Holography and Brownian motion

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution/University.

Pinaki Banerjee

To *Baba*, *Pisimoni*

&

To *Maa* who would've been the proudest person to see this..

The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction. – Sidney Coleman

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Conferences and workshops attended

1. **Strings 2017** at Tel Aviv during June 26-30, 2017.
2. **The Advanced String School** at Haifa on June 18-22, 2017.
3. **Student Talks on Trending Topics in Theory (ST^4) 2017** at Chennai Mathematical Institute during May 8-19, 2017.
4. **New Developments in AdS_3/CFT_2 Holography** at Galileo Galilei Institute for Theoretical Physics in Florence during March 27- April 29, 2017.
5. **Advanced String School 2017** at Puri during February 12-18, 2017.
6. **Indian Strings Meeting 2016** at IISER Pune from Dec 16 -21, 2016.
7. **Workshop on Topics in Three Dimensional Gravity** at International Centre for Theoretical Physics, Trieste from March 21 - March 24, 2016.
8. **Spring School on Superstring Theory and Related Topics** at International Centre for Theoretical Physics, Trieste from March 10 - March 18, 2016.
9. **Strings 2015** at International Centre for Theoretical Sciences, Bangalore from June 22 - 26, 2015.
10. **Advanced Strings School 2015** at International Centre for Theoretical Sciences, Bangalore from June 11 - 18, 2015.
11. **Spring School on Superstring Theory and Related Topics** at International Centre for Theoretical Physics, Trieste from March 25 - April 2, 2015.
12. **Indian Strings Meeting 2014** at Puri from December 15 - 20, 2014.
13. **Advanced String School** at Puri during September 22-28, 2014.

14. **The 8th Asian Winter School on Strings, Particles and Cosmology** at Puri during January 9-18, 2014.

15. **National Strings Meeting 2013** at Indian Institute of Technology, Kharagpur from December 22 - 27, 2013.

16. **Asymptotia : Workshop on Asymptotic Symmetries in Classical and Quantum Gravity** held at Chennai Mathematical Institute, Chennai between 6 - 8 December 2013.

17. **Thermalization : From Glasses to Black Hole** an ICTS Program held during 10-21 Jun 2013 at Indian Institute of Science, Bangalore.

18. **Indian Strings Meeting 2012** at Puri from December 16 - 21, 2012.

19. **Frontiers of High Energy Physics IMSc Golden Jubilee Symposium** held at The Institute of Mathematical Sciences, Chennai between December 10 - 13, 2012.

20. **XXVII SERC Main School on Theoretical High Energy Physics** held at Saha Institute of Nuclear Physics, Kolkata between September 3-22, 2012.

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7. **Conformal blocks, entanglement entropy & heavy states**, talk given at Universidade de Santiago de Compostela, Spain on 28th March, 2016.
8. **Zero Temperature Dissipation and Holography**, talk given at National String Meeting, 2015 at IISER Mohali on 10th Dec, 2015. *(Skype talk)*
9. **Holographic Brownian Motion & Dissipation**, talk given at Strings 2015 at International Centre for Theoretical Sciences on June 24, 2015. *(Gong Show)*
10. **Holographic Brownian Motion & Dissipation**, Gong Show talk given at Advanced Strings School 2015 at International Centre for Theoretical Sciences on June 14, 2015.

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ABSTRACT

This thesis deals with Brownian motion of a charged particle in holographic framework. In the holographic context, dual field theories are usually studied in long-wavelength or hydrodynamic limit which is macroscopic effective description obtained by coarse-graining the more fundamental underlying microscopic physics. Holographic Brownian dynamics is a simple set-up to incorporate fluctuations in AdS/CFT correspondence. The central focus of this thesis is the novel feature of dissipation at zero temperature in this context of holographic Brownian motion. The phenomenon has its origin in radiation reaction of accelerating charged particle. Langevin dynamics has been explored in diverse spacetime dimensions and the coefficient of zero-temperature dissipation has been shown to have a jump in its value at zero temperature. Brownian motion is also studied in presence of finite matter density at zero and small temperature. All these have been explored using analytic techniques.

Contents

Synopsis

Motivation and Introduction

Gauge/gravity duality [1–4] has been remarkably successful in understanding strongly coupled gauge theories at finite temperature. These field theories are usually studied in long-wavelength or hydrodynamic limit which is macroscopic effective description obtained by coarse-graining the more fundamental underlying microscopic physics. It is thus very natural to make steps toward more microscopic non-equilibrium aspects of the duality where fluctuations play very important role. In this thesis we study holographic Brownian motion [5–9] which is a simple set-up to incorporate fluctuations in AdS/CFT correspondence.

The dynamics of a Brownian particle of mass M_0 moving with velocity v in a viscous medium can be described by the famous Langevin equation

$$M_0 \frac{dv}{dt} + \gamma\, v = \xi(t)\,, \tag{1}$$

$$\text{with} \quad \langle \xi(t)\, \xi(t') \rangle = \Gamma\, \delta(t - t')\,, \tag{2}$$

where γ is the viscous drag, ξ is the random force acting on that particle and the constant Γ quantifies the strength of the random force. Eqn. (1.0.2) is a particular form of the fluctuation-dissipation theorem. Kubo [10] has argued that for smaller time scales one would face a contradiction if one works with constant (*i.e.* time independent) γ and Γ.

Thus a proper microscopic theory should be better behaved and one should replace $\Gamma\,\delta(t-t')$ by a more general function $\Gamma(t-t')$ which is less singular than a delta function and γ by $\gamma(t-t')$. The resulting equation is called the generalized Langevin equation,

$$M_0\,\frac{d^2x(t)}{dt^2} + \int_{-\infty}^{t} dt'\;G_R(t,t')\,x(t') = \xi(t) \qquad \langle\xi(t)\,\xi(t')\rangle = i\,G_{\rm sym}(t,t')\,. \tag{3}$$

The retarded and the symmetric Green functions are given by

$$G_{\rm sym}(t,t') = \frac{1}{2}\,\langle\{\xi(t),\xi(t')\}\rangle\ , \tag{4}$$

$$i\,G_R(t,t') = \theta(t-t')\,\langle[\xi(t),\xi(t')]\rangle\ . \tag{5}$$

$G_R(t,t')$ is thus the same as $\gamma(t-t')$ and $iG_{\rm sym}(t,t')$ is the same as $\Gamma(t-t')$. The generalized Langevin equation is easier to interpret in frequency space

$$\left[-M_0\omega^2 + G_R(\omega)\right]x(\omega) = \xi(\omega)\,, \qquad \langle\xi(-\omega)\,\xi(\omega)\rangle = i\,G_{\rm sym}(\omega)\,. \tag{6}$$

If one is interested in small frequencies one can expand $G_R(\omega)$ as

$$G_R(\omega) = -\,i\,\gamma\,\omega - \Delta M\,\omega^2 - i\,\rho\,\omega^3 + \ldots \tag{7}$$

and can straightforwardly interpret the coefficients of different powers of ω as viscous drag, mass-shift etc. Thus the retarded Green function contains almost all the information about the dynamics of the system. All one needs is to compute this Green function. In this thesis we compute this quantity using gauge/gravity duality. Our particular focus is to study dissipation at (and near) zero temperature for various systems. The thesis is organized as following chapters.

Dualities in physics

Duality means equivalence between two seemingly different theories which is rather old concept in physics. The existence of dualities points to a great underlying unity in the structure of theoretical physics. In this chapter we briefly discuss about few important dualities that appear in different branches of theoretical physics particularly in quantum field theories and string theory and some of their typical characteristics.

We start describing dualities in statistical mechanics/field theories : Maxwell duality, Kramers-Wannier duality, bosonization, Montonen-Olive duality, Seiberg-Witten duality. Then briefly discuss few important dualities which involve string theory namely T-duality and S-duality and head towards the most important one for this thesis which connects field theories to string theory - gauge/gravity duality.

String theories, D-branes and AdS/CFT

Historically there have been many hints about gauge/gravity duality - open-closed string duality, large N field theories, the holographic principle, to name a few. In the year 1997, Maldacena conjectured [1] the celebrated AdS/CFT correspondence. This chapter is dedicated to describe Maldacena's original decoupling argument [1] and the recipes to compute Euclidean and Minkowski correlators using the duality [2, 3, 11].

String theory is not a theory of only strings. It contains extended objects which play important role when the string coupling is very large. These are known as D-branes. These objects can be described in two completely different but equivalent ways - either as extended objects where an open string can end on or objects which source closed strings and therefore gravitate. Starting with a stack of N D3 branes and describing them both in terms of open and closed string Maldacena conjectured,

$$\boxed{\mathcal{N} = 4 \text{ SU(N) super Yang-Mills in 4D} \;\equiv\; \text{Type IIB string theory in } \mathrm{AdS}_5 \times \mathrm{S}^5} \tag{8}$$

To use the above mentioned duality quantitatively one should have some precise prescription that relates the field theory quantities to their gravity theory equivalents. Such a prescription has been given in [2, 3] which state that partition function of the QFT coincides with the same of gravity theory.

$$\left\langle \exp\left(\int_{\partial \mathrm{AdS}_5} \phi_0^i O_i \right) \right\rangle_{\mathrm{CFT}} = Z_{\mathrm{QG}}(\phi_0^i)\,, \tag{9}$$

where ϕ^i are bulk fields in gravity theory and O^i are their dual boundary operators in the gauge theory. $Z_{\mathrm{QG}}(\phi_0^i)$ is the partition function of quantum gravity with the boundary conditions that ϕ^i goes to ϕ_0^i on the boundary. The conjecture becomes useful in studying strongly coupled field theories when the gravity theory is 'classical'. In that limit the path integral can be approximated by saddle point. Treating ϕ_0^i as the sources of boundary field theory one can calculate the correlators by taking functional derivative of Z_{QG} with respect to ϕ_0^i.

The above prescription is applicable to obtain Euclidean correlators. The Euclidean signature avoids some complications related to boundary conditions. However in many cases, particularly for finite temperature systems, extraction of Lorentzian-signature AdS/CFT results directly from bulk gravity theory is inevitable and therefore one requires to have some prescription for computing real time correlators directly from gravity. This was done by Son and Starinets [11]. We heavily use their prescription for computing Green function throughout the main part of this thesis. The retarded Green function for stretched string in AdS black hole background

$$ds^2 = \frac{L_{d+1}^2}{z^2}\left(-f(z)\,dt^2 + d\vec{x}^2\right) + \frac{L_{d+1}^2}{z^2}\frac{dz^2}{f(z)}\,, \tag{10}$$

can be obtained by imposing ingoing boundary condition at the horizon of the black hole

$$G_R(\omega) := \lim_{z\to 0} T_0(z)\left(-\frac{z^2}{L_{d+1}^2}\right)\frac{x_\omega{}'(z)}{x_\omega(z)}, \tag{11}$$

where local string tension $T_0(z) = \frac{1}{2\pi l_s^2}\frac{L_{d+1}^4}{z^4}f(z)$; L_{d+1} is the radius of AdS_{d+1}; l_s is string length, $f(z)$ is the blackening function and $x_\omega(z)$ is the solution to the string equation of motion in the bulk background.

The imaginary part of $G_R(\omega)$ contains all the information about dissipation or energy loss in the boundary theory. For example, for the string in bulk it represents energy loss of an external quark in the strongly coupled plasma. In the rest of the thesis we particularly focus on this quantity near and/or at zero temperature.

Brownian motion in 1+1 D

Brownian motion of a heavy quark in strongly coupled medium has been extensively studied using holography [5, 6]. The retarded Green function (noise-noise correlator) has all the information about the generalized Langevin equation of the quark. In this chapter we start with describing Brownian motion in 1+1 dimensions and obtain the exact Schwinger-Keldysh Green function by studying motion of a fundamental string in BTZ black hole background. We obtain the retarded Green function exactly

$$G_R(\omega) = \frac{\mathcal{M}\,\omega}{2\,\pi}\,\frac{(\omega^2 + 4\,\pi^2\,T^2)}{(\omega + i\,\frac{\mathcal{M}}{\sqrt{\lambda}})}\,, \tag{12}$$

where $\mathcal{M} = 2\pi M_0$, λ is the dimensionless coupling in the 1+1 CFT and T is its temperature which is also the Hawking temperature of BTZ black hole in the dual gravity. For small frequencies the Green function goes as

$$G_R(\omega) = -\,i\frac{\sqrt{\lambda}}{2\pi}\,\omega^3\,. \tag{13}$$

We identify the coefficient $\frac{\sqrt{\lambda}}{2\pi}$ as the dissipation at zero temperature. This does not violate boost (Lorentz) invariance because the drag force on a constant velocity quark continues to be zero. This phenomenon have a nice interpretation as radiation due to accelerating charged particle, namely, the quark. The dissipation co-efficient actually matches the bremsstrahlung function $B(\lambda)$ [12–14] for an accelerating heavy quark at leading order in large coupling. Furthermore since the Green function is exact, we can write down an effective membrane action, and thus a Langevin equation, located at a 'stretched horizon' which is placed at an arbitrary finite distance from the original horizon.

Dissipation at $T = 0$ and $T \to 0$

In this chapter we use holographic techniques to study the zero-temperature limit of dissipation for a Brownian particle moving in a strongly coupled CFT at finite temperature in various space-time dimensions [8]. For 1+1 dimensional CFT the dissipative term at zero temperature matches the same quantity near zero temperature both being equal to the bremsstrahlung function $B(\lambda) = \frac{\sqrt{\lambda}}{2\pi}$.

But for higher dimensional theories this is not the case. In particular we compute the dissipative term for the 3+1 dimensional boundary theory near zero temperature (T) for small frequencies (ω) with ω/T held fixed (see figure 5.1) studying dynamics of a stretched string in AdS_5 black hole

$$G_R(\omega)\Big|_{T\to 0} = -\,i\left(\frac{\pi - Log\,4}{4}\right)\frac{\sqrt{\lambda}}{2\pi}\,\omega^3\,. \tag{14}$$

Clearly this does not match $G_R(\omega, T = 0)$ which is independent of the number of space-time dimensions the CFT lives in and is given by

$$G_R(\omega)\Big|_{T=0} = -\,i\,\frac{\sqrt{\lambda}}{2\pi}\,\omega^3\,. \tag{15}$$

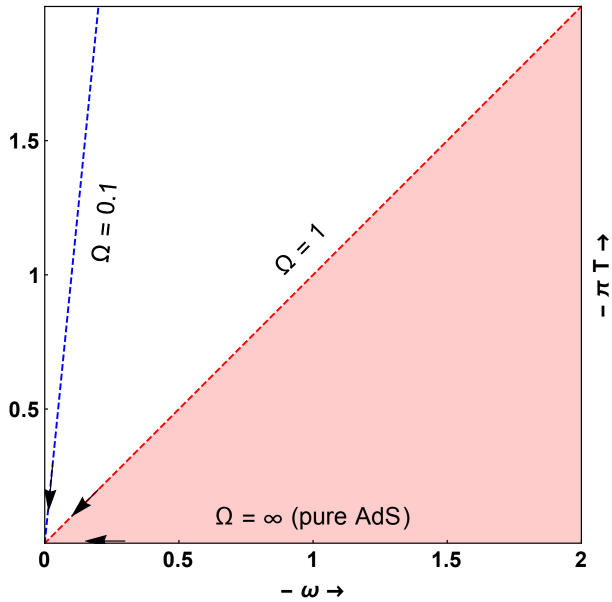

Figure 1. Different ways of taking $T = 0, \omega = 0$ limit. Our analysis and results hold true when $\omega/\pi T$ is a constant and smaller than one *i.e.* for the straight lines (*e.g,* the blue line) in the upper triangular region of the box.

Thus that particular way of taking $T \to 0$ limit is not smooth. This phenomenon appeared to be related to a confinement-deconfinement phase transition at $T = 0$ in the field theory. The result is important in the context of quark-gluon-plasma (QGP) which is always at finite temperature. The analysis suggests, for a quark moving in QGP one should be very careful in using zero temperature results to compute useful quantities (e.g, bremsstrahlung function) however small be the temperature of the system.

Brownian motion at finite density

This chapter deals with Brownian motion at finite density. We study holographic Brownian motion of a heavy charged particle in higher spacetime dimensions ($d \geq 3$) at zero and small temperature in presence of finite density [9]. Our main interest is to understand the dynamics of that particle at (near-) zero temperature which was holographically described by motion of a fundamental string in an (near-) extremal Reissner-Nordström (RN) black hole.

We analytically compute the functional form of retarded Green function to extract the dissipative term at zero temperature following the matching technique in [15],

$$G_R^{(0)}(\omega) = -\frac{\sqrt{\lambda}}{2\pi} \frac{i\,\mu_*^2\,\omega}{(1 + i\,\frac{\omega}{\mu_*}\,a_-^{(0)})}\,, \tag{16}$$

where μ_* is the chemical potential in the boundary theory which is a mass scale dual to the charge of the RN black hole and $a_-^{(0)}$ is an undetermined constant that can be fixed numerically. For small frequencies

$$G_R^{(0)}(\omega) \approx -i\,\frac{\sqrt{\lambda}}{2\pi}\,\mu_*^2\,\omega - \frac{\sqrt{\lambda}}{2\pi}\,\mu_*\,a_-^{(0)}\,\omega^2\,. \tag{17}$$

The zero temperature dissipation goes linear in ω unlike zero density ($\mu = 0$) case [7, 8] where this goes as ω^3. The leading dissipative term is proportional to μ_*^2 *i.e.* energy loss for the charged Brownian particle is more for medium with higher charge density.

We show that the leading dissipative behaviour remains unchanged even at small temperature

$$G_{R,T}^{(0)}(\omega) = -\frac{\sqrt{\lambda}}{2\pi} \frac{i\mu_*^2\,\omega}{(1 + i\frac{\omega}{\mu_*}\,a_-^{(0)})}\,. \tag{18}$$

This Green function can be improved perturbatively in ω and T. The corrections will be in powers of $\frac{\omega}{\mu_*}$ and $\frac{T}{\mu_*}$. The corresponding real coefficients can also be obtained numerically in a systematic fashion.

Conclusions and outlook

In this thesis we have studied dynamics of a heavy charged particle (a quark) in a strongly coupled plasma using holography. Our main focus has been on the zero temperature dissipation of the Brownian particle which has been interpreted as energy loss due to radiation [12–14]. We have studied the Brownian dynamics in diverse spacetime dimensions and also in zero and finite matter density in the plasma. We have used only analytic techniques throughout the thesis and for particular cases have obtained even exact results.

There are several directions that one can make progress in, in the context of holographic Brownian motion. It will be interesting to study holographic renormalization group [16–19] for this type of systems particularly for the 2+1 dimensional bulk where one can solve the equations of motion the fluctuating string in closed form. The zero temperature dissipation of the charged particle is very interesting in its own right. There are several scenarios where this phenomenon can be studied. Some of them are as follows. It would be interesting to investigate the dissipation near zero temperature in 1+1 dimensional CFT at finite matter density by studying stochastic string in a charged BTZ black hole. Our technique for computing Green function at finite density for higher dimensional systems (*i.e.* CFT_d with $d \geq 3$) only requires an AdS_2 factor near the horizon. Therefore it should work even if the UV theory is non-conformal (not asymptotically AdS) but the IR geometry has a AdS_2 factor. For example, instead of D3 branes one can look at D2 or D4 brane geometries [20]. If for some charge density they flow to a AdS_2 then the procedure can be applied. By the same argument it can be also used for some rotating extremal black hole backgrounds. Finally one can explore zero temperature dissipation for anisotropic backgrounds [21, 22] which are more interesting phenomenologically.

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1 Introduction

The ultimate aim of theoretical physics is to explain or rather describe Nature with minimal set of assumptions. Fewer the number of assumptions the more beautiful is the theory. Large number of hypotheses and numerous empirical laws have been proposed to understand different phenomena. According to the current understanding any physical phenomenon can be described by only *four* type of forces or interactions at the fundamental level namely, *electromagnetic force, strong force, weak force* and *gravitational force*. The dream of unification of all four interactions together is the holy grail of theoretical physics. The aim is to write down *a single* theory that will describe all the natural phenomena. One requires a common language or a framework for that unification. Quantum field theory (QFT) provides us with such a language that can express electromagnetic, weak and strong interactions within extreme accuracy to present day available experiments (*E.g,* see 'anomalous magnetic moment' of muon in [23]). But the program of unification is only partially successful since writing down a complete consistent quantum theory of gravity is still elusive (we shall discuss briefly about one candidate theory of quantum gravity in chapter 3).

Thus quantum field theory has been proved to be an extremely useful framework in describing large number of physical phenomena. But we can solve only a very restrictive class of systems using standard QFT toolbox which essentially consists of (a) perturbation theory, (b) lattice gauge theory and (c) integrable systems. These methods have their own limitations. For example, there are many interesting systems (*e.g,* QCD at low energy, high temperature superconductors *etc*) which are intrinsically strongly coupled in nature and cannot be solved perturbatively. Lattice gauge theory has the famous *'sign problem'* which restricts its applicability to systems no matter density and also without real time dependence. The integrability techniques are mostly useful in two dimensions. Therefore the standard field theoretic tools fail for a large class of systems with - (a) strong coupling, (b) time dependence, (c) nonzero density and (d) for systems in higher (more than two) dimensions.

The only way one can attack these 'difficult' strongly coupled problems analytically is via duality (some strong/weak coupling duality, to be more precise). These type of dualities map a strongly coupled system to a weakly coupled one (Chapter 2 contains more detailed discussion on dualities). Thus one can solve the corresponding weakly coupled system perturbatively and interpret the results using the dictionary of that duality. Gauge/gravity duality [1–3] is probably the most useful among all these strong/weak dualities. It maps some strongly coupled field theories to corresponding classical (super-) gravity theories. Performing some classical gravity computation one can extract information about the dual strongly coupled theory.

Gauge/gravity duality has been applied to study numerous strongly coupled systems. The duality has been remarkably successful in understanding strongly coupled gauge theories at finite temperature. Particularly it produces the famous shear viscosity to entropy density ratio [24, 25] $\eta/s = \frac{1}{4\pi}$ for quark-gluon plasma at finite temperature which is very close to the experimental result obtained in RHIC. But all these field theories are usually studied in long-wavelength or hydrodynamic limit which is macroscopic effective description obtained by coarse-graining the more fundamental underlying microscopic physics. It is thus very natural to make steps toward more microscopic non-equilibrium aspects of the duality where fluctuations play very important role. In this thesis we study holographic Brownian motion which is a simple set-up to incorporate fluctuations in AdS/CFT correspondence. Starting with [5, 6] there has been a numerous work addressing different aspects of holographic Brownian motions - for non-relativistic theories [26, 27], in presence of background magnetic field [28], with rotating Plasma [29], in de Sitter space [30], in higher derivative gravity, for anisotropic plasma [21,22] etc. Our main aim in this thesis is to explore dissipation of a charged Brownian particle at or very near zero temperature. This is a novel phenomenon in the context of Brownian motion in holographic settings. We explore this phenomenon in various situations using analytic techniques. Before we discuss all the details about holography and Brownian motion in this context, below we briefly describe the Langevin dynamics of a Brownian particle. Then we discuss how the rest of the thesis is organized which contains the summary of the main results of this thesis as well.

The dynamics of a Brownian particle of mass M_0 moving with velocity v in a viscous medium can be described by the famous Langevin equation

$$M_0 \frac{dv}{dt} + \gamma\, v = \xi(t)\,, \tag{1.0.1}$$

$$\text{with} \quad \langle \xi(t)\, \xi(t') \rangle = \Gamma\, \delta(t - t')\,, \tag{1.0.2}$$

where γ is the viscous drag, ξ is the random force acting on that particle and the constant Γ quantifies the strength of the random force. Eqn. (1.0.2) is a particular form of the fluctuation-dissipation theorem. Kubo [10] has argued that for smaller time scales one would face a contradiction if one works with constant (*i.e.* time independent) γ and Γ. Thus a proper microscopic theory should be better behaved and one should replace $\Gamma\, \delta(t - t')$ by a more general function $\Gamma(t - t')$ which is less singular than a delta function and γ by $\gamma(t - t')$. The resulting equation is called the generalized Langevin equation

$$M_0 \frac{d^2 x(t)}{dt^2} + \int_{-\infty}^{t} dt'\, G_R(t, t')\, x(t') = \xi(t) \qquad \langle \xi(t)\, \xi(t') \rangle = i\, G_{\text{sym}}(t, t')\,. \tag{1.0.3}$$

The retarded and the symmetric Green functions are given by

$$G_{\rm sym}(t,t') = \frac{1}{2}\,\langle\{\xi(t),\xi(t')\}\rangle\ , \tag{1.0.4}$$

$$i\,G_R(t,t') = \theta(t-t')\,\langle[\xi(t),\xi(t')]\rangle\ . \tag{1.0.5}$$

$G_R(t,t')$ is thus the same as $\gamma(t-t')$ and $i\,G_{\rm sym}(t,t')$ is the same as $\varGamma(t-t')$. The generalized Langevin equation is easier to interpret in frequency space

$$\left[-M_0\,\omega^2 + G_R(\omega)\right]x(\omega) = \ \xi(\omega)\,, \qquad\qquad \langle\xi(-\omega)\,\xi(\omega)\rangle = \ i\,G_{\rm sym}(\omega)\,. \tag{1.0.6}$$

If one is interested in small frequencies one can expand $G_R(\omega)$ as

$$G_R(\omega) = -i\,\gamma\,\omega - \varDelta M\,\omega^2 - i\,\rho\,\omega^3 + \ldots \tag{1.0.7}$$

and can straightforwardly interpret the coefficients of different powers of ω as viscous drag, mass-shift etc. Thus the retarded Green function contains almost all the information about the dynamics of the system. All one needs is to compute this Green function. In this thesis we compute this quantity using gauge/gravity duality. Our particular focus is to study dissipation at (and near) zero temperature for various systems. Rest of the thesis is organized as follows.

In chapter 2 we briefly discuss about few important dualities that appear in different branches of theoretical physics particularly in quantum field theories and string theory and some of their typical characteristics. We start describing dualities in statistical mechanics/field theories : Maxwell duality, Kramers-Wannier duality, bosonization, Montonen-Olive duality, Seiberg-Witten duality. Then briefly discuss few important dualities which involve string theory namely T-duality and S-duality and head towards the most important one for this thesis which connects field theories to string theory - gauge/gravity duality.

Historically there have been many hints about gauge/gravity duality - open-closed string duality, large N field theories, the holographic principle, to name a few. In the year 1997, Maldacena conjectured [1] the celebrated AdS/CFT correspondence. Chapter 3 is dedicated to describe Maldacena's original decoupling argument and the recipes to compute Euclidean and Minkowski correlators using the duality.

In chapter 4 we start with describing Brownian motion in 1+1 dimensions and obtain the exact Schwinger-Keldysh Green function by studying motion of a fundamental string in BTZ black hole background. We obtain the retarded Green function exactly using holography. For small frequencies the Green function goes as

$$G_R(\omega) = -\, i\frac{\sqrt{\lambda}}{2\pi}\,\omega^3\,, \tag{1.0.8}$$

where λ is dimensionless coupling of the field theory. We identify the coefficient $\frac{\sqrt{\lambda}}{2\pi}$ as the dissipation at zero temperature. This does not violate boost (Lorentz) invariance because the drag force on a constant velocity quark continues to be zero. This phenomenon have a nice interpretation as radiation due to accelerating charged particle, namely, the quark. The dissipation co-efficient actually matches the bremsstrahlung function $B(\lambda)$ [12–14] for an accelerating heavy quark at leading order in large coupling. Furthermore since the Green function is exact, we can write down an effective membrane action, and thus a Langevin equation, located at a 'stretched horizon' which is placed at an arbitrary finite distance from the original horizon.

In chapter 5 we use holographic techniques to study the zero-temperature limit of dissipation for a Brownian particle moving in a strongly coupled CFT at finite temperature in various space-time dimensions [8]. For 1+1 dimensional CFT the dissipative term at zero temperature matches the same quantity near zero temperature both being equal to the bremsstrahlung function $B(\lambda) = \frac{\sqrt{\lambda}}{2\pi}$. But for higher dimensional theories this is not the case. In particular we compute the dissipative term for the 3+1 dimensional boundary theory near zero temperature (T) for small frequencies (ω) with ω/T held fixed (see figure

5.1) studying dynamics of a stretched string in AdS$_5$ black hole

$$G_R(\omega)\Big|_{T\to 0} = -\,i\left(\frac{\pi - Log\,4}{4}\right)\frac{\sqrt{\lambda}}{2\pi}\,\omega^3\,. \tag{1.0.9}$$

Clearly this does not match $G_R\,(\omega, T\,=\,0)$ which is independent of the number of space-time dimensions the CFT lives in and is given by

$$G_R(\omega)\Big|_{T=0} = -\,i\,\frac{\sqrt{\lambda}}{2\pi}\,\omega^3\,. \tag{1.0.10}$$

Thus that particular way of taking $T\,\to 0$ limit is not smooth. This phenomenon appeared to be related to a confinement-deconfinement phase transition at $T\,=0$ in the field theory. The result is important in the context of quark-gluon-plasma (QGP) which is always at finite temperature. The analysis suggests, for a quark moving in QGP one should be very careful in using zero temperature results to compute useful quantities (e.g, bremsstrahlung function) however small be the temperature of the system.

Chapter 6 deals with Brownian motion at finite density. We study holographic Brownian motion of a heavy charged particle in higher spacetime dimensions ($d\,\geq\,3$) at zero and small temperature in presence of finite density [9]. Our main interest is to understand the dynamics of that particle at (near-) zero temperature which was holographically described by motion of a fundamental string in an (near-) extremal Reissner-Nordström (RN) black hole. We analytically compute the functional form of retarded Green function to extract the dissipative term at zero temperature following the matching technique in [15],

$$G_R^{(0)}(\omega) = -\frac{\sqrt{\lambda}}{2\,\pi}\,\frac{i\,\mu_*^2\,\omega}{(1+i\,\frac{\omega}{\mu_*}\,a_-^{(0)})}\,, \tag{1.0.11}$$

where μ_* is the chemical potential in the boundary theory which is a mass scale dual to the charge of the RN black hole and $a_-^{(0)}$ is an undetermined constant that can be fixed numerically. For small frequencies

$$G_R^{(0)}(\omega) \approx -i\,\frac{\sqrt{\lambda}}{2\pi}\,\mu_*^2\,\omega - \frac{\sqrt{\lambda}}{2\pi}\,\mu_*\,a_-^{(0)}\,\omega^2\,. \tag{1.0.12}$$

The zero temperature dissipation goes linear in ω unlike zero density ($\mu = 0$) case [7, 8] where this goes as ω^3. The leading dissipative term is proportional to μ_*^2 *i.e.* energy loss for the charged Brownian particle is more for medium with higher charge density.

We show that the leading dissipative behaviour remains unchanged even at small temperature

$$G_{R,T}^{(0)}(\omega) = -\frac{\sqrt{\lambda}}{2\pi}\,\frac{i\mu_*^2\,\omega}{(1 + i\frac{\omega}{\mu_*}\,a_-^{(0)})}\,. \tag{1.0.13}$$

This Green function can be improved perturbatively in ω and T. The corrections will be in powers of $\frac{\omega}{\mu_*}$ and $\frac{T}{\mu_*}$. The corresponding real coefficients can also be obtained numerically in a systematic fashion.

In chapter 7 we summarize the main results of the thesis and also discuss about some future research directions. Appendices include some background materials, computational details *etc.* to make the thesis self-contained.

2 Dualities in physics

Duality means equivalence between two seemingly different theories. This is actually a very old concept in physics. In this section we discuss about dualities in quantum field theories and also in string theory. Before we go into the details here are few *typical* characteristics of dualities which we will encounter in this section many times.

Characteristics of dualities

Two sides (theories) of a duality are typically related by following maps.

- Degrees of freedom or the Lagrangian need *not* be same.
- Global symmetries *coincide*.
- Equation of motion $\Longleftrightarrow$ Bianchi identity
- Weak coupling $\Longleftrightarrow$ Strong coupling

The last one typically holds but not always true. When it holds, one calls that a strong-weak duality. We will see in chapter 3 that AdS/CFT is a famous example of strong-weak duality.

Some useful techniques in the context of duality are provided in appendix A.

Dualities in quantum field theory

Here is a list of dualities from quantum field theories and/or statistical mechanics.

1. Maxwell duality
2. Kramers-Wannier duality
3. Bosonization
4. Montonen-Olive duality
5. Seiberg-Witten duality

We will elaborate on one from the list *viz.* the Maxwell duality and comment on other dualities only briefly.

Maxwell duality

The oldest example of duality goes back to Maxwell. The famous equations due to Maxwell for electric field $\vec{E}$, magnetic field $\vec{B}$, charge density ρ and electric current $\vec{J}$ are given by

$$\left.\begin{aligned} \nabla.\vec{E} &= \rho \\ \nabla \times \mathbf{B} &= \vec{J} + \frac{\partial \vec{E}}{\partial t} \end{aligned}\right\} \qquad \Longleftrightarrow \qquad \partial_\mu F^{\mu\nu} = J^\nu\,, \tag{2.1.1}$$

$$\left.\begin{aligned} \nabla.\vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned}\right\} \qquad \Longleftrightarrow \qquad \partial_\mu \widetilde{F}^{\mu\nu} = 0\,, \tag{2.1.2}$$

where $\widetilde{F}^{\mu\nu} := \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ is the Hodge dual to $F^{\mu\nu}$. The equations (2.1.2) are independent of sources whereas (2.1.1) depend on sources. We refer to (2.1.1) as *'Maxwell equations'* and call (2.1.2) as *'Bianchi identities'*.

Notice that, for $\rho = 0$ and $\vec{J} = 0$, $\vec{E} \leftrightarrow -\vec{B}$ is a symmetry. Actually $\vec{E} \leftrightarrow -\vec{B}$ interchanges $F^{\mu\nu} \leftrightarrow \widetilde{F}^{\mu\nu}$ that amounts to interchanging

Dynamical "Maxwell equations" $\longleftrightarrow$ *Geometric* Bianchi identities.

Here we choose a 'trivial' example to illustrate the electro-magnetic duality. Let's consider a bunch of photons (A_μ) and they don't interact with any sources (e.g, electrons). That's the reason we call it a 'trivial' theory

$$Z = \int \mathcal{D}A_\mu e^{-i\frac{1}{4g^2}\int F_{\mu\nu}^2} \delta(\partial_\mu A^\mu)\,. \tag{2.1.3}$$

The delta function ensures we are dealing with only *physical* of degrees of freedom. This is known as a gauge choice. Let's perform a change of variable : $A_\mu \rightarrow F_{\mu\nu}$ i.e, our integration variable will be $F_{\mu\nu}$ instead of A_μ. This implies a Jacobian[1] which is not important for the dynamics of the system and can be taken out of the integral.

$$Z = \text{“Jacobian”} \times \int \mathcal{D}F_{\mu\nu}\, e^{-i\frac{1}{4g^2}\int F_{\mu\nu}^2} \delta(\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma})\,. \tag{2.1.4}$$

We have imposed the Bianchi identities over the path integral. Now our aim is to introduce a Lagrange multiplier C_α into the path integral and integrate out the *dynamical* $F_{\mu\nu}$ to write down a theory for the 'fake variable' C_α. Let's first introduce C_α

$$Z \approx \int \mathcal{D}F_{\mu\nu}\, \mathcal{D}C_\alpha\, e^{-i\int(\frac{1}{4g^2}F_{\mu\nu}^2 + \mathcal{N}\, C_\alpha \epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta})} \delta(\partial_\mu C^\mu)\,. \tag{2.1.5}$$

The $\delta(\partial_\mu C^\mu)$ is there because if one shifts $C_\alpha \rightarrow C_\alpha + \partial_\alpha f$ in the exponent, that does nothing to the integral (extra piece vanishes due to anti-symmetry of $\epsilon^{\alpha\beta\gamma\delta}$) and $\mathcal{N}$ is just normalization factor which is not important for this discussion. Now we integrate by parts

[1]The Jacobians are, in general, rather tricky in path integrals. They can be very important which may lead to *anomaly*. But for this particular example the Jacobian is innocent.

and drop the boundary term[2] assuming 'nice' boundary condition.

$$Z \approx \int \mathcal{D}F_{\mu\nu}\mathcal{D}C_\alpha \, e^{-i\int(\frac{1}{4g^2}F_{\mu\nu}^2 + \mathcal{N}\,\partial_\beta C_\alpha\,\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta})}\delta(\partial_\mu C^\mu)\,. \tag{2.1.6}$$

Notice that due to anti-symmetry of $\epsilon^{\alpha\beta\gamma\delta}$ the term $\partial_\beta C_\alpha$ has to be anti-symmetric in its indices. Therefore one can define this as the *field strength* for the new 'gauge field' C_α : $G_{\alpha\beta} := \partial_\alpha C_\beta - \partial_\beta C_\alpha$.

The above action is a just quadratic in $F_{\mu\nu}$ and therefore we can easily integrate out $F_{\mu\nu}$ to obtain

$$Z \approx \int \mathcal{D}C_\alpha \, e^{-\frac{ig^2}{\mathcal{N}}\int G_{\mu\nu}^2}\,\delta(\partial_\mu C^\mu)\,. \tag{2.1.7}$$

This is almost the same $U(1)$ theory but of a completely different 'gauge field' C_μ. Few remarks in order.

- This is an example of *self duality* where a Maxwell theory goes to another Maxwell theory. But the coupling is inverted $g \to \frac{1}{g}$. This is also an example of strong-weak duality.
- A_μ and C_μ are completely different degrees of freedom. They are related by extremely involved relation.
- Degrees of freedom need not be the same in both sides of a duality. But in this case a gauge field goes to another.

[2]This step is very non-trivial. We are assuming very particular boundary conditions. By 'nice' we mean the field C_α and/or its derivative dies down at the boundary. But some non-trivial boundary condition can give rise to more interesting physics - topological theories.

Kramers-Wannier duality

It relates the partition function of a two-dimensional square-lattice Ising model at a low temperature to that of another Ising model at a high temperature. Using this duality Kramers and Wannier [31] predicted the exact location of the critical point of 2D Ising model in 1941 before Onsager [32] could solve that model exactly in 1944.

Bosonization

In 1+1 dimensions one can map an interacting fermionic system to a system of bosons. *E.g.,* massive Thirring model is dual to sine-Gordan model. This is also an example of strong-weak duality. This duality was uncovered independently by particle physicists Coleman [33] and Mandelstam [34], and condensed matter physicists (Mattis, Luther and others) around 1975.

Montonen-Olive duality

This is generalization of Maxwell duality with magnetic charge and current but in $\mathcal{N} = 4$ SYM [35]. This is again a strong-weak duality and it relates 'elementary particles' of one side to 'monopoles' of the other side.

Seiberg-Witten duality

Similar to Montonen-Olive duality but it is for IR effective theory of $\mathcal{N} = 2$ SUSY theory in $D = 4$ [36]. Unlike $\mathcal{N} = 4$ SYM this theory not conformally invariant in general *i.e.* its beta function runs – more interesting dynamics.

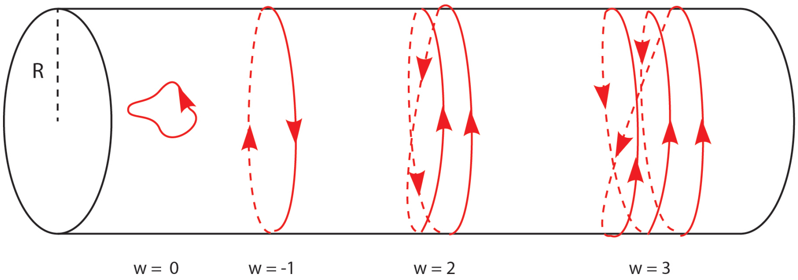


Figure 2.1. Closed strings wrapping a compact direction in different 'windings'.

Dualities in string theory

In string theory there are mainly two important dualities : T-duality and S-duality.

T-duality or 'Target space' duality is very simple and elegant. This is also an example of duality which is *not* strong/weak type. Hints of this duality were noticed in [37, 38] but it was first stated and explicitly shown to be a symmetry by Sathiapalan [39] and also independently by Nair *et al* [40] in 1987. We discuss this in some detail and briefly mention about S-duality.

T-duality

Einstein changed our view of space and time by marrying them. The notion of 'space' and 'time' became rather observer dependent. T-duality goes one step further to completely change our notion of space-time itself. It shows how different objects or probes perceive space-time quite differently. In that sense the notion of space-time itself is an 'emergent' concept. Let's see how T-duality works in string theory. Consider a flat 1+1 dimensional space-time (higher dimensional generalization is straight forward). This is just a plane sheet of paper. Let's compactify the spatial direction to make it an infinite cylinder with radius R (see Fig. 2.1).

First consider a particle (or a field) of mass m moving on this cylinder. Its momentum ($\vec{p}$) has two orthogonal components : along the circle (p_θ) and along the non-compact direction ($p_\perp$). But along the compact direction p_θ has to obey the periodicity condition $e^{i\,p_\theta(2\pi R)} = 1$ *i.e.* $p_\theta = \frac{n}{R}$ where $n \in \mathbb{Z}$. Thus the total momentum

$$\begin{aligned} \vec{p} &= p_\theta\, \boldsymbol{e}_\theta + p_\perp\, \boldsymbol{e}_\perp \\ &= \frac{n}{R}\, \boldsymbol{e}_\theta + p_\perp\, \boldsymbol{e}_\perp\,, \end{aligned} \tag{2.2.1}$$

where $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\perp$ are the corresponding unit vectors. The energy is given by

$$\begin{aligned} E^2 &= p^2 + m^2 \\ &= (p_\perp^2 + m^2) + \frac{n^2}{R^2}\,. \end{aligned} \tag{2.2.2}$$

Notice that, if one takes $R \to 0$ the energy $E \to \infty$, unless $n = 0$. Physically this means when the compact direction is very small the particle can not ‘sense’ or probe that direction and effectively ‘lives’ only in the non-compact dimension.

But what happens if we replace the particle with a closed string? Unlike the particle it can wind around (fig. 2.1) the cylinder. Therefore there will be an extra contribution to the energy from these winding modes.

$$\begin{aligned} E^2 &= p^2 + M^2 + (\text{“winding energy”})^2 \\ &= \left(p_\perp^2 + \frac{\mathbb{N}}{\alpha'}\right) + \frac{n^2}{R^2} + (\text{“winding energy”})^2\,, \end{aligned} \tag{2.2.3}$$

where α' is the string tension and $\mathbb{N}$ indicates the ‘level’ of the tower of closed string states. Winding energy (E_w) of a string which wraps the cylinder w times is given by

$$E_w = \text{length of the string} \times \text{string tension}$$

$$= w \times (2\pi R) \times \frac{1}{2\pi\alpha'}$$
$$= \frac{wR}{\alpha'} \,. \tag{2.2.4}$$

Thus the total energy becomes

$$E^2 = \left(p_\perp^2 + \frac{\mathbb{N}}{\alpha'}\right) + \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} \,. \tag{2.2.5}$$

If we take $R \to 0$ the momentum modes along the compact direction become very 'heavy' as before but at the same time the winding modes become very 'light'! On the other hand if we take $R \to \infty$ momentum modes play the role of 'light' modes and winding modes become 'heavy'. Clearly there is a duality at work here and to be precise if we make the following transformations

$$R \to \frac{\alpha'}{R}$$
$$(n, w) \to (w, n) \,, \tag{2.2.6}$$

the expression for energy remains unaltered. $R := \sqrt{\alpha'} = l_s$ is called the *self-dual* radius. Physics for $R < \alpha'$ is identical to physics with $R > \alpha'$. I just want to point out the following characteristics of T-duality.

