:

$$\sum_{r=0}^{\infty} \frac{\left(\frac{\Delta+\Delta_{12}}{2}\right)_r\left(\frac{\Delta-\Delta_{34}}{2}\right)_r}{r!\,(\Delta-\mu)_r} \sum_{m,p=0}^{\infty} \frac{\left(\frac{\Delta-\Delta_{12}}{2}\right)_{r+m+p}\left(\frac{\Delta+\Delta_{34}}{2}\right)_{r+m+p}}{m!\,p!\,(\Delta)_{2r+m+p}}(-1)^p(z\bar{z})^{r+p}\,(z+\bar{z})^m \tag{4.61}$$

In the next step we extract the coefficients of $(z\bar{z})^q(z+\bar{z})^m$ in both these expressions. For this in the first expression we change $n \to p,\ s \to q-p$ and in the second we change $p \to p,\ r \to q-p$. Then in both the expressions the indices q and m run freely over all non-negative integers and the index p runs over $0, 1, \cdots, q$. The corresponding coefficient for the first expression is:

$$\sum_{p=0}^{q} \frac{\left(\frac{\Delta-\Delta_{12}}{2}-\mu\right)_{q-p}\left(\frac{\Delta+\Delta_{34}}{2}-\mu\right)_{q-p}}{(q-p)!\,(\Delta-\mu)_{q-p}} \frac{\left(\frac{\Delta-\Delta_{12}}{2}\right)_{m+p+q}\left(\frac{\Delta+\Delta_{34}}{2}\right)_{m+p+q}}{(\Delta)_{m+p+q}\,p!\,m!} \frac{\mu+m+2p}{(\mu+m+p)_{q+1}}(-1)^p \tag{4.62}$$

and for the second expression is:

$$\sum_{p=0}^{q} \frac{\left(\frac{\Delta+\Delta_{12}}{2}\right)_{q-p}\left(\frac{\Delta-\Delta_{34}}{2}\right)_{q-p}}{(q-p)!\,(\Delta-\mu)_{q-p}} \frac{\left(\frac{\Delta-\Delta_{12}}{2}\right)_{m+q}\left(\frac{\Delta+\Delta_{34}}{2}\right)_{m+q}}{m!\,p!\,(\Delta)_{m+2q-p}}(-1)^p \tag{4.63}$$

Now the final step is to compare these two expressions (4.62) and (4.63) for arbitrary integers $\{d \geq 1, q \geq 0, m \geq 0\}$. We conjecture that these expressions are identical. We have verified this claim for various special cases exactly, and for large subsets of the integer parameters $\{d \geq 1, q \geq 0, m \geq 0\}$ using Mathematica.

In principle one can put together the conformal wave functions of section 2, and the CGC of appendix A many times over to generate the scalar CPW for any higher point correlation function.

Recovery of scalar CPW*s* in $d \leq 4$

In this section we want to recover the known results for four-point scalar conformal partial waves in $d \leq 4$ from our answer above (4.54). To take limits in d it is convenient to express our answer in different variables, namely $(z, \bar{z})$. If $\mathbf{x} \cdot \mathbf{u} = x \cos\theta$, then we define

$$z = x\,e^{i\theta}; \qquad \bar{z} = x\,e^{-i\theta} \tag{4.64}$$

In terms of these variables our answer for four-point CPW takes the form

$$W^{(d)}_{\Delta,0}(\Delta_i; z,\bar z) = (z\bar z)^{\frac{(\Delta-\Delta_3-\Delta_4)}{2}} \frac{1}{4\pi^{d/2}} \sum_{l,s=0}^{\infty} \frac{(2l+d-2)\left(\frac{\Delta-\Delta_{12}}{2}\right)_{l+s}\left(\frac{\Delta+\Delta_{34}}{2}\right)_{l+s}}{s!\,(d/2)_{l+s}\,(\Delta)_{l+s}\left(\Delta-\frac{d-2}{2}\right)_s}$$
$$\times\left(\frac{\Delta-\Delta_{12}}{2}-\frac{d-2}{2}\right)_s\left(\frac{\Delta+\Delta_{34}}{2}-\frac{d-2}{2}\right)_s (z\bar z)^{s+\frac{l}{2}}\frac{2}{(d-2)}C_l^{\frac{d-2}{2}}\left(\frac{z+\bar z}{2\sqrt{z\bar z}}\right) \tag{4.65}$$

$\underline{d = 4}$

In $d = 4$ the four-point CPW for scalar primaries takes the following form

$$\begin{aligned} W^{(4)}_{\Delta,0}(\Delta_i, z,\bar z) &= \frac{1}{z-\bar z}(z\bar z)^{\frac{1}{2}(\Delta-\Delta_3-\Delta_4)} \sum_{l,s=0}^{\infty} \Gamma(\frac{1}{2}(\Delta-\Delta_{34})+l+s)\Gamma(\frac{1}{2}(\Delta-\Delta_{12})+l+s) \\ &\quad\times\Gamma(\frac{1}{2}(\Delta-\Delta_{12})+s-1)\Gamma(\frac{1}{2}(\Delta-\Delta_{34})+s-1) \\ &\quad\times\frac{(l+1)\Gamma(\Delta)\,\Gamma(\Delta-1)}{s!\,(l+s+1)!\Gamma(\Delta+l+s)\Gamma(\Delta+s-1)}(z^{l+s+1}\bar z^{s}-z^{s}\bar z^{l+s+1}) \\ &= \frac{1}{z-\bar z}(z\bar z)^{\frac{1}{2}(\Delta-\Delta_3-\Delta_4)}\Gamma(\alpha)\,\Gamma(\alpha-1)\Gamma(\beta)\,\Gamma(\beta-1) \\ &\quad[z\,{}_2F_1(\alpha,\beta,\Delta,z)\,{}_2F_1(\alpha-1,\beta-1,\Delta-2,\bar z)-\bar z\,{}_2F_1(\alpha,\beta,\Delta,\bar z)\,{}_2F_1(\alpha-1,\beta-1,\Delta-2,z)] \end{aligned} \tag{4.66}$$

where $\alpha = \frac{1}{2}(\Delta-\Delta_{12})$ and $\beta = \frac{1}{2}(\Delta-\Delta_{34})$. We now demonstrate how to go from the first to the second expression. We start by expanding the answer in power series.

$$z\,{}_2F_1(\alpha,\beta,\Delta,z)\,{}_2F_1(\alpha-1,\beta-1,\Delta-2,\bar z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+m)\Gamma(\alpha+n-1)}{\Gamma(\alpha)\Gamma(\alpha-1)}$$
$$\frac{\Gamma(\beta+m)\Gamma(\beta+n-1)}{\Gamma(\beta)\Gamma(\beta-1)}\frac{\Gamma(\Delta)\Gamma(\Delta-2)}{\Gamma(\Delta+m)\Gamma(\Delta+n-2)}\frac{z^{m+1}\bar z^{n}}{m!n!} \tag{4.67}$$

We now divide the right hand side into three terms with $m+1 > n$, $m+1 < n$ and $m+1 = n$. The piece coming from terms with $m+1 = n$ are real and therefore cancel with the corresponding terms from the complex conjugate combination. The remaining parts are obtained by considering the restricted sums

$$\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}+\sum_{m=0}^{\infty}\sum_{n=m+2}^{\infty} \tag{4.68}$$

Let us consider the conjugate term next:

$$\begin{aligned}\bar{z}\,{}_2F_1(\alpha,\beta,\varDelta,\bar{z})\,{}_2F_1(\alpha-1,\beta-1,\varDelta-2,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varGamma(\alpha+m)\varGamma(\alpha+n-1)}{\varGamma(\alpha)\varGamma(\alpha-1)}\\ \frac{\varGamma(\beta+m)\varGamma(\beta+n-1)}{\varGamma(\beta)\varGamma(\beta-1)}\frac{\varGamma(\varDelta)\varGamma(\varDelta-2)}{\varGamma(\varDelta+m)\varGamma(\varDelta+n-2)}\frac{\bar{z}^{m+1}z^{n}}{m!n!}\end{aligned} \tag{4.69}$$

again we split this into three types of terms as above and drop the term that is real. Then we can split the rest into two types of terms by writing the sum as before in two parts:

$$\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}+\sum_{m=0}^{\infty}\sum_{n=m+2}^{\infty} \tag{4.70}$$

Noticing that the first sum in the first term and the second sum in the second have more z's than $\bar{z}$'s we would like to combine them. In these two we introduce two new variables $m = n + p$ and $n = m + 2 + q$ to replace m and n respectively. Combing these we have:

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{\varGamma(\alpha+n+p)\varGamma(\alpha+n-1)}{\varGamma(\alpha)\varGamma(\alpha-1)}\frac{\varGamma(\beta+n+p)\varGamma(\beta+n-1)}{\varGamma(\beta)\varGamma(\beta-1)}\frac{\varGamma(\varDelta)\varGamma(\varDelta-2)}{\varGamma(\varDelta+n+p)\varGamma(\varDelta+n-2)}\\ &\qquad\times\frac{z^{n+p+1}\bar{z}^{n}}{(n+p)!n!}\\ &-\sum_{m=0}^{\infty}\sum_{q=0}^{\infty}\frac{\varGamma(\alpha+m)\varGamma(\alpha+q+m+1)}{\varGamma(\alpha)\varGamma(\alpha-1)}\frac{\varGamma(\beta+m)\varGamma(\beta+q+m+1)}{\varGamma(\beta)\varGamma(\beta-1)}\frac{\varGamma(\varDelta)\varGamma(\varDelta-2)}{\varGamma(\varDelta+m)\varGamma(\varDelta+q+m)}\\ &\qquad\times\frac{\bar{z}^{m+1}z^{q+m+2}}{m!(q+m+2)!}\end{aligned} \tag{4.71}$$

In the second term we can replace $m \to m-1$ and still sum over the new m from 0 to ∞ as there will be term $(m-1)!$ in the denominator which kills the $m=0$ term. Then

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{\Gamma(\alpha+n+p)\Gamma(\alpha+n-1)}{\Gamma(\alpha)\Gamma(\alpha-1)}\frac{\Gamma(\beta+n+p)\Gamma(\beta+n-1)}{\Gamma(\beta)\Gamma(\beta-1)}\frac{\Gamma(\Delta)\Gamma(\Delta-2)}{\Gamma(\Delta+n+p)\Gamma(\Delta+n-2)} \times\frac{z^{n+p+1}\bar{z}^{n}}{(n+p)!n!}$$

$$-\sum_{m=0}^{\infty}\sum_{q=0}^{\infty}\frac{\Gamma(\alpha+m-1)\Gamma(\alpha+q+m)}{\Gamma(\alpha)\Gamma(\alpha-1)}\frac{\Gamma(\beta+m-1)\Gamma(\beta+q+m)}{\Gamma(\beta)\Gamma(\beta-1)} \times\frac{\Gamma(\Delta)\Gamma(\Delta-2)}{\Gamma(\Delta+m-1)\Gamma(\Delta+q+m-1)}\frac{\bar{z}^{m}z^{q+m+1}}{(m-1)!(q+m+1)!}$$

(4.72)

Now we change dummy variables $n \to s$, $p \to l$ in the first term and $m \to s$ and $q \to l$ in the second term and combine terms to write this as:

$$\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\frac{\Gamma(\alpha+l+s)\Gamma(\alpha+s-1)}{\Gamma(\alpha)\Gamma(\alpha-1)}\frac{\Gamma(\beta+l+s)\Gamma(\beta+s-1)}{\Gamma(\beta)\Gamma(\beta-1)}\frac{\Gamma(\Delta)\Gamma(\Delta-2)}{\Gamma(\Delta+s-2)\Gamma(\Delta+l+s-1)} \times\frac{z^{l+s+1}\bar{z}^{s}}{(s-1)!(l+s)!}\left[\frac{1}{(\Delta+l+s-1)\,s}-\frac{1}{(\Delta+s-2)(l+s+1)}\right]$$

(4.73)

Using

$$\frac{1}{(\Delta+l+s-1)\,s}-\frac{1}{(\Delta+s-2)\,(l+s+1)}=\frac{(\Delta-2)(l+1)}{(\Delta+l+s-1)(\Delta+s-2)(l+s+1)s}$$

(4.74)

This can be seen to be:

$$\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\frac{\Gamma(\alpha+l+s)\Gamma(\alpha+s-1)}{\Gamma(\alpha)\Gamma(\alpha-1)}\frac{\Gamma(\beta+l+s)\Gamma(\beta+s-1)}{\Gamma(\beta)\Gamma(\beta-1)} \times\frac{\Gamma(\Delta)\Gamma(\Delta-1)}{\Gamma(\Delta+s-1)\Gamma(\Delta+l+s)}\frac{z^{l+s+1}\bar{z}^{s}}{s!(l+s+1)!} \tag{4.75}$$

which is precisely the first term in our OWN computation of the block. The remaining two terms are simply conjugates of what we have dealt with so far and therefore are going to reproduce the second term in our OWN computation. Thus it is shown that our answer perfectly matches with the known results [6].

$\underline{d = 3}$

In $d = 3$ the Gegenbauer polynomials we are using to express the answer for four-point scalar CPW becomes the Legendre polynomials, i.e. $C_l^{1/2}(\cos\theta) = P_l(\cos\theta)$. Therefore, our answer reads

$$\begin{aligned} W^{(3)}_{\Delta,0}(\Delta_i;z,\bar{z}) &= (z\bar{z})^{\frac{(\Delta-\Delta_3-\Delta_4)}{2}}\frac{1}{\pi^{3/2}}\sum_{l,s=0}^{\infty}\frac{(l+1/2)\left(\frac{\Delta-\Delta_{12}}{2}\right)_{l+s}\left(\frac{\Delta+\Delta_{34}}{2}\right)_{l+s}}{s!\,(3/2)_{l+s}\,(\Delta)_{l+s}\,(\Delta-1/2)_s} \\ &\times\left(\frac{\Delta-\Delta_{12}}{2}-\frac{1}{2}\right)_s\left(\frac{\Delta+\Delta_{34}}{2}-\frac{1}{2}\right)_s(z\bar{z})^{s+\frac{l}{2}}\,P_l\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right) \end{aligned} \tag{4.76}$$

We are not aware of any closed form for this case. There exists a conjectured formula by [67] where the $d = 3$ 4-point block is written as a single infinite sum over products of pairs of ${}_2F_1$ functions. We have checked that our answer also agrees with [67] to some finite order.

