Non-Perturbative Aspects of Supersymmetric Gauge Theories with Surface Operators

by

RENJAN RAJAN JOHN

PHYS10201105001

The Institute of Mathematical Sciences, Chennai

A thesis submitted to the

Board of Studies in Physical Sciences

In partial fulfilment of requirements

For the Degree of

DOCTOR OF PHILOSOPHY

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Renjan Rajan John

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Renjan Rajan John

LIST OF PUBLICATIONS (INCLUDED IN THIS THESIS)

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2. **Modular and Duality Properties of Surface Operators in $\mathcal{N}=2^\star$ Gauge Theories**,
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2. **Surface operators, chiral rings and localization in $\mathcal{N}=2$ gauge theories**.
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Renjan Rajan John

Dedicated to

Pavan

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ABSTRACT

In this thesis, we study gauge theories with $\mathcal{N} = 2$ supersymmetry in four dimensions. The low energy effective action of these theories on their Coulomb branch is described by a holomorphic function called the prepotential. In the first half, we study linear conformal quiver theories with gauge group SU(2). These theories have an SU(2) gauge group at each node of the quiver, and matter arranged in the fundamental and the bi-fundamental representations, such that at each node the β-function vanishes. To compute the prepotential for these theories, we follow three different approaches. These are (i) the classic Seiberg-Witten approach, in which we consider an M-theory construction of the Seiberg-Witten curve and the associated differential, (ii) equivariant localization as developed by Nekrasov, and (iii) the 2d/4d correspondence of the four dimensional gauge theory with the two dimensional Liouville conformal field theory, as put forward by Alday, Gaiotto, and Tachikawa. Matching the prepotential, we find out the precise map between the various parameters that appear in the three descriptions.

In the latter half of the thesis, we study surface operators in the context of $\mathcal{N} = 2^{\star}$ theories with gauge group SU(N). These theories describe the dynamics of a vector multiplet, and a massive hypermultiplet in the adjoint representation of the gauge group. Surface operators are non-local operators that have support on a two dimensional sub-manifold of the four dimensional spacetime. They are defined by the singularities they induce in the four-dimensional gauge fields, or can be characterized by the two-dimensional theory they support on their world-volume. The infrared dynamics on the world-volume of the two-dimensional surface operator is described by a holomorphic function called the twisted superpotential. Using localization techniques, we obtain the instanton partition function, and thereby the twisted

superpotential of these theories. This involves taking a suitable orbifold of the original action without the surface operator. Imposing constraints from S-duality, we obtain a modular anomaly equation for the coefficients that appear in the mass expansion of the twisted superpotential. Solving the modular anomaly equation at each order, and comparing with the results obtained from localization, we resum the twisted superpotential in a mass series, whose coefficient functions depend on (quasi-) modular forms and elliptic functions of the bare coupling constant and the continuous (complex) parameters that describe the surface operator. This gives us the entire tail of instanton corrections, at each order in the mass expansion. We further show that our results for monodromy defects in the four-dimensional theory, match the effective twisted superpotential that describes the infrared properties of certain two-dimensional sigma models coupled to $\mathcal{N} = 2^\star$ gauge theories. This provides strong evidence for the proposed duality between the two descriptions of surface operators.

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Chapter 0

Synopsis

Symmetry has played an important role in our understanding of modern physics. Relativistic quantum field theories have Poincaré symmetry, which consists of Lorentz and space-time translational symmetries. These are generated by bosonic generators. Supersymmetry is an extension of Poincaré symmetry in which one introduces spinorial generators to the algebra. It is a continuous space-time symmetry that pairs bosons and fermions. The advantages of introducing supersymmetry to the symmetry algebra are manifold. From a particle physicist's point of view, introducing supersymmetry cures quadratic divergences that appear when one computes self-energy Feynman diagrams. From a more theoretical point of view, supersymmetry gives good control over the dynamics of the theory. Supersymmetric theories are closely related to integrable systems, which are exactly solvable. Exact computations which are not possible in non-supersymmetric theories are made possible by supersymmetry. These serve as useful examples to understand various strong-coupling, and non-perturbative effects that appear in non-supersymmetric theories.

In this thesis, we study $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions. These theories stand midway between the most symmetric, and hence tractable

$\mathcal{N} = 4$ theories, and the $\mathcal{N} = 1$ theories, that have the least amount of supersymmetry. The study of $\mathcal{N} = 2$ theories was pioneered by the work of Seiberg and Witten in [1, 2]. Since their work, tremendous progress has been made in the study of these theories. We will make use of three different approaches to our study of $\mathcal{N} = 2$ supersymmetric gauge theories. These include the classic Seiberg-Witten analysis, the techniques of equivariant localization as developed by Nekrasov in [3], and the 2d/4d correspondence put forward by Alday, Gaiotto, and Tachikawa in [4].

Our interest is in $\mathcal{N} = 2$ super-conformal theories that have the added symmetry of conformality. In [5], a large class of super-conformal theories called class $\mathcal{S}$ theories, was realized by the compactification of $(2, 0)$ theory in six dimensions, on Riemann surfaces with punctures. Particular theories of interest to us are i) linear conformal quiver theories that contain products of SU(2) gauge groups. These have matter arranged in the fundamental and the bi-fundamental representations such that the β-function at each node vanishes. These theories are obtained by choosing the Riemann surface on which the 6d theory is compactified to be a Riemann sphere with punctures. We study the low energy effective action of these theories on the Coulomb branch by computing the prepotential, and ii) $\mathcal{N} = 2^\star$ theories which describe the dynamics of a vector multiplet and a single massive hypermultiplet in the adjoint representation of the gauge group, for which the corresponding Riemann surface is a torus with a single puncture. We study the low energy effective action of these theories in the presence of a surface operator by computing the twisted effective superpotential. This describes the infra-red physics of the two dimensional theory that lives on the world-volume of the surface defect. We study modular properties of the twisted superpotential under the constraints imposed by S-duality. This thesis contains the following chapters.

0.1 Seiberg-Witten theory

In this chapter, we briefly describe the Seiberg-Witten theory which sets the platform for the work that is presented in the thesis. Seiberg-Witten theory and its generalization to higher rank gauge groups in [6–8] compute the low energy effective action of $\mathcal{N}=2$ supersymmetric gauge theories on the Coulomb branch where the gauge group $\mathrm{SU}(N)$ is broken down to its maximal torus. The Seiberg-Witten approach is a geometric one, in which the prepotential that describes the effective theory is given in terms of an algebraic curve and an associated differential, called the Seiberg-Witten curve and differential respectively.

In the traditional Seiberg-Witten approach, the vacuum expectation values a_i of the adjoint scalar and the dual variables $a_i^D \equiv \frac{\partial \mathcal{F}}{\partial a_i}$ correspond to period integrals of the Seiberg-Witten differential along a symplectic basis of cycles on the Seiberg-Witten curve.

$$a_i = \int_{\alpha_i} \lambda \quad \text{and} \quad a_j^D = \int_{\beta_j} \lambda \tag{1}$$

The curve and the differential depend on the gauge theory parameters, in particular on the gauge-invariant parameters u_i that are coordinates on the moduli space of vacua. This implies that the set of variables (a_i, a_i^D) are functions of u_i. By inverting the functions $a_i(u)$, the dual periods $a_i^D(u)$ can be written as $a_i^D(a_i)$. Thus we obtain the relation,

$$\frac{\partial a_i^D}{\partial a_i} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} \tag{2}$$

Integrating this formula twice, we obtain $\mathcal{F}$ as a function of the a_i's. However, this procedure is rather cumbersome as the integrals leading to the dual periods a_i^D are often very difficult. In our approach [9], we circumvent doing the β_i set of integrals that appear in (1), and make the computation of the prepotential much more viable.

Our study is in the context of linear super-conformal quiver theories with gauge group SU(2). As shown in Figure 1, these theories contain products of SU(2) gauge groups, and matter arranged in fundamental and bi-fundamental representations such that the β-function at each node vanishes. We focus on cases with a single node, two nodes, and for the massless case arbitrary number of nodes. The Seiberg-Witten curve of these theories cover a Riemann sphere with $n+3$ punctures, where n is the number of nodes in the quiver. The expressions for the curve and the

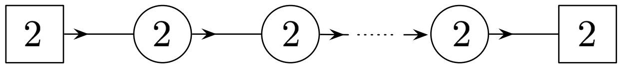

Figure 1: Linear Quiver with gauge group SU(2)

corresponding differential are obtained by considering a system of NS5-D4 branes uplifted to M-theory [10]. The Gaiotto form of the curve when all the gauge groups are SU(2) takes the form,

$$x^2(t) = \frac{\mathcal{P}_{2n+2}(t)}{t^2\,(t-t_1)^2\dots(t-t_n)^2\,(t-1)^2} \tag{3}$$

where t_i's are the positions of the punctures on the Riemann sphere. These are related to the gauge theory couplings as $q_i = t_i/t_{i+1}$, and $\mathcal{P}_{2n+2}(t)$ is a polynomial of degree $(2n+2)$ which depends on the q_i's, the masses, and the Coulomb branch parameters u_i. The Gaiotto formulation of the curve (3) has the advantage that the associated differential is readily given as,

$$\lambda = x(t)dt \tag{4}$$

Our approach to computing the prepotential makes use of what are called Matone's relations. The relevant relation takes the form [11],

$$U_i \equiv \left\langle \mathrm{Tr}\ \Phi_i^2 \right\rangle = q_i \frac{\partial \mathcal{F}}{\partial q_i} \tag{5}$$

It was proposed in [12, 13] that U_i should be identified with the residues of the quadratic differential $x^2(t)dt^2$ at the various punctures of the curve (3),

$$\mathrm{Res}_{t=t_i}\left(x^2(t)\right) = \frac{\partial \mathcal{F}}{\partial t_i} \tag{6}$$

We now illustrate these ideas in the case of the quiver with two nodes, and gauge group SU(2) [9]. The Seiberg-Witten curve in the Gaiotto form is,

$$x^2(t) = \frac{p_6(t)}{t^2\,(t-q_1)^2(t-1)^2(q_2 t-1)^2}\ , \tag{7}$$

where $p_6(t)$ is in general a sixth order polynomial in t, with the coefficient functions depending on the masses, the coupling constants, and the gauge invariant Coulomb branch parameters. The genus 2 Seiberg-Witten curve may be expressed in the hyper-elliptic form,

$$y^2(t) = p_6(t) = c\prod_{i=1}^{6}(t-e_i) \tag{8}$$

where e_i's are the six roots of the polynomial, which are clearly branch points for the function $y(t)$. With a projective transformation we can fix three of them at 0, 1, and ∞. If we call ζ_1, ζ_2 and $\widehat{\zeta}$ the remaining three parameters corresponding to the three independent anharmonic ratios of the e_i's, equation (8) reduces to

$$y^2(t) = c\,t\big(t-1\big)\big(t-\zeta_1\big)\big(t-\zeta_2\big)\big(t-\widehat{\zeta}\,\big)\ . \tag{9}$$

When the curve is put in this form, we can choose a symplectic basis of cycles $\{\alpha_i, \beta^j\}$ on the Riemann sphere as shown in Figure 2, and then proceed to compute the periods of the Seiberg-Witten differential and finally derive the effective prepotential.

However, for generic values of the masses of the matter hypermultiplets this method is not practical since one cannot find the roots of $p_6(t)$ in closed form, and hence only a perturbative approach in the masses is viable. We choose the following mass

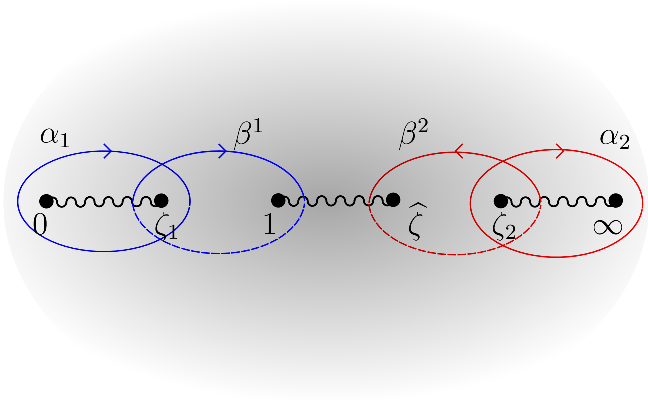

Figure 2: Structure of branch cuts and a basis of cycles for the Riemann surface described by Eq (9)

configuration,

$$m_1 = m_2 = m_3 = m_4 = 0 \quad \text{and} \quad m_{12} = M \tag{10}$$

for which the polynomial $p_6(t)$ in (7) can be factorized. For this mass choice, the curve becomes

$$x^2(t) = \frac{C(t-\zeta_3)(t-\widehat{\zeta})}{t(t-q_1)(q_2 t-1)(t-1)^2} \tag{11}$$

where ζ_3, $\hat{\zeta}$ and C are functions of the couplings constants q_i, the bi-fundamental mass M, and the Coulomb branch parameters u_i. The residue condition (6) takes the form,

$$\mathrm{Res}_{t=q_1}\left(x^2(t)\right) = \frac{\partial \mathcal{F}}{\partial q_1}\,, \qquad \mathrm{Res}_{t=1/q_2}\left(x^2(t)\right) = -q_2^2\,\frac{\partial \mathcal{F}}{\partial q_2}\,, \tag{12}$$

The left hand sides of the above equations are functions of the gauge theory parameters, in particular of u_i. We now compute the α-periods of the Seiberg-Witten differential, and obtain $a_1(U_i)$.

$$
\begin{aligned}
a_1 &= \int_0^{q_1} \sqrt{\frac{C(\zeta_3 - t)(t - \widehat{\zeta})}{t(q_1 - t)(1 - q_2 t)}} \, \frac{dt}{(1-t)} \\
&= \sqrt{U_1} \left(1 - q_1 \frac{U_1 + M^2}{4U_1^2} + q_2 \frac{U_2}{4U_1^2} - q_1 q_2 \, \frac{U_2}{4U_1} \right. \\
&\qquad \left. - \, q_1^2 \, \frac{7U_1^2 - 10U_1U_2 + 3U_2^2 + M^2\big(14U_1 - 6U_2 + 3M^2\big)}{64U_1^2} + \ldots \right) \qquad (13)
\end{aligned}
$$

A similar period integral obtains for us $a_2(U_i)$. Inverting these relations, and integrating over q_1 and q_2 we obtain the prepotential [9],

$$
\begin{aligned}
F &= a_1^2 \log q_1 + a_2^2 \log q_2 + q_1 \, \frac{a_1^2 - a_2^2 + M^2}{2} + q_2 \, \frac{a_2^2 - a_1^2 + M^2}{2} \\
&+ q_1 q_2 \, \frac{a_1^2 + a_2^2 - M^2}{4} + q_1^2 \left(\frac{13a_1^4 - 14a_1^2 a_2^2 + a_2^2}{64a_1^2} + \frac{9M^2}{32} + \frac{M^2(M^2 - 2a_2^2)}{64a_1^2} \right) \\
&+ q_2^2 \left(\frac{13a_2^4 - 14a_1^2 a_2^2 + a_1^2}{64a_2^2} + \frac{9M^2}{32} + \frac{M^2(M^2 - 2a_1^2)}{64a_2^2} \right) + \ldots \, . \qquad (14)
\end{aligned}
$$

0.2 Localization

In this chapter, we describe Nekrasov's equivariant localization [3], and compute the instanton partition function. The instanton partition function is typically divergent, and in order to obtain finite results one considers the Ω-deformed theory which is obtained by a standard dimensional reduction from a suitable theory in an appropriate supersymmetric background. The Ω-background is parameterized by two deformation parameters ϵ_1 and ϵ_2 that break the SO(4) symmetry down to rotations on two planes. By equivariant localization, the moduli space of instantons is localized to isolated points, and the instanton partition function of the quiver is given by,

$$
Z_{\text{inst}} = \sum_{k_i} \int \prod_i \frac{q_i^{k_i}}{k_i!} \prod_{I_i=1}^{k_i} \frac{d\chi_{I_i}}{2\pi \mathrm{i}} \; z^{\text{quiver}}_{\{k_i\}} \; . \qquad (15)
$$

Here $z^{\text{quiver}}_{\{k_i\}}$ gets contributions from the gauge and matter sectors of the theory. The configurations of χ_i that contribute to the integral (15) are in one to one correspondence with a set of Young tableaux $Y = \{Y_i\}$ with $k = \sum_i k_i$ number of boxes. The instanton partition function can be rewritten as

$$Z_{\text{inst}} = 1 + \sum_{Y_i} \prod_i q_i^{|Y_i|} Z_{\{Y_i\}} \ . \tag{16}$$

Here the 1 represents the contribution at zero instanton number, and $|Y_i|$ is the total number of boxes of the i-th Young tableau. The Z_{Y_i} are then calculated using the formalism of group characters. For the 2-node SU(2) quiver at one instanton [9],

$$\begin{aligned}
Z_{(\square,\bullet|\bullet,\bullet)} &= \frac{(2a_1+2a_2+2m_{12}+\epsilon)(2a_1-2a_2+2m_{12}+\epsilon)}{32\,\epsilon_1\epsilon_2\,a_1(-2a_1-\epsilon)} \prod_{f=1}^{2}(2a_1+2m_f+\epsilon) \\
Z_{(\bullet,\square|\bullet,\bullet)} &= \left[Z_{(\square,\bullet|\bullet,\bullet)}\right]_{a_1\to -a_1} \\
Z_{(\bullet,\bullet|\square,\bullet)} &= \frac{(2a_2+2a_1-2m_{12}+\epsilon)(2a_2-2a_1-2m_{12}+\epsilon)}{32\,\epsilon_1\epsilon_2\,a_2(-2a_2-\epsilon)} \prod_{f=3}^{4}(2a_2+2m_f+\epsilon) \\
Z_{(\bullet,\bullet|\bullet,\square)} &= \left[Z_{(\bullet,\bullet|\square,\bullet)}\right]_{a_2\to -a_2}
\end{aligned} \tag{17}$$

where $\epsilon \equiv \epsilon_1+\epsilon_2$. The instanton partition function at 1-instanton is then given by, $Z_1 = q_1\,Z_{1,0} + q_2\,Z_{0,1}$, with

$$Z_{1,0} = Z_{(\square,\bullet|\bullet,\bullet)} + Z_{(\bullet,\ \square,|\bullet,\bullet)} \ , \quad Z_{0,1} = Z_{(\bullet,\bullet|\square,\bullet)} + Z_{(\bullet,\bullet|\bullet,\square)} \ . \tag{18}$$

This can be extended to higher instanton orders to get

$$Z_{\text{inst}} = 1 + \sum_{k_1,k_2} Z_{k_1,k_2}\, q_1^{k_1}\, q_2^{k_2} \tag{19}$$

The non-perturbative prepotential is given by,

$$F_{\text{inst}} = -\epsilon_1\epsilon_2\,\log Z_{\text{inst}} = \sum_{k_1,k_2} F_{k_1,k_2}\, q_1^{k_1}\, q_2^{k_2} \ . \tag{20}$$

Below we tabulate the first few prepotential coefficients F_{k_1,k_2}, in the Nekraso-Shatashvili limit [14] where we set $\epsilon_2 = 0$ and each F_{k_1,k_2} has a further expansion of the form

$$F_{k_1,k_2} = \sum_{n=0}^{\infty} F^{(n)}_{k_1,k_2}\, \epsilon_1^n \,. \tag{21}$$

At order ϵ_1^0 we have [9]

$$\begin{aligned}
F^{(0)}_{1,0} &= \frac{a_1^2 - a_2^2}{2} + \frac{1}{2}\big(m_1 m_2 + 2(m_1+m_2)m_{12} + m_{12}^2\big) + \frac{m_1 m_2\big(m_{12}^2 - a_2^2\big)}{2a_1^2} \\
F^{(0)}_{2,0} &= \frac{13a_1^4 - 14a_1^2a_2^2 + a_2^4}{64a_1^2} + \frac{1}{64}\big(m_1^2 + 16m_1m_2 + m_2^2 + 32(m_1+m_2)m_{12} + 18m_{12}^2\big) \\
&+ \frac{m_1^2m_2^2 + 2\big(m_1^2 + 8m_1m_2 + m_2^2\big)m_{12}^2 + m_{12}^4 + 2a_2^2\big(m_1^2 - 8m_1m_2 + m_2^2 - m_{12}^2\big)}{64a_1^2} \\
&- \frac{3\Big[2m_1^2m_2^2m_{12}^2 + (m_1^2+m_2^2)m_{12}^4 + 2a_2^2(m_1^2m_2^2 - (m_1^2+m_2^2)m_{12}^2) + a_2^4(m_1^2+m_2^2)\Big]}{64a_1^4} \\
&+ \frac{5m_1^2m_2^2\big(m_{12}^4 - 2a_2^2m_{12}^2 + a_2^4\big)}{64a_1^6} \\
F^{(0)}_{1,1} &= \frac{a_1^2 + a_2^2}{4} + \frac{1}{4}\big(m_1m_2 + m_3m_4 + 2(m_1+m_2)(m_3+m_4) - m_{12}^2\big) \\
&+ \frac{m_1m_2\big(m_3m_4 - m_{12}^2 + a_2^2\big)}{4a_1^2} + \frac{m_3m_4\big(m_1m_2 - m_{12}^2 + a_1^2\big)}{4a_2^2} - \frac{m_1m_2m_3m_4m_{12}^2}{4a_1^2a_2^2}
\end{aligned} \tag{22}$$

The other prepotential terms $F_{k,\ell}$ can be obtained from $F_{\ell,k}$ by the operations

$$a_1 \leftrightarrow a_2, \quad (m_1, m_2) \leftrightarrow (m_3, m_4), \quad m_{12} \leftrightarrow -m_{12} \tag{23}$$

This for the particular choice of masses (10) precisely matches the instanton prepotential derived using the Seiberg-Witten curve and differential in the previous section. We have also calculated corrections to the prepotential in the Ω-background.

0.3 2d/4d correspondence

In this chapter, we use the 2d/4d correspondence which relates the instanton partition function of a linear quiver with gauge group $\mathrm{SU}(2)^n$ to the $n+3$ point spherical conformal block in two dimensional Liouville conformal field theory [4],

$$\left\langle V_{\alpha_0}(0) \prod_{i=1}^{n} V_{\alpha_i}(t_i)\, V_{\alpha_{n+1}}(1)\, V_{\alpha_{n+2}}(\infty) \right\rangle_{\{\xi_1,\ldots,\xi_n\}} = \mathcal{N}\ Z_{\mathrm{U}(1)}\ \mathrm{e}^{-\frac{\mathcal{F}_{\mathrm{inst}}}{\epsilon_1\epsilon_2}} \tag{24}$$

We compute the correlator in a specific pair-of-pants decomposition of the $(n+3)$ punctured sphere, and in each internal line only a primary field with specific Liouville momentum and its descendants propagate. We take the degenerate limit in which the $(n+3)$ punctured sphere becomes $(n+1)$ 3-punctured spheres. Relating the conformal block to the instanton partition function of the gauge theory requires a detailed map of the parameters that appear in the Liouville theory to those in the gauge theory. While the ratios of global coordinates on the spheres are mapped to the instanton counting parameters in the gauge theory, the Liouville momenta that flow through the external and internal lines in the conformal block are mapped respectively to the masses and the vacuum expectation values in the gauge theory. The central charge of the Liouville theory is mapped to a particular combination of the Ω-deformation parameters ϵ_1 and ϵ_2. The Liouville theory also contains information about the Seiberg-Witten curve. In order to see this, we consider the conformal block with the additional insertion of energy-momentum tensor $T(z)$,

$$\phi_2(z) \equiv \frac{\left\langle V_{\alpha_0}(0) \prod_{i=1}^{n} V_{\alpha_i}(t_i)\, T(z) V_{\alpha_{n+1}}(1)\, V_{\alpha_{n+2}}(\infty) \right\rangle}{\left\langle V_{\alpha_0}(0) \prod_{i=1}^{n} V_{\alpha_i}(t_i)\, V_{\alpha_{n+1}}(1)\, V_{\alpha_{n+2}}(\infty) \right\rangle} \tag{25}$$

Using the conformal Ward identities, and the operator product expansion of the energy momentum tensor with conformal primaries, we obtain $\phi_2(z)$ to be of the same form as $x^2(z)$ that appears in the Seiberg-Witten curve (3). The Ω-deformed prepotential is also obtained from the Liouville theory by considering a null-vector decoupling equation [15]. We consider the conformal block with the insertion of a specific degenerate primary $\Phi_{2,1}(z)$,

$$\Psi(z) = \left\langle V_{\alpha_0}(0) \prod_{i=1}^{n} V_{\alpha_i}(t_i)\, \Phi_{2,1}(z) V_{\alpha_{n+1}}(1) V_{\alpha_{n+2}}(\infty) \right\rangle \tag{26}$$

$\Psi(z)$ obeys a second order differential equation which in the Nekrasov-Shatashvili limit takes the form of a Schrödinger equation,

$$\left(-\epsilon_1^2 \frac{d^2}{dz^2} + V(z, \epsilon_1)\right) \Psi(z) = 0 \tag{27}$$

The potential takes the form,

$$V(z, \epsilon_1) = V^{(0)}(z) + \epsilon_1 V^{(1)}(z) + \epsilon_1^2 V^{(2)}(z) \tag{28}$$

For the 2-node quiver that we consider, the Schrödinger potential is [9],

$$\begin{aligned}
V^{(0)}(z) &= \phi_2(z) \\
V^{(1)}(z) &= \frac{(m_1+m_2+m_3+m_4)\, q_1 q_2}{2z(z-1)(z-q_1q_2)} + \frac{(m_1+m_2+2m_{12})\, q_1 q_2}{2z(z-q_2)(z-q_1q_2)} + \frac{(m_3+m_4-2m_{12})\, q_2}{2z(z-1)(z-q_2)} \\
V^{(2)}(z) &= -\frac{1}{4z^2} - \frac{1}{4(z-q_1q_2)^2} - \frac{1}{4(z-q_2)^2} - \frac{1}{4(z-1)^2} + \frac{3}{4(z-1)} \\
&\qquad - \frac{\eta_1}{z(z-1)(z-q_2)} - \frac{\eta_2}{z(z-q_2)(z-q_1q_2)}
\end{aligned} \tag{29}$$

where η_1 and η_2 are functions of the instanton counting parameters. Here $V^{(0)}$ has the same expression that appears in the Seiberg-Witten curve of the undeformed theory. In the Nekrasov-Shatashvili limit, $\Psi(z)$ has the following singular behaviour,

$$\Psi(z) = \mathrm{e}^{-\frac{W(z)}{\epsilon_1}} \tag{30}$$

In order to solve (27), we make a WKB-like ansatz

$$W(z) = \int^{z} P\left(z', \epsilon_1\right) dz' \tag{31}$$

and expand P in a power series in ϵ_1, namely $P\left(z, \epsilon_1\right) = \sum_{n=0}^{\infty} \epsilon_1^n P^{(n)}(z)$. With this ansatz, (27) becomes

$$-P\left(z, \epsilon_1\right)^2 + \epsilon_1 \frac{dP\left(z, \epsilon_1\right)}{dz} + V(z, \epsilon_1) = 0, \tag{32}$$

Solving perturbatively in ϵ_1, we obtain

$$\begin{aligned} P^{(0)}(z) &= \sqrt{\phi_2(z)} \\ P^{(0)}(z) &= \frac{1}{2}\frac{d}{dz}\log P^{(0)}(z) + \frac{V^{(1)}(z)}{2P^{(0)(z)}} \\ P^{(2)}(z) &= \frac{P^{(1)'}(z) - P^{(1)^2}(z)}{2P^{(0)}(z)} + \frac{V^{(2)}(z)}{2P^{(0)(z)}} \end{aligned} \tag{33}$$

and so on. Since $P^{(0)}(z)dz$ is simply the Seiberg-Witten differential of the undeformed theory, it is natural to define the deformed Seiberg-Witten differential as

$$\lambda\left(\epsilon_1\right) \equiv P\left(z, \epsilon_1\right)\ dz \tag{34}$$

As in section 0.1, the α period integral is now readily calculated to obtain $a_1(U_i)$,

$$a_1 = a_1^{(0)}(U_i) + \epsilon_1 a_1^{(1)}(U_i) + \epsilon_1^2 a_1^{(2)}(U_i) + \dots \tag{35}$$

A similar period integral is performed to evaluate $a_2(U_i)$. Inverting the expansion of the periods order-by-order in ϵ_1, we determine the ϵ_1 dependence of U_1 and U_2. At each order, the resulting expressions turn out to be integrable, and we recover the prepotential. The zeroth-order term matches our results from the Seiberg-Witten analysis in 0.1. The ϵ_1 corrections precisely match the microscopic results obtained from the Nekrasov partition function via localization methods.

We obtained the prepotential of the simplest SU(2) linear conformal quiver theory using three different approaches. One of our motivations to match the prepotential from these approaches was that it requires us to find the precise map between the gauge theory parameters, the geometric parameters that arise in the M-theory construction, and the parameters in Liouville conformal field theory.

0.4 Surface Operators

In this chapter, we study surface operators in the context of $\mathcal{N} = 2^\star$ gauge theories. $\mathcal{N} = 2^\star$ theories describe the dynamics of a vector multiplet and a massive hypermultiplet in the adjoint representation of the gauge group which we take to be SU(N). In the massless limit, this theory reduce to the $\mathcal{N} = 4$ super Yang-Mills theory, and in the limit in which the mass is decoupled we obtain the pure $\mathcal{N} = 2$ theory. In the Gaiotto construction [5], $\mathcal{N} = 2^\star$ theories are obtained by wrapping M5 branes on a torus with a single puncture.

We study non-local operators called surface operators that have support on a two dimensional plane inside the four dimensional (Euclidean) spacetime. In particular, we parameterize $\mathbb{R}^4 \simeq \mathbb{C}^2$ by two complex variables (z_1, z_2), and place the defect at $z_2 = 0$, filling the z_1 plane. Surface operators can be defined by the transverse singularities they induce in the four dimensional fields, or can be characterized by the two dimensional theory they support on their world-volume. Our interest is in Gukov-Witten like defects [16] which induce the following singular behaviour in the gauge connection A,

$$A = A_\mu \, dx^\mu \simeq -\,\mathrm{diag}\Big(\underbrace{\gamma_1, \cdots, \gamma_1}_{n_1}, \underbrace{\gamma_2, \cdots, \gamma_2}_{n_2}, \cdots, \underbrace{\gamma_M, \cdots, \gamma_M}_{n_M}\Big)\, d\theta \tag{36}$$

as $r \to 0$. Here (r, θ) denotes the set of polar coordinates in the z_2 plane, and the γ_I's constant parameters, where $I = 1, \ldots, M$. The M integers n_I satisfy

$$\sum_{I=1}^{M} n_I = N \tag{37}$$

and define a vector $\vec{n}$ that identifies the type of the surface operator. This vector is related to the breaking pattern of the gauge group on the two-dimensional defect, namely

$$SU(N) \to S\left[U(n_1) \times U(n_2) \times \ldots \times U(n_M)\right] \tag{38}$$

The type $\vec{n} = \{1, 1, \ldots, 1\}$ corresponds to the full surface operator, and the type $\vec{n} = \{1, N-1\}$ corresponds to the simple surface operator. There are two kinds of defects, one realized by M2-branes and the other realized by M5 branes. Our interest is in the latter kind, which from the six dimensional point of view correspond to co-dimension 2 defects. In the presence of a surface operator, one can turn on magnetic fluxes for each factor of the gauge group. Thus the instanton partition function depends on, in addition to the vacuum expectation values of the adjoint scalar, and the adjoint mass, a set of continuous complex parameters z_i that combines the electric and magnetic parameters. A holomorphic function called the twisted effective superpotential $\mathcal{W}$ determines the dynamics of the effective two dimensional theory living on the world-volume of the defects [17, 18]. The instanton partition function is obtained by suitably adapting equivariant localization to the case at hand. This involves taking an orbifold of the original action without surface operator. The logarithm of the resulting partition function exhibits both a four dimensional and a two dimensional singularity in the limit of vanishing deformations. These singularities are encoded respectively in the prepotential $\mathcal{F}$ and in a new function $\mathcal{W}$, the so-called twisted superpotential.

$$\log Z = -\frac{\mathcal{F}\left(a_i, m, \epsilon_1, q\right)}{\epsilon_1 \epsilon_2} + \frac{\mathcal{W}\left(a_i, m, \epsilon_1, q, z\right)}{\epsilon_1} \tag{39}$$

The latter depends, in addition to the vacuum expectation values of the adjoint scalar and the adjoint mass, on the continuous parameters z_i that characterize the defect.

Treating $\mathcal{N} = 2^\star$ theories as massive deformations of the $\mathcal{N} = 4$ super-conformal theory, the prepotential expanded as a power series in the adjoint mass parameter has been shown to be tightly constrained by S-duality in [19]. The coefficient functions are expressed in terms of (quasi)-modular forms of the bare coupling constant, and the first few f_ℓ that appear at $O(m^\ell)$ are given by,

$$\begin{aligned}
f_{\text{odd}} &= 0 \\
f_2 &= \left(m^2 - \frac{\epsilon_1^2}{4}\right)\log(2a) \\
f_4 &= -\frac{1}{48a^2}\left(m^2 - \frac{\epsilon_1^2}{4}\right)^2 E_2 \\
f_6 &= -\frac{1}{5760a^4}\left(\left(m^2 - \frac{\epsilon_1^2}{4}\right)^3\left(5E_2^2 + E_4\right) - 3E_4\left(m^2 - \frac{\epsilon_1^2}{4}\right)^2\epsilon_1^2\right)
\end{aligned} \tag{40}$$

As for the prepotential, the derivatives of the coefficient functions that appear in the mass expansion of the $\mathcal{W}$,

$$\mathcal{W} = \sum_{n=1}^{\infty} w_n m^n \tag{41}$$

with respect to the complex parameters z_i, obey a modular anomaly equation. Here we focus on the SU(2) case which has a single complex parameter z that labels the surface operator. The modular anomaly equation takes the form [20],

$$\frac{\partial w'_\ell}{\partial E_2} + \frac{1}{24}\sum_{n=0}^{\ell-1}\frac{\partial f_{\ell-n}}{\partial a}\frac{\partial w'_n}{\partial a} = 0 \tag{42}$$

This can be solved to obtain the complete quasi-modular dependence at each order in m. Combining results from localization, we can express the coefficient functions in terms of (quasi)-modular forms and elliptic functions of the bare coupling constant τ and the continuous parameters z_i. The first few coefficients in the SU(2) case are given below [20],

$$
\begin{aligned}
w_2' &= \frac{1}{24a}\left(m^2-\frac{\epsilon_1^2}{4}\right)\left(E_2+12\widetilde{\wp}\,\right)\\
w_3' &= \frac{\epsilon_1}{4a^2}\left(m^2-\frac{\epsilon_1^2}{4}\right)\widetilde{\wp}\,'\\
w_4' &= \frac{1}{1152a^3}\left(m^2-\frac{\epsilon_1^2}{4}\right)\left[\left(m^2-\frac{\epsilon_1^2}{4}\right)\left(2E_2^2-E_4+24E_2\widetilde{\wp}+144\widetilde{\wp}^2\right)+6\epsilon_1^2\left(E_4-144\widetilde{\wp}^2\right)\right]\\
w_5' &= \frac{\epsilon_1}{48a^4}\left(m^2-\frac{\epsilon_1^2}{4}\right)\left[\left(m^2-\frac{\epsilon_1^2}{4}\right)\left(E_2+12\widetilde{\wp}\right)\widetilde{\wp}\,'-36\epsilon_1^2\widetilde{\wp}\widetilde{\wp}\,'\right]
\end{aligned}
\tag{43}
$$

and so on. Similar resummation has been done for higher rank gauge groups up to SU(7) for a variety of surface operators corresponding to various decompositions (38) of the gauge group, and very general formulas for the mass coefficients at various orders have been obtained [20]. These contain the entire tail of instanton corrections.

We make contact with known results in the pure gauge theory [21]. The pure gauge theory is obtained by decoupling the adjoint matter hypermultiplet, which is carried out by taking the following limit

$$
m\to\infty,\ \text{ and } q\to 0 \text{ such that } qm^{2N}=(-1)^N\Lambda^{2N} \text{ is finite} \tag{44}
$$

where Λ is the strong coupling scale of the pure $\mathcal{N}=2$ theory. In the presence of a surface operator, this limit must be combined with a scaling prescription for the continuous variables z_i that characterize the defect. In the case when there is only one such parameter $x=\mathrm{e}^{2\pi\mathrm{i}z}$, the scaling is

$$
m\to\infty \quad \text{and} \quad x\to 0 \quad \text{such that} \quad xm^N=(-1)^{p-1}x_0\,\Lambda^N \quad \text{is finite.} \tag{45}
$$

Here $x_0=\mathrm{e}^{2\pi\mathrm{i}z_0}$ is the complex parameter that labels the surface operator in the pure gauge theory. We now restrict ourselves to the SU(2) case which has only one kind of surface operator, and which is labelled by a single complex parameter x. Performing the limits in (44) and (45) on the results obtained for the $\mathcal{N} = 2^\star$ theories, we obtain

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\frac{\partial\mathcal{W}}{\partial z_0} = -a - \frac{\Lambda^2}{2a}\left(x_0 + \frac{1}{x_0}\right) + \frac{\Lambda^4}{8a^3}\left(x_0^2 + \frac{1}{x_0^2}\right) - \frac{\Lambda^6}{16a^5}\left(x_0^3 + x_0 + \frac{1}{x_0} + \frac{1}{x_0^3}\right)\\ + \frac{\Lambda^8}{16a^7}\left(x_0^2 + \frac{1}{x_0^2} + \frac{5}{8}\left(x_0^4 + \frac{1}{x_0^4}\right)\right) + \ldots\end{aligned} \tag{46}$$

We now show that in a specific semi-classical limit, this is identical to the twisted superpotential that one obtains when one couples a two dimensional $\mathbb{CP}^1$ sigma model to the pure $\mathcal{N} = 2$ four dimensional gauge theory.

For the pure gauge theory, the exact (in Λ) expression for the twisted superpotential has already been obtained in [21]. The quantum corrected chiral ring relation for the pure SU(2) gauge theory can be written in the following form [22]:

$$P_2(y) = \Lambda^2\left(x_0 + \frac{1}{x_0}\right)\,, \tag{47}$$

where $P_2(y)$ is the quantum corrected gauge polynomial, given by

$$P_2(y) = y^2 - \tilde{u}\,, \tag{48}$$

$\tilde{u}$ is identified with the gauge invariant parameter on the Coulomb branch, and $x_0 = \mathrm{e}^{2\pi\mathrm{i}z_0}$ is identified with the continuous parameter that labels the surface operator in the pure gauge theory [21, 22]. The variable y is interpreted as the expectation value of the twisted chiral ring element in the two dimensional theory and is related to the twisted superpotential by the relation

$$y = \frac{1}{2\pi\mathrm{i}}\frac{\partial\mathcal{W}}{\partial z_0}\,. \tag{49}$$

Since the effective twisted superpotential generates the quantum chiral ring in the infra-red, it follows that for the simple surface operator $\partial_{z_0}\mathcal{W}$ is simply given by solving for y using (47):

$$\frac{1}{2\pi\mathrm{i}}\frac{\partial\mathcal{W}}{\partial z_0} = -\sqrt{\tilde{u}}\left(1 + \frac{\Lambda^2}{\tilde{u}}\left(x_0 + \frac{1}{x_0}\right)\right)^{\frac{1}{2}} . \tag{50}$$

In order to compare this result with the result we obtained in (46), we have to take a semi-classical limit of this exact result. We consider the limit,

$$|\Lambda|^2 << |\tilde{u}| . \tag{51}$$

In this limit, we can expand the exact expression in a power series in Λ. Expressing $\tilde{u}$ in terms of the classical vacuum expectation value a using Matone's relation, we obtain a perfect match up to two instantons [20]. We have extended the check for simple surface operators for higher rank gauge groups up to SU(7).

0.5 Conclusion

In this thesis, we studied $\mathcal{N} = 2$ theories in four dimensions on their Coulomb branch, with and without the presence of surface operators. The low energy effective action of such theories is described by two holomorphic functions, the prepotential that determines the effective action of the four dimensional theory, and the twisted superpotential which governs the infra-red behaviour of the two-dimensional theory living on the world-volume of the surface operator. We have focused on computing these two functions. In the case of $\mathcal{N} = 2$ super-conformal linear quivers, we computed the effective prepotential via the Seiberg-Witten curve and differential obtained by an M-theory construction, and matched it with the results from equivariant localization, and the 2d/4d correspondence. In doing so, we obtained a detailed map of the various parameters that appear in the three approaches mentioned above. In the latter part of the thesis, we considered surface operators in the context of $\mathcal{N} = 2^{\star}$ theories. The effective superpotential was obtained by equivariant localization, and the microscopic results were resummed into a mass expansion, where at each order in mass, the coefficient functions were made up of (quasi)-modular forms and elliptic functions of the bare coupling constant, and the continuous complex parameters that label the surface operator. These results helped us make contact with known results for the twisted effective superpotential for two dimensional sigma models coupled to four dimensional $\mathrm{SU}(N)$ gauge theories, in a semi-classical limit.

Chapter 1

Essential Supersymmetry

In this chapter, we give a lightning review of $\mathcal{N}=2$ supersymmetry, focusing only on topics that are of relevance to the chapters to follow. We shall mostly stick to the notations and conventions of [23].