1. It is intrinsically stringy – there is no field theoretic analog to this. Strings perceive the spacetime quite differently compared to point particles.

2. This is *not* a strong-weak duality.

S-duality

This is a strong coupling- weak coupling duality in string theory. S-duality in string theory was first proposed Sen [41] in 1994. This duality maps one string theory with coupling

g_s to another string theory with coupling $\frac{1}{g_s}$. For example, type IIB string theory with the coupling constant g_s is equivalent via S-duality to the same string theory with the coupling constant $\frac{1}{g_s}$. Similarly, type I string theory and the SO(32) heterotic string theory are dual to each other.

Gauge/string duality

This duality [1] mixes the above two frameworks *viz.* QFTs and string theory. It was proposed by Juan Maldacena in 1997. It roughly states a particular gravitational theory in asymptotically AdS space-time is equivalent to a certain field theory in one less number of spacetime dimensions. This statement is a very 'coarse grained' version of Maldacena's original conjecture which will be reviewed in detail in chapter 3.

3 Strings, D-branes & Holography

String theories

General theory of relativity is arguably the most beautiful theory written down by human mind. It works remarkably accurately at the astrophysical level. Recent discovery of gravitational waves [42–44] has put it into a even firm footing. But if one tries to write down a quantum field theory of gravity it suffers from UV divergences and turns out to be non-renormalizable. String theory is the most promising candidate for quantum theory of gravity. General relativity comes out naturally from perturbative string theory. After the discovery of dualities and particularly AdS/CFT correspondence string theory seems to be promising even at the non-perturbative level. String theory avoids the UV divergences since it contains objects with finite length, namely the strings, instead of point particles. Thus the theory is described by a two dimensional worldsheet action, instead of one dimensional worldline,

$$S_{NG} = -\frac{1}{2\pi l_s^2}\int d\tau\, d\sigma\, \sqrt{-det(h)}\,. \tag{3.1.1}$$

This action contains a square root and hard to quantize. That's why people usually work with Polyakov action,

$$S_P = \frac{1}{4\pi l_s^2}\int d\tau\, d\sigma\, (-\gamma)^{1/2}\,\gamma^{ab}\,\partial_a X^\mu\,\partial_b X^\nu\,\eta_{\mu\nu}\,, \tag{3.1.2}$$

which can be easily shown to be equivalent to Nambu-Goto action at classical level.

This worldsheet theory represents d free bosons in two dimensions and $\gamma_{ab}(\sigma, \tau)$ is independent metric on the worldsheet. Moreover this action enjoys

1. Reparameterization invariance : $\{\sigma, \tau\} \to \{\tilde{\sigma}(\sigma, \tau), \tilde{\tau}(\sigma, \tau)\}$.

2. Weyl invariance : $\gamma_{ab} \to \gamma'_{ab} = e^{2\omega}\gamma_{ab}$.

After choosing a particular gauge namely 'conformal gauge' the theory is conformally invariant and the worldsheet fields satisfy following free two dimensional wave equation

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu(\tau, \sigma) = 0\,. \tag{3.1.3}$$

So the aim is to solve this string equation of motion with appropriate boundary conditions *i.e.* periodic for closed strings and for open strings the boundary conditions can be Dirichlet or Neumann.

It is well known that if one quantizes the theory without harming worldsheet conformal invariance and Poincaré invariance of the target space (which is spacetime itself) one has to have 26 scalars on the worldsheet which is same as having a 26 dimensional spacetime. Below we focus on the spectrum of the bosonic strings.

Closed string spectrum

Closed strings have two independent modes which are usually called left and right movers. Upon quantization the Fourier modes become creation and annihilation operators. It contains a tachyon and three massless fields namely Dilaton (Φ), Graviton ($G_{\mu\nu}$) and anti-symmetric Kalb-Ramond field ($B_{\mu\nu}$) and then there are infinite number of massive higher spin modes.

Open string spectrum

The open string spectrum is given by a single copy of oscillator. This too has a tachyon and a massless Gauge field (A_μ) and then there are infinite number of massive higher spin modes.

The bosonic string theory is not 'physical' for two reasons (i.) it contains tachyon in its spectrum and (ii.) it doesn't have fermions which are fundamental constituents of matter in our universe. For these reasons one adds fermions to the theory and moreover considers supersymmetric version which is famously known as superstring theory. Again symmetries restrict the spacetime dimensionality of superstring to be 10 and there are no tachyons in its spectrum. There are five consistent superstring theories namely Type I, Type IIA, Type IIB, Heterotic $E_8 \times E_8$ and Heterotic SO (32). They have different spectra. But if one is interested in low energy effective theory one can focus only on the massless modes and forget about the infinite tower of massive higher spin modes since they are suppressed by higher powers of $\frac{1}{\alpha'}$. In this thesis we mainly focus on Type II theories (particularly Type IIB) as the original AdS/CFT conjecture was proposed in Type IIB theory.

Superstring theory	**Low energy massless fields**
Type IIA	$G_{\mu\nu}, B_{\mu\nu}, \Phi$ and $C_\mu, C_{\mu\nu\kappa}, C_{\mu\nu\kappa\sigma\rho}$
Type IIB	$G_{\mu\nu}, B_{\mu\nu}, \Phi$ and $C, C_{\mu\nu}, C_{\mu\nu\kappa\sigma}$

Table 3.1. Low energy massless fields in different superstring theories

Notice that, $G_{\mu\nu}, B_{\mu\nu}, \Phi$ fields which come from the NS-NS sector are present in both theories. But the Ramond-Ramond (antisymmetric) fields are different for different theories - type IIA has 'odd-form' R-R fields where as type IIB contains only 'even-form' R-R fields. A d-form gauge field naturally couples to a $d-1$ dimensional charge. E.g, a $U(1)$

1-form gauge field couples to a worldline of a charged particle as : $\int A_\mu \, dx^\mu$ or 2-form Kalb-Ramond field $B_{\mu\nu}$ naturally couples to string worldsheet as : $\int B_{\mu\nu} \, dx^\mu dx^\nu$.

The natural question that comes to one's mind is what are the corresponding charges for different R-R fields? The answer is D-branes!

D-branes

D-branes [45] appear in string theory in two seemingly different ways.

1. They are lower dimensional objects where an open string can end.

2. These are also solutions to particular supergravity *e.g.* type IIB SUGRA.

The first one is essentially an open string description of D-brane. If one is interested in studying D-branes from this perspective one should study the dynamics or fluctuations of open strings ending on the D-brane. The second one is effectively a closed string description which says that a D-brane as an 'heavy' object sources gravitons and can be described geometrically. In section 3.4 we will see how the equivalence between these two seemingly different descriptions of D-brane leads Maldacena to the AdS/CFT conjecture.

In the last section we stated that D-branes source the R-R fields or equivalently they are the charges of R-R fields.

Superstring theory	**Low energy massless fields**	**Sources coupled to fields**
	$G_{\mu\nu}, B_{\mu\nu}, \Phi$	F1 couples to $B_{\mu\nu}$
Type IIA	$C_\mu, C_{\mu\nu\kappa}, C_{\mu\nu\kappa\sigma\rho}, C_{\mu\nu\kappa\sigma\rho\lambda\gamma}$	D0, D2, D4, D6
Type IIB	$C, C_{\mu\nu}, C_{\mu\nu\kappa\sigma}, C_{\mu\nu\kappa\sigma\rho\lambda}, C_{\mu\nu\kappa\sigma\rho\lambda\gamma\delta}$	D(-1), D1, D3, D5, D7

Table 3.2. D-branes appearing in IIA and IIB theories.

Note that we have added the magnetic dual fields in the table 3.2. For example, $C_{\mu\nu\kappa\sigma\rho\lambda}$ is dual to $C_{\mu\nu}$ and thus D5 brane is magnetic dual to D1 brane. Notice that $C_{\mu\nu\kappa\sigma}$ is self dual field in type IIB theory and correspondingly D3 brane is its own magnetic dual.

Why do we expect AdS/CFT duality?

At first sight the equivalence of gravity with gauge theory in one lower spacetime dimensions might seem very strange mainly for following reasons.

1. Two theories don't even live in same number of spacetime dimensions.
2. One is gauge theory without gravity and other one *is* a gravity theory.

If we look back to few influential discoveries of theoretical physics in last few decades the duality looks more plausible.

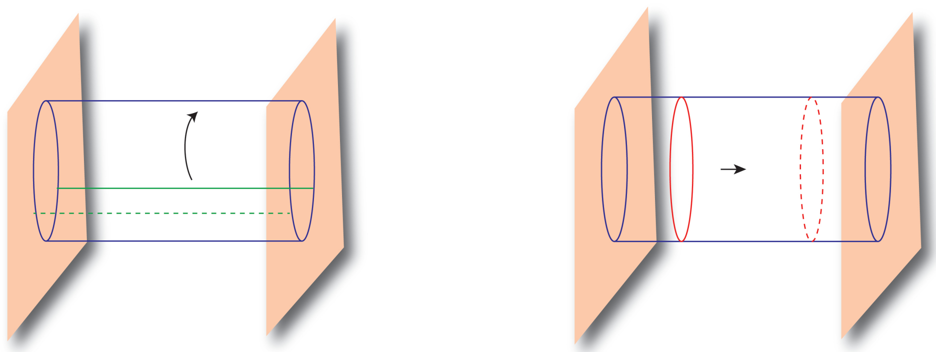
Open string-closed string duality

Closed string spectrum contains : {Graviton + infinite tower massive modes.}

Open string spectrum contains : {Gauge field + infinite tower of massive modes.}

If we are interested only in low energy physics, closed string has 'gravity' in it where as open string contains Yang-Mills 'gauge fields'. Now let's look at the following process in fig. 3.1. One can look at it in two completely different but equivalent ways namely a closed string is being exchanged between the D-branes or an open string is running in a loop between them. Roughly it means,

$$\text{Closed string tree} = \text{Open string loop}.$$



(a) Open string running in a loop **(b)** Closed string is exchanged between D-branes

Figure 3.1. Open string-closed string duality

Therefore one would expect, at least in some particular sense, there should be an equivalence between *gauge theory* and *gravity*.

Large-N gauge theories

It is established that strong nuclear force is described by QCD which is nothing but a Yang-Mills theory with gauge group SU(3). Here three indicates the number of colors. The Yang-Mills coupling undergoes dynamical transmutation and QCD doesn't have a free parameter to play with – QCD is very difficult. What will happen if one works with infinite number of colors instead of only three? This was the question 't Hooft asked in seventies [46]. Actually the theory simplifies[1] a lot. 't Hooft introduced a parameter N which is the number of colors and it then plays the role of a free parameter. The $N \to \infty$ limit is similar to taking $\hbar \to 0$ *i.e,* 'classical' limit of QCD.

For this discussion we shall consider pure $SU(N)$ YM *i.e.* no 'quarks'. But adding 'quarks' is a very straight forward extension. The 'gluons' are adjoint valued elements of

[1]This is in the same spirit in statistical mechanics. When fluctuations are important one way to handle them is to work with a lot of such fluctuating variables. 3-body problem is very difficult but a box of gas with huge number of molecules is easier to handle! Same is true with dimensionality. In lower dimensions there are lot of fluctuations. Mean field theory is easier because one works in infinite dimensions.

$SU(N)$ and the Lagrangian

$$\mathcal{L} = -\frac{1}{4\, g_{YM}^2} \int F^a_{\mu\nu b} F^{\mu\nu b}_a\,, \tag{3.3.1}$$

where $\mathcal{L}$ is a Lorentz scalar since μ, ν indices are contracted and is singlet under $SU(N)$ since a, b indices are contracted. From the Lagrangian it is clear that in this theory,

$$\text{Propagator} \sim g_{YM}^2$$
$$\text{Interaction vertices} \sim \tfrac{1}{g_{YM}^2}\,.$$

We will follow the *double-line* notation what 't Hooft introduced to make the counting easy – replacing each gluon propagator by a quark-antiquark pair (see fig. 3.2).

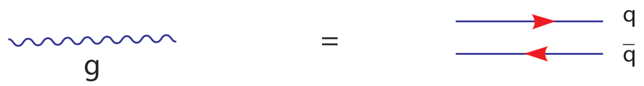


Figure 3.2. The gluon propagator $\sim g_{YM}^2$

In this notation the 3-point and 4-point functions look as follows.

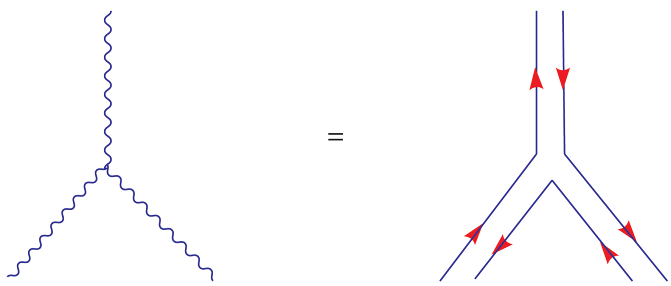


Figure 3.3. 3-pt vertex $\sim \frac{1}{g_{YM}^2}$

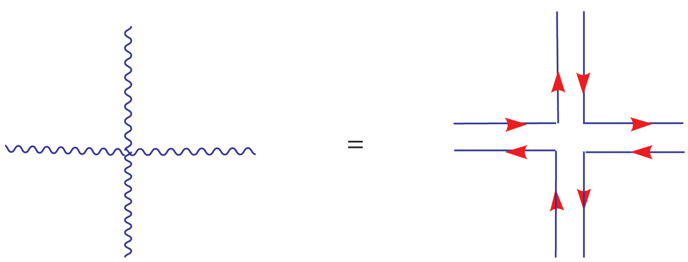


Figure 3.4. 4-pt vertex $\sim \frac{1}{g_{YM}^2}$

Suppose we are interested in vacuum-to-vacuum amplitudes (see fig. 3.5). Our aim is to see how the diagrams scale with N. For that we just need to count the number of propagators and vertices. We know how they scale with the coupling g_{YM}. On top of that whenever we have a *color loop* (color index is summed over) that should correspond to a factor of N, since there are total N colors.

(a) Planar diagram : $\mathcal{O}(N^2)$ **(b)** Non-planar diagram : $\mathcal{O}(N^0)$

Figure 3.5. Vacuum-to-vacuum amplitude in double-line notation

For fig. 3.5a :

# of propagators = 3

# of vertices = 2

# of loops = 3

$$\therefore \quad \text{It scales as} \sim (g_{YM}^2)^3 \frac{1}{(g_{YM}^2)^2} N^3 = (g_{YM}^2 N)\, N^2 \equiv \lambda\, \boldsymbol{N^2} \,.$$

For fig. 3.5b :

# of propagators = 6

# of vertices = 4

# of loops = 2

$$\therefore \quad \text{It scales as} \sim (g_{YM}^2)^6 \frac{1}{(g_{YM}^2)^4} N^2 = (g_{YM}^2 N)^2\, N^0 \equiv \lambda^2\, \boldsymbol{N^0} \,.$$

We have defined[2] a new effective coupling $\lambda := g_{YM}^2 N$ and have extracted the N-dependence.

[2]This λ is called 't Hooft coupling since 't Hooft introduced this quantity. Also keeping $\lambda := g_{YM}^2 N$ *fixed*, with $N \to \infty$ is known as 't Hooft scaling limit for the same reason.

If we keep λ to a fixed value as $N \to \infty$, the fig. 3.5a contributes at $O(N^2)$ whereas fig. 3.5b contributes at $O(N^0)$. Notice that the fig. 3.5a can be drawn on a plane or a sphere and is called *planar* diagram [46]. On the other hand fig. 3.5b can not be drawn on a plane – one requires a torus. This is a *non-planar* diagram.

Therefore at large N and fixed (but small λ) one can schematically write down SU(N) YM vacuum-to-vacuum amplitude as following.

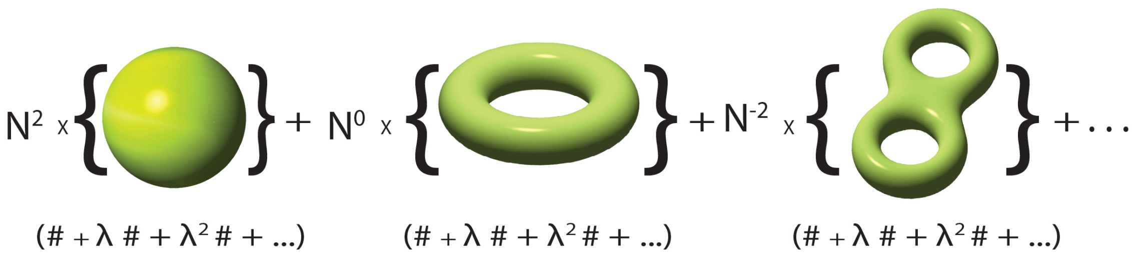


Figure 3.6. Large N expansion of SU(N) gauge theory

Notice that at large N one needs to consider only the planar (or sphere) diagrams but there are still infinitely many such terms since it is a perturbative expansion in λ. At this point someone familiar with string perturbation theory [47, 48] can easily correlate this with perturbative string amplitude which looks as follows,

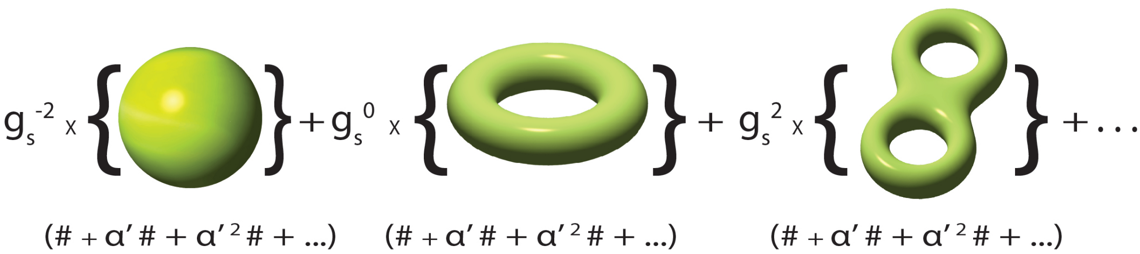


Figure 3.7. Perturbative expansion of closed strings

and formally identify

$$\boxed{\begin{aligned} g_s &\Leftrightarrow \frac{1}{N} \\ \alpha' &\Leftrightarrow \lambda \end{aligned}} \tag{3.3.2}$$

Thus gauge theory at large N and string theory have similar perturbative expansions – it's not very hard to imagine that they can be related to each other.

Holographic principle

The holographic principle, originally proposed by 't Hooft [49], states that the total amount of information inside a volume of space cannot be larger than the amount of information that can be encoded on its boundary. Later Susskind worked on this principle in the context of string theory [50]. We don't really need the details. The expression for Bekenstein-Hawking entropy [51,52] of black hole

$$S_{BH} = \frac{\text{Horizon area}}{4G}\,, \tag{3.3.3}$$

will ring a bell. It scales as the area of its horizon divided by $4G$. Intuitively one can argue why there should be such a principle. Suppose we have some volume of space with some matter in it and we can put more and more matter inside that volume of space, thus increasing the entropy within it. But at some point there will be so much matter inside that it will essentially collapse into a black hole! Thus we cannot increase the entropy of a volume of space indefinitely; we can only increase it until it is equal to the surface area of the volume divided by $4G$. We know that entropy is a measure of information. The more information the volume has, the more entropy it will get. This is the holographic principle which roughly says that the total amount of information inside a volume of space cannot be larger than the amount of information that can be encoded on the boundary of that volume.

Thus in a crude way 'volume' is equivalent to 'its boundary'. And therefore if a $d+1$ dimensional gravity theory is dual to a d dimensional field theory living on its boundary, it shouldn't be so surprising.

The decoupling argument

It is quite clear from the above motivations that there should be some relationship between gauge theory and gravity (string theory). After the discovery of D-branes [45] in mid-nineties there was a rapid development in this direction [53–56]. Finally, in 1997, Maldacena proposed a duality [1] between[3] $\mathcal{N} = 4$ SYM and Type IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$. His argument to reach this conjecture is famously known as *the decoupling argument* and we discuss this in detail in this section.

Different descriptions of same physics

Before discussing about the decoupling argument let's discuss about some simple examples of describing same physical phenomenon using two complementary point of views[4]. This discussion will be very useful in describing Maldacena's decoupling limit.

QED

Let's start with a very basic example from QED. Suppose we want to understand how an electron's moves in presence of a proton. We can treat this problem perturbatively and sum up all possible Feynman diagrams (see fig. 3.8).

The first diagram of fig. 3.8 in position space gives the standard Coulomb potential $V(r) \sim -\frac{1}{r}$. The other diagrams are corrections to this 'classical potential'. There will be many more diagrams as one goes to higher loops. The more number of diagrams one considers the more accurate the description will be. Effectively the extra diagrams change the form of the potential, $V(r) = -\frac{\alpha}{r}[1 + \#\,\alpha\,\frac{e^{-2m_e r}}{(m_e r)^{3/2}} + \ldots]$.

[3] See appendix B and appendix C for more details on AdS space and $\mathcal{N} = 4$ SYM respectively.

[4] In this section we heavily follow [57, 58].

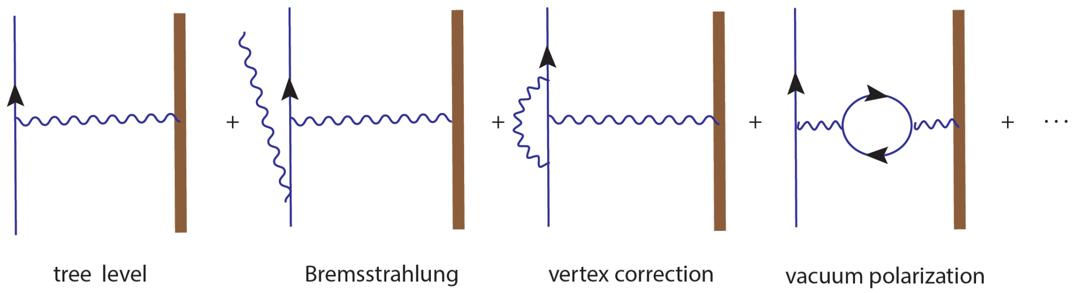


Figure 3.8. Electron's motion near proton

There are two different ways of describing the same phenomenon.

- **Picture I :** The electron and the proton are in vacuum and they are interacting via exchanging photons (see fig. 3.8). Then sum all such Feynman diagrams.
- **Picture II :** Another way of describing the same problem is the following. There is no proton but the electron is moving in a background potential $V(r) = -\frac{\alpha}{r}\,[\,1 + \#\,\alpha\,\frac{e^{-2m_e r}}{(m_e r)^{3/2}} + \ldots]$ (see fig. 3.9).

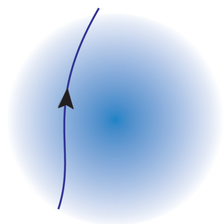


Figure 3.9. Electron moving in a 'background' field

String theory

What can be the analogous picture in string theory? One should replace the electron by an 'elementary' closed string and the 'heavy' proton by a heavy and extended object available in the theory – D-brane.

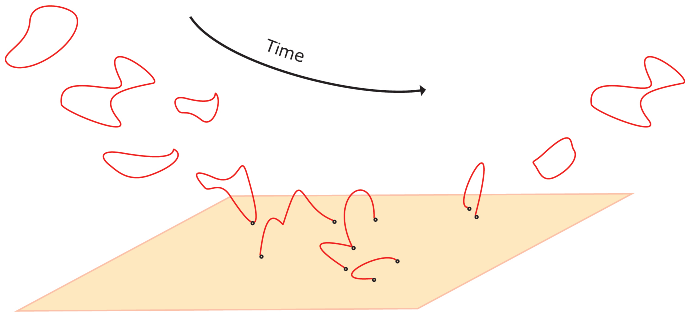


Figure 3.10. A closed string moving near a D-brane.

Again we have two different ways of looking at this phenomenon.

- **Picture I :** First approach would be analogous to summing over Feynman diagrams *i.e.,* studying the scattering of a closed string with a D-brane perturbatively. The string can split into many closed strings or can become an open string on the D-brane and then can further split into many open strings on that brane (see fig. 3.10). Some of the open strings can join the end points on the D-brane and leave the brane as closed strings.

  In the world sheet picture it is easier to keep track of the factors of couplings (similar to counting loops in QED). The number of handle indicates string splitting and number of boundary of the worldsheet signifies the interaction with the D-brane (see Fig 3.11). One needs to sum over all different worldsheet topologies.

- **Picture II :** Here is another equivalent description of the same phenomenon. One can forget about the existence of the D-brane and replace all intermediate effects (Feynman diagrams) by an effective background (see fig. 3.12) in which the closed string moves. In this picture we are considering D-brane as a source of closed strings. The 'coherent state' of large number of closed strings effectively changes the background near the brane.

Notice that 'Picture I' holds true only in the perturbative regime *i.e,* low energy action of open strings on the brane. This is described by SYM theories.

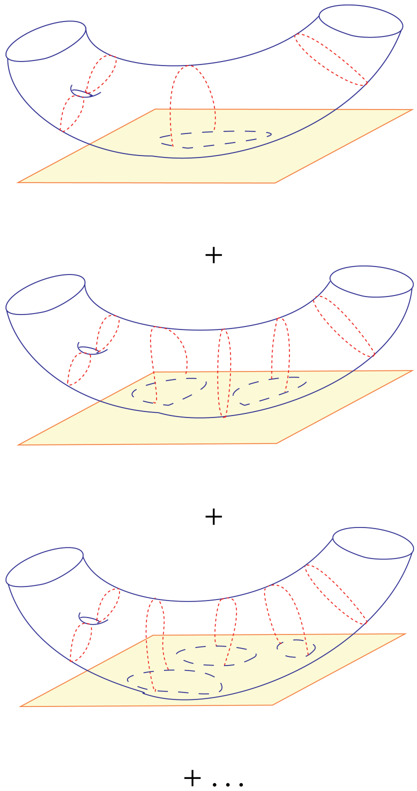


Figure 3.11. Worldsheet picture of closed string - D-brane interaction

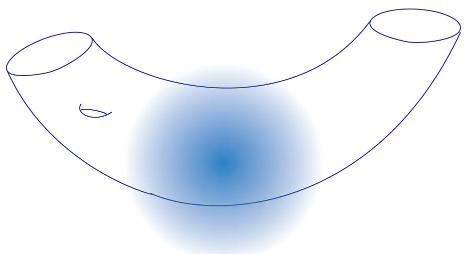


Figure 3.12. A closed string moving near a D-brane.

Whereas in ‘Picture II’ D-brane is the source of closed strings. Since the closed strings change the background this should have some gravitational description.

Maldacena's Argument

In his original paper [1], Maldacena started with a stack of N $D3$ branes (fig. 3.13). One can again describe the system in two alternative ways – (i) by open string dynamics or (ii) by closed string dynamics[5].

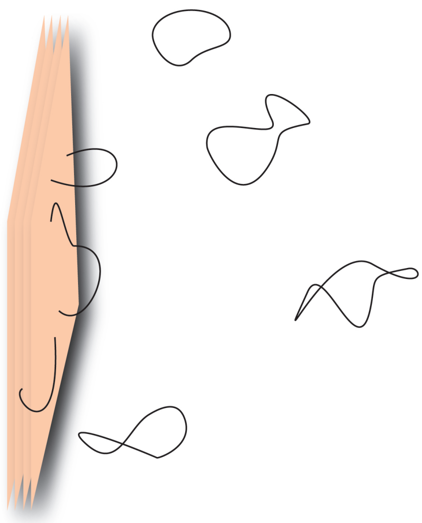


Figure 3.13. A stack of N D3-branes

Picture I

Let's see how one would describe the low energy dynamics of this system from open string perspective. The stack of N branes are described by $\mathcal{N} = 4$ U(N) SYM theory plus higher derivative terms. These higher derivative interactions come due to integrating out all massive open string modes. These are all suppressed by increasing powers of α'.

[5]See fig. 3.11. If one tries to look along a D-brane one can ‘see’ the worldsheet of open string fluctuating. On the other hand if one looks perpendicular to the D-brane one can ‘see’ closed string(s) being emitted (or absorbed) by the D-brane.

Similarly away from the branes (we call it 'bulk') the physics should be described by 10D low energy string theory (type IIB super-string theory since D3 brane appears in IIB theory) which is known as type IIB super-gravity. Again there will be higher derivative interactions suppressed by different powers of α'. And these two theories can interact. So schematically the action for the total system looks as following.

$$S = \underbrace{S_{branes}}_{\text{(SYM + higher derivatives)}} + \underbrace{S_{bulk}}_{\text{(10D SUGRA + higher derivatives)}} + S_{int} \quad . \tag{3.4.1}$$

The 'bulk' and the 'branes' interact gravitationally. Maldacena's main aim was to turning off this interaction by tuning some coupling and to decouple the theories. Notice that if we take $\alpha' \to 0$ keeping g_s and N fixed, it is equivalent to taking Newton's constant $G_N \to 0$ because $\sqrt{G_N} \sim g_s \alpha'^2$. But α' is a dimensionful quantity, therefore it can not be taken to zero. The correct way to take the limit is to make α' smaller compared to the energy (or inverse length) scale one is looking at *i.e.,*

$$\alpha' \, |\vec{k}|^2 \ll 1 \quad \text{or,} \quad \frac{|\vec{x} - \vec{x}'|^2}{\alpha'} \gg 1 \, . \tag{3.4.2}$$

Taking such a limit amounts to turning off all the interactions and all higher derivative terms since they come with positive powers of G_N or α'. Thus we are left with 4 D SYM and 10 D super-gravity which are not talking to each other.

$$\boxed{\text{Picture I :} \quad \mathcal{N} = 4 \text{ SYM in 4 dimensions} \oplus \text{Super-gravity in 10 dimensions.}}$$

Picture II

Following same chain of arguments we want to see the stack of D3 branes a gravitational solution or we should ask, what background do the stack of branes produce?

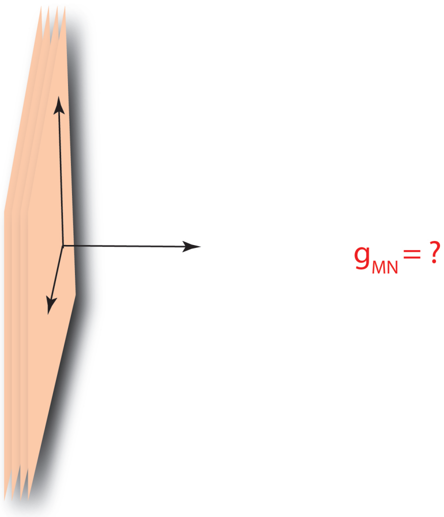


Figure 3.14. Stack of D3-branes as a gravitational solution

This "classical" background should be described by low energy effective action of string theory which for this particular case is type IIB super-gravity (Einstein action + "other fields"). Therefore the aim is to look for a solution or a metric for this stack of N D3 branes (fig. 3.14).

Exploiting the symmetry of the system we start with the following ansatz

$$ds^2 = \eta_{\mu\nu} \frac{dx_\mu \, dx_\nu}{f(r)} + \widetilde{f}(r) \, dx^m \, dx_m \quad \text{where,} \quad r^2 = x^m \, x_m \, . \tag{3.4.3}$$

To satisfy Einstein equations the unknown functions in the ansatz have to have the following form

$$f(r) = \widetilde{f}(r) = \sqrt{1 + \frac{L^4}{r^4}} \quad \text{with,} \;\; L^4 = g_s N \, (4\pi\alpha'^2) \, . \tag{3.4.4}$$

Once we have the metric there are two obvious 'extreme' limits we can look at in this picture II :

(i) $r \to \infty$

(ii) $r \to 0$

Far away from the branes ($r \to \infty$)

Far away from the branes the geometry has to be flat space $\mathbb{R}^{9,1}$

$$ds^2 = dx^\mu dx_\mu + dx^m dx_m \,. \tag{3.4.5}$$

Near the branes ($r \to 0$)

$$ds^2 = \underbrace{\frac{r^2}{L^2}\eta_{\mu\nu}dx_\mu dx_\nu + \frac{L^2}{r^2}dr^2}_{\mathrm{AdS}_5} + \underbrace{L^2 d\Omega_5^2}_{\mathrm{S}^5} \,. \tag{3.4.6}$$

In GR we always talk about observables with respect to particular observers. Let's ask the question what do we mean by 'time'?

$$dx^\mu dx_\mu = -dt^2 + d\vec{x}^2 \,. \tag{3.4.7}$$

This t is just co-ordinate time and it is 'physical' or 'proper' time only for an observer at $r = \infty$. Therefore the natural question arises what is the 'time' for an observer at arbitrary r?

The proper time for an observer is related to co-ordinate time as follows.

$$\begin{aligned} \Delta t_{prop} &= \sqrt{g_{tt}}\,\Delta t \\ &= \frac{r}{L}\Delta t \,. \end{aligned} \tag{3.4.8}$$

On dimensional ground the 'proper' energy

$$\Delta E_{prop} = \frac{L}{r}\Delta E \,. \tag{3.4.9}$$

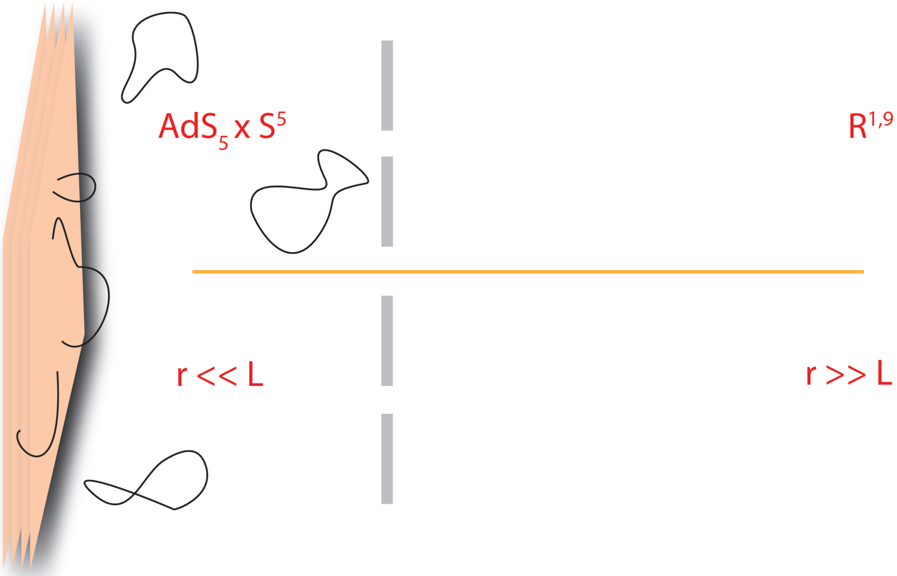


Figure 3.15. The decoupling limit

Notice that for $r \to 0$ there is an infinite red shift. So even if the near the stack of branes the energy E is arbitrarily large[6] for the observer at $r \to \infty$ it is *finite* due to the redshift.

We are interested in 'low energy' physics for the observer at $r \to \infty$. The things near him/her are already low energy *i.e,* 10 D super-gravity and *anything* near $r \to 0$ is also low energy due to huge redshift as discussed above. Here by *anything* we mean *full string theory* in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

$$\boxed{\text{Picture II : } \quad \text{Full type IIB string theory in } \mathrm{AdS}_5 \times \mathrm{S}^5 \oplus \text{Super-gravity in 10 dimensions.}}$$

[6]This is very crucial point. So let's elaborate on it with a simple thought experiment. Suppose A and B are at $r \to \infty$. A is carrying a $10^{100}\, GeV$ 'lamp'. Suddenly A decides to walk towards it. Due to the red shift factor, to B the lamp energy keeps decreasing (*i.e.,* lamp's frequency gets smaller) as A approaches the stack. When A is very close to the branes, such that the redshift factor is 10^{-109} say, to B the lamp's energy is just $1\, eV$. But for A it is still the $10^{100}\, GeV$ lamp! Therefore arbitrarily large energy near the branes is finite energy for the observer far away. The bottom line is, low energy theory for the observer at infinity includes all possible high energy phenomena near the D-branes – full string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

We need to "equate" Picture I and Picture II. Comparing these two description Maldacena conjectured,

$$\boxed{\mathcal{N} = 4 \text{ SYM in 4 dimensions} \equiv \text{Full type IIB string theory in AdS}_5 \times \text{S}^5\,.}$$

The dictionary of parameters

There are two dimensionless parameters in the gauge theory namely g_{YM} and N. As we have discussed before it is more convenient to define dimensionless 't Hooft coupling $\lambda \equiv g_{YM}^2 N$. Thus the gauge theory has two independent dimensionless couplings g_{YM} and λ.

On the other hand the string theory in $\text{AdS}_5 \times \text{S}^5$ has one dimensionless coupling g_s and two dimensionful parameters namely the string length $l_s = \sqrt{\alpha'}$ and the AdS radius L. Thus effectively this theory also has two dimensionless parameters g_s and $\frac{L}{l_s}$. They are related as follows.

$$\boxed{\begin{aligned} g_{YM}^2 &= g_s\,, \\ \lambda \equiv g_{YM}^2 N &= \left(\frac{L}{l_s}\right)^4\,. \end{aligned}} \tag{3.5.1}$$

Planar limit : According to the stronger version of the conjecture, the above matching of parameters holds true for all values of the parameters. Things get simplified if one takes $N \to \infty$ keeping $\lambda = fixed$ *i.e,* $g_{YM}^2 \to 0$. This is famous 't Hooft limit. In this limit, as we have seen in the large N gauge theories, only the planar diagrams in $\mathcal{N} = 4$ SYM contribute since the other non-planar diagrams are suppressed by powers of $1/N$.

Gauge theory	String theory
Small λ Perturbative SYM `(Easy)`	*Small λ i.e, $L \sim l_s$* Highly (stringy) quantum theory `(Hard)`
Large λ Strongly coupled theory `(Hard)`	*Large λ i.e, $L \gg l_s$* Classical SUGRA `(Easy)`

Table 3.3. Different regimes of gauge/gravity duality

Analogously in the string theory side $g_s \to 0$ and $\frac{L}{l_s}$ remains finite which means that string cannot split and join (*i.e,* no 'handles' in the string world sheet).

If we restrict ourselves in this planar limit where $N \to \infty$ and $\lambda = fixed$, there are two possibilities : λ can be *large* or *small*.

Notice that (see table 3.3) when one side of the duality is computationally easy the other side becomes extremely hard to handle. Thus it is not only very difficult to *prove* the duality but it's even hard to *check*. At the same time, due to exactly the same reason, the duality is extremely powerful. One can calculate interesting quantities in strongly coupled quantum field theories by computing corresponding quantities in dual classical gravitational theory.

Generalization to finite temperature and density

Although the AdS/CFT duality was proposed in a very particular setup it is believed to be valid for more generic systems and is usually referred to as gauge/gravity duality. This was first generalized to thermal state of CFT which is dual to a black hole in asymptoti-

Field theory in d-dimensions	**Dual gravity in $(d+1)$-dimensions**
$T = 0$ and $\mu = 0$	Pure AdS$_{d+1}$
$T \neq 0$ and $\mu = 0$	Black hole in AdS$_{d+1}$ (or thermal AdS$_{d+1}$)
$T \neq 0$ and $\mu \neq 0$	Charged black hole in AdS$_{d+1}$
$T = 0$ and $\mu \neq 0$	Extremal charged black hole in AdS$_{d+1}$

Table 3.4. Generalization of gauge/gravity duality for different backgrounds

cally AdS spacetime [4]. Turning on chemical potential in the field theory is equivalent to adding charge to the black hole[7] (see table 3.4). In this thesis we study Brownian motion using all those different dual gravitational backgrounds.

The GKPW prescription

To use the above mentioned duality quantitatively one needs to have a prescription that relates the field theory quantities to their gravity theory equivalents. Such a prescription has been given in [2, 3] which state that partition function of the QFT coincides with the same of gravity theory

$$\left\langle \exp\left(\int_{\partial \mathrm{AdS}_5} \phi_0^i O_i\right)\right\rangle_{\mathrm{CFT}} = Z_{\mathrm{QG}}(\phi_0^i)\,, \tag{3.7.1}$$

where ϕ^i are bulk fields in gravity theory and O^i are their dual boundary operators in the gauge theory. $Z_{QG}(\phi_0^i)$ is the partition function of quantum gravity with the boundary conditions that ϕ^i goes to ϕ_0^i on the boundary. The conjecture becomes useful in studying strongly coupled field theories when the gravity theory is 'classical'. In that limit the path integral can be approximated by saddle point. Treating ϕ_0^i as the sources of boundary

[7]There are also other generalizations *e.g,* for rotating black holes. This is known as Kerr/CFT correspondence [59–61] which is not very well understood.

field theory one can calculate the correlators by taking functional derivative of Z_{QG} with respect to ϕ_0^i.

The above prescription is applicable to obtain Euclidean correlators. The Euclidean signature avoids some complications related to boundary conditions. However in many cases, particularly for finite temperature systems which also appear frequently in this thesis, extraction of Lorentzian-signature AdS/CFT results directly from bulk gravity theory is inevitable (see appendix D to review some properties of different correlators in QFT). Therefore one requires to have some prescription for computing real time correlators directly from gravity. This was done by Son and Starinets [11].

In Euclidean space

Let us first recall the AdS/CFT formulation in Euclidean space [2, 3]. For historical significance and definiteness, we talk about the famous correspondence between $\mathcal{N}=4$ SYM theory and classical gravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$. The Euclidean version of the metric for this compact manifold is given by (see (B.3.6))

$$ds^2 = \frac{L^2}{z^2}(d\tau^2 + d\vec{x}^2 + dz^2) + L^2 d\vec{\Omega}_5^{\,2}\,, \tag{3.7.2}$$

where $z = 0$ corresponds to the boundary of AdS_5 where the four dimensional quantum field theory lives. Consider a scalar field ϕ in the bulk, which is coupled to an operator O on the boundary such that the interaction Lagrangian is ϕO. AdS/CFT correspondence then states

$$\left\langle e^{\int_{\partial M}\phi_0 O}\right\rangle = e^{-S_{cl}[\phi]}, \tag{3.7.3}$$

where $S_{cl}[\phi]$ is the action of classical solution to the equation of motion for ϕ in the bulk metric with the boundary condition : $\quad \phi\big|_{z=0} = \phi_0.$

The metric 3.7.2 corresponds to the zero-temperature field theory. To study field theory at finite temperature, one has to modify it to a non-extremal one (see table 3.4),

$$ds^2 = \frac{L^2}{z^2}\left(f(z)\,d\tau^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right) + L^2 d\vec{\Omega}_5^{\,2}\,, \tag{3.7.4}$$

where $f(z) = 1 - z^4/z_H^4$ and $z_H = (\pi T)^{-1}$. T is Hawking temperature, τ is the Euclidean time co-ordinate which is periodic : $\tau \sim \tau + T^{-1}$ and z is between 0 and z_H.