d = 2

To recover the answer for $d = 2$ we have to take $d \to 2$ limit of our answer. Then we find

$$\begin{aligned} & W^{(2)}_{\Delta,0}(\Delta_i; z, \bar{z}) \\ = \; & (z\bar{z})^{\frac{(\Delta-\Delta_3-\Delta_4)}{2}} \frac{1}{\pi} \sum_{l,s} \frac{\left(\frac{\Delta-\Delta_{12}}{2}\right)_{l+s} \left(\frac{\Delta+\Delta_{34}}{2}\right)_{l+s}}{s!\,(l+s)!\,(\Delta)_{l+s}\,(\Delta)_s} \left(\frac{\Delta-\Delta_{12}}{2}\right)_s \left(\frac{\Delta+\Delta_{34}}{2}\right)_s (z\bar{z})^{s+\frac{l}{2}} \cos(l\theta) \\ = \; & \frac{1}{2\pi}(z\bar{z})^{\frac{(\Delta-\Delta_3-\Delta_4)}{2}} \sum_{l,s} \frac{\left(\frac{\Delta-\Delta_{12}}{2}\right)_{l+s} \left(\frac{\Delta+\Delta_{34}}{2}\right)_{l+s}}{(l+s)!\,(\Delta)_{l+s}} \frac{\left(\frac{\Delta-\Delta_{12}}{2}\right)_s \left(\frac{\Delta+\Delta_{34}}{2}\right)_s}{s!\,(\Delta)_s} (z^{l+s}\bar{z}^s + z^s\bar{z}^{l+s}) \end{aligned} \tag{4.77}$$

where we have used the following identity

$$\lim_{\mu\to 0} \frac{1}{\mu} C^{\mu}_l(\cos\theta) = \frac{2}{l} T_l(\cos\theta) = \frac{2}{l}\cos(l\theta) \tag{4.78}$$

satisfied by the Chebyshev polynomials of the first kind, T_l. Finally performing the summations we get back the familiar answer for scalar CPW in two dimensions (see chapter 3)

$$\begin{aligned} W^{(2)}_{\Delta,0}(\Delta_i; z, \bar{z}) = \; & \frac{1}{\pi}(z\bar{z})^{\frac{(\Delta-\Delta_3-\Delta_4)}{2}} {}_2F_1\left[\left(\frac{\Delta-\Delta_{12}}{2}\right), \left(\frac{\Delta+\Delta_{34}}{2}\right); \Delta;\, z\right] \\ & \times {}_2F_1\left[\left(\frac{\Delta-\Delta_{12}}{2}\right), \left(\frac{\Delta+\Delta_{34}}{2}\right); \Delta;\, \bar{z}\right] \end{aligned} \tag{4.79}$$

d = 1

This case corresponds to $\mu = -1/2$ and the Gegenbauer polynomial for this value of μ

takes the following form:

$$C_l^{-\frac{1}{2}}(\chi) = \delta_{l,0} - \chi\,\delta_{l,1} + \theta(l-2)\frac{1-\chi^2}{l(l-1)}\frac{d}{d\chi}P_{l-1}(\chi) \tag{4.80}$$

Further, in this case all the positions of the operators are simply real numbers. In particular the unit vector $\mathbf{u}$ becomes either 1 or -1. Without loss of generality we take $\mathbf{u} = 1$. Then the argument of the Gegenbauer polynomial in (4.54), $\hat{\mathbf{x}} \cdot \mathbf{u}$ also becomes ± 1 depending the sign of $\mathbf{x}$. For both the cases the Gegenbauer polynomial simplifies to

$$C_l^{(-1/2)}(\pm 1) = \delta_{l0} \mp \delta_{l1} = \delta_{l0} - \text{sign}(\mathbf{x})\,\delta_{l1} \tag{4.81}$$

Then the expression for 4-point CPW splits into two parts as follows

$$\begin{aligned} W^{(1)}_{\Delta,0}(\Delta_i, x) &= (x^2)^{\frac{(\Delta-\Delta_3-\Delta_4)}{2}} \frac{1}{2\sqrt{\pi}} \Bigg[\sum_{s=0}^{\infty} \frac{(\alpha)_s\,(\beta)_s}{s!\,(1/2)_s\,(\Delta)_s\left(\Delta+\frac{1}{2}\right)_s} \left(\alpha+\frac{1}{2}\right)_s \left(\beta+\frac{1}{2}\right)_s x^{2s} \\ &\quad + \text{sign}(x) \sum_{s=0}^{\infty} \frac{(\alpha)_{s+1}\,(\beta)_{s+1}}{s!\,(1/2)_{s+1}\,(\Delta)_{s+1}\left(\Delta+\frac{1}{2}\right)_s} \left(\alpha+\frac{1}{2}\right)_s \left(\beta+\frac{1}{2}\right)_s x^{2s+1} \Bigg] \end{aligned} \tag{4.82}$$

where $\alpha = \frac{1}{2}(\Delta - \Delta_{12})$ and $\beta = \frac{1}{2}(\Delta + \Delta_{34})$. Now using the following identities for Pochhammer symbols

$$(A)_s\left(A+\frac{1}{2}\right)_s = \frac{1}{2^{2s}}\,(2A)_{2s}\,, \qquad (A)_{s+1}\left(A+\frac{1}{2}\right)_s = \frac{1}{2^{2s+1}}\,(2A)_{2s+1} \tag{4.83}$$

for $A \in \{\alpha, \beta, \Delta\}$, and

$$s!\left(\frac{1}{2}\right)_s = \frac{(2s)!}{2^{2s}}, \qquad s!\left(\frac{1}{2}\right)_{s+1} = \frac{(2s+1)!}{2^{2s+1}} \tag{4.84}$$

we can show that the above expression (4.82) for 4-point CPW can be written as a single sum, which can be carried out to yield the answer

$$W^{(1)}_{\Delta,0}(x) \;=\; \frac{1}{2\sqrt{\pi}}\, x^{\Delta-\Delta_3-\Delta_4}\,{}_2F_1\left(2\alpha, 2\beta;\, 2\Delta;\, x\right) \tag{4.85}$$

where $x = |\mathbf{x}|$, and this expression agrees with the known result [24, 103] for the $d = 1$ case.

Chapter 5

Conclusion and outlook

In this thesis we have provided an alternative prescription to compute conformal partial waves of correlators of primaries in arbitrary dimensional CFTs holgraphically in terms of certain gravitational open Wilson networks. The gravitational OWNs are for the gauge field A in the adjoint of the algebra $so(1, d + 1)$ for the AdS_{d+1} gravity written as a gauge theory in its first order Hilbert-Palatini formulation.

The results of this thesis include

- Definitions of the OWNs and establishment of the necessary Ward identities and Casimir equations for their interpretation as CPWs in chapter 2.
- Explicit computation of the OWNs in $d = 1$ and $d = 2$ case for generic OWNs and recovery of many of the known results in chapter 3.
- Explicit computation of the 4-point scalar CPWs using the OPE modules and obtaining the expressions for them in Gegenbauer basis in chapter 4.

Computational techniques have been developed to a state from where the answers associated with a given OWN can be written down using simple Feynman like rules which are easily stated.

For $d = 2$ a similar prescription is provided also by other authors [108]. They use the finite dimensional representations of the conformal algebra and rely on producing the results for the unitary irreducible representations of the Lorentzian algebra through some analyticity properties of CPWs which are functions of conformal weights of the external primaries and as well as the primary (and its descendants) being exchanged in the intermediate channel. In this case the exercise reduces to computing $SL(N)$ matrix elements in finite dimensional representations. Setting $N = 2$ in [108] the known results for the global conformal blocks are obtained. The authors also reproduced the heavy-light Virasoro blocks taking two operators to be heavy (with operator dimension $h/c \gg 1$). For $N = 3$ the $\mathcal{W}_3$ blocks are found and a generalization to arbitrary N has been carried out. The answers match with those found in the literatures [109, 110]. In this approach the primaries having negative conformal dimensions have been considered and finally the answer is analytically continued to positive conformal weights to obtain answers in unitary regime.

In chapter 3 what we computed for the $d = 1$ and $d = 2$ cases are the global blocks – which don not use the fact that the corresponding CFTs have infinite dimensional conformal algebras (Virasoro). To go beyond this large approximation one needs to include the quantum effects of the bulk gauge theory. A prescription for this now exists in the literature [93]. One of the original motivation was to apply our prescription to bulk theories which do not have (or hard to find) second order formulations – such as the higher spin theories. A few results in this direction are now available – following our prescription [99].

The OWNs that we have used is a vey small subset of all possible spin networks. For instance, we have only used the tree level spin networks (no loops). It is natural to ask that the interpretation of all other OWNs are. They all are expected to satisfy Ward identities but no Casimir equations.

In a novel way of implementing the bootstrap in [54, 71] the authors have used the exchange Witten diagrams. It will be interesting to see if one can make use of the OWNs. One may also wonder how the geodesic Witten diagrams are related to the open Wilson line network operators.

In [111] the holographic entanglement entropy was defined using a Wilson-line operator called $\mathcal{W}_{\mathcal{R}}(P, Q)$ connecting the points P and Q and in the two dimensional (non-unitary) representation of $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$ corresponding to $h = \bar{h} = -1/2$ [94] as:

$$S_A = \frac{c}{6} \log \left[\mathcal{W}_{\mathcal{R}}(P, Q)\big|_{P,Q \to boundary} \right]. \tag{5.1}$$

We note that one can define the entanglement entropy using our capped open Wilson lines as

$$S_A = -\frac{c}{6} \log \left[\langle\!\langle h, \bar{h}; j, j | g(P) g^{-1}(Q) | h, \bar{h}; j, -j \rangle\!\rangle \big|_{P,Q \to boundary} \right] \tag{5.2}$$

where we take the representation to be that of $h = \bar{h} = 1/2$ (and $j = 0$) which is a unitary representation of the conformal algebra $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$. Using (the Lorentzian analog of) (3.30) with the UV cutoff $a = le^{-\rho}$ in (5.2) one can immediately recover the famous entanglement entropy formula $S_A = \frac{c}{3} \log \frac{\Delta x}{a}$. We expect this definition (5.2) to reproduce other results of [111] of entanglement entropy relevant to the global AdS_3 and the BTZ black hole contexts as well. The $1/c$ correction in this case needs to be further investigated and interpreted.

One may be interested in computing OWNs in different backgrounds other than pure AdS. For instance, we computed two-point functions in [15] in the BTZ background and recovered the thermal correlators. Other higher point CPWs can also be computed. In $d > 2$ we can have solutions that are not necessarily locally AdS_{d+1}. How to compute the OWN in them and what would be their interpretation? In higher dimension there are Rindler-AdS spaces which are locally AdS. So we can use our techniques to OWNs in them. One expects the answers to resemble thermal CPWs.

As a by product we have given a prescription to compute CGC of infinite dimensional irreducible representations of $so(1, d+1)$ in terms of the conformal three-point functions- which are dictated by Ward identities. We can also generalize the explicit matrix elements of the generators of $so(1, d + 1)$ using conformal wave function representations we have found using the OWNs. For demonstration we give them for scalar modules of $so(1, 4)$ in appendix C.

Generalization of the computational techniques of generic OWNs in arbitrary dimensions is under progress.

An interesting set of further investigations in this directions should include exploring the role of Weight shifting operators [112] in our formalism.

There is an alternative way to compute the Wilson lines using the quantum mechanics of particles in AdS. This approach was followed in [105]. It will be interesting to see how the junctions would work there.

The prescription in [93] to go beyond the large c limit of OWNs in $d = 2$ has been used in study of the self energy of the gravitational point particles in AdS_3 to obtain the $1/c$ corrections to the anomalous dimensions of the operators in CFT_2 [113]. The loop diagrams play an important role there. They exist in all dimensions among the full set of OWNs. It will be interesting to see if they play a similar role in all dimensions.

The correlators and the conformal blocks have been computed using OWNs in higher spin AdS_3 holography [114, 115]. Recently the OWN method to generalized to compute superconformal blocks [116].

We took the boundary to be $\mathbb{R}^2$ in section 3. We can easily take it to be either S^2, $\mathbb{H}^2$ or $S^1 \times \mathbb{R}$. It will be interesting to consider the case of the boundary being any Riemann surface as well.

Of course one would like see if our method gives answers in forms more amenable to potential applications, such as in the bootstrap approach towards the classification of CFTs.

Since our answers are in Gegenbauer polynomial basis it is possible that they may be found more suitable as working with this basis is much more easy (as we have seen in section 4 for example).

It may be of interest to compute objects similar to our OWN*s* in both flat and de Sitter gravity theories. Such diagrams could provide a basis of partial waves for S-matrices for scattering problems in these spaces.

We hope that this program will naturally lend itself to answering dynamical questions as well in CFT*s*.

Appendix A

$SO(1,3)$ group

In this appendix we review some necessary group theory for the computations we have done in the text. Let us begin by setting our conventions of $so(1,3)$ algebra and its representations. We take its generators to be $M_{\mu\nu}$ with the algebra:

$$[M_{\alpha\beta}, M_{\gamma\delta}] = \eta_{\alpha\delta}M_{\beta\gamma} + \eta_{\beta\gamma}M_{\alpha\delta} - \eta_{\alpha\gamma}M_{\beta\delta} - \eta_{\beta\delta}M_{\alpha\gamma} \tag{A.1}$$

where $\mu, \nu, \dots = 0, 1, 2, 3$ and $\eta = \text{diag}\{-1, 1, 1, 1\}$. One way of writing this algebra is as $so(1,3) = su(2) \oplus su(2)$ with the generators:

$$J_1^{(\pm)} = \frac{1}{2}(-iM_{23} \pm M_{01}), \quad J_2^{(\pm)} = \frac{1}{2}(-iM_{31} \pm M_{02}), \quad J_3^{(\pm)} = \frac{1}{2}(-iM_{12} \pm M_{03}) \tag{A.2}$$

with the algebra

$$[J_a^{(\pm)}, J_b^{(\pm)}] = i\,\epsilon_{abc}J_c^{(\pm)}, \quad [J_a^{(\pm)}, J_b^{(\mp)}] = 0. \tag{A.3}$$

Working with unitary representations for each $SU(2)$ factor provides a finite dimensional non-unitary representation of $so(1,3)$. So a general finite dimensional non-unitary representation is labeled by two half-integers (j_1, j_2). We are interested in constructing representations of the "diagonal" $SU(2)$ out of these representations labeled by one single j.

Another way to write the algebra of $so(1,3)$ is as $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$ with the generators:

$$J_1^{(\pm)} = \frac{i}{2}(-iM_{23} \pm M_{01}), \;\; J_2^{(\pm)} = \frac{i}{2}(-iM_{31} \pm M_{02}), \;\; J_0^{(\pm)} = \frac{1}{2}(-iM_{12} \pm M_{03}) \tag{A.4}$$

with the algebra

$$[J_a^{(\pm)}, J_b^{(\pm)}] = i\,\epsilon_{ab}{}^c J_c^{(\pm)}, \;\; [J_a^{(\pm)}, J_b^{(\mp)}] = 0. \tag{A.5}$$

with $\epsilon_{012} = 1$ and $\eta_{ab} = \mathrm{diag}\{-1,1,1\}$ used to raise and lower indices. Working with unitary representations of each $sl(2,\mathbb{R})$ factor provides infinite dimensional but non-unitary representations of $so(1,3)$. These generators of $sl(2,\mathbb{R})$ can be mapped to the standard ones used in the 2d CFT language by defining:

$$\begin{aligned} L_0 &= -J_0^{(+)}, \;\; L_1 = i(J_1^{(+)} + i\,J_2^{(+)}), \;\; , \;\; L_{-1} = -i(J_1^{(+)} - i\,J_2^{(+)}) \\ \bar{L}_0 &= J_0^{(-)}, \;\; \bar{L}_1 = -i(J_1^{(-)} - i\,J_2^{(-)}), \;\; \bar{L}_{-1} = i(J_1^{(-)} + i\,J_2^{(-)}) \end{aligned} \tag{A.6}$$

which satisfy the algebra:

$$[L_m, L_n] = (m-n)L_{m+n}, \;\; [\bar{L}_m, \bar{L}_n] = (m-n)\,\bar{L}_{m+n}, \;\; [L_m, \bar{L}_n] = 0. \tag{A.7}$$

The representation where the generators have the hermiticity properties:

$$L_0^\dagger = L_0, \; L_1^\dagger = L_{-1}; \bar{L}_0^\dagger = \bar{L}_0, \; \bar{L}_1^\dagger = \bar{L}_{-1} \tag{A.8}$$

is the relevant one for us here. We will consider the unitary highest (lowest) weight representation of each of these two $sl(2,\mathbb{R})$ as in, for instance, [94, 97]. The sub-algebra $so(3)$ in $so(1,3)$ is generated by:

$$L_0 - \bar{L}_0 = iM_{12}, \;\; L_1 + \bar{L}_{-1} = iM_{23} + M_{13}, \;\; L_{-1} + \bar{L}_1 = -iM_{23} + M_{13} \tag{A.9}$$

The rest of the generators are

$$-M_{03} = L_0 + \bar{L}_0, \quad M_{01} + iM_{02} = -L_1 + \bar{L}_{-1}, \quad M_{01} - iM_{02} = L_{-1} - \bar{L}_1 \tag{A.10}$$

So the finite dimensional representation of the "local Lorentz algebra" $so(3)$ are thus associated to the finite dimensional non-unitary representation of the twisted diagonal $sl(2,\mathbb{R})$ generated by $L_n - (-1)^n \bar{L}_{-n}$ for $n = -1, 0, 1$.