1.1 Supersymmetry Algebra

The $\mathcal{N}=2$ supersymmetry algebra, in the presence of central charges is,

$$\begin{aligned} \{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\} &= 2\left(\sigma^\mu\right)_{\alpha\dot{\beta}} P_\mu \delta^I_J \ , \\ \{Q^I_\alpha, Q^J_\beta\} &= 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{IJ} Z \ , \\ \{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{\beta}J}\} &= 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{IJ} Z \ . \end{aligned} \tag{1.1}$$

where Q_α and $\bar{Q}_{\dot{\alpha}}$ are the supersymmetry generators that transform respectively in the $\left(\frac{1}{2},0\right)$ and the $\left(0,\frac{1}{2}\right)$ representations of the Lorentz group, the indices I and J run from 1 to 2 for the 2 supersymmetries, and Z is the central charge. In a massive theory, the central charge gives a lower bound on the mass of the states as,

$$M \geq \sqrt{2}|Z| \ . \tag{1.2}$$

1.2 Representing the Algebra on Fields

In ordinary relativistic quantum field theory, various quantum fields carry different representations of the Lorentz group. In the study of supersymmetric quantum field theories, it is convenient to construct fields that carry a representation of the supersymmetry algebra. This requires enhancing the usual space-time to a superspace, which has spinor coordinates (θ^I_α and $\bar{\theta}^I_{\dot{\alpha}}$) in addition to space-time coordinates. The supersymmetry generators then have a natural action on the superspace, just as Lorentz generators have on the space-time. We will discuss the $\mathcal{N}=1$ superspace[1], and generalize it to the $\mathcal{N}=2$ case. Any function of the superspace coordinates $\left(x_\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right)$ is called a superfield. In general, a superfield contains too many component fields to correspond to an irreducible representation of the $\mathcal{N}=1$ algebra. In order to obtain an irreducible representation, one needs to impose supersymmetry invariant constraints on the superfields. We will now see a few examples that are relevant to our discussion.

1.2.1 $\mathcal{N}=1$ Chiral Multiplet

The $\mathcal{N}=1$ chiral multiplet is represented by a chiral superfield Φ which satisfies the condition,

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \ , \tag{1.3}$$

[1]The $\mathcal{N}=1$ algebra is obtained by setting $Z=0$ in (1.1).

where $\bar{D}_{\dot{\alpha}}$ is the super-covariant derivative $-\partial/\partial\bar{\theta}^{\dot{\alpha}} - \mathrm{i}\sigma^{\mu}_{\alpha\dot{\alpha}}\theta^{\alpha}\partial_{\mu}$. This forces the functional dependence of Φ to be $\Phi(y^{\mu},\theta)$, where $y^{\mu} := x^{\mu} + i\theta\sigma^{\mu}\bar{\theta}$. Thus, an $\mathcal{N}=1$ chiral superfield is expanded as,

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \ , \tag{1.4}$$

where ϕ and ψ are the scalar and the fermionic components in the chiral multiplet, and F is an auxiliary field required for off-shell closure of the supersymmetry algebra, and vanishes on-shell. The anti-chiral multiplet can similarly be represented an anti-chiral superfield $\bar{\Phi}$ which is annihilated by the super-covariant derivative D_{α}. Clearly, any arbitrary function of chiral superfields is also a chiral superfield, and similarly for the anti-chiral superfields. A general function of (anti)-chiral superfields is called the super-potential, which has the following expansion,

$$\begin{aligned} \mathcal{W}\left(\Phi_i\right) &= \mathcal{W}\left(\phi_i + \sqrt{2}\theta\psi_i + \theta\theta F_i\right) \\ &= \mathcal{W}\left(\phi_i\right) + \frac{\partial\mathcal{W}}{\partial\phi_i}\sqrt{2}\theta\psi_i + \theta\theta\left(\frac{\partial\mathcal{W}}{\partial\phi_i}F_i - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial\phi_i\partial\phi_j}\psi_i\psi_j\right) \ . \end{aligned} \tag{1.5}$$

In terms of the original variables $(x^{\mu},\theta,\bar{\theta})$, the chiral superfield (1.4) takes the following expansion,

$$\begin{aligned} \Phi(x,\theta,\bar{\theta}) = \phi(x) + i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi - \frac{1}{4}\theta^2\bar{\theta}^2\ \Box\phi \\ + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_{\mu}\psi\sigma^{\mu}\bar{\theta} + \theta\theta F(x) \ . \end{aligned} \tag{1.6}$$

1.2.2 $\mathcal{N}=1$ Vector Multiplet

The $\mathcal{N}=1$ vector multiplet V is a real superfield, i.e. $V = V^{\dagger}$. After imposing the reality condition on a generic superfield, and choosing the Wess-Zumino gauge, V

takes the following form,

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D \ . \tag{1.7}$$

Here V belongs to the adjoint representation of the gauge group, $V = V_a T^a$ where $T^{a\dagger} = T^a$ are the group generators. The gauge field strength is,

$$W_\alpha = \frac{1}{8}\bar{D}^2 \ \mathrm{e}^{2V} D_\alpha \mathrm{e}^{-2V} \ , \tag{1.8}$$

which in terms of the components has the expansion,

$$W_\alpha = T^a \left(-i\lambda^a_\alpha + \theta_\alpha D^a - \frac{\mathrm{i}}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F^a_{\mu\nu} + \theta^2\sigma^\mu D_\mu\bar{\lambda}^a \right) \ , \tag{1.9}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu$, and $D_\mu\bar{\lambda}^a = \partial_\mu\bar{\lambda}^a + f^{abc}A^b_\mu\bar{\lambda}^c$.

1.3 Supersymmetric Actions

The most general $\mathcal{N} = 1$ supersymmetric Lagrangian with both the gauge multiplet and the charged chiral multiplet is,

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\left(\tau \ \mathrm{Tr}\int d\theta \ W^\alpha W_\alpha\right) + \int d^2\theta d^2\bar{\theta} \ \Phi^\dagger \mathrm{e}^{-2V}\Phi + \int d^2\theta \ \mathcal{W} + \int d^2\bar{\theta} \ \mathcal{W} \ , \tag{1.10}$$

where the $\mathcal{N}=1$ chiral superfield Φ is in any given representation, and $\tau = \frac{\theta}{2\pi} + \frac{4\pi \mathrm{i}}{g^2}$ is the complexified gauge coupling. In terms of component fields,

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\theta}{32\pi^2} F^a_{\mu\nu} \widetilde{F}^{a\mu\nu} - \frac{\mathrm{i}}{g^2} \lambda^a \sigma^\mu D_\mu \bar{\lambda}^a + \frac{1}{2g^2} D^a D^a \\
&+ \left(\partial_\mu \phi - \mathrm{i} A^a_\mu T^a \phi\right)^\dagger \left(\partial^\mu \phi - \mathrm{i} A^{a\mu} T^a \phi\right) - \mathrm{i} \bar{\psi} \bar{\sigma}^\mu \left(\partial_\mu \psi - \mathrm{i} A^a_\mu T^a \psi\right) \\
&- D^a \phi^\dagger T^a \phi - \mathrm{i}\sqrt{2} \phi^\dagger T^a \lambda^a \psi + \mathrm{i}\sqrt{2} \bar{\psi} T^a \phi \bar{\lambda}^a + F^\dagger_i F_i \\
&+ \frac{\partial \mathcal{W}}{\partial \phi_i} F_i + \frac{\partial \bar{\mathcal{W}}}{\partial \phi^\dagger_i} F^\dagger_i - \frac{1}{2} \frac{\partial^2 \mathcal{W}}{\partial \phi_i \partial \phi_j} \psi_i \psi_j - \frac{1}{2} \frac{\partial^2 \bar{\mathcal{W}}}{\partial \phi^\dagger_i \partial \phi^\dagger_j} \bar{\psi}_i \bar{\psi}_j \ ,
\end{aligned} \tag{1.11}$$

where $\mathcal{W}$ denotes the scalar component of the superpotential. The on-shell field content of an $\mathcal{N}=2$ vector multiplet is an $\mathcal{N}=1$ chiral multiplet (ϕ, ψ), and an $\mathcal{N}=1$ vector multiplet (λ, A_μ). Thus, the Lagrangian in (1.11) has all the fields, but is not $\mathcal{N}=2$ supersymmetric. $\mathcal{N}=2$ supersymmetry forces all the fields to be in the adjoint representation of the gauge group. Since $\mathcal{N}=2$ supersymmetry treats ψ and λ on the same footing, the form of their kinetic terms in (1.11) suggests that the $\mathcal{N}=2$ Lagrangian is a combination of the chiral superfield, and the vector superfield Lagrangians, with the former multiplied by $\frac{1}{g^2}$. The $\mathcal{N}=2$ supersymmetry sets the superpotential $\mathcal{W}$ to zero, as it couples only to ψ^a. Thus, the full Lagrangian with $\mathcal{N}=2$ supersymmetry is,

$$\begin{aligned}
\mathcal{L} &= \frac{1}{8\pi} \mathrm{Im}\ \mathrm{Tr} \left[\tau \left(\int d^2\theta\ W^\alpha W_\alpha + 2 \int d^2\theta d^2\bar{\theta}\ \Phi^\dagger \mathrm{e}^{-2V} \Phi\right)\right] \\
&= \frac{1}{g^2} \mathrm{Tr}\Bigg(- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + g^2 \frac{\theta}{32\pi^2} F_{\mu\nu} \widetilde{F}^{\mu\nu} + \left(D_\mu \phi\right)^\dagger D^\mu \phi - \frac{1}{2} [\phi^\dagger, \phi]^2 \\
&\qquad - \mathrm{i} \lambda \sigma^\mu D_\mu \bar{\lambda} - \mathrm{i} \bar{\psi} \bar{\sigma}^\mu D_\mu \psi - \mathrm{i}\sqrt{2} [\lambda, \psi] \phi^\dagger - \mathrm{i}\sqrt{2} [\bar{\lambda}, \bar{\psi}] \phi \Bigg) \ .
\end{aligned} \tag{1.12}$$

$\mathcal{N}=2$ supersymmetry is made manifest by going to the $\mathcal{N}=2$ superspace which has two sets of spinor coordinates, $(\theta_\alpha, \bar{\theta}_{\dot{\alpha}}, \widetilde{\theta}_\alpha, \bar{\widetilde{\theta}}_{\dot{\alpha}})$. The $\mathcal{N}=2$ chiral superfield is defined by the constraints,

$$\bar{D}_{\dot{\alpha}}\Psi = 0, \quad \bar{\widetilde{D}}_{\dot{\alpha}}\Psi = 0 \ , \tag{1.13}$$

where D_α and $\widetilde{D}_\alpha$ are the super-covariant derivatives with respect to θ and $\widetilde{\theta}$ respectively. The chiral superfield has an expansion of the form,

$$\Psi = \Psi^{(1)}(\widetilde{y},\theta) + \sqrt{2}\widetilde{\theta}^\alpha \Psi^{(2)}_\alpha(\widetilde{y},\theta) + \widetilde{\theta}^\alpha\widetilde{\theta}_\alpha \Psi^{(3)}(\widetilde{y},\theta) \ , \tag{1.14}$$

where $\widetilde{y}^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} + i\widetilde{\theta}\sigma^\mu\bar{\widetilde{\theta}}$. The $\mathcal{N}=2$ Lagrangian in terms of the $\mathcal{N}=2$ superfield takes the compact form,

$$\mathcal{L} = \frac{1}{4\pi}\text{Im Tr}\int d^2\theta d^2\widetilde{\theta}\ \frac{1}{2}\tau\Psi^2 \ . \tag{1.15}$$

Note that the $\mathcal{N}=2$ supersymmetric action is holomorphic in the fields, and in the couplings, as it depends only on Ψ and τ, and not on $\Psi^\dagger$ and $\bar{\tau}$. In general, $\mathcal{N}=2$ supersymmetry constrains the effective Lagrangian with at most two derivatives, and not more than four fermions as,

$$\mathcal{L} = \frac{1}{4\pi}\text{Im Tr}\int d^2\theta d^2\widetilde{\theta}\mathcal{F}(\Psi) \ , \tag{1.16}$$

where $\mathcal{F}$ is the $\mathcal{N}=2$ prepotential. In $\mathcal{N}=1$ superspace, the effective Lagrangian takes the form,

$$\mathcal{L} = \frac{1}{8\pi}\text{Im}\left(\int d^2\theta\mathcal{F}_{ab}(\Phi)W^{a\alpha}W^b_\alpha + 2\int d^2\theta d^2\bar{\theta}\left(\Phi^\dagger e^{2gV}\right)^a\mathcal{F}_a(\Phi)\right) \ , \tag{1.17}$$

where $\mathcal{F}_a(\Phi) = \frac{\partial\mathcal{F}}{\partial\Phi^a}$, $\mathcal{F}_{ab}(\Phi) = \frac{\partial^2\mathcal{F}}{\partial\Phi^a\partial\Phi^b}$. For the classical $\mathcal{N}=2$ super Yang-Mills action in (1.15), the prepotential is given by,

$$\mathcal{F}_{\text{classical}}(\Psi) = \frac{1}{2}\text{ Tr }\tau\Psi^2 \ . \tag{1.18}$$

1.4 One-loop contribution to the Prepotential

Due to the non-renormalization theorem by Seiberg in [24], the prepotential has been found to be perturbatively exact at one-loop. In this section, we compute this one-loop correction for a pure theory with gauge group $\mathrm{SU}(N_c)$, following [23].

Classically, an $\mathcal{N}=2$ theory has the global symmetry group $\mathrm{SU}(2)_R\times\,\mathrm{U}(1)_R$. While the $\mathrm{SU}(2)_R$ rotates the two θ's into each other, the $\mathrm{U}(1)_R$ gives them a phase. In terms of field content, the fermions form a doublet, and the scalar and the gauge field form singlets under the $\mathrm{SU}(2)_R$. At the quantum level, the $\mathrm{U}(1)_R$ symmetry is broken by the chiral anomaly,

$$\partial_\mu j_5^\mu = -\frac{N_c}{8\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu}\ . \tag{1.19}$$

Thus, under a $\mathrm{U}(1)_R$ transformation, the Lagrangian changes as

$$\delta\mathcal{L}_{\text{eff}} = -\frac{\alpha N_c}{8\pi^2}F\widetilde{F}\ . \tag{1.20}$$

Since $(32\pi^2)^{-1}\int F\widetilde{F}$ is an integer, the $\mathrm{U}(1)_R$ invariance is broken to Z_{4N_c}. Demanding that the Lagrangian changes as above, and focussing only on the relevant terms from (1.12), we obtain,

$$\frac{1}{16\pi}\mathrm{Im}\left[\mathcal{F}''\left(\mathrm{e}^{2\mathrm{i}\alpha}\phi\right)\left(-FF+\mathrm{i}F\widetilde{F}\right)\right] = \frac{1}{16\pi}\mathrm{Im}\ \left[\mathcal{F}''(\phi)(-FF+\mathrm{i}F\widetilde{F})\right] - \frac{\alpha N_c}{8\pi^2}F\widetilde{F}\ . \tag{1.21}$$

For infinitesimal α, we obtain the differential equation,

$$\frac{\partial^3\mathcal{F}}{\partial\phi^3} = \frac{N_c}{\pi}\frac{\mathrm{i}}{\phi}\ , \tag{1.22}$$

which for $N_c = 2$ integrates to give,

$$\mathcal{F}_{\text{1-loop}}(\phi) = \frac{\mathrm{i}}{2\pi}\phi^2 \log\left(\frac{\phi^2}{\Lambda^2}\right) . \tag{1.23}$$

where Λ is a dynamically generated scale.

The one-loop correction to the prepotential may also be obtained from the β-function of the theory. The one-loop β-function of the theory is,

$$\beta(g) = -\frac{dg}{d\ln\Lambda} = -\frac{g^3}{4\pi^2} , \tag{1.24}$$

which upon integration leads to

$$\frac{1}{g^2} = \frac{1}{g_0^2} - \frac{1}{4\pi^2}\log\left(\frac{\Lambda^2}{\phi^2}\right) . \tag{1.25}$$

In order to make contact with the complexified gauge coupling, we rewrite the above as,

$$\frac{4\pi}{g^2} = \frac{4\pi}{g_0^2} - \frac{1}{\pi}\log\left(\frac{\Lambda^2}{\phi^2}\right) , \tag{1.26}$$

which gives,

$$\frac{\partial^2\mathcal{F}}{\partial a^2} \equiv \tau(a) = \tau_0 + \frac{\mathrm{i}}{\pi}\log\left(\frac{\phi^2}{\Lambda^2}\right) , \tag{1.27}$$

which upon integration gives,

$$F_{\text{1-loop}} = \frac{\mathrm{i}}{\pi}\left(-\frac{3}{4}a^2 + a^2\log\left(\frac{\phi^2}{\Lambda^2}\right)\right) . \tag{1.28}$$

Although the prepotential is perturbatively exact at one-loop, it gets corrections at all orders from the non-perturbative sector due to instantons. The contribution of the k instanton sector to the prepotential is,

$$\begin{aligned} \mathrm{e}^{-S_{\text{inst}}} &= \exp\left(-\frac{8\pi^2 k}{g^2}\right) \\ &= \exp\left(-\frac{8\pi^2 k}{g_0^2}\right)\left(\frac{\Lambda}{a}\right)^{4k} \ . \end{aligned} \tag{1.29}$$

Thus, the instanton sector of the prepotential takes the form,

$$F_{\text{inst}} = \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{a}\right)^{4k} a^2 \ . \tag{1.30}$$

For a conformal theory, there is only the bare coupling τ_0, and

$$F_{\text{inst}} \propto \exp(2\pi \mathrm{i} \tau_0) \ . \tag{1.31}$$

The goal of Seiberg and Witten in [1] was to compute the $\mathcal{F}_k$. In the first half of this thesis, we describe various methods to compute these coefficients for a class of theories.

Chapter 2

Seiberg-Witten Theory

Given a quantum field theory, one is often interested in the low energy effective action. Starting from the original bare action with a momentum cut-off given by μ, the standard prescription to obtain the low energy effective action is to compute the Wilsonian effective action. We integrate out modes above a particular mass scale, say Λ, to obtain the effective action $S[\Lambda]$ for the low lying states $(E < \Lambda)$ in the theory. However, such an approach fails when there is a moduli space, and the masses of the states depend on the position on the moduli space. In such cases, the Wilsonian approach fails, as it may lead to integrating out massless modes.

In this chapter, we describe the Seiberg-Witten theory which computes the low energy effective action of $\mathcal{N} = 2$ supersymmetric gauge theories, which are known to have a moduli space. The Seiberg-Witten theory for the pure SU(2) gauge theory introduced in [1], later extended to higher gauge groups in [6–8], and to the cases with matter in [2] concerns itself with understanding the low energy effective action on the Coulomb branch of these theories. We restrict our attention to the pure gauge theory with gauge group SU(2). This presents itself with enough details for all the work that follows.

2.1 Moduli space of vacua

We consider the pure $\mathcal{N}=2$ gauge theory with gauge group SU(2). The Lagrangian (1.12) has the scalar potential,

$$V\left(\phi,\phi^{\dagger}\right)=\frac{1}{g^2}\mathrm{Tr}\left[\phi,\phi^{\dagger}\right]^2 \tag{2.1}$$

Supersymmetry preserving classical vacua do not require ϕ to vanish, but only requires it to be valued in the Cartan sub-algebra of the gauge group, i.e.

$$\phi=\frac{1}{2}\,a\,\sigma_3\ , \tag{2.2}$$

where $a\in\mathbb{C}$, and $\sigma_3=\mathrm{diag}\,(1,-1)$. The vacua breaks the gauge group SU(2) down to U(1), and for obvious reasons this is called the Coulomb branch of the theory. There is a continuous family of vacua labelled by the vacuum expectation value of the adjoint scalar ϕ. The manifold of this family is called the moduli space of vacua, and classically it is $\mathbb{C}$. It is clear that a is not a gauge-invariant parameterization of the moduli space, as there are Weyl reflections that change the vacuum by the action $a\to -a$. Gauge-invariant parameterization of the moduli space is given by,

$$u=\frac{1}{2}\mathrm{Tr}\ \phi^2\ , \tag{2.3}$$

Semi-classically, $u=\frac{a^2}{2}$. However, when one takes into account quantum fluctuations, gauge inequivalent vacua are labelled by,

$$u\equiv\frac{1}{2}\langle\mathrm{Tr}\ \phi^2\rangle\ . \tag{2.4}$$

We now go back to the global symmetry group, discussed in section 1.4. Chiral anomaly breaks the symmetry group $\mathrm{SU}(2)_R\times$ $\mathrm{U}(1)_R$ to $\mathrm{SU}(2)_R\times\mathbb{Z}_8$. Accounting for the center of $\mathrm{SU}(2)_R$ which is also contained in $\mathbb{Z}_8$, the symmetry group breaks into $\mathrm{SU}(2)_R \times \mathbb{Z}_8/\mathbb{Z}_2$. A non-zero vacuum expectation value for the adjoint scalar further breaks the $\mathbb{Z}_8$ to $\mathbb{Z}_4$. The non-trivial action of the global symmetry group on the Coulomb moduli u is as, $u \to -u$. Classically, there is a singularity at $u = 0$, where the broken symmetry on the Coulomb branch is restored, and additional fields go massless.

2.1.1 Metric on the Moduli Space

Let us denote the $\mathcal{N} = 1$ chiral superfield on the Coulomb branch by A. The effective Lagrangian (1.16) on the Coulomb branch is,

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta \frac{\partial \mathcal{F}}{\partial A}\bar{A} + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial A^2} W_\alpha W^\alpha\right] , \tag{2.5}$$

where we have the notation $d^4\theta \equiv d^2\theta d^2\bar{\theta}$. This can be viewed as a sigma-model with a metric on the field space which in turn gives the metric on the moduli space,

$$(ds)^2 = \mathrm{Im}\ \frac{\partial^2 \mathcal{F}(a)}{\partial a^2} da\ d\bar{a} = \mathrm{Im}\ \tau(a)\ da\ d\bar{a}\ , \tag{2.6}$$

where a is the vacuum expectation value of the scalar component of A. As a harmonic function cannot have a minimum, the metric Im $\tau(a)$ is not globally defined. Thus, the requirement of positivity of the metric makes it clear that the description in terms of a is valid only locally.

2.2 An Electric-Magnetic Duality

From the above discussion, it is clear that we require a different description of the moduli space where Im $\tau(a) < 0$. To obtain such a description, we consider terms involving only the gauge fields in the $\mathcal{N} = 2$ Lagrangian (1.12). We have the terms,

$$\frac{1}{32\pi}\text{Im} \int \tau(a) \left(F + i\widetilde{F}\right)^2 = \frac{1}{16\pi}\text{Im} \int \tau(a)(F^2 + i\widetilde{F}F) \ . \tag{2.7}$$

Treating F as the independent field, we implement the Bianchi identity $dF = 0$ by coupling a Lagrange multiplier vector field V_D to a monopole. The monopole satisfies,

$$\epsilon^{0\mu\nu\rho}\partial_\nu F_{\nu\rho} = 8\pi\delta^{(3)}(x) \ . \tag{2.8}$$

The Lagrange multiplier term is then,

$$\frac{1}{8\pi}\int V_{D_\mu}\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = \frac{1}{8\pi}\int \widetilde{F}_D F = \frac{1}{16\pi}\text{Re} \int (\widetilde{F}_D - \text{i}F_D)(F + \text{i}\widetilde{F}) \tag{2.9}$$

where, $F_{D_{\mu\nu}} = \partial_\mu V_{D_\nu} - \partial_\nu V_{D_\mu}$. Adding the Lagrange multiplier term to the action, and integrating over F, we obtain,

$$\frac{1}{32\pi}\text{Im} \int \left(-\frac{1}{\tau}\right)(F_D + i\widetilde{F}_D)^2 = \frac{1}{16\pi}\text{Im} \int \left(-\frac{1}{\tau}\right)\left(F_D^2 + \widetilde{F}_D F_D\right) \ . \tag{2.10}$$

Comparing (2.7) and (2.10), the effect of the duality transformation is to replace the gauge field A_μ which couples to electric charges, by the dual gauge field V_{D_μ} which couples to magnetic charges, and transforms the complexified gauge coupling as,

$$\tau \to \tau_D = -\frac{1}{\tau} \ . \tag{2.11}$$

This is how electric-magnetic duality manifests itself in these theories. The action (1.12) is also invariant under $\tau \to \tau + b \implies \theta \to \theta + 2\pi b$, which requires $b \in \mathbb{Z}$. These two transformations generate the SL(2,$\mathbb{Z}$) duality group of the theory. A general element of SL(2,$\mathbb{Z}$) acts on the coupling as,

$$\tau \to \frac{a\tau + b}{c\tau + d} \ , \tag{2.12}$$

where $ad-bc=1$, and $a,b,c,d\in\mathbb{Z}$. While $\tau\to-\frac{1}{\tau}$ corresponds to a dual description of the theory, $\tau\to\tau+1$ corresponds to an actual symmetry of the theory. We will now find out the magnetic variables a_D corresponding to a. For this, we introduce $h(A)=\partial\mathcal{F}/\partial A$. The complexified gauge coupling is then, $\tau(A)=\partial h(A)/\partial A$. The scalar kinetic energy term in (2.5) becomes $\mathrm{Im}\int d^4\theta\ h(A)\bar{A}$. For the dual theory, we introduce the variables $A_D,\mathcal{F}_D,h_D\left(A_D\right)$ and τ_D. From (2.11), we obtain $A_D=h=\partial\mathcal{F}/\partial A$, and $h_D=-A$. With $a_D=\frac{\partial\mathcal{F}}{\partial a}$, the metric (2.6) on the moduli space takes the completely symmetric form,

$$ds^2=\mathrm{Im}\ da_D\ d\bar{a}=-\frac{i}{2}\left(da_D d\bar{a}-da\ d\bar{a}_D\right)\ . \tag{2.13}$$

We will now identify the class of local parameters in terms of which the metric can be written as above. For this, we introduce the set $a^\alpha=(a_D,a)$, where $\alpha=1,2$, and the antisymmetric tensor $\epsilon_{\alpha,\beta}$ with $\epsilon_{1,2}=1$. The metric may then be rewritten as,

$$ds^2=-\frac{i}{2}\epsilon_{\alpha\beta}\frac{da^\alpha}{du}\frac{d\bar{a}^\beta}{d\bar{u}}du\ d\bar{u}\ . \tag{2.14}$$

The metric has $\mathrm{SL}(2,\mathbb{R})$ invariance which preserves the ϵ tensor, and commutes with complex conjugation. The $\mathrm{SL}(2,\mathbb{R})$ group is generated by,

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\ . \tag{2.15}$$

The latter corresponds to $\tau\to\tau+b\implies\theta\to\theta+2\pi b$ which requires $b\in\mathbb{Z}$. Thus, we again see that the theory enjoys an $\mathrm{SL}(2,\mathbb{Z})$ duality.

2.3 Monodromies on the Moduli Space

In section 2.1.1, we saw that the moduli space of vacua has singularities. Understanding the details of the singularity structure will help us solve the low energy theory. At large $|a|$, the theory is asympotically free, and $u = \frac{1}{2}a^2$. In this regime, the prepotential is approximated by the one-loop answer (1.23),

$$\mathcal{F}(a) = \frac{i}{2\pi} a^2 \log\left(a^2/\Lambda^2\right) \ . \tag{2.16}$$

The dual magnetic variables are then,

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = \frac{2\mathrm{i}a}{\pi} \log\left(\frac{a}{\Lambda}\right) + \frac{\mathrm{i}a}{\pi} \ . \tag{2.17}$$

Making a closed loop on the u-plane around $u = 0$, $\log u \to \log u + 2\pi\mathrm{i}$, and $\log a \to \log a + \mathrm{i}\pi$. Thus we get,

$$\begin{aligned} a_D &\to -a_D + 2a \\ a &\to -a \end{aligned} \tag{2.18}$$

The monodromy matrix acting on $(a_D, a)^T$ is,

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \ . \tag{2.19}$$

The monodromy at infinity signals a non-trivial monodromy at a finite point on the moduli space. Let us now try to understand the number of singularities that might be there. If there is only one more singularity, it has to be at the origin of the moduli space, else the $u \to -u$ symmetry on the u-plane, mentioned in section 2.1 is not respected. However, if there is only one singularity at a finite point on the moduli space, it commutes with the monodromy at ∞, making a a good coordinate over the entire moduli space, in contradiction to what we saw in section 2.1.1. Thus, we require at least two singularities at finite points on the u-plane with non-trivial monodromies around them. The R-symmetry group imposes that these be at $+u_0$ and $-u_0$. Singularities on the moduli space occur due to massive particles going massless at these points. The naive intuition that these particles correspond to gauge bosons leads to inconsistencies. The other massive states are monopoles and dyons, that belong to hypermultiplets, and are BPS states. Seiberg and Witten conjectured that singularities at finite points on the moduli space correspond to these states going massless at the two chosen points. The hypermultiplet with monopoles and dyons does not couple to the fundamental fields in our theory locally. However, from section 2.2, it is possible to go to the dual description where these are locally coupled to the dual fields.

The central charge in (1.2) is given by,

$$Z = an_e + a_D n_m \ , \tag{2.20}$$

where n_e and n_m denote the units of electric and magnetic charges respectively. Let us suppose that at the point u_0 on the moduli space, magnetic monopoles go massless. From the mass formula (1.2), and the form of the central charge Z in (2.20), we have,

$$a_D\left(u_0\right) = 0 \ . \tag{2.21}$$

As mentioned above, unlike electric charges, monopoles do not couple locally to photons, but instead couples to the dual photon field. The low energy theory is $\mathcal{N} = 2$ SQED. From the one-loop beta function, the magnetic coupling is,

$$\tau_D \approx -\frac{\mathrm{i}}{\pi} \ln a_D \ . \tag{2.22}$$

Since a_D is a good coordinate near u_0,

$$a_D \approx c_0(u - u_0) \ , \tag{2.23}$$

where c_0 is come constant. Following the discussion in section 2.2, the electric parameter a is obtained as,

$$a(u) \approx a_0 + \frac{\mathrm{i}}{\pi} a_D \log{(a_D)} = a_0 + \frac{\mathrm{i}}{\pi} c_0 \,(u - u_0) \log{(u - u_0)} \ . \tag{2.24}$$

When u circles around u_0, $\log(u - u_0) \to \log(u - u_0) + 2\pi\mathrm{i}$, and we obtain,

$$\begin{aligned} a_D &\to a_D \\ a &\to a - 2a_D \end{aligned} \tag{2.25}$$

Thus the monodromy matrix associated with a monopole going massless at u_0 is,

$$M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \ . \tag{2.26}$$

The monodromy matrix corresponding to the third singularity is obtained from the relation $M_{u_0} M_{-u_0} = M_\infty$. This relation gives,

$$M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \ . \tag{2.27}$$

The particle that goes massless to generate this singularity is described by the condition, $(n_m, n_e)\, M_{-u_0} = (n_m, n_e)$. This imposes the condition, $n_m = -n_e$ on the charge of the dyon that goes massless at $-u_0$.

2.4 Solution

The Seiberg-Witten solution for the prepotential is described by an algebraic curve and an associated differential. We follow the discussion in [8]. The quantum moduli space of the pure gauge theory with gauge group SU(2) described above, coincides with the moduli space of the elliptic curve,

$$y^2 = (x^2 - u)^2 - \Lambda^4 \ . \tag{2.28}$$

We denote the four zero's of $p(x) = y^2(x)$ by $e_1^+ = -\sqrt{u+\Lambda^2}$, $e_1^- = -\sqrt{u-\Lambda^2}$, $e_2^- = \sqrt{u-\Lambda^2}$ and $e_2^+ = \sqrt{u+\Lambda^2}$. The basis for the homology cycles are as in the figure below. Our goal is to compute the periods,

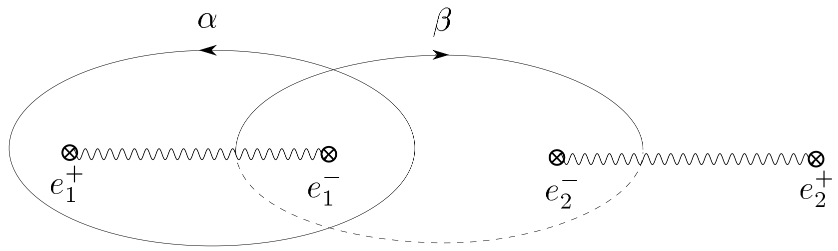

Figure 2.1: The α and the β cycles for the elliptic curve (2.28) in the x-plane

.

$$\begin{pmatrix} \omega_D \\ \omega \end{pmatrix} = \frac{\partial}{\partial u}\pi(u) \equiv \frac{\partial}{\partial u}\begin{pmatrix} a_D \\ a \end{pmatrix} \sim \begin{pmatrix} \oint_\beta \\ \oint_\alpha \end{pmatrix} \cdot \frac{dx}{y(x,u)} \ , \tag{2.29}$$

so that the prepotential is obtained by a simple integration of $a_D(a)$. In order to compute the periods, we use the fact that the periods form a system of solutions of the Picard-Fuchs equation associated with (2.28). The periods are given in terms of hypergeometric functions,

$$
\begin{aligned}
a_D(\alpha) &= \oint_\beta \lambda = \frac{\mathrm{i}}{4}\Lambda(\alpha-1)\ {}_2F_1\left(\frac{3}{4},\frac{3}{4},2;1-\alpha\right)\ , \\
a(\alpha) &= \oint_\alpha \lambda = \frac{1}{1+\mathrm{i}}\Lambda(1-\alpha)^{1/4}\ {}_2F_1\left(-\frac{1}{4},\frac{3}{4},1;\frac{1}{1-\alpha}\right)\ .
\end{aligned} \tag{2.30}
$$

where, $\alpha = \frac{u^2}{\Lambda^4}$. After inverting $a(u)$, and inserting it into $a_D(u)$, the prepotential is obtained by integration with respect to a. We obtain the instanton contribution,

$$
\mathcal{F}_{\text{inst}} = -\frac{\mathrm{i}}{\pi}\sum_{k=1}^{\infty}\mathcal{F}_k\left(\frac{\Lambda}{2a}\right)^{4k} a^2\ , \tag{2.31}
$$

at the first few orders,

$$
\mathcal{F}_1 = \frac{1}{2},\quad \mathcal{F}_2 = \frac{5}{64},\quad \mathcal{F}_3 = \frac{3}{64},\ldots \tag{2.32}
$$

We have obtained the same results using Nekrasov's methods.

In the following, we mention a check of the Seiberg-Witten solution described above, following [25]. Let us for simplicity work at one instanton. We have,

$$
\tau(a) = \frac{\partial^2\mathcal{F}}{\partial a^2} = -\frac{3\mathrm{i}}{16\pi}\frac{\Lambda^4}{a^4}\ . \tag{2.33}
$$

In the effective action on the Coulomb branch,

$$
\mathcal{L}_{eff} \supset \frac{1}{8\pi}\int d^2\theta\ \mathrm{Im}\left(\tau(a)W_\alpha W^\alpha\right)\ . \tag{2.34}
$$

This gives,

$$
\mathcal{L}_{eff} \supset \frac{1}{8\pi}\left(\frac{-3\Lambda^4}{16\pi}\right)\int d^2\theta\ \frac{1}{a^4}W_\alpha W^\alpha \tag{2.35}
$$

We now put to use the fact that a is a chiral superfield, and has the following expansion in $\mathcal{N} = 1$ superspace,

$$a(\theta) = a + \sqrt{2}\theta\psi + \dots \tag{2.36}$$

so that,

$$a^{-4}(\theta) = a^{-4}\left(1 + \frac{\sqrt{2}\theta\psi}{a}\right)^{-4} \sim a^{-4}\left(1 + 20\frac{\theta^2\psi^2}{a^2} + \dots\right) + \dots \tag{2.37}$$

This gives the following contribution to the action,

$$\begin{aligned} \mathcal{L}_{eff} &\supset -\frac{3\Lambda^4}{128\pi^2}\int d^2\theta\ \theta^2\left(\psi^2\lambda^2\right) \times \frac{20}{a^6} \\ &= -\frac{15}{32\pi^2}\frac{\Lambda^4}{a^6}\psi^2\lambda^2 \end{aligned} \tag{2.38}$$

The four point function $\langle\psi\psi\lambda\lambda\rangle$ has been explicitly computed in the background of a single instanton to match this answer.

2.5 Seiberg-Witten curves from M-theory

In this section, following [26], we review the M-theory construction [10] of the Seiberg-Witten curves for $\mathcal{N} = 2$ quiver gauge theories in four dimensions. This will fix our conventions, and set the stage for the explicit calculations in the sections that follow.

We begin with a collection of NS5 branes and D4 branes in Type IIA string theory, arranged as shown in Table 2.1. The first four directions $\{x^0, x^1, x^2, x^3\}$ are longitudinal for both kinds of branes and span the space-time $\mathbb{R}^{1,3}$ where the quiver gauge theory is defined. After compacting the x^5 direction on a circle S^1 of radius R_5, we uplift the system to M-theory by introducing a compact eleventh coordinate x^{10} with radius R_{10}. We finally minimize the world-volume of the resulting M5 branes;

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
NS5 branes	$-$	$-$	$-$	$-$	$-$	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
D4 branes	$-$	$-$	$-$	$-$	$\cdot$	$\cdot$	$-$	$\cdot$	$\cdot$	$\cdot$	$-$

Table 2.1: Type IIA brane configuration: $-$ and $\cdot$ denote longitudinal and transverse directions respectively; the last column refers to the eleventh dimension after the M-theory uplift.

in this way we obtain the Seiberg-Witten curve for a 5-dimensional $\mathcal{N}=1$ gauge theory defined in $\mathbb{R}^{1,3}\times S^1$ which takes the form of a 2-dimensional surface inside the space parameterized by $\{x^4, x^5, x^6, x^{10}\}$. To get the curve for the $\mathcal{N}=2$ theory in four dimensions, we first perform a T-duality along x^5 and then take the limit of small (dual) radius. Thus, in terms of the dual circumference

$$\beta = \frac{2\pi\alpha'}{R_5}\ , \tag{2.39}$$

the 4-dimensional limit corresponds to $\beta\to 0$. Let us now give some details.

2.5.1 Brane solution

We want to engineer a conformal quiver with n SU(2) nodes, two massive fundamental flavors attached to the first node, two massive fundamental flavors attached to the last node and one massive bi-fundamental hypermultiplet between each pair of nodes[1]. To do so we consider a brane system in Type IIA consisting of:

- $n+1$ NS5 branes separated by finite distances along the x^6 direction; we denote them as NS5$_i$ with $i=1,\dots,n+1$.
- Two semi-infinite D4 branes ending on NS5$_1$ and two semi-infinite D4 branes ending on NS5$_{n+1}$; we call them flavour branes.

[1] With this field content, the β-function vanishes for each SU(2) factor; see (A.1).

- Two finite D4 branes stretching between $\mathrm{NS5}_i$ and $\mathrm{NS5}_{i+1}$ for $i = 1, \ldots, n$; we will refer to them as colour branes.

In Fig. 2.2 we have represented, as an example, the set-up for the 2-node quiver theory ($n = 2$).

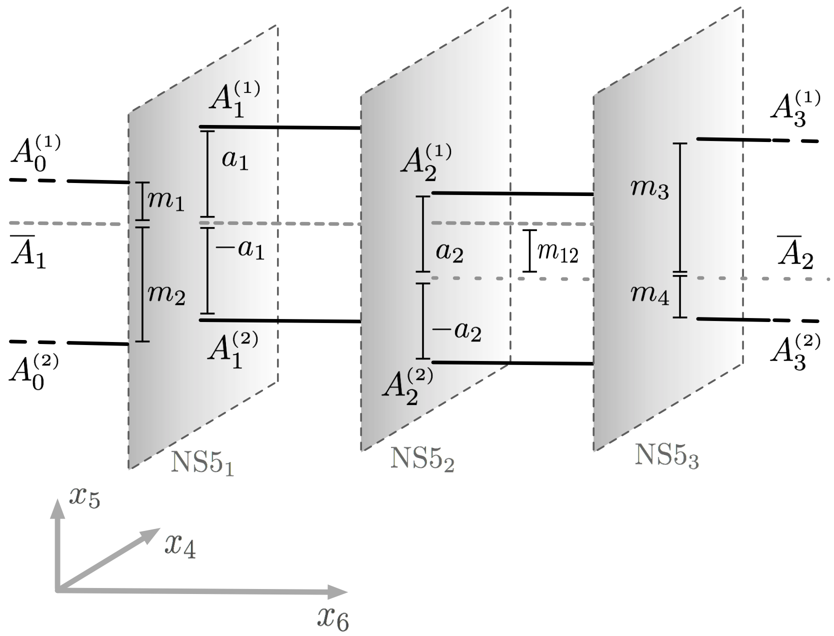

Figure 2.2: NS5 and D4 brane set up for the conformal SU(2) × SU(2) quiver theory

The brane configuration is best described in terms of the complex combinations

$$x^4 + \mathrm{i}\, x^5 \equiv 2\pi\alpha' v \qquad \text{and} \qquad x^6 + \mathrm{i}\, x^{10} \equiv s\ , \tag{2.40}$$

or their exponentials

$$w \equiv \mathrm{e}^{\frac{2\pi\alpha' v}{R_5}} = \mathrm{e}^{\beta v} \qquad \text{and} \qquad t \equiv \mathrm{e}^{\frac{s}{R_{10}}} \tag{2.41}$$

which are single-valued under integer shifts of x^5 and x^{10} along the respective circumferences. Notice that we have introduced factors of α' to assign to v scaling dimensions of a mass; this choice will be particularly convenient for our later purposes. For each NS5$_i$ the variable s_i satisfies the Poisson equation in the v-plane [10]

$$\nabla^2 s_i = f_i \tag{2.42}$$

where the source term in the right hand side describes the pulling on the i-th NS5 brane due to the D4 branes terminating on it from each side. For our configuration this is simply a sum of four δ-functions localized at the relevant D4 positions in the v-plane. We denote the positions of the flavour D4 branes on the left by $\left(A_0^{(1)}, A_0^{(2)}\right)$, those of the flavour D4 branes on the right by $\left(A_{n+1}^{(1)}, A_{n+1}^{(2)}\right)$, and those of the colour D4 branes between NS5$_i$ and NS5$_{i+1}$ by $\left(A_i^{(1)}, A_i^{(2)}\right)$. Since x^5 is compact, we have to take into account also the infinite images of these brane positions and hence the solution of the Poisson equation (2.42) is

$$\begin{aligned}\frac{s_i}{R_{10}} = \sum_{k=-\infty}^{\infty} \Bigg\{ &\log\left[\beta\left(v - A_{i-1}^{(1)}\right) - 2\pi \mathrm{i} k\right] + \log\left[\beta\left(v - A_{i-1}^{(2)}\right) - 2\pi \mathrm{i} k\right] \\ &- \log\left[\beta\left(v - A_{i}^{(1)}\right) - 2\pi \mathrm{i} k\right] - \log\left[\beta\left(v - A_{i}^{(2)}\right) - 2\pi \mathrm{i} k\right]\Bigg\} + \text{const.}\end{aligned} \tag{2.43}$$

for $i = 1, \ldots, n+1$. Using the identity

$$\prod_{k=1}^{\infty}\left(1 + \frac{x^2}{k^2}\right) = \frac{\sinh \pi x}{\pi x} \ , \tag{2.44}$$

and exponentiating the above result, this can be rewritten as

$$\mathrm{e}^{\frac{s_i}{R_{10}}} = t_i \, \frac{\sinh\left(\frac{\beta}{2}\left(v - A_{i-1}^{(1)}\right)\right) \sinh\left(\frac{\beta}{2}\left(v - A_{i-1}^{(2)}\right)\right)}{\sinh\left(\frac{\beta}{2}\left(v - A_{i}^{(1)}\right)\right) \sinh\left(\frac{\beta}{2}\left(v - A_{i}^{(2)}\right)\right)} \ , \tag{2.45}$$

where t_i is related to the integration constant in (2.43). The asymptotic positions of the NS5 branes can be obtained by taking the limits re$v \to -\infty$ (*i.e.* $w \to 0$)

and real$v \to +\infty$ (*i.e.* $w \to \infty$) and are given by

$$\mathrm{e}^{\frac{s_i}{R_{10}}}\Big|_{w\to 0} = t_i \sqrt{\frac{\widetilde{A}^{(1)}_{i-1}\,\widetilde{A}^{(2)}_{i-1}}{\widetilde{A}^{(1)}_{i}\,\widetilde{A}^{(2)}_{i}}} \equiv t_i^{(0)}~, \qquad \mathrm{e}^{\frac{s_i}{R_{10}}}\Big|_{w\to \infty} = t_i \sqrt{\frac{\widetilde{A}^{(1)}_{i}\,\widetilde{A}^{(2)}_{i}}{\widetilde{A}^{(1)}_{i-1}\,\widetilde{A}^{(2)}_{i-1}}} \equiv t_i^{(\infty)}~. \tag{2.46}$$

Here we have introduced tilded variables according to

$$\widetilde{A} = \mathrm{e}^{\beta\, A} \tag{2.47}$$

for any given A.

