Physics of Gravitational Waves in Presence of Positive Cosmological Constant

By

SK JAHANUR HOQUE

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Sk Jahanur Hoque

Dedicated to

The people who inspire me

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LIST OF PUBLICATIONS ARISING FROM THE THESIS

- **Gravitational waves from compact sources in a de Sitter background**

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- **Cosmological Horizon and the Quadrupole Formula in de Sitter background**

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1. 18^{th} January, 2017 ***Cosmological horizon and energy flux in de Sitter background***, The Institute of Mathematical Sciences, India.

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ABSTRACT

Cosmological observations over past couple of decades favour our universe with a tiny positive cosmological constant. Even a tiniest value of cosmological constant, profoundly alters the asymptotic structure of space-time. This suggests that weak gravitational waves should be re-looked at as ripples on the de Sitter background. The thesis deals with different aspects of weak gravitational waves on the de Sitter background. We obtain the gravitational radiation field of compactly supported source. This is given by gauge invariant part of retarded solution, expressed in terms of source multi-pole moments. Employing suitably chosen Fermi normal coordinates in static patch of de Sitter background, we compute the field to the first order in Λ. For contrast, we also present the field in Poincaré patch where the leading correction is in $\sqrt{\Lambda}$. We introduce a gauge invariant quantity, *deviation scalar*, containing polarization information of gravitational waves and compute it in both charts for a comparison of the two fields. The deviation scalar is not the same for these fields and this is attributed to the differently defined source moments in the two charts. Another result is the derivation of power radiated quadrupolar formula for $\Lambda > 0$. Employing Isaacson's effective gravitational stress tensor for rapidly varying source, we obtain quadrupole formula in de Sitter background. Using the Isaacson prescription we show that energy flux of gravitational waves measured at $\mathscr{J}^+$ is the same as that measured across cosmological horizon of compact source. The expression for Isaacson flux also matches with that given by Ashtekar et. al (in covariant phase space formalism) at a coarse grained level. In the last part of the thesis we employ modified quadrupole formula to estimate orbital decay rate of circular orbit in de Sitter background. It confirms the expectation that for compact binary systems, corrections due to cosmological constant are negligible.

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Synopsis

Recently gravitational waves were detected directly for first time, 100 years after their existence was originally predicted by Einstein. Einstein's general theory of relativity postulates that space-time is a dynamic entity on which 'geometry tells matter how to move and matter tells geometry how to curve'. Gravitational waves are ripples on the fabric of space-time which are caused by motion of matter and propagate at the speed of light. Though there was initial controversy about physical existence of gravitational wave itself, Bondi-Sachs [1, 2] provided a detailed analysis of gravitational waves in full non-linear theory along with a definition of energy that is carried away from an isolated system by gravitational radiation. Bondi introduced the term 'News function' - whose absolute square integrated over the sphere at infinity, measures the rate of energy loss by an isolated system. Later Penrose geometrized the results due to Bondi-Sachs via conformal completion technique, identifying Ψ_4^0[1] as radiation field in an 'origin independent' way [3]. Their foundational framework was established in the context of asymptotic flat space-time ($\Lambda = 0$). However by now cosmological observations (e.g. red shift of type Ia supernovae) have established that the universe is undergoing an accelerated expansion which is best explained by a positive Λ. Hence it is reasonable to ask whether the description of gravitational field due to compact isolated systems and the characterization of gravitational radiation obtained for $\Lambda = 0$ can be generalized for $\Lambda > 0$ (de-Sitter). Even a tiniest value of cosmological constant qualitatively changes the structure of null-infinity ($\mathscr{J}^+$) from being a null hypersurface to space-like. Space-like character of $\mathscr{J}^+$

[1] $1/r$ part of $\Psi_4(:= C_{\mu\nu\rho\sigma}n^\mu\bar{m}^\nu n^\rho\bar{m}^\sigma)$ which remains constant along the out-going null geodesic

poses challenges for defining energy, momentum and their fluxes. As emphasized by Penrose the concept of radiation is 'less invariant' in cases when $\mathcal{J}^+$ does not have a null character. Namely the radiative component Ψ_4^0 of the field, may differ for different null geodesics approaching the same point on $\mathcal{J}^+$. Hence, we do not have an invariant notion of gravitational radiation in the non-linear regime, nor an analogue of the 'News function' for asymptotically de Sitter space-time. For time being let us keep aside all issues of full non-linear theory, we will ask simpler question : how does linearized theory work out in de Sitter background? As linearized theory is a good model to study 'far zone' and asymptotic behaviour of compact gravitating system, we hope this may provide some clue in full non-linear analysis.

The next level of question is to identify the physical (gauge invariant) attributes of the wave solution. In the Minkowski background, the linearized Riemann tensor is gauge invariant and consequently the induced geodesic deviation or tidal distortion is a physical effect of the waves. We ask what is the analogous statement of gauge invariant geodesic deviation in de Sitter background? Since gravitational waves are capable of doing work, we could also ask for a measure of the energy carried by the waves. In particular, what are the modifications to the quadrupole formula for radiated power? Can the modifications be obtained as 'small' corrections in powers of the cosmological constant?

Presence of cosmological constant not only imposes theoretical challenges, it has also observational relevance. In the vicinity of astrophysical sources or near ground-based gravitational wave detectors, one can neglect Λ. Signature of Λ may be crucial over the vast distances of source-free regions in which gravitational waves propagate. To get an overview of our region of interest, let us take the example of Schwarzschild-de Sitter universe. In static coordinates, Schwarzschild-de Sitter metric is given by,

$$ds^2 = -\Big(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\Big)\, dt^2 + \frac{dr^2}{\Big(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\Big)} + r^2 d\Omega_2^2 \tag{0.0.1}$$

For $9M^2\Lambda < 1$, the factor $(1 - 2M/r - \Lambda r^2/3)$ is zero at two positive values of r. The smaller of of these values, which we shall denote by r_H can be regarded as the position of the black-hole event horizon, while the larger value r_C represents the position of the cosmological event horizon for observers on world lines of constant r between r_H and r_C. Near the source we have $\frac{\Lambda r^2}{3} \ll \frac{2M}{r}$ and far away from the source we can assume $\frac{2M}{r} \ll 1$. Hence in this particular example our region of interest is

$$2M \ll r \ll \left(\frac{6M}{\Lambda}\right)^{\frac{1}{3}} \tag{0.0.2}$$

It would be more interesting to know if there are significant observable differences, particularly in the light of newly opened window of gravitational wave astronomy. Does Λ induce any significant change in the phase calculation of gravitational waveform template?

This thesis deals with these issues. The first chapter is devoted in obtaining linearized solution in terms of source moments. We present the solution in two different co-ordinate systems, namely Fermi normal co- ordinate (in static patch) and conformal co-ordinate (in future Poincaré patch) of de Sitter background. A gauge invariant quantity, 'deviation scalar' is constructed for comparison. In second chapter we derive quadrupole formula for radiated power in presence of Λ and argue in what context cosmological horizon may be treated as effective infinity of de Sitter. Last chapter contains a summary of results and a discussion of open problems. Requisite mathematical information is included in appendices.

0.1 Weak Gravitational Waves from Compact Sources in de Sitter Background

In this chapter we focus on linearized gravitational waves generated by rapidly varying, distant, spatially compact source on de Sitter background and the induced tidal distortion.

Even for compact astrophysical sources, the extension of $\Lambda = 0$ analysis introduces many nontrivial issues. The background being curved there is no natural choice of global coordinate to define source moments. Secondly due to curvature of de Sitter space-time, the gravitational waves back- scatter and repetitively backscattered waves superimposing on each other produce a tail term even in the first Post Newtonian (PN) order. Thirdly unlike the Minkowski space-time which admits a natural, global Cartesian chart, de Sitter space-time has several charts appropriate for different situations.

The de Sitter space-time defined as the hyperboloid in five dimensional Minkowski space-time, has a '*global chart*' of coordinates $(\tau, \chi, \theta, \phi)$, as shown in figure (1). To be definite, let us take the world tube of the spatially compact source to be around the line AD for all times. In this case source world - tube has future and past time-like infinity in both $\Lambda = 0$ and $\Lambda > 0$ cases, denoted by $i^{\pm}$. However, whereas in the $\Lambda = 0$ case causal future of compact source is the entire Minkwoski space-time, but for $\Lambda > 0$, it is only future Poicaré patch (M_P^+) of full de Sitter. No observer whose word-line is confined to the past Poincaré patch (M_P^-)can see the compact source or detect the radiation it emits. Therefore to study this system, it is sufficient to restrict oneself just to M_P^+ rather than full de Sitter space-time. There are two natural coordinate charts for the future Poincaré patch e.g.. a conformal chart: (η, x^i) and a cosmological chart: (t, x^i). A 'half' of the future Poincaré patch admits a time-like Killing vector and is referred to as a static patch. This is a natural patch for an isolated body or a black hole with a stationary neighbourhood. We present computations in two different charts: suitably defined Fermi Normal Coordinates (FNC) covering a portion of static patch and a conformal chart covering the future Poincaré patch, (see figure (1)). While physical implications should not depend on choice of charts, their explicit computations do depend on the chosen chart. The proto-typical Schwarzschild-de Sitter solution suggests the static patch while spatially flat cosmologies suggest the Poincaré patch. A priori, it is not clear which chart(s) are convenient for what aspect and we present computations for two choices of charts - the FNC and the conformal chart.

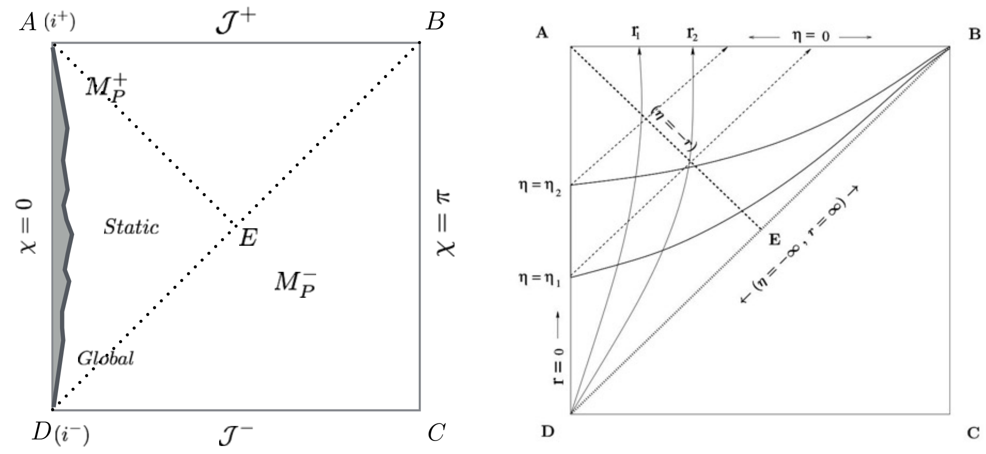

Figure 1. ABCD denotes the Global Chart, ABD (M_P^+) is future Poincaré patch , while AED is a static patch. The angular coordinates θ, ϕ are suppressed. A compact source, denoted by wiggly lines in the figure of the left, is confined near AD with past and future time like infinity i^-, i^+ respectively. Null hyper-surface AE denotes cosmological horizon. Figure on the right shows conformal coordinates in M_P^+. The two dotted lines at 45 degrees, denote the paths of gravitational waves emitted at $\eta = \eta_1, \eta_2$ on the world line at $r = 0$, through the source.

Weak gravitational fields are understood as perturbations about a background specified in the form, $g_{\mu\nu} := \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}$. The physical perturbations are understood as the equivalence classes of solutions of the linearized equation $h_{\mu\nu}$, with respect to the gauge transformations: $\delta h_{\mu\nu}(x) = \mathcal{L}_\xi \bar{g}_{\mu\nu}(x) = \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$. In terms of the trace reversed combination $\tilde{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\bar{h}_{\mu\nu}(\bar{g}^{\alpha\beta} h_{\alpha\beta})$, the linearized equation takes the form,

$$\frac{1}{2}\left[-\bar{\Box}\tilde{h}_{\mu\nu} + \left\{\bar{\nabla}_\mu B_\nu + \bar{\nabla}_\nu B_\mu - \bar{g}_{\mu\nu}(\bar{\nabla}^\alpha B_\alpha)\right\}\right] + \frac{\Lambda}{3}\left[\tilde{h}_{\mu\nu} - \tilde{h}\bar{g}_{\mu\nu}\right] \;=\; 8\pi T_{\mu\nu} \qquad (0.1.1)$$

where, $B_\mu := \bar{\nabla}_\alpha \tilde{h}^\alpha{}_\mu$. The gauge freedom is exploited subsequently to simplify the equation. Two different choices of gauges are presented , each with its advantages and limitations. The first is the transverse, traceless gauge or TT gauge while the second is generalized TT gauge.

0.1.1 Transverse traceless gauge in FNC

The particular choice of arranging $\bar{\nabla}_\alpha \tilde{h}^\alpha_{\ \mu} = 0 = h$, is the transverse, traceless gauge or TT gauge for short. It simplifies the equation (0.1.1) to (for traceless stress tensor),

$$-\frac{1}{2}\bar{\Box}\tilde{h}_{\mu\nu} + \frac{\Lambda}{3}\tilde{h}_{\mu\nu} \ = \ 8\pi T_{\mu\nu} \tag{0.1.2}$$

The equation for the Green function is,

$$\bar{\Box}G^{\alpha\beta}_{\ \ \mu'\nu'}(x,x') - \frac{2\Lambda}{3}G^{\alpha\beta}_{\ \ \mu'\nu'}(x,x') \ = \ -4\pi J^{\alpha\beta}_{\ \ \mu'\nu'}\delta_4(x,x') \ , \ \ \text{where,} \tag{0.1.3}$$

$$J^{\alpha\beta}_{\ \ \mu'\nu'}(x,x') \ := \ \frac{g^\alpha_{\ \mu'}g^\beta_{\ \nu'} + g^\alpha_{\ \nu'}g^\beta_{\ \mu'}}{2} - \frac{1}{4}\bar{g}^{\alpha\beta}(x)\bar{g}_{\mu'\nu'}(x') \ , \ \ \text{and,} \tag{0.1.4}$$

$g^\alpha_{\ \mu'}(x,x')$ denotes the parallel propagator along the geodesic connecting x, x'. The Green's function is obtained using the Hadamard ansatz. The Hadamard ansatz for the retarded Green function for a general wave equation is [4, 5],

$$G^{\alpha\beta}_{\ \ \mu'\nu'}(x,x') \ = \ U^{\alpha\beta}_{\ \ \mu'\nu'}(x,x')\delta_+(\sigma+\epsilon) + V^{\alpha\beta}_{\ \ \mu'\nu'}(x,x')\theta_+(-\sigma-\epsilon) \ , \ \ \text{where} \tag{0.1.5}$$

the space-time points x, x' belong to a *convex normal neighbourhood* with x in the chronological future of x'; $\sigma(x,x')$ is the Synge world function which is half the geodesic distance squared between x and x' ; θ_+, δ_+ are distributions, viewed as functions of x, having support in the chronological future and future light cone of x' respectively. The small parameter ϵ is introduced to permit differentiation of the distribution and is to be taken to zero in the end. As the wave propagates it back scatters due to the curvature of background. This back- scattering introduces tail term, as indicated in the coefficient of θ function in the Hadamard ansatz. The bi-tensors U, V are determined by inserting the ansatz in the equation (0.1.3). This leads to, $U^{\alpha\beta}_{\ \ \mu'\nu'}(x,x')|_{\sigma=0} \ := \ J^{\alpha\beta}_{\ \ \mu'\nu'}|_{\sigma=0}$ and $V^{\alpha\beta}_{\ \ \mu'\nu'}$ is at least of order Λ^2. As we will be computing corrections to order Λ, the inhomogeneous equation gets contribution from sharp propagation only (along light-cone),

$$\tilde{h}^{\alpha\beta}(x) = 4\int_{\text{source}} d^4x' \sqrt{-g(x')}\delta_+(\sigma) J^{\alpha\beta}{}_{\mu'\nu'}(x,x') T^{\mu'\nu'}(x') \tag{0.1.6}$$

$$= 4\int_{\text{source}} d^4x' \sqrt{-g(x')}\delta_+(\sigma) g^{\alpha}{}_{\mu'}(x,x') g^{\beta}{}_{\nu'}(x,x') T^{\mu'\nu'}(x') \tag{0.1.7}$$

To get an explicit form we introduce Fermi Normal Coordinates (FNC) (see fig. 2) in de Sitter background. In terms of the FNC, the metric to first order in the curvature, is given

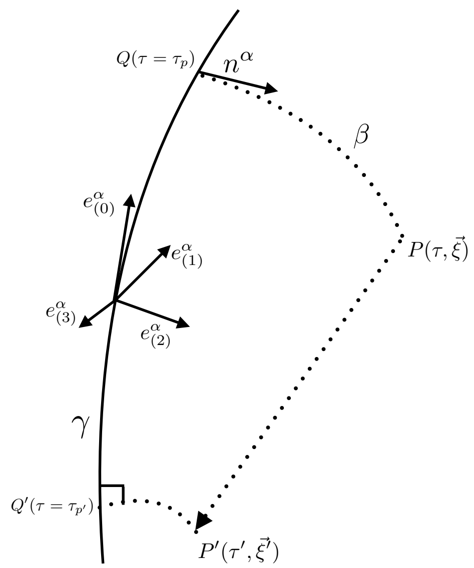

Figure 2. Fermi Normal Coordinates are based on the choice of a time-like reference geodesic γ (here we choose the line AD in fig (1)), which is parametrized by proper time τ. Everywhere on γ we construct an orthonormal tetrad $e^\alpha_{(\mu)}$ such that $e^\alpha_{(0)}$ is aligned with γ's tangent vector and we assume that the orthonormal tetrad is parallel transported along γ. To define the FNC of observation point P off γ, let β be the unique (space-like) geodesic from P, orthogonally meeting γ at a point $Q = \gamma(\tau_P)$, with a unit affine parameter interval. Its tangent vector, n^α at Q can be resolved along the triad of space-like vectors at Q as: $n^\alpha := \xi^i e^\alpha_{(i)}$. Its norm gives the proper distance between P and Q, $s^2 := n^\alpha n^\beta \eta_{\alpha\beta} = \xi^i \xi^j \delta_{ij}$. The FNC of P are then defined to be (τ_P, ξ^i). Similarly we construct FNC at source point P'. The dotted line from P to P' is the unique null geodesic.

as [5],

$$g_{00}(\tau,\vec{\xi}) \;=\; -1+\frac{\Lambda s^2}{3} \;\;,\;\; g_{0i} \;=\; 0 \;\;,\;\; g_{ij} \;=\; \delta_{ij}-\frac{\Lambda}{9}(\delta_{ij}s^2-\xi_i\xi_j)\,. \qquad (0.1.8)$$

Calculating $\sigma(x,x')$ in FNC and and exhausting τ' integration in $\sqrt{-g}$, to the leading order in s'/s eq. (0.1.6) becomes,

$$\tilde{h}^{\mu\nu}(\tau,\vec{\xi}) \;=\; \frac{4}{s}\left(1+\frac{\Lambda s^2}{18}\right)g^{\mu}{}_{\alpha'}(\vec{\xi})g^{\nu}{}_{\beta'}(\vec{\xi})\int d^3\xi'\,\sqrt{g_3(\xi')}T^{\alpha'\beta'}(\tau_{\text{ret}},\vec{\xi}')\,. \qquad (0.1.9)$$

The integral over the source is usually expressed in terms of time derivatives of moments, using the conservation of the stress tensor. To make these integrals well defined, it is convenient and transparent to introduce suitable orthonormal tetrad and convert the coordinate components to frame components. The frame components are coordinate scalars (although they change under Lorentz transformations) and their integrals are well defined. In the FNC chart, there is a natural choice provided by the τ' = constant hypersurface passing through the source world tube. At any point on this hypersurface, we have a unique orthonormal triad obtained from the triad on the reference curve by parallel transport along the spatial geodesic. Explicitly, to order Λ (underlined indices denote frame indices),

$$e^{\alpha'}_{\underline{a}} \;:=\; \left(1+\frac{\Lambda s'^2}{6}\right)\delta^{\alpha'}_{\tau}\delta^{0}_{\underline{a}}+\delta^{\alpha'}_{i}\delta^{j}_{\underline{a}}\left\{\delta^{i}_{\underline{j}}\left(1+\frac{\Lambda s'^2}{18}\right)-\frac{\Lambda}{18}\xi'^i\xi'_{\underline{j}}\right\} \qquad (0.1.10)$$

It is easy to check that $e^{\alpha'}_{\underline{a}}e^{\beta'}_{\underline{b}}g_{\alpha'\beta'}=\eta_{\underline{ab}}$. It follows that,

$$g^{m}{}_{\alpha'}(x)e^{\alpha'}_{\underline{a}}(x') \;\simeq\; \delta^{m}_{\underline{a}}\left(1+\frac{\Lambda s^2}{18}\right)+\frac{\Lambda}{6}\delta^{0}_{\underline{a}}s\,\xi^m-\frac{\Lambda}{18}\delta^{j}_{\underline{a}}\xi_{\underline{j}}\xi^m. \qquad (0.1.11)$$

We define the frame components of the stress tensor through the relation, $T^{\mu\nu}:=e^{\mu}{}_{\underline{a}}e^{\nu}{}_{\underline{b}}\Pi^{\underline{ab}}$. Now substituting for $g^{m}_{\alpha'}e^{\alpha'}_{\underline{a}}(\tau,\vec{\xi},\tau',\vec{\xi}')$ and choosing synchronous gauge, $h_{0\alpha}=0$, we note the spatial components of filed to leading order in s'/s and to $o(\Lambda)$ is given by,

$$
\begin{aligned}
\tilde{h}^{mn}(\tau, \vec{\xi}\,) &= \frac{4}{s}\left(1+\frac{\Lambda s^2}{18}\right)\Bigg[\left(1+\frac{\Lambda s^2}{9}\right)\delta^m_{\ \underline{m}}\delta^n_{\ \underline{n}}\int\sqrt{g_3(\vec{\xi}')}\Pi^{\underline{mn}}(\tau_{\mathrm{ret}},\vec{\xi}') \\
&+\frac{\Lambda s}{6}\left\{\xi^m\delta^n_{\ \underline{n}}\int\sqrt{g_3(\vec{\xi}')}\Pi^{\underline{0n}}+\xi^n\delta^m_{\ \underline{m}}\int\sqrt{g_3(\vec{\xi}')}\Pi^{\underline{0m}}\right\} \\
&-\frac{\Lambda}{18}\left\{\xi^m\delta^n_{\ \underline{n}}\xi_{\underline{k}}\int\sqrt{g_3(\vec{\xi}')}\Pi^{\underline{kn}}+\xi^n\delta^m_{\ \underline{m}}\xi_{\underline{k}}\int\sqrt{g_3(\vec{\xi}')}\Pi^{\underline{km}}\right\}\Bigg]
\end{aligned}
\tag{0.1.12}
$$

This is still not in a convenient form. To express the source integrals in terms of moments, we have to consider the conservation equation. Introducing the notation $\rho := \Pi^{\underline{00}}, \pi := \Pi^{\underline{ij}}\delta_{\underline{ij}}$ and writing conservation equations in terms of frame components, we obtain (the constant tetrad are suppressed),

$$
\partial_\tau \Pi^{\underline{00}} = -\left(1-\frac{\Lambda s^2}{9}\right)\partial_j\Pi^{\underline{0j}}+\frac{\Lambda}{18}\xi_j\xi^i\partial_i\Pi^{\underline{0j}}+\Lambda\Pi^{\underline{0j}}\xi_j \tag{0.1.13}
$$

$$
\partial_\tau \Pi^{\underline{0i}} = -\left(1-\frac{\Lambda s^2}{9}\right)\partial_j\Pi^{\underline{ji}}+\frac{\Lambda}{18}\left\{\xi_j\xi\cdot\partial\,\Pi^{\underline{ji}}+15\Pi^{\underline{ij}}\xi_j+3\rho\xi^i\right\} \tag{0.1.14}
$$

Eliminating $\Pi^{\underline{0i}}$ and using $\pi = \rho$ thanks to the trace free stress tensor, we get the second order conservation equation as,

$$
\partial^2_\tau\rho = \left(1-\frac{2\Lambda s^2}{9}\right)\partial^2_{ij}\Pi^{\underline{ij}}-\frac{\Lambda}{9}\left[\xi^i\xi_j\partial^2_{ik}\Pi^{\underline{jk}}+19\xi_i\partial_j\Pi^{\underline{ij}}+2\xi^j\partial_j\rho+12\rho\right] \tag{0.1.15}
$$

The usual strategy is to define suitable moments of energy density/pressures and taking moments of the above equation, express the integral of $\Pi^{\underline{ij}}$ in terms of the moments and its time derivatives. To maintain coordinate invariance, the moment variable (analogue of x^i in the Minkowski background) must also be a coordinate scalar. Note that in FNC, ξ^i is a contravariant vector. Its frame components naturally provide coordinate scalars. We still have the freedom to multiply these by suitable scalar functions. It is easy to see that the frame components of ξ are the same as the coordinate components at best up to permutations i.e. $\xi^{\underline{i}} := e^{\underline{i}}_{\ j}\xi^j = \delta^{\underline{i}}_{\ j}\xi^j$. It is also true that $g_{ij}\xi^i\xi^j = \xi^{\underline{i}}\xi^{\underline{j}}\delta_{ij} = s^2$. With this understood the moments are defined as

$$\mathcal{M}^{\underline{i_1}\underline{i_2}\cdots\underline{i_n}}(\tau) := \int_{\text{source}} d^3\xi \sqrt{g_3(\vec{\xi})}\zeta^{\underline{i_1}}\cdots\zeta^{\underline{i_n}}\rho(\tau,\vec{\xi}) \ , \ \zeta^{\underline{i}}(\vec{\xi}) := \left(1+\frac{\Lambda s^2}{9}\right)\xi^{\underline{i}}, \tag{0.1.16}$$

where the integration is over the support of the source on the constant$-\tau$ hypersurface. Taking the moments of eq. (0.1.15), solution can be expressed as,

$$\begin{aligned}\tilde{h}^{mn}(\tau,\vec{\xi}) &= \delta^m_{\ \underline{m}}\delta^n_{\ \underline{n}}\left[\left(\frac{2}{s}\partial_\tau^2\mathcal{M}^{\underline{mn}}\right) - \frac{\Lambda}{3s}\left(\xi_{\underline{k}}\frac{\xi^{\underline{m}}\partial_\tau^2\mathcal{M}^{\underline{kn}}+\xi^{\underline{n}}\partial_\tau^2\mathcal{M}^{\underline{km}}}{3} - s^2\partial_\tau^2\mathcal{M}^{\underline{mn}}\right)\right. \\ &\left. +\frac{\Lambda}{s}\left(-\mathcal{M}^{\underline{mn}}+\mathcal{N}^{\underline{mn}}-\frac{1}{2}\delta_{\underline{rs}}\partial_\tau^2\mathcal{M}^{\underline{mnrs}}\right)+\frac{2\Lambda}{3}\left(\xi^{\underline{m}}\partial_\tau\mathcal{M}^{\underline{n}}+\xi^{\underline{n}}\partial_\tau\mathcal{M}^{\underline{m}}\right)\right]\end{aligned} \tag{0.1.17}$$

In this subsection the computation is limited to static patch. But it reflects general implication of curved background, e.g. Hadamard construction of Green function and generalization of definition of moments. For contrast, in the next subsection, we present the computation in conformal chart of Poincaré patch.

0.1.2 Generalized transverse gauge in Poincaré patch

In the conformal chart (η, x^i) the background metric takes the form,

$$ds^2 = \frac{1}{H^2\eta^2}\left[-d\eta^2+\sum_i (dx^i)^2\right] \ , \ \eta\in(-\infty,0) \ , \ H := \sqrt{\frac{\Lambda}{3}}. \tag{0.1.18}$$

While conformal coordinates are convenient in detailed calculations of gravitational perturbations, they are ill-suited for taking the limit $\Lambda \to 0$. To take that limit we have to use cosmological time t, which is related to conformal time η via $\eta := -H^{-1}e^{-Ht}$. The conformally flat form of background in (η, x^i) coordinates leads to a great deal of simplification to the eq. (0.1.1) in generalized TT gauge: $B_\mu = \frac{2\Lambda\eta}{3}\,\tilde{h}_{0\mu}$,

$$-16\pi T_{\mu\nu}\Omega^2 = \Box\tilde{h}_{\mu\nu} - \frac{2}{\eta}\partial_0\tilde{h}_{\mu\nu} - \frac{2}{\eta^2}\left\{\delta^0_\mu\delta^0_\nu\tilde{h}_\alpha^{\ \alpha} - \tilde{h}_{\mu\nu} + \delta^0_\mu\tilde{h}_{0\nu} + \delta^0_\nu\tilde{h}_{0\mu}\right\} \tag{0.1.19}$$

It turns out to be convenient to work with new variables, $\chi_{\mu\nu} := \Omega^{-2}\tilde{h}_{\mu\nu}$. All factors of Ω^2 *and* Λ drop out of the equations and $\chi_{\mu\nu}$ satisfies,

$$
\begin{aligned}
-16\pi T_{\mu\nu} &= \Box\chi_{\mu\nu} + \frac{2}{\eta}\partial_0\chi_{\mu\nu} - \frac{2}{\eta^2}\left(\delta^0_\mu\delta^0_\nu\chi_\alpha{}^\alpha + \delta^0_\mu\chi_{0\nu} + \delta^0_\nu\chi_{0\mu}\right) . && (0.1.20)\\
0 &= \partial^\alpha\chi_{\alpha\mu} + \frac{1}{\eta}\left(2\chi_{0\mu} + \delta^0_\mu\chi_\alpha{}^\alpha\right) \qquad \text{(gauge condition)}. && (0.1.21)
\end{aligned}
$$

The main simplification of eq. (0.1.20) occurs in decomposition of components into $\hat{\chi}(:= \chi_{00} + \chi_i{}^i)$, χ_{0i}, χ_{ij} . It can be shown that the residual gauge freedom is exhausted by setting $\chi_{0i} = 0 = \hat{\chi}(:= \chi_{00} + \chi_i{}^i)$ [6]. The gauge condition (0.1.21) then implies $\partial^0\chi_{00} = 0$ and by choosing it to be zero at some initial η =constant hypersurface we can take $\chi_{00} = 0$ as well. Thus it suffices to focus on the equation (0.1.20) for $\mu, \nu = i, j$,

$$
\Box\chi_{ij} + \frac{2}{\eta}\partial_0\chi_{ij} = -16\pi T_{ij} \quad , \quad \partial_i\chi^i{}_j = 0 = \chi_i{}^i . \tag{0.1.22}
$$

Green's function of the filed χ_{ij} mimics that of minimally coupled scalar field in de Sitter background. Hence from the Hadamard construction corresponding retarded Green function is given by,

$$
G_R(\eta, x; \eta' x') = \frac{\Lambda}{3}\eta\eta'\frac{1}{4\pi}\frac{\delta(\eta - \eta' - |x - x'|)}{|x - x'|} + \frac{\Lambda}{3}\frac{1}{4\pi}\theta(\eta - \eta' - |x - x'|) . \tag{0.1.23}
$$

The particular solution is given by,

$$
\begin{aligned}
\chi_{ij}(\eta, x) &= 16\pi\int_{\text{source}}\frac{d\eta' d^3x'}{\frac{\Lambda}{3}\eta'^2}G_R(\eta, x; \eta' x')T_{ij}(\eta', x') && (0.1.24)\\
&= 4\int d^3x'\frac{\eta}{|x - x'|(\eta - |x - x'|)}\,T_{ij}(\eta', x')\Big|_{\eta'=\eta-|x-x'|}\\
&\quad + 4\int d^3x'\int_{-\infty}^{\eta-|x-x'|}d\eta'\frac{T_{ij}(\eta', x')}{\eta'^2} && (0.1.25)
\end{aligned}
$$

The spatial integration is over the matter source confined to a compact region and is finite. The second term in the eqns. (0.1.25) is the tail term which depends on past history of the source.

As in the previous section, we introduce a tetrad to define the frame components of the stress tensor. The conformal form of the metric suggests a natural choice: ($\sqrt{\Lambda/3} =: H$),

$$f^{\alpha}_{\underline{0}} \ := \ - H\eta(1,\vec{0}) \ , \ f^{\alpha}_{\underline{m}} \ := \ - H\eta\, \delta^{\alpha}_{\underline{m}} \qquad \Longleftrightarrow \qquad f^{\alpha}_{\underline{a}} := -H\eta\delta^{\alpha}_{\underline{a}} \tag{0.1.26}$$

and denote the frame components of the source stress tensor as: $\rho := H^2\eta^2 T_{\underline{00}}$, $\pi := H^2\eta^2 T_{\underline{ij}}\delta^{\underline{ij}}$. Define moment variable $\bar{x}^{\underline{i}} = f^{\underline{i}}_{\ \alpha}x^{\alpha} = -(\eta H)^{-1}\delta^{\underline{i}}_{\ j}x^j := a(t)x^{\underline{i}}$. Two sets of moments are defined as functions of η, or of t defined through $\eta := -H^{-1}e^{-Ht}$ as,

$$Q^{\underline{ij}}(t) := \int_{Source(t)} d^3x\, a^3(t)\rho(t,\vec{x})\bar{x}^{\underline{i}}\bar{x}^{\underline{j}} \ , \ \bar{Q}^{\underline{ij}}(t) := \int_{Source(t)} d^3x\, a^3(t)\pi(t,\vec{x})\bar{x}^{\underline{i}}\bar{x}^{\underline{j}} \, . \tag{0.1.27}$$

In terms of these, the approximated retarded solution is given by,

$$\chi_{ij}(\eta,\vec{x}) \ = \ \frac{1}{r}f_{ij}(\eta_{ret}) + g_{ij}(\eta_{ret}) + \hat{g}_{ij} \qquad \text{with,} \tag{0.1.28}$$

$$f_{ij}(\eta_{ret}) \ := \ \frac{2}{a(\eta_{ret})}\left[\mathcal{L}_T^2 Q_{ij} + 2H\mathcal{L}_T Q_{ij} + H\mathcal{L}_T\bar{Q}_{ij} + 2H^2\bar{Q}_{ij}\right]\Big|_{\eta_{ret}} \ , \tag{0.1.29}$$

$$g_{ij}(\eta_{ret}) \ := \ -2H\left[\mathcal{L}_T^2 Q_{ij} + H\mathcal{L}_T Q_{ij} + H\mathcal{L}_T\bar{Q}_{ij} + H^2\bar{Q}_{ij}\right]\Big|_{\eta_{ret}} \qquad \text{and,} \tag{0.1.30}$$

$$\hat{g}_{ij} \ := \ -2H^2\left[\mathcal{L}_T Q_{ij} + H\bar{Q}_{ij}\right]\Big|_{-\infty} \tag{0.1.31}$$

$\mathcal{L}_T$ denotes the Lie derivative with respect to the time translation Killing vector. In eq. (0.1.28), the first term is the contribution of the so called sharp term while the second and the third terms denote the tail contributions. The tail contribution has separated into a term which depends on retarded time, $(\eta - r)$ only, just as the sharp term does, and the contribution from the history of the source is given by the limiting value at $\eta = -\infty$. This expression is valid as the leading term for $|\vec{x}| \gg |\vec{x}'|$. For future use, we display the derivatives of χ_{ij}. Since χ_{ij} depends on $\vec{x}$ only through r, we need only the derivatives with respect to η and r. On functions of η_{ret}, $\partial_r = -\partial_\eta$ and we can replace the r-derivatives in favor of $\eta-$derivatives. Hence,

$$\partial_\eta \chi_{ij} = \frac{1}{r}\partial_\eta f_{ij} + \partial_\eta g_{ij} \ , \ \partial_r \chi_{ij}(\eta, r) = -\frac{1}{r^2} f_{ij} - \frac{1}{r}\partial_\eta f_{ij} - \partial_\eta g_{ij} = -\frac{f_{ij}}{r^2} - \partial_\eta \chi_{ij} \,. \quad (0.1.32)$$

The two solutions presented in last two subsections were obtained in two different gauges. With a further choice of synchronous gauge, we could restrict the solutions to the spatial components alone. While these conditions fix the gauge completely, physical solutions still have to satisfy 'spatial transversality and trace free' conditions, i.e, $\partial^j \chi^{tt}_{ij} = 0 = \delta^{ij}\chi_{ij}$. As in the case of the Minkowski background, corresponding to each spatial, unit vector $\hat{n}$, define the projectors,

$$P^i{}_j(\hat{n}) := \delta^i{}_j - \hat{n}^i \hat{n}_j \ , \ \Lambda^{ij}{}_{kl} := \frac{1}{2}\left(P^i{}_k P^j{}_l + P^i{}_l P^j{}_k - P^{ij} P_{kl}\right) \,. \quad (0.1.33)$$

Contraction with $\hat{n}$ gives zero and the trace of Λ-projector in either pair of indices vanishes. Given any χ^{kl}, the Λ-projector extracts out physical component $\chi^{ij}_{tt} := \Lambda^{ij}{}_{kl}\chi^{kl}$ which is trace free and is transverse to the unit vector $\hat{n}$. These arguments also hold for FNC solutions. In the next subsection we introduce a gauge invariant quantity , deviation scalar and compute it in both charts for a comparison.

0.1.3 Comparison of deviation scalar

Those two solutions have been obtained in two different gauges and two different charts - how can those be compared? One natural strategy is to construct a gauge invariant physical observable and compute it in both coordinates and for matching one can do coordinate transformation also. As an illustration, we consider a congruence of time like geodesics of the background space-time, as tracked by a freely falling observer and consider the tidal effects of a transient gravitational wave. In flat background deviation acceleration is gauge invariant but this is not so in de Sitter background. Nevertheless it turns out that in de Sitter case we can construct a a gauge invariant quantity, deviation scalar - which is the component of acceleration of one deviation vector along another orthogonal deviation vector. For a particular choice of vectors u, Z, Z' (u denote four velocity of freely falling observer, Z and Z' are two orthogonal deviation vectors), satisfying $u \cdot Z = u \cdot Z' = Z \cdot Z' = 0$, let us define deviation scalar ,

$$D(u, Z, Z') := -R_{\alpha\beta\mu\nu} Z'^{\alpha} u^{\beta} Z^{\mu} u^{\nu} \;=\; -\, C_{\alpha\beta\mu\nu} Z'^{\alpha} u^{\beta} Z^{\mu} u^{\nu} \tag{0.1.34}$$

Vanishing nature of Weyl tensor and its gauge transformation in de Sitter background ensure gauge invariance of $D(u, Z', Z)$ to the linear order in field.

For the same family of background geodesics $\bar{u}^{\alpha}(\eta, x^i) := -H\eta(1, \vec{0})$ and $\bar{u}^{\alpha} = (1 + \Lambda s^2/3,\ \sqrt{\Lambda/3}\,\vec{\xi}\,)$ respectively for conformal chart and FNC, we obtain the deviation scalars to the linear order in field,

$$\begin{aligned} D^{FNC}(u, Z', Z) &= \frac{1}{s}\left(1 - 2Hs + \frac{23}{6}H^2 s^2\right)\Big[\partial_\tau^4 \mathcal{M}^{tt}_{ij} - H^2 s \partial_\tau^3 \mathcal{M}^{tt}_{ij} \\ &\qquad -H^2 \partial_\tau^2 \mathcal{M}^{tt}_{ij} - \frac{3H^2}{4}\partial_\tau^4 \mathcal{M}^{tt}_{ijkl}\delta^{kl}\Big]\hat{Z}'^i \hat{Z}^j \qquad (0.1.35) \\ D^{Conf}(u, Z', Z)\Big|_{o(H^2)} &= \frac{1}{s}\left(1 - 2Hs + \frac{19}{6}H^2 s^2\right)\Big[\mathcal{L}_T^4 Q^{tt}_{ij} + 7H\mathcal{L}_T^3 Q^{tt}_{ij} \\ &\qquad +17H^2 \mathcal{L}_T^2 Q^{tt}_{ij}\Big]\hat{Z}'^i \hat{Z}^j \qquad (0.1.36) \end{aligned}$$

In the conformal chart, we retain terms up to order H^2 only and since the FNC calculation uses traceless stress tensor, we take $\bar{Q}$ moments to equal the Q moments. We have obtained two different looking expressions for the same, gauge invariant deviation scalar. The difference can be attributed to the definition of moments. They have been defined on two different spatial hypersurfaces - the τ =constant in FNC and the η = constant in the conformal chart.

A physical interpretation : The deviation scalar also has a physical interpretation. It is simpler to look at Minkowski background. Let us align deviation vectors along two arms of interferometric detector and $\hat{Z}$ makes an angle ϕ with $\hat{e}_1$ axis of a chosen basis $\{\hat{e}_1, \hat{e}_2\}$

(plane transverse to the wave propagation direction). For these alignment and geodesic congruence of $\bar{u}^\alpha = (1, \vec{0})$ in Minkowski background, it can be shown that $D(u, Z', Z) \approx \hat{Z}'^a A_{ab}(h)\hat{Z}^b$ where a, b take two values and the real, symmetric matrix A_{ab} is traceless. With respect to an arbitrarily chosen basis, $\{\hat{e}_1, \hat{e}_2\}$ in the plane transverse to the wave direction, $\hat{n}$, we can define the '+' and the '×' polarizations by setting the matrix $A := h_+\sigma_3 + h_\times\sigma_1$. It follows that,

$$D(u, Z', Z) = -h_+ sin(2\phi) + h_\times cos(2\phi) \ . \tag{0.1.37}$$

Thus, for a pair of bases $(\hat{e}_1, \hat{e}_2)$ and $(\hat{Z}, \hat{Z}')$, determination of the deviation scalar gives one relation between the amplitudes of the two polarizations. A similar determination at another detector location gives a second relation, thereby providing amplitudes of individual polarizations.

0.2 Quadrupole formula in de Sitter background

Having obtained the gravitational field it is natural to ask the next question what is the modification of quadrupole formula for radiated power for $\Lambda > 0$? It should also be noted that cosmological horizon is un ambiguously defined for a spatially compact source : it is the past light cone of i^+. In this chapter we would like to explore under what conditions and to what extent we may regard cosmological horizon as a "substitute" for the future null infinity. For this we will investigate energy flux of gravitational wave with respect to time translational isometry of background de Sitter metric. There are seven globally defined killing vectors on the Poincaré patch, [7] corresponding to energy, three momenta, and three angular momenta. We focus on the time translation vector field,

$$T \ := \ -H(\eta\partial_\eta + x^i\partial_i) \ , \ \ H = \sqrt{\frac{\Lambda}{3}} \tag{0.2.1}$$

As we would like to explore energy flux near $\mathcal{J}^+$ also, in this section we will restrict to the field solution in future Poincaré patch itself. We employ Isaacson prescription to define effective gravitational stress tensor.

For sources which are sufficiently rapidly varying (relative to the scale set by the cosmological constant), there is an identification of gravitational waves as ripples on a background within the so called 'short wave approximation'. Let L denote the length scale of variation of the background and λ the length scale of the ripple with $\lambda \ll L$. In this context Isaacson defined an effective gravitational stress tensor for the ripples which is gauge invariant to leading order in the ratio of two scale. To obtain Isaacson stress tensor, one begins with an expansion of the form $g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}$ and writes the Einstein equation in source free region as,

$$\begin{aligned} R_{\mu\nu}(\bar{g} + \epsilon h) &= \Lambda(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}) \\ \therefore R^{(0)}_{\mu\nu}(\bar{g}) + \epsilon R^{(1)}_{\mu\nu}(\bar{g}, h) + \epsilon^2 R^{(2)}_{\mu\nu}(\bar{g}, h) &= \Lambda(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}) \end{aligned} \tag{0.2.2}$$

Introduce an averaging over an intermediate scale ℓ, $\lambda \ll \ell \ll L$ which satisfies the properties: (i) average of odd powers of h vanish and (ii) average of space-time divergence of tensors are sub-leading. Taking the average of the above equation gives,

$$\langle R^{(0)}_{\mu\nu} \rangle + \epsilon^2 \langle R^{(2)}_{\mu\nu} \rangle = \Lambda \bar{g}_{\mu\nu}. \tag{0.2.3}$$

Notice that $R^{(2)}$ which is quadratic in h, *can* have L-scale variations and hence non-zero average. Thus it incorporates back reaction of ripple on the background and modify the background equation as,

$$8\pi t_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} + \Lambda \bar{g}_{\mu\nu} \qquad \textit{with}, \tag{0.2.4}$$

$$t_{\mu\nu}(\bar{g}, h) := -\frac{\epsilon^2}{8\pi}\left[\langle R^{(2)}_{\mu\nu} \rangle - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\langle R^{(2)}_{\alpha\beta} \rangle\right] \tag{0.2.5}$$

Calculating this expression to the leading order in ϵ for de Sitter background, we write stress tensor for ripples,

$$t_{\mu\nu} = \frac{1}{32\pi}\langle\partial_\mu\chi^{tt}_{ij}\,\partial_\nu\chi^{ij}_{tt}\rangle\,. \tag{0.2.6}$$

Given a symmetric, conserved stress tensor and time-translational killing vector of background, one can construct conserved current $J^\mu_T := -t^\mu_\nu\,T^\nu$. And from the conservation equation it follows,

$$0 = \int_{\mathcal{V}} d^4x\,\sqrt{\bar{g}}\bar{\nabla}_\mu J^\mu_\xi = \int_{\mathcal{V}} d^4x\partial_\mu(\sqrt{\bar{g}}J^\mu_\xi) = \int_{\partial\mathcal{V}} d\sigma_\mu J^\mu_\xi, \tag{0.2.7}$$

In the subsequent discussion we recall the result of *energy flux* ($\int_\Sigma d\sigma_\mu J^\mu_T$) computation, for various hypersurfaces, Σ's. These, together with the conservation equation (0.2.7) is used to relate power received at $\mathcal{J}^+$ to that crossing the cosmological horizon.

0.2.1 Flux computations

There are three natural classes of hypersurfaces: (a) hypersurfaces of constant physical radial distance, (b) space-like hypersurfaces of constant η and (c) the out-going and in-coming null hypersurfaces and we compute the fluxes for these.

0.2.1.1 Hypersurface of constant, radial physical distance:

These represent the *physical radial distance=const.*, hypersurfaces with $r_{phy} = |\Omega|r := \rho$. They are spanned by the integral curves of the Killing vector T. They are time-like, null and space like according as the physical distance being less than, equal to and greater than the physical distance to the cosmological horizon, namely H^{-1}. For all these family of hypersurfaces, the energy flux integral is expressed as,

$$\int_{\Sigma_\rho} d\Sigma_\alpha J^\alpha_T = -\int_{-\infty}^{\infty} d\tau \int_{S^2} d^2s\rho^2\frac{1}{H\eta}(H\rho,\hat{x}_i)J^\mu_T \tag{0.2.8}$$

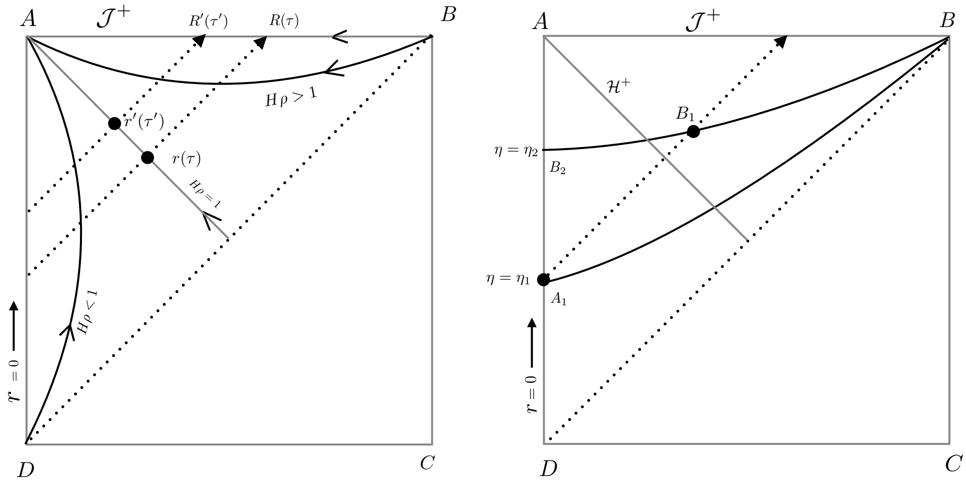

Figure 3. The figure on the left shows the ρ = constant hypersurfaces which are time-like for $H\rho < 1$, null for $H\rho = 1$ and space-like for $H\rho > 1$. The two 45 degree out-going null hypersurfaces intersecting the $\mathcal{H}^+$ and $\mathcal{J}^+$ in the spheres at $r(\tau), r'(\tau'), R(\tau), R'(\tau')$, bound a space-time region. The figure on the right shows the space-like hypersurfaces with constant value of η. The fluxes across the out-going null hypersurfaces turn out to be zero signifying sharp propagation of energy. Hence the energy flux across the portion of the horizon bounded by the spheres at $r(\tau), r'(\tau')$ equals the flux across the portion of the future infinity bounded by the spheres at $R(\tau), R'(\tau')$.

Neglecting $1/r^2$ term in eq. (0.1.32) for rapidly varying source we have

$$J^\alpha_T = -\frac{H}{32\pi a^2}(\eta - r)\left\langle \partial_\eta \chi^{mn}_{tt} \partial_\eta \chi^{tt}_{mn} \right\rangle (1, x^i) \tag{0.2.9}$$

$$\therefore \int_{\Sigma_\rho} d\Sigma_\alpha J^\alpha_T = \int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2 s \left[\frac{1}{8\pi}\right] \left\langle Q^{tt}_{ij} Q^{ij}_{tt} \right\rangle (t_{ret}) \tag{0.2.10}$$

$$\text{where, } \partial_\eta \chi^{tt}_{mn}(\eta_{ret}) = \frac{2}{r}\frac{\eta}{\eta - r} Q^{tt}_{mn} \text{ with,} \tag{0.2.11}$$

$$Q^{tt}_{mn} := \left[\dddot{Q}_{mn} + 3H\ddot{Q}_{mn} + 2H^2\dot{Q}_{mn} + H\ddot{\bar{Q}}_{mn} + 3H^2\dot{\bar{Q}}_{mn} + 2H^3\bar{Q}_{mn}\right]^{tt}(t_{ret})$$

The full flux integral is ρ independent. In particular it means that the total flux across $\mathcal{J}^+$ equals the total flux across the cosmological horizon, $\mathcal{H}^+$.

$$\lim_{\rho\to\infty} \int_{\Sigma_\rho} d\Sigma_\mu J^\mu_T = \int_{\Sigma_{(H\rho=1)}} d\Sigma_\mu J^\mu_T \Leftrightarrow \int_{\mathcal{J}^+} d\Sigma_\mu J^\mu_T = \int_{\mathcal{H}^+} d\Sigma_\mu J^\mu_T . \tag{0.2.12}$$

0.2.1.2 Flux through null hypersurfaces:

There are two families of future directed null hypersurfaces given by $\eta + \epsilon r + \sigma = 0$, see figure (3). For $\epsilon = +1$, these 45 degree lines in the Penrose diagram are parallel to the cosmological horizon while for $\epsilon = -1$, the lines are parallel to the null boundary of the Poincaré patch. We refer to these as the in-coming ($\epsilon = 1$) and out-going ($\epsilon = -1$) null hypersurfaces. The parameter σ labels members of these families. The null normals of these families are of the form $n_\mu = \gamma(1, \epsilon\hat{x}_i)$. The hypersurface integral is given by,

$$\int_{\Sigma_{(\epsilon,\sigma)}} d\Sigma_\mu J^\mu_T = \epsilon \int_{\lambda_1}^{\lambda_2} d\lambda \int_{S^2} d^2 s\,[\gamma H] \left[\frac{(1+\epsilon)}{8\pi}\frac{\eta^2}{\eta - r}\right] \langle Q^{tt}_{mn} Q^{mn}_{tt} \rangle \qquad (0.2.13)$$

It is immediately clear that the flux through the out-going null hypersurfaces (ρ or r increase along these) vanishes indicating sharp propagation of energy. In the $\epsilon = 1$ family, only the cosmological horizon is of interest. For this we have ($\eta = -r$) and we choose the factor $\gamma = -(Hr)^{-1}$ so the null normal matches with the Killing vector (γ is negative as desired for future orientation) and the affine parameter λ matches with the Killing parameter τ. With this choice, the flux in eqn. (0.2.13) matches with that given in eq. (0.2.10) for $H\rho = 1$. Thus, once again, the full flux through cosmological horizon is exactly same as that of $r_{physical} = const$ hypersurfaces.

