Root Multiplicities for Borcherds-Kac-Moody Algebras and Graph Coloring

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As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by G. Arunkumar entitled "Root Multiplicities for Borcherds-Kac-Moody Algebras and Graph Coloring" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

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G. Arunkumar

DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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G. Arunkumar

DEDICATIONS

To

My Advisor

Prof. Sankaran Viswanath,

My Senior

Dr. M. Saravanan,

and

My Junior

Mr. S.P. Murugan.

NEVER STOP DREAMING,
LIFE CAN GO FROM ZERO TO HUNDRED REAL QUICK.

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G.Arunkumar

Contents

Synopsis

In this thesis we explore the connection between root multiplicities of infinite dimensional Lie algebras and graph colorings. We show that certain root multiplicities of Borcherds-Kac-Moody algebras (Borcherds algebras in short) can be obtained in terms of generalized chromatic polynomials of the associated graph.

We use this result together with the Lyndon basis of free Lie algebras to produce a basis of the corresponding root spaces. As a further application, we prove that the generalized chromatic polynomials evaluated at negative integers arise naturally as coefficients of the Hilbert series of tensor powers of the universal enveloping algebra of free partially commutative Lie algebras.

Finally, we study the linear coefficient of the chromatic polynomial of a graph (the so-called *chromatic discriminant*). We state a recurrence formula for this number, first proved using Lie theoretic methods in [32]. We give two purely combinatorial (bijective) proofs of this recurrence using the interpretations of the chromatic discriminant in terms of acyclic orientations and spanning trees without broken circuits.

The results of this thesis appear in [1, 2]. This thesis contains five chapters.

The first chapter deals with the preliminaries.

In chapter 2 we derive an expression for the generalized chromatic polynomial of a graph G in terms of certain root multiplicities of the corresponding Borcherds algebra.

In chapter 3 we address the problem of finding *right normed bases* for (certain) root spaces of Borcherds algebras.

In chapter 4 we establish a connection between the generalized chromatic polynomial of a finite graph G and the Hilbert series of the q-fold tensor product of the universal enveloping algebra of the free partially commutative Lie algebra associated to G.

The absolute value of the coefficient of q in the chromatic polynomial of a graph G is known as the *chromatic discriminant* of G and is denoted $\alpha(G)$. In chapter 5 we give a bijective proof of a recurrence formula for the chromatic discriminant of a graph G.

We now give an extended description of our main results, which appear in chapters 2–5.

The generalized chromatic polynomial and root multiplicities of Borcherds algebras

Borcherds algebras

We recall the definition of Borcherds algebras (also called *generalized Kac–Moody algebras*). For more details, we refer the reader to [5, 14, 16] and the references therein. A real matrix $A = (a_{ij})_{i,j\in I}$ indexed by a finite or countably infinite set, which we identify with $I = \{1, \dots, n\}$ or $\mathbb{Z}_+$, is said to be a *Borcherds–Cartan matrix* if the following conditions are satisfied for all $i, j \in I$:

1. A is symmetrizable

2. $a_{ii} = 2$ or $a_{ii} \leq 0$

3. $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$

4. $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Recall that a matrix A is called symmetrizable if there exists a diagonal matrix $D = \text{diag}(\epsilon_i, i \in I)$ with positive entries such that DA is symmetric. Set $I^{\text{re}} = \{i \in I : a_{ii} = 2\}$ and $I^{\text{im}} = I \backslash I^{\text{re}}$.

The Borcherds algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated to the Borcherds–Cartan matrix A is the Lie algebra generated by $\{e_i, f_i, h_i : i \in I\}$ with the following defining relations:

(R1) $[h_i, h_j] = 0$ for $i, j \in I$

(R2) $[h_i, e_k] = a_{ik} e_k$, $[h_i, f_k] = -a_{ik} f_k$ for $i, k \in I$

(R3) $[e_i, f_j] = \delta_{ij} h_i$ for $i, j \in I$

(R4) $(\text{ad } e_i)^{1-a_{ij}} e_j = 0$, $(\text{ad } f_i)^{1-a_{ij}} f_j = 0$ if $i \in I^{\text{re}}$ and $i \neq j$

(R5) $[e_i, e_j] = 0$ and $[f_i, f_j] = 0$ if $i, j \in I^{\text{im}}$ and $a_{ij} = 0$.

We define $\mathfrak{h}$ to be the span of the h_i; the simple roots are the $\alpha_j \in \mathfrak{h}^*$ satisfying $(\alpha_j, h_i) = a_{ij}$ for all $i, j \in I$. We associate a graph G with the Lie algebra $\mathfrak{g}$ as follows: the vertex set of G is I, and there is an edge between the vertices i, j if $a_{ij} \neq 0$.

Vertex multicoloring

Next, we define the notion of a proper vertex multicoloring of a graph; this is a generalization of the well-known notion of graph coloring. For more details about multicoloring of a graph we refer to [13].

Definition. Let G be an undirected graph, with (possibly infinite) vertex set I and edge set E.

In this thesis , $\mathbf{k} = (k_i : i \in I)$ *will always mean a tuple of non-negative integers in which all but finitely many of the* k_i *are zero.*

For $q \in \mathbb{N}$, we let $[q] := \{1, 2, \cdots, q\}$ and $\mathcal{P}([q])$ be the set of all subsets of $[q]$.

A map $\tau : I \to \mathcal{P}([q])$ satisfying the following conditions will be called a proper vertex multicoloring of G of weight $\mathbf{k}$ with q colors:

(i) For all $i \in I$ we have $|\tau(i)| = k_i$.

(ii) For all $i, j \in I$ such that $\{i, j\} \in E$ we have $\tau(i) \cap \tau(j) = \emptyset$.

We let $\pi^G_{\mathbf{k}}(q)$ be the number of such maps τ. This turns out to be a polynomial in q, and is called the $\mathbf{k}$-*generalized chromatic polynomial* (or simply the generalized chromatic polynomial, when $\mathbf{k}$ is fixed). When I is finite and all $k_i = 1$, this reduces to the usual chromatic polynomial of G.

k-weighted bond lattice

To state our main theorem, we need another definition:

Definition. The $\mathbf{k}$-*weighted bond lattice* $L_G(\mathbf{k})$ of G is the set of all $\mathbf{J} = \{J_1, \ldots, J_k\}$ satisfying the following properties:

(i) $\mathbf{J}$ is a multiset, i.e. we allow $J_i = J_j$ for $i \neq j$.

(ii) Each J_i is a finite multisubset of I, i.e., a finite multiset whose underlying set is a subset of I.

(iii) The subgraph of G induced by the underlying set of J_i is connected for each i.

(iv) The disjoint union of $J_1 \sqcup \cdots \sqcup J_k = \{\underbrace{i, \ldots, i}_{k_i \text{ times}} : i \in I\}$.

Let $\mathbf{J} = \{J_1, J_2, \cdots, J_k\} \in L_G(\mathbf{k})$ and J be a finite multisubset of I. We denote by $D(J, \mathbf{J})$ the number of $1 \leq i \leq k$ such that $J_i = J$.

Main theorem

Now suppose that $\mathfrak{g}$ is a Borcherds algebra and G is its graph, with vertex set I. For a finite multisubset J of I, we let $\beta(J) := \sum_{i \in J} \alpha_i$ and $\text{mult}(\beta(J)) := \dim \mathfrak{g}_{\beta(J)}$. Finally, let $\eta(\mathbf{k}) := \sum_{i \in I} k_i \alpha_i$ and $\varepsilon(\mathbf{k}) := (-1)^{\sum k_i}$.

Given these notions we have our first main theorem, which expresses the generalized chromatic polynomial of G in terms of root multiplicities of $\mathfrak{g}$:

Theorem 0.1. *Assume that* $\mathbf{k}$ *satisfies:*

$$k_i \in \{0, 1\} \text{ for all } i \in I^{\text{re}}. \tag{1}$$

Then, we have:

$$\pi_{\mathbf{k}}^G(q) = \varepsilon(\mathbf{k}) \sum_{\mathbf{J} \in L_G(\mathbf{k})} (-1)^{|\mathbf{J}|} \prod_{J \in \mathbf{J}} \binom{q \, \text{mult}(\beta(J))}{D(J, \mathbf{J})}.$$

where $|\mathbf{J}|$ *is the number of parts (counted with multiplicity) in the partition* $\mathbf{J}$.

Remark 0.2. When $\mathfrak{g}$ is a Kac–Moody algebra and $k_i = 1$ for all i, this reduces to the expression for the chromatic polynomial obtained in [32].

We deduce the following corollary which gives a combinatorial formula for certain root multiplicities.

Corollary 0.3. *If* $\mathbf{k}$ *satisfies* (1), *then:*

$$\text{mult}(\eta(\mathbf{k})) = \sum_{\ell|\mathbf{k}} \frac{\mu(\ell)}{\ell} \left|\pi^G_{\mathbf{k}/\ell}(q)\ [q]\right|, \tag{2}$$

where $\left|\pi^G_{\mathbf{k}}(q)\ [q]\right|$ *denotes the absolute value of the coefficient of* q *in* $\pi^G_{\mathbf{k}}(q)$, μ *is the Möbius function, and* $\ell|\mathbf{k}$ *means that* $\ell|k_i$ *for each* i.

Assume $\mathbf{k}$ satisfies (1) and $k_i \neq 0$ $(k_i = 1)$ for some $i \in I^{re}$, then $\text{mult}(\eta(\mathbf{k})) = \left|\pi^G_{\mathbf{k}}(q)\ [q]\right|$. Note that the same formula holds true if $k_i = 0$ for all $i \in I^{\text{re}}$ and if $\gcd(\{k_i : i \in I\}) = 1$.

We note that if $\mathfrak{g}$ is a Borcherds algebra with $I = I^{im}$, then its positive part $\mathfrak{n}^+$ is isomorphic to the free partially commutative Lie algebra associated to the graph G. We remark that, in view of the previous statement, the above formlula can be used to find the dimensions of certain grade spaces of these Lie algebras. In the following examples we explain this method in detail.

Example

The generalized chromatic polynomial $\pi^G_{\mathbf{k}}(q)$ can be computed explicitly for many families of graphs, and in such cases, equation (2) gives us a way of computing certain root multiplicities of the associated Borcherds algebra. The case of the complete graph is illustrated below.

Let $G = K_n$ be the complete graph with n vertices and $\mathbf{k} = (k_1, \ldots, k_n)$ be a tuple of positive integers. We take $\mathfrak{g}$ to be a Borcherds algebra with graph G and having no real simple roots. In this case, it is well known that the positive part $\mathfrak{n}^+$ of $\mathfrak{g}$ (the Lie subalgebra generated by the $\{e_i : i \in I\}$) is a free Lie algebra.

To compute the generalized chromatic polynomial of G, observe that vertex 1 can receive any k_1 colors from the given q colors. Vertex 2 can receive any color other than the k_1 colors that were assigned to vertex 1. Continuing in this way, we obtain:

$$\pi_{\mathbf{k}}^{K_n}(q) = \binom{q}{k_1}\binom{q-k_1}{k_2}\binom{q-(k_1+k_2)}{k_3}\cdots\binom{q-(k_1+\cdots+k_{n-1})}{k_n}.$$

In particular $|\pi_{\mathbf{k}}^{K_n}(q)[q]| = \frac{(k_1+\cdots+k_n-1)!}{k_1!\cdots k_n!}$. Hence we recover Witt's formula [34] for the dimensions of the graded spaces in a free Lie algebra:

$$\operatorname{mult}\eta(\mathbf{k}) = \frac{1}{\operatorname{ht}\mathbf{k}}\sum_{\ell|\mathbf{k}}\mu(\ell)\frac{(\operatorname{ht}\mathbf{k}/\ell)!}{(\mathbf{k}/\ell)!}$$

where $\operatorname{ht}\mathbf{k} := \sum_{i\in I} k_i$ and $\mathbf{k}! := \prod_{i\in I} k_i!$.

Bases for certain root spaces of Borcherds algebras

If $\mathfrak{g}$ is a free Lie algebra then it has a classical Lyndon basis indexed by Lyndon words [26]. This has been extended to the case of free partially commutative Lie algebras in [19] and [20]. For free Lie algebras we also have a special type of basis known as a *right normed basis* [7]; however, a right normed basis for free partially commutative Lie algebras is not known in general. We note that if $\mathfrak{g}$ is a Borcherds algebra with $I = I^{im}$, then its positive part $\mathfrak{n}^+$ is isomorphic to a free partially commutative Lie algebra. Hence finding right normed bases for root spaces of $\mathfrak{g}$ will in turn give us right normed bases for the corresponding graded spaces of the free partially commutative Lie algebra.

Lyndon basis

We use the notation of the previous chapter. Let $\mathfrak{g}$ be a Borcherds algebra, with graph G. The vertex set of G is I. We let I^* be the free monoid generated by I.

The free partially commutative monoid associated with G is denoted by $M(I,G) := I^*/\sim$, where $\sim$ is generated by the relations

$$ab \sim ba, \quad (a,b) \notin E(G).$$

Fix a word $\mathbf{w} \in M(I,G)$. For $i \in I$, its initial multiplicity in $\mathbf{w}$ is defined to be the largest $k \geq 0$ for which there exists $\mathbf{u} \in M(I,G)$ such that $\mathbf{w} = \mathbf{u}i^k$. We define the *initial alphabet* $\mathrm{IA_m}(\mathbf{w})$ of $\mathbf{w}$ to be the multiset in which each $i \in I$ occurs as many times as its initial multiplicity in $\mathbf{w}$. The underlying set is denoted by $\mathrm{IA}(\mathbf{w})$.

Let $\mathcal{X}_i = \{\mathbf{w} \in M(I,G) : \mathrm{IA_m}(\mathbf{w}) = \{\mathrm{i}\}\}$. We denote by $FL(\mathcal{X}_i)$ the free Lie algebra generated by $\mathcal{X}_i$. Let $\mathcal{X}_i^*$ be the free monoid on $\mathcal{X}$. To each Lyndon word $\mathbf{w} \in \mathcal{X}_i^*$ we associate a Lie word $L(\mathbf{w})$ in $FL(\mathcal{X}_i)$ as follows. If $\mathbf{w} \in \mathcal{X}_i$, then $L(\mathbf{w}) = \mathbf{w}$ and otherwise $L(\mathbf{w}) = [L(\mathbf{u}), L(\mathbf{v})]$, where $\mathbf{w} = \mathbf{uv}$ is the standard factorization of $\mathbf{w}$. The following result can be found in [26] and is known as the Lyndon basis for free Lie algebras.

Proposition 0.4. *The set $\{L(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_i^*$ is a Lyndon word$\}$ forms a basis of $FL(\mathcal{X}_i)$.* □

Remark. This proposition holds true for the free Lie algebra $FL(X)$ associated to any arbitrary set X.

Main theorem

For $\mathbf{w} = i_1\, i_2 \cdots i_r$ in $M(I,G)$, the corresponding *right normed Lie word* is defined by $e(\mathbf{w}) = [e_{i_1}, [e_{i_2}, [\dots [e_{i_{r-2}}, [e_{i_{r-1}}, e_{i_r}]] \dots]]] \in \mathfrak{g}$. Using the Jacobi identity, it is easy to see that the association $\mathbf{w} \mapsto e(\mathbf{w})$ is well defined.

Fix $i \in I$ and let $\mathfrak{g}^i$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\{e(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_i\}$.

By the universal property of $FL(\mathcal{X}_i)$ we have a surjective homomorphism

$$\Phi : FL(\mathcal{X}_i) \to \mathfrak{g}^i, \quad \mathbf{w} \mapsto e(\mathbf{w}) \ \ \forall\, \mathbf{w} \in \mathcal{X}_i. \tag{3}$$

Proposition 0.4 implies that the image of the set $\{L(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_i^* \text{ is a Lyndon word}\}$ under the map Φ generates $\mathfrak{g}^i$. The main theorem of this chapter (Theorem 0.5) shows that suitable subsets of this generating set will in fact form bases for certain root spaces.

To state our theorem, fix a tuple $\mathbf{k}$ of non-negative integers such that $k_i > 0$ (for our fixed i) and satisfying the hypothesis of Theorem 0.1. For our fixed i define:

$$C^i(\mathbf{k}, G) = \{\mathbf{w} \in \mathcal{X}_i^* : \mathbf{w} \text{ is a Lyndon word},\ \operatorname{wt}(\mathbf{w}) = \eta(\mathbf{k})\}$$

where $\operatorname{wt}(\mathbf{w})$ is the tuple which counts the number of times an alphabet from I appeared in $\mathbf{w}$.

