STUDIES IN GENERALIZED CLIFFORD ALGEBRAS,
GENERALIZED CLIFFORD GROUPS AND
THEIR PHYSICAL APPLICATIONS

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PREFACE

This thesis is based upon the work done by me during the period 1971-1975 on Generalized Clifford Algebras and their applications under the guidance of Professor Alladi Ramakrishnan, Director, MATSCIENCE, The Institute of Mathematical Sciences, Madras.

I am extremely grateful to Professor Alladi Ramakrishnan for his constant encouragement, guidance and collaboration during the course of this work. It is a pleasure to record gratefully the benefit derived from useful discussions and collaboration with Professor N.R. Ranganathan, who initiated my research in certain group theoretical aspects of Generalized Clifford algebras and their applications. I am thankful to Professor R. Vasudevan for introducing to me the subject of Weyl's rule and Wigner distribution function in quantum mechanics and useful collaboration in the application of C.C.A's to them. I wish to thank Professor K.H. Mariwala for useful discussions and bringing to my notice often many useful references on the subject of projective representations of groups. I like to thank Mr. K.N.V. Dutt for collaboration in a piece of work on Dirac's positive energy wave equation. With pleasure I thank Professor V. Radhakrishnan, Professor T. R. Santhanam, Dr. A. R. Tekumalla and Dr. C. N. Keshava Murthy for many useful discussions.
Particularly I like to thank very much Dr. A. R. Tekumalla for bringing to my notice the book of Morris Newman on Integral Matrices which helped me in my investigations on projective representations of Abelian groups. I wish to thank Mr. Krishnaswami Alladi for introducing to me certain number theoretic concepts which have been of great use. I take great pleasure in thanking all my colleagues who have helped me in many ways.

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[Signature]
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This thesis presents some recent developments in the study of Generalised Clifford Algebras (G.C.A.'s), their associated structures and their physical applications. These investigations are extensions of the research programme initiated by Alladi Ramakrishnan with the development of L-Matrix Theory dealing with the Grammar of Dirac Matrices and their generalisations. Alladi Ramakrishnan and his collaborators have carried out extensive analysis of G.C.A.'s and their applications to physical problems. Results of these investigations are found in the book 'L-Matrix Theory or Grammar of Dirac Matrices' by Alladi Ramakrishnan.

The new developments presented in this thesis include:

i. Introduction of concepts 'computation matrices' and 'product transforms' and the formulation of Alladi Ramakrishnan's 'tension and mortice coupling method' of representation of G.C.A.'s in terms of them.

ii. Generalisation of a 'matrix decomposition theorem' due to Alladi Ramakrishnan and discussion of his 'new approach to matrix theory' in relation to the work of Hermann Weyl, Wigner and Schrödinger.


iv. Formulation of new group structures called 'Generalised Clifford Groups' (G.C.G.'s) and discussion of their significance in the study of the problem of Bloch electrons in homogeneous magnetic field.
v. A complete, simple and explicit solution to the problem of projective representations of finite abelian groups.

vi. Proposal of a negative energy relativistic wave equation as a counterpart of Dirac's positive energy relativistic wave equation. 


CHAPTER 1 presents a summary of main results of Alladi Ramakrishnan and his collaborators in L-Matrix theory which are essential to the investigations in later Chapters. New concepts 'Commutation Matrices' and 'Product Transforms' are introduced and Alladi Ramakrishnan's 'Tensor and Metric Coupling Method' of representations of Clifford algebras is reformulated in terms of product transforms.

CHAPTER 2 studies a 'Matrix Decomposition Theorem' due to Alladi Ramakrishnan and its generalisations. Significance of these theorems in relation to quantum kinematics is discussed and canonical transformations of conjugate variables, especially affine canonical transformations, are studied using the basic generating elements of G.C.A.

CHAPTER 3 and 4 study some group theoretical aspects of G.C.A. by formulating new group structures called Generalised Clifford Groups (G.C.G.'s). All the irreducible and inequivalent representations are explicitly constructed and studied in detail.
CHAPTER 5. determines explicitly all the inequivalent irreducible representations of all $G_c A'$s associated with thefinite Abelian group $\mathbb{Z}_{m_1} \otimes \cdots \otimes \mathbb{Z}_{m_n}$. This provides a
canonical and simple solution to the problem of explicit determina-
tion of all the inequivalent irreducible projective representations offinite Abelian groups.

CHAPTER 6 studies the problem of Bloch electrons in homo-
geneous magnetic field in the light of above investigations using
the isomorphism of $G_c A'$s and Magnetic translation groups and
arrive at a new and simple version of so-called magnetic Bloch func-
tion convenient for playing the role of Bloch function in study solid
state phenomena in presence of an external homogeneous magnetic
field.

CHAPTER 7 proposes and studies a negative energy relativistic
wave equation as a counter-part to Dirac’s positive energy relativ-
istic wave equation.

CHAPTER 8 and 9 are mathematical appendices, the first dealing
with Clifford’s ‘Commuting quaternion algebras’ viewed in terms of
$\mathbb{L}$-Matrix Theory, and second dealing with Baez’s approach to
Clifford algebra and its generalisation to $G_c A'$s.
CHAPTER 1

FUNDAMENTALS OF L-MATRIX THEORY

L-Matrix theory was initiated by Alladi Ramakrishnan with the objective of understanding the mathematical procedure of obtaining Dirac matrices from the basic Pauli set. This led to a detailed study by him and his collaborators, of the mathematical structure — 'too fundamental to be unnoticed, too consistent to be ignored and much too pretty to be without consequence' — as a common basis of understanding various branches of theoretical physics. L-Matrix theory studies Clifford Algebra, its generalisations and their physical applications using explicit matrix representations.

Briefly the course of major developments in L-Matrix theory can be sketched as follows. Alladi Ramakrishnan developed in a series of definitive papers the viewpoint of Dirac Hamiltonian as a member of a hierarchy of matrices analysing helicity and energy as members of a hierarchy of eigenvalues. He developed further the connection of these hierarchy of matrices — L-matrices — to quaternions, propagators and Cartan spinors. In collaboration with Vasudevan, Sanganathan, Santhosh and Chandrasekharan, he extended this approach to Generalised Clifford algebras and this led to the development of representations of Jordan Algebra, para-Fermi rings, certain polynomial algebras and Unitary groups from the elements of $G_2,4$.

With the realization of a shell structure in L-Matrices he studied weak interaction Hamiltonian in this approach. Following the mathematical logic behind the symmetries associated with the roots of the Unit matrix as a guide to physical thought he interpreted the internal quantum numbers of quarks in terms of roots of unity with an exciting generalization of Cohl-Mann-Nishijima relation. Leaving these interesting aspects to the reference of his book 'L-Matrix Theory of Grammar of Dirac Matrices' let us consider in this chapter the mathematical foundations of L-Matrix theory essential for our investigations in latter chapters.

Clifford Algebra originated in the paper of W.K. Clifford 'Applications of Grassmann's Extensive Algebra' (Proc. L. Math. Soc. 1, 350-352, 1878) in which he generalized Hamilton's 'Quaternions algebra' with three generators to higher dimensions. He called the resulting algebra with n generators as 'n-way geometric algebra', due to its relation to Grassmann's algebra of polynomials in n-dimensional Euclidean space. Clifford algebra became the basis for spin representations of orthogonal groups. The spin representation of the special orthogonal group by means of Clifford algebra was discovered by Lipschitz and then forgotten. Elie Cartan gave the theory of spinors and Cartan's theory was developed by Brunner and Weyl. The first application of spinors to physics was by Pauli who introduced his famous spin matrices. The fully relativistic theory
of the electron spin was discovered by Dirac\textsuperscript{15} who by his famous linear relativistic wave equation showed the connection between spinors and the Lorentz group. Freundental\textsuperscript{16} rediscovered the results of Lipschitz and developed them using the theory of characters. Mathematical aspects and physical applications of spinors were later developed by van der Waerden\textsuperscript{17}, Fierz\textsuperscript{18}, Bargmann and Wigner\textsuperscript{19} and others\textsuperscript{30}.

Clifford algebra and its generalizations can be studied algebraically from the point of view of projective representations of finite Abelian groups. The problem of finding the projective (ray) representations of finite groups was stated and 'general' methods of finding the irreducible representations were given by Schur in a series of papers in 1904-11\textsuperscript{21}. In mathematical literature Generalized Clifford Algebras (G.C.A.'s) were introduced and studied in detail by Morina and Nonomura\textsuperscript{22} while linearizing $m$-th order partial differential equations for $m > 2$. Tamazaki\textsuperscript{23} introduced G.C.A.'s in dealing with projective representations and ring extensions of finite groups and Morris\textsuperscript{24} studied these algebras in explicit detail. Popovici and Turcan\textsuperscript{25} studied these algebra in relation to generalizations of spinor structures. Nonomura\textsuperscript{22} and Morris\textsuperscript{24} determined explicitly the projective representations of $G = \mathbb{Z}_m \oplus \cdots \oplus \mathbb{Z}_m$ (n copies), for some factor systems. But so far no theory exists providing an explicit construction technique for determination of all the projective representations of an arbitrary finite Abelian group $G = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_n}$. (For a detailed discussion of the subject of projective representations of finite groups cf. Morris\textsuperscript{24}). Recent achievement in this direction is due to Backhouse and
Bradley, who have determined explicitly the formula for the
dimension of representation associated with a given factor system
for G. This thesis solves this problem completely in a simple
manner using the techniques of L-Matrix theory.

In physics literature, Weyl discussed the projective
representations of Abelian groups and interpreted the fundamental
laws of quantum kinematics in terms of them. Schwinger developed
his approach more elaborately in relation to measurement algebra in
quantum kinematics and emphasized the significance of these struc-
tures. But in the work of both Weyl and Schwinger these structures
in finite dimensions occur as only intermediate steps in a limiting
process. Only after the development of L-Matrix theory by A. Rama-
krishnan and his collaborators, the physical significance of
these structures in finite dimensions have been realized and developed
extensively. This thesis presents further applications of C.C.A's
to the problem of Bloch electrons in magnetic field. Throughout
this thesis we shall use mainly the techniques of L-Matrix theory
and so first we shall summarize briefly the main results of L-Matrix
theory essential to the investigations in later chapters.

1. L-Matrices and \( \sigma \)-operation method

A linear combination of the three Pauli matrices
\[
L_3^{(2)} = \sum_{J=1}^{3} x_{J} \sigma_{J} \sigma^{(2)} = \begin{pmatrix}
    x_3 & x_1 - ic x_2 \\
    x_1 + ic x_2 & -x_3
\end{pmatrix}
\]
(1.1)
is recognized to be a member of hierarchy of matrices, called
L-matrices, with the property
\[
L^{(2)}_{mn} = \sum_{i=1}^{n} x_i L^{(2)}_{ni}
\]
\[
\left( L^{(2)}_{mn} \right)^2 = \left( \sum_{i=1}^{n} x_i^2 \right) I
\]
\[
L^{(2)}_{ni} L^{(2)}_{nj} = -L^{(2)}_{n,j'}^L L^{(2)}_{n,i} \quad ; \quad i \neq j \quad ; \quad i, j = 1, 2, \ldots, n
\]
\[
\left( L^{(2)}_{ni} \right)^2 = I \quad ; \quad i = 1, 2, \ldots, n
\]

First let \( n = 2r+1 \).  

A method due to Alladi Ramakrishnan obtains \( L^{(3)}_{2r+1} \) from \( L^{(3)}_{2r-1} \) as follows.  In \( L^{(3)}_{2r-1} \) any one of the parameters \( x_1, x_2, x_r \) is replaced by \( x_{2r} \) and the other two parameters are replaced by \( x_{2r-1} \) and \( x_{2r-1} \), where \( I \) is the identity matrix of the same dimension as \( L^{(3)}_{2r-1} \).

\[
L^{(3)}_{2r+1} = \begin{pmatrix}
x_{2r+1}^T & L^{(3)}_{2r-1} - ix_{2r}^T \\
L^{(3)}_{2r-1} + ix_{2r}^T & -x_{2r+1}^T
\end{pmatrix} = \sigma \left( L^{(3)}_{2r-1} \right)
\]

Taking \( x_{2r} = x_1 \); this relation provides explicit representation of \( L^{(3)}_{2}, L^{(3)}_{4}, \ldots \). When \( n = 2r \), any one of the parameters in \( L^{(3)}_{2r+1} \) is put equal to zero. It is seen that at each step of going from \( L^{(3)}_{2r-1} \) to \( L^{(3)}_{2r+1} \), the dimension of \( L \) is doubled and hence

\[
\dim \left( L^{(3)}_{2r+1} \right) = 2^r
\]

Also

\[
\sum_{i=0}^{r-1} \binom{r}{i} \left( L^{(3)}_{2i+1} \right)^2 \left( L^{(3)}_{2(r-i)+1} \right)^2 = 0 \quad (1.6)
\]
or
\[
\dim_\ell (\ell_n^{(2)}) = 2^{\lceil n/2 \rceil}
\]
(1.7)

where \( \lceil n/2 \rceil \) denotes the largest integer part of \( n/2 \).

The irreducibility of the above representation is seen as follows. The matrices \( \ell_{(n,i)} \); \( i = 1, 2, \ldots, n \) called 'generator matrices' are seen to be traceless by the construction formula (1.4).

This is also inferred from (1.3) as follows
\[
\begin{align*}
&\ell_{(n,i)} \ell_{(n,j)} \ell_{(n,j)}^{-1} = -\ell_{(n,j)} \ell_{(n,j)}^{-1} \ell_{(n,j)} = 0 \quad (j = 1, 2, \ldots, n). \\
&\ell_{(n,j)}^{-1} = 0.
\end{align*}
\]

Now consider the case \( n = 2r \). If we form all possible products of all possible powers of the \( \ell_n \)'s the \( 2^{2r} \) matrices are given by
\[
\prod_{i=1}^{2r} \left( \ell_{(\alpha_i, i)} \right)^{k_i}, \quad k_i = 0, 1, \ldots, r.
\]
Consider one of the terms
\[
\prod_{i=1}^{2r} \left( \ell_{(\alpha_i, i)} \right)^{k_i} \quad (i = 1, 2, \ldots, r).
\]
If \( k \) is even
\[
\ell_{(\alpha_{i_1}, i_1)} \ell_{(\alpha_{i_2}, i_2)} \cdots \ell_{(\alpha_{i_r}, i_r)}
\]
then this term anticommutes with any of the
\[
\ell_{(\alpha_{i_r}, i_r)} \quad (r = 1, 2, \ldots, 2r)
\]
and hence by the argument of (1.6) its trace is zero. If \( k \) is odd then any \( \ell_{(\alpha_{i_r}, i_r)} \) which is not contained as a factor in the product
\[
\prod_{i=1}^{2r} \left( \ell_{(\alpha_i, i)} \right)^{k_i}
\]
anticommutes with it. Hence all the members of the set
\[
\left\{ \prod_{i=1}^{2r} \left( \ell_{(\alpha_i, i)} \right)^{k_i} \mid k_i = 0, 1, \ldots, r \right\}
\]
are traceless matrices except the identity matrix corresponding to the case all \( k_i = 0 \). Consider the equation
\[
A = \sum_{(k_1, k_2, \ldots, k_{2r} = 0, 1)} A_{k_1 k_2 \ldots k_{2r}} \left( \ell_{(\alpha_1, 1)} \right)^{k_1} \cdots \left( \ell_{(\alpha_r, 2r)} \right)^{k_{2r}} = 0
\]
(1.9)
it follows
\[ \text{tr} A = 0 \]  (1.10)

and hence \( a_{00\ldots0} = 0 \). Multiplying \( A \) on the left by
\[
\begin{pmatrix}
L^{(2)}_{(\ell_1, \ell_2 Y)} & \cdots & L^{(2)}_{(\ell_1, \ell_1)} \\
\cdot & \cdots & \cdot \\
L^{(2)}_{(\ell_1, \ell_1 Y)} & \cdots & L^{(2)}_{(\ell_1, \ell_1)}
\end{pmatrix}
\]


\[
A = a_{k_1 k_2 \ldots k_{2Y}} \text{I + } (-\ldots) = 0
\]  (1.11)

where \((\ldots)\) is again a sum of traceless terms. So from
\[ T_2 \{ \prod \left( L^{(2)}_{(\ell_i, \ell_i Y)} \right) \} A^2 = 0 \]  (1.12)

it follows
\[ a_{k_1 k_2 \ldots k_{2Y}} = 0 \]  (1.13)

for all \( k_1, \ldots, k_{2Y} \). This means that all the \( 2^{2Y} \) elements
\[
\{ \prod_{i=1}^{2Y} \left( L^{(2)}_{(\ell_i, \ell_1)} \right)^{k_1} \} \quad k_i = 0, 1, \ldots, 1, 2, \ldots, 2^Y
\]
are linearly independent. Since their dimension as given by (1.6) is \( 2^Y \), this representation is irreducible since the condition for irreducibility of a \( D \)-dimensional representation of a set of elements is that the set of matrices generated by taking all possible products of all possible powers of them should contain \( D^2 \) linearly independent matrices.

Considering the case of \( n = 2r+1 \), the product
\[ L^{(2)}_{(\ell_1, \ell_1 Y + 1)} \cdots L^{(2)}_{(\ell_1, \ell_1 Y)} = 0 \]  (1.14)

commutes with all \( L^{(2)}_{(\ell_1, \ell_1)} \).
\[ L^{(1)} \cdot Q = Q \cdot L^{(2)} \quad \lambda = 1, 2, \ldots, \eta. \]  
\[ \text{(1.16)} \]

Hence by Schur's lemma

\[ Q \sim I \]  
\[ \text{(1.16)} \]

where \( \sim \) means proportionality. Or

\[ L^{(2)} \sim L^{(2)} \quad \lambda = 1, 2, \ldots, \eta. \]  
\[ \text{(1.17)} \]

which is obtained by multiplying both sides of (1.14) by \( L^{(2)} \)

\[ 2 \times (2r+1) \]

and using (1.3) and (1.3), (b). Hence in the set of all possible products of all possible powers of \( L^{(2)} \) there are only \( 2^{2r} \) linearly independent elements. Hence \( 2^r \) dimensional representation of \( L^{(2)} \) is irreducible.

2. Property of Clifford Algebra \( \mathbb{C} \)

The \( n \) elements obeying (1.3)
\[ \prod_{\lambda=1}^{n} L^{(2)}(r_{\lambda}, r_{\lambda}), \quad \lambda = 1, 2, \ldots, n \]

generate a set of \( 2^n \) elements taking all possible products of all possible powers of them. In the set
\[ \left\{ \prod_{\lambda=1}^{n} L^{(2)}(r_{\lambda}, r_{\lambda}) \right\}_{k_i \in \{0, 1\}, \lambda = 1, 2, \ldots, n} \]

product of any two elements is defined by
\[ \left( \prod_{j=1}^{m} L^{(2)}(r_{\mu}, r_{\mu}) \right) \left( \prod_{\lambda=1}^{n} L^{(2)}(r_{\lambda}, r_{\lambda}) \right) = \left\{ \prod_{j=1}^{m} L^{(2)}(r_{\mu}, r_{\mu}) \right\}_{k_i \in \{0, 1\}, \lambda = 1, 2, \ldots, n} \]

\[ \text{(1.20)} \]

where \( (k_1 + k_2) \mod 2 = k_1 + k_2 \). Hence the set of elements
\[ \left\{ \prod_{\lambda=1}^{n} L^{(2)}(r_{\lambda}, r_{\lambda}) \right\}_{k_i = 0, 1, \lambda = 1, 2, \ldots, n} \]

form the basis for an algebra called Clifford Algebra \( \mathbb{C} \). The basis of
algebra called Clifford algebra, denoted by \( C_{n}^{(m)} \). The basis of \( C_{n}^{(m)} \) contains a set of \( 2^{m} \) linearly independent matrices when represented by \( 2^{m} \) dimensional matrices as shown above and hence \( C_{n}^{(m)} = \mathbb{M}^{(m)(n)} = \text{the total matrix algebra of dimension } 2^{m} \) over a field containing \( 1 \). The algebra \( C_{n+1}^{(m)} \) can be shown to be a direct sum of two \( C_{n}^{(m)} \) and hence \( C_{n+1}^{(m)} = \mathbb{M}^{(m)} \oplus \mathbb{M}^{(m)} \). (For more details cf. 22).

3. Generalised Clifford Algebra \( C_{n}^{(m)} \) and their properties.

Generalization of L-Matrix hierarchy (1.8) to include matrices obeying

\[
L_{n}^{(m)} = \sum_{i=1}^{n} x_{i} L_{n, i}^{(m)}
\]

(1.19)

\[
\left( \sum_{i=1}^{m} x_{i} L_{n, i}^{(m)} \right)^{m} = \left( \sum_{i=1}^{m} x_{i} \cdot m \right) I
\]

was considered by Alladi Ramakrishnan, R. Vasudevan, K. R. Ranganathan, T. S. Santhanam and P. V. Chandrasekaran as an extension of the problem for \( n = 2 \), and they showed that \( \sigma \)-operation method can be generalized to yield the irreducible representations of (1.19). These matrices (1.19) are associated with the so-called \( \sigma \)-commutation relations

\[
L_{n, i}^{(m)} L_{n, j}^{(m)} = \omega(n) L_{n, j}^{(m)} L_{n, i}^{(m)} ; i, j = 1, 2, \ldots, m
\]

(1.20)

\[
\left( L_{n, i}^{(m)} \right)^{m} = I ; i = 1, 2, \ldots, m
\]

where \( \omega(n) = \exp(\pi i/m) \). If we call (1.8) as Clifford conditions then these are generalized Clifford conditions since for \( n = 2 \), (1.20) is the same as (1.8). From (1.20), (1.19) follows due to the fact...
that
\[
\sum_{\text{all permutations of } i_1, i_2, \ldots, i_m} \frac{m!}{L_{(m, i_1)} \cdot L_{(m, i_2)} \cdots L_{(m, i_m)}} = m! \cdot \prod_{i_1 \neq i_2} \delta_{i_1 i_2} \cdots \delta_{i_1 i_m}
\]

Here we assume that we substitute the values of the indices \(i_1, i_2, \ldots, i_m\) in the symmetric sum containing \(m!\) terms. Then if \(i_1 = i_2 = \ldots = i_m = l\),

\[
 m! \cdot \left( \frac{m!}{L_{(m, i)}} \right)^m = m! \quad \forall i = 1, 2, \ldots, m.
\]

By taking all possible products of all possible powers of the \(n\) generators \(\left\{ L_{(m, i)} \right\}_{i=1}^{n} \), we get a set of \(m^n\) elements \(\left\{ \prod_{i=1}^{k} L_{(m, i)}^{k_i} \right\}_{k_i \geq 0, k_i \leq n} \). Let us consider first the case \(m = 2n\). Let us denote by \(g(k_1, k_2, \ldots, k_n)\) the element \(\left\{ \prod_{i=1}^{n} L_{(m, i)}^{k_i} \right\}\). Obviously

\[
g(0,0,\ldots,0) = I.
\]

Now let us determine whether there are any relationships among the \(m^n\) elements \(\left\{ g(k_1, k_2, \ldots, k_n) \right\}_{k_i \geq 0, k_i \leq n}\) of the type \(g(k_1, k_2, \ldots, k_n) \sim g(l_1, l_2, \ldots, l_n)\).

For some values of \(k_i\) and \(l_i\), such a relationship for a pair of sets of values \(\left\{ k_i \right\}\) and \(\left\{ l_i \right\}\) would imply the relationship
\[
g(k_1, \ldots, k_n) \sim g(l_1, \ldots, l_n) \iff I.
\]

Or from the help of (1.23) it is seen to imply that \(g(k_1, \ldots, k_n) \sim I\). Due to Schur's lemma this would happen if there exists any element \(g(\gamma_1, \gamma_2, \ldots, \gamma_n)\), with at least one \(\gamma_i > 0\), which commutes with all generating elements \(\left\{ L_{(m, i)} \right\}_{i=1}^{n}\).

The condition for this is seen to be

\[
\left( L_{(m, i)} \right)^{-1} g(\gamma_1, \gamma_2, \ldots, \gamma_n) \left( L_{(m, i)} \right) = g(\gamma_1, \gamma_2, \ldots, \gamma_n)
\]

\(\forall i = 1, \ldots, n\).

This leads to the condition
\[
\begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
-1 & 0 & 1 & \cdots & 1 \\
-1 & -1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\vdots \\
\gamma_m
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\mod m
\] (1.28)

or
\[T(I) = (Q) \mod m.
\]

When \(m = 2
\nu\), the matrix \(T\) is invertible since \(|\det T| = 1\) and hence
\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\vdots \\
\gamma_m
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_m
\end{pmatrix}
\mod m.
\] (1.29)

showing that the element \(q(\gamma_1, \gamma_2, \ldots, \gamma_m)\) commutes with all the generators \(I_i \mod m = \gamma_i\). This shows that really there are no relationships of the type sought for. Thus for any \(q(\gamma_1, \gamma_2, \ldots, \gamma_m)\) if \(\gamma_i \neq 0\), \(\forall \gamma_i = 1, 2, \ldots, m\) there exists some element \(\omega(2i, i)\) with the property
\[
\begin{pmatrix}
\omega(2i, i) \\
\omega(2i, i)
\end{pmatrix}
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\vdots \\
\gamma_m
\end{pmatrix}
= 
\begin{pmatrix}
\omega(2i, i) \\
\omega(2i, i)
\end{pmatrix}
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\vdots \\
\gamma_m
\end{pmatrix}
\mod m
\] (1.30)

Hence taking trace of both sides
\[
\frac{1}{Z}(q(\gamma_1, \gamma_2, \ldots, \gamma_m)) = \omega(m) \frac{1}{Z}(q(\gamma_1, \gamma_2, \ldots, \gamma_m))
\] with \(\omega \mod m > 0\)

or
\[
\frac{1}{Z}(q(\gamma_1, \gamma_2, \ldots, \gamma_m)) = 0.
\] (1.31)

Only \[
\frac{1}{Z}(q(0, 0, \ldots, 0)) \neq 0
\]

As before consider the equation
\[
A = \sum_{0 \leq k_1, k_2, \ldots, k_{2\nu} \leq m-1} a_{k_1 k_2 \ldots k_{2\nu}} g(k_1, \ldots, k_{2\nu}) = 0
\] (1.32)

and multiplying it on both sides from left by
\[
g(k_1, \ldots, k_{2\nu}) - 1 \omega(2m - k_1, \ldots, m - k_{2\nu})
\] (1.33)

It follows by taking trace of both sides
\[
a_{k_1, \ldots, k_{2\nu}} = 0 \quad \forall 0 \leq k_1, \ldots, k_{2\nu} \leq m-1.
\] (1.34)

This proves the linear independence of the \(m\) elements
\[
g(k_1, \ldots, k_{2\nu}) \mid 0 \leq k_i \leq m-1
\]

When \(m = 2\nu + 1\), dot \(T = 0\) and hence there exists a relationship of the type proposed. Since rank \(T = 2\nu\), there is only one such relationship possible. One can easily convince himself that this relationship is given by a \(\xi\) noted by Morris\(3\).
\[ q(\nu_{1}, \ldots, \nu_{n}, 1, \nu_{n+1}, \ldots, \nu_{n+m-1}) \sim I \]  

(1.29)  

This means that one of the generators \( \left\{ \begin{pmatrix} (m) \\ x \end{pmatrix} \right\} \) \( x = 1 \ldots 2v+1 \) is expressible as a product of others. Thus the set of all possible products of all possible powers of \( \left\{ \begin{pmatrix} (m) \\ x \end{pmatrix} \right\} \) \( x = 1 \ldots 2v+1 \) splits into a direct sum of sets each containing \( m^{2k} \) linearly independent elements. Thus in both cases of \( \nu = \nu_{1} \) and \( \nu = 2v+1 \) there are only \( m \) linearly independent matrices showing that in general the irreducible dimension is given by

\[ \dim \left( \begin{pmatrix} (m) \\ (\nu, i) \end{pmatrix} \right) = \frac{\nu}{2} \]  

(1.30)  

Incidentally this also establishes that there cannot exist more than \( 2v+1 \) mutually commuting matrices of dimension in the sense of eqn. (1.29a).

The set of all possible products of possible powers of \( \left\{ \begin{pmatrix} (m) \\ (\nu, i) \end{pmatrix} \right\} \) namely \( \left\{ \prod_{i=1}^{\nu} \begin{pmatrix} (m) \\ (\nu, i) \end{pmatrix} R_{i} \mid \emptyset \leq R_{i} \leq \nu_{1}; i = 1 \ldots \nu \right\} \) is easily seen to be the basis of an algebra, the product of basis elements being given by

\[ \left\{ \prod_{i=1}^{\nu} \begin{pmatrix} (m) \\ (\nu, i) \end{pmatrix} R_{i} \right\} \left\{ \prod_{i=1}^{\nu} \begin{pmatrix} (m) \\ (\nu, i) \end{pmatrix} L_{i} \right\} = \omega(\nu) \delta \left\{ \prod_{i=1}^{\nu} \begin{pmatrix} (m) \\ (\nu, i) \end{pmatrix} (k_{i} + l_{i}) \right\} \]  

(1.34)  

\[ \delta \bmod m = -(k_{1}, \ldots, k_{n}) \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} l_{1} \\ l_{2} \\ \vdots \\ l_{n} \end{pmatrix} \]

and

\[ (k_{i} + l_{i}) \bmod m = k_{i} + l_{i} \quad \forall i = 1 \ldots n \]  

(1.35)

This is the so-called Generalized Clifford algebra, one of many such types of structures which we shall study in latter chapters. All these algebras can be introduced from different points of view as projective representations of finite Abelian groups (Weyl, Yamasaki, Morris)...
linearization of certain partial differential equations (Morimura and Nom) or generalization of spinor structure (Popovici and Turtol).

Now we shall describe the generalized co-operation method due to Alladi Ramakrishnan and his collaborators for finding the irreducible representations of this \( G_2G_2 \).

Let

\[
\begin{align*}
\text{a) } L^{(m)}_{(3,1)} &= C(m) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix} \\
\text{b) } L^{(m)}_{(3,2)} &= B(m) = \begin{pmatrix}
1 & \omega(m) & 0 & \cdots & 0 \\
0 & \omega(m) & \omega(m) & \cdots & 0 \\
0 & 0 & \omega(m) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \omega(m)
\end{pmatrix} \\
\text{c) } L^{(m)}_{(3,3)} &= \zeta C(m)^{m-1} B(m) \quad \zeta = \begin{cases} 
\omega(m)^{m} & \text{if } m \text{ is even} \\
1 & \text{if } m \text{ is odd}
\end{cases}
\end{align*}
\]

It is verified that \( \{ L^{(m)}_{(3,i)} \mid i = 1, 2, 3 \} \) obey the co-commutation relations

\[
L^{(m)}_{(3,i)} L^{(m)}_{(3,j)} = \omega(m) L^{(m)}_{(3,j)} L^{(m)}_{(3,i)} \quad i < j
\]

\[
(L^{(m)}_{(3,i)})^{m} = 1 \quad \forall i = 1, 2, 3.
\]

\( \zeta \) is introduced in \( L^{(m)}_{(3,3)} \) when \( m \) is even since

\[
(C(m)^{m-1} B(m))^{m} = \omega(m)^{m} \omega(m)^{m-1} = -I
\]

As in the case of \( C^{(2)}_{m} \), we consider first

\[\text{Henceforth wherever } C(m) \text{ and } B(m) \text{ occur they will refer to these matrices, and when there is no chance of confusion the index } m \text{ will be omitted.}\]
\[
L_3^{(m)} = \sum_{i=1}^{3} x_i L_{(2, i)}^{(m)} = \begin{pmatrix}
\lambda_2 & x_1 & 0 & \ldots & \omega^{(m)} x_2 & \ldots & 0 \\
x_3 & \omega^{(m)} x_2 & x_1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

(1.39)

Now replace \( x_1 \) by \( L_3^{(m)} \) itself and \( x_2 \) and \( x_3 \) by \( x_4 I \) and \( x_5 I \) respectively, where \( I \) is the identity matrix of dimension \( m^2 = \text{dim}(L_3^{(m)}) \). This gives now

\[
L_5^{(m)} = L_{(3, 1)}^{(m)} \otimes \left( \sum_{i=1}^{3} x_i L_{(2, i)}^{(m)} \right) + x_4 L_{(3, 2)}^{(m)} \otimes I + x_5 L_{(3, 3)}^{(m)} \otimes I
\]

\[
= \sum_{i=1}^{5} x_i L_{(5, i)}^{(m)}
\]

(1.40)

Likewise

\[
L_7^{(m)} = L_{(3, 1)}^{(m)} \otimes \left( \sum_{i=1}^{5} x_i L_{(5, i)}^{(m)} \right) + x_6 L_{(3, 2)}^{(m)} \otimes I + x_7 L_{(3, 3)}^{(m)} \otimes I
\]

(1.41)

with \( \text{dim}(I) = \text{dim}(L_5^{(m)}) = m^2 \). In general

\[
L_{2v+1}^{(m)} = 0(L_{2v-1}^{(m)}) = \sum_{i=1}^{2v+1} x_i L_{(2v+1, i)}^{(m)}
\]

\[
= L_{(3, 1)}^{(m)} \otimes \left( \sum_{i=1}^{2v-1} x_i L_{(2v-1, i)}^{(m)} \right) + x_{2v} L_{(3, 2)}^{(m)} \otimes I + x_{2v+1} L_{(3, 3)}^{(m)} \otimes I
\]

(1.42)
with \( \dim I = \dim \left( L_{2V+1}^{(m)} \right) = m^{V-1} \). Explicitly writing

\[
(a) \quad \left( L_{2V+1}^{(m)} \right) (i) = \left( L_{(3,1)}^{(m)} \right) \otimes \left( L_{(2V-1,i)}^{(m)} \right) \quad i = 1, 2, \ldots, 2V-1
\]

\[
(b) \quad L_{(2V+1,2V)}^{(m)} = L_{(3,2)}^{(m)} \otimes I,
\]

\[
(c) \quad L_{(2V+1, 2V+1)}^{(m)} = L_{(3,3)}^{(m)} \otimes I, \quad \text{with} \quad \dim I = m^{V-1}
\]

For \( m = 2 \), (1.43) gives Dirac's procedure of obtaining the anti-commuting elements (for further details cf. Alladi Ramakrishnan). The above procedure gives as shown before,

\[
\dim \left( L_{2V}^{(m)} \right) = \left[ \frac{m}{2} \right]
\]

showing the irreducibility of the above representation. Since \( C_{2V}^{(m)} \) contains as shown before \( m^{2V} \) linearly independent elements as basis

\[
C_{2V}^{(m)} \cong M_{m^V}^V (K)
\]

where \( M_{m^V}^V (K) \) is the total matrix algebra of dimension \( m^V \) over a field \( K \) (containing \( \zeta = \omega(m) \) when \( m \) is even). As earlier shown the set of all possible products of all possible powers of

\[
\left( L_{2V+1}^{(m)} \right)^k, \quad k \in \{ 0, \left( \frac{m^V}{2} \right) - 1 \}
\]

does not contain completely linearly independent elements due to the relation (1.32). This makes this set split into a direct sum of \( m \)
sets each containing \( \binom{m}{2n} \) linearly independent terms. This is seen by writing \( L_{(2n+1,2n+1)}^{(m+1)} \) in terms of the other \( 2^{n} \) elements using (1.3a). Hence we get
\[
C_{(m)}^{(m)} = M_{m}^{(n)} \oplus \cdots \oplus M_{m}^{(n)} \quad (m \text{ copies})
\]
(for more details see \(^{22},^{23},^{24}\)) Raseevski\(^{8}\) has given a complete geometric treatment of Clifford algebra and in Chapter IX we shall attempt a generalization of this approach to Generalized Clifford algebra using a generalization of determinants due to Rangarathan \(^{30}\).

4. Computation Matrices and Product Transforms

Recently Alladi Ramakrishnan \(^{1,2}\) introduced the concepts of computation matrices and product transforms in connection with the e-comutation relations. Consider the relation
\[
L_{(n,i)}^{(m)} L_{(n,j)}^{(m)} = \omega(n) \ t_{ij} \ L_{(n,j)}^{(m)} L_{(n,i)}^{(m)} \quad ; \ i,j = 1, 2, \ldots, n.
\]
(1.47)
Henceforth let us drop the indices \( m \) a nd \( n \) and simply denote
\[
L_{(n,i)}^{(m)} = L_{i}^{m} \quad \text{when there is no chance of confusion anyway. Since}
\]
the integers \( t_{ij} \) satisfy they define an antisymmetric integer matrix as
\[
\quad \text{T} = (t_{ij})
\]
(1.48)
called computation matrix associated with the system of equations (1.47).

Now let us define a product transform on \( L_{i}^{m} \) as
\[
L_{i}^{m} = \prod_{j=1}^{n} L_{j}^{m} \quad ; \quad \delta_{ij}
\]
(1.49)
where \((u_{ij})\) are integers, positive or negative. They define an integer matrix

\[
U = (u_{ij})
\]  

(1.50)

Now determining the commutation relations among \(L^*_i\)'s we get

\[
L^*_i L^*_j = \left( \prod_{k=1}^{n} L^*_k \right) \left( \prod_{l=1}^{n} L^*_l \right)
\]

\[
= (U \tilde{U})_{ij} \left( \prod_{l=1}^{n} L^*_l \right) \left( \prod_{k=1}^{n} L^*_k \right)
\]

\[
= \omega(mn) L^*_i L^*_j L^*_i
\]

\[
= \omega(mn) L^*_i L^*_j L^*_i
\]

\[
\prod_{k=1}^{n} L^*_k
\]

\[
\prod_{l=1}^{n} L^*_l
\]

\[
i,j = 1, 2, \ldots, n.
\]

where

\[
(L^*_i L^*_j) = T^* = UT \tilde{U}.
\]  

(1.51)

Thus the commutation matrix \(T^* = (t^*_{ij})\) associated with the new set of matrices is given by

\[
T^* = UT \tilde{U}.
\]  

(1.52)

If \(|\det U| = 1\) then there exists a unimodular matrix \(V\) such that

\[
V = U^{-1}.
\]  

(1.53)
and
\[ T = V T^* \tilde{V} \]

This implies that there exists a product transform on \( L' \)
\[ L'_i = \prod_{j=1}^{n} L^*_j \quad ; \quad i = 1, 2, \ldots, n. \]

where
\[ (V L'_i) = V \]
such that \( L'_i \)'s have the same commutation matrix \( T \). Actually explicitly writing
\[ L'_i = \prod_{j=1}^{n} \left( \prod_{k=1}^{n} L^*_{j_k} \right)^{u_{j_k}} \]
\[ \sim \prod_{k=1}^{n} \left( \prod_{i=1}^{n} L^*_{i} \right)^{u_{j_k} u_{j_k}} = \prod_{k=1}^{n} L^*_{k} \]
\[ = L^*_i ; \quad \forall i \quad L^*_i = L^*_i \quad ; \quad i = 1, 2, \ldots, n. \]

The \( \tilde{V} \) defines in a sense an inverse transform to \( U \)-transform.

Now the commutation matrix associated with (1.50) is given by

\[ T = \begin{bmatrix}
0 & 1 & \cdots & 1 \\
-1 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & 0 \\
-1 & -1 & \cdots & 1
\end{bmatrix} \]
6. Tensor and Vertex coupling Method in terms of Product transform

Following σ-operation method Alladi Ramakrishnan formulated another equivalent method called Tensor and vertex coupling method for the representation of Clifford algebras. Here we reformulate this method in terms of product transforms introduced above. Consider the commutation relations of the type

\[ L^*_{2i-1} L^*_{2i} = \omega(m) L^*_{2i} L^*_{2i-1} \quad \text{if} \quad i = 1, 2, \ldots, \nu. \]

\[ L^*_{k} L^*_{l} = L^*_{l} L^*_{k} \quad \text{otherwise} \quad ; \quad l, k = 1, 2, \ldots, \nu. \]

Relabelling these as

\[ L^*_{2i-1} = H^i_1 \quad \text{if} \quad i = 1, 2, \ldots, \nu \]

\[ L^*_{2i} = H^i_2 \]

we have

\[ H^i_1 H^i_2 = \omega(m) H^i_2 H^i_1 \quad ; \quad i = 1, 2, \ldots, \nu. \]

\[ H^i_k H^j_k = H^i_k H^j_k \quad ; \quad i \neq j \quad ; \quad l, k = 1, 2. \]

The set of \( \nu \) matrices \( \{ H^i_k \} \quad l = 1, \nu; \quad k = 1, 2 \}

form a set of \( \nu \) pairs, the members of each pair commuting among themselves but members of different pairs commuting with each other. To each pair \( \{ H^i_k \} \quad k = 1, 2 \}

a third one can be attached

\[ H^i_3 = (H^i_1)^{-1} H^i_2 \]

such that the three form a complete set of mutually commuting matrices. Looking at the problem of the irreducible dimensionality of representation the following is clear. If we consider only two pairs then since the second pair commutes with the first one if the first one is represented irreducibly by \( m \times m \) matrices the members of the
second pair are to be multiples of identity matrices by Schur's lemma, but this contradicts the requirement that the second pair is also mutually commuting pair whose irreducible dimension is $n \times n$. Hence the irreducible dimension has to be at least $n^2$. By taking all possible products of all possible powers of the members of the two pairs we get $n^2$ linearly independent elements whose linear independence can be easily proved by arguments similar to the one given above. Thus the irreducible dimension is $n^2$.

In general if we take $\nu$ pairs the set of all possible products of all powers of them consists of $n^{2\nu}$ linearly independent elements thus requiring that the irreducible dimension has to be $n^{\nu}$.

It is very easy to construct these representations as follows

$$H_1^{i_1} = I \otimes \ldots \otimes I \otimes C(m) \otimes I \otimes \ldots \otimes I$$
$$H_2^{i_2} = I \otimes \ldots \otimes I \otimes B(m) \otimes I \otimes \ldots \otimes I$$

where

$$(C(m)B(m) = \omega(m)B(m)C(m))$$

$$\text{dim} \cdot C(m) = \text{dim} \cdot B(m) = \text{dim} \cdot I = m.$$ (1.63)

A linear combination of the three matrices of the same set

$$H^{i} = \sum_{j=1}^{3} x_{j} \cdot H_{j}^{i}$$ (1.64)

has been called 'generalized Helicity Matrices' since in the case $n = 2, \nu = 2$, they are called Helicity matrices in quantum mechanics.

Now in terms of these $\left\{ H_{k}^{i} | i = 1, 2, \ldots, \nu; k = 1, 2 \right\}$ form the set of matrices
\[ L_1 = H_1^\nu \]
\[ L_2 = H_2^\nu \]
\[ L_3 = (H_1^\nu)^{-1} H_2^\nu H_2^{\nu-1} \]
\[ L_4 = (H_1^\nu)^{-1} H_2^\nu H_2^{\nu-1} \]

\[ L_{2\gamma-1} = (H_1^\nu)^{-1} H_2^\nu (H_1^{\nu-1})^{-1} \ldots H_1^{\nu-(\gamma-1)} \]
\[ L_{2\gamma} = (H_1^\nu)^{-1} H_2^\nu (H_1^{\nu-1})^{-1} \ldots H_2 \]

\[ \gamma = 1, 2, \ldots, \nu. \]

These are seen to obey the ordered e-commutation relations
\[ L_i; L_j = \omega(m) L_j; L_i \quad ; \quad i, j = 1, 2, \ldots, 2\nu. \]
\[ i < j \quad \text{procedure} \]

This is the basic idea of tensor and matrix coupling method. The matrices obtained by this procedure are same as those obtained by e-operation method. Converse of this procedure of obtaining $\mathbb{H}$-matrices from $L_i^\nu$ has also been considered in detail by Alladi Ramakrishnan and for $m = 2$, this procedure of obtaining $\mathbb{H}$-matrices from $L_i^\nu$ has been considered by Clifford himself in 1873. We make a comparison of both in Chapter VIII. Now we shall reformulate the above method by using the product transforms. As already noted in (1.68) the commutation matrix associated with the system of ordered e-commutation relations is given by
The \( H \)-matrices defined by (1.60 - 1.61) interpreted in terms of \( L^0 \)-matrices have the commutation matrix

\[
T^* = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 0 & 1 & 0 & \cdots & 0 & 0 \\
-1 & -1 & 0 & 1 & \cdots & 0 & 0 \\
-1 & -1 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & -1 & -1 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & -1 \\
0 & 0 & 0 & 0 & \cdots & -1 & 0 \\
\end{bmatrix}
\]  

(1.67)

and one solution for \( V \) connecting \( T \) and \( T^0 \) is given by

\[
T = V T^* V^T ; \quad |\text{det} \, V| = 1, \quad v_{ij} \in \mathbb{Z}.
\]  

(1.69)

Consequently

\[
V = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & -1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & -1 & -1 \\
1 & 0 & -1 & \cdots & -1 & -1 & \cdots & \cdots \\
0 & 1 & -1 & \cdots & -1 & -1 & \cdots & \cdots \\
\end{bmatrix}
\]  

(1.70)

Hence as shown in IV., the matrices associated with \( T \) can be constructed from those associated with \( T^0 \) by using \( V \) as the kernel of the product transformation as

\[
L^l = \prod_{j=1}^{2V} L_j^* V_{ij} ; \quad l = 1, 2, \ldots, 2V.
\]  

(1.71)
and this reproduces the formula \( (1.66) \) as

\[
L_{2(\nu-1)} = L^*_{2\nu-1} L^*_{2\nu-3} \cdots L^*_{2(\nu-k)+1}
\]

\[
L_{2k} = L_{2\nu-1} L_{2\nu-3} \cdots L_{2(\nu-k)+1}
\]

(1.73)

One has to remember that after getting these \( L^* \)'s they have to be
sometimes norm lined suitably to obey the condition

\[
L^T \cdot \nu = \nu
\]

thereupon this situation will arise. The norming of \( \nu \) in
(1.66) does not matter as we shall see in later chapters. All \( \nu \)'s lead to
the same representation in the above case. This will become clear in
Chapter V. Generalization of the above method to obtain all the irreducible
representations of algebras generated by elements with an
arbitrary commutation matrix \( T \) will be considered in Chapter V leading
to the complete and simple solution of the problem of explicitly deter-
mining all the projective representations of any finite Abelian group.

**Theorem.**

The following matrix decomposition is due to Allied Homogenization. We shall study this theorem and an extension of it due to
Alieni Homogenization and myself in the next chapter.