Difficulties in Minkowski Space

In Minkowski space too one can try to put down the correspondence in following way

$$\left\langle e^{i\int_{\partial M}\phi_0 O}\right\rangle = e^{iS_{cl}[\phi]}\,. \tag{3.7.5}$$

But there are some difficulties with this Minkowski version of the duality. The basic problem is with the boundary condition. In Euclidean case ϕ is uniquely determined by its value at the boundary $z = 0$ and the requirement of regularity at horizon, $z = z_H$. So, the Euclidean correlator is unique. In Minkowski space, unlike the previous case, only the regularity at horizon is insufficient. To pick a solution one has to have a more refined boundary condition there. From physical perspective one important boundary condition is the incoming wave at $z = z_H$. This wave goes inside the horizon but cannot escape from there. But even if we choose such a boundary condition, the Minkowski version (3.7.5) will still be problematic [11].

Action of scalar field in AdS

We start with the AdS part of the metric (3.7.4), which can be written as

$$ds^2 = g_{zz}\,dz^2 + g_{\mu\nu}(z)\,dx^\mu dx^\nu. \tag{3.7.6}$$

Consider a fluctuation of scalar field ϕ on this background space-time. For any curved (d+1) dimensional space-time the action due to scalar field reads

$$S = \int \sqrt{-g}\, d^{d+1}x \left[D^{\mu}\phi\, D_{\mu}\phi + m^2\phi^2)\right], \tag{3.7.7}$$

where μ runs from 0 to d and D_μ is the covariant derivative.

The action for scalar in AdS$_5$ space is

$$S = K \int d^4x \int_{z_B}^{z_H} dz\, \sqrt{-g} \left[g^{zz}\,(\partial_z\phi)^2 + g^{\mu\nu}(\partial_\mu\phi)\,(\partial_\nu\phi) + m^2\phi^2\right], \tag{3.7.8}$$

where K is normalization constant (for dilaton $K = -\pi^3 L^5/4\kappa_{10}^2$, κ_{10} is the 10 dimensional gravitational constant) and m is the mass of the scalar.

The action (3.7.7) can be re-written in the following way

$$S = K \int \sqrt{-g}\, d^4x \int dz \left[D_A(\phi\, D^A\phi) - \phi\, D_A D^A\phi + m^2\phi^2)\right], \tag{3.7.9}$$

where A contains both μ and z.

$$S = \underbrace{K \int \sqrt{-g}\, d^4x \int dz\, [-\phi\,(\Box - m^2)\,\phi]}_{S_{\text{EOM}}} + \underbrace{K \int \sqrt{-g}\, d^4x \int dz\, [D_A(\phi\, D^A\phi)]}_{S_{\text{Boundary}}}. \tag{3.7.10}$$

The equation of motion (EOM) for ϕ is

$$(\Box - m^2)\,\phi = 0 \tag{3.7.11}$$

$$\frac{1}{\sqrt{-g}}\,\partial_z(\sqrt{-g}\, g^{zz}\,\partial_z\phi) + \frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi) - m^2\phi = 0\,. \tag{3.7.12}$$

Since $g^{\mu\nu}(z)$ is only a function of z,

$$\frac{1}{\sqrt{-g}}\,\partial_z(\sqrt{-g}\, g^{zz}\,\partial_z\phi) + g^{\mu\nu}\,\partial_\mu\partial_\nu\phi) - m^2\phi = 0\,. \tag{3.7.13}$$

It has to be solved using the boundary condition at $z = z_B$. Let's take the solution to be

$$\phi(z,x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik.x}\, f_k(z)\, \phi_0(k), \tag{3.7.14}$$

where $\phi_0(k)$ is determined by the following boundary condition

$$\phi(z_B,x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik.x}\, \phi_0(k), \quad \text{with } f_k(z_B) = 1. \tag{3.7.15}$$

Now substituting (3.7.15) into the EOM, (3.7.13)

$$\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g}\, g^{zz}\partial_z f_k) - (g^{\mu\nu}k_\mu k_\nu + m^2) f_k = 0\,. \tag{3.7.16}$$

Boundary conditions

1. $f_k(z_B) = 1$.
2. Satisfies the incoming wave boundary condition at horizon ($z = z_H$).

Boundary action and Green function

Let us look at the action on shell (i.e, when ϕ satisfies the EOM). Clearly from (3.7.10), the action reduces only to a boundary term,

$$\begin{aligned} S_{\text{Boundary}} &= K \int \sqrt{-g}\, d^4x \int dz\, [D_A(\phi D^A \phi)] \\ &= K \int \sqrt{-g}\;\; d\sigma_k(\phi\, D^k\phi), \end{aligned} \tag{3.7.17}$$

where $d\sigma_k$ is a hyper-surface perpendicular to k direction. Now if the surface is chosen to be perpendicular to z direction (as we are integrating over z from $z = z_B$ to $z = z_H$) the action reduces to

$$S_{\text{Boundary}} = K \int \sqrt{-g}\, d\sigma_z \{\phi D^z \phi\} \tag{3.7.18}$$

$$= K \int \sqrt{-g}\, d^4x \{\phi\, g^{zz}\, (\partial_z \phi)\}\Big|_{z_B}^{z_H}. \tag{3.7.19}$$

Substituting (3.7.15) into (3.7.18) and integrating over z we get,

$$S_{\text{Boundary}} = \int \frac{d^4k}{(2\pi)^4} \Big\{\phi_0(-k)\, \mathscr{F}(k,z)\, \phi_0(k)\Big\}\Big|_{z_B}^{z_H} \tag{3.7.20}$$

$$\text{where,} \quad \mathscr{F}(k,z) = K\sqrt{-g}\, g^{zz}\, f_{-k}(z)\, \partial_z f_k(z). \tag{3.7.21}$$

If we want to calculate Green function, we can use equality (3.7.5) and we can find the two point function taking the second derivative of classical action with respect to ϕ_0, the boundary value of ϕ.

Thus using (3.7.20) the Feynman Green's function

$$\tilde{G}(k) = \mathscr{F}(k,z)\Big|_{z_B}^{z_H} - \mathscr{F}(-k,z)\Big|_{z_B}^{z_H}. \tag{3.7.22}$$

The "problematic" Green function

The problem with this Green function is it is *real*. Green functions are complex in general. Noticing the fact that $f_k^*(z) = f_{-k}(z)$ and using the equation of motion (3.7.16), it can be easily shown that imaginary part of $\mathscr{F}(k,z)$

$$\operatorname{Im}\mathscr{F}(k,z) = \frac{K}{2i}\sqrt{-g}\, g^{zz}\left[f_k^*\partial_z f_k - f_k\partial_z f_k^*\right], \tag{3.7.23}$$

is independent of radial co-ordinate z, i.e, $\partial_z \operatorname{Im}\mathscr{F}(k,z) = 0$. Therefore, in each term of (3.7.22), the imaginary parts at horizon $z = z_H$ and at boundary $z = z_B$ cancel each other.

To avoid the problem we can throw the contribution from horizon term. But from reality of field equation one can show, $\mathscr{F}(-k,z) = \mathscr{F}^*(k,z)$ and imaginary parts still cancel and $\tilde{G}(k)$ remains real.

Minkowski space correlators from AdS/CFT

To get the complex retarded Green's function we will follow the prescription by Son and Starinets, which they proposed as a conjecture in [11] and 'derived' later in [62]. The prescription is

$$\tilde{G}_R(k) = -2\,\mathscr{F}(k,z)\Big|_{z=z_B}. \tag{3.8.1}$$

We check the above conjecture for a zero temperature field theory and reproduce the two point functions following [11]. The prescription is as follows.

1. Find a solution to the (3.7.16) with following properties :
   - It equals to 1 at boundary $z = z_B$.
   - For *time-like momenta* : It satisfies incoming boundary condition at horizon. For *space-like momenta* : The solution is regular at horizon.
2. The retarded Green's function is given by $G = -2\,\mathscr{F}_{\partial M}$, where $\mathscr{F}$ is defined as (3.7.20) and only contribution from boundary has to be taken.

Note that Im $\mathscr{F}(k,z)$ (see (3.7.23)) is independent of radial co-ordinate z and therefore one can calculate it at any convenient value of z, in particular at the horizon.

A sample calculation

To see whether the prescription works, let us consider the following situations whose Green functions are already known using other methods. Let us use the above prescription to calculate the retarded (or advanced) Green's function of the operator $\mathcal{O} = \frac{1}{4}F^2$ at zero temperature. Here the action is of minimally coupled massless scalar field in the background AdS$_5$. The horizon is at $z_H = \infty$ and the boundary is at $z_B = 0$. EOM satisfied by the modes is given by

$$f_k''(z) - \frac{3}{z} f_k'(z) - k^2 f_k(z) = 0 \,. \tag{3.8.2}$$

The Euclidean two point function using GKPW prescription has been computed in appendix E. For Minkowski correlator we need to treat spacelike and timelike momenta separately.

For *spacelike momenta i.e,* $k^2 > 0$, we can follow the steps identical to the Euclidean case (see appendix E) and obtain the Green function

$$G_R(k) = \frac{N^2 k^4}{64\,\pi^2} \ln k^2 \,, \quad k^2 > 0 \,. \tag{3.8.3}$$

The extra minus sign is due to the Lorentzian signature.

For *timelike momenta*, we introduce a new variable $q = \sqrt{-k^2}$. The solution to the equation (3.8.2) with above mentioned boundary conditions

$$f_k(z) = \begin{cases} \dfrac{z^2 H_2^{(1)}(qz)}{\epsilon^2 H_\nu^{(1)}(q\epsilon)} & \text{if } \omega > 0, \\[2ex] \dfrac{z^2 H_2^{(2)}(qz)}{\epsilon^2 H_2^{(2)}(q\epsilon)} & \text{if } \omega < 0\,. \end{cases}$$

Note that $f_{-k} = f_k^*$. Calculating $\mathscr{F}$ from (3.7.20) and using the prescription (3.8.1)

$$\tilde{G}_R(k) = \frac{N^2 K^4}{64\,\pi^2}\left(\ln k^2 - i\,\pi \,\mathrm{sgn}\,\omega\right). \tag{3.8.4}$$

Combining (3.8.3) and (3.8.4) we can express the complete retarded Green's function as

$$G_R(k) = \frac{N^2 K^4}{64\,\pi^2}\left(\ln |k^2| - i\,\pi\,\theta(-k^2)\,\mathrm{sgn}\,\omega\right). \tag{3.8.5}$$

As $z \to \infty$, $\mathscr{F}(k, z)$ does not go to zero rather it becomes purely imaginary in that limit.

$$\mathscr{F}(k, z \to \infty) = \frac{i\,N^2 K^4 \,\mathrm{sgn}\,\omega}{128\pi} = \mathrm{Im}\,\mathscr{F}(k, \epsilon)\,. \tag{3.8.6}$$

This we could have guessed from the fact that flux is conserved (3.7.23) and therefore imaginary part of the Green function can be calculated independently - just from the asymptotic behavior of the solution at the horizon.

We can now use the relation (D.1.11) to get the Feynman propagator at zero temperature

$$\tilde{G}_F(k) = \frac{N^2 K^4}{64\pi^2}(\ln |k^2| - i\pi\theta(-k^2))\,. \tag{3.8.7}$$

Evidently, we can obtain the same propagator by Wick rotating the Euclidean correlator

$$\tilde{G}_E(k_E) = -\frac{N^2 K_E^4}{64\pi^2}\ln k_E^2\,. \tag{3.8.8}$$

Thus the prescription gives the correct answer for retarded Green's function at zero temperature.

Above we have checked the prescription at zero temperature with an example. The same procedure can be applied to compute the retarded Green's functions of two dimensional CFT dual to the non-extremal BTZ black hole. And if the result is analytically continued to complex frequencies we can reproduce the well known Matsubara correlators for thermal field theory (see [11] for details).

We heavily use this prescription to calculate retarded Green function for a Brownian particle throughout rest of this thesis. For that particular system we need to study dynamics of a fundamental string in an asymptotically AdS background in place of scalar field.

4 Brownian motion in 1+1 D

In this chapter, we study the motion of a stochastic string in the background of a BTZ black hole. In the 1+1 dimensional boundary theory this corresponds to a very heavy external quark interacting with the fields of a CFT at finite temperature, and describing Brownian motion. The equations of motion for a string in the BTZ background can be solved exactly. Thus we can use holographic techniques to obtain the Schwinger-Keldysh Green function for the boundary theory for the force acting on the quark. We write down the generalized Langevin equation describing the motion of the external particle and calculate the drag and the thermal mass shift. Interestingly we obtain dissipation even at zero temperature for this 1+1 system. Even so, this does not violate boost (Lorentz) invariance because the drag force on a *constant* velocity quark continues to be zero. Furthermore since the Green function is exact, it is possible to write down an effective membrane action, and thus a Langevin equation, located at a 'stretched horizon' at an arbitrary finite distance from the horizon.

The content of this chapter is based on work done with B. Sathiapalan [7].

Introduction

AdS/CFT correspondence [1–4] has been used quite successfully to study thermal properties such as the viscosity of $\mathcal{N} = 4$ super Yang-Mills theory at finite temperature. Dissipation and thermal fluctuation are two sides of the same coin as embodied in the famous fluctuation dissipation (FD) theorem. The study of fluctuations using holographic techniques has been done in several papers [5, 6, 26–28, 63–65] and the fluctuation dissipation theorem has been shown to hold. Different techniques [5, 6] have been used to address this issue. A very versatile technique is in terms of Green functions. Son and Teaney [6] have used holographic techniques to calculate Green functions to address these questions in the context of Brownian motion of a particle such as a quark.

The fluctuation-dissipation theorem in the context of Brownian motion has been studied by Kubo [10, 66] and Mori [67, 68] amongst others. Brownian motion can be described as a stochastic process [69]. In some approximation it is Markovian. If we can assume that velocities at two instants are not correlated, then it is a Markovian process when described in terms of position. Thus one can define a probability $P\left(x(t), t; x(t_0), t_0\right)$ as the conditional probability for the particle to be in position $x(t)$ at time t given that it was at $x(t_0)$ at time t_0. One can also write a Fokker Plank equation for $P(x(t), t; x(t_0), t_0)$. On the other hand if we want a finer description one can use the velocity as the variable defining the Markovian process in terms of $P(v(t), t; v(t_0), t_0)$. This is a good approximation as long as the duration of a collision is very small, which is equivalent to saying that acceleration at different instants is uncorrelated. The Fokker-Planck equation in the velocity description is

$$\frac{\partial P(v,t)}{\partial t} = -\frac{\partial}{\partial v} a_1(v) P + \frac{a_2}{2}\frac{\partial^2 P}{\partial v^2}\,. \tag{4.1.1}$$

Here $a_1 = \frac{\langle \varDelta v\rangle}{\varDelta t}$ and $a_2 = \frac{\langle (\varDelta v)^2\rangle}{\varDelta t}$. Here $\varDelta v$ is the change in velocity in time $\varDelta t$.

One can obtain these from the related Langevin equation

$$m\dot{v} = -\gamma v + \xi(t)\,, \tag{4.1.2}$$

where $\xi(t)$ is the random force that is responsible for the fluctuations, obeying $\langle \xi(t)\xi(t')\rangle = \varGamma\delta(t-t')$ and $\langle \xi(t)\rangle = 0$. $v(t_0) = v_0$ is the initial condition. Thus $a_1 = \langle v(\varDelta t) - v_0\rangle = -\frac{\gamma}{m} v_0 \varDelta t$.

From the solution of the Langevin equation (taking $t_0 = 0$):

$$v(t) = v_0 e^{-\frac{\gamma}{m}t} + \frac{1}{m}\int_0^t e^{-\frac{\gamma}{m}(t-t')}\xi(t')\,dt', \tag{4.1.3}$$

one can obtain $a_2 = \frac{\Gamma}{m^2}$. Thus the Fokker Planck equation becomes

$$\frac{\partial P(v,t)}{\partial t} = \frac{\gamma}{m}\frac{\partial}{\partial v}vP + \frac{\Gamma}{2m^2}\frac{\partial^2 P}{\partial v^2}. \tag{4.1.4}$$

Finally since we know that $P(v) = e^{-\frac{mv^2}{2kT}}$ is a time independent solution of the Fokker-Planck equation we get

$$\Gamma = 2\gamma kT. \tag{4.1.5}$$

This is the fluctuation dissipation theorem in this context, because it relates Γ, the strength of the fluctuation, to γ the strength of the dissipation.

The Langevin equation is much more convenient to work with. To the extent that it assumes that time scales are larger than the microscopic time scale it must fail for very small time scales. As Kubo [10] has shown, stationarity should imply that

$$\frac{d}{dt_0}\langle v(t_0)v(t_0)\rangle = 0 = \langle \dot{v}(t_0)v(t_0)\rangle. \tag{4.1.6}$$

Whereas (4.1.3) gives

$$\langle \dot{v}(t_0)\,v(t_0)\rangle = -\frac{\gamma}{m}\langle v(t_0)\,v(t_0)\rangle \neq 0\,. \tag{4.1.7}$$

The random force ξ represents the effects of the interaction of other degrees of freedom on our particle and the assumption that the correlation time is zero is unphysical. A proper microscopic theory that incorporates these effects should not give this contradiction. Kubo has argued that one can replace $\Gamma\delta(t-t')$ by a more general $\Gamma(t-t')$ which is less singular than a delta function. To see this we modify the Langevin equation to

$$m\dot{v} = -\int_{t_0}^{t} dt'\,\gamma(t-t')\,v(t') + \xi(t)\,. \tag{4.1.8}$$

As long as

$$\lim_{t\to t_0}\int_{t_0}^{t} dt'\,\gamma(t-t')\,v(t') = 0, \tag{4.1.9}$$

the aforementioned contradiction is avoided. Thus

$$\gamma(\omega) = \int_0^\infty dt\, e^{i\omega t}\,\gamma(t) \tag{4.1.10}$$

acquires a non trivial frequency dependence. With this it can be shown that

$$\int_0^\infty dt\, e^{i\omega t}\langle\xi(t_0)\,\xi(t_0+t)\rangle = \Gamma(\omega) = k\,T\,\gamma(\omega)\,. \tag{4.1.11}$$

This is the fluctuation-dissipation theorem that replaces (4.2.18)[1]. In fact more generally fluctuation dissipation theorems can be stated in terms of properties of various two point correlation functions. This is particularly clear in the Schwinger-Keldysh formalism [70,71]. Son and Teaney [6] have shown how the Schwinger-Keldysh Green functions can be obtained holographically and their holographic calculation gives such a frequency dependent correlation function for the noise which satisfies the FD theorems. However if one expands in powers of frequency one cannot see the softening of the delta function. One needs a more non perturbative result.

In addition to obtaining a Langevin equation for the boundary theory at infinity, Son and Teaney [6] also obtained an effective membrane action and a Langevin equation, at a 'stretched' horizon close to the event horizon. However in AdS_5 the equations cannot be solved exactly. Thus the solution had to be worked out as a power series in the frequency. We do an almost identical calculation for the case of the BTZ black hole in AdS_3 where

[1]The factor of 2 has disappeared because in the Laplace transform the integral is from 0 to ∞ and not from $-\infty$ to ∞.

one can solve the bulk equation of motion exactly. This was first shown in [5] where some exact correlators were computed. We then use the techniques of [6] to obtain the Schwinger-Keldysh Green functions exactly. We do indeed find the softening of the delta function that avoids the contradiction pointed out by Kubo [10]. It is interesting that internal consistency at the microscopic level is built into the holographic formalism. (Of course the holographic result is in some sense the leading term in a "strong coupling" expansion i.e, large N and large λ limits of the theory. Departures from large N requires quantum or stringy corrections in the bulk, whereas departure from large λ suggests 'supergravity' is not a good approximation and needs higher derivative corrections to the gravitational theory. Therefore to ensure consistency at higher orders it may be that one has to embed the boundary theory in a string theory.)

We also find the interesting phenomenon of dissipation at zero temperature. This is a little puzzling because at zero temperature one expects the system to have Lorentz invariance and boost invariance would say that a quark moving at a constant velocity cannot possibly feel any drag force. One can indeed check in our case, that even though the Green function does have a dissipative component at zero temperature, the frequency dependence is such that force on a constant velocity quark does continue to be zero. Thus there is nothing unphysical about this. Accelerating quarks can certainly experience dissipation by coupling to the massless degrees of freedom in the conformal field theory - i.e. "radiation" [12, 72, 73]. Dissipation at zero temperature has been reported in the literature earlier [12, 72–82].

We are also able to place the membrane at an arbitrary location without a power series expansion and thus obtain a generalized Langevin equation at an arbitrary location. We believe this may be useful in a holographic RG analysis of this system.

In the path integral approach to the Langevin equation it is manifest that both $\gamma(\omega)$ and $\Gamma(\omega)$ are related to correlation functions of the noise. Γ is related to the symmetric two point function and γ to the retarded two point function. The FD theorem is then a statement of a relation between these two correlation functions and what we find is, as expected, consistent with this theorem.

Rest of this chapter is organized as follows. Section 4.2 is a description of the Langevin equation and its derivation using the Schwinger-Keldysh technique and is a review. In Section 4.3 the retarded Green function is calculated using the usual AdS/CFT prescription. For the BTZ case the Green function can be obtained exactly. This section contains one of the main results of this chapter. The Section 4.4 is mainly a review where we repeat the Son and Teaney derivation of the Schwinger-Keldysh Green functions using holography. This is also then a verification of the FD theorem. The main point of departure is that the various Green functions that make up the Schwinger-Keldysh Green function are all known exactly in the BTZ case. Section 4.5 starts with a brief review of the holographic RG as discussed in [19] and its relevance for our work. It also contains the second main result of this chapter in which, by calculating the bulk to bulk propagators exactly, we obtain an effective 'boundary' action but now with the boundary at an arbitrary location. From the boundary perspective this is like an effective action at an arbitrary point along the RG flow. In Section 4.6 different time scales relevant to Brownian motion have been discussed. Section 4.7 contains some conclusions.

Langevin Dynamics : A Review

Here the Langevin dynamics [83] will be reviewed in brief. Suppose in a viscous medium a very heavy (compared to the masses of the medium particles) particle is moving. Its dynamics will be described by the Langevin equation[2]

$$M_{\text{kin}}\frac{dv}{dt} + \gamma v = \xi(t), \tag{4.2.1}$$

[2] This is actually the small-frequency limit of the generalized Langevin equation (4.1.8). In Section 4.3 it will turn out that one obtains the generalized Langevin equation (4.1.8) from holographic calculation rather than its local version (6.1.1).

$$\text{with} \quad \langle \xi(t)\,\xi(t')\rangle = \Gamma\,\delta(t-t') = 2kT\gamma\,\delta(t-t'), \tag{4.2.2}$$

where $-\gamma v$ is the drag, ξ is the random noise and $M_{\rm kin}$ is the 'renormalized mass' in the thermal medium. Evidently equation (6.1.2) is a statement of fluctuation-dissipation theorem. At the ultimate long time limit we can neglect inertial term in (6.1.1)

$$\gamma v = \xi \tag{4.2.3}$$

and can define the diffusion coefficient as

$$D = \frac{T}{\gamma}\,. \tag{4.2.4}$$

We will see later the dynamics on the stretched horizon (4.5.32) is identical to this over-damped motion (6.1.3).

The aim of this section is to review how to derive Langevin equation from path integral formalism. There are many good references [84, 85] for detail description of this derivation, we will go through this quickly just to fix the notation we will use through out this chapter and we follow mostly the steps sketched in [6]. We can define the partition function for a heavy particle in a heat bath using a Schwinger-Keldysh contour[3] (fig.4.1)

$$Z = \left\langle \int [\mathcal{D}x_1][\mathcal{D}x_2]\; e^{i\int dt_1 M_Q^0 \dot{x}_1^2}\; e^{-i\int dt_2 M_Q^0 \dot{x}_2^2}\; e^{i\int dt_1 \phi_1(t_1)x_1(t_1)}\; e^{-i\int dt_2 \phi_2(t_2)x_2(t_2)} \right\rangle_{\rm bath}, \tag{4.2.5}$$

ϕ_1, ϕ_2 are the heat bath degrees of freedom which act like sources. x_1, x_2 are the fields 'living' on the two different sections 1 and 2 of the time contour. We will see later in the gravity side these are the two types of field those 'live' on the two boundaries, 1 and 2 of the full Kruskal diagram (fig.4.2). Path integral along the vertical portion of the contour

[3]For some recent developments in Schwinger-Keldysh formalism see [86, 87] and see [88–90] for detailed reviews.

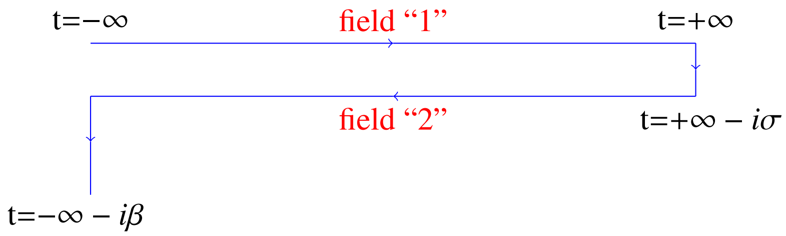


Figure 4.1. Schwinger-Keldysh contour for systems in thermal equilibrium with temperature β^{-1}

gives us average over the thermal density matrix $e^{-\beta H}$. And σ is a free parameter that can take any value. For this discussion we can safely choose $\sigma = 0$ and later we will see that this choice is necessary for '*ra* formalism' which is used extensively in real time thermal field theory literature [63, 85, 88, 91].

For very heavy particle we can consider the forces on the particle is very small compared to inertial term so we can expand it in second order, take the average over bath and make it an exponentiate again to get

$$Z = \int [\mathcal{D}x_1][\mathcal{D}x_2]\, e^{i\int dt_1 M_Q^0 \dot{x}_1^2}\, e^{-i\int dt_2 M_Q^0 \dot{x}_2^2}\, e^{-\frac{1}{2}\int dt dt'\, x_s(t)[\langle\phi(t)\phi(t')\rangle]_{ss'} x_{s'}(t')}\,. \tag{4.2.6}$$

Here the Green function takes a 2×2 matrix form as there are two type of fields and it is contour ordered

$$[\langle\phi(t)\phi(t')\rangle]_{ss'} \equiv i \begin{pmatrix} G_{11}(t,t') & -G_{12}(t,t') \\ -G_{21}(t,t') & G_{22}(t,t') \end{pmatrix}. \tag{4.2.7}$$

Notice that $G_{11}(t,t')$ is the usual time ordered Feynman Green function where as $G_{22}(t,t')$ is anti-time ordered Green function.

In operator language, if we define

$$\phi(t) = e^{iHt}\, \phi(0)\, e^{-iHt}. \tag{4.2.8}$$

The different Green functions are defined as,

$$i\,G_{11}(t,t') = \langle \mathcal{T}\,\phi(t)\,\phi(t')\rangle \tag{4.2.9}$$

$$i\,G_{22}(t,t') = \langle \tilde{\mathcal{T}}\,\phi(t)\,\phi(t')\rangle \tag{4.2.10}$$

$$i\,G_{12}(t,t') = \langle \phi(t')\,\phi(t)\rangle \tag{4.2.11}$$

$$i\,G_{21}(t,t') = \langle \phi(t)\,\phi(t')\rangle\,. \tag{4.2.12}$$

The KMS relation $Tr\,[e^{-\beta H}\,\phi(t)\,\phi(0)] = Tr\,[e^{-\beta H}\,\phi(0)\,\phi(t+i\beta)]$ is easy to prove using cyclicity of the trace and the definition (4.2.8). This implies for the Fourier transform

$$e^{\beta\omega}\int_{-\infty}^{\infty} dt\; e^{i\omega t}\langle Tr\,,[e^{-\beta H}\,\phi(0)\,\phi(t)]\rangle = \int_{-\infty}^{\infty} dt\; e^{i\omega t}Tr\,[e^{-\beta H}\,\phi(t)\,\phi(0)]\,. \tag{4.2.13}$$

In addition to this if we add $i\,G_{11} + i\,G_{22}$ we will get

$$\begin{aligned} i\,G_{11}(t,t') + i\,G_{22}(t,t') &= \langle \mathcal{T}\,\phi(t)\,\phi(t')\rangle + \langle \tilde{\mathcal{T}}\,\phi(t)\,\phi(t')\rangle \\ &= \langle \phi(t)\,\phi(t')\rangle\,\{\theta(t-t') + \theta(t'-t)\} \\ &\quad + \langle \phi(t')\,\phi(t)\rangle\,\{\theta(t'-t) + \theta(t-t')\} \\ &= i\,G_{12}(t,t') + i\,G_{21}(t,t')\,. \end{aligned} \tag{4.2.14}$$

Therefore we can write

$$G_{11} + G_{22} = G_{12} + G_{21}\,. \tag{4.2.15}$$

Note that (4.2.15) is true only for $\sigma = 0$. We will see our Green functions will obey this relation.

Using (4.2.13) and (4.2.15) all these components of the matrix can be expressed in terms of any one Green function, say retarded Green function

$$i\,G_R(t) = \theta(t)\,\langle[\phi(t), \phi(0)]\rangle_{\text{bath}}\ . \tag{4.2.16}$$

Thus for instance we can use

$$\text{Im}\,G_R(\omega) = -\frac{i}{2}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\langle[\phi(t), \phi(0)]\rangle = -\frac{i\,(e^{\beta\omega}-1)}{2}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\,\langle\phi(0)\,\phi(t)\rangle\,, \tag{4.2.17}$$

to write

$$\text{Im}\,G_R(\omega) = -i\tanh\left(\frac{\beta\omega}{2}\right)G_{\text{sym}}(\omega), \tag{4.2.18}$$

where[4] $G_{\text{sym}}(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} dt\,\langle\{\phi(t), \phi(0)\}\rangle\, e^{i\omega t}$.

Now we introduce the previously advertised *ra* formalism. We have already taken $\sigma = 0$. As we are working with very heavy quark the motion will be nearly classical. So, $x_1 \sim x_2$. Therefore we can use some sort of 'centre of mass' coordinates for the particle and for the forces too,

$$x_r = \frac{x_1 + x_2}{2}, \qquad\qquad x_a = x_1 - x_2, \tag{4.2.19}$$

$$\phi_r = \frac{\phi_1 + \phi_2}{2}, \qquad\qquad \phi_a = \phi_1 - \phi_2. \tag{4.2.20}$$

r and a here refer to retarded and advanced respectively and we should remember x_a is a very small quantity for quasi-classical description. Now substituting (4.2.19) and (4.2.20) into the partition function (6.1.5)

$$Z = \int [\mathcal{D}x_r]\,[\mathcal{D}x_a]\; e^{-i\int dt\, M_Q^0\, x_a\,\ddot{x}_r}\; e^{-i\int dt\,dt'[x_a(t)\,i\,G_R(t,t')\,x_r(t') - \frac{1}{2}x_a(t)\,G_{\text{sym}}(t,t')\,x_a(t')]}\,, \tag{4.2.21}$$

[4]According to our definition $G_{\text{sym}}(\omega)$ is purely imaginary.

where the propagators

$$G_{\rm sym}(t,t') = \langle \phi_r(t)\,\phi_r(t')\rangle = \frac{1}{2}\,\langle\{\phi(t),\phi(t')\}\rangle\,, \tag{4.2.22}$$

$$i\,G_R(t,t') = \langle \phi_r(t)\,\phi_a(t')\rangle = \theta(t-t')\,\langle[\phi(t),\phi(t')]\rangle\,. \tag{4.2.23}$$

As we have argued earlier the different Green functions are not independent. In particular the retarded and the symmetric Green function are related as

$$i\,G_{\rm sym}(\omega) = -(1+2\,n_B)\,{\rm Im}\,G_R(\omega), \tag{4.2.24}$$

where $n_B(\omega) = \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}$ is the Bosonic occupation number. This is just a rewriting of (4.2.18).

This is a canonical statement of fluctuation-dissipation theorem. Later in this section we will identify these two Green functions as $\gamma(\omega)$ and $\Gamma(\omega)$ of the equations (4.1.8) and (4.1.11). Now we can write down the path integral in Fourier space

$$Z = \int [\mathcal{D}x_r]\,[\mathcal{D}x_a]\,\exp\left(-i\int\frac{d\omega}{2\pi}x_a(-\omega)[-M_Q^0\omega^2+G_R(\omega)]x_r(\omega)\right)e^{-\frac{1}{2}\int\frac{d\omega}{2\pi}\,x_a(-\omega)[i\,G_{\rm sym}(\omega)]x_a(\omega)}. \tag{4.2.25}$$

We introduce a new random variable which we call ξ in anticipation that it will turn out to be the random noise, by defining

$$e^{-\frac{1}{2}\int\frac{d\omega}{2\pi}x_a(-\omega)\,[i\,G_{\rm sym}(\omega)]\,x_a(\omega)} = \int[\mathcal{D}\xi]\,e^{i\int x_a(-\omega)\,\xi(\omega)}e^{-\frac{1}{2}\int\frac{\xi(\omega)\,\xi(-\omega)}{i\,G_{\rm sym}(\omega)}\frac{d\omega}{2\pi}}\,. \tag{4.2.26}$$

The partition function becomes

$$Z = \int [\mathcal{D}x_r]\,[\mathcal{D}x_a]\,[\mathcal{D}\xi]\,e^{-\frac{1}{2}\int\frac{d\omega}{2\pi}\frac{\xi(\omega)(-\omega)}{i\,G_{\rm sym}(\omega)}}\,\exp\left(-i\int\frac{d\omega}{2\pi}x_a(-\omega)[-M_Q^0\omega^2x_r(\omega)+G_R(\omega)x_r(\omega)-\xi(\omega)]\right). \tag{4.2.27}$$

Integrate out $x_a(-\omega)$ to get a delta function in ω-space

$$Z = \int [\mathcal{D}x_r]\,[\mathcal{D}\xi]\; e^{-\frac{1}{2}\int \frac{d\omega}{2\pi}\frac{\xi(\omega)\,\xi(-\omega)}{i\,G_{\rm sym}(\omega)}}\;\delta_\omega\left[-M_Q^0\omega^2 x_r(\omega) + G_R(\omega)x_r(\omega) - \xi(\omega)\right]. \tag{4.2.28}$$

This partition function is an average over the classical trajectories for the heavy particle under the noise ξ.

$$\left[-M_Q^0\omega^2 + G_R(\omega)\right]x(\omega) = \xi(\omega), \qquad \langle \xi(-\omega)\,\xi(\omega)\rangle = i\,G_{\rm sym}(\omega)\,. \tag{4.2.29}$$

Going back to time space we obtain the generalized Langevin equation

$$M_Q^0\frac{d^2x(t)}{dt^2} + \int_{-\infty}^{t} dt'\; G_R(t,t')\,x(t') = \xi(t)\,, \qquad \langle \xi(t)\,\xi(t')\rangle = i\,G_{\rm sym}(t,t')\,. \tag{4.2.30}$$

$G_R(t,t')$ is thus the same as $\gamma(t-t')$ of Section 4.1 for the choice $t_0 = -\infty$ and $i\,G_{\rm sym}(t,t')$ is the same as $\Gamma(t-t')$.

If the Green function is expanded for small frequencies the coefficient of $\omega^2\left(i.e, \frac{d^2x(t)}{dt^2}\right)$ adds to the mass of the particle and the coefficient of $\omega\left(i.e, \frac{dx(t)}{dt}\right)$ will contributes as the drag term

$$G_R(\omega) = -\Delta M\,\omega^2 - i\,\gamma\,\omega + \ldots \tag{4.2.31}$$

After taking into account the thermal mass correction we define the effective mass

$$M_{\rm kin}(T) = M_Q^0 + \Delta M\,.$$

Then the Langevin equation reads

$$M_{\rm kin}\frac{d^2x}{dt^2} + \gamma\,\frac{dx}{dt} = \xi\,, \tag{4.2.32}$$

$$\text{with}\quad \langle \xi(t)\,\xi(t')\rangle = \Gamma(t-t')\,. \tag{4.2.33}$$

These equations are identical to (6.1.1) and (6.1.2).

Generalized Langevin Equation from Holography

The Einstein-Hilbert action for AdS_3 which has negative cosmological constant, $-\Lambda = \frac{1}{L^2}$ is given by

$$\mathrm{I}_{\mathrm{EH}} = \frac{1}{2\pi}\int \mathrm{d}t\,\mathrm{d}r\,\mathrm{d}x\,\sqrt{-g}\left[R + \frac{2}{L^2}\right] + \mathrm{I}_{\mathrm{B'dy}}\,. \tag{4.3.1}$$

In this units $16\pi G$ is same as 2π. And therefore, $8G = 1$. Since G goes as *length* in 2+1 dimensions this defines a choice of length units.

We write down the action for a string stretching from the horizon ($r = r_H$) towards the AdS boundary and ending on the probe brane at $r = r_B$, in background metric of AdS_3 with a BTZ black hole embedding. The BTZ metric is

$$ds^2 = -\left(\frac{r^2}{L^2} - 8GM\right)\mathrm{d}t^2 + \left(\frac{r^2}{L^2} - 8GM\right)^{-1}\mathrm{d}r^2 + \frac{r^2}{L^2}\mathrm{d}x^2\,. \tag{4.3.2}$$

Let's write this background metric as

$$ds^2 = \frac{r^2}{L^2}\left[-f(b\,r)\,\mathrm{d}t^2 + \mathrm{d}x^2\right] + \frac{L^2\,\mathrm{d}r^2}{f(b\,r)\,r^2}\,, \tag{4.3.3}$$

where r is the canonical choice of coordinate with dimension of length, b is the inverse horizon radius, L is the AdS radius. In our unit, $r_H = b^{-1} = \sqrt{8GM}\,L.$

So, $f(b\,r) = 1 - \frac{8GML^2}{s^2}$. Thus $f(s) = 1 - \frac{1}{s^2}$ and $\pi T = \frac{\sqrt{8GM}}{2L} = \frac{\sqrt{2GM}}{L}$ defines the Hawking temperature. $b = \frac{1}{2\pi T L^2}$ is an alternate expression for b, which can be taken to be the black hole mass parameter.

We write the same metric (4.3.3) with a dimensionless coordinate, $s \equiv b\, r$

$$ds^2 = (2\pi T)^2 L^2 \left[- s^2 f(s)\, \mathrm{d}t^2 + s^2 \mathrm{d}x^2 \right] + \frac{L^2\, \mathrm{d}s^2}{s^2 f(s)}\,. \tag{4.3.4}$$

We want to study the small fluctuation of the string in this non trivial background. The Nambu-Goto action is

$$S = -\frac{1}{2\,\pi\, l_s^2} \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \sqrt{-\det h_{ab}}\,. \tag{4.3.5}$$

Target space coordinates are,

$$X^\mu \equiv (t, s, x)\,.$$

And world sheet coordinates are,

$$\sigma_0 = \tau \quad \text{and} \quad \sigma_1 = \sigma.$$

We will choose (static gauge), $t = \tau$ and $s = \sigma$. Therefore, $x \equiv x(\tau, \sigma) = x(t, s)$ and the induced metric, $h_{ab} = G_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma_a} \frac{\mathrm{d}X^\nu}{\mathrm{d}\sigma_b}$ where $a, b = 0, 1$.

$G_{\mu\nu}$ is the target space metric which is AdS_3-BH for present case.

$$h \equiv \det(h_{ab}) = \begin{vmatrix} G_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} & G_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}X^\nu}{\mathrm{d}\sigma} \\ G_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} & G_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}X^\nu}{\mathrm{d}\sigma} \end{vmatrix}$$

$$= \begin{vmatrix} G_{tt} + G_{xx}\, \dot{x}^2 & G_{xx}\, \dot{x}\, x' \\ G_{xx}\, x' \dot{x} & G_{ss} + G_{xx}\, x'^2 \end{vmatrix}$$

$$h = -(2\pi T)^2 L^4 \left[1 + (2\pi T)^2 s^4 f(s)\, x'^2 - \frac{\dot{x}^2}{f(s)}\right]. \tag{4.3.6}$$

For small fluctuations x' and $\dot{x}$ are very small. So we can write

$$\begin{aligned}\sqrt{-h} &= (2\pi T)\, L^2 \sqrt{1 + (2\pi T)^2\, s^4 f(s)\, x'^2 - \frac{\dot{x}^2}{f(s)}}\\ &\approx (2\pi T)\, L^2 \left[1 + \frac{1}{2}(2\pi T)^2\, s^4 f(s)\, x'^2 - \frac{1}{2}\frac{\dot{x}^2}{f(s)}\right].\end{aligned} \tag{4.3.7}$$

Action for the small fluctuation of string world sheet

$$S = -\frac{(2\pi T)\, L^2}{2\pi l_s^2}\int \mathrm{d}t\, \mathrm{d}s \left[1 + \frac{1}{2}(2\pi T)^2\, s^4 f(s)\, x'^2 - \frac{1}{2}\frac{\dot{x}^2}{f(s)}\right]. \tag{4.3.8}$$

Define mass per unit s

$$m \equiv \frac{(2\pi T)\, L^2}{2\pi l_s^2} = \sqrt{\lambda}\, T, \quad \text{with,} \quad \left(\frac{L}{l_s}\right)^4 \equiv \lambda\,. \tag{4.3.9}$$

And the local tension

$$T_0(s) \equiv \frac{(2\pi T)^3\, L^2}{2\pi l_s^2} f s^4 = 4\sqrt{\lambda}\,\pi^2\, T^3\, f\, s^4 = 4\,\sqrt{\lambda}\,\pi^2\, T^3 s^2\, (s^2 - 1)\,. \tag{4.3.10}$$

Then the action reduces to

$$S = -\int \mathrm{d}t\, \mathrm{d}s \left[m + \frac{1}{2} T_0\, (\partial_s x)^2 - \frac{m}{2f}\, (\partial_t x)^2\right]. \tag{4.3.11}$$

The equation of motion (EOM) can be obtained by varying the action ($\delta S = 0$),

$$0 = -\frac{m}{f}\,\partial_t^2 x + \partial_s\, (T_0(s)\, \partial_s x)\,. \tag{4.3.12}$$

Then the standard way is to write down the EOM in Fourier space

$$\begin{aligned} x(s,t) &= \int \frac{\mathrm{d}\omega}{2\pi}\, e^{i\omega t} f_\omega(s)\, x_0(\omega)\,, \\ x(s=s_B,t) &= \int \frac{\mathrm{d}\omega}{2\pi}\, e^{i\omega t}\, x_0(\omega)\,, \qquad \text{with } f_\omega(s_B)=1\,. \end{aligned} \tag{4.3.13}$$

Therefore the EOM in terms of the modes reduces to

$$\frac{\mathfrak{w}^2}{f} f_\omega(s) + \partial_s\left[f\, s^4\, \partial_s f_\omega(s)\right] = 0\,, \tag{4.3.14}$$

where we have defined $\mathfrak{w} \equiv \omega/2\pi T$.

For our case $f(s) = 1 - \frac{1}{s^2}$, so the EOM reduces to

$$\partial_s^2 f_\omega + \frac{2\,(2s^2-1)}{s\,(s^2-1)}\,\partial_s f_\omega + \frac{\mathfrak{w}^2}{(s^2-1)^2}\, f_\omega = 0\,. \tag{4.3.15}$$

This is an ordinary second order linear differential equation in s. This can be recast into associated Legendre differential equation which one can solve exactly[5]. The general solution to the EOM will be

$$f_\omega(s) = \; C_1\, \frac{P_1^{i\mathfrak{w}}}{s} + C_2\, \frac{Q_1^{i\mathfrak{w}}}{s}\,, \tag{4.3.16}$$

where P_λ^μ and Q_λ^μ are associated Legendre functions and C_1, C_2 are two constants which will be determined by two boundary conditions at the horizon ($s=1$) and at the boundary ($s\to\infty$) of the AdS space. We will impose the following boundary conditions on the modes, f_ω to obtain the retarded Green function as prescribed by Son and Starinets [11]. Actually this 'prescription' has been derived quite rigorously later by van Rees in [92] based on the dictionary of the Lorentzian AdS/CFT as formulated in [93]. Furthermore the application of this formalism to holographic Brownian motion is described in appendix D

[5]The exact solution to this EOM for a stochastic string in BTZ background was obtained earlier by J. de Boer et al. in [5] to calculate some exact correlators.

of [94].