Theorem 0.5. *The set $\{\Phi(L(\mathbf{w})) : \mathbf{w} \in C^i(\mathbf{k}, G)\}$ is a basis of the root space $\mathfrak{g}_{\eta(\mathbf{k})}$. Moreover, if $k_i = 1$, the set*

$$\{e(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_i,\ \operatorname{wt}(\mathbf{w}) = \eta(\mathbf{k})\}$$

forms a right–normed basis of $\mathfrak{g}_{\eta(\mathbf{k})}$ and

$$C^i(\mathbf{k}, G) = \{\mathbf{w} \in \mathcal{X}_i : e(\mathbf{w}) \neq 0,\ \operatorname{wt}(\mathbf{w}) = \eta(\mathbf{k})\}.$$

Sketch of the proof

In chapter 1, we give a combinatorial proof of the above theorem. The following combinatorial model is important for the proof. We set

$$\widetilde{B}^i(\mathbf{k}, G) := \{\mathbf{w} \in M(I, G) : \mathrm{wt}(\mathbf{w}) = \eta(\mathbf{k}) \text{ and } \mathrm{IA}(\mathbf{w}) = \{\mathrm{i}\}\}.$$

Let $\mathbf{w} \in \widetilde{B}^i(\mathbf{k}, G)$, then one can write $\mathbf{w} = \mathbf{w}_1 \cdots \mathbf{w}_{k_i} \in \mathcal{X}_i^*$ in its i-form. We say $\mathbf{w}$ is aperiodic if the elements in the cyclic rotation class of $\mathbf{w}$ are all distinct, i.e. all elements in

$$C(\mathbf{w}) := \{\mathbf{w}_1 \cdots \mathbf{w}_{k_i},\ \mathbf{w}_2 \cdots \mathbf{w}_{k_i}\mathbf{w}_1,\ \cdots,\ \mathbf{w}_{k_i}\mathbf{w}_1 \cdots \mathbf{w}_{k_i-1}\}$$

are distinct. We naturally identify $C^i(\mathbf{k}, G)$ with the set

$$B^i(\mathbf{k}, G) := \{\mathbf{w} \in \widetilde{B}^i(\mathbf{k}, G) : \mathbf{w} \text{ is aperiodic}\}/\sim,$$

where $\mathbf{w} \sim \mathbf{w}' \Leftrightarrow C(\mathbf{w}) = C(\mathbf{w}')$.

The following proposition shows that our generating set has cardinality equal to the dimension of $\mathfrak{g}_{\eta(\mathbf{k})}$, thereby proving Theorem 0.5.

Proposition. *We have*

(i) The root space $\mathfrak{g}_{\eta(\mathbf{k})}$ *is contained in* $\mathfrak{g}^i$.

(ii) Let $\mathbf{w} \in M(I, G)$ *and* $\mathrm{wt}(\mathbf{w}) = \eta(\mathbf{k})$. *Then*

$$e(\mathbf{w}) \neq 0 \Longleftrightarrow \mathrm{IA}_{\mathrm{m}}(\mathbf{w}) = \{\mathrm{i}\}.$$

(iii) We have

$$\mathit{mult}\ \eta(\mathbf{k}) = |B^i(\mathbf{k}, G)| = |C^i(\mathbf{k}, G)|.$$

We finish chapter 1 with the proof of the above proposition.

Generalized chromatic polynomials and free partially commutative Lie algebras

Let Γ be an abelian semigroup with at most countably infinite elements and $\mathfrak{a}$ be a Γ-graded Lie algebra with finite dimensional homogeneous spaces, i.e.

$$\mathfrak{a} = \bigoplus_{\alpha\in\Gamma} \mathfrak{a}_\alpha \quad \text{and} \quad \dim(\mathfrak{a}_\alpha) < \infty \text{ for all } \alpha \in \Gamma.$$

The Γ-grading of $\mathfrak{a}$ induces a $\Gamma \cup \{0\}$-grading on the universal enveloping algebra $\mathbf{U}(\mathfrak{a})$ and we define the Hilbert series:

$$H_\Gamma(\mathbf{U}(\mathfrak{a})) = 1 + \sum_{\alpha\in\Gamma} (\dim \mathbf{U}(\mathfrak{a})_\alpha)\ e^\alpha.$$

Let $\mathfrak{g}$ be a Borcherds algebra with no real simple roots whose associated graph is G. As remarked previously, in this case, $\mathfrak{n}^+$ is isomorphic to the free partially commutative Lie algebra associated to the graph G. Further $\mathfrak{n}^+$ is graded by $\Gamma := Q_+\backslash\{0\}$, where $Q_+ := \oplus_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$, with n the number of vertices of G.

The main result of this chapter states that the Hilbert series of $\mathbf{U}(\mathfrak{n}^+)^{\otimes q}$ is determined by the evaluation of the generalized chromatic polynomials of G at $-q$. More precisely, we have:

Theorem 0.1. *Let $q \in \mathbb{N}$. Then the Hilbert series of $\mathbf{U}(\mathfrak{n}^+)^{\otimes q}$ is given by*

$$H_\Gamma(\mathbf{U}(\mathfrak{n}^+)^{\otimes q}) = \sum_{\alpha\in\Gamma} (-1)^{\mathrm{ht}(\alpha)} \pi^G_\alpha(-q)\ e^\alpha.$$

In particular,

$$\dim(\mathbf{U}(\mathfrak{n}^+)^{\otimes q})_\alpha = (-1)^{\mathrm{ht}(\alpha)}\pi_\alpha^G(-q) \textit{ for all } \alpha \in \Gamma.$$

As an application of Theorem 0.1, we give a different proof of the following reciprocity theorem due to Stanley for chromatic polynomials [29, Theorem 1.2]. For a map $\sigma : I \to \{1, 2, \ldots, q\}$ and an acyclic orientation $\mathcal{O}$ of G, we say $(\sigma, \mathcal{O})$ is a q-compatible pair if for each directed edge $i \to j$ in $\mathcal{O}$ we have $\sigma(i) \geq \sigma(j)$. We prove by an alternate method,

Theorem 0.2. *The number of q–compatible pairs of G is equal to $(-1)^{|I|}\pi_{\alpha(I)}(-q)$. In particular, $(-1)^{|I|}\pi_{\alpha(I)}(-1)$ counts the number of acyclic orientations of G.*

A recurrence formula for the chromatic discriminant of a graph

The absolute value of the coefficient of q in the chromatic polynomial of a graph G is known as the *chromatic discriminant* of G and is denoted $\alpha(G)$. A well-known recurrence formula for $\alpha(G)$ (see for instance [10]) is the following:

$$\alpha(G) = \alpha(G\backslash e) + \alpha(G/e), \tag{4}$$

where e is any edge of G. Here, $G\backslash e$ denotes G with e deleted and G/e denotes the simple graph obtained from G by identifying the two ends of e (i.e., contracting e to a single vertex) and removing any multiple edges that result. This is an immediate consequence of the *deletion-contraction rule* for the chromatic polynomial.

Yet another recurrence formula for $\alpha(G)$ was obtained in [32] via the connection to root multiplicities of Kac–Moody Lie algebras and using the Peterson recurrence formula for these multiplicities.

Proposition. *[32]*

$$2\ e(G)\ \alpha(G) = \sum_{(G_1,G_2)} \alpha(G_1)\ \alpha(G_2)\ e(G_1,G_2). \tag{5}$$

Here the sum ranges over pairs (G_1, G_2) of non-empty induced subgraphs of G whose vertex sets partition the vertex set of G, $e(G)$ is the total number of edges in G and $e(G_1, G_2)$ is the number of edges that straddle G_1 and G_2. We say that an edge e straddles *G_1 and G_2 if one end of e is in G_1 and the other in G_2.*

We give two new proofs of this recurrence which are purely combinatorial. We demonstrate explicit bijections between sets whose cardinalities equal the left and right hand sides of (5.2), using the interpretations of $\alpha(G)$ in terms of (i) acyclic orientations and (ii) spanning trees of G.

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Chapter 1

Preliminaries

1.1 Borcherds algebras

Borcherds–Kac–Moody algebras were introduced by R. Borcherds in [5] as a natural generalization of Kac–Moody algebras.

The structure theory of Borcherds–Kac–Moody algebras is very similar to the structure theory of Kac–Moody algebras; however the main point of difference is that one is allowed to have imaginary simple roots. The most important step in understanding the structure of these algebras is to study roots and root multiplicities; the imaginary roots being the most mysterious ones. *In this thesis we call Borcherds–Kac–Moody algebras as Borcherds algebras in short.* Effective closed formulas for the root multiplicities are unknown in general, except for the affine Kac–Moody algebras and some small rank Borcherds algebras; see for example [6, 18, 24, 30] and the references therein. All these papers deal with some particular examples of small rank Borcherds algebras.

1.2 Definition of Borcherds algebras

We denote the set of complex numbers by $\mathbb{C}$ and, respectively, the sets of integers, non-negative integers, and positive integers by $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$. Unless otherwise stated, all the vector spaces considered in this paper are $\mathbb{C}$–vector spaces.

We recall the definition of Borcherds algebras, also called generalized Kac–Moody algebras. For more details, we refer the reader to [5,14,16] and the references therein. A real matrix $A = (a_{ij})_{i,j\in I}$ indexed by a finite or countably infinite set, which we identify with $I = \{1, \ldots, n\}$ or $\mathbb{Z}_+$, is said to be a *Borcherds–Cartan matrix* if the following conditions are satisfied for all $i, j \in I$:

1. A is symmetrizable
2. $a_{ii} = 2$ or $a_{ii} \leq 0$
3. $a_{ij} \leq 0$ if $i \neq j$ and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$
4. $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Recall that a matrix A is called symmetrizable if there exists a diagonal matrix $D = \mathrm{diag}(\epsilon_i, i \in I)$ with positive entries such that DA is symmetric. Set $I^{\mathrm{re}} = \{i \in I : a_{ii} = 2\}$ and $I^{\mathrm{im}} = I \backslash I^{\mathrm{re}}$. The Borcherds algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated to a Borcherds–Cartan matrix A is the Lie algebra generated by e_i, f_i, h_i, $i \in I$ with the following defining relations:

(R1) $[h_i, h_j] = 0$ for $i, j \in I$

(R2) $[h_i, e_k] = a_{ik}e_k$, $[h_i, f_k] = -a_{ik}f_k$ for $i, k \in I$

(R3) $[e_i, f_j] = \delta_{ij}h_i$ for $i, j \in I$

(R4) $(\mathrm{ad}\ e_i)^{1-a_{ij}}e_j = 0$, $(\mathrm{ad}\ f_i)^{1-a_{ij}}f_j = 0$ if $i \in I^{\mathrm{re}}$ and $i \neq j$

(R5) $[e_i, e_j] = 0$ and $[f_i, f_j] = 0$ if $i, j \in I^{\text{im}}$ and $a_{ij} = 0$.

Remark. Note that there are no further relations of the form

$$(\text{ad } e_i)^{1-a_{ij}} e_j = 0, (\text{ad } f_i)^{1-a_{ij}} f_j = 0$$

for $j \in I^{\text{re}}$, $i \notin I^{\text{re}}$ and $k > 1$.

Remark. If $i \in I$ is such that $a_{ii} = 0$, the subalgebra spanned by the elements h_i, e_i, f_i is a Heisenberg algebra and otherwise this subalgebra is isomorphic to $\mathfrak{sl}_2$ (possibly after rescaling e_i, f_i and h_i).

1.3 Elementary properties of Borcherds algebras

We collect some elementary properties of the Borcherds algebras $\mathfrak{g}$; see [14, Proposition 1.5] for more details and proofs. We have that $\mathfrak{g}$ is $\mathbb{Z}^I$–graded by giving h_i degree $(0, 0, \dots)$, e_i degree $(0, \dots, 0, 1, 0, \dots)$ and f_i degree $(0, \dots, 0, -1, 0, \dots)$ where ± 1 appears at the i-th position. For a sequence $(n_1, n_2, \dots)$, we denote by $\mathfrak{g}(n_1, n_2, \dots)$ the corresponding graded piece; note that $\mathfrak{g}(n_1, n_2, \dots) = 0$ unless finitely many of the n_i are non-zero. Let $\mathfrak{h}$ be the abelian subalgebra spanned by the h_i, $i \in I$ and let $\mathfrak{E}$ be the space of commuting derivations of $\mathfrak{g}$ spanned by the D_i, $i \in I$, where D_i denotes the derivation that acts on $\mathfrak{g}(n_1, n_2, \dots)$ as multiplication by the scalar n_i. Note that the abelian subalgebra $\mathfrak{E} \ltimes \mathfrak{h}$ of $\mathfrak{E} \ltimes \mathfrak{g}$ acts by scalars on $\mathfrak{g}(n_1, n_2, \dots)$ and D_i's are added to make the α_i's linearly independent. Given this, we have a root space decomposition:

$$\mathfrak{g} = \bigoplus_{\alpha \in (\mathfrak{E} \ltimes \mathfrak{h})^*} \mathfrak{g}_\alpha, \text{ where } \mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{E} \ltimes \mathfrak{h}\}. \tag{1.1}$$

Define $\Pi := \{\alpha_i\}_{i\in I} \subset (\mathfrak{E} \ltimes \mathfrak{h})^*$ by $\alpha_j((D_k, h_i)) := \delta_{k,j} + a_{ij}$ and set

$$Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i, \quad Q_+ := \sum_{i\in I} \mathbb{Z}_+\alpha_i.$$

Denote by $\Delta := \{\alpha \in (\mathfrak{E} \ltimes \mathfrak{h})^* \backslash \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ the set of roots, and by Δ_+ the set of roots which are non-negative integral linear combinations of the $\alpha_i's$, called the positive roots. The elements in Π are called the simple roots; we call $\Pi^{\rm re} := \{\alpha_i : i \in I^{\rm re}\}$ the set of real simple roots and $\Pi^{\rm im} = \Pi\backslash\Pi^{\rm re}$ the set of imaginary simple roots. One of the important properties of Borcherds algebras is that $\Delta = \Delta_+ \sqcup -\Delta_+$ and

$$\mathfrak{g}_0 = \mathfrak{h}, \quad \mathfrak{g}_\alpha = \mathfrak{g}(n_1, n_2, \dots), \quad \text{if} \quad \alpha = \sum_{i\in I} n_i\alpha_i \in \Delta.$$

Moreover, we have a triangular decomposition

$$\mathfrak{g} \cong \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) is the free Lie algebra generated by $e_i,\ i \in I$ (resp. $f_i,\ i \in I$) with defining relations

$$(\text{ad } e_i)^{1-a_{ij}} e_j = 0 \text{ (resp. } (\text{ad } f_i)^{1-a_{ij}} f_j = 0) \text{ for } i \in I^{\rm re},\ j \in I \text{ and } i \neq j$$

and

$$[e_i, e_j] = 0 \text{ (resp. } [f_i, f_j] = 0) \text{ for } i, j \in I^{\rm im} \text{ and } a_{ij} = 0.$$

In view of (1.1) we have

$$\mathfrak{n}^\pm = \bigoplus_{\alpha\in\pm\Delta_+} \mathfrak{g}_\alpha.$$

Finally, given $\gamma = \sum_{i\in I} n_i\alpha_i \in Q_+$ (only finitely many n_i are non-zero), we set $\text{ht}(\gamma) := \sum_{i\in I} n_i$. The following lemma has been proved in [16, Corollary 11.13.1] for a finite index set I and the same proof remains valid without any modification for countable I. We will need this result in Chapter 2.

Lemma 1.1. *Let $i \in I^{\mathrm{im}}$ and $\alpha \in \Delta_+ \backslash \{\alpha_i\}$ such that $\alpha(h_i) < 0$. Then $\alpha + j\alpha_i \in \Delta_+$ for all $j \in \mathbb{Z}_+$.* $\square$

Remark. Although Kac–Moody algebras are constructed similarly as Borcherds algebras using generalized Cartan matrices (see [16] for details), the theory of Borcherds algebras includes examples which behave in a very different way from Kac–Moody algebras. The main point of difference is that one is allowed to have imaginary simple roots.