As argued in [10], the difference in the asymptotic positions of the NS5 branes is related to the complexified UV coupling constant of the gauge theory on the color D-branes; more precisely if we define

$$\tau_i = \frac{\theta_i}{\pi} + i\,\frac{8\pi}{g_i^2} \tag{2.48}$$

where θ_i and g_i are the θ-angle and the Yang-Mills coupling for the SU(2) theory of the i-th node, we have

$$\pi \mathrm{i} \tau_i ~\sim~ \frac{s_i - s_{i+1}}{R_{10}}~. \tag{2.49}$$

However, since the distance between the NS5 branes is different in the two asymptotic regions $\exp v \to \pm\infty$, there is some ambiguity in this definition. We fix it as in [10, 26] and use

$$q_i = \mathrm{e}^{\pi \mathrm{i}\tau_i} = \frac{t_i}{t_{i+1}} \quad \text{or, equivalently,} \quad t_i = t_{n+1} \prod_{j=i}^{n} q_j~. \tag{2.50}$$

The overall constant t_{n+1} drops out from all equations and can be set to 1 without any loss of generality. In subsequent sections we will confirm that the above identification of the UV coupling constants is fully consistent with the Nekrasov multi-instanton calculations.

2.5.2 The 5-dimensional curve

The general Seiberg-Witten curve for the 5-dimensional theory defined on the color D4 branes takes the form of a polynomial equation [10] in the t and w variables introduced in (2.41):

$$\sum_{p,q} C_{p,q}\, t^p w^q = 0 \ . \tag{2.51}$$

Since there are always only two D4 branes in each region and in total we have $(n+1)$ NS5 branes, the polynomial in (2.51) must be of degree 2 in w and of degree $(n+1)$ in t. Of course, there are two equivalent ways of writing it. One is:

$$\mathcal{C}_1 : \quad w^2\, Q_2(t) + w\, Q_1(t) + Q_0(t) = 0 \ , \tag{2.52}$$

where the Q's are polynomials in t of degree $(n+1)$; the other is:

$$\mathcal{C}_2 : \quad t^{n+1}\, P_{n+1}(w) + t^n\, P_n(w) + \cdots t\, P_1(w) + P_0(w) = 0 \ , \tag{2.53}$$

where each of the P's is a polynomial of degree 2 in w. Using the known solutions of t when $w \to 0$ or $w \to \infty$, the form $\mathcal{C}_1$ can be written as

$$\mathcal{C}_1 : \quad w^2 \prod_{i=1}^{n+1} \big(t - t_i^{(\infty)}\big) + w\, Q_2(t) + d' \prod_{i=1}^{n+1} \big(t - t_i^{(0)}\big) = 0 \ . \tag{2.54}$$

Having fixed to 1 the coefficient of the highest term $w^2 t^{n+1}$, in (2.54) there are $(n+3)$ undetermined constants in this equation: d' and the $(n+2)$ coefficients of Q_2. On the other hand, using the fact that when $t \to 0$ and $t \to \infty$ there are two flavour branes at $w = (\widetilde{A}_0^{(1)}, \widetilde{A}_0^{(2)})$ and $w = (\widetilde{A}_{n+1}^{(1)}, \widetilde{A}_{n+1}^{(2)})$ respectively, we can write the form $\mathcal{C}_2$ of the curve as

$$\mathcal{C}_2 : \quad t^{n+1} \prod_{\alpha=1}^{2} \big(w - \widetilde{A}_{n+1}^{(\alpha)}\big) + t^n\, P_n(w) + \cdots t\, P_1(w) + d \prod_{\alpha=1}^{2} \big(w - \widetilde{A}_0^{(\alpha)}\big) = 0 \ . \tag{2.55}$$

Again we have fixed to 1 the coefficient of the highest term $w^2\, t^{n+1}$, but in this form there are $(3n+1)$ undetermined parameters: d and the three coefficients for each of the n polynomials P_k's.

Equating the two forms (2.54) and (2.55) allows us to find relations that determine some of the curve parameters: for instance, by comparing the coefficients of $w^2\, t^0$ and $w^0\, t^{n+1}$ in the two expressions we get

$$d = (-1)^{n+1} \prod_{i=1}^{n+1} t_i^{(\infty)} \ , \qquad d' = \widetilde{A}^{(1)}_{n+1} \widetilde{A}^{(2)}_{n+1} \ . \tag{2.56}$$

Similarly, by comparing the coefficients of $w\, t^0$ and $w\, t^{n+1}$ we find that the undetermined polynomial $Q_2(t)$ in (2.54) takes the form

$$Q_2(t) = -\big(\widetilde{A}^{(1)}_{n+1} + \widetilde{A}^{(2)}_{n+1}\big)\, t^{n+1} + \sum_{k=1}^{n} c_k\, t^k + (-1)^n \big(\widetilde{A}^{(1)}_0 + \widetilde{A}^{(2)}_0\big) \prod_{i=1}^{n+1} t_i^{(\infty)} \ . \tag{2.57}$$

Proceeding in a similar way one can fix the coefficients of w^2 and w^0 in the n quadratic polynomials P_i's of (2.55). In the end, all but n parameters in the Seiberg-Witten curve are fixed; the n free coefficients that remain parametrize the Coulomb branch of the $\mathrm{SU}(2)^n$ quiver gauge theory. One subtlety is that the constant terms in (2.54) and (2.55) match only if the following identity is satisfied:

$$\widetilde{A}^{(1)}_0\, \widetilde{A}^{(2)}_0 \prod_{i=1}^{n+1} t_i^{(\infty)} = \widetilde{A}^{(1)}_{n+1}\, \widetilde{A}^{(2)}_{n+1} \prod_{i=1}^{n+1} t_i^{(0)} \ . \tag{2.58}$$

Using the explicit expressions (2.46) for the asymptotic positions of the NS5 branes, we see this is identically satisfied and both sides are equal to $(\widetilde{A}^{(1)}_0\, \widetilde{A}^{(2)}_0\, \widetilde{A}^{(1)}_{n+1}\, \widetilde{A}^{(2)}_{n+1})^{1/2}$. This shows that indeed the two forms $\mathcal{C}_1$ and $\mathcal{C}_2$ of the Seiberg-Witten curve are fully equivalent.

2.5.3 The 4-dimensional curve

We now dimensionally reduce to four dimensions by first performing a T-duality and then taking the limit $\beta \to 0$. To find explicit expressions it necessary to introduce the physical parameters of the 4-dimensional theory and rewrite the geometric positions of the various branes in terms of these. In order to do this, for each pair of colour D4 branes we define the center of mass and relative positions in the v-plane according to

$$A_i^{(1)} = a_i + \bar{A}_i \ , \qquad A_i^{(2)} = -a_i + \bar{A}_i \tag{2.59}$$

for $i = 1, \dots, n$. The relative position a_i is identified with the vacuum expectation value of the adjoint scalar field Φ_i of the i-th SU(2) factor in the quiver theory. Furthermore we remove the global U(1) factor by requiring

$$\bar{A}_1 + \cdots + \bar{A}_n = 0 \ , \tag{2.60}$$

and identify the relative positions of the centers of mass with the physical masses of the bi-fundamental hypermultiplets, *i.e.*

$$m_{i,i+1} = \bar{A}_i - \bar{A}_{i+1} \tag{2.61}$$

for $i = 1, \dots, n-1$. Finally, the physical masses of the fundamental hypermultiplets attached to the first and the last NS5 branes are related to the positions of the flavour D4 branes measured with respect to the first and last center of mass in the v-plane, namely

$$m_1 = A_0^{(1)} - \bar{A}_1 \ , \quad m_2 = A_0^{(2)} - \bar{A}_1 \ , \quad m_3 = A_{n+1}^{(1)} - \bar{A}_n \ , \quad m_4 = A_{n+1}^{(2)} - \bar{A}_n \ . \tag{2.62}$$

All this is displayed in Fig. 2.2 for the case $n = 2$.

Given this set-up, it is rather straightforward to obtain the 4-dimensional Seiberg-Witten curve. However, in general it is not so simple to write explicit expressions in terms of the relevant physical parameters. Thus, we discuss in detail the following three cases:

- the conformal $\mathrm{SU}(2)^n$ quiver with massless hypermultiplets;
- the SU(2) theory with $N_f = 4$ massive fundamental flavours;
- the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ quiver theory with generically massive hypermultiplets.

• The conformal $\mathrm{SU}(2)^n$ quiver

When all matter hypermultiplets are massless the curve equation drastically simplifies. Indeed, all stacks of colour branes have the same center of mass positions, so that (2.60) implies that $\bar{A}_i = 0$ for $i = 1, \ldots, n$. Moreover, setting to zero the four fundamental masses implies that $A_0^{(1)} = A_0^{(2)} = A_{n+1}^{(1)} = A_{n+1}^{(2)} = 0$. Using this, we have

$$t_i^{(0)} = t_i^{(\infty)} = t_i \tag{2.63}$$

where the constants t_i are defined in terms of the gauge couplings q_i according to (2.50). The 5-dimensional curve (2.54) then becomes

$$w^2 \prod_{i=1}^{n+1}(t-t_i) - 2\,w\Big(t^{n+1} - \frac{1}{2}\sum_{k=1}^{n} c_k\, t^k - (-1)^n \prod_{i=1}^{n+1} t_i\Big) + \prod_{i=1}^{n+1}(t-t_i) = 0 \ . \tag{2.64}$$

We now take the 4-dimensional limit $\beta \to 0$ after writing $c_k = c_{k0} + c_{k1}\beta + c_{k2}\beta^2 + \cdots$ and $w = \exp^{\beta v}$. The $\mathcal{O}(\beta^0)$ and $\mathcal{O}(\beta^1)$ terms yield algebraic constraints for c_{k0} and c_{k1} that can be easily solved. Instead, the $\mathcal{O}(\beta^2)$ term leads to the 4-dimensional Seiberg-Witten curve. Writing $v = x\,t$ and setting $t_{n+1} = 1$, the curve becomes

$$x^2(t) = \frac{\mathcal{P}_{n-1}(t)}{t\,(t-t_1)\cdots(t-t_n)(t-1)} \tag{2.65}$$

where $\mathcal{P}_{n-1}(t)$ is a polynomial of degree $n-1$, whose n coefficients parametrize the Coulomb branch of the $\mathrm{SU}(2)^n$ theory. This is precisely the form of the Seiberg-Witten curve discussed in [5].

When the matter multiplets are massive, things become more involved. While it is always quite straightforward to write formal expressions, it is not always immediate to identify the meaning of the various coefficients in terms of the physical parameters of the gauge theory. Thus to avoid clumsy general expressions we discuss in detail the cases with $n = 1$ and $n = 2$.

- **The SU(2) theory with $N_f = 4$**

When $n = 1$ the formulæ (2.59)-(2.62) lead to

$$A_0^{(1)} = m_1 \ , \quad A_0^{(2)} = m_2 \ , \quad A_1^{(1)} = a \ , \quad A_1^{(2)} = -a \ , \quad A_2^{(1)} = m_3 \ , \quad A_2^{(2)} = m_4 \ , \tag{2.66}$$

where a is the vacuum expectation of the adjoint scalar field Φ. Then the curve (2.54) becomes

$$\begin{aligned} w^2\big(t-t_1^{(\infty)}\big)\big(t-t_2^{(\infty)}\big) - w\Big[\big(\widetilde{m}_3+\widetilde{m}_4\big)t^2 - c\,t + \big(\widetilde{m}_1+\widetilde{m}_2\big)t_1^{(\infty)}t_2^{(\infty)}\Big] \\ + \widetilde{m}_3\widetilde{m}_4\big(t-t_1^{(0)}\big)\big(t-t_2^{(0)}\big) = 0 \end{aligned} \tag{2.67}$$

where, according to (2.46) and (2.50),

$$t_1^{(0)} = q\,\sqrt{\widetilde{m}_1\widetilde{m}_2} \ , \quad t_1^{(\infty)} = \frac{q}{\sqrt{\widetilde{m}_1\widetilde{m}_2}} \ , \quad t_2^{(0)} = \frac{1}{\sqrt{\widetilde{m}_3\widetilde{m}_4}} \ , \quad t_2^{(\infty)} = \sqrt{\widetilde{m}_3\widetilde{m}_4} \tag{2.68}$$

and we are using the tilded variables $\widetilde{m}_i$ according to the notation introduced in (2.47). To obtain the 4-dimensional curve we expand w, c and all tilded variables in powers of β. The $\mathcal{O}(\beta^0)$ and $\mathcal{O}(\beta^1)$ terms can be set to zero by suitably choosing the first two coefficients in the expansion of c, while the $\mathcal{O}(\beta^2)$ term yields the Seiberg-Witten curve for the SU(2) $N_f = 4$ theory. The result is [26–28]

$$v^2(t-q)(t-1)-v\Big[(m_3+m_4)t^2-q\sum_{f=1}^{4}m_f\,t+q(m_1+m_2)\Big]+m_3m_4\,t^2+u\,t+q\,m_1m_2=0\,. \tag{2.69}$$

Here we have absorbed all terms linear in t and independent of v by redefining c into a new parameter u. A simple dimensional analysis reveals that u has dimensions of $(\text{mass})^2$. As pointed out in [5] it is a bit arbitrary to define the origin for this u parameter when masses are present. Here we fix such arbitrariness by requiring

$$u\,\Big|_{q\to 0} = a^2\ . \tag{2.70}$$

Shifting away the linear term in v in (2.69) and writing $v = x\,t$, we get [26–28]

$$x^2(t) = \frac{\mathcal{P}_4(t)}{t^2(t-q)^2(t-1)^2} \tag{2.71}$$

where $\mathcal{P}_4(t)$ is a fourth-order polynomial in t of the form

$$\mathcal{P}_4(t) = -u\,t\,(t-q)(t-1) + \mathcal{M}_4(t) \tag{2.72}$$

where we have collected in $\mathcal{M}_4(t)$ all terms that depend on the masses. The explicit expression of this polynomial is given in (B.1). Using it and choosing a specific determination for the square-root, one easily finds

$$\begin{aligned} &\mathrm{Res}_{\,t=0}\,(x(t)) = \frac{m_1-m_2}{2}\ , &&\mathrm{Res}_{\,t=q}\,(x(t)) = \frac{m_1+m_2}{2}\ ,\\ &\mathrm{Res}_{\,t=1}\,(x(t)) = \frac{m_3+m_4}{2}\ , &&\mathrm{Res}_{\,t=\infty}\,(x(t)) = \frac{m_4-m_3}{2}\ . \end{aligned} \tag{2.73}$$

- **The SU(2) $\times$ SU(2) quiver theory**

For a 2-node quiver (see Fig. 2.2), the formulæ (2.59)-(2.62) read

$$\begin{aligned}&A_1^{(0)}=m_1+\frac{m_{12}}{2}\ ,\ \ A_2^{(0)}=m_2+\frac{m_{12}}{2}\ ,\ \ A_1^{(1)}=a_1+\frac{m_{12}}{2}\ ,\ \ A_2^{(1)}=-a_1+\frac{m_{12}}{2}\ ,\\&A_1^{(2)}=a_2-\frac{m_{12}}{2}\ ,\ \ A_2^{(2)}=-a_2-\frac{m_{12}}{2}\ ,\ \ A_1^{(3)}=m_3-\frac{m_{12}}{2}\ ,\ \ A_2^{(3)}=m_4-\frac{m_{12}}{2}\end{aligned} \tag{2.74}$$

where a_1 and a_2 are the vacuum expectation values of the adjoint scalars Φ_1 and Φ_2 of the two SU(2) factors. With this configuration the 5-dimensional curve (2.54) becomes

$$\begin{aligned}&w^2\prod_{i=1}^{3}\big(t-t_i^{(\infty)}\big)-w\,\bigg(\frac{\widetilde{m}_3+\widetilde{m}_4}{\sqrt{\widetilde{m}_{12}}}\,t^3-c_2\,t^2-c_1\,t\\&\qquad\qquad-\sqrt{\widetilde{m}_{12}}\big(\widetilde{m}_1+\widetilde{m}_2\big)\prod_{i=1}^{3}t_i^{(\infty)}\bigg)+\frac{\widetilde{m}_3\widetilde{m}_4}{\widetilde{m}_{12}}\prod_{i=1}^{3}\big(t-t_i^{(0)}\big)=0\end{aligned} \tag{2.75}$$

where the asymptotic values are

$$\begin{aligned}t_1^{(0)}&=t_1\sqrt{\widetilde{m}_1\widetilde{m}_2}\ ,\qquad t_2^{(0)}=t_2\,\widetilde{m}_{12}\ ,\qquad t_3^{(0)}=\frac{1}{\sqrt{\widetilde{m}_3\widetilde{m}_4}}\ ,\\t_1^{(\infty)}&=\frac{t_1}{\sqrt{\widetilde{m}_1\widetilde{m}_2}}\ ,\qquad t_2^{(\infty)}=\frac{t_2}{\widetilde{m}_{12}}\ ,\qquad t_3^{(\infty)}=\sqrt{\widetilde{m}_3\widetilde{m}_4}\end{aligned} \tag{2.76}$$

with

$$t_1=q_1q_2\ ,\qquad t_2=q_2\ . \tag{2.77}$$

We now take the 4-dimensional limit $\beta \to 0$, proceeding as in the previous examples. The resulting Seiberg-Witten curve is

$$
\begin{aligned}
&v^2(t-t_1)(t-t_2)(t-1) \\
&-v\Bigg[(m_3+m_4-m_{12})\,t^3 - \bigg(\Big(\sum_{f=1}^{4} m_f - m_{12}\Big)\,t_1 + (m_3+m_4+m_{12})\,t_2 - m_{12}\bigg)t^2 \\
&\quad + \bigg((m_1+m_2-m_{12})\,t_1 + m_{12}\,t_2 + \Big(\sum_{f=1}^{4} m_f + m_{12}\Big)\,t_1\,t_2\bigg)\,t - (m_1+m_2+m_{12})\,t_1\,t_2\Bigg] \\
&+\bigg[\Big(m_3-\frac{m_{12}}{2}\Big)\Big(m_4-\frac{m_{12}}{2}\Big)\,t^3 - \Big(\frac{m_{12}^2}{4}-u_2\Big)\,t^2 + \Big(\frac{m_{12}^2}{4}-u_1\Big)t_2\,t \\
&\quad - \Big(m_1+\frac{m_{12}}{2}\Big)\Big(m_2+\frac{m_{12}}{2}\Big)\,t_1\,t_2\bigg] = 0\ .
\end{aligned}
\tag{2.78}
$$

Here we have exploited the freedom to redefine the arbitrary coefficients c_1 and c_2 into the parameters u_1 and u_2 for which we require the following classical limit

$$
u_1\,\big|_{q_1,q_2\to 0} = a_1^2 \qquad \text{and} \qquad u_2\,\big|_{q_1,q_2\to 0} = a_2^2\ . \tag{2.79}
$$

In Section 2.7 we will confirm the validity of this requirement.

In order to put the curve in a more convenient form, we shift away the linear term in v in (2.78) and then write $v = x\,t$, obtaining

$$
x^2(t) = \frac{\mathcal{P}_6(t)}{t^2\,(t-t_1)^2(t-t_2)^2(t-1)^2}\ , \tag{2.80}
$$

where $\mathcal{P}_6(t)$ is a polynomial of degree six in t of the form

$$
\mathcal{P}_6(t) = -t\,(u_2\,t - t_2\,u_1)(t-t_1)(t-t_2)(t-1) + \mathcal{M}_6(t) \tag{2.81}
$$

with $\mathcal{M}_6(t)$ containing all mass-dependent terms. The explicit expression of this polynomial is given in (B.3). Using it we find

$$\begin{aligned}\mathrm{Res}_{t=0}\,(x(t)) &= \frac{m_1-m_2}{2}\ , \quad \mathrm{Res}_{t=t_1}\,(x(t))=\frac{m_1+m_2}{2}\ , \quad \mathrm{Res}_{t=t_2}\,(x(t))=m_{12}\ ,\\ \mathrm{Res}_{t=1}\,(x(t)) &= \frac{m_3+m_4}{2}\ , \quad \mathrm{Res}_{t=\infty}\,(x(t))=\frac{m_4-m_3}{2}\ .\end{aligned} \tag{2.82}$$

2.5.4 From the 4-dimensional curve to the prepotential

The spectral curve (2.80) encodes all relevant information about the effective quiver gauge theory through the Seiberg-Witten differential

$$\lambda = x(t)dt\ . \tag{2.83}$$

If we differentiate λ with respect to u_1 and u_2, we get (up to normalizations which are irrelevant for our present purposes)

$$\frac{\partial\lambda}{\partial u_1}\simeq\frac{dt}{y}\ , \qquad \frac{\partial\lambda}{\partial u_2}\simeq\frac{t\,dt}{y}\ , \tag{2.84}$$

where

$$y^2=\mathcal{P}_6(t)\ . \tag{2.85}$$

This is the standard equation defining a genus-2 Riemann surface. Such a surface admits a canonical symplectic basis with two pairs of 1-cycles (α_1,α_2) and (β_1,β_2) whose intersection matrix is $\alpha_i\cap\alpha_j=\beta_i\cap\beta_j=\delta_{ij}$. The periods of the Seiberg-Witten differential λ along these cycles represent the quantities a_i and a_i^D in the effective gauge theory, namely

$$a_i=\frac{1}{2\pi\mathrm{i}}\oint_{\alpha_i}\lambda\ , \qquad a_i^D=\frac{1}{2\pi\mathrm{i}}\oint_{\beta_i}\lambda\ . \tag{2.86}$$

Through these relations, a_i and a_i^D are determined as functions of the u_i's (and, of course, of the UV couplings q_i and of the mass parameters). Inverting these relations, one can express the u_i's in terms of the a_i's and, substituting them into the dual periods, obtain $a_i^D(a)$. Since

$$a_i^D(a) = \frac{\partial F}{\partial a_i} , \tag{2.87}$$

one can reconstruct in this way the prepotential F (up to a-independent terms). By comparing this prepotential with the one obtained from the multi-instanton calculus via localization one can therefore test the validity of the proposed form of the Seiberg-Witten curve.

However, an alternative and more efficient approach has been presented in [12,13] in which the difficult computations of the dual periods a_i^D are avoided and the effective prepotential is directly put in relation with the residues of the quadratic differential $x^2(t)dt^2$ in the following way

$$\mathrm{Res}_{t=t_i}\left(x^2(t)\right) = \frac{\partial \widetilde{F}}{\partial t_i} . \tag{2.88}$$

As we will show in more detail below, assuming this relation and just computing the α-periods of the Seiberg-Witten differential we can readily reconstruct $\widetilde{F}$ from the spectral curve and check that it coincides with the effective prepotential F computed via localization up to mass-dependent but a-independent shifts (so that $\widetilde{F}$ and F encode the same effective gauge couplings); the expression of these shifts is however rather interesting, and we will comment on this in the next sections.

2.6 The SU(2) theory with $N_f = 4$

We show how to derive the effective prepotential for the SU(2) $N_f = 4$ theory starting from the curve (2.71) and the residue formula (2.88) which in this case reads

$$\mathrm{Res}_{t=q}\left(x^2(t)\right) = \frac{\partial \widetilde{F}}{\partial q} \ . \tag{2.89}$$

In doing this we do not only provide a generalization of the results presented in [13], but also set the stage for the discussion of the quiver theory in the next section.

Using the curve (2.71) and the explicit expression of the polynomial $\mathcal{P}_4$ reported in (B.1), the above residue formula leads to

$$q(1-q)\frac{\partial \widetilde{F}}{\partial q} = u - \frac{1-q}{2}\left(m_1^2+m_2^2\right) + \frac{q}{2}\left(m_1+m_2\right)\left(m_3+m_4\right) + q\left(m_1m_2+m_3m_4\right) \ . \tag{2.90}$$

Combining this with the residues (2.73) amounts to rewrite the Seiberg-Witten curve (2.71) as

$$\begin{aligned} x^2(t) = & \frac{(m_1-m_2)^2}{4t^2} + \frac{(m_1+m_2)^2}{4(t-q)^2} + \frac{(m_3+m_4)^2}{4(t-1)^2} - \frac{m_1^2+m_2^2+2m_3m_4}{2t(t-1)} \\ & + \frac{q(q-1)}{t(t-q)(t-1)}\,\frac{\partial \widetilde{F}}{\partial q} \ . \end{aligned} \tag{2.91}$$

We now clarify the meaning of $\widetilde{F}$. Imposing in (2.90) the boundary value (2.70) for u, we easily find

$$q\frac{\partial \widetilde{F}}{\partial q}\bigg|_{q\to 0} = a^2 - \frac{1}{2}\left(m_1^2+m_2^2\right) \ , \tag{2.92}$$

from which we deduce that $\widetilde{F}$ cannot be directly identified with the effective gauge theory prepotential, whose classical term is in fact $F_{\mathrm{cl}} = a^2 \log q$. Therefore, to ensure the proper classical limit we shift $\widetilde{F}$ according to

$$\widetilde{F} = \widehat{F} - \frac{1}{2}\left(m_1^2+m_2^2\right)\log q \ , \tag{2.93}$$

and rewrite (2.90) as

$$q(1-q)\frac{\partial\widehat{F}}{\partial q} = u + \frac{q}{2}\,(m_1+m_2)\,(m_3+m_4) + q\,(m_1m_2+m_3m_4) \ . \tag{2.94}$$

The function $\widehat{F}$ has the correct classical limit, but it is not yet the gauge theory prepotential since it is determined by an equation in which the four masses do not appear on equal footing. There are two independent ways to remedy this and restore complete symmetry among the flavors, namely by redefining $\widehat{F}$ as[2]

$$\mathrm{I})\quad : \qquad \widehat{F} = F_{\mathrm{I}} + \frac{1}{2}\log(1-q)\,(m_1+m_2)\,(m_3+m_4) \ , \tag{2.95}$$

$$\mathrm{II})\quad : \qquad \widehat{F} = F_{\mathrm{II}} - \frac{1}{2}\log(1-q)\,(m_1m_2+m_3m_4) \ . \tag{2.96}$$

In this way, from (2.94) we get

$$\mathrm{I})\quad : \qquad q(1-q)\frac{\partial F_{\mathrm{I}}}{\partial q} \equiv (1-q)\,U_{\mathrm{I}} = u + q\sum_{f<f'} m_f m'_f \ , \tag{2.97}$$

$$\mathrm{II})\quad : \qquad q(1-q)\frac{\partial F_{\mathrm{II}}}{\partial q} \equiv (1-q)\,U_{\mathrm{II}} = u + \frac{q}{2}\sum_{f<f'} m_f m'_f \ . \tag{2.98}$$

The minor difference in the numerical coefficient in front of the mass terms in these two equations is, actually, quite significant. In fact, as we will see, F_{I} is the Nekrasov prepotential for the SU(2) $N_f = 4$ theory, while F_{II} is the SO(8) invariant prepotential that can be derived from the Seiberg-Witten curve of [2] expressed in terms of the UV coupling q.

To verify this statement in an explicit way, we take

$$m_1 = m_2 = m \ , \quad m_3 = m_4 = M \ . \tag{2.99}$$

[2] All other possibilities can be seen as linear combinations of these two. It is interesting to observe that the shifts in (2.97) and (2.98) are directly related to the so-called U(1) dressing factors used in the AGT correspondence [4].

This is a simple choice of masses that allows us to exhibit all non-trivial features of the calculation. With these masses the curve (2.71) becomes

$$x^2(t) = \frac{\mathcal{P}_2(t)}{t(t-1)^2(t-q)^2} \tag{2.100}$$

where

$$\begin{aligned}
\mathcal{P}_2(t) &= -Ct^2 + \Big(u(1+q) - q(m-M)^2 + q^2(m+M)^2\Big)t - q\Big(u - (1-q)\,m^2 + 2qmM\Big) \\
&= C(e_2 - t)(t - e_3)
\end{aligned} \tag{2.101}$$

with

$$C = u + 2qmM - M^2(1-q)\ . \tag{2.102}$$

The expressions of the two roots e_2 and e_3 can be easily obtained by solving the quadratic equation $\mathcal{P}_2(t) = 0$; in the 1-instanton approximation we find[3]

$$\begin{aligned}
e_2 &= q\Big(1 - \frac{m^2}{u} + q\frac{m^2\,(u^2 + M^2u + 2mMu - m^2M^2)}{u^3} + \dots\Big)\ , \\
e_3 &= 1 + \frac{M^2}{u - M^2} + q\frac{M^2\,(m^2M^2 - m^2u - 2mMu - u^2)}{u(u-M^2)} + \dots
\end{aligned} \tag{2.103}$$

The Seiberg-Witten differential associated to the spectral curve (2.100) is

$$\lambda = x(t)dt = \sqrt{\frac{(e_2-t)(t-e_3)}{t}}\ \frac{\sqrt{C}\,dt}{(1-t)(t-q)}\ ; \tag{2.104}$$

it possess four branch points at $t = 0$, e_2, e_3 and ∞ and two simple poles at $t = q$ and 1. This singularity structure is shown in Fig. 2.3. The cross-ratio of the four

[3] Here and in the following, for brevity we explicitly exhibit the results only up to one or two instantons, but we have checked that everything works also for higher instanton numbers.

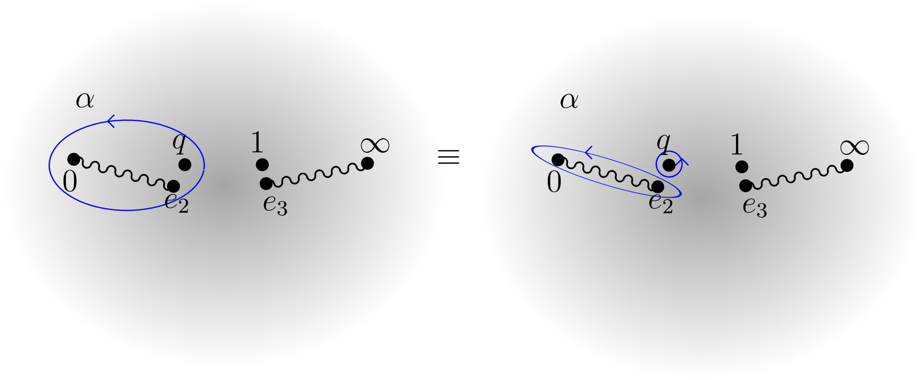

Figure 2.3: Branch cuts and singularities of the α-period of the Seiberg-Witten differential λ of the SU(2) $N_f = 4$ theory

branch points is

$$\zeta = \frac{e_2}{e_3} = q\left(1 - \frac{(m^2+M^2)u - m^2M^2}{u^2}\right) + q^2\left(\frac{(m^2+M^2)u^3 + 2mM(m^2+M^2)u^2 - 2m^2M^2(m+M)^2u + 2m^4M^4}{u^4}\right) + \dots \tag{2.105}$$

In the massless limit, note that the cross ratio reduces to the Nekrasov counting parameter q, as expected. As always, we identify the α-period of the Seiberg-Witten differential with the vacuum expectation value a, namely

$$a = \frac{1}{2\pi \mathrm{i}}\oint_\alpha \lambda = \mathrm{Res}_{t=q}(\lambda) + \frac{\sqrt{C}}{\pi}\int_0^{e_2}\sqrt{\frac{e_2-t}{t}}\,\frac{\sqrt{e_3-t}}{(1-t)(q-t)}\,dt\ . \tag{2.106}$$

It is important to stress that the α-cycle corresponds to a closed contour encircling both the branch cut from 0 to e_2 and the simple pole of λ at $t = q$, see Fig. 2.3. With this prescription, the α-cycle has a smooth limit when the masses are set to zero. This explains the two terms on the right hand side of (2.106): the residue over the pole in $t = q$, which in view of (2.73) is simply m, and the integral over the branch cut. This integral is explicitly evaluated in Appendix C (see in particular (C.9)); in the final result the mass term coming from the residue is canceled and we are left with

$$a=\frac{\sqrt{C(e_3-q)}}{1-q}+\frac{\sqrt{C}}{1-q}\sum_{n,\ell=0}^{\infty}(-1)^{\ell}\binom{1/2}{n+1}\binom{1/2}{n+\ell+1}\frac{e_2^{n+1}\,q^{\ell}}{e_3^{n+\ell+1/2}}$$
$$-\frac{\sqrt{C}}{1-q}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{(n+\ell)}\binom{1/2}{n+1}\binom{1/2}{\ell}\frac{e_2^{n+1}}{e_3^{\ell-1/2}}\,. \tag{2.107}$$

Exploiting the expressions of the roots e_2 and e_3, it is not difficult to realize that the right hand side of (2.107) has an expansion in positive powers of q and that only a finite number of terms contribute to a given order, *i.e.* to a given instanton number. For example, using (2.102) and (2.103), up to one instanton we find

$$a=\sqrt{u}\left(1+q\,\frac{u^2+m^2u+4mMu+M^2u-m^2M^2}{4u^2}+\dots\right)\,, \tag{2.108}$$

which can be inverted leading to

$$u=a^2\left(1-q\,\frac{a^4+m^2a^2+4mMa^2+M^2a^2-m^2M^2}{2a^2}+\dots\right)\,. \tag{2.109}$$

This result allows us to finally obtain the prepotential. Inserting it into (2.97) we get

$$F_{\rm I}-a^2\log q=q\left(\frac{a^2}{2}+\frac{m^2+4mM+M^2}{2}+\frac{m^2M^2}{2a^2}\right)+\dots \tag{2.110}$$

On the right hand side we recognize the 1-instanton prepotential for the SU(2) $N_f=4$ theory obtained in Nekrasov's approach described in Appendix A[4]. This instanton prepotential follows from that of the U(2) theory after decoupling the U(1) contribution and, as is well known, does not possess the SO(8) flavor symmetry of the effective theory; however the terms which spoil this symmetry are all a-independent (like for example the pure mass terms in (2.110)) and therefore are not physical. On

[4]For the explicit expression see for example Section 7 and Appendix D of [29], keeping in mind that $m_i^{\rm there}=\sqrt{2}\,m_i^{\rm here}$.

the other hand, if we insert (2.109) into (2.98) we get

$$F_{\rm II} - a^2 \log q = q \left(\frac{a^2}{2} + \frac{m^2 M^2}{2a^2} \right) + \ldots \tag{2.111}$$

which is the 1-instanton term of the SO(8) invariant prepotential following from the Seiberg-Witten curve of [2]. In this respect it is worth recalling that this curve, differently from (2.71), is parametrized in terms of the IR coupling of the massless theory $Q^{(0)}$ which is related to the UV coupling q by [30]

$$q = \frac{\theta_2^4}{\theta_3^4}\big(Q^{(0)}\big) \ . \tag{2.112}$$

As shown for example in [29, 31], if one rewrites the prepotential derived from the Seiberg-Witten curve in terms of q using (2.112) one can precisely recover the above SO(8) invariant result.

The last ingredient is the perturbative 1-loop contribution which is given by[5]

$$F_{\rm pert} = -2a^2 \log \frac{4a^2}{\Lambda^2} + \frac{1}{4} \sum_{i=1}^{4} \left[(a+m_i)^2 \log \frac{(a+m_i)^2}{\Lambda^2} + (a-m_i)^2 \log \frac{(a-m_i)^2}{\Lambda^2} \right] \ . \tag{2.113}$$

From the complete prepotential $\mathcal{F} = F + F_{\rm pert}$ one obtains the IR effective coupling Q of the massive theory by means of

$$Q = \mathrm{e}^{\pi \mathrm{i} \tau} \qquad \text{with} \qquad \pi \mathrm{i} \tau = \frac{1}{2} \frac{\partial^2 \mathcal{F}}{\partial a^2} \ . \tag{2.114}$$

Notice that both $F_{\rm I}$ and $F_{\rm II}$ lead to the same Q since they only differ by a-independent terms. For our specific mass choice (2.99), up to 1 instanton we find

$$Q = \frac{q}{16} \left(1 - \frac{m^2 + M^2}{a^2} + \frac{m^2 M^2}{a^4} \right) \left(1 + q \, \frac{a^4 + 3m^2 M^4}{2a^4} + \ldots \right) \ . \tag{2.115}$$

[5] See also (A.25), with obvious modifications, in the limit $\epsilon_1, \epsilon_2 \to 0$.

As is well-known, given Q one can obtain the cross-ratio ζ of the four roots e_i of the associated Seiberg-Witten torus by means of the uniformization formula

$$\zeta = \frac{(e_1 - e_2)(e_3 - e_4)}{(e_1 - e_3)(e_2 - e_4)} = \frac{\theta_2^4}{\theta_3^4}(Q) \tag{2.116}$$

which is the massive analogue of the massless relation (2.112). Using (2.115) and expanding the Jacobi θ-functions we find

$$\begin{aligned}\zeta = {}& q\left(1 - \frac{m^2 + M^2}{a^2} + \frac{m^2 M^2}{a^4}\right) \\ & + q^2\left(\frac{m^2 + M^2}{2a^2} - \frac{m^4 + M^4}{2a^6} - \frac{m^2 M^2(m^2 + M^2)}{2a^6} + \frac{m^4 M^4}{a^8}\right) + \ldots\end{aligned} \tag{2.117}$$

It is not difficult to check that this expression exactly agrees with the cross-ratio (2.105) upon using the relations between a and u given in (2.108) and (2.109), thus confirming in full detail the consistency of the calculations.

2.7 The SU(2) $\times$ SU(2) quiver theory

We now consider the 2-node quiver theory whose Seiberg-Witten curve takes the form (see (2.80))

$$x^2(t) = \frac{\mathcal{P}_6(t)}{t^2\,(t - q_1 q_2)^2 (t - q_2)^2 (t - 1)^2}\ , \tag{2.118}$$

where the sixth-order polynomial $\mathcal{P}_6(t)$ is given in (B.3). In the following it will be useful to use yet another form of the curve that can be obtained from (2.118) by performing the rescaling $(x, t) \to (x\, {q_2}^{-1}, t\, q_2)$. This yields

$$x^2(t) = \frac{p_6(t)}{t^2\,(t - q_1)^2 (t - 1)^2 (q_2 t - 1)^2} \tag{2.119}$$

where

$$p_6(t) = \mathcal{P}_6(q_2 t)\, q_2^{-4} = (u_1 - u_2 t)\, t(t-1)(t-q_1)(q_2 t - 1) + \mathcal{M}_6(q_2 t)\, q_2^{-4}\ . \quad (2.120)$$

In this form the two SU(2) factors appear on the same footing and their weak coupling limit is simply described by q_1 and q_2 approaching zero. In this limit the punctured sphere which corresponds to the denominator of (2.119) looks as depicted in Fig. 2.4.