**Theorem.** Any real matrix \( M \) can be expanded uniquely as

\[
M = \sum_{k,l=0}^{\nu-1} \alpha(k,l) b^k c^l
\]

(1.73)

where \( \alpha(k,l) = \alpha^T \) is determined by the relation,

\[
A = S^{-1} R
\]

where

\[
R = \begin{bmatrix}
M_{00} & M_{01} & \cdots & M_{0,m-1} \\
M_{11} & M_{12} & \cdots & M_{1,m-1} \\
\vdots & \vdots & \ddots & \vdots \\
M_{m-1,m-1} & M_{m-1,0} & \cdots & M_{m-1,m-2}
\end{bmatrix}
\]

and
being the inverse of Sylvester matrix $S$

$$S = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & w & \cdots & w^{m-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & w^{m-1} & \cdots & w^{m-1}
\end{bmatrix}$$

$$S^{-1} = \frac{1}{N} S^T$$

$$(S^{-1})_{ij} = \frac{1}{N} \omega^{-ij}$$

$$(S)_{ij} = \omega^{ij}, i,j = 0,1,\ldots,m-1.$$  (2.76)

$\omega = \omega(m)$

Proof of this follows by noting first that $M$ can be written as a linear combination

$$M = M_0 I + M_1 C(m) + \cdots + M_{m-1} C(m)^{m-1} + \cdots + M_{m-1} C(m)^{m-1}$$

where

$$M_l = \begin{bmatrix}
m_l & 0 \\
0 & m_l
\end{bmatrix}$$

and any diagonal matrix $D$ can be written as

$$D = d_0 I + d_1 B(m) + \cdots + d_{m-1} B(m)$$

and

$$(S^{-1})_{ij} D_{ij} = \omega^{-ij}$$

Putting together we arrive at the theorem

Let us put the formula for $A_{kl}$ in the following form for use later

$$A_{kl} = \frac{1}{m^l} \text{Tr} \left( M C(m)^{m-1} B(m)^{-k} \right)$$  (2.79)

or

$$A_{kl} = \frac{\omega(m)^k}{m} \text{Tr} \left( M B(m)^{-k} C(m)^{m-1} \right)$$  (2.80)
Significance of this theorem arises from the fact that it expresses any matrix (operator) in an orthonormal unitary matrix (operator) basis. Weyl's rule and Wigner distribution function used in quantum mechanics are derived from this fact only. Some details of application of these matrices to understand the algebraic structure of quantum kinematics have been treated explicitly by Schwinger and we shall consider these in Chapter II.

Summary of Important Points.

Any Generalized Clifford Algebra \( C_{(p,q)} \) has as its basis the set of all possible products of all possible powers of \( n \) generators obeying the commutation relations

\[
L_i L_j = \omega(m) L_j L_i \quad ; \quad i, j = 1, 2, \ldots, n. 
\]

Then \( k_{ij} = \delta_{ij} \) a \( n \times n \) matrix \( m = m \) this algebra becomes the usual \( C_{(0,0)} \). The anti-symmetric integer matrix \( T \) associated with \( \{L_i^*| i = 1, \ldots, n^2\} \) is called a 'commutation matrix'. If a 'product transform' is defined as

\[
L_i^* = \prod_{j=1}^{n} L_j^* \quad ; \quad i = b, \ldots, n. 
\]

the commutation matrix associated with \( \{L_i^*\} \) say \( T^* \) is related to \( T \) by \( T^* = U^* T U^* \). If \( |U| = 1 \) then \( U^* = U \) relates \( T \) and \( T^* \) by \( T = T^* \).

Thus if any irreducible representation of \( \{L_i^*\} \) is known an irreducible representation of \( \{L_i^2\} \) can be obtained if \( T^* \) a \( n \times n \) matrix are also explicitly known. Irreducibility of the representation of \( \{L_i^2\} \) implies the irreducibility of the representation thus obtained, since the mapping 'product transform' is invertible. The commutation relations and hence \( T \) essentially determine the matrices up to indeterminacy of normalization factors which are to be determined to fit the requirements \( L_i^* L_i = T \).

This product transform provides a powerful tool for representation theory of \( C_{(p,q)} \). For any anti-symmetric integer matrix \( T \), there is a unique
show normal form \[ T^* = U^T \tilde{y}, \text{ with } | \det U | = 1, \] of the simple type \[ T^* = \sum_{i=1}^{s} \begin{bmatrix} 0 & t_i^* \end{bmatrix} \oplus 0_{m_1-2k}, \quad s \leq \left[ \frac{m}{2\nu} \right] = \text{rank of } T^* \]

for which the corresponding set \( \{ L_i^* \}_{i=1}^{n} \) having the commutation relations \[ L_i^* L_j^* = \omega(m)^{tij} L_j^* L_i^*; \quad i,j = 1, \ldots, n; \quad (tij^*) = T^* \]

can be represented easily irreducibly leading to the determination of a irreducible representation of \( \{ L_i \}_{i=1}^{n} \) through the product transform \[ L_i = \prod_{j=1}^{s} L_i^* \] due to the possibility of explicit determination of a corresponding \( V \) and \( V^* \) (Cf. Morrid Neuman*). We shall take up this matter further in Chapter VII.

This new approach to matrix theory due to Alladi Ramakrishnan consists in viewing an $n \times n$ matrix as a linear combination of the $m^2$ matrices $\{ B^R C^l : R, l = 0, \ldots, m-1 \}$ generated by taking all possible products of all possible powers of $B$ and $C$. For some purposes this approach is useful as is demonstrated in the following problems.

(a) To find the number of inequivalent irreducible representations of:

\begin{align*}
(2.1) \quad & L_1 L_2 = \omega(m)^l L_2 L_1 \\
& L_1^m = L_2^m = 1 \quad (l, m) = d
\end{align*}

Let $d = d'$ and $m = m'$; $(l', m') = 1

\begin{align*}
(2.2) \quad & \frac{d}{d'} = \frac{m}{m'}
\end{align*}

Then

\begin{align*}
(2.3) \quad & \omega(m)^l = \exp \left( \frac{2\pi i l m}{m'} \right) = \exp \left( \frac{2\pi i l d'}{d} \right) = \omega(m')^{l'}
\end{align*}

Now consider the relations

\begin{align*}
(2.4) \quad & L_1 L_2 = \omega(m')^{l'} L_2 L_1 \\
& L_1^{m'} = L_2^{m'} = 1 \quad (l', m') = 1.
\end{align*}

These relations are same as (1.23) for $n = m'$, $n = 2$ except that
\(o(m')\) has been replaced by \(\omega(m')^{l'}\) which is also a primitive \(m'\)th root. Hence the irreducible representations of (2.4) are given by the same matrices as in (1.36) except that \(o(m')\) has replaced by \(\omega(m')^{l'}\). Hence \(L_1', L_2'\) are represented by \(m'\)-dimensional \(C\) and \(B\) matrices with \(\omega(m') \rightarrow \omega(m')^{l'}\) or,

\[
L_1' = C(m'), \quad L_2' = B(m')^{l'}
\]

(2.5)

In the relations (2.1) the first determines the structure of the matrix up to the uncertainty of a phase factor and the second is a normalisation condition which shows that the matrices obeying both can always be multiplied by \(m'\)th root of unity. Since the first relation determines the irreducible dimension its equivalence with (2.4a) shows that

\[
L_1 \sim L_1', \quad L_2 \sim L_2'
\]

(2.6)

The possibilities are given completely by

\[
\left\{ L_1 = \omega(m)^{k_1}, L_1', L_2 = \omega(m)^{k_2}, L_2' \mid k_1, k_2 = 0, 1, \ldots, m-1 \right\}
\]

(2.7)

Thus there are \(m^2\) possibilities. But all of these are not inequivalent. If two representations say,

\[
R_1: \left\{ L_1 = \omega(m)^{k_1}, L_1'; L_2 = \omega(m)^{k_2}, L_2' \right\}
\]

\[
R_2: \left\{ L_1 = \omega(m)^{k_1}, L_1'; L_2 = \omega(m)^{k_2}, L_2' \right\}
\]

(2.8)
are equivalent then there exists a nonsingular $m' \times m'$ matrix $S$ such that

$$
\omega(m) L_1 S = S \omega(m) L_1' \tag{2.10}
$$

$$
\omega(m) L_2 S = S \omega(m) L_2' \tag{2.11}
$$

or

$$
L_1 S = \omega(m)^{k_2-k_1} S L_1 = \omega(m)^h S L_1 \tag{2.11a}
$$

$$
L_2 S = \omega(m)^{l_2-l_1} S L_2 = \omega(m)^h S L_2 \tag{2.11b}
$$

Now use the theorem (1.12) to write

$$
S = \sum_{k_1, y=0}^{m-1} a_{k_1} L_1^{k_1} L_2^{y} \tag{2.19}
$$

$$
L_1 \left\{ \sum_{k_1, y=0}^{m-1} a_{k_1} L_1^{k_1} L_2^{y} \right\} = \omega(m)^{h_1} \left\{ \sum_{k_1, y=0}^{m-1} a_{k_1} L_1^{k_1} L_2^{y} \right\} L_1' \tag{2.19a}
$$

$$
L_2 \left\{ \sum_{k_1, y=0}^{m-1} a_{k_1} L_1^{k_1} L_2^{y} \right\} = \omega(m)^{h_2} \left\{ \sum_{k_1, y=0}^{m-1} a_{k_1} L_1^{k_1} L_2^{y} \right\} L_2' \tag{2.19b}
$$

Multiplying the first by $L_1^{-1}$ and second by $L_2^{-1}$ from the left we get

$$
\left\{ \sum_{k_1, y=0}^{m-1} a_{k_1} L_1^{k_1} L_2^{y} \right\} = \omega(m)^{h_1} \left\{ \sum_{k_1, y=0}^{m-1} a_{k_1} \omega(m) - \omega(m') L_1^{k_1} L_2^{y} \right\} L_1' \tag{2.19c}
$$

$$
\left\{ \sum_{k_1, y=0}^{m-1} a_{k_1} L_1^{k_1} L_2^{y} \right\} = \omega(m)^{h_2} \left\{ \sum_{k_1, y=0}^{m-1} a_{k_1} \omega(m') L_1^{k_1} L_2^{y} \right\} L_2' \tag{2.19d}
$$
By the uniqueness of the expansion of $S$ it follows
\[ \alpha_{k\gamma} = \omega(m) \cdot a_{k\gamma} \cdot \omega(m') \cdot l' \gamma = \omega(m) \cdot a_{k\gamma} \cdot \omega(m') \cdot k e' \]
\[ = \omega(m) \cdot a_{k\gamma} - \omega(m) \cdot a_{k\gamma} = \omega(m) \cdot a_{k\gamma}, \quad \forall \alpha_{k\gamma} \neq 0. \tag{2.15} \]

or
\[ h_1 - l \gamma = 0 \mod m \]
\[ h_2 + l k = 0 \mod m \quad \forall k, \gamma \text{ for which } a_{k\gamma} \neq 0 \tag{2.16} \]

If there is only one $\alpha_{k\gamma} \neq 0$ then immediately the solution for $h_1$ and $h_2$ follows
\[ h_1 = \gamma \]
\[ h_2 = -k \quad \text{or} \quad l(m-k) \tag{2.17} \]

But if more than one $\alpha_{k\gamma} \neq 0$ then no solution exists for $h_1$, $h_2$, or stating it conversely for a given $(h_1, h_2)$ the $S$ can only be a multiple of any one of the members of the set
\[ \{ L_1^k L_2^l | k, l = 0, 1, \ldots, m'-1^2 \} \]
and cannot be a sum of more than one term. This can be established by the following argument also. Let
\[ S = \left( \sum_{k,l=0}^{m'-1} a_{k\gamma} L_1^k L_2^l \right). \]
For $S$ to be a sum of more than one term number of
\[ \{ L_1^k L_2^l | k, l = 0, 1, \ldots, m'-1^2 \} \]
those terms must have the same commutation relation with $L_1$ and $L_2$. But if
\[ L_1^k L_2^l \quad \text{and} \quad L_1^{k'} L_2^{l'} \quad (k, l) \neq (k', l') \]
both have same commutation relation with $L_1$ and $L_2$, $(L_1^k L_2^l)^{-1} (L_1^{k'} L_2^{l'})$ must commute with $L_1$ and $L'_2$ or by Schur's lemma $(L_1^k L_2^l)^{-1} (L_1^{k'} L_2^{l'}) = I$
\[ l_1 = l_2, l_1 = l_2 \]
contrary to the initial assumption that
\[ (k_1, l_1) \neq (k_2, l_2). \]
Thus \( k \) can not be a sum of more than one number of
\[ \{ L_1^k L_2^l \mid 0 \leq k, l \leq m' - 1 \} \].
Thus all the \( m^2 \) elements \[ \{ L_1^k L_2^l \mid 0 \leq k, l \leq m' - 1 \} \] have \( m^2 \) different commutation relations with \( L_1 \) and \( L_2 \).

From (3.17) it is seen that for different \( (\gamma_k, k) \) values \( h_1 \) and \( h_2 \) are different and as \( (\gamma, k) \) take values \( (0, 1, \ldots, m' - 1) \) and \( w(m)^{h_1}, w(m)^{h_2} \) each takes all values of the \( m' \)-th roots of unity. Thus all the \( m^2 \) different pairs given by
\[
\{ w(m)^{k_1} L_1^{l_1}, w(m)^{k_2} L_2^{l_2} \mid k_1, k_2 \neq l_1, l_2 \}
\] (3.18)
are equivalent. So \( R_1 \) and \( R_2 \) are equivalent iff
\[ k_1 - k_2, l_1 - l_2 \neq l \]
or
\[ w(m)^{k_1 - k_2} = w(m)^{l_1 - l_2} \]
for some value of \( \gamma, k = 0, 1, \ldots, m' - 1 \). Thus if we start with a representation,
\[
R_1: \{ L_1 = w(m)^{k_1} L_1^{l_1}, L_2 = w(m)^{k_2} L_2^{l_2} \}
\] (3.21)
than the representations given by
\[
\{ L_1 = w(m)^{k_1 + \gamma} L_1^{l_1}, L_2 = w(m)^{k_2 + \gamma} L_2^{l_2} \mid \gamma, k = 0, 1, \ldots, m' - 1 \}
\] (3.22)
are all equivalent. Thus by starting with a particular set of
values \((k_i, l_i)\) out of (2.2) all these \(m_i\) can be classified into a single representation, another set values for \((k_i, l_i)\) is chosen and all those equivalent to it are classified and in this way one arrives at \((m_i^2/m_{i+1}^2) = d_i^2\) inequivalent representations. The above procedure leads to classification of representations as

\[
R_i : \{ L_1 = \omega(m_i) L_i^{k_i} L_2 L_2^{\gamma_i} ; i = 1, 2, \ldots, d_i^2 \}
\]

where \(k_i\) and \(l_i\) take values

\[
k_i, l_i = 0, 1, 2, \ldots, d_i - 1.
\]

Proof: \(\{\omega(m)k, \omega(m)l\}\) should be chosen from the quotient set

\[
\begin{align*}
\{ &1, \omega(m), \ldots, \omega(m)^{m-1}\} \\
\{ &1, \omega(m)^{l}, \ldots, \omega(m)^{l(m-1)}\}
\end{align*}
\]

Now upto rearrangement,

\[
\{ 1, \omega(m), \ldots, \omega(m)^{l(m-1)} \} = \{ 1, \omega(m), \ldots, \omega(m)^{l(m-1)} \}
\]

since

\[
\omega(m)^{l \cdot k} = \omega(m)^{l' \cdot k} \quad 0 \leq l \leq m - 1
\]

has a unique solution for \(k\) by Euler-Fermat theorem

\[
k = r \phi(m_i - 1) \mod m_i
\]

writing

\[
\{ 1, \omega(m), \ldots, \omega(m)^{m-1} \} = \{ 1, \omega(m)^d, \ldots, \omega(m)^{m-d} \}
\]
it follows

\[(2.25) \equiv \{1, \omega(m), \ldots, \omega(m)^{d-1}\}\]

Thus we have found \(d^2 = (\frac{m}{m'})^2\) inequivalent representations of \((2.1)\)

Corresponding to one representation of \((2.4)\). Now yet we do not know whether these \(d^2\) exhaust all possible representations. Suppose there exists another representation inequivalent to those \(\{L_1, L_2\}\) which we denote by \(\{L_1', L_2'\}\). The relations \((2.1)\) imply the following

\[L_1' L_2' = \omega(m) L_2' L_1' \quad \text{or} \quad L_1' L_2' = L_2' L_1'\]

\[L_1' L_2 = \omega(m) L_2' L_1 \quad \text{or} \quad L_1' L_2 = L_2' L_1\]

or \(L_1'\) and \(L_2'\) must commute with both \(L_1\) and \(L_2\) and hence with all possible products of all possible powers of them.

Thus by Schur's lemma

\[L_1' \sim 1, \quad L_2' \sim 1\]

let

\[L_1' = \xi I, \quad L_2' = \eta I\]

Then \((2.30)\) gives

\[(L_1' L_2')^d = \xi^d I, \quad L_1' = \xi^d I = I\]

\[(L_2' L_1')^d = \eta^d I = I\]

So

\[\xi = \exp(2\pi i k/d) = \omega(d)^k \quad k, t = 0, 1, \ldots, d-1\]

\[\eta = \exp(2\pi i t/d) = \omega(d)^t\]
Thus the following holds

\[(\lambda L_1)^{m'} = 1, \quad (\mu L_2)^{m'} = 1,\]

where

\[\lambda^{m'} w(d)^k = 1, \quad \mu^{m'} w(d)^t = 1\]

or

\[\lambda^{m'} = w(d)^{d-k} \quad \text{and} \quad \mu^{m'} = w(d)^{d-t}\]

Thus for any representation of \(L_1, L_2\), there exist some value of \(k\) and \(t\) such that

\[(w(m)^{d-k} L_1)^{m'} = 1, \quad k, t < d.\]

\[(w(m)^{d-t} L_2)^{m'} = 1.\]

Thus \(\{w(m)^{d-k} L_1, w(m)^{d-t} L_2\}\) obey (2.4). As porno\(^3\) and Morris\(^5\) have shown the relations (2.1) have only one inequivalent irreducible representation. Thus there exists some \(S\), such that for \(L_1\) and \(L_2\) for some \(k\) and \(t\)

\[w(m)^{d-k} L_1 S = S L_1,\]

\[w(m)^{d-t} L_2 S = S L_2.\]

or

\[S^{-1} L_1 S = w(m)^{m+k-d} L_1, \quad w(m) \cdot w(m') L_1;\]

\[S^{-1} L_2 S = w(m)^{m+t-d} L_2, \quad w(m) \cdot w(m') L_2;\]

\(s, u < m',\)

\(r, u < d.\)
\[ m - (d - k) = \lambda'd + k \quad \lambda' < d \]
\[ m - (d - t) = \nu'd + u \quad u < d \]  \hspace{1cm} (2.42)

has solutions and then set
\[ s \mod m' = s' \]
\[ v \mod m' = v' \]  \hspace{1cm} (2.43)

By the arguments given above there exists a \( U \) such that
\[ w(m')^{\lambda'} L'_{1} = U L'_{1} U^{-1} \]
\[ w(m')^{v} L'_{2} = U L'_{2} U^{-1} \]  \hspace{1cm} (2.44)

so
\[ (SU)^{-1} L''_{1} SU = w(m)^{\lambda'} L'_{1} \]
\[ (SU)^{-1} L''_{2} SU = w(m)^{v} L'_{2} \quad 0 \leq \gamma, \nu < d - 1 \]  \hspace{1cm} (2.45)

Thus it follows that any representation \( \{ L''_{1}, L''_{2} \} \) is equivalent
to any one of the \( d^2 \) representations \( \{ w(m)^{\lambda'} L'_{1}, w(m)^{v} L'_{2} | 0 \leq \lambda', \nu < d - 1 \} \) where \( \{ L'_{1}, L'_{2} \} \) is the unique (up to equivalence) representation
of \( (2,4) \). Here we have assumed the result of Nunez\(^2\) and Morris\(^3\)
for proving that \( (2.1) \) has \( d = (l, m)^{2} = \frac{m'\nu}{d} \) inequivalent representations
of same dimension \( D = \frac{m'}{(l, m)} = m' \). Later in Chapter IX - V we
shall prove this result a priori in a more general way. The result
that the relations
\[ L_{1} L_{2} = w(m) L_{2} L_{1} \]
\[ L_{1}^{m'} = L_{1} \]
\[ L_{2}^{m'} = L_{2} \]  \hspace{1cm} (3.46)
have only one irreducible representation is known to physicists from the work of Hermann Weyl who used these representations to show that the quantum operators for position and momentum are given by the Schrodinger representation

\[
\begin{align*}
\text{position} & \quad q \rightarrow q \hat{x} = \hat{q} \\
\text{momentum} & \quad p \rightarrow -i\hbar \frac{\partial}{\partial q} = \hat{p}.
\end{align*}
\]

\(\hbar = \) Planck's constant. Thus the uniqueness of representation of (2.46) implies the uniqueness of Schrodinger representation quantum mechanics.

Usually Weyl's form of canonical commutation rule is written as

\[
U_d^* V_p = e^{i\hbar \alpha} V_p U_d
\]

where

\[
U_d = e^{i\alpha \hat{q}}, \quad V_p = e^{i\beta \hat{p}} \quad [\hat{q}, \hat{p}] = i\hbar.
\]

\[
(U_d \psi)(\nu) = \psi(\nu + \hbar \alpha); \quad (V_p \psi)(\nu) = e^{i\beta \nu} \psi(\nu)
\]

Von Neumann proved the uniqueness of Schrodinger representation \((2.46, 2.48)\) up to unitary equivalence and since then many other proofs have been given. For more general discussions on these considerations reference can be made to \(\text{c}\).
b) Understanding of Weyl's rule and Wigner distribution function.

Following the statement of Weyl \(^4\) that we can understand the relations existing in Hilbert space by analogy with or as limiting cases of those existing in spaces of a finite number of dimensions, here we shall understand the mathematical aspects of Weyl correspondence from the matrix decomposition theorems.

Now let \(\nu = \nu + 1\) and let us take our generator matrices \(C\) and \(B\) as follows:

\[
C = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
& 0 & 0 & \cdots & 0 \\
& & 0 & \cdots & 1 \\
& & & \ddots & \vdots \\
& & & & 0
\end{pmatrix}, \quad
B = \begin{pmatrix}
\omega(2\nu+1)^{-1} & \omega(2\nu+1)^{-\nu+1} & 0 \\
\omega(2\nu+1)^{-\nu+1} & 1 & \omega(2\nu+1) \\
0 & \omega(2\nu+1) & \omega(2\nu+1)^{-1}
\end{pmatrix}
\]

(2.51)

Let us choose the basis elements as:

\[
\mathcal{B} = \left\{ \omega(2\nu+1)^{\nu \cdot k \cdot l} B^{k} C^{l} \mid k, l = -\nu, -\nu+1, \ldots, 0, 1, \ldots \nu \right\}
\]

(2.50)

Then any \((2\nu+1) \times (2\nu+1)\) matrix is written uniquely as

\[
M = \sum_{k, l = -\nu}^{\nu} a_{kl} \omega(2\nu+1)^{\nu \cdot k \cdot l} B^{k} C^{l}
\]

(2.53)

where

\[
a_{kl} = \frac{1}{2(2\nu+1)} \text{Tr} \left( M \omega(2\nu+1)^{\nu \cdot k \cdot l} C^{-1} B^{-1} \right)
\]

Let \(X = (x_{\nu}, x_{\nu+1}, \ldots, x_{\nu})\) be any \((2\nu+1)\) dimensional vector. The effect of
\( C^l \) and \( B^k \) on it are given by

\[
(C^l X)_j = \omega(i + l) x_j, \quad (B^k X)_j = \omega(2n+1) x_j, \quad j, j = -n, \ldots, n.
\]

(2.54)

In analogy to this finite dimensional case consider the function space (square integrable) where now \( \phi(q) \) is regarded as one coordinate of the \( \infty \) (non-density) dimensional vector \( \psi \). Now following Weyl consider the relations:

\[
(C^l \psi)(q) = \psi(q + \xi_l \xi), \quad (B^k \psi)(q) = \xi q \psi(q) = \psi(q + \sigma)
\]

(2.55)

which are analogues of (2.54) for \( \gamma \rightarrow \alpha \) in a continuous way. When \( \gamma \rightarrow \infty \) continuously \( l \) and \( k \) are also continuous. So \( C^l \) and \( B^k \) form one parameter continuous groups as \( l, k \) vary and hence by Stone's theorem they can be represented exponentially

\[
C^l \rightarrow e^{i_\xi \hat{P}}, \quad B^k \rightarrow e^{i_\xi \hat{Q}}
\]

(2.56)

Following Weyl, by returning to the infinitesimal transformations from finite transformations \( \hat{P} \) and \( \hat{Q} \) are identified as Schrödinger representation of momentum and position operators namely \(-i \frac{\partial}{\partial q} \) and \( q \) respectively (taking \( \hbar = 1 \)). With this identification (2.56) in the case of infinitesimal dimensional (non-density) vector space, the analogue of (2.55) becomes for any operator in this space

\[
M = \iint a(\tau, \sigma) e^{i_\xi \hat{P}} e^{i_\tau \hat{Q}} e^{i_\sigma \hat{P}} d\sigma d\tau
\]

(2.57)
or using the fact
\[ e^{\frac{i}{2} \sigma \cdot \hat{T}} e^{i \sigma \cdot \hat{P}} e^{i(\hat{T} \cdot \hat{P})} = e \]

when
\[ [\hat{T}, \hat{P}] = i \hbar \]
we have
\[ M = \int \int a(\tau, \sigma) e^{i\tau \hat{T} + i\sigma \cdot \hat{P}} d\tau d\sigma \]

where
\[ a(\tau, \sigma) = \frac{1}{\sqrt{\pi}} \text{Tr} \left\{ M e^{-\frac{i}{2} \sigma \cdot \hat{T}} e^{i \sigma \cdot \hat{P}} e^{i \tau \hat{T}} e^{i \sigma \cdot \hat{P}} \right\} = \frac{1}{\sqrt{\pi}} \text{Tr} \left\{ M e^{i \sigma \cdot \hat{T}} e^{i \tau \hat{T}} e^{i \sigma \cdot \hat{P}} \right\} \]

Now we shall use carefully the language of Dirac delta function to evaluate this trace in position representation of the wave functions. In position representation, \( e^{i \tau \hat{T}} \) is diagonal since
\[ e^{i \tau \hat{T}} \delta(q-q') = e^{i \tau T} \delta(q-q') \]

Hence
\[ \langle q'' | e^{i \tau \hat{T}} | q' \rangle = e^{i \tau T} \delta(q'-q'') \]

We have
\[ \langle q'' | e^{i \sigma \cdot \hat{P}} | q' \rangle = \delta(q'' - q' - \sigma) = \langle q'' | q' + \sigma \rangle \]
It is usual to write this as \( \langle \psi^\prime - \frac{i}{2} \sigma | \psi^\prime + \frac{i}{2} \sigma \rangle \). So we shall write (2.61) as

\[
a(\tau, \delta) = \frac{1}{2\pi} \int \frac{d\psi'}{2\pi} \langle \psi' | M | e^{-i\sigma \hat{p} - i\tau \hat{q}'} \rangle \\
= \frac{1}{2\pi} \int \frac{d\psi'}{2\pi} \langle \psi' | M | e^{-i\sigma \hat{p} - i\tau \hat{q}'} \rangle \langle \psi' | e^{i\sigma \hat{p} + i\tau \hat{q}'} \rangle \\
= \frac{1}{2\pi} \int \frac{d\psi'}{2\pi} \langle \psi' - \frac{i}{2} \sigma | M | \psi' + \frac{i}{2} \sigma \rangle e^{-i\tau \psi'}
\]

(2.68)

This case relation can be obtained directly by taking the limit of relation (1.74) for \( a_{K\ell} \)

\[
a_{K\ell} = \sum_{\gamma} (S^{-1})_{K\ell} R_{\gamma \ell}
\]

(2.66)

or

\[
a_{K\ell} = \frac{1}{2} \sum_{\gamma} \omega^{-k_{\gamma}} e^{-i \frac{K_{\gamma}}{2}}
\]

(2.67)

This becomes

\[
a(\tau, \delta) = \frac{1}{2\pi} \int \frac{d\psi'}{2\pi} e^{-i \tau \psi'} \langle \psi' - \frac{i}{2} \sigma | M | \psi' + \frac{i}{2} \sigma \rangle
\]

(2.69)

as in (2.66). Thus we have shown that the expansion (2.69) of operators (Hilbert–Schmidt operators to be precise) on a Hilbert space of single variable can be understood as a limiting case of Alladi Ramakrishnan theorem (2.53) for finite dimensions.
In classical mechanics each observable attribute of a physical system is represented by a function on the phase space of the system. \(\hat{a}\) suggested the following method of obtaining quantum mechanical operators corresponding to classically observable functions on phase space. In (2.52) if we regard \(\hat{a}\) and \(\hat{p}\) as classical canonical variables, \(q\) and \(p\) then \(\mathcal{M}(q,p)\) is a function in phase space and \(a(\tau, \sigma)\) are Fourier coefficients of \(\mathcal{M}(q,p)\) defined by
\[
a(\tau, \sigma) = \int\int \mathcal{M}(q,p) e^{-i\tau q - i\sigma p} \, dq \, dp
\]
(2.66)

Then \(\hat{a}\) suggested to regard (2.52), with the replacement \(q \rightarrow \hat{q}\), \(p \rightarrow \hat{p}\), as the quantum mechanical operator corresponding to the observable represented classically by the phase space function \(\mathcal{M}(q,p)\). Pool has given a well defined meaning to this expression (2.53) using basic results of measure theory, the theory of Hilbert spaces and harmonic analysis.

If we taking the hermitian conjugate of (2.62)
\[
\mathcal{M}^+ = \sum_{k,l=-\nu}^{+\nu} A_{kl}^* \omega(2\nu+1) C B \rho_1 C \rho - k - l
\]
(2.63)
If \( M = \Pi \) then this shows

\[
a^{*}_{kl} = a_{-k-, -l}
\]

Let \( \bar{A} = (\alpha_{kl}) \) be a real matrix \((k, l = -\nu, \ldots, +\nu)\) then define the double Fourier transform

\[
\tilde{\alpha}_{\sigma \tau} = \sum_{k, l = -\nu}^{+\nu} \alpha_{kl} \omega(2\nu+1)^{-\sigma k + \tau l}
\]

We have

\[
\tilde{\alpha}_{-\sigma \tau} = \sum_{k, l = -\nu}^{+\nu} \alpha_{kl} \omega(2\nu+1)^{-\sigma k + \tau l} = \tilde{\alpha}^{*}_{\sigma \tau}
\]

Hence the matrix

\[
M = \sum_{\sigma \tau = -\nu}^{+\nu} \tilde{\alpha}_{\sigma \tau} \omega(2\nu+1)^{-\sigma k + \tau l} B^{\sigma} C^{\tau}
\]

is hermitian. It is this property of this expansion that to every real matrix \( \bar{A} \) there is associated a unique Hermitian matrix

\[
M = \sum_{\sigma \tau = -\nu}^{+\nu} \sum_{k, l = -\nu}^{+\nu} \alpha_{kl} \omega(2\nu+1)^{-\sigma k + \tau l - \frac{\tau}{2} \sigma \tau} B^{\sigma} C^{\tau}
\]

which is the basis of Weyl correspondence. The elements of the real matrix \( \bar{A} \) associated with a Hermitian matrix \( M \), i.e., the inverse
transform $H \implies A$ is given by the formula

$$a(p, \eta) = \int \left\{ \int \left( \int a(\eta', \eta) e^{-i \pi \eta'} \left< \eta' - \frac{\sigma}{2} | M | \eta' + \frac{\sigma}{2} \right> \right) d\eta' \right\} d\sigma d\eta e^{i \pi \eta + i \sigma}$$

$$= \int d\eta \int d\eta' \int d\tau \left< \eta' - \frac{\sigma}{2} | M | \eta' + \frac{\sigma}{2} \right> e^{i \tau (\eta - \eta')} e^{i \sigma p}$$

$$= \int d\sigma \int d\eta' \left< \eta' - \frac{\sigma}{2} | M | \eta' + \frac{\sigma}{2} \right> e^{i \sigma p} \delta(\eta - \eta')$$

$$= \int d\sigma e^{i \sigma p} \left< \eta - \frac{\sigma}{2} | M | \eta + \frac{\sigma}{2} \right>$$

(2.75)

This expresses the inverse of Weyl transform $-a(p, \eta')$ is the classical observable corresponding to the quantum mechanical operator $H$ and if $H$ is Hermitian $A$ is real. This formula can also be understood as the limiting case of inverse formula of (2.74). We will not treat this point any more in detail. The Wigner quasi-probability distribution function $f_w(\eta, \sigma)$ is obtained by taking the Weyl transform of $M = \langle \psi | \psi \rangle$, the density matrix by substituting in (2.75) $M = \langle \psi | \psi \rangle$ we get the familiar form of the Wigner function

$$f_w(\eta, \sigma) = \int d\sigma' e^{i \sigma' p} \psi(\eta - \frac{\sigma'}{2}) \psi(\eta + \frac{\sigma'}{2})$$

(2.76)

(For very detailed consideration of these refer\(^5\).

Schrödinger\(^9\) developed further the approach of Weyl to consider the algebraic structure of quantum kinematics as a limiting case of
the algebraic structure manifested in finite dimension by the
matrices (operators) \( C \) and \( B \) with \( \pi \)-commutation relation \( C B = \pi B C \).
He shows that the set of matrices
\[ \chi(\ell \ell') = \pi^{-\frac{1}{2}} e^{i\pi \frac{k\ell l'}{m^2}} B^{\ell \ell'} \]
form a complete orthonormal operator basis and therefore together
supply the foundation for a full description of a physical system
possessing \( m \) states. Interpreting in terms of measurement algebra
he says that the properties of \( C \) and \( B \) exhibit the maximum degree
of incompatibility and thus are \( C \) and \( B \) form a complementary pair
of operators. He feels that though the algebraic properties of \( C \) and
\( B \) have been known long since the work of Hermann Weyl, there has been
a lack of appreciation of these operators as generators of a com-
plete operator basis for any \( m \) and of their optimum incompatibility
as summarized in the attribute of complementarity. He also has stressed
that an a priori classification of all possible types of phys-
sical degrees of freedom emerges from these considerations, while
leaving further interesting details to the reference of Schwinger, 9
let us pass on to a generalization of matrix decomposition theorems
due to Alladi Ramakrishnan and myself, which is an explicit version
of Schwinger's suggestion of commutative factorization of the unitary
operator basis, provided by \( B, C \), , for classification of several
degrees of freedom of a physical system with finite number of states.

II. Extended versions of matrix decomposition theorems.

**Theorem** Any \( m \times m \) matrix can be expanded as

\[ M = \sum_{0 \leq k_i, l_i \leq m_i-1} a_{k_i l_i} B(m_i)^{k_i} C(m_i)^{l_i} \otimes \cdots \otimes B(m_{i+1})^{k_{i+1}} C(m_{i+1})^{l_{i+1}} \]

where \( B(m_i), C(m_i) \) are \( B \) and \( C \) matrices of dimension \( m_i \).
obeying \( C(m_i; B(m_i) = \omega(m_i, B(m_i) C(m_i)), C(m_i) m_i = B(m_i) m_i = I \quad \forall i \neq r \)

and \( m = \prod_{i=1}^{r} m_i \). If \( A \) is the matrix of the coefficients

\[
\begin{pmatrix}
 a_{k_1 l_1} & \cdots & a_{k_r l_r}
\end{pmatrix}
\]

in which the element \( a_{k_1 l_1} \ldots a_{k_r l_r} \) occurs in

the position of the element \( d_{k_1 l_1} \ldots d_{k_r l_r} \) in a matrix

\[
D_1 \otimes \cdots \otimes D_r, \quad D_i \text{ being any matrix of dimension } m_i
\]

with elements \( d_{k_i l_i}, 0 \leq k_i, l_i \leq m_i - 1 \), then it is given by the prescription

\[
A = S_i^{-1} R
\]

where

\[
S_i^{-1} = \begin{pmatrix} S_1^{-1} \otimes \cdots \otimes S_r^{-1} \\
S_{i-1}^{(i)} \end{pmatrix}_{k l}
\]

\[
(\omega(m_i))^{-1} \quad 0 \leq k, l \leq m_i - 1.
\]

and \( R \) is the rearranged matrix of \( M \), the rearrangement being done

in \( r \) stages, first rearranging \( M \) as an \( m_1 \times m_1 \) partitioned matrix,

then rearranging each of the constituent matrices as a partitioned matrix of order \( m_2 \times m_2 \) \ldots \ldots and so on till finally the partitioning stops at the \( r \) th stage. Proof of this proposition follows easily repeating the arguments for the case of \( r = 1 \) dealt with in the last chapter. First the matrix \( M \) can be decomposed into a linear sum of matrices as

\[
M = \sum K_{l_1 l_2 \ldots l_r} C(m_1)_{l_1} \otimes \cdots \otimes C(m_r)_{l_r}
\]

where \( K_{l_1 l_2 \ldots l_r} \) are unique diagonal matrices. Then each of these
diagonal matrices can be written as a linear form

$$K l_1 l_2 \ldots l_R = \sum_{0 \leq k_i \leq m_i - 1} A_{k_1 l_1 \ldots k_R l_R} B(m_{l_1})^{k_1} \otimes \ldots \otimes B(m_{l_R})^{k_R} \tag{2.80}$$

Thus follows the theorem and the relation (2.79) prescribes the rearrangement operation \( M \to R \). \( R \) contains the elements of \((K l_1 l_2 \ldots l_R)\) as columns with lexicographic ordering from left to right.

The decomposition of the type (2.77) stops at the unique decomposition \( m = \prod_{l=1}^{r} p_i^{a_i} \) where \( p_i \)'s are distinct primes.

Schrodinger calls the unique quantity \( \sum_{l=1}^{r} a_i = \Omega(m) \) as the number of freedom for a system possessing \( m \) states. Following the same procedure as we did for the case of \( \tau = 1 \), we can go to the limit of each \( m_i \to \infty \) in (2.77). In this also as before one has to make the necessary in-consequential changes of labelling as

\[-V_i \leq k_i, l_i \leq +V_i\]

with \( m_i = 2V_i + 1, \forall i = 1 \ldots r \) and multiplying each

\[B(2V_i + 1)^{k_i} C(2V_i + 1)^{l_i}\]

by \( \omega(2V_i + 1)^{k_i l_i} \). Then following the same steps and associating the correspondence

\[C(m_i)^{l_i} \to e^{i \theta_i \hat{P}_i}, \quad B(m_i)^{k_i} \to e^{i T_i \hat{Q}_i} \]

\[\forall i = 1 \ldots r \] \tag{2.81}

the continuous analogue of (2.77) becomes for any operator \( M \) involving \( \gamma \) variables

invoking
\[ M(p_i, v_i) = \int \cdots \int \prod_{l=1}^r d\tau_i \, d\Phi_i \, a_i(\tau_i, \sigma_i) \, e^{i(\sum_{l=1}^r \tau_i \cdot \Phi_i + \sum \sigma_i \cdot p_i)} \]

\[ a_i(\tau_i, \sigma_i) = \frac{1}{(2\pi)^r} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{r} d\varphi_j \, e^{-\frac{1}{2} \left( \sum_{j=1}^{r} \varphi_j \right)^2} \left\langle \varphi_j - \frac{\varphi_j}{\lambda} \ldots \varphi_j - \frac{\varphi_j}{\lambda} \left| M \right| \varphi_j + \frac{\varphi_j}{\lambda} \right\rangle \]

where \( \left\langle \cdots | M | \cdots \right\rangle \) are the matrix elements of \( M \) in position representation. This becomes analogously the basis for the Weyl rule for several degrees of freedom. Since the arguments and steps are straightforward we omit them.

(iii) Canonical transformations and \( C \) and \( D \) matrices

Let us consider the so-called canonical transformations of a pair of conjugate variables obeying the commutation relation

\[ [q_i, p_i] = i \lambda \].

These transformations \( q_i \rightarrow q'_i(q, p), \ p_i \rightarrow p'_i(q, p) \) are said to be canonical if they leave invariant the commutator i.e.

\[ [q_i, p_i] = [q'_i, p'_i] = i \lambda \].

As soon above any function of \( q, p \) can be expressed in a basis generated by \[ e^{i(\sigma q + \tau p)}, -\infty \leq \sigma, \tau \leq \infty \].

The Heisenberg commutation relation \[ [q, p] = i \lambda \] take the Weyl form as \[ e^{i\vec{m} \cdot \vec{v}} e^{i\vec{v} \cdot \vec{p}} = e^{-i\mu \nu} e^{i\nu \cdot \vec{v}} e^{i\mu \cdot \vec{v}} \]. The transformation

\[ q_i \rightarrow q'_i(q, p), \ p_i \rightarrow p'_i(q, p) \]

would also preserve the Weyl commutation relation.
for them $e^{\imath \mu \hat{a}} e^{\imath \nu \hat{b}} = e^{-\imath \mu \nu \hat{c}} e^{\imath \mu \nu \hat{d}}$. The unique (up to equivalence) of the representation of unitary operators obeying N"{o}rlund commutation relations implies the existence of a unitary transformation $S$ such that $Se^{\imath \mu \hat{a}}S^{-1} = e^{\imath \mu \nu \hat{b}}$, $Se^{\imath \nu \hat{b}}S^{-1} = e^{\imath \mu \nu \hat{a}}$. This can be seen in the finite case also. $C$ and $B$ matrices, as shown before, have only one representation up to equivalence if required to obey $C B = \omega B C$, $C^\mu = B^\nu$. Thus any other matrix representations which are in turn expressible as $\left( \sum_{R,L=0}^{m-1} A_{RL} B^{RL} C^L \right)$ should be equivalent to $C$ and $B$ and thus all the canonical transformations of $C$ and $B$, $B \rightarrow F(C, B), C \rightarrow G(C, B)$ obeying $FG = \omega GF$ should be connected to $C$ and $B$ by unitary equivalence then $m \rightarrow \infty$ this fact implies the existence of unitary equivalence between any (hermitian) representations of the conjugate pair $(\varphi, \hbar \lambda)$ obeying $[\varphi, \lambda] = i \hbar \lambda$. These are very well known facts since the birth of quantum mechanics. We shall use the fact of correspondence $g^\mu \rightarrow e^{\imath \mu \hat{a}}, \varphi \rightarrow e^{\imath \lambda \hat{b}}$ to derive the explicit form of the similarity transformations in case of affine canonical transformations in which $\varphi = k \hat{a} + \ell \hat{b}, \lambda = m \hat{a} + n \hat{b}$ with the condition $|m \ell \ k \ n| = 1$ so that $[\ell, k] = i \hbar \lambda$. Hence we should find the similarity transformation $S$, such that

\begin{align*}
S e^{\imath \mu \hat{a}} S^{-1} &= e^{\imath \mu \nu (\ell \hat{a} + m \hat{b})} \\
S e^{\imath \lambda \hat{b}} S^{-1} &= e^{\imath \mu \nu (k \hat{a} + \ell \hat{b})}
\end{align*}

(2.84)
Splitting the expressions on the right hand side

\[ e^{i\mu(\hat{\lambda} + m\hat{\rho})} = e^{(i\mu^2nm/2)} e^{i\mu\hat{\lambda}} e^{i\mu m\hat{\rho}} \]

\[ e^{i\nu(\hat{\lambda} + k\hat{\rho})} = e^{(i\nu^2kl/2)} e^{i\nu\hat{\lambda}} e^{i\nu k\hat{\rho}} \]

Now we can try this problem as the limiting case of finding the similarity transformation in

\[ SB^N \hat{S}^{-1} = \omega(N)^{\frac{1}{2}} \mu^2 nm B^{\mu \nu} C^\mu \nu \]

\[ SC^N \hat{S}^{-1} = \omega(N)^{\frac{1}{2}} \nu^2 kl B^{\nu \kappa} C^{\nu \kappa} \]

Assume \( N \) to be even then the factors \( \omega(N)^{\frac{1}{2}} \mu^2 nm \) and \( \omega(N)^{\frac{1}{2}} \nu^2 kl \) are necessitated to have the condition \( B^N = C^N = I \) satisfied. So we have to determine the \( \hat{S} \) such that

\[ SB^\mu = \omega(N)^{\frac{1}{2}} \mu^2 nm B^{\mu \nu} C^\mu \nu \hat{S} \]

\[ SC^\nu = \omega(N)^{\frac{1}{2}} \nu^2 kl B^{\nu \kappa} C^{\nu \kappa} \hat{S} \]

\[ \forall \mu, \nu = 0, 1, \ldots, N-1 \]

Writing these equations in terms of matrix elements

\[ \sum_\alpha S_{\alpha \lambda}(C^\nu)_{\lambda \beta} = \omega(N)^{\frac{1}{2}} \nu^2 kl \sum_\lambda (B^{\nu \kappa} C^{\nu \kappa})_{\lambda \alpha} S_{\alpha \beta} \]

\[ \sum_\alpha S_{\alpha \lambda}(B^\mu)_{\lambda \beta} = \omega(N)^{\frac{1}{2}} \mu^2 nm \sum_\lambda (B^{\mu \nu} C^{\mu \nu})_{\lambda \alpha} S_{\alpha \beta} \]
Substituting

\[
(B^\nu)^\lambda_\beta = \omega (N)^{\mu \nu} \delta^\nu_\lambda \delta^\mu_\beta
\]

\[
(C^\nu)^\lambda_\beta = \delta^\nu_\lambda, \beta - \nu
\]

and summing over \( \lambda \) on both sides we arrive at coupled equations for \( S_{\alpha \beta} \) as

\[
S_{\alpha, \beta - \nu} = \omega (N) \quad S_{\alpha + \nu, \beta}
\]

\[
S_{\alpha, \beta} = \omega (N)^{\mu \nu} \quad S_{\alpha + \mu \nu, \beta}
\]

Solving for these equations we obtain

\[
S_{\alpha, \beta} = \omega (N)^{\frac{1}{2}} \left( \frac{k}{m} \alpha^2 + \frac{n}{m} \beta^2 - \frac{2 \alpha \beta}{m} \right)
\]

(2.02)

Considering the limiting case of \( N \to \infty \) by comparing

\[
C^k_B = \omega (N)^{\frac{1}{2}} \frac{k}{m} \epsilon^k_c \epsilon^l_c \quad e^{ik \hat{p}} e^{lr \hat{q}} = e^{ik \hat{p}} e^{lr \hat{q}}
\]

we have to replace \( \omega (N) \to \epsilon^i_i \). The \( n, \alpha, \beta \) are to be interpreted as continuous and \( k, n, \mu \) are also any real numbers. Then

\[
S (\alpha, \beta) = e^{\frac{1}{2} \left( \frac{k}{m} \alpha^2 + \frac{n}{m} \beta^2 - \frac{2 \alpha \beta}{m} \right)}
\]

(2.03)

Replacing \( \alpha \) and \( \beta \) by \( q' \) and \( q \) to be more suggestive, we get

\[
S (q', q) = e^{\frac{1}{2} \left( \frac{k}{m} q'^2 + \frac{n}{m} q^2 - \frac{2 q'q}{m} \right)}
\]

(2.04)
which are the well-known unitary integral transformations which
carry the transformation of operators \( \hat{\varphi} \rightarrow \hat{\varphi}' = \mathbb{M} \hat{\varphi} + m(-i\frac{d}{dq}) \)
and correspondingly \( (-i\frac{d}{dq}) \rightarrow \mathbb{L}\hat{\varphi} + k(-i\frac{d}{dq}) \) such that
\[
|l \ k\rangle = 1.
\]
The Fourier transform corresponds the case \( m' = -1 \), \( l = 1 \), \( k = 0 \).
Then
\[
S(q', q) = e^{i\mathbb{q}^\prime \mathbb{q}}.
\]  
(2.99)

Thus we have achieved our aim of showing that finite dimensional \( C \) and
\( B \) matrices can be used as a powerful tool in dealing with canonical
transformations. Also there has been an attempt by Santanam and
Sethamala\(^{11}\) to construct a quantum mechanics in finite dimension
using \( C \) and \( B \) matrices.