0.2.1.3 Flux through a constant η slice:

The hypersurface Σ_{η_0} defined by $\eta = \eta_0$ is a cosmological slice $\sim \mathbb{R}^3$. It is space-like, with a normal $n_\mu = -|H\eta_0|^{-1}(1, \vec{0})$. The hypersurface integral can be expressed as,

$$\int_{\Sigma_{\eta_0}} d\Sigma_\mu J^\mu_T = \int_{r_1}^{r_2} \frac{dr}{H(r - \eta_0)} \int_{S^2} d^2 s \left[\frac{-1}{8\pi}\right] \langle Q^{tt}_{mn} Q^{mn}_{tt} \rangle \qquad (0.2.14)$$

As the flux across out-going null hypersurface is zero, the flux across two hypersurfaces Σ_{η_1} and Σ_{η_2} cannot be equal, see the right side figure of (3). In the limit $\eta_0 \to 0$ with

$(r_1, r_2) \to (0, \infty)$, the hypersurface becomes $\mathcal{J}^+$ and the integration measure becomes $\frac{dr}{Hr}$. As the hypersurface integral when expressed in terms of the Killing parameter, has a minus sign due to the reversal of the induced orientation. The measures (positive) themselves are related as $\frac{dr}{Hr} = d\tau$, leading to $\int_0^\infty dr/Hr = -\int_{-\infty}^{\infty} d\tau$ and we get,

$$\lim_{\eta_0 \to 0} \int_{\Sigma_{\eta_0}} d\Sigma_\mu J^\mu_T = \lim_{\rho \to \infty} \int_{\Sigma_\rho} d\Sigma_\mu J^\mu_T \ . \tag{0.2.15}$$

In the previous subsections, we assembled fluxes through various hypersurfaces, all having the topology $\varDelta \times S^2$. We considered $\varDelta$ to be a finite interval and also the cases with $\varDelta = \mathbb{R}$. The relevant hypersurfaces have $\rho =$ constant. In all cases, the energy flux integral has the form,

$$\mathcal{F}(a, b) := \int_a^b d\tau \int_{S^2} d^2 s \left[\frac{1}{8\pi}\right] \langle Q^{tt}_{mn} Q^{mn}_{tt} \rangle \tag{0.2.16}$$

The sharp propagation of energy can be used to define the instantaneous emitted power as :

$$\mathcal{P}(\tau) \ := \ \lim_{\delta\tau \to 0} \frac{\mathcal{F}(\tau + \delta\tau, \tau)}{\delta\tau} \ = \ \frac{1}{8\pi} \int_{S^2} d^2 s \langle Q^{tt}_{mn} Q^{mn}_{tt} \rangle \ . \tag{0.2.17}$$

An explicit averaging procedure is illustrated in an appendix. It permits to split four dimensional averaging integral into an integral over ρ =const. hypersurface and a three dimensional flux integral. Averaging integral over ρ =const. hypersurface and angular integral can be done explicitly, leaving the four dimensional averaging integral to a time-averaged quantity only. This allows to compare the energy flux with that of given by Ashtekar et. al in covariant phase space formalism [8].

0.2.2 Cosmological horizon as effective infinity

As mentioned earlier cosmological horizon for spatially compact source is well defined: it is the past light cone of the common point on $\mathcal{J}^+$ where the source world tube converges. To maintain finite physical distance between different component of spatially compact source, world-tube has to converge to a point. Equally well, any observer who remains at a finite physical distance from a compact source for all time must necessarily lie within cosmological horizon. Furthermore, neither any such observer, nor the source has any access to energy which has crossed the horizon. In this context cosmological horizon is special from other $\rho = const.$ surfaces even for non-sharp propagation. Hence cosmological horizon does share physically relevant properties with the future infinity.

Further support for the role of cosmological horizon as 'future null boundary' comes from the computations of the energy flux. For these, we employed the effective ripple stress tensor and showed that the fluxes defined at $\mathcal{J}^+$, also matches with those computed at the horizon.

Momentum flux can also be computed from Isaacson prescription. All the results of energy flux also holds for momentum flux. The main lesson that follows from the analysis is that energy-momentum flux can be evaluated on the cosmological horizon. The same does not hold for angular-momentum flux - as it is well known that Isaacson procedure does not suffice to capture flux of angular momentum even in flat background.

0.3 Discussion and Outlook

The thesis poses the problem of implications of cosmological constant for gravitational waves. It is relevant for both observational and conceptual perspectives and the focus is on the former. In particular we obtain linearized field in terms of source moments. Many non-trivialities regarding solution of non-homogeneous wave equation, e.g. definition of moments, relation of moments with source integral are discussed. Solutions are obtained in two different gauges and in two different charts , namely FNC and conformal coordinate. Computation in conformal coordinate takes the advantage of conformally flat background metric and the solution is valid for all the way up to $\mathcal{J}^+$ of future Poincaré patch, while the FNC is very natural to the local analysis and gives the answer as corrections to the corresponding Minkowski answer. For comparison a gauge invariant quantity, deviation scalar is introduced and computed in both charts. Subsequent flux calculation demonstrates that radiated power can be computed at cosmological horizon and energy-momentum flux propagation is sharp, even though the field has a tail term.

There are several open directions to pursue:

As mentioned earlier concept of radiation field is not well defined, when $\mathcal{J}^+$ is non null. In de Sitter case cosmological horizon is well defined for compact source. Cosmological horizon being a null hypersurface, we do have the Weyl scalar Ψ_4 defined on it, which is independent of the direction of null geodesics meeting the horizon. One of the main results of this thesis is that in the context of energy-momentum flux radiated by spatially compact source in de Sitter background, cosmological horizon can be treated as effective null infinity. Could cosmological horizon be used as an effective $\mathcal{J}^+$ to analyze gravitational waves in full non-linear theory? One can also extend Bondi's non-linear analysis of axis symmetric gravitational wave with an expectation to get analogous 'mass' and 'news function'.

In chapter (0.1) we showed that leading order correction of field in Poincaré patch is in $\sqrt{\Lambda}$ while in FNC it is of order Λ. We suspect $\sqrt{\Lambda}$ behaviour may be an artifact of conformal compactification of de Sitter metric in Poincaré patch. FNC calculation, ignores the global structure of space-time and the answer has a form of perturbation expansion around Minkowski background. One can use conformal FNC [9] to extend the FNC solution beyond cosmological horizon, which may explain the $\sqrt{\Lambda}$ correction. Exactly how this happens remain to be understood.

There are good reasons to expect that the Universe is permeated by a stochastic background of gravitational waves generated in the early Universe (a stochastic background can also emerge from the incoherent superposition of a large number of astrophysical sources). Our early Universe is modelled faithfully by FLRW background. One can study linearized gravitational wave to a FLRW background in presence of positive cosmological constant. Qualitatively our analysis is different due to non compactness of matter sources in FLRW background.

There are two types of observations which can be impacted by the modified quadrupole formula. One is the orbital decay of an inspiraling binary and other is the modification of the waveform at the detector. To the extent that the Hulse-Taylor pulsar observations have already vindicated the quadrupole formula computed in Minkowski background at the accuracy of 10^{-4}, one does not expect the radically different nature of $\mathcal{J}^+$ to play any significant role in such indirect detections. A physically relevant question then is: how are the effects of positive Λ to be estimated quantitatively? To extract the signal of gravitational wave from the noise of detector we need the waveform to a very high PN order. For number of cycles spent in the bandwidth of ground-based detector to be $O(1)$, we need at least 2.5 PN of phase calculation. A more accurate computation is really required in order to exploit optimally the information contained in the output of a ground based interferometer, at least upto 3 PN and better yet to 3.5 PN. One of the key ingredients that feeds into phase computation is that power radiated by gravitational wave. Schematically,

$$\frac{d\phi}{dt} = \frac{d\phi}{dE}\frac{dE}{dt} \tag{0.3.1}$$

First term is computed from orbital parameter of the source, the second term is due to power radiated by gravitational wave. We can now use new power loss formula for positive cosmological constant in this computation. Hence a problem of immediate concern is to estimate Λ corrections in orbital phase of inspiral binary system due to new power loss formula. Are these corrections comparable to higher order PN corrections?

1 Introduction

On September 14, 2015 gravitational wave detectors in Livingston and Hanford recorded 'chirping' of space-time - gravitational waves were detected directly for the first time, 100 years after their existence was originally predicted by Einstein [10]! Einstein's general theory of relativity interprets gravity as curvature of space-time geometry. Space-time geometry is a dynamic entity whose curvature is determined by matter (mass or energy) content of space-time while motion of the matter is dictated by curvature of space-time. One of the crucial predictions of general relativity is the existence of gravitational waves, i.e. ripples on the fabric of space-time which are caused by motion of matter and propagate at the speed of light.

There was initial controversy about physical existence of gravitational waves and energy carried away by them. In 1918 Einstein had derived celebrated quadrupole formula in the *linear approximation* around Minkowski background . He found that the energy carried away by *transverse- traceless* part of gravitational wave is proportional to square of third time derivative of the mass quadrupole moment [11]. The debate was about whether gravitational waves even exist in full, non-linear general relativity [12]. Beyond the linear approximation can one still distinguish *physical* gravitational waves in full, non-linear general relativity? Since in full non-linear theory there is no global coordinate to characterize gravitational wave metric, it was unclear whether gravitational waves were physical phenomena or merely coordinate artifacts. This long standing controversy was finally resolved in the early 1960s by the work of Bondi, Sachs, Penrose and others [1, 3, 13]. The first detailed study of full non-linear gravitational radiation by examining the asymptotic structure of the gravitational field of isolated systems was carried out by Bondi, Burg and Metzner [1]. Bondi and his collaborators provided a detailed analysis of axis symmetric gravitational waves in full non-linear theory along with a definition of energy that is carried away from an isolated system by gravitational radiation. Sachs extended it for generalized case [2]. Bondi and his collaborators performed a systematic expansions of gravitational wave metric along out going null directions. In asymptotic region they matched the gravitational wave metric with axis symmetric time independent generalized metric (Weyl metric) identifying total mass of the system. Bondi introduced the term 'News function' - whose absolute square integrated over the sphere at infinity, measures the rate of energy loss by an isolated system.

Later Penrose geometrized the results due to Bondi-Sachs via conformal completion technique, attaching a boundary to physical space-time [3, 14]. In an asymptotically simple spacetime using appropriate null tetrad, the peeling theorem shows that Ψ_4 falls off as r^{-1}, and more generally that Ψ_n falls off as r^{n-5} (r is understood as affine parameter along null direction) [2, 15]. Hence presence of gravitational radiation is characterized by $\Psi_4^0(t, \theta, \phi) := \lim_{r\to\infty} r\Psi_4$ being nonzero. For asymptotically flat space-time identification of Ψ_4^0 as radiation field is *unambiguous*. The value of Ψ_4^0 is the same for all different null geodesics approaching the same point on $\mathcal{J}^+$ while the other components Ψ_i^0, get mingled with each other. Ψ_4^0 plays key role in extracting physical information of gravitational wave in numerical relativity [16].

It is relevant to mention that in presence of gravitational radiation (non-zero 'news') asymptotic symmetry group is the Bondi-Metzner-Sachs (BMS) group, having a structure very similar to that of the Poincaré group. The only difference is that while the Poincaré group is a semidirect product of the Lorentz group and a 4-dimensional group of translations, the BMS group is the semidirect product of the Lorentz group and an infinite- dimensional 'angle-dependent' translational group, called the group of the super translations. A particularly useful feature of the BMS group is that it admits a unique 4-dimensional normal sub-group of translations, which can be used to obtain unambiguous notions of energy-momentum [13, 17, 18]. Exploiting time translational symmetry of asymptotically flat space-time, Bondi mass for radiating isolated system is defined on the 'cuts' of future null infinity and it measures the energy of space-time at retarded instant of time. When the 'cut' is taken to the past limit, $u := t - r = -\infty$, Bondi mass matches with the ADM mass defined at spatial infinity [19, 20]. Bondi mass is not constant in time, but decreases with retarded time parameter, showing that gravitational radiation carries away positive energy.

All these foundational frameworks of gravitational waves are established in the context of asymptotic flat space- time ($\Lambda = 0$). However by now cosmological observations (e.g. red shift of type Ia supernovae) have established that the universe is undergoing an accelerated expansion which is most simply explained by a positive Λ. Hence it is reasonable to ask whether the description of gravitational field due to compact isolated systems and the characterization of gravitational radiation obtained for $\Lambda = 0$ can be generalized for $\Lambda > 0$ (de-Sitter).

This is nontrivial for following reasons:

- Even a tiniest value of cosmological constant qualitatively changes the structure of null-infinity ($\mathcal{J}^+$) from being a null hypersurface to space-like. All the rich structures of asymptotically flat space-time are not available for $\Lambda > 0$.

- Space-like character of $\mathcal{J}^+$ poses challenges for defining energy, momentum and their fluxes. The asymptotic symmetry group is Diff($\mathcal{J}$) which does not admit a preferred 4-dimensional group of translations. Therefore, notion of energy-momentum carried by gravitational waves is not available. There is no positive energy theorem - energy can take any value near space-like $\mathcal{J}^+$.

- As emphasized by Penrose the concept of radiation is 'less invariant' in cases when

$\mathscr{J}^+$ does not have a null character. Namely the radiative component Ψ_4^0 of the field, may differ for different null geodesics approaching the same point on $\mathscr{J}^+$.

- We do not have an invariant notion of gravitational radiation in the non-linear regime, nor an analogue of the 'News function' for asymptotically de Sitter space-time.

While these issues are highly nontrivial and important, in this thesis we will address simpler question : how does linearized theory work out in de Sitter background? As linearized theory is a good model to study 'far zone' and asymptotic behavior of compact gravitating system, we hope this may provide some clue in full nonlinear analysis. A positive value of Λ introduces constant curvature to the background. Therefore, even for compact astrophysical sources, the intuition from Minkowski background is challenged at every step. We would like to obtain linearized field solution in terms of source moments. A priori it is not clear how to define source moments in curved background. As the background is curved, one may expect some nontrivial tail integral in wave propagation even in first order Post Newtonian level. The next level of question is to identify the physical (gauge invariant) attributes of the wave solution. In the Minkowski background, the linearized Riemann tensor is gauge invariant and consequently the induced geodesic deviation or tidal distortion is a physical effect of the waves. We ask what is the analogous statement of gauge invariant geodesic deviation in de Sitter background?

Since gravitational waves are capable of doing work, we could also ask for a measure of the energy carried by the waves. In particular, what are the modifications to the quadrupole formula for radiated power? The natural strategy for defining a measure of energy through a stress tensor, does not work for gravity. There is simply no gauge invariant, tensorial definition of a gravitational stress tensor. There are two approaches taken for a measure of the *flux of gravitational energy*. One is based on an *effective gravitational stress tensor* tailored for the context wherein there are two widely separated scales, $\lambda \ll L$, of spatio-temporal variations of the metric which are used to identify the

$L-$scale component of the metric as a *background metric* and $\lambda-$scale component as a small *ripple* [21]. The other approach directly defines the *flux of gravitational radiation* in reference to the *null infinity* using the the canonical structure of the space of asymptotically flat/de Sitter solutions of the Einstein equation. This is applicable for all spatially compact sources [22]. In chapter 4, using *effective gravitational stress tensor* and exploiting time-translational symmetry of de Sitter background we derive power radiated quadrupole formula for $\Lambda > 0$. For comparison we also discuss the other approach.

Presence of cosmological constant not only imposes theoretical challenges, it has also observational relevance. Value of cosmological constant being tiny, $10^{-29} gm/cc$ or $10^{-52} m^{-2}$ in the geometrized unit with $G = 1 = c$, one can neglect Λ in the vicinity of astrophysical sources or near ground-based gravitational wave detectors. Signature of Λ may be visible over the vast distances of source-free regions in which gravitational waves propagate. To get a sense of our region of interest, let us take the example of Schwarzschild-de Sitter space-time. In static coordinates, Schwarzschild-de Sitter metric is given by,

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)} + r^2 d\Omega_2^2 \qquad (1.0.1)$$

For $9M^2\Lambda < 1$, the factor $(1 - 2M/r - \Lambda r^2/3)$ is zero at two positive values of r. The smaller of of these values, which we shall denote by r_H can be regarded as the position of the black-hole event horizon, while the larger value r_C represents the position of the cosmological event horizon for observers on world lines of constant r between r_H and r_C. The exact expressions for r_H and r_C are available in literature [23]. For our purpose it suffices to mention that r_H is larger than that of the Schwarzschild radius in Minkowski space and r_C is smaller than the radius of cosmological horizon in pure dS_4. The repulsive nature of de Sitter potential increases the Schwarzschild radius while attractive nature of Schwarzschild potential decreases the de Sitter radius. Hence for observational context our region of interest is

$$r_H \ll r \ll r_C \qquad (1.0.2)$$

Our estimate is compatible with current ground based, advanced LIGO detector for which largest value of r is of order 1 Gpc away (well inside the cosmological radius of $\sqrt{3/\Lambda} \sim$ 5 Gpc).

It would be more interesting to know if there are significant observable differences, particularly in the light of newly opened window of gravitational wave astronomy. Does Λ induce any significant change in orbital decay of binary system or in the phase calculation of gravitational waveform template? We explore some of these aspects in chapter 5.

We use following conventions. Space-time is assumed to be 4-dimensional and its metric has signature $(-,+,+,+)$. Except chapter 5, we set $c = 1, G = 1$. The Riemann tensor is defined as $2\nabla_{[\mu}\nabla_{\nu]}A_\rho = R^\sigma{}_{\rho\nu\mu}A_\sigma$. The Ricci tensor is $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ and the Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu}$. Throughout we denote spatial *transverse-traceless* gauge condition on gravitational perturbation as TT while tt denotes projected components of field which satisfies spatial TT condition to the leading order in $1/r$.

Organization of the thesis is as follows. In chapter 2, we recall some relevant coordinate systems of de Sitter space-time. Chapter 3 is devoted in obtaining linearized solution in terms of source moments. We present the solution in two different co-ordinate systems, namely fermi normal co- ordinate (in static patch) and conformal co-ordinate (in future Poincaré patch) of de Sitter background. A gauge invariant quantity, 'deviation scalar' is constructed for comparison. In chapter 4, we derive power radiated quadrupole formula using Isaacson's effective stress tensor in presence of Λ and argue in what context cosmological horizon may be treated as effective infinity of de Sitter. In chapter 5, we discuss some relevant observable effects of cosmological constant . Last chapter contains a summary of results and a discussion of open problems. Requisite mathematical information is included in appendices, A,B, C,D.

2 Geometry of de Sitter space-time

Any physical space-time satisfies Einstein's field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.0.1}$$

In the absence of matter source one finds the vacuum Einstein equation,

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{2.0.2}$$

It is well known that a maximally symmetric space-time which admits the maximum numbers of Killing vectors, is locally characterized by, $R_{\rho\sigma\mu\nu} = \frac{R}{12}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right)$. This equation is equivalent to $R_{\mu\nu} = \frac{R}{4}g_{\mu\nu}$, hence Riemann tensor is completely determined by Ricci scalar R. It immediately follows from contracted bianchi identity, $(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu})_{;\,\mu} = 0$, that R is constant throughout the space-time, in fact these space-times are homogeneous and isotropic. Therefore, maximally symmetric space-times are solutions of vacuum (i.e. matter free) Einstein equation with $R = 4\Lambda$. For vanishing curvature ($R = 0$), maximally symmetric space-time is simply the Minkowski space-time. The space for $R > 0$ (or $R < 0$) is de Sitter (or anti de Sitter) space-time.

The 4-dimensional de Sitter space, dS_4 can be visualized as the one-sheeted hyperboloid,

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \frac{1}{H^2} \tag{2.0.3}$$

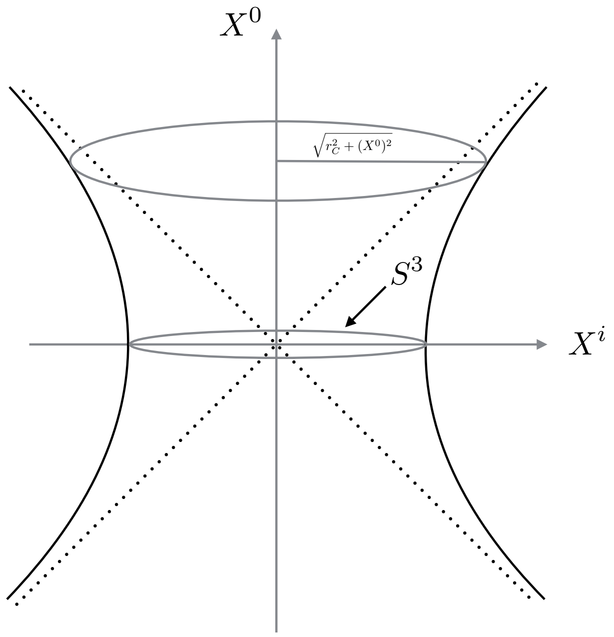

Figure 2.1. De Sitter space-time represented as hyperboloid in five dimensional Minkowski space-time. Each $X^0 = const$ slice represents 3 dimensional sphere with radius $\sqrt{(X^0)^2 + 1/H^2}$. The circumference of the space contracts upto present epoch ($X^0 = 0$) forming a 'bottle-neck' and then expands.

embedded in five-dimensional flat Minkowski space-time $\mathcal{M}^{1+4}$. $\sqrt{(X^0)^2 + 1/H^2}$ is interpreted as radius of 3-dimensional sphere. In fact the hyperboloid is surface of revolution of $\sqrt{(X^0)^2 + 1/H^2}$ around X^0 axis in the ambient space $\mathcal{M}^{1+4}$. $1/H$ is usually interpreted as cosmological length scale that equals $\sqrt{\frac{3}{\Lambda}}$ (often referred to as the Hubble length). As dS_4 is embedded in $\mathcal{M}^{1+4}$, the isometries are characterized by five dimensional rotational group, $SO(1,4)$. Hence full de Sitter space-time has 10 killing fields, same as that of Lorentz generators in 5 dimensional Minkowski space.

2.1 Conformal infinity

In this section we recall some basic concepts of conformal space-time which will be necessary to introduce Penrose diagrams of de Sitter space-time. Following the general formalism, let us consider a physical space-time $(M, g_{\mu\nu})$. The idea is to construct another 'unphysical' space-time $(\hat{M}, \hat{g}_{\mu\nu})$ with a boundary $\mathcal{J}$, such that M is conformal to the interior of $\hat{M}$ with $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. On boundary $\mathcal{J}$, $\Omega = 0$ and $n_\mu := \hat{\nabla}_\mu \Omega$ is nonzero on $\mathcal{J}$. This construction ensures that $(\hat{M}, \hat{g}_{\mu\nu})$ is a conformal completion of physical space-time $(M, g_{\mu\nu})$ in which the boundary $\mathcal{J}$ is at infinity with respect to physical metric $g_{\mu\nu}$. Both space-times have identical local causal structure. Global character of space-time is determined by the norm σ of normal vector to $\mathcal{J}$ [24] . We introduce a normal vector $\hat{n}^\mu$ to boundary $\mathcal{J}$,

$$\hat{n}^\mu = \hat{g}^{\mu\nu}\, \hat{\nabla}_\nu \Omega \quad , \qquad with \;\; \hat{g}_{\mu\nu}\, \hat{n}^\mu\, \hat{n}^\mu = \sigma \;\; . \tag{2.1.1}$$

Norm of of the normal vector σ can be obtained from the relation between conformal and physical Ricci scalars,

$$\hat{g}_{\alpha\beta} \hat{\nabla}^\alpha \Omega\, \hat{\nabla}^\beta \Omega = -\frac{R}{12} + \Omega \left[\frac{1}{2} \hat{\Box}\, \Omega + \frac{\Omega}{12} \hat{R} \right] \implies \sigma := \hat{g}_{\alpha\beta}\, \hat{\nabla}^\alpha \Omega\, \hat{\nabla}^\beta \Omega \,\Big|_{\mathcal{J}} = -\frac{R}{12} \tag{2.1.2}$$

Hence from 2.0.2 the character of conformal infinity $\mathcal{J}$ is determined by sign of Λ, namely

$$\Lambda \begin{cases} < 0 & : \quad \mathcal{J} \text{ is timelike} \\ = 0 & : \quad \mathcal{J} \text{ is null} \\ > 0 & : \quad \mathcal{J} \text{ is spacelike} \end{cases} \tag{2.1.3}$$

2.2 Coordinates and Penrose diagrams

In this section we introduce different co ordinate systems on de Sitter space-time which are important for different physical situations. There are many illuminating reviews on de Sitter Coordinates [25–27]. We summarize it briefly.

2.2.1 'Global' coordinates

Eq. 2.0.3 can be satisfied identically by the relations

$$X^0 = H^{-1}\sinh(Ht)\ , \qquad X^1 = H^{-1}\cosh(Ht)\ \cos\chi, \tag{2.2.1}$$

$$X^2 = H^{-1}\cosh(Ht)\ \sin\chi\ \cos\theta\ , \qquad X^3 = H^{-1}\cosh(Ht)\ \sin\chi\ \cos\theta\ \cos\phi, \tag{2.2.2}$$

$$X^4 = H^{-1}\cosh(Ht)\ \sin\chi\ \cos\theta\ \sin\phi. \tag{2.2.3}$$

In these coordinates the induced line element on dS_4 has the form

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2 \tag{2.2.4}$$

$$= -dt^2 + H^{-2}\cosh^2(Ht)\ \left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{2.2.5}$$

where $-\infty < t < \infty,\ 0 \le \chi \le \pi,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$. θ, χ, ϕ are interpreted as angle variable of S^3. Note that singularities in the metric 2.2.5 at $\chi = 0,\ \chi = \pi$ and $\theta = 0,\ \theta = \pi$ are simply coordinate artifacts that occur with polar coordinates. Apart from these trivial singularities the coordinates $(\tau, \chi, \theta, \phi)$ cover full dS_4 and therefore referred as global coordinates. At fixed t, the line element 2.2.5 describes the spacelike hypersurfaces of 3-dimensional spheres of radius $H^{-1}\cosh\ (Ht)$ which is infinitely large at $t = \pm\infty$ and of minimum length at $t = 0$. From the line element in global coordinates it can be concluded that topology of de Sitter space is $\mathbb{R} \times S^3$.

To study infinity of de Sitter space-time, we introduce a new time coordinate $\tau = 2\tan^{-1}(e^{Ht}) - \frac{1}{2}\pi$ with $-\frac{\pi}{2} < \tau < \frac{\pi}{2}$. In this coordinates the metric becomes,

$$ds^2 = \frac{1}{H^2}\sec^2\tau\left[-d\tau^2 + d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{2.2.6}$$

This suggests that we may identify conformal completion of physical metric as $d\hat{s}^2 = (H^2\cos^2\tau)\ ds^2$. Hence de Sitter space-time is conformal to that portion of the Einstein static universe which is covered by $-\frac{\pi}{2} < \tau < \frac{\pi}{2}$. The Penrose diagram of de Sitter space is accordingly as in right panel of figure (2.2). The condition $\Omega = 0$ determines the hypersurface $\mathcal{J}$. In $(\tau, \chi, \theta, \phi)$ coordinate system this gives $\Omega = H \cos\tau = 0$. Therefore $\tau = +\frac{\pi}{2}$ and $\tau = -\frac{\pi}{2}$ correspond to $\mathcal{J}^+$ and $\mathcal{J}^-$ respectively. This also confirms the space-like character of $\mathcal{J}$ in de Sitter. Each point in this diagram is an S^2, except for the points on $\chi = 0$ and $\chi = \pi$ (where the radius of two sphere vanishes). Unlike Minkowski space-time one of the peculiar features of de Sitter space is that no single observer can access the entire space-time (see Fig. 2.2). An observer at $\chi = 0$, at any moment in time, cannot see

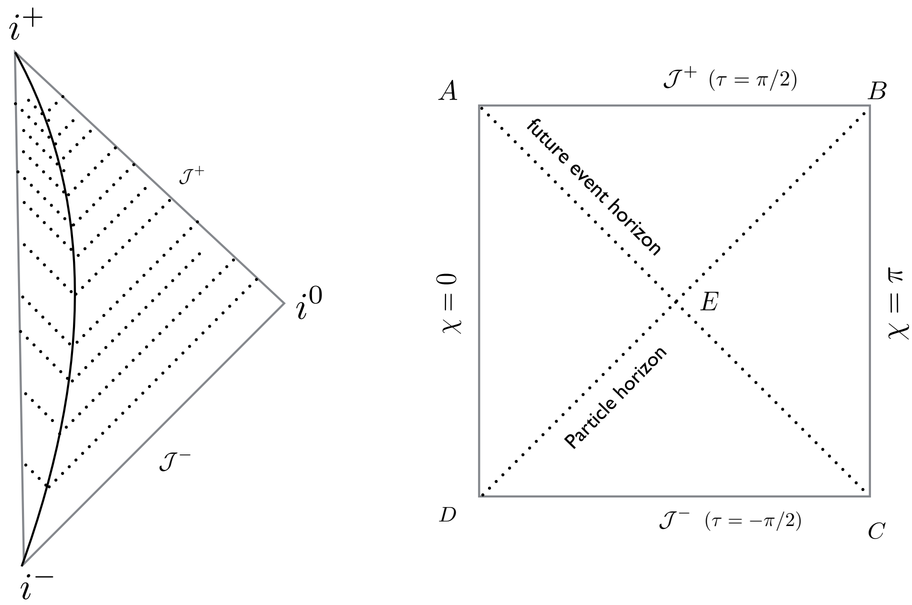

Figure 2.2. Left panel clarifies that any observer can access full Minkowski space-time. While right panel shows an observer along line AD can influence ABD region of space-time and it can access information from ACD portion of space- time.

anything from $\chi = \pi$ line due to *observer's future event horizon*. We will refer this event horizon as cosmological horizon. Further the observer can never send any information to the lower triangle of space-time due to observer's *particle horizon*. In this thesis our focus is on gravitational waves emitted from a compact source on de Sitter background. In our subsequent discussion we always assume that the compact source is confined near the line AD. Hence for our purpose it is sufficient to concentrate on ABD portion of full de Sitter. For that purpose in next section we will introduce cosmological coordinates which is convenient to describe upper half (ABD) of full de Sitter space-time.

In global coordinates axial symmetry is evident, hence $\partial/\partial\phi$ is a killing vector. Unlike Minkowski space-time, $\partial/\partial\tau$ is not a killing vector in global coordinates.

2.2.2 Cosmological coordinates

One can also introduce

$$\begin{aligned} X^0 &= -H^{-1}\sinh(Ht) + Hr^2\frac{e^{Ht}}{2} \quad , \quad X^1 = -H^{-1}\cosh(Ht) - Hr^2\frac{e^{Ht}}{2} \quad , \\ X^i &= e^{Ht}x^i \quad , \quad i = 2,3,4 \quad ; \quad r^2 := \sum_i (x^i)^2 \quad ; \quad t,\ x^i \in \mathbb{R}\,. \end{aligned} \tag{2.2.7}$$

on the hyperboloid. Since t ($H^{-1}\ln H\,(X^0 + X^1)$) is not defined for $X^0 + X^1 \le 0$, these coordinates cover only upper triangular half of the full de Sitter space as shown in figure 2.3. This portion of space- time is referred as future Poincaré patch. In these coordinates line element becomes,

$$ds^2 = -dt^2 + e^{2Ht}\sum_{i=2}^{4}(dx^i)^2 \tag{2.2.8}$$

Some times it is useful to introduce conformal time, $\eta := -H^{-1}\ e^{-Ht}$. In conformal coordinates (η, x, y, z) the metric becomes,

$$ds^2 = \frac{1}{H^2\eta^2}\left[-d\eta^2 + \sum_i (dx^i)^2\right] \ , \ \eta \in (-\infty, 0) \ , \ H := \sqrt{\frac{\Lambda}{3}}. \tag{2.2.9}$$

While conformally flat nature of this metric simplifies the computation of gravitational perturbation in de Sitter background, these coordinates are ill-suited for taking $\Lambda \to 0$ limit. To take this limit we have to work with proper time t and in cosmological coordinates (t, x, y, z), metric (2.2.8) also goes Minkwoski metric as we take $\Lambda \to 0$. In (η, x, y, z) coordinates $\mathcal{J}^+$ is approached as $\eta \to 0_-$ while the $\eta \to -\infty$ corresponds to the Robertson-Walker singularity.

In order to find the Killing vector field of time translations we let $t \rightarrow t + \delta t$ in the above metric. This leads to

$$ds^2 \rightarrow -dt^2 + e^{2Ht}(1 + 2H\,\delta t)\sum_{i=2}^{4}(dx^i)^2 \tag{2.2.10}$$

To make the metric invariant under this time translation the spatial coordinates, x^i, must be dilated $x^i \rightarrow x^i - Hx^i\delta t$. In general a Killing vector ξ^α is an infinitesimal generator of isometry, i.e. for $x^\alpha \rightarrow x^\alpha + \epsilon\xi^\alpha$, the metric remains invariant. Identifying $\epsilon = \delta t$ the Killing vector which generates time translation in the cosmological coordinate system is $\xi^\alpha = (1, -Hx^i)$. It reduces to time translational generator in Minkowski space-time as

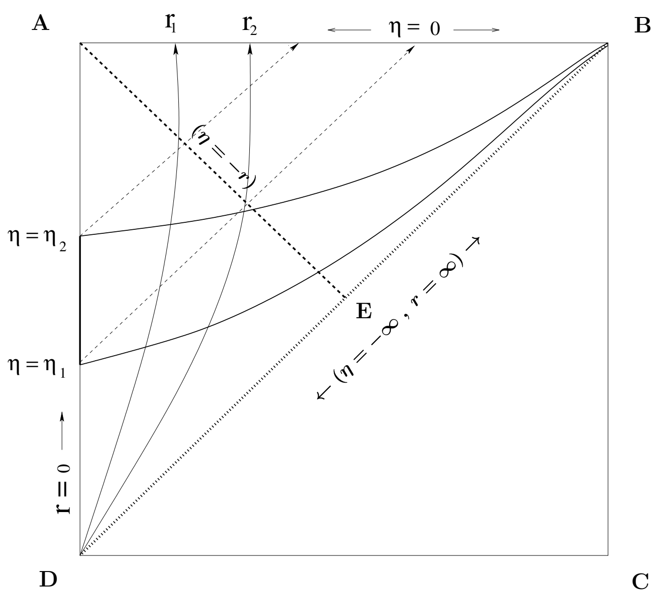

Figure 2.3. The full square is the Penrose diagram of de Sitter space-time with generic point representing a 2-sphere. The future Poincaré patch labeled ABD, is covered by the conformal chart (η, r, θ, ϕ). The line BD does not belong to the chart. The line AB is the *future null infinity*, $\mathcal{J}^+$ and the line AE is the *cosmological horizon*. Two constant η space-like hypersurfaces are shown with $\eta_2 > \eta_1$. The two constant r, time-like hypersurfaces have $r_2 > r_1$. The two dotted lines at 45 degrees, denote the paths of gravitational waves emitted at $\eta = \eta_1, \eta_2$ on the world line at $r = 0$, through the source. During the interval (η_1, η_2), the source is 'active' i.e. varying rapidly enough to be in the detectable range of frequencies. The region AED is a static patch.

$\Lambda \rightarrow 0$. In conformal chart, this time translational Killing vector becomes $T^\mu = -H(\eta, x^i)$. In addition we have 3 spatial translational and 3 spatial rotational killing vectors tangential to $\eta = const$ surface. [7,8].

2.2.3 Static coordinates

The constraint of hyperboloid in 2.0.3 can be expressed in terms of two constraints,

$$-(HX^0)^2 + (HX^1)^2 = 1 - H^2r^2 \tag{2.2.11}$$

$$(HX^2)^2 + (HX^3)^2 + (HX^4)^2 = H^2r^2 \tag{2.2.12}$$

where the first describes a hyperbola and the second a 2-dimensional sphere of radius r. It can be parametrized by

$$X^0 = -H^{-1}\sqrt{1 - H^2r^2}\sinh(Ht) \ , \qquad X^1 = -H^{-1}\sqrt{1 - H^2r^2}\cosh(Ht) \ , \tag{2.2.13}$$

$$X^2 = r\sin\theta\cos\phi \ , \quad X^3 = r\sin\theta\sin\phi \ , \quad X^4 = r\cos\theta. \tag{2.2.14}$$

where $0 \le r \le H^{-1}$ and $-\infty < t < \infty$. Noting that $-X^0 + X^1 \le 0$ and $X^0 + X^1 \le 0$, the coordinates cover only one quarter of the de Sitter space as shown in figure 2.3. In this parametrization line element becomes,

$$ds^2 = -(1 - H^2r^2)\, dt^2 + \frac{dr^2}{1 - H^2r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.2.15}$$

In these coordinates two Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$ are manifest, correspondingly the space-time has axial and time-translational symmetries. Time like killing vector $\partial/\partial t$, becomes null when $r = \frac{1}{H}$. Physically, this null surface corresponds to the cosmological horizon of de Sitter space-time.

It should be noted that 10 dimensional symmetry group of full de Sitter is reduced to a 7 dimensional subgroup in future Poincaré patch and in static patch the symmetry group is even smaller, 4 dimensional. Null hyperplane BD can be thought as adding an additional boundary to full de Sitter Space. Hence symmetry generators which are not tangential to BD are truncated in Poincaré patch, those are not the symmetries of Poincaré pach. Similarly for static patch, killing fields which are not tangential to cosmological horizon (AE) are not the symmetry generator in static patch. It can be seen easily from figure 2.3 spatial translation symmetry of Poincaré patch along $\eta = const$ surface is not available in static patch.

We now have three relevant coordinate systems at our disposal: the global, conformal and static coordinate systems. The global coordinate system is suitable for describing processes comoving with the expansion/contraction of dS_4. The static coordinate system is relevant for processes as observed by one particular observer and it is relevant for describing Schwarschild de Sitter universe. The conformal coordinate system is useful for describing spatially flat cosmological model. While conformal coordinate is extended all the way upto $\mathcal{J}^+$ static coordinates are restricted upto cosmological horizon. To study linearized gravitational perturbation on dS background we will work on future Poincaré patch and static patch.

3 Gravitational Waves from Compact Sources in de Sitter Background

3.1 Introduction

Having introduced different relevant coordinate systems in de Sitter background, at this stage, it is worth noting the different facets of the gravitational fields far away from dynamical sources such as astrophysical bodies. The most basic question is: what is the field due to a spatially compact source in de Sitter background at large separations? The very characterization of compact sources in de Sitter background presumes a source free region where vacuum equations, including the positive cosmological term, hold. Thus at large separations we have a natural split of the field into a background and a small deviation caused by the source. The simplest approach is then to linearize the Einstein equation about de Sitter background and study its solutions, keeping in mind the inherent non-linear nature of the theory and hoping for reliable estimates. The linearized equation is a wave equation with a finite propagation speed. Among these linear waves are also the fields due to sources which are computed from the retarded Green function. The Green functions of course depend on the choice of 'gauge conditions' on the linear fields and their explicit form depends on the choice of coordinate chart on the background space-time. Our focus in this thesis is on sufficiently rapidly varying, distant, spatially compact sources. A spatially compact source has two natural scales - its physical size R and the scale of its time variation T. For R sufficiently small compared to the distance to the source, d, it is essentially the scale T that is relevant for gravitational radiation and we may take the corresponding equivalent length scale as, $\lambda \sim T$. ($c = 1$ units) On the other hand, the curvature scale of the ambient geometry sufficiently far away from the source, provides the scale L. For Minkowski space-time background, $L = \infty$ whereas for non-zero cosmological constant, $L \sim |\Lambda|^{-1/2}$. A sufficiently *rapidly varying* source is one which has its time scale of variation or equivalent spatial scale $\lambda \ll L$ while a source is *distant* if $\lambda/d \ll 1$. For current interferometric detectors, the scale $\lambda \sim 10^4 - 10^5$ meters, the distances d, are in the range of kilo to hundreds of mega parsecs ($\sim 10^{19} - 10^{24}$ meters) while the spatial extents, R, vary over light seconds or less ($\lesssim 10^8$ meters). We would like to note that induced tidal distortions are needed in the direct detection of gravitational waves, while for indirect detection based on energy loss due to gravitational radiation, reliable flux measures are crucial. In this chapter, we focus on the gravitational field and the induced tidal distortion and in the next chapter we will discuss about energy flux carried away by gravitational waves due to rapidly varying spatially compact source.

In obtaining the field due to a compact source, we follow the basic steps which are well known and well understood for Minkowski background: (a) set up the linearized equations, (b) choose a suitable gauge and obtain a retarded Green function, (c) identify the physical solutions for subsequent computation of geodesic deviation and power radiated and (d) relate the physical field to appropriate source multipole moments. At each of these steps, we encounter new features compared to the computations in the Minkowski background.

As we discussed in section (2.2), unlike the Minkowski space-time which admits a natural, global Cartesian chart, de Sitter space-time has several charts, namely global chart, conformal chart and static chart. We present computations in two different charts: suit-

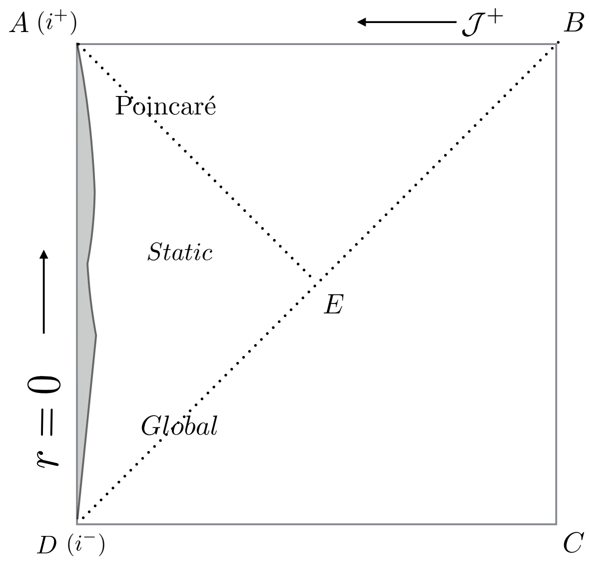

Figure 3.1. ABCD denotes the Global Chart, ABD is a Poincaré patch while AED is a static patch. The angular coordinates, θ, ϕ are suppressed. A compact source is confined near AD with past and future time-like infinity i^-, i^+ respectively.

ably defined *Fermi Normal Coordinates* (FNC) covering the static patch and a conformal chart covering the future Poincaré patch, see figure 3.1.

While physical implications should not depend on choice of charts, their explicit computations do depend on the chosen chart. For convenience as well as for building up intuition, different charts could have different advantages. For instance, the time coordinate of the FNC chart is the Killing parameter of the stationary Killing vector. This reduces the Lie derivative with respect to the Killing vector, to simple coordinate derivative. The metric too is obtained as a Taylor series in the curvature and hence effects due to the cosmological constant can naturally be expected to appear as a power series in Λ. However, this chart is limited to the static patch. By contrast, the conformal chart can be extended all the way upto $\mathcal{J}^+$. Conformally flat nature of background metric in this coordinates considerably simplify the computations. A priori, it is not clear which chart(s) are convenient for what aspect and we present computations for two choices of charts - the FNC and the conformal chart.

After obtaining the linearized equation, the next step is to choose 'a gauge'. The natural choice (also used in the Minkowski background) is the *transverse, traceless (TT) gauge*. But there has been another gauge choice [6], which in the conformal chart simplifies the linearized equations as well as subsequent analysis due to its similarity with the Minkowski space-time. This is a gauge which imposes a variant of the transversality condition. We present the solutions in both gauges. The wave propagation has a *tail* term in both gauges. The TT gauge computations are performed in a FNC system and are restricted to order Λ. The tail term is of order Λ^2. In the second gauge, in a large separation regime, the tail integral can be computed explicitly.

Next, to identify the physical fields, one chooses the so-called *synchronous gauge* which sets all fields with at least one temporal index to be zero. This steps needs a generalization when the background has a curvature and needs a suitable time-like vector field. Fortunately such a generalization is available [28] in a neighbourhood of a Cauchy surface.

In a curved space-time, the notion of *source multipole moments* needs to be defined appropriately. In the Minkowski background, the coordinates of the global chart are *vectors* under spatial rotations on a constant t hypersurface. In a curved background, the local coordinates have no such property. A suitable definition can be constructed using tetrad frame in de Sitter background. This strategy is taken for both FNC and conformal chart. To relate source integral with moments we recast conservation of stress-energy tensor in tetrad frame (similarly one can introduce spin connection to write conservation equation in tetrad frame). In FNC we obtain field in terms of source moments upto the order Λ, while the answer in conformal chart is exact and correction appears in power of $\sqrt{\Lambda}$. In conformal chart, only source quadrupole moment appears in the field solution on the other hand FNC solution depends on additional type of moments.

Having obtained the field in different gauges and in different coordinate systems, it is reasonable to ask how can we compare these two solutions? For all observational perspectives we do not measure the field directly, rather it is important to investigate the rate of change in field. When a transient gravitational waves travel through a region, it stretches and squeezes the fabric of space-time introducing tidal distortion to nearby geodesics. Acceleration of geodesic deviation vector is a measure of tidal distortion. All the interferometric detectors measure these distortions in some form or other. In the Minkowski background, the linearized Riemann tensor is gauge invariant and consequently the induced geodesic deviation or tidal distortion is a physical effect of the waves. In the de Sitter background, the linearized Riemann tensor itself is *not* gauge invariant but thanks to its conformal flatness, certain *deviation scalar* can be constructed which is gauge invariant. We compute this quantity in both FNC and conformal chart for a comparison.

The chapter is organized as follows. In the section 3.2, we recall the linearization procedure together with the associated notion of gauge freedom. We collect the expression for the Ricci tensor up to the quadratic order and give the linearized wave equation for the metric perturbations. We discuss the gauge choices and residual gauge invariance. The section 3.3 is divided in two sub-sections. In the first subsection we choose the usual transverse, traceless gauge. We present the Hadamard form of the retarded Green function and simplify the expression for field due to a localized source, using the Fermi Normal Coordinates (FNC). The leading contribution of the order Λ^0 to the quadrupole field is the same as that in the Minkowski background and we present the order Λ contributions. Here appropriate source moments are defined and the solution in a synchronous gauge is presented. For contrast, in the second sub-section we summarize the computation of the quadrupole field in an alternative gauge [6]. The solution in synchronous gauge is presented in terms of analogously defined source moments. Here, using the cosmological chart, the corrections appear in powers of $\sqrt{\Lambda}$. In section 3.4 we extract out physical solutions of gauge-fixed inhomogeneous equation introducing a local projection operator. In the section 3.5, we present a suitably defined, gauge invariant deviation scalar and compute it for the appropriately projected fields. In the final section 3.6, we summarize and discuss our results. Some of the technical details are given in the appendices A, B, C.

3.2 Linearization about de Sitter background

As mentioned earlier, there are several natural patches and charts available in the de Sitter space-time. To introduce perturbations without referring to coordinates[1], consider a one parameter family of metrics, $g_{\mu\nu}(\epsilon)$ which is differentiable with respect to ϵ at $\epsilon = 0$ and let $\bar{g}_{\mu\nu} := g_{\mu\nu}(0)$ be a given solution of the exact Einstein equation. Define *a perturbation* of the exact solution as: $h_{\mu\nu} := \frac{dg_{\mu\nu}(\epsilon)}{d\epsilon}|_{\epsilon=0}$. As the one parameter families of metrics are varied, we generate the space of perturbations from the corresponding $h_{\mu\nu}$. If every member of the family of metrics solves Einstein equation (with sources and cosmological constant), then the perturbation satisfies a *linear equation* obtained by differentiating the exact equation with respect to ϵ and setting ϵ to zero. Thus every one parameter family of exact solutions of the Einstein equation gives a solution of the linearized equation. The converse is not always true and is known as the linearization instability problem [29]. In our context, this is not a concern. The general covariance of the Einstein equation implies that every one parameter family of metrics, obtained by diffeomorphisms generated by a vector field on a solution to Einstein equation, also solves the equation and leads to a corresponding perturbation satisfying the linearized equation. However, these families give the *same physical space-time*. The corresponding perturbations do *not* give physically distinct, nearby space-times and therefore do not represent *physical perturbations*. These perturbations have the form: $h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}$ where $\mathcal{L}_\xi$ denotes the Lie derivative. To identify the physical perturbations, we have to 'mod out' these perturbations, generated by diffeomorphisms. In other words, physical perturbations are equivalence classes of perturbations:

$$[h_{\mu\nu}] := \left\{ h'_{\mu\nu} / h'_{\mu\nu} = h_{\mu\nu} + \mathcal{L}_\xi \bar{g}_{\mu\nu} \ \forall \text{ vector fields } \xi \right\} .$$

More commonly the expression $h'_{\mu\nu} = h_{\mu\nu} + \mathcal{L}_\xi \bar{g}_{\mu\nu}$ is referred to as a *gauge transformation* and the equivalence classes are of course the gauge invariant perturbations. Thus, by

[1]Some times a coordinate system is presumed in which the metric is split into a background plus small perturbations. This obscures the tensorial nature of the perturbation and can be avoided [30].

definition of gauge transformations, the linearized equation is gauge invariant. While the perturbations are subjected to these gauge transformations, it should be borne in mind that they are tensors with respect to general coordinate transformations.