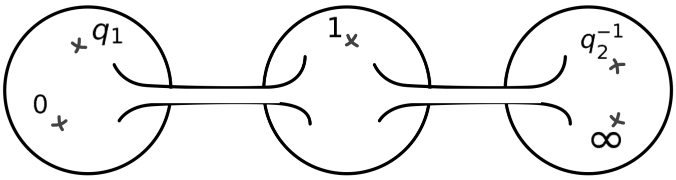

Figure 2.4: Punctured sphere in the weak-coupling limit

In general the polynomial $p_6(t)$ defined in (2.120) is of order 6, and thus the hyperelliptic equation (see (2.85)) identifying the genus-2 Seiberg-Witten curve can be written as

$$y^2(t) = p_6(t) = c\prod_{i=1}^{6}(t - e_i) \quad (2.121)$$

where e_i's are the six roots of the polynomial, which clearly are branch points for the function $y(t)$. With a projective transformation we can always fix three of them, say e_1, e_3 and e_6, in 0, 1 and ∞ and lower by one the degree of the polynomial in the right hand side; if we call ζ_1, ζ_2 and $\widehat{\zeta}$ the remaining three parameters, corresponding to three independent anharmonic ratios of the e_i's, the equation (2.121) reduces to

$$y^2(t) = c\, t\big(t-1\big)\big(t-\zeta_1\big)\big(t-\zeta_2\big)\big(t-\widehat{\zeta}\,\big)\ . \quad (2.122)$$

When the curve is put in this form, we can choose a symplectic basis of cycles $\{\alpha_i, \beta^i\}$ in the Riemann sphere parametrized by the t variable as shown in Fig. 2.5, and then proceed to compute the periods of the Seiberg-Witten differential and finally derive the effective prepotential. However, for generic values of the masses of the matter

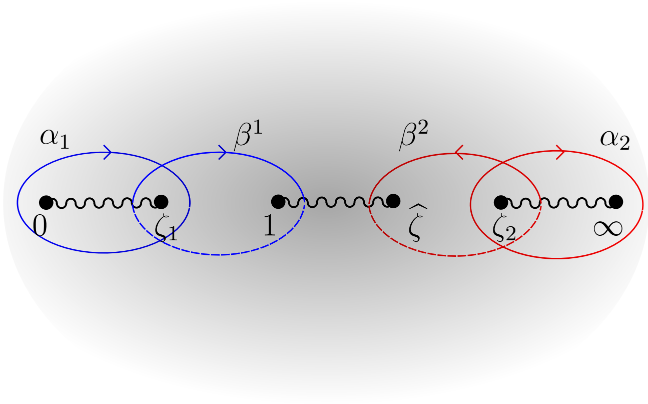

Figure 2.5: Structure of branch cuts and a basis of cycles for the Riemann surface described by Eq (2.122)

hypermultiplets this method is not practical since one is not able to find the roots of $p_6(t)$ in closed form and only a perturbative approach in the masses is viable to derive the effective prepotential. On the other hand we can exploit the residue conditions (2.88), which after the rescalings we have performed, take the form

$$\mathrm{Res}_{t=q_1}\left(x^2(t)\right) = \frac{\partial \widetilde{F}}{\partial q_1}\ , \qquad \mathrm{Res}_{t=1/q_2}\left(x^2(t)\right) = -q_2^2\,\frac{\partial \widetilde{F}}{\partial q_2}\ , \tag{2.123}$$

and through them obtain some information on the prepotential directly from the quadratic differential. Evaluating the residues using the curve equation (2.119), we find

$$\begin{aligned} q_1(1-q_1)(1-q_1q_2)\,\frac{\partial \widetilde{F}}{\partial q_1} \quad = \quad & u_1 - q_1u_2 - \frac{1}{2}\left(m_1^2+m_2^2\right) \\ & + \frac{q_1}{4}\left((m_1+m_2)^2 + (m_1+m_2-m_{12})^2\right) \\ & + \frac{q_1q_2}{2}\,(m_1+m_2)\left(m_{12}+\sum_{f=1}^{4} m_f\right) \\ & - \frac{q_1^2q_2}{4}\left(m_{12}^2 + 2\big(m_1+m_2-m_{12}\big)\sum_{f=1}^{4} m_f + 4m_3m_4\right) \end{aligned} \tag{2.124}$$

$$
\begin{aligned}
q_2(1-q_2)(1-q_1q_2)\,\frac{\partial \widetilde{F}}{\partial q_2} \;&=\; u_2 - q_2 u_1 + m_3 m_4 + \frac{q_2}{4} m_{12}\big(m_{12}+2m_3+2m_4\big) \\
&\quad + \frac{q_1 q_2}{2}\big(m_3+m_4\big)\big(m_1+m_2-m_{12}\big) \\
&\quad - \frac{q_1 q_2^2}{4}\Big(m_{12}^2 + 2m_{12}\sum_{f=1}^{4} m_f + 2\big(m_1+m_2\big)\big(m_3+m_4\big) + 4m_1m_2\Big)\,.
\end{aligned}
\tag{2.125}
$$

Combining these with the residues (2.82) suitably rescaled for the new poles, we can rewrite the curve as

$$
\begin{aligned}
x^2(t) = &\,\frac{(m_1-m_2)^2}{4t^2} + \frac{(m_1+m_2)^2}{4(t-q_1)^2} + \frac{m_{12}^2}{(t-1)^2} + \frac{(m_3+m_4)^2}{4(t-\frac{1}{q_2})^2} \\
&- \frac{m_1^2+m_2^2+2m_3m_4+2m_{12}^2}{2t(t-1)} + \frac{q_1(q_1-1)}{t(t-q_1)(t-1)}\,\frac{\partial \widetilde{F}}{\partial q_1} + \frac{q_2(1-\frac{1}{q_2})}{t(t-1)(t-\frac{1}{q_2})}\,\frac{\partial \widetilde{F}}{\partial q_2}
\end{aligned}
\tag{2.126}
$$

which is a simple generalization of (2.91). We now investigate the meaning of the function $\widetilde{F}$ appearing in the last two terms of (2.126). If we impose the boundary conditions (2.79) on the u_i's, from (2.124) and (2.125) we obtain

$$
\begin{aligned}
q_1\frac{\partial \widetilde{F}}{\partial q_1}\Big|_{q_1,q_2\to 0} &= a_1^2 - \frac{1}{2}\big(m_1^2+m_2^2\big)\ , \\
q_2\frac{\partial \widetilde{F}}{\partial q_2}\Big|_{q_1,q_2\to 0} &= a_2^2 + m_3 m_4\ .
\end{aligned}
\tag{2.127}
$$

Thus, in order to match with the classical prepotential $F_{\text{cl}} = a_1^2 \log q_1 + a_2^2 \log q_2$, we are led to the following redefinition

$$
\widetilde{F} = \widehat{F} - \frac{1}{2}\big(m_1^2+m_2^2\big)\log q_1 + m_3\, m_4 \log q_2\ . \tag{2.128}
$$

Just as we did for the SU(2) $N_f = 4$ theory discussed in Section 2.6, here too we have to make sure that all symmetries of the quiver model are correctly implemented. If we just focus on the first group factor, we obtain an SU(2) theory with coupling q_1 and four effective flavors with masses $\{m_1, m_2, a_2+m_{12}, -a_2+m_{12}\}$. Therefore, according to (2.95) we have to redefine $\widehat{F}$ by the term

$$\frac{1}{2}\,(m_1+m_2)\,(a_2+m_{12}-a_2+m_{12})\log(1-q_1) = (m_1+m_2)\,m_{12}\log(1-q_1)\;. \quad (2.129)$$

Likewise, if we focus on the second group factor, we find an SU(2) theory with coupling q_2 and four effective flavors with masses $\{a_1-m_{12}, -a_1-m_{12}, m_3, m_4\}$; finally if we consider the quiver as whole, we have a "diagonal" SU(2) theory with coupling q_1q_2 and four masses given by $\{m_1, m_2, m_3, m_4\}$. All in all, in order to implement all symmetries of the quiver diagram and its subdiagrams, we must redefine $\widehat{F}$ according to

$$\begin{aligned}\widehat{F} = F &+ (m_1+m_2)\,m_{12}\log(1-q_1) - m_{12}\,(m_3+m_4)\log(1-q_2) \\ &+ \frac{1}{2}\,(m_1+m_2)\,(m_3+m_4)\log(1-q_1q_2)\;.\end{aligned} \quad (2.130)$$

It is interesting to observe that these logarithmic terms are like the U(1) dressing factors commonly used in the context of the AGT correspondence [4]. Quite remarkably, if we combine (2.128) and (2.130), the two very asymmetric equations (2.124) and (2.125) acquire a symmetric structure. Indeed, if we set

$$U_i = q_i\frac{\partial F}{\partial q_i} \qquad \text{for } i=1,2\;, \quad (2.131)$$

then equation (2.124) becomes

$$\begin{aligned}(1-q_1)(1-q_1q_2)\,U_1 = u_1 - q_1u_2 &+ \frac{q_1}{4}\Big(m_{12}\big(m_{12}+2m_1+2m_2\big)+4m_1m_2\Big) \\ &+ \frac{q_1q_2}{2}\Big(\big(m_1+m_2\big)\big(m_{12}+2m_3+2m_4\big)+2m_1m_2\Big) \\ &- \frac{q_1^2q_2}{4}\Big(m_{12}\big(m_{12}+2m_1+2m_2-2m_3-2m_4\big)+4\sum_{f<f'}m_FM_{f'}\Big)\;,\end{aligned} \quad (2.132)$$

while the corresponding equation for U_2 following from (2.125) can be obtained from

(2.132) with the replacements

$$q_1 \leftrightarrow q_2 \ , \quad u_1 \leftrightarrow u_2 \ , \quad (m_1, m_2) \leftrightarrow (m_3, m_4) \ , \quad m_{12} \leftrightarrow -m_{12} \ . \tag{2.133}$$

This is precisely the exchange symmetry that should hold in the 2-node quiver model under consideration. The function F therefore has all the required properties to be identified with the effective prepotential of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ gauge theory. To check this statement in an explicit way, we choose two mass configurations for which the polynomial $p_6(t)$ in (2.119) can be factorized and its roots and period integrals can be explicitly computed. Specifically we consider the following two cases:

$$\mathrm{A}) : \qquad m_1 = m_2 = m \ , \quad m_3 = m_4 = m_{12} = 0 \ , \tag{2.134a}$$

$$\mathrm{B}) : \qquad m_1 = m_2 = m_3 = m_4 = 0 \ , \quad m_{12} = M \ . \tag{2.134b}$$

As we will see, these mass configurations allow us to make the point and exhibit all relevant features while keeping the treatment quite simple.

2.7.1 The IR prepotential from the UV curve

Case A): With the masses (2.134a) the polynomial $p_6(t)$ of the Seiberg-Witten curve becomes

$$p_6(t) = t(t-1)(q_2 t - 1)\Big[(u_1 - u_2 t)(t - q_1) + m^2 q_1 \left(q_1 q_2 t + 1 - q_1 - q_1 q_2\right)\Big] \ . \tag{2.135}$$

If we factorize the term in square brackets we immediately bring the curve to the form (2.122), with $c = -q_2 u_2$ and

$$\zeta_1 = \frac{u_1 + q_1 u_2 + m^2 q_1^2 q_2 - \sqrt{D}}{2u_2} \ , \quad \widehat{\zeta} = \frac{u_1 + q_1 u_2 + m^2 q_1^2 q_2 + \sqrt{D}}{2u_2} \ , \quad \zeta_2 = \frac{1}{q_2} \tag{2.136}$$

where

$$D=\big(u_1-q_1u_2\big)^2+2m^2q_1\Big[q_1q_2u_1+u_2\big(2-2q_1-2q_1q_2+q_1^2q_2\big)\Big]+m^4q_1^4q_2^2\ . \quad (2.137)$$

Then the spectral curve (2.119) reduces to[6]

$$x^2(t)=\frac{-u_2(t-\zeta_1)(t-\widehat{\zeta})}{t(t-1)(t-q_1)^2(q_2t-1)}\ . \quad (2.138)$$

For later purposes it is convenient to invert the relation (2.132) and the corresponding one for U_2 in order express u_1 and u_2 in terms of U_1 and U_2. For the mass configuration (2.134a) we get

$$\begin{aligned}u_1&=\big(1-q_1\big)\,U_1+q_1\big(1-q_2\big)\,U_2-m^2q_1\big(1+q_2\big)\ ,\\ u_2&=\big(1-q_2\big)\,U_2+q_2\big(1-q_1\big)\,U_1-m^2q_1q_2\ .\end{aligned} \quad (2.139)$$

The Seiberg-Witten differential associated to the curve (2.138) is

$$\lambda=x(t)\,dt=\sqrt{\frac{-u_2(t-\zeta_1)(t-\widehat{\zeta})}{t(t-1)(q_2t-1)}}\ \frac{dt}{q_1-t}\ , \quad (2.140)$$

and its singularity structure is shown in Fig. 2.6. The periods of λ along the cycles α_1 and α_2 are identified with the vacuum expectation values a_1 and a_2, respectively. Let us first consider the cycle α_1 and note that it surrounds both the branch cut from 0 to ζ_1 and the pole in $t=q_1$. Thus we have

$$a_1=\frac{1}{2\pi\mathrm{i}}\oint_{\alpha_1}\lambda=\ \mathrm{Res}_{t=q_1}(\lambda)\ \ +\frac{1}{\pi}\int_0^{\zeta_1}\sqrt{\frac{\zeta_1-t}{t}}\ \sqrt{\frac{u_2(\widehat{\zeta}-t)}{(1-t)(1-q_2t)}}\ \frac{dt}{q_1-t}\ . \quad (2.141)$$

The integral over the branch cut can be evaluated as explained in Appendix C (see in particular Eq. (C.12)); it contains a contribution that cancels the residue and the

[6] Note that in the massless limit we have $\zeta_1\to q_1$ and $\widehat{\zeta}\to u_1/u_2$.

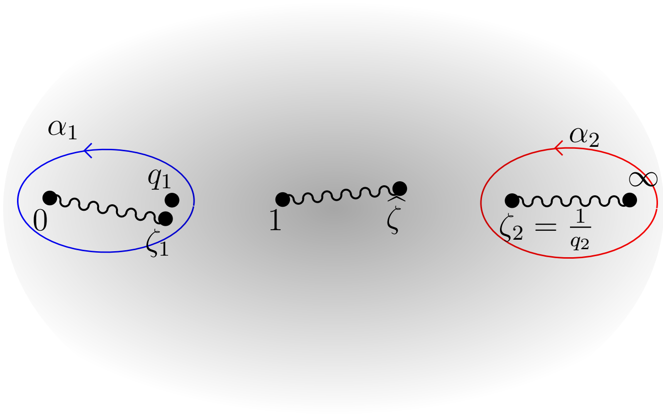

Figure 2.6: Branch cuts and singularities of the α-periods of the Seiberg-Witten differential λ of the conformal $SU(2) \times SU(2)$ quiver

final result for a_1 is

$$a_1 = \sqrt{\frac{u_2(\widehat{\zeta} - q_1)}{(1-q_1)(1-q_1 q_2)}} - \sum_{n,\ell=0}^{\infty} (-1)^n \binom{1/2}{n+1} f_{n+\ell+1}\, \zeta_1^{n+1}\, q_1^{\ell} \tag{2.142}$$

where the f_n's are the coefficients in the following Taylor expansion

$$\sqrt{\frac{u_2(\widehat{\zeta} - t)}{(1-t)(1-q_2 t)}} = \sum_{n=0}^{\infty} f_n\, t^n \ , \tag{2.143}$$

namely

$$f_n = (-1)^n \sqrt{u_2} \sum_{\ell,k=0}^{n} \binom{1/2}{\ell} \binom{-1/2}{k} \binom{-1/2}{n-\ell-k} \frac{q_2^k}{\widehat{\zeta}^{\,\ell-1/2}} \ . \tag{2.144}$$

Using the expressions (2.136) for the roots it is not difficult to check that a_1 has an expansion in positive powers of q_1 and q_2 and that only a finite number of terms contribute to a given instanton number. Substituting in the result the relations

(2.139) we obtain the following weak coupling expansion[7]

$$
\begin{aligned}
a_1 \;=\; \sqrt{U_1}\Bigg(1 &- q_1\,\frac{(U_1-U_2)(U_1+m^2)}{4U_1^2} - q_1q_2\,\frac{(U_1+m^2)U_2}{4U_1^2} \\
&- q_1^2\,\frac{(U_1-U_2)\Big(U_1(7U_1-3U_2)(U_1+2m^2)+3m^4(U_1-5U_2)\Big)}{64U_1^4} + \ldots\Bigg)
\end{aligned}
\tag{2.145}
$$

Let us now turn to the second period a_2 along the cycle α_2. Referring to Fig. 2.6 we have

$$
\begin{aligned}
a_2 = \frac{1}{2\pi\mathrm{i}}\oint_{\alpha_2}\lambda \;&=\; \frac{1}{\pi}\int_{1/q_2}^{\infty}\sqrt{\frac{u_2(t-\zeta_1)(t-\widehat{\zeta})}{t(t-1)(q_2t-1)}}\,\frac{dt}{t-q_1} \\
&= \frac{1}{\pi}\int_0^1\left[\sqrt{\frac{u_2(1-q_2\zeta_1 z)(1-q_2\widehat{\zeta}z)}{(1-q_2z)}}\,\frac{1}{(1-q_1q_2z)}\right]\frac{dz}{\sqrt{z(1-z)}}
\end{aligned}
\tag{2.146}
$$

where the last step simply follows from the change of integration variable: $t \to 1/(q_2z)$. This integral can be computed by expanding the factor in square brackets in powers of z and then using

$$
\int_0^1\frac{z^n dz}{\sqrt{z(1-z)}} = (-1)^n\,\pi\binom{-1/2}{n}\,. \tag{2.147}
$$

Inserting the root expressions (2.136) and exploiting the relations (2.139), we find

$$
a_2 = \sqrt{U_2}\left(1 - q_2\,\frac{U_2-U_1}{4U_2} - q_2^2\,\frac{7U_2^2-10U_1U_2+3U_1^2}{64U_2^2} - q_1q_2\,\frac{U_1+m^2}{4U_2} + \ldots\right). \tag{2.148}
$$

Note that the results (2.145) and (2.148) are perturbative in the instanton counting paramaters q_1 and q_2, but are exact in the mass deformation parameter m. We now

[7]For brevity we display only the results up to two instantons, but we have computed also higher instanton contributions without difficulty.

invert these weak-coupling expansions to obtain

$$U_1 = a_1^2 + q_1\left(\frac{a_1^2-a_2^2}{2} + m^2\,\frac{a_1^2-a_2^2}{2a_1^2}\right) + q_1q_2\left(\frac{a_1^2+a_2^2}{4} + m^2\,\frac{a_1^2+a_2^2}{4a_1^2}\right)$$
$$+\, q_1^2\left(\frac{13a_1^4-14a_1^2a_2^2+a_2^4}{32a_1^2} + m^2\,\frac{9a_1^4-6a_1^2a_2^2-3a_2^4}{16a_1^4} + m^4\,\frac{a_1^4-6a_1^2a_2^2+5a_2^4}{32a_1^6}\right) + \dots\,, \tag{2.149}$$

$$U_2 = a_2^2 + q_2\,\frac{a_2^2-a_1^2}{2} + q_1q_2\left(\frac{a_1^2+a_2^2}{4} + m^2\,\frac{a_1^2+a_2^2}{4a_1^2}\right) + q_2^2\,\frac{13a_2^4-14a_1^2a_2^2+a_1^4}{32a_2^2} + \dots\,. \tag{2.150}$$

These two expressions are integrable, thus leading to the determination of F (up to q-independent terms)

$$\begin{aligned} F \;=\;& a_1^2\log q_1 + a_2^2\log q_2 + q_1\left(\frac{a_1^2-a_2^2}{2} + m^2\,\frac{a_1^2-a_2^2}{2a_1^2}\right) + q_2\,\frac{a_2^2-a_1^2}{2} \\ &+ q_1q_2\left(\frac{a_1^2+a_2^2}{4} + m^2\,\frac{a_1^2+a_2^2}{4a_1^2}\right) + q_2^2\,\frac{13a_2^4-14a_1^2a_2^2+a_1^4}{64a_2^2} \\ &+ q_1^2\left(\frac{13a_1^4-14a_1^2a_2^2+a_2^4}{64a_1^2} + m^2\,\frac{9a_1^4-6a_1^2a_2^2-3a_2^4}{32a_1^4} + m^4\,\frac{a_1^4-6a_1^2a_2^2+5a_2^4}{64a_1^6}\right) + \dots\,. \end{aligned} \tag{2.151}$$

This precisely matches the q-dependent part of the prepotential derived using Nekrasov's localization techniques in the quiver theory when we choose the masses as in (2.134a) and set the Ω-deformation parameters ϵ_i to zero (see Appendix A for details, and in particular (A.17)).

Finally, adding the q-independent 1-loop contribution F_{pert} (see Eq. (A.26)), we may obtain the complete prepotential of the effective theory

$$\begin{aligned} \mathcal{F} &= F + F_{\text{pert}} \\ &= F - 2a_1^2\log\frac{4a_1^2}{\Lambda^2} - 2a_2^2\log\frac{4a_2^2}{\Lambda^2} + \frac{1}{2}(a_1+m)^2\log\frac{(a_1+m)^2}{\Lambda^2} \\ &\quad + \frac{1}{2}(a_1-m)^2\log\frac{(a_1-m)^2}{\Lambda^2} + a_2^2\log\frac{a_1^2}{\Lambda^2} \\ &\quad + \frac{1}{2}(a_1+a_2)^2\log\frac{(a_1+a_2)^2}{\Lambda^2} + \frac{1}{2}(a_1-a_2)^2\log\frac{(a_1-a_2)^2}{\Lambda^2}\,. \end{aligned} \tag{2.152}$$

This result represents a nice check of the spectral curve (2.138) and of the relations (2.123).

Using all our findings so far, we can easily derive the weak-coupling expansions of the roots (2.136) which are

$$
\begin{aligned}
\zeta_1 = q_1\Big(1-\frac{m^2}{a_1^2}\Big)\Bigg(&1+q_1\,m^2\,\frac{a_1^2-a_2^2}{2a_1^4}+q_1q_2\,m^2\,\frac{a_1^2+a_2^2}{4a_1^4}\\
&+q_1^2\,m^2\,\frac{\big(a_1^2-a_2^2\big)\big(5a_1^4+7a_1^2a_2^2+7a_1^2m^2-19a_2^2m^2\big)}{32a_1^8}+\dots\Bigg)\,, \qquad (2.153)\\
\widehat{\zeta} = \frac{a_1^2}{a_2^2}\Bigg(&1-q_1\,\frac{\big(a_1^2-a_2^2\big)\big(a_1^2+m^2\big)}{2a_1^4}-q_2\,\frac{a_1^2-a_2^2}{2a_2^2}+q_1q_2\,\frac{\big(a_1^2-a_2^2\big)\big(a_1^2+m^2\big)}{2a_1^2a_2^2}\\
&-q_1^2\,\frac{\big(a_1^2-a_2^2\big)\big(a_1^2-m^2\big)\big(3a_1^4+a_1^2a_2^2+a_1^2m^2+11a_2^2m^2\big)}{32a_1^8}\\
&+q_2^2\,\frac{\big(a_1^2-a_2^2\big)\big(7a_1^2-11a_2^2\big)}{32a_2^4}+\dots\Bigg)\,, \qquad (2.154)
\end{aligned}
$$

and

$$
\frac{1}{\zeta_2}=q_2\,. \qquad (2.155)
$$

We remark that (2.153) and (2.154) are perturbative in the q's but are exact in the mass parameter.

Case B): Let us now briefly consider the second mass choice (2.134b). In this case the spectral curve (2.119) becomes

$$
x^2(t)=\frac{C(t-\zeta_3)(t-\widehat{\zeta})}{t(t-q_1)(q_2t-1)(t-1)^2} \qquad (2.156)
$$

where

$$
\begin{aligned}
\zeta_3 &= \frac{-4u_1-4u_2+M^2(4-q_1-q_2+2q_1q_2)-4\sqrt{D}}{8C}\,,\\
\widehat{\zeta} &= \frac{-4u_1-4u_2+M^2(4-q_1-q_2+2q_1q_2)+4\sqrt{D}}{8C}\,,
\end{aligned} \qquad (2.157)
$$

with

$$
\begin{aligned}
C &= -u_2 + \frac{3M^2}{4} q_2 - \frac{M^2}{4} q_1 q_2 \ , \\
D &= \frac{1}{16}\big(4u_1 + 4u_2 - M^2(4 - q_1 - q_2 + 2q_1q_2)\big)^2 + C\big(4u_1 - 3M^2 q_1 + M^2 q_1 q_2\big) \ .
\end{aligned}
\tag{2.158}
$$

As in the previous case, it will prove useful to invert the relation (2.132) and the corresponding one for U_2; this leads to

$$
\begin{aligned}
u_1 &= \big(1 - q_1\big)\, U_1 + q_1 \big(1 - q_2\big)\, U_2 - \frac{M^2}{4} q_1 \big(1 + q_2\big) \ , \\
u_2 &= \big(1 - q_2\big)\, U_2 + q_2 \big(1 - q_1\big)\, U_1 - \frac{M^2}{4} q_2 \big(1 + q_1\big) \ .
\end{aligned}
\tag{2.159}
$$

We now compute the α-periods of the Seiberg-Witten differential $\lambda = x(t)dt$, whose singularity structure is similar to the one shown in Fig. 2.6. The main difference is that now $t = q_1$ is a branch-point and not a pole, while $t = 1$ is a pole and not a branch-point. Taking this into account we therefore have

$$
a_1 = \frac{1}{2\pi \mathrm{i}} \oint_{\alpha_1} \lambda = \frac{1}{\pi} \int_0^{q_1} \sqrt{\frac{C(\zeta_3 - t)(t - \widehat{\zeta})}{t(q_1 - t)(1 - q_2 t)}}\, \frac{dt}{(1 - t)} \ . \tag{2.160}
$$

After rescaling $t \to q_1 t$, we can easily compute the integral as discussed in the previous case expanding in powers of t and exploiting (2.147). Making use of the relations (2.159) to express the result in terms of U_i, we obtain

$$
\begin{aligned}
a_1 = \sqrt{U_1}\, \Bigg(1 &- q_1 \frac{U_1 + M^2}{4U_1^2} + q_2 \frac{U_2}{4U_1^2} - q_1 q_2\, \frac{U_2}{4U_1} \\
&- q_1^2\, \frac{7U_1^2 - 10 U_1 U_2 + 3U_2^2 + M^2\big(14U_1 - 6U_2 + 3M^2\big)}{64 U_1^2} + \dots \Bigg) \ .
\end{aligned}
\tag{2.161}
$$

The second period a_2 can be calculated along the same lines and the final result can be obtained from (2.161) by simply exchanging $q_1 \leftrightarrow q_2$ and $U_1 \leftrightarrow U_2$. If we invert these formulæ and then integrate over q_1 and q_2, we get

$$\begin{aligned} F \;=\;& a_1^2 \log q_1 + a_2^2 \log q_2 + q_1 \frac{a_1^2 - a_2^2 + M^2}{2} + q_2 \frac{a_2^2 - a_1^2 + M^2}{2} \\ & + q_1 q_2 \frac{a_1^2 + a_2^2 - M^2}{4} + q_1^2 \Big(\frac{13 a_1^4 - 14 a_1^2 a_2^2 + a_2^2}{64 a_1^2} + \frac{9M^2}{32} + \frac{M^2(M^2 - 2a_2^2)}{64 a_1^2} \Big) \\ & + q_2^2 \Big(\frac{13 a_2^4 - 14 a_1^2 a_2^2 + a_1^2}{64 a_2^2} + \frac{9M^2}{32} + \frac{M^2(M^2 - 2a_1^2)}{64 a_2^2} \Big) + \dots \,. \end{aligned} \tag{2.162}$$

This exactly matches the instanton prepotential derived using Nekrasov's approach in the quiver theory for the particular mass choice (2.134b) as one can see by comparing with (A.17).

Our results provide an explicit check of the UV equation of the Seiberg-Witten curve and of the way in which the IR effective prepotential is explicitly encoded in it; this will be confirmed in Section 3 by exploiting the AGT correspondence [4].

2.7.2 The period matrix and the roots

We now consider another approach to the derivation of the effective gauge theory from the Seiberg-Witten curve, which is based on the computation of the period matrix in terms of the roots of its defining equation (2.121). Taking the standard basis of holomorphic differentials as

$$\omega^i = \frac{t^{i-1}\, dt}{y(t)} \quad \text{for} \;\; i = 1, 2 \;, \tag{2.163}$$

we denote their periods along the cycles described in Fig. 2.5 as follows:

$$\int_{\alpha_j} \omega^i = \big(\Omega_{(1)}\big)^{ij} \;, \qquad \int_{\beta^j} \omega^i = \big(\Omega_{(2)}\big)^{i}_{\;j} \;. \tag{2.164}$$

The period matrix τ of the curve is given by

$$\tau = \Omega_{(1)}^{-1}\,\Omega_{(2)}\ . \tag{2.165}$$

It is a symmetric matrix and has thus three independent entries τ_{11}, τ_{22} and τ_{12}. In terms of these we introduce the quantities

$$Q_1 = \mathrm{e}^{\mathrm{i}\pi\tau_{11}}\ ,\quad Q_2 = \mathrm{e}^{\mathrm{i}\pi\tau_{22}}\ ,\quad \widehat{Q} = \mathrm{e}^{\mathrm{i}\pi\tau_{12}} \tag{2.166}$$

which will be conveniently used in the following. Given the period matrix τ, we introduce the genus-2 θ-constants defined as

$$\theta\begin{bmatrix}\vec{\varepsilon}\\ \vec{\varepsilon'}\end{bmatrix} \equiv \sum_{\vec{n}\in\mathbb{Z}^2} \exp\left\{\pi\mathrm{i}\left[\left(\vec{n}+\tfrac{\vec{\varepsilon}}{2}\right)^t\tau\left(\vec{n}+\tfrac{\vec{\varepsilon}}{2}\right)+\left(\vec{n}+\tfrac{\vec{\varepsilon}}{2}\right)^t\vec{\varepsilon'}\right]\right\}\ , \tag{2.167}$$

where $\vec{\varepsilon},\vec{\varepsilon'}$ are two 2-vectors; in what follows we will only need to consider the case in which these vectors have integer components.

The Thomae formulæ [32] can be used to express[8] the anharmonic ratios ζ_1, ζ_2 and $\widehat{\zeta}$ in terms of the θ-constants. Specifically, one has

$$\zeta_1 = \frac{\theta^2\begin{bmatrix}10\\00\end{bmatrix}\theta^2\begin{bmatrix}11\\00\end{bmatrix}}{\theta^2\begin{bmatrix}01\\00\end{bmatrix}\theta^2\begin{bmatrix}00\\00\end{bmatrix}}\ ,\quad \zeta_2 = \frac{\theta^2\begin{bmatrix}10\\00\end{bmatrix}\theta^2\begin{bmatrix}00\\11\end{bmatrix}}{\theta^2\begin{bmatrix}01\\00\end{bmatrix}\theta^2\begin{bmatrix}11\\11\end{bmatrix}}\ ,\quad \widehat{\zeta} = \frac{\theta^2\begin{bmatrix}00\\11\end{bmatrix}\theta^2\begin{bmatrix}11\\00\end{bmatrix}}{\theta^2\begin{bmatrix}11\\11\end{bmatrix}\theta^2\begin{bmatrix}00\\00\end{bmatrix}}\ . \tag{2.168}$$

Using (2.166) and(2.167), we find that ζ_1, $1/\zeta_2$ and $\widehat{\zeta}$ can be expressed as infinite sums containing positive integer powers of Q_1 and Q_2, and both positive and negative

[8] See for instance [33] and Appendix C of [34].

powers of $\widehat{Q}$. Up to second order in Q_1 and Q_2, we have

$$\zeta_1 = Q_1 \frac{4(\widehat{Q}+1)^2}{\widehat{Q}}\Bigg[1 - Q_1\frac{2(\widehat{Q}+1)^2}{\widehat{Q}} + Q_2\frac{2(\widehat{Q}-1)^2}{\widehat{Q}} - Q_1Q_2\frac{8(\widehat{Q}^2-1)^2}{\widehat{Q}^2} + Q_1^2\frac{3\widehat{Q}^4+10\widehat{Q}^3+18\widehat{Q}^2+10\widehat{Q}+3}{\widehat{Q}^2} + Q_2^2\frac{(\widehat{Q}-1)^2(\widehat{Q}^2-4\widehat{Q}+1)}{\widehat{Q}^2} + \dots\Bigg], \tag{2.169}$$

$$\frac{1}{\zeta_2} = Q_2 \frac{4(\widehat{Q}-1)^2}{\widehat{Q}}\Bigg[1 + Q_1\frac{2(\widehat{Q}+1)^2}{\widehat{Q}} - Q_2\frac{2(\widehat{Q}-1)^2}{\widehat{Q}} - Q_1Q_2\frac{8(\widehat{Q}^2-1)^2}{\widehat{Q}^2} + Q_1^2\frac{(\widehat{Q}+1)^2(\widehat{Q}^2+4\widehat{Q}+1)}{\widehat{Q}^2} + Q_2^2\frac{3\widehat{Q}^4-10\widehat{Q}^3+18\widehat{Q}^2-10\widehat{Q}+3}{\widehat{Q}^2} + \dots\Bigg], \tag{2.170}$$

and

$$\widehat{\zeta} = \frac{(\widehat{Q}+1)^2}{(\widehat{Q}-1)^2}\Bigg[1 - 8(Q_1+Q_2-8Q_1Q_2) + (Q_1^2+Q_2^2)\frac{4(\widehat{Q}^2+8\widehat{Q}+1)}{\widehat{Q}}\dots\Bigg]. \tag{2.171}$$

As is well-known, the period matrix of the Seiberg-Witten curve is identified with the matrix of the coupling constants of the low-energy effective theory, which are expressed in terms of the prepotential $\mathcal{F}$ according to

$$2\pi\mathrm{i}\tau_{ij} = \frac{\partial^2\mathcal{F}}{\partial a_i\partial a_j}\ . \tag{2.172}$$

Using the prepotential (2.152), from (2.172) and (2.166) we get

$$\begin{aligned}
Q_1 = q_1\,\frac{(a_1^2-a_2^2)(a_1^2-m^2)}{16a_1^4}\bigg[1 &+ q_1\Big(\frac{1}{2}-\frac{3m^2a_2^2}{2a_1^4}\Big)-\frac{q_2}{2}\\
&+ q_1^2\Big(\frac{21a_1^4+3a_2^4}{64a_1^4}-m^2\,\frac{21a_1^2a_2^2+15a_2^4}{16a_1^6}+m^4\,\frac{3a_1^4-60a_1^2a_2^2+177a_2^4}{64a_1^8}\Big)\\
&+ q_2^2\,\frac{3a_1^2-3a_2^2}{32a_2^2}+q_1q_2\,\frac{3m^2a_2^2}{2a_1^4}+\ldots\bigg]\,,
\end{aligned} \tag{2.173}$$

$$\begin{aligned}
Q_2 = q_2\,\frac{a_1^2-a_2^2}{16a_2^2}\bigg[1 &- q_1\Big(\frac{1}{2}+\frac{m^2}{2a_1^2}\Big)+\frac{q_2}{2}+q_2^2\,\frac{21a_2^4+3a_1^4}{64a_2^4}\\
&+ q_1^2\Big(\frac{3a_2^2-3a_1^2}{32a_1^2}-m^2\,\frac{9a_2^2-a_1^2}{16a_1^4}+m^4\,\frac{15a_2^2+a_1^2}{32a_1^6}\Big)+\ldots\bigg]\,,
\end{aligned} \tag{2.174}$$

and

$$\begin{aligned}
\widehat{Q} = \frac{a_1+a_2}{a_1-a_2}\bigg[1 &+ q_1\,\frac{m^2a_2}{a_1^3}-q_1q_2\,\frac{m^2a_2}{2a_1^3}-q_2^2\,\frac{a_1^3}{16a_2^3}\\
&- q_1^2\Big(\frac{a_2^3}{16a_1^3}-m^2\,\frac{3a_1^2a_2+6a_2^3}{8a_1^5}-m^4\,\frac{6a_1^2a_2+8a_1a_2^2-15a_2^3}{16a_1^7}\Big)+\ldots\bigg]\,.
\end{aligned} \tag{2.175}$$

These formulæ represent the explicit map between the IR effective couplings and the UV data of the quiver theory. Inserting the above expressions into (2.169)–(2.171) we can derive the corresponding anharmonic ratios ζ_1, $\widehat{\zeta}$ and ζ_2, and find perfect agreement with the expressions in (2.153), (2.154) and (2.155)! The same agreement is found also when we use the second mass configuration (2.134b) and the corresponding prepotential (2.162), thus confirming the validity of the whole picture.

Summarizing, we have verified that the Seiberg-Witten curve is correct since it reproduces the correct prepotential of the low-energy effective field theory. In doing so, we have also found the precise relations between the UV data, namely the instanton expansion parameters q_1, q_2 (which encode the UV gauge couplings) and the Coulomb branch parameters a_1, a_2 on one side, and the IR couplings $\tau_{11}, \tau_{22}, \tau_{12}$ (or equivalently Q_1, Q_2 and $\widehat{Q}$) on the other side. Such relations are given in (2.173)–(2.175) which in turn follow from

$$\zeta_1 = \frac{\theta^2\begin{bmatrix}10\\00\end{bmatrix}\theta^2\begin{bmatrix}11\\00\end{bmatrix}}{\theta^2\begin{bmatrix}01\\00\end{bmatrix}\theta^2\begin{bmatrix}00\\00\end{bmatrix}}(Q)\ ,\quad \frac{1}{\zeta_2} = \frac{\theta^2\begin{bmatrix}01\\00\end{bmatrix}\theta^2\begin{bmatrix}11\\11\end{bmatrix}}{\theta^2\begin{bmatrix}10\\00\end{bmatrix}\theta^2\begin{bmatrix}00\\11\end{bmatrix}}(Q)\ ,\quad \widehat{\zeta} = \frac{\theta^2\begin{bmatrix}00\\11\end{bmatrix}\theta^2\begin{bmatrix}11\\00\end{bmatrix}}{\theta^2\begin{bmatrix}11\\11\end{bmatrix}\theta^2\begin{bmatrix}00\\00\end{bmatrix}}(Q)\ . \tag{2.176}$$

These relations are the genus-2 analogues of the well-known relation [30] that holds in the SU(2) theory with $N_f = 4$ and links the instanton counting parameter q of the UV theory to the effective IR coupling Q (see (2.116) for the massive theory or (2.112) for the massless one). Note that in the SU(2), $N_f = 4$ case, for purely dimensional reasons, the vacuum expectation value of the adjoint scalar cannot appear in the massless UV/IR relation but, as we have just shown, this is no longer the case for quivers with more than one node.

Chapter 3

The Omega background and the 2d/4d correspondence

3.1 Nekrasov and the Omega background

The Seiberg-Witten solution was a giant leap towards our understanding of the low energy physics of $\mathcal{N} = 2$ gauge theories. However, a direct computation of the complete instanton contribution to the prepotential within the gauge theory was missing until Nekrasov came up with his revolutionary work in [3]. We note that there were a few successful attempts [24, 35] at low instanton orders. At higher orders, due to large instanton measures, the integrals involved became extremely hard. In order to compute the instanton corrections, Nekrasov studied the gauge theory in what is called the Ω-background. The Ω-background compactifies the instanton moduli space, and makes it finite. This occurs very often in our computations, and we will very briefly describe it.

$\mathcal{N} = 2$ super Yang-Mills action in four dimensions is obtained from $\mathcal{N} = 1$ super Yang-Mills action in six dimensions after compactification. The undeformed theory is obtained by compactifying, say $x^4 \equiv x^4 + 2\pi R_4$ and $x^5 \equiv x^5 + 2\pi R_5$ in the six dimensional flat space,

$$ds_6^2 = g_{\mu\nu} dx^\mu dx^\nu - dx^a dx^a, \ a = 4, 5 \ . \tag{3.1}$$

The Ω-deformed theory is obtained by choosing the deformed metric,

$$ds_6^2 = g_{\mu\nu}(dx^\mu + V_a^\mu(x) dx^a)(dx^\nu + V_b^\nu(x) dx^b) - dx^c dx^c \ , \tag{3.2}$$

where $V_a^\mu(x) = \Omega^\mu_{a,\nu} x^\nu$ and $\Omega^\mu_{a,\nu}$ are matrices of Lorentz rotations. The deformation is parameterized by ϵ_1 and ϵ_2 as follows,

$$\Omega_{\mu\nu} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & \epsilon_1 \\ 0 & 0 & \epsilon_2 & 0 \\ 0 & -\epsilon_2 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \end{bmatrix} \ . \tag{3.3}$$

The partition function of the deformed gauge theory is given by,

$$Z = \sum_{k=0}^{\infty} Z_k q^k \ , \tag{3.4}$$

where $q \sim \Lambda^{2N}$ is the dynamically generated scale. For conformal theories, q is simply the bare coupling constant exponentiated $q = \mathrm{e}^{2\pi \mathrm{i} \tau_0}$. The contribution of the k^{th} instanton sector to the partition function of the deformed theory is,

$$Z_k = \frac{1}{k!} \frac{\epsilon^k}{(\epsilon_1 \epsilon_2)^k} \oint \prod_{I=1}^{k} \frac{d\phi_I Q(\phi_I)}{P(\phi_I) P(\phi_I + \epsilon)} \prod_{1 \leq I < J \leq k} \frac{\phi_{IJ}^2 (\phi_{IJ}^2 - \epsilon^2)}{(\phi_{IJ}^2 - \epsilon_1^2)(\phi_{IJ}^2 - \epsilon_2^2)} \ , \tag{3.5}$$

where,

$$Q(x) = \prod_{f=1}^{N_f} (x + m_f) \ ,$$
$$P(x) = \prod_{f=1}^{N} (x - a_l) \ , \tag{3.6}$$

and $\phi_{IJ} = \phi_I - \phi_J$, $\epsilon = \epsilon_1 + \epsilon_2$. The poles that contribute to the integral are classified by partitions of the instanton number k, characterized by Young Tableaux. While the Z_k have an essential singularity as we turn off the deformation parameters, the corresponding free energy,

$$\mathcal{F}(a, \epsilon_1, \epsilon_2, q) = -\epsilon_1 \epsilon_2 \log Z(a, \epsilon_1, \epsilon_2, q) \tag{3.7}$$

has a smooth limit. The claim of Nekrasov in [3] is that, this free energy in the limit the deformation parameters vanish, matches the prepotential of the undeformed gauge theory. These results have been succesfully compared with those obtained from the more traditional Seiberg-Witten analysis. For details on the computation, we refer to Appendix A. The ϵ deformed free energy gives the graviphoton corrected prepotential of the gauge theory.