I wish to express my gratitude to Professor P.W. Stone
for useful discussions on Weyl's rule and bringing to my notice
relevant references.

Summary of important points.

In this chapter we have shown that the set of elements \( L_1, L_2 \)
obeying \( L_1 L_2 = \omega(m) L_2 L_1, L_1 = L_2 = 1 \) with \( (l, m) = d \)
has \( \frac{d^2}{(l, m)^2} \) inequivalent irreducible representations of some
dimension \( \frac{m}{d} \) and all these representations are specified by
\[
\{w(m)^{\delta_1} C, w(m)^{\delta_2} B\} 0 \leq \delta_1, \delta_2 \leq \frac{d-1}{2}\]
where \( L_1 \) and \( L_2 \)
are any one set of \( \left( \frac{m}{d} \right) \times \left( \frac{m}{d} \right) \) matrices obeying
\[
CB = \omega(m') L'B C,
\]
\( m' = \frac{m}{d}, \quad l' = \frac{l}{d}.\)
Extension of Alladi Ramakrishnan's theorem determining explicitly representation

\[ M = \sum_{k, l=0}^{m_i-1} A_{k,l} B_{k,l} C_{k,l} \]

has been considered and the procedure of determining explicitly the coefficients in the expansion

\[ M = \sum_{\substack{0 \leq k_i, l_i \leq m_i-1 \quad \text{for } i = 1, \ldots, t}} A_{k_1, l_1, \ldots, k_t, l_t} B(m_1)^{k_1} C(m_1)^{l_1} \otimes \cdots \otimes B(m_t)^{k_t} C(m_t)^{l_t} \]

is given.

Weyl's rule, Wigner distribution function and canonical transformations of conjugate pairs of operators are also discussed.
Chapter 3

Generalized Clifford Groups - I

In this chapter we shall formulate and study a group structure associated with the basis of the generalized Clifford Algebra \( C^{(m)}_{nu} \) generated by the relations

\[
\sum_{i<j}^{n} L_i L_j = \omega(nu) L_i L_i L_i \quad ; \quad L_i^2 = 1 \quad ; \quad i, j = 1, \ldots, n.
\]

(3.1)

This chapter is based mainly on the paper of Rangarajan and myself.

1) Direct group associated with ordinary Clifford Algebra

The basis of Clifford algebra \( C^{(a)}_{nu} \) generated by the relations (3.1) corresponding to the case \( m = 2 \), namely

\[
L_i L_j = -L_j L_i \quad ; \quad i, j = 1, 2, \ldots, n.
\]

(3.2)

consists of \( 2^n \) elements given by

\[
\left\{ \prod_{i=1}^{n} L_i^{r_i} \mid k_i = 0, 1 \right\}
\]

(3.3)

It is easy to see that product of any two elements of this set is \( \pm 1 \) times another element of the set. So this set of elements do not form a group. But it is immediately seen that if we add to this set the set of elements (3.3) multiplied by \(-1\) the set of \( 2^{n+1} \) elements given by

\[
\left\{ \prod_{i=0}^{n} L_i^{r_i} \mid k_i = 0, 1 \ ; \ L_0 = -1 \right\}
\]

(3.4)
form a group. This group has been called a Dirac group. This has been studied in the context of Lie algebra theory by Alladi Ramakrishnan and Raghavan Ramakrishnan. This group was studied earlier by Jordan and Wigner, Pauli, Lévy, Lorentz, Casimir, and others. DeVries and van Santen studied the Dirac matrix group corresponding to \( n = 4 \) in relation to Fierz transformations in theory of weak interactions in elementary particle physics. (cf., also Kahane) The generalization of this group structure for \( n > 2 \) associated with a generalized Clifford algebra \( \mathbb{C}_n \), has been noted very briefly in a study of their properties and representations by Deepak et al. In this paper they have also noted the properties of what we call product transforms associated with commutation matrices but its essential power as a tool of representation theory of generalized Clifford algebras has not been realized. Here in this chapter we study the properties and representations of these group structures in detail.

The number of conjugate classes in the group \( \mathbb{C}_n^2 \) is given by

\[
\frac{2^n}{2^n + 1}
\]

when \( n = 2^n \), and

\[
\frac{2^{n+1}}{2^n + 1}
\]

when \( n = 2^n + 1 \).

**Proof.** When \( n = 2^n \), the elements +1 and -1 are obviously self-conjugate elements. Among the other elements consider one element

\[
q(k_1, \ldots, k_{2^n}) = \prod_{i=1}^{2^n} L_i^{k_i}
\]
If it is a product of odd number of elements \( \prod_{i=1}^{n} L_i \) then all elements not contained in the product anti-commute with it. Hence \( q \) and \( L_{i} q L_{i}^{-1} = -q \) are in the same class. If \( q \) contains an even number of factors \( L_i \) then it anticommutes with all its factor elements \( L_i \). Thus in this case also \( q \) and \(-q\) are in the same class. Hence the set of elements

\[
\{ q, -q \mid q \neq \pm 1 \}
\]

contains

\[
\frac{1}{2} (2^{2^{n+1}} - 2^n) = 2^{2^n} - 1
\]

classes. Including the two self-conjugate elements \( \pm 1 \), the total number of classes in \( D_{2^{2^n}} \) is given by

\[
2 + (2^{2^n} - 1) = 2^{2^n} + 1
\]

(3.8)

When \( n = 2^{2^n} + 1 \), the self-conjugate elements are given by \( \{ 1, -1 \} \), where

\[
\eta = \prod_{i=1}^{2^{n+1}} L_i
\]

(3.9)
since \( \eta \) commutes with all \( L_i \) and hence all products of \( L_i \).

The other elements \( (q^{2^v+2} - 1) \) in number are grouped into \( \frac{1}{2}(q^{2^v+2} - 1) = q^{2^v+1} - 2 \) classes since by the same argument as above each element \( g \) of this set is equivalent to \(-g\).

So total number of classes in this case is

\[
(q^{2^v+1} - 2) + 4 = q^{2^v+1} + 2
\]

By Burnside's theorem the total number of classes is equal to the total number of irreducible inequivalent representations. Hence total number of representations of \( D^2_{2^v} \) are given by

\[
\begin{cases}
q^{2^v+1} + 1 & \text{when } n = 2^v \\
q^{2^v+1} + 2 & \text{when } n = 2^v + 1
\end{cases}
\]

Now let us construct all these representations. When \( d_i \)'s are dimension of the representation then by Burnside's theorem

\[
\sum_{i=1}^{N_{2^v}} d_i^2 = \text{total number of elements in } D^2_{2^v} = |D^2_{2^v}|
\]

Hence it follows from (3.11) and (3.12) that \( D^2_{2^v} \) has \( q^{2^v} \) one dimensional representations and one \( 2^v \) dimensional representation and \( D^2_{2^v+1} \) has \( q^{2^v+1} \) one dimensional representation and 2 \( 2^v \) dimensional representations. The \( 2^n \) one dimensional representation for both \( n = 2^v \) and \( 2^v + 1 \), are given by the representations of the Abelian group

\[
G_i = \mathbb{Z}_{2^n} \times \cdots \times \mathbb{Z}_{2^n} \quad \text{(n copies)}
\]

which is isomorphic to \( D^2_{2^v} \)

\[
D^2_{2^v}/\mathbb{Z}_2 = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \quad \text{(n copies)}
\]
Also it is seen easily that $Z_n$ is the centre of $D_n^2$ and thus $D_n^2$ is the central extension of $Z_2 \otimes \cdots \otimes Z_2$ with $Z_2$ as kernel of extension. Thus all these $Z_n^2$ on dimensional representations of are constructed by taking each $L_i = \pm 1$ independent of each other.

Higher dimensional representations are constructed as follows.

Let $n = 2^\nu$. Then relation (1.4) or (1.44) $a,b,c$ with $n = 2$ provide $2^\nu$ dimensional representations of the $2^\nu$ anticommuting generators. From (3.11) and (3.12) these cannot exist any other representation.

Let $n = 2^\nu + 1$. Then relation (1.4) or (1.44) $a,b,c$ with $n = 2$ provide a $2^\nu$ dimensional representation of the $2^\nu + 1$ anticommuting generators. Let us now put

$$L_i' = -L_i \quad i = 1, 2, \ldots, 2^\nu + 1$$

(3.15)

obviously $\{L_i' \mid i = 1, \ldots, 2^\nu + 1\}$ also provide an irreducible representation of these $2^\nu + 1$ elements. Let us now prove that this representation is inequivalent to $\{L_i \mid i = 1, 2, \ldots, 2^\nu + 1\}$. If there two were equivalent then there exists a nonsingular matrix $S$ such that

$$L_i' = SL_i S^{-1} = -L_i \quad \forall i = 1, 2, \ldots, 2^\nu + 1.$$  

(3.16)

and

$$SL_i = -L_i S \quad \forall i = 1, 2, \ldots, 2^\nu + 1.$$  

(3.17)

But as has been proved earlier in Chapter 1 when $2^\nu + 1$ in $L_i$ are represented by $2^\nu$ dimensional matrices there is a relation

$$L_1 \cdots L_{2^\nu + 1} \sim I$$

(3.17)

and hence (3.16) would imply

$$SL_1 \cdots L_{2^\nu + 1} S = -L_1 \cdots L_{2^\nu + 1} S \quad \text{and} \quad SS^t = -I.$$  

(3.18)
which is absurd. Hence those two representations
\[ s l_i = L_{(2v + 1), i}, \quad i = 1, \ldots, 2v + 3 \] and
\[ s l_i = -L_{(2v + 1), i}, \quad i = 1, \ldots, 2v + 3 \]
are inequivalent. Thus the two irreducible representations have
been found and according to (3.16, 3.17) there is no more representa-
tion. The above facts are known as Pauli's theorem that
has only one irreducible representation of dimension > 1 and
has two such representations of same dimension > 1.

(11) Generalization of Dirac Group or Generalized Clifford Group
associated with Generalized Clifford Algebra.

Considering the algebra \( G^2 \) generated by (3.1) the basis of
this algebra is given by \( m^n \) elements
\[ \left\{ \prod_{i=1}^{n} L_{i}^{k_i} \mid 0 \leq k_i \leq m - 1 \right\} \] (3.10)

It is easy to see that product of any two elements of this set is in-
general only a scalar multiple of another element of the set as
given by (1.36), so that this set of elements (3.30) do not form a
group. But as was done in the case of \( G^2 \) it is easy to form a
group. The following set of \( m^n \) elements
\[ G_{m^n} = \left\{ \prod_{i=1}^{n} L_{i}^{k_i} \mid 0 \leq k_i \leq m - 1, L_0 = \omega(m) \right\} \] (3.20)
form a group which we shall call a generalized clifford group (G.C.G).
It can be called also a Generalized Dirac Group. The choice of 'G.C.G'
will be made clear in the last section of this Chapter.
In this Chapter we shall study in detail $\mathbb{G}_n$, $\mathbb{G}_n^m$ only for the case of $n = \text{prime number}$. Let us denote the group $\mathbb{G}_n$ corresponding to $n$ being a prime number by $\mathbb{G}_n$.

iii) Properties of $\mathbb{G}_n$

Let us denote

$$g(k_0, k_1, \ldots, k_n) = \prod_{i=0}^{n} L_i^{k_i}$$  \hspace{1cm} (3.22)

Then product is defined by

$$g(k_0, k_1, \ldots, k_n) g(j_0, j_1, \ldots, j_n) = g((k+j)_0, (k+j)_1, \ldots, (k+j)_n)$$  \hspace{1cm} (3.23)

where

$$(k+j)_i \mod m = k_i + j_i, \forall i = 1, \ldots, n.$$  \hspace{1cm} (3.24)

and

$$(k+j+l)_0 \mod m = k_0 + j_0 + l_0.$$  \hspace{1cm} (3.25)

$$l_0 \mod m = (m-1) \left[ \begin{array}{cccc} j_1 & j_2 & \cdots & j_n \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \end{array} \right] \left[ \begin{array}{c} k_1 \\ k_2 \\ \vdots \\ k_n \end{array} \right]$$

$$= (m-1) \sum_{x \neq y} j_x k_y.$$  \hspace{1cm} (3.26)

The inverse of an element is given by

$$g(k_0, k_1, \ldots, k_n)^{-1} = g(k_0', k_1', \ldots, k_n')$$  \hspace{1cm} (3.27)

where

$$k_i' = m - k_i \quad \forall i = 1, \ldots, n.$$  \hspace{1cm} (3.28)
\[ k'_0 \mod m = m - k_0 + K \]

\[ K \mod m = (m-1) \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} k'_1 \\ k'_2 \\ \vdots \\ k'_n \end{pmatrix} \]

\[ = (m-1) \sum_{x \leq y} k'_x k'_y \]

You let us determine the number of conjugate classes in it. It follows from \((3.23 - 3.34)\) that

\[ g(y_0, y_1, \ldots, y_m) g(k_0, k_1, \ldots, k_n) g(j_0, j_1, \ldots, j_m)^{-1} \]

\[ = g(k'_0, k'_1, \ldots, k'_n) \]

where

\[ k'_0 \mod m = k_0 + l_0 \]
\[ l_0 \mod m = (j'_1, j'_2, \ldots, j'_m) \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \]

\[ = \sum_{x < y} (j'_{x y} - j'_{y x}) \]

First let \( n = 2 \nu \). For a fixed \((k_0, k_1, \ldots, k_{2\nu})\) let \( k \) be the greatest common division of all \( k_i, \ i = 1, \ldots, 2\nu \). Then by the theory of linear Diophantine equations, the equation

\[ k = \sum_{x=1}^{2\nu} k_x \left\{ \sum_{y=1}^{2\nu} j'_x - \sum_{y=1}^{2\nu} j'_y \right\} = \sum_{x=1}^{2\nu} k_x j'_x \]

has a solution for \( \left\{ j'_x \mid x = 1, \ldots, 2\nu \right\} \).
negative values also. Since
\[
\begin{pmatrix}
\frac{y''}{x}
\end{pmatrix}
=
\begin{pmatrix}
0 & -1 & \cdots & -1 \\
1 & 0 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\frac{y'}{x}
\end{pmatrix}
\]
(3.33)

\[
\left(\begin{array}{c}
f_2' \\
f_3' \\
\vdots \\
f_{2n}'
\end{array}\right)_{x = 1, 2, \ldots, 2n}
\]

have solution given by
\[
\begin{pmatrix}
f_1' \\
f_2' \\
\vdots \\
f_{2n}'
\end{pmatrix}
=
\begin{pmatrix}
0 & -1 & \cdots & -1 \\
+1 & 0 & \cdots & +1 \\
\vdots & \vdots & \ddots & \vdots \\
+1 & 1 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
f_1'' \\
f_2'' \\
\vdots \\
f_{2n}''
\end{pmatrix}
\]
(3.34)

\[
(\tilde{y}_1', \ldots, \tilde{y}_{2n}')
\]
can now be reduced to modulo -m. The resulting set of values \(\{(\tilde{y}_1, \ldots, \tilde{y}_{2n})|\tilde{y}_i \mod m = \tilde{y}_i'; \forall i\}\) provide a solution to the equation
\[
k \mod m = \sum_{x=1}^{2n} k_x \left\{ \sum_{y=1}^{x-1} j_y - \sum_{y=x}^{2n} j'_y \right\}
\]
(3.35)

or
\[
g(k_0, j_1, \ldots, j_{2n}) g(k_0, k_1, \ldots, k_{2n}) g(j_0, j_1, \ldots, j_{2n})^{-1}
\]
(3.36)

with
\[
k_0' \mod m = k_0 + k
\]
(3.37)
\[ k_i \text{ mod. } m' = \sum_{x=1}^{2^v} k_i^x \left\{ \sum_{y=1}^{\nu} y' - \sum_{y'=1}^{\nu} y \right\} \]

Hence

\[ g(y_0, \ldots, y_{2^v}) \gamma g(k_0, k_1, \ldots, k_{2^v}) / g(y_0, \ldots, y_{2^v})^- \]

\[ = g(k', k_1, \ldots, k_{2^v}) \]

with

\[ k'(r) \text{ mod. } m = k_0 + k(r) \]

\[ k(r) \text{ mod. } m = r k \]

when \( m \) is prime, \( k(r) \text{ mod. } m \) takes all values 0, 1, 2, \ldots, \( m-1 \)

as \( r \) takes values 0, 1, 2, \ldots, \( m-1 \). Thus this shows that all the elements of the set

\[ \left\{ g(k_0, k_1, \ldots, k_{2^v}) \mid k_0 = 0, 1, \ldots, m-1 \right\} \]

for fixed \( (k_1, \ldots, k_{2^v}) \) are members of a class when at least one of the \( k_i \)'s is nonzero. In case \( k_0 = 0 \) and \( \forall i = 1, \ldots, 2^v \), then it is obvious that the element \( g(k_0, 0, \ldots, 0) \) is a self conjugate element or it is a class in itself. Thus there are \( m \) self conjugate elements corresponding to \( k_0 = 0, 1, \ldots, m-1 \) in \( g(k_0, 0, \ldots, 0) \). The set of remaining \( m^{2^v} - m \) elements is divided into

\[ (m^{2^v+1} - m') / m = m^{2^v} - 1. \]
classes each containing $n$ elements of the form (3.42). Thus
the total number of classes in $G_{\nu\nu}$ is
\[ N_{\nu\nu}^m = m^{2\nu} - 1 + m \]  
(3.44)

Now let us consider the case of $G_{\nu\nu+1}^m$. For an element
$g(k_0, k_1, \ldots, k_{2\nu})$ to be self conjugate the condition is from (3.31)
(3.45):
\[ 0 \mod m = \sum_{x=1}^{\nu} k_x \sum_{i=1}^{x-1} y_i - \sum_{y_i=x+1}^{n} y_i \]
for all values of \( \{ (y_1, \ldots, y_{\nu}) \mid 0 \leq y_i \leq m-1; \forall i \} \). In the case of
$\nu = \nu \nu$, only the set \( \{ k_0 = 0 \} \forall i \} \) satisfies this condition.

But when $\nu = \nu \nu + 1$, the following set of $n^3$ elements
\[ \{ g(k_0, m-\nu, \ldots, m-\nu, \nu, \ldots, \nu, m-\nu) \mid 0 \leq k_0 \leq n^{\nu-1}, 0 \leq \nu \leq m-1 \} \]
(3.46)
satisfy this condition. But in this the elements with $\nu = 0$ are
\[ \sim \text{I} \]
as is directly seen. Denoting
\[ \eta = g(0, m-\nu, \ldots, m-\nu, \nu, \ldots, \nu, m-\nu) \]
since it commutes with all $g \in G_{\nu\nu+1}^m$, it is $\sim \text{I}$ by Schur's
lemma. It is seen that we can write
\[ L_{\nu\nu+1} = [g(0, m-\nu, \ldots, m-\nu, \nu, \ldots, \nu, m-\nu) \sim \eta]_{\nu\nu-1} \]
(3.47)
\[ = \eta^{m-1} [g(0, m-\nu, \ldots, m-\nu, \nu, \ldots, \nu, m-\nu) \sim \eta]_{\nu\nu-1} \]
Hence in the product \( \prod_{i=0}^{2\nu+1} \frac{L^i}{i!} \) \( L_{2\nu+1} \) can be replaced as a product of the \( L_i \) \( i=1, \ldots, 2\nu+1 \) as

\[
q(k_0, k_1, \ldots, k_{2\nu+1}) = q(k_0, k_1, \ldots, k_{2\nu}, 0) \left\{ \eta^m \left[ q(0, m-1, \ldots, m-1, 1, \ldots, 0) \right]^{m-1} \right\}
\]

\[
= q(k_0, k_1, \ldots, k_{2\nu}, 0)
\]

Thus in the set of group elements \( \mathbb{G}_{2\nu+1}^m \)

\[
\left\{ q(k_0, k_1, \ldots, k_{2\nu+1}) \mid 0 \leq k^i \leq m-1 ; i = 1, \ldots, 2\nu+1 \right\}
\]

(3.30)

each element is a product powers of only \( \omega(m) \) and the \( 2 \nu + 1 \) \( \eta^m \) and hence each element other than self-conjugate elements give rise to a class of \( m^2 \) elements which are multiples of it, by

\[
\left\{ \omega(m)^l \mid 0 \leq l \leq m \right\}
\]

as per the arguments presented in the case of \( \mathbb{G}_{2\nu}^m \). Thus remembering that \( \mathbb{G}_{2\nu+1}^m \) has \( m^2 \) self-conjugate elements (3.30) the total number of class in \( \mathbb{G}_{2\nu+1}^m \) is given by

\[
N' = \left( \frac{m^{2\nu+1}}{(m-1)^2} \right) + m^2 = m^{2\nu+1} + m(m-1)
\]

(3.31)

iv) Representations of \( \mathbb{G}_{2\nu}^m \)

By (3.34) \( \mathbb{G}_{2\nu}^m \) has \( m^{2\nu} + (m-1) \) conjugate classes and hence it must have so many irreducible inequivalent representations. It can be realised that a scheme of \( m^{2\nu} + 1 \)-dimensional representations and \( (m-1) \), \( m^2 \)-dimensional representations fits the requirement that values independent of others.
\[ |G_i| = \text{Order of the group} = \sum_{i=1}^{N} d_i \quad (3.52) \]

where \( d_i \) is the dimension of the \( i \)th representation and \( N \) is the total number of representations. In our case we have

\[ \sum_{i} j^2 + \sum_{i} (m^i)^2 = m^{2v} + (m-1)m^{2v} = m^{2v+1} = |G_{2,1}^{m^v}| \quad (3.53) \]

\section{The dimensional representations}

These representations come from the homomorphism of the generality relations (3.1) with

\[ L^i_j L^j_i = L^j_i L^i_j \quad i,j = 1 \ldots 2^v. \quad (3.54) \]

\[ L^i_i = 1 \quad \forall i = 1 \ldots n \]

or in other words, these are due to the homomorphism of \( G_{2,1}^{m^v} \) to \( \mathbb{Z}_{m^v} \otimes \ldots \otimes \mathbb{Z}_{m^v} \) (\( 2^v \) copies). The kernel of homomorphism is the normal subgroup \( \{ 1, w(m), \ldots w(m)^{2^v} \} \subseteq \mathbb{Z}_m \) or

\[ G_{2,1}^{m^v}/\mathbb{Z}_m \cong \mathbb{Z}_{m^v} \otimes \ldots \otimes \mathbb{Z}_{m^v} \quad (2^v \text{- copies}) \quad (3.55) \]

Thus to generate the one dimensional representations simply set

\[ L^i_j = w(m)^{k_{ij}} \quad j = 1 \ldots 2^v \]

\[ 0 \leq k_{ij} \leq m - 1. \quad (3.56) \]

where \( \tau \) labels the representation. Thus \( \tau = 1 \ldots m^{2v}, m^{2v} \) representations arise from the fact each of the \( L^i_j \)'s can take \( n \) values independent of others.
b) Higher dimensional representations

We have seen in (2.53) that there should be \((m-1)\), \(m\) - dimensional representations. These are recognized to arise from the \((m-1)\) isomorphic \(G_{2,1}\)'s whose generating relations are

\[
L_i L_j = \omega(m)^{k_{ij}} L_j L_i, \quad i, j = 1, \ldots, 2v,
\]

\[
L_i L_{m+1} = 1, \quad \forall i = 1, \ldots, 2v.
\]

\[
l = 1, \ldots, m-1.
\]

Since \(m\) is prime

\[
(l, m) = 1, \quad \forall l = 1, \ldots, m-1.
\]

Let us denote the \(G_{2,1}\)'s generated by (3.57) by \(\big\{ \omega^{(m)}_{2,1}(l) \big| l = 1, \ldots, m-1 \big\}\).

All the \(\omega^{(m)}_{2,1}(l)\) are primitive \(m\)th roots and hence all \(\omega^{(m)}_{2,1}(l)\) are isomorphic. Hence for the group \(G_{2,1}^{m}\), representations of all these algebras provide a representation. Let us consider one \(\omega^{(m)}_{2,1}(l)\). We shall use the product-transform method in Chapter 1 for construction of irreducible representations. The commutation matrix \(T\) associated with the system of relations (3.57) is

\[
T = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 1 \end{pmatrix}
\]

where \(\omega^{(m)}_{2,1}\) is a diagonal matrix of irreducible representation of \(G_{2,1}\) and \(T\) is a primitive nil matrix for all \(l = 1, \ldots, m-1\).
So the skew normal form of $T$ is

$$
T^*_{\xi} = \begin{pmatrix}
0 & b & 0 & 0 \\
-l_0 & 0 & l_1 & 0 \\
0 & 0 & -l_0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = l \begin{pmatrix}
0 & l & 0 & 0 \\
-l & 0 & l_1 & 0 \\
0 & 0 & -l & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

and hence the skew-normal form of the relations (3.67) is

$$
L^*_{\xi_{i-1}} L^*_{\xi_i} = \omega(m)^l L^*_{\xi_i} L^*_{\xi_{i-1}} ; i = 1, \ldots, \nu. 
$$

$$
L^*_{\xi_i} L^*_{\xi_j} = L^*_{\xi_j} L^*_{\xi_i} \quad \text{otherwise.}
$$

Following the procedure in (1.64) it is easy to construct

$$
L^*_{\xi_{i-1}} = \begin{array}{c}
\bigotimes \bigotimes \cdots \bigotimes \\
\bigotimes \bigotimes \cdots \bigotimes \\
\bigotimes \bigotimes \cdots \bigotimes
\end{array}
$$

$$
L^*_{\xi_i} = \begin{array}{c}
\bigotimes \bigotimes \cdots \bigotimes \\
\bigotimes \bigotimes \cdots \bigotimes \\
\bigotimes \bigotimes \cdots \bigotimes
\end{array}
$$

with

$$
C_i B_i = \omega(m)^l B_i C_i ; \quad i = 1, 2, \ldots, \nu.
$$

From (3.62) it follows that

$$
\det C_i B_i = \omega(m)^l d_i \det B_i C_i = \det C_i B_i \cdot \omega(m)^l d_i
$$

$$
\det \omega(m)^l d_i = 1.
$$

where $d_i$ = dimension of irreducible representation of $C_i$ and $B_i$

Since $\omega(m)^l$ is a primitive $m$th root for all $l = 1, \ldots, m-1$.

$$
d_i = m, \quad \forall i = 1, \ldots, \nu.
$$
From (1.36) it is easy to construct the irreducible representation of $C_i$ and $B_i$ as

$$C_i = C^k, \quad B_i = B^T$$

with

$$l \pmod{n'} = k r.$$  \hspace{1cm} (3.66)

Without loss of generality we can take $k = 1, r = l$ and hence

$$L_{2i-1}^* = \begin{bmatrix} \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & 0 & 0 \\ \end{bmatrix}$$

$$L_{2i}^* = \begin{bmatrix} \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & 0 & 0 \\ \end{bmatrix}$$

(3.67)

Now since $T_i$ and $T_i^*$ are identical with $T$ and $T^*$ given in (1.65) and (1.66) respectively except for a multiplicative factor $l$. The $U$ and $V$ matrices are also the same. Hence the solution for $V$ in

$$T_i = V T_i^* V$$

is given by (1.70)

$$V = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \end{bmatrix}$$ \hspace{1cm} (3.68)

Hence by the product transform method outlined in Chapter 1, are represented by

$$L_{ij} = \sum_{y=1}^{N} L_j^* V_{ij}. \quad \sum_{i=1}^{N}$$

$$= \begin{cases} L_{2k-1}^* L_{2k}^* L_{2k-2}^* \cdots L_{2(N-k)+1}^* & \text{if } i = 2k-1 \\
L_{2k-1}^* L_{2k}^* L_{2k-2}^* \cdots L_{2(N-k)+2}^* & \text{if } i = 2k \\
L_{1}^* \cdots \cdots \ \cdots & \text{if } i = 1 \ldots N \end{cases} \hspace{1cm} (3.70)$$
Explicitly writing
\[
L_{2k+1} = c^{m-1} b^l \otimes c^{m-1} b^l \otimes \cdots \otimes c^{m-1} b^l \otimes c^{m} i \otimes \cdots \otimes i
\]
\[
L_{2k} = c^{m-1} b^l \otimes c^{m-1} b^l \otimes \cdots \otimes c^{m-1} b^l \otimes c^{m} b^l \otimes i \otimes \cdots \otimes i
\]
\[
k = 1, 2, \ldots, n.
\]
These also satisfy the condition as can be seen easily
\[
L_i^m = I \quad \forall i = 1, 2, \ldots, 2n
\]
Thus the dimension of these representations is \(m^v\). Corresponding to each value of \(l = 1, \ldots, m-1\), we get the representations of all the relations (3.57), each of them being of dimension \(m^v\). These are obviously inequivalent since they have different algebraic relations and equivalence transformations cannot alter algebraic relations among the set of given matrices. Thus we have obtained all the \((n-1)\), \(m^v\) dimensional representations of \(G_{2n}^m\) in terms of the generators obeying (3.57).

iii) Representations of \(G_{2n+1}^m\)

We can guess from (3.51) that \(G_{2n+1}^m\) should have \(m^{2n+1}\) one-dimensional representations and \(m(n-1)\), \(v\)-dimensional representations since this scheme fits the requirement (3.52) as
\[
\sum 1^2 + \sum (m^v)^2 = m^{2n+1} + m(n-1)m^v = m^v = \left( G_{2n+1}^m \right)
\]
c) The dimensional representations

As in the case of $G^m_{2\nu}$ those arise from the representations of
$\mathbb{Z}_m \otimes \cdots \otimes \mathbb{Z}_n$ ($\nu + 1$ copies) since $G^m_{2\nu}$ is isomorphic
to $G^m_{2\nu + 1}$.

\[ L_i = \omega(m)^{\nu_i} \]
\[ l = 1, 2, \ldots, 2\nu + 1 \]
\[ 0 \leq \nu_i \leq m - 1 \]
\[ r = 1, \ldots, m^{2\nu + 1} \]

Corresponding to $L_i$ the same representation \((3.70)\) in
there, $r$ labels the representations. Since each $L_i$ can take a
value independent of others these arise from
the representations.

3) Higher dimensional representations

Again we apply the product transfer notion. For these
higher dimensional representations arise from the \((n-1)\) isomorphic.

\[ \text{G.C.A.} \quad \mathcal{C}^{(m)} \]

\[ L_i L_{i'} = \omega(m)^{\nu_i} L_{i'} L_i \quad ; \quad i, i' = 1, 2, \ldots, 2\nu + 1 \]
\[ L_i^{m-1} = 1 \quad ; \quad i = 1, 2, \ldots, 2\nu + 1 \]
\[ C^{(m)}_{2\nu + 1} (L) \]

Consider one of the association matrices associated
with \((3.75)\) is

\[ T_{\varepsilon} = \varepsilon \begin{bmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 0 \end{bmatrix} \]

and its transpose then is

\[ T_{\varepsilon}^* = \varepsilon \begin{bmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & \cdots & 0 \\ -1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} \]
and the solution for \( V \) in
\[
T = V T^+ \tilde{V}
\]  
(3.78)

is given by
\[
V = \begin{bmatrix}
\vdots \\
0 & 0 & \cdots & 0 & 1 \\
\vdots & \ddots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \cdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 0 & 1 \\
\end{bmatrix}
\]  
(3.79)

Corresponding to \( T^* \) the skew normal form of the relation (3.78) is
\[
\begin{align*}
1) & \quad L_{2i-1}^* L_{2j}^* = \omega(m) L_{2i}^* L_{2j-1}^* \quad i = 1, 2, \ldots, \nu, \\
& \quad L_{2i-1}^* L_{2j}^* = L_{2i}^* L_{2j}^* \quad \text{otherwise}
\end{align*}
\]  
(3.80)
\[
2) \quad L_{2i}^* L_{2j}^* = L_{2i}^* L_{2j}^* \quad \forall i = 1, 2, \ldots, 2\nu.
\]

The relations (3.80a) are the same as those occurring in the case of \( G_{2\nu}^m \) and hence by the arguments given above their representation is given by the same in dimensional matrices as in (3.71). All these \( G_{2\nu}^m \) matrices are seen to generate a set of \( m^{2\nu} \) linearly independent matrices by following the arguments of Chapter I. So \( L_{2\nu+1}^* \) has to be represented by a scalar which can be taken as \( \tilde{g} \) without loss of generality. Hence corresponding to the product transform with \( V \) in (3.78) the representations of \( L_i^* \)'s are given by
\[
L_i = \prod_{j} L_{i-j}^* \quad \text{for } i = 2k + 1
\]  
(3.81)
\[
= \begin{cases}
L_{2(\nu+1)}^* - L_{2\nu}^* L_{2\nu-1}^* L_{2\nu-2}^* \cdots L_{2(\nu-k)+1}^* i = 2k-1 \\
L_{2(\nu-1)}^* L_{2\nu}^* L_{2\nu-1}^* L_{2\nu-2}^* \cdots L_{2(\nu-k)+2}^* \quad i = 2k \\
L_{2(\nu-1)}^* L_{2\nu}^* L_{2\nu-1}^* L_{2\nu-2}^* \cdots L_{2(\nu-k)+1}^* \quad \forall \nu = 1, 2, \ldots, 1
\end{cases}
\]
Thus these provide a representation of (8.25) for a value of $l$.

By (4.1) and (8.1), there are $n$ inequivalent representations of some dimension $n$ which are explicitly given by

$$\left\{ \omega^{(m)} L_i \mid i = 1, \ldots, 2v + 1 ; r = 0, 1, \ldots, n - 1 \right\}$$

where $L_i$'s are those constructed in (3.32).

**Proof.** Consider two representations

$$\left\{ \omega^{(m)} L_i \mid i = 1, \ldots, 2v + 1 \right\}$$

$$\left\{ \omega^{(m)} L^r \mid i = 1, \ldots, 2v + 1 \right\}$$

These two will be equivalent only if there exists a nonsingular $m^v$-dimensional matrix $S$ such that

$$S \omega^{(m)} L_i = \omega^{(m)} L^r S \quad \forall i = 1, \ldots, 2v + 1$$

or there exists a matrix $S$ such that

$$S L_i = \omega^{(m)} L^r S \quad \forall i = 1, \ldots, 2v + 1$$

This implies that the set of $2v+2$ matrices

$$\left\{ L_1, L_2, \ldots, L_{2v+1}, S \right\}$$

obey mutually commuting relations of the type (3.37) and we can put

$$S = L_{2v+2}$$

But as shown in section 3.1b these $2v+2$ elements must have only one irreducible representation of dimension $m^v+1$ and cannot have a $m^v$ dimensional representation as the validity of (3.36) would imply. Hence there cannot exist a matrix $S$ satisfying (3.35). Thus the two representations (3.36) are inequivalent. Since it is true for all $r = 0, 1, \ldots, n - 1$ in (3.32) the $m$ representations in (3.37) are all inequivalent. Since
reduced modulo \( m \) takes all values \( 0, 1, \ldots, m-1 \) as \( r \) takes values \( 0, 1, \ldots, m-1 \), the \( m \) inequivalent representations (3.32) can be written as

\[
\{ \omega(m)^b L_i^b \mid i = 1, 2, \ldots, 2v+1; b = 0, 1, \ldots, m-1 \}
\]

(3.33)

The above considerations hold for the case of all values of \( l = 1, 2, \ldots, m-1 \) and each algebra \( S_m^{2v+1}(l) \) gives rise to \( m \) inequivalent representations as given by (3.33). Obviously representations of different algebras \( S_m^{2v+1}(l) \) are inequivalent. Thus totally we get \( m(m-1) \) inequivalent irreducible representations of dimension \( m \) each, as is required. Incidentally these considerations prove that \( S_m^{2v+1}(l) \) has only \( m \) inequivalent irreducible representations of dimension \( m \) for each \( l \) when \( m \) is prime.

Let us denote by \( \Gamma_l^m \) the representation of \( G_l^m \) arising from \( C^{(m)}_{2v+1}(l) \) and by \( \Gamma_l^{(l, b)} \) the representation of arising from \( C^{(m)}_{2v+1}(l) \) and corresponding to the phase factor \( \omega(m^b) \) as in (3.36). Let \( \Gamma_0^{(l, b)} \) denote a one-dimensional representation of \( G_l^m \) where \( \beta \equiv (\beta_1, \beta_2, \ldots, \beta_m) \) \( 0 \leq \beta_i \leq m-1, \forall i = 1, \ldots, m \) which corresponds to the choice

\[
L_i^* = \omega(m)_{\beta_i} \quad \forall i = 1, \ldots, m.
\]

(3.37)

Here we like to make an important note about the construction of representations. In considering the group \( G_m \) as an abstract group, the element \( L_0 \) should be interpreted as

\[
L_0 = L_i^{L_j^{L_i^{-1}}} \quad \forall i < j \quad i, j = 1, \ldots, n
\]

(3.38)

and hence if for example \( L_i^{L_j} \) correspond to the generators of then \( L_0 = \omega(m)^L \). In case of 1-dimensional representations always.
iv) Character Tables

We shall illustrate the above considerations by a simple example $G_2^3$. Order of the group $= n^{n+1} = 2^{2^2+1} = 27$. The elements are given by $\{ \omega^{k_0} L_{k_1} L_{k_2} | k_0, k_1, k_2 = 0, 1, 2 \}$. Total number of classes are $n^n + (n-1) = 2^3 + (3-1) = 11$. These are given by:

\[ C_1 = \{ \omega^k L_1 | k = 0, 1, 2 \} \quad C_5 = \{ \omega^k L_2 | k = 0, 1, 2 \} \\
C_2 = \{ \omega^k L_1^2 | k = 0, 1, 2 \} \quad C_6 = \{ \omega^k L_1 L_2 | k = 0, 1, 2 \} \\
C_3 = \{ \omega^k L_2^2 | k = 0, 1, 2 \} \quad C_7 = \{ \omega^k L_1^2 | k = 0, 1, 2 \} \\
C_4 = \{ \omega^k L_2^2 | k = 0, 1, 2 \} \quad C_8 = \{ \omega^k L_2 L_2 | k = 0, 1, 2 \} \\
C_9 = \{ \omega^k | k \} \quad C_{10} = \{ \omega^2 \} \quad C_{11} = \{ \omega^3 \}

(3.39)

where $\omega$ stands for the primitive 3rd root of unity. Number of irreducible inequivalent representations are 11, 2, 3-dimensional and 9 one-dimensional.

The 9 one-dimensional representations are $\Gamma_0^{(0,0)}, \Gamma_0^{(0,1)}$ $\Gamma_0^{(1,0)}, \Gamma_0^{(1,1)}$, $\Gamma_0^{(1,2)}, \Gamma_0^{(2,0)}$, $\Gamma_0^{(2,1)}, \Gamma_0^{(3,2)}$ in the notation introduced above. The two 3-dimensional representations are obtained by setting:

\[ \Gamma_1^{(1)} : L_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} \quad L_0 = \omega(3) \]

(3.39)

\[ \Gamma_1^{(2)} : L_1 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix} \quad L_0 = \omega(3)^2 \]

(3.31)
Let us exhibit the character table of $d_6^2$ explicitly

Representations          Character

3 dimensional

$\Gamma_1$: 0 0 0 0 0 0 0 0 0 0 3 3 3 0
$\Gamma_2$: 0 0 0 0 0 0 0 0 0 0 3 3 3 0

1 dimensional

$\Gamma_1$: 1 1 1 1 1 1 1 1 1
$\Gamma_2$: 1 1 $w$ $w$ $w$ $w$ $w$ $w$ $w$
$\Gamma_3$: 1 1 $w^2$ $w^2$ $w^2$ $w^2$ $w^2$ $w^2$ $w^2$
$\Gamma_4$: 1 1 $w$ $w$ $w$ $w$ $w$ $w$ $w$
$\Gamma_5$: 1 1 $w^2$ $w^2$ $w^2$ $w^2$ $w^2$ $w^2$ $w^2$
$\Gamma_6$: 1 1 $w$ $w$ $w$ $w$ $w$ $w$ $w$
$\Gamma_7$: 1 1 $w^2$ $w^2$ $w^2$ $w^2$ $w^2$ $w^2$ $w^2$
$\Gamma_8$: 1 1 $w$ $w$ $w$ $w$ $w$ $w$ $w$
$\Gamma_9$: 1 1 $w^2$ $w^2$ $w^2$ $w^2$ $w^2$ $w^2$ $w^2$
v) **Direct product representations and Clebsch-Gordan series.**

With the help of the character table it is easy to obtain the Clebsch-Gordan series. We give here briefly the results.

1) \( n = 2^n \).

(a) Let \( \Gamma (r_1) \) and \( \Gamma (r_2) \) be two faithful \( n \)-dimensional representations. Let \( \alpha_{r_1 r_2} \) denote the number of times the representation \( \Gamma (r) \) one dimensional or higher dimensional, occurs in the direct product representation \( \Gamma (r_1) \otimes \Gamma (r_2) \).

The \( n \)

\( i ) \text{ if } 0 \mod n = r_1 + r_2 \)

\[ \alpha_{r_1 r_2} = \begin{cases} 1 & \text{for all the } n^2 \text{ one-dimensional representations,} \\ 0 & \text{for all other higher dimensional representations.} \end{cases} \]  

\( 3.93 \)

\( (ii) \text{ if } 0 \mod n \neq r_1 + r_2 \)

\[ \alpha_{r_1 r_2} = \begin{cases} n \alpha & \text{for } r \mod n = r_1 + r_2 \text{ corresponding to } \Gamma (r) \\ 0 & \text{for all other representations.} \end{cases} \]

(b) If \( \Gamma_0 (\alpha) \) and \( \Gamma_0 (\beta) \) are two one-dimensional representations then \( \Gamma_0 (\alpha) \otimes \Gamma_0 (\beta) \) is given a one-dimensional representation given by \( \Gamma (\alpha + \beta) \) where \( (\alpha + \beta) i \mod m = \alpha_i + \beta_i \) for \( i = 1 \ldots 2n \).

(c) If \( \Gamma (r) \) is an \( n^2 \)-dimensional representation and \( \Gamma_0 (\beta) \) a one-dimensional representation then \( \Gamma (r) \otimes \Gamma_0 (\beta) \) is a faithful representation equivalent to \( \Gamma (r) \) itself.