1. At the *horizon* to impose the ingoing wave boundary condition one has to pick the solution $\frac{P_1^{i\mathfrak{w}}}{s}$ (see appendix F). So,

$$f_\omega^R(s) \sim \frac{P_1^{i\mathfrak{w}}}{s}\,.$$

2. The other condition that it should satisfy at the '*boundary*' (s_B, say, where $s_B \gg 1$) of the AdS space is, $s \to s_B$, $f_\omega^R(s) \to 1$.

$$\begin{aligned} f_\omega^R(s) &= \frac{(1+s)^{i\mathfrak{w}/2}}{(1+s_B)^{i\mathfrak{w}/2}}\,\frac{(1-s)^{-i\mathfrak{w}/2}}{(1-s_B)^{-i\mathfrak{w}/2}}\,\frac{s_B}{s}\,\frac{{}_2F_1(-1,2;1-i\mathfrak{w};\frac{1-s}{2})}{{}_2F_1(-1,2;1-i\mathfrak{w};\frac{1-s_B}{2})} \\ &= \frac{(1+s)^{i\mathfrak{w}/2}}{(1+s_B)^{i\mathfrak{w}/2}}\,\frac{(1-s)^{-i\mathfrak{w}/2}}{(1-s_B)^{-i\mathfrak{w}/2}}\,\frac{s_B}{s}\,\frac{\mathfrak{w}+is}{\mathfrak{w}+is_B}\,. \end{aligned} \tag{4.3.17}$$

Now the retarded correlator $G_R(\omega)$ is defined as

$$G_R^0 \equiv \lim_{s\to s_B} T_0(s)\, f_{-\omega}^R(s)\,\partial_s f_\omega^R(s) = \; - \; M_Q^0\,\omega^2 + G_R(\omega)\,. \tag{4.3.18}$$

M_Q^0 is zero temperature mass of the external particle and the term containing it comes from the 'divergent part' of the boundary limit (*i.e,* $s_B \to \infty$). Our aim is to extract $G_R(\omega)$ and then some interesting physical quantities like viscous drag and mass shift form it.

Here s_B is UV regulator for the field theory and IR regulator from the dual bulk perspective. So to calculate the retarded correlator we should take the limit $s \to s_B$. Taking this limit, from (4.3.17),

$$\partial_s f_\omega^R(s)\Big|_{s\to s_B} \;=\; - \;\frac{\mathfrak{w}\,(s_B\mathfrak{w}+i)}{s_B({s_B}^2-1)(s_B-i\mathfrak{w})}\,, \tag{4.3.19}$$

and using the fact that $f_{-\omega}^R(s_B) = 1$, we obtain

$$G_R^0 = T_0(s)\, f_{-\omega}^R(s)\,\partial_s f_\omega^R(s)\Big|_{s\to s_B}$$

$$= \ - \ 4\sqrt{\lambda}\pi^2 T^3 \, \frac{s_B \mathfrak{w} \, (s_B \mathfrak{w} + i) \, (s_B + i\mathfrak{w})}{(s_B^2 + \mathfrak{w}^2)} \tag{4.3.20}$$

$$= \ - \ 4\sqrt{\lambda}\pi^2 T^3 \, \frac{s_B \mathfrak{w} \, (s_B \mathfrak{w} + i)}{(s_B - i\mathfrak{w})} \, . \tag{4.3.21}$$

Equation (4.3.21) is an exact expression for the retarded force-force correlator for the boundary field theory. Note also that it has a singularity only in the lower half ω-plane as required for a retarded Green function. But it is written in terms of two dimensionless parameters s and $\mathfrak{w}$. To make the scaling behavior of the boundary theory correlator more natural we use the corresponding dimensionful parameters, namely $\omega = 2\pi T \mathfrak{w}$, $r_B = 2\pi T L^2 s_B$.

Now we can introduce a mass scale $\mathcal{M} \equiv \frac{r_B}{l_s^2}$. Actually, as discussed in Section 4.5.1, $\mathcal{M}$ can be treated as a renormalization group (RG) scale for the dual field theory. Therefore we do not have to push $\mathcal{M}$ all the way to infinity[6]. We would rather take the point of view that the parameters of the field theory run with $\mathcal{M}$ such a way that the physical quantities remain unchanged. So the correlator reduces to

$$G_R^0(\omega) = \ - \mathcal{M}\omega \, \frac{(i \, 4\pi^2 T^2 \, \sqrt{\lambda} + \mathcal{M}\omega)}{2\pi \, (\mathcal{M} - i\sqrt{\lambda}\,\omega)} \, . \tag{4.3.22}$$

Absorbing the divergent piece (the leading term in the large $\mathcal{M}$ expansion which goes as $\mathcal{M}\omega^2$) in the definition of the zero temperature mass of the Brownian particle and subtracting it from G_R^0 we can define the retarded boundary Green function, G_R

$$G_R^0 \equiv \ - M_Q^0 \, \omega^2 + G_R(\omega) \, , \tag{4.3.23}$$

$$\text{where,} \quad M_Q^0 = \frac{\mathcal{M}}{2\pi} \tag{4.3.24}$$

$$= \sqrt{\lambda} T \, s_B$$

[6]There is also another physical reason why it shouldn't be pushed all the way to the boundary. In such case the mass of the heavy quark is infinite and there would be no Brownian motion.

$$= \frac{\sqrt{\lambda}}{\pi L^2}\, r_B\,. \tag{4.3.25}$$

As mentioned above, M_Q^0 is nothing but the mass of the string stretching from r_B to 0 in the zero temperature limit. And,

$$G_R(\omega) = -\frac{i\,\sqrt{\lambda}\,\mathcal{M}\,\omega\,(4\,\pi^2\,T^2+\omega^2)}{2\pi\,(\mathcal{M}-i\,\sqrt{\lambda}\,\omega)} \tag{4.3.26}$$
$$= \frac{\mathcal{M}\,\omega}{2\,\pi}\,\frac{(\omega^2+4\,\pi^2\,T^2)}{(\omega+i\,\frac{\mathcal{M}}{\sqrt{\lambda}})}\,. \tag{4.3.27}$$

$G_R(\omega)$ is clearly finite in the $\mathcal{M}\to\infty$ limit.

Expanding G_R (4.3.26) in small frequencies, ω

$$G_R(\omega) \approx \frac{2\,\lambda\,\pi\,T^2}{\mathcal{M}}\omega^2 - i\left(2\,\sqrt{\lambda}\,\pi\,T^2\,\omega + \left(\frac{\sqrt{\lambda}}{2\,\pi} - \frac{2\,(\,\sqrt{\lambda})^3\,\pi\,T^2}{\mathcal{M}^2}\right)\omega^3\right). \tag{4.3.28}$$

Again we know when $G_R(\omega)$ is expanded in small ω it takes the form

$$G_R(\omega) = -i\,\gamma\,\omega - \varDelta M\omega^2 - i\,\rho\,\omega^3 + \ldots \tag{4.3.29}$$

where γ and $\varDelta M$ are the viscous drag and the thermal mass shift for the Brownian particle. Whereas ρ is some higher order 'drag coefficient' as it is known that the imaginary part of the retarded Green function (Im $\mathrm{G}_R(\omega)$) is responsible for dissipation.

Comparing (4.3.28) and (4.3.29) we can identify

$$\gamma = 2\,\sqrt{\lambda}\,\pi\,T^2\,, \tag{4.3.30}$$

$$\varDelta M = -\frac{2\,\lambda\,\pi\,T^2}{\mathcal{M}} \tag{4.3.31}$$
$$= -\,\sqrt{\lambda}\,T\,\frac{1}{s_B}\,, \tag{4.3.32}$$

$$\rho = \frac{\sqrt{\lambda}}{2\pi} - \frac{2(\sqrt{\lambda})^3 \pi T^2}{\mathcal{M}^2}. \tag{4.3.33}$$

Note that the particle's rest mass at zero temperature, M_Q^0 (4.3.24) and its viscous drag, γ (4.3.30) are identical to that of a quark in an $\mathcal{N} = 4$ SYM plasma at finite temperature in 3+1 dimensions [6, 95–97]. But the thermal mass shift, ΔM is vanishingly small for large value of $\mathcal{M}$. We have intentionally kept the $O(s_B^{-1})$ term to look at its leading behavior. ΔM has the correct dimension since s_B is a dimensionless quantity ($s_B = b\, r_B$ and $b \sim length^{-1}$ and $r_B \sim length$).

One can also compare this mass shift with that obtained by considering the change in mass of a *static* string coming from the change in its length due to the presence of a horizon.

$$\begin{aligned} \Delta M &= -\int_0^{r_H} (\text{Tension}).\, \sqrt{-g}\, dr \\ &= -\frac{1}{2\pi l_s^2} \int_0^{r_H} \sqrt{-g_{tt}\, g_{rr}}\, dr \\ &= -\frac{r_H}{2\pi l_s^2} \\ &= -\frac{T L^2}{l_s^2} \\ &= -\sqrt{\lambda}\, T\,. \end{aligned} \tag{4.3.34}$$

Note that it is not quite the same as (4.3.31). In lower dimension systems the effect of fluctuations could be much larger and could explain the discrepancy.

Notice if we take the limit $\mathcal{M} \to \infty$ (ultra-violet limit) and $T \to 0$ we obtain

$$G_R(\omega) = -i\, \frac{\sqrt{\lambda}}{2\pi}\, \omega^3\,, \tag{4.3.35}$$

which doesn't contain any dimensionful parameter other than ω and thus properly describes a conformal field theory at UV fixed point.

We can now put $T = 0$ in (4.3.26) to get

$$G_R(\omega)\Big|_{T=0} = \frac{\mathcal{M}\,\omega^3}{2\pi\,(\omega + i\,\frac{\mathcal{M}}{\sqrt{\lambda}})} = \frac{\mathcal{M}\,\omega^3\left(\omega - i\,\frac{\mathcal{M}}{\sqrt{\lambda}}\right)}{2\pi\left(\omega^2 + \left(\frac{\mathcal{M}}{\sqrt{\lambda}}\right)^2\right)}\,. \tag{4.3.36}$$

The presence of an imaginary part in $G_R(\omega)$ signifies dissipation. Thus an interesting result we get from the expression (4.3.36) is a temperature independent dissipation.

- For low frequency regime ($\omega \ll \mathcal{M}$) at zero temperature we recover (4.3.35) which shows the diffusive behavior,

$$G_R(\omega)\Big|_{T=0} \approx -\,i\frac{\sqrt{\lambda}}{2\pi}\,\omega^3\,. \tag{4.3.37}$$

- If we consider the frequency range $\omega \gg \mathcal{M}$ [7]

$$G_R(\omega)\Big|_{T=0} \approx \frac{\mathcal{M}\,\omega^2}{2\,\pi} - i\,\frac{\mathcal{M}^2\,\omega}{2\pi\,\sqrt{\lambda}} + \ldots \tag{4.3.38}$$

  We see a drag like term proportional to ω. This term strongly suggests that there must be 'drag' for the heavy particle even at zero temperature for this 1+1 d CFT.

At first sight this is puzzling because Lorentz invariance of a theory would say that a quark moving with a constant velocity for *all* time, should not slow down - this would violate boost invariance[8]. In fact the drag force on a particle moving with a constant velocity turns out to be zero as we see below. The drag force $F(t)$ is given by (in frequency space)

$$F(\omega) = G_R(\omega)\,x(\omega)\,. \tag{4.3.39}$$

[7] If we are to think of $\mathcal{M}$ as an effective cutoff of the theory, then we should keep $\omega < \mathcal{M}$. So (4.3.38) is only a formal limit. This result will not be used elsewhere in the thesis.

[8] A Similar phenomenon has been observed for theories with hyperscaling violation [26, 27]. Clearly these backgrounds break Poincaré invariance. For these non-relativistic situations, energy and momentum are conserved but drain into the soft infra-red modes of the theory [27, 98]. Moreover, this mechanism of energy loss is present even at *constant velocity* of the particle! Evidently this phenomenon is quite different from the one we are addressing in this thesis.

For a particle moving at constant velocity $x(t) = vt$. This translates to

$$x(\omega) = -\,iv\,\delta'(\omega)\,. \tag{4.3.40}$$

Since $G_R(\omega = 0) = G'_R(\omega = 0) = 0$, the force is zero. In more detail, since we have a distribution $\delta'(\omega)$, we should consider a smooth function $f(\omega)$ and evaluate the integral:

$$\int \mathrm{d}\omega\, G_R(\omega)\, x(\omega)\, f(\omega) = \int \mathrm{d}\omega\, G_R(\omega)\,(-\,i\,v\,\delta'(\omega))\, f(\omega) = 0\,, \tag{4.3.41}$$

on integrating by parts.

(One can trace this to (4.3.36) which says that $G_R(\omega)$ starts off as ω^3.)

The phenomena of zero temperature dissipation have been noticed in holography [12, 72–76] and many other contexts [77–82]. The physical mechanism that gives rise to energy loss at zero temperature in the relativistic theories was first explained in [12], and then elaborated on in [72, 73]. For accelerated quarks in the vacuum of a CFT, energy and momentum are radiated away by emission of gluonic fields[9]), in analogy with the theory of radiation in classical electrodynamics. This in turns leads to a Liénard-like formula for the rate of energy loss and a generalized Lorentz-Dirac equation that captures the effects of radiation damping. The previous interpretation agrees with the fact that a quark moving with constant velocity does not feel drag force and thus, does not radiate. Moreover, for the Langevin dynamics around accelerated trajectories at zero temperature [65,100], it has also been seen that the stochastic motion of the heavy probe is not due to collisions with the fluid constituents but rather arises due to the random emission of 'gluonic' radiation[10].

We conclude this section with some more remarks on the zero temperature dissipation

[9] Notice that in 1+1 dimensions gauge fields are not dynamical and hence cannot cause radiation. But radiations which are massless degrees of freedom can also be scalars (*e.g.*, see [99] for scalar radiation due to heavy quark rotating in $\mathcal{N} = 4$ SYM in 3+1 dimensions.). There are theoretical and experimental evidences for scalar radiation in 1+1 dimensions particularly, in several condensed matter systems.

[10] This interpretation is further supported by studies of the radiation pattern of a heavy quark [65, 101]. See [102] for a review of all these topics.

term :

- It is *finite* and cannot be renormalized away in the boundary theory by Hermitian counter terms.

- There has been some discussion in the literature on zero temperature dissipation [77–82, 103]. [103] advocate renormalizing this term away by subtracting the contribution from pure AdS which corresponds to a vacuum. While this is certainly a valid option, we do not feel compelled to do this, because as we have seen in this 1+1 dimensional system there is no violation of any physical principle such as Lorentz invariance. Also as the calculations of radiation show, there is a compelling physical reason to expect that it should be there.

- We take the view point that $\mathcal{M}$ is finite because it is to be interpreted as an RG scale. Also as mentioned earlier, as $\mathcal{M} \to \infty$ the particle becomes infinitely heavy. Otherwise there is nothing pathological in our calculation even if $\mathcal{M}$ is infinite.

- There are some 1+1 condensed matter systems [77–82] which exhibit such dissipation (or decoherence) at absolute zero due to zero-point fluctuations.

Schwinger-Keldysh Propagators from Holography

The first two subsections of this section are basically review of how to get Schwinger-Keldysh propagators in the boundary field theories using extended Kruskal structure of the black hole. This is written in terms of the retarded Green functions. Thus combining this with the results of Section 4.3, we immediately obtain the exact Schwinger-Keldysh Green functions for our system.

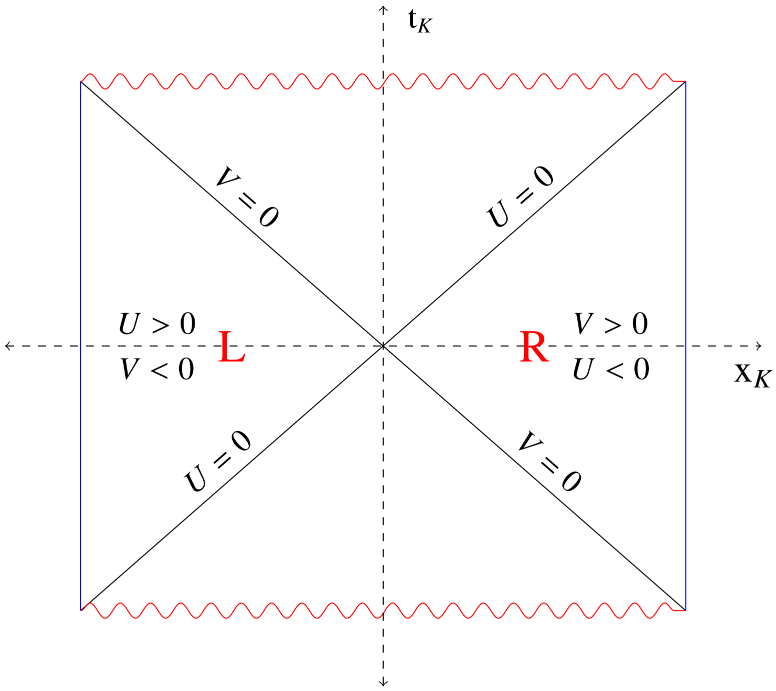


Figure 4.2. AdS space in Kruskal coordinates

Kruskal/Keldysh correspondence

Herzog and Son [62] derived Schwinger-Keldysh propagators from bulk calculation in AdS_5-Schwarzschild metric. They analytically continued [104, 105] the modes of the scalar field from I to II (see figure 4.2). During this procedure only the modes near the horizon are crucial. It is very straight forward to see that their prescription goes through for AdS_3-BTZ background too, as modes near the horizon behave identically. The same method is also applicable to our system with string where modes are functions of frequency (ω) only. Their derivation involved symmetric contour i.e, $\sigma = \beta/2$. As we want to express our result in *ra* formalism we will fix $\sigma = 0$ as before. We just sketch the generic four step AdS/CFT procedure.

I. The EOM for the fluctuating string is solved subjected to the boundary conditions

$$\lim_{s \to s_B} x(\omega, s_1) = x_1^0(\omega)\,, \tag{4.4.1}$$

$$\lim_{s \to s_B} x(\omega, s_2) = x_2^0(\omega)\,. \tag{4.4.2}$$

Here s_1, s_2 are the radial coordinates in L and R regions respectively. Now the general solutions in L and R are

$$x(\omega, s_1) = a(\omega)\, f_\omega(s_1) + b(\omega)\, f^*_\omega(s_1)\,, \tag{4.4.3}$$

$$x(\omega, s_2) = c(\omega)\, f_\omega(s_2) + d(\omega)\, f^*_\omega(s_2)\,. \tag{4.4.4}$$

II. We have four undetermined coefficients in (4.4.3) and (4.4.4) but have only two boundary conditions namely (4.4.1)and (4.4.2). So to specify the solution uniquely we need other two constraints. Imposing horizon boundary conditions we can eliminate two coefficients namely $c(\omega)$ and $d(\omega)$. Near the horizon the ingoing and outgoing modes in the R region behave as

$$e^{-i\omega t} f_\omega(s_1) \sim e^{-i\frac{\omega}{2\pi T}\, log(V)}\,, \tag{4.4.5}$$

$$e^{-i\omega t} f^*_\omega(s_1) \sim e^{+i\frac{\omega}{2\pi T}\, log(-U)}\,. \tag{4.4.6}$$

Now following [62] we will analytically continue the solution from R ($U < 0$, $V > 0$) to L ($U > 0$, $V < 0$) region such that the solution is analytic in lower V-plane and upper U-plane[11]

$$f_\omega(s_1) \to e^{-\omega/2T} f_\omega(s_2)\,, \tag{4.4.7}$$

$$f^*_\omega(s_1) \to e^{+\omega/2T} f^*_\omega(s_2)\,. \tag{4.4.8}$$

[11]This choice is motivated by the fact that in field theory Feynman Green function contains positive energy modes for $t \to \infty$ and negative energy modes for $t \to -\infty$. And the Green function (G_{11}) for the field theory 'living' on the boundary of the R-region should be time ordered one like usual Feynman Green function,G_F in Minkowski space.

Therefore the solution when analytically continued to L region becomes

$$x(\omega, s_2) = a(\omega)\, e^{-\omega/2T} f_\omega(s_2) + b(\omega)\, e^{+\omega/2T} f^*_\omega(s_2)\,. \tag{4.4.9}$$

But as mentioned above this is a special case where the contour is symmetric i.e, $\sigma = \beta/2$. One can generalize this result by starting with $V \to |V|\, e^{-i\theta}$ and $-U \to |U|\, e^{-i(2\pi-\theta)}$ and defining $\sigma \equiv \frac{\theta}{2\pi T}$, then continuing analytically to get

$$f_\omega(s_1) \to e^{-\omega\sigma} f_\omega(s_2) = f_\omega(s_2) \tag{4.4.10}$$

$$f^*_\omega(s_1) \to e^{+\omega/T} e^{-\omega\sigma} f^*_\omega(s_2) = e^{+\omega/T} f^*_\omega(s_2)\,, \tag{4.4.11}$$

where we have taken $\sigma = 0$ as usual. So, the solution in L region reduces to

$$x(\omega, s_2) = a(\omega)\, f_\omega(s_2) + b(\omega)\, e^{+\omega/T} f^*_\omega(s_2)\,. \tag{4.4.12}$$

Imposing the boundary conditions (4.4.1), (4.4.2) into (4.4.3) and (4.4.12) we can solve for $a(\omega)$ and $b(\omega)$

$$a(\omega) = x^0_1(\omega)\,\{1 + n_B(\omega)\} - x^0_2(\omega)\, n_B(\omega) \tag{4.4.13}$$

$$b(\omega) = x^0_2(\omega) n_B(\omega) - x^0_1(\omega) n_B(\omega)\,. \tag{4.4.14}$$

We have *the* solution fully specified by R and L region solutions (4.4.3) and (4.4.12) satisfying necessary boundary conditions at the boundary and the horizon.

III. The next step is to plug this solution into the boundary action

$$S_{\text{b'dy}} = -\frac{T_0(s_B)}{2} \int_{s_1} \frac{\mathrm{d}\omega}{2\pi}\, x_1(-\omega, s_1)\, \partial_s x_1(\omega, s_1) + \frac{T_0(s_B)}{2} \int_{s_2} \frac{\mathrm{d}\omega}{2\pi}\, x_2(-\omega, s_2)\, \partial_s x_2(\omega, s_2)\,, \tag{4.4.15}$$

to get

$$
\begin{aligned}
i\,S_{\text{b'dy}} = -\frac{1}{2}\int\frac{\mathrm{d}\omega}{2\pi}\; & x_1^0(-\omega)\left[i\,\mathrm{Re}\,G_R^0 - (1+2n_B)\,\mathrm{Im}\,G_R^0\right]\,x_1^0(\omega) \\
& + x_2^0(-\omega)\left[-i\,\mathrm{Re}\,G_R^0 - (1+2n_B)\,\mathrm{Im}\,G_R^0\right]\,x_2^0(\omega) \\
& - x_1^0(-\omega)\,\left[-2\,n_B\,\mathrm{Im}\,G_R^0\right]\,x_2^0(\omega) \\
& - x_2^0(-\omega)\,\left[-2\,(1+n_B)\,\mathrm{Im}\,G_R^0\right]\,x_1^0(\omega)\,.
\end{aligned} \tag{4.4.16}
$$

Here retarded Green function is defined as

$$
G_R^0(\omega) \equiv\; T_0(s)\,\frac{f_{-\omega}(s)\,\partial_s f_\omega}{|f_\omega(s)|^2}\bigg|_{s=s_B} \tag{4.4.17}
$$

(as $s \to \infty$, $|f_\omega(s)|^2 \to 1$. So the numerator is already normalized if the probe brane is very close to the boundary).

IV. The last step is to take functional derivative with respect to x_1^0 and/or x_2^0 which are acting like two source terms for the boundary field theory to get the Schwinger-Keldysh propagators

$$
G_{\text{ab}} = \begin{bmatrix} i\,\mathrm{Re}\,G_R^0 - (1+2n_B)\,\mathrm{Im}\,G_R^0 & -2\,n_B\,\mathrm{Im}\,G_R^0 \\ -2\,(1+n_B)\,\mathrm{Im}\,G_R^0 & -i\,\mathrm{Re}\,G_R^0 - (1+2n_B)\,\mathrm{Im}\,G_R^0 \end{bmatrix}. \tag{4.4.18}
$$

G_{ab} is exactly known from the expressions of G_R^0 in (4.3.23) and (4.3.27).

Now we want to express our result in *ra* formalism. So we need to covert x_1, x_2 into x_s, x_a. Then the relations between bulk and the boundary fields reduce to

$$
x_a(\omega, s) = f_\omega^*(s)\,x_a^0(\omega) \tag{4.4.19}
$$

$$
x_r(\omega, s) = f_\omega(s)\,x_s^0(\omega) + i\,(1+2n_B)\,\mathrm{Im}f_\omega(s)\,x_a^0(\omega)\,. \tag{4.4.20}
$$

And the boundary action in this set up becomes

$$S_{\text{b'dy}} = -\frac{T_0(s_B)}{2}\int_{s_B}\frac{\mathrm{d}\omega}{2\pi}x_a(-\omega,s)\,\partial_s x_r(\omega,s) - \frac{T_0(s_B)}{2}\int_{s_B}\frac{\mathrm{d}\omega}{2\pi}x_r(-\omega,s)\,\partial_s x_a(\omega,s)\,. \tag{4.4.21}$$

Plugging (4.4.19) and (4.4.20) into the boundary action as before we end up getting

$$iS_{\text{b'dy}} = -i\int\frac{\mathrm{d}\omega}{2\pi}x_a^0(-\omega)\,[G_R^0(\omega)]\,x_r^0(\omega) - \frac{1}{2}\int\frac{\mathrm{d}\omega}{2\pi}x_a^0(-\omega)\,[iG_{\text{sym}}(\omega)]\,x_a^0(\omega)\,. \tag{4.4.22}$$

Boundary stochastic motion

Now here we want to have the boundary stochastic motion of the string. The partition function for the string can be written as

$$Z = \int[\mathfrak{D}x_1\mathcal{D}x_1^0]\,[\mathfrak{D}x_2\mathcal{D}x_2^0]\,e^{iS_1-iS_2}\,, \tag{4.4.23}$$

where $\mathcal{D}x_1^0$ is a measure for temporal path of the string end point and $\mathfrak{D}x_1$ is a measure for the bulk path integral for the body of the string in R-region of the full Kruskal plane. Similarly $\mathcal{D}x_1^0$ and $\mathfrak{D}x_1$ are defined in L-region.

$$[\mathcal{D}x_1^0] = \prod_t \mathrm{d}x_1^0(t), \qquad [\mathfrak{D}x_1] = \prod_{t,r}\mathrm{d}x_1(t,s)\,. \tag{4.4.24}$$

To obtain the effective action of the string end points we will integrate out all string coordinates inside the bulk. If we do this path integral (over the terms contained in the bracket)

$$\begin{aligned} Z &= \int[\mathcal{D}x_1^0]\,[\mathcal{D}x_2^0]\,\underbrace{[\mathfrak{D}x_1]\,[\mathfrak{D}x_2]\,e^{iS_1-iS_2}} \\ &\equiv \int[\mathcal{D}x_1^0]\,[\mathcal{D}x_2^0]\,e^{iS_{\text{eff}}^0}\,. \end{aligned} \tag{4.4.25}$$

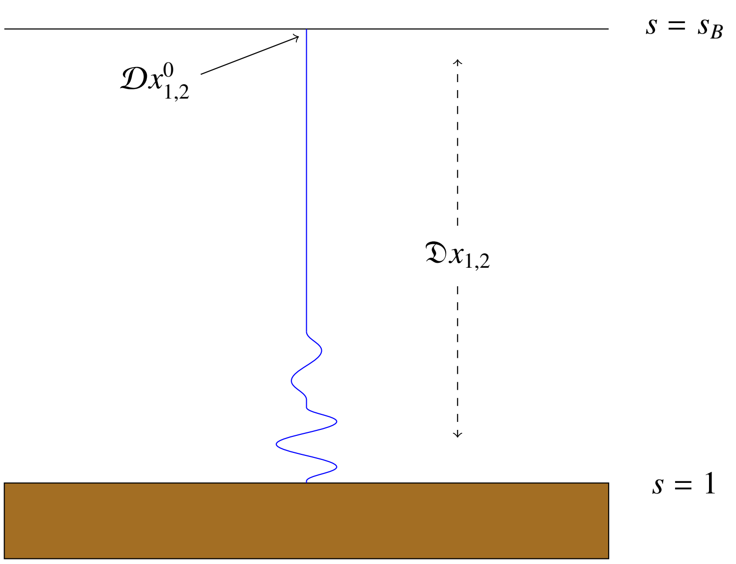


Figure 4.3. Visualizing the boundary stochastic motion of the heavy particle by integrating out all string degrees of freedom.

We have absorbed the field independent determinant in the normalization of the path integral. Now will use the results from previous section where we have already calculated the boundary actions (4.4.15) and (4.4.21). As there is no 'boundary' at the horizon there will be only two boundary terms from the two boundaries of the Kruskal plane

$$S^0_{\rm eff} = -\frac{T_0(s_B)}{2}\int_{s_1}\frac{d\omega}{2\pi}x_1(-\omega,s_1)\,\partial_s x_1(\omega,s_1) + \frac{T_0(s_B)}{2}\int_{s_2}\frac{d\omega}{2\pi}x_2(-\omega,s_2)\,\partial_s x_2(\omega,a_2)\,. \tag{4.4.26}$$

We can easily write down the partition function for the string endpoints in *ra* formalism from (4.4.22)

$$Z = \int [\mathcal{D}x^0_r]\,[\mathcal{D}x^0_a]\,e^{iS^0_{\rm eff}}\,, \tag{4.4.27}$$

$$i\,S^0_{\rm eff} = -i\int\frac{d\omega}{2\pi}\,x^0_a(-\omega)\,[G^0_R(\omega)]\,x^0_r(\omega) - \frac{1}{2}\int\frac{d\omega}{2\pi}\,x^0_a(-\omega)\,[i\,G_{\rm sym}(\omega)]\,x^0_a(\omega)\,. \tag{4.4.28}$$

Notice that the effective partition function of the string end points (4.4.27) is exactly similar to the Fourier space path integral (4.2.25). Therefore we can perform the same procedure of introducing a "noise", ξ to obtain the following equations of motion obeyed by the string end points

$$\left[-M_Q^0\,\omega^2 + G_R(\omega)\right] x_r(\omega) = \xi(\omega)\,, \qquad \langle \xi(-\omega)\,\xi(\omega)\rangle = -\,(1+2n_B)\,\mathrm{Im}\,G_R(\omega)\,. \tag{4.4.29}$$

Here we have used the facts that

$$G_R^0(\omega) = -M_Q^0\,\omega^2 + G_R(\omega), \tag{4.4.30}$$

$$i\,G_{\rm sym}(\omega) = -\,(1+2n_B)\,\mathrm{Im}\,G_R(\omega)\,. \tag{4.4.31}$$

Effective Action at General r : Brownian Motion on Stretched Horizon

Since we have an exact solution one can hope to generalize the membrane paradigm by locating the membrane at *arbitrary* r (r_0, say). This would be in the spirit of a holographic renormalization group (RG) [19, 106] approach to the problem. This would then justify the statement made in Section 4.3 that $\mathcal{M}$ can be interpreted as an RG scale. This is done in this section. We begin with a review of some basic ideas in holographic RG following [19].

Holographic Renormalization Group

A version of the holographic RG that is useful here was discussed in [19] and is reviewed in this section. The main idea is to start with an action, which is the original bulk action supplemented by a boundary action at $r = r_0$, that takes into account the effect of integrating out of the bulk region $r > r_0$. This region $r > r_0$ in the bulk represents the high energy region of the boundary field theory. The so called 'alternative quantization' [107] where the boundary value of the bulk field ϕ is interpreted as the expectation value of a boundary single trace operator rather than as a source for the boundary operator comes in handy in explaining the approach. The boundary action obtained this way can also be interpreted as the generating functional for a different boundary theory that is obtained by the so called 'standard quantization'. Furthermore there is an RG flow from the first boundary action perturbed by a relevant deformation involving double trace operators to the second boundary action.

Thus we begin with

$$S = \int_0^{r_0} \mathrm{d}r\, \mathrm{d}^D x\, \sqrt{-g}\left[-\frac{1}{2}\,\partial_M\phi\,\partial_N\phi\, g^{MN} - V(\phi)\right] + S_B[\phi, r_0]\,. \tag{4.5.1}$$

In the [19] D is the space time dimension of the boundary theory and ϕ fills all of AdS bulk. However we can interpret D for our purposes as the dimension of a brane/string hanging down from the boundary into the center with the other end going into the horizon of the black hole. Thus in our case $\phi(x) = x(r,t)$, $D = 1$ and the action becomes

$$S = \int_0^{r_0} \mathrm{d}r\, \mathrm{d}t\, \sqrt{-g}\left[-\frac{1}{2}\,\left(\partial_t x(r,t)\,\partial_t x(r,t)\, g^{tt} + \partial_r x(r,t)\,\partial_r x(r,t)\, g^{rr}\right) - V(x)\right] + S_B[x, r_0]\,. \tag{4.5.2}$$

This can be compared with (4.3.11) and we see that it is exactly the same with $V(x) = m$, which does not contribute to the equations of motion, and so can be ignored in this discussion.

For our purposes, since we are only interested in the two point function, we can think of the boundary action as

$$S_B[\phi, r_0] = \frac{1}{2}\int_{r=r_0} \mathrm{d}^D k\, \phi(k)\, G_R(k, r_0)\, \phi(-k)\,. \tag{4.5.3}$$

Specializing to our case this becomes:

$$S_B[x, r_0] = \frac{1}{2} \int_{r=r_0} \mathrm{d}\omega \; x(\omega)\, G_R(\omega, r_0)\, x(-\omega)\,. \tag{4.5.4}$$

The parameters of the boundary field theory action are collected here in $G_R(k, r_0)$ and their dependence on the RG scale r_0 is indicated. When we vary ϕ we get the usual bulk equation and also a (boundary) condition at the boundary $r = r_0$. This depends on the boundary action and is:

$$G_R(k, r_0) = -\sqrt{-g}\, g^{rr} \frac{\partial_r \phi_c(r_0)}{\phi_c(r_0)}\,. \tag{4.5.5}$$

Fixing the solution to the equation of motion, a second order differential equation in r, requires specifying $\phi(r_0)$ and $\partial_r\phi(r_0)$. If we specify $\phi(r_0)$ and $\partial_r\phi(r_0)$, G_R is fixed by this boundary condition. In the alternative quantization G_R is the coefficient of the quadratic term in the effective action of the boundary theory. On the other hand if we interpret $S_B(\phi)$ as the generating functional for the boundary theory, $G_R(k, r_0)$ is the Green function of the boundary theory. This is the interpretation that is relevant for us. The Green functions in the two cases are inverses of each other.

One important point is that if the bulk equation of motion is linear, therefore scaling $\phi(r_0)$ by a number just scales the solution everywhere by the same number. Hence G_R is not affected. But this would not be true in a non linear bulk theory. In Sec 4.3 we have a linear approximation to the equation for the string fluctuation. Thus there is no loss of generality in setting $\phi(r_0) = 1$.

In this approach one can write down an RG, [19], that says the total action (evaluated on the solution) cannot depend on r_0. As also shown in [19] the parameters of the boundary action must vary such that the the classical solution is reproduced. Thus solving the RG gives the classical solution. The converse is also true. It is easy to see [19] that if we use the exact classical solution in the action, the RG becomes an identity, because it becomes equivalent to imposing (4.5.5).

The above formalism can be applied to our case where we use (4.5.2) and (4.5.4).

$$G_R(\omega, r_0) = -\sqrt{-g}\, g^{rr} \frac{\partial_r x_c(r_0)}{x_c(r_0)}, \tag{4.5.6}$$

with $\sqrt{-g}\, g^{rr} = T_0$. This is the same as (5.3.6) except that we have not assumed any normalization for the $x_c(r_0)$.

As we change r_0 to r_0', RG demands that one has to change the boundary condition on x and the boundary action so that physical quantities are fixed. In our case since we know the exact solution, we know the boundary condition at r_0': $x(r_0') = x_c(r_0')$ where x_c is the exact classical solution, which has an earlier prescribed boundary value at r_0. We also know the new boundary action. It is given by (4.5.3) where $G_R(r_0')$ is given by (4.5.5), where the RHS is evaluated at r_0'. (Actually for the situation in Section 4.3, the equation for x is linear, and as mentioned above we can just continue to use $x(r_0') = 1$.) The functional form of the Green function does not change - except that all explicit r_0's are replaced by r_0''s. Thus the parameter $\mathcal{M} = \frac{r_0}{l_s^2}$ used in Section 4.3 can be understood as a renormalization scale.

Thus G_R^0 in our case is the correlation function for the random force i.e. we interpret the action involving x as the generating functional for the boundary theory of the random force acting on the quark.

(4.3.22) has a diffusive pole at $-\frac{i\mathcal{M}}{\sqrt{\lambda}}$. This gives an exponential decay time scale for the random force acting on the quark. Being a mass scale it is appropriately proportional to $\mathcal{M}$ the RG scale. From the point of view of the action for x (which is the coordinate of the quark in addition to being the source for ξ), this is a non local quadratic term and cannot be renormalized away by adding local counter terms. For the effective action for ξ the random force acting on the quark, which involves the inverse Green function, this is a zero rather than a pole. However being imaginary, it cannot be renormalized away by a hermitian counter term in the bare action, and furthermore the powers of ω in the denominator would make the counterterm nonlocal. This leads us to conclude that this pole represents a physical effect in the low energy dynamics of the quark.

Placing the Membrane at Arbitrary r

In the previous section we have integrated out all modes of the string to obtain the effective action for the string end points and we end up getting a Langevin equation on the boundary. Here our aim is to obtain such an effective action on a spatial slice at a general value of s (s_0, say). This requires determining the solution to the EOM (4.3.15) exactly which we have already obtained (4.3.17). Then we will choose s_0 very close to s_H to get a Langevin equation on that stretched horizon (a hypothetical membrane which we consider to be very close to the horizon of the black hole).

For this purpose again we write the partition function of string in several parts

$$Z = \int [\mathcal{D}x_1^0 \mathfrak{D}x_1^> \mathcal{D}x_1^{s_0}]\, [\mathcal{D}x_2^0 \mathfrak{D}x_2^> \mathcal{D}x_2^{s_0}]\, [\mathfrak{D}x_1^< \mathfrak{D}x_2^<]\, e^{iS_1^> - iS_2^>}\, e^{iS_1^< - iS_2^<}\,. \tag{4.5.7}$$

As before $\mathcal{D}x_1^0$ is a measure for temporal path of the string end point and $\mathfrak{D}x_1^>$ and $\mathfrak{D}x_1^<$ are the measures for the bulk path integral for the body of the string outside and inside of the spatial slice in R-region and $\mathcal{D}x_1^{s_0}$ denotes the temporal path integral for the string end point on the spatial slice (see fig. 4.4). Whereas $S_1^>$ is the action outside the spatial slice and $S_1^<$ is the action inside the spatial slice. Same is true for L region. This time integrating out the region of the string inside $s = s_0$,

$$Z = \int [\mathcal{D}x_1^0 \mathfrak{D}x_1^> \mathcal{D}x_1^{s_0}]\, [\mathcal{D}x_2^0 \mathfrak{D}x_2^> \mathcal{D}x_2^{s_0}]\, e^{iS_1^> - iS_2^>}\, \underbrace{[\mathfrak{D}x_1^<]\, [\mathfrak{D}x_2^<]\, e^{iS_1^< - iS_2^<}} \tag{4.5.8}$$

$$= \int [\mathcal{D}x_1^0 \mathfrak{D}x_1^> \mathcal{D}x_1^{s_0}]\, [\mathcal{D}x_2^0 \mathfrak{D}x_2^> \mathcal{D}x_2^{s_0}]\, e^{iS_1^> - iS_2^>}\, e^{iS_{\text{eff}}^{s_0}}\,, \tag{4.5.9}$$

where $S_{\text{eff}}^{s_0}$ = The boundary action which passes through $x_1^{s_0}(\omega)$ and $x_2^{s_0}(\omega)$. Notice here the stretched horizon at $s = s_0$ is a boundary. And the older boundary conditions are now

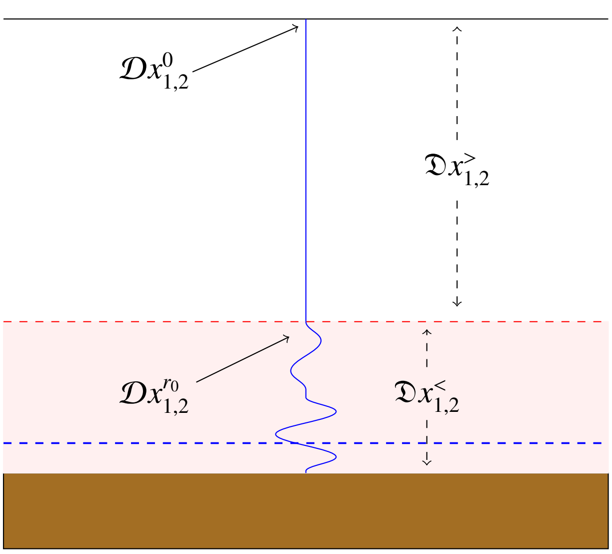


Figure 4.4. Integrating out the string degrees of freedom inside a hypothetical 'membrane' and push it very close to the horizon (i.e, stretched horizon) to obtain a Langevin equation which is overdamped.

applicable at $s = s_0$. So the the bulk fields, $x^<_{1,2}(s, \omega)$ and the boundary fields, $x^{s_0}_{1,2}(\omega)$ are related by

$$x^<_1(s_1 = s^0_1, \omega) = x^{s_0}_1(\omega) \tag{4.5.10}$$

$$x^<_2(s_2 = s^0_2, \omega) = x^{s_0}_2(\omega)\,. \tag{4.5.11}$$

If we use the *ra* basis then the boundary conditions reduce to

$$x^<_a(\omega, s) = f^*_\omega(s)\, x^{s_0}_a(\omega) \tag{4.5.12}$$

$$x^<_r(\omega, s) = f_\omega(s)\, x^{s_0}_r(\omega) + i\,(1 + 2n_B)\,\mathrm{Im} f_\omega(s)\, x^{s_0}_a(\omega)\,. \tag{4.5.13}$$

Going through the same calculation as before and using the fact there is no "boundary" at the horizon we end up getting the membrane effective action

$$S^{s_0}_{\text{eff}} = -\frac{T_0(s_0)}{2}\int_{s^0_1} \frac{\mathrm{d}\omega}{2\pi}\, x^<_1(-\omega, s)\,\partial_s x^<_1(\omega, s) + \frac{T_0(s_0)}{2}\int_{s^0_2} \frac{\mathrm{d}\omega}{2\pi}\, x^<_2(-\omega, s)\,\partial_s x^<_2(\omega, s)\,. \tag{4.5.14}$$

Now if we use the *ra*-coordinates then using (4.5.12) and (4.5.13) we will have

$$i\,S_{\text{eff}} = -\,i \int \frac{\mathrm{d}\omega}{2\pi}\, x_a^{s_0}(-\omega)\, [G_R^{s_0}(\omega)]\, x_r^{s_0}(\omega) - \frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi}\, x_a^{s_0}(-\omega)\, [\, i\, G_{\text{sym}}^{s_0}(\omega)]\, x_a^{s_0}(\omega)\,. \tag{4.5.15}$$

The retarded propagator is defined such that it is normalized at the spatial slice. Now the expression for retarded force-force correlator can be written down for any fixed value of s_0. Using the value of $T_0(s)$ from (4.3.10) and substituting the expression for $f_\omega(s)$ from (4.3.17)

$$G_R^{s_0}(\omega) \equiv T_0(s)\, \frac{f_{-\omega}(s)\,\partial_s f_\omega}{|f_\omega(s)|^2}\bigg|_{s=s_0}$$

$$= -\,\frac{\sqrt{\lambda}\,\pi^2\,T^3}{2}\,\frac{s_0\mathfrak{w}\,(s_0\mathfrak{w}+i)}{(s_0 - i\,\mathfrak{w})} \tag{4.5.16}$$

$$= -\,\mathcal{M}_0\,\omega\,\frac{(i\,\sqrt{\lambda}\,\pi^2\,T^2 + \mathcal{M}_0\,\omega)}{2\pi(\mathcal{M}_0 - i\,\sqrt{\lambda}\,\omega)}\,. \tag{4.5.17}$$

(4.5.16) and (4.5.17) are exact expressions for the retarded propagator on the probe brane which is placed at $s = s_0$ and or equivalently when the field theory is probed at the energy scale $\mathcal{M}_0 = \dfrac{r_0}{l_s^2}$. It trivially reduces to the boundary propagator $G_R^0(\omega)$ as in (4.3.20) when $s_0 \longrightarrow s_B$.