3.2 The 2d/4d Correspondence

We now consider Ω-deformed quiver theories with the goal of both confirming, and extending the results of the previous chapter. We will also exploit the remarkable 2d/4d correspondence proposed by Alday-Gaiotto-Tachikawa (AGT) in [4]. This correspondence states that the Nekrasov partition function of a linear quiver with gauge group $\mathrm{SU}(2)^n$ is directly related to the $(n+3)$-point spherical conformal block in two dimensional Liouville CFT. Let us give some details[1].

3.3 The AGT map

In 2-dimensional Liouville theory with central charge $c = 1 + 6Q^2$, let us consider the conformal block

$$\left\langle \prod_{i=0}^{n+2} V_{\alpha_i}(z_i) \right\rangle_{\{\xi_1,\ldots,\xi_n\}} \tag{3.8}$$

where V_α denotes a primary operator with Liouville momentum α and conformal dimension

$$\Delta_\alpha = \alpha(Q - \alpha) \ . \tag{3.9}$$

In (3.8) the subscript $\{\xi_1, \ldots, \xi_n\}$ means that the correlator is computed in the specific pair-of-pants decomposition of the $(n+3)$-punctured sphere where only the primary field with Liouville momentum ξ_i and dimension Δ_{ξ_i} plus its descendants propagate in the i-th internal line (see Fig. 3.1). Furthermore, we take the degen-

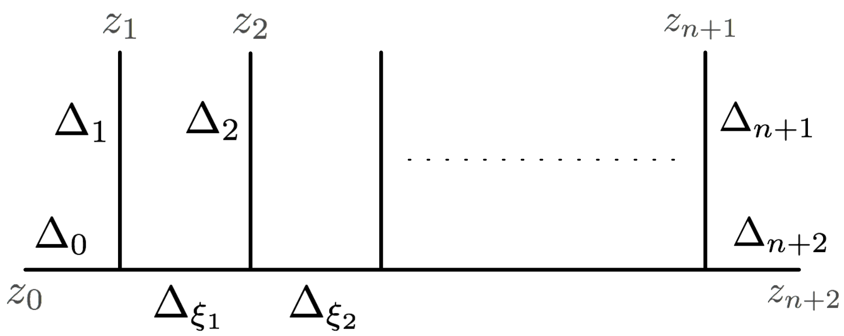

Figure 3.1: Pair-of-pants decomposition of the spherical conformal block with $(n+3)$ punctures

erate limit in which the $(n+3)$-punctured sphere reduces to a sequence of $(n+1)$ 3-punctured spheres connected by n long thin tubes with sewing parameters q_i, as shown in Fig. 3.2. If we denote the local coordinates on each 3-sphere by w_i, then

[1]For a more extended and technical discussion see for example [36] or the review [37]

the sewing procedure requires that

$$\frac{w_{i+1}}{w_i} = q_i \quad \text{with} \quad |q_i| < 0 \ . \tag{3.10}$$

In the local coordinates of each sphere, the punctures are located at $(0, 1, \infty)$; in particular all the unsewn external punctures are at 1 (except for the first and the last one which are at 0 and ∞ respectively). However, if we use the local coordinates of the last sphere as coordinates for the global surface, the sewing relations (3.10) imply that the external punctures of the first n spheres are at

$$t_i = \prod_{j=i}^{n} q_j \qquad \text{for} \qquad i \in \{1, \dots n\} \ . \tag{3.11}$$

This is precisely the same relation we found in (2.50). When written in terms of the

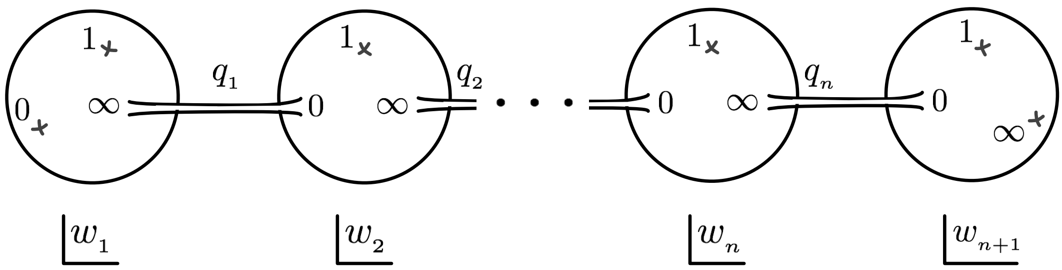

Figure 3.2: Three-punctured spheres connected by long thin tubes, with sewing parameters q_i

t_i's, the conformal block (3.8) becomes [36]

$$\left\langle V_{\alpha_0}(0) \prod_{i=1}^{n} V_{\alpha_i}(t_i)\, V_{\alpha_{n+1}}(1)\, V_{\alpha_{n+2}}(\infty) \right\rangle_{\{\xi_1,\dots,\xi_n\}} = \mathcal{N}\, \mathcal{B}(t_i, \Delta_{\alpha_i}, \Delta_{\xi_i}) \tag{3.12}$$

where the prefactor

$$\mathcal{N} = t_1^{-\Delta_{\alpha_0} - \Delta_{\alpha_1} + \Delta_{\xi_1}} \prod_{i=2}^{n} t_i^{-\Delta_{\xi_{i-1}} - \Delta_{\alpha_i} + \Delta_{\xi_i}} = t_1^{-\Delta_{\alpha_0}} \prod_{i=i}^{n} t_i^{-\Delta_{\alpha_i}} q_i^{\Delta_{\xi_i}} \tag{3.13}$$

originates from the conformal transformations that move the vertices V_{α_i} from 1 to t_i, while $\mathcal{B}(t_i, \Delta_{\alpha_i}, \Delta_{\xi_i})$ contains all other relevant information, including the structure function coefficients and the contribution of all descendants in the internal legs.

According to [4], it is possible to establish a correspondence between the conformal block (3.12) and the partition function of the ϵ-deformed $\mathrm{SU}(2)^n$ quiver theory. To do so, one has to identify q_i with the gauge coupling of the i-th group factor, set

$$Q = \frac{\epsilon_1 + \epsilon_2}{\sqrt{\epsilon_1 \epsilon_2}} \,, \tag{3.14}$$

and choose the Liouville momenta as follows:

$$\begin{aligned}
\alpha_0 &= \frac{Q}{2} + \frac{m_1 - m_2}{2\sqrt{\epsilon_1\epsilon_2}} \,, \qquad \alpha_1 = \frac{Q}{2} + \frac{m_1 + m_2}{2\sqrt{\epsilon_1\epsilon_2}} \,, \\
\alpha_i &= \frac{Q}{2} - \frac{m_{i-1,i}}{\sqrt{\epsilon_1\epsilon_2}} \qquad \text{for } i = 2, \ldots, n \,, \\
\xi_i &= \frac{Q}{2} - \frac{a_i}{\sqrt{\epsilon_1\epsilon_2}} \qquad \text{for } i = 1, \ldots, n \,, \\
\alpha_{n+1} &= \frac{Q}{2} - \frac{m_3 + m_4}{2\sqrt{\epsilon_1\epsilon_2}} \,, \qquad \alpha_{n+2} = \frac{Q}{2} - \frac{m_3 - m_4}{2\sqrt{\epsilon_1\epsilon_2}} \,,
\end{aligned} \tag{3.15}$$

where the m's are the fundamental or bi-fundamental masses of the matter hypermultiplets as discussed in the previous sections, and a_i is the vacuum expectation value of the adjoint scalar of the i-th gauge group. From (3.9) and (3.15) one can check that the conformal dimensions of the various operators are

$$\begin{aligned}
\Delta_{\alpha_0} &= \frac{(\epsilon_1+\epsilon_2)^2 - (m_1 - m_2)^2}{4\epsilon_1\epsilon_2} \,, \qquad \Delta_{\alpha_1} = \frac{(\epsilon_1+\epsilon_2)^2 - (m_1 + m_2)^2}{4\epsilon_1\epsilon_2} \,, \\
\Delta_{\alpha_i} &= \frac{(\epsilon_1+\epsilon_2)^2 - 4m_{i-1,i}^2}{4\epsilon_1\epsilon_2} \qquad \text{for } i = 2, \ldots, n \,, \\
\Delta_{\xi_i} &= \frac{(\epsilon_1+\epsilon_2)^2 - 4a_i^2}{4\epsilon_1\epsilon_2} \qquad \text{for } i = 1, \ldots, n \,, \\
\Delta_{\alpha_{n+1}} &= \frac{(\epsilon_1+\epsilon_2)^2 - (m_3 + m_4)^2}{4\epsilon_1\epsilon_2} \,, \qquad \Delta_{\alpha_{n+2}} = \frac{(\epsilon_1+\epsilon_2)^2 - (m_3 - m_4)^2}{4\epsilon_1\epsilon_2} \,.
\end{aligned} \tag{3.16}$$

The remarkable observation of [4] is that[2],

$$\mathcal{B}(t_i, \Delta_{\alpha_i}, \Delta_{\xi_i}) = Z_{\mathrm{U}(1)}\ \mathrm{e}^{-\frac{F_{\mathrm{inst}}}{\epsilon_1\epsilon_2}} \tag{3.17}$$

where F_{inst} is the Nekrasov instanton prepotential and $Z_{\mathrm{U}(1)}$ ensures the correct decoupling of the U(1) factors. This U(1) contribution can be explicitly computed (see for example [36]) and the result is

$$Z_{\mathrm{U}(1)} = \prod_{i=1}^{n}\prod_{j=i+1}^{n+1}\left(1-\frac{t_i}{t_j}\right)^{-2\alpha_i(Q-\alpha_j)} = \prod_{i=1}^{n}\prod_{j=i+1}^{n+1}(1-q_i\dots q_{j-1})^{-2\alpha_i(Q-\alpha_j)}\ . \tag{3.18}$$

The structure of these U(1) terms is actually quite simple: each factor in (3.18) can be associated to a connected subdiagram with four legs that is obtained by grouping together adjacent nodes of the quiver; the Liouville momenta of the two resulting inner legs determine the exponent [4]. For example, for $n = 1$ we have just one diagram with one node and coupling constant q; its inner legs carry momenta α_1 and α_2, and the corresponding U(1) factor is

$$(1-q)^{-2\alpha_1(Q-\alpha_2)}\ . \tag{3.19}$$

For $n = 2$ we have a subdiagram corresponding to the first node with coupling constant q_1 and inner legs with momenta α_1 and α_2; a subdiagram with coupling constant q_2 and inner legs carrying momenta α_2 and α_3, and finally a diagram with the two nodes combined, which has coupling q_1q_2 and inner legs with momenta α_1 and α_3. Thus the U(1) dressing factor is

$$(1-q_1)^{-2\alpha_1(Q-\alpha_2)}\,(1-q_2)^{-2\alpha_2(Q-\alpha_3)}\,(1-q_1q_2)^{-2\alpha_1(Q-\alpha_3)}\ . \tag{3.20}$$

[2] In our subsequent analysis we ignore the structure function coefficients in the conformal block $\mathcal{B}$. These are related to the 1-loop contribution to the prepotential while our focus is the instanton part.

This structure, which can be easily generalized to higher values of n, bears a clear resemblance with that of the symmetry factors introduced in Sections 2.6 and 2.7 in the redefinition of $\widehat{F}$ (see in particular (2.95) and (2.130)). In fact, the U(1) terms (3.18) can be considered as the proper generalization in the ϵ-deformed theory of the symmetry factors discussed in the previous sections. Finally, combining (3.12) and (3.17), we can write

$$\Big\langle V_{\alpha_0}(0) \prod_{i=1}^{n} V_{\alpha_i}(t_i)\, V_{\alpha_{n+1}}(1)\, V_{\alpha_{n+2}}(\infty)\Big\rangle_{\{\xi_1,\ldots,\xi_n\}} = \mathrm{e}^{-\frac{\widetilde{F}(\epsilon)}{\epsilon_1\epsilon_2}} \tag{3.21}$$

where

$$\widetilde{F}(\epsilon) = -\epsilon_1\epsilon_2 \log \mathcal{N} - \epsilon_1\epsilon_2 \log Z_{\mathrm{U}(1)} + F_{\mathrm{inst}}\ . \tag{3.22}$$

3.4 The UV curve

The 2-dimensional Liouville theory also contains information about the Seiberg-Witten curve of the 4-dimensional quiver gauge theory and its quantum deformation. To see this let us consider the normalized conformal block (3.12) with the insertion of the energy momentum tensor, namely[3]

$$\phi_2^{\epsilon}(z) = \frac{\Big\langle V_{\alpha_0}(0) \prod_{i=1}^{n} V_{\alpha_i}(t_i)\, T(z)\, V_{\alpha_{n+1}}(1) V_{\alpha_{n+2}}(\infty)\Big\rangle}{\Big\langle V_{\alpha_0}(0) \prod_{i=1}^{n} V_{\alpha_i}(t_i) V_{\alpha_{n+1}}(1) V_{\alpha_{n+2}}(\infty)\Big\rangle} \tag{3.23}$$

[3]From now on we simplify the notation by omitting the subscript $\{\xi_1, \ldots, \xi_n\}$ in the correlators.

with $|z| < 1$. As shown in Appendix D, using the conformal Ward identities it is possible to rewrite $\phi_2^\epsilon(z)$ as

$$\phi_2^\epsilon(z) = \frac{\Delta_{\alpha_0}}{z^2} + \sum_{i=1}^{n} \frac{\Delta_{\alpha_i}}{(z-t_i)^2} + \frac{\Delta_{\alpha_{n+1}}}{(z-1)^2} - \frac{\Delta_{\alpha_0} + \sum_{i=1}^{n} \Delta_{\alpha_i} + \Delta_{\alpha_{n+1}} - \Delta_{\alpha_{n+2}}}{z(z-1)}$$
$$+ \sum_{i=1}^{n} \frac{t_i(t_i-1)}{z(z-1)(z-t_i)} \frac{\partial}{\partial t_i} \log \Big\langle V_{\alpha_0}(0) \prod_{i=1}^{n} V_{\alpha_i}(t_i) V_{\alpha_{n+1}}(1) V_{\alpha_{n+2}}(\infty) \Big\rangle \,. \tag{3.24}$$

All terms on the right hand side of this equation are proportional to $1/(\epsilon_1\epsilon_2)$ since both the conformal dimensions Δ's and the logarithm of the conformal block scale in that manner. Thus the following limit

$$\lim_{\epsilon_1,\epsilon_2 \to 0} \big[-\epsilon_1\epsilon_2\, \phi_2^\epsilon(z) \big] \;\equiv\; \phi_2(z) \tag{3.25}$$

is well-defined and non-singular. In this limit only the mass dependent terms of the conformal weights contribute so that one finds

$$\phi_2(z) = \frac{(m_1-m_2)^2}{4z^2} + \frac{(m_1+m_2)^2}{4(z-t_1)^2} + \sum_{i=2}^{n} \frac{m_{i-1,i}^2}{(z-t_i)^2} + \frac{(m_3+m_4)^2}{4(z-1)^2}$$
$$- \frac{m_1^2 + m_2^2 + 2m_3m_4 + 2\sum_{i=2}^{n} m_{i-1,i}^2}{2z(z-1)} + \sum_{i=1}^{n} \frac{t_i(t_i-1)}{z(z-1)(z-t_i)} \frac{\partial \widetilde{F}}{\partial t_i} \tag{3.26}$$

where

$$\widetilde{F} = \lim_{\epsilon_1,\epsilon_2 \to 0} \widetilde{F}(\epsilon) \,. \tag{3.27}$$

$\phi_2(z)$ has the same form of $x^2(z)$ appearing in the expression of the Seiberg-Witten curve of the quiver theories described in the previous sections (see for example (2.91) or (2.126)). Indeed the mass terms are exactly the ones needed to produce the correct residues of the Seiberg-Witten differential and coincide with those we have written for the single node and the two-node quivers in Sections 2.6 and 2.7. Also the other terms have the right structure, and thus what remains to be checked is whether the function $\widetilde{F}$ in (3.26) coincides with the analogous quantity appearing in the Seiberg-

Witten curve. We now do this check in the three cases we have analyzed in more detail.

- **The SU(2) theory with $N_f = 4$**

For the SU(2) theory with $N_f = 4$ things are particularly simple, since in this case there is only a non-trivial puncture at $t_1 = q$ and $\widetilde{F}$ defined in (3.27) becomes

$$\widetilde{F} = a^2 \log q - \frac{1}{2}(m_1^2 + m_2^2) \log q + \frac{1}{2}(m_1 + m_2)(m_3 + m_4) \log(1 - q) + F_{\text{inst}} \ . \quad (3.28)$$

Using (2.93) and (2.95), one can immediately see that this agrees with the function $\widetilde{F}$ appearing in the Seiberg-Witten curve (2.91).

- **The SU(2) $\times$ SU(2) quiver theory**

In the 2-node quiver there are two non-trivial punctures. In the above discussion we have located them at $t_1 = q_1 q_2$ and $t_2 = q_2$, while in the curve derivation of Section 2.7 we have considered a different (though completely equivalent) configuration with punctures at $t_1 = q_1$ and $t_2 = 1/q_2$. Thus, before comparing we have to make the appropriate changes in the prefactor $\mathcal{N}$ which, being directly connected to the factorization of the conformal block in pair-of-pants diagrams, crucially depends on where the non-trivial punctures are located. If we set the punctures at $t_1 = q_1$ and $t_2 = 1/q_2$, we have to use

$$\mathcal{N} = q_1^{-\Delta_{\alpha_0} - \Delta_{\alpha_1} + \Delta_{\xi_1}} \, q_2^{\Delta_{\xi_2} + \Delta_{\alpha_2} - \Delta_{\alpha_3}} \ . \quad (3.29)$$

The corresponding expression for $\widetilde{F}$ is then

$$\begin{aligned}\widetilde{F} =& a_1^2 \log q_1 + a_2^2 \log q_2 - \frac{1}{2}(m_1^2 + m_2^2) \log q_1 + m_3 m_4 \log q_2 \\ &+ m_{12}(m_1 + m_2) \log(1 - q_1) - m_{12}(m_3 + m_4) \log(1 - q_2) \\ &+ \frac{1}{2}(m_1 + m_2)(m_3 + m_4) \log(1 - q_1 q_2) + F_{\text{inst}} \ ,\end{aligned} \tag{3.30}$$

which exactly matches the one appearing in the M-theory derivation of the Seiberg-Witten curve, as one can see using (2.128) and (2.130). This same result can also be obtained from the general expression (3.26) if we notice that under the change of variables that maps $(q_1 q_2, q_2, 1)$ to $(q_1, 1, 1/q_2)$, the term of $\phi_2(z)$ proportional to $1/(z(z-1)$ produces an extra contribution to $\widetilde{F}$ modifying its expression and leading to (3.30).

• **The conformal** $\mathrm{SU}(2)^n$ **quiver**

When all masses are zero, $\widetilde{F}$ in (3.27) is simply

$$\widetilde{F} = \sum_{i=1}^{N} a_i^2 \, \log q_i + F_{\text{inst}} \, . \tag{3.31}$$

Up to 1-loop t-independent contributions, this is precisely the prepotential F of the conformal quiver gauge theory, and thus the corresponding Seiberg-Witten curve can be written as

$$\phi_2(z) = \sum_{i=1}^{n} \frac{t_i(t_i - 1)}{z(z-1)(z-t_i)} \frac{\partial F}{\partial t_i} \ , \tag{3.32}$$

confirming in this case the direct identification of the residues at t_i with the derivatives of the gauge theory prepotential [12, 13]. We can therefore say that the AGT correspondence provides the analogue of the Matone relations [11] for the quiver gauge theory. One can go even further and map the curve (3.32) to that in (2.65) obtained using the M-theory analysis, thus finding the explicit relation between the

Coulomb parameters u_i appearing there and the t_i-derivatives of the prepotential.

3.5 The quiver prepotential from null-vector decoupling

We now present the derivation of the Ω-deformed prepotential for the $\mathrm{SU}(2)^n$ quiver model in the Nekrasov-Shatashvili limit [14] using a null-vector decoupling equation in the Liouville theory introduced in the previous section. The observable we consider is the conformal block obtained by deforming (3.12) with the insertion of the degenerate field $\Phi_{2,1}(z)$ of the Virasoro algebra [38], namely

$$\Psi(z) = \Big\langle V_{\alpha_0}(0) \prod_{i=1}^{n} V_{\alpha_i}(t_i)\, \Phi_{2,1}(z)\, V_{\alpha_{n+1}}(1)\, V_{\alpha_{n+2}}(\infty) \Big\rangle_{\{\xi_1,\ldots,\xi_n\}} \tag{3.33}$$

with $|z| < 1$. The degenerate field $\Phi_{2,1}$ has conformal dimension

$$\Delta_{2,1} = -\frac{1}{2} - \frac{3}{4}\frac{\epsilon_2}{\epsilon_1} \tag{3.34}$$

and satisfies the null-vector condition

$$\frac{\epsilon_1}{\epsilon_2}\frac{d^2\Phi_{2,1}(z)}{dz^2} + \,:\!T(z)\Phi_{2,1}(z)\!:\, = 0 \ . \tag{3.35}$$

This condition implies that $\Psi(z)$ obeys a second order differential equation that can be obtained from the conformal Ward identities as discussed in Appendix D. If we normalize the correlator (3.33) with the unperturbed one (3.21) and write

$$\Psi(z) = \mathrm{e}^{-\frac{\widetilde{F}(\epsilon)}{\epsilon_1\epsilon_2}}\, \Phi(z) \ , \tag{3.36}$$

then the differential equation for $\Psi(z)$ turns into the following differential equation for $\Phi(z)$

$$\left[\frac{\epsilon_1}{\epsilon_2}\frac{\partial^2}{\partial z^2} - \frac{2z-1}{z(z-1)}\frac{\partial}{\partial z} + \sum_{i=1}^{n}\left(\frac{t_i(t_i-1)}{z(z-1)(z-t_i)}\frac{\partial}{\partial t_i} - \frac{1}{\epsilon_1\epsilon_2}\frac{t_i(t_i-1)}{z(z-1)(z-t_i)}\frac{\partial \widetilde{F}(\epsilon)}{\partial t_i}\right) + \frac{\Delta_{\alpha_0}}{z^2}\right.$$
$$\left. + \sum_{i=1}^{n}\frac{\Delta_{\alpha_i}}{(z-t_i)^2} + \frac{\Delta_{\alpha_{n+1}}}{(z-1)^2} - \frac{\Delta_{\alpha_0} + \sum_{i=1}^{n}\Delta_{\alpha_i} + \Delta_{2,1} + \Delta_{\alpha_{n+1}} - \Delta_{\alpha_{n+2}}}{z(z-1)}\right]\Phi(z) \ = 0 \ . \tag{3.37}$$

This equation is well-suited to take the Nekrasov-Shatashvili limit [14] in which $\epsilon_2 \to 0$ with $\epsilon_1 \neq 0$, provided we assume that

$$\Phi(z) = \mathrm{e}^{-\frac{W(z)}{\epsilon_1}} \tag{3.38}$$

where $W(z)$ is regular in ϵ_1. Multiplying (3.37) by $(-\epsilon_1\epsilon_2)$ and sending ϵ_2 to zero, the differential equation simplifies in a few ways: the linear derivatives in z and t_i drop out along with the term proportional to the conformal dimension $\Delta_{2,1}$ of the degenerate field. Furthermore, in the Nekrasov-Shatashvili limit the generalized prepotential $\widetilde{F}(\epsilon)$ in (3.22) becomes

$$\widetilde{F}(\epsilon) \ \to \ \widetilde{F} + \epsilon_1\widetilde{F}^{(1)} + \epsilon_1^2\widetilde{F}^{(2)} \tag{3.39}$$

where the ϵ_1 corrections arise from the explicit ϵ-dependence of the prefactors $\mathcal{N}$ and $Z_{\mathrm{U}(1)}$. Since the terms proportional to the conformal dimensions Δ_{α_i} yield contributions at most of order ϵ_1^2, in the end we obtain the Schroedinger-type differential equation:

$$\left(-\epsilon_1^2\frac{d^2}{dz^2} + V(z,\epsilon_1)\right)\Phi(z) = 0 \ , \tag{3.40}$$

where

$$V(z,\epsilon_1) = V^{(0)}(z) + \epsilon_1\, V^{(1)}(z) + \epsilon_1^2\, V^{(2)}(z) \tag{3.41}$$

with

$$
\begin{aligned}
V^{(0)}(z) &= \phi_2(z) \ , \\
V^{(1)}(z) &= \sum_{i=1}^{n} \frac{t_i(t_i-1)}{z(z-1)(z-t_i)} \frac{\partial \widetilde{F}^{(1)}}{\partial t_i} \ , \\
V^{(2)}(z) &= -\frac{1}{4z^2} - \sum_{i=1}^{n} \frac{1}{4(z-t_i)^2} - \frac{1}{4(z-1)^2} + \frac{n+1}{4z(z-1)} + \sum_{i=1}^{n} \frac{t_i(t_i-1)}{z(z-1)(z-t_i)} \frac{\partial \widetilde{F}^{(2)}}{\partial t_i} \ .
\end{aligned}
\tag{3.42}
$$

Note that $V^{(0)}$ is the Seiberg-Witten curve of the undeformed theory. To solve (3.40) we make a WKB-like ansatz for $\Phi(z)$ writing

$$
W(z) = \int^{z} P(z', \epsilon_1)\, dz' \ , \tag{3.43}
$$

and then expand P in powers of ϵ_1

$$
P(z, \epsilon_1) = \sum_{n=0}^{\infty} \epsilon_1^n \, P^{(n)}(z) \ . \tag{3.44}
$$

Substituting in (3.40) we find

$$
-P(z, \epsilon_1)^2 + \epsilon_1 \frac{dP(z, \epsilon_1)}{dz} + V(z, \epsilon_1) = 0 \ , \tag{3.45}
$$

which in turn can be solved perturbatively in ϵ_1. The first few terms are

$$
P^{(0)}(z) = \sqrt{\phi_2(z)} \ , \tag{3.46a}
$$

$$
P^{(1)}(z) = \frac{1}{2} \frac{d}{dz} \log P^{(0)}(z) + \frac{V^{(1)}(z)}{2P^{(0)}(z)} \ , \tag{3.46b}
$$

$$
P^{(2)}(z) = \frac{P^{(1)'}(z) - P^{(1)^2}(z)}{2P^{(0)}(z)} + \frac{V^{(2)}(z)}{2P^{(0)}(z)} \ , \tag{3.46c}
$$

and so on. Since $P^{(0)}(z)dz$ is simply the Seiberg-Witten differential of the undeformed theory, it is more than natural to define the deformed Seiberg-Witten differential as

$$\lambda(\epsilon_1) = P(z, \epsilon_1)\, dz \ . \tag{3.47}$$

The periods of $\lambda(\epsilon_1)$ along the α_i-cycles can then be interpreted as the a_i's in the deformed theory, namely

$$a_i = \frac{1}{2\pi\mathrm{i}} \oint_{\alpha_i} \lambda(\epsilon_1) = \sum_{n=0}^{\infty} \epsilon_1^n \, a_i^{(n)} \qquad \text{with} \qquad a_i^{(n)} = \frac{1}{2\pi} \oint_{\alpha_i} P^{(n)}(z)\, dz \ . \tag{3.48}$$

Clearly the above integrals depend on the prepotential F and its t_i-derivatives; therefore we can use this information to fix the ϵ_1-dependence of F by demanding consistency, namely by choosing a_i's as independent variables and thus taking them to be constant. Even if it does not seem so at first sight, this procedure is fully equivalent to that used for instance in [39, 40] to obtain the deformed prepotential for the $\mathcal{N} = 2^*$ SU(2) theory or the $\mathcal{N} = 2$ SU(2) theory with $N_f = 4$. Indeed, also in our case the periods a_i which determine the monodromy properties of the wave function $\Phi(z)$, are constant, since the ϵ_1 (and q_i) dependence of the prepotential is fixed precisely to achieve this goal. It is remarkable that the prepotential obtained in this way agrees with the one computed using localization methods in the Nekrasov-Shatashvili limit.

3.5.1 The prepotential from deformed period integrals

We now illustrate the above procedure, focusing on the examples considered in the previous sections.

• **The SU(2) theory with $N_f = 4$**

When $n = 1$, the ϵ_1-terms of the potential in the Schroedinger-type equation are

$$
\begin{aligned}
V^{(1)}(z) &= q\,\frac{(m_1+m_2+m_3+m_4)}{2z(z-q)(z-1)}\ ,\\
V^{(2)}(z) &= -\frac{1}{4z^2}-\frac{1}{4(z-q)^2}-\frac{1}{4(z-1)^2}+\frac{1}{2z(z-1)}+\frac{3q-1}{4z(z-1)(z-q)}\ ,
\end{aligned}
\tag{3.49}
$$

while $V^{(0)}(z)$ is given by the Seiberg-Witten curve $\phi_2(z)$.

To proceed we choose the same mass configuration that we have discussed in Section 2.6, namely $m_1 = m_2 = m$, $m_3 = m_4 = M$, which allows us to write the curve in the factorized form

$$
\phi_2(z) = \frac{C(e_2-z)(z-e_3)}{z(z-1)^2(z-q)^2}\ . \tag{3.50}
$$

Here the roots e_2 and e_3 and the constant C are the same as in (2.102) and (2.103), but they are expressed in terms of the prepotential instead of the Coulomb modulus u.

At order ϵ_1^0, the period has already been calculated in Section 2.6 (see (2.108)); expressing it in terms of $U \equiv q\,\partial F/\partial q$, we have (up to 2 instantons)

$$
\begin{aligned}
a^{(0)} \;=\; & \sqrt{U}\,\Bigg[1-\frac{q}{4}\Big(1+\frac{(m^2+4mM+M^2)}{U}+\frac{m^2M^2}{U^2}\Big)\\
& -\frac{q^2}{64}\Big(7+\frac{14m^2+48mM+14M^2}{U}+\frac{3m^4+16m^3M+60m^2M^2+16mM^3+3M^4}{U^2}\\
& +\frac{6m^2M^2(m^2+8mM+M^2)}{U^3}+\frac{15m^4M^4}{U^4}\Big)+\ldots\Bigg]\ .
\end{aligned}
\tag{3.51}
$$

At order ϵ_1 we have instead

$$
a^{(1)} = \frac{1}{2\pi\mathrm{i}}\oint_\alpha P^{(1)}(z)\,dz = -q\,\frac{m+M}{2\pi\sqrt{C}}\int_0^{e_2}\frac{dz}{\sqrt{z(e_2-z)(e_3-z)}} \tag{3.52}
$$

where in the second step we used (3.46b) and discarded the total derivative term.

This integral can be evaluated as a power series and, up to two instantons, we find

$$a^{(1)} = -q\,\frac{m+M}{2\sqrt{U}}\left[1 + q\,\frac{3U^2 + U(m^2+4mM+M^2) + 3m^2M^2}{4U^2} + \dots\right]. \tag{3.53}$$

Using the formulæ in (3.46) iteratively, we can easily compute the order ϵ_1^2 correction to the period and get

$$\begin{aligned} a^{(2)} = -\,\frac{q}{16U^{\frac{5}{2}}}\Big[&3U^2 + m^2M^2 + \frac{q}{8U^2}\Big(17U^4 + 7U^3(3m^2+8mM+3M^2) \\ &+ 2U^2(m^4+20m^2M^2+M^4) - 5Um^2M^2(m^2-8mM+M^2) + 35m^4M^4\Big) + \dots\Big]. \end{aligned} \tag{3.54}$$

So far, we have calculated the period integral as an expansion of the form

$$a = a^{(0)}(U) + \epsilon_1\,a^{(1)}(U) + \epsilon_1^2\,a^{(2)}(U) + \dots \tag{3.55}$$

We now invert this expression and determine how U should depend on ϵ_1 so that a be a constant. We can do this by writing

$$U = U^{(0)} + \epsilon_1\,U^{(1)} + \epsilon_1^2\,U^{(2)} + \dots \tag{3.56}$$

and demanding consistency order by order in ϵ_1. Once U is computed, we can obtain the deformed prepotential F by integrating it with respect to (the logarithm of) q. The zeroth-order term that we get in this way clearly coincides with (2.110), while the first successive corrections are given by

$$\begin{aligned} F^{(1)} &= q\,(m+M) + \frac{q^2}{2}\,(m+M) + \dots\,, \\ F^{(2)} &= \frac{q}{8}\Big(3 + \frac{m^2M^2}{a^4}\Big) + \frac{q^2}{128}\Big(23 - \frac{m^2+M^2}{a^2} + \frac{2m^4+16m^2M^2+2M^4}{a^4} \\ &\qquad - \frac{15m^2M^2(m^2+M^2)}{a^6} + \frac{21m^4M^4}{a^8}\Big) + \dots \end{aligned} \tag{3.57}$$

These precisely match the microscopic results obtained from the Nekrasov partition function via localization methods.

- **The SU(2)$\times$ SU(2) quiver theory**

When $n=2$ the Schroedinger problem is algebraically more complicated, but still doable. The ϵ_1-corrections of the potential V are

$$
\begin{aligned}
V^{(1)}(z) =& \frac{(m_1+m_2+m_3+m_4)q_1q_2}{2z(z-1)(z-q_1q_2)} + \frac{(m_1+m_2+2m_{12})q_1q_2}{2z(z-q_2)(z-q_1q_2)} + \frac{(m_3+m_4-2m_{12})q_2}{2z(z-1)(z-q_2)} \ , \\
V^{(2)}(z) =& -\frac{1}{4z^2} - \frac{1}{4(z-q_1q_2)^2} - \frac{1}{4(z-q_2)^2} - \frac{1}{4(z-1)^2} + \frac{3}{4z(z-1)} \\
& - \frac{\eta_1}{z(z-1)(z-q_2)} - \frac{\eta_2}{z(z-q_2)(z-q_1q_2)}
\end{aligned}
\tag{3.58}
$$

where

$$
\eta_1 = \frac{(1-2(1+q_1)q_2+3q_1q_2^2)}{2(1-q_1q_2)} \ , \qquad \eta_2 = \frac{q_2(1+5q_1^2q_2-3q_1(1+q_2))}{4(1-q_1q_2)} \ . \tag{3.59}
$$

To proceed we make the simplifying mass choices discussed in Section 2.7, see (2.134).

Case A): In our present conventions the Seiberg-Witten curve takes the factorized form

$$
\phi_2(z) = \frac{-u_2(z-q_2\zeta_1)(z-q_2\widehat{\zeta})}{z(z-1)(z-q_1q_2)^2(z-q_2)} \tag{3.60}
$$

where the various constants are exactly those appearing in (2.136), with the u_i's written in terms of the U_i's using (2.139). Furthermore, with this mass choice the first-order term of the potential simplifies to

$$
V^{(1)}(z) = -\frac{mq_1q_2(1+q_2-2z)}{z(z-1)(z-q_2)(z-q_1q_2)} \ . \tag{3.61}
$$

Using the same basis of α-cycles discussed in Section 2.7, we find that the first correction to the a_1-period takes the form

$$a_1^{(1)} = \frac{1}{2\pi\mathrm{i}}\oint_{\alpha_1} P^{(1)}(z)\,dz = -\frac{mq_1q_2}{2\sqrt{u_2}}\int_0^{q_2\zeta_1}\frac{dz}{\sqrt{z(q_2\zeta_1-z)}}\,\frac{(1+q_2-2z)}{\sqrt{(1-z)(q_2-z)(q_2\widehat{\zeta}-z)}}\ . \tag{3.62}$$

Note that, unlike the case of the undeformed period (2.141), now there are no poles in the integrand and the integral can be done simply by expanding the second factor of (3.62) in powers of z and writing the resulting integrals in terms of Euler β-functions. In this way we find[4]

$$a_1^{(1)} = -\frac{mq_1}{2\sqrt{U_1}}\left[1-q_1\frac{U_1(U_2-3U_1)+m^2(3U_2-U_1)}{4U_1^2}+q_2+\dots\right]. \tag{3.63}$$

The first correction to the a_2 period can be similarly performed and we obtain

$$a_2^{(1)} = -\frac{3m\,q_1q_2}{4\sqrt{U_2}}+\dots\ . \tag{3.64}$$

At order ϵ_1^2 we find

$$\begin{aligned}
a_1^{(2)} &= -\frac{q_1}{16\,U_1^{\frac{5}{2}}}\bigg[3U_1^2-m^2U_2-q_2\Big(5U_1^2+m^2(U_1+U_2)\Big)-\frac{q_1}{8}\Big(17U_1^2-7U_1U_2+2U_2^2\\
&\qquad+\frac{m^2(21U_1^2-24U_1U_2-5U_2^2)}{U_1}+\frac{m^4(2U_1^2-25U_1U_2+35U_2^2)}{U_1^2}\Big)+\dots\bigg]\,,\\
a_2^{(2)} &= -\frac{q_2}{16\sqrt{U_2}}\left[3+5q_1+q_2\frac{2U_1^2-7U_1U_2+17U_2^2}{8U_1^2}+\dots\right].
\end{aligned} \tag{3.65}$$

Inverting the expansion of the periods order-by-order in ϵ_1, we can determine the ϵ_1 dependence of U_1 and U_2. At each order the resulting expressions turn out to be integrable and the prepotential can be recovered. At order ϵ_1^0 we get the same

[4] To keep the expressions compact we only exhibit the results up to 2 instantons. The calculations have been performed for higher instantons numbers as well.

expression as in (2.151), while the corrections of order ϵ_1 and ϵ_1^2 are

$$
\begin{aligned}
F^{(1)} &= m\left(q_1 + \frac{1}{2}q_1^2 + q_1 q_2 + \dots\right) , & (3.66)\\
F^{(2)} &= q_1\,\frac{3a_1^4 - m^2 a_2^2}{8a_1^4} + q_2\,\frac{3}{8} + q_1 q_2\,\frac{7a_1^4 + m^2 a_2^2}{16a_1^4} + q_2^2\,\frac{23a_2^4 - a_1^2 a_2^2 + 2a_1^4}{128a_2^4} & (3.67)\\
&\quad + q_1^2\left(\frac{23a_1^4 - a_1^2 a_2^2 + 2a_2^4}{128a_1^4} - \frac{m^2(a_1^4 + 15a_2^4)}{128a_1^6} + \frac{m^4(2a_1^4 - 15a_1^2 a_2^2 + 21a_2^4)}{128a_1^8}\right) + \dots
\end{aligned}
$$

One can check that this precisely matches the ϵ_1 corrections to the prepotential obtained using Nekrasov's analysis, thus validating the entire picture.

Case B): The Seiberg-Witten curve in this case is

$$
\phi_2(z) = \frac{C(z - q_2\,\zeta_3)(z - q_2\,\widehat{\zeta})}{z(z - q_1 q_2)(z - q_2)^2(z - 1)} \tag{3.68}
$$

where the constants are the same as in (2.157) and (2.158), provided we write the u_i's in terms of the U_i's by means of (2.159). For this mass configuration, the first-order correction to the Schroedinger potential is

$$
V^{(1)}(z) = -\frac{M\,q_2\big(z(1 - q_1) + q_1(1 - q_2)\big)}{z(z - q_1 q_2)(z - q_2)(z - 1)} \ , \tag{3.69}
$$

and the α_i-cycles are unchanged from the undeformed theory. Thus the period integrals are straightforward to perform, leading to the following results

$$
\begin{aligned}
a_1^{(1)} &= \frac{1}{2\pi\mathrm{i}}\oint_{\alpha_1} P^{(1)}(z)\,dz = -\frac{Mq_2}{2\sqrt{C}}\int_0^{q_1 q_2} \frac{dz}{\sqrt{z(q_1 q_2 - z)}}\,\frac{z(1 - q_1) + q_1(1 - q_2)}{\sqrt{(q_2\zeta_3 - z)(q_2\widehat{\zeta} - z)(1 - z)}}\\
&= -\frac{q_1 M}{2\sqrt{U_1}}\left[1 + q_1\,\frac{3U_1 - U_2 + M^2}{4U_1} - \frac{q_2}{2} + \dots\right] . & (3.70)
\end{aligned}
$$

At order ϵ_1^2 we find

$$a_1^{(2)} = -\frac{q_1}{16\sqrt{U_1}}\left[3 + 5q_2 + q_1\,\frac{17U_1^2 - 7U_1U_2 + 2U_2^2 - M^2(21U_1 - 4U_2) - 2M^4}{8U_1^2} + \ldots\right] . \tag{3.71}$$

The period integrals $a_2^{(k)}$ along the α_2-cycle can be obtained from the above expressions by the following symmetry operations

$$U_1 \leftrightarrow U_2 \;, \qquad q_1 \leftrightarrow q_2 \;, \qquad M \leftrightarrow -M \;. \tag{3.72}$$

Inverting as before the map between the a_i's and the U_i's, and integrating with respect to the coupling constants q_i, we find that the first ϵ_1-corrections to the prepotential are

$$\begin{aligned} F^{(1)} &= M(q_1 - q_2) + \frac{M}{2}(q_1^2 - q_2^2) + \ldots \;, \\ F^{(2)} &= \frac{3(q_1 + q_2)}{8} + \frac{7q_1q_2}{16} + q_1^2\,\frac{23a_1^4 - a_1^2a_2^2 + 2a_2^4 - M^2(4a_1^2 + a_2^2) + 2M^4}{128a_1^4} \\ &\quad + q_2^2\,\frac{23a_2^4 - a_1^2a_2^2 + 2a_1^4 - M^2(4a_2^2 + a_1^2) + 2M^4}{128a_2^4} + \ldots \;. \end{aligned} \tag{3.73}$$

This perfectly agrees with the Nekrasov prepotential for this mass configuration.