We are interested in decomposing each of the representations of $so(1,3)$ given by the tensor product of the infinite dimensional unitary representation of each of the $sl(2,\mathbb{R})$ algebras in $so(1,3)$ into a given irreducible representation of the twisted diagonal $sl(2,\mathbb{R})$ sub-algebra.

The fundamental (and the defining representation) of the Lorentz algebra $so(1,3)$ is the vector representation in which we take the generators to be 4×4 real trace-less matrices given by:

$$(M_{ab})^{\alpha}{}_{\beta} = \delta^{\alpha}_{a}\, \eta_{b\beta} - \delta^{\alpha}_{b}\eta_{a\beta} \tag{A.11}$$

with $\eta^{-1}(-M^{\mathrm{T}}_{\mu\nu})\eta = M_{\mu\nu}$. The 6×6 adjoint representation is given by

$$(M_{ab})^{gh}{}_{mn} = \eta_{an}(\delta^g_b\delta^h_m - \delta^g_m\delta^h_b) + \eta_{bm}(\delta^g_a\delta^h_n - \delta^g_n\delta^h_a) \tag{A.12}$$

$$-\eta_{am}(\delta^g_b\delta^h_n - \delta^g_n\delta^h_b) - \eta_{bn}(\delta^g_a\delta^h_m - \delta^g_m\delta^h_a) \tag{A.13}$$

such that one has

$$[M_{ab}, M_{mn}] = \frac{1}{2}(M_{ab})^{gh}{}_{mn}M_{gh} = -\frac{1}{2}(M_{mn})^{gh}{}_{ab}M_{gh} \tag{A.14}$$

Defining $O_{[ab][cd]} = \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}$ and the identity matrix as $I_{[ab]}{}^{[cd]} = \delta^c_a\delta^d_b - \delta^d_a\delta^c_b$ we again have $O^{-1}(-M^{\mathrm{T}})O = M$. At the level of the group elements in the adjoint representation this translates to $R[g^{-1}] = O^{-1}(R[g])^{\mathrm{T}}O$ where $R[g]$ is the 6×6 adjoint representation of the group element g.

There are two quadratic Casimirs of the Lie algebra $so(1,3)$:

- (i) $C_2^{(1)} = M_{0c}M_{0c} - \frac{1}{2}M_{ab}M_{ab}$
- (ii) $C_2^{(2)} = M_{01}M_{23} + M_{23}M_{01} - M_{02}M_{13} - M_{13}M_{02} + M_{03}M_{12} + M_{12}M_{03}$.

Written in terms of the two $sl(2,\mathbb{R})$ factors these read:

$$C_2^{(1)} = C_2 + \bar{C}_2, \quad i\,C_2^{(2)} = C_2 - \bar{C}_2 \tag{A.15}$$

or equivalently

$$C_2 := 2L_0L_0 - \{L_1, L_{-1}\} = \frac{1}{2}[C_2^{(1)} + i\,C_2^{(2)}],\ \bar{C}_2 := 2\bar{L}_0\bar{L}_0 - \{\bar{L}_1, \bar{L}_{-1}\} = \frac{1}{2}[C_2^{(1)} - i\,C_2^{(2)}]$$

Hermitian representations of $sl(2,\mathbb{R})$

Let us review some facts regarding the hermitian representations of the algebra $sl(2,\mathbb{R})$ as in [94]. The quadratic Casimir operator of the algebra $[L_m, L_n] = (m-n)\,L_{m+n}$ is again

$$C_2 = 2L_0L_0 - 2L_0 - 2L_{-1}L_1 = 2L_0L_0 + 2L_0 - 2L_1L_{-1} \tag{A.16}$$

Consider a state $|h,0\rangle$ which is a highest weight state: $L_1|h,0\rangle = 0$ and $L_0|h,0\rangle = h|h,0\rangle$ with $C_2|h,0\rangle = 2h(h-1)|h,0\rangle$. The rest of the states in this representation can be obtained by successively operating with L_{-1} starting from the highest weight state, that is $|h,n\rangle \sim L_{-1}^n|h,0\rangle$. Thus we have states in this highest weight representation given by $|h,n\rangle$ such that

$$L_0|h,n\rangle = (h+n)|h,n\rangle, \quad C_2|h,n\rangle = 2h(h-1)|h,n\rangle. \tag{A.17}$$

which in-turn imply

$$
\begin{aligned}
L_{-1}|h,n\rangle &= \sqrt{(2h+n)(n+1)}\;|h,n+1\rangle, \\
L_{1}|h,n\rangle &= \sqrt{n(2h+n-1)}\;|h,n-1\rangle. \qquad \text{(A.18)}
\end{aligned}
$$

For a positive h there is no state $|h,n\rangle$ with a non-negative integer n which is annihilated by L_{-1} and so they are all infinite dimensional and unitary.

Clebsch-Gordan coefficients

We need some more group theory - namely, the Clebsch-Gordan coefficients that appear in the decomposition of the tensor product of two unitary representations of $sl(2,\mathbb{R})$ algebra into other unitary irreducible ones. We take the basis states of the tensor product of two irreducible representations of $sl(2,\mathbb{R})$ labeled by (the non-negative real numbers) h_1 and h_2 by $|h_1,n_1\rangle \otimes |h_2,n_2\rangle = |h_1,h_2;n_1,n_2\rangle$ which diagonalize $\{C_2^{(1)},C_2^{(2)},L_0^{(1)},L_0^{(2)}\}$ with the eigen values $\{2h_1(h_1-1),2h_2(h_2-1),h_1+n_1,h_2+n_2\}$ respectively. The generators of $sl(2,\mathbb{R})$ act on the tensor product as

$$
L_n = L_n^{(1)} \otimes \mathbb{I} + \mathbb{I} \otimes L_n^{(2)}\,. \qquad \text{(A.19)}
$$

We make a change of basis to states $|h_1,h_2;h,n\rangle$ which diagonalize $\{C_2^{(1)},C_2^{(2)},C_2,L_0\}$ with eigen values $\{2h_1(h_1-1),2h_2(h_2-1),2h(h-1),h+n\}$ respectively. We also have

$$
\begin{aligned}
L_0|h_1,h_2;h,n\rangle &= (h+n)|h_1,h_2;h,n\rangle \\
L_{-1}|h_1,h_2;h,n\rangle &= \sqrt{(2h+n)(n+1)}\;|h_1,h_2;h,n+1\rangle, \\
L_{1}|h_1,h_2;h,n\rangle &= \sqrt{n(2h+n-1)}\;|h_1,h_2;h,n-1\rangle. \qquad \text{(A.20)}
\end{aligned}
$$

Taking the matrix element of the operator $L_0 - L_0^{(1)} - L_0^{(2)}$ as

$$\langle h_1, h_2; h, n | L_0 - L_0^{(1)} - L_0^{(2)} | h_1, h_2; n_1, n_2 \rangle \tag{A.21}$$

$$= (h + n - h_1 - n_1 - h_2 - n_2)\langle h_1, h_2; h, n | h_1, h_2; n_1, n_2 \rangle = 0 \tag{A.22}$$

Imposes the condition

$$h - h_1 - h_2 = n_1 + n_2 - n. \tag{A.23}$$

Given that n_1, n_2, n are integer the allowed values of h are discrete and integer spaced. On the other hand considering the action of L_1 gives:

$$\begin{aligned}
&\sqrt{n(2h + n - 1)}\, \langle h_1, h_2; n_1, n_2 | h_1, h_2; h, n - 1 \rangle \\
&\quad = \quad \sqrt{(n_1 + 1)(2h_1 + n_1)}\, \langle h_1, h_2; n_1 + 1, n_2 | h_1, h_2; h, n \rangle \\
&\qquad + \sqrt{(n_2 + 1)(2h_2 + n_2)}\, \langle h_1, h_2; n_1, n_2 + 1 | h_1, h_2; h, n \rangle
\end{aligned} \tag{A.24}$$

and the action of L_{-1} gives:

$$\begin{aligned}
&\sqrt{(n + 1)(2h + n)}\, \langle h_1, h_2; n_1, n_2 | h_1, h_2; h, n + 1 \rangle \\
&\quad = \quad \sqrt{n_1(2h_1 + n_1 - 1)}\, \langle h_1, h_2; n_1 - 1, n_2 | h_1, h_2; h, n \rangle \\
&\qquad + \sqrt{n_2(2h_2 + n_2 - 1)}\, \langle h_1, h_2; n_1, n_2 - 1 | h_1, h_2; h, n \rangle
\end{aligned} \tag{A.25}$$

when we write

$$|h_1, h_2; h, n\rangle = \sum_{m_1, m_2 = 0}^{\infty} |h_1, h_2, m_1, m_2\rangle \langle h_1, h_2; m_1, m_2 | h_1, h_2; h, n \rangle \tag{A.26}$$

and when $|h_1, h_2, m_1, m_2\rangle$ and $|h_1, h_2; h, n\rangle$ are basis vectors of unitary representations (as it is when $h_1, h_2, h > 0$) one should impose

$$\sum_{h,n} \langle h_1, h_2; n_1, n_2 | h_1, h_2; h, n\rangle \langle h_1, h_2; m_1, m_2 | h_1, h_2; h, n\rangle = \delta_{n_1,m_1}\delta_{n_2,m_2} \tag{A.27}$$

Later on we will need explicit expressions for these CGC*s*. So let us derive them here (see also [98]). To proceed let us first write the CGC $\langle h_1, h_2; n_1, n_2 | h_1, h_2; h, n\rangle$ as

$$\langle h_1, h_2; n_1, n_2 | h_1, h_2; h_3, n_3\rangle = \frac{1}{\prod_{i=1}^{3} \sqrt{k_i!(\Gamma(2h_i + k_i)}} f(k_1, k_2; k_3) \tag{A.28}$$

where the f's satisfy

$$k_3(2h_3 + k_3 - 1) f(k_3 - 1) - f(k_1 + 1) - f(k_2 + 1) = 0 \tag{A.29}$$

$$f(k_3 + 1) - k_1(2h_1 + k_1 - 1) f(k_1 - 1) - k_2(2h_2 + k_2 - 1) f(k_2 - 1) = 0 \tag{A.30}$$

modulo $h_1 + k_1 + h_2 + k_2 - h_3 - k_3$ (as a consequence of the Kronecker delta in the CGC). There are two cases we have to consider carefully: $h_1 + h_2 - h_3 \geq 0$ and $h_1 + h_2 - h_3 \leq 0$. The CGC*s* are:

$$\begin{aligned} f(k_1, k_2; k_3) &\sim \Gamma(2h_2 + k_1 + k_2)\Gamma(k_3 - k_2) k_2!\, \delta_{h_1+k_1+h_2+k_2-h_3-k_3} \\ &\quad \times {}_3F_2\left(\begin{matrix} -k_1, -k_3, h_3 + h_1 - h_2 \\ 1 + k_2 - k_3, 1 - 2h_2 - k_1 - k_2 \end{matrix}; 1 \right); \text{ for } h_i + h_j \geq h_k \text{ or } h_3 + h_1 \leq h_2 \\ &\sim (-1)^{k_1} \frac{\Gamma(2h_2 + k_1 + k_2) k_2!}{\Gamma(k_2 - k_3 + 1)}\, \delta_{h_1+k_1+h_2+k_2-h_3-k_3} \\ &\quad \times {}_3F_2\left(\begin{matrix} -k_1, -k_3, h_3 + h_1 - h_2 \\ 1 + k_2 - k_3, 1 - 2h_2 - k_1 - k_2 \end{matrix}; 1 \right) \quad \text{for} \quad h_1 + h_2 \leq h_3 \\ &\sim (-1)^{k_3} \frac{\Gamma(k_3 - k_2) k_2!}{\Gamma(h_1 - h_2 - h_3 + k_3 + 1)}\, \delta_{h_1+k_1+h_2+k_2-h_3-k_3} \\ &\quad \times {}_3F_2\left(\begin{matrix} -k_1, -k_3, h_3 + h_1 - h_2 \\ 1 + k_2 - k_3, 1 - 2h_2 - k_1 - k_2 \end{matrix}; 1 \right) \quad \text{for} \quad h_2 + h_3 \leq h_1 \end{aligned} \tag{A.31}$$

Substituting them (any of the above four cases) into the left hand side of (A.29) ((A.30)) comes out to be proportional to $\delta_{h_1+k_1+h_2+k_2-h_3-k_3+1}$ ($\delta_{h_1+k_1+h_2+k_2-h_3-k_3-1}$) times

$$de_3F_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};1\right)+a(c-d)_3F_2\left(\begin{matrix}a+1,b+1,c\\d+1,e+1\end{matrix};1\right)+d(a-e)_3F_2\left(\begin{matrix}a,b+1,c\\d,e+1\end{matrix};1\right)$$

where $a=-k_3$, $b=-k_1-1$, $c=h_3+h_1-h_2$, $d=-2h_2-k_1-k_2$ and $e=1+k_2-k_3$ ($a=-k_1$, $b=-k_3-1$, $c=h_3+h_1-h_2=k_2-k_3-k_1-1$, $d=k_2-k_3$ and $e=1-2h_2-k_1-k_2$) for either case $h_1+h_2\geq h_3$ or $h_1+h_2\leq h_3$. Happily this vanishes identically (at least for when either a or b is a negative integer as it is the case in our case)

$$de_3F_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};1\right) = a(d-c)_3F_2\left(\begin{matrix}a+1,b+1,c\\d+1,e+1\end{matrix};1\right)+d(e-a)_3F_2\left(\begin{matrix}a,b+1,c\\d,e+1\end{matrix};1\right) \tag{A.32}$$

To see this we begin by applying the identity [117]

$$a(b-c)\ {}_3F_2\left(\begin{matrix}a+1,a_2,a_3\\b+1,c+1\end{matrix};z\right)-c(b-a)\ {}_3F_2\left(\begin{matrix}a,a_2,a_3\\b+1,c\end{matrix};z\right)+b(c-a)\ {}_3F_2\left(\begin{matrix}a,a_2,a_3\\b,c+1\end{matrix};z\right)=0$$

at $z=1$[1] to the right hand side of (A.32) to give

$$a(d-c)\left[\frac{e(d-a)}{a(d-e)}\ {}_3F_2\left(\begin{matrix}a,b+1,c\\d+1,e\end{matrix};1\right)-\frac{d(e-a)}{a(d-e)}\ {}_3F_2\left(\begin{matrix}a,b+1,c\\d,e+1\end{matrix};1\right)\right]$$
$$+d(e-a)\ {}_3F_2\left(\begin{matrix}a,b+1,c\\d,e+1\end{matrix};1\right)$$

which further simplifies to

$$\frac{1}{a(d-e)}\left[ae(d-a)(d-c)\ {}_3F_2\left(\begin{matrix}a,b+1,c\\d+1,e\end{matrix};1\right)-ad(e-a)(e-c)\ {}_3F_2\left(\begin{matrix}a,b+1,c\\d,e+1\end{matrix};1\right)\right] \tag{A.33}$$

[1] we note here that in general the hypergeometric function ${}_3F_2$ is not well defined at $z=1$, but we will be using it only for the case where $a=-k_3\pm1$, $b=-k_1\pm1$, $c=h_3+h_1-h_2\pm1$, $d=-2h_2-k_1-k_2\pm1$ and $e=1\pm k_2-k_3$ in which case ${}_3F_2$ is a polynomial and the limit is easily taken.