While the linearization can be specified in a coordinate free manner, explicit computation of solutions needs coordinates to be introduced. In practice, one begins by writing $g_{\mu\nu}(\epsilon, x) \approx \bar{g}_{\mu\nu}(x) + \epsilon h_{\mu\nu}(x)$ and obtains the linearized equation by substituting this in the full equation and keeping terms to order ϵ. Since we consider perturbations of the source free de Sitter solution, the matter stress tensor is of order ϵ while the cosmological constant is of order ϵ^0. Under an infinitesimal diffeomorphism generated by a vector field $\xi^\mu(x)$, $x'^\mu = x^\mu - \epsilon\xi^\mu(x)$, the Lie derivative of the background metric, $\bar{g}_{\mu\nu}(x)$, is given by $\mathcal{L}_\xi \bar{g}_{\mu\nu} = \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$. Here the $\bar{\nabla}$ denotes the covariant derivative with the Riemann-Christoffel connection of $\bar{g}$ and $\xi_\mu := \bar{g}_{\mu\nu}\xi^\nu$. The gauge transformations thus take the form: $h'_{\mu\nu}(x) = h_{\mu\nu}(x) + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$.

We begin by summarizing the expansions of the connection and Ricci tensor to $o(h^2)$.

In the following, the indices are raised and lowered using the background metric which is taken to be a maximally symmetric one. Background quantities carry an overbar.

$$
\begin{aligned}
g^{\mu\nu} &= \bar{g}^{\mu\nu} - \epsilon h^{\mu\nu} + \epsilon^2 h^\mu{}_\alpha h^{\alpha\nu} && (3.2.1)\\
\Gamma^\lambda{}_{\mu\nu} &= \bar{\Gamma}^\lambda{}_{\mu\nu} + \epsilon\left[\frac{1}{2}\bar{g}^{\lambda\alpha}(\bar{\nabla}_\nu h_{\alpha\mu} + \bar{\nabla}_\mu h_{\alpha\nu} - \bar{\nabla}_\alpha h_{\mu\nu})\right] \\
&\quad -\epsilon^2\left[\frac{1}{2}h^{\lambda\alpha}(\bar{\nabla}_\nu h_{\alpha\mu} + \bar{\nabla}_\mu h_{\alpha\nu} - \bar{\nabla}_\alpha h_{\mu\nu})\right] && (3.2.2)
\end{aligned}
$$

$$
\begin{aligned}
R_{\mu\nu} &= \bar{R}_{\mu\nu} + \epsilon R^{(1)}_{\mu\nu} + \epsilon^2 R^{(2)}_{\mu\nu} \\
R^{(1)}_{\mu\nu} &= -\frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu h + \frac{1}{2}\left(\bar{\nabla}_\mu\bar{\nabla}_\alpha h^\alpha{}_\nu + \bar{\nabla}_\nu\bar{\nabla}_\alpha h^\alpha{}_\mu\right) \\
&\quad + \frac{1}{2}\left(\bar{R}_{\mu\alpha}h^\alpha{}_\nu + \bar{R}_{\nu\alpha}h^\alpha{}_\mu\right) + \bar{R}_{\mu\alpha\beta\nu}h^{\alpha\beta} \qquad h := \bar{g}^{\alpha\beta}h_{\alpha\beta}\ ; && (3.2.3)\\
R^{(2)}_{\mu\nu} &= \frac{1}{2}h^{\alpha\beta}\left[\bar{\nabla}_\nu\bar{\nabla}_\mu h_{\alpha\beta} + \bar{\nabla}_\alpha\bar{\nabla}_\beta h_{\mu\nu} - \bar{\nabla}_\alpha\bar{\nabla}_\mu h_{\beta\nu} - \bar{\nabla}_\alpha\bar{\nabla}_\nu h_{\beta\mu}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{4}\left\{2\bar{\nabla}_\alpha h^\alpha_{\ \beta} - \bar{\nabla}_\beta h\right\}\left\{\bar{\nabla}_\mu h_\nu^{\ \beta} + \bar{\nabla}_\nu h_\mu^{\ \beta} - \bar{\nabla}^\beta h_{\mu\nu}\right\} \\
&+\frac{1}{4}\left(\bar{\nabla}_\mu h^{\alpha\beta} + \bar{\nabla}^\alpha h^\beta_{\ \mu} - \bar{\nabla}^\beta h^\alpha_{\ \mu}\right)\left(\bar{\nabla}_\nu h_{\alpha\beta} + \bar{\nabla}_\alpha h_{\beta\nu} - \bar{\nabla}_\beta h_{\alpha\nu}\right) \qquad (3.2.4)
\end{aligned}
$$

$$
\begin{aligned}
\bar{g}^{\mu\nu} R^{(1)}_{\mu\nu} &= -\bar{\Box} h + \bar{\nabla}_\mu \bar{\nabla}_\nu h^{\mu\nu} \, , \qquad (3.2.5) \\
G_{\mu\nu} + \Lambda g_{\mu\nu} &= [\bar{G}_{\mu\nu} + \Lambda \bar{g}_{\mu\nu}] + [G^{(1)}_{\mu\nu} + \Lambda h_{\mu\nu}] \\
G^{(1)}_{\mu\nu} + \Lambda h_{\mu\nu} &= -\frac{1}{2}\bar{\Box} h_{\mu\nu} - \frac{1}{2}\left(\bar{\nabla}_\mu \bar{\nabla}_\nu - \bar{g}_{\mu\nu}\bar{\Box}\right) h + h_{\mu\nu}\left(\Lambda - \frac{1}{2}\bar{R}\right) \\
&\quad + \frac{1}{2}\left(\bar{\nabla}_\mu \bar{\nabla}_\alpha h^\alpha_{\ \nu} + \bar{\nabla}_\nu \bar{\nabla}_\alpha h^\alpha_{\ \mu} - \bar{g}_{\mu\nu}\left(\bar{\nabla}_\alpha \bar{\nabla}_\beta h^{\alpha\beta}\right)\right) \\
&\quad + \bar{R}_{\mu\alpha\beta\nu} h^{\alpha\beta} + \frac{1}{2}\left(\bar{R}_{\mu\alpha} h^\alpha_{\ \nu} + \bar{R}_{\nu\alpha} h^\alpha_{\ \mu} + \bar{g}_{\mu\nu}\bar{R}^{\alpha\beta} h_{\alpha\beta}\right) \qquad (3.2.6)
\end{aligned}
$$

The expressions simplify further for the *maximally symmetric* solution of the background equation, $\bar{G}_{\mu\nu} + \Lambda \bar{g}_{\mu\nu} = 0$. Maximal symmetry implies $\bar{R}_{\mu\alpha\beta\nu} = K(\bar{g}_{\mu\beta}\bar{g}_{\nu\alpha} - \bar{g}_{\mu\nu}\bar{g}_{\alpha\beta})$ while the background equation fixes $K = \Lambda/3$ and the linearized equation becomes[2],

$$
\begin{aligned}
&-\frac{1}{2}\bar{\Box} h_{\mu\nu} - \frac{1}{2}\left(\bar{\nabla}_\mu \bar{\nabla}_\nu - \bar{g}_{\mu\nu}\bar{\Box}\right) h + \frac{\Lambda}{3} h_{\mu\nu} + \frac{\Lambda}{6}\bar{g}_{\mu\nu} h \\
&\quad + \frac{1}{2}\left(\bar{\nabla}_\mu \bar{\nabla}_\alpha h^\alpha_{\ \nu} + \bar{\nabla}_\nu \bar{\nabla}_\alpha h^\alpha_{\ \mu} - \bar{g}_{\mu\nu}\left(\bar{\nabla}_\alpha \bar{\nabla}_\beta h^{\alpha\beta}\right)\right) = 8\pi T_{\mu\nu}. \qquad (3.2.7)
\end{aligned}
$$

It is customary and convenient to use the trace-reversed combination: $\tilde{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} h$. Denoting, $B_\mu := \bar{\nabla}_\alpha \tilde{h}^\alpha_{\ \mu}$, in terms of the tilde variables, the linearized equation takes the form,

$$
\frac{1}{2}\left[-\bar{\Box}\tilde{h}_{\mu\nu} + \left\{\bar{\nabla}_\mu B_\nu + \bar{\nabla}_\nu B_\mu - \bar{g}_{\mu\nu}(\bar{\nabla}^\alpha B_\alpha)\right\}\right] + \frac{\Lambda}{3}\left[\tilde{h}_{\mu\nu} - \tilde{h}\bar{g}_{\mu\nu}\right] = 8\pi T_{\mu\nu} \qquad (3.2.8)
$$

This equation describes generation of gravitational perturbation by source term $T_{\mu\nu}$, for order of magnitude consistency we will assume it to be a linearized source, i.e. to the order of ϵ. The divergence of the left hand side, $\bar{\nabla}^\mu[LHS]_{\mu\nu}$ is identically zero and thus

[2] From now on, the background is taken to be the de Sitter space-time with $\Lambda > 0$ and the units are chosen so that $G = 1 = c$.

source tensor is conserved automatically as it should be. For $\Lambda = 0$ the equation goes over to the flat background equation. Under the gauge transformations, $\tilde{h}_{\mu\nu}$ transforms as,

$$\delta\tilde{h}_{\mu\nu}(x) = \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu - \bar{g}_{\mu\nu}\bar{\nabla}_\alpha\xi^\alpha,$$

and the linearized equation (3.2.8) is explicitly invariant under these as it should be. It is well known that availing this freedom, it is possible to impose the *transversality condition*, $\bar{\nabla}_\alpha\tilde{h}^\alpha{}_\mu = 0$ (this is also referred as Dedonder Gauge or Lorentz gauge). The trace can be further gauged away [31] in the absence of sources (or for traceless stress tensor). The particular choice of arranging $\bar{\nabla}_\alpha\tilde{h}^\alpha{}_\mu = 0 = h$, is the *transverse, traceless gauge* or TT gauge for short. It simplifies the equation (3.2.8) to (for traceless stress tensor),

$$-\frac{1}{2}\bar{\Box}\tilde{h}_{\mu\nu} + \frac{\Lambda}{3}\tilde{h}_{\mu\nu} = 8\pi T_{\mu\nu} \tag{3.2.9}$$

The transversality condition does not fix the gauge completely, it still allows *residual gauge transformations* generated by vector fields ξ^μ satisfying,

$$\delta(\bar{\nabla}_\mu\tilde{h}^\mu{}_\nu) = \bar{\nabla}_\mu(\delta\tilde{h}^\mu{}_\nu) = \bar{\Box}\xi_\nu + \bar{R}_{\alpha\nu}\xi^\alpha = (\bar{\Box} + \Lambda)\xi_\nu = 0 \tag{3.2.10}$$

It should be noted that residual gauge transformation also implies, the gauge solutions $\delta\tilde{h}_{\mu\nu}$ satisfies wave equation with transversality condition in a source free region,

$$\bar{\Box}(\delta\tilde{h}_{\mu\nu}) - \frac{2\Lambda}{3}\left[(\delta\tilde{h}_{\mu\nu}) - (\delta\tilde{h})\bar{g}_{\mu\nu}\right] = 0 \tag{3.2.11}$$

To prove this we have used commutator of covariant derivatives in maximally symmetric background. This gauge freedom allows us to fix the gauge further and in effect we can impose the following gauge conditions on the field $h_{\mu\nu}$,

$$transverse: \quad \bar{\nabla}_\alpha\tilde{h}^\alpha{}_\mu = 0 \tag{3.2.12}$$

$$traceless: \quad h = 0$$

$$\textit{synchronous}: \quad t^{\mu} h_{\mu\nu} = 0 \ ,$$

where t^{μ} is a suitable future-directed time-like vector. In addition, the trace (zero or non-zero) is to be preserved, then ξ^{μ} must further satisfy, $\bar{\nabla}_{\alpha}\xi^{\alpha} = 0$ and this is consistent with eq. (3.2.10).

While it is common to choose the TT gauge, it is also possible to make a different choice of gauge [6] in the Poincaré patch of the de Sitter space-time. This will be done in the subsection 3.3.2 below.

The task now is to obtain the particular solution of the linearized, inhomogeneous equation (3.2.8), and extract the *physical solutions* i.e. solutions satisfying conditions which leaves *no* gauge transformations possible, in the source free region. Within the perturbative framework, this is obtained at the leading order by using a suitable Green function for the linearized equation on the de Sitter background. The *retarded Green functions* will be determined after some gauge fixing simplifying the equation (3.2.8).

3.3 The Retarded Green function

There have been several computations of two point functions for scalar, vector and tensor fields on de Sitter background [6, 31–33]. We will consider two retarded Green functions. In the subsection 3.3.1, we impose first the transversality condition and then also the tracelessness condition. We refer to these as the *transverse gauge* and the *TT gauge* respectively. In the subsection 3.3.2, following [6], we choose a gauge which changes the transversality condition by making its right hand side non-zero. We refer to it as *generalized transverse gauge*. With the tracelessness condition imposed, we refer to it as *generalized-TT* gauge. The two computations will provide different views of the physical solutions, in particular the form of the manifestation of the Λ dependence. The computations in the transverse gauge, employing the Hadamard construction [4], follow reference [5] while the generalized transverse gauge computations are based on [6].

3.3.1 The Transverse and the TT gauge

Imposing the transversality condition, $B_\mu = 0$ in eqn. (3.2.8) gives,

$$\bar{\Box}\tilde{h}_{\mu\nu} - \frac{2\Lambda}{3}\left[\tilde{h}_{\mu\nu} - \tilde{h}\bar{g}_{\mu\nu}\right] = -16\pi T_{\mu\nu} \tag{3.3.1}$$

One cannot put $\tilde{h} = 0$ to solve inhomogeneous equation unless the source is traceless itself. Hence it is convenient to separate the trace part of the equation and construct the retarded Green function in the transverse, traceless (TT) gauge directly with traceless part of source. Taking the trace of the above equation, gives an equation for the trace, $\tilde{h}$,

$$(\bar{\Box} + 2\Lambda)\tilde{h} = -16\pi T \quad , \quad T := \bar{g}_{\mu\nu}T^{\mu\nu} \,. \tag{3.3.2}$$

Subtracting $\frac{1}{4}\bar{g}_{\mu\nu}\times$ eqn.(3.3.2) from eqn.(3.3.1), we get

$$\bar{\Box}\tilde{h}'_{\mu\nu} - \frac{2\Lambda}{3}\tilde{h}'_{\mu\nu} = -16\pi T'_{\mu\nu} \quad , \quad \tilde{h}'_{\mu\nu} := \tilde{h}_{\mu\nu} - \frac{1}{4}\tilde{h}\bar{g}_{\mu\nu} \, , \; T'_{\mu\nu} := T_{\mu\nu} - \frac{1}{4}T\bar{g}_{\mu\nu} \,. \tag{3.3.3}$$

The equation (3.3.2) for $\tilde{h}$ is a scalar equation and its solution is determined by a corresponding Green function with a source which is the trace of the stress tensor. However we know that in the source free region, we *can* make a gauge transformation to set the $\tilde{h}$ to zero. Hence, in the region of observational interest, we can gauge away the effect of the trace T. With this understood, we take $\tilde{h} = 0$ which gives $\tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu}$ and use the traceless $T'_{\mu\nu}$ as the source. For notational simplicity, we drop the prime from the stress tensor. Thus, we focus on the TT gauge equation (3.2.9) with a *trace-free stress tensor* as the source.

The equation for the Green function is,

$$\bar{\Box} G^{\alpha\beta}{}_{\mu'\nu'}(x,x') - \frac{2\Lambda}{3} G^{\alpha\beta}{}_{\mu'\nu'}(x,x') = -4\pi J^{\alpha\beta}{}_{\mu'\nu'}\delta_4(x,x') \ , \ \text{where,} \tag{3.3.4}$$

$$J^{\alpha\beta}{}_{\mu'\nu'}(x,x') := \frac{g^{\alpha}{}_{\mu'}g^{\beta}{}_{\nu'} + g^{\alpha}{}_{\nu'}g^{\beta}{}_{\mu'}}{2} - \frac{1}{4}\bar{g}^{\alpha\beta}(x)\bar{g}_{\mu'\nu'}(x') \ , \ \text{and,} \tag{3.3.5}$$

$g^{\alpha}{}_{\mu'}(x,x')$ denotes the *parallel propagator* along the geodesic connecting x, x'. The tensor $J^{\alpha\beta}{}_{\mu'\nu'}$ is symmetric and traceless in the pairs of indices $\alpha\beta$ and $\mu'\nu'$. The Green's function is obtained using the *Hadamard ansatz*.

The Hadamard ansatz for the retarded Green function for *a general* wave equation is [4],

$$G^{\alpha\beta}{}_{\mu'\nu'}(x,x') = U^{\alpha\beta}{}_{\mu'\nu'}(x,x')\delta_+(\sigma+\epsilon) + V^{\alpha\beta}{}_{\mu'\nu'}(x,x')\theta_+(-\sigma-\epsilon) \ , \ \text{where} \tag{3.3.6}$$

the space-time point x belongs to a *convex normal neighbourhood* of x' (set of points such that any two points have a unique geodesic lying entirely within the neighbourhood) ; $\sigma(x,x')$ is the Synge world function which is half the geodesic distance squared between x and x' [5, 34]; θ_+, δ_+ (see figure 3.2) are distributions, viewed as functions of x, having support in the chronological future and future light cone of x' respectively. The small

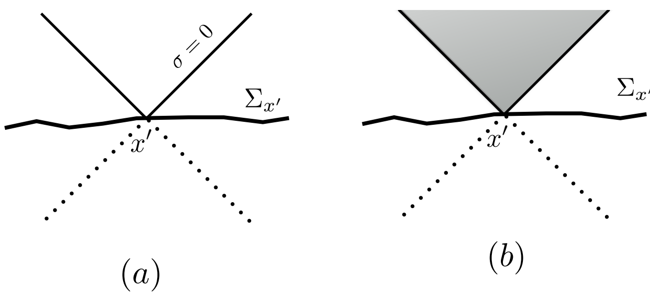

Figure 3.2. Light-cone distributions in Hadamard construction of Green's function. The grey regions or lines denote a non-zero value for the distributions. The space-like hypersurface $\Sigma_{x'}$ contains the point x'. (a) Light-cone delta function, $\delta_+(\sigma(x,x'))$ is supported on the future light-cone of x'. (b) The light-cone step function, $\theta_+(-\sigma(x,x'))$ equals one inside the forward light-cone.

parameter ϵ is introduced to permit differentiation of the distribution and is to be taken to zero in the end. The bi-tensors U, V are determined by inserting the ansatz in the equation (3.3.4).

Using the relation $\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\sigma\bar{\nabla}_\beta\sigma = 2\sigma$ and the distributional identities [5]:

$$\begin{aligned} (\sigma+\epsilon)\delta'(\sigma+\epsilon) = -\delta(\sigma+\epsilon) \quad &, \quad (\sigma+\epsilon)\delta''(\sigma+\epsilon) = -2\delta'(\sigma+\epsilon) \\ \text{as } \epsilon \to 0: \ \epsilon\delta'(\sigma+\epsilon) \to 0 \quad &, \quad \epsilon\delta''(\sigma+\epsilon) \to 2\pi\delta_4(x,x') \ , \end{aligned} \tag{3.3.7}$$

leads to four equations by equating the coefficients of $\theta(-\sigma), \delta(\sigma), \delta'(\sigma)$ and $\delta_4(x,x')$ to zero. The respective equations are:

$$\bar{\Box}V^{\alpha\beta}{}_{\mu'\nu'}(x,x') - \frac{2\Lambda}{3}V^{\alpha\beta}{}_{\mu'\nu'}(x,x') = 0 \qquad , \quad \sigma(x,x') < 0; \tag{3.3.8}$$

$$2\sigma^\lambda\bar{\nabla}_\lambda V^{\alpha\beta}{}_{\mu'\nu'} + (\bar{\Box}\sigma - 2)V^{\alpha\beta}{}_{\mu'\nu'} = \bar{\Box}U^{\alpha\beta}{}_{\mu'\nu'} - \frac{2\Lambda}{3}U^{\alpha\beta}{}_{\mu'\nu'} \qquad , \quad \sigma(x,x') = 0; \tag{3.3.9}$$

$$\left(2\sigma^\lambda\bar{\nabla}_\lambda + (\bar{\Box}\sigma - 4)\right)U^{\alpha\beta}{}_{\mu'\nu'} = 0 \qquad , \quad \sigma(x,x') = 0; \tag{3.3.10}$$

$$\left[U^{\alpha\beta}{}_{\mu'\nu'}\right] = \left[J^{\alpha\beta}{}_{\mu'\nu'}\right] = \delta^{(\alpha'}_{\mu'}\delta^{\beta')}_{\nu'} - \frac{1}{4}\bar{g}^{\alpha'\beta'}\bar{g}_{\mu'\nu'} \qquad , \quad x = x' \ . \tag{3.3.11}$$

In the above, the quantity enclosed within square brackets denotes its *coincidence limit* - evaluation for $x = x'$ and super(sub)script on σ denotes its covariant derivative.

The last two equations uniquely determine $U^{\alpha\beta}{}_{\mu'\nu'}(x,x')$ on the light cone through x' while the first two equations uniquely determine $V^{\alpha\beta}{}_{\mu'\nu'}(x,x')$ inside and on the light cone through x'. The cosmological constant appears explicitly in these two equations.

Determination of $U^{\alpha\beta}{}_{\mu'\nu'}$:

Equation (3.3.10) is a homogeneous, first order, linear differential equation and its solution is completely determined by the initial condition provided by eqn. (3.3.11). Noting that $\sigma^\lambda\bar{\nabla}_\lambda$ on the parallel propagator and the metric gives zero, we get $\sigma^\lambda\bar{\nabla}_\lambda J^{\alpha\beta}{}_{\mu'\nu'} = 0$.

Hence the ansatz $U^{\alpha\beta}{}_{\mu'\nu'}(x,x') := J^{\alpha\beta}{}_{\mu'\nu'}\tilde{U}(x,x')$ in eqs.(3.3.10)and (3.3.11) leads to

$$\left(2\sigma^{\alpha}\bar{\nabla}_{\alpha} + (\bar{\Box}\sigma - 4)\right)\tilde{U} = 0 \ , \quad [\tilde{U}] = 1 \tag{3.3.12}$$

To solve this equation, it is useful to mention that $\sigma^{\alpha}(x,x')$ is proportional to the tangent vector to the unique geodesic at x (see figure(3.3)). This geodesic is affinely parameterized by λ and a displacement along the geodesic is described by $dx^{\alpha} = (\sigma^{\alpha}/\lambda)\, d\lambda$. Therefore the first term in eqn. (3.3.12) can be expressed as $2\lambda\, d\tilde{U}/d\lambda$. Using the identity $4 = \bar{\Box}\sigma + \bar{\nabla}_{\alpha}(\ln \varDelta(x,x'))\,\sigma^{\alpha}$, the second term can be written as $-\varDelta^{-1}(\lambda\, d\varDelta/d\lambda)\,\tilde{U}$, where $\varDelta(x,x')$

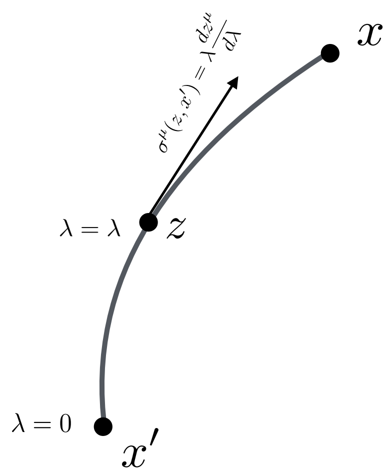

Figure 3.3. Points x' and x are linked by a unique geodesic. The geodesic is parametrized by λ and $t^{\mu} := dz^{\mu}/d\lambda$ is its tangent vector. First derivative of Synge's world function is a rescaled tangent vector on the geodesic, i.e. $\sigma^{\mu}(z,x') = \lambda\, dz^{\mu}/d\lambda$.

is the (scalarised) Van Vleck determinant or Van Vleck bi-scalar defined as, $\varDelta(x,x') := -\det(-\sigma_{\alpha\beta'}(x,x'))/\sqrt{g(x)g(x')}$, with g in the denominator denoting the modulus of the determinant of the metric [5]. Hence eqn. (3.3.12) becomes,

$$2\lambda\frac{d\tilde{U}}{d\lambda} - \varDelta^{-1}\left(\lambda\frac{d\varDelta}{d\lambda}\right)\tilde{U} = 0 \tag{3.3.13}$$

$$\implies \frac{d}{d\lambda}(2\ln\tilde{U} - \ln\varDelta) = 0 \tag{3.3.14}$$

It follows that $\tilde{U}^2/\varDelta$ is constant along geodesic and it must be equal to its initial point x';

i.e. $\tilde{U}^2/\Delta = [\tilde{U}^2/\Delta] = 1$ as coincidence limit of $\tilde{U}$ and Δ are both unity. Hence solution of eqn. (3.3.12) is

$$\tilde{U}(x,x') = \sqrt{\Delta(x,x')} \tag{3.3.15}$$

The bi-scalar $\tilde{U}$, being de Sitter invariant, depends on x, x' only through the world function $\sigma(x,x')$ which means that value of $\tilde{U}$ along the light cone is same as its value in the coincidence limit i.e. $\tilde{U}|_{\sigma=0} = [\tilde{U}] = 1 (= \Delta(x,x')|_{\sigma=0})$ and we need the solution only on the light cone. Thus,

$$U^{\alpha\beta}{}_{\mu'\nu'}(x,x')|_{\sigma=0} \ := \ J^{\alpha\beta}{}_{\mu'\nu'}|_{\sigma=0}.$$

Determination of $V^{\alpha\beta}{}_{\mu'\nu'}$ **:**

We *cannot* similarly factor out $J^{\alpha\beta}{}_{\mu'\nu'}$ from $V^{\alpha\beta}{}_{\mu'\nu'}(x,x')$. The reason is that equation (3.3.9) is an *inhomogeneous* equation and the tensor structure of its right hand side is *not* the same as that of $U^{\alpha\beta}{}_{\mu'\nu'}$. Using the solution $U^{\alpha\beta}{}_{\mu'\nu'}(x,x') = J^{\alpha\beta}{}_{\mu'\nu'}(x,x')\sqrt{\Delta(x,x')}$, to order $(\sigma)^2$, we find [5],

$$\begin{aligned}\left(\bar{\Box} - \frac{2\Lambda}{3}\right)U^{\alpha\beta}{}_{\mu'\nu'} \ &= \ \left\{-\frac{\Lambda}{6}(4 - \bar{\Box}\sigma) - \frac{\Lambda^2\sigma}{9}\right\}J^{\alpha\beta}{}_{\mu'\nu'} \\ &+\frac{\Lambda^2}{18}\Big\{\bar{g}^{\alpha\beta}\sigma_{\mu'}\sigma_{\nu'} + \sigma^\alpha\sigma^\beta\bar{g}_{\mu'\nu'} - \sigma(g^\alpha{}_{\mu'}g^\beta{}_{\nu'} + g^\alpha{}_{\nu'}g^\beta{}_{\mu'}) \\ &\qquad + \left(g^\alpha{}_{\mu'}\sigma^\beta\sigma_{\nu'} + g^\beta{}_{\mu'}\sigma^\alpha\sigma_{\nu'} + g^\alpha{}_{\nu'}\sigma^\beta\sigma_{\mu'} + g^\beta{}_{\nu'}\sigma^\alpha\sigma_{\mu'}\right)\Big\}\end{aligned} \tag{3.3.16}$$

$$:= \ \Phi(\sigma)J^{\alpha\beta}{}_{\mu'\nu'} + \frac{\Lambda^2}{18}K^{\alpha\beta}{}_{\mu'\nu'} + o(\sigma^3) \tag{3.3.17}$$

Note that the bi-tensor $K^{\alpha\beta}{}_{\mu'\nu'}$ is traceless just as the bi-tensor $J^{\alpha\beta}{}_{\mu'\nu'}$.

Noting the coincidence limits: $[\bar{\Box}\sigma] = 4, [\sigma] = 0, [\sigma^\alpha] = 0$, we see that, $[\Phi] = 0 = [K^{\alpha\beta}{}_{\mu'\nu'}]$ and hence the coincidence limit of the left hand side vanishes.

The coincidence limit of the equation (3.3.9) then implies $[V^{\alpha\beta}{}_{\mu'\nu'}(x,x')] = 0$. However, this does *not* imply $V^{\alpha\beta}{}_{\mu'\nu'}(x,x')|_{\sigma=0} = 0$. To order σ^2, we can write,

$$V^{\alpha\beta}{}_{\mu'\nu'}(x,x') := \tilde{V}_1(\sigma)J^{\alpha\beta}{}_{\mu'\nu'} + \tilde{V}_2(\sigma)K^{\alpha\beta}{}_{\mu'\nu'} \ .$$

Plugging this ansatz to the left hand side of eqn. (3.3.9) and equating to (3.3.17), we obtain two inhomogeneous differential equations for the bi-scalars $\tilde{V}_1, \tilde{V}_2$,

$$2\sigma^{\lambda}\bar{\nabla}_{\lambda}\tilde{V}_1 + (\bar{\Box}\sigma - 2)\tilde{V}_1 = \Phi \tag{3.3.18}$$

$$2\sigma^{\lambda}\bar{\nabla}_{\lambda}\tilde{V}_2 + (\bar{\Box}\sigma - 2)\tilde{V}_2 + 4\tilde{V}_2 = \frac{\Lambda^2}{18} \tag{3.3.19}$$

In the intermediate step we have used $\sigma^{\lambda}\bar{\nabla}_{\lambda}J^{\alpha\beta}_{\mu'\nu'} = 0$ and $\sigma^{\lambda}\bar{\nabla}_{\lambda}K^{\alpha\beta}_{\mu'\nu'} = 2K^{\alpha\beta}_{\mu'\nu'}$ The coincidence limits of these equations, combined with $[\Phi] = 0$ leads to $[\tilde{V}_1] = 0$ and $[\tilde{V}_2] = \frac{\Lambda^2}{108}$. Once again, these values determine these bi-scalars everywhere on the light cone. Hence, to order σ^2,

$$V^{\alpha\beta}_{\mu'\nu'}(x,x')|_{\sigma=0} = \frac{\Lambda^2}{108}K^{\alpha\beta}_{\mu'\nu'}|_{\sigma=0} \tag{3.3.20}$$

With this characteristic data on the light-cone, $V^{\alpha\beta}_{\mu'\nu'}(x,x')$ can be determined uniquely by eq. (3.3.8). This shows clearly that the data for characteristic evolution off the light cone is non-zero and hence the tail term is non-zero as well. Equally well, it also shows that the tail term is *at least of order* Λ^2. The Green function is then given by,

$$G^{\alpha\beta}_{\mu'\nu'}(x,x') = J^{\alpha\beta}_{\mu'\nu'}(x,x')\delta_+(\sigma) + V^{\alpha\beta}_{\mu'\nu'}\theta_+(-\sigma).$$

In TT gauge we will be computing corrections to order Λ and hence *we do not compute the effect of the tail term in this section*. From now on, we restrict to the sharp propagation term only and only the *trace-free* part of the source stress tensor contributes.

Using the sharp term of the Green function above, the solution to the inhomogeneous equation becomes,

$$\tilde{h}^{\alpha\beta}(x) = 4\int_{\text{source}} d^4x'\,\sqrt{-g(x')}\,\delta_+(\sigma)J^{\alpha\beta}_{\mu'\nu'}(x,x')T^{\mu'\nu'}(x') \tag{3.3.21}$$

$$= \quad 4 \int_{\text{source}} d^4x' \, \sqrt{-g(x')} \delta_+(\sigma) g^{\alpha}_{\ \mu'}(x,x') g^{\beta}_{\ \nu'}(x,x') T^{\mu'\nu'}(x') \qquad (3.3.22)$$

In the second line we have substituted for $J^{\alpha\beta}_{\ \ \mu'\nu'}$ and used the fact that the stress tensor is trace-free and symmetric.

Introducing RNC and FNC :

Till now we have not specified any coordinate system. In a generic curved background it is convenient to perform perturbative calculation in *Riemann Normal Coordinates* (RNC) or *Fermi Normal Coordinates* (FNC). At different stages of calculation we employ RNC and FNC as per simplification of computation and we will express final result in terms of FNC. These coordinate charts are based on the choice of a time-like reference curve γ, a reference point P_0 on it and an orthonormal tetrad, E^{α}_a at P_0 such that E^{α}_0 equals the normalised tangent to γ, at P_0. To be definite, let us take the world tube of the spatially compact source to be around the line AD of the figure 3.1. The line AD is a time-like geodesic and we naturally choose the reference curve, γ, to be this line. Denoting the proper time along γ by τ, we choose $P_0 = \gamma(\tau = 0)$, as the reference point. Let E^{α}_a denote an orthonormal tetrad at P_0 chosen such that E^{α}_0 is the normalized, geodesic tangent to γ. Fermi transport the tetrad along γ (which is same as parallel transport since γ is a geodesic). Thus we have an orthonormal tetrad, $e^{\alpha}_{\ a} e^{\beta}_{\ b} \bar{g}_{\alpha\beta} = \eta_{ab}$, with $e^{\alpha}_{\ 0}$ equal to the geodesic tangent to γ, all along $\gamma(\tau)$. The corresponding orthonormal co-tetrad is denoted as $e^{a}_{\ \alpha}$. It follows that, all along $\gamma(\tau)$, $\bar{g}_{\alpha\beta} = \eta_{\alpha\beta}$ and the Christoffel connection is zero. With these choices, the *Fermi Normal Coordinates* (FNC) and the *Riemann Normal Coordinates* (RNC) are set up as follows (see Fig. 3.4).

To define the Fermi coordinates of a point P off γ, let β be the unique (space-like) geodesic from P, *orthogonally* meeting γ at a point $Q = \gamma(\tau_P)$, with a unit affine parameter interval. Its tangent vector, n^{α} at Q can be resolved along the triad of space-like vectors at Q as: $n^{\alpha} := \xi^i e^{\alpha}_{\ i}$. Its norm gives the proper distance between P and Q, $s^2 := n^{\alpha} n^{\beta} \eta_{\alpha\beta} = \xi^i \xi^j \delta_{ij}$. The FNC of P are then defined to be (τ_P, ξ^i). Evidently, for points along γ, the spatial

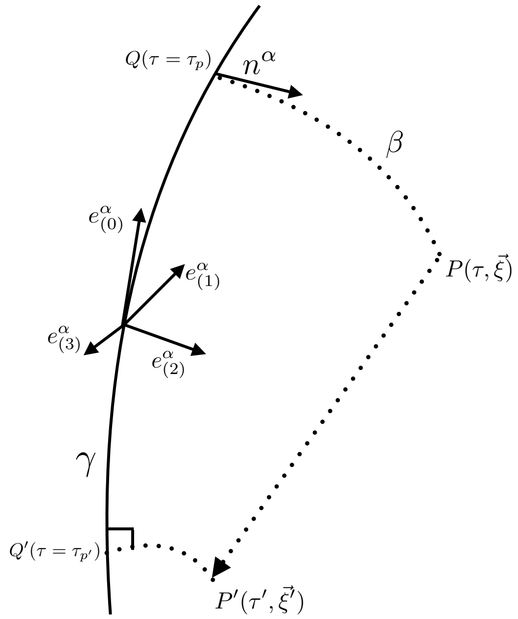

Figure 3.4. The definition of Fermi Normal Coordinates. The dotted line from P to P' is the unique null geodesic for which the parallel propagator is computed in appendix B. The geodesics P_0P, P_0P' are used in setting up the Riemann Normal Coordinates. P and P' denote the observation and the source points respectively.

coordinates ξ^i, are zero. To define the RNC for the same point P as above, construct the unique geodesic starting from reference point P_0 and reaching P in a unit affine parameter interval. This fixes the geodesics tangent vector N^α at P_0. The normal coordinates of P, X^a, are then defined through $N^\alpha := X^a E^\alpha_{\ a}$. We will use them in intermediate computations.

Generally, the FNC and the RNC have a domain consisting of points P which have the required unique geodesics from the reference curve/point. By examining the geodesic equation in the global chart it is easy to see that the RNC's and the FNC's would be valid in the static patch (see also [35]). *In effect, the computations of this subsection are restricted to the static patch.*

Our task is to evaluate the terms in the integrand of eqn. (3.3.22). The final answer will be expressed in terms of the FNC introduced above.

Computation of $\sigma(x, x')$:

Let P, P' denote the observation point and a source point respectively. With the base point P_0, we get a geodesic triangle P_0PP' with the $P'P$ geodesic being null and future directed. Let X^a, X'^a denote the RNCs of P and P' respectively. In terms of the RNC set up in this manner, we have to obtain $\sigma(P', P)$. For this we follow chapter II of [34].

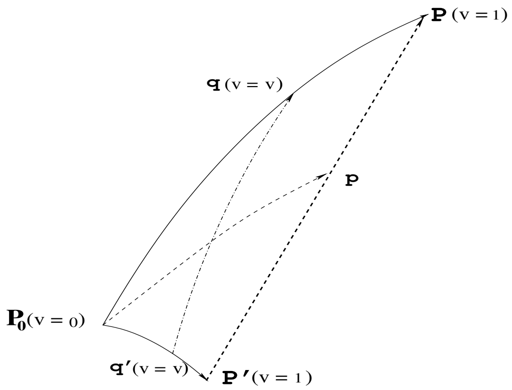

Figure 3.5. All lines are the unique geodesics in the Riemann normal neighbourhood. As the point p slides between P' and P, a two dimensional surface is generated.

The idea is to construct a surface spanning family of geodesics (figure 3.5), interpolating between the geodesics P_0P, P_0P', all originating at P_0 and ending on a point p on the geodesic connecting $P'P_0$. Each of these have their affine parameters, v's, running from 0 to 1. Choose points q' and q on the geodesics P_0P' and P_0P respectively and having the same value of affine parameter, $0 \leq v \leq 1$. The world function $\sigma(q', q)$ depends *only* on v and gives the desired answer for $v = 1$. When the Riemann tensor is small, i.e. can be treated as order 1 (different from the orders used in the metric expansion), the $\sigma(q', q)$ is expressed as a Taylor expansion, in v, to third order together with the remainder. This gives,

$$\begin{aligned}\sigma(P', P) &= \sigma(P_0, P') + \sigma(P_0, P) - \left(g^{\alpha\beta} \frac{\partial\sigma(y, P')}{\partial y^\alpha} \frac{\partial\sigma(y, P)}{\partial y^\beta} \right)\Bigg|_{P_0} \\ &\qquad + \frac{1}{6}\int_0^1 dv (1-v)^3 \frac{D^4\sigma(q', q)}{Dv^4} \; ; \qquad \leftarrow \quad \text{vanishes for flat space.}\end{aligned} \tag{3.3.23}$$

The term in the second line comes from the remainder in the Taylor expansion and contains the modifications due to non-zero Riemann tensor. This is computed to the first order in the curvature. For maximally symmetric space-time, the computation simplifies. The steps are sketched in the appendix (A) and here is the final result expressed in terms of the RNCs of P, P':

$$2\sigma(P, P') = (X - X') \cdot (X - X') - \frac{\Lambda}{9}\left\{(X \cdot X)(X' \cdot X') - (X \cdot X')^2\right\} + o(\Lambda^2) \quad (3.3.24)$$

Here the dot product is the Minkowski dot product, $X \cdot Y := \eta_{ab} X^a Y^b$ etc.

At this stage we convert the above expression from RNC to FNC. The coordinate transformation between the RNC and the FNC is given by [36],

$$\begin{aligned} X^0(\tau, \vec{\xi}) &= \tau + \tau \frac{R^0{}_{ij0} + R^0{}_{ji0}}{6} \xi^i \xi^j + \cdots \\ &= \tau\left(1 - \frac{\Lambda s^2}{9}\right) \end{aligned} \quad (3.3.25)$$

$$\begin{aligned} X^i(\tau, \vec{\xi}) &= \xi^i + \frac{R^i{}_{0j0}}{6} \xi^j \tau^2 + \frac{R^i{}_{jk0}}{3} \xi^j \xi^k \tau \\ &= \xi^i \left(1 - \frac{\Lambda \tau^2}{18}\right) \end{aligned} \quad (3.3.26)$$

In the second lines we have used the de Sitter curvature.

Substitution in (3.3.24) leads to,

$$\begin{aligned} 2\sigma(\tau, \vec{\xi}, \tau', \vec{\xi}') &= \left\{-(\tau - \tau')^2 + (\vec{\xi} - \vec{\xi}')^2\right\} + \frac{\Lambda}{9}\left\{(\tau - \tau')^2(\vec{\xi}^2 + \vec{\xi}'^2 + \vec{\xi} \cdot \vec{\xi}') - (\tau \vec{\xi}' + \tau' \vec{\xi})^2 \right. \\ &\quad \left. - (-\tau^2 + \vec{\xi}^2)(-\tau'^2 + \vec{\xi}'^2) + (-\tau\tau' + \vec{\xi} \cdot \vec{\xi}')^2\right\} \end{aligned} \quad (3.3.27)$$

Solving the $\delta_+(\sigma)$:

We have to solve the $\delta_+(\sigma(P, P'))$ for τ' and eliminate the $d\tau'$ integration in eqn. (3.3.22).

From the property of delta functional

$$\delta_+(\sigma(\tau,\vec{\xi},\tau',\vec{\xi}')) = \frac{\delta(\tau'-\tau'_{ret})}{\partial\sigma/\partial\tau'\big|_{\tau'=\tau'_{ret}}} \tag{3.3.28}$$

The solution for $\sigma = 0$ is sought in the form of $\tau'_{ret} = \tau_0 + \Lambda\tau_1$. The τ_0 is determined by vanishing of the first braces in eq. (3.3.27) and the retarded condition picks out one solution, namely, $\tau_0 = \tau - |\vec{\xi} - \vec{\xi}'|$. The full solution is obtained as,

$$\begin{aligned}
\tau'_{\rm ret} &:= \tau_0 + \Lambda\tau_1 \qquad\qquad \text{where,} \\
\tau_0 &= \tau - |\vec{\xi}-\vec{\xi}'| \qquad\qquad (3.3.29)\\
\tau_1 &= -\frac{1}{18}\frac{1}{|\vec{\xi}-\vec{\xi}'|}\Big\{|\vec{\xi}-\vec{\xi}'|^2(\vec{\xi}^2+\vec{\xi}'^2+\vec{\xi}\cdot\vec{\xi}') - (\tau(\vec{\xi}+\vec{\xi}') - |\vec{\xi}-\vec{\xi}'|\vec{\xi})^2 \\
&\quad -(-\tau^2+\vec{\xi}^2)\left(\vec{\xi}'^2 - (\tau - |\vec{\xi}-\vec{\xi}'|)^2\right) + (-\tau^2 + \tau|\vec{\xi}-\vec{\xi}'| + \vec{\xi}\cdot\vec{\xi}')^2\Big\} \qquad (3.3.30)
\end{aligned}$$

Now we introduce the approximation that the source size is much smaller than its distance from observers i.e. $\vec{\xi}'^2 \ll \vec{\xi}^2 \leftrightarrow s' \ll s$ [3]. With this assumption,

$$|\vec{\xi}' - \vec{\xi}| \approx s\sqrt{1 + \frac{s'^2}{s^2} - 2\hat{\xi}\cdot\hat{\xi}'\frac{s'}{s}} \approx s\ ,$$

and keeping only the leading term in powers of s, we get,

$$\tau'_{\rm ret} := \tau - \left(s + \frac{\Lambda}{18}s^3\right) =: \tau - \bar{s}(s) =: \tau_{\rm ret}\ . \tag{3.3.31}$$

From this, it follows that,

$$\frac{\partial\sigma(\tau,\vec{\xi},\tau',\vec{\xi}')}{\partial\tau'}\bigg|_{\tau'=\tau-s-\frac{\Lambda s^3}{18}} \approx -s\left(1 - \frac{\Lambda s^2}{18}\right) \implies \left|\frac{\partial\sigma}{\partial\tau'}\right|^{-1} \approx \frac{1}{s}\left(1 + \frac{\Lambda}{18}s^2\right)\ . \tag{3.3.32}$$

[3] The spatial coordinates are proportional to the *proper distance* along the corresponding spatial geodesics. This distance is related to but not equal to the 'physical distance' equaling the scale factor times the co-moving distance. Explicit relation is given in eqn. (C.0.15). Nevertheless, $s' \ll s$, reflects the assumption of source size being much smaller than the distance to the observer. The cosmological horizon bounding the static chart is at a 'physical distance' of $\sqrt{3/\Lambda}$ and all our s', s are within the static chart.

Note that the $\tau_{\rm ret}(\tau, s)$ defined above in terms of $\bar{s}$ reflects the non-Minkowskian metric and exactly corresponds to the light-cone in FNC.

Metric and its determinant in FNC:

In terms of the FNC, the metric to first order in the curvature, is given as [5],

$$g_{00}(\tau, \vec{\xi}) \ = \ -1 + \frac{\Lambda s^2}{3} \ , \ g_{0i} \ = \ 0 \ , \ g_{ij} \ = \ \delta_{ij} - \frac{\Lambda}{9}(\delta_{ij}s^2 - \xi_i\xi_j) \ . \tag{3.3.33}$$

The metric is *static* and its determinant is given by,

$$\sqrt{-g}|_{\rm FNC} \ \approx \ 1 - \frac{5}{18}\Lambda s^2 \ = \ \left(1 - \frac{\Lambda s^2}{6}\right)\left(1 - \frac{\Lambda s^2}{9}\right) . \tag{3.3.34}$$

The second factor is the square root of the determinant of the induced metric on a constant τ hypersurface. The metric being static (independent of τ with $g_{0i} = 0$) also means that ∂_τ is the stationary Killing vector in the FNC chart.

At this stage we recall that in the Minkowski background, a simplification is achieved by further imposing the *synchronous gauge* condition, $\tilde{h}_{0\alpha} = 0$ which removes the residual gauge freedom of the TT gauge completely and we are left with only the physical solution: the components $\tilde{h}_{ij}$ satisfying $\partial^i\tilde{h}_{ij} = 0 = \delta^{ij}\tilde{h}_{ij}$. Is such a simplification available in the de Sitter background?

As a matter of fact, it is a general result [28], that in a globally hyperbolic space-time, given any Cauchy surface, Σ, the normalised, time-like geodesic vector field, η^α orthogonal to the Cauchy surface allows us to impose the synchronous gauge condition $\tilde{h}_{\alpha\beta}\eta^\beta = 0$ in a *normal neighbourhood* of Σ. The vector field also provides us with a convenient way to identify the physical components of the solution.

For the static patch we are working in, the hypersurface of constant τ corresponding to the horizontal line through the point 'E' of figure 3.1 is a Cauchy surface and the required η^α field can be constructed easily to order Λ. For instance, let $\tau = \tau_0$ be the surface Σ_0, with a normalized normal given by $\hat{n}^\alpha := (1 + \frac{\Lambda}{6}s^2)\delta^\alpha_0$. Then the vector field η is determined as the solution of an initial value problem:

$$0 = \eta^\beta(\partial_\beta\eta^\alpha + \Gamma^\alpha{}_{\beta\gamma}\eta^\gamma) \ , \quad \eta^\alpha|_{\Sigma_0} = \hat{n}^\alpha \implies \tag{3.3.35}$$

$$\eta^\alpha = \delta^\alpha{}_0 + \frac{\Lambda}{3}\left(\frac{s^2}{2}\delta^\alpha{}_0 + (\tau - \tau_0)\,\xi^i\delta^\alpha{}_i\right) + o(\Lambda^2) \tag{3.3.36}$$

From this it follows that in the synchronous gauge,

$$\tilde{h}^{\alpha\beta}\eta_\beta = 0 \ \Rightarrow \ \tilde{h}^{00} = \frac{\Lambda}{3}(\tau - \tau_0)\tilde{h}^{0i}\xi_i \ , \ \tilde{h}^{0i} = \frac{\Lambda}{3}(\tau - \tau_0)\tilde{h}^{ij}\xi_j \, . \tag{3.3.37}$$

Clearly, $\tilde{h}^{00} \sim o(\Lambda^2)$ and can be set to be zero while $\tilde{h}^{0i}$ is completely determined by $\tilde{h}^{ij}$. It will turn out in the next section that *for 'tt-projected'* $\tilde{h}^{ij}$, $\tilde{h}^{0i} = 0$. Therefore, we now specialise to the spatial components, $\mu = m, \nu = n$.

Field in terms of source integral :

Let us recall from appendix B, null parallel propagator in FNC to the leading order of Λ is given by (B.0.9),

$$g^m{}_{\alpha'}(\tau, \vec{\xi}, \tau'_{\rm ret}, \vec{\xi}') \approx \delta^m{}_{\alpha'} + \frac{\Lambda s^2}{18}\left[\delta^m{}_{\alpha'} + 3\delta^0{}_{\alpha'}\frac{\xi^m}{s} - \delta^j{}_{\alpha'}\frac{\xi_j\xi^m}{s^2}\right] . \tag{3.3.38}$$

Note that it is independent of the source point $(\tau', \vec{\xi}')$, thanks to the leading s'/s approximation. It is also independent of τ.