The other point we want to emphasis here is that the retarded Green function (4.3.27) which is derived using holography incorporates the "softening of delta function" to avoid the contradiction described in section 4.1.

$$\lim_{t\to t_0}\int_{t_0}^{t}\mathrm{d}t'\;\gamma(t') = \lim_{t\to t_0}\int_{t_0}^{t}\mathrm{d}t'\int_{-\infty}^{\infty}\mathrm{d}\omega\; e^{-i\omega t'}\gamma(\omega) \tag{4.5.18}$$

$$= -\lim_{t\to t_0}\int_{t_0}^{t}\mathrm{d}t'\int_{-\infty}^{\infty}\mathrm{d}\omega\; e^{-i\omega t'}\,\frac{\mathcal{M}\omega}{2\pi}\,\frac{(\omega^2+\pi^2\,T^2)}{(\omega + i\,\frac{\mathcal{M}}{\sqrt{\lambda}})}\,. \tag{4.5.19}$$

One can perform the contour integral for ω to pick up the residue at $\omega = -\,i\,\frac{\mathcal{M}}{\sqrt{\lambda}}$. So the corresponding integral

$$\lim_{t\to t_0}\int_{t_0}^{t} \mathrm{d}t'\; 2\pi i\, e^{-\frac{\mathcal{M}}{\sqrt{\lambda}}t'}\; \frac{\mathcal{M}(-i\frac{\mathcal{M}}{\sqrt{\lambda}})}{2\pi}\left(\left(-i\frac{\mathcal{M}}{\sqrt{\lambda}}\right)^2+\pi^2T^2\right)\to 0\,. \tag{4.5.20}$$

This shows that our Green function (4.5.17) is consistent with (4.1.9).

From (4.5.9) it is evident that after we integrate out the string coordinates inside $s = s_0$ we have partition functions of two haves of the Kruskal plane and which are coupled by the membrane effective action, $S^{s_0}_{\text{eff}}$. Now will be shown (and already been mentioned) that a part ($G^{s_0}_{\text{sym}}$) of this 'coupling' introduces thermal noise. It can be done following exactly same procedure of invoking a horizon noise for the second part of $e^{iS^{s_0}_{\text{eff}}}$ in the partition function

$$e^{-\frac{1}{2}\int\frac{\mathrm{d}\omega}{2\pi}\; x_a^{s_0}(-\omega)\,[iG^{s_0}_{\text{sym}}(\omega)]\,x_a^{s_0}(\omega)} = \int[\mathcal{D}\xi^{s_0}]\; e^{i\int x_a^{s_0}(-\omega)\,\xi^{s_0}(\omega)}\; e^{-\frac{1}{2}\int\frac{\xi^{s_0}(\omega)\,\xi^{s_0}(-\omega)}{iG^{s_0}_{\text{sym}}(\omega)}} \tag{4.5.21}$$

$$\text{with,}\quad \langle\xi^{s_0}(-\omega)\,\xi^{s_0}(\omega)\rangle = i\,G^{s_0}_{\text{sym}}(\omega) = -(1+2n_B)\,\mathrm{Im}\,G^{s_0}_R(\omega)\,. \tag{4.5.22}$$

We have computed the contribution coming from the boundary action namely $S^{s_0}_{\text{eff}}$. Now in order to calculate the partition function (4.5.9) we need to look at the bulk contributions. In *ra* basis this bulk action reduces to

$$\begin{aligned} iS^{>}_1 - iS^{>}_2 &= -i\int\frac{\mathrm{d}\omega}{2\pi}\,ds\left[T_0(s)\,\partial_s x^{>}_a(-\omega,s)\,\partial_s x^{>}_r(\omega,s) - \frac{m\,\omega^2\,x^{>}_r(\omega,s)\,x^{>}_a(-\omega,s)}{f}\right]\\ &= -i\int\frac{\mathrm{d}\omega}{2\pi}\,x^{>}_a(-\omega,s)\,[T_0(s)\,\partial_s x^{>}_r(\omega,s)]\Big|^{s=s_B}_{s=s_0}\\ &\quad - i\int\frac{\mathrm{d}\omega}{2\pi}\,ds\,x^{>}_a(-\omega,s)\left[-\partial_s(T_0(s)\,\partial_s x^{>}_r(\omega,s)) - \frac{m\,\omega^2\,x^{>}_r(\omega,s)}{f}\right]. \end{aligned} \tag{4.5.23}$$

From (4.5.15), (4.5.21) and (4.5.23) we can finally write

$$iS^{>}_1 - iS^{>}_2 + iS^{s_0}_{\text{eff}} = -\,i\int_{s_B}\frac{\mathrm{d}\omega}{2\pi}\,x^0_a(-\omega,s)\,[T_0(s_B)\,\partial_s x^{>}_r(\omega,s)]$$

$$
\begin{aligned}
&- i \int_{s_0} \frac{\mathrm{d}\omega}{2\pi}\, x_a^{s_0}(-\omega, s)\,[-T_0(s_0)\,\partial_s x_r^{>}(\omega, s) + G_R^{s_0}(\omega)\, x_r^{s_0}(\omega) - \xi^{s_0}(\omega)] \\
&- i \int \frac{\mathrm{d}\omega}{2\pi}\, ds\, x_a^{>}(-\omega, s)\left[-\partial_s\left(T_0(s)\,\partial_s x_r^{>}(\omega, s)\right) - \frac{m\,\omega^2\, x_r^{>}(\omega, s)}{f}\right].
\end{aligned}
\tag{4.5.24}
$$

The path integral reduces to

$$
\begin{aligned}
Z &= \int [\mathcal{D}x_r^0 \mathfrak{D}x_r^{>} \mathcal{D}x_r^{s_0}]\,[\mathcal{D}\xi^{s_0}]\, e^{-\frac{1}{2}\int \frac{\xi^{s_0}(\omega)\,\xi^{s_0}(-\omega)}{-(1+2n_B)\,\mathrm{Im}G_R^{s_0}(\omega)}}\, \underbrace{[\mathcal{D}x_a^0 \mathfrak{D}x_a^{>} \mathcal{D}x_a^{s_0}]\, e^{iS_1^{>} - iS_2^{>} + iS_{\mathrm{eff}}^{s_0}}} \\
&= \int [\mathcal{D}x_r^0 \mathfrak{D}x_r^{>} \mathcal{D}x_r^{s_0}]\,[\mathcal{D}\xi^{s_0}]\, e^{-\frac{1}{2}\int \frac{\xi^{s_0}(\omega)\,\xi^{s_0}(-\omega)}{-(1+2n_B)\,\mathrm{Im}G_R^{s_0}(\omega)}}\, \delta_\omega\left[-T_0(s_B)\,\partial_s x_r^{>}(\omega, s)\right]_{s=s_B} \\
&\qquad \delta_\omega\left[-\partial_s(T_0(s)\partial_s x_r^{>}(\omega, s)) - \frac{m\,\omega^2\, x_r^{>}(\omega, s)}{f}\right] \\
&\qquad \delta_\omega[-T_0(s_0)\,\partial_s x_r^{>}(\omega, s) + G_R^{s_0}(\omega)\, x_r^{s_0}(\omega) - \xi^{s_0}(\omega)]_{s=s_0}\,.
\end{aligned}
\tag{4.5.25}
$$

We have integrated over the terms inside the bracket viz, $[\mathcal{D}x_a^0]$, $[\mathfrak{D}x_a^{>}]$ and $[\mathcal{D}x_a^{s_0}]$. This path integral (4.5.25) leads to three equations for boundary end point, the horizon end point and the body of the string.

I. The boundary end point dynamics is governed by the deterministic equation which just tells us this end point is free,

$$
T_0(s_B)\,\partial_s x_r^{>}(\omega, s) = 0\,. \tag{4.5.26}
$$

II. The body of the string, as expected, satisfies the equation of motion

$$
\left[\partial_s\left(T_0(s)\,\partial_s x_r^{>}(\omega, s)\right) + \frac{m\,\omega^2\, x_r^{>}(\omega, s)}{f}\right] = 0\,. \tag{4.5.27}
$$

III. And the end point on the $s = s_0$ membrane obeys the stochastic equation of motion

$$
T_0(s_0)\,\partial_s x_r^{>}(\omega, s) + \xi^{s_0}(\omega) = G_R^{s_0}(\omega)\, x_r^{s_0}(\omega) \tag{4.5.28}
$$

$$\text{with,}\quad \langle \xi^{s_0}(-\omega)\,\xi^{s_0}(\omega)\rangle = -\,(1+2n_B)\,\mathrm{Im}\,G_R^{s_0}(\omega)\,. \tag{4.5.29}$$

This is the Langevin dynamics describing the endpoint of the string living on the membrane at $s = s_0$.

So far in this section everything was for arbitrary s_0. Now for the sake of completeness we follow Son and Teaney [6] again to re-derive the overdamped motion on the stretched horizon and also the boundary FD equation for AdS_3-BH which is identical to their (AdS_5-BH) calculation. For that purpose we want to move this membrane very close to the horizon i.e, $s_0 = 1+\epsilon$; ϵ is very small (see fig. 4.4). Putting this value of s_0 into (4.5.16) one obtains retarded Green function on the stretched horizon

$$\begin{aligned} G_R^H(\omega) &= -\lim_{\epsilon\to 0}\frac{\sqrt{\lambda}\,\pi^2\,T^3}{2}\left[\frac{i\,(1+2\epsilon)\,(\mathfrak{w}+\mathfrak{w}^3)}{1+2\epsilon+\mathfrak{w}^2}+\frac{2\,\epsilon\,\mathfrak{w}^2}{1+2\epsilon+\mathfrak{w}^2}\right] \\ &\sim -i\,\gamma\,\omega\,. \end{aligned} \tag{4.5.30}$$

Here we have assumed that frequency is very small i.e, $\mathfrak{w} \ll 1$. The other point to notice is the 'inertial term' is suppressed by an extra factor of ϵ. And as $\epsilon \to 0$ the mass of the string end point on the stretched horizon,

$$M_Q^H \equiv \frac{2\epsilon}{1+2\epsilon}\frac{\sqrt{\lambda}\,T^2}{2} \to 0\,. \tag{4.5.31}$$

(Expanding the real part of $G_R^H(\omega)$ in ϵ we get correction to mass, $\Delta M \sim O(\epsilon^2)$.)

Therefore from (4.5.28) and (4.5.30) one obtains the Langevin equation on the stretched horizon

$$T_0(s_H)\,\partial_s x_r^{>}(\omega,s) + \xi^h(\omega) = -i\,\omega\,\gamma\,x_r^h(\omega) \tag{4.5.32}$$

$$\text{with,}\quad \langle \xi^h(-\omega)\,\xi^h(\omega)\rangle = (1+2n_B)\,\gamma\,\omega\,. \tag{4.5.33}$$

This is the overdamped motion of the horizon end point as discussed in [6], where the first term signifies the pulling of the end point by the string outside the horizon. Vanishing of M_Q^h (4.5.31) is the reason behind getting an overdamped Langevin dynamics.

Our next task is to investigate how the fluctuations on this membrane is transmitted to the boundary through the string dynamics such that the boundary end point satisfies a Langevin equation (4.4.29). In other words, we wish to have a relationship between ξ^h and ξ^0 and using this we want to show the fluctuation-dissipation for boundary fluctuations, ξ^0. To proceed let's consider the behavior of the solution near the AdS boundary

$$x(\omega, s) = x_0(\omega)\, f_\omega(s) + \xi^0(\omega) \left[\frac{\mathrm{Im} f_\omega(s)}{-\mathrm{Im}\, G_R(\omega)} \right] , \tag{4.5.34}$$

where $f_\omega(s)$ is non-normalizable and $\mathrm{Im}\, f_\omega(s)$ is a normalizable mode. $-\mathrm{Im}\, G_R(\omega)$ is just a normalization such that $\xi^0(\omega)$ can be recognized as the boundary fluctuation. Now if substitute this (4.5.34) into the equation describing boundary dynamics (4.5.26) we obtain expected Brownian equation for the boundary end point

$$[-M_Q^0\, \omega^2 + G_R(\omega)]\, x_0(\omega) = \xi^0(\omega) . \tag{4.5.35}$$

To get the fluctuation-dissipation relation for ξ^0 we use

$$-i\, \omega\, \gamma = T_0(s_H)\, \frac{f_{-\omega}(s_H)\, \partial_s f_\omega(s_H)}{|f_\omega(s_H)|^2} , \tag{4.5.36}$$

to re-write the equation at the stretched horizon dynamics (4.5.32) as

$$\frac{\xi^0(\omega)}{-\mathrm{Im}\, G_R(\omega)} T_0(s_H)\, [f_\omega(s_H)\, \partial_s \mathrm{Im}\, f_\omega - \mathrm{Im} f_\omega(s_H)\, \partial_s f_\omega(s_H)] + f_\omega(s_H)\, \xi^H(\omega) = 0 . \tag{4.5.37}$$

But the term in the square bracket is the Wronskian (G.0.3) of the two solutions and using $T_0(s)\, W(s) = \; + \mathrm{Im}\, G_R(\omega)$ (G.0.4) we have desired relation

$$\xi^0(\omega) = f_\omega(s_H)\, \xi^H(\omega) . \tag{4.5.38}$$

For the AdS$_3$-BH system we are considering we can exactly calculate $f_\omega(s_H)$ from (4.3.17) and equation (4.5.38) reduces to

$$\xi^0(\omega) = \left(1 - i\frac{\omega}{\pi T}\right)\xi^H(\omega)\,. \tag{4.5.39}$$

Once we have obtained this relation (4.5.38) we can use the horizon fluctuation-dissipation theorem(4.5.33) and (4.5.36) to get

$$\langle \xi^0(-\omega)\,\xi^0(\omega)\rangle = -(1 + 2n_B)\,\mathrm{Im}\,G_R(\omega)\,. \tag{4.5.40}$$

This is the statement of boundary fluctuation-dissipation theorem.

Different Time Scales

Brownian motion is usually characterized mainly by two time scales [10] : relaxation time (t_r) and collision time (t_c). Apart from these two there is another time scale called mean free path time (t_{mfp}). These different time scales come very naturally in kinetic theory of fluids. But in the holographic context when one considers classical gravity in the bulk the dual field theory is inevitably strongly coupled. In the same spirit the fluid that contains the quark in present context is strongly coupled. As a consequence, as we will see in this section, all those intuitive notions from kinetic theory don't go through in this case of holographic Brownian motion. First of all we will define the different time scales mentioned above.

- The *relaxation time* is a time scale which separates ballistic regime where the Brownian particle moves inertially (displacement $\sim$ time) from diffusive regime where it undergoes a random walk (displacement $\sim \sqrt{\text{time}}$). This is the time taken by the system to thermalize. In the small frequency regime (4.3.29) (using (4.3.24)) we can write the Langevin equation (4.2.29) as

$$\left[-\frac{\mathcal{M}}{2\pi}\omega^2 - i\gamma\,\omega + \ldots\right] x(\omega) = \xi(\omega)\,. \tag{4.6.1}$$

One obtains usual ballistic motion when the inertial term dominates over the diffusive term i.e, $\frac{\mathcal{M}}{2\pi}\,\omega^2 \gg \gamma\,\omega$. Evidently one gets one characteristic frequency when these two terms are of equal strength,

$$\omega \sim \frac{\gamma}{\mathcal{M}} \approx \frac{\sqrt{\lambda}\,T^2}{\mathrm{M_{kin}}}\,. \tag{4.6.2}$$

Consequently the corresponding time scale (relaxation time)

$$t_r \sim \frac{\mathrm{M_{kin}}}{\sqrt{\lambda}\,T^2}\,. \tag{4.6.3}$$

- The *collision time* is defined as a time scale over which random noise is correlated or in other words it's the time elapsed in a single collision. It measures how much $\gamma(t-t')$ deviates from $\delta(t-t')$. From (4.3.27) we obtain

$$\gamma(t) \equiv G_R(t) = \frac{\mathcal{M}^2}{\sqrt{\lambda}}\left(-\frac{\mathcal{M}^2}{\lambda} + 4\,\pi^2\,T^2\right) e^{-\frac{\mathcal{M}}{\sqrt{\lambda}}t}\,. \tag{4.6.4}$$

In (4.6.4) $\frac{\sqrt{\lambda}}{\mathcal{M}}$ naturally comes out to be a time scale. This is the 'memory time' which fixes the width of $\gamma(t)$. Hence it determines the duration of collision (t_c),

$$t_c = \frac{\sqrt{\lambda}}{\mathcal{M}} \approx \frac{\sqrt{\lambda}}{\mathrm{M_{kin}}}\,. \tag{4.6.5}$$

One can observe that this is the pole of the retarded Green function (4.3.27) at $\omega = -i\frac{\mathcal{M}}{\sqrt{\lambda}}$ that fixes the collision time scale[12].

[12]Note that we have used Dirichlet boundary condition (standard quantization) (4.3.17) on the string modes at the boundary. On the other hand, if one uses Neumann boundary condition (alternative quantization), presumably one would get the retarded Green function, $\tilde{G}_R = G_R^{-1}$ (see appendix A of [19]). These two different boundary conditions describe two completely different dual CFTs. Therefore, for the latter

From (4.6.3) and (4.6.5)

$$\frac{t_r}{t_c} \sim \left(\frac{\mathrm{M_{kin}}}{\sqrt{\lambda}\,T}\right)^2 . \tag{4.6.6}$$

For 'dilute' fluid one expects $t_r \gg t_c$. But from (4.6.6) it is clear that for strongly coupled fluids for which $\lambda \gg 1$ this relation is not necessarily true.

- The *mean free path time* is the time elapsed between two consecutive collisions of the Brownian particle. As argued in [5] to obtain t_{mfp} one needs to compute the 4-point correlation function. Therefore it will be suppressed by a factor of $\frac{1}{\sqrt{\lambda}}$ compared to t_c,

$$t_{mfp} \sim \frac{1}{\sqrt{\lambda}} t_c \approx \frac{1}{\mathrm{M_{kin}}} . \tag{4.6.7}$$

Again from (4.6.7) and (4.6.5) $t_{mfp} \gg t_c$ does not necessarily hold for $\lambda \gg 1$.

Conclusions

We have studied the Brownian diffusion of a particle in one dimension using the holographic techniques. The holographic dual is a BTZ black hole with a string. We have used the Green function techniques of Son and Teaney [6]. Since the differential equation can be solved exactly we find an exact Green function and an exact (generalized) Langevin equation.

Some interesting features :

- We show that the exact generalized Langevin equation, which is valid on short time scales also, does not suffer from the inconsistency that is associated with the usual

CFT the corresponding time scale is determined by the *zero* of our Green function $G_R(\omega)$ i.e, $\omega = -i2\pi T$ and thus $t_c \sim \frac{1}{T}$. For example, in [5] Neumann boundary condition is used and the collision time, $t_c \sim \frac{1}{T}$.

Langevin equation that has a delta function for the drag term.

- We also find that the temperature dependent mass correction is zero (in the limit that the UV cutoff is taken to infinity) unlike in the higher dimensional cases.

- There is also a temperature independent dissipation at all frequencies. At high frequencies it is a drag term. This does not violate Lorentz invariance as the force on a quark moving with constant velocity for all time continues to be zero. This has already been studied in higher dimensional systems and is due to radiation [12, 72–76]. It is noteworthy that a temperature independent dissipation in one dimension has also been noted in the condensed matter literature [77–82].

- Once again because an exact Green function is available, the 'stretched horizon' can in fact be placed at an arbitrary radius and an effective action obtained.

5 Dissipation at $T = 0$ and $T \to 0$

We use holographic techniques to study the zero-temperature limit of dissipation for a Brownian particle moving in a strongly coupled CFT at finite temperature in various space-time dimensions. The dissipative term in the boundary theory for $\omega \to 0$, $T \to 0$ with ω/T held small and fixed, does not match the same at $T = 0$, $\omega \to 0$. Thus the $T \to 0$ limit is not smooth for $\omega < T$. This phenomenon appears to be related to a confinement-deconfinement phase transition at $T = 0$ in the field theory.

The material presented in this chapter is based on work done with B. Sathiapalan [8].

Introduction

The motion of an external heavy quark in a non-Abelian gauge theory has been studied in a number of papers (see [5–7, 9, 72, 73, 76, 95] and references there in). One motivation for this are the experimental results that came out of the Relativistic Heavy Ion Collider (RHIC). The suggestion that the quark gluon plasma is 'strongly coupled' with a very small value of $\frac{\eta}{s}$ came from experiments. The calculation of $\frac{\eta}{s}$ using AdS/CFT techniques [1–4] gave the small value of $\frac{1}{4\pi}$ [24]. This gave impetus to holographic techniques for understanding the quark gluon plasma. On the dual gauge theory side exact calculations have been done for $\mathcal{N}$=4 super Yang-Mills theory - calculations that make heavy use of supersymmetry. One can hope that at finite temperature the deconfined QCD quark gluon plasma behaves qualitatively like strongly coupled $\mathcal{N}$=4 super Yang-Mills theory. The "running" effective coupling constant of QCD *presumably* is large in the field configurations that dominate the quark gluon plasma and therefore well approximated by a *strongly* coupled super Yang-Mills theory. At finite temperature since supersymmetry is broken anyway and fermions and scalars effectively become massive, one can also presume that supersymmetry and the presence of adjoint fermions and scalars, does not invalidate the approximation. The success of the holographic calculation provides some indirect justification for all this.

Many calculations in super Yang-Mills have been done at zero temperature where supersymmetry is exact. In flat space, at all non zero temperatures, the theory is in the same Coulomb phase (unlike QCD) and therefore calculations done at zero temperature might be thought to still be of some relevance for the quark gluon plasma. On the dual gravity side this corresponds to pure AdS_5. On the gravity side it is a little easier to explore the finite temperature regime [4]. This corresponds to non-extremal D3-branes. In fact the $\frac{\eta}{s}$ calculation involves calculating η and s separately at finite temperature.

Here we study the $T \to 0$ limit of the finite temperature calculations on the gravity side. This limit is subtle for reasons that will become clear later. The motivation for this study comes from [7] where the Langevin equation describing Brownian motion[1] of a stationary heavy quark in 1+1 CFT at finite temperature was studied using the gravity dual, which is a BTZ black hole (see section 4). The calculation was done using the holographic Schwinger-Keldysh method worked out in [6]. The calculation can be done exactly (unlike in 3+1 dimensions). One of the interesting results is that there is a drag force (dissipating energy) on the fluctuating external quark *even at zero temperature*. This was identified as being due to radiation [12, 72, 73, 76, 100, 101, 108–110]. This force term in

[1]Brownian motion of a heavy quark in quark-gluon-plasma was first described using holographic techniques in [5, 6].

the Langevin equation was of the form

$$F(\omega) = -i\frac{\sqrt{\lambda}}{2\pi}\omega^3 x(\omega)\,. \tag{5.1.1}$$

If one calculates the integrated energy loss one finds ($x(t) = \int \frac{d\omega}{2\pi}x(\omega)e^{-i\omega t}$):

$$\begin{aligned}\varDelta E = \int_{-\infty}^{\infty} F(t)\dot{x}(t)dt &= \int \frac{d\omega}{2\pi}F(w)(i\omega)x(-\omega) \\ &= \int \frac{d\omega}{2\pi}(-i)\frac{\sqrt{\lambda}}{2\pi}\omega^3 x(\omega)(i\omega)x(-\omega) \\ &= \frac{\sqrt{\lambda}}{2\pi}\int \frac{d\omega}{2\pi}\omega^4 x(\omega)x(-\omega)\,. \end{aligned} \tag{5.1.2}$$

While the above calculation was done by taking the $T \to 0$ of a finite temperature calculation in BTZ, the same result is obtained for pure AdS in all dimensions, as we shall see later in this chapter (see section 5.2). The energy radiated by an accelerating quark has been calculated using other techniques (also holographic) by Mikhailov [12] and the answer obtained is

$$\varDelta E = \frac{\sqrt{\lambda}}{2\pi}\int dt\, a^2\,, \tag{5.1.3}$$

which on Fourier transforming gives exactly the same result. In fact the coefficient $\frac{\sqrt{\lambda}}{2\pi}$ is essentially the bremsstrahlung function[2] $B(\lambda, N)$ ($2\pi B(\lambda, N) = \frac{\sqrt{\lambda}}{2\pi}$) identified in [13] as occurring in many other physical quantities (such as the cusp anomalous dimension introduced by Polyakov [114]).

It is thus interesting to check whether the same result is also obtained as one takes $T \to 0$ in 3+1 dimensions. In fact the $T \to 0$ limit is a little tricky because of singularities. Taking $T \to 0$ in the finite T theory can be done if one keeps $\frac{\omega}{T}$ small. Thus the limit where $\omega \to 0$ and $T \to 0$ is well defined and can be calculated perturbatively. (We shall

[2] see [14, 111–113] and references there in for more details about bremsstrahlung function in supersymmetric theories.

see later that all the dimensionful quantities are measured in the unit of quark mass, $\mathcal{M}$). The result for dissipation in this (DC) limit can be compared with that calculated for the $T = 0$ result of pure AdS for $\omega \to 0$. The results do not agree. Thus we find that pure AdS ($T = 0$) results cannot be assumed to be close to small T results as one would naively have assumed. This is not surprising given that there is a Hawking Page transition [115] at exactly zero temperature in the Poincaré patch description of Schwarzschild-AdS. (In global AdS where space is S^3, this happens at a finite temperature.) But this does raise questions of the relevance of the zero temperature calculations in $\mathcal{N}$=4 super Yang-Mills for comparison with data taken from experiments such as RHIC.

In this chapter we study in a general way, the zero temperature limit of some theories in 3+1 dimensions using holography. We show how one can do a perturbation in T - however this has to be done *about a solution that is singular as* $T \to 0$. This does not go smoothly to the $T = 0$ result of pure AdS.

The rest of this chapter is organized as follows. In section 5.2 we discuss dissipation at exactly zero temperature by studying dynamics of a long fundamental string in pure AdS space-time. We check whether the dissipation very close to zero temperature smoothly matches the same at absolute zero in section 5.3 by changing the background geometry for the moving string to AdS-black holes. In section 5.4 we try to interpret our results. The section 5.5 summarizes the main results of this chapter. The perturbative technique used in AdS_5-BH case is also applied for studying string in BTZ background as a check of applicability of the method in appendix H.

Dissipation at Zero Temperature ($T = 0$)

A Brownian particle dissipates energy at zero temperature only by radiating soft or massless modes (photons, gluons etc). The dual background where the string moves is a pure

AdS space.

$$ds^2 = -\frac{r^2}{L^2}\,dt^2 + \frac{L^2}{r^2}\,dr^2 + \frac{r^2}{L^2}\,d\vec{x}^{\,2}\,, \tag{5.2.1}$$

where L is AdS radius and $\vec{x} \equiv (x^1, x^2 \ldots x^{d-1})$.

A stochastic string in this pure AdS background is exactly solvable in arbitrary dimensions. To compute the retarded Green's function and the dissipative term from that we need to study string dynamics in (5.2.1).

We shall be eventually working with linearized Nambu-Goto action. Therefore after choosing the static gauge, without loss of generality, we can pick up any one transverse direction ($x^1 \equiv x$, say) and fix others ($x^2, x^3 \ldots x^{d-1}$) to zero. Essentially we are looking at a three dimensional slice of AdS$_{d+1}$. So, X(σ, τ) is a map to (τ, r, x). The Nambu-Goto action for small fluctuation in space and in time reduces to

$$\begin{aligned} S &= -\frac{1}{2\pi l_s^2}\int dt\,dr\sqrt{1 + \dot{x}^2 + \frac{r^4}{L^4}\,x'^2} \\ &\approx -\int dt\,dr\left[1 + \frac{m}{2}\dot{x}^2 + \frac{1}{2}T_0(r)\,x'^2\right], \end{aligned} \tag{5.2.2}$$

where $m = \frac{1}{2\pi l_s^2}$ and $T_0(r) = \frac{r^4}{2\pi l_s^2 L^4}$. Varying the action we get the EOM which in frequency space reads

$$f''_\omega(r) + \frac{4}{r}f'_\omega(r) + \frac{L^4\omega^2}{r^4}f_\omega(r) = 0\,, \tag{5.2.3}$$

where $x(r,t) = \int \frac{d\omega}{2\pi}e^{-i\omega t}f_\omega(r)x_0(\omega)$ and $x_0(\omega)$ is the boundary value[3] of $x(r,t)$ such that $f_\omega(r_B) = 1$.

This is a linear second order ordinary differential equation with following two linearly

[3] Notice that $r = r_B$ is the boundary of the geometry. This is IR cutoff for the bulk and UV cutoff for the dual field theory. For large but finite value of r_B the quark is very heavy but has a finite mass, and therefore, a detectable Brownian motion

independent solutions

$$f^{(1)}_\omega(r) = \frac{e^{-i\frac{L^2\omega}{r}}(r + iL^2\omega)}{r} \tag{5.2.4}$$

$$f^{(2)}_\omega(r) = \frac{e^{i\frac{L^2\omega}{r}}(r - iL^2\omega)}{r} . \tag{5.2.5}$$

- As we want to compute *retarded* Green's function we pick $f^{(2)}_\omega(r)$ which is *ingoing*[4] at $r = 0$.

- The boundary condition, $f_\omega(r) \to 1$ as $r \to r_B$, fixes the solution to be

$$f_\omega(r) = \frac{r_B}{r} \frac{e^{+i\frac{L^2\omega}{r}}(r - iL^2\omega)}{e^{+i\frac{L^2\omega}{r_B}}(r_B - iL^2\omega)} . \tag{5.2.6}$$

We just use these modes in calculating the on-shell action and obtain the retarded Green's function [11, 62]

$$\begin{aligned} G^0_R(\omega) &\equiv \lim_{r\to r_B} T_0(r)\, f_{-\omega}(r)\, \partial_r f_\omega(r) \\ &= -\frac{r_B^2\,\omega^2}{2\,\pi\, l_s^2}\frac{1}{(r_B - i\,L^2\,\omega)} \\ &= -i\,\frac{\mathcal{M}^2\,\omega^2}{2\pi\,\sqrt{\lambda}}\frac{1}{(\omega + i\,\frac{\mathcal{M}}{\sqrt{\lambda}})} , \end{aligned} \tag{5.2.7}$$

where we have used $\mathcal{M} = \frac{r_B}{l_s^2}$ and $\sqrt{\lambda} = \frac{L^2}{l_s^2}$. As stated in chapter 4, $\mathcal{M}$ and λ behave like Wilsonian cutoff scale and dimensionless coupling for the field theory respectively. Also $\mathcal{M}$ is essentially the energy stored in the string as it is stretched from the horizon to the boundary of the AdS and can be interpreted as the mass of the external quark.

For $\mathcal{M} \to \infty$, $G^0_R(\omega) = -\frac{\mathcal{M}\omega^2}{2\pi}$ which is divergent. We can renormalize the Green's function by absorbing the UV divergent piece to define the zero temperature mass of the

[4]Notice that $e^{-i\omega t}e^{i\frac{L^2\omega}{r}} = e^{-i\omega(t-\frac{L^2}{r})}$. To keep the phase unchanged, for increasing t, r must decrease. So the wave is ingoing.

quark, i.e, $M_Q^0 = \frac{\mu}{2\pi}$ and obtain the *renormalized* Green's function

$$G_R(\omega) \equiv G_R^0(\omega) + \frac{\mathcal{M}\omega^2}{2\pi} = \frac{\mathcal{M}\omega^3}{2\pi}\frac{1}{(\omega + i\,\frac{\mathcal{M}}{\sqrt{\lambda}})}\,. \tag{5.2.8}$$

The renormalized Green's function is UV finite by construction. If we take $\mathcal{M} \to \infty$ (or we can take ω very small)

$$G_R(\omega) \to -\,i\,\frac{\sqrt{\lambda}}{2\pi}\,\omega^3\,. \tag{5.2.9}$$

This is purely dissipative term which is independent of temperature.

Dissipation near Zero Temperature ($T \to 0$)

To describe a Brownian particle moving in a d-dimensional space-time at finite temperature holographically one needs to consider a fundamental string in (d+1)-dimensional dual geometry with a black hole. In this section also we work in the Poincaré patch of AdS_{d+1}-Black hole geometry[5].

We start with the AdS_{d+1}-black brane metric [95]

$$ds^2_{d+1} = L^2\left[-h(u)\,dt^2 + \frac{\mathrm{d}u^2}{h(u)} + u^2\mathrm{d}\vec{x}^2\right], \tag{5.3.1}$$

where $h(u) = u^2\left(1 - \left(\frac{u_H}{u}\right)^d\right)$ with $u_H = \frac{4\pi T}{d}$ and u has dimension of energy.

We choose $d = 2$ and 4 for illustration. The aim is to check whether the dissipative terms match smoothly to that of the zero temperature case as we take $T \to 0$.

[5]We essentially repeat the calculation we did in chapter 4 but with the metric 5.3.1. Note that here we directly work with coordinate r which has a dimension of length.

BTZ Black hole Background

A string in (2+1) dimension is *exactly* solvable even in presence of a (BTZ) black hole [5, 7]. We work in (5.3.1) for $d = 2$ but with r-coordinate where $r \equiv L^2 u$ has dimension of length.

$$ds^2 = -\frac{r^2}{L^2}\left(1 - \frac{(2\pi T L^2)^2}{r^2}\right)\mathrm{d}t^2 + \frac{L^2}{r^2}\frac{\mathrm{d}r^2}{\left(1 - \frac{(2\pi T L^2)^2}{r^2}\right)} + \frac{r^2}{L^2}\mathrm{d}\vec{x}^2 \,. \tag{5.3.2}$$

Using the holographic prescription for Minkowski space [6, 11] one obtains the exact retarded Green's function. Here are the key steps (see chapter 4 for details).

Choosing the static gauge, we study small fluctuations of the string from the Nambu-Goto action

$$S \approx -\int \mathrm{d}t\,\mathrm{d}r\left[m + \frac{1}{2}T_0\,(\partial_r x)^2 - \frac{m}{1 - \left(\frac{2\pi T L^2}{r}\right)^2}(\partial_t x)^2\right], \tag{5.3.3}$$

where $m \equiv \frac{1}{2\pi l_s^2}$ and $T_0(r) \equiv \frac{1}{2\pi l_s^2}\frac{r^4}{L^4}\left[1 - \left(\frac{2\pi T L^2}{r}\right)^2\right]$.

Now varying this action one obtains the EOM in frequency space

$$f_\omega''(r) + \frac{2(2r^2 - 4\pi^2 T^2 L^4)}{r(r^2 - 4\pi^2 T^2 L^4)}f_\omega'(r) + \frac{L^4\omega^2}{(r^2 - 4\pi^2 T^2 L^4)^2}f_\omega(r) = 0\,, \tag{5.3.4}$$

which is exactly solvable and the solution is

$$f_\omega(r) = C_1\frac{P_1^{\frac{i\omega}{2\pi T}}\left(\frac{r}{2\pi T L^2}\right)}{r} + C_2\frac{Q_1^{\frac{i\omega}{2\pi T}}\left(\frac{r}{2\pi T L^2}\right)}{r}\,,$$

where P and Q are associated Legendre functions.

Now choosing ingoing boundary condition (to obtain retarded Green's function) at the horizon and fixing $f_\omega(r_B) = 1$ one obtains the required solution

$$f^R_\omega(r) = \frac{(1-\frac{r}{2\pi TL^2})^{-i\omega/4\pi T}}{(1-\frac{r_B}{2\pi TL^2})^{-i\omega/4\pi T}}\frac{(1+\frac{r}{2\pi TL^2})^{i\omega/4\pi T}}{(1+\frac{r_B}{2\pi TL^2})^{i\omega/4\pi T}}\frac{r_B}{r}\frac{(L^2\omega+ir)}{(L^2\omega+ir_B)}\,. \tag{5.3.5}$$

Now from the on-shell action we can read off the Green's function

$$G^0_R \equiv \lim_{r\to r_B} T_0(r) f^R_{-\omega}(r)\partial_r f^R_\omega(r) = -\frac{\mathcal{M}\omega}{2\pi}\frac{\mathcal{M}\omega + i\sqrt{\lambda}\,4\pi^2T^2}{\mathcal{M}-i\sqrt{\lambda}\,\omega} \tag{5.3.6}$$

where we have used previously defined mass scale $\mathcal{M}$ and the dimensionless parameter λ.

For $\mathcal{M}\to\infty$, $G^0_R(\omega) = -\frac{\mathcal{M}\omega^2}{2\pi}$ which is again divergent. We can renormalize the Green's function as before by absorbing the UV divergent piece to define the zero temperature mass of the quark, i.e, $M^0_Q = \frac{\mathcal{M}}{2\pi}$ and obtain the renormalized Green's function

$$G_R(\omega) \equiv G^0_R(\omega) + \frac{\mathcal{M}\omega^2}{2\pi} = \frac{\mathcal{M}\omega}{2\pi}\frac{(\omega^2+4\pi^2T^2)}{(\omega + i\frac{\mathcal{M}}{\sqrt{\lambda}})}\,. \tag{5.3.7}$$

Near zero temperature

$$G_R(\omega)\Big|_{T\to 0} = \frac{\mathcal{M}\omega^3}{2\pi}\frac{1}{(\omega+i\frac{\mathcal{M}}{\sqrt{\lambda}})}\,. \tag{5.3.8}$$

This is identical to the retarded Green's function for zero temperature system (5.2.8). Evidently for small frequency

$$G_R(\omega) \approx -i\frac{\sqrt{\lambda}}{2\pi}\,\omega^3\,. \tag{5.3.9}$$

From this calculation black hole background seems to match smoothly pure AdS space as one takes $T\to 0$. The obvious question comes to one's mind is whether this coefficient of zero temperature dissipation is universal and independent of the dimensionality of space time. Actually we will see in the next section that this is not really the case in general. Reason being the Poincaré patch of BTZ black hole is, strictly speaking, AdS_3 at finite temperature for all practical purposes. That's why it smoothly matches the pure AdS result as one takes $T \to 0$. But AdS_{d+1}-BH with $d > 2$ is a 'genuine' black hole background and therefore the limit might not be smooth.

AdS_5 Black Hole Background

Now let us check if the dissipation coefficients for higher dimensional black holes in AdS space near zero temperature match the exact zero temperature coefficient. But those are not exactly solvable. As an example we will demonstrate it for AdS_5 black hole [6].

We can write down the metric in r-coordinate, as before, fixing $d = 4$ in (5.3.1)

$$ds^2 = -\frac{r^2}{L^2}\left(1 - \frac{(\pi T L^2)^4}{r^4}\right)\mathrm{d}t^2 + \frac{L^2}{r^2}\frac{\mathrm{d}r^2}{\left(1 - \frac{(\pi T L^2)^4}{r^4}\right)} + \frac{r^2}{L^2}\,\mathrm{d}\vec{x}^2\,, \tag{5.3.10}$$

with $\vec{x} \equiv (x^1, x^2, x^3)$.

The EOM[6] for the string

$$F''_\omega(r) + \frac{4r^3}{(r^4 - \pi^4 T^4 L^8)}F'_\omega(r) + \frac{\omega^2 L^4 r^4}{(r^4 - \pi^4 T^4 L^8)^2}F_\omega(r) = 0\,, \tag{5.3.11}$$

can not be solved exactly. As it's an ordinary second order linear differential equation it has two linearly independent solutions.

Near the horizon

$$F_\omega(r) \sim \left(1 - \frac{\pi^4 T^4 L^8}{r^4}\right)^{\pm i\frac{\Omega}{4}}\,. \tag{5.3.12}$$

[6]Notice that for this case $m \equiv \frac{1}{2\pi l_s^2}$ and $T_0(r) \equiv \frac{1}{2\pi l_s^2}\frac{r^4}{L^4}\left[1 - \left(\frac{\pi T L^2}{r}\right)^4\right]$. But eventually we want to calculate Green's function for $T \to 0$. Therefore for both BTZ and AdS-BH background we can practically use $T_0(r) \equiv \frac{1}{2\pi l_s^2}\frac{r^4}{L^4}$.

Near the boundary

$$F_\omega(r) = \left(1 + \frac{\Omega^2}{2r^2} + \ldots\right) + \frac{\chi(\Omega)}{r^3}\left(1 - \frac{\Omega^2}{10r^2} + \ldots\right), \tag{5.3.13}$$

contains a non-normalizable and a normalizable mode. Here $\Omega \equiv \frac{\omega}{\pi T}$.

But still one can solve it in perturbation expansion[7] in small frequency

$$F_\omega(r) = \left(1 - \frac{\pi^4 T^4 L^8}{r^4}\right)^{-i\frac{\Omega}{4}} \left(1 - i\Omega f_1(r) - \Omega^2 f_2(r) + i\Omega^3 f_3(r) + \ldots\right). \tag{5.3.14}$$

Putting this ansatz into (5.3.11) we get hierarchy of differential equations. Solving them order by order in $\Omega \equiv \frac{\omega}{\pi T}$ in principle one can obtain the unknown functions, $f_i(r)$ where $i = 1, 2, 3, \ldots$

Few useful remarks on the perturbative solution before we actually obtain it.

1. This type of perturbative solution has been calculated in [6] by Son and Teaney. As the authors were mostly interested in finite temperature phenomena they computed the Green's function up to ω^2 term. Here we show that their solution can be used even for $T \to 0$ and also we compute the Green's function up to ω^3 term which indicates the zero temperature dissipation.

2. The solution we obtain is a perturbation in Ω and T. But finally we are interested in $T \to 0$ limit. Clearly this limit is pathological for any finite ω. Only way we can make sense of the solution is by taking both

$$\omega, T \to 0 \;\text{ with }\; \Omega \text{ held fixed (and small).}$$

   The temperature independent term that we are interested in is the coefficient of $\Omega^3 T^3$ in this solution.