**Proof.** \( \Gamma (r) \) corresponds to choice of \( L_0 = \omega (m) r \) and \( \Gamma_0 (\beta) \) corresponds to the one-dimensional representation generated by the
choice \( L_i = \omega(m)^{\beta_i} \) \( \forall i = 1 \ldots 2v \). The direct product

\[ \Gamma_1^{(r)} \otimes \Gamma_2^{(s)} \]

corresponds to the choice of generators \( \{ \omega(m)^{\beta_i} L_i \mid i = 1 \ldots 2v \} \)

\[ L_0 = \omega(m)^{r \beta} \]

This representation is equivalent to \( \Gamma^{(r)} \) itself if we can find a non-singular matrix \( M \) such that

\[ L_i \cdot M = M \cdot \omega(m)^{\beta_i} L_i \quad \forall i = 1 \ldots 2v \]

(3.94)

Taking

\[ M = \prod_{i=1}^{2v} L_i \cdot k_i \]

(3.95)

we have the condition as

\[
\begin{pmatrix}
0 & 1 & \cdots & 1 \\
-1 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & 0 \\
-1 & -1 & \cdots & 0 \\
\end{pmatrix}
\begin{pmatrix}
k_1 \\
k_2 \\
\vdots \\
k_{2v} \\
\end{pmatrix}
= 
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_{2v} \\
\end{pmatrix}
\mod m
\]

(3.96)

The inverse transformation always exists giving a solution for \( k_i \)'s

namely

\[
\begin{pmatrix}
k_1 \\
k_2 \\
\vdots \\
k_{2v} \\
\end{pmatrix}
\mod m = 
\begin{pmatrix}
0 & -1 & +1 & \cdots & -1 \\
-1 & 0 & -1 & \cdots & +1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & \cdots & -1 & +1 \\
-1 & -1 & \cdots & 0 & +1 \\
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_{2v} \\
\end{pmatrix}
\]

(3.97)

This proves the statement (c)

2) \( n = 2^2 + 1 \)

\[ [r_1, l_1; r_2, k] \]

a) Let \( [r_1, t] \) denote the number of times the
representation \( \Gamma^{(r, l)} \) occurs in the direct product representation \( \Gamma^{(r_1, l_1)} \otimes \Gamma^{(r_2, l_2)} \) if

(i) \( 0 \mod n = r_1 + r_2 \) then \( \Gamma^{(r_1, l_1)} \otimes \Gamma^{(r_2, l_2)} \) contains all the \( n^{\text{th}} \) one dimensional representations which have the element \( g(m-1, s-1), \ldots, m-1, l, \ldots, 1, m-1 \) represented by \( e(m)^b \), \( e \mod n = l + k \)

(ii) \( 0 \mod n \neq r_1 + r_2 \)

\[
\begin{align*}
\begin{bmatrix} r_1 & r_2 & r \end{bmatrix} & \begin{bmatrix} n^v \text{ for } r = \mod n = r_1 + r_2 \\
0 \text{ for other representations}
\end{bmatrix} \\
\begin{bmatrix} y & t \end{bmatrix} & \begin{bmatrix} t \mod n = l + k \end{bmatrix}
\end{align*}
\]

(b) The direct product of two one dimensional representations \( \Gamma^{(r_1, l_1)} \) and \( \Gamma^{(r_2, l_2)} \) is again a one dimensional representation \( \Gamma^{(r_1 + r_2, l_1 + l_2)} \) with \( (\alpha + \beta)_d \mod n = \alpha_d + \beta_d \)

(c) The direct product representation \( \Gamma^{(r, l)} \otimes \Gamma^{(r_0, l_0)} \) is equivalent to \( \Gamma^{(r + r_0, l + l_0)} \) where \( t \) is uniquely determined by \( \Lambda \) and \( \Sigma \). Proof: The representation \( \Gamma^{(r, l)} \otimes \Gamma^{(r_0, l_0)} \) corresponds to the choice of generators

\[
\begin{align*}
\omega(m)^{b_1 + p_1} L_i & \quad i = 1, \ldots, 2v + 1 \\
L_0 & = \omega(m)_v^2
\end{align*}
\]

and this is equivalent to \( \Gamma^{(r, l)} \) if there is a nonsingular matrix \( M \) such that

\[
\begin{align*}
\omega(m)^{b_1 + p_1} L_i & \quad M = M \omega(m)^{b_1 + p_1} L_i \quad i = 1, \ldots, 2v + 1 \\
L_0 & = \omega(m)_v^2
\end{align*}
\]

Taking \( M \) as \( \prod_{i=1}^{2v+1} L_i^{b_1} \), the condition for (3.99) is
\[
\begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
-1 & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & 0 & 1 \\
-1 & -1 & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
k_1 \\
k_2 \\
\vdots \\
r_{2N+1}
\end{pmatrix} =
\begin{pmatrix}
\beta_1 + l - t \\
\beta_2 + l - t \\
\vdots \\
\beta_{2N+1} + l - t
\end{pmatrix} \mod m.
\]

(3.100)

This does not admit a solution for \( k_i \)'s when \( \beta_i \)'s and \( l \) are fixed and \( t \) is arbitrary. Hence let us suppose \( k_{2N+1} = 0 \). Then we have two conditions:

\[-(k_1 + k_2 + \cdots + k_{2N}) = \beta_{2N+1} + l - t \mod m \]

(3.101)

\[
\begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
-1 & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & 0 & 1 \\
-1 & -1 & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
k_1 \\
k_2 \\
\vdots \\
r_{2N}
\end{pmatrix} =
\begin{pmatrix}
\beta_1 + l - t \\
\beta_2 + l - t \\
\vdots \\
\beta_{2N} + l - t
\end{pmatrix} \mod m.
\]

(3.102)

From (3.102) we find

\[-(k_1 + k_2 + \cdots + k_{2N}) = (-\beta_1 + \beta_2 - \cdots + \beta_{2N})
\]

\[= (\beta_{2N+1} + l - t) \text{ by (3.101)}
\]

or

\[t = (\beta_1 - \beta_2 + \beta_3 - \cdots - \beta_{2N} + \beta_{2N+1} + l)
\]

(3.104)

Now with this value of \( t \) substituted in (3.102) solution for \( k_i \)'s can be obtained uniquely as given in (3.97). Hence this proves (c).

vi) Generalised Clifford group \( C_n^m \) and the group of linear transformations leaving invariant

\[\sum_{i=1}^{m} (x^i)^m \text{ for } m > 2\]

It is well known (ref: Boerner (11)) that in the case of \( C_n^{(2)} \)-ordinary Clifford algebra, if we define
\[ S_{jkl} = \frac{1}{2} \mathcal{L}^{-j}_{kl} \quad ; \quad j, k, l = 1 \ldots n. \] 

(3.105)

then they satisfy

\[
[S_{jkl}, S_{mnl}] = \delta_{kr} S_{jnl} + \delta_{jl} S_{kmn} - \delta_{kl} S_{jm} - \delta_{jl} S_{km} = S_{jkl} S_{kmn} - S_{jm} S_{kmn} - S_{jl} S_{kmn}
\]

(3.106)

which is the structure of the infinitesimal ring of the \((n+1)\) dimensional rotation group. Thus the connection of Clifford algebra with the rotation group which forms the basis of the so-called spin representations of orthogonal groups is well known. Let us describe the situation as follows. Let the L-Matrix

\[
L(x) = \sum_{i=1}^{n} x_i L_i
\]

(3.107)

be associated with the n-dimensional vector \(x = (x_1, \ldots, x_n)\) with coordinates \((x_1, \ldots, x_n)\). Then after a rotation of the space, let the coordinates of \(x = x' = (x'_1, \ldots, x'_n)\) in the new frame. Then let the corresponding L-matrix be

\[
L' = \sum_{i=1}^{n} x'_i L_i = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} x_j \right) L_i
\]

(3.108)

where the old and new coordinates are related by the orthogonal transformation \(A\)

\[
x'_i = \sum_{j=1}^{n} a_{ij} x_j \quad ; \quad AA^T = I \quad ; \quad \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x'_i^2
\]

(3.109)
we have
\[ L^{'1} = L^{'2} = \left( \sum_{i=1}^{n} x_i^{('1)} \right) I = \left( \sum_{i=1}^{n} x_i^{('2)} \right) I \]  
(3.10)

Since
\[ L^{'1} = \sum_{i=1}^{n} x_i^{('1)} L_i = \sum_{i=1}^{n} x_i^{('1)} \left( \sum_{i=1}^{n} a_{ij} L_i \right) = \sum_{i=1}^{n} x_i^{('1)} L_i \]  
(3.111)

L_i also satisfy
\[ L_i^{('1)} L_i^{('2)} = -L_i^{('2)} L_i^{('1)} ; \; k, j = 1, \ldots, n. \]
\[ L_i^{('1)} = I ; \; \forall j = 1, \ldots, n. \]  
(3.122)

But since L_i have only one representation in case of \( n = 2v \)
and only two in case of \( n = 2v + 1 \) by Pauli's theorem the following
equations must have solution for S
\[ a) \quad L' = S^{-1} L S \quad \text{if} \quad n = 2v. \]  
(3.123)

\[ b) \quad L' = \pm S^{-1} L S \quad \text{if} \quad n = 2v + 1. \]

Thus the group of matrices S induced by rotations A of the
space form a group homomorphic to the rotation group which has been
called a Clifford group (cf. Kahan (8), H. Freudenthal and H. de Vries (12)).

Dirac group \( D_2 \) is a subgroup of this Clifford group, and corres-
ponds to only permutations and reflections of the basis vectors.

Now in the case of generalized Clifford algebra there exists a similar
connection between the group of linear transformations leaving in-
viant the expression
\[ \sum_{i=1}^{n} x_i^{('1)} \]  
(3.114)
and the group \( G^n_m \) which we have called a generalized Clifford group. This is the reason for calling \( G^n_m \) a G.G.1., rather than Generalized Dirac Group. Now we shall describe this relationship in detail. As Ham(13) has observed the set of all linear transformations leaving invariant the expression \( \sum_{i=1}^{n} x_i^m \) for \( n > 2 \) is a finite group of order \( n^n \) and these are given by the transformations

\[
\begin{pmatrix}
  x_1' \\
  x_2' \\
  \vdots \\
  x_n'
\end{pmatrix} = \begin{pmatrix}
  \omega(m) x_1 \\
  \omega(m) x_2 \\
  \vdots \\
  \omega(m) x_n
\end{pmatrix} P \begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
\]

or

\[
(x') = S P (x)
\]

where \( P \) is any \( n \times n \) permutation matrix and \( 0 \leq \lambda_i \leq n-1 \)
\( \forall i = 1, \ldots, n \). Now consider the subgroup of the above transformations with \( P = \mathbf{I} \) = identity matrix. Consider analogous to (3.107) the matrix

\[
L(x) = \sum_{i=1}^{n} x_i L_i
\]

where

\[
L_i L_j = \omega(m) L_j L_i \quad (i, m) = 1
\]

then

\[
L(x)^m = \left( \sum_{i=1}^{n} x_i^m \right) \mathbf{I}
\]
The transformation \( (x) \mapsto (x') = S(x) \) is represented by the transformation

\[
L(x) = \sum_{i=1}^{n} x_i L_i \rightarrow L'(x) = \sum_{i=1}^{n} x_i' L_i = \sum_{i=1}^{n} x_i (\omega^{k_i} L_i)
\]

\[ (3.119) \]

Here also there are two cases corresponding to whether \( n \) is even or odd. When \( n = 2 \nu \) similar to \((3.113a)\)

\[
L'(x) = M^{-1} L(x) M
\]

\[ (3.120) \]

where

\[
M = \prod_{i=0}^{2\nu} L_i^{k_i}
\]

in which \( \{ k_i \mid i=0, \ldots, 2\nu \} \) are fixed uniquely by \( S \) as discussed in \((v,1,c)\) and \( k_0 \) can take all \( m \) values \( 0,1,\ldots,m-1 \). In \( \mathcal{G}_m \) all these elements form a class.

When \( n = 2\nu + 1 \), analogous to the case \((3.113b)\) we have

\[
L'(x) = K M^{-1} L(x) M
\]

\[ (3.121) \]

where

\[
K = \omega(m)^{t \mod m} = \sum_{k=1}^{t \mod m} (-)^{k+1} \chi_k
\]

\[ (3.122) \]

and

\[
M = \prod_{i=0}^{2\nu} L_i^{k_i}
\]

\[ (3.123) \]

with \( \{ k_i \mid i=1, \ldots, 2\nu \} \) being fixed uniquely by \( S \) as discussed in \((v,3a)\) and \( 0 \leq k_0 \leq m-1 \). This set of elements \( K \) form a class in \( \mathcal{O}, \mathcal{G}, \mathcal{O} \). Thus just like Clifford group provides double valued representations of the orthogonal group leaving invariant the quadratic expression \( \sum_{i=1}^{n} x_i^2 \), \( \mathcal{O}, \mathcal{G}, \mathcal{O} \) provides \( m \)-valued representation of the group proper (excluding permutations) transformations leaving
invariant the expression $\sum_{i=1}^{\tilde{n}} x_i^{m_i}$ for $m > 2$.

Summary of important points.

We call the group of elements $G_{n_n}^m = \{ \prod_{i=0}^{r} L_i^{k_i} \mid 0 \leq k_i \leq m-1 \}$ as a Generalized Clifford group $(G_0, G_1, G_2)$ if the generators $\{ L_i \mid i = 1 \ldots n \}$ obey

$$L_i L_j = L_0 L_j L_i \quad \forall i, j = 1 \ldots n \quad s_{L_i} = 1, \quad L_0 L_j = L_j L_0 \quad \forall i = 0, 1, \ldots n.$$  

When $n$ is assumed to be a prime number $\frac{m_n}{m_{n+1}}$ has the following properties.

- Total number of elements $= m_{n+1}$.
- Total number of conjugate classes.

Hence the total number of irreducible inequivalent representations is $m_{n+1}(m_{n-1})$ when $n = 2^N$ and $m_{n+1} m_{n+1}(m_{n-1})$ when $n = 2^{N+1}$.

All the representations are obtained by assuming the various permitted values of $L_0 = \exp\left(2\pi i l/n\right)$; $l = 0, 1, \ldots, n-1$. When $l = 0$, $\chi_{mn}$ one-dimensional representations of $G_{mn}$ are generated for both the cases $m = n^2$ and $n = 2^{N+1}$ by the representations of $\mathbb{Z}_m \otimes \cdots \otimes \mathbb{Z}_m$ (n copies) where $\mathbb{Z}_m$ is the cyclic group of order $m$. For each value of $l = 1, 2, \ldots, n-1$, there is one irreducible representation of dimension $m_l$ when $n = 2^N$. In the case of $n = 2^{N+1}$, each value of $l = 1, 2, \ldots, n-1$ corresponds to a generalized Clifford algebra having $m$ inequivalent irreducible representations of same dimension $m_l$. This chapter determines explicitly all the inequivalent irreducible representations and also studies direct product representations.
CHAPTER 4

GENERALIZED CLIFFORD GROUPS - XI

In this chapter we shall study in detail the properties and representations of \( \mathbb{C}_n \) when \( n \) is any integer. This chapter is based mainly on the paper of Rao and myself.

(i) Properties of \( \mathbb{C}_n \) when \( n \) is not a prime number.

As defined in the previous chapter \( \mathbb{C}_n \) is a group of \( n^{n+1} \) elements

\[
G_n = \left\{ \prod_{i=0}^{n} L_i^{k_i} \mid 0 \leq k_i \leq n-1 \right\}
\]

where \( L_i \)'s obey the commutation relations

\[
L_i L_j = L_j L_i, \quad i < j, \quad \forall L_i, L_j = L_0, L_1, \ldots, L_n
\]

As before denoting by

\[
g(k_0, k_1, \ldots, k_n) = \prod_{i=0}^{n} L_i^{k_i}
\]

the product and inverse of elements are given by the same formula (3.2.2-25). Let us now count the number of classes in this group.

First let \( n = 2^k \). In this case there exists no relation of linear dependence among the elements \( \{ g(0, k_1, \ldots, k_n) \mid 0 \leq k_i \leq n-1 \} \)

Let \( D = \{ d_1, d_2, \ldots, d_{\tau(m)} \} \) be the set of all divisors of \( n \).
in ascending order, \( \tau(m) \) denotes the number of divisors of \( m \).

An element \( q_j = (q_0, j_1, \ldots, j_m) \) obeys

\[
q_j^{d_j} = \begin{cases} 
+1 & \text{if } d_j \text{ is odd} \\
-1 & \text{if } d_j \text{ is even}
\end{cases}
\]  

(4.4)

where \( d_j \in \mathbb{D} \) and

\[
v_k d_j \equiv 0 \pmod{m} \quad \forall k = 1, 2, \ldots, m
\]  

(4.5)

**Proof.** Let \( v'_k \)'s be such that \( \forall k, v'_k d_j \equiv 0 \pmod{m} \). Then

\[
q_j^{d_j} = \prod_{k=1}^{m} L_k i \frac{1}{2} (d_j - 1) d_j K
\]  

(4.6)

where

\[
K = \begin{pmatrix} j_1 & \cdots & j_{2m} \\ 0 & 0 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & j_0
\end{pmatrix}
\]  

(4.7)

Since \( L_0^n = 1, L_0 = 1 \). Hence if \( d_j = 2 \delta + 1 \)

\[
q_j^{d_j} = \frac{1}{2} 2 \delta (2 \delta + 1) K s_d = L_0^{d_j} = +1
\]  

(4.8)

If \( d_j = 2 \delta \) then

\[
q_j^{d_j} = \frac{1}{2} K (2 \delta - 1) 2 \delta = \frac{1}{2} K (2 \delta - 1) d_j
\]  

(4.9)

For a given element \( q(0, j_1, \ldots, j_m) \) let us choose \( d_j \) to be the minimum \( d \in \mathbb{D} \) satisfying (4.4) and (4.5). It is easy to see that for any element \( q(0, j_1, \ldots, j_m) \) such a \( d_j \in \mathbb{D} \) exists. Then
we have

\[ q^{d_j} = \pm 1 \Rightarrow q_k (q_j^{d_j})^{-1} = (q_k, q_j^{d_j})^{-1} = \pm 1 \quad (4.10) \]

or

\[ q_k q_j q_k^{-1} = L_0 q_j \quad \text{for any } k \quad (4.11) \]

where

\[ q_j = \frac{m}{d_j} \quad 0 \leq r_k \leq d_j - 1. \quad (4.12) \]

when we vary \( q_k \in \mathbb{Z}_m \) overall elements then we obtain a set

\[ \{ L_0 q_j \} \]

which are in a class. If \( d_j \) is nonprime then there is a possibility that \( \{ L_0 q_j \} \) may not contain all the \( d_j \)-th roots of unity which arises when \((r_k, d_j) = \gcd(r_k, d_j) = 1\), for all \( r_k \). In that case let \( e_j \) be the greatest common divisor of \( \{ r_k, d_j \} \) and \( \{ r_k, d_j \} \). Then the set will contain only all \( \frac{d_j}{e_j} \)-th roots of unity and this would imply that \( q_j^{(d_j/e_j)} \) commutes with all \( q_k \in \mathbb{Z}_m \). By Schur's lemma this means that which is in contradiction to the fact that \( d_j \) was the minimum divisor satisfying this condition. Hence the set \( \{ L_0 q_j \} \) contains \( d_j \) distinct elements corresponding to \( 0 \leq r_k \leq d_j - 1 \). This shows that the set of \( n \) elements

\[ \{ q_l^{(d_j/2)} \mid 0 \leq j_0 \leq m-1 \} \quad (4.13) \]

are partitioned in \( \frac{m}{d_j} \) classes each containing \( d_j \) elements.
Explicitly the classes are given by

\[
\left\{ \left. g \left( y_0^\prime, y_1^\prime, \ldots, y_{2\nu}^\prime \right) \right| \tau = 0, 1, \ldots, d_j - 1 \right\}
\]

\( y_0^\prime = 0, 1, \ldots, \nu_j - 1 \)

(4.13)

Let \( N_j \) be the total number of elements with \( y_0^\prime = 0 \) obeying the relation (4.4) with \( d_j \) as the minimum for a \( d_j \in D \). Then the total number of classes in \( G_{2\nu}^m \)

\[
N_{2\nu}^m = \sum_{j=1}^{T(m)} N_j \left( \frac{m}{d_j} \right)
\]

(4.14)

Obviously \( N_1 = 1 \), \( N_{T(m)} = \left\{ m^{2\nu} - \sum_{j=1}^{T(m)-1} N_j \right\} \). For others

\[
N_j = d_j^{2\nu} - \sum_{s < j} N_s e(s)
\]

where

\[
e(s) = \begin{cases} 
1 & \text{if } d_s < d_j \\
0 & \text{if } d_s = d_j
\end{cases}
\]

(4.16)

These follow from the observation that the condition

\[
j \cdot d_j = 0 \mod m, \quad \forall \ k = 1, \ldots, 2\nu
\]

(4.17)

is satisfied whenever \( 0 \leq \frac{j}{d_j} \leq d_j - 1, \forall \ k = 1, \ldots, 2\nu \) and thus there are \( d_{2\nu}^m \) solutions for this condition. Substituting (4.16) in (4.14), we have

\[
N_{2\nu}^m = \sum_{j=1}^{T(m)} \left( \frac{m}{d_j} \right) \left\{ d_j^{2\nu} - \sum_{s \leq j} N_s e(s) \right\}
\]

(4.18)
In our paper\textsuperscript{1} it was conjectured that
\begin{equation}
N_{\frac{m}{l}} = \sum_{l=1}^{m} (l, m) \frac{2^l}{2^m} = \sum_{l=1}^{m} d_l \frac{2^l}{2^m}
\end{equation}
\text{d}_l is the greatest common divisor of \text{l} and \text{m}. This implies the identity
\begin{equation}
\sum_{j=1}^{(m)} \left( \frac{m}{d_j} \right) \left\{ \frac{d_j^{2^l}}{2^m} - \sum N_\frac{a}{b} \chi(s) \right\} = \sum_{l=1}^{m} (l, m) \frac{2^l}{2^m}
\end{equation}
\text{This identity was first proved recently by Krishnaswamy Alladi\textsuperscript{2}.}

Following his proof we shall reformulate it using a matrix approach to the underlying number theoretic problem\textsuperscript{3}.

Let us denote \text{N}_j by \text{N}_d where \text{d} | \text{m}. The suffix \text{j} in \text{d}_j is unnecessary since all \text{d}_j's are distinct. Then the definition (4.16) can be written as
\begin{equation}
\sum_{\text{d'} | \text{d}} \text{N}_{\text{d'}} = \text{d}^{2^m}
\end{equation}
where summation is over all \text{d'} | \text{d}.

Möbius inversion formula tell us that if an arithmetical function \text{f}_n is defined by
\begin{equation}
\sum_{\text{d} | \text{m}} \text{f}_\frac{\text{d}}{\text{m}} = \text{g}_n
\end{equation}
then \text{f}_n is uniquely given by
\begin{equation}
\text{f}_n = \sum_{\text{d} | \text{n}} g\left(\frac{n}{d}\right) \mu(d) = \sum_{\text{d} | \text{n}} \frac{\mu\left(\frac{n}{d}\right)}{d} g(d)
\end{equation}

I wish to thank Professor Paul Erdős for very encouraging and useful discussion on this identity. Also I wish to thank Professor Kirschhern University of New South Wales, School of Mathematics, Australia for a similar proof of the identity\textsuperscript{4}. The proof of Mr. Krishnaswamy Alladi applies to more general context and consequently he has used this identity in other number theoretic problems\textsuperscript{5}.
where $\mu(n)$ is the Möbius function

$$
\mu(n) = \begin{cases} 
1 & \text{if } n = 1 \\
(-1)^k & \text{if } n = \prod_{i=1}^{k} p_i^k, \ p_i's \ are \ distinct \ primes \\
0 & \text{otherwise}
\end{cases}
$$

(4.34)

Applied to (4.23) this gives

$$
N_d = \sum_{d' \mid d} \left( \frac{d}{d'} \right)^{2\nu} \mu(d') = \sum_{d' \mid d} \mu \left( \frac{d}{d'} \right) d'^{-2\nu}
$$

(4.35)

Let us now introduce matrices $A$ and $S$ such that

$$
\Delta_{d,d'} = \begin{cases} 
1 & \text{if } d' \mid d \\
0 & \text{otherwise}
\end{cases} 
\quad 1 \leq d', d \leq \infty
$$

(4.36)

$$
S_{ij} = \delta_{ij} 
\quad 1 \leq i, j \leq \infty
$$

(4.37)

Associated to any arithmetic function $A_n$ let us introduce a matrix $A$ such that

$$
A_{d,d'} = \begin{cases} 
A \left( \frac{d}{d'} \right) & \text{if } d' \mid d \\
0 & \text{if } d' \not\mid d
\end{cases}
$$

(4.38)

Then (4.23) can be written as

$$
AF = G
$$

(4.39)

where $F$, $G$ are column vectors with $f_n$ and $g_n$ as elements

(4.39) implies

$$
F = \mu G
$$

(4.40)

where $\mu$ is the matrix associated with Möbius function $\mu(n)$ defined by (4.34). From (4.33) and (4.40) it follows

$$
A^{-1} = \mu
$$

(4.41)
Euler's function \( \phi(n) \) is defined as the number of integers less than

\[
\sum_{d \leq n} \phi(d) = \sum_{d \leq n} \frac{n}{d} \mu\left(\frac{n}{d}\right)
\]

(4.38)

It obeys the identity

\[
\sum_{d \mid n} \phi(d) = n
\]

(4.39)
or using Möbius inversion

\[
\phi(n) = \sum_{d \mid n} \frac{n}{d} \mu\left(\frac{n}{d}\right)
\]

(4.34)

Let an \( d \mid n \). Then

\[
\phi\left(\frac{n}{d}\right) = \sum_{d \mid \frac{n}{d}} \frac{n}{d} \mu\left(\frac{n}{d}\right) = \sum_{d \mid \frac{n}{d}} \frac{n}{d} \mu\left(\frac{d'}{e}\right) ; \quad d' = 2d
\]

(4.35)
or writing in matrix form

\[
\Phi = S \Delta S^{-1} \mu
\]

(4.36)

where \( \Phi \) is the matrix associated with Euler \( \phi(n) \). From (4.31) it is clear that

\[
\Delta N = D
\]

(4.37)

where \( N \) is the column vector with elements \( (1 \leq d \leq \infty) \) and \( D \) is the column vector with elements \( D_d = d \) \( (1 \leq d \leq \infty) \) (4.36) gives

\[
\Phi \Delta N = S \Delta S^{-1} N
\]

(4.38)

Hence

\[
\Phi \Delta N = S \Delta S^{-1} N
\]

(4.39)

writing in terms of elements

\[
\begin{pmatrix}
S & \Delta \\
\end{pmatrix}
\]
\[
\sum \phi \left( \frac{n}{d} \right) D_d = \sum_{d \mid n} \frac{n}{d} N_d
\]
\[
\sum \phi \left( \frac{n}{d} \right) d^{2\nu} = \sum_{d \mid n} \frac{n}{d} N_d
\]

We have from the definition of \( \phi(n) \)
\[
\phi \left( \frac{n}{d} \right) = \sum_{(d', \frac{n}{d}) = 1} 1 = \sum_{l < d, \frac{n}{d} = d'} 1
\]

Hence (4.42) gives
\[
\sum \phi \left( \frac{n}{d} \right) d^{2\nu} = \sum_{l=1}^{n} \sum_{d \mid n} \frac{n}{d} N_d
\]

With \( n_1 = n \) this proves the identity (4.30). Incidentally the general nature of the identity (4.41) has to be noticed. Any two sequences \( R_1 \) related by (4.37) will obey (4.41) and the identity (4.30) is a special case of this. Thus the total number of classes in \( G_\nu^m \) is
\[
\sum_{l=1}^{m} (l, m)^{2\nu}
\]

In the case of \( G^m_\nu+1 \) the set of \( m^{2\nu+2} \) group elements
\[
\{ q(k_0, k_1, \ldots, k_{2\nu+1}) \mid \forall i, 0 \leq k_i \leq m-1 \}
\]
can be written as a sum of \( \nu \) subsets:
\[
\left\{ \left( \prod_{i=0}^{2\nu} k_i \right) \mid 0 \leq k_i \leq m-1 \right\}, \left( \prod_{i=0}^{2\nu} k_i \right) \left[ \prod_{i=0}^{2\nu+1} l_i \right] \mid 0 \leq k_i \leq m-1 \right\}, \ldots,
\]
\[
\left( \prod_{i=0}^{2\nu} k_i \right) \left[ \prod_{i=0}^{2\nu} l_i \right] \mid 0 \leq k_i \leq m-1 \}
\]
\[
\begin{align*}
\quad = \sum_{l=1}^{m^2} \left( g(k_0, k_1, \ldots, k_{2m}) \right) & \\
\quad = \sum_{l=1}^{m^2} \left( g(k_0, k_1, \ldots, k_{2m}) \right) & \\
= \left\{ \left( g(k_0, k_1, \ldots, k_{2m}) \right) \right\} & \\
\end{align*}
\]

(4.44)

Now as we have seen earlier (ref. chapter 1.) there exists a relationship among the \( L_i \) namely

\[
\eta = g(0, m-1, m-1, \ldots, m-1, 1, \ldots, 1, m-1) - I
\]

(4.45)

and hence \( L_{2^v+1} \) can be replaced as a product of powers of other \( L_i \) so that each subset of (4.44) given in brackets contains only product of powers of \( 2^v, L_i \) . Or the set \( G_{2^v+1}^m \) splits into a group of \( m \) subsets each isomorphic to \( G_{2^v}^m \). Hence the total number of classes in \( G_{2^v+1}^m \) is \( m \) times the number of classes in \( G_{2^v}^m \). Thus denoting by \( N_{2^v}^m \) the number of classes in \( G_{2^v}^m \), we have

\[
N_{2^v}^m = \sum_{l=1}^{m^2} (l, m) 2^v
\]

(4.46)

(4.47)

(11) \textbf{Representations of } \( G_{2^v}^m \) has \( m^{2^v+1} \) elements and \( \sum_{l=1}^{m^2} (l, m)^{2^v} \) conjugate classes.
It is seen that
\[ m^{2\nu+1} = \sum_{l=1}^{m} \left( \frac{m}{(l, m)} \right)^{2\nu} \] 
\[ = \sum_{m \text{ times}} m^{2\nu} = m^{2\nu+1} \]  
(4.48)

Thus we can guess that there should be for each \( 1 \leq l \leq m \), \((l, m)^{2\nu}\) representations of dimension \( \left( \frac{m}{(l, m)} \right)^{2\nu} \). As done for \( G_{1} \), we should look for these representations in the \( m \) inequivalent isomorphic algebras \( C_{\mathbb{C}}(a), 1 \leq l \leq m \) generated by

\[ L_{i} L_{j} = \omega(m) L_{j} L_{i}, \quad i, j = 1, 2, \ldots, 2\nu. \]  
(4.49)

\[ L_{i}^{m} = 1, \quad \forall i = 1, 2, \ldots, 2\nu, \quad 1 \leq l \leq m \]  
(4.50)

respectively.

a) one-dimensional representations

These correspond to the case \( \nu = m \) or \( \nu = 0 \). Then (4.49) becomes

\[ L_{i} L_{j} = L_{j} L_{i}, \quad i, j = 1, 2, \ldots, 2\nu \]  
(4.50)

\[ L_{i}^{m} = 1, \quad \forall i = 1, 2, \ldots, 2\nu \]  
(4.51)

These being the generating relations one of the Abelian group \( G \cong \mathbb{Z}_{m} \otimes \cdots \otimes \mathbb{Z}_{m} (2\nu \text{- copies}) \) as before these representations arise from the homomorphism of \( G_{2\nu}^{m} \) to \( \mathbb{Z}_{m} \otimes \cdots \otimes \mathbb{Z}_{m} (2\nu \text{- copies}) \)
Thus all these representations totally \((l,m') = (m/m) = m\) one dimensional representations arise by putting

\[ L_i = \omega(m) e^{T_{k'i}}; \quad 0 \leq T_{k'i} \leq m - 1; \quad i = 1, \ldots, 2m. \]  \((4.60)\)

where \(k\) labels the representations \(k = 1, \ldots, m\) in all these representations.

b) Higher dimensional representations

Consider now \((4.40)\) for \(l > 0\). For each \(1 \leq l \leq m - 1\) find the representations. Let

\[ (l,m) = \frac{p}{q}, \quad \Omega_l = \frac{m}{p}, \quad \Omega = \frac{k}{l}, \quad (\Omega_l, \Omega_v) = 1 \]  \((4.61)\)

Then \((4.60)\) becomes

\[ L_i L_j = \omega(\Omega_v)^{T_{k'i} T_{k'j}} L_{i'} L_{j'}; \quad l, j = 1, \ldots, 2m. \]  \((4.62)\)

When \(\Omega = 1\) the problem reduces to the one considered in Chapter II -a). There we showed that the elements \(L_1, L_2\) obeying

\[ L_1 L_2 = \omega(m) L_2 L_1, \quad L_{i'} L_{i'} = 1; \quad (l,m) = d \]

have \(2^2\) inequivalent irreducible representations of dimension \(m' = \frac{m}{d}\) which are given by

\[ \{ \omega(m) e^{\xi' i} L_1', \omega(m) e^{\xi' i} L_2' \}; \quad i = 1, \ldots, d^2; \quad 0 \leq \xi, \xi' \leq d - 1; \]  \((4.61)\)

\(L_1'\) and \(L_2'\) being the unique (up to equivalence) representations of

\[ L_1' L_2' = \omega(m) L_2' L_1', \quad L_{i'} L_{i'} = 1; \quad m' = m; \quad l' = \frac{k}{d}; \quad (\Omega, m') = 1. \]  \((4.63)\)

Uniqueness of \(L_1', L_2'\) was assumed in proving this result. Now due to the above group theoretical considerations the uniqueness of \(L_1', L_2'\)
atomic proved as follows. Considering the group \( G_{12}^{m'} \) for which \( L'_i, L'_j \) provide a representation corresponding to \( L' \) such that \( (i', m') = 1 \), there should be only \( (v', m') = 1 \) representation of dimension \( m' \). Thus for \( L'_i \), \( L'_{j} \) there cannot be any other representation.

Now corresponding to (4.32) consider the relations

\[
L'_i L'_j = \omega(q_{12})^{ij} L'_j L'_i \quad \forall i, j = 1, 2, \ldots, 2\nu.
\]

(4.33)

These relations would provide a representation for \( G_{12-\nu}^{2\nu} \) and for those there should be only \( (q_{12}, \tau_{12}) = 1 \) representation. Thus (4.33) should have only one representation. Based on this fact following the same type of arguments as in Chapter II a, we can prove that (4.34) have \( p_{2\nu}^{c2\nu} \) representations of dimension \( q_{c}^{2\nu} \) which are given by

\[
\{ (\omega(q_{c}^{2\nu}))_{ij} | i = 1, \ldots, 2\nu \} j = 1, \ldots, p_{2\nu}^{2\nu}, 0 \leq b_{j} \leq p_{c} - 1 \}
\]

(4.56) being the unique (up to equivalence) representations of (4.56). Here we shall follow a simpler reasoning to arrive at this result. Rewrite (4.54) as

a) \( L'_i L'_j = \omega(q_{12})^{ij} L'_j L'_i \quad \forall i, j = 1, \ldots, 2\nu \)

(4.56)

b) \( L'^{\nu_{c}}_{i} = (L_{i} q_{c})^{\nu_{c}} = 1 \quad \forall i = 1, \ldots, 2\nu \)

From a) it follows that \( L'_{i} q_{c}^{\nu_{c}}, \forall i = 1, \ldots, 2\nu \), commute with each other so that they are \( \sim I \). By b) it follows that \( L'^{q_{c}^{\nu_{c}}} \) can
take any of the \( p_k \), \( p_k^* \) roots of unity. So

\[
L_i^{\nu^k} = \omega(p_k)^{t_k} I \quad 0 \leq t_k \leq p_k - 1.
\]

(4.57)

Hence with \( L_i = \xi L_i^j \) where \( L_i^{\nu^k} = 1 \), \( L_i L_j^{\nu^k} = \omega \nu_1^k \nu_2^k \nu_3^k \), \( 0 \leq t_k \leq p_k - 1 \) or \( \xi = \omega(p_k) \), \( t_k/\nu_2^k = \omega(p_k)^{t_k} \)

\[
0 \leq t_k \leq p_k - 1.
\]

Thus

\[
L_i^{\nu^k} = \omega(p_k)^{t_k} L_i^j \quad \forall t_k = 1 \ldots 2
\]

(4.58)

\( L_i^{\nu^k} \)'s having only one representation, there are not any other representations possible for \( L_i^{\nu^k} \).

For the representations of \( L_i^{\nu^k} \)'s obeying the relations (4.58) we can use the method of product transforms detailed in Chapters I and III. The commutation matrix \( T \) associated with (4.58) is

\[
T = \tau_k \begin{bmatrix}
\omega(p_k)^{t_k} & 0 & \cdots & 0 \\
0 & \omega(p_k)^{t_{k-1}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \omega(p_k)^{t_1}
\end{bmatrix}
\]

(4.59)

and hence its canonical form is

\[
T^* = \tau_k \begin{bmatrix}
\omega(p_k)^{t_k} & 0 \\
0 & \omega(p_k)^{t_{k-1}} \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

(4.60)

associated with the relations...
\[ L'_{j} L'_{i} = \omega(q_k)^{j} L'_{j} L'_{i} \quad ; \quad i=1 \ldots \nu \]

\[ L'_{i} L'_{j} = \begin{cases} L'_{j} L'_{i} & \text{otherwise} \\ \text{otherwise} & \end{cases} \]

Following the same arguments as in the representation of (3.57) we get the representation of \( L'_{i} \) as

\[ L'_{i} = \prod_{j=1}^{\nu} L'_{j} \]

using the same \( V \) in (3.60) or explicitly using the formula (3.71)

\[ L'_{i} = \begin{cases} C_{q'-1}^{q-1} B \times \cdots \times C_{q'-1}^{q-1} B \times C \times I \times \cdots \times I \\ \text{for } k=1 \ldots \nu \end{cases} \]

Summarizing the representations of \( L'_{i} \) obeying (4.64) are given by

\[ \left\{ (\omega(m)^{8,i} L'_{i} \mid i=1 \ldots 2^\nu) \mid j=1 \ldots 2^\nu \quad ; \quad 0 \leq 8, j \leq p - 1^{2} \right\} \]

where \( j \) labels the representations and \( L'_{i} \) are given by (4.63). Thus there are totally \( p^{2^\nu} = (l, m)^{2^\nu} \) inequivalent representations of the dimension \( q_{e}^{l} = [m / (l, m)]^{\nu} \). \( L_{0} \) has to be taken as \( \omega(m)^{l} \) corresponding to (4.54). Thus corresponding to each value of \( 1 \leq l \leq m-1 \), there are \( (l, m)^{2^\nu} \) inequivalent irreducible representations of dimension \( [m / (l, m)]^{\nu} \) respectively. Thus we have got them all the representations of \( G_{m}^{2^\nu} \).
In this case we found we have \( m \sum_{l=1}^{m} (l, m) 2^l \) classes. Let us write now

\[
m^{2^l + 2} = \sum_{l=1}^{m} (l, m)^{2^l+1} \left( \frac{m}{(l, m)} \right)^{2^l+1}
\]

\[
= \sum_{l=1}^{m} (l, m) \left( \frac{m}{(l, m)} \right)^{2^l} \left( \frac{m}{(l, m)} \right)^{2^l+1} \left( \frac{m}{(l, m)} \right)^2
\]

\[
= \sum_{l=1}^{m} m(l, m)^{2^l} \left( \frac{m}{(l, m)} \right)^{2^l+1} \left( \frac{m}{(l, m)} \right)^2
\]

This tells us that there should be \( m(l, m)^{2^l} \left( \frac{m}{(l, m)} \right)^{2^l+1} \left( \frac{m}{(l, m)} \right)^2 \) representations of dimension \( \left( \frac{m}{(l, m)} \right)^{2^l} \) corresponding to each \( l \), \( 1 \leq l \leq m \).

a) One dimensional representations.

These correspond to the case of \( l = m \) or \( 0 \). Corresponding to this there should be \( m^{2^l+1} \) representations. As in earlier cases it is obvious that these arise due to the homomorphism of \( G_{2^l+1} \) to \( \mathbb{Z}_m \oplus \cdots \oplus \mathbb{Z}_m \) \((2^l+1 \text{ copies})\). All these representations are obtained by putting \( L_i = \omega(m)^{Y_{kl}}, 0 \leq Y_{kl} \leq m-1 \); \( i=1 \ldots 2^{l+1} \) where \( k \) is the representation index \( k=1 \ldots m \).

b) Higher dimensional representations.

As usual consider the relations...
\[
L_i L_j = \omega(m)^{\ell} L_j L_i ; \quad i, j = 1, \ldots, 2\nu + 1.
\]
\[
L_i \ell = \ell ; \quad \forall i = 1, \ldots, 2\nu + 1.
\]
(4.66)

For \(1 \leq \ell \leq m - 1\). If \((\ell, m) = p_\ell\), then letting \(v_\ell = \frac{m}{p_\ell}, \quad r_\ell = \frac{\ell}{p_\ell}\)

to have

\[
L_i L_j = \omega(v_\ell)^{r_\ell} L_j L_i ; \quad i, j = 1, \ldots, 2\nu + 1.
\]
\[
L_i \ell = \left( \ell v_\ell \right)^{r_\ell} = 1 ; \quad \forall i = 1, \ldots, 2\nu + 1. \quad (r_\ell, v_\ell) = 1
\]
(4.67)

Letting \(L_i^\prime\)'s obey

\[
L_i^\prime L_j^\prime = \omega(v_\ell)^{r_\ell} L_j^\prime L_i^\prime ; \quad i, j = 1, \ldots, 2\nu + 1.
\]
(4.68)

\[
(L_i^\prime \ell)^{r_\ell} = 1 ; \quad \forall i = 1, \ldots, 2\nu + 1.
\]

\(L_i^\prime\)'s obeying (4.67) are given the representation

\[
\left\{ (L_i = \omega(m)^{\ell v_\ell}, L_i^\prime) \mid \ell = 1, \ldots, 2\nu + 1 ; \quad 0 \leq v_\ell \leq p_\ell - 1 \right\}
\]
(4.69)

Following the same arguments leading to the similar result (4.66) in the case of \(2\nu\), \(L_i^\prime\)'s. Now the difference between the two cases \(G_{2\nu}^m\) and \(G_{2\nu + 1}^m\) lies in the fact that \(L_i^\prime\)'s have been more than one representations. While dealing with \(G_{2\nu + 1}^m\), it was shown that the relations (3.75) exactly identical to (4.66) except for the replacement \(v_\ell = l, q_\ell = m\), have \(m\) inequivalent irreducible
representations given by

\[ \left\{ \omega(m) \right\} L_i'' \mid i = 1, \ldots, 2N+1, \; \delta = 0, 1, \ldots, m - 1 \]  

(4.75)

where \( L_i'' \) are given by (cf. 3.79 and 3.81)

\[ L_i'' = \prod_{j=1}^{2N+1} L_j'' \quad \forall i \neq j \]  

(4.76)

with

\[ V = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \]

(4.77)

\[ L_j^* \text{ satisfying} \]

a) \( L_{\omega^{-1}} L_\omega = \omega(m) L_{\omega^{-1}} L_\omega^{-1} \) \( \forall i \neq 1 \)

b) \( L_j^* L_j^* = L_j^* L_j^* \) otherwise \( \forall i, j \neq 1 \)

(4.78)

and having representations

\[ L_{\omega^{-1}}^* = I \otimes \cdots \otimes \hat{C} \otimes I \otimes \cdots \otimes I \]

\[ L_\omega^* = I \otimes \cdots \otimes \hat{B} \otimes I \otimes \cdots \otimes I \]

\[ L_{2N+1}^* = I \otimes \cdots \otimes I \otimes \cdots \otimes I = \text{identity} \]

(4.79)

Now just taking \( \omega = \gamma \), \( m = \gamma \) we get for \( L_i'' \) and \( \gamma \)

\[ \left\{ \omega(\gamma) \right\} L_i'' \mid i = 1, \ldots, 2N+1, \; \delta = 0, 1, \ldots, m - 1 \]  

(4.75)
where $L^{''}_{\lambda}$ are given by (4.71) in which the $C$ and $D$ matrices are $\sqrt{\lambda}$-dimensional matrices and $\lambda$ is to be replaced by $\lambda_e$. By considering the group $G_{2\nu+1}$ for which (4.66) would generate representations corresponding to $\lambda_e$, there should be only $(\nu_e, \nu_e) = \nu_e$ representations. This shows that (4.66) cannot be more than $\sqrt{\lambda}$ representations which are given by (4.75). Now going back to (4.66) corresponding, each of the $\sqrt{\lambda}$ representations of $L^{''}_{\lambda}$ there are $P_{\lambda_e}^{2\nu+1}$ representations and hence there totally there are $q_{\lambda_e}^2 P_{\lambda_e}^{2\nu+1}$ representations for the relations (4.66). Thus summarising all the $P_{\lambda_e}^{\nu_e} \cdot m$ representations of (4.66) are given by

$$L^{''}_{\lambda} = \omega(m) \left[ \begin{array}{c} L^{''} \\
\end{array} \right]$$

$$0 \leq \nu_e \leq q_{\lambda_e} - 1, \quad 0 \leq k \leq q_{\lambda_e} - 1$$

(4.76)

$L^{''}_{\lambda}$ being given by (4.71) and $(\lambda, m) = P_{\lambda_e}^{\nu_e}, q_{\lambda_e} = \frac{m}{P_{\lambda_e}}, \nu_e = \frac{1}{P_{\lambda_e}}$

Considering each value of $\lambda = 1 \ldots m - 1$ all the corresponding relations of the type (4.66) respectively provide all the higher dimensional representations of the generators of corresponding to (4.66).

(iv) Examples.

We shall illustrate the above considerations by means of two examples.

(a) $G_{2\nu}^4$: The total number of elements $N$ is $4^{2\nu+1} = 64$ and the number of classes is $\sum (l, 4)^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$. Therefore
exist \((4,4)^2 = 16\) one dimensional representations corresponding
to \(L = 4\), which arise from the representations of \(\mathbb{Z}_4 \otimes \mathbb{Z}_4\).
These are given by taking 
\[ L_1 = \omega(4)^{i}, \quad L_2 = \omega^{i}, \quad (\gamma, \lambda = 0, 1, 2, 3) \] 
\[ L_3 = 1 \] 
Corresponding to the roots \(\omega(4)^1\) and \(\omega(4)^3\) which are primitive
there are four dimensional representations for each which are given
by taking the generators as
\[ L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = C(4) \] 
\[ L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega(4) & 0 & 0 \\ 0 & 0 & \omega(4) & 0 \\ 0 & 0 & 0 & \omega(4) \end{pmatrix} \] 
\[ L_3 = \omega(4) \]
and
\[ L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = C(4) \] 
\[ L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega(4)^3 & 0 & 0 \\ 0 & 0 & \omega(4)^3 & 0 \\ 0 & 0 & 0 & \omega(4)^3 \end{pmatrix} \] 
\[ L_3 = \omega(4)^3 \]
respectively. Corresponding to the value of \(L = 2\), there are
\((4,4)^2 = 4\) \(\frac{m}{l-m} = \frac{q}{2} = 4\)-dimensional representations, these
are now given according to the formula \((4.114)\) by
\[ \left\{ L_{i'}^{\gamma'} = \omega(4)^{i'\gamma'} L_{i}^{\gamma}, \quad i = 1, 2, \quad \frac{4}{4} = 1, 2, 3, 4 \right\} \] 
where \(L_{i}', L_{\gamma}'\) obey
\[ L_{i}^{\gamma}' L_{\gamma}^{i'} = \omega(4)^{i'\gamma'} L_{\gamma}^{i'} L_{i}^{\gamma} = -L_{\gamma}^{i'} L_{i}^{\gamma} \] 
\[ L_{i}^{\gamma} L_{i}^{\gamma} = \omega(4)^{i'\gamma'} L_{i}^{\gamma} L_{i}^{\gamma} = L_{i}^{\gamma} L_{i}^{\gamma} \]
or

\[ L_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad L_2' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  

(4.61)

(b) \( G_3^+: \) Total number of elements = \( 4^{2+1} = 256 \) and number of classes = \( 4 \times \sum_{b=1}^{4} (1,4)^2 = 88 \) are present in \( G_3^+ \). There are \( 4 \times (4 \times 4)^3 = 64 \) one dimensional representations given by the representations of 

\( \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \). Corresponding to each of the primitive roots \( \omega(4) \), and \( \omega^3(4) \) there are 4 inequivalent representations of dimension 4. These are given respectively by \( \omega(4)^b \), according to formula (4.7d)

\[ \{ L_i = \omega(4)^b L_{i'} \mid i = 1, 2, 3; \quad 0 \leq b \leq 3; \quad L_0 = \omega(4)^3 \} \]  

and

\[ \{ L_i = \omega(4)^b L_{i''} \mid i = 1, 2, 3; \quad 0 \leq b \leq 3; \quad L_0 = \omega(4)^3 \} \]  

\[ \{ L_i = \omega(4)^b L_{i'} \mid i = 1, 2, 3; \quad 0 \leq b \leq 3; \quad L_0 = \omega(4)^3 \} \]  

where

\[ L_1' = C(4), \quad L_2' = B(4), \quad L_3' = C(4)^{-1} B(4) \]  

(4.84)

and

\[ L_1'' = C(4), \quad L_2'' = B(4)^3, \quad L_3'' = C(4)^{-1} B(4)^3 \]  

(4.85)

There are \( 4^2 = 16 \) two dimensional representations which are given by, according to formula (4.7b),

\[ \{ L_i = \omega(4)^b i L_{i'} \mid i = 1, 2, 3; \quad s_i = 0, 1 \} \]  

(4.86)

and

\[ \{ L_i = -\omega(4)^b i L_{i'} \mid i = 1, 2, 3; \quad s_i = 0, 1 \} \]
where $L_1', L_2', L_3'$ are

$$L_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = c(x); \quad L_2' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = b(x); \quad L_3' = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i(c(x))^{-1} g(x)$$

(4.87)

(iv) Direct product representations

Here we shall make some general observations about direct product representations. Considering $G_{m}^{\nu}$ if $\Gamma(L)$ and $\Gamma(T)$ correspond to two faithful representations corresponding to roots $\omega(m)^L$ and $\omega(m)^T$ respectively then the direct product representation $\Gamma(L) \otimes \Gamma(T)$ has generators as $L_i^\gamma(l+t) = L_i^\gamma(l) \otimes L_i^\gamma(t)$ and hence the commutation relations are

$$L_i^\gamma(l+t) L_j^\gamma(l+t) = \omega(m)^{l+t} L_j^\gamma(l+t) L_i^\gamma(l+t); \quad i,j = 1, \ldots, 2\nu.$$

Let $\Gamma \mod m = (l+t)$, (4.88) shows that $\Gamma(L) \otimes \Gamma(T)$ contains only $\Gamma(r)$.