Now we have assembled all the terms in equation (3.3.22). The τ' integration exhausts the first factor in the $\sqrt{-g}$ and we get,

$$\tilde{h}^{\mu\nu}(\tau, \vec{\xi}) = \frac{4}{s}\left(1 + \frac{\Lambda s^2}{18}\right) g^\mu{}_{\alpha'}(\vec{\xi}) g^\nu{}_{\beta'}(\vec{\xi}) \int d^3\xi' \sqrt{g_3(\xi')}T^{\alpha'\beta'}(\tau_{\rm ret}, \vec{\xi}') \, . \tag{3.3.39}$$

The integral over the source is usually expressed in terms of time derivatives of moments, using the conservation of the stress tensor. To make these integrals well defined, it is convenient and transparent to introduce suitable orthonormal tetrad and convert the coordinate components to frame components. The frame components are coordinate scalars (although they change under Lorentz transformations of the frame) and their integrals are well defined. In the FNC chart, there is a natural choice provided by the $\tau' =$ constant hypersurface passing through the source world tube. At any point on this hypersurface, we have a unique orthonormal triad obtained from the triad on the reference curve by parallel transport along the spatial geodesic. The unit normal, n^α, together with this triad, $e^\alpha_{\ m}, m = 1,2,3$, provide the frame, $e^\alpha_{\ a}$. Explicitly, to order Λ,

$$n^\tau(\tau,\vec{\xi}') \ = \ 1 + \frac{\Lambda s'^2}{6} \ , \ n^i = 0 \quad , \quad e^\tau_m(\tau,\vec{\xi}') \ = \ 0 \ , \ e^i_m(\tau,\vec{\xi}') \ = \ \left(1 + \frac{\Lambda s'^2}{18}\right)\delta^i_m - \frac{\Lambda}{18}\xi'^i\xi'_m \ ,$$

In more compact form (*underlined indices denote frame indices*),

$$e^{\alpha'}_{\underline{a}} \ := \ \left(1 + \frac{\Lambda s'^2}{6}\right)\delta^{\alpha'}_{\ \tau}\delta^0_{\underline{a}} + \delta^{\alpha'}_i\delta^j_{\underline{a}}\left\{\delta^i_{\underline{j}}\left(1 + \frac{\Lambda s'^2}{18}\right) - \frac{\Lambda}{18}\xi'^i\xi'_{\underline{j}}\right\} \tag{3.3.40}$$

It is easy to check that $e^{\alpha'}_{\underline{a}} e^{\beta'}_{\underline{b}} g_{\alpha'\beta'} = \eta_{\underline{ab}}$. It follows that,

$$g^m_{\ \alpha'}(x)e^{\alpha'}_{\underline{a}}(x') \ \simeq \ \delta^m_{\underline{a}}\left(1 + \frac{\Lambda s^2}{18}\right) + \frac{\Lambda}{6}\delta^0_{\ \underline{a}}s\,\xi^m - \frac{\Lambda}{18}\delta^j_{\ \underline{a}}\xi_{\underline{j}}\xi^m. \tag{3.3.41}$$

Defining the frame components of the stress tensor through the relation, $T^{\mu\nu} := e^\mu_{\ \underline{a}}e^\nu_{\ \underline{b}}\Pi^{\underline{ab}}$. and substituting for $g^m_{\alpha'}e^{\alpha'}_{\underline{a}}(\tau,\vec{\xi},\tau',\vec{\xi}')$, we obtain the final expression for the solution in the synchronous gauge, to leading order in s'/s and to $o(\Lambda)$ as,

$$\begin{aligned} \tilde{h}^{mn}(\tau,\vec{\xi}\,) \ &= \ \frac{4}{s}\left(1 + \frac{\Lambda s^2}{18}\right)\Bigg[\left(1 + \frac{\Lambda s^2}{9}\right)\delta^m_{\ \underline{m}}\delta^n_{\ \underline{n}}\int\sqrt{g_3(\vec{\xi}')}\Pi^{\underline{mn}}(\tau_{\rm ret},\vec{\xi}') \\ &+\frac{\Lambda s}{6}\left\{\xi^m\delta^n_{\ \underline{n}}\int\sqrt{g_3(\vec{\xi}')}\Pi^{\underline{0n}} + \xi^n\delta^m_{\ \underline{m}}\int\sqrt{g_3(\vec{\xi}')}\Pi^{\underline{0m}}\right\} \\ &-\frac{\Lambda}{18}\left\{\xi^m\delta^n_{\ \underline{n}}\xi_{\underline{k}}\int\sqrt{g_3(\vec{\xi}')}\Pi^{\underline{kn}} + \xi^n\delta^m_{\ \underline{m}}\xi_{\underline{k}}\int\sqrt{g_3(\vec{\xi}')}\Pi^{\underline{km}}\right\}\Bigg] \end{aligned} \tag{3.3.42}$$

The stress tensor is a function of $(\tau_{\rm ret}, \vec{\xi}')$, $\tau_{\rm ret}$ being defined in equation (3.3.31). The terms in the second and the third line will drop out when a suitable tt (transverse, traceless) projection is applied to the above solution to extract its *gauge invariant* content, in section 3.5. Each of these integrals over the source on $\tau' =$ constant hypersurface, are well defined and give a quantity which is a function of the retarded time and carry only the frame indices. The explicit factors of the mixed-indexed δ's are constant triad which serve to convert the integrated quantities from frame indices to coordinate indices.

To express the source integrals in terms of moments, we have to consider the conservation equation.

Conservation equation:

The conservation equation is $\partial_\mu T^{\mu\nu} = -\Gamma^\mu_{\ \mu\lambda}T^{\lambda\nu} - \Gamma^\nu_{\ \mu\lambda}T^{\mu\lambda}$ and we have computed the connection in FNC in equation (B.0.6). Recalling that the stress tensor is trace free, $T^{\mu\nu}\bar{g}_{\mu\nu} = T^{\mu\nu}(\eta_{\mu\nu} + \delta g_{\mu\nu}) = (-T^{00} + T^j_{\ j}) + T^{\mu\nu}\delta g_{\mu\nu} = 0$, we eliminate the spatial trace by using $T^j_{\ j} = T^{00} - T^{\mu\nu}\delta g_{\mu\nu}$. The second term is order Λ. To within our approximation and *momentarily suppressing the primes on the coordinates*, we find,

$$\partial_0 T^{00} + \partial_i T^{i0} = \frac{11\Lambda}{9} T^{0i}\xi_i \tag{3.3.43}$$

$$\partial_0 T^{0i} + \partial_j T^{ji} = \frac{\Lambda}{9}(7T^{ij}\xi_j + T^{00}\xi^i) \tag{3.3.44}$$

Taking second derivatives and eliminating T^{0i} we get,

$$\partial^2_{ij} T^{ij} = \partial^2_0 T^{00} + \frac{\Lambda}{9}\left\{10T^{00} + \xi^j\partial_j T^{00} + 18\xi_i\partial_j T^{ij}\right\} \tag{3.3.45}$$

Introducing the notation, $\rho := \Pi^{\underline{00}}, \pi := \Pi^{\underline{ij}}\delta_{\underline{ij}}$, we express the the coordinate components of the stress tensor in terms of the frame components as,

$$T^{00} = \delta^0_{\ \underline{0}}\delta^0_{\ \underline{0}}\left(1 + \frac{\Lambda s^2}{3}\right)\rho; \tag{3.3.46}$$

$$
\begin{aligned}
T^{0i} &= \delta^0_{\underline{0}}\left[\left(1+\frac{2\Lambda s^2}{9}\right)\delta^i{}_{\underline{j}} - \frac{\Lambda}{18}\xi^i\xi_{\underline{j}}\right]\Pi^{\underline{0j}} \\
T^{ij} &= \left[\left(1+\frac{\Lambda s^2}{9}\right)\delta^i{}_{\underline{k}}\delta^j{}_{\underline{l}} - \frac{\Lambda}{18}\left(\delta^i{}_{\underline{k}}\xi^j\xi_{\underline{l}} + \delta^j{}_{\underline{l}}\xi^i\xi_{\underline{k}}\right)\right]\Pi^{\underline{kl}}
\end{aligned}
$$

In terms of the frame components, the conservation equations take the form (the constant tetrad are suppressed),

$$
\partial_\tau \Pi^{\underline{00}} = -\left(1-\frac{\Lambda s^2}{9}\right)\partial_j\Pi^{\underline{0j}} + \frac{\Lambda}{18}\xi_j\xi^i\partial_i\Pi^{\underline{0j}} + \Lambda\Pi^{\underline{0j}}\xi_j \tag{3.3.47}
$$

$$
\partial_\tau \Pi^{\underline{0i}} = -\left(1-\frac{\Lambda s^2}{9}\right)\partial_j\Pi^{\underline{ji}} + \frac{\Lambda}{18}\left\{\xi_j\xi\cdot\partial\,\Pi^{\underline{ji}} + 15\Pi^{\underline{ij}}\xi_j + 3\rho\xi^i\right\} \tag{3.3.48}
$$

Eliminating $\Pi^{\underline{0i}}$ and using $\pi = \rho$ thanks to the trace free stress tensor, we get the second order conservation equation as,

$$
\partial^2_\tau\rho = \left(1-\frac{2\Lambda s^2}{9}\right)\partial^2_{ij}\Pi^{\underline{ij}} - \frac{\Lambda}{9}\left[\xi^i\xi_j\partial^2_{ik}\Pi^{\underline{jk}} + 19\xi_i\partial_j\Pi^{\underline{ij}} + 2\xi^j\partial_j\rho + 12\rho\right] \tag{3.3.49}
$$

Field in terms of moment variable :

The usual strategy is to define suitable moments of energy density/pressures and taking moments of the above equation, express the integral of $\Pi^{\underline{ij}}$ in terms of the moments and its time derivatives. To maintain coordinate invariance, the moment variable (analogue of x^i in the Minkowski background) must also be a *coordinate scalar*. Note that in FNC (as in RNC), ξ^i is a contravariant vector. Its frame components naturally provide coordinate scalars. We still have the freedom to multiply these by suitable scalar functions. It is easy to see that the frame components of ξ are the same as the coordinate components at best up to permutations i.e. $\xi^{\underline{i}} := e^{\underline{i}}_j\xi^j = \delta^{\underline{i}}{}_j\xi^j$. It is also true that $g_{ij}\xi^i\xi^j = \xi^{\underline{i}}\xi^{\underline{j}}\delta_{ij} = s^2$. Hence suitable functions of s^2 would qualify to be considered as coordinate scalars.

In equation (3.3.42), we need $\int d^3\xi'\sqrt{g_3(\vec{\xi}')}$. To get this from the equation (3.3.49), we introduce a moment variable $\zeta^i(\vec{\xi})$ and *define* moments of ρ as,

$$\mathcal{M}^{\underline{i_1 i_2 \cdots i_n}}(\tau) \ := \ \int_{\text{source}} d^3\xi \sqrt{g_3(\vec{\xi})}\zeta^{\underline{i_1}} \cdots \zeta^{\underline{i_n}} \rho(\tau, \vec{\xi}) \ , \ \zeta^{\underline{i}}(\vec{\xi}) := \left(1 + \frac{\Lambda s^2}{9}\right)\xi^{\underline{i}} \, , \quad (3.3.50)$$

where the integration is over the support of the source on the constant$-\tau$ hypersurface.

Multiplying the equation (3.3.49) by $\sqrt{g_3(\vec{\xi}')}\zeta^{\underline{i_1}} \ldots \zeta^{\underline{i_n}}$, and integrating over the source, we get,

$$\begin{aligned}\ddot{\mathcal{M}}^{\underline{i_1 \cdots i_n}} = & \int d^3\xi \, \Pi^{\underline{ij}} \partial^2_{ij} \left(\left(1 + \frac{(n-3)\Lambda s^2}{9}\right)\xi^{\underline{i_1 \cdots i_n}}\right) - \frac{\Lambda}{9}\left[\int d^3\xi \, \Pi_{\underline{j}}^{\ \underline{k}} \partial^2_{ik} \left(\xi^{\underline{i}}\xi^{\underline{j}}\xi^{\underline{i_1 \cdots i_n}}\right)\right. \\ & \left. +19 \int d^3\xi \, \Pi^{\underline{ij}} \partial_j \left(\xi_{\underline{i}}\xi^{\underline{i_1 \cdots i_n}}\right) + 2 \int d^3\xi \, \rho \partial_j \left(\xi^{\underline{j}}\xi^{\underline{i_1 \cdots i_n}}\right) + 12 \int d^3\xi \, \rho \xi^{\underline{i_1 \cdots i_n}}\right] \quad (3.3.51)\end{aligned}$$

There are no factors of $(1 - \Lambda s^2/9)$ in the terms enclosed by the square brackets since there is already an explicit pre-factor of Λ.

The first few moments satisfy,

$$\begin{aligned}\partial^2_\tau \mathcal{M} &= \frac{\Lambda}{3}\mathcal{M} \, , && (\text{'Mass non-conservation'}) \ (3.3.52) \\ \partial^2_\tau \mathcal{M}^{\underline{i}} &= \frac{2\Lambda}{3}\mathcal{M}^{\underline{i}} + \frac{2\Lambda}{3}\int d^3\xi \, \Pi^{\underline{ij}}\xi_{\underline{j}} \, , && (\text{'Momentum non-conservation'}) \ (3.3.53) \\ \partial^2_\tau \mathcal{M}^{\underline{ij}} &= 2\int_{\text{source}} d^3\xi \sqrt{g_3(\vec{x})} \, \Pi^{\underline{ij}} + \Lambda \mathcal{M}^{\underline{ij}} + \Lambda \int d^3\xi \xi_{\underline{k}} \left(\Pi^{\underline{ki}}\xi^{\underline{j}} + \Pi^{\underline{kj}}\xi^{\underline{i}}\right) && (3.3.54)\end{aligned}$$

There are additional types of integrals over $\Pi^{\underline{0n}}$ and $\xi \cdot \Pi \cdot \xi$ in equation (3.3.42). But these come with an explicit factor of Λ which simplifies the calculation. These can be expressed in terms of different moments using both the second order conservation equation, (3.3.49) and the first order one, (3.3.47). In particular, taking the fourth moment and tracing over a pair gives (to order $(\Lambda)^0$),

$$\begin{aligned}\int d^3\xi \, \xi_{\underline{k}}(\Pi^{\underline{km}}\xi^{\underline{n}} + \Pi^{\underline{kn}}\xi^{\underline{m}}) &= \frac{1}{4}\left[\delta_{\underline{rs}}\partial^2_\tau \mathcal{M}^{\underline{mnrs}} - 2\mathcal{M}^{\underline{mn}} - 2\mathcal{N}^{\underline{mn}}\right] \quad \text{where,} \\ \mathcal{N}^{\underline{mn}} &:= \int d^3\xi \sqrt{g_3(\vec{\xi})} \, \Pi^{\underline{mn}} s^2 \, . \quad (3.3.55)\end{aligned}$$

Likewise, taking the first moment of equation (3.3.47), we get

$$\int d^3\xi \sqrt{g_3(\vec{\xi})}\Pi^{0\underline{n}} \;=\; \partial_\tau \mathcal{M}^{\underline{n}} \,. \tag{3.3.56}$$

Collecting all these, we write the solution in the form,

$$\begin{aligned}
\tilde{h}^{mn}(\tau,\vec{\xi}) \;=\; \delta^m{}_{\underline{m}}\delta^n{}_{\underline{n}} \; & \Bigg[\left(\frac{2}{s}\partial_\tau^2\mathcal{M}^{\underline{mn}}\right) - \frac{\Lambda}{3s}\left(\xi_{\underline{k}}\frac{\xi^{\underline{m}}\partial_\tau^2\mathcal{M}^{\underline{kn}} + \xi^{\underline{n}}\partial_\tau^2\mathcal{M}^{\underline{km}}}{3} - s^2\partial_\tau^2\mathcal{M}^{\underline{mn}}\right) \\
& + \frac{\Lambda}{s}\left(-\mathcal{M}^{\underline{mn}} + \mathcal{N}^{\underline{mn}} - \frac{1}{2}\delta_{\underline{rs}}\partial_\tau^2\mathcal{M}^{\underline{mnrs}}\right) + \frac{2\Lambda}{3}\left(\xi^{\underline{m}}\partial_\tau\mathcal{M}^{\underline{n}} + \xi^{\underline{n}}\partial_\tau\mathcal{M}^{\underline{m}}\right)\Bigg]
\end{aligned} \tag{3.3.57}$$

The moments on the right hand side are all evaluated at the retarded τ and we have displayed the constant triad. The constant triad plays no role here but a similar one in the next subsection is important.

Remarks : There are several noteworthy points.

(1) The leading term has exactly the same form as for the usual flat space background. The correction terms involve the first, the second and the fourth moments as well as a new type of moment $\mathcal{N}^{\underline{mn}}$. We will see in the next section that the term involving ξ^i will drop out in a tt projection.

(2) There are terms which have no time derivative of any of the moments and hence can have constant (in time) field. This is a new feature not seen in the Minkowski background. A priori, such a term is permitted even in the Minkowski background. For instance, if $\partial_\tau T^{\alpha\beta} = 0$ i.e. the source is *static*, then eqn. (3.3.39) or eqn. (3.3.42) imply that $\partial_\tau\tilde{h}^{mn} = 0$ and hence the solution *can* have a $\tau-$independent piece. However, in this case ($\Lambda = 0$), the conservation equation (3.3.54) equation relates the field to double $\tau-$ derivative of the quadrupole moment which vanishes for a static source. It reflects the physical expectation that a static source does not radiate. Does this expectation change in a curved background?

In a general curved background, 'staticity' could be defined in a coordinate invariant manner only if there is a time-like Killing vector, say, T. A source would then be called static if the Lie derivative of the stress tensor vanishes, $\mathcal{L}_T T^{\alpha\beta} = 0$. In the de Sitter background, in the static patch we are working in, the stationary Killing vector is precisely ∂_τ. Hence, from the definition of moments (3.3.50, 3.3.40), it follows that for a static source, $\partial_\tau T^{\alpha\beta} = 0$, all its moments would be $\tau-$independent. However, the conservation equations (3.3.52) for the zeroth moment[4] contradicts this, unless $\mathcal{M}$ itself vanishes. Hence we *cannot* even have strictly static (test) sources in a curved background. Thus, in the specific case of the de Sitter background, the non-derivative terms in the equation (3.3.54), do *not* indicate the possibility of time independent field $\tilde{h}^{mn}$.

For a very slowly varying source - so that we can neglect the derivative terms - we can have a left over, slowly varying field, falling off as $\sim \Lambda/s$. Such a field has a very long wavelength and is not 'radiative' in the static patch. To isolate radiative fields, one should probe the vicinity of the null infinity which is beyond the extent of the static patch. For typical rapidly changing sources, ($\lambda \ll s$) the $\tau-$derivative terms dominate over these terms and in the context of present focus, we *drop them hereafter*.

The remaining terms that survive the tt projection, all have second order $\tau-$derivative. Similar features also arise in the Cosmological chart in the next subsection.

(3) The mass conservation equation can be immediately integrated and have exponentially growing and decaying components. The scale of this time variation is $\sim (\Lambda)^{-1/2}$ which is extremely slow, about the age of the universe. These equations do *not* depend on the Green's function at all and are just consequences of the matter conservation equation for *small* curvature. We are working in a static patch of the space-time, so the time variation is not driven by the time dependence of the background geometry. It is the background *curvature* that is responsible for the changes in the matter distribution and hence its moments. In effect, this confirms that test matter *cannot* remain static in a *curved* background even if the background is static. In a flat background, there is no work done on the test matter and hence the sources' mass and linear momenta are conserved

[4] The non-zero curvature always does 'work' on the test matter and the 'mass of the matter' alone is not conserved.

(the zeroth and the first moment are time independent).

For contrast, in the next subsection, we present the computation in the conformal chart of Poincaré patch. [6]. This subsection has the tail contribution explicitly available and the correction terms are in powers of $\sqrt{\Lambda}$.

3.3.2 Generalized Transverse Gauge in Poincaré patch

The computation takes advantage of the conformally flat form of the metric in the conformal chart and makes a choice of a generalized transverse gauge to simplify the linearized equation. We summarize them for convenience and present the radiative solution. Recall from previous chapter 2.2.2 in the conformal chart (see figure 2.3), de Sitter metric takes the form,

$$ds^2 = \frac{1}{H^2\eta^2}\left[-d\eta^2 + \sum_i (dx^i)^2\right] \ , \ \eta \in (-\infty, 0) \ , \ H := \sqrt{\frac{\Lambda}{3}}. \tag{3.3.58}$$

The conformally flat form leads to a great deal of simplification. In this chart, the de Sitter d'Alembertian can be conveniently expressed in terms of the Minkowski d'Alembertian leading to simplification of linearized wave eqn. (3.2.8),

$$\begin{aligned} -16\pi T_{\mu\nu} = {} & \Omega^{-2}\left[\Box\tilde{h}_{\mu\nu} + \frac{2}{\eta}\left\{\left(\delta^0_\mu\partial^\sigma\tilde{h}_{\sigma\nu} + \delta^0_\nu\partial^\sigma\tilde{h}_{\sigma\mu}\right) + \left(-\partial_0\tilde{h}_{\mu\nu} + \partial_\mu\tilde{h}_{0\nu} + \partial_\nu\tilde{h}_{0\mu}\right)\right\}\right. \\ & \left. + \frac{2}{\eta^2}\left\{\delta^0_\mu\delta^0_\nu\tilde{h}_{\alpha\beta}\eta^{\alpha\beta} + \eta_{\mu\nu}\tilde{h}_{00} + 2\left(\delta^0_\mu\tilde{h}_{0\nu} + \delta^0_\nu\tilde{h}_{0\mu}\right)\right\}\right] - \left(\frac{2\Lambda}{3}\right)\left[\tilde{h}_{\mu\nu} - \eta_{\mu\nu}\tilde{h}_{\alpha\beta}\eta^{\alpha\beta}\right] \\ & - \left\{\left(\partial_\mu B_\nu + \partial_\nu B_\mu - \eta_{\mu\nu}\partial^\alpha B_\alpha\right) + \frac{2}{\eta}\left(\delta^0_\mu B_\nu + \delta^0_\nu B_\mu\right)\right\} \ , \ \Omega^2 := \frac{1}{H^2\eta^2} = \frac{3}{\Lambda\eta^2} \ . \end{aligned} \tag{3.3.59}$$

While the transverse gauge will eliminate the B_μ terms, it still keeps the linearized equation in a form that mixes different components of $\tilde{h}_{\mu\nu}$. A different choice of B_μ achieves decoupling of the components $\tilde{h}_{0i}$ and $\tilde{h}_{ij}$ ($\tilde{h}_{00}$ component involves in trace) . Taking B_μ

of the form $f(\eta)\tilde{h}_{0\mu}$, shows that for the choice $f(\eta) := \frac{2\Lambda}{3}\eta$, the equation simplifies to [6],

$$-16\pi T_{\mu\nu}\Omega^2 = \Box\tilde{h}_{\mu\nu} - \frac{2}{\eta}\partial_0\tilde{h}_{\mu\nu} - \frac{2}{\eta^2}\left\{\delta^0_\mu\delta^0_\nu\tilde{h}_\alpha^{\ \alpha} - \tilde{h}_{\mu\nu} + \delta^0_\mu\tilde{h}_{0\nu} + \delta^0_\nu\tilde{h}_{0\mu}\right\} \quad , \quad \text{with} \tag{3.3.60}$$

$$0 = \partial^\alpha\tilde{h}_{\alpha\mu} + \frac{1}{\eta}\delta^0_\mu\tilde{h}_\alpha^{\ \alpha} \quad , \quad \tilde{h}_\alpha^{\ \alpha} := \tilde{h}_{\alpha\beta}\eta^{\alpha\beta} \qquad \text{(gauge fixing condition)} \tag{3.3.61}$$

From now on in this subsection, the tensor indices are raised/lowered with the Minkowski metric.

It turns out to be convenient to work with new variables, $\chi_{\mu\nu} := \Omega^{-2}\tilde{h}_{\mu\nu}$. All factors of Ω^2 *and* Λ drop out of the equations and $\chi_{\mu\nu}$ satisfies [6],

$$-16\pi T_{\mu\nu} = \Box\chi_{\mu\nu} + \frac{2}{\eta}\partial_0\chi_{\mu\nu} - \frac{2}{\eta^2}\left(\delta^0_\mu\delta^0_\nu\chi_\alpha^{\ \alpha} + \delta^0_\mu\chi_{0\nu} + \delta^0_\nu\chi_{0\mu}\right) . \tag{3.3.62}$$

$$0 = \partial^\alpha\chi_{\alpha\mu} + \frac{1}{\eta}\left(2\chi_{0\mu} + \delta^0_\mu\chi_\alpha^{\ \alpha}\right) \qquad \text{(gauge condition).} \tag{3.3.63}$$

Under the gauge transformations generated by a vector field ξ^μ, the $\chi_{\mu\nu}$ transform as,

$$\delta\chi_{\mu\nu} = (\partial_\mu\underline{\xi}_\nu + \partial_\nu\underline{\xi}_\mu - \eta_{\mu\nu}\partial^\alpha\underline{\xi}_\alpha) - \frac{2}{\eta}\eta_{\mu\nu}\underline{\xi}_0 \quad , \quad \underline{\xi}_\mu := \Omega^{-2}\xi_\mu = \eta_{\mu\nu}\xi^\nu . \tag{3.3.64}$$

The gauge condition (3.3.63) is preserved by the transformation generated by a vector field ξ^μ satisfying residual gauge transformation,

$$\Box\underline{\xi}_\mu + \frac{2}{\eta}\partial_0\underline{\xi}_\mu - \frac{2}{\eta^2}\delta^0_\mu\underline{\xi}_0 = 0 \tag{3.3.65}$$

and the equation (3.3.62) is invariant under the gauge transformations generated by these restricted vector fields.

The main simplification of eqn.(3.3.62) occurs in decomposition of field components into

$\hat{\chi}(:= \chi_{00} + \chi^i{}_i)$, χ_{0i}, χ_{ij} as follows,

$$\Box\left(\frac{\hat{\chi}}{\eta}\right) = -\frac{16\pi\,\hat{T}}{\eta} \tag{3.3.66}$$

$$\Box\left(\frac{\chi_{0i}}{\eta}\right) = -\frac{16\pi\,T_{0i}}{\eta} \tag{3.3.67}$$

$$\Box\,\chi_{ij} + \frac{2}{\eta}\partial_0\,\chi_{ij} = -16\pi\,T_{ij} \tag{3.3.68}$$

To exhaust residual gauge freedom, let us investigate equation of motion satisfied gauge field $\delta\hat{\chi}$,

$$\Box\,(\delta\hat{\chi}) \;\; = 4\left[\Box\left(\partial_0\underline{\xi}_0 - \frac{\underline{\xi}_0}{\eta}\right)\right] = -\frac{2}{\eta}\partial_0\,(\delta\hat{\chi}) + \frac{2}{\eta^2}\,(\delta\hat{\chi}) \implies \Box\left(\frac{\delta\hat{\chi}}{\eta}\right) = 0 \tag{3.3.69}$$

Hence gauge field $\delta\hat{\chi}$ satisfies the wave equation outside the source. Therefore using residual gauge transformation of $\underline{\xi}_0$, we can set $\hat{\chi} = 0$ outside the matter source. Similarly χ_{0i} can be gauged away [6]. The gauge condition (3.3.63) then implies $\partial^0\chi_{00} = 0$ and by choosing it to be zero at some initial η =constant hypersurface we can take $\chi_{00} = 0$ as well. Thus the *physical solutions*, satisfy conditions: $\partial^i\chi_{ij} = 0 = \chi^i{}_i$ and it suffices to focus on the equation (3.3.62) for $\mu, \nu = i, j$,

$$\Box\chi_{ij} + \frac{2}{\eta}\partial_0\chi_{ij} = -16\pi T_{ij} \quad , \quad \partial_i\chi^i{}_j = 0 = \chi_i{}^i\,.$$

The corresponding equation for retarded Green function is defined by

$$\left(\Box + \frac{2}{\eta}\partial_0\right)G_R(\eta, x; \eta', x') = -\frac{\Lambda}{3}\eta^2\delta^4(x - x')\,. \tag{3.3.70}$$

Green's function of the filed χ_{ij} mimics that of minimally coupled scalar field in de Sitter background. Hence from the Hadamard construction corresponding retarded Green function is given by, [6],

$$G_R(\eta, x; \eta' x') = \frac{\Lambda}{3}\eta\eta' \frac{1}{4\pi}\frac{\delta(\eta - \eta' - |x - x'|)}{|x - x'|} + \frac{\Lambda}{3}\frac{1}{4\pi}\theta(\eta - \eta' - |x - x'|) . \tag{3.3.71}$$

The particular solution is given by,

$$\chi_{ij}(\eta, x) = 16\pi \int_{\text{source}} \frac{d\eta' d^3x'}{\frac{\Lambda}{3}\eta'^2} G_R(\eta, x; \eta' x') T_{ij}(\eta', x') \tag{3.3.72}$$

$$\begin{aligned} &= 4\int d\eta' d^3x' \frac{\eta}{\eta'}\frac{\delta(\eta - \eta' - |x - x'|)}{|x - x'|} T_{ij}(\eta', x') \\ &\quad + 4\int d\eta' d^3x' \frac{1}{\eta'^2}\theta(\eta - \eta' - |x - x'|) T_{ij}(\eta', x') \end{aligned} \tag{3.3.73}$$

$$\begin{aligned} &= 4\int d^3x' \frac{\eta}{|x - x'|(\eta - |x - x'|)} T_{ij}(\eta', x')\Big|_{\eta' = \eta - |x - x'|} \\ &\quad + 4\int d^3x' \int_{-\infty}^{\eta - |x - x'|} d\eta' \frac{T_{ij}(\eta', x')}{\eta'^2} \end{aligned} \tag{3.3.74}$$

The spatial integration is over the matter source confined to a compact region and is finite. The second term in the eqns. (3.3.73, 3.3.74) is the *tail term*.

It is possible to put the solution in the same form as in the case of flat background, in terms of suitable Fourier transforms with respect to η [6]. However, we work with the $(\eta, \vec{x})$-space.

For $|\vec{x}\,| \gg |\vec{x}\,'|$, we can approximate $|\vec{x} - \vec{x}\,'| \approx r := |\vec{x}\,|$. This allows us separate out the $\vec{x}'$ dependence from the $\eta - |x - x'|$. In the first term, this leads to the spatial integral over $T_{ij}(\eta - r, x')$ while in the second term, we can interchange the order of integration again leading to the same spatial integral. The spatial integral of T_{ij} can be simplified using moments. This is done through the matter conservation equation using the conformally flat form of the metric,

$$\partial^\mu T_{\mu 0} + \frac{1}{\eta}\left(T_{00} + T_i{}^i\right) = 0 \quad , \quad \partial^\mu T_{\mu i} + \frac{2}{\eta} T_{0i} = 0 . \tag{3.3.75}$$

Taking derivatives of these equations to eliminate T_{0i}, we get,

$$\partial^i\partial^j T_{ij} \ = \ \partial_\eta^2 T_{00} - \frac{1}{\eta}\partial_\eta(T_{00} + T_i{}^i) + \frac{3}{\eta^2}(T_{00} + T_i{}^i) - \frac{2}{\eta}\partial_\eta T_{00} \tag{3.3.76}$$

As in the previous section, we introduce a tetrad to define the frame components of the stress tensor. The conformal form of the metric suggests a natural choice: ($\sqrt{\Lambda/3} =: H$),

$$f^\alpha_{\underline{0}} \ := \ -H\eta(1, \vec{0}) \ , \ f^\alpha_{\underline{m}} \ := \ -H\eta\ \delta^\alpha_{\ \underline{m}} \qquad \Longleftrightarrow \qquad f^\alpha_{\underline{a}} := -H\eta\delta^\alpha_{\ \underline{a}} \tag{3.3.77}$$

The corresponding components of the stress tensor are given by,

$$\rho \ := \ P_{\underline{00}} = T_{\alpha\beta} f^\alpha_{\underline{0}} f^\beta_{\underline{0}} \ = H^2\eta^2 T_{00}\delta^0_{\ \underline{0}}\delta^0_{\ \underline{0}} \ , \ P_{\underline{ij}} \ := \ T_{\alpha\beta} f^\alpha_{\underline{i}} f^\beta_{\underline{j}} \ = \ H^2\eta^2 T_{ij}\delta^i_{\ \underline{i}}\delta^j_{\ \underline{j}} \ ; \tag{3.3.78}$$

$$P_{\underline{0i}} := T_{\alpha\beta} f^\alpha_{\underline{0}} f^\beta_{\underline{i}} \ , \ \pi := P_{\underline{ij}}\delta^{\underline{ij}} \ . \tag{3.3.79}$$

In terms of these, the conservation equations take the form (suppressing the constant tetrad),

$$0 \ = \ \partial_\eta P^{\underline{00}} + \partial_i P^{\underline{i0}} - \frac{1}{\eta}\left(3P^{\underline{00}} + \pi\right) \tag{3.3.80}$$

$$0 \ = \ \partial_\eta P^{\underline{0i}} + \partial_j P^{\underline{ji}} - \frac{1}{\eta} 4P^{\underline{0i}} \tag{3.3.81}$$

It is convenient to go over to the cosmological chart, $(t, \vec{x})$ and convert the ∂_η to ∂_t using the definitions: $\eta := -H^{-1}e^{-Ht}$. This leads to $\partial_\eta \ = \ e^{Ht}\partial_t \ \ := \ \ a(t)\partial_t$.

$$0 \ = \ \partial_t\rho + \frac{1}{a}\partial_i P^{\underline{0i}} + H(3\rho + \pi) \ ; \tag{3.3.82}$$

$$0 \ = \ \partial_t P^{\underline{0i}} + \frac{1}{a}\partial_j P^{\underline{ij}} + 4HP^{\underline{0i}} \ ; \tag{3.3.83}$$

$$0 \ = \ \partial_t^2\rho - \frac{1}{a^2}\partial^2_{ij} P^{\underline{ij}} + 8H\partial_t\rho + H\partial_t\pi + 5H^2(3\rho + \pi) \tag{3.3.84}$$

As before, we define the moments of the two rotational scalars, ρ, π, by integrating over the source distribution at η =constant hypersurface. The determinant of the induced metric on these hypersurfaces is $a^3(\eta)$. The tetrad components of the moment variable are given by, $\bar{x}^{\underline{i}} := f^{\underline{i}}_{\alpha} x^{\alpha} = -(\eta H)^{-1}\delta^{\underline{i}}{}_{j} x^j = a(t) x^{\underline{i}}$. The moments are defined by,

$$Q^{\underline{i_1}\cdots\underline{i_n}}(t) := \int_{Source(t)} d^3x\, a^3(t)\rho(t,\vec{x})\bar{x}^{\underline{i_1}}\cdots\bar{x}^{\underline{i_n}} \ , \tag{3.3.85}$$

$$, \ \bar{Q}^{\underline{i_1}\cdots\underline{i_n}}(t) := \int_{Source(t)} d^3x\, a^3(t)\pi(t,\vec{x})\bar{x}^{\underline{i_1}}\cdots\bar{x}^{\underline{i_n}} \ . \tag{3.3.86}$$

Taking second moment of the eqn. (3.3.76) and lowering the frame indices we get,

$$\int d^3x a^3(t) P_{\underline{ij}}(t,x) = \frac{1}{2}\left[\partial_t^2 Q_{\underline{ij}} - 2H\partial_t Q_{\underline{ij}} + H\partial_t \bar{Q}_{\underline{ij}}\right] \tag{3.3.87}$$

Let us write the solution, eq.(3.3.74), in terms of the cosmological chart, incorporating the approximation $|\vec{x}\,'| \ll |\vec{x}\,|$.

$$\chi_{ij}(\eta,x) = 4\frac{\eta}{r(\eta-r)}\int d^3x' T_{ij}(\eta',x')\Big|_{\eta'=\eta-r} + 4\int d^3x' \int_{-\infty}^{\eta-r} d\eta' \frac{T_{ij}(\eta',x')}{\eta'^2} \tag{3.3.88}$$

Define the *retarded time*, t_{ret}, through $(\eta - r) := -H^{-1}e^{-Ht_{\text{ret}}}$ and set $\bar{a} := a(t_{\text{ret}})$. Then we have, $\eta = -(aH)^{-1}$, $(\eta - r) = -(\bar{a}H)^{-1}$. Using these,

$$\frac{\eta}{\eta - r} T_{ij}(\eta - r, x') = a(t)^{-1}\bar{a}^3 P_{\underline{ij}}(t_{\text{ret}}, x') \ , \ d\eta' \frac{1}{\eta'^2} T_{ij}(\eta', x') = H^2 dt' a^3(t') P_{\underline{ij}}(t', x') \tag{3.3.89}$$

All terms involve only the $\int a^3 P_{ij}$ which is obtained above. With these, the solution takes the form,

$$\begin{aligned}\chi_{\underline{ij}}(t,r) \approx & \frac{2}{r\,a(t)}\left[\partial_{t'}^2 Q_{\underline{ij}} - 2H\partial_{t'} Q_{\underline{ij}} + H\partial_{t'}\bar{Q}_{\underline{ij}}\right]\Big|_{t_{\text{ret}}} + 2H^2\left\{\partial_{t'} Q_{\underline{ij}} - 2HQ_{\underline{ij}} + H\bar{Q}_{\underline{ij}}\right\}\Big|_{t_{\text{ret}}} \\ & -2H^2\left\{\partial_{t'} Q_{\underline{ij}} - 2HQ_{\underline{ij}} + H\bar{Q}_{\underline{ij}}\right\}\Big|_{-\infty}\end{aligned} \tag{3.3.90}$$

We have restored the constant triad and used the definition: $\chi_{\underline{ij}} := \delta_{\underline{i}}{}^{i}\delta_{\underline{j}}{}^{j}\chi_{ij}$. The first term in the equation (3.3.90) is the contribution of the sharp term and the remaining terms are from the tail. The tail contribution has separated into a term which depends on the retarded time just as the sharp term does, and the contribution from the history is given by the limiting value of the last line.

This expression is valid *as a leading term for* $|\vec{x}\,| \gg |\vec{x}\,'|$. (For Hulse-Taylor system, the physical size is about 3 light-seconds and it is about 20,000 light years away, giving $|x'|/|x| \sim 10^{-12}$.) We work with this expression in the following and suppress the $\approx$ sign.

We write $a^{-1} = \bar{a}^{-1}(\frac{\bar{a}}{a}) = \bar{a}^{-1}(1 - Hr\bar{a}) = \bar{a}^{-1} - rH$ in the first term to make manifest the dependence on retarded time t_{ret}. The solution is then expressed as,

$$\begin{aligned} \chi_{\underline{ij}}(t,r) \;\approx\; & \frac{2}{r\,\bar{a}}\left\{\partial_t^2 Q_{\underline{ij}} - 2H\partial_t Q_{\underline{ij}} + H\partial_t \bar{Q}_{\underline{ij}}\right\} \\ & -2H\left\{\partial_t^2 Q_{\underline{ij}} - 3H\partial_t Q_{\underline{ij}} + H\partial_t \bar{Q}_{\underline{ij}} + 2H^2 Q_{\underline{ij}} - H^2\bar{Q}_{\underline{ij}}\right\} \\ & -2H^2\left.\left\{\partial_{t'} Q_{\underline{ij}} - 2HQ_{\underline{ij}} + H\bar{Q}_{\underline{ij}}\right\}\right|_{-\infty} \end{aligned} \tag{3.3.91}$$

Remarks:

(1) In the conformal chart, there is no explicit dependence on the cosmological constant and it is not a suitable chart for exploring the subtle limit of vanishing cosmological constant [8, 22]. Hence we changed to the cosmological chart and exhibited the solution with explicit powers of H. Although the solution in eqn. (3.3.74) showed the presence of a tail term as an integral over the history of the source, in the final expression the field depends only on the properties of the source at the retarded time t_{ret} which was defined through $(\eta - r)$ except for the limiting value in the last line.

(2) Unlike the FNC chart, here the *tail contribution* has moments without a time derivative which naively indicates that for 'static' sources, there could be a non-zero field. A coordinate invariant way of specifying staticity of a source is to refer to the Killing parameter of a stationary Killing vector in its vicinity, eg, $T\cdot\partial := -H(\eta\partial_\eta + x^i\partial_i) = \partial_t - Hx^i\partial_i$ (This also equals the ∂_τ of the FNC). A static source satisfies $\mathcal{L}_T T_{\mu\nu} = T\cdot\partial T_{\mu\nu} - 2HT_{\mu\nu} = 0$. Explic-

itly, $\mathcal{L}_T f^\alpha_{\underline{a}} = 0$ and hence for a static source, $\mathcal{L}_T P_{\underline{ab}} = 0$. Furthermore, the Lie derivative of the moment variable $x^{\underline{i}} = ax^i$ also vanishes as does that of the volume element. Hence, $\mathcal{L}_T Q_{\underline{ij}} = 0$. Since the moments are coordinate scalars and independent of spatial coordinates, their Lie derivative is just ∂_t. Hence, for static sources, $\partial_t Q_{\underline{ij}} = 0 = \partial_t \bar{Q}_{\underline{ij}}$ (indeed *all* moments will be independent of t). For constant moments, there is a cancellation between the terms in the second line of equation (3.3.90) and the field vanishes. The boundary term at $t = -\infty$ is essential for this cancellation.

However, the conservation equations for the zeroth and the first moment are,

$$\partial_t Q + H\bar{Q} \ = \ 0 \quad , \quad \partial_t^2 Q_{\underline{i}} + H\partial_t \bar{Q}_{\underline{i}} - H^2 \left(Q_{\underline{i}} - \bar{Q}_{\underline{i}}\right) \ = \ 0 \tag{3.3.92}$$

The equation for the zeroth moment can be derived directly from (3.3.82). These again show that in a curved background, test matter cannot remain static.

For very slowly varying moments, the sharp contribution is negligible while the tail has a contribution, *not* falling off as r^{-1}. In FNC, the slowly varying contribution is in the sharp term, but could not be thought of as 'radiation'. The absence of such a contribution in the sharp term in eqn. (3.3.90), suggests that the slowly varying sharp term of FNC (eqn. 3.3.57), would not survive as 'radiation' at $\mathcal{J}^+$, though of course this cannot be analysed within FNC chart. The surviving tail contribution has been thought of as inducing a linear memory effect in [37].

The contribution from $t = -\infty$ boundary, is in any case a constant and does not play any role in any physical observables which typically involve time derivatives. With this understood, we now suppress this boundary contributions.

(3) To link with [8], the final step involves replacement of ∂_t by the Lie derivative with respect to the stationary Killing vector. Using $\mathcal{L}_T Q_{\underline{ij}}(t_{\text{ret}}) = (\partial_t - Hx^i\partial_i)(t_{\text{ret}})Q_{\underline{ij}}(t_{\text{ret}}) =$

$\partial_{t_{\rm ret}} Q_{\underline{ij}}(t_{\rm ret})$ and $\mathcal{L}_T \delta_i^{\ \underline{i}} = T \cdot \partial \delta_i^{\ \underline{i}} - H\delta_i^{\ \underline{i}}$, we get

$$\mathcal{L}_T Q_{ij} = \mathcal{L}_T(\delta_i^{\ \underline{i}}\delta_j^{\ \underline{j}} Q_{\underline{ij}}) = (\mathcal{L}_T \delta_i^{\ \underline{i}}\delta_j^{\ \underline{j}})Q_{\underline{ij}} + \delta_i^{\ \underline{i}}\delta_j^{\ \underline{j}}\partial_{t_{\rm ret}} Q_{\underline{ij}} = \partial_{t_{\rm ret}} Q_{ij} - 2HQ_{ij} \,. \qquad (3.3.93)$$

This is where the constant triad plays a role, unlike in the FNC chart where $\mathcal{L}_T = \partial_\tau$ on all tensors. It should be noted that the staticity condition $\partial_t Q_{\underline{ij}} = 0$, in terms of coordinate indices becomes $\mathcal{L}_T Q_{ij} = -2HQ_{ij}$. With these translations, our solution in equation (3.3.91) takes the forms,

$$\begin{aligned} \chi_{ij}(t,r) &= \frac{2}{r\bar{a}}\delta_i^{\ \underline{i}}\delta_j^{\ \underline{j}}\left[\partial_t^2 Q_{\underline{ij}} - 2H\partial_t Q_{\underline{ij}} + H\partial_t \bar{Q}_{\underline{ij}}\right](t) \qquad (3.3.94)\\ &\quad -2H\delta_i^{\ \underline{i}}\delta_j^{\ \underline{j}}\left[\partial_t^2 Q_{\underline{ij}} - 3H\partial_t Q_{\underline{ij}} + 2H^2 Q_{\underline{ij}} + H\partial_t \bar{Q}_{\underline{ij}} - H^2 \bar{Q}_{\underline{ij}}\right](t), \\ &= \frac{2}{r\bar{a}}\left[\mathcal{L}_T^2 Q_{ij} + 2H\mathcal{L}_T Q_{ij} + H\mathcal{L}_T \bar{Q}_{ij} + 2H^2 \bar{Q}_{ij}\right] \qquad (3.3.95)\\ &\quad -2H\left[\mathcal{L}_T^2 Q_{ij} + H\mathcal{L}_T Q_{ij} + H\mathcal{L}_T \bar{Q}_{ij} + H^2 \bar{Q}_{ij}\right] \end{aligned}$$

The derivatives are evaluated at $t = t_{ret}$. Both terms have the same derivatives of moments appearing in them and on combining, lead to a coefficient of the form $((r\bar{a})^{-1} - H)$. Thus in each order in H, the effect of the tail is to reduce the amplitude. The equation (3.3.95) matches with the solution given by Ashtekar et al. [8] and the $\Lambda \to 0$ limit of the solution goes over to the Minkowski background solution.

3.4 Extracting physical solution

The two solutions presented above were obtained in two different gauges. With a further choice of synchronous gauge, we could restrict the solutions to the spatial components alone. While these conditions fix the gauge completely, these spatial components still have to satisfy certain ‘spatial transversality and trace free’ conditions. The inhomogeneous solutions obtained above do *not* satisfy these conditions and hence do not represent solutions of the original linearized Einstein equation. Their dependence on the retarded time and the ‘radial’ coordinate however, offers an easy way to construct solutions which *do* satisfy these spatial-TT conditions [36]. In flat background, this is achieved by the *algebraic* tt-projector (defined below) and the method extends to the de Sitter background as well.

For χ_{ij}, the spatial-TT conditions have the form: $\partial^j\chi_{ji} = 0 = \delta^{ij}\chi_{ij}$ which have exactly the same form as in the case of the Minkowski background. To deduce their form for the $\tilde{h}^{ij}$ consider $\tilde{h}^{\mu\nu}$ satisfying the TT gauge condition and the synchronous gauge condition: $\bar{\nabla}_\mu\tilde{h}^{\mu\nu} = 0 = \tilde{h}^{\mu\nu}\bar{g}_{\mu\nu}$, $\tilde{h}^{\alpha 0} = \frac{\Lambda}{3}(\tau - \tau_0)\tilde{h}^{\alpha i}\xi^j\delta_{ij}$. These imply, $\tilde{h}^{\mu\nu} = h^{\mu\nu}$,

$$h^{00} = o(\Lambda^2) \ , \ h^{0i} = \frac{\Lambda}{3}(\tau - \tau_0)h^{ij}\xi^k\delta_{jk} \quad ; \quad h^{ij}\delta_{ij} = -\frac{\Lambda}{9}h^{ij}\xi_i\xi_j \ . \tag{3.4.1}$$

Furthermore,

$$\begin{aligned}
\bar{\nabla}_\mu h^{\mu\nu} &= \partial_\mu h^{\mu\nu} + \bar{\Gamma}^\mu{}_{\mu\lambda}h^{\lambda\nu} + \bar{\Gamma}^\nu{}_{\mu\lambda}h^{\mu\lambda} = 0 & (3.4.2)\\
\implies \partial_\mu h^{\mu\nu} &= \frac{5\Lambda}{9}h^{i\nu}\xi_i + \frac{2\Lambda}{3}\delta^\nu_0 h^{0i}\xi_i + \frac{2\Lambda}{9}\delta^\nu_i\left(h^{ij} - h^{kl}\delta_{kl}\delta^{ij}\right)\xi_j & (3.4.3)\\
\implies \partial_j(h^{ji}\xi_i) &= -\frac{\Lambda}{3}(\tau - \tau_0)\partial_\tau(h^{ij}\xi_i\xi_j) + \frac{5\Lambda}{9}\xi_i\xi_j h^{ji} \qquad (\text{for } \nu = 0) & (3.4.4)\\
\text{and} \quad \partial_j h^{ji} &= -\frac{\Lambda}{3}(\tau - \tau_0)\partial_\tau(h^{ij}\xi_j) + \frac{4\Lambda}{9}\xi_j h^{ji} \qquad (\text{for } \nu = i) & (3.4.5)
\end{aligned}$$

Multiplying (3.4.5) by ξ_i, subtracting from (3.4.4) and using (3.4.1) implies that $h^{ij}\xi_i\xi_j = 0 = h^{ij}\delta_{ij}$. The $\xi_i\times$(3.4.4) then implies that $\partial_j(h^{ij}\xi_i) = 0$, satisfying eqn. (3.4.4) identically. *Provided* $h^{ij}\xi_j = 0$, the spatial transversality condition, $\partial_j h^{ij} = 0$ will be satisfied. The tt-projector defined below will ensure $h^{ij}\xi_j = 0$ to the *leading order in* s^{-1}. Hence the spatial transversality will also hold for the projected h^{ij} to the leading order in s^{-1}. The projector being local (algebraic) in space-time while the spatial TT conditions being non-local (differential), the projector ensures the condition only for large s. Elsewhere, the condition must be satisfied by adding solutions of the homogeneous wave equation. However, we need the explicit forms of the solution only in the large s regions for which the projector suffices.

As in the case of the Minkowski background, corresponding to each spatial, unit vector $\hat{n}$, define the projectors,

$$P^i{}_j(\hat{n}) \ := \ \delta^i{}_j - \hat{n}^i\hat{n}_j \ , \ \ \Lambda^{ij}{}_{kl} \ := \ \frac{1}{2}\left(P^i{}_k P^j{}_l + P^i{}_l P^j{}_k - P^{ij}P_{kl}\right) \ . \tag{3.4.6}$$

Contraction with $\hat{n}$ gives zero and the trace of Λ-projector in either pair of indices vanishes. From any X^{kl}, the Λ-projector gives $X^{ij}_{tt} := \Lambda^{ij}{}_{kl}X^{kl}$ which is trace free and is transverse to the unit vector $\hat{n}$. For the FNC fields we choose $\hat{n}^i := \vec{\xi}^i/s$ and for the conformal chart fields we choose $\hat{n}^i = -H\eta\,\vec{x}^i/r$. When $\tilde{h}^{ij}_{TT}$ is substituted in the eqn. (3.4.5) the condition reduces to the 'spatial transversality', $\partial_j\tilde{h}^{ij}_{TT} = 0$. The χ^{TT}_{ij} also satisfies the same condition: $\partial^j\chi^{TT}_{ij} = 0$.

Since $\hat{n}$ is a radial unit vector, It follows that, $\partial_j\Lambda^{ij}{}_{kl} = \frac{1}{2r}(P^i{}_k\hat{n}_l + P^i{}_l\hat{n}_k)$ which is down by a power of r (or of s for FNC). Therefore to the leading order in r^{-1}, $\partial_j\tilde{h}^{ij}_{tt} = \Lambda^{ij}{}_{kl}\partial_j\tilde{h}^{kl}$.

Noting that the retarded solutions have a form $\sim f^{ij}(\tau - s)/s$, we get,

$$\partial_j\left[\frac{f^{ij}(\tau - s)}{s}\right] = -\frac{1}{s^2}\xi_j\left(\partial_\tau f^{ij} + s^{-1}f^{ij}\right) \approx -\hat{\xi}_j\partial_\tau\left[\frac{f^{ij}(\tau - s)}{s}\right] + o(\frac{1}{s^2}) \ , \ \hat{\xi}_j := s^{-1}\xi_j \ .$$

It follows immediately that to the leading order in s^{-1} (or r^{-1}), $\partial_j\tilde{h}^{ij}_{TT} \approx -\partial_\tau(\hat{\xi}_j\tilde{h}^{ij}_{TT}) = 0$ (and likewise $\partial^j\chi^{TT}_{ij} = 0$). Note that although to begin with, the spatial TT conditions in FNC look different from those of the conformal chart, they have the same form after the corresponding Λ-projections. Thus, for the $\Lambda-$projected h^{ij} too, $h^{\alpha 0} = 0, \forall\alpha$. There is no plane wave assumption or spatial Fourier transform needed for this projection. Of course, the $\Lambda-$projector only ensures that the gauge conditions are satisfied to the *leading order* in $r^{-1}(s^{-1})$. These Λ-projected field represent physical perturbations and gauge invariant observables of interest can be computed using these. From now on, in this chapter the solutions will be in the synchronous gauge and with tt projection implicit: $\tilde{h}^{\tau\beta} = 0$, $\tilde{h}^{ij} \leftrightarrow \tilde{h}^{ij}_{tt}$ and $\chi_{\eta\alpha} = 0$, $\chi_{ij} \leftrightarrow \chi^{tt}_{ij}$. In particular $\tilde{h}^{ij} = h^{ij}$.