Now we apply the identity [117]

$$b\,{}_3F_2\left(\begin{matrix}a, a_2, a_3\\ b, b_2\end{matrix}; z\right) - a\,{}_3F_2\left(\begin{matrix}a+1, a_2, a_3\\ b+1, b_2\end{matrix}; z\right) + (a-b)\,{}_3F_2\left(\begin{matrix}a, a_2, a_3\\ b+1, b_2\end{matrix}; z\right) = 0$$

at $z = 1$ to both terms of (A.33) to yield

$$\begin{aligned}&\frac{de\,[(d-a)(d-c)-(e-a)(e-c)]}{b(d-e)}\,{}_3F_2\left(\begin{matrix}a, b, c\\ d, e\end{matrix}; 1\right)\\ &-\frac{e(d-a)(d-c)(d-b)}{b(d-e)}\,{}_3F_2\left(\begin{matrix}a, b, c\\ d+1, e\end{matrix}; 1\right)\\ &+\frac{d(e-a)(e-c)(e-b)}{b(e-d)}\,{}_3F_2\left(\begin{matrix}a, b, c\\ d, e+1\end{matrix}; 1\right)\end{aligned} \tag{A.34}$$

Finally we apply the identity [117]

$$\begin{aligned}&(a+(a_2+a_3-d-e)z)\,{}_3F_2\left(\begin{matrix}a, a_2, a_3\\ d, e\end{matrix}; z\right) + \frac{(d-a)(d-a_2)(d-a_3)}{d(d-e)}\,{}_3F_2\left(\begin{matrix}a, a_2, a_3\\ d+1, e\end{matrix}; z\right)\\ &+\frac{(e-a)(e-a_2)(e-a_3)}{e(e-d)}\,{}_3F_2\left(\begin{matrix}a, a_2, a_3\\ d, e+1\end{matrix}; z\right) - a(1-z)\,{}_3F_2\left(\begin{matrix}a+1, a_2, a_3\\ d, e\end{matrix}; z\right) = 0\end{aligned}$$

to (A.34) at $z = 1$ to get

$$\frac{de}{b}\,[(d+e-a-c)+(a+b+c-d-e)]\;{}_3F_2\left(\begin{matrix}a, b, c\\ d, e\end{matrix}; 1\right) = de\,{}_3F_2\left(\begin{matrix}a, b, c\\ d, e\end{matrix}; 1\right)$$

which proves our claim.

Next we will establish two important relations satisfied by the CG coefficients of $sl(2,\mathbb{R})$ and the unitary representations of the algebra elements. First of them is

$$(L_n^{(h_1)})_{k_1k_1'}C^{h_1h_2;h}_{k_1'k_2k_3} + (L_n^{(h_2)})_{k_2k_2'}C^{h_1h_2;h}_{k_1k_2'k_3} - C^{h_1h_2;h}_{k_1k_2k_3'}(L_n^{(h_3)})_{k_3'k_3} = 0 \tag{A.35}$$

where $n = -1, 0, 1$. To verify this first note that the matrix elements of the generators L_n

are:

$$
\begin{aligned}
(L_0^{(h)})_{k,k'} &= (h+k)\delta_{k,k'}, \\
(L_1^{(h)})_{k,k'} &= \sqrt{(k+1)(2h+k)}\;\delta_{k,k'-1}, \\
(L_{-1}^{(h)})_{k,k'} &= \sqrt{k(2h+k-1)}\;\delta_{k,k'+1}
\end{aligned}
\tag{A.36}
$$

Taking the case of $n = 0$ implies that the CGCs vanish unless $h_1 + k_1 + h_2 + k_2 = h + k$. It is easily seen that for $n = -1, 1$ this identity follows simply from the recursion relations satisfied by the CGCs. The second identity is

$$
R_{h_1}[g(x)]_{k_1k'_1}\; R_{h_2}[g(x)]_{k_2,k'_2}\; C^{h_1h_2;h}_{k'_1k'_2;k'_3}\; R_{h_3}[g^{-1}(x)]_{k'_3k_3} = C^{h_1h_2;h}_{k_1k_2;k_3} \tag{A.37}
$$

where $R_h[g(x)]$ is the representation of the group element $g(x)$ in the lowest weight representation of $sl(2,\mathbb{R})$ labeled by h. We will now prove it for an element of the $SL(2,\mathbb{R})$ group of the form $g(x) = e^{\omega^a(x)L_a}$ (summed over $a = -1, 0, 1$). Then we can write $R_h[g(x)] = \sum_{n=0}^{\infty}\frac{1}{n!}\omega^a(L_a^{(h)})^n$ where $L_a^{(h)}$ is the representation of the generator L_a in representation labeled by h. To establish this identity we look at terms involving a fixed number of parameters ω^a. To the order $O(\omega_a^0)$ the right hand side is already taken care of. At the $O(\omega_a^1)$ the terms sum to zero from (A.35). At the $O(\omega_a^2)$ we have

$$
\begin{aligned}
&\omega^a\omega^b[\frac{1}{2}(L_a^{(1)}L_b^{(1)})_{k_1k'_1}C(k'_1) + \frac{1}{2}(L_a^{(2)}L_b^{(2)})_{k_2k'_2}C(k'_2) + \frac{1}{2}C(k'_3)(L_a^{(3)}L_b^{(3)})_{k'_3k_3} \\
&+(L_a^{(1)})_{k_1k'_1}(L_b^{(2)})_{k_2k'_2}C(k'_1,k'_2) - (L_a^{(1)})_{k_1k'_1}C(k'_1,k'_3)(L_b^{(3)})_{k'_3k_3} - (L_a^{(2)})_{k_2k'_2}C(k'_2,k'_3)(L_b^{(3)})_{k'_3k_3}] \\
=\;& \frac{1}{2}\omega^a(L_a^{(1)})_{k_1k''_1}\,\omega^b[(L_b^{(1)})_{k''_1k'_1}C(k'_1) + (L_b^{(2)})_{k_2k'_2}C(k''_1,k'_2) - C(k''_1,k'_3)(L_b^{(3)})_{k'_3k_3}] \\
&+\frac{1}{2}\omega^a(L_a^{(2)})_{k_2k''_2}\,\omega^b[(L_b^{(2)})_{k''_2k'_2}C(k'_2) + (L_b^{(1)})_{k_1k'_1}C(k'_1,k''_2) - C(k''_2,k'_3)(L_b^{(3)})_{k'_3k_3}] \\
&+\frac{1}{2}\omega^b[C(k'_3)(L_b^{(3)})_{k'_3k''_3} - (L_b^{(1)})_{k_1k'_1}C(k'_1,k''_3) - (L_b^{(2)})_{k_2k'_2}C(k'_2,k''_3)]\omega^a(L_a^{(3)})_{k''_3k_3}
\end{aligned}
$$

And using the identity (A.35) each of the three terms vanishes. One can generalise this to any higher order in powers of ω_a's establishing the identity we claimed. The term of $O(\omega^{n+1})$ is

$$\sum_{n_1+n_2+n_3=n+1} \frac{(-1)^{n_3}}{n_1!n_2!n_3!}[(\omega^a L_a^{(1)})^{n_1}]_{k_1k_1'}[(\omega^b L_b^{(2)})^{n_2}]_{k_2k_2'}C(k_1',k_2';k_3')[(\omega^c L_c^{(3)})^{n_3}]_{k_3'k_3}$$

$$= \frac{1}{n+1}\omega^d (L_d^{(1)})_{k_1k_1'}$$

$$\sum_{n_1+n_2+n_3=n} \frac{(-1)^{n_3}}{n_1!n_2!n_3!}\Big\{(\omega^a L_a^{(1)})^{n_1}]_{k_1'k_1''}[(\omega^b L_b^{(2)})^{n_2}]_{k_2k_2'}C(k_1'',k_2';k_3')[(\omega^c L_c^{(3)})^{n_3}]_{k_3'k_3}$$

$$+[(\omega^a L_a^{(1)})^{n_1}]_{k_1k_1'}[(\omega^b L_b^{(2)})^{n_2}]_{k_2'k_2''}C(k_1',k_2'';k_3')[(\omega^c L_c^{(3)})^{n_3}]_{k_3'k_3}$$

$$-\ [(\omega^a L_a^{(1)})^{n_1}]_{k_1k_1'}[(\omega^b L_b^{(2)})^{n_2}]_{k_2k_2'}C(k_1',k_2';k_3'')[(\omega^c L_c^{(3)})^{n_3}]_{k_3''k_3'}\Big\}$$

Therefore assuming the relation to be true at n^{th} order means it is true at order $n+1^{th}$ order. This establishes the identity we want.

Appendix B

$SO(1, d+1)$ CGC

Here we record results of CGC of the representations considered in the text for the algebra $so(1, d+1)$. These are needed to compute the CPW for higher-point functions from OWNs. Before we present the detailed derivation of the CGC from the 3-point functions we collect a few facts about the irreducible representations of $so(d)$ which we will use in the extraction of the Clebsch-Gordan coefficients.

A finite dimensional irreducible representation of $so(d)$ is uniquely defined by its highest weight $[\mu_1, \mu_2, ...\mu_k]$ with

$$\begin{aligned} &\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{k-1} \geq |\mu_k| \quad \text{for } d = 2k \\ &\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{k-1} \geq \mu_k \geq 0 \ \text{ for } d = 2k+1 \end{aligned} \tag{B.1}$$

The components μ_i are either simultaneously integers (tensorial representations) or half-integers (spinorial representations). We only consider symmetric traceless representations of $so(d)$ as these are the only relevant ones for the scalar CGC of $so(d+1, 1)$. These could be represented on the Hilbert space H of square integrable function on S^{d-1}. The Hilbert space can be decomposed into an orthogonal sum of subspaces H^l of homogenous polynomials of degree l in d variables. We introduce a complete orthonormal basis $|l, \mathbf{M}\rangle$

on H^l, where $\mathbf{M} = (m_{d-2}, m_{d-3}, ..., m_2, m_1)$ label these basis states provided they fulfil:

$$l = m_{d-1} \geq m_{d-2} \geq \cdots \geq m_2 \geq |m_1| \quad m_1 \in \mathbb{Z} \quad m_i \in \mathbb{Z}_{>0} \quad i \geq 2 \tag{B.2}$$

The dimension of the space H is $d_l = (2l + d - 2)\frac{(l+d-3)!}{l!(d-2)!}$ - the number of independent components of a general symmetric traceless tensor of rank l in d dimensions. The matrix elements of the representation D^l read:

$$D^l_{\mathbf{M}\mathbf{M}'}(g) = \langle l, \mathbf{M}|D^l(g)|l, \mathbf{M}'\rangle \tag{B.3}$$

In particular,

$$D^l_{\mathbf{M}\,\mathbf{0}}(g) = \frac{1}{\sqrt{d_l}}\, N^d_{l\mathbf{M}} \prod_{k=1}^{d-2} C^{m_k+k/2}_{m_{k+1}-m_k}\, \cos(\Phi_{k+1})\, \sin^{m_k}(\Phi_{k+1})\, e^{im_1\Phi_1} \tag{B.4}$$

where $N^d_{l\mathbf{M}}$ is the normalisation w.r.t the Haar measure on $so(d)$, $C^n_\lambda(z)$ are the Gegenbauer polynomials. The angles $0 \leq \Phi_1 \leq 2\pi$ and $0 \leq \Phi_i \leq \pi$ for $i \neq 1$ can be identified with the Euler angles of a rotation g which maps the north pole $a = (0, \cdots, 0, 1) \in \mathbb{R}^d$ to an arbitrary point on S^{d-1}. Then the hyperspherical harmonics on S^{d-1} are defined as follows:

$$|e\rangle = D^l(g)|a\rangle, \quad Y_{l\mathbf{M}}(e) = \langle e|l,\, \mathbf{M}\rangle, \quad \langle a|l,\, \mathbf{M}\rangle = \sqrt{\frac{d_l}{V_d}}\,\delta_{\mathbf{M}\,\mathbf{0}} \tag{B.5}$$

where $V_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the volume of unit S^{d-1} sphere. Therefore, we get

$$Y_{l\mathbf{M}}(e) \quad = \quad \sqrt{\frac{d_l}{V_d}}\; D^{l*}_{\mathbf{M}\,\mathbf{0}}(g) \tag{B.6}$$

We finally list the following properties of hyperspherical harmonics which can be easily derived using the definitions given above:

1. $Y^*_{l\mathbf{M}}(e) = (-1)^{m_1} Y_{l\bar{\mathbf{M}}}(e) \quad \text{where} \quad \bar{\mathbf{M}} = (m_{d-2}, \cdots, m_2, -m_1).$

2. $Y_{l_1\,\mathbf{M}_1}(e) Y_{l_2\,\mathbf{M}_2}(e) = \sum_{l_3,\mathbf{M}_3} \binom{l_1\,l_2\,l_3}{\mathbf{M}_1\,\mathbf{M}_2\,\mathbf{M}_3} \binom{l_1\,l_2\,l_3}{\mathbf{0}\,\mathbf{0}\,\mathbf{0}} Y^*_{l_3\,\mathbf{M}_3}(e)$

3. $\binom{l_1\,l_2\,l_3}{0\,0\,0} = 0$ unless $l_1 + l_2 + l_3$ is an even integer and $l_3 = |l_1 - l_2|, \cdots, l_1 + l_2$.