[7]The -ve sign in the exponent is chosen in (5.3.14) which is ingoing at the horizon. Because we are interested to calculate the retarded Green's function.

It is important to note, all dimensionful quantifies in this theory are measured in terms of the cutoff scale $\mathcal{M}$ ($= \frac{r_B}{l_s^2}$) defined before which is also interpreted as the mass of the external quark. Therefore whenever we say $\omega, T \to 0$ with Ω held fixed we mean $\omega/\mathcal{M} \to 0$ and $T/\mathcal{M} \to 0$ such a way that ω/T is fixed small number. E.g, say, $\omega/\mathcal{M} = 10^{-7}$, $T/\mathcal{M} = 10^{-6}$, thus $\omega/T = 0.1$ which is smaller than one.

3. Actually, as we will see below, we don't need to obtain all the unknown functions, $f_i(r)$, explicitly by performing complicated integrals. Rather we need only the *residues* of those integrals at the horizon to fix the coefficient of the homogeneous solutions.

Perturbative solution in AdS$_5$-BH

Just for convenience we work with z co-ordinate, where $z \equiv \frac{L^2}{r}$. Obtaining results in the r variable is straightforward. As we have discussed the EOM (5.3.10) for the string in AdS$_5$-BH

$$F''_\omega(z) + \frac{2(1+\pi^4T^4z^4)}{z(1-\pi^4T^4z^4)}F'_\omega(z) + \frac{\omega^2}{(1-\pi^4T^4z^4)^2}F_\omega(z) = 0\,, \tag{5.3.15}$$

is not exactly solvable and to obtain the solution that is ingoing at the horizon we need to use the following ansatz.

$$\begin{aligned} F^R_\omega(z) &= \left(1-\pi^4T^4z^4\right)^{-i\frac{\Omega}{4}} H(z)\,, \\ \text{where }\; H(z) &= 1 - i\Omega h_1(z) - \Omega^2 h_2(z) + i\Omega^3 h_3(z) + \ldots \end{aligned} \tag{5.3.16}$$

with $\Omega \equiv \frac{\omega}{\pi T}$. The differential equation $H(z)$ satisfies is given by

$$H''(z) - \frac{2\left(1+\pi^3T^3z^4(\pi T - i\omega)\right)}{z\,(1-\pi^4\,T^4\,z^4)}H'(z) + \frac{\omega\left(\omega + \pi^2T^2z^2(\omega + i\pi T)\left(\pi^2T^2z^2+1\right)\right)}{(\pi^2T^2z^2+1)\,(1-\pi^4T^4z^4)}H(z) = 0\,. \tag{5.3.17}$$

Notice that by choosing the ansatz (5.3.16) we have taken care of the singular near horizon part of the full solution by the pre-factor $\left(1-\pi^4T^4z^4\right)^{-i\frac{\Omega}{4}}$. Our strategy would be to substitute the ansatz (5.3.16) into (5.3.17) and at each order in Ω we demand that the solution to (5.3.17) is *regular* at the *horizon*. In other words, the full solution to (5.3.15) at any order in Ω behaves like

$$\left(1-\pi^4T^4z^4\right)^{-i\frac{\Omega}{4}} \times \left(\text{Regular function at } z=\frac{1}{\pi T}\right). \tag{5.3.18}$$

Again, we are interested in calculating temperature independent dissipative term in the Green's function. Therefore on dimensional ground we need to look for the coefficient of ω^3 term in the Green's function as the Green's function itself has mass dimension three. In other words, if one takes zero temperature limit of the Green's function only ω^3 term survives. For that one needs to take $T \to 0$ limit of the solution. But clearly the solution is a perturbation series in $\Omega = \frac{\omega}{\pi T}$. Therefore the only way one can make sense of this solution near zero temperature is to take both $T \to 0$ and $\omega \to 0$ (compared to $\mathcal{M}$) keeping the perturbation parameter, Ω fixed and small ($\Omega < 1$) such that the series converges.

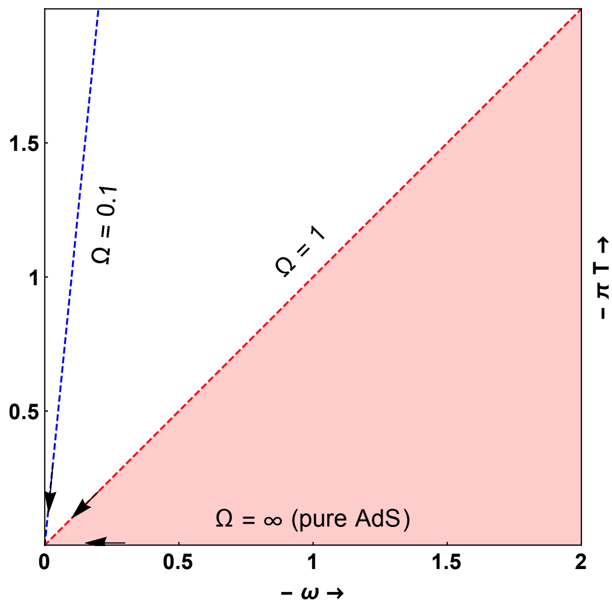


Figure 5.1. Different ways of taking $T = 0, \omega = 0$ limit.

Pictorially, there are many ways one can reach ($\omega = 0, T = 0$) i.e, the origin on the ω-T plane (see figure 5.1). Our solution makes sense when $\omega/\pi T$ is a constant and smaller than one. The shaded region is outside the domain of validity of our solution. Thus all our analysis and results hold true for the straight lines in the upper half of the box.

Also notice that pure AdS ($T = 0$) is along the x-axis. Therefore eventually we will be comparing these two ways (Ω = small and $\Omega = \infty$) of taking limits.

<u>Solution up to $O(\Omega)$</u> :

$$H(z) = 1 - i\Omega h_1(z)\,. \tag{5.3.19}$$

Substituting this ansatz into (5.3.15) we obtain the differential equation for $h_1(z)$

$$h_1''(z) + \underbrace{\frac{2(1+\pi^4T^4z^4)}{z(1-\pi^4T^4z^4)}}_{p_1(z)} h_1'(z) = \underbrace{\frac{\pi^4T^4z^2}{(1-\pi^4T^4z^4)}}_{q_1(z)}\,. \tag{5.3.20}$$

Let's cast this into a first order differential equation defining $y_1(z) \equiv h_1'(z)$ and consequently $y_1'(z) \equiv h_1''(z)$

$$y_1'(z) + p_1(z)y_1(z) = q_1(z)\,. \tag{5.3.21}$$

One can introduce integrating factor $I_1(z) = \exp\left(\int p_1(z)\mathrm{d}z\right)$ to obtain

$$\begin{aligned} y_1(z) &= \frac{c_1}{I_1(z)} + \frac{1}{I_1(z)}\int^z I_1(x)q_1(x)\mathrm{d}x \\ &= \underbrace{\frac{c_1z^2}{1-\pi^4T^4z^4}}_{y_1^h(z)} + \underbrace{\frac{\pi^4T^4z^3}{1-\pi^4T^4z^4}}_{y_1^p(z)}\,. \end{aligned} \tag{5.3.22}$$

We will see that the homogeneous part of the solution ($y_i^h(z)$) is identical in each order in Ω up to the undetermined coefficient (c_i). This coefficient is fixed by demanding the regularity of f_i at the horizon.

$$f_1(z) = \int y_1^h(z)\mathrm{d}z + \int y_1^p(z)\mathrm{d}z \tag{5.3.23}$$
$$\equiv h_1^h(z) + h_1^p(z)\,. \tag{5.3.24}$$

The requirement that $h_1(z)$ has to be regular sets the coefficient of $Log(1 - \pi T z)$ to zero. One can explicitly calculate the integrals and from that expression sort out the required coefficient. For this case

$$h_1^h(z) = \frac{c_1}{4\pi^3 T^3}\left\{-2tan^{-1}(\pi T z) - Log(1 - \pi T z) + Log(1 + \pi T z)\right\}, \tag{5.3.25}$$
$$h_1^p(z) = -\frac{1}{4}Log(1 - \pi^4 T^4 z^4)$$
$$= -\frac{1}{4}\left\{Log(1 - \pi T z) + Log(1 + \pi T z) + Log(1 + \pi^2 T^2 z^2)\right\}. \tag{5.3.26}$$

Clearly setting the coefficient of $Log(1 - \pi T z)$ to zero we get

$$c_1 = -\pi^3 T^3\,. \tag{5.3.27}$$

And the solution at this order becomes

$$h_1(z) = \frac{1}{2}tan^{-1}(\pi T z) - \frac{1}{2}Log(1 + \pi T z) + \frac{1}{4}Log(1 + \pi^2 T^2 z^2)\,. \tag{5.3.28}$$

But there is another way in which we can fix the coefficient without doing the integrals. The only potential singular term in $h_1(z)$ at the horizon appears as $Log(1 - \pi T z)$. This type of terms are originated from the terms of the form $\frac{1}{(1-\pi T z)}$ in $y_1(z)$. Therefore fixing the coefficient of $Log(1 - \pi T z)$ in $f_1(z)$ to zero is equivalent to setting the residue of $y_1(z)$ at $z = \frac{1}{\pi T}$ to zero. This is much easier way when the integrals get complicated as we go in higher orders.

Solution up to $O(\Omega^2)$:

$$H(z) = 1 - i\Omega h_1(z) - \Omega^2 h_2(z)\,. \tag{5.3.29}$$

Notice that $h_1(z)$ is already known from (5.3.28). Substituting this ansatz into (5.3.17) we get the differential equation for $h_2(z)$. Again one can cast that into a first order differential equation

$$y_2'(z) + p_2(z) y_2(z) = q_2(z)\,, \tag{5.3.30}$$

where $p_2(z) = p_1(z)$. Therefore integrating factor $I_2(z) = I_1(z)$.

$$\begin{aligned} y_2(z) &= \frac{c_2}{I_2(z)} + \frac{1}{I_2(z)}\int^z I_2(x) q_2(x)\mathrm{d}x \\ &= \underbrace{\frac{c_2 z^2}{1 - \pi^4 T^4 z^4}}_{y_2^h(z)} + \underbrace{\frac{1}{I_2(z)}\int^z I_2(x) q_2(x)\mathrm{d}x}_{y_2^p(z)}\,. \end{aligned} \tag{5.3.31}$$

Now making the residue of $y_2(z)$ at $z = \frac{1}{\pi T}$ to vanish we can fix

$$c_2 = \pi^3 T^3\,. \tag{5.3.32}$$

The solution at this order

$$\begin{aligned} h_2(z) = \frac{1}{32}[&4\{-4 + tan^{-1}(\pi T z) - Log(1 + \pi T z)\}\{tan^{-1}(\pi T z) - Log(1 + \pi T z)\} \\ &- 4\{2 + tan^{-1}(\pi T z) - Log(1 + \pi T z)\}Log(1 + \pi^2 T^2 z^2) + Log(1 + \pi^2 T^2 z^2)^2]\,. \end{aligned} \tag{5.3.33}$$

Solution up to $O(\Omega^3)$:

$$H(z) = 1 - i\Omega h_1(z) - \Omega^2 h_2(z) + i\Omega^3 h_3(z)\,, \tag{5.3.34}$$

where h_1, h_2 are known from (5.3.28) and (5.3.33). The differential equation for $h_3(z)$ or rather $y_3(z) \equiv h_3'(z)$

$$y_3'(z) + p_3(z)y_2(z) = q_3(z)\,, \tag{5.3.35}$$

where $p_3(z) = p_1(z)$. Therefore integrating factor $I_3(z) = I_1(z)$, as before.

$$\begin{aligned} y_3(z) &= \frac{c_3}{I_3(z)} + \frac{1}{I_3(z)} \int^z I_3(x) q_3(x) \mathrm{d}x \\ &= \underbrace{\frac{c_3 z^2}{1 - \pi^4 T^4 z^4}}_{y_3^h(z)} + \underbrace{\frac{1}{I_3(z)} \int^z I_3(x) q_3(x) \mathrm{d}x}_{y_3^p(z)}\,. \end{aligned} \tag{5.3.36}$$

Now making the residue of $y_3(z)$ at $z = \frac{1}{\pi T}$ to vanish we obtain

$$c_3 = \left(\frac{\pi - Log\,4}{4}\right)\pi^3 T^3\,. \tag{5.3.37}$$

The functional form of $y_3(z)$ is complicated and therefore it's difficult to obtain an explicit expression for $h_3(z)$ unlike the lower order functions. But as we are interested in zero temperature limit of the Green's function, we only need to know the full solution for small Ω

$$\begin{aligned} F_\omega^R(z) &= \left(1 - \pi^4 T^4 z^4\right)^{-i\frac{\Omega}{4}} \left(1 - i\Omega h_1(z) - \Omega^2 h_2(z) + i\Omega^3 h_3(z)\right) \\ &\approx \left(1 + \frac{i}{4}\pi^3 \omega T^3 z^4\right)\left\{1 - \frac{i\omega}{\pi T}\left(-\frac{1}{3}\pi^3 T^3 z^3\right) - \frac{\omega^2}{\pi^2 T^2}\left(-\frac{1}{2}\pi^2 T^2 z^2\right)\right. \\ &\qquad\qquad \left. - \frac{i\omega^3}{\pi^3 T^3}\left(\frac{\pi - Log\,4}{12}\right)\pi^3 z^3 T^3\right\}. \end{aligned} \tag{5.3.38}$$

In the zero temperature ($T \to 0$) limit[8]

$$F^R_\omega(z)\Big|_{T\to 0} = 1 + \frac{\omega^2 z^2}{2} + i\frac{\omega^3 z^3}{3}\left(\frac{\pi - Log\,4}{4}\right). \tag{5.3.39}$$

In r co-ordinate

$$F^R_\omega(r)\Big|_{T\to 0} = 1 + \frac{\omega^2 L^4}{2r^2} + i\frac{\omega^3 L^6}{3r^3}\left(\frac{\pi - Log\,4}{4}\right). \tag{5.3.40}$$

The retarded Green's function at $T \to 0$ (with $\omega \to 0$ and Ω = fixed) can be calculated using the solution (5.3.40)

$$\begin{aligned} G^0_R &\equiv \lim_{r\to r_B} T_0(r)\, F^R_{-\omega}(r)\, \partial_r F^R_\omega(r) \\ &= \lim_{r\to r_B} \frac{1}{2\pi l_s^2}\frac{r^4}{L^4}\left(-\frac{\omega^2 L^4}{r^3} - i\frac{\omega^3 L^6}{4r^4}(\pi - Log\,4)\right) \\ &= -\frac{\mathcal{M}\omega^2}{2\pi} - i\frac{\sqrt{\lambda}}{2\pi}\left(\frac{\pi - Log\,4}{4}\right)\omega^3 . \end{aligned} \tag{5.3.41}$$

Therefore the renormalized Green's function

$$G_R(\omega) \equiv G^0_R + \frac{\mathcal{M}\omega^2}{2\pi} = -i\frac{\sqrt{\lambda}}{2\pi}\left(\frac{\pi - Log\,4}{4}\right)\omega^3 . \tag{5.3.42}$$

It is evident that this zero temperature dissipation term is *not* same as that of pure AdS case and actually off by a factor of $\frac{\pi - Log\,4}{4}$.

Also the significance of the result is that for arbitrarily small T (say $1\,\mu$K), with $\omega < T$, there is a T - *independent* coefficient for ω^3. For example, at $T = 1\,\mu$K and at $T = 1\,n$K, the coefficient *is* the same. This is something of experimental relevance since in experiments one never reaches $T = 0$ but works in very low temperature regime.

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[8] It's worth mentioning here the analysis is valid in the regime ω and T very small with $\frac{\omega}{T}$ (< 1) fixed. Thus here $T \to 0$ essentially means taking T arbitrarily small with keeping fixed $\frac{\omega}{T} < 1$. Therefore one cannot take $T \to 0$ in strict mathematical sense because then one is forced to take $\omega = 0$ and it is well known that there can be no radiation if $\omega = 0$ (which is equivalent to DC).

Discussions

Whenever a charged particle accelerates or decelerates it radiates energy which is known as bremsstrahlung effect. We discuss about dissipation at and near zero temperature. Naturally this zero temperature dissipation finds its origin in this bremsstrahlung phenomenon. One can notice that this dissipative force term ($\mathcal{F}_{diss}$) goes as a cubic power in frequency

$$\mathcal{F}_{diss}(\omega) \sim -i\sqrt{\lambda}\,\omega^3 x(\omega)\,. \tag{5.4.1}$$

In real space this each $i\omega$ gives a time derivative and therefore the above force law reduces to

$$\mathcal{F}_{diss}(t) \sim \sqrt{\lambda}\,\dddot{x}(t) = \sqrt{\lambda}\,\dot{a}\,, \tag{5.4.2}$$

$\dot{a}$ here quantifies the rate of change in acceleration and is often called jerk or jolt. This formula is very similar to that of "Abraham-Lorentz force" [116] in classical electrodynamics for a charged particle with charge q

$$\mathcal{F}_{rad}(t) = \frac{2}{3}q^2\,\dot{a}\,. \tag{5.4.3}$$

This is the force that an accelerating charged particle feels in the recoil from the emission of radiation. Only the effective coupling is different for the holographic case. This 'coupling'($\sqrt{\lambda}$) is essentially the bremsstrahlung function $B(\lambda, N)$. The corrections in λ and N can also be computed for particular known theories (see [14, 111–113]).

The main aim is to understand how this bremsstrahlung function behaves near zero temperature. We work in Poincaré patch of AdS-black hole on the gravity side. We notice that for higher dimensional cases (we performed calculations in AdS_5-BH) value of this function[9] near $T \to 0$ doesn't match that of at $T = 0$. Strictly speaking, this result is obtained in a particular regime of the parameter space namely $\omega/\mathcal{M} \to 0$ and $T/\mathcal{M} \to 0$ such a way that $\Omega < 1$. Thus the domain of validity of our analysis is spanned by family of straight lines (see figure 5.1) which end at the origin and with slope greater than one (i.e, $\frac{\pi T}{\omega} > 1$). It is important to note, $\omega \to 0$ limit for the solution to the pure AdS case is actually represented by the x-axis of figure 5.1. Therefore effectively we are comparing two different ways[10] of taking $T = 0, \omega = 0$ limit, 'almost vertical lines'(AdS-BH) and 'the horizontal line' (pure AdS). They don't match. Actually this result is not unexpected as there is a Hawking-Page phase transition at zero temperature in Poincaré patch of black holes in AdS.

Therefore the result suggests that one should not use the $T = 0$ theory (which always has $\Omega = \infty$) as an approximation to small temperature physics, when Ω is small i.e, $\omega < T$. Thus for example if $T = 10^{-6}$ and $\omega = 10^{-7}$ compared to $\mathcal{M}$, we cannot use the $T = 0$ theory. This is something that can be very crucial in the context of quark-gluon-plasma (QGP). QGP is always at finite temperature and therefore dissipation term more specifically the bremsstrahlung function $B(\lambda)$ is not continuously connected to the zero temperature background (pure AdS) result, at least for small frequencies ($\Omega < 1$). Therefore one must use the AdS-BH background to compute those quantifies even at very small temperature.

Unlike the higher dimensional case, we see that for a particle in 1+1 dimensional field theory the bremsstrahlung functions match smoothly at $T = 0$ (see also appendix H). The possible reason behind this phenomenon is hidden in the corresponding dual geometries namely AdS_3 (for $T = 0$) and BTZ (at $T \neq 0$). A BTZ black hole is just an orbifold

[9]Of course we are working in leading order in large N and large λ. To compute corrections one needs to work with some known supersymmetric UV theories (e.g, ABJM, $\mathcal{N} = 4$ SYM)

[10]It is worthwhile noting that the lines that end on the origin and are contained inside the shaded region are also valid ways of taking $T = 0, \omega = 0$ limit. But our analysis doesn't work in that region. Because in that region $\Omega > 1$ and therefore our perturbative expansion (5.3.16) breaks down. Thus we cannot say much about high frequency domain with this analysis. But since the coefficient is non zero for $\omega < T$, it is reasonable to assume that it will not vanish suddenly when $\omega > T$. So one would expect similar dissipative behavior even for the shaded triangle in 5.1.

of AdS_3 and therefore locally AdS_3. One can only distinguish the former from the latter by studying global properties. BTZ in Poincaré patch is no different than AdS_3 at finite temperature (also called thermal AdS_3) unlike the higher dimensional black holes in AdS which are 'genuine' black hole backgrounds.

Conclusions

We summarize the main points here :

- We have studied Brownian motion in various space time dimensions with the help of holographic Green's function computation. In each case we obtain dissipation at zero temperature due to radiation from accelerated charged Brownian particle.
- As long as we are considering dynamics of the quark at zero temperature, that is the string in the dual gravity theory moves in a pure AdS spacetime, the coefficients of dissipation for arbitrary space time dimensions are identical. The value of the coefficient is $\frac{\sqrt{\lambda}}{2\pi}$ and can be identified with $B(\lambda, N)$ [13] (actually $2\pi B(\lambda, N) = \frac{\sqrt{\lambda}}{2\pi}$) as occurring in many other physical quantities (such as the cusp anomalous dimension introduced by Polyakov [114]).
- Even the coefficients match for string in AdS_3 and in BTZ as we take $T \to 0$. This is because BTZ in Poincaré patch is nothing more than a thermal AdS_3.
- For higher dimensions the coefficients at $T = 0$ and $T \to 0$ don't match for small frequencies ($\Omega < 1$). Here we are effectively comparing *infinite* Ω to *finite* and *small* Ω and they turn out to be different although both refer to the same region around $\omega = 0, T = 0$. Thus one should be careful in using pure AdS for calculating near zero temperature quantities (e.g, $B(\lambda)$) for very low (near zero) frequencies, i.e. $\Omega < 1$. Even if the temperature is very very small (unless it's exactly zero) one should not use $T = 0$ results as the $T = 0$ and $T \to 0$ systems are described by completely

different theories. We have shown this phenomenon via explicit computation by studying a string dynamics in AdS_5-BH background. The corresponding coefficient comes out to be $\frac{\sqrt{\lambda}}{2\pi}\frac{\pi - Log\,4}{4}$. This phenomenon might have its origin in the Hawking-Page transition at $T = 0$ in Poincaré patch.

6 Brownian motion at finite density

This chapter deals with Brownian motion of a heavy charged particle at zero and small (but finite) temperature in presence of finite density. We are primarily interested in the dynamics at (near) zero temperature which is holographically described by motion of a fundamental string in an (near-) extremal Reissner-Nordström black hole. We analytically compute the functional form of retarded Green's function for small frequencies and extract the dissipative behavior at and near zero temperature.

This chapter is based on [9].

Introduction

AdS/CFT or more generally gauge/gravity duality [1–4] has been serving as a great weapon in a theoretician's armory to study strongly coupled systems analytically for almost last two decades. Although for most of the cases its predictions are qualitative, there are instances (see for example the famous η/s computation in [24]) when it relates very formal theoretical frame work to real life experiments. Since its discovery this duality has glued many phenomena appearing in apparently different branches of physics together. Studying Brownian motion of a heavy particle using classical gravity technique is one such example [5, 6] where holography relates a statistical system to a gravitational one. The dual gravity description involves a long fundamental string

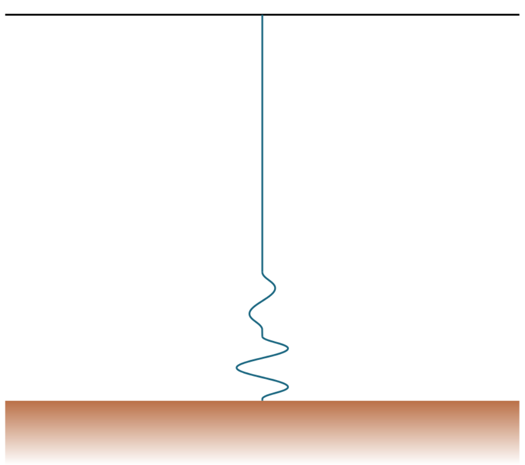

Figure 6.1. Gravity set up describing Brownian motion

stretching from the boundary of the AdS space into the black hole horizon. Numerous works [7, 8, 21, 22, 74, 75, 95, 110, 117] have been done elaborating on different aspects of this set-up. Integrating out[1] the whole string in that background gives an effective description of the heavy particle at the boundary. Its dynamics is governed by a Langevin equation. For a particle with mass[2] M which is moving with velocity v the Langevin equation reads

$$M\frac{dv}{dt} + \gamma v = \xi(t)\,, \tag{6.1.1}$$

$$\text{with} \quad \langle \xi(t)\xi(t')\rangle = \Gamma\delta(t-t')\,, \tag{6.1.2}$$

where γ is the viscous drag, ξ is the random force on the particle and Γ quantifies the strength of the 'noise' (i.e, random force). The Second equation is one of the many avatars of celebrated fluctuation-dissipation theorem. One can write down a generalized

[1]We mostly follow the Green's function language of [6] to describe Brownian motion.

[2]We will see that this is actually 'renormalized' mass. The correction to the *bare mass* (M_0) of the Brownian particle will come from the retarded Green's function.

version[3] of this equation

$$M_0 \frac{d^2x(t)}{dt^2} + \int_{-\infty}^{t} dt' \, G_R(t,t')x(t') = \xi(t) \,, \qquad \langle \xi(t)\xi(t') \rangle = iG_{\rm sym}(t,t') \,. \tag{6.1.3}$$

$G_R(t,t')$ is thus the same as $\gamma(t-t')$ for the choice of the lower limit of the integral, $t_0 = -\infty$ and $iG_{\rm sym}(t,t')$ is the same as $\Gamma(t-t')$.

In frequency space the generalized Langevin equation takes the following form

$$\left[-M_0\,\omega^2 + G_R(\omega)\right] x(\omega) = \xi(\omega) \,, \qquad \langle \xi(-\omega)\,\xi(\omega) \rangle = i\,G_{\rm sym}(\omega) \,. \tag{6.1.4}$$

If the retarded Green's function, $G_R(\omega)$ is expanded for small frequencies the coefficient of $\omega^2 \left(i.e, \frac{d^2x(t)}{dt^2}\right)$ adds to the bare mass of the particle and the coefficient of $\omega \left(i.e, \frac{dx(t)}{dt}\right)$ will show off as the drag term[4]

$$G_R(\omega) = -i\gamma\omega - \varDelta M\omega^2 + \ldots \tag{6.1.5}$$

After defining the 'renormalized' mass as

$$M \equiv M_0 + \varDelta M \,, \tag{6.1.6}$$

this generalized Langevin equations (6.1.4) (up to $O(\omega^2)$) take the standard form (6.1.1) and (6.1.2).

From the above discussion it is quite clear that if we are interested in studying dissipation for a Brownian particle we just need to compute the retarded Green's function, $G_R(\omega)$.

[3] See, for example, [6, 7] for a review of path integral derivation of this generalized Langevin equation. Also notice that this equation is written in terms of the *bare mass* (M_0) of the Brownian particle.

[4] More terms with higher powers in ω will also be generated in this small frequency expansion. Their interpretations are outside the scope of standard Langevin equation (6.1.1). But from properties of Green's functions it is well known that imaginary part of retarded Green's function causes dissipation. Thus odd powers in ω are responsible for dissipation. Actually the ω^3 term signifies dissipation at zero temperature [7, 8] in absence of matter density.

We can calculate this quantity using different holographic techniques [11, 15] depending on the physical systems.

In chapter 5 Brownian motion for a heavy quark in 1+1 dimensions was studied following [6] which used the prescription of [11, 62] to compute the boundary Green's function. The calculations were done in BTZ black hole background where the system is exactly solvable. The main result was to obtain dissipation for the heavy quark even at zero temperature. The result might look very counter intuitive and unphysical at first sight because at zero temperature the thermal fluctuations go to zero and therefore the Brownian particle should stop dissipating energy. But this zero temperature dissipation has its origin in radiation of an accelerated charged particle. The force term[5] in the Langevin equation at zero temperature was of the form

$$F(\omega) = -i\,\frac{\sqrt{\lambda}}{2\pi}\,\omega^3\,x(\omega)\,. \tag{6.1.7}$$

Therefore the integrated energy loss is given by

$$\varDelta E = \frac{\sqrt{\lambda}}{2\pi}\int dt\,a^2\,. \tag{6.1.8}$$

It is known from classical electrodynamics that the energy loss[6] due to radiation is proportional to square of the acceleration (a). See [14,65,72,73,76,100–102,108–110,113,118] for related works.

Dissipation at zero temperature is a fascinating phenomenon. Its emergence from the retarded Green's function signifies that $G_R(\omega)$ actually contains information at the 'quan-

[5]This force formula is same as 'Abraham-Lorentz force' [116] in classical electrodynamics only its 'coupling constant' (which is $\frac{q^2}{6\pi\epsilon_0}$ for a particle with charge q and ϵ_0 is the vacuum permittivity) is different. This similarity is remarkable since in our holographic context the boundary theory is highly non-linear unlike electrodynamics!

[6]This formula for an accelerated quark was obtained first by Mikhailov in [12] by a very different approach. The factor $\frac{\sqrt{\lambda}}{2\pi}$ is essentially the Bremsstrahlung function $B(\lambda, N)$ ($2\pi B(\lambda, N) = \frac{\sqrt{\lambda}}{2\pi}$) identified in [13] as occurring in many other physical quantities (such as the cusp anomalous dimension introduced by Polyakov [114]).

tum' level (by 'quantum' here we mean dynamics at $T = 0$). Brownian motion of a particle is usually studied at finite temperature. The system is driven by fluctuations which are thermal in nature and therefore if the temperature is taken to zero that $G_R(\omega)$ must vanish too. But the $G_R(\omega)$ we obtain from holography contains information of both thermal and quantum fluctuations for the boundary theory. Although at finite T thermal fluctuations dominate over the quantum ones at very small T the latter ones are much more important. The main aim here is to understand how a heavy particle's (quark's) motion at finite density (chemical potential $\mu \neq 0$) is described at and near $T = 0$. The dual gravity theory should contain an (near-) extremal[7] charged black hole. (See [26, 119] for some results in this set up).

For high temperature regime ($\mu \ll T$) the effect of the charge of the non-extremal black hole can be neglected at the leading order and $G_R(\omega)$ can be computed in small μ and small ω expansions using the methods followed in [6, 11].

In this chapter we would like to see how the system behaves near $T = 0$. Therefore the other limit $\mu \gg T$ i.e, the low temperature regime is of more interest to us. We will see that in this regime usual perturbation techniques for small T and small ω won't work because of a double pole for the ω^2 term in the string equation of motion in the extremal black hole background. Due to this double pole, near horizon dynamics is extremely sensitive to ω. To get around this problem we will adopt the *matching technique*[8] described in [15] where the authors studied non-Fermi liquids using holography.

The rest of the chapter is organized as follows. In section 6.2 we quickly review the Reissner-Nordström (RN) black hole in asymptotically AdS space time. The main purpose is to spell out the notations and conventions that we will be following through out

[7]The zero temperature dissipation for a theory dual to pure AdS$_{d+1}$ and black holes in AdS$_{d+1}$ has been calculated in chapter 5. Just on dimensional ground $G_R(\omega) \sim -i\omega^3$. The coefficient depends on the background. The cause of this dissipation being the radiation due to accelerated quark.

[8]This matching technique is familiar to string theorists from the brane absorption calculations that led to the discovery of AdS/CFT correspondence. For example see [55, 120]. Maldacena used similar technique in his famous decoupling argument [1].

rest of the thesis. The section 6.3 contains the analytic computation of retarded Green's function at zero temperature using matching technique. We also list some of its properties in detail. The retarded Green's function at small but finite temperature is analyzed in section 6.4. We mainly discuss how $G_R(\omega)$ gets small T corrections. Section 6.5 summarizes the main results and their interpretations, assumptions we make and some future directions. Section 6.6 contains some concluding remarks.

AdS-RN black hole background

AdS-RN black hole[9] is a solution to Einstein-Maxwell equation with a negative cosmological constant,

$$S_{EM} = \frac{1}{2\kappa^2}\int d^{d+1}x\,\sqrt{-g}\left[R + \frac{d(d-1)}{L_{d+1}^2} + \frac{L_{d+1}^2}{g_F^2}F_{MN}F^{MN}\right]. \tag{6.2.1}$$

R is the Ricci scalar, g_F is the dimensionless gauge coupling in the bulk. L_{d+1} is a length scale (known as AdS radius) and κ^2 is Newton's constant. Notice that we can always redefine the gauge field by absorbing the dimensionless coupling g_F^2 into F_{MN}. Thus we can fix g_F to one without loss of generality. The (d+1) dimensional metric and gauge field that satisfy the corresponding equations of motion are given by

$$ds^2 = \frac{L_{d+1}^2}{z^2}\left(-f(z)\,dt^2 + d\vec{x}^2\right) + \frac{L_{d+1}^2}{z^2}\frac{dz^2}{f(z)}, \tag{6.2.2}$$

where,

$$f(z) = 1 + Q^2 z^{2d-2} - M z^d$$

$$A_t(z) = \mu\left(1 - \frac{z^{d-2}}{z_0^{d-2}}\right).$$

[9]The solution we will be working with has planer horizon with topology $\mathbb{R}^{d-1}$. Therefore it is really a black brane rather than a black hole.

Q, M, z_0 are constant parameters which are black hole charge, black hole mass and horizon radius respectively. μ is the chemical potential, $\vec{x} \equiv (x^1, x^2 \ldots x^{d-1})$ and z is the radial coordinate in the bulk such that the boundary of this space is at $z = 0$.

Notice that if we put $f(z) = 1$ we get back pure AdS_{d+1} in Poincaré patch. This non trivial function $f(z)$ indicates that the physics changes as we move along the radial direction.

At the horizon : $f(z_0) = 0$. Therefore we can express M as

$$M = z_0^{-d} + Q^2\, z_0^{d-2}\,. \tag{6.2.3}$$

Now Q can be expressed in terms of chemical potential (μ)

$$Q = \sqrt{\frac{2\,(d-2)}{d-1}}\,\frac{\mu}{z_0^{d-2}}\,. \tag{6.2.4}$$

And the Hawking temperature

$$T = \frac{d}{4\pi z_0}\left(1 - \frac{d-2}{d}\,Q^2\, z_0^{2d-2}\right). \tag{6.2.5}$$

Actually Q, M and z_0 are related to charge density, energy density and entropy density in the boundary theory respectively. Q is charge density up to some numbers. Let's introduce a new length scale z_* to express Q as

$$Q := \sqrt{\frac{d}{d-2}}\,\frac{1}{z_*^{d-1}}\,. \tag{6.2.6}$$

We also define $\mu_* = \frac{1}{z_*}$. Note that $z_* \geq z_0$ to avoid the naked singular geometry. (This is equivalent to $M \geq Q$ condition.)

There are two distinct situations possible : *Extremal* ($T = 0$) and *Non-extremal* ($T \neq 0$).

Extremal black hole

When the Hawking temperature is zero the black hole is called extremal. Extremal black hole contains maximum possible charge. The 'blackening function' becomes

$$f(z) = 1 + \frac{d}{d-2}\frac{z^{2d-2}}{z_*^{2d-2}} - \frac{2(d-1)}{d-2}\frac{z^d}{z_*^d}\,. \tag{6.2.7}$$

Near horizon region for the extremal black hole becomes $\mathrm{AdS}_2 \times \mathbb{R}^{d-1}$

$$ds^2 = \frac{L_2^2}{\zeta^2}(-dt^2 + d\zeta^2) + \mu_*^2 L_{d+1}^2 d\vec{x}^2 \tag{6.2.8}$$

$$A_t(\zeta) = \frac{1}{\sqrt{2d(d-1)}}\frac{1}{\zeta}\,, \tag{6.2.9}$$

where $\zeta := \frac{z_*^2}{d(d-1)(z_*-z)}$, L_2 is the radius[10] of the AdS_2 and is related to L_{d+1} by the following relation

$$L_2 = \frac{1}{\sqrt{d(d-1)}}L_{d+1}\,. \tag{6.2.10}$$

Non-extremal black hole

Generically charged black holes have non-vanishing temperature. We will be interested in studying Brownian motion at finite density and finite temperature (T) but with $T \ll \mu$. We want to be in this regime because the near horizon geometry will become AdS_2-BH $\times \mathbb{R}^{d-1}$,

$$ds^2 = \frac{L_2^2}{\zeta^2}\left(-g(\zeta)dt^2 + \frac{d\zeta^2}{g(\zeta)}\right) + \mu_*^2 L_{d+1}^2 d\vec{x}^2 \tag{6.2.11}$$

$$A_t(\zeta) = \frac{1}{\sqrt{2d(d-1)}}\frac{1}{\zeta}(1 - \frac{\zeta}{\zeta_0})\,, \tag{6.2.12}$$

[10] Note that $L_2 < L_{d+1}$ for $d \geq 3$.

where $g(\zeta) := (1 - \frac{\zeta^2}{\zeta_0^2})$, $\zeta_0 := \frac{z_*^2}{d(d-1)(z_* - z_0)}$ and the corresponding temperature, $T = \frac{1}{2\pi\zeta_0}$. For $T \approx \mu$ this nice structure breaks down.

Brownian motion at zero temperature

To understand Brownian motion of a heavy charged particle in some strongly coupled field theory in d-dimensions which has a gravity dual one needs to study the dynamics of a long string in the dual gravity background [5, 6]. Therefore to explore the same Brownian motion at zero temperature and finite density one needs to study a string in an extremal charged black hole. This section contains the main analysis and results of the chapter.

Green's function by matching solutions

In this Einstein-Maxwell theory an elementary string cannot couple to the gauge field, A_M. It can only couple to the background metric G_{MN}. We consider geometries with vanishing Kalb-Ramond field, B_{MN}. For this zero temperature case G_{MN} can be read off from the extremal BH background (6.2.2) with $f(z)$ given in (6.2.7).

The string dynamics is given by standard Nambu-Goto action

$$S_{NG} = -\frac{1}{2\pi l_s^2} \int d\tau\, d\sigma\, \sqrt{-h}\,, \tag{6.3.1}$$

where l_s is the string length and h_{ab} is the induced metric on the world sheet

$$h_{ab} = G_{MN}\, \partial_a X^M\, \partial_b X^N\,. \tag{6.3.2}$$

We choose to work in static gauge,

$$\tau \equiv t \text{ and } \sigma \equiv \zeta\,.$$

Also we can choose one particular direction, say x_1 (we call this simply x for brevity), along which the world sheet fluctuates.

$$x \equiv x\,(\tau, \sigma) = x\,(t, \zeta)\,. \tag{6.3.3}$$

To understand the dynamics of the string we need to use the full background metric (6.2.2) with the "blackening factor" given in (6.2.7). Varying the Nambu-Goto action

$$S_{NG} = -\frac{1}{2\pi\, l_s^2} \int dt\, dz\, \frac{L_{d+1}}{z^2} \left[1 + \frac{1}{2}\, f(z)\, x'^2 - \frac{1}{2 f(z)}\, \dot{x}^2\right], \tag{6.3.4}$$

we obtain the equation of motion (EOM) in frequency space

$$x''_\omega(z) + \frac{\frac{d}{dz}(\frac{f(z)}{z^2})}{\frac{f(z)}{z^2}}\, x'_\omega(z) + \frac{\omega^2}{[f(z)]^2}\, x_\omega(z) = 0\,, \tag{6.3.5}$$

where we have used $x(z,t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t} x_\omega(z)$.

Now to obtain $G_R(\omega)$ the standard procedure would be to solve this equation and obtain it from the on shell action. But this procedure involves a few subtleties [15]. Firstly this differential equation is not exactly solvable. Even if we are interested in $G_R(\omega)$ for very small frequencies ($\omega \ll \mu$) we cannot directly perform a perturbation expansion in small ω. Because at zero temperature the $f(z)$ has a double zero at the horizon (extremal limit) and consequently ω^2 term in the equation of motion generates a double pole at the horizon. Thus this singular term dominates at the horizon irrespective of however small ω we choose.

To get around this difficulty we closely follow the matching technique in [15]. At first we isolate the 'singular' near horizon region from the original background. We already know that the near horizon geometry is given by $\mathrm{AdS}_2 \times \mathbb{R}^{d-1}$ (6.2.8). This is referred to as IR/inner region and the rest of the space time as UV/outer region. We can solve the string EOM exactly in this IR region and therefore the treatment will be non-perturbative in ω. On the other hand the ω-dependence in UV region can be treated perturbatively as there is no more ω-sensitivity. The main task is to match the solutions over these two regions. The overlap between these to regions is near the boundary ($\zeta \to 0$) of the AdS_2 where

$$\frac{1}{\mu} \ll \zeta \ll \frac{1}{\omega}\,, \tag{6.3.6}$$

$$\text{with} \quad z_*^2 \frac{\omega^2}{f(z)} \sim \omega^2 \zeta^2 \quad \text{is very } \textit{small},$$

$$\text{and} \quad \mu\,\zeta \sim \frac{z_*}{z_* - z} \quad \text{remains } \textit{large}.$$

The last two expressions ensure that the ω dependent term becomes small in EOM and we are still near the horizon respectively.

• Inner region

For the string in $\mathrm{AdS}_2 \times \mathbb{R}^{d-1}$ (6.2.8)

$$\begin{aligned} \sqrt{-h} &= \frac{L_2^2}{\zeta^2} \sqrt{1 + \frac{L_{d+5}^2}{L_2^2}\, \mu_*^2\, \zeta^2\, (x'^2 - \dot{x}^2)} \\ &\approx \frac{L_2^2}{\zeta^2} \left[1 + \frac{1}{2}\, d\, (d-1)\, \mu_*^2\, \zeta^2\, (x'^2 - \dot{x}^2) \right]. \end{aligned} \tag{6.3.7}$$

The Nambu-Goto action

$$S_{NG} = -\frac{L_2^2}{2\pi l_s^2} \int dt\, d\zeta \left[\frac{1}{\zeta^2} + \frac{1}{2} d(d-1)\, \mu_*^2\, (x'^2 - \dot{x}^2) \right]. \tag{6.3.8}$$

Varying this action we get a very simple EOM for the string which is that of a free wave equation

$$x'' - \ddot{x} = 0\,. \tag{6.3.9}$$

To solve this linear EOM, the standard way is to go to the Fourier space

$$x(\zeta, t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, x_\omega(\zeta)\,. \tag{6.3.10}$$

The equation of motion reduces to

$$x''_\omega(\zeta) + \omega^2\, x_\omega(\zeta) = 0\,. \tag{6.3.11}$$

This is very well known differential equation with two independent solutions

$$x_\omega(\zeta) = e^{\pm i\omega\zeta}\,. \tag{6.3.12}$$

As we are interested in calculating retarded Green's function we need to pick the one which is *ingoing* at the horizon ($\zeta \to \infty$). It's easy to see that $e^{+i\omega\zeta}$ is ingoing at the horizon. Once we pick the right solution at the horizon we need to expand that near the boundary($\zeta = 0$) of the IR geometry i.e, $\mathrm{AdS}_2 \times \mathbb{R}^{d-1}$

$$x_\omega(\zeta) = 1 + i\,\omega\,\zeta\,, \qquad \text{near } \zeta = 0\,. \tag{6.3.13}$$

The ratio of normalizable to non-normalizable mode fixes the Green's function for the IR geometry

$$\mathscr{G}_R(\omega) = i\omega\,. \tag{6.3.14}$$

(We will see in (6.3.20) that for string in AdS_{d+1} non-normalizable and normalizable modes go as z^0 and z^3 respectively. Where as in $\mathrm{AdS}_2 \times \mathbb{R}^{d-1}$ (6.3.13) they go as z^0 and z^1.)