3.5 Constructing gauge invariant observable

Physical solutions of field variable have been obtained in two different gauges and two different charts - how can those be compared? One natural strategy is to construct gauge invariant physical observable and compute it in both coordinates and for matching one can do coordinate transformation also. As an illustration, let us consider the deviation induced in the nearby geodesics, as tracked by a freely falling observer. Thus we consider a congruence of time like geodesics of the *background space-time* and consider the tidal effects of a transient gravitational wave. Let us concentrate on deviation acceleration which incorporates tidal distortion of nearby geodesics. For a freely falling observer with four velocity u^α, acceleration of geodesic deviation vector Z^α is given by,

$$a^\mu = -R^\mu_{\nu\rho\sigma}\, u^\nu Z^\rho u^\sigma \tag{3.5.1}$$

Under infinitesimal diffeomorphism $x^\mu \to x'^\mu = x^\mu - \epsilon\xi^\mu$, deviation acceleration changes as

$$\delta(a^\mu) = -\Big[\delta(R^\mu{}_{\nu\rho\sigma})\, Z^\rho u^\nu u^\sigma + R^\mu{}_{\nu\rho\sigma}\, \delta(u^\nu Z^\rho u^\sigma)\Big] \tag{3.5.2}$$

while to the linear order in perturbation this change becomes,

$$\delta(a^{\,\mu})^{(1)} = -\Big[\delta(R^{\,\mu}{}_{\nu\rho\sigma})^{(1)}\, \bar{Z}^\rho \bar{u}^\nu \bar{u}^\sigma + \bar{R}^\mu{}_{\nu\rho\sigma}\, \delta(u^\nu Z^\rho u^\sigma)^{(1)}\Big] \tag{3.5.3}$$

$$= -\Big[\mathcal{L}_\xi(\bar{R}^{\,\mu}{}_{\nu\rho\sigma})\, \bar{Z}^\rho \bar{u}^\nu \bar{u}^\sigma + \bar{R}^\mu{}_{\nu\rho\sigma}\, \delta(u^\nu Z^\rho u^\sigma)^{(1)}\Big] \tag{3.5.4}$$

In the last line we use $\delta h_{\mu\nu} := \bar{\nabla}_\mu\, \xi_\nu + \bar{\nabla}_\nu\, \xi_\mu$ to prove $\delta(R^{\,\mu}{}_{\nu\rho\sigma})^{(1)} = \mathcal{L}_\xi(\bar{R}^{\,\mu}{}_{\nu\rho\sigma})$ [21]. Vanishing Riemann tensor in flat background implies that deviation acceleration is gauge invariant to the leading order in perturbation. As background Riemann tensor is nonzero for de Sitter space-time, deviation acceleration is not gauge invariant.

To construct a gauge invariant quantity in de Sitter background, we begin with the observation that for *all* space-times satisfying $R_{\mu\nu} = \Lambda g_{\mu\nu}$ (which include the de Sitter background as well as its linearized perturbations in source free regions) and for vectors u, Z, Z' satisfying $u \cdot Z = u \cdot Z' = Z \cdot Z' = 0$, the definition of the Weyl tensor implies,

$$C_{\alpha\beta\mu\nu} - R_{\alpha\beta\mu\nu} = -\frac{\Lambda}{3}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}) \quad \Rightarrow \tag{3.5.5}$$

$$R_{\alpha\beta\mu\nu}Z'^{\alpha}u^{\beta}Z^{\mu}u^{\nu} = C_{\alpha\beta\mu\nu}Z'^{\alpha}u^{\beta}Z^{\mu}u^{\nu} + \frac{\Lambda}{3}\{(u \cdot u)(Z' \cdot Z) - (u \cdot Z')(u \cdot Z)\} \tag{3.5.6}$$

$$\therefore D(u, Z, Z') := -R_{\alpha\beta\mu\nu}Z'^{\alpha}u^{\beta}Z^{\mu}u^{\nu} = -C_{\alpha\beta\mu\nu}Z'^{\alpha}u^{\beta}Z^{\mu}u^{\nu} \tag{3.5.7}$$

As in eq. (3.5.4), one can obtain gauge transformation of $D(u, Z', Z)$. It is evident that $D(u, Z', Z)$ is gauge invariant to the leading order in perturbation. This is because the gauge transform of the Weyl tensor for the background is zero and the gauge transform of the $Z'uZu$ factor (it depends on the perturbation through the normalizations) does not contribute since the Weyl tensor of the de Sitter background itself is zero. Notice that $D(u, Z, Z')$ is symmetric in $Z \leftrightarrow Z'$ and is the component of acceleration of one deviation vector Z, along another orthogonal deviation vector.

A suitably chosen congruence of time-like geodesics, $u^{\alpha}\partial_{\alpha}$, provides a required pair of orthogonal deviation vectors for the gauge invariant observable $D(u, Z', Z)$ which we now refer to as *deviation scalar*. Since deviation vectors are always defined with respect to a geodesic congruence, we leave the argument u implicit and restore it in the final expressions. The deviation scalar is related to Weyl scalars as noted in [38]. We compute this for the $\Lambda-$projected solutions given in (3.3.57, 3.3.91). Note that the dot products in the above equations, involve the perturbed metric, $(\bar{g} + h)_{\mu\nu}$. For the explicit choices that we will make below, we denote the observer and the deviation vectors in the form: $u = \bar{u} + \delta u, Z = \bar{Z} + \delta Z, Z' = \bar{Z}' + \delta Z'$ with the 'barred' quantities normalised using the background metric while the 'delta' quantities are treated as of the same order as the perturbed field. Thus,

$$\bar{u}\cdot\bar{\nabla}\bar{u}^\alpha = 0 \quad , \quad \bar{u}\cdot\partial\bar{Z}^\alpha = \bar{Z}\cdot\partial\bar{u}^\alpha \quad , \quad \bar{u}\cdot\partial\bar{Z}'^\alpha = \bar{Z}'\cdot\partial\bar{u}^\alpha \ ;$$

$$\bar{u}\cdot\bar{u} = -1 \quad , \quad \bar{u}\cdot\bar{Z} = \bar{u}\cdot\bar{Z}' = \bar{Z}'\cdot\bar{Z} = 0 \ . \tag{3.5.8}$$

The ‘delta’ quantities have to satisfy conditions so that the full quantities satisfy the requisite orthogonality relations with respect to the perturbed metric.

The deviation scalar is then given by,

$$\begin{aligned}
-D(Z',Z) &:= \left(\bar{g}_{\alpha\beta}+h_{\alpha\beta}\right)\left(\bar{R}^\alpha{}_{\lambda\mu\nu}+R^{(1)\alpha}{}_{\lambda\mu\nu}\right)\left(\bar{Z}'^\beta\bar{u}^\lambda\bar{Z}^\mu\bar{u}^\nu+\delta(Z'^\beta u^\lambda Z^\mu u^\nu)\right)\\
&= \bar{g}_{\alpha\beta}\bar{R}^\alpha{}_{\lambda\mu\nu}\bar{Z}'^\beta\bar{u}^\lambda\bar{Z}^\mu\bar{u}^\nu+\bar{g}_{\alpha\beta}R^{(1)\alpha}{}_{\lambda\mu\nu}\,\bar{Z}'^\beta\bar{u}^\lambda\bar{Z}^\mu\bar{u}^\nu\\
&\quad +\bar{g}_{\alpha\beta}\bar{R}^\alpha{}_{\lambda\mu\nu}\,\delta\left(Z'^\beta u^\lambda Z^\mu u^\nu\right)+h_{\alpha\beta}\bar{R}^\alpha{}_{\lambda\mu\nu}\bar{Z}'^\beta\bar{u}^\lambda\bar{Z}^\mu\bar{u}^\nu
\end{aligned} \tag{3.5.9}$$

$$\begin{aligned}
\therefore D(Z',Z) &= -R^{(1)\alpha}{}_{\lambda\mu\nu}\,\bar{Z}'_\alpha\bar{u}^\lambda\bar{Z}^\mu\bar{u}^\nu+\frac{\Lambda}{3}\bar{g}_{\alpha\beta}(\bar{Z}'^\beta\delta Z^\alpha+\delta Z'^\beta\bar{Z}^\alpha)+\frac{\Lambda}{3}h_{\alpha\beta}\bar{Z}'^\alpha\bar{Z}^\beta\\
&= -R^{(1)\alpha}{}_{\lambda\mu\nu}\,\bar{Z}'_\alpha\bar{u}^\lambda\bar{Z}^\mu\bar{u}^\nu+\frac{\Lambda}{3}\delta\left(g_{\alpha\beta}Z'^\alpha Z^\beta\right)
\end{aligned} \tag{3.5.10}$$

Here, $R^{(1)}$ refers to the Riemann tensor linear in $h_{\mu\nu}$. In eqn. (3.5.9), the first term vanishes thanks to the properties of the barred quantities, while in the third term, only one factor has a delta-quantity. The only contributions that survive in the third and the fourth terms are the ones with $\bar{u}^2 = -1$. These terms combine (note the full metric in the last term) and (3.5.10) reflects this. Next,

$$R^{(1)\alpha}{}_{\lambda\mu\nu} = \bar{\nabla}_\mu\Gamma^{(1)\alpha}{}_{\nu\lambda}-\bar{\nabla}_\nu\Gamma^{(1)\alpha}{}_{\mu\lambda} \tag{3.5.11}$$

$$\Gamma^{(1)\alpha}{}_{\nu\lambda} = \frac{1}{2}\bar{g}^{\alpha\beta}\left(\bar{\nabla}_\lambda h_{\beta\nu}+\bar{\nabla}_\nu h_{\beta\lambda}-\bar{\nabla}_\beta h_{\nu\lambda}\right); \tag{3.5.12}$$

$$\begin{aligned}
\therefore D(u,Z',Z) &= -\frac{1}{2}\left[\bar{Z}'^\alpha\bar{u}^\lambda\bar{Z}^\mu\bar{u}^\nu\left(\bar{\nabla}_\mu\bar{\nabla}_\lambda h_{\alpha\nu}-\bar{\nabla}_\mu\bar{\nabla}_\alpha h_{\nu\lambda}-\bar{\nabla}_\nu\bar{\nabla}_\lambda h_{\alpha\mu}+\bar{\nabla}_\nu\bar{\nabla}_\alpha h_{\mu\lambda}\right.\right.\\
&\qquad\left.\left.+[\bar{\nabla}_\mu,\bar{\nabla}_\nu]h_{\alpha\lambda}\right)\right]+\frac{\Lambda}{3}\delta\left(g_{\alpha\beta}Z'^\alpha Z^\beta\right) \ .
\end{aligned} \tag{3.5.13}$$

Evaluating the commutator in the last term within the square brackets, we write it as, $\frac{\Lambda}{3}h_{\alpha\beta}\bar{Z}'^\alpha\bar{Z}^\beta$. To proceed further, we need to make choice of the congruence, the deviation vectors and the delta-quantities. This is done in the respective charts.

A natural class of time-like geodesics of the background geometry, is suggested in the conformal chart. From the appendix eqn (C.0.5), we know that the curves $x^i = x^i_0$ are time-like geodesics. The corresponding, normalised velocity is given by $\bar{u}^\alpha(\eta, x^i) := -H\eta(1, \vec{0})$. The same family of geodesics is given in FNC as, $\bar{u}^\alpha = (1 + \Lambda s^2/3, \sqrt{\Lambda/3}\,\vec{\xi}\,)$. From now on, we will use $\Lambda/3 =: H^2$ for ease of comparison.

FNC chart of the Static patch:

From the explicit choice of the freely falling observer, we get the following consequences:

$$\bar{u}\cdot\bar{Z} = 0 \;\Rightarrow\; \bar{Z}^0 = H\vec{\xi}\cdot\vec{\bar{Z}} \;,\; \bar{u}\cdot\bar{Z}' = 0 \;\Rightarrow\; \bar{Z}'^0 = H\vec{\xi}\cdot\vec{\bar{Z}}' \;; \tag{3.5.14}$$

$$\vec{\xi}\cdot\vec{\bar{Z}} = 0 = \vec{\xi}\cdot\vec{\bar{Z}}' \;\Rightarrow\; \bar{Z}^0 = 0 = \bar{Z}'^0 \;; \tag{3.5.15}$$

$$\bar{Z}\cdot\bar{Z}' = 0 \;\Rightarrow\; \vec{\bar{Z}}'\cdot\vec{\bar{Z}} = 0 \;; \tag{3.5.16}$$

$$\bar{u}\cdot\partial\bar{Z}^\alpha = \bar{Z}\cdot\partial\bar{u}^\alpha \;\Rightarrow\; \bar{u}\cdot\partial\bar{Z}^i \;=\; H\bar{Z}^i \;,\; \bar{Z}'\cdot\partial\bar{u}^\alpha \;\Rightarrow\; \bar{u}\cdot\partial\bar{Z}'^i \;=\; H\bar{Z}'^i \;; \tag{3.5.17}$$

$$\partial_\tau\bar{Z}^i \;=\; H(\bar{Z}^i - \vec{\xi}\cdot\vec{\partial}\bar{Z}^i) \;,\; \partial_\tau\bar{Z}'^i \;=\; H(\bar{Z}'^i - \vec{\xi}\cdot\vec{\partial}\bar{Z}'^i) \;; \tag{3.5.18}$$

In the second equation above, we have made the further choice namely, the spatial parts of $\bar{Z}, \bar{Z}'$ are orthogonal to the radial direction $\vec{\xi}$ as well.

The idea is to bring the deviation vectors across the derivatives. Using the properties above and $\tilde{h}_{ij}\bar{u}^i = 0$ which holds thanks to the tt-projection, the equation (3.5.13) gives,

$$D(Z', Z) - \frac{\Lambda}{3}\delta\left(g_{\alpha\beta}Z'^\alpha Z^\beta\right) \;=\; \left[\frac{1}{2}(\bar{u}\cdot\bar{\nabla})^2 - H(\bar{u}\cdot\bar{\nabla})\right](\tilde{h}^{tt}_{ij}\bar{Z}'^i\bar{Z}^j)\,. \tag{3.5.19}$$

The second term on the left hand side of the above equation vanishes.

To see this, we collect the equations satisfied by the $\delta-$quantities.

$$g_{\alpha\beta}u^\alpha u^\beta = -1 \;\Rightarrow\; \bar{u}_\alpha\delta u^\alpha = 0 \tag{3.5.20}$$

$$g_{\alpha\beta}u^\alpha Z^\beta = 0 \;\Rightarrow\; \bar{u}_\alpha\delta Z^\alpha + \bar{Z}_i\delta u^i = 0$$

$$g_{\alpha\beta}u^\alpha Z'^\beta = 0 \;\Rightarrow\; \bar{u}_\alpha \delta Z'^\alpha + \bar{Z}'_i \delta u^i = 0$$

$$u \cdot \nabla u^\alpha = 0 \;\Rightarrow\; \bar{u} \cdot \bar{\nabla} \delta u^\alpha = -\delta u \cdot \bar{\nabla} \bar{u}^\alpha \tag{3.5.21}$$

$$u \cdot \partial Z^\alpha - Z \cdot \partial u^\alpha = 0 \;\Rightarrow\; \bar{u} \cdot \bar{\nabla} \delta Z^\alpha = \bar{Z}^i \bar{\nabla}_i \delta u^\alpha + \delta Z \cdot \bar{\nabla} \bar{u}^\alpha - \delta u \cdot \bar{\nabla} \bar{Z}^\alpha$$

$$u \cdot \partial Z'^\alpha - Z' \cdot \partial u^\alpha = 0 \;\Rightarrow\; \bar{u} \cdot \bar{\nabla} \delta Z'^\alpha = \bar{Z}'^i \bar{\nabla}_i \delta u^\alpha + \delta Z' \cdot \bar{\nabla} \bar{u}^\alpha - \delta u \cdot \bar{\nabla} \bar{Z}'^\alpha$$

$$g_{\alpha\beta} Z^\alpha Z'^\beta = 0 \;\Rightarrow\; \bar{Z}_\alpha \delta Z'^\alpha + \bar{Z}'_i \delta Z^i + \tilde{h}^{tt}_{ij} \bar{Z}'^i \bar{Z}^j = 0 \tag{3.5.22}$$

The equations (3.5.20) serve to give the zeroth components of the $\delta-$vectors in terms of their spatial components. The equations (3.5.21) are evolution equations along the geodesic for the $\delta-$vectors and preserve the previous three equations. The last equation (3.5.22), is needed for the gauge invariance of the deviation scalar. The spatial components of $\delta-$vectors are still free. Demanding that the (3.5.22) is preserved along the observer geodesic leads to,

$$(\bar{Z}'_i \bar{Z}^j + \bar{Z}_i \bar{Z}'^j) \bar{\nabla}_j \delta u^i \;=\; -(\bar{u} \cdot \bar{\nabla} - 2H)(\tilde{h}^{tt}_{ij} \bar{Z}'^i \bar{Z}^j) \tag{3.5.23}$$

Here we have used the evolution equations for $\delta Z, \delta Z'$, equation (3.5.22) as well as $\bar{\nabla}_j \bar{u}^i = H\delta^i{}_j + o(H^3)$. This equation together with the evolution equation for δu^i, can be taken to restrict δu^i and we are still left free with the $\delta Z^i, \delta Z'^i$ subject only to the (3.5.22). This equation precisely sets the second term on the left hand side of eq. (3.5.19) to zero.

Thus we obtain the deviation scalar as a simple expression,

$$D(u, Z', Z) \;=\; \left[\frac{1}{2}(\bar{u} \cdot \bar{\nabla})^2 - H(\bar{u} \cdot \bar{\nabla})\right] Q \quad , \quad Q := (\tilde{h}^{tt}_{ij} \bar{Z}'^i \bar{Z}^j) \quad \text{with,} \tag{3.5.24}$$

$$\bar{u} \cdot \bar{\nabla} Q \;=\; \bar{u} \cdot \partial Q \;=\; \Big((1 + H^2 s^2)\partial_\tau + H\xi^i \partial_i\Big) Q \,.$$

For subsequent comparison, it is more convenient to take the deviation vectors across the derivatives, using $\bar{u}\cdot\bar{\nabla}\bar{Z}^i = H\bar{Z}^i$ etc. The deviation scalar is then given by,

$$D(u,Z',Z) = \bar{Z}'^i\bar{Z}^j\left[\frac{1}{2}(\bar{u}\cdot\bar{\nabla})^2 + H(\bar{u}\cdot\bar{\nabla})\right]\tilde{h}^{tt}_{ij} \quad \text{with,} \tag{3.5.25}$$

$$\bar{u}\cdot\bar{\nabla}\tilde{h}^{tt}_{ij} = \bar{u}\cdot\partial\tilde{h}^{tt}_{ij} + o(H^3)\,. \tag{3.5.26}$$

Substituting the solution (3.3.57) gives,

$$D(u,Z',Z) = \frac{1}{s}\left[\left(1-2Hs+\frac{7}{2}H^2s^2\right)\partial^4_\tau\mathcal{M}^{tt}_{ij} - H^2s\partial^3_\tau\mathcal{M}^{tt}_{ij} - H^2\partial^2_\tau\mathcal{M}^{tt}_{ij} - \frac{3H^2}{4}\partial^4_\tau\mathcal{M}^{tt}_{ijkl}\delta^{kl}\right]\bar{Z}'^i\bar{Z}^j \tag{3.5.27}$$

The τ derivatives are evaluated at the retarded time, $(\tau - \bar{s}(s))$, defined in eqn. (3.3.31).

The Conformal chart of the Poincaré patch:

For the solution in the generalized transverse gauge, the full metric has the form $g_{\mu\nu} = \Omega^2(\eta_{\mu\nu}+\chi_{\mu\nu})$, $\Omega^2 = 3\Lambda^{-1}\eta^{-2} = H^{-2}\eta^{-2}$. We can then use the Weyl transformation property of the Riemann tensor and obtain the full curvature in terms of the curvature of $(\eta+\chi)$ metric plus extra terms depending on *derivatives* of $\ln(\Omega)$. From these derivatives, Λ drops out and the full curvature (and hence the relative acceleration) is completely independent of Λ. Explicitly,

$$R_{\alpha\lambda\mu\nu}[\Omega^2(\eta+\chi)] = \Omega^2\left[R_{\alpha\lambda\mu\nu}[\eta+\chi] + \frac{1}{\eta^2}\left\{\hat{g}_{\alpha\mu}\hat{g}_{\nu\lambda} - \hat{g}_{\alpha\nu}\hat{g}_{\mu\lambda}\right\} + \frac{1}{\eta}\left\{\hat{g}_{\alpha\nu}\hat{\Gamma}^0_{\ \mu\lambda} - \hat{g}_{\alpha\mu}\hat{\Gamma}^0_{\ \nu\lambda} + \hat{g}_{\mu\lambda}\hat{g}_{\alpha\beta}\hat{\Gamma}^\beta_{\ 0\nu} - \hat{g}_{\nu\lambda}\hat{g}_{\alpha\beta}\hat{\Gamma}^\beta_{\ 0\mu}\right\}\right] \text{where,} \tag{3.5.28}$$

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \chi_{\mu\nu} \quad , \quad \hat{\Gamma}^\alpha_{\ \mu\nu}\Big|_{o(\chi)} = \frac{1}{2}\left(\partial_\nu\chi^\alpha_{\ \mu} + \partial_\mu\chi^\alpha_{\ \nu} - \partial^\alpha\chi_{\mu\nu}\right)$$

The definition of the deviation scalar and its invariance remains the same. We also choose the same geodesic congruence in the background space-time so that $u^\alpha = -H\eta\delta^\alpha_{\ 0}$. As before we choose two mutually orthogonal deviation vectors, Z, Z' and write $D(Z',Z) =$

$-R_{\alpha\beta\mu\nu}Z'^{\alpha}u^{\beta}Z^{\mu}u^{\nu}$. Using the Weyl transformation given above, we write

$$\hat{D}(\hat{Z}',\hat{Z}) = \Big[R_{\alpha\lambda\mu\nu}[\hat{g}] + \frac{1}{\eta^2}\left\{\hat{g}_{\alpha\mu}\hat{g}_{\nu\lambda} - \hat{g}_{\alpha\nu}\hat{g}_{\mu\lambda}\right\} \\ + \frac{1}{\eta}\left\{\hat{g}_{\alpha\nu}\hat{\Gamma}^{0}{}_{\mu\lambda} - \hat{g}_{\alpha\mu}\hat{\Gamma}^{0}{}_{\nu\lambda} + \hat{g}_{\mu\lambda}\hat{g}_{\alpha\beta}\hat{\Gamma}^{\beta}{}_{0\nu} - \hat{g}_{\nu\lambda}\hat{g}_{\alpha\beta}\hat{\Gamma}^{\beta}{}_{0\mu}\right\}\Big]\hat{Z}'^{\alpha}\hat{u}^{\lambda}\hat{Z}^{\mu}\hat{u}^{\nu}, \tag{3.5.29}$$

where we have defined new *scaled variables* as: $u^{\alpha} := |\Omega|^{-1}\hat{u}^{\alpha}$, $Z^{\alpha} := |\Omega|^{-1}\hat{Z}^{\alpha}$, $Z'^{\alpha} := |\Omega|^{-1}\hat{Z}'^{\alpha}$ and $D(Z',Z) := \Omega^{-2}\hat{D}(\hat{Z}',\hat{Z})$. This removes all the explicit factors of Ω^2 and we get an expression for the scaled deviation scalar, defined by perturbations about *Minkowski* background, with explicit additional terms.

For notational simplicity, we will suppress the 'hat's in the following, and restore them in the final equation. The background quantities, denoted by 'overbars' refer to the Minkowski metric and the corresponding $\delta-$quantities are treated as the of the same order as the perturbation χ^{tt}_{ij}. In particular, $\bar{u}^{\alpha} = \delta^{\alpha}{}_{0}$, $\bar{u}\cdot\partial\bar{u}^{\alpha} = 0$, $\bar{u}\cdot\partial\bar{Z}^{\alpha} = \bar{Z}\cdot\partial\bar{u}^{\alpha}$ and similarly for $\bar{Z}'^{\alpha}$. Proceeding exactly as before, we deduce:

$$\bar{u}^i = \bar{Z}^0 = \bar{Z}'^0 = \bar{Z}^i\bar{Z}'^j\delta_{ij} = 0 \quad , \quad \partial_\eta\bar{Z}^i = \partial_\eta\bar{Z}'^i = 0\,; \tag{3.5.30}$$

$$\delta u^0 = 0 = \delta Z^0 - \bar{Z}_i\delta u^i = \delta Z'^0 - \bar{Z}'_i\delta u^i \quad , \quad \bar{Z}'_i\delta Z^i + \bar{Z}_i\delta Z'^i + \chi_{ij}\bar{Z}'^i\bar{Z}^j = 0\,; \tag{3.5.31}$$

$$\partial_\eta\delta Z^{\alpha} = \bar{Z}^i\partial_i\delta u^{\alpha} - \delta u^i\partial_i\bar{Z}^i \quad , \quad \partial_\eta\delta Z'^{\alpha} = \bar{Z}'^i\partial_i\delta u^{\alpha} - \delta u^i\partial_i\bar{Z}'^i\,; \tag{3.5.32}$$

$$(\bar{Z}'_i\bar{Z}^j + \bar{Z}_i\bar{Z}'^j)\partial_j\delta u^i = -\partial_\eta(\chi_{ij}\bar{Z}'^i\bar{Z}^j) \ , \ \partial_\eta\delta u^i = 0\,. \tag{3.5.33}$$

As before, demanding preservation of the last of the normalization conditions in (3.5.31) under η evolution, gives conditions of δu^i given in equation (3.5.33). These are used in simplifying eqn. (3.5.29). The η^{-2} term of this equation vanishes as before while the η^{-1} coefficient gives only one contribution. In the first term, $R(\hat{g})$ gets replaced by $R^{(1)}$ which is linear in χ^{tt}_{ij}. This leads to (restoring the 'hats'),

$$\hat{D}(\hat{Z}',\hat{Z}) = \frac{1}{2}\left(\partial^2_\eta\chi^{tt}_{ij} - \frac{1}{\eta}\partial_\eta\chi^{tt}_{ij}\right)\bar{\hat{Z}}'^i\bar{\hat{Z}}^j\,. \tag{3.5.34}$$

Noting that χ_{ij} is a function of η only through $\eta_{\rm ret} = \eta - r$, we can replace ∂_η by $\partial_{\eta_{\rm ret}} =: \partial_{\bar{\eta}}$. Going to the cosmological chart via the definitions $\eta = -H^{-1}e^{-Ht} =: -H^{-1}a^{-1}$ and $\bar{\eta} := \eta - r := -H^{-1}\bar{a}^{-1}$ which defines the retarded time $\bar{t}$ through $\bar{a} = a(\bar{t})$, we replace $\bar{\partial}_\eta = \bar{a}\partial_{\bar{t}}$. This leads to,

$$\hat{D}(\hat{Z}', \hat{Z}) \;=\; \frac{\bar{\hat{Z}}'^i\bar{\hat{Z}}^j}{2}\,\bar{a}^2\left(\partial_{\bar{t}}^2 + H\left(1+\frac{a}{\bar{a}}\right)\partial_{\bar{t}}\right)\chi_{ij}^{tt} \quad \text{with } \chi_{ij} \text{ from equation (3.3.94)} .$$

To express the deviation scalar in terms of the Killing time τ, we observe that on scalars, $\mathcal{L}_T f = T\cdot\partial f$ while on tensorial functions of the retarded time,

$$\mathcal{L}_T Q_{ij}(\bar{t}) = \Big((\partial_t - Hx^i\partial_i)(\bar{t})\Big)\,(\partial_{\bar{t}}Q_{ij}(\bar{t})) - \;2HQ_{ij}(\bar{t}) \;, \quad \text{with} \quad (\partial_t - Hx^i\partial_i)(\bar{t}) = 1.$$

After a straightforward computation, we get,

$$\begin{aligned}\hat{D}(u, \hat{Z}', \hat{Z}) = \left(\bar{\hat{Z}}'^i\bar{\hat{Z}}^j\right)\left(\frac{\bar{a}^2}{ra}\right)\Big[&\mathcal{L}_T^4 Q_{ij}^{tt} + \;6H\mathcal{L}_T^3 Q_{ij}^{tt} + 11H^2\mathcal{L}_T^2 Q_{ij}^{tt} + 6H^3\mathcal{L}_T Q_{ij}^{tt} \\ &+ H\mathcal{L}_T^3\bar{Q}_{ij}^{tt} + 6H^2\mathcal{L}_T^2\bar{Q}_{ij}^{tt} + 11H^3\mathcal{L}_T\bar{Q}_{ij}^{tt} + 6H^4\bar{Q}_{ij}^{tt}\Big]. \end{aligned} \tag{3.5.35}$$

To compare the deviation scalars computed above, we need to ensure that we use the 'same' deviation vectors. Since the same observer is used, the deviation vectors are defined the same way with the only exception of their normalization. So let us use[5] *normalised deviation vectors*: $Z^i := \gamma\hat{Z}^i$ where $\hat{Z}^i\hat{Z}^j\delta_{ij} = 1$. Then $Z^2 = 1$ determines γ. In the FNC, $\gamma = (1 + H^2s^2/6)$ whereas in the conformal chart $\gamma = |\Omega|^{-1}$. Thus, in the conformal chart, the hatted deviation vectors are already normalized. In the FNC, we need to replace the deviation vectors by $(1 + H^2s^2/6)\times\hat{Z}$ and in the conformal chart, we write $\hat{D}(\hat{Z}', \hat{Z}) = \Omega^2 D(Z', Z)$. In the conformal chart, we retain terms up to order H^2 only and since the FNC calculation uses traceless stress tensor, we take $\bar{Q}$ moments to equal the Q moments. The two expressions are given below. (Recall that in FNC, $\mathcal{L}_T$ on all tensors

[5] We now suppress the overbars on the deviation vectors to avoid cluttering.

reduces to ∂_τ.)

$$
\begin{aligned}
D^{FNC}(u,Z',Z) &= \frac{1}{s}\left(1+\frac{H^2s^2}{3}\right)\left[\left(1-2Hs+\frac{7}{2}H^2s^2\right)\partial_\tau^4\mathcal{M}^{tt}_{ij}-H^2s\partial_\tau^3\mathcal{M}^{tt}_{ij}\right. \\
&\qquad \left. -H^2\partial_\tau^2\mathcal{M}^{tt}_{ij}-\frac{3H^2}{4}\partial_\tau^4\mathcal{M}^{tt}_{ijkl}\delta^{kl}\right]\hat{Z}'^i\hat{Z}^j \\
&= \frac{1}{s}\left(1-2Hs+\frac{23}{6}H^2s^2\right)\left[\partial_\tau^4\mathcal{M}^{tt}_{ij}-H^2s\partial_\tau^3\mathcal{M}^{tt}_{ij}\right. \\
&\qquad \left. -H^2\partial_\tau^2\mathcal{M}^{tt}_{ij}-\frac{3H^2}{4}\partial_\tau^4\mathcal{M}^{tt}_{ijkl}\delta^{kl}\right]\hat{Z}'^i\hat{Z}^j
\end{aligned} \tag{3.5.36}
$$

$$
\begin{aligned}
D^{Conf}(u,Z',Z)\Big|_{o(H^2)} &= \left(\frac{\bar{a}^2}{a^2}\frac{1}{ra}\right)\Bigg|_{o(H^2)}\left[\mathcal{L}_T^4Q^{tt}_{ij}+\ 7H\mathcal{L}_T^3Q^{tt}_{ij}+\ 17H^2\mathcal{L}_T^2Q^{tt}_{ij}\right. \\
&\qquad \left. +17H^3\mathcal{L}_TQ^{tt}_{ij}+6H^4Q^{tt}_{ij}\right]\hat{Z}'^i\hat{Z}^j \\
&= \frac{1}{s}\left(1-2Hs+\frac{19}{6}H^2s^2\right)\left[\mathcal{L}_T^4Q^{tt}_{ij}+7H\mathcal{L}_T^3Q^{tt}_{ij}\right. \\
&\qquad \left. +17H^2\mathcal{L}_T^2Q^{tt}_{ij}\right]\hat{Z}'^i\hat{Z}^j
\end{aligned} \tag{3.5.37}
$$

Equations (3.5.27) and (3.5.35) give the deviation scalars in the two charts. The comparable expressions are given in (3.5.36, 3.5.37). These are obtained for the specific choice of the congruence of the de Sitter background: $\bar{u}^\alpha(\eta,x^i) := -H\eta(1,\vec{0})$.

We have obtained two different looking expressions for the same, gauge invariant deviation scalar. The difference can be attributed to the definition of moments. They have been defined on two different spatial hypersurfaces - the τ =constant in FNC and the η = constant in the conformal chart. In the conformal chart solution there is no truncation of powers of H (in the leading 'r' approximation) and it includes the contribution of both the sharp and the tail terms. By contrast, the FNC chart computation is obtained as an expansion in H only up to the quadratic order. Furthermore, it includes only the contribution of the sharp term. While it is possible to relate the frame components of the stress tensor in the two charts, the relation among the moments is non-trivial and is not obtained here.

A physical interpretation :

We have defined a gauge invariant quantity and illustrated how to compute it. It depends on a time-like geodesic congruence and two mutually orthogonal deviation vectors. At the linearized level, it also depends on the $\hat{n}$ direction used in the tt projection. What information about the wave does it contain? To see this, consider the simpler case of *Minkowski* background, choose the congruence so that $\bar{u}^\alpha = (1, \vec{0})$. It follows that at the linearized level, the quantity

$$A_{\alpha\beta}(\eta_{\mu\nu} + h_{\mu\nu}) := -R_{\alpha\mu\beta\nu}(\eta_{\mu\nu} + h_{\mu\nu})u^\mu u^\nu \approx -R^{(1)}_{\alpha\mu\beta\nu}(h)\bar{u}^\mu \bar{u}^\nu = -R^{(1)}_{\alpha 0\beta 0}(h) \tag{3.5.38}$$

is symmetric in $\alpha \leftrightarrow \beta$ and spatial i.e. $A_{00} = 0 = A_{0i}$. When the transient wave $h_{\mu\nu}$ is in synchronous gauge and tt projected, the matrix $A_{ij}(h^{tt}_{kl})$ is also transverse. This is because, the $\Lambda-$projector can be taken across the derivatives up to terms down by powers of r. Explicitly,

$$A_{ij}(h^{tt}) \approx \frac{1}{2}\Lambda_{ij}{}^{kl}(\hat{n})\partial_0^2 h_{kl} \ , \quad A_{ij}\delta^{ij} = 0 \ . \tag{3.5.39}$$

Since the deviation vectors too are taken to be transverse, in effect the deviation scalar reduces to $D(u, Z', Z) \approx \hat{\hat{Z}}'^a A_{ab}(h)\hat{\hat{Z}}^b$ where a, b take two values and the real, symmetric matrix A_{ab} is traceless. With respect to an arbitrarily chosen basis, $\{\hat{e}_1, \hat{e}_2\}$ in the plane transverse to the wave direction, $\hat{n}$, we can define the '+' and the '$\times$' polarizations by setting the matrix $A := h_+\sigma_3 + h_\times\sigma_1$. If $\hat{\hat{Z}}$ makes an angle ϕ with $\hat{e}_1$, then the unit deviation vectors are given by, $\hat{\hat{Z}} = (cos(\phi), sin(\phi))$, $\hat{\hat{Z}}' = (-sin(\phi), cos(\phi))$. It follows that,

$$D(u, Z', Z) = -h_+ sin(2\phi) + h_\times cos(2\phi) \ . \tag{3.5.40}$$

Thus, for a pair of bases $(\hat{e}_1, \hat{e}_2)$ and $(\hat{\hat{Z}}, \hat{\hat{Z}}')$, determination of the deviation scalar gives one relation between the amplitudes of the two polarizations. A similar determination at another detector location gives a second relation, thereby providing amplitudes of individual polarizations.

A natural choice for $\hat{e}_1, \hat{e}_2$ would be the unit vectors provided by the Right Ascension/ Declination coordinate system used by astronomers, at the $\hat{n}$ direction. The basis of unit deviation vectors could be constructed in many ways. For instance, using the wave direction $\hat{n}$ and one of the arms of the interferometer which form a plane. Its unit normal may be taken as $\hat{\hat{Z}}$ and then, $\hat{n} \times \hat{\hat{Z}}$ can be taken as $\hat{\hat{Z}}'$. To avoid the exceptional case where the wave is incident along the chosen arm of the interferometer, one could repeat the procedure with the other arm. The construction gives ϕ at the detector location. Suffice it to say that measurement of deviation scalar for appropriate deviation vectors, at two or more detectors would constitute a measurement of the amplitudes of individual polarizations of a gravitational wave. To be useful in observations, the deviation scalar must be computed for congruence related to specific interferometer (earth based ones are not in free fall, the space based ones would be) and related to the waveform.

3.6 Summary and discussion

In the introduction we mentioned that for $\Lambda = 0$ case, there is a well-developed theory of isolated systems and gravitational radiation in full, nonlinear general relativity that plays a vital role in a number of areas of gravitational physics. Extension to this theory for even a tiny positive cosmological imposes some serious conceptual obstacles. In this chapter we have observed that even for linear theory extension of $\Lambda = 0$ analysis introduce unforeseen difficulties, namely multiple charts, gauges, definition of source moments, identification of physical perturbations, construction of gauge invariant observables. We considered two different charts (FNC and conformal) and two different gauge choices (TT and Generalized-TT), defined the corresponding synchronous gauges to identify the physical components and these were expressed in terms of the appropriately defined source moments. Even a smallest cosmological constant (positive or negative), immediately brings up the more than one 'natural' choices of charts in a given patch. Quite apart from the qualitatively distinct structure of the respective $\mathcal{J}^+$, even the local (near source) analysis reveals different issues to be faced. The FNC is very natural to the local analysis and goes through the same way for AdS as well. It naturally gives the answer as corrections to the corresponding Minkowski answer, in *powers of* Λ. This is also seen the Bondi-Sachs chart [38]. From the intuition from Minkowski background analysis, neighbourhood of infinity is the natural place for characterising *radiation* in a gauge invariant manner. Then the conformal chart (for de Sitter) is a natural choice. And here the corrections to the Minkowski answer are obtained in *powers of* $\sqrt{\Lambda}$. This difference in the powers of Λ, was seen in the solutions obtained in equations (3.3.57, 3.3.91). However it is meaningless to compare the gauge fixed fields. For this purpose the gauge invariant deviation scalar was computed and compared. The manifest dependence of the corrections on Λ does distinguish a local (neighbourhood of source) form from the one in the asymptotic region.

For the class of sources we have assumed (rapidly varying and distant), it seems sufficient to confine attention to a region maximally up to the cosmological horizon. The physical distance (eg luminosity distance) from the source to the cosmological horizon, eg $\eta = -r$ in the conformal chart, is $\sqrt{3/\Lambda\eta^2}\; r = \sqrt{3/\Lambda}$. This contains typical, currently detectable sources and thus should suffice for estimation. We obtained the corresponding fields, to order Λ, using Fermi Normal coordinates based near the compact source and is given in eqn.(3.3.57). For a subsequent comparison, we also computed the field in the conformal/cosmological charts. It is given in equation (3.3.91). In the Minkowski background analysis, tail terms appear at higher orders of perturbations and these are understood to be due to scattering off the curvature generated at the lower orders. In the de Sitter background, curvature effects are felt by the perturbations at the linear order itself. This is manifested in both the gauges. In the generalized transverse gauge, the tail term is explicitly available and plays a crucial role at the null infinity [8]. In the TT gauge however, the tail term itself is order Λ^2 and within the FNC patch does not seem likely to give significant contribution by cumulative effects. However this remains to be computed explicitly.

As a by product of expressing the retarded solution in terms of the source moments, we also saw (not surprisingly) that the 'mass' (zeroth moment) and the 'momentum' (first moment) are not conserved, thanks to the curvature of the de Sitter background. More generally, it also implied that static (test) sources cannot exist in curved background. This is just a consequence of the conservation equation in a curved background, quite independent of any gravitational waves.

In the Minkowski background geodesic deviation acceleration, to the linearized order, is gauge invariant and is used to infer the wave form. In a general curved background, its gauge invariance is lost. However, for a conformally flat background, component of a deviation vector along another, orthogonal deviation vector defines a gauge invariant function, $D(u, Z, Z')$, which we termed as deviation scalar. In the simpler context of flat background, we saw that its measurement at two or more detectors would give the amplitudes of individual polarizations. Its determination could provide useful information on polarization of gravitational waves even for non-zero cosmological constant. We computed the deviation scalar, for the solutions given in the two charts. The expressions obtained (3.5.36,3.5.37) are different. The comparison is expected to be possible when both charts overlap and only up to order $\Lambda \sim H^2$. In FNC, we have computed only the sharp term. However, it is not clear if the 'sharp' contribution can be identified in a chart independent manner. So in the conformal chart, we took the full field and restricted its contribution to order Λ. While we compute the same observable, a chart dependence, or more precisely a dependence on the spatial hypersurface, enters through the definition of source moments. There is also a choice of moment variable involved ($\zeta^{\underline{i}}$ in FNC). Thus, the solutions are given in terms of source moments which are defined on *different spatial hypersurfaces*. As such they cannot be compared immediately. An explicit model system for which the two different moments are computed, should help clarify some of these aspects and show the equality of the deviation scalar computed in two ways. This needs to be checked.

To conclude, linearization about the de Sitter background provides a simplified arena for an extension of the computational steps from a flat background to a curved background. The weak gravitational waves can be computed as corrections in powers of the cosmological constant. There is a gauge invariant observable that could provide information about the amplitudes of the two polarizations. More precise computations at least for a model source are needed for a quantitative estimate of corrections to the waveforms. If the $\Lambda-$corrections could be identified from the signal, it could provide an independent measurement of the cosmological constant.

4 Quadrupole Formula in de Sitter Background

4.1 Introduction

Two years after the completion of general relativity, in 1918, Einstein derived power radiated quadrupole formula in Minkowski background. He found that to the leading order the emitted power is proportional to square of the third time derivatives of mass quadrupole moment of the source. This would cause binary system to shrink slowly. Such a secular change in orbital period of Hulse-Taylor binary pulsar was confirmed by observation to the accuracy of 10^{-3}, thereby providing an indirect affirmation of gravitational waves. Einstein's theory also passed with flying colors in the direct observation of newly opened gravitational wave astronomy - gravitational wave template prdicted by Einstein's theory matches with observed signal. All these theoretical frameworks assume a vanishing cosmological constant. In chapter 3, we obtained linearized field solution in presence of positive cosmological constant. Hence it is natural to ask what is the modification of power radiated quadrupole formula for $\Lambda > 0$? Can the modifications be obtained as "small" corrections in powers of cosmological constant? To the next level, one can also ask does Λ induce any significant observable effect in orbital decay of binary system or in the gravitational wave template? In this chapter we will derive quadrupole formula for $\Lambda > 0$ using *Isaacson's effective stress tensor*. For comparison we also present energy flux associated with linearized gravitational field computed in covariant phase space framework [8].

Linearized solution obtained in conformal chart of future Poincaré patch can be extended all the way upto $\mathscr{J}^+$. $\mathscr{J}^+$ is ultimate destination of outgoing massless field and is the natural place to identify radiation. However the space-like character of the $\mathscr{J}^+$ in de Sitter space poses challenges for defining energy and its flux. Let us recall that the cleanest articulation of 'infinity' arises in the conformal completion of physical space-times. Conformal completion preserves the light cone structure of the physical space-time and captures the notion of approaching infinity along many directions as well as with different speeds. It naturally identifies boundary components, $\mathscr{J}^\pm$ where time-like and null geodesics 'terminate'. The causal nature of these boundary components is determined by the asymptotic form of 'source-free' equations: $\mathscr{J}^\pm$ are null when $\Lambda = 0$ and space-like for $\Lambda > 0$ (time-like for $\Lambda < 0$). These boundary components serve to define out-going (in-coming) fields as those solutions of the asymptotic equations that have suitably finite limiting values on $\mathscr{J}^+(\mathscr{J}^-)$. It is then a result that the Weyl tensor of out-going fields evaluated along out-going null geodesics, has a definite pattern of fall-off in inverse powers of an affine parameter along the null geodesics (the peeling-off theorem) [3, 39]. This enables one to identify the leading term as representing gravitational radiation (far field of a source), in a coordinate invariant manner. It is conveniently described in terms of the Weyl scalars which are defined with respect to a suitable null tetrad. When $\mathscr{J}^+$ is null, a null tetrad at a point $p \in \mathscr{J}^+$ is uniquely determined (modulo real scaling and rotation) by the tangent vector ℓ^μ of an outgoing null geodesic reaching p, *and* the null normal n^μ, satisfying $\ell \cdot n = -1$. Clearly as the null geodesic changes its direction, ℓ changes but not n and hence the Weyl scalar Ψ_4 ($:= C_{\mu\nu\rho\sigma} n^\mu \bar{m}^\nu n^\rho \bar{m}^\sigma$) remains unchanged. Its non-zero value can be taken as showing the presence of gravitational radiation. This feature is lost when the $\mathscr{J}^+$ is space-like. Now the null vector n^μ, with $\ell \cdot n = -1$, is chosen to be in the plane defined by ℓ^μ and the (time-like) normal N^μ. Clearly, as ℓ^μ changes, so does n^μ and then *none* of the Weyl scalars is invariant. An invariant characterization of gravitational radiation is no longer available [3].

Though identification of radiation field is *ambiguous* due to space-like character of $\mathscr{J}^+$, one can still pose the question of energy /angular momentum carried away by gravitational wave from a spatially compact radiating source in de Sitter background. As we mentioned in subsection 2.2.2, future Poincaré patch of de Sitter space has seven globally defined killing vectors - 3 spatial rotation, 3 spatial translation, 1 time translation. Using these isometries of Poincaré patch one can expect corresponding conserved quantities. One of the definition of such conserved quantities is based on the covariant phase space framework [22, 40]. In the context of the linearized theory, it exploits the phase space structure of the space of solutions and defines a manifestly gauge invariant and conserved 'Hamiltonian' corresponding to every *isometry* of the background space-time. Although the 'Hamiltonian' defined on each $\eta = const.$ space-like hypersurface of the Poincaré patch, the simplest expressions result for evaluation at $\mathscr{J}^+(\eta = 0)$. Being defined as an integral over a hypersurface, it is also referred to as a *flux*.

Thus we see that the null infinity provides a characterization of the radiation (up to origin dependence for $\Lambda \neq 0$) which can transport energy-momentum-angular momentum away from a spatially compact gravitating source. One other implication of this characterization at infinity is that the energy/momentum/angular momentum which *is* received at the future null infinity is *necessarily lost* from the source under the assumption of *no incoming radiation* boundary condition. It is this last implication that follows also for the cosmological horizon of a spatially compact source: energy/momentum/angular momentum that crosses the cosmological horizon is also lost forever from the source. Note that the worldlines of different components of a spatially extended (but compact) source must reach the same point on $\mathscr{J}^+$ to maintain a *finite* physical separation among them. It follows that that the cosmological horizon for a spatially compact source is well defined: it is the past light cone of the common point on $\mathscr{J}^+$ where the source world tube converges. Equally well, *any observer* who remains at a finite physical distance from the compact source for all times, must necessarily lie within the cosmological horizon i.e. within the static patch. We would like to explore to what extent and under what conditions may we regard the cosmological horizon as a "substitute" for the *future null infinity*.

We visualize gravitational wave as a ripple propagating through a slowly changing background. As mentioned earlier in de Sitter case characteristic length scale is set by cosmological constant, $\Lambda^{-1/2} \sim 10^{26}m \sim 5Gpc$. And gravitational wave is characterized by its wavelength which inherently depends on frequency of sources. The 'time' used in identifying characteristic frequency depends on choice of coordinates or reference frames. But in a typical astrophysical situation there are preferred frame, e.g. 'asymptotic rest frame' of the source of the waves, and these permit λ and L to be defined with adequate precession for astrophysical discussion [41]. For sources which are sufficiently rapidly varying (relative to the scale set by the cosmological constant), there is a clear distinction between ripple and background within the so called *short wave approximation* [21, 42]. Furthermore, it is possible to define an *effective gravitational stress tensor*, $t_{\mu\nu}$ for the ripples in this context. For vanishing Λ, it is symmetric, conserved and gauge invariant. For non-zero Λ it is *not* gauge invariant but the gauge violations are suppressed by powers of $\sqrt{\Lambda}$. It is very convenient to have such a stress tensor to define and compute fluxes of energy and momenta carried by the ripples across any hypersurface. We will use this to show that for the retarded solution given in [6, 8, 43], the flux of energy-momentum across the cosmological horizon exactly equals the corresponding flux across the $\mathcal{J}^+$ and also equals the flux computed in [8] at a coarse grained level (See equation (4.4.3). The instantaneous power received at infinity matches with that crossing the horizon.

The chapter is organised as follow.

In section 4.2, we recall the solution at the linearized level for which the fluxes will be evaluated. We specify the *exact retarded* solution $\mathcal{X}_{ij}$, the *approximated retarded* solution χ_{ij}, and its *algebraically projected transverse, traceless* part χ^{tt}_{ij} that is used throughout. Section 4.3 is divided into five subsections. In subsection 4.3.1 we summarise the covariant phase space framework and recall the definitions of the fluxes and quadrupole power from [8]. In subsection 4.3.2, we discuss the Isaacson prescription adapted to the presence of the cosmological constant. Since this constructs a symmetric, conserved and gauge invariant (to *leading order*) stress tensor, we get conserved currents for each Killing vector of the background space-time. In the subsection 4.3.3, we present computations of the energy flux for the χ^{tt}_{ij} across various hypersurfaces. In particular we show that the fluxes across the out-going null hypersurfaces are zero, implying for example, that the energy propagation is sharp. Subsection 4.3.4 contains the fluxes for the momentum and the angular momentum. In the 4.3.5 we discuss how the (differential) transverse, traceless decomposition can be used in place the algebraic projection. In section 4.4, we discuss its applications and show that the quadrupole power can be computed at the cosmological horizon. The final section 4.5 concludes with a discussion. Appendix D is included to illustrate an averaging procedure.