1.4 Weyl group of a Borcherds algebra

We denote by $R = Q \otimes_{\mathbb{Z}} \mathbb{R}$ the real vector space spanned by Π. There exists a symmetric bilinear form on R given by $(\alpha_i, \alpha_j) := \epsilon_i a_{ij}$ for $i, j \in I$. For $i \in I^{\mathrm{re}}$, define the linear isomorphism $\mathbf{s}_i$ of R by

$$\mathbf{s}_i(\lambda) := \lambda - \lambda(h_i)\alpha_i = \lambda - 2\frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i, \quad \lambda \in R.$$

The Weyl group W of $\mathfrak{g}$ is the subgroup of $\mathrm{GL}(R)$ generated by the simple reflections $\mathbf{s}_i$, $i \in I^{\mathrm{re}}$. W is a Coxeter group with canonical generators $\mathbf{s}_i, i \in I^{\mathrm{re}}$. Moreover the above bilinear form is W–invariant. We denote by $\ell(w) := \min\{k \in \mathbb{N} : w = \mathbf{s}_{i_1} \cdots \mathbf{s}_{i_k}\}$ the length of $w \in W$ and any decomposition $w = \mathbf{s}_{i_1} \cdots \mathbf{s}_{i_k}$ with $k = \ell(w)$ is called a reduced expression. We denote by $\Delta^{\mathrm{re}} = W(\Pi^{\mathrm{re}})$ the set of real roots and $\Delta^{\mathrm{im}} = \Delta \backslash \Delta^{\mathrm{re}}$ the set of imaginary roots. Equivalently, a root α is imaginary if and only if $(\alpha, \alpha) \leq 0$ and else real. We can extend $(.,.)$ to a symmetric form on $(\mathfrak{E} \ltimes \mathfrak{h})^*$ satisfying $(\lambda, \alpha_i) = \lambda(\epsilon_i h_i)$ and also $\mathbf{s}_i$ to a linear isomorphism of $(\mathfrak{E} \ltimes \mathfrak{h})^*$ by

$$\mathbf{s}_i(\lambda) = \lambda - \lambda(h_i)\alpha_i, \quad \lambda \in (\mathfrak{E} \ltimes \mathfrak{h})^*.$$

Let ρ be any element of $(\mathfrak{C} \ltimes \mathfrak{h})^*$ satisfying $2(\rho, \alpha_i) = (\alpha_i, \alpha_i)$ for all $i \in I$. The following denominator identity has been proved in [5], see also [14, Theorem 3.16].

The denominator identity

$$U := \sum_{w \in W} (-1)^{\ell(w)} \sum_{\gamma \in \Omega} (-1)^{\mathrm{ht}(\gamma)} e^{w(\rho - \gamma) - \rho} = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha} \tag{1.2}$$

where Ω is the set of all $\gamma \in Q_+$ such that γ is a finite sum of mutually orthogonal distinct imaginary simple roots. Note that $0 \in \Omega$ and $\alpha \in \Omega$, if α is an imaginary simple root.

Chapter 2

Generalized chromatic polynomials and root multiplicities of Borcherds algebras

The results of this chapter have appeared in [2].

The aim of this chapter is to give a combinatorial formula for certain root multiplicities of Borcherds algebras using tools from algebraic graph theory.

In what follows we associate a graph G to a given Borcherds algebra $\mathfrak{g}$ and establish a connection between certain root multiplicities of $\mathfrak{g}$ and the generalized chromatic polynomials of G. The main results of this chapter are Theorem 2.1 and the closed form formula stated in Corollary 2.10.

2.1 Vertex multicoloring of G

In this section we discuss the notion of proper vertex multicoloring of a graph, which is a generalization of the usual notion of graph coloring. For more details about multicoloring of a graph we refer to [13]. Let G be a graph with vertex set I (finite or infinite). For a tuple of non-negative integers $\mathbf{k} = (k_i : i \in I)$, we define $\operatorname{supp}(\mathbf{k}) := \{i \in I : k_i \neq 0\}$ and $\operatorname{ht}\mathbf{k} := \sum_{i\in I} k_i$ whenever $\operatorname{supp}(\mathbf{k})$ is finite.

Definition 2.1. Let G be an undirected graph, with (possibly infinite) vertex set I, edge set E and $\{i, j\}$ denotes the edge between the nodes i and j. Let $\mathbf{k} = (k_i : i \in I)$ be a tuple of non-negative integers, such that all but finitely many of the k_i are zero.

For $q \in \mathbb{N}$, we let $[q] := \{1, 2, \cdots, q\}$ and $\mathcal{P}([q])$ be the set of all subsets of $[q]$.

A map $\tau : I \to \mathcal{P}([q])$ satisfying the following conditions will be called a proper vertex multicoloring of G of weight $\mathbf{k}$ with q colors:

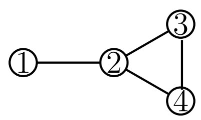
(i) For all $i \in I$ we have $|\tau(i)| = k_i$.

(ii) For all $i, j \in I$ such that $\{i, j\} \in E$ we have $\tau(i) \cap \tau(j) = \emptyset$.

We let $\pi^G_{\mathbf{k}}(q)$ be the number of such maps τ. This turns out to be a polynomial in q, and is called the $\mathbf{k}$-*generalized chromatic polynomial* (or simply the generalized chromatic polynomial, when $\mathbf{k}$ is fixed). When I is finite and all $k_i = 1$, this reduces to the usual chromatic polynomial of G.

Example 2.2. We consider the graph with numbered vertices :

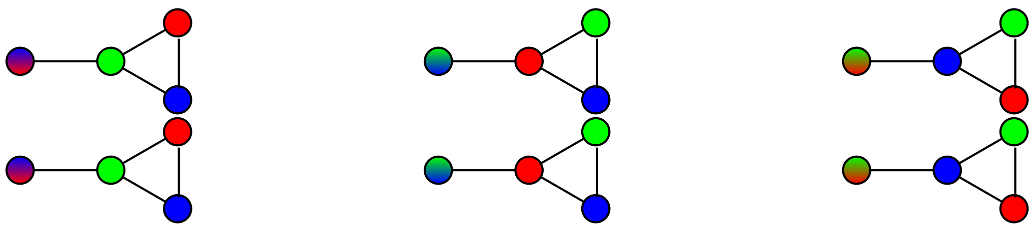
Figure 2.1: An example



We allow 3 colors, say blue, red and green and fix $\mathbf{k} = (2, 1, 1, 1)$.

Below we have listed all proper vertex multicolorings

Figure 2.2: Graph Multicoloring



Multicoloring plays an important role in algebraic graph theory. One of its important applications is to the problem of scheduling dependent jobs on multiple machines. When all jobs have the same execution times, this is modeled by a graph coloring problem and as a graph multicoloring problem for arbitrary execution times. The vertices in the graph represent the jobs and an edge in the graph between two vertices forbids scheduling these jobs simultaneously. For more details and examples we refer to [13].

2.2 Generalized chromatic polynomial $\pi^G_{\mathbf{k}}(q)$

We retain the notations of the previous section. The generalized chromatic polynomial $\pi^G_{\mathbf{k}}(q)$ has the following well-known description [13]. We denote by $P_m(\mathbf{k}, G)$ the set of all ordered m-tuples $(S_1, \dots, S_m)$ such that:

(i) each S_i is a non-empty independent subset of I, i.e. no two vertices have an edge between them,

(ii) the disjoint union of $S_1, \cdots, S_m$ is equal to the multiset $\{\underbrace{i, \dots, i}_{k_i \text{ times}} : i \in I\}$.

Then we have

$$\pi^G_{\mathbf{k}}(q) = \sum_{m \geq 0} |P_m(\mathbf{k}, G)| \binom{q}{m}. \tag{2.1}$$

2.3 Ordinary and Generalized chromatic polynomials

There is a close relationship between the ordinary chromatic polynomials and the generalized chromatic polynomials. We have

$$\pi_{\mathbf{k}}^G(q) = \frac{1}{\mathbf{k}!}\,\chi(G(\mathbf{k}), q) = \frac{1}{\mathbf{k}!}\,\pi_{\mathbf{1}}^{G(\mathbf{k})}(q)$$

where $\chi(G(\mathbf{k}), q) = \pi_{\mathbf{1}}^{G(\mathbf{k})}(q)$ is the chromatic polynomial of the graph $G(\mathbf{k})$ and $\mathbf{k}! = \prod_{i\in I} k_i!$. The graph $G(\mathbf{k})$ is the join of G with respect to $\mathbf{k}$ which is constructed as follows: For each $j \in \operatorname{supp}(\mathbf{k})$, take a clique (complete graph) of size k_j with vertex set $\{j^1, \ldots, j^{k_j}\}$ and join all vertices of the r–th and s–th cliques if $\{r, s\} \in E(G)$. Note that since $|\operatorname{supp}\mathbf{k}| < \infty$, $G(\mathbf{k})$ is a finite graph, even if G is not.

2.4 Graph associated to a Borcherds algebra

Let $A = (a_{ij})_{i,j\in I}$ be a Borcherds-Cartan matrix defined in (1.2) and let $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding Borcherds algebra. We associate a graph G to $\mathfrak{g}$ as follows: G has vertex set I with an edge between two vertices i and j if $a_{ij} \neq 0$ for $i, j \in I$, $i \neq j$. G is a simple (finite or infinite) graph; we call it the graph of $\mathfrak{g}$.

A finite subset $S \subseteq I$ is said to be *connected* if the subgraph induced by S is connected.

2.5 Bond Lattice

Let $\mathfrak{g}$ be a Borcherds algebra and G be its graph. We make the following important assumption on $\mathbf{k}$ for the rest of the thesis:

$$k_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I^{\mathrm{im}}, \quad k_i \in \{0,1\} \text{ for } i \in I^{\mathrm{re}}. \tag{2.2}$$

To state our main theorem, we need the following definition:

Definition 2.3. Let $\mathbf{k}$ satisfy (2.2). The $\mathbf{k}$-*weighted bond lattice* $L_G(\mathbf{k})$ of G is the set of all $\mathbf{J} = \{J_1, \dots, J_k\}$ satisfying the following properties:

(i) $\mathbf{J}$ is a multiset, i.e. we allow $J_i = J_j$ for $i \neq j$.

(ii) Each J_i is a multisubset of I, i.e., a multiset whose underlying set is a subset of I.

(iii) The subgraph of G induced by the underlying set of J_i is connected for each i.

(iv) The disjoint union of $J_1 \dot{\sqcup} \cdots \dot{\sqcup} J_k = \{\underbrace{i, \dots, i}_{k_i \text{ times}} : i \in I\}$.

Let $\mathbf{J} = \{J_1, J_2, \cdots, J_k\} \in L_G(\mathbf{k})$ and J be a finite multisubset of I. We denote by $D(J, \mathbf{J})$ the number of $1 \leq i \leq k$ such that $J_i = J$.

For a finite multisubset J of I, we let $\beta(J) := \sum_{i \in J} \alpha_i$ and $\mathrm{mult}(\beta(J)) := \dim \mathfrak{g}_{\beta(J)}$. Finally, let $\eta(\mathbf{k}) := \sum_{i \in I} k_i \alpha_i$ and $\varepsilon(\mathbf{k}) := (-1)^{\sum k_i}$.

We record the following lemma which will be needed later.

Lemma 2.4. *Let $\mathbf{k}$ satisfy (2.2). Let $\mathcal{P}$ be the collection of multisets $\gamma = \{\beta_1, \dots, \beta_r\}$ (we allow $\beta_i = \beta_j$ for $i \neq j$) such that each $\beta_i \in \Delta_+$ and $\beta_1 + \cdots + \beta_r = \eta(\mathbf{k})$. The map $\Psi : L_G(\mathbf{k}) \to \mathcal{P}$ defined by $\{J_1, \dots, J_k\} \mapsto \{\beta(J_1), \dots, \beta(J_k)\}$ is a bijection.*

Proof. If $\alpha \in \left(Q_+ \cap \sum_{j\in\Pi^{\mathrm{re}}} \mathbb{Z}_{\leq 1}\alpha_j\right)$ is non-zero and the support of α is connected, then $\alpha \in \Delta_+^{\mathrm{re}}$. Moreover, if $\alpha \in \Delta_+$ and $\alpha_i \in \Pi^{\mathrm{im}}$ is such that the support of $\alpha + \alpha_i$ is connected, then by Lemma 1.1 we have that $\alpha + \alpha_i \in \Delta_+$. This shows that each $\beta(J_r)$ is a positive root and hence the map is well–defined. The map is obviously injective and since $\alpha \in \Delta_+$ implies that α has connected support, we also obtain that Ψ is surjective. □

2.6 Main theorem

The following is our main theorem which provides a Lie theoretic interpretation of generalized chromatic polynomials.

Theorem 2.1. *Let G be the graph of a Borcherds algebra $\mathfrak{g}$ and let $\mathbf{k}$ be as in (2.2). Then*

$$\pi_{\mathbf{k}}^G(q) = \varepsilon(\mathbf{k}) \sum_{\mathbf{J}\in L_G(\mathbf{k})} (-1)^{|\mathbf{J}|} \prod_{J\in\mathbf{J}} \binom{q\,\mathrm{mult}(\beta(J))}{D(J,\mathbf{J})}$$

where $|\mathbf{J}|$ is the number of parts (counted with multiplicity) in the partition $\mathbf{J}$.

Remark. The above theorem is a generalization of [32, Theorem 1.1] where the authors considered the special case when $\mathfrak{g}$ is a Kac–Moody algebra and $k_i = 1$ for all $i \in I$.

The rest of this chapter is devoted to the proof of Theorem 2.1.

2.7 Main lemma

Let W be the Weyl group of $\mathfrak{g}$ [Section 1.4] and let $w \in W$. We fix a reduced expression $w = \mathbf{s}_{i_1}\cdots\mathbf{s}_{i_k}$ and let $I(w) = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$. Note that $I(w)$ is independent of the choice of the reduced expression of w. Let Ω be the set of all $\gamma \in Q_+$ such that γ is a finite sum of mutually orthogonal distinct imaginary simple roots (see equation (1.2)). For $\gamma \in \Omega$ we set $I(\gamma) = \{\alpha \in \Pi^{im} : \alpha \text{ is a summand of } \gamma\}$. Note that α is a summand of such γ iff $\alpha \in \operatorname{supp} \gamma$. We naturally identify $I(w)$ and $I(\gamma)$ as subsets of the vertex set I of G and by using this identification we define

$$\mathcal{J}(\gamma) = \{w \in W \backslash \{e\} : I(w) \cup I(\gamma) \text{ is an independent set in } G\}.$$

Note that $\mathcal{J}(0)$ gives the set of independent subsets of Π^{re}. The following lemma is the generalization of [32, Lemma 2.3] in the setting of Borcherds algebras.

Lemma 2.5. *Let $w \in W$ and $\gamma \in \Omega$. We write $\rho - w(\rho) + w(\gamma) = \sum_{\alpha \in \Pi} b_\alpha(w, \gamma)\alpha$. Then we have*

(i) $b_\alpha(w,\gamma) \in \mathbb{Z}_+$ *for all* $\alpha \in \Pi$ *and* $b_\alpha(w,\gamma) = 0$ *if* $\alpha \notin I(w) \cup I(\gamma)$,

(ii) $I(w) = \{\alpha \in \Pi^{\mathrm{re}} : b_\alpha(w,\gamma) \geq 1\}$ *and* $b_\alpha(w,\gamma) = 1$ *if* $\alpha \in I(\gamma)$,

(iii) If $w \in \mathcal{J}(\gamma)$, *then* $b_\alpha(w,\gamma) = 1$ *for all* $\alpha \in I(w) \cup I(\gamma)$ *and* $b_\alpha(w,\gamma) = 0$ *else,*

(iv) If $w \notin \mathcal{J}(\gamma) \cup \{e\}$, *then there exists* $\alpha \in \Pi^{\mathrm{re}}$ *such that* $b_\alpha(w,\gamma) > 1$.