4. $\binom{l\,l'\,0}{\mathbf{M}\,\mathbf{M}'\,0} = \frac{(-1)^{l-m1}}{\sqrt{d_l}} \delta_{l\,l'} \delta_{\mathbf{M}\,\mathbf{M}'}$

5. $\sum_{\{\mathbf{m}_i\}} (-1)^{(\mathbf{m}_2)_1} \binom{l_1\,l_3\,L_2}{\mathbf{m}_1\,\mathbf{m}_3\,\mathbf{M}_2} \binom{l_1\,l_2\,L_3}{\bar{\mathbf{m}}_1\,\mathbf{m}_2\,\mathbf{M}_3} \binom{l_2\,l_3\,L_1}{\bar{\mathbf{m}}_2\,\bar{\mathbf{m}}_3\,\mathbf{M}_1} = (-1)^{l_2+L_2-L_3} \binom{L_3\,L_1\,L_2}{\bar{\mathbf{M}}_3\,\bar{\mathbf{M}}_1\,\mathbf{M}_2} \begin{Bmatrix} L_1 & L_2 & L_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}$

$so(1, d+1)$ CGC for Scalar irreps

The prescription of [14] for the 3-point function of scalar primaries is to extract the leading term, *i.e,* the coefficient of $e^{-\rho(\Delta_1+\Delta_2+\Delta_3)}$ term – in the boundary limit of

$$\langle\!\langle \Delta_1 | g(x_1) | \Delta_1, \mathbf{m}_1 \rangle \, \langle\!\langle \Delta_2 | g(x_2) | \Delta_2, \mathbf{m}_2 \rangle \, C^{\Delta_1,\Delta_2;\Delta_3}_{\mathbf{m}_1,\mathbf{m}_2;\mathbf{m}_3} \langle \Delta_3, \mathbf{m}_3 | g^{-1}(x_3) | \Delta_3 \rangle\!\rangle \tag{B.7}$$

We show that this quantity satisfies the conformal Ward identity. To see this we note the following identities:

$$\begin{aligned} g(x)\, M_{AB} &= l^\mu_{AB}(x) \partial_\mu g(x) + \frac{1}{2} M_{bc} g(x) \left[\omega^{bc}_\mu(x) l^\mu_{AB}(x) + (R[g(x)])^{bc}{}_{AB} \right] \\ M_{AB} g^{-1}(x) &= -l^\mu_{AB}(x) \partial_\mu g^{-1}(x) + \frac{1}{2} \left[\omega^{bc}_\mu(x) l^\mu_{AB}(x) + (R[g(x)])^{bc}{}_{AB} \right] g^{-1}(x) M_{bc} \end{aligned} \tag{B.8}$$

where the $l^\mu_{AB}(x)$ are the components of the Killing vector of the background geometry (4.1) carrying the indices of the corresponding $so(1, d+1)$ algebra generator $M_{AB} \in \{M_{0a}, M_{ab}\}$ of the left hand side. Next we consider:

$$\begin{aligned} &\langle\!\langle \Delta_1 | g(x_1) M_{AB} | \Delta_1, \mathbf{m}_1 \rangle \, \langle\!\langle \Delta_2 | g(x_2) | \Delta_2, \mathbf{m}_2 \rangle \, C^{\Delta_1,\Delta_2;\Delta_3}_{\mathbf{m}_1,\mathbf{m}_2;\mathbf{m}_3} \langle \Delta_3, \mathbf{m}_3 | g^{-1}(x_3) | \Delta_3 \rangle\!\rangle \\ &+ \langle\!\langle \Delta_1 | g(x_1) | \Delta_1, \mathbf{m}_1 \rangle \, \langle\!\langle \Delta_2 | g(x_2) M_{AB} | \Delta_2, \mathbf{m}_2 \rangle \, C^{\Delta_1,\Delta_2;\Delta_3}_{\mathbf{m}_1,\mathbf{m}_2;\mathbf{m}_3} \langle \Delta_3, \mathbf{m}_3 | g^{-1}(x_3) | \Delta_3 \rangle\!\rangle \\ &- \langle\!\langle \Delta_1 | g(x_1) | \Delta_1, \mathbf{m}_1 \rangle \, \langle\!\langle \Delta_2 | g(x_2) | \Delta_2, \mathbf{m}_2 \rangle \, C^{\Delta_1,\Delta_2;\Delta_3}_{\mathbf{m}_1,\mathbf{m}_2;\mathbf{m}_3} \langle \Delta_3, \mathbf{m}_3 | M_{AB} g^{-1}(x_3) | \Delta_3 \rangle\!\rangle \end{aligned} \tag{B.9}$$

which vanishes identically as a consequence of the recursion relation (2.16) for the CGC. On the other hand using the identities (3) above and the fact that the scalar cap is killed by M_{ab}'s we see that the OWN for the 3-point function (B.7) is invariant under simultaneous transformation of the three bulk points (x_1, x_2, x_3) under any AdS_{d+1} isometry. This in turn implies the conformal Ward identity in the limit of the external points x_i approaching the boundary. Therefore, the question of finding the CGC is translated into finding expressions for the quantities $\langle\!\langle \Delta|g(x)|\Delta, \mathbf{m}\rangle$ and $\langle \Delta, \mathbf{m}|g^{-1}(x)|\Delta\rangle\!\rangle$ in the large radius limit, and then amputating them from the corresponding 3-point function.

We extract CG coefficients for $so(1, d+1)$ for three scalars from the three-point function for scalar primary operators amputating the out-going and in-going legs we had found earlier. The out-going and in-going legs takes the following forms respectively

$$\begin{aligned}\lim_{\rho\to\infty} e^{\rho\Delta}\langle\!\langle \Delta|g(x)|\Delta;\{l,\mathbf{m},s\}\rangle &= \frac{2^{l+2s}}{A_{l,s}}\frac{\Gamma(\Delta+s+l)\,\Gamma(\Delta+s-\mu)}{\Gamma(\Delta)\,\Gamma(\Delta-\mu)}\,M^l_{\mathbf{m}}(\mathbf{x})\,(x^2)^{-\Delta-l-s}\\ &= \left[\frac{(\Delta)_{l+s}(\Delta-\mu)_s}{(\mu+1)_{l+s}s!}\right]^{1/2}(x^2)^{-\Delta-l-s}\,M^l_{\mathbf{m}}(\mathbf{x})\end{aligned} \tag{B.10}$$

and

$$\begin{aligned}\lim_{\rho\to\infty} e^{\rho\Delta}\langle \Delta;\{l,\mathbf{m},s\}|g^{-1}(y)|\Delta\rangle\!\rangle &= \frac{A_{l,s}}{2^{l+2s}}\frac{(y^2)^s}{(s)!\,\Gamma(l+s+3/2)}\,M^{l^*}_{\mathbf{m}}(\mathbf{y})\\ &= \left[\frac{(\Delta)_{l+s}(\Delta-\mu)_s}{(\mu+1)_{l+s}s!}\right]^{1/2}(y^2)^s\,M^{l^*}_{\mathbf{m}}(\mathbf{y})\end{aligned} \tag{B.11}$$

The 3-point function of the scalar primary operators with conformal dimensions Δ_1, Δ_2 and Δ_3

$$\frac{1}{|x_2-x_1|^{\Delta_1+\Delta_2-\Delta_3}\,|x_3-x_2|^{\Delta_2+\Delta_3-\Delta_1}\,|x_3-x_1|^{\Delta_1+\Delta_3-\Delta_2}} \tag{B.12}$$

can be expanded as

$$\left(4\pi^{d/2}\right)^3 \prod_{i=1}^{3} \sum_{l_i=0}^{\infty} \sum_{s_i=0}^{\infty} \sum_{\mathbf{m}_i} \frac{(\varDelta_{12}/2)_{l_1+s_1} (\varDelta_{12}/2-\mu)_{s1}}{(\mu+1)_{l_1+s_1} s_1!} \frac{(\varDelta_{23}/2)_{l_2+s_2} (\varDelta_{23}/2-\mu)_{s2}}{(\mu+1)_{l_2+s_2} s_2!}$$

$$\times \frac{(\varDelta_{31}/2)_{l_3+s_3} (\varDelta_{31}/2-\mu)_{s3}}{(\mu+1)_{l_3+s_3} s_3!} (x^2)^{-\varDelta_1-l_1-l_3-s_1-s_3} (y^2)^{-\varDelta_{23}/2-l_2+s_1-s_2} (z^2)^{s_2+s_3}$$

$$\times M^{l_1}_{\mathbf{m}_1}(\mathbf{x})\, M^{l_3}_{\mathbf{m}_3}(\mathbf{x})\, M^{l_1^*}_{\mathbf{m}_1}(\mathbf{y})\, M^{l_2}_{\mathbf{m}_2}(\mathbf{y})\, M^{l_2^*}_{\mathbf{m}_2}(\mathbf{z})\, M^{l_3^*}_{\mathbf{m}_3}(\mathbf{z}) \tag{B.13}$$

where

$$\varDelta_{12} \equiv \varDelta_1 + \varDelta_2 - \varDelta_3, \quad \varDelta_{23} \equiv \varDelta_2 + \varDelta_3 - \varDelta_1, \quad \varDelta_{31} \equiv \varDelta_3 + \varDelta_1 - \varDelta_2 \tag{B.14}$$

We use the following identities:

$$M^{l}_{\mathbf{m}}(\mathbf{x}) M^{l'}_{\mathbf{m}'}(\mathbf{x}) = \sum_{L,M} \begin{pmatrix} l\; l'\; L \\ \mathbf{m}\; \mathbf{m}'\; \mathbf{M} \end{pmatrix} \begin{pmatrix} l\; l'\; L \\ \mathbf{0\,0\,0} \end{pmatrix} (x^2)^{\frac{l+l'-L}{2}} M^{L^*}_{\mathbf{M}}(\mathbf{x}) \tag{B.15}$$

$$M^{l^*}_{\mathbf{m}}(\mathbf{x}) = (-\mathbf{1})^{\mathbf{m}_1} \mathbf{M}^{\mathbf{l}}_{\bar{\mathbf{m}}}(\mathbf{x}) \tag{B.16}$$

where, $\bar{\mathbf{m}} = (m_{n-2}, ..., m_2, -m_1)$ to rewrite the product of spherical harmonics in the summand as

$$M^{l_1}_{\mathbf{m}_1}(\mathbf{x}) M^{l_3}_{\mathbf{m}_3}(\mathbf{x}) M^{l_1^*}_{\mathbf{m}_1}(\mathbf{y}) M^{l_2}_{\mathbf{m}_2}(\mathbf{y}) M^{l_2^*}_{\mathbf{m}_2}(\mathbf{z}) M^{l_3^*}_{\mathbf{m}_3}(\mathbf{z})$$

$$= (-1)^{m_1+m_2+m_3} M^{l_1}_{\mathbf{m}_1}(\mathbf{x}) M^{l_3}_{\mathbf{m}_3}(\mathbf{x}) M^{l_1}_{\bar{\mathbf{m}}_1}(\mathbf{y}) M^{l_2}_{\mathbf{m}_2}(\mathbf{y}) M^{l_2}_{\bar{\mathbf{m}}_2}(\mathbf{z}) M^{l_3}_{\bar{\mathbf{m}}_3}(\mathbf{z})$$

$$= (-1)^{m_2} \prod_{i=1}^{3} \sum_{L_i,M_i} (-1)^{M_2} \begin{pmatrix} l_1\; l_3\; L_2 \\ \mathbf{0\,0\,0} \end{pmatrix} \begin{pmatrix} l_1\; l_2\; L_3 \\ \mathbf{0\,0\,0} \end{pmatrix} \begin{pmatrix} l_2\; l_3\; L_1 \\ \mathbf{0\,0\,0} \end{pmatrix}$$

$$\times \begin{pmatrix} l_1\, l_3\, L_2 \\ \mathbf{m}_1\, \mathbf{m}_3\, \mathbf{M}_2 \end{pmatrix} \begin{pmatrix} l_1\, l_2\, L_3 \\ \bar{\mathbf{m}}_1\, \mathbf{m}_2\, \mathbf{M}_3 \end{pmatrix} \begin{pmatrix} l_2\, l_3\, L_1 \\ \bar{\mathbf{m}}_2\, \bar{\mathbf{m}}_3\, \mathbf{M}_1 \end{pmatrix}$$

$$\times (x^2)^{\frac{l_1+l_3-L_2}{2}} (y^2)^{\frac{l_1+l_2-L_3}{2}} (z^2)^{\frac{l_2+l_3-L_1}{2}} M^{L_2}_{\mathbf{M}_2}(\mathbf{x})\, M^{L_3}_{\mathbf{M}_3}(\mathbf{y})\, M^{L_1}_{\mathbf{M}_1}(\mathbf{z}) \qquad \text{(B.17)}$$

Inserting the above relation in the summand and performing the $\{m_i\}$ summations we get

$$\begin{aligned} &\sum_{\{\mathbf{m}_i\}} (-1)^{m_2} \begin{pmatrix} l_1\, l_3\, L_2 \\ \mathbf{m}_1\, \mathbf{m}_3\, \mathbf{M}_2 \end{pmatrix} \begin{pmatrix} l_1\, l_2\, L_3 \\ \bar{\mathbf{m}}_1\, \mathbf{m}_2\, \mathbf{M}_3 \end{pmatrix} \begin{pmatrix} l_2\, l_3\, L_1 \\ \bar{\mathbf{m}}_2\, \bar{\mathbf{m}}_3\, \mathbf{M}_1 \end{pmatrix} \\ &= (-1)^{l_2+L_2-L_3} \begin{pmatrix} L_3\, L_1\, L_2 \\ \bar{\mathbf{M}}_3\, \bar{\mathbf{M}}_1\, \mathbf{M}_2 \end{pmatrix} \begin{Bmatrix} l_2 & l_1 & L_3 \\ L_2 & L_1 & l_3 \end{Bmatrix} \\ &= (-1)^{l_2+L_2-L_3} \begin{pmatrix} L_3\, L_1\, L_2 \\ \bar{\mathbf{M}}_3\, \bar{\mathbf{M}}_1\, \mathbf{M}_2 \end{pmatrix} \begin{Bmatrix} L_1 & L_2 & L_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \end{aligned} \qquad \text{(B.18)}$$