4.2 Preliminaries

In this chapter we will compute energy flux carried away by gravitational waves due to compact sources. As discussed in 3.1 to study the compact source, it is sufficient to focus on future Poinace patch of full de Sitter space. We will compute energy flux on different 3 dimensional hypersurfaces, particularly on cosmological horizon and on $\mathcal{J}^+$. As we will explore energy flux near $\mathcal{J}^+$, it is apt to work in conformal chart (η, x^i) of future Poincaré patch. Hence in the present context, the background space-time is taken to be the *Poincare patch* of the de Sitter space-time (see figure 2.3) which admits a conformally flat form of the background metric in coordinates (η, x^i),

$$ds^2 \;=\; \frac{1}{H^2\eta^2}\left[-d\eta^2 + \sum_i (dx^i)^2\right] \;,\;\; \eta \in (-\infty, 0) \;,\;\; H := \sqrt{\frac{\Lambda}{3}}. \qquad (4.2.1)$$

As mentioned in subsection 2.2.2, there are seven globally defined Killing vectors on Poincaré patch, corresponding to energy, 3 momenta and 3 angular momenta. They are given by (up to constant scaling):

$$
\begin{aligned}
\text{Generator of time translation} &: \ T^{\mu} = -H(\eta, x^{i}) \ \leftrightarrow \ T_{\mu} = \frac{1}{H\eta^2}(\eta, -x_i) && (4.2.2)\\
\text{Generators of space translation} &: \ \xi^{\mu}_{(j)} = (0, \delta^{i}_{j}) \ \leftrightarrow \ \xi_{(j)\mu} = \frac{1}{H^2\eta^2}(0, \eta_{ji}) && (4.2.3)\\
\text{Generators of space rotations} &: \ L^{\mu}_{(j)} = (0, \epsilon_{jk}{}^{i}x^{k}) \ \leftrightarrow \ L_{(j)\mu} = \frac{1}{H^2\eta^2}(0, \epsilon_{jki}x^{k}) && (4.2.4)
\end{aligned}
$$

We focus on the *time translation vector field*,

$$
T \ := \ -H(\eta\partial_{\eta} + x^{i}\partial_{i}) \tag{4.2.5}
$$

It is time-like in the static patch (hence generating 'time translations') and space-like beyond the cosmological horizon in the Poincaré patch. In particular it is space-like and tangential to $\mathcal{J}^{+}$ and null on the cosmological horizon.

Let us recall from subsection (3.3.2), after exhausting all gauge freedoms, the *exact retarded field solution* in de Sitter background is given by (3.3.74),

$$
\begin{aligned}
\mathcal{X}_{ij}(\eta, x) \ &= \ 4\int d^3x' \frac{\eta}{|x-x'|(\eta-|x-x'|)} \, T_{ij}(\eta', x')\Big|_{\eta'=\eta-|x-x'|} \\
&\quad + 4\int d^3x' \int_{-\infty}^{\eta-|x-x'|} d\eta' \frac{T_{ij}(\eta', x')}{\eta'^2}
\end{aligned} \tag{4.2.6}
$$

Physical solutions are the (spatial) TT part of the solution above, i.e. $\partial^{i}\mathcal{X}_{ij} = 0 = \mathcal{X}^{i}_{i}$. For $|\vec{x}| \gg |\vec{x}\,'|$, the so *approximated retarded solution* is given by,

$$
\begin{aligned}
\mathcal{X}_{ij} \ &= \ \chi_{ij}(\eta, x) + o(r^{-1}) \qquad , \quad \text{with} \\
\chi_{ij}(\eta, x) \ &:= \ 4\frac{\eta}{r(\eta - r)}\int d^3x' T_{ij}(\eta', x')\Big|_{\eta'=\eta-r} + 4\int_{-\infty}^{\eta-r} d\eta' \frac{1}{\eta'^2}\int d^3x' T_{ij}(\eta', x')
\end{aligned} \tag{4.2.7}
$$

We will work with the approximated solution. Note that χ_{ij} depends on $\vec{x}$ only through $r = |\vec{x}|$. The first term is the contribution of the so called sharp term while the second term the tail contributions. The tail contribution can be separated into a term which depends on retarded time, $(\eta - r)$ only, just as the sharp term does, and the contribution from the history of the source is given by the limiting value at $\eta = -\infty$.

In terms of source moments, the approximated retarded solution is given by (3.3.91),

$$\begin{aligned}
\chi_{ij}(\eta, \vec{x}) &= \frac{1}{r} f_{ij}(\eta_{ret}) + g_{ij}(\eta_{ret}) + \hat{g}_{ij} && \text{with,} \quad (4.2.8)\\
f_{ij}(\eta_{ret}) &:= \frac{2}{a(\eta_{ret})}\left[\mathcal{L}_T^2 Q_{ij} + 2H\mathcal{L}_T Q_{ij} + H\mathcal{L}_T \bar{Q}_{ij} + 2H^2 \bar{Q}_{ij}\right] , && (4.2.9)\\
g_{ij}(\eta_{ret}) &:= -2H\left[\mathcal{L}_T^2 Q_{ij} + H\mathcal{L}_T Q_{ij} + H\mathcal{L}_T \bar{Q}_{ij} + H^2 \bar{Q}_{ij}\right] && \text{and,} \quad (4.2.10)\\
\hat{g}_{ij} &:= -2H^2\left[\mathcal{L} Q_{ij} + H\bar{Q}_{ij}\right]\Big|_{-\infty} && (4.2.11)
\end{aligned}$$

All moments are evaluated at the retarded $\eta_{ret} := (\eta - r)$ and $\mathcal{L}_T$ denotes the Lie derivative with respect to the time translation Killing vector. $a(\eta_{ret}) := -(H\eta_{ret})^{-1}$ and $\mathcal{L}_T$ denotes the Lie derivative with respect to the time translation Killing vector. This expression is valid *as the leading term for* $|\vec{x}| \gg |\vec{x}'|$. There is no TT label on these expressions.

For future use in section 4.3.3, we display the derivatives of χ_{ij}. Since χ_{ij} depends on $\vec{x}$ only through r, we need only the derivatives with respect to η and r. On functions of η_{ret}, $\partial_r = -\partial_\eta$ and we can replace the r-derivatives in favour of $\eta-$derivatives. Hence,

$$\partial_\eta \chi_{ij} = \frac{1}{r}\partial_\eta f_{ij} + \partial_\eta g_{ij} \ , \quad \partial_r \chi_{ij}(\eta, r) = -\frac{1}{r^2} f_{ij} - \frac{1}{r}\partial_\eta f_{ij} - \partial_\eta g_{ij} = -\frac{f_{ij}}{r^2} - \partial_\eta \chi_{ij} \, . \quad (4.2.12)$$

There is a well known *algebraic projection* method to construct spatial tensors which satisfy the spatial TT condition *to the leading order in* r^{-1}. Since the approximated solution is also valid to $o(r^{-1})$, we may use this convenient method.

For the unit vectors $\hat{x}$ denoting directions, projectors are defined as in eq. (3.4.6),

$$P_i^{\ j} \ := \ \delta_i^{\ j} - \hat{x}_i\hat{x}^j \ , \ \Lambda_{ij}^{\ \ kl} \ := \ \frac{1}{2}(P_i^{\ k}P_j^{\ l} + P_i^{\ l}P_j^{\ k} - P_{ij}P^{kl}) \ , \ \chi^{tt}_{ij} \ := \ \Lambda_{ij}^{\ \ kl}\chi_{kl} \quad (4.2.13)$$

We have used the notation of ‘tt’ to refer to the algebraically projected transverse, traceless part as in [8]. Noting that on χ_{ij} the spatial derivative is $\partial^j = \hat{x}^j\partial_r$, it follows that,

$$\partial_\eta(\chi^{tt}_{ij}) \ = \ (\partial_\eta\chi_{ij})^{tt} \ , \ \partial_r(\chi^{tt}_{ij}) = (\partial_r\chi_{ij})^{tt} \ , \quad (4.2.14)$$

$$\partial_m(\chi^{tt}_{ij}) \ = \ (\partial_m\Lambda_{ij}^{\ \ kl})\chi_{kl} + \hat{x}_m(\partial_r\chi_{ij})^{tt} \quad (4.2.15)$$

$$\therefore \partial^j(\chi^{tt}_{ij}) \ = \ \hat{x}^j\Lambda_{ij}^{\ \ kl}\partial_r\chi_{ij} + (\partial^j\Lambda_{ij}^{\ \ kl})\chi_{kl} \ = \ 0 + o(r^{-1}) \ ; \quad \text{where we used,} \quad (4.2.16)$$

$$\partial_m\Lambda_{ij}^{\ \ kl} \ = \ -\frac{1}{r}\left[\hat{x}_i\Lambda_{mj}^{\ \ \ kl} + \hat{x}_j\Lambda_{mi}^{\ \ \ kl} + \hat{x}^k\Lambda_{ijm}^{\ \ \ \ l} + \hat{x}^l\Lambda_{ijm}^{\ \ \ \ k}\right] \ = \ o(r^{-1}) \ . \quad (4.2.17)$$

The tracelessness of χ^{tt}_{ij} is manifest and hence χ^{tt}_{ij} satisfies the spatial TT condition to $o(r^{-1})$.

Using the derivatives of χ_{ij} given in (4.2.12), we can write,

$$\partial_\mu\chi^{tt}_{ij} \ = \ (\partial_\eta\chi^{tt}_{ij}, \ \hat{x}_m\partial_r\chi^{tt}_{ij} + (\partial_m\Lambda_{ij}^{\ \ kl})\chi_{kl}) \quad (4.2.18)$$

$$= \ (\partial_\eta\chi^{tt}_{ij})(1, \ -\hat{x}_m) - \left(\frac{f^{tt}_{ij}}{r^2}\right)(0, \ \hat{x}_m) \quad (4.2.19)$$

$$-\frac{1}{r}(0, \left[\hat{x}_i\Lambda_{mj}^{\ \ \ kl} + \hat{x}_j\Lambda_{mi}^{\ \ \ kl} + \hat{x}^k\Lambda_{ijm}^{\ \ \ \ l} + \hat{x}^l\Lambda_{ijm}^{\ \ \ \ k}\right]\chi_{kl}). \quad (4.2.20)$$

The first term is proportional to a null vector. The second term is proportional to the space-like, radial vector. The third is again a space-like vector. Both the second and the third terms are down by a power of r relative to χ_{ij} and therefore also relative to the first term. We will see later in the calculation of the fluxes that for energy and momentum, the second and the third terms can be neglected. However for flux of angular momentum, the third term is crucial. *When* the second and the third terms can be neglected, the effective gravitational stress tensor turns out to correspond to an *out-going null dust* with energy density proportional to $\langle\partial_\eta\chi_{mn}\partial_\eta\chi^{mn}\rangle$.

4.3 Conserved Quantities

4.3.1 Covariant phase space framework

Consider the space C of a class of solutions of the full Einstein equation, satisfying stipulated boundary condition. At each point of this space, the linearized solutions provide tangent vectors. Under certain conditions, it is possible to define a *pre-symplectic form* on the tangent spaces. Every infinitesimal diffeomorphism of the space-time, with suitable asymptotic behaviour, induces a vector field on C. Some of these lie in the kernel of the pre-symplectic form and constitute 'gauge directions' while the remaining ones constitute (asymptotic) symmetries shared by the stipulated class of solutions. Modding out by the gauge directions (null space of the pre-symplectic form), one imparts a symplectic structure to the space of solutions, now denoted as $\Gamma \sim C/gauge$. Under favourable conditions, the vector fields on C corresponding to the asymptotic symmetries descend to Γ and generate infinitesimal *canonical transformations*. Their generating functions, or 'Hamiltonians', are candidates for representing energy, momenta, angular momenta etc [40].

In [8, 22], this strategy is applied to the space of fully gauge fixed solutions of the linearized equation and we summarise it below. Since linearization is always done about a background, space-time transformations which leave the background invariant i.e. isometries of the background, leave the covariant phase space itself invariant. In the present context, the Hamiltonians corresponding to the 7 isometries are the proposed definitions of energy, linear momentum and angular momentum.

Explicitly, C denotes the solutions of the equation (4.3.1) together with the gauge fixing conditions (4.3.2):

$$\Box\chi_{ij} + \frac{2}{\eta}\partial_\eta\chi_{ij} = 0 \tag{4.3.1}$$

$$\partial^i \chi_{ij} = 0 \ , \ \chi_{ij}\delta^{ij} = 0 \tag{4.3.2}$$

Since the linearized equation is a hyperbolic equation, suitable classes of solutions can be chosen by suitably restricted initial data on η =constant (Cauchy) surfaces of the Poincaré patch.

For two elements $\chi, \underline{\chi} \in C$, define the two form ω on C by [22] ,

$$\omega(\chi, \underline{\chi}) := \frac{a^2(\eta)}{32\pi} \int_{\Sigma_\eta} d^3x (\chi_{ij}\partial_\eta \underline{\chi}_{kl} - \underline{\chi}_{kl}\partial_\eta \chi_{ij})\, \delta^{ik}\delta^{jl} \qquad \text{with,} \ a := -\frac{1}{H\eta} \tag{4.3.3}$$

where, Σ_η denotes the cosmological slice defined by a constant value η. The integrals is actually independent of the hypersurface. Also, a label 'TT' signifying that the χ's are *transverse, traceless*, is suppressed. This definition of the symplectic form however, is not convenient for going to $\mathcal{J}^+$ by taking the limit $\eta \to 0_-$. It can be translated in terms of the *electric part* of the perturbed Weyl tensor, $\mathcal{E}_{ij} := -(H\eta)^{-1}[{}^{(1)}C^0{}_{j0i}] = \frac{1}{2H\eta^2}(\partial_\eta \chi_{ij} + \eta\, \nabla^2 \chi_{ij}) = \frac{1}{2H\eta}(\partial_\eta^2 - \frac{1}{\eta}\partial_\eta)\chi_{ij}$, leading to [8],

$$\omega(\chi, \underline{\chi}) = \frac{1}{16\pi H} \int_{\Sigma_\eta} d^3x (\chi_{ij}\underline{\mathcal{E}}_{kl} - \underline{\chi}_{kl}\mathcal{E}_{ij})\, \delta^{ik}\delta^{jl} \tag{4.3.4}$$

For a Killing vector K of the de Sitter background, we have the tangent vector $h^{(K)}_{ij} := \mathcal{L}_K h_{ij} = a^2(\mathcal{L}_K \chi_{ij} + 2(a^{-1}\mathcal{L}_K a)\chi_{ij}) =: a^2 \chi^{(K)}_{ij}$, generating a canonical transformation. The corresponding Hamiltonian function is given by,

$$H_K := -\frac{1}{2}\omega(h, h^{(K)}) = -\frac{1}{2}\omega(\chi, \chi^{(K)}) \tag{4.3.5}$$

For the time translation Killing vector T, $a^{-1}\mathcal{L}_T a = H$, so that $\chi^{(T)}_{ij} = \mathcal{L}_T \chi_{ij} + 2H\chi_{ij}$. Furthermore, $\mathcal{E}^{(T)}_{ij} = \mathcal{L}_T \mathcal{E}_{ij} - H\mathcal{E}_{ij}$ and the corresponding Hamiltonian, now referring to it as the *energy* E_T, is obtained as,

$$E_T := -\frac{1}{2}\omega(\chi, \chi^{(T)}) = -\frac{1}{32\pi H}\int_{\Sigma_\eta} d^3x(\chi_{ij}\mathcal{E}^{(T)}_{kl} - \chi^{(T)}_{kl}\mathcal{E}_{ij})\, \delta^{ik}\delta^{jl} \tag{4.3.6}$$

$$= -\frac{1}{32\pi H}\int_{\Sigma_\eta} d^3x(\chi_{ij}\, \mathcal{L}_T\mathcal{E}_{kl} - \mathcal{E}_{kl}\mathcal{L}_T\chi_{ij} - 3H\chi_{ij}\mathcal{E}_{kl})\, \delta^{ik}\delta^{jl} \tag{4.3.7}$$

This integral is independent of the choice of Σ_η and is conveniently performed on $\mathcal{J}^+ = \Sigma_0$. The Killing vector T^μ also has a smooth limit to $\mathcal{J}^+$, $T\big|_{\mathcal{J}^+} \to -H(x\partial_x + y\partial_y + z\partial_z)$ which is tangential to $\mathcal{J}^+$. On $\mathcal{J}^+$ the equation (4.3.7) simplifies to,

$$E_T = \frac{1}{16\pi H}\int_{\mathcal{J}^+} d^3x\, \mathcal{E}_{kl}\, (\mathcal{L}_T\chi_{ij} + 2H\chi_{ij})\, \delta^{ik}\delta^{jl} \tag{4.3.8}$$

Here we have integrated by parts and used the vanishing of the Weyl tensor at $\mathcal{J}^+$ for de Sitter metric [8, 22]. Now using $(\mathcal{L}_T\chi_{ij} + 2H\chi_{ij})\big|_{\mathcal{J}^+} = T^m\partial_m\chi_{ij}$,

$$E_T = \frac{1}{16\pi H}\int_{\mathcal{J}^+} d^3x\, \mathcal{E}_{kl}\, (T^m\partial_m\chi_{ij})\, \delta^{ik}\delta^{jl} \tag{4.3.9}$$

$$= \frac{1}{32\pi H^2}\int_{\mathcal{J}^+} d^3x \left[\frac{1}{\eta}(\partial^2_\eta - \frac{1}{\eta}\partial_\eta)\mathcal{X}_{kl}\right]^{TT} (T^m\partial_m\mathcal{X}_{ij})^{TT}\, \delta^{ik}\delta^{jl} \tag{4.3.10}$$

In the last line we have used equation of motion and to emphasize it, have replaced $\chi_{ij} \to \mathcal{X}_{ij}$ in source free region and restored the TT label [8]. Both $\mathcal{E}_{kl}$ and $T^m\partial_m\chi_{ij}$ have smooth limit on $\mathcal{J}^+$.

Now for the approximated solution given in (4.2.7), the energy flux turns out to be given by [8],

$$E_T = \frac{1}{8\pi}\int_{\mathcal{J}^+} d\tau\; d^2s[Q_{kl}\, Q^{TT}_{ij}]\delta^{ik}\delta^{jl}, \tag{4.3.11}$$

where, Q_{ij} denotes the 'radiation field' on $\mathcal{J}^+$, expressed in terms of source moments and is given by[1]

$$Q^{TT}_{mn} \;:=\; \left[\dddot{Q}_{mn} + 3H\ddot{Q}_{mn} + 2H^2\dot{Q}_{mn} + H\ddot{\bar{Q}}_{mn} + 3H^2\dot{\bar{Q}}_{mn} + 2H^3\bar{Q}_{mn}\right]^{TT}(t_{ret})\,, \tag{4.3.12}$$

with the overdot denoting the Lie derivative $\mathcal{L}_T$. It may appear that the last term without an overdot implies non-zero flux even for 'time independent' sources. This is not so. A static source is defined by the condition, $\mathcal{L}_T T_{\mu\nu} = 0$. This implies that $\mathcal{L}_T Q_{ij} = -2HQ_{ij}$ and ditto for $\bar{Q}_{ij}$. When this is used, the energy flux is indeed zero for static sources [8,43].

The instantaneous *power* received on $\mathcal{J}^+$ at 'τ' is given by,

$$P(\tau) \;:=\; \frac{1}{8\pi}\int_{S^2} d^2s[Q^{ij}\,Q^{TT}_{ij}](-r(\tau))\,. \tag{4.3.13}$$

This expression is not manifestly positive. Manifestly positive expressions for the flux and the power are also available in [8] and are given by,

$$E_T \;=\; \frac{1}{2\pi}\int_{\mathcal{J}^+} d\tau\, d^2s\left[\partial_r\mathcal{M}^{TT}_{ij}\right]\left[\partial_r\mathcal{M}^{ij}_{TT}\right]\,, \quad \text{where} \tag{4.3.14}$$

$$\mathcal{M}^{TT}_{ij}(\eta - r) \;:=\; \int d^3x' T^{TT'}_{ij}(\eta - r, \vec{x}')\;; \tag{4.3.15}$$

$$P(\tau) \;=\; \frac{1}{2\pi}\int_{S^2} d^2s\left[\partial_r\mathcal{M}^{TT}_{ij}\right]\left[\partial_r\mathcal{M}^{ij}_{TT}\right](-r(\tau)) \tag{4.3.16}$$

In the definition of $\mathcal{M}^{TT}_{ij}$, the TT' on the stress tensor on the right hand side denotes transversality with respect to the $\vec{x}'$ argument. The $\mathcal{M}^{TT}_{ij}$ has no simple relation to the various source moments and its radial derivative is distinct from the Q^{TT}_{ij}.

For completeness, we also give the expressions for the momentum and angular momentum fluxes [8]:

$$P_j \;=\; \frac{1}{16\pi H}\int_{\mathcal{J}^+} d^3x\mathcal{E}^{mn}\mathcal{L}_{\xi_j}\chi^{TT}_{mn} \;=\; 0\,; \tag{4.3.17}$$

[1] The Q_{ij} above is same as $\mathcal{R}_{ij}$ of [8]. We introduce *our* $\mathcal{R}_{ij}$ notation below equation (4.3.72). It is related to radial derivative of the $\mathcal{M}^{TT}_{ij}$ defined in equation (4.3.15).

$$\begin{aligned} J_j &= -\frac{1}{8\pi H}\int_{\mathcal{J}^+} d^3x \mathcal{E}^{mn}\mathcal{L}_{L_j}\chi^{TT}_{mn} \\ &= \frac{1}{4\pi}\int_{\mathcal{J}^+} d\tau d^2s\, \epsilon_{jmn}Q^{nl}\left[\ddot{Q}_l{}^m + H\dot{Q}_l{}^m + H\dot{\bar{Q}}_l{}^m + H^2\bar{Q}_l{}^m\right]^{TT} \end{aligned} \tag{4.3.18}$$

The momentum flux is zero because the integrand is linear in x_j (parity odd) and in the angular momentum flux, the second factor is proportional to the tail term.

4.3.2 Isaacson Prescription

In the previous subsection we saw a definition of *total energy* of radiation field of compactly supported sources in equation (4.3.11). The radiated power, received at infinity, is given in equation (4.3.13). In this subsection we recall an alternative framework, based on a 'short wavelength expansion' [21, 42], for a restricted class of sources but with the benefit of a symmetric, conserved, suitably gauge invariant *effective gravitational stress tensor*.

Conceptually, the framework is somewhat different from perturbation about a *fixed*, given background solution. It envisages construction of a class of solutions for which there exists a coordinate system in which the metric components display two widely separated temporal/spatial scales of variation. The slowly varying (or long wavelength) component is taken as the *background* component and the fast (or short wavelength) component whose amplitude is small compared to that of the background, is identified as the *ripple* component. These statements are manifestly coordinate dependent, but existence of a coordinate system with sufficiently large domain admitting such an identification, itself is a physical property. The calculational scheme is again iterative but now allows for both the background and the ripple components to be corrected. To make such a separation, an *averaging scheme* is introduced. It splits the Einstein equation into two separate, coupled equations for the background and the ripple. These equations provide a definition of the effective gravitational stress tensor.

Let L denote the length scale of variation of the background and λ the length scale of the ripple with $\lambda \ll L$. The length scale R of the spatially compact source satisfies $R \lesssim \lambda$. For example, if the waves are produced by a source with a characteristic frequency $f \sim (period)^{-1} \sim \lambda^{-1}$. The 'time' used in identifying the characteristic frequency is either the time coordinate in a class of coordinate systems or in more favourable cases the time coordinate provided by a stationary Killing vector of the background geometry. In the present context of de Sitter background having a stationary Killing vector $T := -H(\eta\partial_\eta + x^i\partial_i)$, such a situation can be easily envisaged.

Returning to a general context, one begins with an expansion of the form $g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}$ and writes the Einstein equation as,

$$\begin{aligned} R_{\mu\nu}(\bar{g} + \epsilon h) &= \Lambda(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}) + 8\pi\epsilon(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}T_{\alpha\beta}) \\ \therefore R^{(0)}_{\mu\nu}(\bar{g}) + \epsilon R^{(1)}_{\mu\nu}(\bar{g}, h) + \epsilon^2 R^{(2)}_{\mu\nu}(\bar{g}, h) &= \Lambda(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}) + 8\pi\left\{\epsilon T_{\mu\nu} - \frac{1}{2}(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu})\right. \\ &\qquad \left.(\bar{g}^{\alpha\beta} - \epsilon h^{\alpha\beta} + \epsilon^2 h^{\alpha\rho}h_{\rho}^{\ \beta})(\epsilon T_{\alpha\beta})\right\} \end{aligned} \tag{4.3.19}$$

Introduce an averaging over an intermediate scale ℓ, $\lambda \ll \ell \ll L$ which satisfies the properties [42, 44]: (i) average of odd powers of h vanish and (ii) average of space-time divergence of tensors are sub-leading. The average of course leaves the L-scale variations intact, in particular average of $g_{\mu\nu}$ equals $\bar{g}_{\mu\nu}$. For simplicity, we will assume that the average of matter stress tensor is zero i.e. it has only λ-scale variations. Taking the average of the above equation gives,

$$\langle R^{(0)}_{\mu\nu}\rangle + \epsilon^2\langle R^{(2)}_{\mu\nu}\rangle = \Lambda\bar{g}_{\mu\nu}. \tag{4.3.20}$$

Notice that $R^{(2)}$ which is quadratic in h, *can* have L-scale variations and hence non-zero average (two high frequency modes can generate a low frequency mode). Nonlinearity of general relativity comes into play here explicitly, $\langle R^{(2)}_{\mu\nu}\rangle$ incorporates back reaction of ripple on the background. Its λ-scale variations are suppressed by a power of ϵ relative to $R^{(1)}$. We note that $\langle R^{(0)}_{\mu\nu}\rangle = R^{(0)}_{\mu\nu}$ and use $R^{(2)}_{\mu\nu} - \langle R^{(2)}_{\mu\nu}\rangle \approx (R^{(2)}_{\mu\nu})_{\lambda-\mathrm{scale}}$. Subtracting the eqn.(4.3.20) from the eqn.(4.3.19) we get (to order ϵ),

$$R^{(1)}_{\mu\nu} + \epsilon(R^{(2)}_{\mu\nu})_{\lambda-\mathrm{scale}} \approx R^{(1)}_{\mu\nu} \;=\; \Lambda h_{\mu\nu} + 8\pi\left(T_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}T^{\alpha\beta}\bar{g}_{\alpha\beta}\right) \tag{4.3.21}$$

In terms of first order Einstein tensor, this is exactly the linearized equation we had before. The equation (4.3.20) can likewise be rearranged and now includes an additional source term for the equation determining the background metric $\bar{g}_{\mu\nu}$. The rearranged equations take the form,

$$\begin{aligned} 8\pi T_{\mu\nu} &= G^{(1)}_{\mu\nu} + \Lambda h_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}\left(\bar{g}_{\mu\nu}R^{(1)} - \bar{g}_{\mu\nu}h^{\alpha\beta}\bar{R}_{\alpha\beta} + h_{\mu\nu}\bar{R}\right) + \Lambda h_{\mu\nu} && (4.3.22)\\ 8\pi t_{\mu\nu} &= \bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} + \Lambda\bar{g}_{\mu\nu} \qquad \mathit{with,} && (4.3.23)\\ t_{\mu\nu}(\bar{g}, h) &:= -\frac{\epsilon^2}{8\pi}\left[\langle R^{(2)}_{\mu\nu}\rangle - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\langle R^{(2)}_{\alpha\beta}\rangle\right] && (4.3.24) \end{aligned}$$

The equation (4.3.22) is exactly the same linearized Einstein we had before for the weak field $h_{\mu\nu}$ and every term of it has a scale of variation λ. However, the equation (4.3.23) for the background is modified due to back reaction. Although it has terms of order ϵ^2, every term has a scale of variation L. In this problem there are two separate dimensionless small parameters, namely amplitude of ripple ϵ and $\epsilon' := \lambda/L << 1$, which is introduced to make a clear separation between ripple and background. If we now recognise that for $\lambda-$scale variation, $\partial h \sim \lambda^{-1}h$ and $\epsilon' := \lambda/L$ is taken to be of the same order as ϵ (consistent with background eqn. (4.3.20)), then the effective stress tensor which has a leading term of the form $(\partial h)^2$, is of the order $(\epsilon/\epsilon')^2 \sim 1/L^2 \sim o(1)$ and is thus included in the equation.

The effective stress tensor defined in equation (4.3.24) is manifestly symmetric and is covariantly conserved since divergence of the right hand side of (4.3.23) vanishes identically. Modulo the subtleties of its tensor nature and gauge invariance under $\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi\bar{g}_{\mu\nu}$, it is well qualified to be identified as gravitational stress tensor. An averaging procedure constructing a tensor has been given by Isaacson [21, 45] and an explicit illustrative computation is given in the appendix. To verify gauge invariance of $t_{\mu\nu}$ to leading order in ϵ, we need to be more explicit about the gauge transformations [21].

Recall that the gauge transformation involves derivatives of ξ_μ and for a consistency with the background plus ripple split, the gauge transformation should also be restricted to preserve it. There are two possibilities for the generator: (i) ξ is comparable with h and slowly varying, and (ii) ξ is order ϵh but is rapidly varying so that its derivative becomes order h. The gauge transformation of $t_{\mu\nu}$ has terms of the form $\Lambda\langle h\nabla\xi\rangle$ (which vanish identically for Minkowski background). For the ξ of type (i), the average vanishes since the enclosed quantity is rapidly varying and for ξ of type (ii), the averaged quantity is order ϵ. But $t_{\mu\nu}$ itself is $o(1)$ and hence *gauge invariance of $t_{\mu\nu}$ is ensured to the leading order* [21].

The calculational scheme begins by *selecting* a solution of the background equation (4.3.23) ignoring the effective stress tensor. This is just the vacuum Einstein equation with cosmological constant. In the present context, we choose the maximally symmetric, de Sitter space-time. Using this in the ripple equation (4.3.22), determines linearized solutions, $h_{\mu\nu}$ after a suitable gauge choice and choice of boundary conditions. In the present context, the focus is on the retarded solution. This solution gives the effective gravitational stress tensor of equation (4.3.24). This is fed back into the background equation to construct a new, corrected background. This will still have $L-$scale variations since the effective stress tensor varies on the scale L. The corrected background is used in the ripple equation to get corrections to the ripple and so on. Since the equations for the background and the ripple have consistent scales of variations, the background and ripple characters are maintained. Within this scheme, the effective gravitational stress tensor finds its justification. Once we have a conserved and symmetric stress tensor, for every Killing vector of the background, we get a conserved *current* and the corresponding conserved *charge*[2]. It

[2]The conserved charge from a current represents a *flux* for the stress tensor.

turns out that the energy momentum computed using the effective gravitational stress tensor for ripples over Minkowski background, agrees with the quadrupole formula obtained by other methods, thereby strengthening the interpretation.

As an aside, we make a numerical estimate comparing the effective stress tensor and the cosmological constant. Typical estimate for the wave amplitude (dimensionless) at LIGO detectors is around 10^{-21} or smaller. The highest frequency, f, detectable is about 10^4 Hz $\sim 3 \times 10^{-5}$ meter^{-1} in the $c = 1$ units. In the geometrized units we are using, $t_{\mu\nu} \sim (hf)^2 \sim 10^{-51}$ meter^{-2}. Although tiny, this is very much comparable to the cosmological constant! Thus, unlike the Minkowski background which would be inconsistent with the sourcing by gravitational waves, the de Sitter background at least has a chance of being self consistent.

To evaluate the effective stress tensor, we need to carry out the expansion of Ricci tensor to order ϵ^2. In terms of the trace-reversed $\tilde{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\,(\bar{g}^{\alpha\beta}h_{\alpha\beta})$, this is given by,

$$\begin{aligned} R^{(2)}_{\mu\nu} \; = \; & \left[-\frac{1}{4}\bar{\nabla}_\mu\tilde{h}^{\alpha\beta}\bar{\nabla}_\nu\tilde{h}_{\alpha\beta} - \frac{1}{2}\tilde{h}_\mu{}^\alpha\bar{\Box}\tilde{h}_{\alpha\nu} + \frac{2}{3}\Lambda\tilde{h}_\mu{}^\alpha\tilde{h}_{\alpha\nu}\right] + \left[-\frac{1}{2}B_\mu B_\nu\right] \\ & + \left[\tilde{h}\left\{\frac{1}{4}\left(\bar{\Box}\tilde{h}_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu\tilde{h}\right) - \frac{\Lambda}{6}\tilde{h}_{\mu\nu}\right\}\right] + \text{total derivative terms.} \end{aligned} \quad (4.3.25)$$

The expression has been arranged in a suggestive form in the anticipation that it will be evaluated for gauge fixed solution.

The second group of terms containing $B_\mu = \bar{\nabla}_\alpha\tilde{h}^\alpha{}_\mu$, which eventually vanish when the *synchronous gauge* condition, $\tilde{h}_{0\alpha} = 0$ is imposed. Likewise, the third group of terms vanish when the trace free condition, $\tilde{h} = 0$, is imposed. The total divergence terms are not displayed as they will be negligible under averaging. Only the first group of terms, with the sum over α, β restricted to sum over spatial indices i, j, survive for gauge fixed solutions.

The Ricci scalar, $R^{(2)} := \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}$ is obtained as,

$$R^{(2)}_{TT} = \tilde{h}^{\alpha\beta}\left\{-\frac{1}{4}\bar{\Box}\tilde{h}_{\alpha\beta} + \frac{2}{3}\Lambda\tilde{h}_{\alpha\beta}\right\} + \text{total derivative terms} \tag{4.3.26}$$

Next, the $\bar{\Box}$ terms are eliminated using the source-free linearized equation (3.2.8), since we are interested in the ripple stress tensor. Substitution leads to,

$$-8\pi t_{\mu\nu} = \epsilon^2\left\langle\left[-\frac{1}{4}\bar{\nabla}_\mu\tilde{h}^{\alpha\beta}\bar{\nabla}_\nu\tilde{h}_{\alpha\beta} + \frac{\Lambda}{3}\tilde{h}_\mu{}^\alpha\tilde{h}_{\alpha\nu} - \frac{\Lambda}{4}\bar{g}_{\mu\nu}\left(\tilde{h}^{\alpha\beta}\tilde{h}_{\alpha\beta}\right)\right]\right\rangle \tag{4.3.27}$$

This expression reduces to the stress tensor for the Minkowski background by taking $\nabla_\mu \to \partial_\mu$ and dropping the last two terms. However, for the ripple, $\partial\tilde{h} \sim \lambda^{-1}\tilde{h} \sim \epsilon^{-1}\tilde{h}$. The connection terms in the covariant derivatives are order $\tilde{h}$. Hence, to the leading order in $\epsilon \sim \lambda/L$, all the terms without *derivatives of the ripple*, can be dropped and we are back to the same expression for the Minkowski background. Notice that the leading term has no ϵ.

This is further simplified in the conformal coordinates by substituting $\tilde{h}_{\alpha\beta} = \Omega^2\chi_{\alpha\beta}$ and also expressing the covariant derivatives in terms of the Minkowski derivatives. Once again, dropping the derivatives of the conformal factor and using the synchronous gauge and the trace-free condition on the field, we write the gauge invariant stress tensor for ripples,

$$t_{\mu\nu} = \frac{1}{32\pi}\langle\partial_\mu\mathcal{X}^{TT}_{ij}\ \partial_\nu\mathcal{X}^{ij}_{TT}\rangle\ . \tag{4.3.28}$$

We will refer to this as the *ripple stress tensor*. We have used $\mathcal{X}^{TT}_{ij}$ to emphasize that the ripple stress tensor is defined for *fully gauge fixed solutions* of the linearized equation.

We will compute this for the 'tt' projected, approximated retarded solution, χ^{tt}_{ij}. In the subsection 4.3.5, we will discuss how the computations change when $\chi^{tt}_{ij} \to \mathcal{X}^{TT}_{ij}$.

Given *any* symmetric, conserved stress tensor, for every Killing vector of the background space-time, ξ^μ, the current $J^\mu_\xi \sim T^\mu{}_\nu\xi^\nu$, is covariantly conserved. Such a current is defined to within a constant multiple since the Killing vector is defined to within a constant multiple, including a sign. An integral curve of a *time-like* Killing vector can represent an observer and for a future directed worldline, the observer measures, say, the matter energy density to be $T^{\mu\nu}\xi_\mu\xi_\nu$ which is *positive* for normal matter. The observer also expects the energy momentum current representing the flow of energy-momentum to obey special relativity i.e. should be a vector which is time-like/null and future directed. Thus $\xi_\mu J^\mu_\xi < 0$ must also hold (our metric signature is (-, + , + , +)). This implies that the energy-momentum *current*, J^μ_ξ, must be defined with sign: $J^\mu_\xi := -T^\mu{}_\nu\xi^\nu$. This fixes the *sign* for a future directed time-like Killing vector. The normalization can only be done for *one* Killing observer and this is chosen as per the context. We adopt this definition for the time translation Killing vector $T = -H(\eta\partial_\eta + x^i\partial_i)$.

For the time translation Killing vector field, $T^\nu\partial_\nu$ involves only the η and r derivatives since $x^i\partial_i = r\partial_r$. These pass through the $\Lambda-$projector. For the space translation along j^{th} direction, we have ∂_j which does act on the $\Lambda-$projector. In the present context where derivatives of the ripple dominate over (ripple/r), the derivative of the projector can be neglected and we write, $\partial_j\chi^{tt}_{mn} \approx \hat{x}_j\partial_r\chi^{tt}_{mn}$. For generators of rotation however the situation is different. Once again we get two term from the ∂_i, but now the $\epsilon_{jki}x^k\hat{x}^i\partial_r\chi^{mn} = 0$! and we can no longer neglect the derivative of the projector. With these understood, we write the the corresponding currents, $J^\mu_\xi = -\frac{a^{-2}}{32\pi}\langle\partial^\mu\chi^{mn}_{tt}\ \partial_\nu\chi^{tt}_{mn}\rangle\xi^\nu$. Note that the ripple stress tensor has been defined as a covariant rank 2 tensor and hence there is the factor of $a^{-2} = H^2\eta^2$ since the index μ has been raised. The currents are given by,

$$a^2 J^\eta_T = -\frac{H}{32\pi}\left\{\eta\langle\partial_\eta\chi^{mn}_{tt}\ \partial_\eta\chi^{tt}_{mn}\rangle + r\langle\partial_\eta\chi^{mn}_{tt}\ \partial_r\chi^{tt}_{mn}\rangle\right\} \tag{4.3.29}$$

$$a^2 J^i_T = \frac{H}{32\pi}\left\{\eta\langle\hat{x}^i\partial_r\chi^{mn}_{tt}\ \partial_\eta\chi^{tt}_{mn}\rangle + r\langle\hat{x}^i\partial_r\chi^{mn}_{tt}\ \partial_r\chi^{tt}_{mn}\rangle\right\} \tag{4.3.30}$$

$$a^2 J^\eta_{\xi_j} = \frac{1}{32\pi}\langle\partial_\eta\chi^{mn}_{tt}\ \hat{x}_j\partial_r\chi^{tt}_{mn}\rangle \quad , \quad a^2 J^i_{\xi_j} = -\frac{1}{32\pi}\langle\hat{x}^i\partial_r\chi^{mn}_{tt}\ \hat{x}_j\partial_r\chi^{tt}_{mn}\rangle \tag{4.3.31}$$

$$a^2 J^\eta_{L_j} = -\frac{1}{16\pi}\epsilon_{jmn}\hat{x}^m\langle\partial_\eta\chi^{nl}_{tt}\ \chi_{lk}\hat{x}^k\rangle \quad , \quad a^2 J^i_{L_j} = \frac{1}{16\pi}\epsilon_{jmn}\hat{x}^m\langle\hat{x}^i\partial_r\chi^{nl}_{tt}\ \chi_{lk}\hat{x}^k\rangle \tag{4.3.32}$$

The unit vectors within the angular brackets have come from the spatial derivatives while those outside the brackets come from the Killing vector. It is shown in the appendix (eq. D.0.9) that for the averaging regions far away from the source, the *unit vectors can be taken across the angular brackets and we will do so in the subsequent expressions.*

Notice that for the energy and momentum currents (4.3.30, 4.3.31), both fields have the 'tt' label whereas for the angular momentum current (4.3.32), the second factor does *not* have the tt label. The entire contribution to the angular momentum current comes from the derivative of the $\Lambda-$projector. The contribution from the derivative of the field vanishes since the field (without the projector) is spherically symmetric. In all these equations we may use $\partial_r\chi_{mn} = -\partial_\eta\chi_{mn} - \frac{f_{mn}}{r^2}$ from (4.2.12).

We note in passing that *if* the $\frac{f_{mn}}{r^2}$ can be neglected compared to $\partial_\eta\chi_{mn}$, then the currents corresponding to the generators of time and space translations, both become *proportional* to the vector $(1, x^i/r)$ which is null and J^μ_T is also future directed. Both energy and momentum propagate along this direction.

Finally we note the integral form of the conservation equation. Let $\mathcal{V}$ denote a space-time region with a boundary $\partial\mathcal{V}$. Then it follows that,

$$0 = \int_{\mathcal{V}} d^4x\sqrt{\bar{g}}\bar{\nabla}_\mu J^\mu_\xi = \int_{\mathcal{V}} d^4x\partial_\mu(\sqrt{\bar{g}}J^\mu_\xi) = \int_{\partial\mathcal{V}} d\sigma_\mu J^\mu_\xi, \tag{4.3.33}$$

where $d\sigma_\mu$ is the oriented volume element of the boundary $\partial\mathcal{V}$.

In the next subsection we evaluate the *energy flux*, $\int_\Sigma d\sigma_\mu J^\mu_T$, for various hyper-surfaces, Σ's. These, together with the conservation equation (4.3.33) will be used to relate power received at $\mathcal{J}^+$ to that crossing the cosmological horizon. In the following subsection, we will present the fluxes for momentum and angular momentum.

4.3.3 Flux computations

We present flux calculations for three classes of hypersurfaces: (a) hypersurfaces of constant physical radial distance, (b) space-like hypersurfaces of constant η and (c) the out-going and in-coming null hypersurfaces.

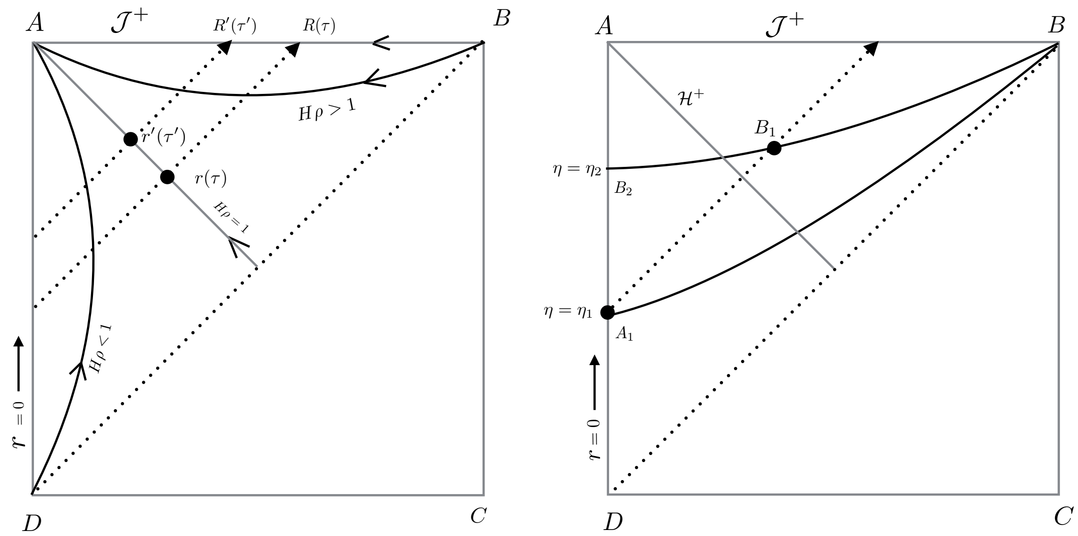

Figure 4.1. The figure on the left shows the ρ = constant hypersurfaces which are time-like for $H\rho < 1$, null for $H\rho = 1$ and space-like for $H\rho > 1$. The two 45 degree out-going null hypersurfaces intersecting the $\mathcal{H}^+$ and $\mathcal{J}^+$ in the spheres at $r(\tau), r'(\tau'), R(\tau), R'(\tau')$, bound a space-time region. The figure on the right shows the space-like hypersurfaces with constant value of η. The fluxes across the out-going null hypersurfaces turn out to be zero signifying sharp propagation of energy-momentum and angular momentum. Hence the energy flux across the portion of the horizon bounded by the spheres at $r(\tau), r'(\tau')$ equals the flux across the portion of the future infinity bounded by the spheres at $R(\tau), R'(\tau')$.

4.3.3.1 Hypersurface of constant, radial physical distance:

These hypersurfaces are time-like, null and space-like according as the physical distance being less than, equal to and greater than the physical distance to the cosmological horizon, namely H^{-1}. They are spanned by the integral curves of the Killing vector T.

This Killing vector is special because in the static patch, it is time-like and its integral curves represent Killing observers. Denoting $x^i := r\hat{x}^i,\ \hat{x}^i\hat{x}^j\delta_{ij} = 1$, in general, its integral curves are given by $\eta(\tau) = \eta_* e^{-H\tau}, r(\tau) = r_* e^{-H\tau}, \hat{x}^i = \hat{x}^i_*$. Evidently, along each curve, $\rho := r/(-H\eta) = r_*/(-H\eta_*)$ is constant. This also represents the *physical radial distance*, $r_{phy} := |\Omega| r$. Each particular curve is labelled by ρ and the two angular coordinates $\hat{x}^i_*$. We compute the flux across the hypersurface Σ_ρ, defined by $r_{phy} = \rho$. This surface is coordinatized by the Killing parameter τ and the usual spherical angles θ, ϕ represented by the unit vectors $\hat{x}^i$. These hypersurfaces are topologically $\Sigma_\rho \sim \Delta\tau \times S^2$ and their embedding is given by,

$$\eta(\tau, \theta, \phi) = \eta_* e^{-H\tau} \ , \ x = r_* e^{-H\tau} sin\theta cos\phi \ , \ y = r_* e^{-H\tau} sin\theta sin\phi \ , \ z = r_* e^{-H\tau} cos\theta \ ,$$

with $r_* + H\rho\eta_* = 0$.

The induced metric is given by $h_{ab} = diag(H^2\rho^2 - 1, \rho^2, \rho^2 sin^2\theta)$. This has Lorentzian signature for $H\rho < 1$ (inside the static patch), is degenerate for $H\rho = 1$ (the cosmological horizon) and Euclidean signature for $H\rho > 1$ (beyond the cosmological horizon). The measure factor for the non-null cases is given by $\sqrt{|det\ h_{ab}|} = \sqrt{|1 - H^2\rho^2|}\rho^2 sin\theta$ while on the cosmological horizon it is given by $\sqrt{h_{\underline{ab}}} = \rho^2 sin\theta$. Here $\underline{a}, \underline{b}$ denote the 'transverse' coordinates θ, ϕ. In the non-null case, the unit normal is given by $n_\mu = \frac{\epsilon}{|H\eta|}|1 - H^2\rho^2|^{-1/2}(H\rho, x_i/r) \leftrightarrow n^\mu = \epsilon|H\eta||1 - H^2\rho^2|^{-1/2}(-H\rho, x^i/r)$. Here $\epsilon = +1$ for *time-like* Σ_ρ ($H\rho < 1$) and $\epsilon = -1$ for *space-like* Σ_ρ ($H\rho > 1$). On the cosmological horizon, we *choose* the normal to be: $n_\mu = -|H\eta|^{-1}(H\rho, x_i/r) \leftrightarrow n^\mu = -|H\eta|(-H\rho, x^i/r)$, so that $n^\mu = T^\mu$ is future directed. Introduce $N^\mu := (-H\rho, \hat{x}^i)$, so that the normal for non-null cases is expressed as $n^\mu = \epsilon|H\eta||1 - H^2\rho^2|^{-1/2}N^\mu$. Note that the n^μ is the same for the space-like and the null hypersurfaces, $\Sigma_{\rho \geq H^{-1}}$. For the time-like hypersurface, the n^μ points in the opposite direction. However, the induced orientation on Σ_ρ is also reversed as the hypersurface changes from being space-like to being time-like. Hence, in *all cases,* $H\rho > 0$, $n^\mu\sqrt{h} = -|H\eta|N^\mu\rho^2\ sin\theta$ and the hypersurface integral is expressed as,

$$\int_{\Sigma_\rho} d\Sigma_\alpha J^\alpha_T \quad = \quad -\int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2s\, \rho^2(-|H\eta(\tau)|N^\mu)(-t_{\mu\nu}T^\nu) \qquad \text{with} \qquad (4.3.34)$$

$$N^{\mu}t_{\mu\nu}T^{\nu} \;=\; -Hr(\tau)\left\{t_{\eta\eta} + \hat{x}^i t_{ij}\hat{x}^j - \hat{x}^i t_{i\eta}\left((H\rho)^{-1} + H\rho\right)\right\} \tag{4.3.35}$$

The minus sign in front of the hypersurface integral is because the orientation defined by the Killing parameter and the angles, is *negative* relative to that defined by the r and the angles. The $sin\theta$ is absorbed in d^2s. The minus sign in the last parentheses is due to the definition $J_\mu = -t_{\mu\nu}T^\nu$. In the second line, we have also used $-H\rho\eta = r$ valid on Σ_ρ.

Substituting for the ripple stress tensor, and taking the unit vectors $\hat{x}$ outside of the angular bracket as mentioned before, the expression within the braces in eq. (4.3.35) becomes,

$$\{\,\} = \frac{1}{32\pi}\left\{\left\langle\partial_\eta\chi_{mn}\,\partial_\eta\chi^{mn} + \partial_r\chi_{mn}\,\partial_r\chi^{mn}\right\rangle - \frac{1+H^2\rho^2}{H\rho}\left\langle\partial_r\chi_{mn}\,\partial_\eta\chi^{mn}\right\rangle\right\} \tag{4.3.36}$$

In the above, χ_{ij} is the tt part of the solution given in eq. (4.2.8). The tt projection introduces angle dependence in the χ^{tt}_{ij}, however equation (4.3.34) needs only η and r derivatives.