Combining the results for the two different mass configurations with the symmetry that exchanges the two gauge groups, the associated masses and coupling constants, we can therefore claim that the results following from the null-vector decoupling equation are completely consistent with the Ω-deformed prepotential obtained from localization in the Nekrasov-Shatashvili limit.

Chapter 4

Surface Operators in Gauge Theories

4.1 Introduction

The study of how a quantum field theory responds to the presence of defects is a very important subject, which has received much attention in recent years especially in the context of supersymmetric gauge theories. In this section we study a class of two-dimensional defects, also known as surface operators, on the Coulomb branch of the $\mathcal{N} = 2^\star$ SU(N) gauge theory in four dimensions[1]. Such surface operators can be introduced and analyzed in different ways. They can be defined by the transverse singularities they induce in the four-dimensional fields [16, 42], or can be characterized by the two-dimensional theory they support on their world-volume [21, 22].

A convenient way to describe four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry is to consider M5 branes wrapped on a punctured Riemann surface [5, 10]. From the point of view of the six-dimensional $(2, 0)$ theory on the M5 branes, surface operators can be realized by means of either M5′ or M2 branes giving rise,

[1] For a review of surface operators see [41].

respectively, to codimension-2 and codimension-4 defects. While a codimension-2 operator extends over the Riemann surface wrapped by the M5 brane realizing the gauge theory, a codimension-4 operator intersects the Riemann surface at a point. Codimension-2 surface operators were systematically studied in [43] where, in the context of the of the $4d/2d$ correspondence [4], the instanton partition functions of $\mathcal{N} = 2$ SU(2) super-conformal quiver theories with surface operators were mapped to the conformal blocks of a two-dimensional conformal field theory with an affine sl(2) symmetry. These studies were later extended to SU(N) quiver theories whose instanton partition functions in the presence of surface operators were related to conformal field theories with an affine sl(N) symmetry [44]. The study of codimension-4 surface operators was pioneered in [38] where the instanton partition function of the conformal SU(2) theory with a surface operator was mapped to the Virasoro blocks of the Liouville theory, augmented by the insertion of a degenerate primary field. Many generalizations and extensions of this have been considered in the last few years [45–52].

Here we study $\mathcal{N} = 2^{\star}$ theories in the presence of surface operators. The low-energy effective dynamics of the bulk four-dimensional theory is completely encoded in the holomorphic prepotential which at the non-perturbative level can be very efficiently determined using localization [3] along with the constraints that arise from S-duality. The latter turn out to imply [53, 54] a modular anomaly equation [55] for the prepotential, which is intimately related to the holomorphic anomaly equation occurring in topological string theories on local Calabi-Yau manifolds [56–59][2]. Working perturbatively in the mass of the adjoint hypermultiplet, the modular anomaly equation allows one to resum all instanton corrections to the prepotential into (quasi)-modular forms, and to write the dependence on the Coulomb branch parameters in terms of particular sums over the roots of the gauge group, thus

[2]Modular anomaly equations have been studied in various contexts, such as the Ω-background [30, 53, 54, 60–65], the $4d/2d$ correspondence [39, 40, 66], SQCD theories with fundamental matter [53, 54, 67–69] and in $\mathcal{N} = 2^{\star}$ theories [19, 53, 54, 70–72].

making it possible to treat any semi-simple algebra [19, 70].

In this section we apply the same approach to study the effective twisted superpotential which governs the infrared dynamics on the world-volume of the two-dimensional surface operator in the $\mathcal{N} = 2^{\star}$ theory. For simplicity, we limit ourselves to SU(N) gauge groups and consider half-BPS surface defects that, from the six-dimensional point of view, are codimension-2 operators. These defects introduce singularities characterized by the pattern of gauge symmetry breaking, *i.e.* by a Levi decomposition of SU(N), and also by a set of continuous (complex) parameters. In [73] it has been shown that the effect of these surface operators on the instanton moduli action is equivalent to a suitable orbifold projection which produces structures known as ramified instantons [73–75]. The moduli spaces of these ramified instantons were already studied in [76] from a mathematical point of view, in terms of representations of a quiver that can be obtained by performing an orbifold projection of the usual ADHM moduli space of the standard instantons. In Section 4.2 we explicitly implement such an orbifold procedure on the non-perturbative sectors of the theory realized by means of systems of D3 and D(-1) branes [77, 78]. In Section 4.3 we carry out the integration on the ramified instanton moduli via equivariant localization. The logarithm of the resulting partition function exhibits both a $4d$ and a $2d$ singularity in the limit of vanishing Ω deformations[3]. The corresponding residues are regular in this limit and encode, respectively, the prepotential $\mathcal{F}$ and the twisted superpotential $\mathcal{W}$. The latter depends, in addition to the Coulomb vacuum expectation values and the adjoint mass, on the continuous parameters of the defect.

In Section 4.4 we show that, as it happens for the prepotential, the constraints arising from S-duality lead to a modular anomaly equation for $\mathcal{W}$. In Section 4.5, we solve this equation explicitly for the SU(2) theory and prove that the resulting $\mathcal{W}$ agrees with the twisted superpotential obtained in [39] in the framework of the $4d/2d$

[3]We actually calculate the effective superpotential in the Nekrasov-Shatashvili limit [14] in which only one of the Ω-deformation parameters is turned on.

correspondence with the insertion of a degenerate field in the Liouville theory. Since this procedure is appropriate for codimension-4 defects [38], the agreement we find supports the proposal of a duality between the two classes of defects recently put forward in [79]. In Section 4.6, we turn our attention to generic surface operators in the SU(N) theory and again, order by order in the adjoint mass, solve the modular anomaly equations in terms of quasi-modular elliptic functions and sums over the root lattice.

We also consider the relation between our findings and what is known for surface defects defined through the two-dimensional theory they support on their world-volume. In [21] the coupling of the sigma-models defined on such defects to a large class of four-dimensional gauge theories was investigated and the twisted superpotential governing their dynamics was obtained. Simple examples for pure $\mathcal{N}=2$ SU(N) gauge theory include the linear sigma-model on $\mathbb{CP}^{N-1}$, that corresponds to the so-called simple defects with Levi decomposition of type $\{1, N-1\}$, and sigma-models on Grassmannian manifolds corresponding to defects of type $\{p, N-p\}$. The main result of [21] is that the Seiberg-Witten geometry of the four-dimensional theory can be recovered by analyzing how the vacuum structure of these sigma-models is fibered over the Coulomb moduli space. Independent analyses based on the $4d/2d$ correspondence also show that the twisted superpotential for the simple surface operator is related to the line integral of the Seiberg-Witten differential over the punctured Riemann surface [38]. In Section 4.7, we test this claim in detail by considering first the pure $\mathcal{N}=2$ gauge theory. Since this theory can be recovered upon decoupling the massive adjoint hypermultiplet, we take the decoupling limit on our $\mathcal{N}=2^\star$ results for $\mathcal{W}$ and precisely reproduce those findings. Furthermore, we show that for simple surface defects the relation between the twisted superpotential and the line integral of the Seiberg-Witten differential holds prior to the decoupling limit, *i.e.* in the $\mathcal{N}=2^\star$ theory itself. The agreement we find provides evidence for the proposed duality between the two types of descriptions of the surface operators.

4.2 Instantons and surface operators in $\mathcal{N} = 2^\star$ SU(N) gauge theories

The $\mathcal{N} = 2^\star$ theory is a four-dimensional gauge theory with $\mathcal{N} = 2$ supersymmetry that describes the dynamics of a vector multiplet and a massive hypermultiplet in the adjoint representation. It interpolates between the $\mathcal{N} = 4$ super Yang-Mills theory, to which it reduces in the massless limit, and the pure $\mathcal{N} = 2$ theory, which is recovered by decoupling the matter hypermultiplet. We will consider for simplicity only special unitary gauge groups SU(N). As is customary, we combine the Yang-Mills coupling constant g and the vacuum angle θ into the complex coupling

$$\tau = \frac{\theta}{2\pi} + \mathrm{i}\,\frac{4\pi}{g^2}\ , \tag{4.1}$$

on which the modular group SL$(2,\mathbb{Z})$ acts in the standard fashion:

$$\tau \quad \to \quad \frac{a\tau + b}{c\tau + d} \tag{4.2}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. In particular under S-duality we have

$$S(\tau) = -\frac{1}{\tau}\ . \tag{4.3}$$

The Coulomb branch of the theory is parametrized by the vacuum expectation value of the adjoint scalar field ϕ in the vector multiplet, which we take to be of the form

$$\langle\phi\rangle = \mathrm{diag}(a_1, a_2, \cdots, a_N) \qquad \text{with} \quad \sum_{u=1}^{N} a_u = 0\ . \tag{4.4}$$

The low-energy effective dynamics on the Coulomb branch is entirely described by a single holomorphic function $\mathcal{F}$, called the prepotential, which contains a classical term, a perturbative 1-loop contribution and a tail of instanton corrections. The latter can be obtained from the instanton partition function

$$Z_{\text{inst}} = \sum_{k=0}^{\infty} q^k \, Z_k \tag{4.5}$$

where

$$q = \mathrm{e}^{2\pi \mathrm{i} \tau} \tag{4.6}$$

and Z_k is the partition function in the k-instanton sector that can be explicitly computed using localization methods[4]. For later purposes, it is useful to recall that the weight q^k in (4.5) originates from the classical instanton action

$$S_{\text{inst}} = -2\pi \mathrm{i}\tau \left(\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \mathrm{Tr}\, F \wedge F \right) = -2\pi \mathrm{i}\tau\, k \tag{4.7}$$

where in the last step we used the fact that the second Chern class of the gauge field strength F equals the instanton charge k. Hence, the weight q^k is simply $\mathrm{e}^{-S_{\text{inst}}}$.

Let us now introduce a surface operator which we view as a non-local defect D supported on a two-dimensional plane inside the four-dimensional (Euclidean) space-time (see Appendix F for more details). In particular, we parametrize $\mathbb{R}^4 \simeq \mathbb{C}^2$ by two complex variables (z_1, z_2), and place D at $z_2 = 0$, filling the z_1-plane. The presence of the surface operator induces a singular behavior in the gauge connection A, which has the following generic form [43, 73]:

$$A = A_\mu \, dx^\mu \simeq -\,\mathrm{diag}\Big(\underbrace{\gamma_1, \cdots, \gamma_1}_{n_1}, \underbrace{\gamma_2, \cdots, \gamma_2}_{n_2}, \cdots, \underbrace{\gamma_M, \cdots, \gamma_M}_{n_M} \Big)\, d\theta \tag{4.8}$$

as $r \to 0$. Here (r, θ) denotes the set of polar coordinates in the z_2-plane, and the

[4] Our conventions are such that $Z_0 = 1$.

γ_I's are constant parameters, where $I = 1, \cdots, M$. The M integers n_I satisfy

$$\sum_{I=1}^{M} n_I = N \tag{4.9}$$

and define a vector $\vec{n}$ that identifies the type of the surface operator. This vector is related to the breaking pattern of the gauge group (or Levi decomposition) felt on the two-dimensional defect D, namely

$$\mathrm{SU}(N) \to \mathrm{S}\big[\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \cdots \times \mathrm{U}(n_M)\big] \ . \tag{4.10}$$

The type $\vec{n} = \{1, 1, \cdots, 1\}$ corresponds to what are called full surface operators, originally considered in [43]. The type $\vec{n} = \{1, N-1\}$ corresponds to simple surface operators, while the type $\vec{n} = \{N\}$ corresponds to no surface operators and hence will not be considered.

In the presence of a surface operator, one can turn on magnetic fluxes for each factor of the gauge group (4.10) and thus the instanton action can receive contributions also from the corresponding first Chern classes. This means that (4.7) is replaced by [38, 42, 43, 73]

$$S_{\mathrm{inst}}[\vec{n}] = -2\pi\mathrm{i}\tau\left(\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}\, F \wedge F\right) - 2\pi\mathrm{i}\sum_{I=1}^{M} \eta_I \left(\frac{1}{2\pi}\int_D \mathrm{Tr}\, F_{\mathrm{U}(n_I)}\right) \tag{4.11}$$

where η_I are constant parameters. As shown in detail in Appendix F, given the behavior (4.8) of the gauge connection near the surface operator, one has

$$\begin{aligned} &\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}\, F \wedge F = k + \sum_{I=1}^{M} \gamma_I\, m_I \ , \\ &\frac{1}{2\pi}\int_D \mathrm{Tr}\, F_{\mathrm{U}(n_I)} = m_I \end{aligned} \tag{4.12}$$

with $m_I \in \mathbb{Z}$. As is clear from the second line in the above equation, each m_I represents the flux of the U(1) factor in each subgroup $\mathrm{U}(n_I)$ in the Levi decomposition (4.10); furthermore, these fluxes satisfy the constraint

$$\sum_{I=1}^{M} m_I = 0 \ . \tag{4.13}$$

Using (4.12), we easily find

$$S_{\rm inst}[\vec{n}] = -2\pi{\rm i}\tau\, k - 2\pi{\rm i}\sum_{I=1}^{M}\big(\eta_I + \tau\,\gamma_I\big)m_I = -2\pi{\rm i}\tau\, k - 2\pi{\rm i}\,\vec{t}\cdot\vec{m} \tag{4.14}$$

where in the last step we have combined the electric and magnetic parameters (η_I, γ_I) to form the M-dimensional vector

$$\vec{t} = \{t_I\} = \{\eta_I + \tau\,\gamma_I\} \ . \tag{4.15}$$

This combination has simple duality transformation properties under $\mathrm{SL}(2,\mathbb{Z})$. Indeed, as shown in [42], given an element $\mathcal{M}$ of the modular group the electromagnetic parameters transform as

$$(\gamma_I, \eta_I) \quad \to \quad (\gamma_I, \eta_I)\mathcal{M}^{-1} = \big(d\,\gamma_I - c\,\eta_I, a\,\eta_I - b\,\gamma_I\big) \ . \tag{4.16}$$

Combining this with the modular transformation (4.2) of the coupling constant, it is easy to show that

$$t_I \quad \to \quad \frac{t_I}{c\tau + d} \ . \tag{4.17}$$

In particular under S-duality we have

$$S(t_I) = -\frac{t_I}{\tau} \ . \tag{4.18}$$

Using (4.14), we deduce that the weight of an instanton configuration in the presence of a surface operator of type $\vec{n}$ is

$$\mathrm{e}^{-S_{\mathrm{inst}}[\vec{n}]} = q^k \, \mathrm{e}^{2\pi \mathrm{i} \, \vec{t}\cdot\vec{m}} \ , \tag{4.19}$$

so that the instanton partition function can be written as

$$Z_{\mathrm{inst}}[\vec{n}] = \sum_{k,\vec{m}} q^k \, \mathrm{e}^{2\pi \mathrm{i} \, \vec{t}\cdot\vec{m}} \, Z_{k,\vec{m}}[\vec{n}] \ . \tag{4.20}$$

In the next section, we will describe the computation of $Z_{k,\vec{m}}[\vec{n}]$ using equivariant localization.

4.3 Partition functions for ramified instantons

As discussed in [73], the $\mathcal{N} = 2^*$ theory with a surface defect of type $\vec{n} = \{n_1, \cdots, n_M\}$, which has a six-dimensional representation as a codimension-2 surface operator, can be realized with a system of D3-branes in the orbifold background

$$\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_M \times \mathbb{C} \times \mathbb{C} \tag{4.21}$$

with coordinates (z_1, z_2, z_3, z_4, v) on which the $\mathbb{Z}_M$-orbifold acts as

$$(z_2, z_3) \to (\omega \, z_2, \omega^{-1} \, z_3) \ , \quad \text{where } \omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{M}} \ . \tag{4.22}$$

Like in the previous section, the complex coordinates z_1 and z_2 span the four-dimensional space-time where the gauge theory is defined (namely the world-volume of the D3-branes), while the z_1-plane is the world-sheet of the surface operator D that sits at the orbifold fixed point $z_2 = 0$. The (massive) deformation which leads from the $\mathcal{N} = 4$ to the $\mathcal{N} = 2^*$ theory takes place in the (z_3, z_4)-directions. Finally, the v-plane corresponds to the Coulomb moduli space of the gauge theory.

Without the $\mathbb{Z}_M$-orbifold projection, the isometry group of the ten-dimensional background is SO(4) $\times$ SO(4) $\times$ U(1), since the D3-branes are extended in the first four directions and are moved in the last two when the vacuum expectation values (4.4) are turned on. In the presence of the surface operator and hence of the $\mathbb{Z}_M$-orbifold in the (z_2, z_3)-directions, this group is broken to

$$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1) \ . \tag{4.23}$$

In the following we will focus only on the first four U(1) factors, since it is in the first four complex directions that we will introduce equivariant deformations to apply localization methods. We parameterize a transformation of this $\mathrm{U}(1)^4$ group by the vector

$$\vec{\epsilon} = \{\epsilon_1, \frac{\epsilon_2}{M}, \frac{\epsilon_3}{M}, \epsilon_4\} = \{\epsilon_1, \hat{\epsilon}_2, \hat{\epsilon}_3, \epsilon_4\} \tag{4.24}$$

where the $1/M$ rescalings in the second and third entry, suggested by the orbifold projection, are made for later convenience. If we denote by

$$\vec{l} = \{l_1, l_2, l_3, l_4\} \tag{4.25}$$

the weight vector of a given state of the theory, then under $\mathrm{U}(1)^4$ such a state transforms with a phase given by $\mathrm{e}^{2\pi\mathrm{i}\,\vec{l}\cdot\vec{\epsilon}}$, while the $\mathbb{Z}_M$-action produces a phase $\omega^{l_2-l_3}$.

On top of this, we also have to consider the action of the orbifold group on the Chan-Paton factors carried by the open string states stretching between the D-branes. There are different types of D-branes depending on the irreducible representation of $\mathbb{Z}_M$ in which this action takes place. Since there are M such representations, we have M types of D-branes, which we label with the index I already used before. On a D-brane of type I, the generator of $\mathbb{Z}_M$ acts as ω^I, and thus the Chan-Paton factor of a string stretching between a D-brane of type I and a D-brane of type J

transforms with a phase ω^{I-J} under the action of the orbifold generator.

In order to realize the split of the gauge group in (4.10), we consider M stacks of n_I D3-branes of type I, and in order to introduce non-perturbative effects we add on top of the D3's M stacks of d_I D-instantons of type I. The latter support an auxiliary ADHM group which is

$$\mathrm{U}(d_1) \times \mathrm{U}(d_2) \times \cdots \times \mathrm{U}(d_M) \ . \tag{4.26}$$

In the resulting D3/D(-1)-brane systems there are many different sectors of open strings depending on the different types of branes to which they are attached. Here we focus only on the states of open strings with at least one end-point on the D-instantons, because they represent the instanton moduli [77, 78] on which one eventually has to integrate in order to obtain the instanton partition function.

Let us first consider the neutral states, corresponding to strings stretched between two D-instantons. In the bosonic Neveu-Schwarz sector one finds states with $\mathrm{U}(1)^4$ weight vectors

$$\{\pm1,0,0,0\}_0 \ , \ \ \{0,\pm1,0,0\}_0 \ , \ \ \{0,0\pm1,0\}_0 \ , \ \ \{0,0,0\pm1\}_0 \ , \ \ \{0,0,0,0\}_{\pm1} \ , \tag{4.27}$$

where the subscripts denote the charge under the last U(1) factor of (4.23). They correspond to space-time vectors along the directions z_1, z_2, z_3, z_4 and v, respectively. In the fermionic Ramond sector one finds states with weight vectors

$$\left\{\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}\right\}_{\pm\frac{1}{2}} \tag{4.28}$$

with a total odd number of minus signs due to the GSO projection. They correspond to anti-chiral space-time spinors[5].

[5]Of course one could have chosen a GSO projection leading to chiral spinors, and the final results would have been the same.

It is clear from (4.27) and (4.28) that the orbifold phase $\omega^{l_2-l_3}$ takes the values ω^0, ω^{+1} or ω^{-1} and can be compensated only if one considers strings of type I-I, I-$(I+1)$ or $(I+1)$-I, respectively. Therefore, the $\mathbb{Z}_M$-invariant neutral moduli carry Chan-Paton factors that transform in the $(\mathbf{d}_I, \bar{\mathbf{d}}_I)$, $(\mathbf{d}_I, \bar{\mathbf{d}}_{I+1})$ or $(\mathbf{d}_{I+1}, \bar{\mathbf{d}}_I)$ representations of the ADHM group (4.26).

Let us now consider the colored states, corresponding to strings stretched between a D-instanton and a D3-brane or vice versa. Due to the twisted boundary conditions in the first two complex space-time directions, the weight vectors of the bosonic states in the Neveu-Schwarz sector are

$$\left\{\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, 0, 0\right\}_0 \tag{4.29}$$

while those of the fermionic states in the Ramond sector are

$$\left\{0, 0, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\right\}_{\pm\frac{1}{2}} . \tag{4.30}$$

Assigning a negative intrinsic parity to the twisted vacuum, both in (4.29) and in (4.30) the GSO-projection selects only those vectors with an even number of minus signs. Moreover, since the orbifold acts on two of the twisted directions, the vacuum carries also an intrinsic $\mathbb{Z}_M$-weight. We take this to be $\omega^{-\frac{1}{2}}$ when the strings are stretched between a D3-brane and a D-instanton, and $\omega^{+\frac{1}{2}}$ for strings with opposite orientation. Then, with this choice we find $\mathbb{Z}_M$-invariant bosonic and fermionic states either from the $3/(-1)$ strings of type I-I, whose Chan-Paton factors transform in the $(\mathbf{n}_I, \bar{\mathbf{d}}_I)$ representation of the gauge and ADHM groups, or from the $(-1)/3$ strings of type I-$(I+1)$, whose Chan-Paton factors transform in the $(\mathbf{d}_I, \bar{\mathbf{n}}_{I+1})$ representation, plus of course the corresponding states arising from the strings with opposite orientation.

In Appendix G we provide a detailed account of all moduli, both neutral and colored, and of their properties in the various sectors. It turns out that the moduli action, which can be derived from the interactions of the moduli on disks with at least a part of their boundary attached to the D-instantons [78], is exact with respect to the supersymmetry charge Q of weight

$$\{+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2}\}_{-\frac{1}{2}} \ . \tag{4.31}$$

Therefore Q can be used as the equivariant BRST-charge to localize the integral over the moduli space provided one considers $\mathrm{U}(1)^4$ transformations under which it is invariant. This corresponds to requiring that

$$\epsilon_1+\hat{\epsilon}_2+\hat{\epsilon}_3+\epsilon_4=0 \ . \tag{4.32}$$

Thus we are left with three equivariant parameters, say ϵ_1, $\hat{\epsilon}_2$ and ϵ_4; as we will see, the latter is related to the (equivariant) mass m of the adjoint hypermultiplet of $\mathcal{N}=2^*$ theory.

As shown in Appendix G, all instanton moduli can be paired in Q-doublets of the type $(\varphi_\alpha,\psi_\alpha)$ such that

$$Q\,\varphi_\alpha=\psi_\alpha\ ,\quad Q\,\psi_\alpha=Q^2\varphi_\alpha=\lambda_\alpha\,\varphi_\alpha \tag{4.33}$$

where λ_α are the eigenvalues of Q^2, determined by the action of the Cartan subgroup of the full symmetry group of the theory, namely the gauge group (4.10), the ADHM group (4.26), and the residual isometry group $\mathrm{U}(1)^4$ with parameters satisfying (4.32) in such a way that the invariant points in the moduli space are finite and isolated. The only exception to this structure of Q-doublets is represented by the neutral bosonic moduli with weight

$$\{0,0,0,0\}_{-1} \tag{4.34}$$

transforming in the adjoint representation $(\mathbf{d}_I, \bar{\mathbf{d}}_I)$ of the ADHM group $\mathrm{U}(d_I)$, which remain unpaired. We denote them as χ_I, and in order to obtain the instanton partition function we must integrate over them. In doing so, we can exploit the $\mathrm{U}(d_I)$ symmetry to rotate χ_I into the maximal torus and write it in terms of the eigenvalues $\chi_{I,\sigma}$, with $\sigma = 1, \cdots, d_I$, which represent the positions of the D-instantons of type I in the υ-plane. In this way we are left with the integration over all the $\chi_{I,\sigma}$'s and a Cauchy-Vandermonde determinant

$$\mathcal{V} = \prod_{I=1}^{M} \prod_{\sigma,\tau=1}^{d_I} \left(\chi_{I,\sigma} - \chi_{I,\tau} + \delta_{\sigma\tau}\right) . \tag{4.35}$$

More precisely, the instanton partition function in the presence of a surface operator of type $\vec{n}$ is defined by

$$Z_{\mathrm{inst}}[\vec{n}] = \sum_{\{d_I\}} \prod_{I=1}^{M} q_I^{d_I}\, Z_{\{d_I\}}[\vec{n}] \quad \text{with} \quad Z_{\{d_I\}}[\vec{n}] = \frac{1}{d_I!} \int \prod_{\sigma=1}^{d_I} \frac{d\chi_{I,\sigma}}{2\pi \mathrm{i}}\, z_{\{d_I\}} \tag{4.36}$$

where $z_{\{d_I\}}$ is the result of the integration over all Q-doublets which localizes on the fixed points of Q^2, and q_I is the counting parameter associated to the D-instantons of type I. With the convention that $z_{\{d_I=0\}} = 1$, we find

$$z_{\{d_I\}} = \mathcal{V} \prod_{\alpha} \left[\lambda_\alpha\right]^{(-)^{F_\alpha+1}} , \tag{4.37}$$

where the index α labels the Q-doublets and λ_α denotes the corresponding eigenvalue of Q^2. This contribution goes to the denominator or to the numerator depending upon the bosonic or fermionic statistics ($F_\alpha = 0$ or 1, respectively) of the first component of the doublet. Explicitly, using the data in Tab. G.1 of Appendix G and the determinant (4.35), we find

$$
\begin{aligned}
z_{\{d_I\}} = & \prod_{I=1}^{M} \prod_{\sigma,\tau=1}^{d_I} \frac{(\chi_{I,\sigma} - \chi_{I,\tau} + \delta_{\sigma,\tau})\,(\chi_{I,\sigma} - \chi_{I,\tau} + \epsilon_1 + \epsilon_4)}{(\chi_{I,\sigma} - \chi_{I,\tau} + \epsilon_4)\,(\chi_{I,\sigma} - \chi_{I,\tau} + \epsilon_1)} \\
& \times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \prod_{\rho=1}^{d_{I+1}} \frac{(\chi_{I,\sigma} - \chi_{I+1,\rho} + \epsilon_1 + \hat{\epsilon}_2)\,(\chi_{I,\sigma} - \chi_{I+1,\rho} + \hat{\epsilon}_2 + \epsilon_4)}{(\chi_{I,\sigma} - \chi_{I+1,\rho} - \hat{\epsilon}_3)\,(\chi_{I,\sigma} - \chi_{I+1,\rho} + \hat{\epsilon}_2)} \\
& \times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \prod_{s=1}^{n_I} \frac{\big(a_{I,s} - \chi_{I,\sigma} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2) + \epsilon_4\big)}{\big(a_{I,s} - \chi_{I,\sigma} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\big)} \\
& \times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \prod_{t=1}^{n_{I+1}} \frac{\big(\chi_{I,\sigma} - a_{I+1,t} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2) + \epsilon_4\big)}{\big(\chi_{I,\sigma} - a_{I+1,t} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\big)}
\end{aligned}
\tag{4.38}
$$

where $d_{M+1} = d_1$, $n_{M+1} = n_1$ and $a_{M+1,t} = a_{1,t}$. The integrations in (4.36) must be suitably defined and regularized. The standard prescription [19,70,80] is to consider $a_{I,s}$ to be real, and close the contours in the upper-half $\chi_{I,\sigma}$ -planes with the choice,

$$
\mathrm{Im}\,\epsilon_4 \gg \mathrm{Im}\,\hat{\epsilon}_3 \gg \mathrm{Im}\,\hat{\epsilon}_2 \gg \mathrm{Im}\,\epsilon_1 > 0 \ , \tag{4.39}
$$

and enforce (4.32) at the very end of the calculations.

In this way one finds that these integrals receive contributions from the poles of $z_{\{d_I\}}$, which are in fact the critical points of Q^2. Such poles can be put in one-to-one correspondence with a set of N Young tableaux $Y = \{Y_{I,s}\}$, with $I = 1, \cdots, M$ and $s = 1, \cdots n_I$, in the sense that the box in the i-th row and j-th column of the tableau $Y_{I,s}$ represents one component of the critical value:

$$
\chi_{I+(j-1)\,\mathrm{mod}\,M,\sigma} = a_{I,s} + \left((i-1) + \tfrac{1}{2}\right)\epsilon_1 + \left((j-1) + \tfrac{1}{2}\right)\hat{\epsilon}_2 \ . \tag{4.40}
$$

Note that in this correspondence, a single tableau accounts for $d_I!$ equivalent ways of relabeling $\chi_{I,\sigma}$.

4.3.1 Summing over fixed points and characters

Summing over the Young tableaux collections Y we get all the non-trivial critical points corresponding to all possible values of $\{d_I\}$. Eq. (4.40) tells us that we get a distinct $\chi_{I,\sigma}$ for each box in the j-th column of the tableau $Y_{I+1-j \bmod M, s}$. Relabeling the index j as

$$j \to J + j\,M \ , \tag{4.41}$$

with $J = 1, \dots M$, we have

$$d_I(Y) = \sum_{J=1}^{M} \sum_{s=1}^{n_{I+1-J}} \sum_{j} Y_{I+1-J,s}^{(J+jM)} \ , \tag{4.42}$$

where $Y_{I,s}^{(j)}$ denotes the height of the j-th column of the tableau $Y_{I,s}$, and the subscript index $I+1-J$ is understood modulo M.

The instanton partition function (4.36) can thus be rewritten as a sum over Young tableaux as follows

$$Z_{\text{inst}}[\vec{n}] = \sum_{Y} \prod_{I=1}^{M} q_I^{d_I(Y)}\, Z(Y) \tag{4.43}$$

where $Z(Y)$ is the residue of $z_{\{d_I\}}$ at the critical point Y. This is obtained by deleting in (4.38) the denominator factors that yield the identifications (4.40), and performing these identifications in the other factors. In other terms,

$$Z(Y) = \mathcal{V}(Y) \prod_{\alpha\,:\,\lambda_\alpha(Y)\neq 0} [\lambda_\alpha(Y)]^{(-)^{F_\alpha+1}} \ , \tag{4.44}$$

where $\mathcal{V}(Y)$ and $\lambda_\alpha(Y)$ are the Vandermonde determinant and the eigenvalues of Q^2 evaluated on (4.40).

A more efficient way to encode the eigenvalues $\lambda_\alpha(Y)$ is to employ the character of the action of Q^2, which is defined as follows

$$X_{\{d_I\}} = \sum_{\alpha} (-)^{F_\alpha} \mathrm{e}^{\mathrm{i}\lambda_\alpha} \ . \tag{4.45}$$

If we introduce

$$V_I = \sum_{\sigma=1}^{d_I} \mathrm{e}^{\mathrm{i}\chi_{I,\sigma} - \frac{\mathrm{i}}{2}(\epsilon_1+\hat{\epsilon}_2)} \ , \quad W_I = \sum_{s=1}^{n_I} \mathrm{e}^{\mathrm{i}a_{I,s}} \tag{4.46}$$

and

$$T_1 = \mathrm{e}^{\mathrm{i}\epsilon_1} \ , \quad T_2 = \mathrm{e}^{\mathrm{i}\hat{\epsilon}_2} \ , \quad T_4 = \mathrm{e}^{\mathrm{i}\epsilon_4} \ , \tag{4.47}$$

we can write the contributions to the character from the various Q-doublets as in the last column of Tab. G.1 in Appendix G. Then, by summing over all doublets and adding also the contribution of the Vandermonde determinant, we obtain

$$X_{\{d_I\}} = (1-T_4) \sum_{I=1}^{M} \left[-(1-T_1)V_I^* V_I + (1-T_1)V_{I+1}^* V_I T_2 + V_I^* W_I + W_{I+1}^* V_I T_1 T_2 \right] \ . \tag{4.48}$$

As we have seen before, through (4.42) and (4.40) each set Y determines both the dimensions $d_I(Y)$ and the eigenvalues $\lambda_\alpha(Y)$. Thus, the character $X(Y)$ associated to a set of Young tableaux is obtained from $X_{\{d_I\}}$ by substituting (4.40) into the definitions of V_I, namely

$$V_I = \sum_{J=1}^{M} \sum_{s=1}^{n_{I+1-J}} \mathrm{e}^{\mathrm{i}a_{I+1-J,s}} T_2^J \sum_{(i,J+jM)\in Y_{I+1-J,s}} T_1^{i-1} T_2^{jM-1} \ . \tag{4.49}$$

By analyzing $X(Y)$ obtained in this way we can extract the explicit expression for the eigenvalues $\lambda_s(Y)$ and finally write the instanton partition function. This procedure is easily implemented in a computer program, and yields the results we will use in the next sections. In Appendix (G.1), as an example, we illustrate these computations for the SU(2) gauge theory.

In our analysis we worked with the moduli action that describes D-branes probing the orbifold geometry. An alternative approach works with the resolution of the orbifold geometry [81,82]. This involves analyzing a gauged linear sigma-model that describes a system of D1 and D5-branes in the background $\mathbb{C}\times\mathbb{C}/\mathbb{Z}_M\times T^\star S^2\times\mathbb{R}^2$. One then uses the localization formulas for supersymmetric field theories defined on the 2-sphere [83, 84] to obtain exact results. This is a very powerful approach since it also includes inherently stringy corrections to the partition function arising from world-sheet instantons [81]. The results for the instanton partition function of the $\mathcal{N}=2^\star$ theory in the presence of surface operators obtained in [82] are equivalent to our results in (4.38).

4.3.2 Map between parameters

One of the key points that needs to be clarified is the map between the microscopic counting parameters q_I which appear in (4.43), and the parameters (q, t_I) which were introduced in Section 4.2 in discussing SU(N) gauge theories with surface operators. To describe this map, we start by rewriting the partition function (4.36) in terms of the total instanton number k and the magnetic fluxes m_I of the gauge groups on the surface operator which are related to the parameters d_I as follows [43, 73]:

$$d_1 = k\ ,\quad d_{I+1} = d_I + m_{I+1}\ . \tag{4.50}$$

Therefore, instead of summing over $\{d_I\}$ we can sum over k and $\vec{m}$ and find

$$Z_{\text{inst}}[\vec{n}] = \sum_{k,\vec{m}} (q_1\cdots q_M)^k\,(q_2\cdots q_M)^{m_2}\,(q_3\cdots q_M)^{m_3}\cdots(q_M)^{m_M}\,Z_{k,\vec{m}}[\vec{n}] \tag{4.51}$$

Furthermore, if we set

$$
\begin{aligned}
q_I &= \mathrm{e}^{2\pi\mathrm{i}(t_I - t_{I+1})} \quad \text{for} \quad I \in \{2, \dots M-1\} \ , \\
q_M &= \mathrm{e}^{2\pi\mathrm{i}(t_M - t_1)} \quad \text{and} \quad q = \prod_{I=1}^{M} q_I \ ,
\end{aligned}
\tag{4.52}
$$

we easily get

$$
Z_{\text{inst}}[\vec{n}] = \sum_{k,\vec{m}} q^k \mathrm{e}^{2\pi\mathrm{i}\sum_{I=2}^{M} m_I(t_I - t_1)}\, Z_{k,\vec{m}} = \sum_{k,\vec{m}} q^k\, \mathrm{e}^{2\pi\mathrm{i}\,\vec{t}\cdot\vec{m}}\, Z_{k,\vec{m}}[\vec{n}] \tag{4.53}
$$

where in the last step we introduced m_1 such that that $\sum_I m_I = 0$ (see (4.13)) in order to write the result in a symmetric form. This is precisely the expected expression of the partition function in the presence of a surface operator as shown in (4.20) and justifies the map (4.52) between the parameters of the two descriptions. From (4.53) we see that only differences of the parameters t_I appear in the partition function so that it may be convenient to use as independent parameters q and the $(M-1)$ variables

$$
z_J = t_J - t_1 \quad \text{for} \quad J \in \{2, \dots M\} \ . \tag{4.54}
$$

This is indeed what we are going to see in the next sections where we will show how to extract relevant information from the the instanton partition functions described above.

4.3.3 Extracting the prepotential and the twisted superpotential

The effective dynamics on the Coulomb branch of the four-dimensional $\mathcal{N} = 2^\star$ gauge theory is described by the prepotential $\mathcal{F}$, while the infrared physics of the two-dimensional theory defined on the world-sheet of the surface operator is governed by the twisted superpotential $\mathcal{W}$. The non-pertubative terms of both $\mathcal{F}$ and $\mathcal{W}$ can be derived from the instanton partition function previously discussed, by considering its behavior for small deformation parameters ϵ_1 and ϵ_2 and, in particular, in the so-called Nekrasov-Shatashvili limit [14].

To make precise contact with the gauge theory quantities, we set

$$\epsilon_4 = -m - \frac{\epsilon_1}{2} \tag{4.55}$$

where m is the mass of the adjoint hypermultiplet, and then take the limit for small ϵ_1 and ϵ_2. In this way we find [43]:

$$\log Z_{\text{inst}}[\vec{n}] \;\simeq\; -\frac{\mathcal{F}_{\text{inst}}(\epsilon_1)}{\epsilon_1\epsilon_2} + \frac{\mathcal{W}_{\text{inst}}(\epsilon_1)}{\epsilon_1} + \mathcal{O}(\epsilon_2) \tag{4.56}$$

The two leading singular contributions arise, respectively, from the (regularized) equivariant volume parts coming from the four-dimensional gauge theory and from the two-dimensional degrees of freedom supported on the surface defect D. This can be understood from the fact that, in the Ω-deformed theory, the respective super-volumes are finite and given by [41, 85]:

$$\int_{\mathbb{R}^4_{\epsilon_1,\epsilon_2}} d^4x\, d^4\theta \longrightarrow \frac{1}{\epsilon_1\epsilon_2} \qquad \text{and} \qquad \int_{\mathbb{R}^2_{\epsilon_1}} d^2x\, d^2\theta \longrightarrow \frac{1}{\epsilon_1}\,. \tag{4.57}$$

The non-trivial result is that the functions $\mathcal{F}_{\text{inst}}$ and $\mathcal{W}_{\text{inst}}$ defined in this way are analytic in the neighborhood of $\epsilon_1 = 0$. As an illustrative example, we now describe in some detail the SU(2) theory.

SU(2)

When the gauge group is SU(2), the only surface operators are of type $\vec{n} = \{1, 1\}$, the Coulomb branch is parameterized by

$$\langle \phi \rangle = \mathrm{diag}(a, -a) \ , \tag{4.58}$$

and the map (4.52) can be written as

$$q_1 = \frac{q}{x} \ , \quad q_2 = x = \mathrm{e}^{2\pi \mathrm{i}\, z} \tag{4.59}$$

where, for later convenience, we have defined $z = (t_2 - t_1)$. Using the results presented in Appendix G.1 and their extension to higher orders, it is possible to check that the instanton prepotential arising from (4.56), namely

$$\mathcal{F}_{\mathrm{inst}} = -\lim_{\epsilon_2 \to 0} \Big(\epsilon_1 \epsilon_2 \log Z_{\mathrm{inst}}[1, 1] \Big) \tag{4.60}$$

is, as expected, a function only of the instanton counting parameter q and not of x. Expanding in inverse powers of a, we have

$$\mathcal{F}_{\mathrm{inst}} = \sum_{\ell=1}^{\infty} f_\ell^{\mathrm{inst}} \tag{4.61}$$

where $f_\ell \sim a^{2-\ell}$. The first few coefficients of this expansion are

$$\begin{aligned}
f_{2\ell+1}^{\mathrm{inst}} &= 0 \quad \text{for } \ell = 0, 1, \cdots \ , \\
f_2^{\mathrm{inst}} &= -\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big(2q + 3q^2 + \frac{8}{3}q^3 + \cdots\Big) \ , \\
f_4^{\mathrm{inst}} &= \frac{1}{2a^2}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)^2\Big(q + 3q^2 + 4q^3 + \cdots\Big) \ , \\
f_6^{\mathrm{inst}} &= \frac{1}{16a^4}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)^2\Big(2\epsilon_1^2\, q - 3\big(4m^2 - 7\epsilon_1^2\big)q^2 - 8\big(8m^2 - 9\epsilon_1^2\big)q^3 + \cdots\Big) \ .
\end{aligned} \tag{4.62}$$

One can check that this precisely agrees with the Nekrasov-Shatashvili limit of the prepotential derived for example in [53, 54]. This complete match is a strong and non-trivial check on the correctness and consistency of the whole construction.