Proof. We start proving (i) and (ii) by induction on $\ell(w)$. If $\ell(w) = 0$, the statement is obvious. So let $\alpha \in \Pi^{\mathrm{re}}$ such that $w = s_\alpha u$ and $\ell(w) = \ell(u) + 1$. Then

$$\begin{aligned} \rho - w(\rho) + w(\gamma) &= \rho - s_\alpha u(\rho) + s_\alpha u(\gamma) \\ &= \rho - u(\rho) + u(\gamma) + 2\frac{(\rho, u^{-1}\alpha)}{(\alpha,\alpha)}\alpha - 2\frac{(\gamma, u^{-1}\alpha)}{(\alpha,\alpha)}\alpha. \end{aligned} \tag{2.3}$$

So by our induction hypothesis we know $\rho - u(\rho) + u(\gamma)$ has the required property and since $\ell(w) = \ell(u) + 1$, we also know $u^{-1}\alpha \in \Delta^{\mathrm{re}} \cap \Delta_+$. Note that $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$ for all $\alpha_i \in \Pi^{\mathrm{re}}$ implies $2\frac{(\rho, u^{-1}\alpha)}{(\alpha,\alpha)} \in \mathbb{N}$. Furthermore, γ is a sum of imaginary simple roots and $a_{ij} \leq 0$ whenever $i \neq j$. Hence $-2\frac{(\gamma, u^{-1}\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}_+$ and the proof of (i) and (ii) is done, since $I(w) = I(u) \cup \{\alpha\}$. If $w \in \mathcal{J}(\gamma)$ and $\alpha \in I(w) \cup I(\gamma)$, we have

$(\rho, u^{-1}\alpha) = (\rho, \alpha) = \frac{1}{2}(\alpha, \alpha)$ and $(\gamma, u^{-1}\alpha) = (\gamma, \alpha) = 0$. So part (iii) follows from (2.3) and an induction argument on $\ell(w)$ since $I(w) = I(u) \cup \{\alpha\}$ and $u \in \mathcal{J}(\gamma)$. It remains to prove part (iv), which will be proved again by induction. If $w = s_\alpha$ we have

$$\rho - w(\rho) + w(\gamma) = \alpha + \gamma - 2\frac{(\gamma, \alpha)}{(\alpha, \alpha)}\alpha.$$

Since $w \notin \mathcal{J}(\gamma) \cup \{e\}$ we get $-2\frac{(\gamma,\alpha)}{(\alpha,\alpha)} \in \mathbb{N}$. For the induction step we write $w = s_\alpha u$. We have $s_\alpha \notin \mathcal{J}(\gamma) \cup \{e\}$ or $u \notin \mathcal{J}(\gamma) \cup \{e\}$. In the latter case we are done by using the induction hypothesis and (2.3). Otherwise we can assume that $u \in \mathcal{J}(\gamma) \cup \{e\}$ and hence $(\gamma, u^{-1}\alpha) = (\gamma, \alpha) < 0$ and $2\frac{(\rho,u^{-1}\alpha)}{(\alpha,\alpha)} \in \mathbb{N}$. Now interpreting this in (2.3) gives the result. □

2.8 Denominator identity and the generalized chromatic polynomial

The following proposition is an easy consequence of Lemma 2.5 and will be needed in the proof of Theorem 2.1. Recall that U is the sumside (left hand side expression) of the denominator identity (1.2).

Proposition 2.6. *Let $q \in \mathbb{Z}$. We have*

$$U^q[e^{-\eta(\mathbf{k})}] = (-1)^{\mathrm{ht}(\eta(\mathbf{k}))}\ \pi^G_{\mathbf{k}}(q),$$

where $U^q[e^{-\eta(\mathbf{k})}]$ denotes the coefficient of $e^{-\eta(\mathbf{k})}$ in U^q.

Proof. If $q = 0$, then there is nothing to prove. So assume that $0 \neq q \in \mathbb{Z}$. We have

$$U^q = \sum_{k \geq 0} \binom{q}{k} (U - 1)^k, \quad \text{where } \binom{q}{k} = \frac{q(q-1)\cdots(q-(k-1))}{k!}.$$

Note that

$$\binom{-q}{k} = (-1)^k \binom{q+k-1}{k}, \quad \text{for } q \in \mathbb{N}.$$

From Lemma 2.5 we get

$$w(\rho) - \rho - w(\gamma) = -\gamma - \sum_{\alpha \in I(w)} \alpha, \quad \text{for } w \in \mathcal{J}(\gamma) \cup \{e\}$$

and thus $(U-1)^k[e^{-\eta(\mathbf{k})}]$ is equal to

$$\left(\sum_{w \in \mathcal{J}(0)} (-1)^{\ell(w)} e^{-\sum_{\alpha \in I(w)} \alpha} + \sum_{\gamma \in \Omega \setminus \{0\}} (-1)^{\mathrm{ht}(\gamma)} \sum_{w \in \mathcal{J}(\gamma) \cup \{e\}} (-1)^{\ell(w)} e^{-\gamma - \sum_{\alpha \in I(w)} \alpha} \right)^k [e^{-\eta(\mathbf{k})}].$$

Hence the coefficient is given by

$$\sum_{\substack{(\gamma_1, \ldots, \gamma_k) \\ (w_1, \ldots, w_k)}} (-1)^{\sum_{i=1}^k \mathrm{ht}(\gamma_i)} (-1)^{\ell(w_1 \cdots w_k)},$$

where the sum ranges over all k–tuples $(\gamma_1, \ldots, \gamma_k) \in \Omega^k$ (repetition is allowed) and $(w_1, \ldots, w_k)$ such that

- $w_i \in \mathcal{J}(\gamma_i) \cup \{e\}, \ 1 \leq i \leq k,$
- $I(w_1) \,\dot\sqcup \cdots \dot\sqcup\, I(w_k) = \{\alpha_i : i \in I^{\mathrm{re}}, k_i = 1\},$
- $I(w_i) \cup I(\gamma_i) \neq \emptyset$ for each $1 \leq i \leq k,$
- $\gamma_1 + \cdots + \gamma_k = \sum_{i \in I^{\mathrm{im}}} k_i \alpha_i.$

It follows that $\big(I(w_1) \cup I(\gamma_1), \ldots, I(w_k) \cup I(\gamma_k)\big) \in P_k\big(\mathbf{k}, G\big)$ and each element is obtained in this way. So the sum ranges over all elements in $P_k(\mathbf{k}, G)$. Since $w_1 \cdots w_k$ is a Coxeter element we get

$$(-1)^{\ell(w_1 \cdots w_k)} = (-1)^{|\{i \in I^{\mathrm{re}} : k_i = 1\}|},$$

and hence $(U-1)^k[e^{-\eta(\mathbf{k})}]$ is equal to $(-1)^{\mathrm{ht}(\eta(\mathbf{k}))}|P_k(\mathbf{k},G)|$ which finishes the proof.

□

2.9 Proof of the main theorem

Now we are able to prove Theorem 2.1 by using the denominator identity (1.2). Proposition 2.6 and (1.2) together imply that the generalized chromatic polynomial $\pi^G_{\mathbf{k}}(q)$ is given by the coefficient of $e^{-\eta(\mathbf{k})}$ in

$$(-1)^{\mathrm{ht}(\eta(\mathbf{k}))}\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{q\dim\mathfrak{g}_\alpha}. \tag{2.4}$$

Expanding (2.4) and using Lemma 2.4 finishes the proof.

2.10 Root multiplicity formula for Borcherds algebras

In this section we prove a corollary of Theorem 2.1 which gives a combinatorial formula for certain root multiplicities. We consider the algebra of formal power series $\mathcal{A}:=\mathbb{C}[[X_i : i\in I]]$. For a formal power series $\zeta\in\mathcal{A}$ with constant term 1, its logarithm $\log(\zeta)=-\sum_{k\geq 1}\frac{(1-\zeta)^k}{k}$ is well–defined.

Corollary. *We have*

$$\operatorname{mult}\eta(\mathbf{k})=\sum_{\ell|\mathbf{k}}\frac{\mu(\ell)}{\ell}\;|\pi^G_{\mathbf{k}/\ell}(q)[q]|, \tag{2.5}$$

where $|\pi^G_{\mathbf{k}}(q)[q]|$ denotes the absolute value of the coefficient of q in $\pi^G_{\mathbf{k}}(q)$ and μ is the Möbius function.

Proof. We consider U as an element of $\mathbb{C}[[e^{-\alpha_i} : i \in I]]$. From the proof of Proposition 2.6 we obtain that the coefficient of $e^{-\eta(\mathbf{k})}$ in $-\log U$ equals

$$(-1)^{\mathrm{ht}(\eta(\mathbf{k}))} \sum_{k \geq 1} \frac{(-1)^k}{k} |P_k(\mathbf{k}, G)|$$

which by (2.1) is equal to $|\pi^G_{\mathbf{k}}(q)[q]|$ by a stright forward computation. Now applying $-\log$ to the right hand side of the denominator identity (1.2) gives

$$\sum_{\substack{\ell \in \mathbb{N} \\ \ell | \mathbf{k}}} \frac{1}{\ell} \text{ mult } \eta\left(\mathbf{k}/\ell\right) = |\pi^G_{\mathbf{k}}(q)[q]|. \tag{2.6}$$

The corollary is now an easy consequence of the Möbius inversion formula. □

Remark. The generalized chromatic polynomial $\pi^G_{\mathbf{k}}(q)$ can be computed explicitly for many families of graphs and hence (2.5) gives an effective method to compute the root multiplicities; the case of complete graphs and trees is treated at the end of this section.

2.11 Examples

1. Let $G = K_n$ be the complete graph with n vertices and $\mathbf{k} = (k_1, \dots, k_n)$ be a tuple of positive integers. Vertex 1 can receive any k_1 colors from the given q colors. Vertex 2 cannot receive those k_1 colors that were assigned to vertex 1 and that is the only restriction that we have. Hence vertex 2 can receive any k_2 colors from the remaining $q - k_1$ colors. Similarly the vertex 3 can receive any k_3 colors from the remaining $q - (k_1 + k_2)$ colors. Continuing in this way, we get that the generalized chromatic polynomial is given by

$$\pi^{K_n}_{\mathbf{k}}(q) = \binom{q}{k_1}\binom{q-k_1}{k_2}\binom{q-(k_1+k_2)}{k_3} \cdots \binom{q-(k_1+\cdots+k_{n-1})}{k_n}.$$

In particular $|\pi_{\mathbf{k}}^{K_n}(q)[q]| = \frac{(k_1+\cdots+k_n-1)!}{k_1!\cdots k_n!}$. Hence we recover Witt's formula proved in [34]:

$$\text{mult }\eta(\mathbf{k}) = \frac{1}{\text{ht }\mathbf{k}} \sum_{\ell|\mathbf{k}} \mu(\ell) \frac{(\text{ht }\mathbf{k}/\ell)!}{(\mathbf{k}/\ell)!}.$$

2. Let $G = T_n$ be a tree with n vertices and $\mathbf{k} = (k_1, \dots, k_n)$ a tuple of positive integers. Assume that the vertex set $I = \{1, \dots, n\}$ of G is ordered in such a way that the vertex i is a leaf of the subgraph of G spanned by the vertices $\{i, i+1, \dots, n\}$. Further, let i' the unique vertex adjacent to i for each $1 \leq i \leq n-1$. We denote by G' the subgraph obtained from G by deleting vertex 1 and claim that each vertex multicoloring of G' gives $\binom{q-k_{1'}}{k_1}$ distinct vertex multicolorings of G. Fix a multicoloring of G', then vertex 1 is colored with k_1 distinct multicolors. Now it is easy to see that vertex 1 cannot be colored by the colors which are used to color vertex $1'$ and this is the only restriction that we have. Hence we can choose any k_1 colors among the $q - k_{1'}$ remaining colors to color vertex 1. This proves that

$$\pi_{\mathbf{k}}^{T_n}(q) = \binom{q - k_{1'}}{k_1} \pi_{\mathbf{k}'}^{G'}(q),$$

where $\mathbf{k}' = (k_2, \dots, k_n)$. Now we can repeat this procedure and obtain together with $\pi_{k_n}^{\{n\}}(q) = \binom{q}{k_n}$ that

$$\pi_{\mathbf{k}}^{T_n}(q) = \binom{q - k_{1'}}{k_1}\binom{q - k_{2'}}{k_2} \cdots \binom{q - k_{(n-1)'}}{k_{n-1}}\binom{q}{k_n}.$$

In particular we get a Witt type formula for trees:

$$\text{mult }\eta(\mathbf{k}) = \sum_{\ell|\mathbf{k}} \mu(\ell) \binom{\frac{k_1+k_{1'}}{\ell} - 1}{\frac{k_1}{\ell}}\binom{\frac{k_2+k_{2'}}{\ell} - 1}{\frac{k_2}{\ell}} \cdots \binom{\frac{k_{n-1}+k_{(n-1)'}}{\ell} - 1}{\frac{k_{n-1}}{\ell}} (1/k_n).$$

3. Let G be a graph obtained from the complete graph K_n by removing the edges $(1, j+1), \dots, (1, n)$ for some $2 \leq j \leq n-1$ and $\mathbf{k} = (k_1, \dots, k_n)$ be a tuple of

positive integers. Vertex 1 can receive any k_1 colors from the given q colors. As before vertex 2 cannot receive those k_1 colors that were assigned to vertex 1 and that is the only restriction that we have. Hence vertex 2 can receive any k_2 colors from the remaining $q - k_1$ colors. Continuing this way, we see that vertex j can receive any k_j colors from the remaining $q-(k_1+\cdots+k_{j-1})$ colors and vertex $j+1$ cannot receive those $k_2 + \cdots + k_j$ colors that were assigned to vertices $2, \ldots, j$ and that is the only restriction that we have since the vertex $j+1$ is not connected with the vertex 1. So vertex $j+1$ can receive any k_{j+1} colors from the remaining $q-(k_2+\cdots+k_j)$ colors. Similarly vertex $j+2$ cannot receive those $k_2+\cdots+k_{j+1}$ colors that were assigned to vertices $2, \ldots, j+1$. So vertex $j+2$ can receive any k_{j+2} colors from the remaining $q-(k_2+\cdots+k_{j+1})$ colors. Again continuing this way we see that vertex n cannot receive those $k_2 + \cdots + k_{n-1}$ colors that were assigned to vertices $2, \ldots, n-1$. So vertex n can receive any k_n colors from the remaining $q - (k_2 + \cdots + k_n)$ colors. Thus the generalized chromatic polynomial is given by

$$\pi_{\mathbf{k}}^G(q) = \binom{q}{k_1}\binom{q-k_1}{k_2}\cdots\binom{q-(k_1+\cdots+k_{j-1})}{k_j}\binom{q-(k_2+\cdots+k_j)}{k_{j+1}} \binom{q-(k_2+\cdots+k_{j+1})}{k_{j+2}}\cdots\binom{q-(k_2+\cdots+k_{n-1})}{k_n}.$$

In particular,

$$|\pi_{\mathbf{k}}^G(q)[q]| = \frac{(k_1+\cdots+k_j-1)!(k_2+\cdots+k_n-1)!}{(k_2+\cdots+k_j-1)!k_1!\cdots k_n!}.$$

Hence we get a Witt type formula for the graph G:

$$\text{(2.7)} \qquad \text{mult } \eta(\mathbf{k}) = \sum_{\ell|\mathbf{k}} \frac{\mu(\ell)}{\ell} \frac{(\frac{k_1}{\ell}+\cdots+\frac{k_j}{\ell}-1)!(\frac{k_2}{\ell}+\cdots+\frac{k_n}{\ell}-1)!}{(\frac{k_2}{\ell}+\cdots+\frac{k_j}{\ell}-1)!(\frac{k_1}{\ell})!\cdots(\frac{k_n}{\ell})!}.$$

4. As our final example, we shall apply Corollary 2.10 to an important example of a Borcherds algebra $\mathfrak{g}(M)$ defined in [15, Section 5.3]. It is well known that if $\mathfrak{c}$ denotes the center of $\mathfrak{g}(M)$, then $\mathfrak{g}(M)/\mathfrak{c}$ is the Monster Lie algebra (see for example [15]). Let $c(n)$ be the coefficients of the q–expansion of the normalized modular invariant $j(q) = q^{-1} + \sum_{n\geq 1} c(n)q^n$ and let $I = \{-1, 1_1, 1_2, \ldots, 1_{c(1)}, 2_1, \ldots, 2_{c(2)}, \ldots\}$ be the countably infinite set where each i occurs $c(i)$ times. We denote by M the symmetric matrix indexed by I where the (i_j, k_l)–th entry equals $-i-k$. Thus M is of the form:

$$M = \left(\begin{array}{c|ccc|ccc|c} 2 & 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots \\ \hline 0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\ \hline -1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\ -1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\ \hline \vdots & & \vdots & & & \vdots & & \end{array}\right)$$

Given $\mathbf{k} = (k_i : i \in I)$, we know that mult $\eta(\mathbf{k})$ can be easily computed from the formula (2.5) by computing the generalized chromatic polynomial of the corresponding graph. We remind the reader that our formula is applicable provided $k_{-1} \leq 1$. It is clear that the subgraph of the graph of $\mathfrak{g}(M)$ spanned by supp$(\mathbf{k})$ is obtained from a complete graph by removing a finite number of edges from the fixed vertex -1. We have calculated the generalized chromatic polynomials for graphs of this type in the previous example. Hence equation (2.7) is a formula for calculating the root multiplicities of the Monster Lie algebra. However it is not an effective formula in the sense that it does not allow for easy computation. Further, the modularity property of these Lie algebras [5] is not evident in this formula.

Chapter 3

Bases for certain root spaces of Borcherds algebras

The results of this chapter have appeared in [2].

In this chapter we will use the combinatorics of Lyndon words to construct a basis for the root space $\mathfrak{g}_{\eta(\mathbf{k})}$ associated to the root $\eta(\mathbf{k})$ (where again all coefficients of the real simple roots are assumed to be less or equal to one). Indeed we will give a family of bases, one for each fixed index $i \in I$.