Now the three-point function takes the form

$$\left(4\pi^{d/2}\right)^3 \prod_{i=1}^{3} \sum_{l_i=0}^{\infty} \sum_{s_i=0}^{\infty} \sum_{\mathbf{m}_i} \frac{(\Delta_{12}/2)_{l_1+s_1}(\Delta_{12}/2-\mu)_{s1}}{(\mu+1)_{l_1+s_1} s_1!} \frac{(\Delta_{23}/2)_{l_2+s_2}(\Delta_{23}/2-\mu)_{s2}}{(\mu+1)_{l_2+s_2} s_2!}$$

$$\times \frac{(\Delta_{31}/2)_{l_3+s_3}(\Delta_{31}/2-\mu)_{s3}}{(\mu+1)_{l_3+s_3} s_3!} \prod_{i=1}^{3} \sum_{L_i, \mathbf{M}_i} (-1)^{l_2+L_2-L_3+M_2}$$

$$(x^2)^{-\Delta_1-l_1-l_3-s_1-s_3+\frac{l_1+l_3-L_2}{2}} (y^2)^{-\Delta_{23}/2-l_2+s_1-s_2+\frac{l_1+l_2-L_3}{2}} (z^2)^{s_2+s_3+\frac{l_2+l_3-L_1}{2}} M^{L_2}_{\mathbf{M}_2}(\mathbf{x})\, M^{L_3}_{\mathbf{M}_3}(\mathbf{y})\, M^{L_1}_{\mathbf{M}_1}(\mathbf{z})$$

$$\begin{pmatrix} l_1\, l_3\, L_2 \\ \mathbf{0\,0\,0} \end{pmatrix} \begin{pmatrix} l_1\, l_2\, L_3 \\ \mathbf{0\,0\,0} \end{pmatrix} \begin{pmatrix} l_2\, l_3\, L_1 \\ \mathbf{0\,0\,0} \end{pmatrix} \begin{pmatrix} L_3\, L_1\, L_2 \\ \bar{\mathbf{M}}_3\, \bar{\mathbf{M}}_1 \mathbf{M}_2 \end{pmatrix} \begin{Bmatrix} L_1 & L_2 & L_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}$$

which can also be written as

$$\left(4\pi^{d/2}\right)^3 \prod_{i=1}^{3} \sum_{l_i=0}^{\infty} \sum_{s_i=0}^{\infty} \sum_{\mathbf{m}_i} \frac{(\Delta_{12}/2)_{l_1+s_1}(\Delta_{12}/2-\mu)_{s1}}{(\mu+1)_{l_1+s_1} s_1!} \frac{(\Delta_{23}/2)_{l_2+s_2}(\Delta_{23}/2-\mu)_{s2}}{(\mu+1)_{l_2+s_2} s_2!}$$

$$\frac{(\Delta_{31}/2)_{l_3+s_3}(\Delta_{31}/2-\mu)_{s3}}{(\mu+1)_{l_3+s_3} s_3!} \prod_{i=1}^{3} \sum_{L_i, M_i} (-1)^{l_2+L_2-L_3}$$

$$(x^2)^{-\Delta_1-l_1-l_3-s_1-s_3+\frac{l_1+l_3-L_2}{2}}\,(y^2)^{-\Delta_{23}/2-l_2+s_1-s_2+\frac{l_1+l_2-L_3}{2}}\,(z^2)^{s_2+s_3+\frac{l_2+l_3-L_1}{2}}\,M^{L_2}_{\bar{\mathbf{M}}_2}(\mathbf{x})\,M^{L_3^*}_{\bar{\mathbf{M}}_3}(\mathbf{y})\,M^{L_1^*}_{\bar{\mathbf{M}}_1}(\mathbf{z})$$

$$\begin{pmatrix} l_1\, l_3\, L_2 \\ \mathbf{0\,0\,0} \end{pmatrix}\begin{pmatrix} l_1\, l_2\, L_3 \\ \mathbf{0\,0\,0} \end{pmatrix}\begin{pmatrix} l_2\, l_3\, L_1 \\ \mathbf{0\,0\,0} \end{pmatrix}\begin{pmatrix} L_3\, L_1\, L_2 \\ \bar{\mathbf{M}}_3\,\bar{\mathbf{M}}_1\mathbf{M}_2 \end{pmatrix}\begin{Bmatrix} L_1 & L_2 & L_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}$$

Note that the $so(d)$ 3 j- coefficients $\binom{l\,l'\,L}{0\,0\,0}$ is non-vanishing only when $l + l' - L$ is even integer. This suggests to change the following variables as

$$l_1 + l_3 = 2K_2 + L_2, \quad l_1 + l_2 = 2K_3 + L_3, \quad l_2 + l_3 = 2K_1 + L_1 \tag{B.19}$$

i.e.

$$\begin{aligned} l_1 &= \frac{L_2 + L_3 - L_1}{2} + K_2 + K_3 - K_1 \\ l_2 &= \frac{L_3 + L_1 - L_2}{2} + K_3 + K_1 - K_2 \\ l_3 &= \frac{L_1 + L_2 - L_3}{2} + K_1 + K_2 - K_3 \end{aligned} \tag{B.20}$$

Then the powers of x^2, y^2, z^2 becomes (excluding the powers within the spherical harmonics)

$$(x^2)^{-\Delta_1-L_2-K_2-s_1-s_3}\,(y^2)^{-\frac{\Delta_{23}}{2}-\frac{L_3+L_1-L_2}{2}+s_1-s_2-K_1+K_2}\,(z^2)^{s_2+s_3+K_1} \tag{B.21}$$

respectively. Comparing with the legs we want to amputate from the three-point function we make the following change of variables in the summand

$$K_2 + s_1 + s_3 = S_2 \tag{B.22}$$

$$K_1 + s_2 + s_3 = S_1 \tag{B.23}$$

$$-\frac{\Delta_{23}}{2} - \frac{L_3 + L_1 - L_2}{2} + s_1 - s_2 - K_1 + K_2 = S_3 \tag{B.24}$$

The last one of the above relations impose the following selection rule

$$(\Delta_2 + L_1 + 2S_1) + (\Delta_3 + L_3 + 2S_3) = (\Delta_1 + L_2 + 2S_2) \tag{B.25}$$

So S_3 is not an independent variable and s_3 is undetermined in terms of new variables. We call it $s_3 = S$. In terms of the new variables the three-point function becomes

$$\frac{\left(4\pi^{d/2}\right)^3}{\Gamma(\Delta_{12}/2)\Gamma(\Delta_{23}/2)\Gamma(\Delta_{31}/2)\Gamma(\Delta_{12}/2-\mu)\Gamma(\Delta_{23}/2-\mu)\Gamma(\Delta_{31}/2-\mu)} \times \prod_{i=1}^{3}\sum_{L_i=0}^{\infty}\sum_{S_i=0}^{\infty}\sum_{M_i}$$

$$\delta(\Delta_2 + L_1 + 2S_1 + \Delta_3 + L_3 + 2S_3 - \Delta_1 - L_2 - 2S_2)\sum_{K_3=0}^{\infty}\sum_{K_1=0}^{S_1}\sum_{K_2=0}^{S_2}\sum_{S=0}^{\min(S_2-K_2,S_1-K_1)}$$

$$\frac{\Gamma(\Delta_2 + L_3 + S_1 + S_3 + K_3 - S - K_1)\,\Gamma(\Delta_{12}/2 + S_2 - K_2 - S - \mu)}{\Gamma(\frac{L_2+L_3-L_1}{2} + K_3 - K_1 + S_2 - S + d/2)\,(S_2 - K_2 - S)!}$$

$$\times\frac{\Gamma(K_3 - K_2 + S_2 - S_3 - S)\,\Gamma(\Delta_{23}/2 + S_1 - K_1 - S - \mu)}{\Gamma(\frac{L_3+L_1-L_2}{2} + K_3 - K_2 + S_1 - S + d/2)\,(S_1 - K_1 - S)!}$$

$$\times\frac{\Gamma(\Delta_3 + L_1 + S_1 + S_3 - S_2 + K_1 + K_2 - K_3 + S)\,\Gamma(\Delta_{31}/2 + S - \mu)}{\Gamma(\frac{L_1+L_2-L_3}{2} + K_1 + K_2 - K_3 + S + d/2)\,S!}$$

$$\times(-1)^{\frac{L_1+L_2-L_3}{2}+K_3+K_1-K_2}\begin{pmatrix}\frac{L_2+L_3-L_1}{2}+K_2+K_3-K_1, & \frac{L_1+L_2-L_3}{2}+K_1+K_2-K_3, & L_2\\ 0 & 0 & 0\end{pmatrix}$$

$$\times\begin{pmatrix}\frac{L_2+L_3-L_1}{2}+K_2+K_3-K_1, & \frac{L_3+L_1-L_2}{2}+K_3+K_1-K_2, & L_3\\ 0 & 0 & 0\end{pmatrix}$$

$$\times\begin{pmatrix}\frac{L_3+L_1-L_2}{2}+K_3+K_1-K_2, & \frac{L_1+L_2-L_3}{2}+K_1+K_2-K_3, & L_1\\ 0 & 0 & 0\end{pmatrix}$$

$$\times\begin{Bmatrix}L_1 & L_2 & L_3\\ \frac{L_2+L_3-L_1}{2}+K_2+K_3-K_1 & \frac{L_3+L_1-L_2}{2}+K_3+K_1-K_2 & \frac{L_1+L_2-L_3}{2}+K_1+K_2-K_3\end{Bmatrix}$$

$$\times\begin{pmatrix}L_3\,L_1\,L_2\\ \mathbf{M}_3\,\mathbf{M}_1\,\mathbf{M}_2\end{pmatrix}(x^2)^{-\Delta_1-L_2-S_2}(y^2)^{S_1}(z^2)^{S_3}\,M^{L_2^*}_{-\mathbf{M}_2}(\mathbf{x})\,M^{L_3}_{-\mathbf{M}_3}(\mathbf{y})\,M^{L_1}_{-\mathbf{M}_1}(\mathbf{z}) \tag{B.26}$$

where we have arranged the order as well as the limits of the summations appropriately. According to our prescription the three-point function can be recovered as

$$\prod_{i=1}^{3}\sum_{L_i=0}^{\infty}\sum_{S_i=0}^{\infty}\sum_{\mathbf{M}_i}\langle\!\langle \Delta_1|g(x)|\Delta_1;\{L_2,\mathbf{M}_2,S_2\}\rangle\,\langle\Delta_2;\{L_1,\mathbf{M}_1,S_1\}|g^{-1}(y)|\Delta_2\rangle\!\rangle$$
$$\langle\Delta_3;\{L_3,\mathbf{M}_3,S_3\}|g^{-1}(z)|\Delta_3\rangle\!\rangle\, C^{(\Delta_2,S_1),(\Delta_3,S_3);(\Delta_1,S_2)}_{(L_1,\mathbf{M}_1),(L_3,\mathbf{M}_3);(L_2,\mathbf{M}_2)} \quad \text{(B.27)}$$

where $C^{(\Delta_2,S_1),(\Delta_3,S_3);(\Delta_1,S_2)}_{(L_1,\mathbf{M}_1),(L_3,\mathbf{M}_3);(L_2,\mathbf{M}_2)}$ is $so(1,d+1)$ CG coefficient. Comparing above with the three-point function we write

$$C^{(\Delta_2,S_1),(\Delta_3,S_3);(\Delta_1,S_2)}_{(L_1,\mathbf{M}_1),(L_3,\mathbf{M}_3);(L_2,\mathbf{M}_2)}$$
$$= \frac{\left(4\pi^{d/2}\right)^3\,\delta(\Delta_2+L_1+2S_1+\Delta_3+L_3+2S_3-\Delta_1-L_2-2S_2)}{\Gamma(\Delta_{12}/2)\Gamma(\Delta_{23}/2)\Gamma(\Delta_{31}/2)\Gamma(\Delta_{12}/2-\mu)\Gamma(\Delta_{23}/2-\mu)\Gamma(\Delta_{31}/2-\mu)}\times$$
$$\times\left[\frac{\Gamma(\Delta_1+L_2+S_2)\,\Gamma(\Delta_1+S_2-\mu)}{\Gamma(\Delta_1)\Gamma(\Delta_1-\mu)\Gamma(L_2+S_2+d/2)S_2!}\right]^{1/2}\left[\frac{\Gamma(\Delta_2+L_1+S_1)\,\Gamma(\Delta_2+S_1-\mu)}{\Gamma(\Delta_2)\Gamma(\Delta_2-\mu)\Gamma(L_1+S_1+d/2)S_1!}\right]^{1/2}$$
$$\times\left[\frac{\Gamma(\Delta_3+L_3+S_3)\,\Gamma(\Delta_3+S_3-\mu)}{\Gamma(\Delta_3)\Gamma(\Delta_3-\mu)\Gamma(L_3+S_3+d/2)S_3!}\right]^{1/2}\sum_{K_3=0}^{\infty}\sum_{K_1=0}^{S_1}\sum_{K_2=0}^{S_2}\sum_{S=0}^{min(S_2-K_2,S_1-K_1)}$$
$$\frac{\Gamma(\Delta_2+L_3+S_1+S_3+K_3-S-K_1)\,\Gamma(\Delta_{12}/2+S_2-K_2-S-\mu)}{\Gamma(\frac{L_2+L_3-L_1}{2}+K_3-K_1+S_2-S+d/2)\,(S_2-K_2-S)!}$$
$$\times\frac{\Gamma(K_3-K_2+S_2-S_3-S)\,\Gamma(\Delta_{23}/2+S_1-K_1-S-\mu)}{\Gamma(\frac{L_3+L_1-L_2}{2}+K_3-K_2+S_1-S+d/2)\,(S_1-K_1-S)!}$$
$$\times\frac{\Gamma(\Delta_3+L_1+S_1+S_3-S_2+K_1+K_2-K_3+S)\,\Gamma(\Delta_{31}/2+S-\mu)}{\Gamma(\frac{L_1+L_2-L_3}{2}+K_1+K_2-K_3+S+d/2)\,S!}$$
$$\times(-1)^{\frac{L_1+L_2-L_3}{2}+K_3+K_1-K_2}\begin{pmatrix}\frac{L_2+L_3-L_1}{2}+K_2+K_3-K_1, & \frac{L_1+L_2-L_3}{2}+K_1+K_2-K_3, & L_2\\ 0 & 0 & 0\end{pmatrix}$$
$$\times\begin{pmatrix}\frac{L_2+L_3-L_1}{2}+K_2+K_3-K_1, & \frac{L_3+L_1-L_2}{2}+K_3+K_1-K_2, & L_3\\ 0 & 0 & 0\end{pmatrix}$$

$$
\times \begin{pmatrix} \frac{L_3+L_1-L_2}{2} + K_3 + K_1 - K_2, & \frac{L_1+L_2-L_3}{2} + K_1 + K_2 - K_3, & L_1 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
\times \begin{Bmatrix} L_1 & L_2 & L_3 \\ \frac{L_2+L_3-L_1}{2} + K_2 + K_3 - K_1 & \frac{L_3+L_1-L_2}{2} + K_3 + K_1 - K_2 & \frac{L_1+L_2-L_3}{2} + K_1 + K_2 - K_3 \end{Bmatrix}
$$

$$
\times \begin{pmatrix} L_3 & L_1 & L_2 \\ \mathbf{M}_3 & \mathbf{M}_1 & \mathbf{M}_2 \end{pmatrix} \tag{B.28}
$$

Appendix C

Matrix elements

We compute matrix elements of $so(1,4)$ generators in the orthonormal basis we found. The dilatation generator acts on a basis state as:

$$D|\Delta;\{l,m,s\}\rangle = (\Delta + l + 2s)|\Delta;\{l,m,s\}\rangle \tag{C.1}$$

Therfore

$$\langle\Delta;\{l',m',s'\}|\,D\,|\Delta;\{l,m,s\}\rangle = (\Delta + l + 2s)\,\delta_{ll'}\,\delta_{mm'}\,\delta_{ss'} \tag{C.2}$$

To find the matrix elements for the translation and special conformal generators we have to rewrite the generators in terms of spherical tensor using the identities:

$$M^1_{-1}(\mathbf{P}) = \sqrt{\frac{3}{8\pi}}\,(P_1 - i\,P_2) \tag{C.3}$$

$$M^1_0(\mathbf{P}) = \sqrt{\frac{3}{4\pi}}\,P_3 \tag{C.4}$$

$$M^1_1(\mathbf{P}) = -\sqrt{\frac{3}{8\pi}}\,(P_1 + i\,P_2) \tag{C.5}$$

as

$$P_1 = \sqrt{\frac{2\pi}{3}}\left(M^1_{-1}(\mathbf{P}) - M^1_1(\mathbf{P})\right) \quad \text{(C.6)}$$

$$P_2 = -i\sqrt{\frac{2\pi}{3}}\left(M^1_{-1}(\mathbf{P}) + M^1_1(\mathbf{P})\right) \quad \text{(C.7)}$$

$$P_3 = \sqrt{\frac{4\pi}{3}}\, M^1_0(\mathbf{P}) \quad \text{(C.8)}$$

Similarly the special transformation generators K_i's can be rewritten in terms of $M^1_m(\mathbf{K})$ with $m \in \{-1, 0, 1\}$.