Let us now consider the non-perturbative superpotential, which according to (4.56) is

$$\mathcal{W}_{\text{inst}} = \lim_{\epsilon_2 \to 0} \left(\epsilon_1 \log Z_{\text{inst}}[1,1] + \frac{\mathcal{F}_{\text{inst}}}{\epsilon_2} \right) . \tag{4.63}$$

Differently from the prepotential, $\mathcal{W}_{\text{inst}}$ is, as expected, a function both of q and x. If we expand it as

$$\mathcal{W}_{\text{inst}} = \sum_{\ell=1}^{\infty} w_\ell^{\text{inst}} \tag{4.64}$$

with $w_\ell^{\text{inst}} \sim a^{1-\ell}$, using the results of Appendix G.1 we find

$$\begin{aligned} w_1^{\text{inst}} = -\Big(m - \frac{\epsilon_1}{2}\Big) \Big[&\Big(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \Big) + \Big(\frac{1}{x} + 2 + x + \cdots \Big) q \\ &+ \Big(\frac{1}{2x^2} + \frac{1}{x} + 3 + \cdots \Big) q^2 + \cdots \Big] , \end{aligned} \tag{4.65a}$$

$$\begin{aligned} w_2^{\text{inst}} = -\frac{1}{a}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \Big[&\Big(\frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{2} + \cdots \Big) + \Big(\frac{x}{2} - \frac{1}{2x} + \cdots \Big) q \\ &- \Big(\frac{1}{2x^2} + \frac{1}{2x} + \cdots \Big) q^2 + \cdots \Big] , \end{aligned} \tag{4.65b}$$

$$\begin{aligned} w_3^{\text{inst}} = -\frac{\epsilon_1}{a^2}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \Big[&\Big(\frac{x}{4} + \frac{x^2}{2} + \frac{3x^3}{4} + x^4 + \cdots \Big) + \Big(\frac{1}{4x} + \frac{x}{4} + \cdots \Big) q \\ &+ \Big(\frac{1}{2x^2} + \frac{1}{4x} + \cdots \Big) q^2 + \cdots \Big] , \end{aligned} \tag{4.65c}$$

and so on. For later convenience we explicitly write down the logarithmic derivatives with respect to x, namely

$$w_1' = -\Big(m - \frac{\epsilon_1}{2}\Big)\left[\Big(x + x^2 + x^3 + x^4 + \cdots\Big) - \Big(\frac{1}{x} - x + \cdots\Big)q - \Big(\frac{1}{x^2} + \frac{1}{x} + \cdots\Big)q^2 + \cdots\right], \tag{4.66a}$$

$$w_2' = -\frac{1}{a}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\left[\Big(\frac{x}{2} + x^2 + \frac{3x^3}{2} + 2x^4 + \cdots\Big) + \Big(\frac{x}{2} + \frac{1}{2x} + \cdots\Big)q + \Big(\frac{1}{x^2} + \frac{1}{2x} + \cdots\Big)q^2 + \cdots\right], \tag{4.66b}$$

$$w_3' = -\frac{\epsilon_1}{a^2}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\left[\Big(\frac{x}{4} + x^2 + \frac{9x^3}{4} + 4x^4 + \cdots\Big) - \Big(\frac{1}{4x} - \frac{x}{4} + \cdots\Big)q - \Big(\frac{1}{x^2} + \frac{1}{4x} + \cdots\Big)q^2 + \cdots\right] \tag{4.66c}$$

where $w_\ell' := x\,\frac{\partial}{\partial x}\big(w_\ell^{\text{inst}}\big)$. In the coming sections we will show that these expressions are the weak-coupling expansions of combinations of elliptic and quasi-modular forms of the modular group $\mathrm{SL}(2,\mathbb{Z})$.

4.4 Modular anomaly equation for the twisted superpotential

In [53, 54] it has been shown for the $\mathcal{N} = 2^\star$ SU(2) theory that the instanton expansions of the prepotential coefficients (4.62) can be resummed in terms of (quasi-) modular forms of the duality group $\mathrm{SL}(2,\mathbb{Z})$ and that the behavior under S-duality severely constrains the prepotential $\mathcal{F}$ which must satisfy a modular anomaly equation. This analysis has been later extended to $\mathcal{N} = 2^\star$ theories with arbitrary classical or exceptional gauge groups [19, 65, 70], and also to $\mathcal{N} = 2$ SQCD theories with fundamental matter [67, 68]. In this section we use a similar approach to study how S-duality constrains the form of the twisted superpotential $\mathcal{W}$.

For simplicity and without loss of generality, in the following we consider a full surface operator of type $\vec{n} = \{1, 1, \cdots, 1\}$ with electro-magnetic parameters $\vec{t} = \{t_1, t_2, \cdots, t_N\}$. Indeed, surface operators of other type correspond to the case in which these parameters are not all different from each other and form M distinct sets, namely

$$\vec{t} = \Big\{ \underbrace{t_1, \ \ldots, \ t_1,}_{n_1} \ \underbrace{t_2, \ \ldots, \ t_2,}_{n_2} \ \cdots, \ \underbrace{t_M, \ \ldots, t_M}_{n_M} \Big\} \ . \tag{4.67}$$

Thus they can be simply recovered from the full ones with suitable identifications.

Before analyzing the S-duality constraints it is necessary to take into account the classical and the perturbative 1-loop contributions to the prepotential and the superpotential.

The classical contribution

Introducing the notation $\vec{a} = \{a_1, a_2, \cdots, a_N\}$ for the vacuum expectation values, the classical contributions to the prepotential and the superpotential are given respectively by

$$\mathcal{F}_{\text{class}} = \pi \mathrm{i} \tau \, \vec{a} \cdot \vec{a} \tag{4.68}$$

and

$$\mathcal{W}_{\text{class}} = 2\pi \mathrm{i} \, \vec{t} \cdot \vec{a} \ . \tag{4.69}$$

Note that if we use the tracelessness condition (4.4), $\mathcal{W}_{\text{class}}$ can be rewritten as

$$\mathcal{W}_{\text{class}} = 2\pi \mathrm{i} \sum_{I=2}^{N} z_I \, a_I \tag{4.70}$$

where z_I is as defined in (4.54).

These classical contributions have very simple behavior under S-duality. Indeed

$$S\big(\mathcal{F}_{\text{class}}\big) = -\,\mathcal{F}_{\text{class}}\ , \tag{4.71a}$$

$$S\big(\mathcal{W}_{\text{class}}\big) = -\,\mathcal{W}_{\text{class}}\ . \tag{4.71b}$$

To show these relations one has to use the S-duality rules (4.3) and (4.18), and recall that

$$S\big(\vec{a}\big) = \vec{a}_{\mathrm{D}} := \frac{1}{2\pi\mathrm{i}}\frac{\partial\mathcal{F}}{\partial\vec{a}} \qquad \text{and} \qquad S\big(\vec{a}_{\mathrm{D}}\big) = -\vec{a}\ , \tag{4.72}$$

which for the classical prepotential simply yield $S(\vec{a}) = \tau\,\vec{a}$.

The 1-loop contribution

The 1-loop contribution to the partition function of the Ω-deformed gauge theory in the presence of a full surface operator of type $\{1, 1, \cdots, 1\}$ can be written in terms of the function

$$\gamma(x) := \log\Gamma_2(x|\epsilon_1,\epsilon_2) = \frac{d}{ds}\left(\frac{\Lambda^s}{\Gamma(s)}\int_0^\infty dt\,\frac{t^{s-1}\mathrm{e}^{-tx}}{(\mathrm{e}^{-\epsilon_1 t}-1)(\mathrm{e}^{-\epsilon_2 t}-1)}\right)\bigg|_{s=0}\ , \tag{4.73}$$

where Γ_2 is the Barnes double Γ-function and Λ an arbitrary scale. Indeed, as shown for example in [82], the perturbative contribution is

$$\log Z_{\text{pert}}[1,1,\cdots,1] = \sum_{\substack{u,v=1\\u\neq v}}^{N}\Big[\gamma\big(a_{uv} + \lceil\tfrac{v-u}{N}\rceil\epsilon_2\big) - \gamma\big(a_{uv} + m + \tfrac{\epsilon_1}{2} + \lceil\tfrac{v-u}{N}\rceil\epsilon_2\big)\Big] \tag{4.74}$$

where $a_{uv} = a_u - a_v$, and the ceiling function $\lceil y\rceil$ denotes the smallest integer greater than or equal to y. The first term in (4.74) represents the contribution of the vector multiplet, while the second term is the contribution of the massive hypermultiplet. Expanding (4.74) for small $\epsilon_{1,2}$ and using the same definitions (4.56) used for the instanton part, we obtain the perturbative contributions to the prepotential and the superpotential in the Nekrasov-Shatashvili limit:

$$
\begin{aligned}
\mathcal{F}_{\text{pert}} &= -\lim_{\epsilon_2\to 0}\Big(\epsilon_1\epsilon_2\log Z_{\text{pert}}[1,1,\cdots,1]\Big)\ ,\\
\mathcal{W}_{\text{pert}} &= \lim_{\epsilon_2\to 0}\Big(\epsilon_1\log Z_{\text{pert}}[1,1,\cdots,1]+\frac{\mathcal{F}_{\text{pert}}}{\epsilon_2}\Big)\ .
\end{aligned}
\tag{4.75}
$$

Exploiting the series expansion of the γ-function, one can explicitly compute these expressions and show that $\mathcal{F}_{\text{pert}}$ precisely matches the perturbative prepotential in the Nekrasov-Shatashvili limit obtained in [19, 65], while the contribution to the superpotential is novel. For example, in the case of the SU(2) theory we obtain

$$
\mathcal{F}_{\text{pert}} = \frac{1}{2}\Big(m^2-\frac{\epsilon_1^2}{4}\Big)\log\Big(\frac{4a^2}{\Lambda^2}\Big)-\frac{1}{48a^2}\Big(m^2-\frac{\epsilon_1^2}{4}\Big)^2-\frac{1}{960a^4}\Big(m^2-\frac{\epsilon_1^2}{4}\Big)^2\Big(m^2-\frac{3\epsilon_1^2}{4}\Big)+\cdots\ ,
\tag{4.76a}
$$

$$
\mathcal{W}_{\text{pert}} = -\frac{1}{4a}\Big(m^2-\frac{\epsilon_1^2}{4}\Big)-\frac{1}{96a^3}\Big(m^2-\frac{\epsilon_1^2}{4}\Big)^2-\frac{1}{960a^5}\Big(m^2-\frac{\epsilon_1^2}{4}\Big)^2\Big(m^2-\frac{3\epsilon_1^2}{4}\Big)+\cdots\ .
\tag{4.76b}
$$

Note that, unlike the prepotential, the twisted superpotential has no logarithmic term[6]. Furthermore, it is interesting to observe that

$$
\mathcal{W}_{\text{pert}} = -\frac{1}{4}\frac{\partial F_{\text{pert}}}{\partial a}\ .
\tag{4.77}
$$

4.4.1 S-duality constraints

We are now in a position to discuss the constraints on the twisted superpotential arising from S-duality. Adding the classical, the perturbative and the instanton terms described in the previous sections, we write the complete prepotential and

[6] This fact is due to the superconformal invariance, and is no longer true in the pure $\mathcal{N}=2$ SU(2) gauge theory, for which we find

$$
\mathcal{W}_{\text{pert}} = -\Big(2-2\log\frac{2a}{\Lambda}\Big)a+\frac{\epsilon_1^2}{24a}-\frac{\epsilon_1^4}{2880a^3}+\frac{\epsilon_1^6}{40320a^5}+\cdots\ .
$$

superpotential in the Nekrasov-Shatashvili limit as

$$\begin{aligned}
\mathcal{F} &= \mathcal{F}_{\text{class}} + \mathcal{F}_{\text{pert}} + \mathcal{F}_{\text{inst}} = \pi \mathrm{i} \tau\, \vec{a} \cdot \vec{a} + \sum_{\ell=1}^{\infty} f_\ell(\tau, \vec{a}) \ , \\
\mathcal{W} &= \mathcal{W}_{\text{class}} + \mathcal{W}_{\text{pert}} + \mathcal{W}_{\text{inst}} = 2\pi \mathrm{i} \sum_{I=2}^{N} z_I\, a_I + \sum_{\ell=1}^{\infty} w_\ell(\tau, z_I, \vec{a})
\end{aligned} \tag{4.78}$$

where for later convenience, we have kept the classical terms separate. The quantum coefficients f_ℓ and w_ℓ scale as $a^{2-\ell}$ and $a^{1-\ell}$, respectively, and account for the perturbative and instanton contributions. While f_ℓ depend on the coupling constant τ, the superpotential coefficients w_ℓ are also functions of the surface operator variables z_I, as we have explicitly seen in the SU(2) theory considered in the previous section.

The coefficients f_ℓ have been explicitly calculated in terms of quasi-modular forms in [19, 65] and we list the first few of them in Appendix H. Their relevant properties can be summarized as follows:

- All f_ℓ with ℓ odd vanish, while those with ℓ even are homogeneous functions of $\vec{a}$ and satisfy the scaling relation [7]

$$f_{2\ell}(\tau, \lambda\, \vec{a}) = \lambda^{2-2\ell}\, f_{2\ell}(\tau, \vec{a}) \ . \tag{4.79}$$

  Since the prepotential has mass-dimension two, the $f_{2\ell}$ are homogeneous polynomials of degree 2ℓ, in m and ϵ_1.

- The coefficients $f_{2\ell}$ depend on the coupling constant τ only through the Eisenstein series $E_2(\tau)$, $E_4(\tau)$ and $E_6(\tau)$, and are quasi-modular forms of SL(2,$\mathbb{Z}$) of weight $2\ell - 2$, such that

$$f_{2\ell}\Big(-\tfrac{1}{\tau}, \vec{a}\Big) = \tau^{2\ell-2}\, f_{2\ell}(\tau, \vec{a})\Big|_{E_2 \to E_2 + \delta} \tag{4.80}$$

[7] To be precise, one should also scale $\Lambda \to \lambda\Lambda$ in the logarithmic term of f_2.

where $\delta = \frac{6}{\pi \mathrm{i} \tau}$. The shift δ in E_2 is due to the fact that the second Eisenstein series is a quasi-modular form with an anomalous modular transformation (see (E.4)).

- The coefficients $f_{2\ell}$ satisfy a modular anomaly equation

$$\frac{\partial f_{2\ell}}{\partial E_2} + \frac{1}{24} \sum_{n=1}^{\ell-1} \frac{\partial f_{2n}}{\partial \vec{a}} \cdot \frac{\partial f_{2\ell-2n}}{\partial \vec{a}} = 0 \tag{4.81}$$

which can be solved iteratively.

Using the above properties, it is possible to show that S-duality acts on the prepotential $\mathcal{F}$ in the Nekrasov-Shatashvili limit as a Legendre transform [19, 70].

Let us now turn to the twisted superpotential $\mathcal{W}$. As we have seen in (4.71), S-duality acts very simply at the classical level but some subtleties arise in the quantum theory. We now make a few important points, anticipating some results of the next sections. It turns out that $\mathcal{W}$ receives contributions so that the coefficients w_ℓ do not have a well-defined modular weight. However, these anomalous terms depend only on the coupling constant τ and the vacuum expectation values $\vec{a}$. In particular, they are independent of the continuous parameters z_I that characterize the surface operator. For this reason it is convenient to consider the z_I derivatives of the superpotential:

$$\mathcal{W}^{(I)} := \frac{1}{2\pi \mathrm{i}} \frac{\partial \mathcal{W}}{\partial z_I} = a_I + \sum_{\ell=1}^{\infty} w_\ell^{(I)}(\tau, z_I, \vec{a}) \tag{4.82}$$

where, of course, $w_\ell^{(I)} := \frac{1}{2\pi \mathrm{i}} \frac{\partial w_\ell}{\partial z_I}$.

Combining intuition from the classical S-duality transformation (4.71b) with the fact that the z_I-derivative increases the modular weight by one, and introduces an extra factor of $(-\tau)$ under S-duality, we are naturally led to propose that

$$S\big(\mathcal{W}^{(I)}\big) = \tau\,\mathcal{W}^{(I)}~. \tag{4.83}$$

This constraint can be solved if we assume that the coefficients $w_\ell^{(I)}$ satisfy the following properties (which are simple generalizations of those satisfied by f_ℓ):

- They are homogeneous functions of $\vec{a}$ and satisfy the scaling relation

$$w_\ell^{(I)}(\tau, z_I, \lambda\,\vec{a}) = \lambda^{1-\ell}\,w_\ell^{(I)}(\tau, z_I, \vec{a})~. \tag{4.84}$$

  Given that the twisted superpotential has mass-dimension one, it follows that $w_\ell^{(I)}$ must be homogeneous polynomials of degree ℓ in m and ϵ_1.

- The dependence of $w_\ell^{(I)}$ on τ and z_I is only through linear combinations of quasi-modular forms made up with the Eisenstein series and elliptic functions with total weight ℓ, such that

$$w_\ell^{(I)}\Big(-\tfrac{1}{\tau}, -\tfrac{z_I}{\tau}, \vec{a}\Big) = \tau^\ell\,w_\ell^{(I)}(\tau, z_I, \vec{a})\Big|_{E_2\to E_2+\delta}~. \tag{4.85}$$

We are now ready to discuss how S-duality acts on the superpotential coefficients $w_\ell^{(I)}$. Recalling that

$$S(\vec{a}) = \vec{a}_{\mathrm{D}} := \frac{1}{2\pi\mathrm{i}}\,\frac{\partial\mathcal{F}}{\partial\vec{a}} = \tau\,\vec{a} + \frac{1}{2\pi\mathrm{i}}\,\frac{\partial f}{\partial\vec{a}} = \tau\Big(\vec{a} + \frac{\delta}{12}\,\frac{\partial f}{\partial\vec{a}}\Big) \tag{4.86}$$

where $f = \mathcal{F}_{\mathrm{pert}} + \mathcal{F}_{\mathrm{inst}}$, we have

$$\begin{aligned} S\big(w_\ell^{(I)}\big) &= w_\ell^{(I)}\Big(-\tfrac{1}{\tau}, -\tfrac{z_I}{\tau}, \vec{a}_{\mathrm{D}}\Big) = \tau^\ell\,w_\ell^{(I)}(\tau, z_I, \vec{a}_{\mathrm{D}})\Big|_{E_2\to E_2+\delta} \\ &= \tau\,w_\ell^{(I)}\Big(\tau, z_I, \vec{a} + \tfrac{\delta}{12}\,\tfrac{\partial f}{\partial\vec{a}}\Big)\Big|_{E_2\to E_2+\delta} \end{aligned} \tag{4.87}$$

where in the last step we exploited the scaling behavior (4.84) together with (4.86). Using this result in (4.82) and formally expanding in δ, we obtain

$$\begin{aligned}\frac{1}{\tau}\,S\big(\mathcal{W}^{(I)}\big) &= \mathcal{W}^{(I)}\Big(\tau, z_I, \vec{a} + \tfrac{\delta}{12}\,\tfrac{\partial f}{\partial \vec{a}}\Big)\Big|_{E_2\to E_2+\delta}\\ &= \mathcal{W}^{(I)} + \delta\left(\frac{\partial\mathcal{W}^{(I)}}{\partial E_2} + \frac{1}{12}\frac{\partial\mathcal{W}^{(I)}}{\partial\vec{a}}\cdot\frac{\partial f}{\partial\vec{a}}\right) + \mathcal{O}(\delta^2)\ .\end{aligned} \tag{4.88}$$

The constraint (4.83) is satisfied if

$$\frac{\partial\mathcal{W}^{(I)}}{\partial E_2} + \frac{1}{12}\frac{\partial\mathcal{W}^{(I)}}{\partial\vec{a}}\cdot\frac{\partial f}{\partial\vec{a}} = 0\ , \tag{4.89}$$

which also implies the vanishing of all terms of higher order in δ. This modular anomaly equation can be equivalently written as

$$\frac{\partial w_\ell^{(I)}}{\partial E_2} + \frac{1}{12}\sum_{n=0}^{\ell-1}\frac{\partial f_{\ell-n}}{\partial\vec{a}}\cdot\frac{\partial w_n^{(I)}}{\partial\vec{a}} = 0 \tag{4.90}$$

where we have defined $w_0^{(I)} = a_I$.

In the next sections we will solve this modular anomaly equation and determine the superpotential coefficients $w_\ell^{(I)}$ in terms of Eisenstein series and elliptic functions; we will also show that by considering the expansion of these quasi-modular functions we recover precisely all instanton contributions computed using localization, thus providing a very strong and highly non-trivial consistency check on our proposal (4.83) and on our entire construction. Since the explicit results are quite involved in the general case, we will start by discussing the SU(2) theory.

4.5 Surface operators in $\mathcal{N} = 2^\star$ SU(2) theory

We now consider the simplest $\mathcal{N} = 2^\star$ theory with gauge group SU(2) and solve in this case the modular anomaly equation (4.90). A slight modification from the earlier discussion is needed since for SU(2) the Coulomb vacuum expectation value of the adjoint scalar field takes the form $\langle\phi\rangle = \mathrm{diag}(a, -a)$ and the index I used in the previous section only takes one value, namely $I = 2$. Thus we have a single z-parameter, corresponding to the unique surface operator we can have in the theory, and (4.82) becomes

$$\mathcal{W}' := \frac{1}{2\pi\mathrm{i}}\frac{\partial\mathcal{W}}{\partial z} = -a + \sum_{\ell=1}^{\infty} w'_\ell \tag{4.91}$$

with $w'_\ell := \frac{1}{2\pi\mathrm{i}}\frac{\partial w_\ell}{\partial z}$, while the recurrence relation (4.90) becomes

$$\frac{\partial w'_\ell}{\partial E_2} + \frac{1}{24}\sum_{n=0}^{\ell-1}\frac{\partial f_{\ell-n}}{\partial a}\,\frac{\partial w'_n}{\partial a} = 0 \tag{4.92}$$

with the initial condition $w'_0 = -a$. The coefficient w_1 and its z-derivative w'_1 do not depend on a and are therefore irrelevant for the IR dynamics on the surface operator. Moreover, w'_1 drops out of the anomaly equation and plays no role in determining w'_ℓ for higher values of ℓ. Nevertheless, for completeness, we observe that if we use the elliptic function

$$h_1(z|\tau) = \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial z}\log\theta_1(z|\tau) \tag{4.93}$$

where $\theta_1(z|\tau)$ is the first Jacobi θ-function, and exploit the expansion reported in (E.16), comparing with the instanton expansion (4.66a) obtained from localization, we are immediately led to,

$$w'_1 = \Big(m - \frac{\epsilon_1}{2}\Big)\Big(h_1 + \frac{1}{2}\Big)\,. \tag{4.94}$$

By expanding h_1 to higher orders one can "predict" all higher instanton contributions to w_1'. We have checked that these predictions perfectly match the explicit results obtained from localization methods involving Young tableaux with up to six boxes.

The first case in which the modular anomaly equation (4.92) shows its power is the case $\ell = 2$. Recalling that the prepotential coefficients f_n with n odd vanish, we have

$$\frac{\partial w_2'}{\partial E_2} + \frac{1}{24}\frac{\partial f_2}{\partial a}\frac{\partial w_0'}{\partial a} = 0 \ . \tag{4.95}$$

Using the initial condition $w_0' = -a$, substituting the exact expression for f_2 given in (H.1) and then integrating, we get

$$w_2' = \frac{1}{24a}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\,\big(E_2 + \text{modular term}\big) \ . \tag{4.96}$$

At this juncture, it is important to observe that the elliptic and modular forms of $\mathrm{SL}(2,\mathbb{Z})$, which are allowed to appear in the superpotential coefficients, are polynomials in the ring generated by the Weierstraß function $\wp(z|\tau)$ and its z-derivative $\wp'(z|\tau)$, and by the Eisenstein series E_4 and E_6. These basis elements have weights $2, 3, 4$ and 6 respectively. We refer to Appendix E for a collection of useful formulas for these elliptic and modular forms and for their perturbative expansions. Since w_2' must have weight 2, the modular term in (4.96) is restricted to be proportional to the Weierstraß function, namely

$$w_2' = \frac{1}{24a}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\,\Big(E_2 + \alpha\,\frac{\wp}{4\pi^2}\Big) \tag{4.97}$$

where α is a constant. Therefore our proposal works only if by fixing a *single* parameter α we can match *all* the microscopic contributions to w_2' computed in the previous sections. Given the many constraints that this requirement puts, it is not at all obvious that it works. But actually it does! Indeed, using the expansions of E_2 and $\widetilde{\wp} = \frac{\wp}{4\pi^2}$ given in (E.2) and (E.17) respectively, and comparing with (4.66b), one finds a perfect match if $\alpha = 12$. Thus, the *exact* expression of w_2' is

$$w_2' = \frac{1}{24a}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\,\Big(E_2 + 12\,\widetilde{\wp}\Big)\ . \tag{4.98}$$

We have checked up to order six that the all instanton corrections predicted by this formula completely agree with the microscopic results obtained from localization.

Let us now consider the modular anomaly equation (4.92) for $\ell = 3$. In this case since w_1' is a-independent and the coefficients f_n with n odd vanish, we simply have

$$\frac{\partial w_3'}{\partial E_2} = 0 \tag{4.99}$$

According to our proposal, w_3' must be an elliptic function with modular weight 3, and in view of (4.99), the only candidate is the derivative of the Weierstraß function $\wp'$. By comparing the expansion (E.18) with the semi-classical results (4.66c) we find a perfect match and obtain

$$w_3' = \frac{\epsilon_1}{4a^2}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\widetilde{\wp}'\ . \tag{4.100}$$

Again we have checked that the higher order instanton corrections predicted by this formula agree with the localization results up to order six.

A similar analysis can done for higher values of ℓ without difficulty. Obtaining the anomalous behavior by integrating the modular anomaly equation, and fixing the coefficients of the modular terms by comparing with the localization results, after a bit of elementary algebra, we get

$$w_4' = \frac{1}{1152a^3}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\big(2E_2^2 - E_4 + 24E_2\,\widetilde{\wp} + 144\widetilde{\wp}^2\big) + 6\,\epsilon_1^2\big(E_4 - 144\widetilde{\wp}^2\big)\Big]\,,$$

$$w_5' = \frac{\epsilon_1}{48a^4}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\big(E_2 + 12\widetilde{\wp}\big)\widetilde{\wp}' - 36\,\epsilon_1^2\,\widetilde{\wp}\,\widetilde{\wp}'\Big]\,, \tag{4.101}$$

$$\begin{aligned} w_6' = \frac{1}{138240a^5}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[&\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)^2\big(20E_2^3 - 11E_2E_4 - 4E_6 + 240E_2^2\,\widetilde{\wp} - 60E_4\,\widetilde{\wp} \\ &+ 2160E_2\,\widetilde{\wp}^2 + 8640\widetilde{\wp}^3\big) + 2\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\epsilon_1^2\,\big(39E_2E_4 + 56E_6 + 1440E_4\,\widetilde{\wp} \\ &- 6480E_2\,\widetilde{\wp}^2 - 120960\widetilde{\wp}^3\big) - 240\,\epsilon_1^4\,\big(E_6 + 27E_4\,\widetilde{\wp} - 2160\widetilde{\wp}^3\big)\Big]\,, \end{aligned}$$

and so on. The complete agreement with the microscopic localization results of the above expressions provides very strong and highly non-trivial evidence for the validity of the modular anomaly equation and the S-duality properties of the superpotential, and hence of our entire construction.

Exploiting the properties of the function h_1 defined in (4.93) and its relation with the Weierstraß function (see Appendix E), it is possible to rewrite the above expressions as total z-derivatives. Indeed, we find

$$\begin{aligned} w_2' &= \frac{1}{2a}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)h_1'\,, \\ w_3' &= \frac{\epsilon_1}{4a^2}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)h_1''\,, \\ w_4' &= \frac{1}{48a^3}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\big(E_2\,h_1 - h_1''\big) + 6\,\epsilon_1^2\,h_1''\Big]'\,, \\ w_5' &= \frac{\epsilon_1}{8a^4}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)(h_1')^2 + \frac{\epsilon_1^2}{2}\big(E_2 - 6h_1'\big)h_1'\Big]'\,. \end{aligned} \tag{4.102}$$

We have checked that the same is also true for w_6' (and for a few higher coefficients as well), which however we do not write explicitly for brevity. Of course this is to be expected since they are the coefficients of the expansion of the derivative of the superpotential. The latter can then be simply obtained by integrating with respect to z and fixing the integration constants by comparing with the explicit localization results. In this way we obtain[8]

$$\mathcal{W} = -2\pi\mathrm{i}z\,a + \sum_n w_n \tag{4.103}$$

with

$$\begin{aligned}
w_2 &= \frac{1}{2a}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)h_1\ , \\
w_3 &= \frac{\epsilon_1}{4a^2}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)h_1'\ , \\
w_4 &= \frac{1}{48a^3}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\big(E_2\,h_1 - h_1''\big) + 6\,\epsilon_1^2\,h_1'' + \frac{1}{2}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\big(E_2 - 1\big)\Big]\ , \\
w_5 &= \frac{\epsilon_1}{8a^4}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)(h_1')^2 + \frac{\epsilon_1^2}{2}\big(E_2 - 6h_1'\big)h_1' \\
&\qquad\qquad + \frac{1}{96}\Big(m^2 - \frac{9\epsilon_1^2}{4}\Big)\big(E_2^2 - E_4\big)\Big]\ ,
\end{aligned} \tag{4.104}$$

and so on. Note that, as anticipated in the previous section, the coefficients w_n do not have a homogeneous modular weight.

4.5.1 Relation to CFT results

So far we have studied the twisted superpotential and its z-derivative as semi-classical expansions for large a. However, we can also arrange these expansions in terms of the deformation parameter ϵ_1. For example, using the results in (4.98), (4.100) and (4.101), we obtain

$$\mathcal{W}' = -a + \sum_{n=0}^{\infty} \epsilon_1^n\,\mathcal{W}_n' \tag{4.105}$$

[8] We neglect the a-independent terms originating from (4.94) since they are irrelevant for the infrared dynamics on the defect.

where

$$
\begin{aligned}
\mathcal{W}_0' &= \frac{m^2}{24a}\big(E_2+12\widetilde{\wp}\big)+\frac{m^4}{1152a^3}\big(2E_2^2-E_4+24E_2\,\widetilde{\wp}+144\widetilde{\wp}^2\big)+\frac{m^6}{138240a^5}\big(20E_2^3\\
&\qquad -11E_2E_4-4E_6+240E_2^2\,\widetilde{\wp}-60E_4\,\widetilde{\wp}+2160E_2\,\widetilde{\wp}^2+8640\widetilde{\wp}^3\big)+\mathcal{O}\big(a^{-7}\big)~,\\
\mathcal{W}_1' &= \frac{m^2}{4a^2}\,\widetilde{\wp}'+\frac{m^4}{48a^4}\big(E_2+12\widetilde{\wp}\big)\widetilde{\wp}'+\mathcal{O}\big(a^{-6}\big)~,\\
\mathcal{W}_2' &= -\frac{1}{96a}\big(E_2+12\wp\big)-\frac{m^2}{2304a^3}\big(2E_2^2-13E_4+24E_2\,\widetilde{\wp}+1872\widetilde{\wp}^2\big)\\
&\qquad -\frac{m^4}{110592a^5}\big(12E_2^3-69E_2E_4-92E_6+144E_2^2\,\widetilde{\wp}-2340E_4\,\widetilde{\wp}\\
&\qquad\qquad +11664E_2\,\widetilde{\wp}^2+198720\widetilde{\wp}^3\big)+\mathcal{O}\big(a^{-7}\big)~,\\
\mathcal{W}_3' &= -\frac{1}{16a^2}\,\widetilde{\wp}'-\frac{m^2}{96a^4}\big(E_2+84\widetilde{\wp}\big)\widetilde{\wp}'+\mathcal{O}\big(a^{-6}\big)~,
\end{aligned}
\tag{4.106}
$$

and so on. Quite remarkably, up to a sign flip $a \to -a$, these expressions precisely coincide with the results obtained in [39] from the null-vector decoupling equation for the toroidal 1-point conformal block in the Liouville theory.

We would like to elaborate a bit on this match. Let us first recall that in the so-called AGT correspondence [4] the toroidal 1-point conformal block of a Virasoro primary field V in the Liouville theory is related to the Nekrasov partition function of the $\mathcal{N}=2^\star$ SU(2) gauge theory. In [38] it was shown that the insertion of the degenerate null-vector $V_{2,1}$ in the Liouville conformal block corresponds to the partition function of the SU(2) theory in the presence of a surface operator. In the semi-classical limit of the Liouville theory (which corresponds to the Nekrasov-Shatashvili limit $\epsilon_2 \to 0$), one has [38, 39]

$$
\langle V(0)V_{2,1}(z)\rangle_{\text{torus}} \simeq \mathcal{N}\,\exp\Big(-\frac{\mathcal{F}}{\epsilon_1\epsilon_2}+\frac{\mathcal{W}(z)}{\epsilon_1}+\cdots\Big)~,
\tag{4.107}
$$

where $\mathcal{N}$ is a suitable normalization factor. In [39] the null-vector decoupling equation satisfied by the degenerate conformal block was used to explicitly calculate the prepotential $\mathcal{F}$ and the z-derivative of the twisted effective superpotential $\mathcal{W}'$ for the $\mathcal{N} = 2^\star$ SU(2) theory, which fully agrees with the one we have obtained using the modular anomaly equation and localization methods. It is important to keep in mind that the insertion of the degenerate field $V_{2,1}$ in the Liouville theory corresponds to the insertion of a surface operator of codimension-4 in the six-dimensional $(2,0)$ theory. In the brane picture, this defect corresponds to an M2 brane ending on the M5 branes that wrap a Riemann surface and support the gauge theory in four dimensions. On the other hand, as explained in the introduction, the results we have obtained using the orbifold construction and localization pertain to a surface operator of codimension-2 in the six dimensional theory, corresponding to an M5$'$ intersecting the original M5 branes. The equality between our results and those of [39] supports the proposal of a duality between the two types of surface operators in [79]. This also supports the conjecture of [86], based on [44, 87, 88], that in the presence of simple surface operators the instanton partition function is insensitive to whether they are realized as codimension-2 or codimension-4 operators. In Section 4.7.1 we will comment on such relations in the case of higher rank gauge groups and will also make contact with the results for the twisted chiral rings when the surface defect is realized by coupling two-dimensional sigma-models to pure $\mathcal{N} = 2$ SU(N) gauge theory.

4.6 Surface operators in $\mathcal{N} = 2^\star$ SU(N) theories

We now generalize the previous analysis to SU(N) gauge groups. As discussed in Section 4.2, in the higher rank cases there are many types of surface operators corresponding to the different partitions of N. We start our discussion from simple surface operators of type $\{1, (N-1)\}$.

4.6.1 Simple surface operators

In the case of the simple partition $\{1,(N-1)\}$, the vector $\vec{t}$ of the electro-magnetic parameters characterizing the surface operator takes the form

$$\vec{t} = \Big\{ \; t_1, \;\; \underbrace{t_2, \;\; \ldots, \;\; t_2}_{N-1} \; \Big\} \; . \tag{4.108}$$

Correspondingly, the classical contribution to the twisted effective superpotential becomes

$$\mathcal{W}_{\text{class}} = 2\pi\mathrm{i}\,\vec{t}\cdot\vec{a} = 2\pi\mathrm{i}\left(a_1\,t_1 + t_2\sum_{i=2}^{N} a_i\right) = -2\pi\mathrm{i}\,z\,a_1 \tag{4.109}$$

where we have used the tracelessness condition on the vacuum expectation values and, according to (4.54), have defined $z = t_2 - t_1$.

When quantum corrections are included, one finds that the coefficients w'_ℓ of the z-derivative of the superpotential satisfy the modular anomaly equation (4.90). The solution of this equation proceeds along the same lines as in the SU(2) case, although new structures, involving the differences $a_{ij} = a_i - a_j$, appear. We omit details of the calculations and merely present the results. As for the SU(2) theory, the coefficients can be compactly written in terms of modular and elliptic functions, particularly the second Eisenstein series and the function h_1 defined in (4.93). For clarity, and also for later convenience, we indicate the dependence on z but understand the dependence on τ in h_1. The first few coefficients w'_ℓ are

$$w'_2 = \Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{j=2}^{N} \frac{h'_1(z)}{a_{1j}} \,, \tag{4.110a}$$

$$w'_3 = \epsilon_1 \Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{j=2}^{N} \frac{h''_1(z)}{a_{1j}^2} + \frac{1}{2}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big(m + \frac{\epsilon_1}{2}\Big) \sum_{j\neq k=2}^{N} \frac{h''_1(z)}{a_{1j}\, a_{1k}} \,, \tag{4.110b}$$

$$\begin{aligned} w'_4 = \frac{1}{6}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\left(E_2\, h'_1(z) - h'''_1(z)\right) + 6\,\epsilon_1^2 h'''_1(z)\,\Big] \sum_{j=2}^{N} \frac{1}{a_{1j}^3} \\ + \epsilon_1 \Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big(m + \frac{\epsilon_1}{2}\Big) \sum_{j\neq k=2}^{N} \frac{h'''_1(z)}{a_{1j}^2\, a_{1k}} \\ + \frac{1}{6}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big(m + \frac{\epsilon_1}{2}\Big)^2 \sum_{j\neq k\neq \ell=2}^{N} \frac{h'''_1(z)}{a_{1j}\, a_{1k}\, a_{1\ell}} \,, \end{aligned} \tag{4.110c}$$

and so on. We have explicitly checked the above formulas against localization results up to SU(7) finding complete agreement. It is easy to realize that for $N = 2$ only the highest order poles contribute and the corresponding expressions precisely coincide with the results in the previous section. In the higher rank cases, there are also contributions from structures with lesser order poles that are made possible because of the larger number of Coulomb parameters. Furthermore, we observe that there is no pole when a_j approaches a_k with $j, k = 2, \cdots, N$.

It is interesting to observe that the above expressions can be rewritten in a suggestive form using the root system Φ of SU(N). The key observation is that using the vector $\vec{t}$ defined in (4.108) we can select a subset of roots $\Psi \subset \Phi$ such that their scalar products with the vector $\vec{a}$ of the vacuum expectation values produce exactly all the factors of a_{1j} in the denominators of (4.110). Defining

$$\Psi = \left\{ \vec{\alpha} \in \Phi \;\middle|\; \vec{\alpha}\cdot\vec{t} + z = 0 \right\} \,, \tag{4.111}$$

one can verify that for any $\vec{\alpha} \in \Psi$, the scalar product $\vec{\alpha}\cdot\vec{a}$ is of the form a_{1j}.

Therefore, w_2' in (4.110a) can be written as

$$w_2' = \Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{\vec{\alpha}\in\Psi} \frac{h_1'(-\vec{\alpha}\cdot\vec{t})}{\vec{\alpha}\cdot\vec{a}} = \Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{\vec{\alpha}\in\Psi} \frac{h_1'(\vec{\alpha}\cdot\vec{t})}{\vec{\alpha}\cdot\vec{a}} \tag{4.112}$$

where in the last step we used the fact that h_1' is an even function. Similarly the other coefficients in (4.110) can also be rewritten using the roots of SU(N). Indeed, introducing the subsets of Ψ defined as [9]

$$\begin{aligned}
&\Psi(\vec{\alpha}) = \big\{\vec{\beta}\in\Psi \ \big|\ \vec{\alpha}\cdot\vec{\beta} = 1\big\} \ , \\
&\Psi(\vec{\alpha},\vec{\beta}) = \big\{\vec{\gamma}\in\Psi \ \big|\ \vec{\alpha}\cdot\vec{\gamma} = \vec{\beta}\cdot\vec{\gamma} = 1\big\} \ ,
\end{aligned} \tag{4.113}$$

we find that w_3' in (4.110b) becomes

$$\begin{aligned}
w_3' = &-\epsilon_1\Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{\vec{\alpha}\in\Psi} \frac{h_1''(\vec{\alpha}\cdot\vec{t})}{(\vec{\alpha}\cdot\vec{a}\,)^2} \\
&- \frac{1}{2}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big(m + \frac{\epsilon_1}{2}\Big) \sum_{\vec{\alpha}\in\Psi}\sum_{\vec{\beta}\in\Psi(\vec{\alpha})} \frac{h_1''(\vec{\alpha}\cdot\vec{t})}{(\vec{\alpha}\cdot\vec{a}\,)\,(\vec{\beta}\cdot\vec{a}\,)} \ ,
\end{aligned} \tag{4.114}$$

while w_4' in (4.110c) is

$$\begin{aligned}
w_4' = &\frac{1}{6}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[\Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{\vec{\alpha}\in\Psi} \frac{E_2\, h_1'(\vec{\alpha}\cdot\vec{t}) - h_1'''(\vec{\alpha}\cdot\vec{t})}{(\vec{\alpha}\cdot\vec{a}\,)^3} + 6\,\epsilon_1^2 \sum_{\vec{\alpha}\in\Psi} \frac{h_1'''(\vec{\alpha}\cdot\vec{t})}{(\vec{\alpha}\cdot\vec{a}\,)^3}\ \Big] \\
&+ \epsilon_1\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big(m + \frac{\epsilon_1}{2}\Big) \sum_{\vec{\alpha}\in\Psi}\sum_{\vec{\beta}\in\Psi(\vec{\alpha})} \frac{h_1'''(\vec{\alpha}\cdot\vec{t})}{(\vec{\alpha}\cdot\vec{a}\,)^2\,(\vec{\beta}\cdot\vec{a}\,)} \\
&+ \frac{1}{4}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big(m + \frac{\epsilon_1}{2}\Big)^2 \Big[\sum_{\vec{\alpha}\in\Psi}\sum_{\vec{\beta}\neq\vec{\gamma}\in\Psi(\vec{\alpha})} \frac{h_1'''(\vec{\alpha}\cdot\vec{t})}{(\vec{\alpha}\cdot\vec{a}\,)\,(\vec{\beta}\cdot\vec{a}\,)\,(\vec{\gamma}\cdot\vec{a}\,)} \\
&\qquad\qquad - \frac{1}{3}\sum_{\vec{\alpha}\in\Psi}\sum_{\vec{\beta}\in\Psi(\vec{\alpha})}\sum_{\vec{\gamma}\in\Psi(\vec{\alpha},\vec{\beta})} \frac{h_1'''(\vec{\alpha}\cdot\vec{t})}{(\vec{\alpha}\cdot\vec{a}\,)\,(\vec{\beta}\cdot\vec{a}\,)\,(\vec{\gamma}\cdot\vec{a}\,)}\Big] \ .
\end{aligned} \tag{4.115}$$

We observe that the two sums in the last two lines of (4.115) are actually equal to each other and thus, taking into account the numerical factors, they exactly

[9] These definitions are analogous to the ones used in [19, 70] to define the root lattice sums appearing in the prepotential; see also (H.7).

reproduce the last line of (4.110c). However, for different sets of roots the two sums are different and lead to different structures. Thus, for reasons that will soon become clear, we have kept them separate even in this case.