Since $\mathfrak{g}_{\eta(\mathbf{k})} \neq 0$ *implies that* $supp(\mathbf{k}) := \{i \in I : k_i \neq 0\}$ *is connected, we will assume without loss of generality for the rest of this chapter that* I *is connected and* $I = supp(\mathbf{k})$ *and in particular* I *is finite.* We freely use the notations introduced in the previous chapters.

3.1 The free partially commutative monoid

Let us first fix some notations. Let I^* be the free monoid generated by I. Note that I^* has a total order given by the lexicographical order. The free partially commutative monoid associated with G is denoted by $M(I,G) := I^*/\sim$, where $\sim$ is generated by the relations

$$ab \sim ba, \quad \{a,b\} \notin E(G).$$

Remark. The motivation to look at these monoids has come from the fact: $[e_i, e_j] = 0$ in $\mathfrak{g}$ if $a_{ij} = 0$.

We associate to each element $a \in M(I,G)$ the unique element $\tilde{a} \in I^*$ which is the maximal element in the equivalence class of a with respect to the lexicographical order. A total order on $M(I,G)$ is then given by

$$a < b \quad :\Leftrightarrow \quad \tilde{a} < \tilde{b}. \tag{3.1}$$

Let $\mathbf{w} \in M(I,G)$ and write $\mathbf{w} = i_1 \cdots i_r$. Define $|\mathbf{w}| = r$, $i(\mathbf{w}) = |\{j : i_j = i\}|$ for all $i \in I$ and $\mathrm{supp}(\mathbf{w}) = \{i \in I : i(\mathbf{w}) \neq 0\}$. The weight of $\mathbf{w}$ is denoted by

$$\mathrm{wt}(\mathbf{w}) = \sum_{i \in I} i(\mathbf{w}) \alpha_i.$$

For $i \in I$, its initial multiplicity in $\mathbf{w}$ is defined to be the largest $k \geq 0$ for which there exists $\mathbf{u} \in M(I,G)$ such that $\mathbf{w} = \mathbf{u} i^k$. We define the *initial alphabet* $\mathrm{IA}_{\mathrm{m}}(\mathbf{w})$ of $\mathbf{w}$ to be the multiset in which each $i \in I$ occurs as many times as its initial multiplicity in $\mathbf{w}$. The underlying set is denoted by $\mathrm{IA}(\mathbf{w})$.

For example $\mathrm{IA}_{\mathrm{m}}(1233) = \{3,3\}$, and $\mathrm{IA}(1233) = \{3\}$ for the complete graph $G = K_3$.

The right normed Lie word associated with $\mathbf{w}$ is defined by

$$e(\mathbf{w}) := [e_{i_1}, [e_{i_2}, [\cdots [e_{i_{r-2}}, [e_{i_{r-1}}, e_{i_r}]] \cdots]]] \in \mathfrak{g}. \tag{3.2}$$

Using the Jacobi identity and the other defining relations of $\mathfrak{g}$, it is easy to see that the association $\mathbf{w} \mapsto e(\mathbf{w})$ is well defined.

3.2 Right subword connectedness and initial alphabets

The following lemma is straightforward.

Lemma. *Let* $\mathbf{w} = i_1 \cdots i_r \in M(I, G)$. *Then* $|\mathrm{IA_m}(\mathbf{w})| = 1$ *if and only if* $\mathbf{w}$ *satisfies the following condition:*

$$\textit{given any } 1 \leq k < r \textit{ there exists } k+1 \leq j \leq r \textit{ such that } (i_k, i_j) \in E(G)$$

or equivalently $i_{r-1} \neq i_r$ *and the subgraph generated by* $i_k, \ldots, i_r$ *is connected for any* $1 \leq k < r$.

□

3.3 i-form of a commuting class

For $i \in I$, we introduce the so-called i–form of a given word $\mathbf{w} \in M(I, G)$.

Proposition. *Let* $\mathbf{w} \in M(I, G)$ *with* $\mathrm{IA}(\mathbf{w}) = \{\mathrm{i}\}$. *Then there exists unique* $\mathbf{w}_1, \ldots, \mathbf{w}_{i(\mathbf{w})} \in M(I, G)$ *such that*

1. $\mathbf{w} = \mathbf{w}_1 \cdots \mathbf{w}_{i(\mathbf{w})}$
2. $\mathrm{IA_m}(\mathbf{w_j}) = \{\mathrm{i}\}, \;\; i(\mathbf{w}_j) = 1$ *for each* $1 \leq j \leq i(\mathbf{w})$.

Proof. We prove this result by induction on $i(\mathbf{w})$. If $i(\mathbf{w}) = 1$, there is nothing to prove; so assume that $i(\mathbf{w}) > 1$. We choose an expression $\mathbf{w} = i_1 \cdots i_{r-1} i \in M(I, G)$

of $\mathbf{w}$ such that $i_k = i$, $i_\ell \neq i$ for all $k < \ell < r$ and k is minimal with this property. To be more precise, if $\mathbf{w} = i'_1 \cdots i'_{r-1} i$ is another expression of $\mathbf{w}$, with $i_{k'} = i$ and $i_\ell \neq i$ for all $k' < \ell < r$, then $k' \geq k$. We set $\mathbf{w}_{i(\mathbf{w})} = i_{k+1} \cdots i_{r-1} i$. It is clear that $\mathrm{IA_m}(\mathrm{w_{i(w)}}) = \{\mathrm{i}\}$ and $\mathbf{w} = \mathbf{u}\mathbf{w}_{i(\mathbf{w})}$ where $\mathbf{u} = i_1 \cdots i_k$. The minimality of k implies that $\mathrm{IA}(\mathbf{u}) = \{i\}$. Since $i(\mathbf{u}) = i(\mathbf{w}) - 1$ we get by induction

$$\mathbf{u} = \mathbf{w}_1 \cdots \mathbf{w}_{i(\mathbf{w})-1}$$

such that $\mathrm{IA_m}(\mathrm{w_j}) = \{\mathrm{i}\}$ for each $1 \leq j \leq i(\mathbf{w}) - 1$.

Now we prove the uniqueness part; assume that $\mathbf{w} = \mathbf{w}'_1 \cdots \mathbf{w}'_{i(\mathbf{w})} = \mathbf{u}'\mathbf{w}'_{i(\mathbf{w})}$ is another expression such that $\mathrm{IA_m}(\mathrm{w'_j}) = \{\mathrm{i}\}$ for all $1 \leq j \leq i(\mathbf{w})$. Suppose $\mathbf{w}_{i(\mathbf{w})} \neq \mathbf{w}'_{i(\mathbf{w})}$ in I^*, which is only possible if there exists i_p in $\mathbf{u}$, say $i_p \in \mathbf{w}_p$ with $p < i(\mathbf{w})$ which we can pass through $i_{p+1}, i_{p+2}, \ldots, i_{t-1}, i_t = i \in \mathbf{w}_p$. This contradicts $|\mathrm{IA_m}(\mathrm{w_p})| = 1$. Hence $\mathbf{w}_{i(\mathbf{w})} = \mathbf{w}'_{i(\mathbf{w})}$ and the rest follows again by induction. $\square$

The factorization of $\mathbf{w}$ in Proposition 3.3 is called the i–form of $\mathbf{w}$.

3.4 A combinatorial model for the root multiplicity

Here we will recall the combinatorics of Lyndon words, and using this we define an important combinatorial model. For more details about Lyndon words we refer the reader to [26].

Consider the set $\mathcal{X}_i = \{\mathbf{w} \in M(I,G) : \mathrm{IA_m}(\mathrm{w}) = \{\mathrm{i}\}\}$ and recall that $\mathcal{X}_i$ (and hence $\mathcal{X}_i^*$) is totally ordered using (3.1). We denote by $FL(\mathcal{X}_i)$ the free Lie algebra generated by $\mathcal{X}_i$. A non–empty word $\mathbf{w} \in \mathcal{X}_i^*$ is called a Lyndon word if it satisfies one of the following equivalent definitions:

- $\mathbf{w}$ is strictly smaller than any of its proper cyclic rotations
- $\mathbf{w} \in \mathcal{X}_i$ or $\mathbf{w} = \mathbf{uv}$ for Lyndon words $\mathbf{u}$ and $\mathbf{v}$ with $\mathbf{u} < \mathbf{v}$.

There may be more than one choice of $\mathbf{u}$ and $\mathbf{v}$ with $\mathbf{w} = \mathbf{uv}$ and $\mathbf{u} < \mathbf{v}$ but if $\mathbf{v}$ is of maximal possible length we call it a standard factorization.

To each Lyndon word $\mathbf{w} \in \mathcal{X}_i^*$ we associate a Lie word $L(\mathbf{w})$ in $FL(\mathcal{X}_i)$ as follows. If $\mathbf{w} \in \mathcal{X}_i$, then $L(\mathbf{w}) = \mathbf{w}$ and otherwise $L(\mathbf{w}) = [L(\mathbf{u}), L(\mathbf{v})]$, where $\mathbf{w} = \mathbf{uv}$ is the standard factorization of $\mathbf{w}$. The following result can be found in [26] and is known as the Lyndon basis for free Lie algebras.

Proposition 3.1. *The set* $\{L(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_i^* \text{ is a Lyndon word}\}$ *forms a basis of* $FL(\mathcal{X}_i)$. □

Let $\mathfrak{g}^i$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\{e(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_i\}$ where $e(\mathbf{w})$ is as in (3.2) . By the universal property of $FL(\mathcal{X}_i)$ we have a surjective homomorphism

$$\Phi : FL(\mathcal{X}_i) \to \mathfrak{g}^i, \quad \mathbf{w} \mapsto e(\mathbf{w}) \ \ \forall \, \mathbf{w} \in \mathcal{X}_i. \tag{3.3}$$

Using Proposition 3.1 we immediately get that the image of the above defined set $\{L(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_i^* \text{ is a Lyndon word}\}$ under the map Φ generates $\mathfrak{g}^i$. It is natural to ask if in fact this procedure gives a basis for the root space $\mathfrak{g}_{\eta(\mathbf{k})}$; the main theorem of this chapter gives an answer to this question. Set

$$C^i(\mathbf{k}, G) = \{\mathbf{w} \in \mathcal{X}_i^* : \mathbf{w} \text{ is a Lyndon word}, \ \mathrm{wt}(\mathbf{w}) = \eta(\mathbf{k})\}, \quad \iota(\mathbf{w}) = \Phi \circ \mathrm{L}(\mathbf{w}).$$

Theorem 3.1. *The set* $\{\iota(\mathbf{w}) : \mathbf{w} \in C^i(\mathbf{k}, G)\}$ *is a basis of the root space* $\mathfrak{g}_{\eta(\mathbf{k})}$. *Moreover, if* $k_i = 1$, *the set*

$$\{e(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_i, \ \mathrm{wt}(\mathbf{w}) = \eta(\mathbf{k})\}$$

forms a right–normed basis of $\mathfrak{g}_{\eta(\mathbf{k})}$ *and*

$$C^i(\mathbf{k},G)=\{\mathbf{w}\in\mathcal{X}_i : e(\mathbf{w})\neq 0,\ \mathrm{wt}(\mathbf{w})=\eta(\mathbf{k})\}.$$

So if $k_i=1$, the above theorem implies that the root space $\mathfrak{g}_{\eta(\mathbf{k})}$ has a very special type of basis, namely the set of non-zero right-normed Lie words $e(\mathbf{w})$ of weight $\eta(\mathbf{k})$ (cf. 3.2). Hence, in this case $e(\mathbf{w})$ is either zero or a basis element.

3.5 Main proposition

Here we prove Theorem 3.1 with the help of the following proposition. We set

$$\widetilde{B}^i(\mathbf{k},G):=\{\mathbf{w}\in M(I,G):\mathrm{wt}(\mathbf{w})=\eta(\mathbf{k})\text{ and }\ \mathrm{IA}(\mathbf{w})=\{\mathrm{i}\}\}\,.$$

Let $\mathbf{w}\in\widetilde{B}^i(\mathbf{k},G)$ and write $\mathbf{w}=\mathbf{w}_1\cdots\mathbf{w}_{k_i}\in\mathcal{X}_i^*$ in its i-form (see §3.3). We say $\mathbf{w}$ is aperiodic if the elements in the cyclic rotation class of $\mathbf{w}$ are all distinct, i.e. all elements in

$$C(\mathbf{w}):=\{\mathbf{w}_1\cdots\mathbf{w}_{k_i},\mathbf{w}_2\cdots\mathbf{w}_{k_i}\mathbf{w}_1,\cdots,\mathbf{w}_{k_i}\mathbf{w}_1\cdots\mathbf{w}_{k_i-1}\}$$

are distinct. We naturally identify $C^i(\mathbf{k},G)$ with the set

$$B^i(\mathbf{k},G):=\{\mathbf{w}\in\widetilde{B}^i(\mathbf{k},G):\mathbf{w}\text{ is aperiodic}\}/\sim,$$

where $\mathbf{w}\sim\mathbf{w}'\Leftrightarrow C(\mathbf{w})=C(\mathbf{w}')$. The following proposition is crucial for the proof of Theorem 3.1.

Proposition. *We have*

(i) The root space $\mathfrak{g}_{\eta(\mathbf{k})}$ *is contained in* $\mathfrak{g}^i$*.*

(ii) Let $\mathbf{w} \in M(I,G)$ *and* $\mathrm{wt}(\mathbf{w}) = \eta(\mathbf{k})$. *Then*

$$e(\mathbf{w}) \neq 0 \Longleftrightarrow \mathrm{IA_m}(\mathbf{w}) = \{\mathrm{i}\}.$$

(iii) We have

$$mult\ \eta(\mathbf{k}) = |B^i(\mathbf{k}, G)|.$$

The proof of the above proposition is postponed to the next section. We first show how this proposition proves Theorem 3.1. Since $\mathfrak{g}_{\eta(\mathbf{k})}$ is contained in $\mathfrak{g}^i$ we get with Proposition 3.1 and (3.3) that $\{\iota(\mathbf{w}) : \mathbf{w} \in C^i(\mathbf{k}, G)\}$ is a spanning set for $\mathfrak{g}_{\eta(\mathbf{k})}$ of cardinality equal to $|C^i(\mathbf{k}, G)|$. Therefore Proposition 3.5 (iii) shows that this is in fact a basis. So in the special case when $k_i = 1$ we get that $\{e(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_i,\ \mathrm{wt}(\mathbf{w}) = \eta(\mathbf{k})\}$ is a basis. In order to finish the theorem we have to observe when a Lie word $e(\mathbf{w})$ is non–zero, which is exactly answered by part (ii) of the proposition.

3.6 Proof of Proposition 3.5(i)

This part of the proof is an easy consequence of the following lemma.

Lemma. *Fix an index* $i \in I$. *Then the root space* $\mathfrak{g}_{\eta(\mathbf{k})}$ *is spanned by all right normed Lie words* $e(\mathbf{w})$, *where* $\mathbf{w} \in M(I,G)$ *is such that* $\mathrm{wt}(\mathbf{w}) = \eta(\mathbf{k})$ *and* $\mathrm{IA_m}(\mathbf{w}) = \{\mathrm{i}\}$.

Proof. We fix $\mathbf{w} = i_1 \cdots i_r$ and claim that any element

$$e(\mathbf{w}, k) = \big[[[[e_{i_1}, e_{i_2}], e_{i_3}] \cdots, e_{i_k}], [e_{i_{k+1}}, [e_{i_{k+2}}, \dots [e_{i_{r-1}}, e_{i_r}]]]\big],\ 0 \leq k < r$$

can be written as a linear combination of right normed Lie words $e(\mathbf{w}')$ with $\mathbf{w}' = j_1 \cdots j_{r-1} i$. The claim finishes the proof since $e(\mathbf{w}) = e(\mathbf{w}, 0)$ and $e(\mathbf{w}') \neq 0$ only if

$\mathrm{IA_m}(\mathbf{w}') = \{\mathrm{i}\}$. If $k = r-1$ this follows immediately using repeatedly $[x,y] = -[y,x]$ for all $x, y \in \mathfrak{g}$. If $k < r-1$ we get from the Jacobi identity

$$e(\mathbf{w},k) = e(\mathbf{w},k+1) + \Big[e_{i_{k+1}}, \big[[[[e_{i_1}, e_{i_2}], e_{i_3}] \cdots, e_{i_k}], [e_{i_{k+2}}, \dots [e_{i_{r-1}}, e_{i_r}]]\big]\Big]$$

$$= e(\mathbf{w},k+1) + \big[e_{i_{k+1}}, e(\widetilde{\mathbf{w}}, k+1)\big] \tag{3.4}$$

where $\widetilde{\mathbf{w}} = (i_1, \dots, \widehat{i_{k+1}}, \cdots, i_r)$. An easy induction argument shows that each term in (3.4) has the desired property. □

3.7 Proof of Proposition 3.5(ii)

Now we want to analyze when a right normed Lie word $e(\mathbf{w})$ is non–zero. One has for $\mathbf{w} \in M(I,G)$, $e(\mathbf{w}) \neq 0$ implies $|\mathrm{IA_m}(\mathbf{w})| = 1$. Indeed we prove that the converse is also true.