Let us compute the matrix element:

$$\begin{aligned}
&\langle \mathit{\Delta}; \{l', m', s'\}|\, M^1_{-1}(\mathbf{P})\, |\mathit{\Delta}; \{l, m, s\}\rangle \\
&= \frac{1}{A_{l,s}} \langle \mathit{\Delta}; \{l', m', s'\}|(P^2)^s\, M^1_{-1}(\mathbf{P})\, M^l_m(\mathbf{P})\, |\mathit{\Delta}\rangle \\
&= \frac{1}{A_{l,s}} \sum_L \sqrt{\frac{3(2l+1)}{4\pi(2L+1)}} \langle 1,\, 0;\, l,\, 0|\, 1,\, l;\, L,\, 0\rangle \langle 1,\, -1;\, l,\, m|\, 1,\, l;\, L,\, m-1\rangle \\
&\qquad \times \langle \mathit{\Delta}; \{l', m', s'\}|(P^2)^{s+\frac{1+l-L}{2}}\, M^L_{m-1}(\mathbf{P})\, |\mathit{\Delta}\rangle
\end{aligned} \quad \text{(C.9)}$$

where we have used the identity (B.15). Now using the definition of the basis states we find

$$\begin{aligned}
&\langle \mathit{\Delta}; \{l', m', s'\}|\, M^1_{-1}(\mathbf{P})\, |\mathit{\Delta}; \{l, m, s\}\rangle \\
&= \frac{1}{A_{l,s}} \sum_L \sqrt{\frac{3(2l+1)}{4\pi(2L+1)}} \langle 1,\, 0;\, l,\, 0|\, 1,\, l;\, L,\, 0\rangle \langle 1,\, -1;\, l,\, m|\, 1,\, l;\, L,\, m-1\rangle \\
&\qquad \times A_{L,\, m-1,\, s+\frac{1+l-L}{2}} \left\langle \mathit{\Delta}; \{l', m', s'\} \middle| \mathit{\Delta}; \left\{L,\, m-1,\, s + \frac{1+l-L}{2}\right\}\right\rangle \\
&= \frac{1}{A_{l,s}} \sum_L \sqrt{\frac{3(2l+1)}{4\pi(2L+1)}} \langle 1,\, 0;\, l,\, 0|\, 1,\, l;\, L,\, 0\rangle \langle 1,\, -1;\, l,\, m|\, 1,\, l;\, L,\, m-1\rangle \\
&\qquad \times A_{L,\, s+\frac{1+l-L}{2}}\, \delta_{Ll'}\, \delta_{m'(m-1)}\, \delta\left(s + \frac{1+l-L}{2} - s'\right)
\end{aligned} \quad \text{(C.10)}$$

Performing the sum over L with $\delta_{Ll'}$ we find

$$\langle\varDelta;\{l',m',s'\}|\,M^1_{-1}(\mathbf{P})\,|\varDelta;\{l,m,s\}\rangle$$

$$= \frac{A_{l',\,s+\frac{1+l-l'}{2}}}{A_{l,s}}\sqrt{\frac{3(2l+1)}{4\pi(2l'+1)}}\,\langle 1,\,0;\,l,\,0|\,1,\,l;\,l',\,0\rangle\,\langle 1,\,-1;\,l,\,m|\,1,\,l;\,l',\,m-1\rangle$$
$$\times\,\delta_{m'(m-1)}\,\delta\left(s+\frac{1+l-l'}{2}-s'\right) \qquad \text{(C.11)}$$

Similarly

$$\langle\varDelta;\{l',m',s'\}|\,M^1_{1}(\mathbf{P})\,|\varDelta;\{l,m,s\}\rangle$$

$$= \frac{A_{l',\,s+\frac{1+l-l'}{2}}}{A_{l,s}}\sqrt{\frac{3(2l+1)}{4\pi(2l'+1)}}\,\langle 1,\,0;\,l,\,0|\,1,\,l;\,l',\,0\rangle\,\langle 1,\,1;\,l,\,m|\,1,\,l;\,l',\,m+1\rangle$$
$$\times\,\delta_{m'(m+1)}\,\delta\left(s+\frac{1+l-l'}{2}-s'\right) \qquad \text{(C.12)}$$

and

$$\langle\varDelta;\{l',m',s'\}|\,M^1_{0}(\mathbf{P})\,|\varDelta;\{l,m,s\}\rangle$$

$$= \frac{A_{l',\,s+\frac{1+l-l'}{2}}}{A_{l,s}}\sqrt{\frac{3(2l+1)}{4\pi(2l'+1)}}\,\langle 1,\,0;\,l,\,0|\,1,\,l;\,l',\,0\rangle\,\langle 1,\,0;\,l,\,m|\,1,\,l;\,l',\,m\rangle$$
$$\times\,\delta_{mm'}\,\delta\left(s+\frac{1+l-l'}{2}-s'\right) \qquad \text{(C.13)}$$

Similarly we compute the matrix elements for $M^1_m(\mathbf{K})$ for the special conformal transformatio generators. For example,

$$\langle\varDelta;\{l',m',s'\}|\,M^1_{0}(\mathbf{K})\,|\varDelta;\{l,m,s\}\rangle$$

$$= \frac{1}{A_{l',s'}}\langle\varDelta|(K^2)^{s'}\,M^{l'}_{m'}(\mathbf{K})M^1_0(\mathbf{K})\,|\varDelta;\{l,m,s\}\rangle$$

$$= \frac{1}{A_{l',s'}} \sum_L \sqrt{\frac{3(2l'+1)}{4\pi(2L+1)}} \langle 1,\, 0;\, l',\, 0 | 1,\, l';\, L,\, 0 \rangle \langle 1,\, 0;\, l',\, m' | 1,\, l';\, L,\, m' \rangle$$

$$\times \langle \varDelta | (K^2)^{s' + \frac{1+l'-L}{2}} \, M^L_{m'}(\mathbf{K}) \, | \varDelta; \{l, m, s\} \rangle$$

$$= \frac{1}{A_{l',s'}} \sum_L \sqrt{\frac{3(2l'+1)}{4\pi(2L+1)}} \langle 1,\, 0;\, l',\, 0 | 1,\, l';\, L,\, 0 \rangle \langle 1,\, 0;\, l',\, m' | 1,\, l';\, L,\, m' \rangle$$

$$\times A_{L,\, s' + \frac{1+l'-L}{2}} \, \delta_{Ll} \, \delta_{mm'} \, \delta\left(s' + \frac{1 + l' - L}{2} - s \right) \qquad \text{(C.14)}$$

Therefore

$$\langle \varDelta; \{l', m', s'\} | \, M^1_0(\mathbf{K}) \, | \varDelta; \{l, m, s\} \rangle$$

$$= \frac{A_{l,\, s' + \frac{1+l'-l}{2}}}{A_{l',s'}} \sqrt{\frac{3(2l'+1)}{4\pi(2l+1)}} \langle 1,\, 0;\, l',\, 0 | 1,\, l';\, l,\, 0 \rangle \langle 1,\, 0;\, l',\, m' | 1,\, l';\, l,\, m' \rangle$$

$$\times \delta_{mm'} \, \delta\left(s' + \frac{1 + l' - l}{2} - s \right) \qquad \text{(C.15)}$$

Similarly we can compute the matrix elements for other spherical harmonics, $M^1_m(\mathbf{K})$ using the above results and the following replacements

$$(\, l \longleftrightarrow l' \,), \qquad (\, m \longleftrightarrow m' \,), \qquad (\, s \longleftrightarrow s' \,) \Longrightarrow (\, \mathbf{P} \longleftrightarrow \mathbf{K} \,) \qquad \text{(C.16)}$$

Now we consider the matrix elements for the rotation generators which takes the familiar forms

$$\langle \varDelta; \{l', m', s'\} | \, J_0 \, | \varDelta; \{l, m, s\} \rangle = m \, \delta_{ll'} \, \delta_{mm'} \, \delta_{ss'} \qquad \text{(C.17)}$$

and

$$\langle \varDelta; \{l', m', s'\} | \, J_\pm \, | \varDelta; \{l, m, s\} \rangle = \sqrt{(l \mp m)(l \pm m + 1)} \, \delta_{ll'} \, \delta_{m'(m\pm 1)} \, \delta_{ss'} \qquad \text{(C.18)}$$

Bibliography

[1] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113–1133, arXiv:hep-th/9711200 [hep-th]. [Adv. Theor. Math. Phys.2,231(1998)].

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett.* **B428** (1998) 105–114, arXiv:hep-th/9802109 [hep-th].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[4] F. A. Dolan and H. Osborn, *Conformal four point functions and the operator product expansion*, *Nucl. Phys.* **B599** (2001) 459–496, arXiv:hep-th/0011040 [hep-th].

[5] F. A. Dolan and H. Osborn, *Conformal partial waves and the operator product expansion*, *Nucl. Phys.* **B678** (2004) 491–507, arXiv:hep-th/0309180 [hep-th].

[6] F. A. Dolan and H. Osborn, *Conformal Partial Waves: Further Mathematical Results*, arXiv:1108.6194 [hep-th].

[7] E. Hijano, P. Kraus, E. Perlmutter, and R. Snively, *Witten Diagrams Revisited: The AdS Geometry of Conformal Blocks*, *JHEP* **01** (2016) 146, arXiv:1508.00501 [hep-th].

[8] M. P. Blencowe, *A Consistent Interacting Massless Higher Spin Field Theory in D = (2+1)*, *Class. Quant. Grav.* **6** (1989) 443.

[9] A. Achucarro and P. K. Townsend, *A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories*, *Phys. Lett.* **B180** (1986) 89. [,732(1987)].

[10] E. Witten, *(2+1)-Dimensional Gravity as an Exactly Soluble System*, *Nucl. Phys.* **B311** (1988) 46.

[11] S. W. MacDowell and F. Mansouri, *Unified Geometric Theory of Gravity and Supergravity*, *Phys. Rev. Lett.* **38** (1977) 739. [Erratum: Phys. Rev. Lett.38,1376(1977)].

[12] L. Freidel, K. Krasnov, and R. Puzio, *BF description of higher dimensional gravity theories*, *Adv. Theor. Math. Phys.* **3** (1999) 1289–1324, `arXiv:hep-th/9901069 [hep-th]`.

[13] L. Freidel and A. Starodubtsev, *Quantum gravity in terms of topological observables*, `arXiv:hep-th/0501191 [hep-th]`.

[14] A. Bhatta, P. Raman, and N. V. Suryanarayana, *Holographic Conformal Partial Waves as Gravitational Open Wilson Networks*, *JHEP* **06** (2016) 119, `arXiv:1602.02962 [hep-th]`.

[15] A. Bhatta, P. Raman, and N. V. Suryanarayana, *Scalar Blocks as Gravitational Wilson Networks*, *JHEP* **12** (2018) 125, `arXiv:1806.05475 [hep-th]`.

[16] Y. Nakayama and H. Ooguri, *Bulk Locality and Boundary Creating Operators*, *JHEP* **10** (2015) 114, `arXiv:1507.04130 [hep-th]`.

[17] H. Verlinde, *Poking Holes in AdS/CFT: Bulk Fields from Boundary States*, `arXiv:1505.05069 [hep-th]`.

[18] M. Miyaji, T. Numasawa, N. Shiba, T. Takayanagi, and K. Watanabe, *Continuous Multiscale Entanglement Renormalization Ansatz as Holographic Surface-State*

Correspondence, *Phys. Rev. Lett.* **115** no. 17, (2015) 171602, arXiv:1506.01353 [hep-th].

[19] B. Czech, L. Lamprou, S. McCandlish, B. Mosk, and J. Sully, *A Stereoscopic Look into the Bulk*, *JHEP* **07** (2016) 129, arXiv:1604.03110 [hep-th].

[20] J. de Boer, F. M. Haehl, M. P. Heller, and R. C. Myers, *Entanglement, holography and causal diamonds*, *JHEP* **08** (2016) 162, arXiv:1606.03307 [hep-th].

[21] K. B. Alkalaev and V. A. Belavin, *From global to heavy-light: 5-point conformal blocks*, *JHEP* **03** (2016) 184, arXiv:1512.07627 [hep-th].

[22] S. Terashima, *AdS/CFT Correspondence in Operator Formalism*, *JHEP* **02** (2018) 019, arXiv:1710.07298 [hep-th].

[23] Z. Wen and J. Avery, *Some properties of hyperspherical harmonics*, *Journal of Mathematical Physics* **26** no. 3, (1985) 396–403, https://doi.org/10.1063/1.526621. https://doi.org/10.1063/1.526621.

[24] J. Qiao and S. Rychkov, *A tauberian theorem for the conformal bootstrap*, *JHEP* **12** (2017) 119, arXiv:1709.00008 [hep-th].

[25] J. Polchinski, *Scale and Conformal Invariance in Quantum Field Theory*, *Nucl. Phys.* **B303** (1988) 226–236.

[26] S. Ferrara, A. F. Grillo, and R. Gatto, *Tensor representations of conformal algebra and conformally covariant operator product expansion*, *Annals Phys.* **76** (1973) 161–188.

[27] A. M. Polyakov, *Nonhamiltonian approach to conformal quantum field theory*, *Zh. Eksp. Teor. Fiz.* **66** (1974) 23–42. [Sov. Phys. JETP39,9(1974)].

[28] G. Mack, *Duality in quantum field theory*, *Nucl. Phys.* **B118** (1977) 445–457.

[29] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, *JHEP* **12** (2008) 031, arXiv:0807.0004 [hep-th].

[30] F. Caracciolo and V. S. Rychkov, *Rigorous Limits on the Interaction Strength in Quantum Field Theory*, *Phys. Rev.* **D81** (2010) 085037, arXiv:0912.2726 [hep-th].

[31] R. Rattazzi, S. Rychkov, and A. Vichi, *Central Charge Bounds in 4D Conformal Field Theory*, *Phys. Rev.* **D83** (2011) 046011, arXiv:1009.2725 [hep-th].

[32] D. Poland and D. Simmons-Duffin, *Bounds on 4D Conformal and Superconformal Field Theories*, *JHEP* **05** (2011) 017, arXiv:1009.2087 [hep-th].

[33] R. Rattazzi, S. Rychkov, and A. Vichi, *Bounds in 4D Conformal Field Theories with Global Symmetry*, *J. Phys.* **A44** (2011) 035402, arXiv:1009.5985 [hep-th].

[34] A. Vichi, *Improved bounds for CFT's with global symmetries*, *JHEP* **01** (2012) 162, arXiv:1106.4037 [hep-th].

[35] D. Poland, D. Simmons-Duffin, and A. Vichi, *Carving Out the Space of 4D CFTs*, *JHEP* **05** (2012) 110, arXiv:1109.5176 [hep-th].

[36] S. Rychkov, *Conformal Bootstrap in Three Dimensions?*, arXiv:1111.2115 [hep-th].

[37] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, *Solving the 3D Ising Model with the Conformal Bootstrap*, *Phys. Rev.* **D86** (2012) 025022, arXiv:1203.6064 [hep-th].

[38] P. Liendo, L. Rastelli, and B. C. van Rees, *The Bootstrap Program for Boundary* CFT_d, *JHEP* **07** (2013) 113, arXiv:1210.4258 [hep-th].

[39] S. El-Showk and M. F. Paulos, *Bootstrapping Conformal Field Theories with the Extremal Functional Method*, Phys. Rev. Lett. **111** no. 24, (2013) 241601, arXiv:1211.2810 [hep-th].

[40] F. Gliozzi, *More constraining conformal bootstrap*, Phys. Rev. Lett. **111** (2013) 161602, arXiv:1307.3111 [hep-th].