• Outer region

For the outer region we need to solve the full EOM (6.3.5). But now as we are away from the 'dangerous' near horizon region we can do a small frequency expansion. At the leading order we can put $\omega = 0$. Let's say that the (6.3.5) has two independent solutions $\eta_+^{(0)}$ and $\eta_-^{(0)}$ for $\omega = 0$. We can fix there behavior near the horizon ($z = z_*$) and near the boundary ($z = 0$) by solving this equation near those regions.

Near horizon

Near $z = z_*$

$$f(z) \approx d(d-1)\,\frac{(z_* - z)^2}{z_*^2}\,. \tag{6.3.15}$$

The EOM reduces to

$$x_\omega''(z) - \frac{2}{z_* - z}\,x_\omega'(z) = 0\,. \tag{6.3.16}$$

The two independent solutions are

$$c \;\text{ and }\; \frac{z_*}{z_* - z}\,, \tag{6.3.17}$$

where c is some constant which can be chosen to be 1. Since we need to match the inner and the outer solutions near $z = z_*$ let's express these independent solutions in terms of ζ,

$$\eta_+^{(0)} \to \left(\frac{\zeta}{z_*}\right)^0, \quad \eta_-^{(0)} \to \left(\frac{\zeta}{z_*}\right)^1. \tag{6.3.18}$$

Near boundary

Near the boundary, $z = 0$ we can approximate $f(z) \approx 1$ and consequently the EOM

$$x''_\omega(z) - \frac{2}{z}\, x'_\omega(z) = 0\,. \tag{6.3.19}$$

The solutions near $z = 0$ will behave as

$$\eta_+^{(0)} \approx a_+^{(0)}\left(\frac{z}{z_*}\right)^0 + b_+^{(0)}\left(\frac{z}{z_*}\right)^3, \tag{6.3.20}$$

$$\eta_-^{(0)} \approx a_-^{(0)}\left(\frac{z}{z_*}\right)^0 + b_-^{(0)}\left(\frac{z}{z_*}\right)^3. \tag{6.3.21}$$

Notice that $a_\pm^{(0)}, b_\pm^{(0)}$ are not independent but related by Wronskian. We will use this information below to fix one of those coefficients.

Matching the solutions

We have some solutions to the full EOM in patches. All we need to do to obtain the Green's function is to determine the outer solution by matching it to the inner solution in the overlap region. Then expand that solution near $z = 0$ to compute the ratio of normalizable to non-normalizable mode.

Let's do the matching first. From (6.3.13) and (6.3.18) we can express the outer solution as

$$x_\omega(z) = \eta_+^{(0)}(z) + \mathscr{G}_R(\omega)\, z_*\, \eta_-^{(0)}(z)\,. \tag{6.3.22}$$

Notice that so far we have been using solutions to the UV equation which are 0^{th}-order in ω (as we have put $\omega = 0$). But in principle we can systematically add higher order corrections in ω. In that improved version the outer solution will be given by

$$x_\omega(z) = \eta_+(z) + \mathscr{G}_R(\omega)\, z_*\, \eta_-(z)\,, \tag{6.3.23}$$

$$\text{where} \quad \eta_\pm(z) = \eta_\pm^{(0)}(z) + \omega^2\, \eta_\pm^{(2)}(z) + \omega^4\, \eta_\pm^{(4)}(z) + \dots \tag{6.3.24}$$

And as before near boundary, $z = 0$

$$\eta_\pm \approx a_\pm \left(\frac{z}{z_*}\right)^0 + b_\pm \left(\frac{z}{z_*}\right)^3\,, \tag{6.3.25}$$

$$\text{where} \quad a_\pm = a_\pm^{(0)} + \omega^2\, a_\pm^{(2)} + \omega^4\, a_\pm^{(4)} + \dots \tag{6.3.26}$$

$$b_\pm = b_\pm^{(0)} + \omega^2\, b_\pm^{(2)} + \omega^4\, b_\pm^{(4)} + \dots \tag{6.3.27}$$

Note that $a_\pm, b_\pm$ are all *real* coefficients because the UV equation (6.3.5) and the boundary condition (6.3.18) at $z = z_*$ are both real. Also the perturbation in frequency are in even powers in ω as (6.3.5) contains only ω^2.

Finally to obtain the retarded Green's function we expand the outer solution (6.3.23) near the boundary ($z = 0$)

$$x_\omega(z) = A(\omega)\left(\frac{z}{z_*}\right)^0 + B(\omega)\left(\frac{z}{z_*}\right)^3\,, \tag{6.3.28}$$

$$A(\omega) = a_+ + \mathscr{G}_R(\omega)\, z_*\, a_-\,,$$

$$B(\omega) = b_+ + \mathscr{G}_R(\omega)\, z_*\, b_-\,.$$

Green's function of the boundary theory is given by (see [6–8])

$$G_R(\omega) := \lim_{z\to 0} T_0(z)\left(-\frac{z^2}{L_{d+1}^2}\right)\frac{x_\omega{}'(z)}{x_\omega(z)}\,, \tag{6.3.29}$$

where

$$T_0(z) = \frac{1}{2\pi l_s^2}\frac{L_{d+1}^4}{z^4}\left[1+\frac{d}{d-2}\left(\frac{z}{z_*}\right)^{2d-2}-\frac{2(d-1)}{d-2}\left(\frac{z}{z_*}\right)^{d}\right], \tag{6.3.30}$$

is identified as local string tension which comes from the z-dependent normalization of the boundary action. Since we are interested in boundary Green's function

$$T_0(z) \approx \frac{1}{2\pi l_s^2}\frac{L_{d+1}^4}{z^4}\,, \tag{6.3.31}$$

and consequently the retarded Green's function

$$\begin{aligned} G_R(\omega) &\approx \lim_{z\to 0} -\frac{1}{2\pi l_s^2}\frac{L_{d+1}^2}{z^2}\frac{\eta_+{}'(z)+\mathcal{G}_R(\omega)\,z_*\,\eta_-{}'(z)}{\eta_+(z)+\mathcal{G}_R(\omega)\,z_*\,\eta_-(z)} \\ &= \lim_{z\to 0} -\frac{L_{d+1}^2}{2\pi l_s^2}\frac{1}{z^2}\frac{3b_+\,(\frac{z}{z_*})^2\,\frac{1}{z_*}+\mathcal{G}_R(\omega)\,3b_-\,(\frac{z}{z_*})^2}{[a_+ +\mathcal{G}_R(\omega)\,z_* a_-]+[b_+ +\mathcal{G}_R(\omega)\,z_* b_-]\,(\frac{z}{z_*})^3} \\ &= -\frac{\sqrt{\lambda}}{2\pi}\frac{3}{z_*^3}\left[\frac{b_+ +\mathcal{G}_R(\omega)\,z_* b_-}{a_+ +\mathcal{G}_R(\omega)\,z_* a_-}\right]. \end{aligned} \tag{6.3.32}$$

Therefore finally,

$$\boxed{G_R(\omega) = -\frac{\sqrt{\lambda}}{2\pi}\frac{3}{z_*^3}\left[\frac{b_+ +\mathcal{G}_R(\omega)\,z_* b_-}{a_+ +\mathcal{G}_R(\omega)\,z_* a_-}\right].} \tag{6.3.33}$$

We have introduced a dimensionless quantity $\lambda := \frac{L_{d+1}^4}{l_s^4}$ which behaves like a coupling constant in the boundary field theory. Since we are working in supergravity limit in the bulk $L_{d+1} \gg l_s$ and therefore $\lambda \gg 1$ i.e, the boundary theory is strongly coupled. The expression (6.3.33) is the main result of this chapter. Below we analyze this in detail.

Properties of the Green's function

- The expression (6.3.33) depends on two sets of data.

   1. $\{a_\pm, b_\pm\}$: These constants come from solving EOM in the outer region. Therefore they depend on the geometry of the outer region. In this sense they are *non-universal* UV data.
   2. $\mathcal{G}_R(\omega)$: This depends only on the IR region which always contains AdS_2 independent of the full UV theory. This is *universal* IR data.

- As we have already pointed out the UV data $(a_\pm, b_\pm)$ are always *real*. Whereas the IR data $(\mathcal{G}_R(\omega))$ is in general *complex*. Therefore the dissipation is always controlled by the IR data. Actually all non-analytic[11] behavior enters into $G_R(\omega)$ from $\mathcal{G}_R(\omega)$.

- In principle $a_\pm^{(2n)}, b_\pm^{(2n)}$ are fixed by (numerically) solving the EOM in UV region in ω^2 perturbation.

- The interesting thing to notice is that the (6.3.5) with $\omega = 0$ allows a constant solution. From the boundary condition (6.3.18) at $z = z_*$, we see that

$$\eta_+^{(0)} = 1\,. \tag{6.3.34}$$

  This value of $\eta_+^{(0)}$ will continue to solve the EOM (6.3.5) with $\omega = 0$ for the outer region $z_* \geq z \geq 0$. So near the boundary ($z = 0$) we have (from (6.3.25))

$$\eta_+ \approx a_+^{(0)} \left(\frac{z}{z_*}\right)^0 + b_+^{(0)} \left(\frac{z}{z_*}\right)^3 = 1\,. \tag{6.3.35}$$

[11] There is no non-integer powers of ω for the system we are considering. Therefore there is no branch cuts but $G_R(\omega)$ can only have poles at particular ω-values.

This fixes

$$a_+^{(0)} = 1, \quad b_+^{(0)} = 0\,. \tag{6.3.36}$$

We can fix one more coefficient by equating the generalized Wronskian[12] at the boundary and at the horizon. We get (see appendix I for details)

$$b_-^{(0)} = \frac{1}{3}\,. \tag{6.3.37}$$

The 0^{th}-order Green's function reduces[13] to

$$\begin{aligned} G_R^{(0)}(\omega) &= -\frac{L_{d+1}^2}{2\pi l_s^2}\frac{3}{z_*^3}\frac{b_+^{(0)} + \mathcal{G}_R(\omega) z_* b_-^{(0)}}{a_+^{(0)} + \mathcal{G}_R(\omega) z_* a_-^{(0)}} \\ &= -\frac{\sqrt{\lambda}}{2\pi}\frac{i\mu_*^2\,\omega}{(1 + i\frac{\omega}{\mu_*}\,a_-^{(0)})}\,. \end{aligned} \tag{6.3.38}$$

The form of $G_R^{(0)}(\omega)$ ensures[14]

$$G_R^{(0)}(\omega) = 0\,, \quad \text{as} \;\; \omega \to 0\,.$$

The real and imaginary parts of $G_R^{(0)}(\omega)$ are plotted (see Fig. 6.2 and Fig. 6.3) for particular values of the parameters : $\lambda = 50,\ \mu_* = 5$ and $a_-^{(0)} = 10$.

---

[12]The generalized Wronskian of a 2nd order homogeneous ODE with two independent solutions ϕ_1 and ϕ_2 is defined as

$$\begin{aligned} W(z) &\equiv e^{\int^z P(t)dt}[\phi_1\partial_z\phi_2 - \phi_2\partial_z\phi_1] \\ &= \sqrt{-g}g^{zz}[\phi_1\partial_z\phi_2 - \phi_2\partial_z\phi_1]\,. \end{aligned}$$

[13]There is no principle that tells us that the all the coefficients of Green's function (even in 0^{th}-th order in ω) should be determined by analytic methods. Due to the simplicity of this particular differential equation we can fix few of them analytically. In general one needs numerical techniques to fix all of them.

[14]Instead of a fluctuating string if one considers bulk Fermionic field (not world sheet field) in the same geometry, $a_\pm^{(n)}, b_\pm^{(n)}$ are functions of momentum k. For certain value of $k = k_F$, say, $a_+^{(0)} = 0$. At this value of momentum $G_R^{(0)}(\omega, k_F)$ becomes singular at $\omega = 0$. This indicates the *Fermi surface*.

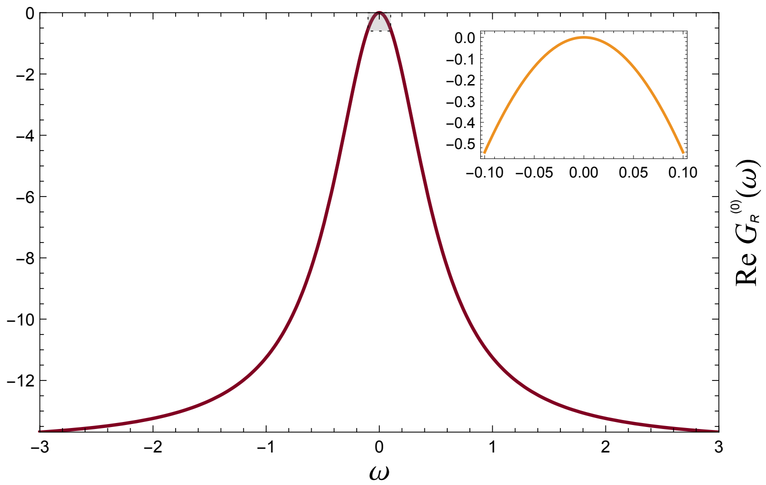

Figure 6.2. Real part of $G_R^{(0)}(\omega)$ with $\mu_* = 5$

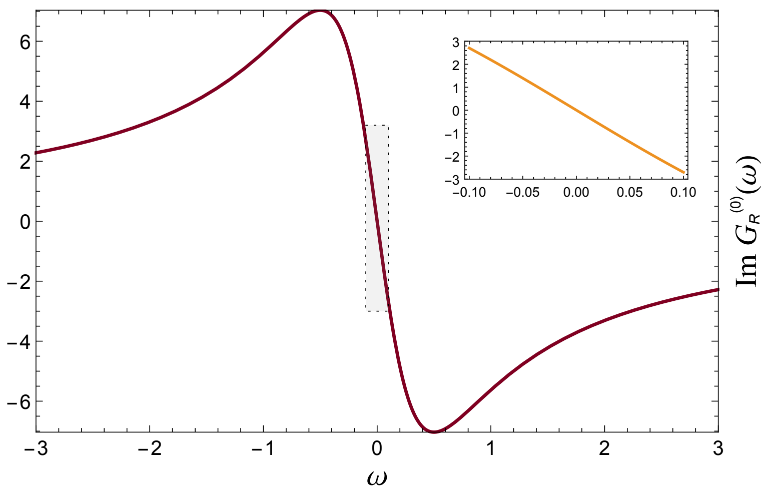

Figure 6.3. Imaginary part of $G_R^{(0)}(\omega)$ with $\mu_* = 5$

For small frequency

$$\begin{aligned} G_R^{(0)}(\omega) &= -i\frac{\sqrt{\lambda}}{2\pi}\mu_*^2\,\omega(1 - i\frac{\omega}{\mu_*}a_-^{(0)} + \ldots) \\ &\approx -i\,\frac{\sqrt{\lambda}}{2\pi}\mu_*^2\,\omega - a_-^{(0)}\,\frac{\sqrt{\lambda}}{2\pi}\mu_*\,\omega^2\,. \end{aligned} \tag{6.3.39}$$

This is also consistent with Langevin equation (6.1.5) as $G_R(\omega)$ expansion starts with $-i\omega$. Note that, for small frequencies, the zero temperature dissipation goes linear in ω (see Fig. 6.3) unlike $\mu = 0$ case [7,8] where this goes as ω^3. The fact that $G_R(\omega)$ is linear in ω comes from the fact that the effective AdS_2 dimension (which can be read off from (6.3.18)) of the 'quark operator' is *one* (*i.e.* $\Delta = 1$).

The leading dissipative term is proportional to μ_*^2. This result indicates that energy loss for the charged Brownian particle is more for medium with higher charge density.

Brownian motion at finite temperature

To study Brownian motion at finite but very small temperature we need to follow the same steps as before. But now the inner region will become a non-extremal (rather near extremal) black hole (6.2.2) background.

Green's function by matching solutions

In this section we will go through the same procedure of matching functional form of the solutions in inner and outer regions. There will be few modifications to the zero temperature $G_R(\omega)$.

- **Inner region**

The metric in this region is black hole[15] in $AdS_2 \times \mathbb{R}^{d-1}$ (6.2.11).

The Nambu-Goto action

$$S_{NG} = -\frac{L_2^2}{2\pi l_s^2} \int dt\, d\zeta \left[\frac{1}{\zeta^2} + \frac{1}{2} d(d-1)\, \mu_*^2 \left(g(\zeta)\, x'^2 - \frac{1}{g(\zeta)}\, \dot{x}^2 \right) \right]. \tag{6.4.1}$$

[15]This black hole is related to AdS_2 geometry by a co-ordinate transformation [121, 122] (combined with a gauge transformation) that acts as a conformal transformation on the boundary of AdS_2. So correlators can be obtained directly from AdS_2 correlators via conformal transformation.

Varying this action we obtain the EOM in frequency space

$$x''_\omega(\zeta) + \frac{2\,\zeta}{\zeta^2 - \zeta_0^2}\, x'_\omega(\zeta) + \frac{\zeta_0^4\,\omega^2}{(\zeta^2-\zeta_0^2)^2}\, x_\omega(\zeta) = 0\,. \tag{6.4.2}$$

This EOM can be solved exactly. The two independent solutions[16] are :

$$x_\omega(\zeta) = e^{\pm\, i\,\zeta_0\,\omega\,\tanh^{-1}\left(\frac{\zeta}{\zeta_0}\right)}. \tag{6.4.3}$$

Again we are interested in retarded Green's function so we pick the solution which is ingoing at the horizon ($\zeta = \zeta_0$)

$$e^{+\, i\,\zeta_0\,\omega\,\tanh^{-1}\left(\frac{\zeta}{\zeta_0}\right)}. \tag{6.4.4}$$

Once we have the ingoing solution we need to expand it near the boundary($\zeta = 0$) of near horizon geometry

$$x^R_\omega(\zeta) = 1 + i\,\omega\,\zeta\,. \tag{6.4.5}$$

We can now read off the Green's function in IR region

$$\mathcal{G}_{R,T}(\omega) = i\,\omega\,. \tag{6.4.6}$$

This is identical to the zero temperature case (6.3.14).

We have discussed earlier the dissipative part of $G_R(\omega)$ comes solely from the IR Green's function. For this particular problem $\mathcal{G}_{R,T}(\omega) = \mathcal{G}_R(\omega) = i\omega$. Therefore T-dependence can only creep in via the expansion coefficients $(a_\pm, b_\pm)$.

[16] Notice the same $\zeta_0\, tanh^{-1}\left(\frac{\zeta}{\zeta_0}\right)$ factor appears in the conformal transformation from AdS_2 to AdS_2-BH (see [122]).

- **Outer region**

This outer region analysis will be almost identical to that of the zero temperature case. One just has to be careful about the coefficients $(a_\pm, b_\pm)$ which are now *temperature dependent*, in general. Therefore we can skip repeating the analysis and directly write down the Green's function at finite temperature following the zero temperature case (see section 6.3)

$$G_{R,T}(\omega) = -\frac{\sqrt{\lambda}}{2\pi}\frac{3}{z_*^3}\left[\frac{b_+(\omega,T) + \mathcal{G}_{R,T}(\omega)\, z_*\, b_-(\omega,T)}{a_+(\omega,T) + \mathcal{G}_{R,T}(\omega)\, z_*\, a_-(\omega,T)}\right]$$
$$= -\frac{\sqrt{\lambda}}{2\pi}\frac{3}{z_*^3}\left[\frac{b_+(\omega,T) + i\,\omega\, z_*\, b_-(\omega,T)}{a_+(\omega,T) + i\,\omega\, z_*\, a_-(\omega,T)}\right]. \tag{6.4.7}$$

If we consider only the leading order in ω (i.e, putting $\omega = 0$ in the EOM), even for the non-extremal case, $x_\omega = const.$ is again a solution. As before we can normalize it to one. By the same argument as in zero temperature case

$$a_+^{(0)} = 1, \quad b_+^{(0)} = 0\,. \tag{6.4.8}$$

Therefore the leading order Green's function is identical to that of zero temperature case (6.3.38)

$$G^{(0)}_{R,T}(\omega) = -\frac{\sqrt{\lambda}}{2\pi}\frac{i\,\mu_*^2\,\omega}{(1 + i\frac{\omega}{\mu_*}\,a_-^{(0)})}\,. \tag{6.4.9}$$

This Green's function can be improved by solving the (6.3.5) perturbatively in ω and T. Actually the corrections will be in powers of $\frac{\omega}{\mu_*}$ and $\frac{T}{\mu_*}$. The corresponding real coefficients can also be obtained numerically in a systematic fashion.

Discussions

We have studied in detail the important properties of the retarded Green's function we obtained from the matching technique. It has a nice structure in terms of frequency (and also in temperature). We discussed that the dissipative (in general non-analytic) part of the system is determined by the near horizon behavior i.e, the IR data of the system. On the other hand the near boundary behavior i.e the UV data is always some analytic expansion in nature. Actually these facts are compatible with our field theoretic and geometric intuitions.

For generic many body systems we know that IR physics can show non-analytic behavior but UV physics can only give analytic corrections to that. From Renormalization Group (RG) point of view this matching technique can be thought of as matching UV to IR physics at some intermediate energy scale fixed by the chemical potential (μ_*).

Geometrically also this is expected. Dissipation is caused due to energy or 'modes' disappearing into 'something'. In the bulk picture this can only happen near the horizon of a black hole where the modes fall into the black hole and never come back. Whereas near boundary geometry is very smooth and therefore no non-analytic behavior can be expected from that UV region.

It is worth mentioning that the leading order dissipative term at zero temperature is linear[17] in frequency unlike the zero density situations [7, 8] where it starts with cubic term (ω^3). Therefore this is actually the *drag* term associated to the velocity of the charged particle rather than the acceleration of the same. A particle moving at constant velocity at zero temperature can dissipate energy for this set-up since the presence of finite charge density breaks Lorentz symmetry of the boundary theory explicitly. Nevertheless there will

[17] This linear dependence in frequency comes from the fact that effective AdS_2 dimension (see (6.3.18)) of the 'quark operator' is *one* (*i.e.* $\Delta = 1$) and is very crucial. Due to this particular low frequency behaviour the dissipative structure is qualitatively same at zero and finite temperature. If the dimension has been different from *one*, the small ω expansion in (6.3.39) at zero temperature would have started with a different power (i.e. not linear) and the story would have been different from the $T \neq 0$ result.

be dissipation due to acceleration of the charged particle as radiation at the subleading order. Expanding $G_R^{(0)}(\omega)$ (6.3.38) in small frequency one can obtain the Bremsstrahlung function $B(\lambda)$ by collecting the coefficient of ω^3,

$$B(\lambda) = \frac{\sqrt{\lambda}}{2\pi} \left(a_-^{(0)}\right)^2 . \tag{6.5.1}$$

$a_-^{(0)}$ can be fixed by numerical technique. But this is obtained solving the string EOM (6.3.5) only upto $O(\omega^0)$. It will get corrections for higher orders in ω^2 that can be taken into account systematically.

It will be interesting to use this matching technique to system with small chemical potential (μ) and then take that to zero. Then the results obtained in chapter 5 can be checked. But there is a subtlety. All the analysis in this chapter heavily rely on the near horizon AdS_2 factor which appears for *large* chemical potential and *small* temperature. For $T \gtrsim \mu$ the 'nice' inner region structure breaks down. Thus one cannot use this method for zero density situation at least in a straightforward manner.

Conclusions

We have used holographic technique to study Langevin dynamics of a heavy particle moving at finite charge density. We have studied this by computing retarded Green's function via solution matching technique. Here are the main results.

- Analytic form of retarded Green's function for small frequencies has been obtained at zero temperature.
- The drag force at zero temperature shows up as the leading contribution at small frequencies.
- It is also been sketched how the retarded Green's function gets corrections due to

small (but finite) temperature. The leading dissipative part (drag) remains identical to that in the zero temperature case.

- The drag term grows quadratically with the chemical potential i.e, loss in energy of the Brownian particle is more for medium with higher charge density.

- The leading contribution to the Bremsstrahlung function $B(\lambda)$ is obtained with an unknown co-efficient $a_{-}^{(0)}$ which can be fixed by numerical method. Its corrections in $\frac{\omega}{\mu_*}$ and $\frac{T}{\mu_*}$ can be computed systematically.

7 Conclusions & outlook

In this thesis we have studied dynamics of a heavy charged particle (a quark) in a strongly coupled plasma using holography. Our main focus has been on the zero temperature dissipation of the Brownian particle which is interpreted as energy loss due to radiation [12–14]. We have studied the Brownian dynamics in diverse spacetime dimensions and also in zero and finite matter density in the plasma. We have used only analytic techniques throughout the thesis and for particular cases have obtained even exact results.

In the first part of the thesis we have studied the Brownian diffusion of a particle in 1+1 dimensions using the holographic techniques. The holographic dual is a BTZ black hole with a string. We find an exact Green function and an exact (generalized) Langevin equation.

- We show that the exact generalized Langevin equation, which is valid on short time scales also, does not suffer from the inconsistency that is associated with the usual Langevin equation that has a delta function for the drag term.
- There is also a temperature independent dissipation at all frequencies. At high frequencies it is a drag term. This does not violate Lorentz invariance as the force on a quark moving with constant velocity for all time continues to be zero. This has already been studied in higher dimensional systems and is due to radiation.
- Since an exact Green function is available, the 'stretched horizon' can in fact be placed at an arbitrary radius and an effective action obtained which has a nice holo-

graphic RG interpretation.

Next we explore dissipation of a Brownian particle near and at zero temperature. The motivation came from our earlier analysis in 1+1 dimensions where we noticed dissipation $T \to 0$ and $T = 0$ match *i.e,* dissipation has smooth $T = 0$ limit. Here we list our main results.

- As long as we are considering dynamics of the quark at zero temperature, that is the string in the dual gravity theory moves in a pure AdS spacetime, the coefficients of dissipation for arbitrary space time dimensions are identical. The value of the coefficient is $\frac{\sqrt{\lambda}}{2\pi}$ and can be identified with the bremsstrahlung function $B(\lambda, N)$.

- For higher dimensions (AdS_{d+1} with $d > 2$) the dissipation coefficient at $T = 0$ and $T \to 0$ don't match for small frequencies ($\frac{\omega}{\pi T} < 1$). The coefficient for $T \to 0$ comes out to be $\left(\frac{\pi - Log\, 4}{4}\right) \frac{\sqrt{\lambda}}{2\pi}$ instead of $\frac{\sqrt{\lambda}}{2\pi}$. We interpret this phenomenon as possible signature of Hawking-Page transition at $T = 0$.

Finally we have studied Langevin dynamics of a heavy particle moving at finite charge density. This analysis is valid for field theories in more than two spacetime dimensions *i.e,* for AdS_{d+1} with $d > 3$.

- We derive analytic form of retarded Green's function for small frequencies at zero temperature. The drag force at zero temperature shows up as the leading contribution. Even if we turn on small (but finite) temperature leading dissipative part (which is drag) remains unchanged.

- The drag term grows quadratically with the chemical potential i.e, loss in energy of the Brownian particle is more for medium with higher charge density.

- The leading contribution to the Bremsstrahlung function $B(\lambda)$ is obtained with an unknown co-efficient which can be fixed by numerical method.

There are several directions that one can make progress in, in the context of holographic Brownian motion.

- It will be interesting to study holographic renormalization group [16–19] for this type of systems particularly for the 2+1 dimensional bulk where one can solve the equations of motion the fluctuating string in closed form.
- Most of the methods applied in this thesis can possibly be generalized to fluctuating higher dimensional extended objects *i.e,* D-branes in AdS spacetime. These systems can be very useful in exploring different non-equilibrium phenomena in the context of gauge/gravity duality.
- The zero temperature dissipation of the charged particle is very interesting in its own right. There are several scenarios where this phenomenon can be studied. Some of them are as follows.
  - It would be interesting to investigate the dissipation near zero temperature in 1+1 dimensional CFT at finite matter density by studying stochastic string in a charged BTZ black hole.
  - Our technique for computing Green function at finite density for higher dimensional systems (*i.e.* CFT_d with $d \geq 3$) only requires an AdS_2 factor near the horizon. Therefore it should work even if the UV theory is non-conformal (not asymptotically AdS) but the IR geometry has a AdS_2 factor. For example, instead of D3 branes one can look at D2 or D4 brane geometries [20]. If for some charge density they flow to a AdS_2 then the procedure can be applied. By the same argument it can be also used for some rotating extremal black hole backgrounds.
  - Finally one can explore zero temperature dissipation for anisotropic backgrounds [21, 22] which are more interesting phenomenologically.

- All our results are valid for *large* chemical potential and *small* temperature. If one is interested in studying Brownian motion in the opposite regime this technique can not be used. The reason being for $\mu \sim T$ the 'nice' inner region structure breaks down. In that case the small μ corrections can be computed using same techniques used in [6, 7] but for a charged black hole background in AdS with very small charge.

- For a particular theory at *finite density* but at *zero temperature* if one can independently compute the Bremsstrahlung function, then that can be compared with the result obtained in our method. The standard and well known method of computing the Bremsstrahlung function is using supersymmetric localization technique (see e.g, [13, 14, 113]). But one would face following challenges to apply this technique at finite density. Firstly one needs to, if possible, turn on background fields corresponding to finite density while preserving enough supersymmetry. Secondly, and more specific to the computation of the Bremsstrahlung function, finite density breaks conformal invariance. Some of the steps in computing the Bremsstrahlung function use explicitly conformal symmetry. Although the Bremsstrahlung function must exist for non-conformal theories, it may no longer be controlled by localization.

- In this thesis we have performed all the computations in the dual bulk geometry. It will be extremely interesting if all these results can be obtained directly from conformal field theory calculations. One needs to work with CFT in Minkowski space and it will be particularly interesting to see if rich 2D CFT tools can be used to model Brownian motion.

A Some useful tricks & results

Here we list some very useful techniques and results in the context of dualities in chapter. 2.1.

Lagrange multipliers

In classical physics to impose constraint on a system one introduces Lagrange multiplier. Let's describe a particular dynamical system to illustrate how to deal with Lagrange multiplier.

Consider a free scalar field ϕ that can take value only on a unit sphere in field space i.e, ϕ satisfies the constraint $\phi^2 = 1$. The Lagrangian which describes this free scalar field on a sphere is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \Lambda(\phi^2 - 1)\,. \tag{A.1.1}$$

We have introduced Lagrange multiplier Λ which is a non-dynamical field since its conjugate momentum vanishes. The EOM of Λ is

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\Lambda} &= 0\,,\\ \phi^2 - 1 &= 0\,,\\ \phi^2 &= 1\,.\end{aligned} \tag{A.1.2}$$

This shows that our Lagrangian correctly incorporates the constraint on the field and the constraint equation comes from the EOM of the Lagrange multiplier.

So far this was all classical. To do the same thing in quantum theory one just needs to introduce same Λ into the path integral. A free un-constrained scalar field in d dimensions is represented by the following partition function

$$\mathcal{Z} = \int \mathcal{D}\phi \, e^{i S[\phi]} \, , \tag{A.1.3}$$

where $S[\phi] = \int d^{d+1}x \, [\frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi]$.

To impose the constraint one has to introduce a 'fake' variable Λ into the partition function

$$\mathcal{Z} = \int \mathcal{D}\phi \, \mathcal{D}\Lambda \, e^{i S[\phi] - i \int d^{d+1}x \, \Lambda (\phi^2 - 1)} \, . \tag{A.1.4}$$

"Solvable" path integrals

Path integrals are usually tricky objects. Only a very special subclass of them namely those with quadratic action can be exactly solved.

$$\mathcal{Z} = \int \mathcal{D}\phi \, e^{-\int \phi \hat{\theta} \phi} \, , \tag{A.2.1}$$

where $\hat{\theta}$ is the kinetic energy operator or inverse propagator *e.g,* $\hat{\theta} = (\Box - m^2)$ for free massive scalar field. And for this particular case

$$\mathcal{Z} = \int \mathcal{D}\phi \, e^{-\int \phi \hat{\theta} \phi} = \det \hat{\theta} \, . \tag{A.2.2}$$

The result $\det \hat{\theta}$ is a formal expression. It can be realized by diagonalizing $\hat{\theta}$ and then multiplying all its eigenvalues.

Gaussian is special

In the previous section we discussed that we can exactly compute only those path integrals whose integrands are Gaussian in the field variables. Gaussian has one more special property namely Fourier transform of a Gaussian is also a Gaussian

$$\frac{1}{2\pi}\int d\omega\, e^{i\,\omega t}\, e^{-\frac{t^2}{2\sigma^2}} = \sqrt{\frac{\sigma}{2\pi}}\, e^{-\frac{\sigma^2\omega^2}{2}}. \tag{A.3.1}$$

The interesting fact to notice here is the 'width' or the standard deviation of the Gaussian gets inverted *i.e,* $\sigma \to \frac{1}{\sigma}$ after the Fourier transformation. Thus if the Fourier transformation is performed at the level of path integrals, σ may be interpreted as a 'coupling' for some theory. Then roughly speaking this transformation effectively maps a 'weakly coupled' theory of 'field' t with coupling constant σ to a 'strongly coupled' one of 'field' ω with coupling $\frac{1}{\sigma}$. (For a more concrete example see Maxwell duality in section 2.1.) We also make use of similar property in chapter 4 while introducing Gaussian noise ξ into the system (see (4.2.26)).

B Anti-de Sitter spacetime

In this appendix we discuss about AdS spacetime in detail (see [123, 124]). We start with studying lower dimensional spaces then build on them and talk about AdS_{d+1} spacetime in different co-ordinates. Choice of coordinate is very crucial in studying particular problem in holography.

Spaces with constant curvature

Let's first consider two dimensional spaces (not space-time) with constant curvature.

Sphere (S^2)

A sphere is defined by the surface which satisfies the following constraint in 3D Euclidean space,

$$\begin{aligned} X^2 + Y^2 + Z^2 &= L^2\,, \\ ds^2 &= dX^2 + dY^2 + dZ^2\,, \end{aligned} \tag{B.1.1}$$

where L is the radius of the sphere. One can parametrize the 2-sphere by there intrinsic co-ordinates (θ, ϕ)

$$X = L\, sin\, \theta\, cos\, \phi$$

$$
\begin{aligned}
Y &= L\, sin\,\theta\; sin\,\phi \\
Z &= L\, cos\,\theta\,. \qquad \text{(B.1.2)}
\end{aligned}
$$

In (θ, ϕ) coordinates the metric reduces to

$$ds^2 = L^2\,(d\theta^2 + sin^2\theta\, d\phi^2)\,. \qquad \text{(B.1.3)}$$

- S^2 has SO(3) invariance of the ambient space.
- S^2 is *homogeneous* : Any point on S^2 can be mapped to other point on S^2 by SO(3).
- S^2 has constant positive curvature $\frac{2}{L^2}$. This is also called the Ricci scalar.

Hyperbolic space (H^2)

This is a space of constant negative curvature. A hyperbolic space[1] is defined by the surface which satisfies the following constraint in 3D Minkowski space. As H^2 is embedded in Minkowski space rather than Euclidean space it's hard to visualize.

$$
\begin{aligned}
X^2 + Y^2 - Z^2 &= -L^2 \\
ds^2 &= dX^2 + dY^2 - dZ^2\,. \qquad \text{(B.1.4)}
\end{aligned}
$$

One can parametrize the H^2 by its intrinsic co-ordinates (θ, ϕ)

$$X = L\, sinh\,\rho\; cos\,\phi$$

[1]Hyperbolic space (H^2) is not to be confused with *hyperboloid* which is embedded in Euclidean space with identical constraint equation

$$
\begin{aligned}
X^2 + Y^2 - Z^2 &= -L^2 \\
ds^2 &= dX^2 + dY^2 + dZ^2\,.
\end{aligned}
$$

Notice that hyperboloid is *not* homogeneous because it does not respect the SO(3) invariance of the ambient space.

$$Y = L\,sinh\,\rho\;sin\,\phi$$

$$Z = L\,cosh\,\rho\,. \tag{B.1.5}$$

In (ρ, ϕ) coordinates the metric reduces to

$$ds^2 = L^2\,(d\rho^2 + sinh^2\rho\;d\phi^2)\,. \tag{B.1.6}$$

- H^2 has SO(1,2) invariance of the ambient space.
- H^2 is *homogeneous* : Any point on H^2 can be mapped to other point on H^2 by SO(1,2) i.e, 'Lorentz transformation'[2].
- H^2 has constant negative curvature $-\frac{2}{L^2}$.

Space-times with constant curvature

Now we move on to space-times with constant curvature. We will discuss two spacetimes AdS_2 and dS_2 which are analogous to H^2 and S^2 respectively. Our focus will be mostly on AdS spacetime.

de Sitter spacetime (dS_2)

The de Sitter spacetime is a spacetime with constant positive curvature. It (specially dS_4) frequently appears in the context of dark energy in cosmology. dS_2 spacetime is defined by

$$-Z^2 + X^2 + Y^2 = +L^2$$

$$ds^2 = -dZ^2 + dX^2 + dY^2\,. \tag{B.2.1}$$

[2]Strictly speaking this is not Lorentz transformation since we are essentially dealing with a space *not* a space-time.

It has the SO(1,2) invariance. Again we can parametrize dS_2

$$\begin{aligned} X &= L\, cosh\,\tilde{t}\, cos\,\theta \\ Y &= L\, cosh\,\tilde{t}\, sin\,\theta \\ Z &= L\, sinh\,\tilde{t}\,. \end{aligned} \tag{B.2.2}$$

The metric becomes

$$ds^2 = L^2\,(-\,d\tilde{t}^2 + cosh^2\tilde{t}\, d\theta^2)\,. \tag{B.2.3}$$

Notice that (see (B.1.1) and (B.2.1)) if we Euclideanize the Z-direction i.e, define $Z_E = i\,Z$ then

$$\mathrm{dS}_2 \xrightarrow{\text{Euclideanize}} \mathrm{S}^2\,.$$

Anti de Sitter spacetime (AdS_2)

AdS_2 can be represented by the surface which satisfies the following constraint in flat space with *two* timelike directions.

$$\begin{aligned} -Z^2 - X^2 + Y^2 &= -L^2 \\ ds^2 &= -dZ^2 - dX^2 + dY^2\,, \end{aligned} \tag{B.2.4}$$

where L is the AdS radius. One can parametrize AdS_2 by its intrinsic co-ordinates

$$\begin{aligned} Z &= L\, cosh\,\rho\, cos\,\tilde{t} \\ X &= L\, cosh\,\rho\, sin\,\tilde{t} \\ Y &= L\, sinh\,\rho\,. \end{aligned} \tag{B.2.5}$$

The metric reduces to

$$ds^2 = L^2\,(-cosh^2\rho\, d\tilde{t}^{\,2} + d\rho^2)\,. \tag{B.2.6}$$

AdS_2 has SO(2,1) invariance. Notice that the coordinate $\tilde{t}$ is a periodic variable $\tilde{t} \in (0, 2\pi]$ but $\frac{\partial}{\partial \tilde{t}}$ is the timelike direction. Therefore this is the closed timelike direction and this allows closed timelike curves. To avoid this pathology one goes to the universal cover of this space where $\tilde{t} \in (-\infty, \infty)$. We call this AdS spacetime.

Also note that (see (B.1.4) and (B.2.4)) if we Euclideanize the X-direction i.e, define $X_E = i\,X$ then

$$\mathrm{AdS}_2 \xrightarrow{\text{Euclideanize}} \mathrm{H}^2\,.$$

Actually one can easily generalize the metric to arbitrary dimensions with the following embedding.

$$\begin{aligned} -X_0^2 - X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 &= -L^2 \\ ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2\,. \end{aligned} \tag{B.2.7}$$

(For $d = 1$ one gets back the AdS_2.) AdS_{d+1} has SO(2,d) symmetry[3] which contains SO(d) as a subgroup. One can use the this rotational invariance to parametrize AdS_{d+1} with standard unit sphere (S^{p-1}) coordinates $\omega_i\,(i = 1, \ldots, p)$

$$\begin{aligned} X_0 &= L\,cosh\,\rho\;cos\,\tilde{t} \\ X_{p+2} &= L\,cosh\,\rho\;sin\,\tilde{t} \\ X_i &= L\,sinh\,\rho\;\omega_i\,. \end{aligned} \tag{B.2.8}$$

[3]It is worthwhile to note that this is the symmetry group of d-dimensional conformal field theories (CFT_d). This is a nice observation that hints towards the AdS/CFT correspondence

The metric becomes

$$ds^2 = L^2\,(-cosh^2\rho\, d\tilde{t}^2 + d\rho^2 + sinh^2\rho\, d\,\Omega_{d-1}^2)\,. \tag{B.2.9}$$

This metric is said to be in *global co-ordinates*. One can change variables to express the same spacetime in different co-ordinates.

AdS$_{d+1}$ in different co-ordinates

So far we have worked only with global co-ordinates (B.2.9). But people work with different co-ordinates depending on situations.

Static co-ordinates

Let's define $\tilde{r} \equiv \sinh\rho$. Then (B.2.9) reduces to

$$\frac{ds^2}{L^2} = -(\tilde{r}^2 + 1)\, d\tilde{t}^2 + \frac{d\tilde{r}^2}{r^2 + 1} + \tilde{r}^2\, d\,\Omega_{d-1}^2\,. \tag{B.3.1}$$

This coordinate is useful for studying AdS black holes.

Conformal co-ordinates

Define $tan\,\theta \equiv sinh\,\rho$. Here θ takes value between $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. The metric becomes conformally flat,

$$\frac{ds^2}{L^2} = \frac{1}{cos^2\theta}\left(-\,d\tilde{t}^2 + d\theta^2 + sin^2\theta\, d\,\Omega_{d-1}^2\right)\,. \tag{B.3.2}$$

Note that $\theta = \pm\frac{\pi}{2}$ represent the conformal boundary of the AdS spacetime. This is where the dual field theory 'lives'.

Poincaré co-ordinates

The Poincaré co-ordinates are defined as

$$
\begin{aligned}
X_0 &= \frac{L\,\bar{r}}{2}\left(\bar{x}_i^{\,2} - \bar{t}^2 + \frac{1}{\bar{r}^2} + 1\right) \\
X_{d+1} &= L\,\bar{r}\,\bar{t} \\
X_i &= L\,\bar{r}\,\bar{x}_i \\
X_d &= \frac{L\,\bar{r}}{2}\left(\bar{x}_i^{\,2} - \bar{t}^2 + \frac{1}{\bar{r}^2} - 1\right).
\end{aligned} \tag{B.3.3}
$$

The metric (B.2.9) reduces to

$$
\frac{ds^2}{L^2} = -\bar{r}^2\, d\bar{t}^2 + \frac{d\bar{r}^2}{\bar{r}^2} + \bar{r}^2\, d\,\vec{\bar{x}}^2_{d-1}\,. \tag{B.3.4}
$$

Note that all the co-ordinates namely $\bar{r}, \bar{t}, \vec{\bar{x}}$ are dimensionless. It is always useful to express metrics in dimensionful co-ordinates. Let's define the following dimensionful co-ordinates.

$$
r = L\,\bar{r}, \quad t = L\,\bar{t}, \quad \vec{x} = L\,\vec{\bar{x}}.
$$

The metric reduces to

$$
ds^2 = -\frac{r^2}{L^2}\,dt^2 + \frac{L^2}{r^2}\,dr^2 + \frac{r^2}{L^2}\,d\,\vec{x}^2_{d-1}\,. \tag{B.3.5}
$$

One can make one more change of variable $z = \frac{L^2}{r}$ to obtain

$$
ds^2 = \frac{L^2}{z^2}\,(-dt^2 + dz^2 + d\,\vec{x}^2_{d-1})\,. \tag{B.3.6}
$$

This is the most used co-ordinate system in the literature. This co-ordinate system makes the boundary Poincaré symmetry manifest (i.e, for any fixed value of $z = z_0$, say, the metric is that of Minkowski spacetime.). This is called a Poincaré patch because the coordinate doesn’t cover the whole AdS spacetime. Poincaré patch in Euclidean signature covers the entire AdS spacetime just like the global coordinates.