Eliminating $\partial_r\chi_{ij}$ using equation (4.2.12), we write,

$$\begin{aligned}
\{\,\} &= \frac{1}{32\pi}\left[\left\langle\partial_\eta\chi_{mn}\partial_\eta\chi^{mn}\right\rangle\frac{(1+H\rho)^2}{H\rho} + \left\langle\frac{f_{mn}}{r^2}\partial_\eta\chi^{mn}\right\rangle\frac{(1+H\rho)^2}{H\rho} + \left\langle\frac{f_{mn}f^{mn}}{r^4}\right\rangle\right] \\
&= \frac{1}{32\pi}\frac{(1+H\rho)^2}{H\rho}\left[\left\langle\partial_\eta\chi_{mn}\partial_\eta\chi^{mn} + \frac{f_{mn}}{r^2}\partial_\eta\chi^{mn}\right\rangle + \frac{H\rho}{(1+H\rho)^2}\left\langle\frac{f_{mn}f^{mn}}{r^4}\right\rangle\right] \\
\therefore \int_{\Sigma_\rho} d\Sigma_\alpha J^\alpha_T &= \int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2s\,\left[-\rho^2H^2\eta(\tau)r(\tau)\right]\{\,\}
\end{aligned} \tag{4.3.37}$$

The approximated solution χ_{ij}, is valid for (source dimension)/(distance to the source) $\ll 1$. This is consistent with the assumption that the $\lambda/r \ll 1$. Furthermore, the source being rapidly changing, $\lambda H \ll 1$, it follows that $f_{mn}/r^2 \ll \dot{f}_{mn}/r$. Hence we drop f_{mn}/r^2 terms. With this, {} takes a simple quadratic form $\frac{1}{32\pi}(1+H\rho)^2(H\rho)^{-1}\langle\partial_\eta\chi_{mn}\partial_\eta\chi^{mn}\rangle$.

To compute ∂_η we recall, $\eta_{ret} = \eta - r := -H^{-1}e^{-Ht_{ret}} := -(Ha(t_{ret}))^{-1}$ and use,

$$\partial_\eta f_{ij}(\eta_{ret}) = \partial_{\eta_{ret}} f_{ij}(\eta_{ret}) = a(\eta_{ret})\partial_{t_{ret}} f_{ij}(t_{ret}) = a(\eta_{ret})\left(\mathcal{L}_T + 2H\right)f_{ij}(t_{ret})$$

This leads to (overdot denoting $\mathcal{L}_T$),

$$
\begin{aligned}
\partial_\eta \chi_{mn}(\eta_{ret}) &= \frac{1}{r}\partial_\eta f_{mn}(\eta_{ret}) + \partial_\eta g_{mn}(\eta_{ret}) \\
&= \frac{a(t_{ret})}{r}\left(\mathcal{L}_T f_{mn} + 2H f_{mn}\right) + a(t_{ret})\left(\mathcal{L}_T g_{mn} + 2g_{mn}\right) && (4.3.38) \\
\mathcal{L}_T f_{mn} &= \frac{2}{a(t_{ret})}\left[\dddot{Q}_{mn} + H\ddot{Q}_{mn} - 2H^2\dot{Q}_{mn} + H\ddot{\bar{Q}}_{mn} + H^2\dot{\bar{Q}}_{mn} - 2H^3\bar{Q}_{mn}\right] && (4.3.39) \\
\mathcal{L}_T g_{mn} &= -2H\left[\dddot{Q}_{mn} + H\ddot{Q}_{mn} + H\ddot{\bar{Q}}_{mn} + H^2\dot{\bar{Q}}_{mn}\right] \qquad \Rightarrow && (4.3.40) \\
\partial_\eta \chi^{tt}_{mn}(\eta_{ret}) &= \frac{2}{r}\frac{\eta}{\eta - r}\mathcal{Q}^{tt}_{mn} \quad \text{with,} && (4.3.41) \\
\mathcal{Q}^{tt}_{mn} &:= \left[\dddot{Q}_{mn} + 3H\ddot{Q}_{mn} + 2H^2\dot{Q}_{mn} + H\ddot{\bar{Q}}_{mn} + 3H^2\dot{\bar{Q}}_{mn} + 2H^3\bar{Q}_{mn}\right]^{tt}(t_{ret}) && (4.3.42)
\end{aligned}
$$

Here we have also used $(1 - a(t_{ret})rH) = \frac{\eta}{\eta - r}$. Collecting all expressions, we write the flux through a segment of $r_{phy} = \rho$ hypersurface in a convenient form as,

$$
\begin{aligned}
\int_{\Sigma_\rho} d\Sigma_\alpha J^\alpha_T &= \int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2 s\left[-\rho^2 H^2 \eta(\tau) r(\tau)\right]\left[\frac{(1+H\rho)^2}{32\pi H\rho}\right]\left\langle\left[\frac{2}{r}\frac{\eta}{\eta - r}\right]^2 \mathcal{Q}^{tt}_{ij}\mathcal{Q}^{ij}_{tt}\right\rangle \\
&= \int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2 s\left[\frac{1}{8\pi}\right]\left\langle\mathcal{Q}^{tt}_{ij}\mathcal{Q}^{ij}_{tt}\right\rangle(t_{ret}) && (4.3.43)
\end{aligned}
$$

In the appendix, we show that for large ρ, the expression within the square brackets inside the angular brackets, can be taken outside. Then, using $r = -H\rho\eta$ which is valid over the hypersurface $\forall\rho \in \mathbb{R}^+$, we see that the explicit dependence on ρ (for large enough ρ) disappears from the integrand but there is an implicit dependence on ρ and τ through t_{ret}. If however, the $\tau-$integration is extended over its full range, $(-\infty, \infty)$, then the integral *is* independent of ρ as well. Hence, *for sufficiently large ρ, all Killing observers infer the same energy flux in the limit $(\tau_1, \tau_2) \to (-\infty, \infty)$.*

The ρ independence of the full flux integral in particular means that the total flux across $\mathcal{J}^+$ equals the total flux across the cosmological horizon, $\mathcal{H}^+$.

$$
\lim_{\rho\to\infty}\int_{\Sigma_\rho} d\Sigma_\mu J^\mu_T = \int_{\Sigma_{(H\rho=1)}} d\Sigma_\mu J^\mu_T \Leftrightarrow \int_{\mathcal{J}^+} d\Sigma_\mu J^\mu_T = \int_{\mathcal{H}^+} d\Sigma_\mu J^\mu_T . \tag{4.3.44}
$$

4.3.3.2 Flux through a constant η slice:

The hypersurface Σ_{η_0} defined by $\eta = \eta_0$ is a cosmological slice $\sim \mathbb{R}^3$. It is space-like, with a normal $n_\mu = -|H\eta_0|^{-1}(1, \vec{0}) \leftrightarrow n^\mu = |H\eta_0|(1, \vec{0})$ which is future directed. We choose a finite portion of it with $r \in [r_1, r_2]$. The hypersurface is topologically $\Delta r \times s^2$. Choosing the (r, θ, ϕ) coordinates on the hypersurface, the embedding is given by

$$\eta(r, \theta, \phi) = \eta_0 \ , \ x = r \ sin\theta \ cos\phi \ , \ y = r \ sin\theta \ sin\phi \ , \ z = r \ cos\theta \ .$$

The induced metric is given by $h_{ab} = (H\eta_0)^{-2} diag(1, r^2, r^2 sin^2\theta)$ giving $\sqrt{|det \ h_{ab}|} = |H\eta_0|^{-3} r^2 sin\theta$. Denoting $N^\mu := (1, \vec{0})$, the hypersurface integral is given by,

$$\begin{aligned}
\int_{\Sigma_{\eta_0}} d\Sigma_\mu J^\mu_T &= \int_{r_1}^{r_2} dr \int_{S^2} d^2s \ r^2 a^2(\eta_0) \left(-N^\mu t_{\mu\nu} T^\nu\right) \qquad \text{with} && (4.3.45) \\
N^\mu t_{\mu\nu} T^\nu &= (-H)\left(t_{\eta\eta}\eta + t_{\eta i} x^i\right) \\
&= \frac{-H}{32\pi}\left(\eta \langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn} \rangle + x^i \langle \partial_\eta \chi_{mn} \ \partial_i \chi^{mn} \rangle\right) && (4.3.46) \\
&= \frac{-H}{32\pi}\left((\eta - r)\langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn} \rangle - \left\langle \frac{f_{mn}}{r} \partial_\eta \chi^{mn} \right\rangle\right) && (4.3.47) \\
\therefore \int_{\Sigma_{\eta_0}} d\Sigma_\mu J^\mu_T &= -\frac{1}{32\pi H^2 \eta_0^2} \int_{r_1}^{r_2} dr \int_{S^2} d^2s \ r^2 \left\{ \frac{1}{a(\eta_{ret})} \langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn} \rangle \right\} && (4.3.48) \\
&\approx \int_{r_1}^{r_2} \frac{dr}{H(r - \eta_0)} \int_{S^2} d^2s \left[\frac{-1}{8\pi}\right] \langle Q^{tt}_{mn} Q^{mn}_{tt} \rangle && (4.3.49)
\end{aligned}$$

By the same reasoning as before, we have dropped the $\frac{f_{mn}}{r}$ and also used equation (D.0.12). In the limit $\eta_0 \to 0$ with $(r_1, r_2) \to (0, \infty)$, the hypersurface becomes $\mathcal{J}^+$ and the integration measure becomes $\frac{dr}{Hr}$. The limit $\eta \to 0$ is thus finite.

As noted earlier, the hypersurface integral when expressed in terms of the Killing parameter, has a minus sign due to the reversal of the induced orientation. The measures (positive) themselves are related as $\frac{dr}{Hr} = d\tau$, leading to $\int_0^\infty dr/Hr = -\int_{-\infty}^\infty d\tau$ and we get,

$$\lim_{\eta_0 \to 0} \int_{\Sigma_{\eta_0}} d\Sigma_\mu J^\mu_T = \lim_{\rho \to \infty} \int_{\Sigma_\rho} d\Sigma_\mu J^\mu_T \ . \qquad (4.3.50)$$

4.3.3.3 Flux through null hypersurfaces:

There are two families of *future directed* null hypersurfaces given by $\eta + \epsilon r + \sigma = 0$, see figure (4.1). For $\epsilon = +1$, these 45 degree lines in the Penrose diagram are parallel to the cosmological horizon while for $\epsilon = -1$, the lines are parallel to the null boundary of the Poincaré patch. We refer to these as the *in-coming* ($\epsilon = 1$) and *out-going* ($\epsilon = -1$) null hypersurfaces. The parameter σ labels members of these families.

The null normals of these families are of the form $n_\mu = \gamma(1, \epsilon\hat{x}_i) \leftrightarrow n^\mu = (H\eta)^2\gamma(-1, \epsilon\hat{x}^i)$, where γ is to be chosen suitably and should be *negative* for future directed hypersurfaces. Choosing coordinates (λ, θ, ϕ) on a null hypersurface, its embedding may be taken as $\eta(\lambda), r(\lambda)$ with identity mapping of the angles. Here λ is an affine parameter of the null geodesics generating the null hypersurfaces. The induced metric is obtained as $h_{ab} = (H\eta)^{-2}diag(0, r^2, r^2 sin^2\theta)$. Note that the orientation of the hypersurfaces, relative to that defined by (r, θ, ϕ) is the same for the out-going hypersurfaces and opposite for the in-coming hypersurfaces. The hypersurface integral is then given by ($N^\mu := (-1, \epsilon\hat{x}^i)$),

$$
\begin{aligned}
\int_{\Sigma_{(\epsilon,\sigma)}} d\Sigma_\mu J^\mu_T &= -\epsilon\int_{\lambda_1}^{\lambda_2} d\lambda \int_S^2 d^2s\left[\frac{r^2}{H^2\eta^2}\right](H^2\eta^2)\gamma(-N^\mu t_{\mu\nu}T^\nu) \ , && (4.3.51)\\
N^\mu t_{\mu\nu}T^\nu &= -H\left(-t_{\eta\eta}\eta - t_{\eta j}r\hat{x}^j + \epsilon\hat{x}^i t_{i\eta}\eta + \epsilon r\hat{x}^i t_{ij}\hat{x}^j\right)\\
&= -\frac{H}{32\pi}\left(-\eta\langle\partial_\eta\chi_{mn}\partial_\eta\chi^{mn}\rangle + (\epsilon\eta - r)\langle\partial_\eta\chi_{mn}\partial_r\chi^{mn}\rangle + \epsilon r\langle\partial_r\chi_{mn}\partial_r\chi^{mn}\rangle\right)\\
&= -\frac{H}{32\pi}(1+\epsilon)(r-\eta)\langle\partial_\eta\chi^{tt}_{mn}\partial_\eta\chi^{mn}_{tt}\rangle + o(r^{-2}) && (4.3.52)\\
&= -\frac{H}{32\pi}(1+\epsilon)(r-\eta)\left[\frac{4}{r^2}\frac{\eta^2}{(\eta-r)^2}\right]\langle Q^{tt}_{mn}Q^{mn}_{tt}\rangle + o(r^{-2}) && (4.3.53)\\
\therefore \int_{\Sigma_{(\epsilon,\sigma)}} d\Sigma_\mu J^\mu_T &= \epsilon\int_{\lambda_1}^{\lambda_2} d\lambda\int_{S^2} d^2s\,[\gamma H]\left[\frac{(1+\epsilon)}{8\pi}\frac{\eta^2}{\eta-r}\right]\langle Q^{tt}_{mn}Q^{mn}_{tt}\rangle && (4.3.54)
\end{aligned}
$$

As before, we have dropped the f_{mn}/r^2 terms from $\partial_r\chi_{mn}$ and used the equation (D.0.12).

It is immediately clear that the flux through the out-going null hypersurfaces (ρ or r increase along these) vanishes. In the $\epsilon = 1$ family, only the cosmological horizon is of interest. For this we have ($\eta = -r$) and we *choose* the factor $\gamma = -(Hr)^{-1}$ so the null normal matches with the Killing vector (γ is negative as desired for future orientation) and the affine parameter λ matches with the Killing parameter τ. With this choice, the flux in eqn. (4.3.54) matches with that given in eq. (4.3.43) for $H\rho = 1$. Thus, once again, the full flux through cosmological horizon is exactly same as that of $r_{physical} = const$ hypersurfaces.

Remarks:

All three calculations consistently have the same $[+1/8\pi]$ factor, with integrals oriented along the stationary Killing vector.

It is surprising at first that the flux through $\eta - r = constant$ hypersurfaces is zero, which indicates sharp propagation of the energy, even though the retarded solution has a tail contribution. This can be seen more directly as follows. Let us recast eqn. (4.3.37) as

$$\int_{\Sigma_\rho} d\Sigma_\alpha J^\alpha_T = \frac{1}{32\pi}\int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2s \left\langle r^2(\eta - r)^2 \left(\frac{1}{\eta}\partial_\eta \chi_{ij}\right)\left(\frac{1}{\eta}\partial_\eta \chi_{ij}\right)\right\rangle \tag{4.3.55}$$

where we have neglected the $1/r^2$ terms and have used $r = -H\rho\eta$.

Now in taking the η derivative, contribution of the tail term in (4.2.7) cancels out, leaving only the contribution from the sharp term:

$$\frac{1}{\eta}\partial_\eta \chi_{ij} = \frac{4}{r\,\eta_{ret}}\partial_\eta \int d^3x' T_{ij}\,(\eta - r, x')\,. \tag{4.3.56}$$

Hence, though the field solution has a tail term, the energy propagation is sharp along the light cone. This is also supported from vanishing energy flux across out-going null surfaces.

4.3.4 Momentum and angular momentum fluxes

For the same three classes of hypersurfaces, we present the momentum and angular momentum fluxes. We already have the measures for these hypersurfaces as well as the currents given in (4.3.31, 4.3.32). The full fluxes, only to the leading order in r^{-1}, are given by,

$$\Sigma_\rho \;:\; -\int_{-\infty}^{\infty} d\tau \int_{S^2} d^2 s \rho^2 \frac{1}{H\eta}(H\rho, \hat{x}_i) J^\mu \tag{4.3.57}$$

$$\Sigma_{\eta_0} \;:\; \int_0^{\infty} dr \int_{S^2} d^2 s \left[\frac{r^2}{|H\eta_0|^3}\right]\left[\frac{1}{H\eta_0}\right](1, \vec{0}) J^\mu \tag{4.3.58}$$

$$\Sigma_{(\epsilon,\sigma)} \;:\; -\epsilon \int_{\lambda_1}^{\lambda_2} d\lambda \int_{S^2} d^2 s \left[\frac{r^2 \gamma}{|H\eta_0|^2}\right](1, \epsilon\hat{x}_i) J^\mu \tag{4.3.59}$$

Momentum fluxes: The momentum current is given by,

$$J^\mu_{\xi_j} \;=\; -\frac{a^{-2}}{32\pi}\left[\frac{2}{r}\frac{\eta}{\eta - r}\right]^2 \langle Q^{mn}_{tt}\, Q^{tt}_{mn}\rangle(\hat{x}_j)(1, -\hat{x}^i) \tag{4.3.60}$$

Dotting with the n_μ produces a rotational scalar and the average is a rotational scalar too. Then the angular integration with $\hat{x}_j$ vanishes, in all three cases. Hence, the momentum flux is zero across the three classes of hypersurfaces.

Angular Momentum fluxes: Replacing $\partial_r \chi^{mn}_{tt} \approx -\partial_\eta \chi^{mn}_{tt}$, we can write the angular momentum current as,

$$J^\mu_{L_j} \;=\; -\frac{a^{-2}}{16\pi}\left[\frac{2}{r}\frac{\eta}{\eta - r}\right]\left[\epsilon_{jmn}\hat{x}^m \hat{x}^k \langle Q^{nl}_{tt}\, \chi_{kl}\rangle\right](1, \hat{x}^i)$$

The fluxes then take the form,

$$\Sigma_\rho \;:\; -\frac{\rho}{8\pi}\int_{-\infty}^{\infty} d\tau \int_{S^2} d^2 s \left[\epsilon_{jmn}\hat{x}^m \hat{x}^k \langle Q^{nl}_{tt}\, \chi_{kl}\rangle\right] \tag{4.3.61}$$

$$\Sigma_{\eta_0} \;:\; \frac{1}{8\pi H^2 |\eta_0|}\int_0^{\infty} dr \frac{r}{r - \eta_0}\int_{S^2} d^2 s \left[\epsilon_{jmn}\hat{x}^m \hat{x}^k \langle Q^{nl}_{tt}\, \chi_{kl}\rangle\right] \tag{4.3.62}$$

$$\Sigma_{(\epsilon,\sigma)} \;:\; -\epsilon\frac{1+\epsilon}{8\pi}\int_{\lambda_1}^{\lambda^2} d\lambda(-\gamma)\frac{r\,\eta}{\eta - \;r}\int_{S^2} d^2 s \left[\epsilon_{jmn}\hat{x}^m \hat{x}^k \langle Q^{nl}_{tt}\, \chi_{kl}\rangle\right] \tag{4.3.63}$$

Consider the average. The function enclosed in averaging is product of the $\Lambda-$projector containing angular dependence and a function having dependence on (η, r). The averaging can then be split into averaging over a cell $\Delta\omega$ in the angular coordinates around the direction $\hat{r}$ and averaging over a cell in the (η, r) plane, see equation (D.0.9). Thus, we write,

$$\langle Q^{nl}_{tt}\,\chi_{kl}\rangle(\eta, r, \hat{r}) \;=\; \left[\frac{1}{\Delta\omega}\int_{\Delta\omega} d^2s'\Lambda^{nl}{}_{rs}(\hat{r}')\right]\left[\langle Q^{rs}\chi_{kl}\rangle(\eta, r)\right] \tag{4.3.64}$$

$$= \Lambda^{nl}{}_{rs}(\hat{r})\langle Q^{rs}\chi_{kl}\rangle(\eta, r) \tag{4.3.65}$$

The angular integration over the sphere can be done explicitly:

$$\int_{S^2} d^2s\,\epsilon_{jmn}\hat{x}^m\hat{x}^k\Lambda^{nl}{}_{rs}(\hat{r})\langle Q^{rs}\chi_{kl}(\eta, r)\rangle \;=\; \frac{8\pi}{15}\epsilon_{jmn}\langle Q^{nl}{\chi_l}^m\rangle(\eta, r). \tag{4.3.66}$$

This is to be integrated over the Killing parameter τ or r or λ for the three classes of hypersurfaces. The average is now over an (η, r) cell.

This integration in the flux expressions above, can be expressed in terms of the Killing parameter τ and then they all take the same form *provided* for Σ_{η_0} we consider the $\eta_0 \approx 0 \to \mathcal{J}^+$ and for the null hypersurface we choose the cosmological horizon, $\mathcal{H}^+$ $(\epsilon = +1, \eta = -r)$:

$$(\text{Angular Momentum Flux})_j \;=\; -\frac{1}{15}\int_{-\infty}^{\infty} d\tau\; a(\eta(\tau))\; r(\tau)\epsilon_{jmn}\langle Q^{nl}{\chi_l}^m\rangle \tag{4.3.67}$$

The radiation field Q^{nl} is given in equation (4.3.12) but without the tt label and,

$$\begin{aligned}\chi_{lm} \;=\; & \frac{2}{ra(\eta)}\left[\ddot{\bar{Q}}_{lm} + 2H\dot{\bar{Q}}_{lm} + H\dot{\bar{Q}}_{lm} + 2H^2\bar{Q}_{lm}\right](\eta_{ret}) \\ & +2H^2\left[\dot{\bar{Q}}_{lm} + H\bar{Q}_{lm}\right](\eta_{ret}) - 2H^2\left[\dot{\bar{Q}}_{lm} + H\bar{Q}_{lm}\right](-\infty)\,.\end{aligned} \tag{4.3.68}$$

This flux does *not* have a finite limit to $\mathcal{J}^+$ due to the *tail term* in ${\chi_l}^m$ and does *not* match with the flux given by [8]. It is finite along the $\mathcal{H}^+$ though. It does *not* match with the correct angular momentum flux in the flat space limit as well and it is well known [42,46] that the Isaacson effective stress tensor does not suffice to capture the flux of angular momentum. The sharp propagation property still holds in the sense that the flux across out-going null hypersurface is zero.

4.3.5 Extending from 'tt' to 'TT'

We have used the algebraic 'tt' projection on the approximated, retarded solution. How would the results change if we were to use the 'TT' decomposition of the exact solution prior to the $|\vec{x}'|/|\vec{x}| \ll 1$ approximation? For this we note a few points.

It is easy to see that the TT part of the retarded solution is given by [8],

$$\begin{aligned} X_{ij}^{TT}(\eta, x) &= 4\int d^3x' \frac{\eta}{|x-x'|(\eta-|x-x'|)} T_{ij}^{TT'}(\eta', x')\Big|_{\eta'=\eta-|x-x'|} \\ &\quad + 4\int d^3x' \int_{-\infty}^{\eta-|x-x'|} d\eta' \frac{T_{ij}^{TT'}(\eta', x')}{\eta'^2} \end{aligned} \tag{4.3.69}$$

where the TT' refers to the *second* argument of the stress tensor. This follows by checking that the divergence, ∂_x^i, of the right hand side converts into the divergence, $\partial_{x'}^i$ on the second argument of the stress tensor. For this relation, it is important to have the exact $|x - x'|$ dependence and that the source has compact support. This is not true for the approximated solution χ_{ij}^{TT}.

We can now consider the solution (4.3.69) for $|x| \gg |x'|$, and replace $|x - x'| \approx r$ which simplifies the source integral. We denote this approximated expression as χ_{ij}^{TT}. This satisfies the transversality condition to $o(r^{-1})$ only[3]. Furthermore, since the transverse, traceless part of the stress tensor *drops out of its conservation equation*, we cannot directly express $\int_{source} T_{ij}^{TT'}$ in terms of correspondingly defined moments. Nevertheless, we do get,

$$\partial_\eta \chi_{ij}^{TT}(\eta, x) = 4\frac{\eta}{r(\eta-r)}\partial_\eta \mathcal{M}_{ij}^{TT}, \ \mathcal{M}_{ij}^{TT}(\eta - r) := \int d^3x' T_{ij}^{TT'}(\eta - r, x') \tag{4.3.70}$$

[3] Extracting the TT part and making the approximation for $|\vec{x}| \gg |\vec{x}'|$, do not commute i.e. $[(X_{ij})_{approx}]^{TT} \neq ([X_{ij}]^{TT})_{approx}$. This is so because the ∂^j of the l.h.s. is always zero by definition while that of the r.h.s. is non-zero in general.

$$\partial_m \chi^{TT}_{ij}(\eta, x) = 4\frac{\hat{x}_m}{r}\left(\frac{\eta}{\eta - r}\partial_r \mathcal{M}^{TT}_{ij} - \frac{\mathcal{M}^{TT}_{ij}}{r}\right) = -\hat{x}_m \partial_\eta \chi^{TT}_{ij} - 4\frac{\hat{x}_m}{r^2}\mathcal{M}^{TT}_{ij} \quad (4.3.71)$$

$$\therefore \partial_r \chi^{TT}_{ij} = -\partial_\eta \chi^{TT}_{ij} - 4\frac{\mathcal{M}^{TT}_{ij}}{r^2} \quad (4.3.72)$$

The equation (4.3.72) has the same form as eq.(4.2.12). The equation (4.3.70) has the same form as eq.(4.3.41) which introduced the radiation field Q^{tt}_{ij}. We can thus introduce a new 'radiation field', $\mathcal{R}^{TT}_{ij} := 2\partial_\eta \mathcal{M}^{TT}_{ij}$. With this, the form of the expressions for fluxes will remain the same with $Q^{tt}_{ij} \to \mathcal{R}^{TT}_{ij}$. Note that unlike Q^{tt}_{ij}, $\mathcal{R}^{TT}_{ij}$ does *not* have a simple relation to the source moments defined earlier. Nevertheless, it shares the important property with Q^{tt}_{ij}, namely, it too is a function of $\eta - r$ alone. This enables the space-time averaging to be reduced to averaging over ρ =constant hypersurfaces, as shown in eqn. (D.0.13).

$$\langle \partial_\eta \chi^{TT}_{mn} \partial_\eta \chi^{mn}_{TT} \rangle (t, r, \hat{r}) = 4\frac{a^2(\bar{t}_0)}{\rho_0^2}\langle \mathcal{R}^{TT}_{ij}\, \mathcal{R}^{ij}_{TT} \rangle (\bar{t}_0, \hat{r}) \quad (4.3.73)$$

In the next section we restrict to the energy fluxes and see two applications of the conservation equation and the sharp propagation property.

4.4 Implications of conservation equation and sharp propagation

In the previous subsection, we assembled fluxes through various hypersurfaces, all having the topology $\Delta \times S^2$. We considered Δ to be a finite interval and also the cases with $\Delta = \mathbb{R}$. The relevant hypersurfaces have $\rho =$ constant. In all cases, the energy flux integral had the form,

$$\mathcal{F}(a, b) := \int_a^b d\tau \int_{S^2} d^2 s \left[\frac{1}{8\pi}\right] \langle Q^{tt}_{mn} Q^{mn}_{tt} \rangle =: \int_a^b d\tau \langle F \rangle (\tau) \,. \quad (4.4.1)$$

As shown in the appendix, equations (D.0.12), the angular brackets denote averaging over $\tau-$intervals and a trivial averaging over the angular intervals. Since the angular average is trivial, we have taken the angular integration across the averaging and denoted the integration over the sphere by $\langle F\rangle(\tau)$. Using the mean value theorem, we write,

$$\int_a^b d\tau\langle F\rangle(\tau) = \langle F\rangle(c)(b-a) = \int_{c-\delta}^{c+\delta} d\tau\ F\ \frac{(b-a)}{2\delta} \quad , \quad c \in (a,b)\ . \tag{4.4.2}$$

Let us choose (a,b) to be an averaging interval i.e. $(b-a) = 2\delta$. Recall that the averaged quantities are slowly varying i.e. $\langle F\rangle$ is varying only over the scale $L \gg 2\delta$ and thus essentially constant over the averaging interval. Therefore, we can *choose* $c = (a+b)/2$ possibly making a small error. But then the right hand side of the last equality in the above equation becomes $\int_a^b d\tau F(\tau)$. In effect, for integral over an averaging interval, we can drop the angular brackets in equation (4.4.1).

For $a \ll 0, b \gg 0$, the τ integral can be replaced by a sum with each sub-interval, $[a_k, b_k]$ being an averaging interval. Using the above argument, we can write,

$$\mathcal{F}(a,b) \approx \sum_k \int_{a_k}^{b_k} d\tau \int_{S^2} d^2s \left[\frac{1}{8\pi}\right] Q^{tt}_{mn} Q^{mn}_{tt} \tag{4.4.3}$$

However, the averaging $\tau-$intervals cannot be made arbitrarily finer and the Riemann sum cannot be taken to the integral. Hence, flux integral over an averaged integrand matches with the flux integral over an *un*-averaged integrand only at a *coarse grained level*. The same arguments also hold for $Q^{tt}_{mn} \to \mathcal{R}^{TT}_{mn}$ and *then* the fluxes defined using the averaged stress tensor match with the expressions (4.3.14) at a *coarse grained level*.

By judicious choices of hypersurfaces comprising the boundary $\partial\mathcal{V}$ of a space-time region $\mathcal{V}$, we can relate different fluxes using the conservation equation (4.3.33). The sharp propagation of energy comes in very useful. We note two of its implications.

(1) The flux across two hypersurfaces Σ_{η_1} and Σ_{η_2} *cannot* be equal, see the right side figure

of (4.1).

Let $\eta_2 > \eta_1$. Let Σ_{η_1} meet the $r = 0$ line at A_1. Let the out-going null hypersurface through A_1 intersect the Σ_{η_2} in a S^2 at B_1 with the radial coordinate being r_1. The three hypersurfaces Σ_{η_1}, the out-going null hypersurface and the hypersurface Σ_{η_2} bounded by the sphere at B_1 enclose a space-time region, A_1BB_1. By the conservation equation (4.3.33), the sum of the fluxes through these bounding hypersurfaces must vanish. But the flux through the out-going null hypersurface vanishes as shown before. Hence the fluxes through Σ_{η_1} and the partial hypersurface Σ_{η_2} between B_1 and B, must be equal. However, this leaves the contribution of the flux through the 'remaining' portion of the Σ_{η_2} hypersurface between B_2 and B_1. Hence the result. Alternatively, one can also see this explicitly by writing the full flux through the two hypersurfaces using the expression given in equation (4.3.49) and matching the integrands along the out-going null hypersurface. Evidently, the full flux through $\Sigma_{\eta\neq 0}$ is also not equal to that through $\mathcal{J}^+$. Physically this is understandable since the hypersurface at a later value of η receives energy emitted *after* the earlier value of η. The null infinity of course records *all* the energy emitted by the source and so does the cosmological horizon. We also conclude that the total flux at $\mathcal{J}^+$ computed by Ashtekar et al, as given in eq.(4.3.14), matches (at coarse grained level) with that given in equation (4.3.49) (with $Q \to \mathcal{R}$) *only* for $\eta = 0$. Note that unlike the spatial slices Σ_η, all hypersurfaces $\Sigma_{\rho>0}$ intercept all the emitted energy.

(2) The sharp propagation of energy can also be used to infer the *instantaneous emitted power*. Consider two out-going null hypersurfaces intersecting the cosmological horizon in spheres with radii $r(\tau)$ and $r'(\tau')$. The same hypersurfaces intersect the null infinity at corresponding spheres at $R(\tau)$ and $R'(\tau')$, see the left side figure in (4.1). For $\tau' > \tau$, we have $r'(\tau') < r(\tau)$ and $R'(\tau') < R(\tau)$. By the conservation equation and sharp propagation, the flux integral over the portion bounded by the spheres R, R' on $\mathcal{J}^+$ and the flux integral over the portion bounded by the spheres $r(\tau), r'(\tau')$ on the $\mathcal{H}^+$, are equal. Taking $\tau' = \tau + \delta\tau$, the integral becomes $\delta\tau\times$ the integral over the sphere at $r(\tau)$. The *emitted power*

is then defined by dividing the flux integral by $\delta\tau$ and taking the limit. Thus we get the instantaneous power as:

$$\mathcal{P}(\tau) \;:=\; \lim_{\delta\tau\to 0}\frac{\mathcal{F}(\tau+\delta\tau,\tau)}{\delta\tau} \;=\; \frac{1}{8\pi}\int_{S^2} d^2s\langle\mathcal{R}^{TT}_{ij}\mathcal{R}^{ij}_{TT}\rangle\,. \tag{4.4.4}$$

This is manifestly positive.

This is very similar to the definition given by Ashtekar et al [8] in the form of equation (4.3.16) *except that* the integrand is an average over τ and angular windows. The power is usually averaged over a few periods. If this is done to the power expression in [8], it will match with the above expression, again at a coarse grained level.

The upshot is that the quadrupole power is gauge invariant and can be computed at the cosmological horizon.

4.5 Discussion and Summary

We have dealt with two aspects namely the role of the cosmological horizon and the use of ripple stress tensor to compute energy flux of gravitational waves emitted by rapidly changing, distant sources.

A question regarding the validity of the 'short wavelength approximation' near $\mathcal{J}^+$ arises due to the understanding that the physical wavelength will diverge near the future null infinity thanks to the scale factor $a(t)$ Let us recall that background plus ripple decomposition is based on the expectation: $\partial_\alpha\bar{g}_{\mu\nu}\sim\bar{g}_{\mu\nu}/L$ and $\partial_\alpha h_{\mu\nu}\sim h_{\mu\nu}/\lambda$. In the cosmological chart, the non-zero coordinate derivatives of the background are: $\partial_t\bar{g}_{ij} = 2H\bar{g}_{ij}\sim\bar{g}_{ij}/L$. For the retarded solution we have,

$$\frac{\partial_t h_{ij}}{h_{ij}} \;=\; \partial_t[\ell n(a^2(t)\chi_{ij})] = 2H + \partial_t\eta\partial_\eta\ell n(\chi_{ij}) = 2H + \frac{\partial_\eta\ell n(\chi_{ij})}{a(t)} \sim \frac{1}{L}+\frac{1}{a(t)\lambda}\,, \tag{4.5.1}$$

$$\frac{\partial_k h_{ij}}{h_{ij}} \;=\; \partial_k\ell n(\chi_{ij}) = \hat{r}_k\partial_r\ell n(\chi_{ij}) \approx -\hat{r}_k\partial_\eta\ell n(\chi_{ij}) \sim \frac{\hat{r}_k}{\lambda} \tag{4.5.2}$$

The first equation shows that the $t-$derivative of the perturbation does *not* satisfy the premise, near $\mathcal{J}^+$ thanks to the presence of the scale factor. The second equation however does *not* have the scale factor and the ripple indeed has short scale of spatial variation. Interestingly, in the calculation of the fluxes, spatial components of the ripple stress tensor (and hence the spatial derivatives of the perturbation) do contribute since all Killing vectors are space-like near $\mathcal{J}^+$ and the 'short wavelength approximation' can justifiably be used.

As noted in the introduction, the cosmological horizon is unambiguously defined for a spatially compact source. This follows because worldlines with finite physical separation at every η must converge to i^+, the point A of figure (2.3). If Δ denotes the physical radial distance corresponding to the radial coordinate difference δ, then $\Delta^2 = \frac{\delta^2}{H^2\eta^2}$. To maintain Δ^2 to be finite as $\eta \to 0_-$, we must have $\delta^2 \sim \alpha^2\eta^2 + 0(|\eta|^3)$ near i^+. This identifies δ with $-\alpha\eta$ or $\alpha = H\rho$. Thus, the worldlines approach i^+ along the ρ =constant hypersurfaces. The cosmological horizon is then the past lightcone of i^+. The same argument also shows that any observer, who remains at finite physical distance away from the source must remain confined within the cosmological horizon. Furthermore, neither any such observer, nor the source has any access to energy/momentum which has crossed the horizon. This statement holds equally well for non-sharp propagation also (see figure 4.2). Hence cosmological horizon does share physically relevant properties with the future infinity.

Further support for the role of cosmological horizon as future null infinity comes from the computations of the energy momentum fluxes. For these, we employed the effective ripple stress tensor and showed that the fluxes defined at $\mathcal{J}^+$ by more geometric methods, also matched (at a coarse grained level) with those computed at the horizon. In particular the quadrupole power can be evaluated at the horizon. In full, non-linear general relativity, $\mathcal{H}^+$ may well serve as an *effective* $\mathcal{J}^+$ to analyze gravitational waves. Cosmological horizon being null, one can define Bondi-like charges and fluxes across $\mathcal{H}^+$ with an ex-

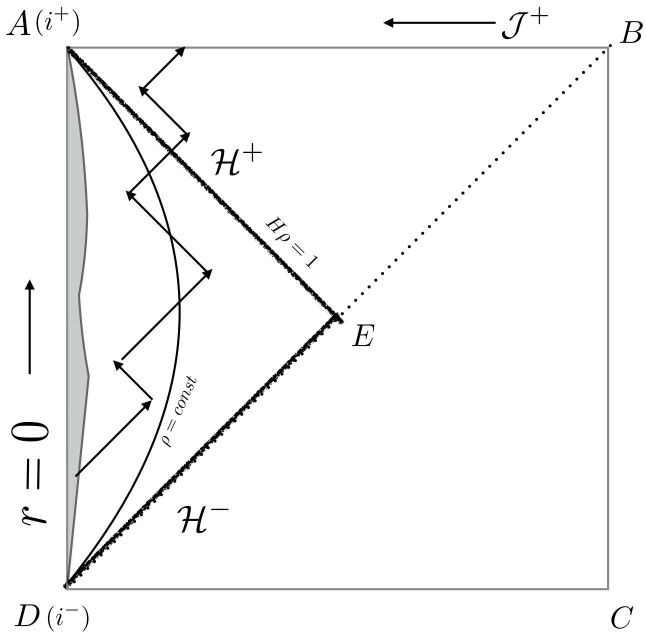

Figure 4.2. Gray region denotes compact source with future and past time-like infinity $i^{\pm}$ confined near $r \to 0$. A compact source has distinguished cosmological horizon $\mathcal{H}^+$, namely past lightcone of i^+. $\mathcal{H}^-$ denotes particle horizon of observer. For sharp propagation, energy flux across large enough $\rho = const.$ surface is independent of ρ. Hence in the context of sharp propagation of energy, large $\rho = const.$ surfaces are indistinguishable. But in the non-linear case, when the propagation is non-sharp, cosmological horizon ($\rho = H^{-1}$) is well distinguished from other $\rho = const$ surfaces (less than H^{-1}). For non-sharp propagation also once radiation crossed the cosmological horizon, it can never come back, it will propagate all the way upto $\mathcal{J}^+$. But for other $\rho = const$ surfaces, once radiation has crossed the surface, it can also come back at late time.

pectation to get balance law. The observation that the cosmological horizon is a Killing horizon and hence an isolated horizon should be helpful in this regard.

One of the concerns of space-like $\mathcal{J}^+$ of de Sitter is positivity of energy. Killing vector being space-like near $\mathcal{J}^+$ energy flux can have either sign across $\mathcal{J}^+$. In our computation we assume that compact source is well inside the cosmological horizon where the killing vector is time- like, hence the energy contribution from the radiating source is strictly positive. Using retarded solution of the field we also discard any negative contribution of energy across DE (see figure 4.2). Hence for spatially compact sources in linearized theory energy contribution is positive, as reflected in energy flux computations across different hypersurfaces.

The ripple stress tensor, although limited to short wavelength regimes (which covers most common sources), provides a convenient picture of energy momentum flows much like those for matter. There is a shortcoming of the ripple stress tensor - it does not capture the angular momentum flux correctly. A clearer understanding of this failure is lacking at present.

5 Observable effect of cosmological constant

In this chapter we illuminate some possible observable effects of cosmological constant due to new quadrupole formula derived in previous chapter. It is interesting to ask does this new quadrupole formula introduce any observable effect in orbital decay of binary system or does it give any significant impact on orbital phase evolution of inspiraling chirp signal which is now-a-days routinely used to detect gravitational wave? Let us begin with the linearized field expression (in coordinate indices) (3.3.90) ,

$$\begin{aligned}\chi_{ij}(t,r) \approx \frac{2}{r\,a(t)}\left[\partial_{t'}^2 Q_{ij} - 2H\partial_{t'}Q_{ij} + H\partial_{t'}\bar{Q}_{ij}\right]\Big|_{t_{\rm ret}} + 2H^2\left\{\partial_{t'}Q_{ij} - 2HQ_{ij} + H\bar{Q}_{ij}\right\}\Big|_{t_{\rm ret}} \\ -2H^2\left\{\partial_{t'}Q_{ij} - 2HQ_{ij} + H\bar{Q}_{ij}\right\}\Big|_{-\infty}\end{aligned} \tag{5.0.1}$$

It should be noted that the first term is due to sharp propagation and the last two terms are due to tail contribution. On $\mathcal{J}^+(\eta = 0)$ sharp term vanishes, only tail term contributes. The assumption of staticity $\mathcal{L}_T T_{\mu\nu} = 0$, at distant past implies $(\mathcal{L}_T + 2H)\, Q_{ij}\big|_{t=-\infty} = 0 = \partial_t Q_{ij}\big|_{t=-\infty}$. Hence on $\mathcal{J}^+$ filed takes the form,

$$\chi_{ij}\,\Big|_{\mathcal{J}^+} \approx 2H^2\left\{\partial_{t'}Q_{ij} - 2HQ_{ij} + H\bar{Q}_{ij}\right\}\Big|_{t_{\rm ret}} + 4H^3(Q_{ij} - \bar{Q}_{ij})\,\Big|_{t=-\infty} \tag{5.0.2}$$

Therefore no matter how small Λ is the effect of tail term is crucial at late time. The presence of tail term induces 'memory effect' to gravitational wave detectors. When a GW without memory passes through a detector, it causes oscillatory deformations but eventually returns the detector to its initial state. After a GW with memory has passed through an idealized detector (one that is truly freely-falling), it causes a permanent deformation - leaving a 'memory' of the waves to the detector. It has been argued in different literatures that low frequency band detectors, i.e. LISA, Pulsar Timing Array could be sensitive to memory effect [47, 48]. Recently there are several studies on gravitational wave memory on de Sitter space time also [49, 50]. Correction term due presence of cosmological constant in memory effect is needed to be investigated.

For a typical rapidly varying compact source second derivative term of mass quadrupole moment is the leading term. The first derivative and non-derivative terms are down by λ/L_B and $(\lambda/L_B)^2$ respectively. For current ground based interferometric detector $\lambda \sim 10^3 - 10^4$ meters and background scale is fixed by cosmological constant, $L_B \sim 1/\sqrt{\Lambda} \sim 10^{26}$ meters. Hence first derivative term is down by $10^{-22} - 10^{-23}$ to the leading term. To the leading order field can be written as,

$$\chi_{ij}(t,r) \approx \frac{2}{ra(t)}\partial_t^2 Q_{ij} \approx \frac{2}{r_{ph}}\partial_t^2 Q_{ij} \tag{5.0.3}$$

This is exactly the Minkowski answer with r replaced by physical radial distance r_{ph}. Hence in the field expression for a rapidly varying source, presence of Λ is manifested only in the scale factor which changes r to r_{ph}. As we discussed earlier we do not measure field directly. For direct observation tidal distortion is an observable quantity while for indirect detection, change in radius and eccentricity of a binary system due to gravitational wave emission is crucial. In the next section we will discuss about orbital decay of a circular binary orbit due to new quadrupole formula.

5.1 Quadrupole power radiation from inspiraling binary in presence of cosmological constant

At first we would like to compute power radiated by an inspiralling binary system due to new quadrupole formula,

$$\mathcal{P}(\tau) = \frac{1}{8\pi} \int_{S^2} d^2S \, \langle \mathcal{Q}^{tt}_{ij} \, \mathcal{Q}^{ij}_{tt} \rangle \tag{5.1.1}$$

where $\mathcal{Q}_{ij} := \left[\dddot{Q}_{ij} + 3H\ddot{Q}_{ij} + 2H^2\dot{Q}_{ij} + H\ddot{\bar{Q}}_{ij} + 3H^2\dot{\bar{Q}}_{ij} + 2H^3\bar{Q}_{ij}\right](t_{ret})$. These dots denote lie derivative with respect to killing vector, $\mathcal{L}_T$. We would like to express this quantity in proper time coordinate t, of matter source . Hence using $\mathcal{L}_T := \partial_t - 2H$ we can express $\mathcal{Q}_{ij}$ as ,

$$\mathcal{Q}_{ij} = \left[\partial_t^3 Q_{ij} - 3H\partial_t^2 Q_{ij} + 2H^2\partial_t Q_{ij} + H\partial_t^2\bar{Q}_{ij} - H^2\partial_t\bar{Q}_{ij}\right](t_{ret}). \tag{5.1.2}$$

Now using Λ projection, $\mathcal{Q}^{tt}_{ij}\mathcal{Q}^{ij}_{tt} := \Lambda^{kl}_{ij}\mathcal{Q}_{kl}\Lambda^{ij}_{mn}\mathcal{Q}^{mn} = \Lambda^{kl}_{mn}\mathcal{Q}_{kl}\mathcal{Q}^{mn}$, eq. (5.1.1) can be written as,

$$\mathcal{P}(t) = \frac{1}{5}\mathcal{Q}_{ij}\mathcal{Q}^{ij} - \frac{1}{15}\mathcal{Q}^2, \tag{5.1.3}$$

where $\mathcal{Q} := \delta^{ij}\mathcal{Q}_{ij}$. In deriving this expression we have used the identity for Λ projector, $\int d^2S\,\Lambda^{ij}_{kl} = \frac{2\pi}{15}\left[11\delta^i_k\delta^j_l - 4\delta^{ij}_{kl} + \delta^i_l\delta^j_k\right]$. For a weakly stressed system, as for Newtonian fluids, pressure π can be neglected compared to the energy density ρ, so we can neglect the pressure quadrupole moment terms in radiated power formula [8].

5.1.1 Mass quadrupole moment of binary in a circular orbit and orbital decay

We will assume that a binary system is separated at r_{ph} = *constant* (r = *const*. surface does not have compact support in de Sitter background). As source moment is defined in tetrad variable, we will attach the tetrad system of conformal chart to the centre of mass of the system and set $r_{ph} = 0$ to the centre of mass coordinate. Therefore in the centre of mass frame of binary, this system is equivalent to an effective one body problem with reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ following r_{ph} = *const*. trajectory. As near the matter source we can neglect the effect of cosmological constant, we will further make the assumption that the conservative dynamics of binary is completely governed by Newtonian potential (presently we focus on special case of circular orbit). It should be noted that moments are defined in the tetrad frame (3.3.77) and in terms coordinates it measures physical distances. We choose the time direction of tetrad frame along r_{ph} = *const* surface and a triad frame $(\bar{x}, \bar{y}, \bar{z})$ is attached such that the orbit is restricted to $(\bar{x}, \bar{y})$ and is given by,

$$\bar{x}(t) = r_{ph}\ \cos(\omega_s t + \frac{\pi}{2})\ ,\quad \bar{y}(t) = r_{ph}\ \sin(\omega_s t + \frac{\pi}{2})\ ,\quad \bar{z}(t) = 0 \tag{5.1.4}$$

Hence the mass quadrupole moment of the binary system can be written in a matrixform [51],

$$Q^{ij} = \mu\,\frac{r_{ph}^2}{2}\begin{bmatrix} 1-\cos(2\omega_s t) & -\sin 2\omega_s t & 0 \\ -\sin 2\omega_s t & 1+\cos(2\omega_s t) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence neglecting the pressure quadrupole moment terms, power radiated by binary system can be expressed as,

$$\mathcal{P}(t) = \frac{1}{5}Q_{ij}Q^{ij} - \frac{1}{15}Q^2 \tag{5.1.5}$$

$$\approx \frac{32\mu^2\, r_{ph}^4\, \omega_s^2}{5}\Big(\omega_s^4 + \frac{5}{4}H^2\omega_s^2 + \frac{H^4}{4}\Big) \tag{5.1.6}$$

It should be noted that trace of $Q_{ij} = 0$ and contribution from odd power of H exactly cancels. For circular orbit substituting $\omega_s = \sqrt{\frac{m_1+m_2}{r_{ph}^3}}$. Hence power loss due to gravitational radiation from a circular binary orbit can be expressed as,

$$\mathcal{P} \approx \frac{32}{5}\frac{\mu^2 G^4\,(m_1+m_2)^3}{c^5 r_{ph}^5}\left[1+\frac{5}{4}\frac{H^2 r_{ph}^3}{G(m_1+m_2)}+\frac{1}{4}\frac{H^4 r_{ph}^6}{G^2(m_1+m_2)^2}\right] \tag{5.1.7}$$

For $\Lambda \to 0$, this formula reduces to the classical formula for quadrupolar power radiation of a circular orbit [52]. In Newtonian approximation energy of circular orbit can be written as,

$$E_{orbit} = -\frac{G(m_1+m_2)\,\mu}{2r_{ph}} \tag{5.1.8}$$

$$\implies \frac{dr_{ph}}{dt} = \frac{2r_{ph}^2}{G\,(m_1+m_2)\,\mu}\frac{dE_{orbit}}{dt} \tag{5.1.9}$$

$$= -\frac{64}{5}\frac{G^3\,\mu\,(m_1+m_2)^2}{c^5\,r_{ph}^3}\left[1+\frac{5}{4}\frac{H^2 r_{ph}^3}{G(m_1+m_2)}+\frac{1}{4}\frac{H^4 r_{ph}^6}{G^2(m_1+m_2)^2}\right] \tag{5.1.10}$$

In the last line we have assumed that the binary system is loosing energy, entirely due to gravitational quadrupolar radiation. It should be noted that correction terms due to Λ in quadrupolar power radiation and orbital decay rate of circular orbit have the expansion parameter $\frac{H^2 r_{ph}^3}{G(m_1+m_2)}$. As mentioned in [53], a compact binary that coalesces after passing through the last stable orbit is a powerful source of gravitational waves, we assume $r_{ph} = r_{LSO} = 3R_S$. For simplicity assume $m_1 = m_2 = 1M_\odot$ and using Schwarzschild radius of sun, $R_S = 2GM_\odot = 3\times 10^3 meters$,

$$\frac{H^2 r_{ph}^3}{G(m_1+m_2)} = \frac{H^2 R_S^3}{R_S}\times 27 \approx H^2 R_S^2 \times 10 \approx 10^{-52}\times 10^6 \times 10 \approx 10^{-45} \tag{5.1.11}$$

This can also be seen from eqn. (5.1.6) - correction terms are down by H^2/ω_s^2, i.e. for a typical rapidly varying source, $(\lambda/L_B)^2 \sim 10^{-44} - 10^{-46}$. These corrections are too small to generate any significant effect in indirect observation, namely orbital decay of binary system. The observation of orbital decay of the Hulse-Taylor binary pulsar has established that the decay is fully consistent with the energy radiated as per the quadrupole formula in Minkowski background. Current accuracy level of this observation is at 10^{-3}. Hence correction terms due to Λ are utterly negligible in this context. From dimensional analysis one may argue that correction terms due to Λ should be $\sqrt{\Lambda} \times lengthscale$. Naturally two length scales are available - one is observational distance and another is source dimension (or equivalently background and wavelength of ripple). It should be noted that though linearized field expression depends on observational length scale, energy flux is independent of obsertvational length scale. Hence only available length scale is orbital length scale which enters into the expression via definition of source quadrupole moments. Therefore all the non trivialities of de Sitter space is harmless in orbital decay of binary system.