Since we have

$$\dim \Gamma(L) \otimes \Gamma(T) = \left(\frac{m}{(l,m)}\right)^\nu \times \left(\frac{m}{(t,m)}\right)^\nu$$

$$\dim \Gamma(r) = \left(\frac{m}{(r,m)}\right)^\nu \tag{4.89}$$

the number of times $\Gamma(r)$ is contained is

$$\alpha(r; l, t) = \frac{m^{2\nu}}{(l,m)^\nu (t,m)^\nu} \times \frac{m^\nu}{(r,m)^\nu}$$

$$= \frac{m^\nu (r,m)^\nu}{(l,m)^\nu (t,m)^\nu} = \left(\frac{m(r,m)}{(l,m)(t,m)}\right)^\nu \tag{4.90}$$
But there are \((r, m)^{2\nu}\) inequivalent representations corresponding to \(\omega(\mu)\). Which of these occur in the \(\Gamma(r; l, t)\) representations is interesting but difficult to analyse. In the case of 

\[ G^{m}_{l, t}, \]

the same result \((4, 30)\) holds as is seen easily.

(v) A different approach to the representation problem of \(G^{m}_{n}\).

In a direct approach to the representation problem of \(G^{m}_{n}\), by counting the number of classes and guessing at the representations we have seen that the analysis of class structure is a difficult affair, involving tricky number theoretic considerations. Avoiding this and having a unified approach to both the cases \(n = 2\nu\) and \(n = 2\nu + 1\) is possible utilizing the generalized matrix decomposition theorems developed in Chapter II. From these we know that any \(d^{\nu}\)

dimensional matrix can be represented as a linear sum of \(d^{2\nu}\) linearly independent matrices given by the set

\[ \{C^{k_1}_{B_{1}} \otimes \ldots \otimes C^{k_{d'}}_{B_{d'}} \mid 0 \leq k_i, l_i \leq d - 1\} \]

where \(C\) and \(B\) are \(d\)-dimensional matrices obeying \(CB = \omega(d)\) \(BC\) \(C = B = 1\). The method of product transforms gives the irreducible representations of \(L_i L_j = \omega(d)^{\nu} L_j L_i\); \(L_i d^\nu I\) \(\forall i, j = 1, \ldots, m\) as some

\[ L_i \in \{L_i \mid i = 1, \ldots, n\} \]

one representation gets by this process then due to the normalization condition \(L_i^m = 1, \forall i\), the \(m^n\) different sets \(\{\omega(m)^{2\nu} L_i \mid i = 1, \ldots, n; \ell_i = 0, 1, \ldots, n - 1; r = 1, m^n\}\) are also representations. But all of these may or may not be inequivalent.
If two of these representations are equivalent then there should be a non-singular matrix $S$ of dimension $d^v$ connecting the two representations by equivalence transforms which can be written as linear combination of the $d^{2v}$ matrices $\left\{ \prod_{i=1}^{2v} C_{k_i}^i B_{l_i}^i \right\}$. But since each of these $d^{2v}$ basis matrices induces different phase factors on similarity transformation on $\left\{ L_{k_i} \right\}$ two representations will be equivalent if and only if the set of phase factor differences is a member of the class of $d^{2v}$ sets of phase factors generated by these. Hence out of the $m^n$ possible representations there are

\[
\left( \frac{m^v}{d^{2v}} \right)
\]

inequivalent representations. When $v = 2v$ there are

\[
\left( \frac{m}{d} \right)^{2v}
\]

representations and when $v = 2v + 1$ there are

\[
m \left( \frac{m}{d} \right)^{2v}
\]

representations as was found earlier. The phase factors corresponding to different inequivalent representations can be found explicitly by taking the quotient set of the set of all $m^n$ sets of phase factors by the set of $d^{2v}$ sets of phase factors corresponding to equivalent representations. The completeness of all these representations i.e., non-existence of any other representation, can be then proved by considering the group $\mathbb{G}_m^n$ and showing that the relation $|G_{m^n}^m| = \sum d_i^2$ is satisfied exactly. In the next chapter we shall explain in detail this procedure and use it to obtain all the inequivalent projective representations of the finite abelian group $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_n}$.

(vi) In certain limiting cases of $G_m^n$ when $m \to \infty$ in the representations of the group $G_m^n$ we get relations of the type
Thus \( L_i \) can take all values \( \exp(2\pi i \xi) \), with rational \( \xi \in [0, 1) \).

Consider a particular value of \( \xi = \frac{j}{p} \); \((j, p) = 1\) corresponding to this case we have

\[
L_i L_j = \omega(p)^{ij} L_j L_i \quad ; \quad i, j = 1 \ldots n
\]

\[
\lim_{m \to \infty} L_i = 1 \quad ; \quad \forall i = 1 \ldots n
\]

Then the matrices obeying

\[
L_i L_j = \omega(p)^{ij} L_j L_i \quad ; \quad i, j = 1 \ldots n
\]

\[
L_i = 1 \quad ; \quad \forall i = 1 \ldots n
\]

are represented by \( p^v \)-dimensional matrices and there are \( \left( \frac{m}{p} \right) ^{2v} \) or \( m \left( \frac{m}{p} \right) ^{2v} \) inequivalent representations for \( m = 2v \) and \( m = 2v + 1 \) respectively which are given by

\[
\left\{ \omega(m) \frac{d_{ji}}{\vartheta(i, j = 1 \ldots 2v, 0 \leq \delta_{ji} \leq \left( \frac{m}{p} - 1 \right)} ; r = 1 \ldots \left( \frac{m}{p} \right)^{2v} \right\}
\]

and

\[
\left\{ \omega(m) \frac{d_{ji} + m \delta_{ji}}{p} \frac{m^{t+1}}{r = 1 \ldots \left( \frac{m}{p} \right)^{2v+1}} ; \frac{r \delta_{ji} + m \delta_{ji}}{p} \frac{m^{t+1}}{0 \leq t \leq p-1} \right\}
\]
\[ L_i L_j = \omega(p) L_{ij} L_i \quad ; i, j = 1 \ldots n \]
\[ L_i p = 1 \quad \forall i = 1 \ldots n \]

(4.96)

Since (4.92) is the limiting case of (4.93) as \( m \to \infty \) according to (4.94) and (4.95) there are infinite representations. In this case as \( \lim_{m \to \infty} \omega(m)^2 \eta_i \quad (0 \leq \eta_i \leq \frac{m}{p} - 1) \) become quasi-continuous and take on all values \( \exp(2\pi i \eta) \), \( 0 \leq \eta < \frac{2}{p} \). Thus the representations of (4.92) are given by

\[ \left\{ \exp(2\pi i \eta_i) L_i \right\} \quad i = 1 \ldots 2n \quad ; 0 \leq \eta_i < \frac{2}{p} \]

(4.97)

and

\[ \left\{ \exp 2\pi i (\eta_i + \frac{t}{p}) L_i \right\} \quad i = 1 \ldots 2n+1 \quad ; 0 \leq \eta_i < \frac{2}{p} \quad ; 0 \leq t \leq p-1 \]

(4.98)

for \( n = 2n \) and \( n = 2n+1 \) respectively. Thus representations of

\[ \lim_{m \to \infty} G^m \]

are given by (4.97) and (4.98) corresponding to each value of \( 0 \leq \xi = \frac{t}{p} < 1 \) for \( L_0 \).

Next let us consider the relations

\[ L_i L_j = \exp(2\pi i \xi) L_{ij} L_i \quad ; i, j = 1 \ldots n \]

(4.99)

where \( \xi \) is any irrational \( \xi < 1, > 0 \).

We have

\[ L_i L_j^m = \exp(2\pi i m \xi) L_{ij} L_i^m \quad \forall i = 1 \ldots n \]
But since $\eta \pmod{1} = \eta \xi$ is always irrational for any $m \leq \infty$, no finite $m$ exists such that $L_j:1 = \eta \xi$ for any $j$. Thus the group of elements generated by products of $L_0 = \exp(2\pi i \xi)$, $L_1, L_2, \ldots, L_n$, is a free infinite discrete group $G$ and if $H$ is any normal subgroup of it then $|H| = \infty$ as well as $|G/H| = \infty$ such groups have been called non-type I groups and representation theory of such groups is not well-developed. Using the analogy of the relations (4.100) with the usual relations of the type (4.96) we can develop formally a representation taking the basic $C$ and $D$ matrices, out of which $L_i$'s are built by direct products, to be now

$$C_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 1 & \exp(2\pi i \xi) & \cdots & 0 \\ \exp(-2\pi i \xi) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

(4.101)

It is seen evidently that these are representations only in the sense of a limiting case of $C_0 = \lim_{m \to \infty} C_m$ and $B_0 = \lim_{m \to \infty} B_m$ where $C_m$ and $B_m$ are $m$ dimensional matrices where $m$ is denominator of a rational approximation to $\xi$.

(viii) More general $\omega$-commutation relations and associated commutative structures.

So far we considered ordered $\omega$-commutation relations with

$$L_i L_j = \omega L_j L_i$$

when $i, j = 1 \ldots n$. When we have general $\omega$-commutation relations of the type
\[ L_i L_j = \omega(m) L_j L_i \quad ; \quad i, j = 1, \ldots, n. \]  

and corresponding normalization conditions as

\[ L_i^m v_i = 1 \quad ; \quad \forall i = 1, \ldots, n. \]  

(4.102)

The consistency of the two require

\[ \omega(m) L_i^m = \omega(m) L_i^m v_i, \quad \forall i, j = 1, \ldots, n. \]  

(4.103)

For all these relations to be satisfied the \( \{ t_{ij}, m_i, m_j, m \mid i, j = 1, \ldots, n \} \) should be restricted as follows

\[ \forall i, j, \quad t_{ij} = \chi_{ij}, \quad (\chi_{ij}, k_{ij}) = 1 \quad ; \quad k_{ij} = (m_i, m_j) \]  

(4.104)

\[ m = \text{l.c.m. of all } k_{ij}. \]

Such type of such relationships as (4.102-103) generate a Generalised Clifford algebra with a basis \( \left\{ \prod_{i=1}^{m_i} L_i^{k_i} \mid 0 \leq k_i \leq m_i - 1 \right\} \) of elements. These basis elements are seen to form a 'ray group' i.e. groups with multiplication property

\[ \chi y = \bar{\chi}(\chi y) Z \quad ; \quad \chi, y, Z \in G \]  

(4.105)

By completing the basis, adding more elements, one can readily form a vector group which obeys the multiplication rule \( \chi y = Z \); \( \chi, y, Z \in G \).

Such a group structure \( G \) would contain \( (m \prod_{i=1}^{n} m_i) \) elements which are given by

\[ \left\{ \prod_{i=0}^{n} L_i^{k_i} \mid 0 \leq k_i \leq m_i - 1 \quad ; \quad 0 \leq k_0 \leq m - 1 \right\} \]  

(4.106)
Denoting an element \( \{ \sum_{i=0}^{n} i k_i \} \) by \( q(k_0, k_1, \ldots, k_n) \) the product of two such elements is given by

\[
q(k_0, k_1, \ldots, k_n) \cdot q(j_0, j_1, \ldots, j_n) = q((k+j)_0, (k+j)_1, \ldots, (k+j)_n)
\]

where

\[
(k+j)_i \mod m_i = k_i + j_i \quad \forall i = 1 \ldots n.
\]

\[
(k+j)_0 \mod m = k_0 + j_0 + t_0
\]

with

\[
t_0 \mod m = (k_1, \ldots, k_n) \begin{bmatrix} t_{12} & t_{13} & \cdots & t_{1n} \\ 0 & t_{23} & \cdots & t_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & t_{n-1n} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ j_n \end{bmatrix}
\]

From this we have

\[
q(j_0, j_1, \ldots, j_n) \cdot q(k_0, k_1, \ldots, k_n) \cdot q(j_0, j_1, \ldots, j_n)^{-1} = q(K, k_1, \ldots, k_n)
\]

where

\[
K \mod m = k_0 + S
\]

\[
S \mod m = (k_1, \ldots, k_n) \begin{bmatrix} 0 & t_{12} & t_{13} & \cdots & t_{1n} \\ -t_{12} & 0 & t_{23} & \cdots & t_{2n} \\ \vdots & & \ddots & \vdots & \vdots \\ -t_{1n} & -t_{2n} & -t_{3n} & \cdots & 0 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ j_n \end{bmatrix}
\]

Now compared to the case of \( M_i = m_i, \forall i = 1 \ldots n \), the counting of classes becomes still more difficult and a direct analysis is seldom
possible. But there are very interesting number theoretic problems arising here. In the case of \( G_{2m}^m \) we found that an element \( g(k_1, k_2, \ldots, k_{2m}) \) obeying \( q^{k_r} \equiv 1 \pmod{p} \) with \( p \) being minimum gives rise to \( \binom{m}{p} \) classes and if \( N_p \) is the number of such elements for a given \( p \mid m \) then there is an interesting identity

\[
N_p = \sum_{p \mid m} \frac{m}{p} (p, m) \cdot n \cdot \nu \cdot \left( \frac{k_1 \cdots k_{2m}}{m} \right) \pmod{p}, \quad \forall i = 1 \ldots m
\]

Actually \( N_p \), being the number of \( 2m \)-tuples \( (k_1 \cdots k_{2m}) \), \( k_i \leq m, \forall i = 1 \ldots m \) obeying \( k_i \equiv 0 \pmod{m}, \forall i = 1 \ldots m \) with \( p \) as minimum it is the so-called Jordan's totient function

\[
J_{2m}(p) = \text{no. of } 2m \text{-tuples such that the greatest common divisor of } (k_1 \cdots k_{2m}, m) = \frac{m}{p}
\]

which is a generalization of the Euler function \( \phi(p) = \text{no. of } 1 \leq k \leq m \quad \text{with } (k, m) = \frac{m}{p} \).

In the above case when \( m_i \)'s are different for \( i = 1 \ldots 2m \) as similar analysis of class structure should yield a generalization of Jordan's function of the type \( J_{2m}^*(p) = \text{no. of } 2m \text{-tuples } (k_1 \cdots k_{2m}) \quad 1 \leq k_i \leq m_i, \forall i \) such that \( k_i \equiv 0 \pmod{m_i}, \forall i \) and a corresponding identity should result by equating the number of classes and number of representations (found otherwise, since even if a formula is obtained for number of classes usually it is a sum over certain set of integers which is again a difficult affair).

In the case of the group \( G \) generated by \( (4.190-193) \) the commutation matrix is

\[
T = \begin{bmatrix}
0 & t_{12} & t_{13} & \cdots & t_{1n} \\
-t_{12} & 0 & t_{23} & \cdots & t_{2n} \\
& & & \ddots & \vdots \\
-t_{1n-1} & t_{1n-2} & -t_{3n-2} & \cdots & 0
\end{bmatrix}
\]
and let its show canonical form be

$$T = \begin{bmatrix} 0 & t_1 & & & \\ -t_1 & 0 & t_2 & & \\ & & 0 & b & \\ & & -b & 0 & e_b \\ & & & -e_b & 0 \end{bmatrix}$$  \tag{4.128}

As in the case of $G_n^m$, all the representations of the $G_n$ must arise from the $m$ values for $L_0 = \omega(m) \ell$; $0 \leq \ell \leq m-1$ in $L_i L_j = L_0 L_i L_j + \epsilon_{ij}; i, j = 1, \ldots, n$ or these arise from relations associated with commutation matrices $\{ L_i T \mid L = 0, 1, \ldots, m-1 \}$. Corresponding to $L = 0$, the one dimensional representations arise which are obtained by putting $L_0 = 1$ and $L_i = \omega(m) \ell_i$; $0 \leq \ell_i \leq m-1$. Thus there are $\frac{m^2}{2} m_n$ one dimensional representations. As explained in the next Chapter corresponding any $1 \leq \ell \leq m-1$, there are $\left( \frac{m^2}{2} m_n \right) \left( \frac{m^2}{2} (m, \ell i) \right)$ representations of dimension $\left( \frac{m^2}{2} m_n \right) \left( \frac{m^2}{2} (m, \ell i) \right)$. Thus there are totally $N_n = \sum_{\ell=0}^{m-1} \left( \frac{m^2}{2} m_n \right) \left( \frac{m^2}{2} (m, \ell i) \right)^2 \left( \frac{m^2}{2} (m, \ell i) \right) = \left( \frac{m^2}{2} m_n \right) \sum_{\ell=0}^{m-1} \left( \frac{m^2}{2} (m, \ell i) \right)^2$ representations and hence so many classes. When $m_i = m$; $i = 1, \ldots, n$, we have $\ell = \nu$, $\ell i = 1, \ldots, \nu$ for both $n = 2 \nu$ and $n = 2 \nu + 1$ and $N_n = \sum_{\ell=0}^{m-1} (m, \ell)^{2 \nu}$ for $n = 2 \nu$. $N_n = m \sum_{\ell=0}^{m-1} (m, \ell)^{2 \nu}$ for $n = 2 \nu + 1$.

In these types of structures the case of $n = 3$ has become important in the study of the problem of Bloch electrons in homogeneous magnetic field and this will be considered in detail in Chapter VII.
Continuing the matrix version of the Möbius inversion \((4.29)\) let us observe the following interesting number theoretic relations. Generally if an arithmetic function \(f(n)\) defined through the Dirichlet product
\[
\sum_{d \mid n} g\left(\frac{n}{d}\right) f(d) = h(n)
\]
when \(g(n)\) and \(h(n)\) are known, we can write it in matrix form
\[
G(F) = (H)
\]
where \(G\) is the matrix associated to function \(g(n)\) by
\[
G_{nd} = \begin{cases} 
  g\left(\frac{n}{d}\right) & \text{if } d \mid n \\
  0 & \text{if } d \not\mid n
\end{cases}
\]
and \((F)\) and \((H)\) are the column vectors with elements as \(f(n)\) and \(h(n)\) respectively. Noting that \(G\) is a lower triangular matrix with \(G_{mn} = g\left(\frac{m}{n}\right)\) for \(m \leq n\), the condition for invertibility of \(G\) becomes \(g(1) \neq 0\). Hence if \(g(1) \neq 0\) then we can find \(G^{-1}\) and write
\[
(F) = G^{-1}(H)
\]
Remarkably \(G^{-1}\) is also a matrix of the same type as \(G\) namely,
\[
G^{-1}_{nd} = \begin{cases} 
  \#0 & \text{if } d \mid n \\
  0 & \text{if } d \not\mid n
\end{cases}
\]
Thus we can write
\[
f(n) = \sum_{d \mid n} g^{-1}\left(\frac{n}{d}\right) h(d)
\]
where \(g^{-1}\left(\frac{n}{d}\right) = G^{-1}_{nd} \cdot 1_{d \mid n}\). Without going into the details of proof.
let us assert that

\[ q^{-1}(n) = \sum_{s=1}^{n-1} \frac{(-1)^s}{q^{(1)}(s+1)} \left\{ \sum_{i=1}^{s!} \frac{1}{\prod_{i=1}^{t} (s_i)!} \left[ \prod_{i=1}^{t} g(n_i)^{s_i} \right] \right\} \]

where \( \sum \) denotes summation over all \( s_i \) such that \( \sum s_i = s \) and \( \prod_{i=1}^{t} n_i = n \) with distinct \( n_i \).

Let us also observe the following dual to the Möbius inversion formula. Analogous to the matrix relation

\[ G(F) = (H) \]

leading to the Dirichlet product of arithmetic functions let us write

\[ \mathcal{E} G = \mathcal{H} \]

where \( \mathcal{E} \) is a row vector and \( \mathcal{H} \) is also. Written in terms of matrix elements this means a product

\[ \sum_{d=1}^{\infty} g(d) f(m) = h(n) \]

Then as before if \( g(1) \neq 0 \) it follows that there is an inversion for \( f \) in terms of \( h \) and \( g \) namely

\[ f(n) = \sum_{d=1}^{\infty} q^{-1}(d) h(m) \]

If \( \omega \) is trouble some one can have equally well,
\[
\begin{align*}
&\sum_{d=1}^{M} \frac{q(d)}{n} f(d, n) = h(n); \quad n = 1, \ldots, M \\
&\sum_{d=1}^{M} q^{-1}(d) f(n) = f(n); \quad n = 1, \ldots, M.
\end{align*}
\]

Finding \( G^{-1} \) from \( G \) through involving lengthy computation is possible since \( G \) has a triangular structure and hence the elements of \( G^{-1} \) up to any desired row can be obtained considering only those elements in \( G \) up to that particular row.

When \( q^{-1}(r) = 0 \) let us call the arithmetic function as singular which does not have an inverse in the above sense. But following the development of pseudo-inverses or generalized inverses for singular matrices considered by Noor, Penrose, and C. R. Rao (4) - one should be able to arrive at pseudo-inverses of arithmetic functions and corresponding inversion formulas for Dirichlet type products of arithmetic functions. Further developments in this direction will be published elsewhere.

Summary of important points.

The \( G_{m}^{n} \) has the following properties when \( m \) is any integer. Total number of elements = \( m^{n+1} \). Total number of conjugate classes = \( \sum_{v=1}^{m} (\lambda, m)^{v} \), for \( n = a \), and

\[
\sum_{\lambda=1}^{m} \lambda^{v} \text{ for } n = a v + 1,
\]

where \( (\lambda, m) = g.c.d \) of \( \lambda \) and \( m \),
Just as in the case of $G_n^m$, for prime $n$, all the representations arise due to the various permitted values of given by

$$\sum_{k=0}^{L} \exp \left( \frac{2\pi i k}{m} \right) \text{ for } 0 \leq k \leq m-1.$$  

Corresponding to $l = 0$ are the $m^n$ one-dimensional representations, same as those of $\mathbb{Z}_m \otimes \cdots \otimes \mathbb{Z}_m$ (n copies) for both $n = 2^l$ and $n = 2^{l+1}$. When $l \neq 0$, there are $m^v_l(l, m)^{(v)}$ representations of same dimension $m^v_l(l, m)^{(v)}$ for $n = 2^{l+1}$ and for $n = 2^l$, there are $(l, m)^{(v)}$ representations of same dimension $m^v_l(l, m)^{(v)}$. All these inequivalent irreducible representations have been explicitly constructed in this chapter and direct product representations are studied. The limiting case of $m \to \infty$ in $G_n^m$ is studied and also $o, L, G -$ type of group structures associated to the generating relations

$$L_i L_j = L_i^b_{ij} L_j L_i; L_0 L_j = L_j L_0; L_i^m = 1, \text{ for } i, j = 1, \ldots, n.$$  

are analysed, and $L_0$ and $b_{ij}$ are such that $L_0^m = L_0 = 1$. Thus

$$L_0^{b_{ij}} = \exp \left( \frac{2\pi i b_{ij}}{m^i} \right) = \exp \left( \frac{2\pi i b_{ij}}{k_{ij}} \right), (b_{ij}, k_{ij}) = 1, \forall i, j = 1, \ldots, n.$$  

Letting

$$m = \text{lcm} \{k_{ij} \}$, $L_0 = L_0^{b_{ij}} \right) \text{ one-dimensional representations same as those of } \mathbb{Z}_m \otimes \cdots \otimes \mathbb{Z}_m \text{ and higher dimensional representations arise corresponding to all the (n-1) values of } L_0 = \exp \left( \frac{2\pi i b_{ij}}{m} \right).$$  

For a particular value of $l$ there are

$$\left( \prod_{i=1}^{n} m_i \right) / \left( \prod_{i=1}^{n} m_i, t_c \right),$$  

representations of dimension

$$\left( \prod_{i=1}^{n} \frac{m_i}{(m_i, t_i)} \right)^a,$$

where $t_i$'s are elements of the skew normal form of $T = (t_{ij})$. 
CHAPTER 6

Projective Representations of Abelian Groups

Here we develop a simple procedure for explicitly determining all the irreducible inequivalent projective representations of finite abelian groups using the concept of ‘product transform’ introduced by us recently. This procedure is a generalization of ‘tensornormice coupling method’ of representation of Clifford algebra, studied by Alladi Ramakrishnan during the development by him and his collaborators, of L-Matrix Theory dealing with properties and applications of Clifford algebra and its generalizations.

Projective representations of groups were first studied by Schur in a series of definitive papers and general methods of representations were developed. But surprisingly even for Abelian finite groups there does not exist any explicit procedure of determining all the irreducible inequivalent projective representations. Except in a few special cases of groups of small order and simple factor sets application of little group technique of Wigner or the same so called induced representation technique of Frobenius, Mackey and Clifford becomes too complicated to handle. Hence special simpler methods to deal with particular group structures are of interest. In this connection Generalized Clifford algebras were introduced and studied in different fashions by Noriga and Nono, Kanazaki, Harris and Popovic and Turco. By now it is clear from their work that linear representations of all C.G.A’s associated with a finite Abelian group \( \mathbb{Z}_m \otimes \cdots \otimes \mathbb{Z}_{m_n} \) provide all the projective representations of \( G \). But so far only certain special C.G.A’s have been studied explicitly since the work of Hermann Weyl.
the first used basic relations of a C.C.A in interpreting the fundamental laws of quantum mechanics as ray (projective) representations of Abelian groups and proving the uniqueness of Schrödinger's representation of position and momentum operators in quantum mechanics. He has solved the problem of projective representations of continuous Abelian groups in terms of canonically conjugate pairs of operators obeying Heisenberg commutation relations. But in the case of finite Abelian groups till recently there is no explicit procedure. The basic principle of the following method is essentially the same as that used by Weyl in the case of continuous groups, but a great difference arises due to the finite nature of the group bringing in difficulties in details. Only recently Backhouse and Bradley have determined the dimensions of representations for arbitrary choice of factor systems. Our procedure determines all the representations explicitly in terms of simple matrices.

1. **Projective representations of finite Abelian groups and commutative Clifford algebras.**

First let us recall some elements of Schur's theory of projective representations. A representation \( D \) of a group \( G \) is called a projective representation when

\[
D(q_i)D(q_j) = \xi(q_i,q_j)D(q_iq_j), \quad \forall q_i,q_j \in G
\]

\[
D(q_0) = I
\]

where \( q_0 \) is the identity element of \( G \). \( \xi(q_i,q_j) \)'s are elements of the field of complex numbers \( \mathbb{C} \). For us, then \( \xi(q_i,q_j) = 1, \forall q_i,q_j \in G \) is the ordinary or linear representation of \( G \). Associativity of \( G \) requires

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\[
\zeta(g_i, g_j) \zeta(g_i, g_j) = \zeta(g_i, g_j, g_k) \zeta(g_i, g_j) \\
\forall g_i, g_j, g_k \in G
\]

(5.2)

Also,

\[
\zeta(g_o, g_j) = \zeta(g_i, g_o) = \zeta(g_o, g_o) = 1 \\
\forall g_i, g_j \in G
\]

(5.3)

The set \( \zeta \) is called a factor set of \( G \) in \( G^\ast \) and two factor sets \( \zeta, \eta \) are equivalent if there exists a mapping \( \mu : G \to G^\ast \) such that \( \mu(g_o) = 1 \) and

\[
\eta(g_i, g_j) = \frac{\mu(g_i) \mu(g_j)}{\mu(g_i g_j)} \zeta(g_i, g_j) \forall g_i, g_j \in G
\]

(5.4)

Let \( M(G, G^\ast) \) denote the set of all factor sets of \( G \) in \( G^\ast \) and it is an Abelian group with the product \( \zeta \circ \eta \).

A factor set \( \zeta \in M(G, G^\ast) \) is said to be normalized if \( \zeta(g_i, g_i) = 1 \forall g_i \in G \), in which case \( D(g_i)^{-1} = D(g_i^{-1}) \). Any factor set \( \zeta \in M(G, G^\ast) \) can be normalized by an equivalence transformation (5.4) with \( \mu(g_i) = \frac{1}{\sqrt{\zeta(g_i, g_i)}} \forall g_i \in G \). It is known that for any \( \zeta \in M(G, G^\ast) \) there is a projective representation.

The relation of equivalence of two factor sets given by (5.4) is easily verified to be an equivalence relation on \( M(G, G^\ast) \). Let \( \{\zeta\} \) denote the equivalence class on which contains \( \zeta \in M(G, G^\ast) \).

The set of all equivalence classes denoted by \( H^2(G, G^\ast) \) is called a finite Abelian group with the product \( \{\zeta\} \{\eta\} = \{\zeta \circ \eta\}, \forall \zeta, \eta \in M(G, G^\ast) \) and is called the cohomology multiplier of \( G \) in \( G^\ast \) A group \( G^\ast \), called a representation group, can be constructed by central extension of \( G \) with \( H^2(G, G^\ast) \) as kernel of the extension such that

\[
G^\ast / H^2(G, G^\ast) \cong G, H^2(G, G^\ast) \in \mathbb{Z}(G)
\]
of \( \sigma \) is obtained as a linear representation of \( G^\sigma \). (For a detailed account of the subject cf. Morris). Let us consider a finite Abelian group \( G \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n} \) where \( \mathbb{Z}_{m_i} \) is a cyclic group of order \( m_i \). Any element \( g_i \in G \) can be written as \( \prod_{i=1}^{n} e_i^{k_i} \), \( 1 \leq k_i \leq m_i \), \( \forall i = 1, \ldots, n \) where \( e_i \) 's are generators of \( \sigma \) obeying

\[
\begin{align*}
    e_i^{m_i} &= 1, & \forall i = 1, \ldots, n \\
    e_i e_j &= e_j e_i, & \forall i, j = 1, \ldots, n
\end{align*}
\]

Let \( \xi \) be a factor set. Define another associated set \( \omega_\xi \) by

\[
\omega_\xi(g_i, g_j) = \frac{\xi(g_i, g_j)}{\xi(g_j, g_i)}, \quad \forall g_i, g_j \in G
\]

It is easy to see that \( \omega_\xi \equiv \omega_\eta \) if \( \xi \) and \( \eta \) are equivalent. Thus \( \omega_\xi \) denotes the \( \omega \)-factor set for the entire class \( \{\xi\} \in H^2(G, \mathbb{C}^*) \). For Abelian groups the projective representations \( \beta \) can be characterised in terms of \( \omega \)-factor sets as

\[
\beta(g_i) \beta(g_j) = \xi(g_i, g_j) \beta(g_i g_j) = \xi(g_i, g_j) \beta(g_i) \beta(g_j)
\]

\[
= \xi(g_i, g_j) \xi(g_j, g_i) \xi(g_j, g_i)^{-1} \beta(g_i) \beta(g_j)
\]

\[
= \omega_\xi(g_i, g_j) \xi(g_j, g_i) \beta(g_i) \beta(g_j), \quad \forall g_i, g_j \in G
\]

\[
\kappa = 1, 2, \ldots, |H^2(G, \mathbb{C}^*)|\]

where \( |H^2(G, \mathbb{C}^*)| \) denotes the order of \( H^2(G, \mathbb{C}^*) \). It follows from (5.6) and (5.7) that

\[
\omega(g_i, g_j) \omega(g_i, g_j, g_k) = \omega(g_i, g_j, g_k) \omega(g_i, g_j)
\]

\[
\forall g_i, g_j, g_k \in G
\]

(5.8)
\[ \omega(g_i, g_j) = \omega(g_i, g_0) = \omega(g_i, g_i^{-1}) = \omega(g_j, g_i) = 1 \quad \forall g_i \in G \]  \\
\[ \omega(g_i, g_j) = \omega(g_j, g_i)^{-1} \quad \forall g_i, g_j \in G \]  \\
(5.8) \\
(5.9)

(5.8) - (5.9) give
\[ \omega(g_i, g_j, g_k) = \omega(g_i, g_j) \omega(g_j, g_k) \quad \forall g_i, g_j, g_k \in G \]
\[ \omega(g_i, g_i) = 1 \quad \forall g_i \in G \]  \\
(5.10)

Hence the set \( \{ \omega_k(e_i, e_j) | e_j = 1, \ldots, n \} \) completely specifies the set \( \{ \omega(g_i, g_j)^{g_k} | g_i, g_j \in G \} \). Since a representation \( D_x \) of \( G \) for the factor set \( G \) obeys (5.7) as a first stage of determining \( D_x \), we can consider the problem of determining of a representation \( L \omega_x \) such that
\[ L(g_i) L(g_j) = \omega_x(g_i, g_j) L(g_j) L(g_i) \quad \forall g_i, g_j \in G \]  \\
(5.11)

Denoting \( L(e_i) = L_i, \forall i = 1, \ldots, n \), it is enough to determine first \( L_i \)'s since \( L(g_i), \forall g_i \in G \) can be constructed from these as the following procedure will make it clear. These \( L_i \)'s obey from (5.10)
\[ L_i L_j = \omega_{ij} L_j L_i \quad ; \quad i, j = 1, \ldots, n \]  \\
(5.12)

denoting \( \omega_{ij} = \omega_x(e_i, e_j) \). From (5.1) it follows that we can write
\[ D(g_i, g_j) = \omega(g_i, g_j) \omega_x(g_i) D(g_j) \]  \\
(5.13)
and hence
\[
D\left(\prod_{i=1}^{n} e_i \cdot \mathbf{r}_i\right) = \mathbf{r}_1 \cdot \prod_{i=2}^{n} \mathbf{e}_i \cdot \mathbf{r}_i \cdot D(e_1) D\left(\prod_{i=2}^{n} e_i \cdot \mathbf{r}_i\right)
\]
(5.15)

and so on, finally leading to
\[
D\left(\prod_{i=1}^{n} e_i \cdot \mathbf{r}_i\right) = \prod_{i=1}^{n} \left\{ \prod_{s=1}^{n} \left( e_s \cdot \prod_{i=s+1}^{n} e_i \cdot \mathbf{r}_i \right) \right\} \times
\]
\[
\left( \prod_{i=1}^{n} D(e_i)^{\mathbf{r}_i} \right)
\]
(5.16)

Thus
\[
D(e_i^{m_i}) = \left\{ \prod_{k_i=1}^{m_i} \mathbf{r}_i \right\} \times D(e_i)^{m_i} = I, \quad \text{by (5.1)}
\]
(5.17)

Defining
\[
K_i := \left( \prod_{k_i=1}^{m_i} \mathbf{r}_i \right)^{-1} \left( \prod_{k_i=1}^{m_i} \mathbf{r}_i \right)^{K_i} = \left( \prod_{k_i=1}^{m_i} \mathbf{r}_i \right)^{-1} \left( \prod_{k_i=1}^{m_i} \mathbf{r}_i \right)
\]
(5.18)

we have
\[
D(e_i)^{m_i} = K_i \cdot I
\]
(5.19)

and \(K_i\)'s are uniquely fixed by \(\mathbf{r}_i\). A recent theorem due to Baillie and Bradley is of importance for further considerations. They have shown that if \(D_1\) and \(D_2\) are two representations of \(G\)
with the same factor set \( \zeta \), then there exist a unitary transformation \( U \) and a linear character \( \chi \) such that
\[
U^{-1} D_1(g_i) U = \chi(g_i) D_2(g_i) \quad \forall g_i \in G
\]
\[
\chi(g_i) \chi(g_j) = \chi(g_i g_j) \quad \forall g_i, g_j \in G
\]
(5.22)

\( \chi \) is thus a one-dimensional linear representation of \( G \), and is generated by
\[
\chi(e_i) = e \times 1 \left( \prod_{i=1}^{m_i} l_i \right) \quad 1 \leq l_i \leq m_i
\]
(5.22)

There are only \( \prod_{i=1}^{n} m_i \) possible choices of \( \chi \) and hence by the above theorem all the inequivalent (linearly) projective representations of \( G \) for a given factor set \( \zeta \) are contained in the set of representations given by
\[
\left\{ \chi_\gamma(g_i) D(g_i) \mid \forall g_i \in G \right\}
\]
where \( \left\{ D(g_i) \mid \forall g_i \in G \right\} \) is any representation of \( G \) for the given factor set \( \zeta \). All these representations may not be inequivalent and we have to choose them by some method which we shall consider later. Hence without loss of generality we can take \( D(e_i) = L_i \), \( \forall i = 1 \ldots n \) which generates a representation of \( G \) for the factor set \( \zeta \) by (5.16). From (5.13)

\[
L_i L_j = \omega_{ij} L_j L_i \quad \forall i, j = 1 \ldots n
\]
\[
L_i^m = K_i I \quad \forall i = 1 \ldots n
\]
(5.23)
This shows that \( W_{ij} \) cannot take arbitrary values. They must satisfy
\[
W_{ij} \cdot m^i = W_{ji} \cdot m^j = 1
\]
\[\text{(5.24)}\]
or
\[
W_{ij} = \exp\left( \frac{2\pi i}{K_{ij}} \right), \quad (\psi_{ij}, \psi_{ij}) = 1
\]
\[\text{(5.25)}\]
\[
K_{ij} \cdot l_{ij}, \quad l_{ij} = (m^i, m^j); \quad \forall i, j = 1, \ldots, n
\]
These show that since \( W_{ij} \) can take \( l_{ij} \) values given by
\[
\{ \exp\left( \frac{2\pi i}{K_{ij}} \right) l_{ij} \mid 1 \leq l_{ij} \leq l_{ij} \}
\]
there are totally \( \prod_{i} l_{ij} \) c-factor sets and hence so many are the equivalence classes of c-factor sets in \( M(G, \mathbb{C}^*) \), i.e., order of \( H^2(G, \mathbb{C}^*) \) given by
\[
|H^2(G, \mathbb{C}^*)| = \left( \prod_{i} l_{ij} \right) = \left\{ \prod_{i} l_{ij} \mid (m^i, m^j) \right\}
\]
\[\text{(5.26)}\]
Defining a matrix \( W \) by \( W_{ij} = \psi_{ij} \), \( \forall i, j = 1, \ldots, n \), it is a Hermitian matrix, \( W^\dagger = W \), and all the \( \left( \prod_{i} l_{ij} \right) \) matrices \( W \) form the group \( \sim H^2(G, \mathbb{C}^*) \) under the Hadamard product \( (W_1 \odot W_2)_{ij} = (W_1)_{ij} W_2_{ij} \) \( \forall i, j = 1, \ldots, n \). We saw that corresponding to any factor set \( \xi \), the representation can be specified in terms of representations of \( L_i \) 's obeying \( (5.33) \) and all the inequivalent representations are contained in a set of \( \left( \prod_{i} m_i \right) \) representations which arise from taking different values for \( \left\{ \chi(\xi_i) \right\} \) in \( \{ L_i = \chi(\xi_i) \tilde{L}_i, \forall i = 1, \ldots, n \} \) where \( \left\{ \tilde{L}_i \right\} \) is any one representation of \( (5.33) \), and \( \chi(\xi_i) = \exp\left( 2\pi \frac{\tilde{l}_i}{m_i} \right) \)
Taking two sets of values for \( \{ \chi_i(e_i) \} \), say \( \{ \chi_1 \}, \{ \chi_2 \} \) the two representations would be equivalent if there exists a matrix \( S \) such that

\[
S \chi_i(e_i) \tilde{L}_i S^{-1} = \chi_i(e_i) \tilde{L}_i, \quad \forall i = 1 \ldots n
\]

or

\[
S \tilde{L}_i S^{-1} \tilde{L}_i^{-1} = \chi_i(e_i) \chi_i(e_i)^{-1} = \psi(e_i) I
\]

This implies that two representations with \( \{ \chi_1(e_i) \} \) and \( \{ \chi_2(e_i) \} \) as sets of coefficients will be equivalent iff the set \( \psi = \{ \chi_2(e_i) \chi_1(e_i)^{-1} \} \) is a member of the set of sets

\[
\bar{\Psi} = \{ (SL_i S^{-1} \tilde{L}_i^{-1} | i = 1 \ldots n) \forall S \in M_D(\mathbb{C}), \exists S \tilde{L}_i S^{-1} \tilde{L}_i^{-1} I \}
\]

where \( M_D(\mathbb{C}) \) is the total matrix algebra over \( \mathbb{C} \) of dimension \( D \) = dimension of representation of \( \tilde{L}_i \). So if this set \( \bar{\Psi} \) is determined then the sets \( \{ \chi(e_i) \} \) belonging to inequivalent representations can be identified with members of different classes under the equivalence relation that \( \{ \chi_1(e_i) \} \) is equivalent to \( \{ \chi_2(e_i) \} \) if \( \{ \chi_1(e_i) \chi_2(e_i)^{-1} \} \in \bar{\Psi} \). Then the number of inequivalent representations is given by

\[
\frac{(\prod_{i=1}^n m_i)}{\mid \bar{\Psi} \mid}
\]

where \( \mid \bar{\Psi} \mid \) is the number of elements in \( \bar{\Psi} \).

Thus our primary task is to determine a representation of \( L_i^2 \) obeying (5,23) and the corresponding set \( \bar{\Psi} \). In (5,23) the second relation \( L_i^m_i = K_i I \), \( i = 1 \ldots n \) is a normalization condition in all and for a given factor set \( \xi_i, K_i \) are fixed uniquely and
hence without loss of generality instead of (5.23) we can consider the problem of representation of

\[
L_i L_j = \omega_{ij} L_j L_i \quad ; \quad i, j = 1 \ldots n
\]

\[
L_i^{m_i} = 1 \quad ; \quad i = 1 \ldots n
\]

\[
\omega_{ij} = \omega_{ji}^{m_i} \quad ; \quad i, j = 1 \ldots n
\]

It is seen easily that the set \( \{ \prod_{i=1}^{n} L_i^{k_i} \mid 0 \leq k_i \leq m_i - 1 \} \) forms the basis for an algebra. The product of two elements of the basis is given by

\[
\left\{ \prod_{i=1}^{n} L_i^{k_i} \right\} \left\{ \prod_{i=1}^{n} L_i^{t_i} \right\} = \left\{ \prod_{j=1}^{n} \left( \prod_{l=0}^{m_n-j-1} \omega_{n-j-1,j}^{k_{n-j-1,j}} \right) \right\} \times \left\{ \prod_{i=1}^{n} L_i^{(k_i+t_i)} \right\}
\]

where \( (k_i + t_i) \mod m_i = k_i + t_i \; ; i = 1 \ldots n \). This algebra has been called a 'Generalized Clifford Algebra'. When \( m_i = 2 \), \( \forall i = 1 \ldots n \) and \( \omega_{ij} = -1 \); \( \forall i, j = 1 \ldots n \) this algebra is seen to be the Clifford algebra \( \mathbb{C}_n \) generated by \( n \) mutually anticommuting elements.

So far representation theory of \( C_n \) is only with \( m_i = m, \forall i = 1 \ldots n \) and \( \omega_{ij} = \exp \left( \frac{2\pi i j}{m} \right), \forall i, j = 1 \ldots m \) have been considered in detail (2, 9, 11, 16, 17, 18).

11) Representation theory.

Let us rewrite the equations (5.23) as

\[
L_i L_j = \omega(m)_{ij} L_j L_i \quad ; \quad i, j = 1 \ldots n
\]

\[
t_{ij} = -t_{ji} \quad ; \quad L_i^{m_i} = 1 \quad ; \quad i = 1 \ldots n
\]
\[ w(m) = \exp \left( \frac{2\pi i}{m} \right) \]

\[ w_{ij} = \exp \left( \frac{2\pi i}{m} \cdot \frac{k_{ij}}{k_{ij}} \right) ; \quad (\pi_{ij}, k_{ij}) = 1 \]

\[ k_{ij} \mid k_{ij} ; \quad l_{ij} = (m_i, m_j) \]

\[ m = \text{lcm} \left\{ k_{ij} \mid \forall i, j = 1, \ldots, n \right\} \]

Then defining an antisymmetric integer matrix \( T \) by

\[ T_{ij} = t_{ij} ; \quad i, j = 1, \ldots, n \]

The product (5.30) becomes

\[
\left( \prod_{i=1}^{m} L_i \right) \left( \prod_{i=1}^{l} L_i \right) = w(m) \sum_{v=1}^{m} S(\mathbf{k}, \mathbf{t}) \left( \prod_{i=1}^{m} L_i \right) \left( \prod_{v=1}^{n} L_i \right)
\]

where

\[ (k_i + t_i) \mod m_i = k_i + t_i ; \quad i = 1, \ldots, n \]

and

\[ S(\mathbf{k}, \mathbf{t}) \mod m = (k_1, k_2, \ldots, k_m) \begin{pmatrix} o & t_1 & t_3 & \ldots & t_{2n} \\ o & o & t_3 & \ldots & t_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ o & o & o & \ldots & t_{m-1} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix} \]

It follows from (5.34) that

\[
\left( \prod_{i=1}^{n} L_i \right) \left( \prod_{i=1}^{l} L_i \right) = w(m) \sum_{v=1}^{m} S(\mathbf{t}, \mathbf{k}) \left( \prod_{i=1}^{n} L_i \right) \left( \prod_{i=1}^{l} L_i \right)
\]
Substituting this in the right hand side of (5.24)

\[
\left( \prod_{i=1}^{m} L_{i} \right) \left( \prod_{i=1}^{m} L_{i}^{*} \right) = w(m) \left( \prod_{i=1}^{m} L_{i} \right) \left( \prod_{i=1}^{m} L_{i}^{*} \right)
\]

where

\[
S[k, t] = S(k, t) - S(t, k)
\]

\[
= (k_1, k_2, \ldots, k_m) \begin{bmatrix} \cdots & 0 & \cdots \end{bmatrix} \begin{bmatrix} t_1 \cr t_2 \cr \vdots \cr t_m \end{bmatrix} = k_1 t_1 + k_2 t_2 + \cdots + k_m t_m.
\]

(5.39)

Let us call the antisymmetric integer matrix \( T \), the commutation matrix associated with the set \( \{ L_i \} \).

Let \( U = (u_{ij}) \); \( i, j = 1 \ldots n \) be an integer matrix. Then let the following be the definition of a 'product transform' from the set \( \{ L_i \} \) to \( \{ L_i^* \} \):

\[
L_i^* = \prod_{k=1}^{m} L_{i_k}^* ; \quad i = 1, \ldots, n
\]

(5.40)

Any \( L_k \) may be rewritten as \( L_k \) such that \( 0 \leq s_{m_k-1} + s_{m_k-1} \leq 1 \).

Using (5.36) the commutation relations of \( L_i^* \) are

\[
L_i^* L_j = w(m) \quad L_j^* L_i \quad \forall j = 1, \ldots, n.
\]

(5.41)

where \( u_{ij} \) denotes the \( i \)-th row of \( U \). Denoting \( S[u_{ij}, u_{ij}] = t_{ij} \)
it is seen that \( t^*_y = -t^*_i \) and \( T^* = (t^*_y) \) is the commutation matrix associated with the set \( \{ L^*_i \} \). We have

\[
T^* = U T \bar{U}
\]

(5.49)

\[\text{Note: If } |\text{det} U| = 1 \text{, there exists the integer matrix } U^{-1} = V = (v_{ij}) \text{ such that}
\]

\[
T = V T^* \bar{V}
\]

(5.43)

This means that the set \( \{ L^*_i \} \) defined by

\[
L^*_i = \prod_{j=1}^{n} L^*_j \quad V_{ij} \quad i = 1 \ldots n
\]

(5.44)

has the same commutation matrix \( T \) as the set \( \{ L_i \} \). Actually substituting for \( L^*_j \) in (5.44) from (5.40) it can be seen that

\[
L^*_i = c_i L_i \quad i = 1 \ldots n
\]

(5.45)

where \( c_i \)'s are some \( m \)th roots of unity. This implies that if \( \{ L_i \} \) are irreducible so are \( \{ L^*_i \} \) and vice versa. Because if \( \{ L^*_i \} \) become reducible \( \{ L_i \} \) are not initially the relation (5.45) would imply \( \{ L^*_i \} \) are reducible contradictory to initial condition. Only when \( |\text{det} U| = 0 \) \( \{ L^*_i \} \) become reducible starting with irreducible \( \{ L_i \} \) and then inverse transformation

\[
\{ L^*_i \} \to \{ L_i \}
\]

is impossible. We are not interested in this case.