C $\mathcal{N} = 4$ super Yang-Mills

Here we briefly describe different features of $\mathcal{N} = 4$ super Yang-Mills theory. For more details see for example, [125, 126].

The Lagrangian

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions was written down for the first time in [127, 128] in 1976 by applying the method of dimensional reduction to $\mathcal{N} = 1$ super Yang-Mills in ten dimensions. For $\mathcal{N} = 4$ supersymmetric gauge theory in four dimensions the gauge multiplet is the only possible multiplet (unlike $\mathcal{N} = 1$ or $\mathcal{N} = 2$ theories) and is given by $(A_\mu, \Psi^a_\alpha, \Phi^i)$ where A_μ is a vector field, Ψ^a_α with $(a = 1, \dots 4)$ are Weyl spinors and Φ^i with $(i = 1 \dots 6)$ are real scalars. And the Lagrangian is given by

$$\begin{aligned}\mathcal{L} = tr\Bigg\{ &-\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - i \sum_a \bar{\Psi}_a \bar{\sigma}^\mu D_\mu \Psi_a - \sum_i D_\mu \Phi^i D^\mu \Phi^i \\ &+ \sum_{a,b,i} g\, C^{ab}_i \Psi_a [\Phi^i, \Psi_b] + \sum_{a,b,i} g\, \bar{C}_{iab} \bar{\Psi}^a [\Phi^i, \bar{\Psi}^b] + \frac{g^2}{2} \sum_{i,j} [\Phi^i, \Phi^j]^2 \Bigg\}, \end{aligned} \tag{C.1.1}$$

where g is the coupling constant, θ_I is the so-called instanton angle, $F_{\mu\nu}$ is the usual field strength of the gauge field, D_μ is the usual gauge-covariant derivative, $\tilde{F}$ is the Hodge dual of F, and C^{ab}_i are the structure constants of R-symmetry $SU(4)_R$. The trace is over the gauge indices to make the action gauge invariant. The full action is invariant under the $\mathcal{N} = 4$ supersymmetry transformations.

Dimensional reduction of $\mathcal{N} = 1$ SYM

The Lagrangian of $\mathcal{N} = 4$ SYM in $D = 4$ may look complicated but it arises naturally if one dimensionally reduce ten dimensional $\mathcal{N} = 1$ SYM to four dimensions and this is how it was first obtained [127, 128]. Let's start with that 10D theory

$$S_{10} = \int d^{10}x \, tr \left(-\frac{1}{2g^2} F_{MN}F^{MN} + i\, \Psi \Gamma^M D_M \Psi \right) , \tag{C.2.1}$$

where $M, N = 0, 1, \ldots, 9\,; \quad \mu, \nu = 0, 1, 2, 3$ and Ψ is a 16 component real (Majorana-Weyl) spinor. It has 8 degrees of freedom (DOF). The gauge field in 10 D also has 8 degrees of freedom, as it should be the case due to supersymmetry. Γ^M are the ten dimensional gamma matrices. Here the trace actually means

$$tr\, \Psi \not{D} \Psi \equiv \sum_{M=0}^{9} \sum_{b=1}^{N} \sum_{a=1}^{N} \sum_{B=1}^{N} \sum_{A=1}^{N} (\Psi_A)_{ab}\, \Gamma^M_{AB} \, (\partial_M \Psi_B + i\, [A_M, \Psi_B])_{ba} \ , \tag{C.2.2}$$

where M is a Lorentz index, A, B are number of spinors and a, b represent color indices.

Dimensionally reducing this theory from ten to four dimensions on a torus (T^6) involve the following steps.

- Nothing depends on co-ordinates $x^4, x^5, \ldots, x^9$. Therefore $\partial_{x_4}, \ldots, \partial_{x_9}$ must vanish.
- The gauge field should be decomposed as $A_M = (A_\mu, \Phi_i)$ and the gamma matrices as $\Gamma^M = (\gamma^\mu, \gamma^i)$.

If we follow the above 'rules'

$$\begin{aligned} F_{MN} &= \partial_\mu A_\nu - \partial_\nu A_\mu + i\,[A_\mu, A_\mu] + \cancelto{0}{\partial_i \Phi_j} - \cancelto{0}{\partial_j \Phi_i} \\ &\quad + i\,[\Phi_i, \Phi_j] + \partial_\mu \Phi_i - \cancelto{0}{\partial_i \Phi_\mu} + i[A_\mu, \Phi_i]\,, \\ &= F_{\mu\nu} + i[\Phi_i, \Phi_j] + D_\mu \Phi_i\,, \end{aligned}$$

$$-F_{MN}F^{MN} = -F_{\mu\nu}F^{\mu\nu} - D_\mu\Phi_i D^\mu\Phi_i + [\Phi_i,\Phi_j][\Phi_i,\Phi_j]\,, \tag{C.2.3}$$

$$\begin{aligned} i\bar{\Psi}\not{D}\Psi &= i\bar{\Psi}\gamma^\mu\left\{\partial_\mu\Psi + [A_\mu,\Psi]\right\} + i\,\bar{\Psi}\gamma^i\left\{\cancelto{0}{\partial_i\Psi} + [\Phi_i,\Psi]\right\} \\ &= i\bar{\Psi}\gamma^\mu\left\{\partial_\mu\Psi + [A_\mu,\Psi]\right\} + i\,\bar{\Psi}\gamma^i[\Phi_i,\Psi]\,. \end{aligned} \tag{C.2.4}$$

Adding RHS of (C.2.3) and (C.2.4) and keeping track of the coupling constants one obtains the $\mathcal{N} = 4$ SYM action in 4D (C.1.1). Notice that the γ^is which are in the compactified T^6 play the tole of Yukawa couplings in this theory.

Symmetries

The above Lagrangian is $\mathcal{N} = 4$ Poincaré supersymmetry invariant by construction. From dimensional analysis

$$\begin{gathered} [A_\mu] = [\Phi^i] = 1, \quad [\Psi_a] = 3/2\,, \\ \therefore \quad [g] = [\theta_I] = 0, \end{gathered}$$

we can see that the theory is also classically scale invariant theory since all fields are massless and there is no dimensionful parameter. It is well known that relativistic theories which posses both Poincaré symmetry and scale invariance are actually invariant under enhanced symmetry group called the conformal symmetry. For four spacetime dimensions the group is SO(2,4) ~ SU(2,2). Furthermore, the $\mathcal{N} = 4$ Poincaré supersymmetry and conformal invariance combines themselves to an even more larger symmetry group known as superconformal symmetry which is given by the supergroup SU(2,2|4). Remarkably the theory remains scale invariant even after quantization. Its β-function vanishes to all order in perturbation and it is believed to vanish even non-perturbatively. Thus the superconformal group SU(2, 2|4) is a symmetry even at the quantum mechanical level.

This UV finiteness makes $\mathcal{N} = 4$ SYM a very special quantum field theory.

In addition to superconformal symmetry as described above this theory enjoys a discrete global symmetry known as S-duality or Montonen-Olive duality (see chapter 2). This symmetry can be described better by combining the coupling constant (g) and instanton angle (θ_I) into a single complex coupling as follows

$$\tau \equiv \frac{\theta_I}{2\pi} + \frac{4\pi i}{g^2}\,.$$

The quantized theory is invariant under $\tau \to \tau + 1$. Montonen-Olive conjectured the theory to be invariant under a full SL(2,$\mathbb{Z}$) symmetry group which is realized as following.

$$\tau \to \frac{a\tau + b}{c\tau + d}\,.$$

where $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$. Note that when $\theta_I = 0$ this duality transformation relates $g \to \frac{1}{g}$ which is equivalent to exchanging weak coupling and strong coupling.

Phases

To study the phases of the theory one needs to analyze the potential energy term of $\mathcal{N} = 4$ SYM

$$V(\Phi) = -\frac{g^2}{2} \sum_{i,j=1}^{6} tr\,[\Phi^i, \Phi^j]^2\,. \tag{C.4.1}$$

This is a sum of positive terms. The ground state obtained when

$$[\Phi^i, \Phi^j] = 0\,.$$

This criterion can be satisfied by following two different ways.

- $\langle \Phi^i \rangle = 0$ for all $i = 1, ..., 6$.

  The gauge algebra and the superconformal symmetry SU(2, 2|4) are unbroken. This is known as *superconformal phase*.

- $\langle \Phi^i \rangle \neq 0$ for at least one i.

  Superconformal symmetry is spontaneously broken since the non-zero vacuum expectation value of $\langle \Phi^i \rangle$ sets a scale. This is known as *Coulomb phase*.

D Correlators in QFT

The central theme of this thesis is thermal Green functions and particularly computing them from gravity theory. Here we review some general well known properties about different Green functions that appear in usual QFT (see also [11]) .

In Minkowski space

Let $\hat{O}$ be a local, Bosonic operator in a finite temperature quantum field theory. Retarded and advanced propagators for $\hat{O}$ are defined by

$$G_R(k) = -i\int d^D x\, e^{-ik.x}\,\theta(t)\,\langle[\hat{O}(x),\hat{O}(0)]\rangle\,, \tag{D.1.1}$$

$$G_A(k) = i\int d^D x\, e^{-ik.x}\,\theta(-t)\,\langle[\hat{O}(x),\hat{O}(0)]\rangle\,. \tag{D.1.2}$$

Here $g^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. The different Green functions are not independent.

$$\begin{aligned}
G^R(k)^* &= i\int d^D x\, e^{ik.x}\,\theta(t)\,\langle[\hat{O}(x),\hat{O}(0)]\rangle^* \\
&= i\int d^D x\, e^{ik.x}\,\theta(t)\,\left\{\langle\hat{O}^\dagger(0)\,\hat{O}^\dagger(x)\rangle - \langle\hat{O}^\dagger(x)\,\hat{O}^\dagger(0)\rangle\right\} \\
&= -i\int d^D x\, e^{ik.x}\,\theta(t)\,\langle[\hat{O}(x),\hat{O}(0)]\rangle \quad (O'\text{s are Hermitian}) \\
&= G^R(-k)\,.
\end{aligned} \tag{D.1.3}$$

$$
\begin{aligned}
G^A(k) &= i \int d^D x\, e^{-ik.x}\, \theta(-t)\, \langle [\hat{O}(x), \hat{O}(0)] \rangle \\
&= i \int d^D x\, e^{-ik.x}\, \theta(-t)\, \langle [\hat{O}(0), \hat{O}(-x)] \rangle \quad \text{(space time translational invariance)} \\
&= i \int d^D x\, e^{ik.x}\, \theta(t)\, \langle [\hat{O}(0), \hat{O}(x)] \rangle \quad (x \to -x) \\
&= -\, i \int d^D x\, e^{ik.x}\, \theta(t)\, \langle [\hat{O}(x), \hat{O}(0)] \rangle \\
&= G^R(k)^* \,. \qquad \text{(D.1.4)}
\end{aligned}
$$

$$
\text{Therefore,} \quad G^R(k)^* = G^R(-k) = G^A(k)\,. \qquad \text{(D.1.5)}
$$

For *parity invariant* systems, $\operatorname{Re} G_{R,A}$ are even functions of $\omega \equiv k^0$ and $\operatorname{Im} G_{R,A}$ are odd functions of ω.

Let's consider symmetrized Wightman function

$$
G(k) = \frac{1}{2} \int d^D x\, e^{-ik.x}\, \langle \hat{O}(x)\, \hat{O}(0) + \hat{O}(0)\, \hat{O}(x) \rangle. \qquad \text{(D.1.6)}
$$

All other correlators can be written in terns of G^R, G^A and G. As an useful example, Feynman propagator

$$
\begin{aligned}
G_F(k) &= -i \int d^D x\, e^{-ik.x}\, \langle | T\{\hat{O}(x)\, \hat{O}(0)\} | \rangle \qquad \text{(D.1.7)} \\
&= \frac{1}{2}[G_R(k) + G_A(k)] - i\, G(k). \qquad \text{(D.1.8)}
\end{aligned}
$$

From the spectral representation of G_R and G one can show

$$
G(k) = -\coth \frac{\omega}{2T} \operatorname{Im} G^R(k). \qquad \text{(D.1.9)}
$$

For known $G^R(k)$ we can compute

$$
G_F(k) = \operatorname{Re} G_R(k) + i \coth \frac{\omega}{2T} \operatorname{Im} G_R(k). \qquad \text{(D.1.10)}
$$

So, at zero temperature (D.1.10) becomes

$$G_F(k) = \text{ Re } G_R(k) + i\ sgn\ (\omega)\ \text{ Im } G_R(k); \quad at\ T = 0. \tag{D.1.11}$$

Taking the limit $\omega \to 0$ in (D.1.9), we get another useful formula

$$G(0, \vec{k}) = -\lim_{\omega \to 0} \frac{2T}{\omega} \text{ Im } G_R(k) = 2iT \frac{\partial}{\partial \omega} G_R(\omega, \vec{k}) \Big|_{\omega = 0}. \tag{D.1.12}$$

In Euclidean space

In Euclidean space one has to normally deal with Matsubara propagator,

$$G_E\,(k_E) = \int d^D x_E\, e^{-ik_E.x_E}\, \langle\, T_E\, \{\hat{O}(x_E)\, \hat{O}(0)\}\rangle\,, \tag{D.2.1}$$

where T_E denotes Euclidean time ordering. The Matsubara propagators are defined only at discrete values of the frequency ω_E. For Bosonic $\hat{O}$ they are multiples of $2\pi T$. We can always relate the Euclidean and Minkowski propagators. The retarded propagator $G_R(k)$ (as a function of ω) can always be continued analytically to the whole upper half plane and at complex values of ω equal to $2\pi i\, T\, n$, reduces to the Euclidean propagator,

$$G_R(2\pi i\, T\, n,\, \vec{k}) = -\, G_E(2\pi\, T\, n,\, \vec{k}). \tag{D.2.2}$$

Similarly, if we analytically continue the advanced propagator to the lower half plane, gives Matsubara propagator at $\omega = -2\pi i T n$,

$$G_R(-2\pi i\, T\, n,\, \vec{k}) = -\, G_E(-2\pi\, T\, n,\, \vec{k}). \tag{D.2.3}$$

In particular, for $n = 0$ one gets

$$G_R(0, \vec{k}) = G_A(0, \vec{k}) = -\, G_E(0, \vec{k})\,. \tag{D.2.4}$$

E Euclidean correlators at zero temperature from AdS/CFT

Here we review the computation of Euclidean two point function of a CFT operator O using the AdS/CFT correspondence. This sample computation is for the sake of completeness and we closely follow [11].

From GKPW prescription

$$\langle e^{\int_{\partial M} \phi_0 O} \rangle = e^{-S_E[\phi]} \,, \tag{E.0.1}$$

where $S_E[\phi]$ is classical gravity action and ϕ_0 is boundary value of bulk field ϕ.

At $T = 0$, $M = \mathrm{AdS}_5 \times \mathrm{S}^5$ (no black hole in the bulk). Euclidean AdS_5 metric in Poincare patch is

$$ds_5^2 = \frac{L^2}{z^2} \, (dz^2 + d\vec{x}^{\,2}) \,, \tag{E.0.2}$$

where $\vec{x}$ are coordinates in R^4. The action of massive scalar field on this background

$$S_E = K \int d^4x \int_{z_B=\epsilon}^{z_H=\infty} dz \, \sqrt{g} \left[g^{zz} \, (\partial_z \phi)^2 + g^{\mu\nu} \, (\partial_\mu \phi) \, (\partial_\nu \phi) + m^2 \phi^2 \right] , \tag{E.0.3}$$

where $K = \frac{\pi L^5}{4 \kappa_{10}^2}$ and κ_{10} is ten dimensional gravitational constant.

$$S_E = K\int d^4x\int dz\left(\frac{L^2}{z^2}\right)^{\frac{5}{2}}\left\{\frac{z^2}{L^2}(\partial_z\phi)^2+\frac{z^2}{L^2}(\partial_i\phi)^2+m^2\phi^2\right\} \tag{E.0.4}$$

$$=\frac{\pi^3L^8}{4\kappa_{10}^2}\int dz\int d^4x\, z^{-3}\left((\partial_z\phi)^2+(\partial_i\phi)^2+\frac{L^2m^2}{z^2}\phi^2\right\}. \tag{E.0.5}$$

Using Fourier representation of the field

$$\phi(z,x)=\int\frac{d^4k}{(2\pi)^4}\,e^{ik.x}\,f_k(z)\,\phi_0(k)\,, \tag{E.0.6}$$

and integrating over $\vec{x}$ coordinates the action reduces to

$$S_E = 4\kappa_{10}^2\pi^3L^8\int dz\int\frac{d^4k}{(2\pi)^4}\frac{1}{z^3}\left[\partial_zf_k\,\partial_zf_{-k}+k^2f_k\,f_{-k}+\frac{L^2m^2}{z^2}f_k\,f_{-k}\right]\phi_0(k)\,\phi_0(-k)\,. \tag{E.0.7}$$

f_k satisfies the EOM

$$f_k''(z)-\frac{3}{z}f_k'(z)-\left(k^2+\frac{m^2L^2}{z^2}\right)=0\,. \tag{E.0.8}$$

This EOM can be solved exactly and the general solution is

$$\phi_k(z)=A\,z^2\,I_\nu(kz)+B\,z^2\,I_{-\nu}(kz)\,, \tag{E.0.9}$$

where $\nu=\sqrt{4+m^2L^2}$ and $I_\nu(kz)$ is modified Bessel functions of first kind.

The solution should be regular at $z=\infty$ and equals to 1 at $z=\epsilon$. Therefore

$$f_k(z)=\frac{z^2\,K_\nu(kz)}{\epsilon^2\,K_\nu(k\epsilon)}\,. \tag{E.0.10}$$

On shell, the action reduces to the boundary term

$$S_E = \frac{\pi^3 L^8}{4\,\kappa_{10}^2} \int \frac{d^4k\, d^4k'}{(2\pi)^8}\, \phi_0(k)\, \phi_0(k')\, \mathscr{F}(z,k,k')\Big|_{\epsilon}^{\infty} . \tag{E.0.11}$$

The two point function is given by

$$\begin{aligned}
\langle O(k)\, O(k')\rangle =& Z^{-1} \frac{\delta^2 Z[\phi_0]}{\delta\phi_0(k)\, \delta\phi_0(k')}\Big|_{\phi_0=0} \\
=& -2\, \mathscr{F}(z,k,k')\Big|_{\epsilon}^{\infty} \\
=& -(2\pi)^4\, \delta^4(k+k')\, \frac{\pi^3 L^8}{2\kappa_{10}^2}\, \frac{f_{k'}(z)\, \partial_z f_k(z)}{z^3}\Big|_{\epsilon}^{\infty} .
\end{aligned} \tag{E.0.12}$$

From (E.0.11)

$$\langle O(k)\, O(k')\rangle = -\frac{\pi^3 L^8}{2\kappa_{10}^2}\, \epsilon^{2(\Delta-d)}\, (2\pi)^4\, \delta^4(k+k')\, k^{2\nu}\, 2^{1-2\nu}\, \frac{\Gamma(1-\nu)}{\Gamma(\nu)} + \ldots \tag{E.0.13}$$

where dots denote terms analytic in k and / or those vanishing in the $\epsilon \to 0$ limit. Substituting the value $\kappa_{10} = 2\pi^{\frac{5}{2}} L^4/N$ from [53] ,

$$\langle O(k)\, O(k')\rangle = -\frac{N^2}{8\pi^2} \epsilon^{2(\Delta-4)} (2\pi)^4 \delta^4(k+k') \frac{k^{2\Delta-4}\, \Gamma(3-\Delta)}{2^{2\Delta-5}\, \Gamma(\Delta-2)} . \tag{E.0.14}$$

For integer Δ the propagator,

$$\langle O(k)\, O(k')\rangle = -\frac{(-1)^{\Delta}}{(\Delta-3)!}\, \frac{N^2}{8\pi^2} (2\pi)^4\, \delta^4(k+k')\, \frac{k^{2\Delta-4}}{2^{2\Delta-5}}\, \ln k^2 . \tag{E.0.15}$$

For massless case ($\Delta = 4$),

$$\langle O(k)\, O(k')\rangle = -\frac{N^2}{64\,\pi^4}\, (2\pi)^4\, \delta^4(k+k')\, k^4\, \ln k^2 . \tag{E.0.16}$$

F Associated Legendre differential equation and its solutions

The associated Legendre differential equation is a generalization of the Legendre differential equation and is given by

$$\frac{d}{dz}\left[(1-z^2)\frac{dy}{dz}\right]+\left[\lambda(\lambda+1)+\frac{\mu^2}{1-z^2}\right]y=0\,, \tag{F.0.1}$$

which can be written

$$(1-z^2)y''-2zy'+\left[\lambda(\lambda+1)-\frac{\mu^2}{1-z^2}\right]y=0\,. \tag{F.0.2}$$

P_λ^μ and Q_λ^μ are the two linearly independent solutions to the associated Legendre D.E. The solutions P_λ^μ to this equation are called the associated Legendre polynomials (if λ is an integer), or associated Legendre functions of the first kind (if is not an integer). Similarly, Q_λ^μ is a Legendre function of the second kind. These functions may actually be defined for general complex parameters and argument. In particular they can be expressed in terms of hypergeometric functions and gamma functions

$$P_\lambda^\mu(z)=\frac{1}{\Gamma(1-\mu)}\left[\frac{1+z}{1-z}\right]^{\mu/2}{}_2F_1\left(-\lambda,\lambda+1;1-\mu;\frac{1-z}{2}\right) \tag{F.0.3}$$

$$Q_\lambda^\mu(z)=\frac{\sqrt{\pi}\,\Gamma(\lambda+\mu+1)}{2^{\lambda+1}\Gamma(\lambda+3/2)}\frac{1}{z^{\lambda+\mu+1}}\left(1-z^2\right)^{\frac{\mu}{2}}{}_2F_1\left(\frac{\lambda+\mu+1}{2},\frac{\lambda+\mu+2}{2};\lambda+\frac{3}{2};\frac{1}{z^2}\right). \tag{F.0.4}$$

For the EOM of the string (4.3.15) the general solution will be

$$f_\omega(s) = C_1 \frac{P_1^{i\mathfrak{w}}}{s} + C_2 \frac{Q_1^{i\mathfrak{w}}}{s} \,. \tag{F.0.5}$$

From above expressions (F.0.3) and (F.0.4) we get,

$$P_1^{i\mathfrak{w}}(s) = \frac{1}{\Gamma(1-i\mathfrak{w})} \left[\frac{1+s}{1-s}\right]^{i\mathfrak{w}/2} {}_2F_1\left(-1, 2; 1-i\mathfrak{w}; \frac{1-s}{2}\right)$$

$$Q_1^{i\mathfrak{w}}(s) = \frac{\sqrt{\pi}\,\Gamma(2+i\mathfrak{w})}{4\Gamma(5/2)} \frac{1}{s^{(2+i\mathfrak{w})}} \left(1-s^2\right)^{\frac{i\mathfrak{w}}{2}} {}_2F_1\left(\frac{2+i\mathfrak{w}}{2}, \frac{3+i\mathfrak{w}}{2}; \frac{5}{2}; \frac{1}{s^2}\right) .$$

Near the horizon $f(s) = 1 - \frac{1}{s^2} \to 0$ i.e, $s \to 1$. So the dominant behavior of the solution near the horizon will be of the form,

$$f_\omega(s) \sim (s-1)^{\pm\alpha} \,.$$

From above two solutions it is evident that $\frac{P_1^{i\mathfrak{w}}(s)}{s} \sim (1-s)^{-i\mathfrak{w}/2}$ and $\frac{Q_1^{i\mathfrak{w}}(s)}{s} \sim (1-s)^{i\mathfrak{w}/2}$. Now,

$$e^{-i\omega t}\,(1-s)^{-i\mathfrak{w}/2} \sim e^{-i\omega\left\{t + \frac{1}{2\pi T}\, ln(s-1)\right\}} \,.$$

Notice that near the horizon $s \to 1$, $ln(s-1)$ goes more and more negative. Now to keep the phase fixed t must increase. That means this wave moves towards the horizon with increment of time. So, $\frac{P_1^{i\mathfrak{w}}}{s} \sim (1-s)^{-i\mathfrak{w}/2}$ is the desired *incoming wave* solution. And by the same token $\frac{Q_1^{i\mathfrak{w}}}{s} \sim (1-s)^{i\mathfrak{w}/2}$ is the *outgoing wave* solution. To get retarded propagator one has to choose $\frac{P_1^{i\mathfrak{w}}}{s}$.

G Bulk to bulk correlators

If we have retarded Green function in the boundary theory we can always construct other Green functions. As we know the solution to the string EOM exactly we can build the exact retarded bulk-to-bulk correlator, $G_{\text{ret}}(\omega, s, \tilde{s})$ which is in-falling at the horizon and normalizable at the boundary. In section 4.3 we had $f_\omega(s)$ as a solution to the wave equation (4.3.14) and so was $f^*_\omega(s)$. As it was a linear differential equation any linear combination of them e.g, $\text{Im} f_\omega(s) = \frac{f_\omega(s) - f^*_\omega(s)}{2i}$ is also a solution. But we had chosen them such that $f_\omega(s)$ and $f^*_\omega(s) \to 1$ as $s \to \infty$. Therefore $\text{Im} f_\omega(s)$ is a normalizable solution to that wave equation (4.3.14). Thus the retarded bulk-to-bulk correlator is defined as

$$G_{\text{ret}}(\omega, s, \tilde{s}) = \frac{\text{Im} f_\omega(s) f_\omega(\tilde{s}) \theta(s, \tilde{s}) + f_\omega(s) \text{Im} f_\omega(\tilde{s}) \theta(\tilde{s}, s)}{T_0(\tilde{s}) W_{\text{ret}}(\tilde{s})}\,, \tag{G.0.1}$$

$$W_{\text{ret}}(\tilde{s}) \equiv \text{Im} f'_\omega(\tilde{s}) f_\omega(\tilde{s}) - f'_\omega(\tilde{s}) \text{Im} f_\omega(\tilde{s})\,. \tag{G.0.2}$$

Now from equation (4.3.17) (taking $s_B \to \infty$) we obtain the Wronskian as

$$W_{\text{ret}}(\tilde{s}) = \frac{\mathfrak{w}^3 + \mathfrak{w}}{\tilde{s}^2 - \tilde{s}^4}\,. \tag{G.0.3}$$

The interesting thing to notice that the Wronskian depends on $\tilde{s}$ in such a way that $T_0(\tilde{s}) = \frac{\sqrt{\lambda}\pi^2 T^3}{2}\, \tilde{s}^2(\tilde{s}^2 - 1)$ cancels that $\tilde{s}$-dependence. Therefore the denominator of (G.0.1)

$$T_0(\tilde{s}) W_{\text{ret}}(\tilde{s}) = \; -\frac{1}{2}\pi^2 \sqrt{\lambda}\, T^3 \left(\mathfrak{w}^3 + \mathfrak{w}\right) = \; + \text{Im}\, G_R(\omega)\,, \tag{G.0.4}$$

becomes independent of $\tilde{s}$. We can write (G.0.1) as

$$G_{\text{ret}}(\omega, s, \tilde{s}) \equiv G^{+}_{\text{ret}}(\omega, s, \tilde{s})\, \theta(s, \tilde{s}) + G^{-}_{\text{ret}}(\omega, s, \tilde{s})\, \theta(\tilde{s}, s)\,. \tag{G.0.5}$$

Finally using (4.3.17) and (G.0.4) we obtain

$$\begin{aligned} G^{+}_{\text{ret}}(\omega, s, \tilde{s}) = \left(1 - s^2\right)^{-\frac{i\omega}{2\pi T}} \left(\frac{1+\tilde{s}}{1-\tilde{s}}\right)^{\frac{i\omega}{2\pi T}} (\pi \tilde{s} T - i\omega)\, e^{-\frac{\omega}{T}} \\ \frac{(\omega + i\pi s T)\,(1+s)^{\frac{i\omega}{\pi T}} + e^{\omega/T}\,(1-s)^{\frac{i\omega}{\pi T}}\,(\omega - i\pi s T)}{\pi\sqrt{\lambda} s \tilde{s} T^2 \omega\, \left(\pi^2 T^2 + \omega^2\right)}\,, \end{aligned} \tag{G.0.6}$$

$$\begin{aligned} G^{-}_{\text{ret}}(\omega, s, \tilde{s}) = \left(1 - \tilde{s}^2\right)^{-\frac{i\omega}{2\pi T}} \left(\frac{1+s}{1-s}\right)^{\frac{i\omega}{2\pi T}} (\pi s T - i\omega)\, e^{-\frac{\omega}{T}} \\ \frac{(\omega + i\pi \tilde{s} T)\,(1+\tilde{s})^{\frac{i\omega}{\pi T}} + e^{\omega/T}\,(1-\tilde{s})^{\frac{i\omega}{\pi T}}\,(\omega - i\pi \tilde{s} T)}{\pi\sqrt{\lambda} s \tilde{s} T^2 \omega\, \left(\pi^2 T^2 + \omega^2\right)}\,. \end{aligned} \tag{G.0.7}$$

H Perturbative solution in BTZ

We already mentioned that EOM for a string in BTZ can be solved exactly and hence one can compute exact retarded Green's function for the Brownian particle. Actually we have done the same in 5.3.2. There we have seen that the zero temperature dissipation coefficient is $\dfrac{\sqrt{\lambda}}{2\pi}$. On the other hand, the EOM of string in AdS$_5$-BH is not exactly solvable and therefore we adopted a perturbative technique to compute the above mentioned coefficient. In 5.3.2 we got a different value for that coefficient. In this section we apply the same perturbative method for a string in BTZ and show that the same result for zero temperature dissipation is reproduced. This is just to show that the perturbative approach and the associated limits indeed work.

Our aim is to perturbatively solve (5.3.4) which in z-coordinate looks

$$f''_\omega(z) + \frac{2}{z(1-4\pi^2T^2z^2)}f'_\omega(z) + \frac{\omega^2}{(1-4\pi^2T^2z^2)^2}f_\omega(z) = 0\,, \tag{H.0.1}$$

using the following ansatz

$$f^R_\omega(z) = \left(1-4\pi^2T^2z^2\right)^{-i\frac{\Omega}{2}}\left(1 - i\Omega h_1(z) - \Omega^2 h_2(z) + i\Omega^3 h_3(z) + \ldots\right), \tag{H.0.2}$$

where in this section $\Omega \equiv \frac{\omega}{2\pi T}$ (notice the extra factor of two).

Again we will be working in the regime where

$$\omega, T \to 0 \;\text{ with }\; \Omega \text{ held fixed (and small).}$$

Next step is to solve for the unknown functions, $h_1(x), h_2(x), h_3(x)$ etc. recursively. We repeat the same procedure as in (5.3.2).

Solution up to $O(\Omega)$:

$$f_\omega(z) = \left(1 - 4\pi^2 T^2 z^2\right)^{-i\frac{\Omega}{2}} (1 - i\Omega h_1(z)) . \tag{H.0.3}$$

Substituting this ansatz into (H.0.1) we obtain the differential equation for $h_1(z)$

$$h_1''(z) - \underbrace{\frac{2}{z(1 - 4\pi^2 T^2 z^2)}}_{p_1(z)} h_1'(z) = -\underbrace{\frac{4\pi^2 T^2}{(1 - 4\pi^2 T^2 z^2)}}_{q_1(z)} . \tag{H.0.4}$$

Let's cast this into a first order differential equation defining $y_1(z) \equiv f_1'(z)$ and consequently $y_1'(z) \equiv f_1''(z)$

$$y_1'(z) + p_1(z) y_1(z) = q_1(z) . \tag{H.0.5}$$

Introducing the integrating factor $I_1(z) = \exp\left(\int p_1(z) \mathrm{d}z\right)$

$$\begin{aligned} y_1(z) &= \frac{c_1}{I_1(z)} + \frac{1}{I_1(z)} \int^z I_1(x) q_1(x) \mathrm{d}x \\ &= \underbrace{\frac{c_1 z^2}{1 - 4\pi^2 T^2 z^2}}_{y_1^h(z)} + \underbrace{\frac{4\pi^2 T^2 z}{1 - 4\pi^2 T^2 z^2}}_{y_1^p(z)} . \end{aligned} \tag{H.0.6}$$

The homogeneous part of the solution ($y_i^h(z)$) will again be identical in each order in Ω up to the undetermined coefficient (c_i). This coefficient is fixed by demanding the regularity of h_i at the horizon,

$$h_1(z) = \int y_1^h(z) \mathrm{d}z + \int y_1^p(z) \mathrm{d}z \tag{H.0.7}$$

$$\equiv h_1^h(z) + h_1^p(z) . \tag{H.0.8}$$

Demanding regularity for $h_1(z)$ fixes the coefficient of $Log(1 - 2\pi Tz)$ to zero. One can explicitly calculate the integrals and from that expression sort out the required coefficient. For this case

$$h_1^h(z) = c_1\left[-\frac{z}{4\pi^2T^2} + \frac{1}{16\pi^3T^3}\left\{Log(1+2\pi Tz) - Log(1-2\pi Tz)\right\}\right], \tag{H.0.9}$$

$$\begin{aligned} h_1^p(z) &= -\frac{1}{2}Log(1-4\pi^2T^2z^2) \\ &= -\frac{1}{2}\left\{Log(1+2\pi Tz) + Log(1-2\pi Tz)\right\} . \end{aligned} \tag{H.0.10}$$

Clearly setting the coefficient of $Log(1 - 2\pi Tz)$ to zero we get

$$c_1 = -\,8\pi^3T^3 .$$

And the solution at this order becomes

$$h_1(z) = \frac{1}{2}\left[4\pi Tz - 2Log(1+2\pi Tz)\right] . \tag{H.0.11}$$

As has been argued before we can equivalently set the residue of $y_1(z)$ at $z = \frac{1}{2\pi T}$ to zero to fix the value of c_1.

Solution up to $O(\Omega^2)$:

$$f_\omega(z) = \left(1 - 4\pi^2T^2z^2\right)^{-i\frac{\Omega}{2}}\left(1 - i\Omega h_1(z) - \Omega^2h_2(z)\right), \tag{H.0.12}$$

where $h_1(z)$ is already known from (H.0.11). The differential equation for $h_2(z)$ reduces to

$$y_2'(z) + p_2(z)y_2(z) = q_2(z)\,, \tag{H.0.13}$$

where $p_2(z) = p_1(z)$ and integrating factor $I_2(z) = I_1(z)$.

$$
\begin{aligned}
y_2(z) &= \frac{c_2}{I_2(z)} + \frac{1}{I_2(z)} \int^z I_2(x) q_2(x) \mathrm{d}x \\
&= \underbrace{\frac{c_2 z^2}{1 - 4\pi^2 T^2 z^2}}_{y_2^h(z)} + \underbrace{\frac{1}{I_2(z)} \int^z I_2(x) q_2(x) \mathrm{d}x}_{y_2^p(z)} \,.
\end{aligned} \tag{H.0.14}
$$

Now making the residue of $y_2(z)$ at $z = \frac{1}{2\pi T}$ to vanish we can fix

$$
c_2 = 8\pi^3 T^3 \,.
$$

The solution at this order

$$
h_2(z) = \frac{1}{2} \left[-4\pi T z + \{Log(1 + 2\pi T z)\}^2 \right] . \tag{H.0.15}
$$

<u>Solution up to $O(\Omega^3)$</u> :

$$
f_\omega(z) = \left(1 - 4\pi^2 T^2 z^2\right)^{-i\frac{\Omega}{2}} (1 - i\Omega h_1(z) - \Omega^2 h_2(z) + i\Omega^3 h_3(z)) \,, \tag{H.0.16}
$$

where h_1, h_2 are known from (H.0.11) and (H.0.19). The differential equation for $h_3(z)$ or rather $y_3(z) \equiv h_3'(z)$

$$
y_3'(z) + p_3(z) y_2(z) = q_3(z) \,, \tag{H.0.17}
$$

where $p_3(z) = p_1(z)$ and thus integrating factor $I_3(z) = I_1(z)$, as before.

$$
\begin{aligned}
y_3(z) &= \frac{c_3}{I_3(z)} + \frac{1}{I_3(z)} \int^z I_3(x) q_3(x) \mathrm{d}x \\
&= \underbrace{\frac{c_3 z^2}{1 - 4\pi^2 T^2 z^2}}_{y_3^h(z)} + \underbrace{\frac{1}{I_3(z)} \int^z I_3(x) q_3(x) \mathrm{d}x}_{y_3^p(z)} \,.
\end{aligned} \tag{H.0.18}
$$

Fixing residue of $y_3(z)$ at $z = \frac{1}{2\pi T}$ to vanish we obtain

$$c_3 = 0\,.$$

The functional form of $h_3(z)$ is simple unlike the higher dimensional case,

$$h_3(z) = \pi T z\{Log(1+2\pi T z)\}^2 - \frac{1}{6}\{Log(1+2\pi T z)\}^3\,. \tag{H.0.19}$$

Therefore in this perturbative expansion the full solution becomes

$$\begin{aligned} f^R_\omega(z) = \left(1-4\pi^2T^2z^2\right)^{-i\frac{\Omega}{2}}\Big[1 - i\Omega(\frac{1}{2}(4\pi T z - 2Log(1+2\pi T z))) - \Omega^2(\frac{1}{2}(-4\pi T z + \{Log(1+2\pi T z)\}^2)) \\ + i\Omega^3(\pi T z\{Log(1+2\pi T z)\}^2 - \frac{1}{6}\{Log(1+2\pi T z)\}^3)\Big]\,. \end{aligned} \tag{H.0.20}$$

Evidently in the zero temperature limit (with very small frequency)

$$f^R_\omega(z)\Big|_{T\to 0} = 1 + \frac{\omega^2 z^2}{2} + i\frac{\omega^3 z^3}{3}\,. \tag{H.0.21}$$

In r co-ordinate

$$f^R_\omega(r)\Big|_{T\to 0} = 1 + \frac{\omega^2 L^4}{2r^2} + i\frac{\omega^3 L^6}{3r^3}\,. \tag{H.0.22}$$

The retarded Green's function at $T = 0$ can be calculated using this solution

$$\begin{aligned} G^0_R &\equiv \lim_{r\to r_B} T_0(r)\, f^R_{-\omega}(r)\,\partial_r f^R_\omega(r) \\ &= \lim_{r\to r_B} \frac{1}{2\pi l_s^2}\frac{r^4}{L^4}\left(-\frac{\omega^2 L^4}{r^3} - i\frac{\omega^3 L^6}{r^4}\right) \\ &= -\frac{\mu\omega^2}{2\pi} - i\frac{\sqrt{\lambda}}{2\pi}\,\omega^3\,. \end{aligned} \tag{H.0.23}$$

Thus the renormalized Green's function

$$G_R(\omega) \equiv G_R^0 + \frac{\mu\omega^2}{2\pi} = -i\frac{\sqrt{\lambda}}{2\pi}\,\omega^3\,. \tag{H.0.24}$$

This matches identically with the leading term in small frequency expansion of (5.3.8).

I Fixing the coefficient $b_-^{(0)}$

A field $\psi(z)$ satisfying a second order linear homogeneous differential equation

$$\psi''(z) + P(z)\,\psi'(z) + Q(z)\,\psi(z) = 0\,, \tag{I.0.1}$$

where $P(z)$ and $Q(z)$ are real the generalized Wronskian is defined as

$$W(\psi_1,\psi_2;z) := e^{\int^z P(t)dt}[\psi_1\partial_z\psi_2 - \psi_2\partial_z\psi_1] \tag{I.0.2}$$

$$= \sqrt{-g}g^{zz}[\psi_1\partial_z\psi_2 - \psi_2\partial_z\psi_1]\,, \tag{I.0.3}$$

where ψ_1 and ψ_2 are two solutions of (I.0.1). The interesting fact about this $W(z)$ is it is independent of z

$$\partial_z W(\psi_1,\psi_2;z) = 0\,.$$

Therefore we can write, $\quad W(\psi_1,\psi_2;z) \equiv W(\psi_1,\psi_2)\,.$

Equation (6.3.5) is exactly of the form (I.0.1). We know how its two independent solutions behave at the horizon ($z = z_*$) and at the boundary ($z = 0$). The generalized Wronskian

$$W(\psi_1,\psi_2) = \left(\frac{L_{d+1}^2}{z^2}\right)^{\frac{d+1}{2}} \frac{f(z)}{L_{d+1}^2} z^2(\psi_1\partial_z\psi_2 - \psi_2\partial_z\psi_1) \tag{I.0.4}$$

$$= \left(\frac{L_{d+1}}{z}\right)^{d-1} (\psi_1\partial_z\psi_2 - \psi_2\partial_z\psi_1)\,, \tag{I.0.5}$$

is independent of z.

$$\therefore \quad W(\psi_1, \psi_2)\Big|_{z=0} = W(\psi_1, \psi_2)\Big|_{z=z_*} . \tag{I.0.6}$$

For extremal case,

$$f(z) = 1 + \frac{d}{d-2}\left(\frac{z}{z_*}\right)^2 - \frac{2d-2}{d-2}\left(\frac{z}{z_*}\right)^d .$$

- $f(z)\Big|_{z=0} = 1$.
- $f(z)\Big|_{z=z_*} \approx d(d-1)\,\frac{(z_*-z)^2}{z_*^2}$.

$$\begin{aligned} \text{LHS of (I.0.6)}: \quad W(\psi_1, \psi_2)\Big|_{z=0} &= \frac{1}{z^2}(\eta_+^{(0)}\partial_z\eta_-^{(0)} - \eta_-^{(0)}\partial_z\eta_+^{(0)}) \\ &= \frac{3}{z_*^3}(a_+^{(0)}b_-^{(0)} - a_-^{(0)}b_+^{(0)}) . \end{aligned} \tag{I.0.7}$$

$$\begin{aligned} \text{RHS of (I.0.6)}: \quad W(\psi_1, \psi_2)\Big|_{z=z_*} &= \frac{d(d-1)(z_*-z)^2}{z^2 z_*^2}(\eta_+^{(0)}\partial_z\eta_-^{(0)} - \eta_-^{(0)}\partial_z\eta_+^{(0)}) \\ &= \frac{d(d-1)(z_*-z)^2}{z^2 z_*^2}\left[1.\, \partial_z\left(\frac{\zeta}{z_*}\right) - \left(\frac{\zeta}{z_*}\right)\partial_z(1)\right] \\ &= \frac{1}{z_*^3} . \end{aligned} \tag{I.0.8}$$

Equating (I.0.7) and (I.0.8)

$$a_+^{(0)}b_-^{(0)} - a_-^{(0)}b_+^{(0)} = \frac{1}{3} ,$$

and substituting $b_+^{(0)} = 0$ and $a_+^{(0)} = 1$, we get

$$b_-^{(0)} = \frac{1}{3} .$$

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