6 Conclusion and Outlook

Gravitational waves in full non-linear theory are well-understood for asymptotically flat space-time. Asymptotic flatness presumes a vanishing cosmological constant. However the concordance model of cosmology favors a universe with a tiny positive cosmological constant. Even a tiniest value of positive cosmological constant profoundly alters the asymptotic structure of space-time, forcing a relook at a theory of gravitational radiation. Still there is no satisfactory framework to describe gravitational wave in full non-linear regime for $\Lambda > 0$. The thesis poses the problem of implications of cosmological constant for gravitational waves in a linearized context. It is relevant for both observational and conceptual perspectives and the focus is on the former. In first part of the thesis, chapter 3, we obtained linearized inhomogeneous field solution in terms of source moments. Many non-trivialities regarding solution of non-homogeneous wave equation, e.g. definition of moments, relation of moments with source integral were discussed. Using Hadamard construction of retarded Green's function we expressed the field as an integral over source term. Relating source integral with moments is also non trivial. To get coordinate invariance, we defined moments with respect to a suitably chosen tetrad frame and recasted the conservation equations in that frame. Employing suitably chosen FNC in the static patch of de Sitter background we computed the field in terms of moments to first order in Λ. Though FNC is a natural choice for curved background, it can not be extended beyond cosmological horizon (i.e. beyond static patch). To study the behavior of field near future null infinity ($\mathcal{J}^+$) and for cosmological context we also presented the field in the Poincaré patch where the leading correction is of order $\sqrt{\Lambda}$. For comparison a gauge invariant quantity, deviation scalar was introduced and computed in both charts. In chapter 4, we explored energy flux associated with linearized gravitational field in de Sitter background. Using Isaacson's stress tensor in the special context of two widely separated scales between background and ripple (rapidly varying compact source), we obtained power radiated quadrupole formula for $\Lambda > 0$. An explicit averaging process allows to compare the energy flux with that of given by Ashtekar et. al in covariant phase space formalism [8]. It is shown that though the field has a tail term, energy propagation is sharp. Sharp propagation of energy and energy conservation are manifested in defining quadrupole power at cosmological horizon. Hence in the context of energy propagation cosmological horizon may serve as an effective boundary of de Sitter. In chapter 5 we discussed about some observable effects of cosmological constant and concluded that correction term due to cosmological constant in orbital decay of a binary system is negligible.

There are several open directions to pursue:

As mentioned earlier concept of radiation field is not well defined, when $\mathscr{J}^+$ is non-null. In de Sitter case cosmological horizon is well defined for compact source. Cosmological horizon being a null hypersurface, we do have the Weyl scalar Ψ_4 *unambiguously* identified on it. One of the main results of this thesis is that in the context of energy-momentum flux radiated by spatially compact source in de Sitter background, cosmological horizon can be treated as effective null infinity. Could cosmological horizon be used as an effective $\mathscr{J}^+$ to analyze gravitational waves in full non-linear theory? One can also extend Bondi's nonlinear analysis of axis symmetric gravitational wave with an expectation to get analogous 'mass' and 'news function'. Though some of the recent works have been done in this direction [54, 55], still it needs to be explored.

In chapter 3, we showed that leading order correction of field in Poincaré patch is in $\sqrt{\Lambda}$ while in FNC it is of order Λ. We suspect $\sqrt{\Lambda}$ behavior may be an artifact of conformal compactification of de Sitter metric in Poincaré patch. FNC calculation, ignores the global structure of space-time and the answer has a form of perturbation expansion around Minkowski background. One can use conformal FNC [9] to extend the FNC solution beyond cosmological horizon, which may explain the $\sqrt{\Lambda}$ correction. Exactly how this happens remains to be understood.

There are good reasons to expect that the Universe is permeated by a stochastic background of gravitational waves generated in the early Universe (a stochastic background can also emerge from the incoherent superposition of a large number of astrophysical sources). Our early Universe is modelled faithfully by FLRW background. One can study linearized gravitational wave to a FLRW background in presence of positive cosmological constant. Qualitatively our analysis is different due to non compactness of matter sources in FLRW background. One can also consider compact sources in FLRW background.

There are two types of observations which can be impacted by the modified quadrupole formula. One is the orbital decay of an inspiraling binary and other is the modification of the waveform at the detector. Hulse-Taylor pulsar observations have already vindicated the quadrupole formula computed in Minkowski background at the accuracy of 10^{-3} and we already discussed in chapter 5, the lowest order correction term due to Λ is of the order of 10^{-45}. Hence impact of Λ in orbital decay is negligible in the context of current accuracy of observations. To extract the signal of gravitational wave from the noise of detector we need the waveform to a very high PN order. For number of cycles spent in the bandwidth of ground-based detector to be $O(1)$, we need at least 2.5 PN of phase calculation. A more accurate computation is really required in order to exploit optimally the information contained in the output of a ground based interferometer, at least upto 3 PN and better yet to 3.5 PN. One of the key ingredients that feeds into phase computation is that power radiated by gravitational wave. Schematically,

$$\frac{d\phi}{dt} \approx \frac{d\phi}{dE}\frac{dE}{dt} \tag{6.0.1}$$

First term is computed from orbital parameter of the source, the second term is due to power radiated by gravitational wave. In the previous chapter we have seen Λ correction in qudrupolar radiation by a binary system is negligible. PN correction in orbital phasing formula is obtained in the expansion parameter of v/c, where v is the relative velocity of binary. For a typical astrophysical system $v/c \sim 0.2 - 0.3$. Hence we expect that the change in orbital phasing formula due to Λ will be insignificant in comparision with higher order PN correction.

Appendices

A Triangle law for World function

We sketch the steps that go in the computation of the world function between the observation event and a source event, $\sigma(P', P)$, given in the equation (3.3.24). In reference to the figure (3.5), we want to compute: $\phi := \frac{1}{3}\int_0^1 dv(1-v)^3 \frac{D^4\sigma(q',q)}{Dv^4}$ [34].

Let u denote the parameter along the geodesics connecting q', q as they vary along the geodesics P_0P' and P_0P. These geodesics are all parameterized such that they begin at $q'(u_1, v)$ and end at $q(u_2, v)$. In general $\sigma(q', q)$ is a function of u_1, u_2, v. But since u_1, u_2 are the same for all such pairs, we have $\sigma(q', q) = \sigma(v)$. Therefore,

$$\frac{D\sigma(v)}{Dv} = \frac{dx'^\alpha}{dv}\frac{\partial\sigma}{\partial x'^\alpha} + \frac{dx^\alpha}{dv}\frac{\partial\sigma}{\partial x^\alpha} := \sigma_{\alpha'}V^{\alpha'} + \sigma_\alpha V^\alpha\ . \tag{A.0.1}$$

The V's denote the tangent vectors at the respective end points while the prime on the component labels indicate which end point is implied. The suffix on the σ denote the covariant derivative at the corresponding point. Since σ is a (bi-)scalar, its covariant derivative equals the partial derivative.

The second and higher derivatives of σ with respect to v are computed similarly, noting that $\frac{DV^\alpha}{Dv} = \frac{DV^{\alpha'}}{Dv} = 0$ since $P_0 \to P', P_0 \to P$ are both geodesics and v is the affine parameter along them. We also note the property of the world function [34], $\sigma_{\alpha'\beta} = \sigma_{\beta\alpha'}$. This leads to,

$$\frac{D^2\sigma}{Dv^2} = \sigma_{\alpha'\beta'}V^{\alpha'}V^{\beta'} + \sigma_{\alpha\beta}V^\alpha V^\beta + 2\sigma_{\alpha'\beta}V^{\alpha'}V^\beta \tag{A.0.2}$$

$$\begin{aligned}\frac{D^4\sigma}{Dv^4} &= \sigma_{\alpha'\beta'\mu'\nu'}V^{\alpha'}V^{\beta'}V^{\mu'}V^{\nu'} + 4\sigma_{\alpha'\beta'\mu'\nu}V^{\alpha'}V^{\beta'}V^{\mu'}V^{\nu} \\ &+6\sigma_{\alpha'\beta'\mu\nu}V^{\alpha'}V^{\beta'}V^{\mu}V^{\nu} + 4\sigma_{\alpha'\beta\mu\nu}V^{\alpha'}V^{\beta}V^{\mu}V^{\nu} \\ &+\sigma_{\alpha\beta\mu\nu}V^{\alpha}V^{\beta}V^{\mu}V^{\nu} \end{aligned} \tag{A.0.3}$$

We have not written the third derivative as we do not need it.

The desired world function is, using Taylor expansion with a remainder, about P_0 ($v = 0$),

$$\begin{aligned}\sigma(P', P) = \sigma(\bar{v}) &= \sigma(0) + \bar{v}\left.\frac{D\sigma}{Dv}\right|_0 + \frac{1}{2}\bar{v}^2\left.\frac{D^2\sigma}{Dv^2}\right|_0 + \frac{1}{6}\bar{v}^3\left.\frac{D^3\sigma}{Dv^3}\right|_0 \\ &+\frac{1}{6}\int_0^1 dv(1-v)^3\frac{D^4\sigma(q', q)}{Dv^4} \end{aligned} \tag{A.0.4}$$

It is known that $\sigma(0) = 0 = \frac{D\sigma}{Dv}(0) = \frac{D^3\sigma}{Dv^3}(0)$. The coincidence limits of the second derivatives of σ are given by, $[\sigma_{\alpha'\beta'}] = [\sigma_{\alpha\beta}] = g_{\alpha\beta}$ and $[\sigma_{\alpha'\beta}] = [\sigma_{\alpha\beta'}] = -g_{\alpha'\beta} = -g_{\alpha\beta'}$ and $\bar{v}V^{\alpha'} = -g^{\alpha'\beta'}\sigma_{\beta'}$ and $\bar{v}V^{\alpha} = +g^{\alpha\beta}\sigma_{\beta}$ [34]. This leads to,

$$\begin{aligned}\bar{v}^2\left.\frac{D\sigma}{Dv^2}\right|_0 &= g_{\alpha\beta}(\bar{v}V^{\alpha})(\bar{v}V^{\beta}) + g_{\alpha'\beta'}(\bar{v}V^{\alpha'})(\bar{v}V^{\beta'}) - 2g_{\alpha'\beta}(\bar{v}V^{\alpha'})(\bar{v}V^{\beta}) \\ &= g^{\alpha\beta}\sigma_{\alpha}\sigma_{\beta} + g^{\alpha'\beta'}\sigma_{\alpha'}\sigma_{\beta'} + 2g^{\alpha'\beta}\sigma_{\alpha'}\sigma_{\beta} \\ &= 2\sigma(P_0, P) + 2\sigma(P_0, P') - 2\sigma_{\alpha}(P_0, P')\sigma^{\alpha}(P_0, P) \end{aligned} \tag{A.0.5}$$

In the last line, we have used $2\sigma = g^{\alpha\beta}\sigma_{\alpha}\sigma_{\beta}$. Substituting in eqn. (A.0.4), we get,

$$\begin{aligned}\sigma(P', P) &= \sigma(P_0, P') + \sigma(P_0, P) - \left.\left(g^{\alpha\beta}\frac{\partial\sigma(y, P')}{\partial y^{\alpha}}\frac{\partial\sigma(y, P)}{\partial y^{\beta}}\right)\right|_{P_0} \\ &+\frac{1}{6}\int_0^1 dv(1-v)^3\frac{D^4\sigma(q', q)}{Dv^4} \end{aligned} \tag{A.0.6}$$

To compare with the triangle law, we denote, $\overrightarrow{PQ}^2 := 2\sigma(P, Q)$. Then the above equation can be written as,

$$\overrightarrow{P'P}^2 = \overrightarrow{P_0P'}^2 + \overrightarrow{P_0P}^2 - 2\overrightarrow{P_0P'}\cdot\overrightarrow{P_0P} + \phi \tag{A.0.7}$$

To evaluate ϕ, we need to evaluate the fourth order covariant derivatives of the world function. These are obtained in terms of the parallel propagator and integrals of curvature. To state the result, we introduce the notation:

$$\text{Parallel propagator:} \quad X^{\alpha}_{\parallel}(p) := g^{\alpha}{}_{\beta'}(p', p) X^{\beta'}_{\parallel}(p') \ \text{ where } \ V^{\gamma}\nabla_{\gamma}X^{\alpha}_{\parallel} = 0\,. \tag{A.0.8}$$

$$\text{Symmetrized Riemann:} \quad S_{\alpha\beta\mu\nu} := -\frac{1}{3}\left(R_{\alpha\mu\beta\nu} + R_{\alpha\nu\beta\mu}\right)\,. \tag{A.0.9}$$

The *parallel propagator*, $g^{\alpha}{}_{\beta'}(p', p)$ is a bi-tensor and its indices are raised/lowered by the metric at the respective points.

It is convenient to introduce a tetrad basis, $E^{\alpha}{}_{a}, E^{a}{}_{\alpha}$, at p' and define it at p by parallel transporting it along the geodesic from p' to p. The parallel propagator is then given by $g^{\alpha}{}_{\beta'}(p', p) = E^{\alpha}{}_{a}(p)E^{a}{}_{\beta'}(p')$. Denoting the components with respect to these parallelly transported tetrad by Latin indices, the second order covariant derivatives of the world function are given by (equation 97 of [34]),

$$\sigma_{a'b'}(q', q) = g_{a'b'}(q') + \frac{3}{2}\frac{1}{u_2 - u_1}\int_{u_1}^{u_2} du(u_2 - u)^2 S_{abcd}(u)U^c U^d(u) \tag{A.0.10}$$

$$\sigma_{a'b}(q', q) = g_{a'b}(q') + \frac{3}{2}\frac{1}{u_2 - u_1}\int_{u_1}^{u_2} du(u_2 - u)(u - u_1) S_{abcd}(u)U^c U^d(u) \tag{A.0.11}$$

$$\sigma_{ab}(q', q) = g_{ab}(q') + \frac{3}{2}\frac{1}{u_2 - u_1}\int_{u_1}^{u_2} du(u - u_1)^2 S_{abcd}(u)U^c U^d(u) \tag{A.0.12}$$

Note that the tetrad components of the parallel propagator are just η_{ab} while the tetrad components of the geodesic tangent vectors, U^a are constant along the geodesics and may be taken out of the integration. These expression have corrections at the second order in curvature.

The fourth covariant derivatives have a similar form but now involve covariant derivatives of the symmetrized Riemann tensor. In our context of maximally symmetric background, all these covariant derivatives of the Riemann tensor vanish and the expressions simplify drastically. In particular, the third covariant derivatives are all absent as they involve the covariant derivatives of the Riemann tensor and the index distribution also gets restricted thanks to the symmetries of the Riemann tensor. This leads to (equation 117 of [34]),

$$\sigma_{a'b'c'd'}(q',q) = \frac{3}{(u_2-u_1)^3}\int_{u_1}^{u_2} du(u_2-u)^2 S_{abcd}(u)\,, \tag{A.0.13}$$

$$\sigma_{a'b'c'd}(q',q) = -\frac{3}{(u_2-u_1)^3}\int_{u_1}^{u_2} du(u_2-u)^2 S_{abcd}(u)\,, \tag{A.0.14}$$

$$\sigma_{a'b'cd}(q',q) = \frac{3}{(u_2-u_1)^3}\int_{u_1}^{u_2} du(u_2-u)^2 S_{abcd}(u)\,. \tag{A.0.15}$$

These again have correction at the second order in curvature. Note that the tetrad components refer to the tetrad derived from an arbitrary choice at q', by parallel transport along the geodesic $q' \to q$.

In section 3.3, we choose a tetrad at the base point of the RNC, P_0 and set it up elsewhere by parallel transporting along the geodesics emanating from P_0. This gives the tetrad $E^{\alpha'}_{\ a'}$ at q'. However the tetrad at q, $E^{\alpha}_{\ a}$, is not equal to $\tilde{E}^{\alpha}_{\ a}$ - the one obtained from $E^{\alpha'}_{\ a'}$ by parallel transport along $q' \to q$ geodesic. They are related through the holonomy group element along the closed curve $q \to P_0 \to q' \to q$: $\tilde{E}^{a}_{\ \alpha} = H_{\alpha}^{\ \beta} E^{a}_{\ \beta}$. Because of the smallness of the curvature, $H_{\alpha}^{\ \beta}$ differs from the identity element by a term of order Λ. In short, the error committed in replacing the tetrad components of curvature relative to the $q' \to q$ parallelly transported tetrad, by those derived from tetrad at P_0, will be of second order in the curvature, i.e. order Λ^2.

With this understood, we regard all the tetrad components in the fourth covariant derivatives to be relative to the tetrad derived from P_0. The equation (136) of [34] then gives,

$$\phi = \phi_0 = \frac{3}{(u_2-u_1)^3}\int_0^1 dw(1-w)^3 \int_{u_1}^{u_2} du \left\{(u_2-u)^2 + (u-u_1)^2\right\} \times \left\{S_{a'b'cd}\bar{v}^4 V^{a'}V^{b'}V^{c}V^{d}\right\}(u,w). \tag{A.0.16}$$

The tetrad components of the symmetrized Riemann tensor simplify further thanks to the maximal symmetry.

$$S_{abcd}(u,w) = -\frac{1}{3}\left(R_{\alpha\mu\beta\nu} + R_{\alpha\nu\beta\mu}\right) E^{\alpha}_{\ a} E^{\beta}_{\ b} E^{\mu}_{\ c} E^{\nu}_{\ d}(u,w) \ . \tag{A.0.17}$$

$$= -\frac{\Lambda}{9}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\mu\beta} + g_{\alpha\beta}g_{\nu\mu} - g_{\alpha\mu}g_{\nu\beta}\right) E^{\alpha}_{\ a} E^{\beta}_{\ b} E^{\mu}_{\ c} E^{\nu}_{\ d}(u,w) \tag{A.0.18}$$

$$= -\frac{\Lambda}{9}\left[2(E_a \cdot E_b)(E_c \cdot E_d) - (E_a \cdot E_d)(E_b \cdot E_c) - (E_a \cdot E_c)(E_b \cdot E_d)\right]$$

$$= -\frac{\Lambda}{9}\left[2\eta_{ab}\eta_{cd} - \eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right] \because \text{(orthonormality of the tetrad.)} \tag{A.0.19}$$

Consequently, the symmetrized Riemann tensor comes out of the integrals. The vectors $V^a, V^{a'}$ are independent of u because they come from the expansion of $\sigma(v)$ and are independent of v since they are geodesic tangents and refer to the parallelly transported tetrad. The terms enclosed in the second pair of braces, come out of the integration and we get,

$$\phi = \frac{3}{(u_2 - u_1)^3}\left[\int_0^1 dw(1-w)^3 \int_{u_1}^{u_2} du\left\{(u_2 - u)^2 + (u - u_1)^2\right\}\right] \times \left\{S_{a'b'cd}\bar{v}^4 V^{a'} V^{b'} V^c V^d\right\} \tag{A.0.20}$$

$$= \left[\frac{1}{2}\right]\left\{S_{a'b'cd}X^{a'}X^{b'}X^cX^d\right\} \quad , \ \bar{v}V^* =: X^* (= \text{ corresponding RNC }) \tag{A.0.21}$$

$$= -\frac{\Lambda}{9}\left(X^2X'^2 - (X \cdot X')^2\right) \ . \tag{A.0.22}$$

Notice that the reference to the choice of the tetrad, $E^{\alpha}_{\ a}$ has disappeared.

B Calculation of the parallel propagator

In the main text we needed the parallel propagator $g^{\mu}{}_{\alpha'}(x, x')$ along the null geodesic from the observation point P to a source point P'. To this end, introduce an arbitrary tetrad $e^{\mu}_{a}(P)$ and its inverse co-tetrad $e^{a}_{\alpha}(P)$ which is parallel transported along the null geodesic. These will drop out at the end. The parallel propagator is then given by,

$$g^{\mu}{}_{\alpha}(x, x') = e^{\mu}{}_{a}(x) e^{a}{}_{\alpha}(x') .$$

The geodesic satisfies the equation,

$$\frac{d^2 x^{\mu}}{d\lambda} + \Gamma^{\mu}{}_{\alpha\beta}(x(\lambda)) \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda} = 0 \ ; \ x^{\mu}(0) = x^{\mu}(P) := \hat{x}^{\mu} \ , \ \dot{x}^{\mu}(0) = \hat{t}^{\mu} .$$

The parallel transported co-tetrad satisfies the equation,

$$\frac{de^{a}_{\alpha}}{d\lambda} - \Gamma^{\gamma}{}_{\alpha\beta} \frac{dx^{\beta}}{d\lambda} e^{a}_{\gamma} = 0 \ ; \ e^{a}{}_{\alpha}(0) = e^{a}{}_{\alpha}(P) := \hat{e}^{a}{}_{\alpha} .$$

These are solved by Taylor expanding in the affine parameter λ and determining the coefficients. Denoting the evaluations at $\lambda = 0$ by hatted quantities, we write,

$$e^{a}{}_{\alpha}(\lambda) = \hat{e}^{a}{}_{\alpha} + \lambda \dot{e}^{a}{}_{\alpha}(0) + \frac{\lambda^2}{2} \ddot{e}^{a}{}_{\alpha}(0) \cdots \tag{B.0.1}$$

$$x^\mu(\lambda) \;=\; \hat{x}^\mu + \lambda\hat{t}^\mu + \frac{\lambda^2}{2}\left(-\hat{\Gamma}^\mu{}_{\alpha\beta}\hat{t}^\alpha\hat{t}^\beta\right) + \frac{\lambda^3}{6}\left(-\partial_\gamma\hat{\Gamma}^\mu{}_{\alpha\beta}\hat{t}^\alpha\hat{t}^\beta\hat{t}^\gamma\right) + \cdots \qquad \text{(B.0.2)}$$

In the last equation we have used the geodesic equation. By differentiating the geodesic equation, the higher order terms in $x^\mu(\lambda)$ are determined. We note that the connection is order Λ and linear in coordinates. So more than the first derivative of the connection is not needed. In the Taylor expansion of x^μ, we have shown only the terms to order Λ. Substituting these expansions in the parallel transport equation, determines the solution as,

$$e^a{}_\alpha(\lambda) \;=\; \hat{e}^a{}_\mu\left[\delta^\mu_\alpha + (\lambda\hat{t}^\beta)\hat{\Gamma}^\mu{}_{\alpha\beta} + \frac{1}{2}(\lambda\hat{t}^\gamma)(\lambda\hat{t}^\beta)\partial_\gamma\hat{\Gamma}^\mu{}_{\alpha\beta}\right] \qquad \text{(B.0.3)}$$

From the Taylor expansions of x^μ and $e^a{}_\alpha$, we eliminate $\lambda\hat{t}$ and obtain the parallel tetrad in terms of the coordinates. To the linear order in Λ, this simply replaces $\lambda\hat{t}^\beta$ by $(x' - x)^\beta$. The parallel propagator is then given by,

$$g^\mu{}_{\alpha'}(P, P') \;=\; \hat{\delta}^\mu_{\alpha'} + \hat{\Gamma}^\mu{}_{\alpha'\beta'}(x' - x)^{\beta'} + \frac{1}{2}\partial_{\gamma'}\hat{\Gamma}^\mu{}_{\alpha'\beta'}(x' - x)^{\gamma'}(x' - x)^{\beta'} + o(\Lambda^2) \qquad \text{(B.0.4)}$$

We have used primed indices for notational consistency for bi-tensors. The hatted quantities are the coincidence limits. Notice that the arbitrary tetrad introduced at the beginning has disappeared. We have not used any specific property of the Fermi or Riemann normal coordinates, except for the order Λ.

Now we use Riemann-Christoffel connection in FNC, together with its derivative to compute parallel propagator. Noting that the FNC metric (3.3.33) is of the same form as the perturbation about flat metric, $g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$ with $\delta g_{00} = \Lambda s^2/3, \delta g_{0i} = 0, \delta g_{ij} = -\frac{\Lambda}{9}(\delta_{ij}\vec{\xi}^2 - \xi_i\xi_j)$, we obtain,

$$\Gamma^\mu{}_{\alpha\beta} \;=\; \frac{1}{2}\left(\partial_\alpha\delta g^\mu{}_\beta + \partial_\beta\delta g^\mu{}_\alpha - \partial^\mu\delta g_{\alpha\beta}\right) . \qquad \text{(B.0.5)}$$

Using,

$$\delta g^{\mu}{}_{\alpha} = -\frac{\Lambda s^2}{3}\delta^{\mu}_{0}\delta^{0}_{\alpha} - \frac{\Lambda}{9}(\delta^{\mu}_{i}\delta^{i}_{\alpha})(\xi^{j}\xi_{j}) + \frac{\Lambda}{9}(\delta^{\mu}_{i}\xi^{i})(\delta^{j}_{\alpha}\xi_{j}) ,$$

we get,

$$\begin{aligned} \Gamma^{\mu}{}_{\alpha\beta} = \frac{\Lambda}{18}\Big[&-6\left\{\delta^{\mu}_{0}(\delta^{0}_{\alpha}\delta^{i}_{\beta}\xi_{i} + \delta^{0}_{\beta}\delta^{i}_{\alpha}\xi_{i}) + \delta^{0}_{\alpha}\delta^{0}_{\beta}\delta^{\mu}_{i}\xi^{i}\right\} \\ &- 2\left\{\delta^{\mu}_{i}(\delta^{i}_{\alpha}\delta^{j}_{\beta}\xi_{j} + \delta^{i}_{\beta}\delta^{j}_{\alpha}\xi_{j}) - 2\delta^{i}_{\alpha}\delta^{i}_{\beta}\delta^{\mu}_{j}\xi^{j}\right\}\Big] \end{aligned} \tag{B.0.6}$$

We also obtain the derivative of the connection,

$$\begin{aligned} \partial_{\gamma}\Gamma^{\mu}{}_{\alpha\beta} = \frac{\Lambda}{18}\Big[&-6\left\{\delta^{\mu}_{0}(\delta^{0}_{\alpha}\delta^{i}_{\beta}\delta_{\gamma i} + \delta^{0}_{\beta}\delta^{i}_{\alpha}\delta_{\gamma i}) + \delta^{0}_{\alpha}\delta^{0}_{\beta}\delta^{\mu}_{i}\delta^{i}_{\gamma}\right\} \\ &- 2\left\{\delta^{\mu}_{i}(\delta^{i}_{\alpha}\delta^{j}_{\beta}\delta_{\gamma j} + \delta^{i}_{\beta}\delta^{j}_{\alpha}\delta_{\gamma j}) - 2\delta^{i}_{\alpha}\delta^{i}_{\beta}\delta^{\mu}_{j}\delta^{j}_{\gamma}\right\}\Big] \end{aligned} \tag{B.0.7}$$

Employing these the expression for parallel propagator (B.0.4) turns out to be,

$$\begin{aligned} g^{m}{}_{\alpha'}(x,x') = \delta^{m}{}_{\alpha'} - \frac{\Lambda}{18}\Big[&\delta^{m}{}_{\alpha'}\,(\vec{\Delta x}^{\,2} + 2\vec{\xi}\cdot\vec{\Delta x}) + \Delta x^{m}\,(3\delta^{0}{}_{\alpha'}\,\Delta x^{0} - \delta^{j}{}_{\alpha'}\Delta x_{j} + 2\delta^{j}{}_{\alpha'}\xi_{j}) \\ &+\xi^{m}(6\delta^{0}_{\alpha'}\Delta x^{0} - 4\delta^{j}{}_{\alpha'}\Delta x_{j})\Big] \end{aligned} \tag{B.0.8}$$

where $\vec{\Delta x} = \vec{\xi'} - \vec{\xi}$ and $\Delta x^0 = \xi'^0 - \xi^0 = \tau' - \tau \approx -s$. We need the parallel propagator at the retarded time and in the regime of $s \gg s'$. Keeping only the leading powers in s'/s, the expressions simplify and we obtain the parallel propagator as,

$$g^{m}{}_{\alpha'}(\tau,\vec{\xi},\tau'_{\text{ret}},\vec{\xi'}) \approx \delta^{m}{}_{\alpha'} + \frac{\Lambda s^2}{18}\left[\delta^{m}{}_{\alpha'} + 3\delta^{0}{}_{\alpha'}\frac{\xi^m}{s} - \delta^{j}{}_{\alpha'}\frac{\xi_j\xi^m}{s^2}\right] . \tag{B.0.9}$$

C FNC $\leftrightarrow$ Conformal chart transformations

We used two different charts in presenting the quadrupole field, the FNC restricted to the static patch and the conformal coordinates covering the Poincare patch which overlaps with the static patch. To relate these two sets of coordinates, (τ, ξ^i) and (η, x^i), consider the geodesic equation in the conformal coordinates. In conformal coordinates,

$$ds^2 = \frac{\alpha^2}{\eta^2}\left[-d\eta^2 + \sum_i (dx^i)^2\right] , \quad \alpha^2 = \frac{3}{\Lambda} ; \tag{C.0.1}$$

$$\Gamma^0_{\ 00} = -\frac{1}{\eta} , \quad \Gamma^0_{\ 0j} = 0 , \quad \Gamma^0_{\ ij} = -\frac{\delta_{ij}}{\eta} , \tag{C.0.2}$$

$$\Gamma^i_{\ 00} = 0 , \quad \Gamma^i_{\ 0j} = -\frac{1}{\eta}\delta^i_j , \quad \Gamma^k_{\ ij} = 0 . \tag{C.0.3}$$

The geodesic equation splits as,

$$0 = \frac{d^2\eta}{d\lambda^2} - \frac{1}{\eta}\left(\frac{d\eta}{d\lambda}\right)^2 - \frac{\delta_{ij}}{\eta}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}, \tag{C.0.4}$$

$$0 = \frac{d^2x^i}{d\lambda^2} - \frac{2}{\eta}\frac{d\eta}{d\lambda}\frac{dx^i}{d\lambda} ; \qquad \Rightarrow \frac{d\vec{x}}{d\lambda} = \eta^2\vec{C} ,$$

$$\therefore \vec{x}(\lambda) = \vec{C}\int_0^\lambda d\lambda'\eta^2(\lambda') + \vec{x}_0 \qquad \text{where } \vec{C} \text{ is a constant vector, and} \tag{C.0.5}$$

$$0 = \frac{d^2\eta}{d\lambda^2} - \frac{1}{\eta}\left(\frac{d\eta}{d\lambda}\right)^2 - \vec{C}^2\eta^3 . \tag{C.0.6}$$

The choice $\vec{x}_0 = \vec{0}$ corresponds to 'radial' geodesics.

To define FNC, we have to choose one time-like geodesic whose proper time provides the time coordinate, τ. We choose this to be the line AD in figure 3.1. This corresponds to the choice $\vec{x}_0 = 0$ and $\vec{C} = 0$. The η equation can be immediately integrated to give the *reference geodesic* as:

$$\eta_*(\tau) = -\sqrt{\frac{3}{\Lambda}} e^{-\tau\sqrt{\Lambda/3}} \quad , \quad \vec{x}_*(\tau) = \vec{0} \, . \tag{C.0.7}$$

For future convenience, we have chosen an integration constant to be $-\sqrt{3/\Lambda}$ while the integration constant in the exponent is determined by the proper time condition (norm = -1) which makes τ to be one of the FNC.

To determine ξ^i coordinates, we consider spatial geodesics, emanating orthogonally from the reference geodesic. Clearly, we consider a radial geodesic, $\vec{x}_0 = \vec{0}$ and defining $\vec{x}(\sigma) := \hat{C}r(\sigma)$ where $\hat{C} := \vec{C}/|\vec{C}|$. The geodesic is determined by solving the equation for $\eta(\sigma)$ with initial conditions reflecting the orthogonality, $-d_\tau\eta_*\, d_\sigma\eta + d_\tau r_*\, d_\sigma r = 0$,

$$d_\sigma^2\eta - \frac{(d_\sigma\eta)^2}{\eta} - \vec{C}^2\eta^3 = 0 \ , \ \eta(0) = \eta_*(\tau) \ , \ d_\sigma\eta(0) = 0 \ , \ r(0) = 0 \ , \ d_\sigma r(0) = \gamma \, . \tag{C.0.8}$$

Let P be the point with conformal coordinates (η_P, r_P) and FNC (τ, s). Taking the norm of the initial tangent vector to be s^2, the pairs of coordinates are related as:

$$\eta_P := \eta(\sigma = 1) \ , \ r_P := r(\sigma = 1) \ , \ s^2 = \frac{3}{\Lambda\eta^2(0)}\gamma^2.$$

Using the first integral of the r-equation, we get

$$d_\sigma r(0) = |\vec{C}|\eta^2(0) = \gamma = \sqrt{\frac{\Lambda}{3}}|\eta(0)|s = s e^{-\tau\sqrt{\Lambda/3}} \Rightarrow |\vec{C}| = s\frac{\Lambda}{3}e^{\tau\sqrt{\Lambda/3}} \, . \tag{C.0.9}$$

To obtain (η_P, r_P), we need to solve the $\eta-$equation.

For this, we first take out a scale ζ by defining $\eta(\sigma) := \zeta y(\sigma)$ which gives $y'' - y'^2/y -$

$|\vec{C}|^2\zeta^2 y^3 = 0$ and choosing $\zeta = \eta_*(\tau)$, we get $|\vec{C}|^2\zeta^2 = \Lambda s^2/3 =: \epsilon$. The desired coordinates are then given by,

$$r_P \ := \ r(\sigma = 1) \ = se^{-\tau\sqrt{\Lambda/3}} \int_0^1 d\sigma' y^2(\sigma') \tag{C.0.10}$$

$$\eta_P \ := \ \eta(\sigma = 1) \ = \ -\sqrt{\frac{3}{\Lambda}} e^{-\tau\sqrt{\Lambda/3}} y(\sigma = 1) \quad \text{with,} \tag{C.0.11}$$

$$0 \ = \ y'' - \frac{y'^2}{y} - \epsilon y^3 \quad , \quad y(0) = 1 \ , \ y'(0) \ = \ 0 \quad , \quad \epsilon \ := \ \frac{\Lambda}{3}s^2 \tag{C.0.12}$$

To order ϵ, the solution for $y(\sigma) := y_0(\sigma) + \epsilon y_1(\sigma)$ is obtained as, $y(\sigma) = 1 + \epsilon\sigma^2/2$ which leads to the coordinate transformation,

$$r(\tau, s) \ = \ se^{-\tau\sqrt{\Lambda/3}}\left(1 + \frac{\Lambda s^2}{9}\right) \quad , \quad \eta(\tau, s) \ = \ -\sqrt{\frac{3}{\Lambda}} e^{-\tau\sqrt{\Lambda/3}}\left(1 + \frac{\Lambda s^2}{6}\right) \tag{C.0.13}$$

For inverting the transformation, it is more convenient to use the combinations: $a(\eta) := -\sqrt{3/\Lambda}\eta^{-1}$, $A(\tau) := e^{\tau\sqrt{\Lambda/3}}$ so that,

$$r(A, s) = \frac{s}{A}\left(1 + \frac{\Lambda}{9}s^2\right) \quad , \quad a(A, s) \ = \ A\left(1 - \frac{\Lambda}{6}s^2\right) \tag{C.0.14}$$

$$s(a, r) \ = \ (ra)\left(1 + \frac{\Lambda}{18}(ra)^2\right) \quad , \quad A(a, r) \ = \ a\left(1 + \frac{\Lambda}{6}(ra)^2\right) \tag{C.0.15}$$

Note that ra is *a physical distance* such as the commonly used luminosity distance in cosmology while s is also a physical distance but along a spatial geodesic. The equation (C.0.15) gives the relation between them.

From these relations, it is easy to verify that the stationary Killing vector field,

$$-\sqrt{3/\Lambda}T \ := \ \eta\partial_\eta + x^i\partial_i \ = \ \eta\partial_\eta + r\partial_r \ = \ \partial_\tau \tag{C.0.16}$$

For completeness, we list the transformations between the conformal chart and the FNC chart in the static patch, up to order H^2..

$$\eta(\tau,\xi^i) \;:=\; -\frac{e^{-H\tau}}{H}\left(1+\frac{H^2s^2}{2}\right) \;,\; x^i(\tau,\xi^i) \;:=\; \xi^i e^{-H\tau}\left(1+\frac{H^2s^2}{3}\right) \quad \text{(C.0.17)}$$

$$e^{-H\tau}(\eta,x^i) \;:=\; -\eta H\left(1-\frac{r^2}{2\eta^2}\right) \;,\; \xi^i(\eta,x^i) \;:=\; -\frac{x^i}{\eta H}\left(1+\frac{r^2}{6\eta^2}\right) \quad \text{(C.0.18)}$$

With these, it can be checked that the two metrics go into each other.

D An averaging procedure

In the main body we specified an averaging procedure by stipulating its properties namely, (i) average of odd powers of h vanishes and (ii) average of space-time divergence is sub-leading. This was then used to simplify the expression for the ripple stress tensor. An averaging procedure satisfying these properties is indeed given by Isaacson [21]. We will use the same one and give more explicit details in the present context.

Isaacson defines the *space-time average* of a tensor by using the *parallel propagator bi-tensor,* $g_{\mu}^{\ \mu'}(x, x')$ as:

$$\langle X_{\mu\nu}\rangle(x) \ := \ \frac{\int_{cell} d^4x' \sqrt{|g(x')|} g_{\mu}^{\ \mu'}(x, x') g_{\nu}^{\ \nu'}(x, x') X_{\mu'\nu'}(x')}{\int_{cell} d^4x' \sqrt{|g(x')|}} \ . \tag{D.0.1}$$

In the present context, we need average of the stress tensor for ripples due to an retarded solution which has certain explicit form. We will use this information to choose suitable integration variables and corresponding 'cell' denoting the averaging region. Because of this, we have not used any weighting function as given by Isaacson [21].

To keep track of the powers of H, we begin by going from the conformal chart (η, x^i) to the cosmological chart (t, x^i), $\eta := -H^{-1}e^{-Ht}$ with the spatial coordinates unchanged. In the cosmological chart:

$$\begin{aligned} Metric \ &: \ ds^2 \ = \ -dt^2 + a^2(t)(\delta_{ij}dx^i dx^j) \quad , \quad a(t) := e^{Ht} \qquad \text{(D.0.2)} \\ Connection \ &: \ \Gamma^t_{\ tt} = 0 \ , \ \Gamma^t_{\ tj} = 0 \ , \ \Gamma^t_{\ ij} = Ha^2(t)\delta_{ij} \end{aligned}$$

$$\Gamma^{i}{}_{tt} = 0 \ , \ \Gamma^{i}{}_{tj} = H\delta^{i}{}_{j} \ , \ \Gamma^{i}{}_{jk} = 0 \, . \tag{D.0.3}$$

The parallel propagator is computed in terms of the parallel transport of an arbitrary co-tetrad (or tetrad): $g_{\mu}{}^{\mu'}(x, x') := e_{\mu}{}^{a}(x)e_{a}{}^{\mu'}(x')$. The averaging region is small enough that for a cell around a point P with coordinates x^{α}, there is unique geodesic to points P' with coordinates x'^{α}. The parallel transported co-tetrad is obtained using Taylor expansions of the co-tetrad, the affine connection and the coordinates along the geodesic, in terms of its affine parameter and eliminating the affine parameter afterwards in favour of the coordinate differences $\Delta x^{\alpha} := x'^{\alpha} - x^{\alpha}$. Details may be seen in the appendix B. There is a slight difference from appendix B, since that calculation was given in the context of Fermi normal coordinates where the connection is already of order H^2 while in the cosmological chart, the connection is of order H. The final expressions are:

$$e_{\mu}{}^{a}(x') = \hat{e}_{\lambda}{}^{a}\left[\delta_{\mu}{}^{\lambda} + \hat{\Gamma}_{\mu\alpha}{}^{\lambda}\Delta x^{\alpha} + \frac{1}{2}\left(\widehat{\partial_{\rho}\Gamma}_{\mu\sigma}{}^{\lambda} + \hat{\Gamma}_{\mu\sigma}{}^{\alpha}\hat{\Gamma}_{\alpha\rho}{}^{\lambda}\right)\Delta x^{\rho}\Delta x^{\sigma}\right] \tag{D.0.4}$$

$$g_{\mu}{}^{\mu'}(x, x') = \delta_{\mu}{}^{\mu'} - \hat{\Gamma}_{\mu\alpha}{}^{\mu'}\Delta x^{\alpha} - \frac{1}{2}\left(\widehat{\partial_{\rho}\Gamma}_{\mu\sigma}{}^{\mu'} - \hat{\Gamma}_{\alpha\rho}{}^{\mu'}\hat{\Gamma}_{\sigma\mu}{}^{\alpha}\right)\Delta x^{\rho}\Delta x^{\sigma} \tag{D.0.5}$$

In the above, the hatted quantities are evaluated at x.

The connection dependent terms are linear and quadratic in $H\Delta x$. Although the coordinate differences are much larger than the length scale λ they are much smaller than H^{-1}. Hence, these terms can be neglected and *effectively the parallel propagator reduces to just the Kronecker delta.* For purposes of illustration of averaging, this suffices. It remains to integrate the $X_{\mu'\nu'}$ over the cell and as noted in the main text in the paragraph below equation (4.3.43), the components of the ripple stress tensor are essentially determined in terms of $\partial_{\eta}\chi^{tt}_{ij} = 2\frac{\eta}{r}\frac{Q^{tt}_{ij}(\eta-r)}{\eta-r}$ or alternatively in terms of $\partial_{\eta}\chi^{TT}_{ij} = 2\frac{\eta}{r}\frac{\mathcal{R}^{TT}_{ij}(\eta-r)}{\eta-r}$.

The angular dependence is introduced due to the 'tt' part, eg as is explicit in the $\Lambda_{ij}{}^{kl}(\hat{r})$ projector. The (η, r) dependence has a convenient factorised form. It is thus natural to change the integration variables from (η, r) to $(\bar{t}, \rho)$, where $\bar{t}$ is the retarded synchronous time defined through, $\eta - r := -H^{-1}e^{H\bar{t}}$ and $H\rho := -\frac{r}{\eta}$ defines ρ. For definiteness, consider the average,

$$\langle \partial_\eta \chi^{tt}_{mn} \partial_\eta \chi^{mn}_{tt} \rangle(t, r, \hat{r}) \ := \ \frac{\int_{cell} dt\, dr\, r^2\, d^2s\, a^3(t) \partial_\eta \chi^{tt}_{mn}(\bar{t}) \partial_\eta \chi^{mn}_{tt}(\bar{t})}{\int_{cell} dt\, dr\, r^2\, d^2s\, a^3(t)} \tag{D.0.6}$$

Here $\hat{r}$ denotes a point on S^2 (a spatial direction). We will specify the cell after changing over to $(\bar{t}, \rho, \hat{r})$.

From the definitions, we arrive at the coordinate transformations,

$$\begin{aligned} \eta(\bar{t}, \rho) &= -\frac{1}{H(1 + H\rho)} e^{-H\bar{t}} \quad , \quad r(\bar{t}, \rho) = \frac{\rho}{1 + H\rho} e^{-H\bar{t}} \quad \Rightarrow \\ a(t) &= a(\bar{t})(1 + H\rho) \qquad , \text{ with } \ a(t) := e^{Ht} \ . \end{aligned} \tag{D.0.7}$$

The Jacobian of transformation is $\frac{\partial(t,r)}{\partial(\bar{t},\rho)} = \{a(\bar{t})(1 + H\rho)\}^{-1}$. We choose the cell so that $\bar{t} \in [\bar{t}_0 - \delta, \bar{t}_0 + \delta]$ and $\rho \in [\rho_0 - \varDelta, \rho_0 + \varDelta]$ and $\hat{r} \in \varDelta\omega$. The coordinate windows $\delta, \varDelta$ and $\sqrt{r^2 \varDelta\omega}$ are several times the ripple scale while $(\bar{t}_0, \rho_0)$ are the transforms of (t, r). In terms of these choices, the average becomes,

$$\langle \partial_\eta \chi^{tt}_{mn} \partial_\eta \chi^{mn}_{tt} \rangle(t, r, \hat{r}) \ \ := \ \ \frac{\int_{\bar{t}_0 - \delta}^{\bar{t}_0+\delta} d\bar{t} \ \int_{\rho_0 - \varDelta}^{\rho_0+\varDelta} d\rho\, \rho^2 \int_{\varDelta\omega} d^2s \ \left[4 \frac{a^2(\bar{t})}{\rho^2} Q^{tt}_{mn}(\bar{t}) Q^{mn}_{tt}(\bar{t})\right]}{\int_{\bar{t}_0 - \delta}^{\bar{t}_0+\delta} d\bar{t} \ \int_{\rho_0 - \varDelta}^{\rho_0+\varDelta} d\rho\, \rho^2 \int_{\varDelta\omega} d^2s} \tag{D.0.8}$$

Consider the angular integration. The angular dependence arises in taking the 'tt' part of the solution $\chi_{ij}(\eta, r)$. For illustration purpose, consider r to be sufficiently large so that we can use the ${\varLambda_{ij}}^{kl}(\hat{r})$ projector, giving $\partial_\eta \chi^{tt}_{ij} \partial_\eta \chi^{ij}_{tt} \sim {\varLambda_{ij}}^{kl} \partial_\eta \chi^{ij} \partial_\eta \chi_{kl}$. For large r, the angular coordinate windows are $\sim \lambda/r \ll 1$. Using the mean value theorem in the angular integration in the numerator, we get

$$\frac{\int_{\varDelta\omega} d^2s(\hat{r}') {\varLambda_{ij}}^{kl}(\hat{r}')}{\int_{\varDelta\omega} d^2s(\hat{r}')} \ \approx \ {\varLambda_{ij}}^{kl}(\hat{r}) \ . \tag{D.0.9}$$

In effect, the $\varLambda-$projector comes out of the averaging and the *angular average trivializes.*

Of the remaining integrations, the ρ integration can be done explicitly and is independent of Δ to the leading order in Δ/ρ_0. Thus, in the numerator of (D.0.8) we get,

$$\begin{aligned}
\int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} \int_{\rho_0-\Delta}^{\rho_0+\Delta} d\rho \;\; 4\, a^2(\bar{t}) Q^{ij}(\bar{t})\, Q_{kl}(\bar{t}) &\approx 8\Delta \int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} \;\; a^2(\bar{t}) Q^{ij}(\bar{t})\, Q_{kl}(\bar{t}) \\
&= 8\Delta \int_{-\delta}^{\delta} dy\, a^2(\bar{t}_0 + y) Q^{ij} Q_{kl}(\bar{t}_0 + y) \\
&= 8\Delta a^2(\bar{t}_0) \int_{-\delta}^{\delta} dy\, a^2(y) Q^{ij} Q_{kl}(\bar{t}_0 + y) \\
&:= (8\Delta) a^2(\bar{t}_0)(2\delta)\langle Q^{ij} Q_{kl}\rangle_{\bar{t}}(\bar{t}_0) \qquad \text{(D.0.10)}
\end{aligned}$$

In the third line, we have used a^2 being an exponential function and in the last line we have *defined* the average over the retarded time around $\bar{t}_0$ and put the suffix on the angular bracket as a reminder.

The $\langle\rangle_{\bar{t}}$ averaging has the extra factor of $a^2(y)$. However, over the integration domain $(-\delta, \delta)$, we can approximate $a^2(y) \approx 1 + 2Hy + \cdots$ and neglect $o(Hy)$ terms since $H\delta \sim k\lambda/L \sim k\epsilon \ll 1$. The extra factor thus introduces a small deviation from the usual averaging without the extra factor and *we neglect it henceforth and the reminder suffix, $\bar{t}$ is also suppressed.*

In the denominator we get,

$$\int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} \int_{\rho_0-\Delta}^{\rho_0+\Delta} d\rho\, \rho^2 \;\approx\; 2\rho_0^2 \Delta \int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} \;=\; (2\Delta)(2\delta)\rho_0^2 \qquad \text{(D.0.11)}$$

Combining equations (D.0.8, D.0.9, D.0.10, D.0.11), we get

$$\langle \partial_\eta \chi^{tt}_{mn} \partial_\eta \chi^{mn}_{tt}\rangle(t, r, \hat{r}) \;=\; 4\frac{a^2(\bar{t}_0)}{\rho_0^2}\langle Q^{tt}_{ij}\, Q^{ij}_{tt}\rangle(\bar{t}_0, \hat{r}) \qquad \text{(D.0.12)}$$

In the last equation, we have combined the averaging over retarded time and the (trivial) angular average. We have also inserted the Λ−projector. The averaging over *a space-time cell* has been reduced to averaging over *a 3-dimensional cell on a ρ =constant hypersur-*

face. The pre-factor on the right hand side of the above equation exactly equals the last square bracket in the first line of the equation (4.3.43). In effect, the $(\frac{2\eta}{r(\eta-r)})^2$ factor has come out of the averaging.

We can also reduce the space-time average to a hypersurface average for $\partial_\eta \chi^{TT}_{ij} \partial_\eta \chi^{ij}_{TT}$. Following the same steps as from eqn.(D.0.6) onwards, we will arrive at eqn.(D.0.8) with $Q^{tt}_{mn} \to \mathcal{R}^{TT}_{mn}$. We cannot do the angular averaging as before, but we don't need to. Crucially, the ρ dependence has factored out exactly as before and the average over ρ gives ρ_0^{-2} as before. The $\bar{t}$ averaging too gives $a^2(\bar{t}_0)$ and we get the desired result,

$$\langle \partial_\eta \chi^{TT}_{mn} \partial_\eta \chi^{mn}_{TT} \rangle(t, r, \hat{r}) \quad = \quad 4\frac{a^2(\bar{t}_0)}{\rho_0^2} \langle \mathcal{R}^{TT}_{ij}\ \mathcal{R}^{ij}_{TT} \rangle(\bar{t}_0, \hat{r}) \tag{D.0.13}$$

We can relate the averaging over the retarded time, $\bar{t}$, to the averaging over the Killing time, τ along the $\rho = \rho_0$ curve. From the coordinate transformation, we have $\eta - r = -H^{-1}e^{-H\bar{t}}$ while along $\rho = \rho_0$ Killing trajectory, $\eta - r = (\eta_* - r_*)e^{-H\tau} = -(H^{-1}e^{-H\bar{t}_*})e^{-H\tau}$. Hence, $\bar{t} = \tau + \bar{t}_*$ and the temporal averaging is related to averaging over a Killing time. Note that the averaging cell being bounded by two hypersurfaces of constant retarded times, the temporal averaging may be evaluated along the source worldline, $r = 0$ *or* along the Killing trajectory on $\mathcal{J}^+$.

We also have mixed and spatial components of the ripple stress tensor. These involve $\partial_i \chi^{tt}_{mn} \approx \hat{x}_i \partial_r \chi^{tt}_{mn} \approx -\hat{x}_i \partial_\eta \chi^{tt}_{mn}$. While taking the average, the $\hat{x}_i$ can be taken out of the average since the angular coordinate windows are of very small size $\sim \lambda/r$. This allows us to take $\hat{x}^i$ across the angular averages and replace all components of the ripple stress tensor by $t_{\eta\eta}$ in the conformal chart or by t_{00} in the cosmological chart.

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