Thus we notice that if an irreducible representation of \( \{ L^*_i \} \) is known then an irreducible representation of \( \{ L_i \} \) can be found readily if \( T^* \) and \( T \) are related by the relation (5.43) must with

\[
|\text{det} V| = 1
\]

and if \( V \) can be explicitly determined. Now a basic theorem on antisymmetric integer matrices comes to curiously. We follow
the treatment of Harris Reuten. According to this theorem for any \( n \times n \) antisymmetric integer matrix \( T \), there is an unique skew normal form given by

\[
U + \tilde{U} = T^* = \sum_{i=1}^{s} \bigoplus (0, h_i) + O_{n-1}\n
(5.43)
\]

where \( h_i | h_{i+1}, \ 1 \leq i \leq s-1 \), \( s = \text{rank of } T \), and \( O_{n-2s} \) is a null matrix of order \( n-2s \). The matrix \( U \) is an integer matrix, \( \det U = 1 \) and \( U \) can always be explicitly constructed to satisfy (5.43) (cf. Appendix D).

A set of matrices \( \{L_i^*\} \) having \( T^* \) as their commutation matrix obey the relations

\[
L_{i-1}^* L_i^* = \omega(m) h_i L_{i-1}^* \quad ; \quad i = 1, \ldots, s
\]

\[
L_k^* L_l^* = L_k^* L_l^* \quad \text{otherwise} \quad ; \quad k, l = 1, \ldots, n
\]

(5.47)

Let us put

\[
L_k^* = I_1 \otimes I_2 \otimes \cdots \otimes I_{i-1} \otimes C_i \otimes I_{i+1} \otimes \cdots \otimes I_s \quad \{i = 1, \ldots, s \}
\]

\[
L_k^* = I_1 \otimes I_2 \otimes \cdots \otimes I_{i-1} \otimes B_i \otimes I_{i+1} \otimes \cdots \otimes I_s \quad \{k = 2s+1, \ldots, m \}
\]

(5.45)

where

\[
y_i = (h_i, m) \quad z_i = m / y_i \quad \chi_i = h_i / y_i \quad \frac{h_i}{m} = \frac{\chi_i}{z_i} \quad (\chi_i, z_i) = 1
\]

(5.49)
Taking trace on both sides it follows immediately that the coefficient of the identity matrix $A_{00...0} = 0$. Multiplying on both sides by $(\prod_{k=1}^{n} L_{kk}^{-1})$ and taking trace it follows $A_{i_1i_2...i_n} = 0$, since in the product on the left hand side $A_{i_1i_2...i_n}$ becomes the coefficient of the identity matrix. Thus all these matrices $D^2$ totally:

$$\prod_{k=1}^{n} L_{kk}^{-1} C_i B_j \otimes \cdots \otimes C_s B_s^{-1} B_s^{-1} \mid 0 \leq j \leq n \leq 2^i - 1; \quad i = 1, \ldots, 2^s$$

are linearly independent and form a basis for $M_{D^2}(C)$, the full matrix algebra of dimension $D^2$ over $C$. Hence this representation is irreducible. Explicit versions of such matrix decomposition theorems have been studied by Alladi Srivastava and Alladi Ramakrishnan and myself.

Let $U^{-1} = \tilde{V}$, which is also a unimodular integer matrix, be constructed to satisfy

$$T = \tilde{V} \tilde{T}^* \tilde{V}$$  \hspace{1cm} (5.53)

Though $U$ and hence $V$ are not unique it does not matter for us since we need only one representation to start with to construct all the others which differ only in phase factors as we have seen already. Thus we form the representation

$$L_i' = \prod_{j=1}^{n} L_j^{+} v_{ij}$$

$$= C_1 B_1 \otimes C_2 B_2 \otimes \cdots \otimes C_s B_s^{-1} B_s^{-1}$$  \hspace{1cm} (5.53)

where $(v_{i1}, \ldots, v_{in})$ is the $i$th row of $\tilde{V}$. As seen earlier
\[
C_i = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & \cdots & 0
\end{pmatrix} = \text{cyclic matrix of dimension } Z_i
\]

\[
B_i = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
\omega(z_i) x_i & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \omega(z_i) x_i (z_i - 1)
\end{pmatrix}
\]

\[
C_i z_i = B_i z_i = I_i ; \quad C_i B_i = \omega(z_i) x_i B_i C_i ; \quad \forall i = 1 \ldots s.
\]

\[
\begin{array}{c}
I_i = \text{Identity matrix of dimension } Z_i \\
\end{array}
\]

Hence

\[
\text{dim}(L_i^*) = \prod_{i=1}^{s} Z_i = \prod_{i=1}^{s} \frac{m}{(m_i, m)} = D .
\]

(5.60)

It is easily verified that \( \{ L_i^* \} \) given by (5.60) faithfully obey the relation (5.47).

All possible products of all possible powers of \( \{ L_i^* \} \) are given by the set of

\[
\left\{ \prod_{k=1}^{n} L_k^{x_k} \middle| 0 \leq x_k \leq z_i - 1 ; i = 1 \ldots s ; \quad x_k = 0 \quad \forall k = 2s + 1 \ldots n \right\}
\]

In this set except the identity matrix all are traceless. Consider the equation

\[
\sum_{\text{sum over all } y_k}\left[ \prod_{k=1}^{n} L_k^{x_k y_k} \right] = 0
\]

(5.61)
\{ L_i^j \} obey faithfully the required relations

\[
L_i^j \cdot L_j^i = \omega(m) \cdot t_{ij} \cdot L_j^i \cdot L_i^j \quad i,j = 1, \ldots, n.
\] (5.54)

Since \( \omega(m) = \omega_{ij} \) for \( i,j = 1, \ldots, n \) are such that \( \omega_{ij} \cdot m_i^j = 1 \) for \( j = 1, \ldots, n \) commutes with all \( L_j^i \) for \( j = 1, \ldots, n \) and hence by Schur's lemma

\[
L_i^i = P_i \cdot I, \quad \forall i = 1, \ldots, n.
\] (5.55)

where \( P_i \)'s can be calculated from (5.53). So now define

\[
L_i^j = (P_i \cdot m_i^j) \cdot L_i^j, \quad \forall i = 1, \ldots, n
\]

\[
\big\{ L_i^j \big\} = \left\{ \frac{1}{2} \sum_{j=1}^{n} v_{ij} \cdot y^j \right\} \cdot \prod_{j=1}^{n} \omega(\omega(m)) \cdot \bigoplus_{j=1}^{n} c_j \cdot B_j
\] (5.56)

which obey the required relations

\[
L_i^j \cdot L_j^i = \omega(m) \cdot t_{ij} \cdot L_j^i \cdot L_i^j \quad i,j = 1, \ldots, n
\]

\[
L_i^i = 1, \quad i = 1, \ldots, n.
\] (5.57)

The irreducibility of \( \{ L_i^j \} \) follows from the irreducibility of \( \{ L_i^* \} \).

Consider now the set of \( \Phi = \left\{ (x_i^{(m)} \cdot L_i^j \cdot x_i^{(m)})^l \right\} \) all of which satisfy (5.55). We found earlier that all the inequivalent representations of the generators of \( G \) for the factor set \( \gamma_i \) associated with the \( n \)-factor set
generated by \( \{ \omega_{ij} \} \), are contained in this set \( \Phi \) and the representations with the sets of coefficients \( \{ x^{(a)}_i \} \) and \( \{ x^{(b)}_i \} \) were found to be equivalent iff the set \( \{ \psi_{i}^{(l)} \} x^{(b)}_i x^{(a)}_i^{-1} \} \) is a member of the set of sets of coefficients

\[
\Psi = \left\{ (SL_i S_j^{-1} L_i^{-1}) \mid i = 1 \ldots n \right\} \forall S \in M_{D_n}(C), \exists L_i S_i^{-1} L_i^{-1} \sim 1 \quad \forall i = 1 \ldots n
\]

Now any \( S \in M_{D_n}(C) \) can be written as a linear span of the \( D_n \) matrices \( \{ S_{(j)} = S_{j1} \ldots S_{jn} = \prod_{i=1}^{n} C_{i}^{j} B_{i}^{j} \mid 0 \leq j \leq 2^{n-1} \} \)

We observe that

\[
S_{(j)} L_i S_{(j)}^{-1} \sim L_i S_{(j)} L_i S_{(j)}^{-1} = (S_{(j)} L_i S_{(j)}^{-1} L_i S_{(j)}^{-1}) (S_{(j)} L_i S_{(j)}^{-1} L_i S_{(j)}^{-1})^{-1} = (S_{(j)} L_i S_{(j)}^{-1} L_i S_{(j)}^{-1}) (S_{(j)} L_i S_{(j)}^{-1} L_i S_{(j)}^{-1})^{-1}
\]

These imply that if

\[
S_{(j)} L_i S_{(j)}^{-1} L_i^{-1} = S_{(j')} L_i S_{(j')}^{-1} L_i^{-1} ; \forall i = 1 \ldots n
\]

then

\[
S_{(j)}^{-1} S_{(j')} L_i S_{(j)} S_{(j')}^{-1} L_i^{-1} = I ; \forall i = 1 \ldots n
\]
or by Schur's Lemma: $S_{(j')}^{-1} S_{(j)} \neq I$

This means that

$$\left( S_{(j)} L_i S_{(j')}^{-1} L_i^{-1} \right) \neq \left( S_{(j')} L_i S_{(j)}^{-1} L_i^{-1} \right)$$

(5.60)

then $(j) \neq (j')$ or when any $s \in M \mathbb{D}$ satisfies

$$s L_i s^{-1} L_i^{-1} = \Psi_i I \quad \forall s L_i = \Psi_i L_i s; \forall i = 1 \ldots n$$

(5.60)

Thus the set $\Psi$ contains only distinct elements and are given by

$$\Psi = \left\{ (S_{(j)} L_i \theta_{(j')}^{-1} L_i^{-1} | i = 1 \ldots n) \mid (j) \equiv (j_1, j_2, \ldots, j_{2s}) \right\}$$

$$0 \leq j_{2i-1}, j_{2i} \leq Z_i - 0 \quad \forall i = 1 \ldots s$$

(5.61)

where

$$S^* \left[ \begin{array}{c} x_{i} \\ \psi_i \end{array} \right] \mod m = \left( y_{1}, y_{2}, \ldots, y_{2s}, 0, 0, \ldots \right) \left[ \begin{array}{c} T^* \end{array} \right] \left( \begin{array}{c} \psi_i \\ \psi_{i2} \end{array} \right)$$

(5.62)
Defining the equivalence relation that two coefficient sets \( \{ x^{(1)}_{i} \} \) and \( \{ x^{(2)}_{i} \} \) are equivalent if \( \{ x^{(1)}_{i} x^{(2)}_{i}^{-1} \} \in \mathcal{P} \), the entire set of all coefficient sets \( \{ x_{i}^{(n)} \} \) \( x_{i}^{(n)} = \omega(m)^{n} \), \( 0 \leq n \leq m_{i} - 1 \), \( i = 1 \ldots n \) \( \prod_{i=1}^{n} \frac{m_{i}}{} \) is partitioned into \( \prod_{i=1}^{n} \frac{m_{i}}{} \) equivalence classes. Thus factorizing \( \chi \) as

\[
\chi = \left\{ \chi^{(1)}_{\Psi}, \chi^{(2)}_{\Psi}, \ldots , \chi^{(N)}_{\Psi} \right\} \text{ and } N = \left( \frac{m}{\prod_{i=1}^{n} m_{i}} \right) / D^{2} \tag{6.63}
\]

all the inequivalent representations are given by the set

\[
\Phi^{\ast} = \left[ \left( \chi_{i}^{(n)} I_{i} \right| i = 1 \ldots n \right) \bigg| \gamma = 1 \ldots N = \left( \frac{m}{\prod_{i=1}^{n} m_{i}} \right) / D^{2} \right] \tag{6.64}
\]

Thus the total number of irreducible inequivalent projective representations of \( G \) for the factor set \( \gamma \) is

\[
N = \left( \frac{m}{\prod_{i=1}^{n} m_{i}} \right) / D^{2} \tag{6.65}
\]

where \( D \) is the dimension of the irreducible representations, same for all the representations. \( D \) is given by

\[
D = \prod_{i=1}^{A} \frac{m_{i}}{k_{i} m_{i}} \tag{14}
\]

which is the formula obtained by Backhouse and Sen-Brody. 

Finally in writing the representation of $G$ in terms of $L_i$, one has to normalize them to satisfy (5.23)

$$L_i^m = K_i I; \quad i = 1 \ldots n \quad (5.23)$$

$L_i$ is obtained in (5.53) will have to be multiplied by $(K_i)^{-1} m_i$ respectively.

This completes the task of finding the projective representations of $G$ for the given factor set $\zeta$ since once the representations of generators $\{L_i\}$ are found they have to be just substituted in the formula (5.16) to obtain the representations of other group elements.

Let us now consider the Schur representation group $G^*$ for $G$ constructed by central extension of $G$ with the Schur multiplier $H^2(G, C^*)$ as the kernel of extension. This group can be described by the following generating relations

$$L_i L_j = L_j L_i^{-1} L_{ij}^{-1} = L_{ij}; \quad i, j = 1 \ldots n$$

$$L_{ij}^m = L_{ij}^m = 1; \quad i, j = 1 \ldots n$$

$$L_{ij} L_k = L_k L_{i j} \quad \forall i, j, k = 1 \ldots n$$

$$L_i^m = 1; \quad i = 1 \ldots n.$$  

(5.62)

Choosing the factor set $\zeta$ such that $K_i = 1, \forall i = 1 \ldots n$ as can be achieved by redefining $\zeta$ as

$$\zeta'$$

(5.63)

The corresponding $m$-factor sets are the same. The group $G^*$ generated by (5.63) has the elements

$$\{ \prod_{ij \in \zeta} L_{ij}^{m_{ij}} L_k^{m_k}; 0 \leq m_{ij} \leq (m_i, m_j)^{-1}; 0 \leq m_k \leq m_{ij} - 1; \forall i, j, k = 1 \ldots n \}$$
Thus the order of $G^*$ is

$$|G^*| = \prod_{i=1}^{n} (m_i, m_j) \prod_{i=1}^{n} m_i = |H^2(G, \mathbb{C}^*)| \triangleright G^*$$

It is easy to see that the normal subgroup

$$H = \sum_{i<j} \frac{\prod_{i=1}^{n} m_i}{\prod_{i=1}^{n} (m_i, m_j)}$$

$$H \cong H^2(G, \mathbb{C}^*)$$

and $H \subseteq Z(G^*)$. Thus $G^*$ is the central extension of $G$ with $H^2(G, \mathbb{C}^*)$ as kernel of extension i.e.,

$$G^*/H^2(G, \mathbb{C}^*) \cong G.$$ The relations $L_{ij} m_i = L_{ij} m_j = 1$ and

$$L_{ij} L_{ik} = L_{ik} L_{ij}, \forall i, j, k = 1 \ldots n$$

show that $L_{ij}$'s are scalars and $L_{ij}$ has to be an $m_i m_j$ th root of unity where $L_{ij} = (m_i, m_j)$. Thus all the representations of (5.69) are obtained by the possible

$$\prod_{i<j} (m_i, m_j)$$

choices for the set $\{ L_{ij} | i, j = 1 \ldots n \}$. We found that if a particular choice is made as

$$L_{ij} = \exp(2\pi i t_{ij}/m) | i, j = 1 \ldots n$$

there are

$$\frac{\prod_{i=1}^{n} m_i}{\prod_{i=1}^{n} (m_i, m_j)} \frac{\prod_{i=1}^{n} m_i}{\prod_{i=1}^{n} (m_i, m_j)}$$

inequivalent irreducible representations of the same dimension $D$ where $\{ h_i, s \}$ etc. are uniquely fixed by

$$(t_{ij})$$. Then summing the squares of the dimensions of all the representations of $G^*$, corresponding to each choice of $\{ L_{ij} \}$ we get

$$\sum_{i<j} \frac{\prod_{i=1}^{n} m_i}{\prod_{i=1}^{n} (m_i, m_j)} \frac{\prod_{i=1}^{n} m_i}{\prod_{i=1}^{n} (m_i, m_j)}$$

where

$$\sum_{i<j} \frac{\prod_{i=1}^{n} m_i}{\prod_{i=1}^{n} (m_i, m_j)} \frac{\prod_{i=1}^{n} m_i}{\prod_{i=1}^{n} (m_i, m_j)}$$

inequivalent irreducible representations of the same dimension $D$ where $\{ h_i, s \}$ etc. are uniquely fixed by

$$(t_{ij})$$. Then summing the squares of the dimensions of all the representations of $G^*$, corresponding to each choice of $\{ L_{ij} \}$ we get
\[
\left( \prod_{i=1}^{n} m_i \right) \left( \prod_{j=i}^{n} (m_i, m_j) \right) = |G|^{\ast} \]

as is required by Burnside's theorem, showing that no more representations can occur. From this one can find the number of conjugate classes in \( G \) as

\[
\sum_{\text{all choices of } \{ L_{ij} \}} \left( \prod_{i=1}^{n} m_i \right) \left( \prod_{i=1}^{s} \frac{m_i}{(L_{ii}^2, m_i)} \right)^2 = \left( \prod_{i=1}^{n} m_i \right) \sum_{\text{all choices of } \{ L_{ij} \}} \prod_{i=1}^{s} (n, h_i)^2 / m
\]

As is evidently seen this is a very a complicated expression, impossible to obtain directly by counting the classes in \( G \).

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APPENDIX (cf. Morris Newman)

(i) How to obtain an $n \times n$ integer matrix $D_n$ with a given row of $n$ integers $(x_1, \ldots, x_n)$ as its first row and determinant $\delta_n = \text{the greatest common divisor of } (x_1, \ldots, x_n)$?

The theorem is that it is always possible to do so. We proceed by induction on $n$. For $n = 1$ the theorem is trivial for $n = 1$. Let $n = 2$. Then two integers $\rho, \sigma$ may be determined so that $\rho x_1 - \sigma x_2 = \delta_2$ and thus we may choose

$$D_2 = \begin{bmatrix} x_1 & x_2 \\ \sigma & \rho \end{bmatrix}$$

Now suppose the theorem true for $n-1, n \geq 3$ and let $D_{n-1}$ be an $n-1 \times n-1$ integer matrix with the first row $(x_1, x_2, \ldots, x_{n-1})$ and determinant $\delta_{n-1} = (x_1, x_2, \ldots, x_{n-1})$. Since

$$\delta_n = \left( (x_1, x_2, \ldots, x_{n-1}), x_n \right) = \left( \delta_{n-1}, x_n \right)$$

we can find integers $\rho, \sigma$ such that $\rho \delta_{n-1} - \sigma x_n = \delta_n$ put

$$D_n = \begin{bmatrix} D_{n-1} & \begin{bmatrix} x_n \\ \sigma \\ \rho \end{bmatrix} \\ \begin{bmatrix} x_1 \sigma & x_2 \sigma & \cdots & x_{n-1} \sigma \\ \delta n_{n-1} & \delta_{n-1} & \cdots & \delta_{n-1} \end{bmatrix} \end{bmatrix}$$

Then $\det D_n = \delta_n$

How to obtain a unimodular integer matrix $U$ such that $T = UT^*U$ where $T$ is antisymmetric integer matrix and $T^*$ is its unique skew normal form.
Since $T \neq 0$, a non-zero element may be brought to the first row by a permutation of the rows and a corresponding permutation of the columns. Therefore we may assume without loss of generality that the first row of $T$ contains a non-zero element which cannot occur in the $(1,1)$ position of course. Since $T$ is antisymmetric, define

$$(t_{11}, t_{13}, \ldots, t_{1n}) = \delta$$

and let $x_2, x_3, \ldots, x_n$ be integers such that

$$\sum_{j=2}^{n} a_{1j} x_j = \delta$$

Since $\delta \neq 0$ certainly $(x_2, x_3, \ldots, x_n) = 1$. We may determine a unimodular matrix $D_{n-1}$ with its first column as $(x_2, x_3, \ldots, x_n)^T$.

Then let $D_n = I_1 \oplus D_{n-1}$. Put

$$T^{(1)} = D_n^T T D_n = \begin{bmatrix} 0 & t_{12} & t_{13} & \cdots \\ -t_{12} & 0 & & \\ & & \ddots & \vdots \\ & & & 0 & t_{12} & t_{13} \end{bmatrix}$$

Then a brief computation shows that $t_{12} = \delta$ and that $t_{13}, t_{14}, \ldots, t_{1n}$.

Now add $-t_{1j}^{(1)} / t_{12}^{(1)}$ times row 2 to row $j'$ and then add $-t_{1j}^{(1)} / t_{12}^{(1)}$ times column 2 to column $j'$, $3 \leq j' \leq n$. The result is that $T^{(1)}$ gets transformed to a matrix say, $T^{(2)}$, whose first row is $(0, t_{12}, 0, 0, \ldots, 0)$. The procedure is now repeated with the submatrix obtained by striking out the first row and column until a tridiagonal matrix, say $T^{(3)}$, is reached so that
\[ T^{(3)} = \begin{bmatrix} 0 & t^{(3)}_{12} & \cdots & t^{(3)}_{1n} \\ t^{(3)}_{21} & 0 & \cdots & t^{(3)}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t^{(3)}_{n1} & t^{(3)}_{n2} & \cdots & 0 \end{bmatrix} \]

where \( t^{(3)}_{12} = k^{(3)}_{12} \neq 0 \). There are now two possibilities, either \( t^{(3)}_{12} \) divides every element of \( T^{(3)} \) or it does not. If not, add row 2, 3, \ldots, n to row 1 and column 2, 3, \ldots, n to column 1. The first row of the matrix to obtained is

\[
(0, t^{(3)}_{12}, t^{(3)}_{23}, \ldots, t^{(3)}_{n-1,n-1}, t^{(3)}_{n,n})
\]

Now greatest common divisor \( \gamma \) of the first row elements is

\( \gamma = (t^{(3)}_{12}, t^{(3)}_{23}, \ldots, t^{(3)}_{n,n}) \)

and \( \gamma \) contains fewer prime divisors than \( t^{(3)}_{12} \). As in the first part of the procedure, \( t^{(3)}_{12} \) may now be replaced by \( \gamma \) and the entire process repeated until a triple diagonal matrix is obtained having \( \gamma \) in the \((1,2)\) position. Thus in a finite number of steps a triple diagonal matrix, \( T^{(4)} \), congruent to \( T \) may be obtained in which \((1,2)\) divides every element of the matrix i.e. \( t^{(4)}_{12} \mid t^{(4)}_{ij} \)

\( 1 \leq i, j \leq n \). Add \( t^{(4)}_{23} / t^{(4)}_{12} \) times row 1 to row 3, and then add \( t^{(4)}_{12} / t^{(4)}_{13} \) times column 1 to column 3. Then \( T \) becomes congruent to a matrix of the form

\[
\begin{pmatrix} 0 & t^{(3)}_{12} \\ -t^{(3)}_{12} & 0 \end{pmatrix} \oplus E
\]
where \( t_1 \) divide every element of \( E \). The process is now repeated with \( E \) until it becomes congruent to a matrix

\[
\begin{pmatrix}
0 & t_2 \\
-t_2 & 0
\end{pmatrix} + E'
\]

carrying on this one finally arrives at a skew normal form

\[
\sum_{i=1}^{s} \begin{pmatrix} 0 & t_i \\ -t_i & 0 \end{pmatrix} \oplus O_{n-2s} ; \quad s \leq \left[ \frac{n}{2} \right]
\]

and by keeping track of all the operations performed on \( T \), one can construct the matrix \( U \), which is the product of all those operations preserving the order, such that \( U^T \tilde{\Omega} T = T^* \). It can be shown to be unique and \( t_i \)'s are nothing but invariant factors of \( T \). But \( U \) is not unique and as is seen easily and \( |\det U| = 1 \) follows from the observation that at each step of the above process the transformation matrices applied to \( T \) on both sides are unimodular. Since that \( t_i \)'s are invariant factors they can be specified as \( t_i = \frac{k_i}{k_{i-1}} \) where \( k_i \) is the greatest common divisor of the \( u_i-1 \times u_i-1 \) minors of \( T \) and \( k_{i-1} \) is the greatest common divisor of \( u_i \times u_i \) minors of \( T \).

**Summary of Important Points.**

All the inequivalent irreducible projective representations of \( G \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n} \) are obtained as follows. Let \( t_i \)'s denote the generators of \( G \) obeying \( e_i e_j = e_j e_i; \quad \forall i, j = 1, \ldots, n \)

\( e_i^{m_i} = 1; \quad i = 1, \ldots, n \). Then for a given factor let \( \tau \) such that

\[
\tau(\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}) = \tau(\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k) \tau(\mathbf{g}_i, \mathbf{g}_k)
\]

\( \forall \mathbf{g}_i, \mathbf{g}_j, \mathbf{g} \in G \).
the projective representations are given by

\[ D\left( \prod_{i=1}^{n} e_i^{k_i} \right) = \left[ \prod_{s=1}^{n} \left( \prod_{s=1}^{\delta_{s}} \zeta(e_s, e_i^{k_i}) \right) \right] \left( \prod_{i=1}^{n} L_i \right) \]

where

\[ L_i = \chi_i \left( \prod_{R=1}^{m} \frac{1}{e_i^{e_i^{k_i}} - \exp \left[ \frac{2\pi i (m_i - 1)}{m_i} \sum_{j=1}^{s_i} v_{i,j-1} v_{i,j} \right]} \right) \]

\[ \times \left[ \prod_{y=1}^{2} \bigotimes_{y=1}^{s_y} C_{y}^{y} \right] \]

\[ C_{y}^{y} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \]

\[ B_{y}^{y} = \begin{pmatrix} \exp \left( 2\pi i x_{y}/z_{y} \right) & 0 \\ 0 & \exp \left( 2\pi i x_{y} (z_{y} - 1)/z_{y} \right) \end{pmatrix} \]

\[ x_{y} = h_{y}/(h_{y}, m), \quad z_{y} = m/(h_{y}, m) = \dim C_{y}^{y} = \dim B_{y}^{y}, \quad y = 1, \ldots, s \]

\[ \Psi_{i,j} = V \] is a unitary matrix such that

\[ T = V T^{*} \tilde{T} \]

\[ T^{*} = \sum_{y=1}^{s} \theta \left( 0, h_{y} \right) \theta \left( 0, n - 2s \right) \]
\[ T = (e_{ij}) \text{ is the antisymmetric integer matrix defined by} \]

\[ e_{ij} e_{ji}^{-1} = \omega_{ij} \]

if \( \omega_{ij} = \exp(2\pi i \frac{r_j}{m_j}) \) with \( \langle r_j, k_j \rangle = 1 \)

then let \( \omega_{ij} = \exp(2\pi i \frac{r_j}{m_j}) \) where \( m = \text{l.c.m}(K_{ij} \mid i,j = 1\ldots n) \)

Thus \( \dim D = \{n/m_i \} \times \{ \frac{1}{m_i} m_i \} \) \( \Rightarrow \) \( \dim D = \{n/m_i \} \times \{ \frac{1}{m_i} m_i \} \) \( \Rightarrow \) \( \dim D = \{n/m_i \} \times \{ \frac{1}{m_i} m_i \} \)

sets of values for the set \( \{\{n/m_i \} \times \{ \frac{1}{m_i} m_i \} \} \mid i = 1\ldots n \) and each set of values, labelled by \( \tau = 1\ldots \langle \frac{1}{m_i} m_i \rangle \), gives rise to different (inequivalent) representations. These values of the sets \( \{\{n/m_i \} \times \{ \frac{1}{m_i} m_i \} \} \mid i = 1\ldots n \) are obtained by the relation

\[ \left\{ \left( \chi_i^{(\tau)} \right) \mid i = 1\ldots n \right\} \tau = 1\ldots \langle \frac{1}{m_i} m_i \rangle \]

\[ \left\{ \left( \chi_i^{(\tau)} \right) \mid i = 1\ldots n \right\} \tau = 1\ldots \langle \frac{1}{m_i} m_i \rangle \]

\[ \left\{ \left( \chi_i^{(\tau)} \right) \mid i = 1\ldots n \right\} \tau = 1\ldots \langle \frac{1}{m_i} m_i \rangle \]

\[ S^* \left[ \chi, \nu \right] \mod m = \left( \begin{array}{cccc} v_1 & v_2 & \cdots & v_s \end{array} \right) \]

\[ S^* \left[ \chi, \nu \right] \mod m = \left( \begin{array}{cccc} v_1 & v_2 & \cdots & v_s \end{array} \right) \]

\[ 0 \leq f_1, \ldots, f_s \leq z_i - 1 \text{ ; } \forall i = 1\ldots s \]
and the \( \otimes \) means \( \{ A, B, \ldots \} \otimes \{ A', B', \ldots \} = \{ A \circ A', A \circ B', \ldots, B \circ A', B \circ B', \ldots \} \)

where the product \( \circ \) of the sets \( A \) and \( B \) is defined by \( A \circ B = \{ A_i \otimes B_i \mid i = 1, \ldots, n \} \).

Thus there are \( \left( \prod_{i=1}^{n} m_i \right)/D^2 = \left( \prod_{i=1}^{n} m_i \right) \left( \prod_{j=1}^{s} (m, h_j)^2 \right) / m_{\phi} \) inequivalent representations of dimension \( D \) for the given factor set \( \xi \).

This chapter also studies the Schur representation group \( G \) of \( G \), which is a central extension of \( G \) with the Schur multiplier group \( H^2(G, \mathbb{C}^*) \subseteq \mathbb{Z}(G^*) \) as the kernel of extension.
CHAPTER 6

THREE OFBloch ELECTRONS IN HOMOGENEOUS MAGNETIC FIELD AND

GENERALIZED CLIFFORD GROUPS

The aim of this chapter is to present a simple version of the theory of Bloch electrons in homogeneous magnetic fields using Generalized Clifford groups. This is an extension of our earlier work in this direction.

The knowledge of the behavior of Bloch electrons in a magnetic field is fundamental to the study of various phenomena in solids under the influence of external magnetic fields such as galvanomagnetic, thermomagnetic, magnetic acoustic and magneto-optical effects, microwave absorption in magnetic fields and the de Haas-van Alphen effect etc. which supply very important information about the electronic structure of solids. Almost all relevant effects except a few like magnetic breakdown can be well understood in terms of semi-classical theory.

Attempts at quantum theory of the problem started with Landau and Peierls and further investigations by Luttinger, Kohn, Bloch, Warman and Proskin, Roth and Zehborn and confirmed Casimir's quantization rule and close connection of the quantum theory to the electron orbits of semiclassical theory was established. But since this relation is established by using non-convergent expansions in powers of field strength it becomes useless at high field strengths. In extremely high magnetic fields
the basic electronic structure of the solid will strongly depend on the field strength independent study of the problem of Bloch electron in magnetic field was initiated by Harper, Brown, Slack, Fischbeek, Opolchenskiy, and Van and others elucidated the basic symmetry of the problem and new approaches to the solution of the Schrödinger equation were studied. So far some general properties of eigenvalues and eigenfunctions of the Hamiltonian have been stated but unfortunately, as Fischbeek observes, till now all these results have been of no practical interest. Very strange features of the problem such as the dependence of energies and wavefunctions on certain spasmodically changing quantities associated with monotonically varying field strengths remain unexplained (Cf. Fischbeek for an excellent review of the present state of affairs and extensive bibliography).

The group theoretical approach to the problem brings in algebraic structures which we have called Generalised Clifford Groups in earlier chapters. We shall use this isomorphism of G.C.G. with magnetic translation group \((N,T,E)\) - the symmetry group of the Hamiltonian of the Bloch electron in homogeneous magnetic field - to derive in a very simple manner more complete and improved results, than those of Brown, Slack, Fischbeek and others on the properties of eigenfunctions and eigenvalues of the Hamiltonian. We make an attempt, unlike others, to study both the cases of rational and irrational magnetic fields at the same time, though as it has been shown already, mathematically accurate statements can be made only in the rational case. Hence our results for the irrational case are of
only of explorative and tentative nature due to the inherent
mathematical difficulties involved. Our main aim has been to
derive the Magnetic Bloch Theorem in the simplest form so that by
analogy with the role of the ordinary (zero field) Bloch theorem
methods of energy band structure calculations may be developed.
First let us recall some relevant facts of the theory of electrons
in free lattice potential.

1) Electrons in periodic potential and Bloch functions

The Schrödinger equation of an electron in a periodic poten-
tial is given by

\[ \mathcal{H}_o \psi(\mathbf{r}) = E \psi(\mathbf{r}) \]

\[ \mathcal{H}_o = \left\{ \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}) \right\} \]

\[ \mathbf{p} = -i \hbar \nabla \]

\[ V(\mathbf{r} + \mathbf{R}) = V(\mathbf{r}) \quad V \mathbf{R} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2 + n_3 \mathbf{a}_3 \] (6.1)

where \( m \) is the mass of the electron, \( (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \) are primitive
vectors of a unit cell of the lattice and \( n_1, n_2, n_3 \) are integers.

The group of lattice translation operators

\[ T = \left\{ T(R) = \exp \left\{ \frac{i}{\hbar} \mathbf{R} \cdot \mathbf{p} \right\} \mid \mathbf{R} \in \mathbb{L} \right\} \]

\[ T(R) \psi(\mathbf{r}) = \psi(\mathbf{r} + \mathbf{R}) \quad \mathbb{L} = \left\{ \sum_{i=1}^{3} n_i \mathbf{a}_i \mid n_1, n_2, n_3 \in \mathbb{Z} \right\} \] (6.2)

Commute with the Hamiltonian \( \mathcal{H}_o \), i.e.

\[ [T(R) \mathcal{H}_o - \mathcal{H}_o T(R)] \psi(\mathbf{r}) = 0 \quad \forall T(R) \in T \] (6.3)

Hence by the well-known Wigner's theorem on applications of group
theory to quantum mechanics, eigenfunctions of $H_0$ can be chosen simultaneously as also basis functions of irreducible representations of the group $G$. The group $G$ is Abelian i.e.,

$$T(R_1)T(R_2) = T(R_2)T(R_1) = T(R_1 + R_2) \quad \forall T(R_1), T(R_2) \in G,$$

and hence its irreducible representations are one dimensional, given by

$$\Gamma_k : \{ \Gamma_k(R) = \exp(i k \cdot R) \mid \forall R \in \mathbb{R}^3 \} \quad (6.4)$$

where $k = y_1 k_1 + y_2 k_2 + y_3 k_3$ labels the representations, $k_1, k_2, k_3$ being primitive vectors of reciprocal space, defined by

$$k_1 = 2\pi a_1 a_3 \frac{a_2}{\Omega}, \quad k_2 = 2\pi a_3 a_1 \frac{a_2}{\Omega}, \quad k_3 = 2\pi a_1 a_2 \frac{a_3}{\Omega} \quad (6.5)$$

$\Omega = (a_1 \wedge a_2) \cdot a_3$ and $y_1, y_2, y_3$ are any real numbers. Let $L'$ denote the set of all vectors of the reciprocal lattice $L' = \{ G_i = \sum_{l=1}^{3} m_l k_l \mid m \in \mathbb{Z} \}$. Then

$$\Gamma_k = \Gamma_k + G_i \quad \forall G_i \in L' \quad (6.6)$$

Corresponding to the representation $\Gamma_k$, the basis functions can be obtained, using the technique of projection operators, which gives the particular functions of a representation of a group $G$ by the formula.
where $\Delta(q)_{jj}$ is the $(jj)$-th element of the matrix $D(q)$, $\psi$ is an arbitrary function of the space on which $G$ acts, and $\xi$ is a normalisation constant. Applying this to the present case we get

$$
\psi_k(I) = \xi \sum_{R \in \Lambda} \Gamma_k(R) \tau(R) \psi(I + R)
$$

$$
= \xi \sum_{R} \exp(-iK \cdot R) \psi(I + R)
$$

$$
= \xi \exp(iK \cdot I) \sum_{R} \exp(-iK \cdot [I + R]) \psi(I + R)
$$

$$
= \xi \exp(iK \cdot I) U_k(I)
$$

(6.8)

$\xi$ being a normalisation constant such that $\int \psi_k(I) \psi^* (I) d^2 I = 1$, $\psi(I)$ is an arbitrary function and $U_k(I) = \sum_{R} \exp(-iK \cdot [I + R]) \psi(I + R)$ is a periodic function with the same period as $\psi(I)$.

(6.9)

Also it is seen easily that

$$
\psi_{k+G}(I) = \psi_k(I) ; \forall G \in \Lambda'
$$

(6.10)
I being the symmetry group of \( H_0 \), solutions of \( H_0 \) are
members of the same form as \( \psi_{K}(r) \) which is a product of \( e^{i (k \cdot r)} \)
and a periodic function of \( r \) with the same period as \( V(r) \).
This is known as Bloch theorem. Substituting \( \psi_{K}(x) \) in the
Schrödinger equation (6.11) we get an equation for \( U_K(r) \),

\[
\left\{ \frac{1}{2m} \left( p + \frac{\hbar}{2m} n \right)^2 + V(r) \right\} U_{nK}(r) = E_{nK}(r) U_{nK}(r)
\]

(6.11)

\[
\psi_{nK}(x) = N e^{i (k \cdot r)} U_{nK}(r)
\]

(6.12)

where \( E_{nK}(r) \) is the eigenvalue and \( U_{nK}(r) \) is the corresponding
eigensolution. \( n \) is called a band index and the full eigensolution \( \psi_{nK}(r) \)
called a Bloch function is given by (6.12). \( k \) is called crystal
momentum of electron and due to (6.10) it follows

\[
E_{n}(k + \mathbf{G}) = E_{n}(k) ; \forall \mathbf{G} \in \Gamma
\]

(6.13)

Thus in knowing \( E_{n}(k) \), \( k \) need be specified only over a con-
veniently chosen (to reflect the symmetry of the crystal) unit cell
of the reciprocal space, called Brillouin Zone. Equations (6.12) and
(6.13) constitute the main theory we seek to generalize to the
case of presence of an external homogeneous magnetic field.

(11) **Bloch electrons in homogeneous magnetic field.**

The time independent Schrödinger equation for Bloch electron
in an external homogeneous magnetic field \( B \) is given by, (dis-
regarding spin)
\[ H_m \psi(\mathbf{r}) = \frac{i}{\hbar} \left[ \mathbf{p} + \frac{e}{c} \mathbf{A} \right] \psi(\mathbf{r}) + V(\mathbf{r}) \]

\[ B = \mathbf{\nabla} \wedge \mathbf{A} \]

without loss of generality the vector potential \( \mathbf{A} \) can be chosen in symmetric gauge as

\[ \mathbf{A} = \frac{\mathbf{B} \wedge \mathbf{r}}{\mathbf{r}} \]

Though \( \mathbf{B} \) is constant since \( \mathbf{A}(\mathbf{r} + \mathbf{R}) \neq \mathbf{A}(\mathbf{r}) \), \( H_m \) is not invariant under the group \( T \) of lattice translation operators \((6.2)\). The new symmetry operators for \( H_m \), called magnetic translation operators were first studied group theoretically by Brown and then by Zak, Fischbein and others. The origin of these operators can be traced to the work of Peierls. First it is noticed that, for the gauge \((6.16)\),

\[ \left[ \begin{pmatrix} \mathbf{p} + \frac{e}{c} \mathbf{A} \\ \mathbf{r} \end{pmatrix}, \begin{pmatrix} \mathbf{p} - \frac{e}{c} \mathbf{A} \\ \mathbf{r} \end{pmatrix} \right] = 0 \]

Then it follows that

\[ \left[ \begin{pmatrix} \frac{\mathbf{R} \cdot (\mathbf{p} - \frac{e}{c} \mathbf{A})}{} \\ c \end{pmatrix}, H_m \right] = 0, \quad \forall \mathbf{R} \in L. \]

The following set of relations are easy to verify

(a) \( \begin{pmatrix} \mathbf{p} + \frac{e}{c} \mathbf{A} \\ \mathbf{r} \end{pmatrix} \wedge \begin{pmatrix} \mathbf{p} + \frac{e}{c} \mathbf{A} \\ \mathbf{r} \end{pmatrix} = -\frac{ie}{c} \mathbf{B} = -i \hbar^2 \beta \)

(b) \( \begin{pmatrix} \mathbf{p} - \frac{e}{c} \mathbf{A} \\ \mathbf{r} \end{pmatrix} \wedge \begin{pmatrix} \mathbf{p} - \frac{e}{c} \mathbf{A} \\ \mathbf{r} \end{pmatrix} = \frac{ie}{c} \mathbf{B} = i \hbar^2 \beta \)

(c) \[ \mathbf{R} \cdot \begin{pmatrix} \mathbf{p} - \frac{e}{c} \mathbf{A} \\ \mathbf{r} \end{pmatrix}, \mathbf{R}' \cdot \begin{pmatrix} \mathbf{p} - \frac{e}{c} \mathbf{A} \\ \mathbf{r} \end{pmatrix} \] = \( i \hbar^2 \beta \cdot (\mathbf{R} \wedge \mathbf{R}') \)
Also, in the gauge (4.10),

$$
\left[ \mathbf{R} \cdot \mathbf{P}, \mathbf{R} \cdot \mathbf{A} \right] = -\frac{i}{2} \hbar^2 \mathbf{R} \cdot (\mathbf{R} \wedge \mathbf{R}') \tag{4.10}
$$

ie $$\left[ \mathbf{R} \cdot \mathbf{P}, \mathbf{R} \cdot \mathbf{A} \right] = 0.$$ and hence

$$
\exp \left\{ \frac{i}{\hbar} \mathbf{R} \cdot \mathbf{P} \right\} \exp \left\{ \frac{-i}{\hbar} \frac{e}{c} \mathbf{R} \cdot \mathbf{A} \right\} = \exp \left\{ \frac{i}{\hbar} \mathbf{R} \cdot (\mathbf{P} - \frac{e}{c} \mathbf{A}) \right\}
$$

due to

$$
e^A e^B = e^{A+B} ; \quad [A, B] = 0 \tag{4.10}
$$

The set of operators \( \{ \tau(\mathbf{R}) = \exp \left\{ \frac{i}{\hbar} \mathbf{R} \cdot (\mathbf{P} - \frac{e}{c} \mathbf{A}) \right\}; \forall \mathbf{R} \in L \} \)

comprising with \( H_m \) are called magnetic translation operators.

Due to (4.10) c we have

$$
\tau(\mathbf{R}) \tau(\mathbf{R}') = \exp \left\{ \frac{-i}{\hbar} \frac{e}{c} \mathbf{P} \cdot (\mathbf{R} \wedge \mathbf{R}') \right\} \tau(\mathbf{R} + \mathbf{R}') = \exp \left\{ \frac{-i}{\hbar} \frac{e}{c} \mathbf{P} \cdot (\mathbf{R} \wedge \mathbf{R}') \right\} \tau(\mathbf{R}') \tau(\mathbf{R})
$$

\( \forall \mathbf{R}, \mathbf{R}' \in L. \tag{4.11} \)

This shows that the set \( \{ \tau(\mathbf{R}); \forall \mathbf{R} \in L \} \) does not form a group and in exact analogy with the formation of G.C.G. from basis elements of G.C.A., one has to form a group only by adding more elements to the set which are multiples of \( \tau(\mathbf{R}) \) by phase factors of the type

$$
\exp \left\{ \frac{-i}{\hbar} \frac{e}{c} \mathbf{P} \cdot (\mathbf{R} \wedge \mathbf{R}') \right\} \text{ such a group } \{ \exp (i\phi) \tau(\mathbf{R}) \} \text{ has been called a magnetic translation group by Brown, Zak, Fischbach and others.}
$$
We shall follow a different presentation of the group using generators. For this list

\[ \tau_i = e^{i\beta_k^i}, \quad i = 1, 2, 3 \]

\[ \therefore \tau_i \tau_j = e^{i\beta_k^i \tau_j \tau_i}, \quad (i, j, k = 1, 2, 3) \]

in cyclic order.

Then

\[ \tau_1^{-n_1} \tau_2^{-n_2} \tau_3^{-n_3} = e^{i\frac{2\pi}{\eta} (n_1\beta_1 - n_2\beta_2 + n_3\beta_3)} \tau(R) \]

\[ \forall R = n_1 a_1 + n_2 a_2 + n_3 a_3 \in \mathbb{L}. \]

Thus any \( \tau(R) \) is a multiple of a phase factor and product of powers of \( \{\tau_i\}_{i=1}^3 \). Thus we shall consider the group generated by \( \tau_i \)’s for the purpose of obtaining the symmetry adapted functions. There is no loss of generality in this in choosing the smallest lattice vector in the direction of the magnetic field \( \mathbf{B} \) to be \( a_3 \) and write \( \mathbf{B} = \frac{\Phi \pi}{\eta} a_3 \). Then (6.23) get much simplified. Also let us first consider \( \eta \) to be a rational number \( = \frac{A}{N} \) with \( (S, N) = 1 \). Then the field \( \mathbf{B} \) is said to be rational. To make the group finite for convenience one can impose boundary conditions on the solutions consistent with the relations (6.23) just like imposition of Born–von Karman conditions in the free Bloch electron case \( \psi(I) = \psi(I + NR), \forall R \in \mathbb{L} \) where \( N \) is a very large
integer $\rightarrow \sigma$. This condition makes the crystal momentum to vary quasicontinuously through rational values $K = y_1 K_1 + y_2 K_2 + y_3 K_3$, $y_1, y_2, y_3 \in \mathbb{Q}$.

Assuming the field to be rational $\beta = \frac{2\pi i}{h} \sigma \in \mathbb{Q}$ and imposing the boundary conditions $H(0) = 1$, $\forall H(0) \in H$, $2N | M, M \sim 0$, the group $\Gamma \equiv \{ \exp \left( -\frac{2\pi i \cdot M \cdot \delta}{2N} \right) \gamma_1 \gamma_2 \gamma_3 | 0 \leq M \leq 2N-1, \ 0 \leq \delta_i \leq M-1 \} \forall i = 1, 2, 3$ is a finite group of order $2N M^3$. As an abstract group this is isomorphic to the group $\tilde{G}$ generated by the relations

$$
L_1 L_2 L_3 = L_0^2 \quad L_2 L_3 L_1 = L_0^2
$$

$$
L_3 L_1 = L_1, L_3, \quad L_2 L_3 = L_3 L_2, \quad L_0 = 1,
$$

$$
L_3 L_1 L_0 = L_1 L_3 L_0; L_i = 1, 2, 3
$$

$$(\prod_{i=0}^{2N-1} L_i M_i)^{M} = 1, \quad \forall R_i = 0, 1, \ldots, M-1, \quad 2N | M.
$$

which is one of the types of groups we have called Generalized Clifford groups. The correspondence is $\Gamma_i = L_i^{-1} \Gamma_i L_i, \ i = 1, 2, 3$. From their representation theory developed in earlier chapters we know that all the representations are obtained by faithfully representing the $N$ sets of relations (6.25) arising by varying

$$
L_1 L_2 L_3 L_1^{-1} L_2^{-1} L_3^{-1} = L_0 = \sqrt{N} \quad \text{over all its $N$ possible values}\{ \exp \left( \frac{2\pi i \ell}{N} \right) \}
$$

$0 \leq \ell \leq N-1 \}$. In each of these $N$ cases the representations are of the same dimension which depends on the value of $L_0$, chosen and differ only in phase factors. One can easily count the different representations arising from different phase factors. If we choose a particular value of $L_0$, say $\exp \left\{ \frac{2\pi i \ell}{N} \right\}$ then $L_0$ has two values for $\ell = 0, 1, \ldots, N-1$.