Lemma. *The right normed Lie word $e(\mathbf{w})$ with $\mathrm{wt}(\mathbf{w}) = \eta(\mathbf{k})$ is non–zero if and only if $|\mathrm{IA_m}(\mathbf{w})| = 1$.*

Proof. Using Lemma 3.2 we prove that, for $\mathbf{w} = i_1 \cdots i_r \in M(I,G)$, the right normed Lie word $e(\mathbf{w})$ is non–zero if and only if $\mathbf{w}$ satisfies the following condition:

$$\text{given any } 1 \leq k < r \text{ there exists } k+1 \leq j \leq r \text{ such that } [e_{i_k}, e_{i_j}] \neq 0. \tag{3.5}$$

The only if part will be proven by induction on r, where the initial step $r = 2$ obviously holds. So assume that $r > 2$ and $e(\mathbf{w})$ satisfies (3.5). We choose $p \in \{1, \dots, r-1\}$ to be minimal such that $i_j \neq i_p$ for all $j > p$. Note that p exists, since $i_{r-1} \neq i_r$. Let $I(p) = \{p_1, \dots, p_{k_{i_p}}\}$ be the elements satisfying $i_p = i_{p_j}$ for $1 \leq j \leq k_{i_p}$. Note that $p_j \neq p$ implies $p_j < p$ by the choice of p. For a subset $S \subset I(r)$ let $\mathbf{w}(S)$ be the tuple obtained from $\mathbf{w}$ by removing the vertices i_q for

$q \in S$. The proof considers two cases.

Case 1: We assume that $p < r-1$. We first show that $e(\mathbf{w}(S))$ satisfies (3.5). If (3.5) is violated, we can find a vertex i_k such that $k < p$, $i_k \neq i_p$ and $[e_{i_k}, e_{i_\ell}] = 0$ for all $\ell > k$ with $i_\ell \neq i_p$. We choose k maximal with that property. By the minimality of p we get $i_k \in \{i_{k+1}, \ldots, i_r\} \backslash \{i_p\} \supset \{i_{r-1}, i_r\}$, say $i_k = i_m$ for some $k+1 \leq m \leq r$. If $m > p$, we get $[e_{i_k}, e_{i_t}] = [e_{i_m}, e_{i_t}] \neq 0$ for some $k+1 \leq t \leq r$ with $i_t \neq i_p$, since $e(\mathbf{w})$ satisfies (3.5) and $p < r-1$. This is not possible by the choice of i_k. Hence $m < p$ and the maximality of k implies the existence of a vertex i_t such that $t > m$, $i_t \neq i_p$ and $[e_{i_k}, e_{i_t}] = [e_{i_m}, e_{i_t}] \neq 0$, which is once more a contradiction. Hence $e(\mathbf{w}(S))$ satisfies (3.5) and is therefore non–zero by induction.

We consider the element $(\text{ad } f_{i_p})^{k_{i_p}} e(\mathbf{w})$. If i_p is a real node, by our assumption on $\mathbf{k}$ we have $k_{i_p} = 1$ and therefore

$$(\text{ad } f_{i_p})e(\mathbf{w}) = -(\alpha_{i_p+1}+\cdots+\alpha_{i_r})(h_{i_p})e(\mathbf{w}(\{p\})) = (a_{i_p,i_p+1}+\cdots+a_{i_p,i_r})e(\mathbf{w}(\{p\})).$$

Since $(a_{i_p,i_p+1} + \cdots + a_{i_p,i_r}) < 0$ ($e(\mathbf{w})$ satisfies (3.5)) and $e(\mathbf{w}(\{p\})) \neq 0$ by the above observation we must have $e(\mathbf{w}) \neq 0$. If i_p is not a real node then we have $a_{i_p,s} \leq 0$ for all $s \in I$. We get

$$(\text{ad } f_{i_r})^{k_{i_r}} e(\mathbf{w}) = Ce(\mathbf{w}(I(r))),$$

for some non-zero constant C. Again we deduce $e(\mathbf{w}) \neq 0$.

Case 2: We assume that $p = r-1$. In this case the rank of our Lie algebra is two. So $i_j \in I = \{1, 2\}$ for all $1 \leq j \leq r$. If $I^{\text{re}} \neq \emptyset$, say $1 \in I^{\text{re}}$, we can finish the proof as follows. If $k_2 = 1$, there is nothing to prove. Otherwise, since $e(\mathbf{w})$ satisfies (3.5) we must have (up to a sign) $e(\mathbf{w}) = (\text{ad } e_2)^{k_2} e_1$. Now $k_2 > 1$ implies $2 \in I^{\text{im}}$ and thus $\mathfrak{g}_{\eta(\mathbf{k})} \neq 0$ which forces $e(\mathbf{w}) \neq 0$. So it remains to consider the case when

$I^{\mathrm{re}} = \emptyset$. In this case $\mathfrak{n}^+$ is the free Lie algebra generated by e_1, e_2 and the lemma is proven. □

3.8 A recursion formula

The rest of this chapter is dedicated to the proof of Proposition 3.5(iii). First we prove that $|\widetilde{B}^i(\mathbf{k}, G)|$ satisfies a recursion relation which is similar to the one in (2.6). More precisely,

Proposition. *We have*

$$|\widetilde{B}^i(\mathbf{k}, G)| = \sum_{\ell|\mathbf{k}} \frac{k_i}{\ell} \left| B^i\left(\frac{\mathbf{k}}{\ell}, G\right) \right|.$$

Proof. There exists a subset $\widehat{B}^i\,(\mathbf{k}, G)$ of $\widetilde{B}^i\,(\mathbf{k}, G)$ such that

$$\{\mathbf{w} \in \widetilde{B}^i(\mathbf{k}, G) : \mathbf{w} \text{ is aperiodic}\} = \dot{\bigsqcup_{\mathbf{w}\in\widehat{\mathbf{B}}^{\mathbf{i}}(\mathbf{k},\mathbf{G})}} C(\mathbf{w}).$$

We clearly have $|\widehat{B}^i\,(\mathbf{k}, G)\,| = |B^i\,(\mathbf{k}, G)\,|$. Let $\ell \in \mathbb{Z}_+$ such that $\ell|\mathbf{k}$ and let $k = \frac{k_i}{\ell}$. We consider the map $\Phi_\ell : \widehat{B}^i\left(\frac{\mathbf{k}}{\ell}, G\right) \to \widetilde{B}^i(\mathbf{k}, G)$ defined by

$$\mathbf{w} = \mathbf{w}_1 \cdots \mathbf{w}_k \; (i\text{-form}) \mapsto \underbrace{(\mathbf{w}_1 \cdots \mathbf{w}_k) \cdots (\mathbf{w}_1 \cdots \mathbf{w}_k)}_{\ell-\text{times}}.$$

Since $\mathbf{w}$ is aperiodic it follows that from the uniqueness of the i–form that $\Phi_\ell(\mathbf{w})$ has exactly k distinct elements in its cyclic rotation class. Choose another divisor ℓ' of $\mathbf{k}$ and an element $\mathbf{w}' \in \widehat{B}^i\left(\frac{\mathbf{k}}{\ell'}, G\right)$ such that $C(\Phi_\ell(\mathbf{w})) \cap C(\Phi_{\ell'}(\mathbf{w}')) \neq \emptyset$. We set $k' = \frac{k_i}{\ell'}$ and assume without loss of generality that $k \geq k'$; say $k = pk' + t$ with

$0 < t \leq k'$, $p \in \mathbb{Z}_+$. Let $\mathbf{v} = \mathbf{v}_1 \cdots \mathbf{v}_k \in C(\mathbf{w})$ and $\mathbf{v}' = \mathbf{v}'_1 \cdots \mathbf{v}'_{k'} \in C(\mathbf{w}')$ such that

$$\underbrace{\mathbf{v}\mathbf{v}\cdots\mathbf{v}}_{\ell-\text{times}} = \underbrace{\mathbf{v}'\mathbf{v}'\cdots\mathbf{v}'}_{\ell'-\text{times}} \in C(\Phi_\ell(\mathbf{w})) \cap C(\Phi_{\ell'}(\mathbf{w}')).$$

By the uniqueness property we get $\mathbf{v}_1 \cdots \mathbf{v}_k = \mathbf{v}_{k-t+1} \cdots \mathbf{v}_k \mathbf{v}_1 \cdots \mathbf{v}_{k-t}$. Since $\mathbf{v}$ is aperiodic we must have $t = k \leq k'$ and thus $\ell = \ell'$. Using once more the uniqueness property we obtain that $C(\mathbf{w}) = C(\mathbf{w}')$ and hence $\mathbf{w} = \mathbf{w}'$. It follows that

$$|\widetilde{B}^i(\mathbf{k}, G)| \geq \sum_{\ell | \mathbf{k}} \frac{k_i}{\ell} \left| B^i\left(\frac{\mathbf{k}}{\ell}, G\right)\right|.$$

Now we prove that any element in $\mathbf{w} \in \widetilde{B}^i(\mathbf{k}, G)$ can be obtained by this procedure, i.e. we have to show that there exists $\ell \in \mathbb{Z}_+$ with $\ell | \mathbf{k}$ such that $\mathrm{Im}(\Phi_\ell) \cap C(\mathbf{w}) \neq \emptyset$. In what follows we construct an element in the aforementioned intersection. Let $\mathbf{w} = \mathbf{w}_1 \cdots \mathbf{w}_{k_i}$ (i-form) $\in \widetilde{B}^i(\mathbf{k}, G)$ arbitrary. Note that the statement is clear if $\mathbf{w}$ is aperiodic. So let $1 \leq r < s \leq k_i$ such that

$$\mathbf{w}_r \cdots \mathbf{w}_{k_i} \mathbf{w}_1 \cdots \mathbf{w}_{r-1} = \mathbf{w}_s \cdots \mathbf{w}_{k_i} \mathbf{w}_1 \cdots \mathbf{w}_{s-1}$$

and suppose that $k := s - r$ is minimal with this property. Write $k_i = \ell k + t$, $0 \leq t < k$. Again by the uniqueness of the i–form we obtain

$$\begin{aligned}
&\mathbf{w}_1 = \mathbf{w}_{k+1} = \cdots = \mathbf{w}_{\ell k+1} = \mathbf{w}_{k_i-k+1},\\
&\mathbf{w}_2 = \mathbf{w}_{k+2} = \cdots = \mathbf{w}_{\ell k+2} = \mathbf{w}_{k_i-k+2},\\
&\vdots\vdots\vdots\\
&\mathbf{w}_t = \mathbf{w}_{k+t} = \cdots = \mathbf{w}_{\ell k+t} = \mathbf{w}_{k_i-k+t},\\
&\mathbf{w}_{t+1} = \mathbf{w}_{k+t+1} = \cdots = \mathbf{w}_{(\ell-1)k+t+1} = \mathbf{w}_{k_i-k+t+1},\\
&\vdots\vdots\vdots\\
&\mathbf{w}_k = \mathbf{w}_{2k} = \cdots = \mathbf{w}_{\ell k} = \mathbf{w}_{k_i}.
\end{aligned}$$

Thus $\mathbf{w}_1 = \cdots \mathbf{w}_{k_i} = \mathbf{w}_{t+1} = \cdots \mathbf{w}_{k_i}\mathbf{w}_1 \cdots \mathbf{w}_t$ and the minimality of k implies $t = 0$. Therefore,

$$\mathbf{w} = \underbrace{(\mathbf{w}_1 \cdots \mathbf{w}_k) \cdots (\mathbf{w}_1 \cdots \mathbf{w}_k)}_{\ell-\text{times}}.$$

If $\mathbf{w}_1 \cdots \mathbf{w}_k$ is aperiodic we can choose the unique element in $\widehat{B}^i\left(\frac{\mathbf{k}}{\ell}, G\right) \cap C(\mathbf{w}_1 \cdots \mathbf{w}_k)$ whose image is obviously an element of $\mathrm{Im}(\Phi_\ell) \cap C(\mathbf{w})$. If not, we continue the above procedure with $\mathbf{w}_1 \cdots \mathbf{w}_k$. This completes the proof. □

Remark. From the proof of Proposition 3.8 we get that any element of $B^i(\mathbf{k}, G)$ is aperiodic if $k_i, i \in I$ are relatively prime.

3.9 Proof of the theorem for the case $\mathbf{k} = \mathbf{1}$

In this section we prove Theorem 3.1 for the special case $\mathbf{k} = \mathbf{1} = (k_i = 1 : i \in I)$ which will be needed later to prove Proposition 3.5(iii). We need the notion of an acyclic orientation for this. An acyclic orientation of a graph is an assignment of a direction to each edge that does not form any directed cycle. A node j is said to be a sink of an orientation if no arrow is incident to j at its tail. We denote by $O_i(G)$ the set of acyclic orientations of G with unique sink i. Greene and Zaslavsky [12, Theorem 7.3] proved a connection between the number of acyclic orientations with unique sink and the chromatic polynomial. In particular, up to a sign

(3.6) $$|O_i(G)| = \text{ the linear coefficient of the chromatic polynomial of } G.$$

Now we are able to prove,

Proposition. *The set*

$$\{e(\mathbf{w}) : \mathbf{w} \in \widetilde{B}^i(\mathbf{1}, G)\}$$

is a basis of the root space $\mathfrak{g}_{\eta(\mathbf{1})}$*.*

Proof. Given any $\mathbf{w} \in \widetilde{B}^i(\mathbf{1}, G)$, we associate an acyclic orientation $G^{\mathbf{w}}$ as follows. Draw an arrow from the node u to v $(u \neq v)$ if and only if u appears on the left of v in $\mathbf{w}$. Note that the assignment $\mathbf{w} \mapsto G^{\mathbf{w}}$ is well defined, i.e. $G^{\mathbf{w}}$ does not depend on the choice of an expression of $\mathbf{w}$. It is easy to see that the node i is the unique sink for the acyclic orientation $G^{\mathbf{w}}$. Thus the assignment $\mathbf{w} \mapsto G^{\mathbf{w}}$ defines a well–defined injective map $\widetilde{B}^i(\mathbf{1}, G) \to O_i(G)$. Since $\{e(\mathbf{w}) : \mathbf{w} \in \widetilde{B}^i(\mathbf{1}, G)\}$ spans $\mathfrak{g}_{\eta(\mathbf{1})}$ we obtain together with Corollary 2.10

$$|O_i(G)| \geq |\widetilde{B}^i(\mathbf{1}, G)| \geq \dim \mathfrak{g}_{\eta(\mathbf{1})} = |\pi^G_{\mathbf{1}}(q)[q]|.$$

Now the proposition follows from (3.6). □

We discuss an example,

Example 3.2. Let G be the triangle graph and $\mathbf{k} = (1, 1, 1)$. Hence $\eta(\mathbf{k})$ is the unique non–divisible imaginary root of the affine Kac–Moody algebra $\mathfrak{g}$; it is well–known that the corresponding root space is two dimensional. We have $\widetilde{B}^3(\mathbf{k}, G) = \{123, 213\}$ and hence a basis is given by the right–normed Lie words

$$[e_1, [e_2, e_3]], \quad [e_2, [e_1, e_3]].$$

As a corollary of the above proposition we get,

Corollary. *The number of acyclic orientations of* G *with unique sink* $i \in I$ *is equal to mult* $\eta(\mathbf{1})$*.* □

3.10 Join of a graph and an another recursion formula

We use the join graph associated to the pair $(G, \mathbf{k})$, defined in the Section 2.3, to determine the cardinality of $\widetilde{B}^i(\mathbf{k}, G)$. The notion of join of a graph is the important tool in this section and hence we are giving here the definition of the same again. For each $j \in I$, take a clique (complete graph) of size k_j with vertex set $\{j^1, \dots, j^{k_j}\}$ and join all vertices of the r–th and s–th clique if $\{r, s\} \in E(G)$. The resulting graph is called the join of G with respect to $\mathbf{k}$, denoted by $G(\mathbf{k})$.