[41] F. Kos, D. Poland, and D. Simmons-Duffin, *Bootstrapping the $O(N)$ vector models*, JHEP **06** (2014) 091, arXiv:1307.6856 [hep-th].

[42] L. F. Alday and A. Bissi, *The superconformal bootstrap for structure constants*, JHEP **09** (2014) 144, arXiv:1310.3757 [hep-th].

[43] D. Gaiotto, D. Mazac, and M. F. Paulos, *Bootstrapping the 3d Ising twist defect*, JHEP **03** (2014) 100, arXiv:1310.5078 [hep-th].

[44] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, *Solving the 3d Ising Model with the Conformal Bootstrap II. c-Minimization and Precise Critical Exponents*, J. Stat. Phys. **157** (2014) 869, arXiv:1403.4545 [hep-th].

[45] Y. Nakayama and T. Ohtsuki, *Approaching the conformal window of $O(n) \times O(m)$ symmetric Landau-Ginzburg models using the conformal bootstrap*, Phys. Rev. **D89** no. 12, (2014) 126009, arXiv:1404.0489 [hep-th].

[46] Y. Nakayama and T. Ohtsuki, *Five dimensional $O(N)$-symmetric CFTs from conformal bootstrap*, Phys. Lett. **B734** (2014) 193–197, arXiv:1404.5201 [hep-th].

[47] F. Kos, D. Poland, and D. Simmons-Duffin, *Bootstrapping Mixed Correlators in the 3D Ising Model*, JHEP **11** (2014) 109, arXiv:1406.4858 [hep-th].

[48] Y. Nakayama and T. Ohtsuki, *Bootstrapping phase transitions in QCD and frustrated spin systems*, *Phys. Rev.* **D91** no. 2, (2015) 021901, arXiv:1407.6195 [hep-th].

[49] M. F. Paulos, *JuliBootS: a hands-on guide to the conformal bootstrap*, arXiv:1412.4127 [hep-th].

[50] J.-B. Bae and S.-J. Rey, *Conformal Bootstrap Approach to O(N) Fixed Points in Five Dimensions*, arXiv:1412.6549 [hep-th].

[51] C. Beem, M. Lemos, P. Liendo, L. Rastelli, and B. C. van Rees, *The* $\mathcal{N} = 2$ *superconformal bootstrap*, *JHEP* **03** (2016) 183, arXiv:1412.7541 [hep-th].

[52] S. M. Chester, S. S. Pufu, and R. Yacoby, *Bootstrapping* $O(N)$ *vector models in* $4 < d < 6$, *Phys. Rev.* **D91** no. 8, (2015) 086014, arXiv:1412.7746 [hep-th].

[53] C. Beem, L. Rastelli, and B. C. van Rees, *More* $\mathcal{N} = 4$ *superconformal bootstrap*, *Phys. Rev.* **D96** no. 4, (2017) 046014, arXiv:1612.02363 [hep-th].

[54] R. Gopakumar, A. Kaviraj, K. Sen, and A. Sinha, *A Mellin space approach to the conformal bootstrap*, *JHEP* **05** (2017) 027, arXiv:1611.08407 [hep-th].

[55] Y. Nakayama, *Bootstrap experiments on higher dimensional CFTs*, *Int. J. Mod. Phys.* **A33** no. 07, (2018) 1850036, arXiv:1705.02744 [hep-th].

[56] D. Pappadopulo, S. Rychkov, J. Espin, and R. Rattazzi, *OPE Convergence in Conformal Field Theory*, *Phys. Rev.* **D86** (2012) 105043, arXiv:1208.6449 [hep-th].

[57] M. Hogervorst and S. Rychkov, *Radial Coordinates for Conformal Blocks*, *Phys. Rev.* **D87** (2013) 106004, arXiv:1303.1111 [hep-th].

[58] D. Simmons-Duffin, *The Conformal Bootstrap*, in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields

and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015, pp. 1–74. 2017. arXiv:1602.07982 [hep-th]. https://inspirehep.net/record/1424282/files/arXiv:1602.07982.pdf.

[59] L. Iliesiu, F. Kos, D. Poland, S. S. Pufu, D. Simmons-Duffin, and R. Yacoby, *Bootstrapping 3D Fermions*, *JHEP* **03** (2016) 120, arXiv:1508.00012 [hep-th].

[60] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, *Spinning Conformal Correlators*, *JHEP* **11** (2011) 071, arXiv:1107.3554 [hep-th].

[61] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, *Spinning Conformal Blocks*, *JHEP* **11** (2011) 154, arXiv:1109.6321 [hep-th].

[62] D. Simmons-Duffin, *Projectors, Shadows, and Conformal Blocks*, *JHEP* **04** (2014) 146, arXiv:1204.3894 [hep-th].

[63] L. Iliesiu, F. Kos, D. Poland, S. S. Pufu, D. Simmons-Duffin, and R. Yacoby, *Fermion-Scalar Conformal Blocks*, *JHEP* **04** (2016) 074, arXiv:1511.01497 [hep-th].

[64] M. S. Costa, T. Hansen, J. Penedones, and E. Trevisani, *Radial expansion for spinning conformal blocks*, *JHEP* **07** (2016) 057, arXiv:1603.05552 [hep-th].

[65] M. S. Costa, T. Hansen, J. Penedones, and E. Trevisani, *Projectors and seed conformal blocks for traceless mixed-symmetry tensors*, *JHEP* **07** (2016) 018, arXiv:1603.05551 [hep-th].

[66] S. Giombi, S. Prakash, and X. Yin, *A Note on CFT Correlators in Three Dimensions*, *JHEP* **07** (2013) 105, arXiv:1104.4317 [hep-th].

[67] M. Hogervorst, *Dimensional Reduction for Conformal Blocks*, *JHEP* **09** (2016) 017, arXiv:1604.08913 [hep-th].

[68] G. Mack, *D-dimensional Conformal Field Theories with anomalous dimensions as Dual Resonance Models*, *Bulg. J. Phys.* **36** (2009) 214–226, arXiv:0909.1024 [hep-th].

[69] G. Mack, *D-independent representation of Conformal Field Theories in D dimensions via transformation to auxiliary Dual Resonance Models. Scalar amplitudes*, arXiv:0907.2407 [hep-th].

[70] A. L. Fitzpatrick and J. Kaplan, *AdS Field Theory from Conformal Field Theory*, *JHEP* **02** (2013) 054, arXiv:1208.0337 [hep-th].

[71] R. Gopakumar, A. Kaviraj, K. Sen, and A. Sinha, *Conformal Bootstrap in Mellin Space*, *Phys. Rev. Lett.* **118** no. 8, (2017) 081601, arXiv:1609.00572 [hep-th].

[72] J. Penedones, *Writing CFT correlation functions as AdS scattering amplitudes*, *JHEP* **03** (2011) 025, arXiv:1011.1485 [hep-th].

[73] A. L. Fitzpatrick, J. Kaplan, J. Penedones, S. Raju, and B. C. van Rees, *A Natural Language for AdS/CFT Correlators*, *JHEP* **11** (2011) 095, arXiv:1107.1499 [hep-th].

[74] V. Gonçalves, J. Penedones, and E. Trevisani, *Factorization of Mellin amplitudes*, *JHEP* **10** (2015) 040, arXiv:1410.4185 [hep-th].

[75] E. D'Hoker and D. Z. Freedman, *Supersymmetric gauge theories and the AdS / CFT correspondence*, in *Strings, Branes and Extra Dimensions: TASI 2001: Proceedings*, pp. 3–158. 2002. arXiv:hep-th/0201253 [hep-th].

[76] D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, *Correlation functions in the CFT(d) / AdS(d+1) correspondence*, *Nucl. Phys.* **B546** (1999) 96–118, arXiv:hep-th/9804058 [hep-th].

[77] E. Hijano, P. Kraus, E. Perlmutter, and R. Snively, *Semiclassical Virasoro blocks from AdS$_3$ gravity*, *JHEP* **12** (2015) 077, arXiv:1508.04987 [hep-th].

[78] V. A. Belavin and R. V. Geiko, *Geodesic description of Heavy-Light Virasoro blocks*, *JHEP* **08** (2017) 125, arXiv:1705.10950 [hep-th].

[79] P. Kraus, A. Maloney, H. Maxfield, G. S. Ng, and J.-q. Wu, *Witten Diagrams for Torus Conformal Blocks*, *JHEP* **09** (2017) 149, arXiv:1706.00047 [hep-th].

[80] M. Nishida and K. Tamaoka, *Geodesic Witten diagrams with an external spinning field*, *PTEP* **2017** no. 5, (2017) 053B06, arXiv:1609.04563 [hep-th].

[81] E. Dyer, D. Z. Freedman, and J. Sully, *Spinning Geodesic Witten Diagrams*, *JHEP* **11** (2017) 060, arXiv:1702.06139 [hep-th].

[82] K. Tamaoka, *Geodesic Witten diagrams with antisymmetric tensor exchange*, *Phys. Rev.* **D96** no. 8, (2017) 086007, arXiv:1707.07934 [hep-th].

[83] M. Nishida and K. Tamaoka, *Fermions in Geodesic Witten Diagrams*, arXiv:1805.00217 [hep-th].

[84] T. Fukuyama and K. Kamimura, *Gauge Theory of Two-dimensional Gravity*, *Phys. Lett.* **160B** (1985) 259–262.

[85] K. Isler and C. A. Trugenberger, *Gauge theory of two-dimensional quantum gravity*, *Phys. Rev. Lett.* **63** (Aug, 1989) 834–836. https://link.aps.org/doi/10.1103/PhysRevLett.63.834.

[86] A. H. Chamseddine and D. Wyler, *Topological Gravity in (1+1)-dimensions*, *Nucl. Phys.* **B340** (1990) 595–616.

[87] C. Teitelboim, *Gravitation and hamiltonian structure in two spacetime dimensions*, *Physics Letters B* **126** no. 1, (1983) 41 – 45. http://www.sciencedirect.com/science/article/pii/0370269383900126.

[88] R. Jackiw, *Lower Dimensional Gravity*, *Nucl. Phys.* **B252** (1985) 343–356.

[89] A. Alekseev, L. D. Faddeev, and S. L. Shatashvili, *Quantization of symplectic orbits of compact Lie groups by means of the functional integral*, *J. Geom. Phys.* **5** (1988) 391–406.

[90] L. Freidel, J. Kowalski-Glikman, and A. Starodubtsev, *Particles as Wilson lines of gravitational field*, *Phys. Rev.* **D74** (2006) 084002, arXiv:gr-qc/0607014 [gr-qc].

[91] J. D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, *Commun. Math. Phys.* **104** (1986) 207–226.

[92] E. Witten, *Gauge Theories and Integrable Lattice Models*, *Nucl. Phys.* **B322** (1989) 629–697.

[93] A. L. Fitzpatrick, J. Kaplan, D. Li, and J. Wang, *Exact Virasoro Blocks from Wilson Lines and Background-Independent Operators*, *JHEP* **07** (2017) 092, arXiv:1612.06385 [hep-th].

[94] R. Jackiw and V. P. Nair, *Relativistic wave equations for anyons*, *Phys. Rev.* **D43** (1991) 1933–1942.

[95] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1999.

[96] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997. http://www-spires.fnal.gov/spires/find/books/www?cl=QC174.52.C66D5::1997.

[97] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, *Annals Math.* **48** (1947) 568–640.

[98] W. J. Holman and L. C. Biedenharn, *Complex angular momenta and the groups su(1, 1) and su(2)*, *Annals of Physics* **39** no. 1, (1966) 1 – 42. http://www.sciencedirect.com/science/article/pii/0003491666901357.

[99] A. Hegde, P. Kraus, and E. Perlmutter, *General Results for Higher Spin Wilson Lines and Entanglement in Vasiliev Theory*, *JHEP* **01** (2016) 176, arXiv:1511.05555 [hep-th].

[100] D. Melnikov, A. Mironov, and A. Morozov, *On skew tau-functions in higher spin theory*, *JHEP* **05** (2016) 027, arXiv:1602.06233 [hep-th].

[101] E. Perlmutter, *Virasoro conformal blocks in closed form*, *JHEP* **08** (2015) 088, arXiv:1502.07742 [hep-th].

[102] S. Datta and J. R. David, *Higher Spin Quasinormal Modes and One-Loop Determinants in the BTZ black Hole*, *JHEP* **03** (2012) 079, arXiv:1112.4619 [hep-th].

[103] D. J. Gross and V. Rosenhaus, *All point correlation functions in SYK*, *JHEP* **12** (2017) 148, arXiv:1710.08113 [hep-th].

[104] G. A. Kerimov and I. A. Verdiev, *CLEBSCH-GORDAN COEFFICIENTS OF THE GROUPS SO(P,1)*, *Rept. Math. Phys.* **20** (1984) 247–254.

[105] A. Castro, N. Iqbal, and E. Llabrés, *Wilson Lines and Ishibashi states in* AdS_3/CFT_2, arXiv:1805.05398 [hep-th].

[106] G. Junker, *Explicit evaluation of coupling coefficients for the most degenerate representations of so(n)*, *Journal of Physics A: Mathematical and General* **26** no. 7, (1993) 1649. http://stacks.iop.org/0305-4470/26/i=7/a=021.

[107] H. S. Cohl, *On a generalization of the generating function for gegenbauer polynomials*, *Integral Transforms and Special Functions* **24** no. 10, (2013)

807–816, https://doi.org/10.1080/10652469.2012.761613.
https://doi.org/10.1080/10652469.2012.761613.

[108] M. Besken, A. Hegde, E. Hijano, and P. Kraus, *Holographic conformal blocks from interacting Wilson lines*, *JHEP* **08** (2016) 099, arXiv:1603.07317 [hep-th].

[109] K. Papadodimas and S. Raju, *Correlation Functions in Holographic Minimal Models*, *Nucl. Phys.* **B856** (2012) 607–646, arXiv:1108.3077 [hep-th].

[110] V. Fateev and S. Ribault, *The Large central charge limit of conformal blocks*, *JHEP* **02** (2012) 001, arXiv:1109.6764 [hep-th].

[111] J. de Boer and J. I. Jottar, *Entanglement Entropy and Higher Spin Holography in AdS_3*, *JHEP* **04** (2014) 089, arXiv:1306.4347 [hep-th].

[112] D. Karateev, P. Kravchuk, and D. Simmons-Duffin, *Weight Shifting Operators and Conformal Blocks*, *JHEP* **02** (2018) 081, arXiv:1706.07813 [hep-th].

[113] M. Besken, A. Hegde, and P. Kraus, *Anomalous dimensions from quantum Wilson lines*, arXiv:1702.06640 [hep-th].

[114] Y. Hikida and T. Uetoko, *Correlators in higher-spin AdS_3 holography from Wilson lines with loop corrections*, *PTEP* **2017** (2017) 113B03, arXiv:1708.08657 [hep-th].

[115] Y. Hikida and T. Uetoko, *Conformal blocks from Wilson lines with loop corrections*, *Phys. Rev.* **D97** no. 8, (2018) 086014, arXiv:1801.08549 [hep-th].

[116] Y. Hikida and T. Uetoko, *Superconformal blocks from Wilson lines with loop corrections*, arXiv:1806.05836 [hep-th].

[117] WolframResearch, *HypergeometricPFQ : Relations between contiguous functions*, *The Wolfram Functions Site* . http://functions.wolfram.com/HypergeometricFunctions/Hypergeometric3F2/17/02/01/.