$\mathbb{Q} = \text{set of rational numbers (real)}$. 
\[ \pm \exp \left( \frac{i \pi l}{N} \right) \cdot L_3 \text{ commutes with both } L_1 \text{ and } L_2 \text{ and hence all group elements. Thus in irreducible representations it has to be a scalar obeying } L_3^m = 1 \text{ or } L_3 \text{ has } N \text{ values } \left\{ \exp \left( \frac{i \pi k_3 N}{M} \right) \right\} \]

where \( 0 \leq k_3 \leq M - 1 \). Since \( L_1^M = L_2^M = 1 \), to each of them can be attached \( N \) phase factors \( \left\{ \exp \left( \frac{i \pi k_1 N}{M} \right) \right\} \), \( 0 \leq k_1 \leq M - 1 \), and \( \left\{ \exp \left( \frac{i \pi k_2 N}{M} \right) \right\} \), \( 0 \leq k_2 \leq M - 1 \). Thus there are totally \( 2^3 \) possible different representations of same dimension with only difference of phase factors. But among these, as we have often seen in earlier chapters, two representations would be equivalent if one differs from the other only in the phase factors of \( L_1 \) and \( L_2 \) by any powers of \( L_1 L_2 L_1^{-1} L_2^{-1} = \exp \left( \frac{2i \pi l}{N} \right) = \exp \left( \frac{2i \pi l'}{N'} \right) \text{ with } (l', N') = 1 \) then \( \exp \left( \frac{2i \pi l}{N} \right) = \exp \left( \frac{2i \pi l'}{N'} \right) \), \( (l', N') = 1 \) the dimension of representation \( N' \), and out of the \( N \) choices for the phase factors of \( L_1 \) and \( L_2 \), the above condition of equivalence implies that phase factors of each \( L_1 \) and \( L_2 \) can take only \( \left( M / N' \right) \) values \( \left\{ \exp \left( \frac{i \pi k l_1 N'}{M} \right) \right\} \), \( 0 \leq k \leq \left( M / N' \right) - 1 \} \) corresponding to inequivalent representations.

Since \( \left( M / N' \right) \geq 2 \left( M / N' \right)^2 = 2 \left( M / N' \right)^2 \) inequivalent irreducible representations for \((N, 2M)\) corresponding to a choice of \( L_1 L_2 L_1^{-1} L_2^{-1} = \exp \left( \frac{2i \pi l}{N} \right) = \exp \left( \frac{2i \pi l'}{N'} \right) \), \( (l', N') = 1 \). All representations of \( \mathcal{G} \) are obtained by varying \( \lambda = 0, 1, \ldots, N - 1 \). The set of magnetic translation operators \( \mathcal{L} \) faithfully correspond to relations...
(6.85) with a choice

\[
L_i L_j^{-1} L_i^{-1} L_j = \exp \left( - \frac{2\pi i k}{N} \right), \quad (k,k) = 1 
\]

\[
L_0 = + \exp \left( - \frac{2\pi i k}{N} \right). 
\]

Corresponding to this choice there should be representations of dimension $D$ which are specified by the choices

\[
L_i = \exp \left( \frac{2\pi i k_i}{N} \right)L_i^*, \quad 0 \leq k_i \leq M - 1 
\]

\[
L_{ij} = \exp \left( \frac{2\pi i k_j}{N} \right)L_{ij}^*, \quad 0 \leq k_j \leq M - 1 
\]

\[
L_3 = \exp \left( \frac{2\pi i k_3}{N} \right)I, \quad 0 \leq k_3 \leq M - 1 
\]

\[
L_0 = + \exp \left( - \frac{2\pi i k}{N} \right) 
\]

where $L_i^*$, $L_{ij}^*$ are any two irreducible $n \times n$ matrices obeying

\[
L_i^* L_{ij}^* = \exp \left( - \frac{2\pi i k}{N} \right)L_{ij}^* L_i^* 
\]

Let us choose well known matrices

\[
L_1^* = B_N^* = \begin{pmatrix}
1 & \omega(N) & \omega(N)^2 & \cdots & \omega(N)^{N-1} \\
\omega(N) & \omega(N)^2 & \cdots & \omega(N) & 0 \\
\omega(N)^2 & \omega(N)^3 & \cdots & \omega(N)^2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\omega(N)^{N-1} & \omega(N)^{N-2} & \cdots & \omega(N)^{N-1} & 0
\end{pmatrix} 
\]

(6.86)

with

\[
\omega(N) = \exp \left( \frac{2\pi i}{N} \right) 
\]

and

\[
L_2^* = C_N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix} 
\]

(6.88)

\[
(B_N^*) = C_N = 1.
\]
Then from (6.24)

\[ T_{\tau}(R) = \exp \left( \frac{2\pi i m_{3} 2 \tau^{3}}{2N} \right) T_{1}^{m_{1}} T_{2}^{m_{2}} T_{3}^{m_{3}} \]

\[ = \exp \left( \frac{i\pi m_{1} m_{2}}{N} \right) \exp \left( 2\pi i \left[ \frac{m_{1} k_{1}}{M} + \frac{m_{2} k_{2}}{M} + \frac{m_{3} k_{3}}{M} \right] \right) B_{N}^{m_{1}} C_{N}^{m_{2}} \]

\[ = \exp \left( \frac{i\pi m_{1} m_{2}}{N} \right) \exp \left( i\frac{\mathbf{K} \cdot \mathbf{R}}{N} \right) \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & \exp \left( i\frac{\mathbf{K} \cdot \mathbf{R}}{N} \right) \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \exp \left( i\frac{\mathbf{K} \cdot \mathbf{R}}{N} \right) \end{pmatrix} \]

(6.26)

where \( \mathbf{K} = \frac{1}{M} (k_{1} \mathbf{K}_{1} + k_{2} \mathbf{K}_{2} + k_{3} \mathbf{K}_{3}) \)

or

\[ \left( T_{\tau}^{R} \right)^{j \ell} = \exp \left( \frac{i\pi m_{1} m_{2}}{N} \right) \exp \left( i\frac{\mathbf{K} \cdot \mathbf{R}}{N} \right) \exp \left( i\frac{k_{1} \cdot \mathbf{R}}{N} \right) \delta^{j}_{\ell} \delta_{j, (l-n_{2})} \]

\[ = \exp \left( \frac{i\pi m_{1} m_{2}}{N} \right) \exp \left( i \left[ k_{1} \right] \frac{\mathbf{K}_{1} \cdot \mathbf{R}}{N} \right) \delta^{j}_{\ell, (l-n_{2})} \]

\[ j, \ell = 0, 1, \ldots, N-1; (l-n_{2}) \text{ mod } N = l-n_{2} \]

\( \forall \mathbf{R} = n_{1} \mathbf{a}_{1} + n_{2} \mathbf{a}_{2} + n_{3} \mathbf{a}_{3} \in L \).

\( \mathbf{K} \) is label the representations and if we now take the limit \( M \to \infty \), \( \mathbf{K} \) varies quasiconsistently in a subspace of the Brillouin zone, Magnetic Brillouin zone containing the set of all reciprocal space vectors \( \{ \mathbf{k} = k_{1} \mathbf{a}_{1} + k_{2} \mathbf{a}_{2} + k_{3} \mathbf{a}_{3} \} \) where

\[ 0 \leq k_{1} \leq \frac{1}{N}, 0 \leq k_{2} \leq \frac{1}{N}, 0 \leq k_{3} \leq \frac{1}{N} \]
Shape of the magnetic Brillouin zone can also be taken conveniently. Thus the representations $\Gamma_{K'}$ and $\Gamma_{K''}$ are equivalent if

$$K' = K + \mathbf{G}_m$$

where

$$\mathbf{G}_m \in \mathbf{L}'_m = \left\{ \frac{1}{N} \left( m_1 \mathbf{k}_1 + m_2 \mathbf{k}_2 + m_3 \mathbf{k}_3 \right) \mid m_i \in \mathbb{Z} \right\}$$

Correspondingly the generalization of (6.12) is

$$E_m (K + \mathbf{G}_m) = E_n (K), \quad \forall \mathbf{G}_m \in \mathbf{L}'_m \quad (6.32)$$

where $n$ is the band index. Let us now construct the symmetry adapted functions corresponding to the representation $\Gamma_{K'}$. Before proceeding further to find the partner functions for $\Gamma_{K'}$ of our problem let us notice that for other representations of the group $\Gamma$, considered as isomorphic to $\mathcal{G}$, the projection operator

$$P_j = \prod_{\mathbf{G} \in \mathcal{G}} \langle \Gamma_j \mathbf{D}(\mathbf{G}) \rangle$$

vanishes identically $\forall j$. This happens as follows. In $\Gamma$ each $\tau (\mathbf{R})$ occurs $2N$ times as multiples of the $2N$, $2N$th roots of unity due to the phase factors

$$\left\{ \exp \left( -2\pi i \frac{m \mathbf{G}}{2N} \right) \right\}_{0 \leq m \leq 2N - 1} \quad (6.3)$$. If a representation corresponds to a choice of $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3 \mapsto \mathbf{L}_\alpha \mapsto \mathbf{L}_0 \mapsto \exp \left( -2\pi i \mathbf{G} / N \right)$ with $\alpha \neq \beta$ or even

$$\mathbf{L}_0 = \exp \left( -2\pi i \mathbf{G} / 2N \right)$$

and $\alpha = \beta$, for each $\tau (\mathbf{R})$ in the summation in $P_j$ coefficients add up to 0. Hence it follows that

$$P_j \equiv 0, \quad \forall j$$

Using the projection operation, the zeroth function is given by
\[ \psi_{k,0}(x) = \sum_{\mathcal{L}} \sum_{\mathbb{Z}^3} \exp \left( -\frac{2\pi i m}{2N} \right) \left( t_k(R) \right)_{00} \right \} \exp \left( -\frac{2\pi i m}{2N} \right) \tau(R) \psi(x) \]  

On summation over \( m \) this gives

\[ \psi_{k,0}(x) = 2NF \sum_{\mathcal{L}} \sum_{\mathbb{Z}^3} \exp \left\{ -i \pi n_m \right\} \exp \left\{ -i \left( k \cdot \tau(R) \right) \right\} \exp \left\{ \frac{\pi i}{h} \left( R \cdot \left( p - e \right) \right) \right\} \psi(x) \]  

where \( \mathcal{L} = \{ n_1 \alpha_1 + n_2 \alpha_2 + n_3 \alpha_3 \mid n_1, n_2, n_3 \in \mathbb{Z}^3 \} \)

Now using (6.19) \( \psi_{k,0}(x) \) becomes apart from normalization

\[ \psi_{k,0}(x) = \sum_{\mathcal{L}} \exp \left\{ -i \pi n_m \right\} \exp \left\{ -i \left( k + e / c \right) \cdot \tau(R) \right\} \psi(x) \]  

where \( \psi(x) \) is an arbitrary function. This function has a remarkable property analogous to the Bloch functions (6.13). We shall prove that it can be written as

\[ \psi_{k,0}(x) = \exp \left( i k \cdot x \right) \psi_{k,0}^*(x) \]  

(6.35)
where

\[ \tau(R') U^*_{K,0}(I) = \exp \left\{ -i \tau m_i s^3 \right\} U^*_{K,0}(I) \]

\[ \forall R' = n'_1 a_1 + n'_2 a_2 + n'_3 a_3 \in \mathbb{L} \]

Proof

\[ \psi_{K,0}(x) = \sum_{R \in \mathbb{L}} \exp \left\{ -i \tau m_i n_2 s^3 \right\} \exp \left\{ -i K \cdot (x + R) \right\} \phi_{(x + R)} \]

\[ = \exp \left( i K \cdot x \right) \sum_{R \in \mathbb{L}} \exp \left\{ -i \tau m_i n_2 s^3 \right\} \exp \left\{ -i K \cdot (x + R) \right\} \phi_{(x + R)} \]

\[ = \exp \left( i K \cdot x \right) \psi_{K,0}(I) \]

\[ U^*_{K,0}(I+R') = \sum_{R \in \mathbb{L}} \exp \left\{ -i \tau m_i n_2 s^3 \right\} \exp \left\{ -i K \cdot (I+R+R') \right\} \phi_{(I+R') \cdot (I+R+R')} \]

where \( R' = n'_1 a_1 + n'_2 a_2 + n'_3 a_3 \)

Let \( \tilde{R} = R + R' \). Substituting \( R = \tilde{R} - R' \) and \( \tilde{n}_i = n_i + n'_i \), \( i = 1, 2, 3 \)

\[ U^*_{K,0}(I+R') = \sum_{R'' \in \mathbb{L}} \exp \left\{ -i \tau m_i \right\} \left( \tilde{n}_1 n_2 - \tilde{n}_2 n_1, \tilde{n}_2 n_3 - \tilde{n}_3 n_2, \tilde{n}_3 n_1 - \tilde{n}_1 n_3 \right) \phi_{(I+R')} \exp \left\{ i K \cdot (I+R') \right\} \phi_{(I+R+R')} \]

\[ \chi U(I+R') \]
\[
\begin{align*}
= \exp \left\{ \frac{ie}{2\hbar c} B \wedge T \cdot R \right\} \sum_{R' \in \mathcal{L}} \exp \left\{ -i \pi s (n,m_2'' - n,m_2' - n,m_2 - n,m_2') \right\} x \\
\exp \{ -ie \cdot R \cdot R' \} \exp \{ -i K \cdot (I + R')^3 \} \exp \frac{ie}{2\hbar c} B \wedge T \cdot R' \right\} \mathcal{L}
\end{align*}
\]

\[
\begin{align*}
&= \exp \left\{ \frac{ie}{2\hbar c} B \wedge T \cdot R \right\} \sum_{R'' \in \mathcal{L}} \exp \left\{ -i \pi s (n,m_2'' - n,m_2') \right\} x \\
&\quad \exp \{ -i K \cdot (I + R')^3 \} \exp \frac{ie}{2\hbar c} B \wedge T \cdot R' \right\} \mathcal{L}
\end{align*}
\]

\[
\begin{align*}
&= \exp \left\{ \frac{ie}{2\hbar c} B \wedge T \cdot R \right\} \exp \left\{ i \pi s (n,m_2'') \right\} \sum_{R'' \in \mathcal{L}} \exp \left\{ -i \pi s (n,m_2'') \right\} x \\
&\quad \exp \{ -i K \cdot (I + R')^3 \} \exp \frac{ie}{2\hbar c} B \wedge T \cdot R' \right\} \mathcal{L}
\end{align*}
\]

\[
\begin{align*}
\exp \left\{ -i R', \frac{eA}{\hbar} \right\} U^{*}_{K,0} (I + R') &= \exp \left\{ \frac{i}{\hbar} R', (P - \frac{e}{\hbar} A)^3 \right\} U^{*}_{K,0} (I) \\
&= T(R') \ U^{*}_{K,0} (I) = \exp \left\{ -i \pi s (n,m_2'') \right\} U^{*}_{K,0} (I) \\
&\quad \forall R' \in \mathcal{L}
\end{align*}
\]

Once we have found the zeroth partner function the other partner functions can be obtained by applying to it the operator
\[ \{ \exp \left( -i \frac{K \cdot a_2}{2} \right) T(a_2)^{\frac{3}{2}} \} \]

Successively since this operator is represented by cyclic matrix \( C_n \), implying that the corresponding operator will change the \( j \)-th function to \( j+1 \)-th function. The inverse operation will change \( j \)-th function to \( j-1 \)-th function. One can get all the partner functions directly from the operation (6.36). We shall follow the procedure to obtain a clearer interpretation of the functions. Hence using this we get

\[
\psi_{K,j}^{(r)}(x) = \exp \left( i \frac{K \cdot j a_2}{2} \right) T\left( -j a_2 \right) \psi_{K,0}^{(r)}(x)
\]

\[
= \exp \left\{ i \frac{K \cdot j a_2}{2} \right\} \exp \left\{ \frac{1}{2} \left( -j a_2 \cdot A \right) X \right\} \psi_{K,0}^{(r)}(x)
\]

\[
= \exp \left\{ i \frac{K \cdot j a_2}{2} \right\} \psi_{K,j}^{(r)}(x) \quad j = 0, 1, \ldots, N-1
\]  

(6.37)

\[
\psi_{K,j}^{(r)}(x) = \psi_{K,j}^{(r)}(x) \quad \forall j = 0, 1, \ldots, N-1
\]  

(6.38)

Also one can write

\[
\psi_{K,j}^{(r)}(x) = \sum \exp \left\{ -i \frac{K \cdot (x + R)}{2} \right\} \psi_{K,j}^{(r)}(x)
\]

\[
R = a_2 + \left( n_1 a_1 + (n_2 N - j) a_2 + n_2 \right)
\]

\[
= \exp \left\{ -i \frac{K \cdot R}{2} \right\} \psi_{K,j}^{(r)}(x) \quad \forall j = 0, 1, \ldots, N-1
\]

(6.39)

Analogous to the property (6.36) we have

\[
\tau(R') \psi_{K,j}^{(r)}(x) = \exp \left\{ -i \frac{K \cdot R'}{N} \right\} \psi_{K,j}^{(r)}(x) \quad \forall R' \in \mathbb{R}
\]

(6.40)

Proof is exactly same as that of (6.36).
Summarizing, we have found the Magnetic Bloch functions — symmetry adapted functions for Bloch electron in homogeneous magnetic field — to be given by the formula:

\[
\psi_{k,j}(x) = \xi \exp \left\{ i K \cdot x \right\} \left[ \sum_{j'} u_{k,j'}^*(x) \right] ; j = 0, 1, \ldots, N-1
\]

\[
u_{k,j}^*(y) = \sum \exp \left\{ -i \pi \frac{m_1 y}{a_1} \frac{m_2 y}{a_2} \right\} \exp \left\{ -i \frac{\beta}{2} A \cdot R \cdot x \right\}
\]

\[
R = (n_1 a_1 + (n_2 N - \frac{k}{N}) a_2 + n_3 a_3)
\]

\[
\exp \left\{ -i K \cdot (y + R) \right\} \left[ \sum_{j'} u_{k,j'}(y) \right]
\]

\[
u_{k,j}^*(y) = u_{k,j+y}(y) \quad \forall y \in \mathbb{Z}
\]

where \( \beta = \frac{e B}{\hbar c} = \frac{\omega \tau \Gamma}{\hbar c} \) and \( K \) varies in the Magnetic Brillouin Zone which is \( \frac{1}{N^2} \) times the usual Brillouin Zone and in which the points in the \((\mathbf{k}, \mathbf{G})\) plane differ by only \( \frac{1}{N}(k_1 a_1 + k_2 a_2), 0 \leq k_1, k_2 < 1 \). Also as proved earlier the energy band function obeys,

\[
E_n (k + G_m) = E_n (k) + G_m = \frac{1}{N} (m_1 k_1 + m_2 k_2 + m_3 k_3)
\]

\[m_1, m_2, m_3 \in \mathbb{Z}\]

In the limit \( \beta = 0 \), we can choose \( s = 0, N-1 \), and there is only one function corresponding to \( j = 0 \) (6.41) shows that this is exactly the Bloch function.
Now substituting (6.41) in the Schrödinger equation (6.14) the equation for \( u_{k,j}^*(x) \) is easily to be seen exactly similar to (6.11).

\[
\left[ \frac{1}{2m} \left( -\frac{\hbar^2}{\kappa} \right) + \frac{e}{c} A \cdot \mathbf{E} + V(x) \right] u_{n,k,j}^*(x) = E_n(k) u_{n,k,j}^*(x)
\]

\( j = 0, 1, \ldots, N-1 \)

\( j \) is only a degenerate index and is not involved in energy value.

Finally let us have a closer look at the functions (6.41). They can be written in the following fashion bringing out their remarkable similarity with the usual Bloch functions

\[
\psi_{k,j}^*(x) = e^{\sum_{R} \exp \left\{ -\frac{i}{\hbar} \phi(x, -y, n, \mathbf{a}, n_2 \mathbf{a}_2, n_3 \mathbf{a}_3, (1+R)) \right\}}
\]

\[
\exp \left\{ -\frac{i}{\hbar} \mathbf{k} \cdot (1+R) \right\} u((1+R))
\]

where \( \phi(x, -y, n, \mathbf{a}, n_2 \mathbf{a}_2, n_3 \mathbf{a}_3, (1+R)) \) is the flux of the magnetic field through the polygon connected by the vectors \((1+R)\).

Thus we see that except for the change in the range of values and introduction of a phase function these magnetic Bloch functions have not been realized in this remarkable version. Thus we believe that the establishment of this closest connection between the Bloch functions in field free case and in presence of magnetic field makes the above form the most suitable as starting point for study of solid state phenomena in presence of an external homogeneous magnetic field. We shall return to this point in the conclusion.

(iii) Free electrons in homogeneous magnetic field - Landau Levels:

The case of free electrons in homogeneous magnetic field corresponds to the limit \( V(x) = 0 \). Then the magnetic translation
group becomes continuous with $\mathbb{R}$ in $\mathfrak{r}(\mathbb{R})$ varying over the entire space. Letting $\mathbb{R} = \frac{e}{h c}$ to be in the $a_3$ direction as before the commutation relations between the magnetic translation operators become

$$\tau(x)\tau(x') = \exp \frac{j}{\mathbb{R}}(x \wedge x') \cdot \mathbb{R}^3 \tau(x')\tau(x)$$

where we have replaced $\mathbb{R}$ by $\mathbb{X}$ to denote their continuous nature. Thus for any $\mathbb{X}$ the phase factors $\exp \frac{j}{\mathbb{R}}(x \wedge x') \cdot \mathbb{R}^3$ vary continuously taking all values on the unit circle in complex plane. If we have relations of the type

$$UV = \exp(2\pi i \eta)UV$$

then taking determinants on both sides

$$\det U \det V = \exp(2\pi i \eta d) \det V \det U$$

$d$ being the dimension of the representation. If $\eta$ is such that $\exp(2\pi i \eta d) \neq 1$ for any $d < \infty$ then $U$ and $V$ must have infinite dimensions. Hence in (6.42) also, since the phase factors $\exp(-i \mathbb{X} \wedge \mathbb{X}' \cdot \mathbb{R})$ assume all values when $\mathbb{X}$ and $\mathbb{X}'$ vary, for many values of $\mathbb{X}$ and $\mathbb{X}'$ this condition happens and hence dimension of representation of the magnetic translation group is $\infty$. Thus the parameter $N \rightarrow \infty$ (6.43) due to $n_1, n_3$ being continuous and $N = \infty$ the summation over $n_1, n_3$ is to be replaced by integration over them and $n_2$ has to be removed. Also since $N = \infty$, the Magnetic Brillouin zone shrinks to the origin in the $(\mathbf{a}_1, \mathbf{a}_2)$ plane and hence the different representations are specified by only the 3rd component for $\mathbf{K}$. This third component $k_3$ of $\mathbf{K}$ can take all values from $-\infty$ to $+\infty$ due to the continuous
nature of the group. This can be seen from the analogous case for discrete group in equation (6.37) considering an one parameter continuous Abelian group the representations are given by (6.35)
\[
\Gamma_k : \{ \Gamma_k = \left\{ \exp i k \cdot x \right\} | -\infty \leq x \leq \infty \}
\]
\[
\Gamma_{k+g} : \{ \Gamma_{k+g} = \exp \left( i (k+g) \cdot x \right) \} | -\infty \leq x \leq \infty \}
\]
It is obvious \( \Gamma_{k+g} \neq \Gamma_k \) for any value of \( g > 0 \) since there is no solution to the condition \( \exp (i g \cdot x) = 1, \forall x, g > 0 \). Hence using (6.41), and taking the primitive vector \( a_1, a_2, a_3 \) as unit orthogonal vectors,
\[
\Psi_{k_3, j}(x) = \xi \exp \left( i k_3 x_3 \right) x
\]
\[
\int \int d x'_1 d x'_3 \exp \left( \frac{i}{2} \beta x_2 (x_1 + x'_1) \right) \exp \left( \frac{i}{2} \beta x_2 (x_1 + x'_1) \right) \exp \left( -i k_2 (x_1 + x'_1) \right) \psi(x_1 + x'_1, x_1 - y, x_2 + x'_3)
\]
\[
= \xi \exp \left( i k_3 x_3 - \frac{1}{2} \beta x_2 x_1 + i j \beta x_1 \right) x
\]
\[
\int \int d x'_1 d x'_3 \exp \left[ \frac{i}{2} \beta x_2 (x_1 + x'_1) \right] \exp \left[ \frac{i}{2} \beta x_2 (x_1 + x'_1) \right] \exp \left( -i k_2 (x_1 + x'_1) \right) \psi(x_1 + x'_1, x_1 - y, x_2 + x'_3)
\]
The integrations over \( x'_1, x'_3 \) wipe out the \( x_1, x_3 \) dependence leading to
\[
\Psi_{k_3, j}(x) = \xi \exp \left( i k_3 x_3 + i j \beta x_1 \right) \exp \left( -\frac{1}{2} \beta x_2 x_2 \right) \phi(x_2 - j)
\]
where \( \phi(x, y) \) is an arbitrary function of \((x, y)\) only. The jth function was obtained in (6.37) by the operation of
\[ \epsilon \exp (i k \cdot y a / 2) \]
where \( k \) is an integer. But in this case \( k \) can assume all values since \( \epsilon \) is a symmetry operator for all \( -\infty \leq k \leq \infty \). Hence \( k \) is a continuous index. Since \( k \) is only a label for the rows of the matrix representation it is a degenerate index with respect to eigenvalue. Writing \( k' = k_1 \), we get the familiar Landau solution
\[ \Psi_{k_3, k_1}(x) = \psi \exp \left\{ \frac{i}{2} \beta x \cdot x_3^2 \right\} \left[ \phi(x, y, k_1) \right] \]
(6.47)
where \( \phi(x, y, k_1) \) is determined by requiring \( \psi \) to be the solution of the Schrödinger equation for free electron in homogeneous magnetic field
\[ \frac{1}{2m} \left( \frac{\hbar}{c} \right)^2 \phi(x) = \frac{e}{\hbar} \epsilon \cdot x \phi(x) - \frac{1}{2m} \left( \frac{\hbar}{c} \right)^2 \phi(x) \]
(6.48)
The results are well known Landau levels
\[ E_{k_3, k_1} = \left( n + \frac{1}{2} \right) \hbar \omega + \frac{\hbar^2 k_1^2}{2m} \]
(6.49)
\[ \omega = \frac{eB}{mc} \]
Thus we have demonstrated the correct behaviour of the magnetic Bloch functions (6.41) in both the limiting cases of
\[ \epsilon = 0 \quad \text{and} \quad V(x) = 0 \]
iv. Irrational Magnetic Fields.

The magnetic field is said to be irrational if
\[ \lambda = \frac{e}{\hbar c} = \frac{\sqrt{11} \eta a_3}{\Omega} \]
(6.50)
and $\eta$ is irrational. Let us use formally the relation (6.41) to construct the symmetry adapted functions in this case, simply replacing $\frac{N}{\Omega}$ by $\eta$. Also where ever $N$ occurs explicitly it has to be put equal to $\infty$. This is seen from noting that the irreducible representations of generators $\Gamma_i$'s have to be of infinite dimension due to their commutation relation $\Gamma_i \Gamma_j = \exp(-i\pi \eta) \Gamma_j \Gamma_i$. This makes the summation over $\eta_2$ disappear. Only here one gets into serious doubts. We shall explore the implications of this assumption. Certainly $\Omega$ cannot be made a finite integer $< \infty$, however large one may think of it. Then

$$
\psi_{K,j}(I) = \xi \exp \left\{ i \frac{K}{2} \cdot I \frac{3}{2} \right\} u^*_{K,j}(I)
$$

$$
u^*_{K,j}(I) = \exp \left( \frac{1}{2} \beta \cdot I \cdot \alpha \right) \times \sum \exp \{ i \Gamma_1 \Gamma_2 \eta \eta \} \exp \left( -\frac{i}{2} \beta \cdot I \cdot \alpha \right) \times \sum \exp \left( -i \frac{K}{2} \cdot (I + R) \right) \nu(I + R)
$$

$$
R = n_1 a_1 - j a_2 + n_2 a_2
$$

$$
\xi = 0, \pm 1, \pm 2, \ldots, \pm \infty
$$

(6.61)

$j$ takes only discrete values since the symmetry group $\Gamma$ is still discrete and in (6.37) only for five discrete $\eta$ values we got degenerate functions. Thus degeneracy of each level is now countable (infinitary). Only in determining the possible values of $K$ one gets into difficulty. The third component of $K$ can assume all values between 0 and $K_3$ and two representations with $K$ and $K'$ would be equivalent if $K - K' = m_3 K_3$, $m_3 \in \mathbb{Z}$. This follows from the fact
that the set of magnetic translation operators in the third
direction \( \pi (n_3 \sigma) \) form a discrete Abelian group and commute
with all the operators of the entire group \( T \). The above func-
tions correspond to the representations, in analogy with (6.30)

\[
\tau_K(R) = \exp(i\pi n_1 n_2) \exp(iK \cdot R) b_{\alpha \eta}^n c_\eta^{n_2}
\]

(6.32)

with

\[
B_{\alpha \eta}^n = \begin{pmatrix}
1 & \exp(i\eta K \cdot R) \\
0 & \exp(i2\eta K \cdot R)
\end{pmatrix}
\]

(6.33)

both of \( \infty \times \infty \) dimensions and \( c_\infty \) is thought of as \( \lim_{N \to \infty} C_N \).

Then \( c_\alpha b_{\alpha \eta}^n = \exp(i\eta K \cdot R) b_{\alpha \eta}^n c_\alpha \). One can think of the algebra
of \( \infty \times \infty \) dimensional discrete indexed matrices to be spanned by a
set of matrices \( \{ C_\alpha^R B(R) \}_{0, \pm 1, \pm 2, \cdots} \) where

\[
C_\infty^{-k} = (C_\alpha^{-1})^R = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}^R \quad C_\infty^k = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}^k
\]

(6.34)

\[
B(l) = \begin{pmatrix}
\exp(2\pi i l) & 0 \\
0 & \exp(4\pi i l)
\end{pmatrix}
\]

(6.34)
Then any \( a \times a \) matrix \( \mathbf{S} = \int d\mathbf{r} \sum_{\kappa} A_\kappa(r) C_\kappa B_\kappa(r) \) is finding all possible values of \( \mathbf{S} B_\eta \mathbf{S}^{-1} \mathbf{B}_\eta^{-1} \) and \( \mathbf{S} C_\eta \mathbf{S}^{-1} \mathbf{C}_\eta^{-1} \) varying \( \mathbf{S} \) over such matrices which make these quantities scalars (i.e., restricting \( \mathbf{S} \) to the set \( \{ \mathbf{C}_\eta \mathbf{B}(r) \mathbf{C}_\eta^{-1} \mid \kappa = 0, 1, \ldots, \infty, 0 \leq \kappa \leq 1 \} \) as proved in IIInd Chapter in a similar situation) we find that two sets \( \{ \exp(i\mathbf{k} \cdot \mathbf{r}) \mathbf{B}_\eta \mathbf{B}_\eta \}, \exp(i\mathbf{k} \cdot \mathbf{r}) \mathbf{B}_\eta \mathbf{B}_\eta \} \) \( \{ \exp(i\mathbf{k} \cdot \mathbf{r}) \mathbf{B}_\eta \mathbf{B}_\eta \}, \exp(i\mathbf{k} \cdot \mathbf{r}) \mathbf{B}_\eta \mathbf{B}_\eta \} \) will be equivalent if \( \mathbf{k}^\prime = \mathbf{k} + (m + \lambda) \mathbf{k}_1 + \lambda \mathbf{k}_2 \) where \( \lambda \) is any real number and \( \mu \) is any linear sum of elements of the set \( \{ \mathbf{v} \text{ (mod 1)} \} = m \mathbf{\eta} \mid m \in \mathbb{Z} \} \) with integer coefficients. Thus we should expect in this case

\[
E_{n\kappa}(\mathbf{k} + (m + \lambda) \mathbf{k}_1 + \lambda \mathbf{k}_2 + m_3 \mathbf{k}_3) = E_{n\kappa}(\mathbf{k}) \tag{6.58}
\]

where \( m, m_3 \) are any integers, \( \lambda \) is any real number and \( \mu \) is any linear sum of the elements of the set \( \{ \mathbf{v} \text{ (mod 1)} = m \mathbf{\eta} \mid m \in \mathbb{Z} \} \) with integer coefficients. This shows that \( \mathbf{k} \) does not contain the second component just like in the free electron in homogeneous magnetic field. Thus it is seen that when the field becomes irrational the magnetic Brillouin zone assumes very odd shape described by (6.58).

IV. Conclusion.

The important message of this chapter is that in presence of an external homogeneous magnetic field \( \mathbf{B} \) the group theoretical problem requires the eigenfunctions to be of the following form which
we shall call Magnetic Bloch function

$$\psi_{k,j}(\mathbf{r}) = \frac{1}{\sqrt{V}} \exp(i\mathbf{k} \cdot \mathbf{r}) \sum_{\mathbf{R}} \exp \left\{ -i \left[ \mathbf{k} \cdot (\mathbf{r} + \mathbf{R}) + \frac{\mu}{\hbar c} \mathbf{A}(\mathbf{r} + \mathbf{R}) \right] \right\} \Psi(\mathbf{r} + \mathbf{R})$$

Method of calculation of approximate energy band function $E_n(k)$

for many crystals are mainly based on choosing trial functions to satisfy the boundary conditions implied by the symmetry of the Hamiltonian. So usually one constructs function which are to be of the Bloch type. Hence Bloch functions are important starting points for understanding many phenomena. So far in the case of solids under a homogeneous magnetic field, the magnetic Bloch functions have not been in a form bringing out the similarity with the usual Bloch function in a striking fashion so that its usage as a substitute for Bloch functions in dealing with problems in presence of magnetic field can be developed extensively. For example in the tight binding approximation method of calculation of energy bands in solids the arbitrary function $\Psi(\mathbf{r})$ in the Bloch seen is taken to be one of the core orbitals. Then calculating the average value of the Hamiltonian with this trial function one gets a picture of the band spreading of the core energy levels as the effect of the crystalline field. Though each type of approximation is valid only under certain circumstances one has to be satisfied with the best of that one can get out of these since an exact solution of the Schrödinger equation is impossible.

Hence in replacing tight binding method in presence of magnetic field one can take $\Psi(\mathbf{r})$ to be a core orbital of the individual atom in presence of that magnetic field and calculate the band energies. Similarly one can perhaps try to adapt other methods also with the striking similarity between the usual Bloch functions and magnetic
Bloch functions providing a guide to such developments. Further developments in these directions are in progress and successful results would be published elsewhere. Due to lack of space I would conclude this chapter omitting the discussions on certain other phenomena such as motion of free electron wave packets in homogeneous magnetic fields etc., which I had planned earlier to include.

**Summary of important points.**

According to Bloch theorem the wave functions of an electron in a periodic potential \( V(x) \), obeying the Schrödinger equation

\[
\left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right\} \psi(x) = E \psi(x) , \quad V(x+R) = V(x); \quad \forall R \in L.
\]

can be represented as

\[
\psi_k(x) = \xi \exp(i k \cdot x) U_k(x)
\]

where \( U_k(x) \) is a periodic function

\[
U_k(x+R) = U_k(x)
\]

This property of the eigenfunctions arises from the fact that the Hamiltonian \( \mathcal{H}_0 \) is invariant under the lattice translation group of operators

\[
T \equiv \{ T(R) = \exp\left(\frac{i}{\hbar} \mathbf{R} \cdot \mathbf{P}\right) \mid \forall \mathbf{R} \in L^3 \}
\]

When there is an external magnetic field the so called magnetic translation group given by

\[
\mathcal{T} \equiv \{ T(R) = \exp\left(\frac{i}{\hbar} \mathbf{R} \cdot (\mathbf{L} - \frac{e}{c} \mathbf{A})\right) \mid \forall \mathbf{R} \in L^3 \}
\]
becomes isomorphic to what we have called Generalized Clifford group. Using this fact we construct the representations using the theory developed in earlier chapters and by the use of standard projection operator techniques the symmetry adapted functions are derived in the following form, called magnetic Bloch functions

\[ \Psi_{k,j}(\mathbf{r}) = \xi \exp(\mathbf{i} \mathbf{k} \cdot \mathbf{r}) \sum \exp \left\{ -\mathbf{i} \left[ \mathbf{k} \cdot (\mathbf{r} + \mathbf{R}) + \frac{e}{\hbar c} \Phi(\mathbf{r}, \mathbf{R}) \right] \right\} \mathbf{r} \]

\[ \Phi(\mathbf{r}, \mathbf{R}) = \begin{bmatrix} n_1 a_1 + (n_2 N - j) a_2 + n_3 a_3 \\ n_1 a_1 + (n_2 N - j) a_2 + n_3 a_3 \end{bmatrix} \]

\[ U(\mathbf{r} + \mathbf{R}) \]

\[ j = 0, 1, \ldots, N-1 \] when \( \mathbf{R} = \frac{\hbar c}{e \hbar} \frac{2 \pi n}{N} \mathbf{a}_3 \)

\[ \Phi(\mathbf{r}, \mathbf{R}) = \text{flux of the field } \mathbf{B} \text{ through the polygon of vectors} \]

\[ (x, -y a_2, n_1 a_1, n_2 N a_2, n_3 a_3, -(x + \mathbf{R})) \]

\[ U(\mathbf{r}) \text{ is an arbitrary function of } \mathbf{r} \]

\[ K = k_1 a_1 + k_2 a_2 + k_3 a_3, \quad 0 \leq k_1 \frac{1}{N}, 0 \leq k_3 \leq 1. \]

\[ {\text{Rational magnetic field}} \quad \mathbf{B} = \frac{\hbar c}{e} \frac{2 \pi n}{N} \mathbf{a}_3 \quad \text{and irrational field} \]

\[ \mathbf{B} = \frac{\hbar c}{e} \frac{2 \pi n}{l} \mathbf{a}_3 \]

are discussed in a unified fashion and the shapes of the Brillouin zones for both cases are derived. The form
of the magnetic Bloch functions given above has not been noticed so far and its striking similarity to the usual Bloch functions suggests a generalisation of the methods of energy band structure calculations by simple appropriate replacement of the role of Bloch function in them. The results for irrational magnetic field case are explorative and tentative due to certain probably correct assumptions made in view of the inherent mathematical difficulties.

\[
L = \sum_{i=1}^{A} \mathbf{L}_i
\]

are given in algebraic notation of \( \mathbf{C} \), where the one-electron equations are

\[
L_x \mathbf{E}_y + L_y \mathbf{E}_x = \delta_{y,x} \mathbf{I}, \quad \mathbf{I} = \mathbf{1} \mathbf{1} \mathbf{1}
\]

These equations are transformed by the relations

\[
L = L_x \mathbf{L}_x + L_y \mathbf{L}_y + L_z \mathbf{L}_z
\]

This chapter is based on a recent paper written by us in collaboration with (1988).
CHAPTER 7.

A NEGATIVE ENERGY RELATIVISTIC WAVE EQUATION

We shall demonstrate that by a simple change in the structure of the internal variables of Dirac's positive energy relativistic wave equation\(^1\), we can arrive at a negative energy relativistic wave equation admitting only negative energy solutions. To achieve this we shall use a simple fact of Clifford algebra, studied in relation to an eigenvalue problem of Alladi Ramakrishnan\(^2\). The eigenvalue of an eigenvector of the matrix

\[
L = \sum_{i=1}^{4} x_i L_i
\]

where \(L_i\) are 4 x 4 matrices generators of \(\mathbb{C}_{+}^{2}\), obeying the anticommutation relations

\[
L_i L_j + L_j L_i = \delta_{ij} I, \quad i, j = 1, 2, 3, 4.
\]

Changes sign from + to - when the eigenvector is transformed by the matrix

\[
L_5 = L_1 L_2 L_3 L_4
\]

This chapter is based on a recent paper written by me in collaboration with Dutte\(^3\).

(4) Dirac's positive energy relativistic wave-equation

In 1928 Dirac\(^1\) proposed a positive energy relativistic wave equation for particles of non-zero rest mass, allowing only positive values for the energy unlike the usual relativistic wave equations, originating in Dirac's famous equation of 1928\(^4\), which are symmetrical
between positive and negative energies. As has been pointed out by Dirac, the new equation formally very similar to the old equation for the electron but the physical significance is very different. In particular he has shown that the new equation gives integral values for the spin.

The new wave equation for a particle with unit rest mass reads

\[
\left( \frac{\partial}{\partial x_0} + \sum_{\gamma=1}^{3} \frac{\partial}{\partial x_{\gamma}} \right) (\gamma) \psi = 0
\]

(taking \( \hbar = c = 1 \))

\[
(\gamma) \psi = 0 ; \quad \gamma, l = 1, 2, 3.
\]

\[
\beta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}
\]

\[
(\gamma) = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix}
\]

\(
(\gamma_1, \gamma_2 = p_1, \gamma_2 = p_2 = q_4)
\) are two pairs of dynamical variables describing internal degrees of freedom involving two harmonic oscillators and having the commutation relations

\[
[\gamma_a, \gamma_b] = \gamma_a \gamma_b - \gamma_b \gamma_a = i \beta_{ab} \]

\( \alpha, \beta = 1, 2, 3, 4. \)
and the wave function \( \psi \) is a one component function of two commuting \( q_0 \), say \( q_1 \) and \( q_2 \), as well as of the four \( x_i \). Thus \( \psi \) is a column matrix with four elements \( a_1 \psi, a_2 \psi, a_3 \psi, a_4 \psi \) and the \( 4 \times 4 \) matrices \( a_k \) are to be multiplied into this column matrix in the usual way.

Putting \( \partial^\mu \equiv \partial/\partial x^\mu \) and \( a_0 = 1 \), the wave equation (7.1) can be written exactly as

\[
\left( \sum_{\mu=0}^3 \alpha^\mu \partial^\mu + \beta \right) \psi = 0 \tag{7.10}
\]

or putting \( i \partial^\mu = p^\mu \), these become

\[
(p_0 - \sum_{\gamma=1}^3 \alpha^\gamma p_\gamma + \xi \beta) \psi = 0 \tag{7.12}
\]

(nuice used here is \( 1, -1, -1, -1 \)) with a particular choice of \( \alpha^\gamma \) as

\[
\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\tag{7.12}
\]

These equations become

\[
\begin{pmatrix}
\begin{pmatrix} p_0 & -p_1 & -p_2 & p_3 \\ 0 & p_0 & p_2 & p_1 \end{pmatrix} & \begin{pmatrix} a_1 \psi \\ a_2 \psi \\ a_3 \psi \\ a_4 \psi \end{pmatrix} = p_0 \begin{pmatrix} a_1 \psi \\ a_2 \psi \\ a_3 \psi \\ a_4 \psi \end{pmatrix}
\end{pmatrix}
\tag{7.13}
\]

Taking a solution corresponding to an eigenstate of momentum and energy we can consider \( p_0, p_1, p_2, p_3 \) in (7.13) as real numbers.
obeying the equation \( \sum_{\mu=0}^{3} p_{\mu} = 1 \), which is the relativistic condition naturally arising out of a necessary condition on the solution \( \psi \) that it must satisfy the de Broglie equation \( \left( \frac{3}{2} \sum_{\mu=0}^{3} \partial_{\mu} d^{\mu} \right) + 1 \psi = 0 \) for all values of its internal variables (cf. Dirac's solution).

Hence \( p_{0} = \pm \sqrt{1 + p_{1}^{2} + p_{2}^{2} + p_{3}^{2}} = \pm |p_{0}| \). Taking \( p_{0} = \pm |p_{0}| \), putting \( \psi(q_{1}, x) = \phi(q_{1}) e^{i x \mu} (-\frac{2}{\sum_{\mu=0}^{3} p_{\mu} x_{\mu}}) \), we get

\[
(\frac{\partial}{\partial q_{1}}) \phi(q_{1}) = u_{1}^{\dagger} \xi_{1} + u_{2}^{\dagger} \xi_{2},
\]

(7.14)

where \( u_{1}^{\dagger} \) and \( u_{2}^{\dagger} \) are two independent eigenvectors of \([D]\) both corresponding to the eigenvalue \( +|p_{0}| \) and \( \xi_{1}, \xi_{2} \) are arbitrary functions of \( q_{11} \) and \( q_{12} \) only. It is to be remembered that the matrix \([D]\) satisfies \([D] = |p_{0}| I\) and has corresponding to each eigenvalue \( +|p_{0}| \) and \( \pm |p_{0}| \) two linearly independent eigenvectors, since \([D]\) is hermitian and hence diagonalizable.