We consider the set of vertices $\{i_1, \dots, i_{k_i}\}$

$$M(i_1, \dots, i_{k_i}) := \widetilde{B}^{i^1}(\mathbf{1}, G(\mathbf{k})) \,\dot{\sqcup} \cdots \dot{\sqcup}\, \widetilde{B}^{i^{k_i}}(\mathbf{1}, G(\mathbf{k}))$$

and define a map $\phi : M(i_1, \dots, i_{k_i}) \to \widetilde{B}^i(\mathbf{k}, G)$ by sending the vertices of the j–th clique $\{j^1, \dots, j^{k_j}\}$ to the vertex j for all $j \in I$. This map is clearly surjective. Let $\mathbf{w} \in M(i_1, \dots, i_{k_i})$ and let $\sigma \in S_{k_j}$, $j \in I$ which acts on $\mathbf{w}$ by permuting the entries in the j–th clique. It is easy to show that $|\mathrm{IA}_{\mathrm{m}}(\sigma(\mathbf{w}))| = 1$ and thus $\sigma(\mathbf{w}) \in M(i_1, \dots, i_{k_i})$. Therefore there is a natural action of the symmetric group

$$\mathbf{S_k} = \prod_{j \in I} S_{k_j} \quad \text{on} \quad M(i_1, \dots, i_{k_i}).$$

It is clear that this action induces a bijective map,

$$\overline{\phi} : M(i_1, \dots, i_{k_i})/\mathbf{S_k} \to \widetilde{B}^i(\mathbf{k}, G).$$

We will conclude our discussion by proving that the above action is free. Let $\sigma \in \mathbf{S_k}$ and $\mathbf{w} \in M(i_1, \dots, i_{k_i})$ such that $\sigma\mathbf{w} = \mathbf{w}$. If σ is not the identity we can find $j \in I$ and $1 \leq \ell < s \leq k_j$ such that $\sigma(\ell) > \sigma(s)$. So if j^ℓ appears on the left of j^k in $\mathbf{w}$, then j^k appears on the left of j^ℓ in $\sigma(\mathbf{w})$. Since $\sigma(\mathbf{w}) = \mathbf{w}$, this is only possible if $(j^\ell, j^k) \in E(G(\mathbf{k}))$ which is a contradiction. Thus we have proved that

$$|\widetilde{B}^i(\mathbf{k}, G)| = \sum_{r=1}^{k_i} \frac{1}{\mathbf{k}!} |\widetilde{B}^{i^r}(\mathbf{1}, G(\mathbf{k}))|. \tag{3.7}$$

3.11 Proof of Proposition 3.5(iii)

We complete the proof of Proposition 3.5(iii). We have

$$\begin{aligned}
\sum_{\ell|\mathbf{k}} \frac{1}{\ell} |B^i\,(\mathbf{k}, G)\,| &= \frac{1}{k_i} |\widetilde{B}^i(\mathbf{k}, G)| && \text{by Proposition 3.8} \\
&= \frac{1}{k_i} \sum_{r=1}^{k_i} \frac{1}{\mathbf{k}!} |\widetilde{B}^{i^r}(\mathbf{1}, G(\mathbf{k}))| && \text{by (3.7)} \\
&= \frac{|\pi_{\mathbf{1}}^{G(\mathbf{k})}(q)[q]|}{\mathbf{k}!} && \text{by Proposition 3.9} \\
&= |\pi_{\mathbf{k}}^{G}(q)[q]| \\
&= \sum_{\ell|\mathbf{k}} \frac{1}{\ell} \text{mult } \eta\left(\frac{\mathbf{k}}{\ell}\right) && \text{by (2.6).}
\end{aligned}$$

Now induction on g.c.d. of $\mathbf{k}$ implies that $|B^i\,(\mathbf{k}, G)\,| = \text{mult } \eta(\mathbf{k})$ for all $\mathbf{k}$.

Example 3.3. We finish this section by determining a basis for the root space $\mathfrak{g}_{\eta(\mathbf{k})}$ where G is the graph defined in Example 2.1 and $\mathbf{k} = (2, 1, 1, 1)$. The following is a list of acyclic orientations of $G(\mathbf{k})$ with unique sink $i = 2$

Figure 3.1: Acyclic orientations

So we have mult $\eta(\mathbf{k}) = 2$ and if $i = 2$, the following right–normed Lie words form a basis of $\mathfrak{g}_{\eta(\mathbf{k})}$

$$[e_4,[e_3,[e_1,[e_1,e_2]]]],\ \ [e_3,[e_4,[e_1,[e_1,e_2]]]].$$

Similarly if $i=1$ we have the basis

$$[e_1,[e_3,[e_4,[e_2,e_1]]]],\ \ [e_1,[e_4,[e_3,[e_2,e_1]]]].$$

Chapter 4

The Hilbert series of tensor powers of the Universal enveloping algebra of Free partially commutative Lie algebras

The results of this chapter have appeared in [2].

In this chapter we study the evaluation of the generalized chromatic polynomial at negative integers. We show that these numbers show up in the Hilbert series of the q-fold tensor product of the universal enveloping algebra associated to free partially commutative Lie algebras. We use this interpretation to give a simple Lie theoretic proof of Stanley's reciprocity theorem of chromatic polynomials [29].

In what follows, we assume that the Borcherds algebra $\mathfrak{g}$ has only imaginary simple roots. In this case, it is easy to see that $\mathfrak{n}^+$ is isomorphic to the free partially commutative Lie algebra associated to G, which is by definition the Lie algebra freely generated by $e_i,\ i \in I$ and defining relations $[e_i, e_j] = 0$, whenever there is no edge between the vertices i and j.

4.1 Hilbert series

We first recall the definition of Hilbert series. Let Γ be a semigroup with at most countably infinite elements and $\mathfrak{a}$ be a Γ–graded Lie algebra with finite–dimensional homogeneous spaces, i.e.

$$\mathfrak{a} = \bigoplus_{\alpha\in\Gamma} \mathfrak{a}_\alpha \quad \text{and} \quad \dim(\mathfrak{a}_\alpha) < \infty \text{ for all } \alpha \in \Gamma.$$

The Γ–grading of $\mathfrak{a}$ induces a $\Gamma \cup \{0\}$–grading on the universal enveloping algebra $\mathbf{U}(\mathfrak{a})$ and we define the Hilbert series as

$$H_\Gamma(U(\mathfrak{a})) = 1 + \sum_{\alpha\in\Gamma} (\dim \mathbf{U}(\mathfrak{a})_\alpha)\ e^\alpha.$$

The proof of the following proposition is standard using the Poincaré–Birkhoff–Witt theorem.

Proposition 4.1. *The Hilbert series of* $\mathbf{U}(\mathfrak{a})$ *is given by*

$$H_\Gamma(\mathbf{U}(\mathfrak{a})) = \frac{1}{\prod\limits_{\alpha\in\Gamma} (1 - e^\alpha)^{\dim \mathfrak{a}_\alpha}}.$$

□

4.2 Hilbert series of $\mathbf{U}(\mathfrak{n}^+)$

Set $\Gamma = Q_+ \backslash \{0\}$. Since $\mathfrak{n}^+$ is Γ–graded, the following statement is a straightforward application of the denominator identity given in (1.2).

Corollary 4.2. *The Hilbert series of* $\mathbf{U}(\mathfrak{n}^+)$ *is given by*

$$H_\Gamma(\mathbf{U}(\mathfrak{n}^+)) = \left(\sum_{\gamma\in\Omega}(-1)^{\mathrm{ht}(\gamma)}e^\gamma\right)^{-1}.$$

Proof. Since all the simple roots of $\mathfrak{g}$ are imaginary, the denominator identity (1.2) (applied to $\mathfrak{n}^+$) becomes

$$\sum_{\gamma\in\Omega}(-1)^{\mathrm{ht}(\gamma)}e^\gamma = \prod_{\alpha\in\Delta_+}(1-e^\alpha)^{\dim\mathfrak{g}_\alpha}$$

Using Proposition 4.1 we see that

$$H_\Gamma(\mathbf{U}(\mathfrak{n}^+)) = \frac{1}{\prod\limits_{\alpha\in\Gamma}(1-e^\alpha)^{\dim\mathfrak{g}_\alpha}} = \frac{1}{\prod\limits_{\alpha\in\Delta_+}(1-e^\alpha)^{\dim\mathfrak{g}_\alpha}}$$

where the last equality follows from $\Delta_+ = \{\alpha\in Q_+\backslash\{0\} : \dim\mathfrak{g}_\alpha \neq 0\}$. Now the result is immediate from the denominator identity. $\square$

4.3 Hilbert series and the generalized chromatic polynomial

In this section, we show that the Hilbert series of the q–fold tensor product of the universal enveloping algebra $\mathbf{U}(\mathfrak{n}^+)$ is determined by the evaluation of the generalized chromatic polynomial at $-q$. We naturally identify $\alpha\in Q_+$ with a tuple of non-negative integers. We prove,

Theorem 4.1. *Let* $q\in\mathbb{N}$. *Then the Hilbert series of* $\mathbf{U}(\mathfrak{n}^+)^{\otimes q}$ *is given by*

$$H_\Gamma(\mathbf{U}(\mathfrak{n}^+)^{\otimes q}) = \sum_{\alpha\in Q_+}(-1)^{\mathrm{ht}(\alpha)}\pi^G_\alpha(-q)\ e^\alpha.$$

where $\pi^G_\alpha(q)$ *is the generalized chromatic polynomial defined in section 2.2.*

In particular,

$$\textit{dim } (\mathbf{U}(\mathfrak{n}^+)^{\otimes q})_\alpha = (-1)^{\mathrm{ht}(\alpha)}\pi^G_\alpha(-q), \quad \textit{for all } \alpha \in Q_+.$$

Proof. We obviously have $H_\Gamma(\mathbf{U}(\mathfrak{n}^+)^{\otimes q}) = H_\Gamma(\mathbf{U}(\mathfrak{n}^+))^q$. From Proposition 2.6, we get

$$\left(\sum_{\gamma\in\Omega}(-1)^{\mathrm{ht}(\gamma)}e^\gamma\right)^{-q} = \sum_{\alpha\in Q_+}(-1)^{\mathrm{ht}(\alpha)}\pi^G_\alpha(-q)\ e^\alpha. \tag{4.1}$$

A straightforward application of Corollary 4.2 finishes the proof. □

4.4 Values of the generalized chromatic polynomial at negative integers

The following is an immediate consequence of Theorem 4.1. For a multisubset S of I we let $\alpha(S) = \sum_{i\in S}\alpha_i$.

Corollary 4.3. *Let* $q \in \mathbb{N}$ *and* $\alpha = \sum_{i\in I} k_i\alpha_i \in Q^+$. *Then the generalized chromatic polynomial of* G *satisfies,*

$$\pi^G_\alpha(-q) = \sum_{(S_1,\dots,S_q)} \pi^G_{\alpha(S_1)}(-1)\cdots\pi^G_{\alpha(S_q)}(-1)$$

where the sum runs over all ordered partitions $(S_1,\dots,S_q)$ *of the multiset*

$$\{\underbrace{i,\dots,i}_{k_i \text{ times}} : i \in I\}. \tag{4.2}$$

Proof. By Theorem 4.1, we get dim $(\mathbf{U}(\mathfrak{n}^+)^{\otimes q})_\alpha = (-1)^{\mathrm{ht}(\alpha)}\pi^G_\alpha(-q)$. But

$$(\mathbf{U}(\mathfrak{n}^+)^{\otimes q})_\alpha = \bigoplus_{(S_1,\dots,S_q)} \mathbf{U}(\mathfrak{n}^+)_{\alpha(S_1)} \otimes \cdots \otimes \mathbf{U}(\mathfrak{n}^+)_{\alpha(S_q)}$$

where the sum runs over all ordered partitions $(S_1, \dots, S_q)$ of the multiset (4.2). Again by Theorem 4.1 we know that dim $(\mathbf{U}(\mathfrak{n}^+))_{\alpha(S_i)} = (-1)^{\mathrm{ht}(\alpha(S_i))}\pi^G_{\alpha(S_i)}(-1)$ for all S_i. Putting all this together we get the desired result. □

4.5 Stanley's reciprocity theorem

We give another application of Theorem 4.1, namely we show how this can be used to give a different proof of Stanley's reciprocity theorem of chromatic polynomials [29, Theorem 1.2] (see Theorem 4.2). Note that the universal enveloping algebra of $\mathfrak{n}^+$ is isomorphic to the free associative algebra generated by e_i, $i \in I$ with relations $e_ie_j = e_je_i$ for all $(i,j) \notin E(G)$. *Assume for the rest of this chapter that G is a finite graph with vertex set I.*

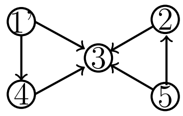
An acyclic orientation of G is an assignment of arrows to its edges such that there are no directed cycles in the resulting directed graph. Suppose we are given an acyclic orientation $\mathcal{O}$ of G. Let

$$e_{\mathcal{O}} = e_{i_1} \cdots e_{i_n}$$

be the unique monomial in $\mathbf{U}(\mathfrak{n}^+)$ determined by the following two conditions: $(i_1, \dots, i_n)$ is a permutation of I and if we have an arrow $i_r \to i_s$ in $\mathcal{O}$, then we require $r < s$. See the below image for an example.

Clearly, this gives a bijective correspondence between the set of acyclic orientations of G and a vector space basis of $\mathbf{U}(\mathfrak{n}^+)_{\alpha(I)}$. For a map $\sigma : I \to \{1, 2, \dots, q\}$,

Figure 4.1: Acyclic orientation corresponds to the word $e_1e_4e_5e_2e_3$



we say $(\sigma, \mathcal{O})$ is a q–compatible pair if for each directed edge $i \to j$ in $\mathcal{O}$ we have $\sigma(i) \geq \sigma(j)$.

Theorem 4.2. *The number of q–compatible pairs of G is equal to $(-1)^{|I|}\pi_{\alpha(I)}(-q)$. In particular, $(-1)^{|I|}\pi_{\alpha(I)}(-1)$ counts the number of acyclic orientations of G.*

Proof. Let $(\sigma, \mathcal{O})$ be a q–compatible pair. For $\mathcal{O}$ we consider the unique monomial

$$e_{\mathcal{O}} = e_{i_1} \cdots e_{i_n}.$$

Set $S_{q-i+1} = \{v \in S : \sigma(v) = i\}$ for $1 \leq i \leq q$ and let $\mathcal{O}_i$ the acyclic orientation of the subgraph spanned by S_i which is induced by $\mathcal{O}$. Then $(S_1, \ldots, S_q)$ is a partition of S and since $(\sigma, \mathcal{O})$ is q–compatible pair, it is easy to see that

$$e_{\mathcal{O}} = e_{\mathcal{O}_1} \cdots e_{\mathcal{O}_q}.$$

On the other hand, suppose $(S_1, \ldots, S_q)$ is a partition of S and let $\mathcal{O}_i$ be an acyclic orientation of the subgraph spanned by S_i for $1 \leq i \leq q$. We extend the acyclic orientations to an acyclic orientation of G. If $(i,j) \in E(G)$ such that $i \in S_r$ and $j \in S_k$ and $r < k$, then we give the orientation $i \to j$. This defines an acyclic orientation $\mathcal{O}$ of G. Further, define $\sigma : S \to \{1, \cdots, q\}$ by $\sigma(S_i) = q - i + 1$ for all $1 \leq i \leq q$. Then it is easy to see that $(\sigma, \mathcal{O})$ is a q–compatible pair. Thus we proved,

$$\# \text{ of } q\text{–compatible pairs of } G = \sum_{(S_1,\ldots,S_q)} (-1)^{|S_1|}\pi^G_{\alpha(S_1)}(-1) \cdots (-1)^{|S_q|}\pi^G_{\alpha(S_q)}(-1)$$

where the sum runs over all $(S_1, \ldots, S_q)$ partitions of S. Now Corollary 4.3 completes the proof.

□

4.6 Dimension formula for the graded spaces of $\mathfrak{n}^+$

We consider the height grading on $\mathfrak{n}^+$

$$\mathfrak{n}^+ = \bigoplus_{k=1}^{\infty} \mathfrak{g}_k, \qquad \mathfrak{g}_k = \bigoplus_{\substack{\text{ht } \alpha = k \\ \alpha \in \Delta_+}} \mathfrak{g}_\alpha$$

and show that in this case the Lucas polynomials show up as coefficients of the Hilbert series when the complement graph is triangle free. The height grading naturally arises in the study of lower central series of right-angled Artin groups; for more details see [25, Theorem 3.4].