Taking

\[
u_{1}^{\dagger} = \begin{pmatrix}
1 & 0 \\
0 & \frac{-p_{1} + i}{p_{2}}
\end{pmatrix}
\]

\[
u_{2}^{\dagger} = \begin{pmatrix}
1 & 0 \\
0 & \frac{p_{2}}{p_{1} - i}
\end{pmatrix}
\]

(7.15)

we get a set of equations, taking explicitly \( q'_{1} = -i \frac{\partial}{\partial q_{1}} \), \( q'_{2} = -i \frac{\partial}{\partial q_{2}} \),

\[
q_{11} \psi = (|p_{0}| + p_{3}) \xi_{1}, \quad q_{12} \psi = (|p_{0}| + p_{3}) \xi_{2},
\]

\[
-\frac{i}{\delta q_{1}} \frac{\partial \psi}{\partial q_{1}} = (-p_{1} + i) \xi_{1} + p_{2} \xi_{2}, \quad -\frac{i}{\delta q_{2}} \frac{\partial \psi}{\partial q_{2}} = p_{2} \xi_{1} + (p_{1} + i) \xi_{2}.
\]

(7.16)

Substituting \( \xi_{1} = q_{11} \psi / (|p_{0}| + p_{3}) \), \( \xi_{2} = q_{12} \psi / (|p_{0}| + p_{3}) \) in the second set and integrating, one readily gets Dirac's solution.
\[
\Psi(q) = K \exp \left\{ -\frac{1}{2} \left[ q_1^2 + q_2^2 + i \vec{p}_1 \cdot (q_1 - q_2) - 2i \cdot p_2 \cdot q_1 \cdot q_2 \right] \right\} \times (|p_0| + |p_3|)^{-\frac{1}{2}} \]

(7.17)

and hence

\[
\Psi_+ = K \exp \left\{ -\frac{1}{2} \left[ q_1^2 + q_2^2 + i \vec{p}_1 \cdot (q_1 - q_2) - 2i \cdot p_2 \cdot q_1 \cdot q_2 \right] \right\} \times (|p_0| + |p_3|)^{-\frac{1}{2}} \exp \left\{ -\frac{3}{4} \sum_{\mu=0}^{2} p^\mu \cdot x^\mu \right\} \]

(7.18)

Corresponding to the negative energy eigenvalue \(-|p_0|\) we can proceed similarly by taking \(u_-^t\) and \(u_-^s\) instead of \(u_+^t\) and \(u_+^s\) in (7.14). It is seen that \(u_-^t\) and \(u_-^s\) are given by simply changing from \(+|p_0|\) to \(-|p_0|\) in (7.15) leading thus to the negative energy solution

\[
\Psi_- = K \exp \left\{ -\frac{1}{2} \left[ q_1^2 + q_2^2 + i \vec{p}_1 \cdot (q_1 - q_2) - 2i \cdot p_2 \cdot q_1 \cdot q_2 \right] \right\} \times (|p_0| + |p_3|)^{-\frac{1}{2}} \exp \left\{ -\frac{3}{4} \sum_{\mu=0}^{2} p^\mu \cdot x^\mu \right\} \]

(7.19)

when energy is positive \((|p_0| + |p_3|) > 0\) always and \(\Psi_+\) is normalizable and hence physically permissible. But when energy is \(-ve\), \((-|p_0| + |p_3|) < 0\) always making \(\Psi_- \sim \exp \left\{ \frac{1}{2} (q_1^2 + q_2^2) \right\}\) which is not normalizable and not physically permissible.

(ii) A negative energy wave equation

Now we shall derive the negative counterpart of the above equation which will admit only negative energy solution as physically
Following (7.3) let us define
\[ \Gamma = \alpha_1 \alpha_2 \alpha_3 \beta = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \]
(7.39)

Then \[ \Gamma^2 = -I \] and \[ \Gamma [D] \Gamma^{-1} = -[D] \] where \[ [D] = \begin{pmatrix} \sum_{\gamma=1}^{3} \alpha_\gamma r_{\gamma} \beta \end{pmatrix} \]
and the wave equation (7.11) is written as
\[ [D] (\Psi) \psi = p_0 (\Psi) \psi \]
(7.20)

Hence
\[ \Gamma [D] \Gamma^{-1} (\Psi) \psi = +p_0 (\Psi) \psi \]

or
\[ [D] (\Gamma (\Psi)) = -p_0 (\Gamma (\Psi)) \]
(7.21)

This shows that if \( \Psi \) satisfies (7.21) corresponding to an energy eigenvalue \( p_0 \), then \( \Gamma (\Psi) \) satisfies (7.21) corresponding to an energy eigenvalue \( -p_0 \). Thus by a transformation \( \Psi \rightarrow (\Psi) = (k) \)
we arrive at an equation similar to (7.21) or (7.4) but with only negative energy solution being normalizable and hence physically permissible. This negative energy relativistic wave equation would then read
\[ (p_0 - \sum_{\gamma=1}^{3} \alpha_\gamma r_{\gamma} + \beta^2) (k) \psi = 0 \]
(7.30)

or which is same as
\[ \left\{ \frac{d}{dx_0} + \sum_{\gamma=1}^{3} \alpha_\gamma \frac{d}{dx_\gamma} + \beta^2 \right\} (k) \psi = 0 \]
(7.31)

where
\[ (k) \equiv (\Gamma (\Psi)) = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} \]
(7.32)
Now \( k^2 \) obey

\[
[k_{a1}, k_{b1}] = -i \beta_{a1} b_{b1}, \quad a, b = 1, 2, 3, 4.
\] (7.30)

with the same matrix \( \beta \). Hence (7.34) would describe a particle with only negative energy states, when the internal variables obey the commutation relations (7.36) instead of (7.9). Perhaps this may be physically interpreted that the internal oscillators are moving backward in time. Thus the wave equation

\[
\left\{ \frac{\partial}{\partial x_0} + \sum_{\gamma=1}^{3} \alpha_{\gamma} \frac{\partial}{\partial x_{\gamma}} + \beta \right\} (q) \psi = 0
\] (7.37)

with

\[
[v_{a1}, v_{b1}] = i \beta_{a1} b_{b1} ; \quad \beta = \begin{pmatrix}
8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-200 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

would describe a particle with only positive energy states if \( \varepsilon = +1 \) (Dirac) and would describe a particle with if only negative energy states if \( \varepsilon = -1 \). The solutions of the negative energy equation are given by, exactly similar to (7.18) and (7.19) except for the interchange their roles,

\[
\psi_\sigma = K \exp \left\{ -i \frac{1}{2} \left( q_1^2 + q_2^2 + i \left[ q_1 - q_2 \right] - 2x_1 p_1 q_1 q_2 \right) \right\} \exp \left\{ -i \sum_{\mu=0}^{3} \frac{1}{\mu!} \left( q_{1\mu} \right) \mu \frac{q_1^2 + q_2^2}{\mu^2} \right\}
\] (7.38)

\[
\psi_+ = K \exp \left\{ -i \frac{1}{2} \left[ q_1^2 + q_2^2 + i \left( q_1^2 - q_2^2 \right) - 2x_1 p_2 q_1 q_2 \right] \right\} \exp \left\{ -i \sum_{\mu=0}^{3} \frac{1}{\mu!} \left( q_{1\mu} \right) \mu \frac{q_1^2 + q_2^2}{\mu^2} \right\}
\] (7.39)

Clearly it is seen that only \( \psi_\sigma \) is normalizable and hence physically permissible. Starting with the equation (7.31) with \( q_j \)'s obeying now
Instead of (7.30),

\[ [q_a, q_b] = -i \beta_\alpha \delta_{\alpha \beta} \quad ; \quad \alpha, \beta = 1, 2, 3, 4. \]  \tag{7.30}

Using an explicit representation

\[ q_3 = i \frac{\partial}{\partial q_1} ; \quad q_4 = \frac{i}{2} \frac{\partial}{\partial q_2} \]  \tag{7.31}

the solutions (7.28) and (7.29) can be obtained in exactly similar in manner to the derivation of (7.18) and (7.19) and hence we do not repeat this procedure of solution of the negative energy equation here.

(iii) Relativistic invariance and spin

We shall follow Dirac in discussing the relativistic invariance of the equation (7.31). It is seen that both cases \( \alpha = \pm 1 \) can be treated simultaneously in the same manner as follows. Equation (7.20) can be written as

\[ \left( \sum_{\mu = 0}^{3} \gamma_\mu \delta_{\mu}^{M-1} \right) (\psi) \psi = 0 \]  \tag{7.32}

where \( \gamma_\mu = \beta \alpha_\mu \) and \( \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2 \sigma_\mu \nu \); \( \mu, \nu = 0, 1, 2, 3 \). (7.32) is not manifestly relativistic since the four \( \gamma_\mu \) cannot be regarded as a 4-vector as \( \gamma_0 \) is skew while \( \gamma_1, \gamma_2, \gamma_3 \) are symmetrical. Hence let us consider the original form (7.10) in the satisfying

\[ \alpha_\mu \beta \alpha_\nu + \alpha_\nu \beta \alpha_\mu = 2 \beta \sigma_\mu \nu \]; \( \mu, \nu = 0, 1, 2, 3 \). \tag{7.33}

Applying on infinitesimal Lorentz Transformation

\[ x^*_\mu = x_\mu + \sum_{\nu = 0}^{3} a_\mu \delta \nu x_\nu \]  \tag{7.34}

\[ \delta x^*_\mu = \delta x_\mu + \sum_{\nu = 0}^{3} a_\mu \delta \nu \delta \nu \]; \( \mu = 0, 1, 2, 3 \).
the $\{a_{\mu\nu}\}$ being infinitesimal coefficients such that $a_{\mu\nu} = -a_{\nu\mu}$.

The equation (7.10) then gives to first order

$$\left\{ \sum_{\mu=0}^{3} a_{\mu} \left( \partial_{\mu}^* - \frac{2}{3} a_{\mu} \partial_{\nu}^* \right) \right\} \psi = 0$$

or

$$\left\{ \sum_{\mu=0}^{3} \left( a_{\mu} + \frac{2}{3} a_{\mu} a_{\nu} \right) \partial_{\mu}^* \right\} \psi = 0$$

Defining

$$N = \frac{1}{4} \sum_{\sigma, \tau=0}^{3} a_{\rho} \beta a_{\sigma}$$

$$N = N$$

we have

$$\partial_{\mu} \beta N - N \beta \partial_{\mu} = -\sum_{\nu=0}^{3} a_{\mu} a_{\nu}$$

Thus multiplying (7.36) by $(1 - N \beta)$ on the left we get

$$\left\{ \sum_{\mu=0}^{3} a_{\mu} + (1 - N \beta) \partial_{\mu}^* \right\} \psi = 0$$

or

$$\left( \sum_{\mu=0}^{3} a_{\mu} \partial_{\mu}^* + \beta \right) (1 - N \beta) \psi = 0$$

where

$$(q^*_{\sigma}) = (1 - N \beta) (q)$$

Thus the wave equation takes the same form in the new system of coordinates with $(q^*_{\sigma})$ replacing $(q)$. The four $q^*_{\sigma}$ are linear functions of the four $q_{\sigma}$ with real coefficients (on account of $a_{\mu}$ and $\beta$ having only real elements). The new $q^*_{\sigma}$ satisfy the
comutation relations

\[
[q_{\alpha}^{\dagger}, q_{\beta}^{\dagger}]=i\epsilon_{\alpha \beta} \delta_{\alpha \beta}; \quad \alpha, \beta=0,1,2,3,4.
\] (7.42)

to the first order, thus having the same properties as the \(q_1^{\dagger}\).

This shows that the form of the wave equation is unchanged by an
infinitesimal, and thus also finite Lorentz transformations not
involving reflections.

There is a unitary transformation connecting \(q_1^{\dagger}\) and \(q_1\).

Letting

\[
W = \sum_{\alpha, \beta=0}^{4} q_{\alpha} N_{\alpha \beta} q_{\beta}.
\] (7.43)

we have

\[
Wq_{\alpha} - q_{\alpha}W = -2i\epsilon \sum_{\mu=0}^{4} \gamma_\mu N_{\mu \alpha} q_{\mu}.
\] (7.44)

or

\[
W(q) - (q)W = -2i\epsilon \beta N(q)
\]

an equation in which every term is a column matrix. Then to the
first order (7.41) becomes

\[
(q_1^{\dagger}) = (1 - \frac{1}{2} \gamma_5 \epsilon W)(q) \left(1 + \frac{1}{2} \gamma_5 \epsilon W\right).
\] (7.46)

Having seen that the negative energy equation (7.37) with \(\epsilon = -1\) is
exactly similar to the positive energy equation (7.37) with \(\epsilon = +1\)
except for the change of \(\epsilon\) from \(+1 \rightarrow -1\), with regard to the
relativistic covariance we shall, without repeating the contents of
Dirac's paper, state that the spin operators are given by

\[
S_{\sigma} = \epsilon \left\{ -\frac{1}{4} \gamma_5 (\bar{q} \gamma_\sigma q) + \frac{1}{2} \gamma_5 \gamma_\sigma \right\}
\] (7.47)

Explicitly with the choice of \(\gamma^i\) in (7.12) we have

\[
S_{01} = \frac{1}{4} (q_1 q_2 - q_3 + q_4) \quad S_{12} = \frac{1}{2} (q_1 q_3 - q_2 q_4)
\]

\[
S_{02} = \frac{1}{2} (q_1 q_4 - q_2 q_3) \quad S_{23} = \frac{1}{2} (q_1 q_2 + q_3 q_4)
\]

\[
S_{03} = \frac{1}{2} (q_1 q_3 + q_2 q_4) \quad S_{31} = \frac{1}{4} (q_1 q_4 - q_2 q_3)
\] (7.49)
\[ S_{23}^2 + S_{31}^2 + S_{12}^2 = \frac{1}{16} (q_1^2 + q_2^2 + q_3^2 + q_4^2) - \frac{1}{4} \]  \hspace{1cm} (7.50)

Defining the magnitude \( s \) of spin according to quantum mechanics by

\[ s(s+1) = S_{23}^2 + S_{31}^2 + S_{12}^2 \]

we find

\[ s = \frac{1}{4} (q_1^2 + q_2^2 + q_3^2 + q_4^2) - \frac{1}{2} \]  \hspace{1cm} (7.51)

The eigenvalues of \( \frac{1}{2} (q_1^2 + q_3^2) \) and \( \frac{1}{2} (q_2^2 + q_4^2) \) are \( m + \frac{1}{2} \) and \( m' + \frac{1}{2} \), with \( m \) and \( m' \) non-negative integers. Thus eigenvalues of \( s \) are \( \frac{1}{2} (m + m') \) which are always integers. These results apply to both the case \( \varepsilon = \pm 1 \), thus showing that the case \( \varepsilon = -1 \) would perfectly well describe a particle with only negative energy states and integer spin. In both cases value of the spin \( s \) depends on the wave function and thus on the momentum of the particle. The quantities \( S_{\mu\sigma} (\mu, \sigma = 0,1,2,3) \) provide a representation of the infinitesimal operators of the Lorentz group. They are associated mathematically with four more quantities

\[ S_{\mu\nu} = -S_{\nu\mu} = \frac{1}{4} \varepsilon (\psi) d_\mu (\psi) ; \mu = 0,1,2,3 \]  \hspace{1cm} (7.52)

The ten quantities \( S_{\alpha\beta} = -S_{\beta\alpha} (\alpha, \beta = 0,1,2,3,5) \) then provide a representation of the \( 5 \times 5 \) de Sitter group as discovered by Dirac in 1958 for the case of \( \varepsilon = +1 \). We observe that in both cases of \( \varepsilon = \pm 1 \), this is true as can be verified directly.
\[ [S_{\alpha c}, S_{\alpha c}] = S_{\beta c} \quad \text{for} \quad \alpha = 0 \quad \text{for} \quad \alpha = 1, 2, 3 \]

which are the commutation relations of the infinitesimal operators of the $3 + 2$ de Sitter group, the group of rotations of five real variables $x_0, x_1, x_2, x_3, x_5$ which leave the quadratic form $x_1^2 + x_2^2 + x_3^2 - x_0^2 - x_5^2$ invariant.

I am very grateful to Professor Dirac for his comment on the equation (7.34) that it would correctly describe a particle with only negative energy states and would be the counterpart to his positive energy equation (7.4). But he feels that the equation (7.34) would not have physical application. Of course, one can not predict the failure of a theory simply because it seems improbable at a time comes when nature reveals the existence of a particle with only positive energy states described exactly by Dirac's equation should one not search for its probable counterpart?

Summary of important points

Dirac's new relativistic wave equation for a particle of unit mass ($\hbar = c = 1$) is

\[
\left( \frac{\partial}{\partial x_0} + \sum_{\gamma=1}^{3} \alpha_\gamma \frac{\partial}{\partial x_\gamma} + \beta \right) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \psi (q_1, q_2, q_3) = 0
\]

\[
\mu = 0, 1, 2, 3
\]

with

\[ [q_\alpha, q_{\beta \mu}] = \delta_{\alpha \beta} \alpha \mu; \quad \alpha, \beta = 1, 2, 3, 4 \]

and

\[
\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} ; \quad \alpha_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ; \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} ; \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]
This allows only positive energy states as physically permissible.

By changing the commutation relations of internal variables as

\[ \beta_{ab} \]

\[ [q_a, q_b] = -i \beta_{ab}, \quad a, b = 1, 2, 3, 4. \]

We observe that the same equation describes particles with only negative energy states.
CHAPTER 2

ON CLIFFORD'S COMPLEX QUATERNION ALGEBRA

In his fundamental paper "Applications of Grassman's extensive algebra" (American Journal of Mathematics Pure and Applied Vol.1, pp.360-383 (1878), W.K. Clifford\(^1\) shows that his \(2^{n+1}\) - way geometric algebra - now called the Clifford algebra \(\mathbb{C}_2^{n+1}\) is a compound of \(2^n\) quaternion algebras, the units of which are commutative with one another. In the development of 2-matrix such commuting structures, called Generalized Helicity matrices have been built by Alladi Ramakrishnan and his collaborators\(^2\). Extended version of matrix decomposition theorems due to Alladi Ramakrishnan and myself considered in Chapter II are also associated with such structures. Here we shall see that these are generalizations of Clifford result.

Hamilton's quaternions apart from being of great mathematical importance as a generalization of the field of complex numbers, have often been considered to be of importance to physics. It was recognised early that special relativity can be elegantly written in quaternion notation\(^4\). But it never gained any popularity due to the greater convenience of using tensors. With the invention of spin of electron and Dirac equation there was again a renewed attempt to introduce the quaternions\(^5\) but spinor formalism took over. Possibility of replacing complex numbers by quaternions in quantum mechanics has been studied by several authors\(^6\) and also there have been attempts to describe elementary particles by means of quaternions\(^7\).
There seems to be no where any reference to Clifford's commuting quaternion algebras. A careful study of his original paper gives a clue to the concept of tensor product and his commuting structures are most easily understood in terms of tensor products in the modern mathematical language.

According to Clifford if \( Q^y_l(q) \) and \( Q^{y'}_j(q') \) are two different types of quaternions associated with the clifford algebra then

\[
Q^y_l(q) Q^{y'}_j(q') = Q^{y'}_j(q') Q^y_l(q) \quad \text{for} \quad l \neq j, \quad \gamma = \sigma \neq \gamma',
\]

\[
Q^y_l(q) Q^{y'}_j(q') \neq Q^{y'}_j(q') Q^y_l(q) \quad \text{if} \quad \gamma \neq \gamma'.
\]  

(3.1)

where \( q \) and \( q' \) are 4-tuples \((q_0, q_1, q_2, q_3)\) and \((q'_0, q'_1, q'_2, q'_3)\) respectively.

The irreducible representations of a quaternion in terms of Pauli matrices are given by

\[
q(q) = q_0 \; e + q_1 \; i + q_2 \; j + q_3 \; k
\]

\[
= \begin{pmatrix}
q_0 & -i q_3 & -i q_2 & q_1
\end{pmatrix}
\begin{pmatrix}
q_0 & -i q_3 & -i q_2 & q_1
\end{pmatrix} = q_0 \; I + \sum_{l=1}^{3} q_l \; e_i \; e_l
\]

(3.2)

with the units \((e, i, j, k)\) obeying

\[
e^2 = i^2 = j^2 = k^2 = -1, \quad e_i e_l = \delta_{ij}
\]

(3.3)

Clifford algebra \( C_n^{(2)} \) has generating relations

\[
\begin{align*}
& e_i e_j = -e_j e_i, \quad i j = 1, \ldots, n \\
& e_i^2 = 1, \quad i = 1, \ldots, n.
\end{align*}
\]

(3.4)
Quaternion algebra is same as the Clifford algebra \( C\_2^{(2)} \) except for a difference in phase factors of the generators \( \epsilon_1, \epsilon_2 \) which can be identified as \( \epsilon_1 = i, \epsilon_2 = i, \), \( \epsilon_3 = i \). Clifford constructed algebraic structures obeying (3.1) starting from the Clifford algebra \( C\_m^{(2)} \).

Following Alladi Ramakrishnan and his collaborators, we shall consider the construction of mutually commuting subalgebras of the Generalized Clifford algebra \( C\_m^{(n)} \) which is generated by the relations

\[
\epsilon_i \epsilon_j = \omega(n) \epsilon_j \epsilon_i, \quad i, j = 1, \ldots, 2n.
\]

(3.9)

The entire basis of the algebra \( C\_m^{(n)} \) is

\[
\left\{ \prod_{i=1}^{2n} \epsilon_i^{k_i}, \quad 0 \leq k_i \leq m-1 \right\}
\]

(3.9)

Now define iteratively

\[
\epsilon_{\pm 1}^* = H_{\pm 1} = (H_{\pm 1}^\dagger)^{m-1} (H_{\pm 1}^\dagger)^{m-1} \cdots (H_{\pm 1}^\dagger)^m - 2(\nu - 1) + 1
\]

(3.10)

\[
\epsilon_{2\nu}^* = H_{2\nu} = (H_{2\nu}^\dagger)^m (H_{2\nu}^\dagger)^m \cdots (H_{2\nu}^\dagger)^m - 2(\nu - 1) + 2
\]

(3.10)

\[
H_{\pm 1}^\dagger H_{\pm 1}^\dagger = \omega(n) H_{\pm 1}^\dagger H_{\pm 1}^\dagger, \quad i = 1, \ldots, \nu.
\]

(3.7)

Each pair \( \{ H_{\pm 1}^\dagger, H_{\pm 1}^\dagger \} \) generates a subalgebra with a basis \( \{ H_{\pm 1}^\dagger, H_{\pm 1}^\dagger \} \), \( 0 \leq k_1, k_2 \leq m-1 \) and elements of one subalgebra commute with the elements of another. If we substitute the explicit matrix representations of \( \epsilon_i \) given in Chapter I, we see that

\[
H_{\pm 1} = \begin{pmatrix} \mathbf{I} & \cdots & \mathbf{I} & \mathbf{X} & \cdots & \mathbf{X} \\ \mathbf{X} & \cdots & \mathbf{X} & \mathbf{I} & \cdots & \mathbf{I} \end{pmatrix}
\]

(3.8)
and hence the subalgebras generated by \( \{ H_{i_1}^c, H_{i_2}^c \} \) \( c = 1 \cdots \nu \)
each isomorphic to \( C_{\nu}^{(m)} \). When the direct product representation is used it is easy to see that any element of \( C_{\nu}^{(m)} \) is a linear combination of matrices which are direct products of basic matrices or using (8.8) one has that any element of \( C_{\nu}^{(m)} \) is a linear combination of a set of basic matrices \( \{ \prod_{i=1}^{\nu} H_{i}^{k_{i}}, H_{j}^{l_{j}} \} \). Any such element \( \sum_{k_{i}, l_{i} = 0}^{m_{i}-1} \alpha_{k_{i}} l_{i} - k_{i} l_{j} H_{i}^{k_{i}}, H_{j}^{l_{j}} \) can be written, due to the commutativity of different pairs \( \{ H_{i}^{k_{i}}, H_{j}^{l_{j}} \} \), \( i = 1 \cdots \nu \), as

\[
\sum_{k_{i}, l_{i} = 1} \left\{ \sum H_{i}^{l_{i}}, W_{a}^{k_{i} l_{i}} \alpha_{k_{i}} l_{i}, \& l_{2} - k_{i} l_{j} \right\} \prod_{i=1}^{\nu} H_{i}^{k_{i}}, H_{j}^{l_{j}}.
\]

This means that elements of \( C_{\nu}^{(m)} \) can be considered as elements of \( C_{\nu-2}^{(m)} \) with coefficients as elements of \( C_{\nu}^{(m)} \). Again elements of \( C_{\nu-2}^{(m)} \) can be considered as elements of \( C_{\nu}^{(m)} \) with coefficients as elements in \( C_{\nu-4}^{(m)} \) and so on. Thus the algebra \( C_{\nu}^{(m)} \) can be considered as a 'compound' of \( \nu \) commuting subalgebras all isomorphic to \( C_{\nu}^{(m)} \). When \( \nu = 2 \), \( C_{2}^{(m)} \) is the quaternion algebra and hence Clifford's proposition immediately follows that \( C_{4}^{(m)} \) is a 'compound' of \( \nu \) commuting quaternion algebras. In terms of matrix representation this means that any \( m^{\nu} \times m^{\nu} \) matrix may be regarded as a compound of \( \nu \), \( m \)-dimensional matrices. First we can regard it as a \( n \times n \) matrix where its elements are \( m^{\nu-1} \times m^{\nu-1} \) matrices which are again \( n \times n \) matrices of \( m^{\nu-2} \times m^{\nu-2} \) matrices and so on. These are the well known results of partitioning of matrices. This is the basic principle behind the generalized matrix decomposition theorem considered in Chapter II.
Here generally one can consider any $N \times N$ matrix algebra as a compound of many matrix algebras of different dimensions if $N$ is a composite integer. This is achieved by representing the two basic generators of the basis of $C(N)$, say $C(N), B(N)$ obeying $C(N)B(N) = \omega(N)B(N)C(N)$ as direct product of smaller dimensional $C, B$ matrices. We shall consider such a commutative factorisation of the basis, as Schwaiger calls it, corresponding to the factorisation of $N$ as $\prod_{i=1}^{r} p_{i}^{\alpha_i}$, where $p_{i}$'s are distinct primes, and $\alpha_i > 0, \forall i = 1, \ldots, r$. We can put

\begin{align*}
C(N) &= C(p_{i}^{\alpha_i}) \otimes \cdots \otimes C(p_{r}^{\alpha_r}) \\
B(N) &= B(p_{i}^{\alpha_i}) \otimes \cdots \otimes B(p_{r}^{\alpha_r})
\end{align*}

where $C(p_{i}^{\alpha_i}) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ of dimension $p_{i}^{\alpha_i}$, $B(p_{i}^{\alpha_i}) = \begin{pmatrix} \omega(p_{i}^{\alpha_i}) & 0 \\ 0 & \omega(p_{i}^{\alpha_i})^{-1} \end{pmatrix}$

\begin{align*}
\omega(p_{i}^{\alpha_i}) &= \exp \left( \frac{2\pi i}{p_{i}^{\alpha_i}} \right) \\
C(p_{i}^{\alpha_i})B(p_{i}^{\alpha_i}) &= \omega(p_{i}^{\alpha_i})B(p_{i}^{\alpha_i})C(p_{i}^{\alpha_i}) ; \forall i = 1, \ldots, r \\
B(p_{i}^{\alpha_i})p_{i}^{\alpha_i} &= C(p_{i}^{\alpha_i})p_{i}^{\alpha_i} = 1.
\end{align*}

If $\beta_{i}$'s are chosen such that

\[ N \sum_{i=1}^{T} \beta_{i} \beta_{i}^{*} = 1 \cdot \text{mod} \cdot N. \]

then $C(N), B(N)$ obey

\[ C(N)B(N) = \omega(N)B(N)C(N) \]
\[
\prod_{i=1}^{r} \omega(L_{i}^{a_{i}})^{L_{i}} = \exp \left( 2\pi i \sum_{i=1}^{r} \frac{\beta_{i}}{L_{i}^{a_{i}}} \right) = \exp \left( \frac{2\pi i L_{i}^{a_{i}}}{N} \right) = \omega(N)
\]

Due to (2.11)

In general one can choose

\[
C(N) = \prod_{i=1}^{r} C(L_{i}^{a_{i}})^{\mu_{i}} B(L_{i}^{a_{i}})^{\nu_{i}}
\]

\[
B(N) = \prod_{i=1}^{r} C(L_{i}^{a_{i}})^{\nu_{i}} B(L_{i}^{a_{i}})^{\delta_{i}}
\]

where

\[
N \sum_{i=1}^{r} \left| \frac{\mu_{i}}{L_{i}^{a_{i}}} \right| = 1 \mod N.
\]

Since prime number decomposition of an integer is unique and the maximum possible starting from this one can easily show that if \( N = n_{1} n_{2} \cdots n_{k} \) where \( n_{i}^{a_{i}} \)'s are not necessarily powers of primes, \( C(N) \), \( B(N) \) can be expressed as invarient direct product of powers of \( C(n_{i}) \)'s and \( B(n_{i}) \)'s \( i = 1 \cdots k \). These results leads to the fact of partitioning that \( n \times n \) matrix algebra can be regarded as a sum compound of matrix algebras of dimensions \( n_{1}, n_{2} \cdots n_{k} \) if \( N = n_{1} n_{2} \cdots n_{k} \). For the corresponding C.C.A., \( C_{2}^{(N)} \) this implies that \( C_{2}^{(N)} \) can be regarded as a sum compound of commuting algebras isomorphic to \( (C_{2}^{(n_{1})}, \cdots, C_{2}^{(n_{k})}) \).

Of all these results the original result of Clifford that \( C_{2}^{(N)} \) is a compound of \( N \) commuting quaternion algebra is perhaps of some interest for physics especially for attempts to generalize quantum mechanics by enlarging the underlying number field from
complex numbers to quaternions. The principal conceptual difficulty realized in such generalizations is in the theory of composite systems where the ordinary tensor product fails due to the non-commutativity of the quaternions. This prevents the formation of a composite system in such a way that all the observables associated with one of the systems commute with all the observables of the other system. Now in the light of the fact that a Clifford algebra can be viewed as a compound of commuting quaternion algebras, perhaps it is worthwhile to investigate the possibility of using commuting quaternions in the description of independent systems and forming composite systems by usual tensor product. It should be noted that all these commuting quaternion algebras are not totally independent — there is a common thread — they can be built out of elements of a Clifford algebra.

**Summary of Important Points.**

Expressing any matrix of dimension \( N = \prod_{i=1}^{r} n_i \) in a basic given by
\[
\mathbb{C}^{(N)}_{2} \cong \mathbb{M}_{N}
\]

the total \( N \times N \) matrix algebra can be thought of as a 'compound' of \( r \) \( \mathbb{C}^{(n_i)}_{2} \) by
\[
\left\{ \mathbb{C}^{(n_i)}_{2} \right\}_{i=1}^{r}
\]

In terms of matrices this means that any, \( N \times N \) matrix can be thought of as a compound of matrices of order \( n_1, n_2, \ldots, n_r \) where \( N = \prod_{i=1}^{r} n_i \); i.e. first \( N \times N \) the matrix can be thought of as a \( n_1 \times n_1 \) matrix (partitioning) with each of its elements as an \( n_2 \times n_2 \) matrix, with each of its elements as an \( n_3 \times n_3 \) matrix, and so on. When \( N = 2^r \),
\[
\begin{align*}
\mathbb{C}^{(N)}_{2} & \cong \mathbb{M}_{2^r} \\
& \cong \mathbb{C}^{(2^r)}_{2^r}
\end{align*}
\]

and hence the Clifford algebra \( \mathbb{C}^{(2^r)}_{2^r} \).
generated by \( 2^r \) antisymmetric elements is a compound of Clifford algebras \( \mathbb{C}^{(2^n)}_2 \) which are Hamilton's quaternion algebras.

Thus in 1878, in his fundamental paper on what we now call as Clifford algebra, W.K. Clifford proved that the Clifford algebra \( \mathbb{C}^{(2^n)}_{2^r} \) is a compound of \( 2^r \) commuting quaternion algebras!
CHAPTER 2.

ON GENERALIZATION OF CERTAIN GEOMETRICAL ASPECTS OF CLIFFORD ALGEBRA TO GENERALIZED CLIFFORD ALGEBRA

(1) Since the time of Clifford it is well known that a Clifford algebra $C_{\mathbb{R}}^{(2)}$ can be regarded as an algebra of aggregates of scalar, vector, and polyvectors in the $n$-dimensional Euclidean space. This is what made Clifford call his algebra a geometric algebra.

Since then there have been many investigations by Lipschitz, Cartan, Brouwer and Weyl, Freudenthal, and others on the geometrical aspects of Clifford algebra which became the foundation of the theory of spinors. Popovici and Turtel have considered certain generalization of spinor structures using generalized Clifford algebra. Rassudov has given a beautiful account of theory of spinors using entirely geometrical approach to Clifford algebra and he has also given a representation theory of Clifford algebras different from others. Alladi Ramakrishnan and his collaborators have considered generalization of his representation theory, but it is incomplete. In this chapter we shall first record an 'algebraic' (since I have failed to see any 'geometry' behind it) generalization of the geometric approach to Clifford algebra, to a generalization of Rassudov's theory of representation of Clifford algebras for Generalized Clifford algebras. The generators of Clifford algebra $C_{\mathbb{R}}^{(2)}$, \( \{ L_{i} | i = 1, \ldots, n \} \) obey

\[
L_{i} L_{j} + L_{j} L_{i} = 2 \delta_{ij} \quad \forall i, j = 1, \ldots, n.
\]
Associated with a vector \( \mathbf{a} \equiv (a^1, a^2, \ldots, a^n) \) in \( n \)-dimensional complex Euclidean space \( \mathbb{R}^n \), is an element of \( C_n \):

\[
\mathbf{a} = \sum_{i=1}^{n} a^i L_i
\]

(9.2)

Then the norm of the vector \( \mathbf{a} \), \( \| \mathbf{a} \| \), is defined by:

\[
\| \mathbf{a} \|^2 = \sum_{i=1}^{n} (a^i)^2 = \mathbf{a} \cdot \mathbf{a}
\]

(9.3)

Thus the element \( \sum_{i=1}^{n} a^i L_i \in C_n \) is regarded as a vector in \( \mathbb{R}^n \).

The scalar product of two vectors becomes:

\[
(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^{n} a^i b^i = \frac{1}{2} (\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}) = \frac{1}{2} \{ \mathbf{a}, \mathbf{b} \}
\]

(9.4)

where \( \mathbf{a} = \sum_{i=1}^{n} a^i L_i \) and \( \mathbf{b} = \sum_{i=1}^{n} b^i L_i \). Two orthogonal vectors have \( (\mathbf{a}, \mathbf{b}) = \frac{1}{2} \{ \mathbf{a}, \mathbf{b} \} = 0 \) and hence the basic relations (9.1) denote the orthogonality of the basis \( \{ L_i | i = 1 \cdots n \} \).

A totally antisymmetric tensor of rank \( k \) is called a simple \( k \)-vector if it is obtained as a skew-product of \( k \) vectors

\[
A_{i_1, i_2, \ldots, i_k} = \frac{1}{k!} \begin{vmatrix} p^1_{i_1} & p^1_{i_2} & \cdots & p^1_{i_k} \\ p^2_{i_1} & p^2_{i_2} & \cdots & p^2_{i_k} \\ \vdots & \vdots & \ddots & \vdots \\ p^k_{i_1} & p^k_{i_2} & \cdots & p^k_{i_k} \end{vmatrix}
\]

(9.5)

symbolically the \( k \)-vector \( \mathbf{a} \) is written as

\[
\mathbf{a} = P[ p_1, \ldots, p_k ]
\]

(9.6)
To this is associated the element

$$ a = \sum_{1 \leq j < k \leq n} a_{i_1 i_2 \cdots i_k} L_{i_1} L_{i_2} \cdots L_{i_k} $$  \hspace{1cm} (0.7) $$

where \( L_{i_1} L_{i_2} \cdots L_{i_k} \) represents a basic poly-vector, a linear combination of a scalar \( a^0 \), vector \( \{a^i\} \), bivector \( \{a^{i_1 i_2}\} \),

an \( n \)-vector \( \{a_{i_1 i_2 \cdots i_n}\} \) is called an aggregate and is written as

$$ A = a^0 I + \sum_{i=1}^{n} a^i L_i + \sum_{i,j=1}^{n} a^{ij} L_i L_j + \cdots + \sum_{i_1, i_2, \cdots, i_n = 1} a^{i_1 i_2 \cdots i_n} L_{i_1} L_{i_2} \cdots L_{i_n} $$  \hspace{1cm} (0.8) $$

The product of two basic poly-vectors being again a poly-vector products of two aggregates is well defined and is again another aggregate. These aggregates are thus elements of an algebra and this algebra is called Clifford algebra. (For more details cf. Rusevski.)

If we choose \( \{L_i | i = 1, \ldots, n\} \) to obey generalized Clifford condition

$$ L_{i'} L_{j'} = \omega(m) L_{j'} L_{i'} ; \quad \omega(m) = \exp(\pi i m / n) \quad i', j' = 1, \ldots, n. $$  \hspace{1cm} (0.9) $$

instead of (0.1) to the corresponding element of (0.8) namely

$$ A = \sum_{i=1}^{n} a^{i} L_{i} \quad \text{can be associated a norm} \quad \| a \| \text{ such that} $$

$$ (\| a \|_m)^m = a^m = \sum_{i=1}^{n} (a^{i})^m. $$  \hspace{1cm} (9.10) $$
\[ \| \alpha \|_{m} = \left( \frac{\sum_{i} (a_{i})^{m}}{m} \right) \]

The basis of a generalized Clifford algebra \((\mathbb{C}, \mathbb{C}, A)\)
\[ C_{m}^{(m)} \] generates \((9.9)\) is given by \[ \{ \prod_{k=1}^{n} L_{i_{k}}, \ 0 \leq i_{k} \leq m-1 ; \forall k = 1 \ldots n \} \]
and elements of it are aggregates of the type
\[
A = a_{0} + \sum_{i=1}^{n} a_{i} L_{i} + \sum_{i,j} a_{ij} L_{i} L_{j} + \sum_{i,j,k} a_{ijk} L_{i} L_{j} L_{k} + \cdots
\]
\[
- \sum_{i_{1}, \ldots, i_{m-1}} a_{i_{1} \ldots i_{m-1}} L_{i_{1}} L_{i_{2}} \cdots L_{i_{m-1}}
\]
in which maximum \((m-1)\) indices can take the same value. In the case of \( C_{m}^{(2)} \) the polyvector \( \alpha = P[k_{1}, \ldots, k_{m}] \) can be represented as
\[
\alpha = \begin{vmatrix}
\sum_{i} p_{1}^{i} L_{i} & \sum_{i} p_{2}^{i} L_{i} & \cdots & \sum_{i} p_{m}^{i} L_{i} \\
\sum_{i} p_{1}^{i} L_{i} & \sum_{i} p_{2}^{i} L_{i} & \cdots & \sum_{i} p_{m}^{i} L_{i} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i} p_{1}^{i} L_{i} & \sum_{i} p_{2}^{i} L_{i} & \cdots & \sum_{i} p_{m}^{i} L_{i}
\end{vmatrix}
\]
which is the exterior product or Grassmann product of the vectors \( (k_{1}, \ldots, k_{m}) \).
Now using a generalization of the determinant due to Ranganathan, the basic components of the aggregate $\mathbf{A}$ (9.11) can be written in a similar form. If we define following Ranganathan

\[
\det_\lambda \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix} = \sum_{i=1}^{n} \lambda^{i-1} a_{i1} \cdot \mathbf{A}_{i1}(\lambda)
\]

where $\mathbf{A}_{i1}(\lambda)$ is the usual cofactor of $a_{1i}$ with the difference that $\lambda$ replaces $(-1)$ in its evaluation when $\lambda = -1$ and $\mathbf{A}$ becomes the usual determinant. For example

\[
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a(ek + \lambda fh) + \lambda b(dh + \lambda fg) + \lambda^2 c (dh + \lambda eg)
\]

Then the quantity

\[
W[k_1, \ldots, k_n] = \begin{vmatrix} \sum p_i^1 L_i \\ \vdots \\ \sum p_i^n L_i \end{vmatrix}
\]

generalizes (9.12). This becomes

\[
W[k_1, \ldots, k_n] = \sum_{i_1, i_2, \ldots, i_n=1}^{n} a_{i_1 i_2 \cdots i_n} W[L_{i_1}, L_{i_2}, \ldots, L_{i_n}]
\]
there
\[ a^{i_1i_2\ldots i_k} = \begin{vmatrix} p_1^{i_1} & p_1^{i_2} & \cdots & p_1^{i_k} \\ \vdots & \vdots & \ddots & \vdots \\ p_k^{i_1} & p_k^{i_2} & \cdots & p_k^{i_k} \end{vmatrix} X \begin{pmatrix} N_{i_1i_2\ldots i_k} \end{pmatrix}^{-1} \]
\[ \omega(m) \]

\[ N_{i_1i_2i_3} L_{i_1i_2} L_{i_1i_3} L_{i_2i_3} = \begin{vmatrix} L_{i_1} & L_{i_2} & \cdots & L_{i_k} \\ \vdots & \vdots & \ddots & \vdots \\ L_{i_1} & L_{i_2} & \cdots & L_{i_k} \end{vmatrix} \omega(m) = W \begin{vmatrix} L_{i_1} & L_{i_2} & \cdots & L_{i_k} \end{vmatrix} \]

(9.17)

The aggregate \( A \) can be rewritten as
\[ A = a^0 + \sum_{i_1, i_2} \alpha^{i_1i_2} W \begin{vmatrix} L_{i_1} & L_{i_2} \end{vmatrix} + \cdots + \sum_{i_1, i_2, \ldots, i_{m-1}} \alpha^{i_1i_2\ldots i_{m-1}} W \begin{vmatrix} L_{i_1} & L_{i_2} & \cdots & L_{i_{m-1}} \end{vmatrix} \]

(9.18)

in which an index can get repeated only up to \((m-1)\) times. Product of two quantities of the type (9.15) is well defined and is again a quantity of the same type. Thus this demonstrates an 'algebraic' generalization of the geometrical aspect of Clifford algebra. May be these quantities (9.15) can also be considered as 'geometric' in a space with the \( \| \cdot \|_m \) norm. But unlike the case of Euclidean space with \( m = 3 \), there does not exist continuous group of \( \| \cdot \|_m \) norm preserving transformations of \( a = \sum_{i=1}^{N} a^i L_i \). The group
of $\| \cdot \|_{\nu}$ norm preserving transformations are trivial
diagonal and permutation transformations as established by
Morinaga and Hono and we have already discussed them in
Chapter III.

(ii) Now we shall give the correct generalised version of
Rasovskii's approach to representation of $G.C.A_{\nu}$ generated by
(9.9). To represent $C^{(m)\nu}_{\nu}$ we start with the basis of $C^{(m)\nu}_{\nu}$. Any element of $C^{(m)\nu}_{\nu}$ is written as

\begin{equation}
\Lambda_{\nu} = \sum_{i_1, i_2, \ldots, i_{\nu}} \lambda_{i_1, i_2, \ldots, i_{\nu}} L_{i_1} L_{i_2} \cdots L_{i_{\nu}}
\end{equation}

(9.19)

Associate with the set of $\nu$ basis elements $\left\{ \frac{L_{i_1} \cdots L_{i_{\nu}}}{\nu} \right\}$ linearly independent vectors of an $\nu$-dimensional vector space. Thus $\Lambda_{\nu}$ is associated with an $\nu$-dimensional vector

\begin{equation}
\Lambda_{\nu} = \sum_{i_1, i_2, \ldots, i_{\nu}} \lambda_{i_1, i_2, \ldots, i_{\nu}} \nu
\end{equation}

(9.20)

where $\left\{ \nu, i_1, i_2, \ldots, i_{\nu} \right\}$ span the $\nu$-dimensional space. Now define a set of $\nu$ operators (matrices) $L_{1, L_{2, \ldots, L_{\nu}}}$ by the following relations

\begin{align}
a) \quad \Lambda_{\nu} L'_{\nu} = & \Lambda_{\nu} ; \quad i = 1, \ldots, \nu \\
b) \quad \zeta L_{i} \Lambda_{\nu} = & L_{i} \Lambda_{\nu} ; \quad i = 1, \ldots, \nu
\end{align}

(9.21)

\[\zeta = \begin{cases} \omega(m)^{\frac{1}{2}} & \text{if } m \text{ is even} \\ 1 & \text{if } m \text{ is odd} \end{cases}\]
where
\[ \Lambda^*_w = \sum_{m=1}^{m-1} \omega(m) \sum_{i_1, i_2, \ldots, i_v = 0} \Lambda^{i_1, i_2, \ldots, i_v}_{L_1, L_2, \ldots, L_v} \]

\[ (k \cdot \text{mod} m = i_1 + i_2 + \cdots + i_v) \]

Then the \( 2^v \) operators (matrices) obey the generating relations of \( C^{(m)}_{2^v} \)

\[ L_i \cdot L_j = \omega(m) L_j \cdot L_i \; ; \; i, j = 1, 2, \ldots, 2^v \]

\[ L_i \cdot L_i^{*} = \Lambda^{i, i}_{L_i} = 1 \; ; \; i = 1, \ldots, 2^v \]

**Proof.**

For \( i, j \leq v \):

\[ L_i \cdot L_j \cdot \Lambda^*_w = \Lambda^*_w L_i \cdot L_j + L_j \cdot L_i \cdot \Lambda^*_w \]

If \( i < j \):

\[ L_i \cdot L_j \cdot \Lambda^*_w = \omega(m) L_j \cdot L_i \cdot \Lambda^*_w \]

\[ L_i \cdot L_j = \omega(m) L_j \cdot L_i \]

For \( i, j \geq v \):

\[ L_i \cdot L_j \cdot \Lambda^*_w = \omega(m) L_j \cdot L_i \cdot \Lambda^*_w \]

\[ L_i \cdot L_j \cdot \Lambda^*_w = \omega(m) L_j \cdot L_i \cdot \Lambda^*_w \]

\[ L_i \cdot L_j = \omega(m) L_j \cdot L_i \]

\[ L_i \cdot L_j = \omega(m) L_j \cdot L_i \]

\[ L_i \cdot L_j = \omega(m) L_j \cdot L_i \]}
If \( i < j \),
\[
\mathbf{L}^i \cdot \mathbf{L}^j = \omega(m) \mathbf{L}^i \cdot \mathbf{L}^j
\]

\[
\therefore \mathbf{L}^i \cdot \mathbf{L}^j = \omega(m) \mathbf{L}^j \cdot \mathbf{L}^i
\]

For \( i \leq \nu, j \geq \nu \),
\[
\mathbf{L}^i \cdot \mathbf{L}^j \mathbf{A}^\nu = (\zeta \mathbf{L}^i \cdot \mathbf{L}^j \mathbf{A}^\nu) \mathbf{L}^i \cdot \mathbf{L}^j
\]
\[
\mathbf{L}^j \cdot \mathbf{L}^i \mathbf{A}^\nu = \zeta \mathbf{L}^j \cdot \mathbf{L}^i (\mathbf{L}^i \cdot \mathbf{L}^j \mathbf{A}^\nu)^* \mathbf{L}^i \cdot \mathbf{L}^j
\]
\[
= \omega(m^{-1}) \zeta \mathbf{L}^j \cdot \mathbf{L}^i \mathbf{A}^\nu \mathbf{L}^i \cdot \mathbf{L}^j
\]
\[\therefore \mathbf{L}^i \cdot \mathbf{L}^j = \omega(m) \mathbf{L}^j \cdot \mathbf{L}^i\]

Thus in all cases,
\[\mathbf{L}^i \cdot \mathbf{L}^j = \omega(m) \mathbf{L}^j \cdot \mathbf{L}^i \quad i \leq j \]

Since the generators of \( \mathbf{C}_\nu^m \), \( \{ \mathbf{L}^i \}_{i=-\nu}^{\nu} \) obey \( \mathbf{L}^i = 1 \quad \forall i \), all these \( \{ \mathbf{L}^i \}_{i=-\nu}^{\nu} \) obey automatically \( \mathbf{L}^i = 1 \), \( \forall i = 1 \ldots 2\nu \), due to the construction (3.24).

Summary of Important Points

An incomplete attempt has been made to obtain a generalization of the geometrical interpretation of Clifford algebra to suit Generalized Clifford Algebra. Racovehii's theory of representation of Clifford algebras has been generalized to obtain the representation of \( \mathbf{GCA}_n \), which is an improvement of an earlier version by Alladi Ramakrishnan and his collaborators.


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