In the rest of this chapter we relate $M_k := \dim \mathfrak{g}_k$ to the independent set polynomial of G defined by

$$I(X, G) := \sum_{i \geq 0} c_i X^i$$

where $c_0 = 1$ and $c_i, i \geq 1$ is the number of independent subsets of size i. Using

$$U = \prod_{k \geq 1} (1 - X^k)^{M_k} = \sum_{j \geq 0} (-1)^j \; c_j \; X^j$$

we get by comparing coefficients of X^k on both sides in $-\log\,(U)$ that

$$N_k = \frac{1}{k} \sum_{d|k} \frac{k}{d} \; M_{\frac{k}{d}} = \frac{1}{k} \sum_{d|k} d \; M_d,$$

where

$$N_k := (-1)^{k+1} c_k + \frac{1}{2}(-1)^{k+2} \left\{ \sum_{\substack{j_1+j_2=k \\ j_1,j_2 \geq 1}} c_{j_1} c_{j_2} \right\} + \tag{4.3}$$

$$\frac{1}{3}(-1)^{k+3} \left\{ \sum_{\substack{j_1+j_2+j_3=k \\ j_1,j_2,j_3 \geq 1}} c_{j_1} c_{j_2} c_{j_3} \right\} + \cdots \tag{4.4}$$

Using the Möbius inversion formula we recover a result of [8, Theorem 3],

$$M_k = \sum_{d|k} \frac{\mu(d)}{d} \, N_{\frac{k}{d}}. \tag{4.5}$$

4.7 Lucas polynomials

In this section, we consider a graph G such that G^{c} (the complement graph of G) is triangle free and determine explicitly the numbers N_k. In this case, these numbers are related to the Lucas polynomials $\langle \ell \rangle_{s,t}$ in two variables s and t. The Lucas polynomials are defined by the recurrence relation

$$\langle \ell \rangle_{s,t} = s\langle \ell - 1 \rangle_{s,t} + t\langle \ell - 2 \rangle_{s,t} \quad \text{for} \quad \ell \geq 2$$

with the initial values $\langle 0 \rangle_{s,t} = 2$ and $\langle 1 \rangle_{s,t} = s$. There is a closed formula (see [31, Proposition 2.1])

$$\langle \ell \rangle_{s,t} = \sum_{j \geq 0} \frac{\ell}{\ell - j} \binom{\ell - j}{j} t^j s^{\ell - 2j}.$$

Proposition. *Let G be a graph such that G^{c} is triangle free. Let v be the number of vertices of G^{c} and e the number of edges of G^{c}. Then for $k \geq 1$, we have*

$$N_k = \frac{1}{k} \langle k \rangle_{v,-e}, \quad M_k = \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) \langle d \rangle_{v,-e}.$$

Proof. We have $I(-X,G) = 1 - vX + eX^2$. Hence applying $-\log$ we get,

$$\sum_{k=1}^{\infty} \frac{(vX)^k}{k}(1-(e/v)X)^k = \sum_{k=1}^{\infty} \frac{1}{k}\left(\sum_{r=0}^{k}(-1)^r \binom{k}{r} e^r v^{k-r} X^{k+r}\right).$$

The coefficient of X^k in $-\log\,(1 - vX + eX^2)$ is

$$N_k = \sum_{\substack{p+r=k \\ p\geq r\geq 0}} \frac{(-1)^r}{p}\binom{p}{r} e^r v^{p-r} = \sum_{j\geq 0} \frac{(-1)^j}{k-j}\binom{k-j}{j} e^j v^{k-2j} = \frac{1}{k}\langle k\rangle_{v,-e}.$$

The rest follows from (4.5).

□

Chapter 5

Recurrence formula for the chromatic discriminant of a graph

The results of this chapter have appeared in [1].

Let G be a simple graph and let $\chi(G, q)$ denote its chromatic polynomial. The absolute value of the coefficient of q in $\chi(G, q)$ is known as the *chromatic discriminant* of the graph G [22, 28] and is denoted $\alpha(G)$. It is an important graph invariant with numerous algebraic and combinatorial interpretations. For instance, letting v_0 denote a fixed vertex of the graph G, it is well known that each of the following sets has cardinality $\alpha(G)$:

1. Acyclic orientations of G with unique sink at v_0 [12],
2. Maximum G-parking functions relative to v_0 [3],
3. Minimal v_0-critical states [11, Lemmas 14.12.1 and 14.12.2],
4. Spanning trees of G without broken circuits [4],
5. Conjugacy classes of Coxeter elements in the Coxeter group associated to G [9, 23, 27],

6. Multilinear Lyndon heaps on G [20, 21, 33].

In addition, $\alpha(G)$ is also equal to the dimension of the root space corresponding to the sum of all simple roots in the Kac–Moody Lie algebra associated to G [2, 32].

We have the following recurrence formula for $\alpha(G)$ (see for instance [10]) which is an immediate consequence of the well-known *deletion-contraction rule* for the chromatic polynomial:

$$\alpha(G) = \alpha(G \backslash e) + \alpha(G/e), \tag{5.1}$$

where e is any edge of G. Here, $G\backslash e$ denotes G with e deleted and G/e denotes the simple graph obtained from G by identifying the two ends of e (*i.e., contracting e to a single vertex*) and removing any multiple edges that result.

Yet another recurrence formula for $\alpha(G)$ was obtained in [32] using its connection to root multiplicities of Kac–Moody Lie algebras. To state this, we introduce some notation: for a graph G, we let $V(G)$ and $E(G)$ denote its vertex and edge sets respectively. We say that the ordered pair (G_1, G_2) is an *ordered partition of* G, if G_1 and G_2 are non-empty subgraphs of G whose vertex sets form a partition of $V(G)$, i.e., they are disjoint and their union is $V(G)$. When we don't care about the ordering of G_1, G_2, we call the set $\{G_1, G_2\}$ an *unordered partition of* G. We say that an edge e *straddles* G_1 and G_2 if one end of e is in G_1 and the other in G_2.

We then have:

Proposition. *[32]*

$$2\ e(G)\ \alpha(G) = \sum_{\substack{(G_1,G_2) \\ \textit{ordered partition of } G}} \alpha(G_1)\ \alpha(G_2)\ e(G_1, G_2). \tag{5.2}$$

Here $e(G)$ is the total number of edges in G, $e(G_1, G_2)$ is the number of edges that straddle G_1 and G_2, and the sum ranges over ordered partitions of G.

We note that the recurrence formula (5.2) does not seem to follow directly from (5.1). In [32], (5.2) was derived from the *Peterson recurrence formula* [17] for root multiplicities of Kac–Moody Lie algebras. The goal of this paper is to give a purely combinatorial (bijective) proof of (5.2).

To construct a bijective proof, we need sets whose cardinalities are the left and right hand sides of (5.2). We in fact give two bijective proofs, starting with the interpretations of $\alpha(G)$ in terms of acyclic orientations and spanning trees.

5.1 Acyclic orientations with unique fixed sink

In this section we give a bijective proof of the recurrence formula (5.2) in terms of acyclic orientations.

We recall that an acyclic orientation of G is an assignment of arrows to its edges such that there are no directed cycles in the resulting directed graph. A sink in an acyclic orientation is a vertex which only has incoming arrows. The set of all acyclic orientations of G is denoted $\mathcal{A}(G)$. For a vertex v_0 of G, the set of all acyclic orientations in which v_0 is the unique sink is denoted $\mathcal{A}(G, v_0)$. It is well-known that the cardinality of $\mathcal{A}(G, v_0)$ is independent of v_0 and equals $\alpha(G)$ [12]. The following characterization of $\mathcal{A}(G, v_0)$ is immediate.

Lemma. *Fix a vertex v_0 of G and let $\lambda \in \mathcal{A}(G)$. Then $\lambda \in \mathcal{A}(G, v_0)$ if and only if for every $u_0 \in V(G)$, there is a directed path in λ from u_0 to v_0.*

This motivates the following:

Definition. Given a vertex v_0 and an acyclic orientation λ of G, let $V(\lambda, v_0)$ denote the set of all vertices u_0 for which there is a directed path in λ from u_0 to v_0. We call this the set of v_0-reachable vertices in λ.

We record the following simple observation:

Lemma. *Let x be a vertex of G.*

(a) If $x \notin V(\lambda, v_0)$, then $x \notin V(\lambda, u_0)$ for all $u_0 \in V(\lambda, v_0)$.

(b) In particular, an edge joining u_0 and x with $u_0 \in V(\lambda, v_0)$ and $x \notin V(\lambda, v_0)$ is directed from u_0 to x in λ.

Our next goal is to construct sets A and B whose cardinalities are respectively equal to the left and right hand sides of (5.2) and to exhibit a bijection between them. To this end, we first consider the set $\vec{E}$ of oriented edges of G; an element of $\vec{E}$ is an edge of G with an arrow marked on it (in one of two possible ways). Thus $\vec{E}$ has cardinality $2\,e(G)$. If $\vec{e}$ is an element of $\vec{E}$ corresponding to an edge joining vertices u_0 and v_0 with the arrow pointing from u_0 to v_0, we call u_0 the *tail* of $\vec{e}$ and v_0 its *head*.

We now define A to be the set consisting of pairs $(\vec{e}, \lambda) \in \vec{E} \times \mathcal{A}(G)$ such that the head of $\vec{e}$ is the unique sink of λ. For a fixed $\vec{e}$, there are $\alpha(G)$ choices for λ since λ ranges over $\mathcal{A}(G, q)$ where q is the head of $\vec{e}$. It is now clear that A has cardinality exactly $2\,e(G)\,\alpha(G)$.

To define B, we first take an ordered partition (G_1, G_2) of G. Let $E(G_1, G_2)$ denote the set of edges straddling G_1 and G_2. Let $B(G_1, G_2)$ denote the set of triples $(e, \lambda_1, \lambda_2)$ where $e \in E(G_1, G_2)$, say e joins p_1 and p_2 with p_i a vertex of G_i, $i = 1, 2$, and λ_i is an acyclic orientation of G_i with unique sink at p_i, $i = 1, 2$. Arguing as before, one concludes that $B(G_1, G_2)$ has cardinality $\alpha(G_1)\,\alpha(G_2)\,e(G_1, G_2)$. We now let B denote the disjoint union of the $B(G_1, G_2)$ over all ordered partitions (G_1, G_2) of G. It clearly has cardinality equal to the right hand side of (5.2).

We now define a map $\varphi : A \to B$ which will turn out to be the bijection we seek. Given $(\vec{e}, \lambda) \in A$, let u_0 and v_0 denote the tail and head of $\vec{e}$ respectively. Note that $\lambda \in \mathcal{A}(G, v_0)$. Let $V_1 = V(\lambda, u_0)$ denote the set of u_0-reachable vertices in λ (definition 5.1) and let $V_2 = V(G) \backslash V_1$. Observe that $u_0 \in V_1$ and $v_0 \in V_2$.

For $i = 1, 2$, let G_i denote the subgraphs of G induced by V_i, and let λ_i denote the restriction of λ to G_i.

We claim that λ_1 has a unique sink at u_0 and λ_2 has a unique sink at v_0. The first assertion follows simply from Lemma 5.1. For the second assertion, observe that if $x \in V_2 \subset V(G)$, then there is a directed path from x to v_0 in λ. Since $x \notin V(\lambda, u_0)$, Lemma 5.1(a) implies that no vertex of this directed path can lie in V_1. In other words this directed path is entirely within G_2, and we are again done by Lemma 5.1.

Let e denote the undirected edge joining p and q. We have thus shown that the triple $(e, \lambda_1, \lambda_2)$ is in $B(G_1, G_2) \subset B$. We define $\varphi(\vec{e}, \lambda) = (e, \lambda_1, \lambda_2)$.

To see that φ is a bijection, we describe its inverse map. Let (G_1, G_2) be an ordered partition of G; given a triple $(e, \lambda_1, \lambda_2) \in B(G_1, G_2)$, we construct an acyclic orientation λ of G as follows: on G_1 and G_2, we define λ to coincide with λ_1 and λ_2 respectively. It only remains to define an orientation for the straddling edges (this includes e); we orient all of them pointing from G_1 towards G_2, i.e., such that their tails are in G_1 and their heads in G_2. We let $\vec{e}$ denote the edge e with the above orientation.

We claim $(\vec{e}, \lambda) \in A$. First observe that λ is in fact acyclic; since λ extends λ_i for $i = 1, 2$, any directed cycle of λ must necessarily involve vertices from both G_1 and G_2. But this is impossible since all straddling edges point the same way, from G_1 towards G_2. Let u_0, v_0 denote the tail and head of $\vec{e}$. It remains to show that λ has a unique sink at v_0, or equivalently, by Lemma 5.1, that there is a directed path in λ from any vertex x to v_0. This is clear if x is a vertex of G_2. If x is in G_1, we have a directed path in λ_1 from x to u_0. Now the edge $\vec{e}$ is directed from u_0 to v_0; concatenating this directed path with $\vec{e}$ produces a directed path from x to v_0 in λ as required. We define the map $\psi : B \to A$ by $\psi(e, \lambda_1, \lambda_2) = (\vec{e}, \lambda)$.

Observe that for the λ defined above, the set of u_0-reachable vertices is exactly $V(G_1)$. This is because edges straddling G_1 and G_2 point away from G_1, so no vertex of G_2 is u_0-reachable. This implies that $\varphi \circ \psi$ is the identity map on B. Further, it readily follows from Lemma 5.1(b) that $\psi \circ \varphi$ is the identity map on A. This establishes that φ is a bijection. □

5.2 Spanning trees without broken circuits

In this section we give another bijective proof of the recurrence formula (5.2), this time using the fact that $\alpha(G)$ counts the number of spanning trees of G without broken circuits.

Definition. Let σ be a total ordering on the set $E(G)$ of edges of G. Given a circuit in G, it has a unique maximum edge with respect to σ; the set of edges obtained by deleting this edge from the circuit is called a *broken circuit* relative to σ. The set of all broken circuits relative to σ is denoted $B_G(\sigma)$.

Let $S_G(\sigma)$ be the set of all spanning trees of G that contain no broken circuits relative to σ. It is well-known that the cardinality of $S_G(\sigma)$ is independent of the choice of σ, and equals $\alpha(G)$ [4].

Given a total ordering σ on $E(G)$, let $\max(\sigma)$ denote the maximum edge in $E(G)$. The following lemma is immediate.

Lemma. *Any spanning tree in $S_G(\sigma)$ contains the edge $\max(\sigma)$.*

In the sequel, we will fix for each edge e, a total order σ_e on $E(G)$ for which $\max(\sigma_e) = e$. We will write $B_G(e)$ and $S_G(e)$ for the sets $B_G(\sigma_e)$ and $S_G(\sigma_e)$ respectively.

We now proceed to prove (5.2) in the following equivalent form:

$$e(G)\,\alpha(G) = \sum_{\substack{\{G_1,G_2\} \\ \text{unordered partitions of } G}} \alpha(G_1)\,\alpha(G_2)\,e(G_1,G_2). \tag{5.3}$$

We first define the set A to consist of pairs (e,T) where e is an edge and $T \in S_G(e)$; from the above discussion, A has cardinality $e(G)\,\alpha(G)$.

Next, we define the set B. Given an unordered partition $\{G_1,G_2\}$ of G, define $B(\{G_1,G_2\})$ to be the set of pairs $(e,\{T_1,T_2\})$ where e is an edge that straddles G_1 and G_2 and $T_i \in S_{G_i}(e)$ for $i = 1,2$. Here, $S_{G_i}(e)$ is the set of spanning trees of G_i which contain no broken circuits relative to the total order σ_e restricted to the edges of G_i. We let B denote the disjoint union of $B(\{G_1,G_2\})$ as $\{G_1,G_2\}$ ranges over unordered partitions of G. Clearly B has cardinality equal to the right hand side of (5.3).

We define maps $\varphi : A \to B$ and $\psi : B \to A$ as follows:

Given $(e,T) \in A$, e occurs in T in view of Lemma 5.2. Deleting e from the spanning tree T will result in a pair of trees T_1, T_2 with vertex sets V_1 and V_2. Let G_i denote the subgraph induced by V_i, $i = 1,2$; clearly $\{G_1,G_2\}$ is an unordered partition of G and e straddles the G_i. Observe that since the total order on $E(G_i)$ is defined as the restriction of the total order σ_e on $E(G)$, T_i will contain no broken circuits of G_i for $i = 1,2$, i.e., $T_i \in S_{G_i}(e)$. We set $\varphi(e,T) = (e,\{T_1,T_2\})$.

For the inverse map ψ, let $(e,\{T_1,T_2\}) \in B$. Define T to be the spanning tree of G obtained by adding the edge e to the union of T_1 and T_2. To prove that T contains no broken circuits relative to σ_e, observe that any broken circuit of T cannot lie entirely within T_1 or T_2, and must hence contain the edge e. But e is the maximum edge relative to σ_e, so this cannot be a broken circuit by definition. Thus $(e,T) \in A$, and we define $\psi(e,\{T_1,T_2\}) = (e,T)$.

It is straightforward to check that φ and ψ are indeed inverse maps. $\square$.

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