4.6.2 Surface operators of type $\{p, N-p\}$

We now discuss a generalization of the simple surface operator in which we still have a single complex variable z as before, but the type is given by the following vector

$$\vec{t} = \Big\{ \underbrace{t_1, \quad \ldots, \quad t_1,}_{p} \ \underbrace{t_2, \quad \ldots, \quad t_2}_{N-p} \Big\} \,. \tag{4.116}$$

In this case, using the tracelessness condition on the vacuum expectation values, the classical contribution to the superpotential is

$$\mathcal{W}_{\text{class}} = 2\pi\mathrm{i}\bigg(t_1 \sum_{i=1}^{p} a_i + t_2 \sum_{j=p+1}^{N} a_j \bigg) = -2\pi\mathrm{i}\, z \sum_{i=1}^{p} a_i \tag{4.117}$$

where again we have defined $z = t_2 - t_1$.

It turns out that the quantum corrections to the z-derivatives of the superpotential are given exactly by the *same* formulas (4.112), (4.114) and (4.115) in which the only difference is in the subsets of the root system Φ that have to be considered in the lattice sums. These subsets are still defined as in (4.111) and (4.113) but with the vector $\vec{t}$ given by (4.116). We observe that in this case the two last sums in (4.115) are different. We have verified these formulas against the localization results up to SU(7) finding perfect agreement. The fact that the superpotential coefficients can be formally written in the same way for all unitary groups and for all types with two entries, suggests that probably universal formulas should exist for surface operators with more than two distinct entries in the $\vec{t}$-vector. This is indeed what happens as we will show in the next subsection.

4.6.3 Surface operators of general type

A surface operator of general type corresponds to splitting the SU(N) gauge group as in (4.10) which leads to the following partition of the Coulomb parameters

$$\vec{a} = \Big\{ \underbrace{a_1, \ \cdots \ a_{n_1},}_{n_1} \ \underbrace{a_{n_1+1}, \ \cdots \ a_{n_1+n_2},}_{n_2} \ \cdots, \ \underbrace{a_{N-n_M+1}, \ \ldots a_N}_{n_M} \Big\} \ , \tag{4.118}$$

and to the following $\vec{t}$-vector

$$\vec{t} = \Big\{ \underbrace{t_1, \ \cdots, \ t_1,}_{n_1} \ \underbrace{t_2, \ \cdots, \ t_2,}_{n_2} \ \cdots, \ \underbrace{t_M, \ \cdots, t_M}_{n_M} \Big\} \tag{4.119}$$

with

$$\sum_{I=1}^{M} n_I = N \ . \tag{4.120}$$

In this case we therefore have several variables z_I defined as in (4.54), and several combinations of elliptic functions evaluated at different points. However, if we use the root system Φ of SU(N) the structure of the superpotential coefficients is very similar to what we have seen before in the simplest cases. To see this, let us first define the following subsets [10] of Φ:

$$\begin{aligned}
\Psi_{IJ} &= \big\{\vec{\alpha} \in \Phi \ \big| \ \vec{\alpha} \cdot \vec{t} + z_I - z_J = 0\big\} \ , \\
\Psi_{IJ}(\vec{\alpha}) &= \big\{\vec{\beta} \in \Psi_{IJ} \ \big| \ \vec{\alpha} \cdot \vec{\beta} = 1\big\} \ , \\
\Psi_{IJ}(\vec{\alpha}, \vec{\beta}) &= \big\{\vec{\gamma} \in \Psi_{IJ} \ \big| \ \vec{\alpha} \cdot \vec{\gamma} = \vec{\beta} \cdot \vec{\gamma} = 1\big\}
\end{aligned} \tag{4.121}$$

which are obvious generalizations of the definitions (4.111) and (4.113). Then, writ-

[10]When $J = 1$ one must take $z_1 = 0$.

ing

$$\mathcal{W}^{(I)} = \frac{1}{2\pi \mathrm{i}} \frac{\partial \mathcal{W}}{\partial z_I} = a_{I_1} + \cdots a_{I_{n_I}} + \sum_{\ell} w_\ell^{(I)} , \tag{4.122}$$

for $I = 2, \cdots , M$, we find that the first few coefficients $w_\ell^{(I)}$ are given by

$$w_2^{(I)} = \Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{J\neq I} \sum_{\vec{\alpha}\in\Psi_{IJ}} \frac{h_1'(\vec{\alpha}\cdot\vec{t}\,)}{\vec{\alpha}\cdot\vec{a}} , \tag{4.123}$$

$$\begin{aligned} w_3^{(I)} = & -\epsilon_1 \Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{J\neq I} \sum_{\vec{\alpha}\in\Psi_{IJ}} \frac{h_1''(\vec{\alpha}\cdot\vec{t}\,)}{(\vec{\alpha}\cdot\vec{a}\,)^2} \\ & - \frac{1}{2}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big(m + \frac{\epsilon_1}{2}\Big) \sum_{J\neq I} \sum_{\vec{\alpha}\in\Psi_{IJ}} \sum_{\vec{\beta}\in\Psi_{IJ}(\vec{\alpha})} \frac{h_1''(\vec{\alpha}\cdot\vec{t}\,)}{(\vec{\alpha}\cdot\vec{a}\,)\,(\vec{\beta}\cdot\vec{a}\,)} , \end{aligned} \tag{4.124}$$

$$\begin{aligned} w_4^{(I)} = & \frac{1}{6}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\bigg[\Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{J\neq I} \sum_{\vec{\alpha}\in\Psi_{IJ}} \frac{E_2\, h_1'(\vec{\alpha}\cdot\vec{t}\,) - h_1'''(\vec{\alpha}\cdot\vec{t}\,)}{(\vec{\alpha}\cdot\vec{a}\,)^3} \\ & \qquad + 6\,\epsilon_1^2 \sum_{J\neq I} \sum_{\vec{\alpha}\in\Psi_{IJ}} \frac{h_1'''(\vec{\alpha}\cdot\vec{t}\,)}{(\vec{\alpha}\cdot\vec{a}\,)^3}\bigg] \\ & + \epsilon_1 \Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big(m + \frac{\epsilon_1}{2}\Big) \sum_{J\neq I} \sum_{\vec{\alpha}\in\Psi_{IJ}} \sum_{\vec{\beta}\in\Psi_{IJ}(\vec{\alpha})} \frac{h_1'''(\vec{\alpha}\cdot\vec{t}\,)}{(\vec{\alpha}\cdot\vec{a}\,)^2\,(\vec{\beta}\cdot\vec{a}\,)} \\ & + \frac{1}{4}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big(m + \frac{\epsilon_1}{2}\Big)^2 \bigg[\sum_{J\neq I} \sum_{\vec{\alpha}\in\Psi_{IJ}} \sum_{\vec{\beta}\neq\vec{\gamma}\in\Psi_{IJ}(\vec{\alpha})} \frac{h_1'''(\vec{\alpha}\cdot\vec{t}\,)}{(\vec{\alpha}\cdot\vec{a}\,)\,(\vec{\beta}\cdot\vec{a}\,)\,(\vec{\gamma}\cdot\vec{a}\,)} \\ & \qquad - \frac{1}{3} \sum_{J\neq I} \sum_{\vec{\alpha}\in\Psi_{IJ}} \sum_{\vec{\beta}\in\Psi_{IJ}(\vec{\alpha})} \sum_{\vec{\gamma}\in\Psi_{IJ}(\vec{\alpha},\vec{\beta})} \frac{h_1'''(\vec{\alpha}\cdot\vec{t}\,)}{(\vec{\alpha}\cdot\vec{a}\,)\,(\vec{\beta}\cdot\vec{a}\,)\,(\vec{\gamma}\cdot\vec{a}\,)}\bigg] \\ & + \Big(m^2 - \frac{\epsilon_1^2}{4}\Big)^2 \sum_{J\neq K\neq I} \sum_{\vec{\alpha}\in\Psi_{IJ}} \sum_{\vec{\beta}\in\Psi_{IK}(\vec{\alpha})} \frac{h_1'(\vec{\alpha}\cdot\vec{t}\,)\, h_1'(\vec{\alpha}\cdot\vec{t} - \vec{\beta}\cdot\vec{t}\,)}{(\vec{\alpha}\cdot\vec{a}\,)\,(\vec{\beta}\cdot\vec{a}\,)\,(\vec{\alpha}\cdot\vec{a} - \vec{\beta}\cdot\vec{a}\,)} \end{aligned} \tag{4.125}$$

where the summation indices $J, K, \cdots$, take integer values from 1 to M. One can explicitly check that these formulas reduce to those of the previous subsections if $M = 2$. We have verified these expressions in many cases up to SU(7), always finding agreement with the explicit localization results. Of course it is possible to write down similar expressions for the higher coefficients $w_\ell^{(I)}$, which however become more and more cumbersome as ℓ increases. Given the group theoretic structure of these formulas, it is tempting to speculate that they may be valid for the other simply laced groups of the ADE series as well, similarly to what happens for the analogous expressions of the prepotential coefficients [19]. It would be interesting to verify whether this happens or not.

4.7 Duality between surface operators

In this section we establish a relation between our localization results and those obtained when the surface defect is realized by coupling two-dimensional sigma-models to the four dimensional gauge theory. When the surface operators are realized in this way, the twisted chiral ring has been independently obtained by studying the two-dimensional $(2,2)$ theories [17, 18] and related to the Seiberg-Witten geometry of the four dimensional gauge theory [21, 22]. Building on these general results, we extract the semi-classical limit and compare it with the localization answer, finding agreement.

In order to be explicit, we will consider only gauge theories without Ω-deformation, and begin our analysis by first discussing the pure $\mathcal{N}=2$ theory with gauge group $\mathrm{SU}(N)$; in the end we will return to the $\mathcal{N}=2^\star$ theory.

4.7.1 The pure $\mathcal{N}=2$ SU(N) theory

The pure $\mathcal{N}=2$ theory can be obtained by decoupling the adjoint hypermultiplet of the $\mathcal{N}=2^\star$ model. More precisely, this decoupling is carried out by taking the following limit (see for example [65])

$$m \to \infty \quad \text{and} \quad q \to 0 \quad \text{such that} \quad q\, m^{2N} = (-1)^N \Lambda^{2N} \quad \text{is finite,} \tag{4.126}$$

where Λ is the strong coupling scale of the pure $\mathcal{N}=2$ theory. In presence of a surface operator, this limit must be combined with a scaling prescription for the continuous variables that characterize the defect. For surface operators of type $\{p, N-p\}$, which possess only one parameter $x = \mathrm{e}^{2\pi\mathrm{i}\,z}$, this scaling is

$$m \to \infty \quad \text{and} \quad x \to 0 \quad \text{such that} \quad x\, m^N = (-1)^{p-1} x_0\, \Lambda^N \quad \text{is finite.} \tag{4.127}$$

Here $x_0 = \mathrm{e}^{2\pi\mathrm{i}\,z_0}$ is the parameter that labels the surface operator in the pure theory à la Gukov-Witten [16, 21, 22, 42].

Performing the limits (4.126) and (4.127) on the localization results described in the previous sections, we obtain

$$\mathcal{W}' = \sum_{i=1}^{p} \mathcal{W}'_i \tag{4.128}$$

where

$$\mathcal{W}'_i = -a_i - \Lambda^N \Big(x_0 + \frac{1}{x_0}\Big) \prod_{j\neq i}^{N} \frac{1}{a_{ij}} - \frac{\Lambda^{2N}}{2} \Big(x_0^2 + \frac{1}{x_0^2}\Big) \frac{\partial}{\partial a_i} \Big(\prod_{j\neq i}^{N} \frac{1}{a_{ij}^2}\Big) + \mathcal{O}(\Lambda^{3N}) \ . \tag{4.129}$$

We have explicitly verified this expression in all cases up to SU(7), and for the low rank groups we have also computed the higher instanton corrections[11]. With some simple algebra one can check that, despite the appearance, $\mathcal{W}'$ is not singular for $a_i \to a_j$ when both i and j are $\leq p$ or $> p$. This fact follows from the residue condition satisfied by the original expression of the superpotential in the $\mathcal{N} = 2^\star$ theory (see the remarks after (4.125)). Furthermore, one can verify that

$$\sum_{i=1}^{N} \mathcal{W}'_i = 0 \tag{4.130}$$

[11] For example, for SU(2) and $p=1$ we find

$$\mathcal{W}'_1 = -a - \frac{\Lambda^2}{2a}\Big(x_0 + \frac{1}{x_0}\Big) + \frac{\Lambda^4}{8a^3}\Big(x_0^2 + \frac{1}{x_0^2}\Big) - \frac{\Lambda^6}{16a^5}\Big(x_0^3 + x_0 + \frac{1}{x_0} + \frac{1}{x_0^3}\Big) + \frac{\Lambda^8}{128a^7}\Big(5x_0^4 + 8x_0^2 + \frac{8}{x_0^2} + \frac{5}{x_0^4}\Big) + \mathcal{O}(\Lambda^{10})$$

where $a = a_1$.

as a consequence of the tracelessness condition on the vacuum expectation values.

We now show that this result is completely consistent with the exact twisted chiral ring relation obtained in [21]. For the pure $\mathcal{N}=2$ SU(N) theory with a surface operator parameterized by x_0, the twisted chiral ring relation takes the form [21]

$$\mathcal{P}_N(y)-\Lambda^N\left(x_0+\frac{1}{x_0}\right)=0 \tag{4.131}$$

with

$$\mathcal{P}_N(y)=\prod_{i=1}^{N}\left(y-e_i\right) \tag{4.132}$$

where e_i are the quantum corrected expectation values of the adjoint scalar. They reduce to a_i in the classical limit $\Lambda\to 0$ and parameterize the quantum moduli space of the theory. The e_i, which satisfy the tracelessness condition

$$\sum_{i=1}^{N}e_i=0\ , \tag{4.133}$$

were explicitly computed long ago in the 1-instanton approximation in [89, 90] by evaluating the period integrals of the Seiberg-Witten differential and read

$$e_i=a_i-\Lambda^{2N}\frac{\partial}{\partial a_i}\Big(\prod_{j\neq i}\frac{1}{a_{ij}^2}\Big)+\mathcal{O}\big(\Lambda^{4N}\big)\ . \tag{4.134}$$

The higher instanton corrections can be efficiently computed using localization methods [91–94], but their expressions will not be needed in the following.

Inserting (4.134) into (4.132) and systematically working order by order in Λ^N, it is possible to show that the N roots of the chiral ring equation (4.131) are

$$y_i=a_i+\Lambda^N\Big(x_0+\frac{1}{x_0}\Big)\prod_{j\neq i}^{N}\frac{1}{a_{ij}}+\frac{\Lambda^{2N}}{2}\Big(x_0^2+\frac{1}{x_0^2}\Big)\frac{\partial}{\partial a_i}\Big(\prod_{j\neq i}^{N}\frac{1}{a_{ij}^2}\Big)+\mathcal{O}\big(\Lambda^{3N}\big) \tag{4.135}$$

for $i=1,\cdots,N$. Comparing with (4.129), we see that, up to an overall sign, y_i

coincide with the derivatives of the superpotential $\mathcal{W}_i'$ we obtained from localization. Therefore, we can rewrite the left hand side of (4.131) in a factorized form and get

$$\prod_{i=1}^{N}\left(y+\mathcal{W}_i'\right)-\mathcal{P}_N(y)+\Lambda^N\left(x_0+\frac{1}{x_0}\right)=0 \tag{4.136}$$

This shows a perfect match between our localization results and the semi-classical expansion of the chiral ring relation of [21], and provides further non-trivial evidence for the equivalence of the two descriptions. Let us elaborate a bit more on this. According to [21], a surface operator of type $\{p, N-p\}$ has a dual description as a Grassmannian sigma-model coupled to the SU(N) gauge theory, and all information about the twisted chiral ring of the sigma-model is contained in two monic polynomials, Q and $\widetilde{Q}$ of degree p and $(N-p)$ respectively, given by

$$Q(y)=\sum_{\ell=0}^{p} y^{\ell}\,\mathcal{X}_{p-\ell}\ ,\qquad \widetilde{Q}(y)=\sum_{k=0}^{N-p} y^{k}\,\widetilde{\mathcal{X}}_{N-p-k}\ . \tag{4.137}$$

with $\mathcal{X}_0=\widetilde{\mathcal{X}}_0=1$. Here, $\mathcal{X}_\ell$ are the twisted chiral ring elements of the Grassmannian sigma-model, and in particular

$$\mathcal{X}_1=\frac{1}{2\pi\mathrm{i}}\frac{\partial\mathcal{W}}{\partial z_0} \tag{4.138}$$

where $\mathcal{W}$ is the superpotential of the surface operator of type $\{p, N-p\}$. The polynomial $\widetilde{Q}$ encodes the auxiliary information about the "dual" surface operator obtained by sending $p\to(N-p)$. The crucial point is that, according to the proposal of [21], the two polynomials Q and $\widetilde{Q}$ satisfy the relation

$$Q(y)\,\widetilde{Q}(y)-\mathcal{P}_N(y)+\Lambda^N\left(x_0+\frac{1}{x_0}\right)=0\ . \tag{4.139}$$

Comparing with (4.136), we are immediately led to the following identifications [12]

$$Q(y) = \prod_{i=1}^{p} \left(y + \mathcal{W}_i'\right) , \qquad \widetilde{Q}(y) = \prod_{j=p+1}^{N} \left(y + \mathcal{W}_j'\right) . \tag{4.140}$$

Thus, using (4.138) and (4.128), we find

$$\frac{1}{2\pi\mathrm{i}} \frac{\partial \mathcal{W}}{\partial z_0} = \sum_{i=1}^{p} \mathcal{W}_i' = \mathcal{W}' . \tag{4.141}$$

This equality shows that our localization results for the superpotential of the surface operator of type $\{p, N-p\}$ in the pure SU(N) theory perfectly consistent with the proposal of [21], thus proving the duality between the two descriptions. All this is also a remarkable consistency check of the way in which we have extracted the semi-classical results for the twisted chiral ring of the Grassmannian sigma-model and of the twisted superpotential we have computed.

4.7.2 The $\mathcal{N} = 2^\star$ SU(N) theory

Inspired by the previous outcome, we now analyze the twisted chiral ring relation for simple operators in $\mathcal{N} = 2^\star$ theories using the Seiberg-Witten curve and compare it with our localization results for the undeformed theory. To this aim, let us first recall from Section 4.6.1 (see in particular (4.110) with $\epsilon_1 = 0$) that for a simple surface operator corresponding to the following partition of the Coulomb parameters

$$\Big\{ \; a_i, \; \underbrace{\{a_j \text{ with } j \neq i\}}_{N-1} \; \Big\} , \tag{4.142}$$

[12] We have chosen a specific ordering in which the first p factors correspond to the first p vacuum expectation values a_i; of course one could as well choose a different ordering by permuting the factors.

the z-derivative of the superpotential is

$$\begin{aligned}\mathcal{W}_i' = -a_i + m^2 \sum_{j\neq i} \frac{h_1'}{a_{ij}} + \frac{m^3}{2} \sum_{j\neq k\neq i} \frac{h_1''}{a_{ij}\, a_{ik}} \\ + \frac{m^4}{6}\left(\sum_{j\neq i} \frac{E_2\, h_1' - h_1'''}{a_{ij}^3} + \sum_{j\neq k\neq \ell\neq i} \frac{h_1'''}{a_{ij}\, a_{ik}\, a_{i\ell}}\right) + \mathcal{O}\big(m^5\big)\ .\end{aligned} \tag{4.143}$$

Let us now see how this information can be retrieved from the Seiberg-Witten curve of the $\mathcal{N} = 2^\star$ theories. As is well known, in this case there are two possible descriptions (see [72] for a review). The first one, which we call the Donagi-Witten curve [95], is written naturally in terms of the modular covariant coordinates on moduli space, while the second, which we call the d'Hoker-Phong curve [96], is written naturally in terms of the quantum corrected coordinates on moduli space. As shown in [72], these two descriptions are linearly related to each other with coefficients depending on the second Eisenstein series E_2.

Since our semi-classical results have been resummed into elliptic and quasi-modular forms, we use the Donagi-Witten curve, which for the SU(N) gauge theory is an N-fold cover of an elliptic curve. It is described by the pair of equations:

$$Y^2 = X^3 - \frac{E_4}{48} X + \frac{E_6}{864}\ , \qquad F_N(y, X, Y) = 0\ . \tag{4.144}$$

The first equation describes an elliptic curve and thus we can identify (X, Y) with the Weierstraß function and its derivative (see (E.11)). More precisely we have

$$\begin{aligned} X &= -\widetilde{\wp} = -h_1' + \frac{1}{12}\, E_2\ , \\ Y &= \frac{1}{2}\, \widetilde{\wp}' = \frac{1}{2}\, h_1'' \end{aligned} \tag{4.145}$$

The second equation in (4.144) contains a polynomial in y of degree N which encodes the modular covariant coordinates A_k on the Coulomb moduli space of the gauge theory:

$$F_N(y,X,Y) = \sum_{k=0}^{N} (-1)^k A_k\, P_{N-k}(y,X,Y) \tag{4.146}$$

where P_k are the modified Donagi-Witten polynomials introduced in [72]. The first few of them are [13]:

$$\begin{aligned}
&P_0 = 1\ , \qquad P_1 = y\ ,\\
&P_2 = y^2 - m^2\, X\ , \qquad P_3 = y^3 - 3\, y\, m^2\, X + 2\, m^3\, Y\ ,\\
&P_4 = y^4 - 6\, m^2\, y^2\, X + 8\, y\, m^3\, Y - m^4\Big(3\, X^2 - \frac{1}{24}\, E_4\Big)\ .
\end{aligned} \tag{4.147}$$

On the other hand, the first few modular covariant coordinates A_k are (see [72]):

$$\begin{aligned}
A_2 &= \sum_{i<j} a_i a_j + \frac{m^2}{12}\binom{N}{2} E_2 + \frac{m^4}{288}\big(E_2^2 - E_4\big)\sum_{i\neq j}\frac{1}{a_{ij}^2} + \mathcal{O}(m^6)\ ,\\
A_3 &= \sum_{i<j<k} a_i a_j a_k - \frac{m^4}{144}\big(E_2^2 - E_4\big)\sum_i\sum_{j\neq i}\frac{a_i}{a_{ij}^2} + \mathcal{O}(m^6)\ ,\\
A_4 &= \sum_{i<j<k<\ell} a_i a_j a_k a_\ell + \frac{m^2}{12}\binom{N-2}{2} E_2 \sum_{i<j} a_i a_j + \frac{m^4}{48} E_2^2\\
&\quad + \frac{m^4}{288}\big(E_2^2 - E_4\big)\Big[\sum_{i<j}\sum_{k\neq\ell}\frac{a_i a_j}{a_{k\ell}^2} + 3\sum_i\sum_{j\neq i}\frac{a_i^2}{a_{ij}^2} - \binom{N}{2}\Big] + \mathcal{O}(m^6)\ ,
\end{aligned} \tag{4.148}$$

and so on.

We now have all the necessary ingredients to proceed. First of all, using the above expressions and performing the decoupling limits (4.126) and (4.127), one can check that the Donagi-Witten equation $F_N = 0$ reduces to the twisted chiral ring relation (4.131) of the pure theory. Of course this is not a mere coincidence; on the contrary it supports the idea that the Donagi-Witten equation actually encodes also the twisted chiral ring relation of the simple codimension-4 surface operators of the $\mathcal{N} = 2^\star$ theories. Secondly, working order by order in the hypermultiplet mass m,

[13] The E_4 term in P_4 is one of the modifications which in [72] were found to be necessary and is crucial also here.

one can verify that the N roots of the Donagi-Witten equation are given by

$$\begin{aligned} y_i = a_i - m^2 \sum_{j\neq i} \frac{h_1'}{a_{ij}} - \frac{m^3}{2} \sum_{j\neq k\neq i} \frac{h_1''}{a_{ij}\, a_{ik}} \\ - \frac{m^4}{6}\Bigg(\sum_{j\neq i} \frac{E_2\, h_1' - h_1'''}{a_{ij}^3} + \sum_{j\neq k\neq \ell\neq i} \frac{h_1'''}{a_{ij}\, a_{ik}\, a_{i\ell}} \Bigg) + \mathcal{O}\big(m^5\big) \ . \end{aligned} \tag{4.149}$$

Remarkably, this precisely matches, up to an overall sign, the answer (4.143) for the simple codimension-2 surface operator we have obtained using localization. Once again, we have exhibited the equivalence of twisted chiral rings calculated for the two kinds of surface operators. Furthermore, we can rewrite the Donagi-Witten equation in a factorized form as follows

$$\prod_{i=1}^{N} \big(y + \mathcal{W}_i'\big) - F_N(y, X, Y) = 0 \tag{4.150}$$

which is the $\mathcal{N} = 2^\star$ equivalent of the pure theory relation (4.136).

4.7.3 Some remarks on the results

The result we obtained from the twisted superpotential in the case of simple operators is totally consistent with the proposal given in the literature for simple codimension-4 surface operators labeled by a single continuous parameter z, whose superpotential has been identified with the line integral of the Seiberg-Witten differential of the four-dimensional gauge theory along an open path [38]:

$$\mathcal{W}(z) = \int_{z*}^{z} \lambda_{SW} \tag{4.151}$$

where $z*$ is an arbitrary reference point. Indeed, in the Donagi-Witten variables, the differential is simply $\lambda_{SW}(z) = y(z)\, dz$. Given that the Donagi-Witten curve is an N-fold cover of the torus, the twisted superpotential with the classical contribution proportional to a_i can be obtained by solving for $y(z)$ and writing out the solution on the ith branch.

As we have seen in the previous subsection, the general identification in (4.151) works also in the pure $\mathcal{N} = 2$ theory, once the parameters in the Seiberg-Witten differential are rescaled by a factor of Λ^N [21]. This rescaling can be interpreted as a renormalization of the continuous parameter that labels the surface operator [97].

The agreement we find gives further evidence of the duality between defects realized as codimension-2 and codimension-4 operators that we have already discussed in Section 4.5.1, where we showed the equality of the twisted effective superpotential computed in the two approaches for simple defects in the SU(2) theory. We have extended these checks to defects of type $\{p, N-p\}$ in pure $\mathcal{N} = 2$ theories, and to simple defects in $\mathcal{N} = 2^\star$ theories with higher rank gauge groups. All these support the proposal of [79] based on a "separation of variables" relation.

Chapter 5

Conclusions

In the first half of this thesis, we considered $\mathrm{SU}(2)^n$ super-conformal linear quiver gauge theories, with special emphasis on the $n = 1, 2$ cases. In this study, we followed three different approaches based on (i) the analysis of the Seiberg-Witten curves, (ii) equivariant localization, and (iii) the AGT correspondence.

Starting from the Seiberg-Witten curves obtained from the M-theory lift of a system of NS5-D4 branes, we derived the instanton expansion of the prepotential. Here we used a generalized residue prescription, along the lines suggested in [12,13], together with global symmetry considerations. We also showed that the cross-ratios of the branch points of the Seiberg-Witten curve, which depend on the UV parameters of the theory, can be expressed in terms of Θ-constants with period matrix τ_{ij}, which encodes the IR gauge couplings. This confirmed the nice geometric interpretation of the Nekrasov counting parameters.

We then considered the AGT correspondence, and showed that the classical Seiberg-Witten curve encoded in this approach matches the one obtained via the M-theory analysis. Within this framework, we also investigated the Ω-deformed quiver theory, in the Nekrasov-Shatashvili limit. The deformed periods a_i can be written as integrals of a deformed Seiberg-Witten differential. From this expression we extracted the expansion of the prepotential to second order in the deformation parameter, and matched this with the microscopic results à la Nekrasov.

To compare the results obtained from the two approaches, the key point is to express all parameters in terms of gauge theory data, which are the masses and the bare coupling constants associated with each gauge group. In the M-theory approach, the parameters are geometric, and are related to the positions of the constituent branes that engineer the quiver gauge theory. In the Liouville theory, the parameters are the central charge of the CFT, and the Liouville momenta of the primary operators involved in the AGT correspondence. After working out the detailed map between the various parameters, we identified the quantum mechanical system that governs the infrared dynamics of the $\mathrm{SU}(2)^n$ quiver gauge theory in the Nekrasov-Shatashvili limit, for the cases $n = 1, 2$. This allowed us to calculate the prepotential of the gauge theory.

In the second half of this thesis, we studied surface operators on the Coulomb branch of the four dimensional $\mathcal{N} = 2^\star$ theory with gauge group $\mathrm{SU}(N)$ focusing on the superpotential $\mathcal{W}$. This superpotential, which describes the effective two-dimensional dynamics on the defect world-sheet, receives non-perturbative contributions, which we calculated using equivariant localization. Exploiting the constraints arising from the non-perturbative $\mathrm{SL}(2,\mathbb{Z})$ symmetry, we showed that in a semi-classical regime in which the mass of the adjoint hypermultiplet is much smaller than the classical Coulomb branch parameters, (derivatives of the) twisted superpotential satisfy a modular anomaly equation. The coefficients functions in the mass expansion are linear combinations of elliptic and quasi-modular forms of a given weight. The twisted superpotential can be written in a very general and compact form in terms of suitable restricted sums over the root lattice of the gauge algebra.

The match of our localization results with the ones obtained in [21] by studying the coupling with two-dimensional sigma models is a non-trivial check of our methods. It also provides evidence for the proposed duality between codimension-2 and codimension-4 surface operators in [79]. Further evidence is given by the match of the twisted superpotentials in the $\mathcal{N} = 2^\star$ theory, which we proved for the simple surface operators using the Donagi-Witten curve of the model.

Appendix A

Nekrasov prepotential for quiver gauge theories

We consider $\mathcal{N} = 2$ quiver theories with a gauge group of the form $\prod_i \mathrm{SU}(N_i)$, and a matter content specified by the numbers $\{n_i\}$ of hypermultiplets in the fundamental representation of $\mathrm{SU}(N_i)$, and by the numbers $\{c_{ij}\}$ of bi-fundamental hypermultiplets which are fundamental under $\mathrm{SU}(N_i)$ and anti-fundamental under $\mathrm{SU}(N_j)$. The β-function coefficient for each $\mathrm{SU}(N_i)$ factor is given by

$$\beta_i = -2N_i + \sum_j N_j(c_{ij} + c_{ji}) + n_j \ . \tag{A.1}$$

We restrict our attention to conformal theories such that the β-function vanishes for every node. The basic quantity of interest is the multi-instanton partition function which, using localization [3, 98], reduces to

$$Z_{\text{inst}} = \sum_{k_i} \int \prod_i \frac{q_i^{k_i}}{k_i!} \prod_{I_i=1}^{k_i} \frac{d\chi_{I_i}}{2\pi\mathrm{i}} \ z_{\{k_i\}}^{\text{quiver}} \ . \tag{A.2}$$

Here we adopt the same conventions used in [99] (see in particular Appendix A). For instance, in the (k_1, k_2) instanton sector of a 2-node quiver theory we have

$$z^{\text{quiver}}_{k_1,k_2} = z^{\text{gauge}}_{k_1}\, z^{\text{gauge}}_{k_2}\, z^{\text{fund}}_{k_1}\, z^{\text{fund}}_{k_2}\, z^{\text{bi-fund}}_{k_1,k_2} \ . \tag{A.3}$$

where, in a rather obvious notation, the various factors represent the contributions of the different multiplets. As shown in [3,98] (see also [100,101]), the configurations of χ_{I_i} which contribute to the integrals in (A.2) can be put in one-to-one correspondence with a set Young tableaux $Y = \{Y_i\}$ containing a total number $k = \sum_i k_i$ of boxes, and the instanton partition function can be rewritten as

$$Z_{\text{inst}} = 1 + \sum_{Y_i} \prod_i q_i^{|Y_i|} Z_{\{Y_i\}} \ . \tag{A.4}$$

Here, the 1 represents the contribution at zero instanton number, $|Y_i|$ is the total number of boxes of the i-th Young tableau.

There is an algorithmic way to calculate the Z_{Y_i}'s, using the formalism of group characters, which now we briefly describe. For a given node i, we introduce the characters associated to the gauge, flavour and instanton symmetries, namely:

$$W_i = \sum_{u_i=1}^{N_i} \mathrm{e}^{i a_{u_i}} \ , \quad W_{F,i} = \sum_{f_i=1}^{n_i} \mathrm{e}^{-i\left(m_{f_i} + \frac{1}{2}(\epsilon_1+\epsilon_2)\right)} \ , \quad V_i = \sum_{I_i=1}^{k_i} \mathrm{e}^{i\left(\chi_{I_i} - \frac{1}{2}(\epsilon_1+\epsilon_2)\right)} \ , \tag{A.5}$$

where the m's are the masses of the fundamental hypermultiplets while ϵ_1 and ϵ_2 are the parameters of the Ω-background [3, 98]. In addition to these, we also have the characters associated to the Lorentz group, which are given by

$$T_1 = \mathrm{e}^{i\epsilon_1} \ , \qquad T_2 = \mathrm{e}^{i\epsilon_2} \ . \tag{A.6}$$

For a quiver model specified by the data $\{n_i, c_{ij}\}$, the character for a given tableau

Y is expressed in terms of the fundamental characters (A.5) as follows:

$$T_Y = \sum_{i,j} t_{ij}\, T_{ij} - T_F\,, \tag{A.7}$$

with

$$\begin{aligned} t_{ij} &= \delta_{ij} - c_{ij}\, \mathrm{e}^{i\left(m_{ij}-\frac{1}{2}(\epsilon_1+\epsilon_2)\right)}\,, \\ T_{ij} &= -V_i V_j^*(1-T_1)(1-T_2) + W_i V_j^* + V_i W_j^* T_1 T_2\,, \\ T_F &= \sum_i V_i W_{F,i}^* \end{aligned} \tag{A.8}$$

where m_{ij} is the mass of the bi-fundamental hypermultiplets. Notice that the combination m_{ij}, ϵ_1 and ϵ_2 that appears in t_{ij} is such that a flip in the orientation of an arrow, which exchanges c_{ij} and c_{ji}, can be reabsorbed in the redefinition $m_{ij} \leftrightarrow -m_{ji}$ to leave Z_Y invariant. In what follows, we will often use the notation $\widehat{m} = m + \frac{1}{2}(\epsilon_1+\epsilon_2)$.

We now focus on the SU(2) $\times$ SU(2) quiver. The field content of this model is specified by $c_{12}=1$, $c_{21}=0$, $n_1=2$ and $n_2=2$. The vacuum expectation values for the two SU(2) factors are a_1 and a_2. Using the notation $T_x = \mathrm{e}^{i\,x}$, the fundamental characters (A.5) are given by

$$\begin{aligned} V_1 &= T_{a_1} \sum_{(r,s)\in Y_{a_1}} T_1^{r-1} T_2^{s-1} + T_{-a_1} \sum_{(r,s)\in Y_{-a_1}} T_1^{r-1} T_2^{s-1}\,, \\ V_2 &= T_{a_2} \sum_{(r,s)\in Y_{a_2}} T_1^{r-1} T_2^{s-1} + T_{-a_2} \sum_{(r,s)\in Y_{-a_2}} T_1^{r-1} T_2^{s-1}\,, \\ W_1 &= T_{a_1} + T_{-a_1}\,, \qquad W_{F,1} = T_{-\widehat{m}_1} + T_{-\widehat{m}_2}\,, \\ W_2 &= T_{a_2} + T_{-a_2}\,, \qquad W_{F,2} = T_{-\widehat{m}_3} + T_{-\widehat{m}_4}\,. \end{aligned} \tag{A.9}$$

For the quiver at hand, from (A.7) and (A.8) we find

$$T_Y = T_{11} - T_{\widehat{m}_{12}} T_1^{-1} T_2^{-1} T_{12} + T_{22} - V_1\big(T_{\widehat{m}_1} + T_{\widehat{m}_2}\big) - V_2\big(T_{\widehat{m}_3} + T_{\widehat{m}_4}\big)\,. \tag{A.10}$$

T_Y can be explicitly calculated for a given arrangement of Young tableaux $Y = \{Y_i\}$ and, from the exponents of its various terms, one can read off the corresponding instanton partition function $Z_{\{Y_i\}}$. For instance, in the one-instanton sector we find

$$\begin{aligned}
Z_{(\square,\bullet|\bullet,\bullet)} &= \frac{(2a_1+2a_2+2m_{12}+\epsilon)(2a_1-2a_2+2m_{12}+\epsilon)}{32\,\epsilon_1\epsilon_2\,a_1(-2a_1-\epsilon)}\prod_{f=1}^{2}(2a_1+2m_f+\epsilon)\ ,\\
Z_{(\bullet,\square|\bullet,\bullet)} &= \Big[Z_{(\square,\bullet|\bullet,\bullet)}\Big]_{a_1\to -a_1}\ ,\\
Z_{(\bullet,\bullet|\square,\bullet)} &= \frac{(2a_2+2a_1-2m_{12}+\epsilon)(2a_2-2a_1-2m_{12}+\epsilon)}{32\,\epsilon_1\epsilon_2\,a_2(-2a_2-\epsilon)}\prod_{f=3}^{4}(2a_2+2m_f+\epsilon)\ ,\\
Z_{(\bullet,\bullet|\bullet,\square)} &= \Big[Z_{(\bullet,\bullet|\square,\bullet)}\Big]_{a_2\to -a_2}\ ,
\end{aligned} \tag{A.11}$$

where we have defined

$$\epsilon = \epsilon_1 + \epsilon_2\ . \tag{A.12}$$

The 1-instanton partition function is then given by $Z_1 = q_1\,Z_{1,0} + q_2\,Z_{0,1}$, with

$$Z_{1,0} = Z_{(\square,\bullet|\bullet,\bullet)} + Z_{(\bullet,\square|\bullet,\bullet)}\ ,\quad Z_{0,1} = Z_{(\bullet,\bullet|\square,\bullet)} + Z_{(\bullet,\bullet|\bullet,\square)}\ . \tag{A.13}$$

In the same way one can calculate the higher instanton contributions, and obtain the instanton partition function,

$$Z_{\text{inst}} = 1 + \sum_{k_1,k_2} Z_{k_1,k_2}\,q_1^{k_1}\,q_2^{k_2} \tag{A.14}$$

and the non-perturbative prepotential

$$F_{\text{inst}} = -\epsilon_1\epsilon_2\,\log Z_{\text{inst}} = \sum_{k_1,k_2} F_{k_1,k_2}\,q_1^{k_1}\,q_2^{k_2}\ . \tag{A.15}$$

Below we tabulate the first few prepotential coefficients F_{k_1,k_2} computed along the lines described above. We write the results in the Nekrasov-Shatashvili limit where we set $\epsilon_2 = 0$ and each F_{k_1,k_2} has a further expansion of the form

$$F_{k_1,k_2} = \sum_{n=0}^{\infty} F^{(n)}_{k_1,k_2}\,\epsilon_1^n \ . \tag{A.16}$$

At order ϵ_1^0 we have

$$F^{(0)}_{1,0} = \frac{a_1^2 - a_2^2}{2} + \frac{1}{2}\big(m_1 m_2 + 2(m_1+m_2)m_{12} + m_{12}^2\big) + \frac{m_1 m_2\big(m_{12}^2 - a_2^2\big)}{2a_1^2} \ , \tag{A.17a}$$

$$\begin{aligned}
F^{(0)}_{2,0} = {} & \frac{13a_1^4 - 14a_1^2 a_2^2 + a_2^4}{64a_1^2} + \frac{1}{64}\big(m_1^2 + 16m_1m_2 + m_2^2 + 32(m_1+m_2)m_{12} + 18m_{12}^2\big) \\
& + \frac{m_1^2m_2^2 + 2\big(m_1^2 + 8m_1m_2 + m_2^2\big)m_{12}^2 + m_{12}^4 + 2a_2^2\big(m_1^2 - 8m_1m_2 + m_2^2 - m_{12}^2\big)}{64a_1^2} \\
& - \frac{3\Big[2m_1^2m_2^2m_{12}^2 + (m_1^2+m_2^2)m_{12}^4 + 2a_2^2(m_1^2m_2^2 - (m_1^2+m_2^2)m_{12}^2) + a_2^4(m_1^2+m_2^2)\Big]}{64a_1^4} \\
& + \frac{5m_1^2m_2^2\big(m_{12}^4 - 2a_2^2m_{12}^2 + a_2^4\big)}{64a_1^6} \ ,
\end{aligned} \tag{A.17b}$$

$$\begin{aligned}
F^{(0)}_{1,1} = {} & \frac{a_1^2 + a_2^2}{4} + \frac{1}{4}\big(m_1m_2 + m_3m_4 + 2(m_1+m_2)(m_3+m_4) - m_{12}^2\big) \\
& + \frac{m_1m_2\big(m_3m_4 - m_{12}^2 + a_2^2\big)}{4a_1^2} + \frac{m_3m_4\big(m_1m_2 - m_{12}^2 + a_1^2\big)}{4a_2^2} - \frac{m_1m_2m_3m_4m_{12}^2}{4a_1^2a_2^2} \ .
\end{aligned} \tag{A.17c}$$

At order ϵ_1^1 we simply have

$$F^{(1)}_{1,0} = \frac{1}{2}(m_1 + m_2 + 2m_{12}) \ , \tag{A.18a}$$

$$F^{(1)}_{2,0} = \frac{1}{4}(m_1 + m_2 + 2m_{12}) \ , \tag{A.18b}$$

$$F^{(1)}_{1,1} = m_1 + m_2 + m_3 + m_4 \ . \tag{A.18c}$$

Finally, at order ϵ_1^2 we find

$$F_{1,0}^{(2)} = \frac{3}{8} + \frac{m_1 m_2 (m_{12}^2 - a_2^2)}{8a_1^4} \ , \tag{A.19a}$$

$$\begin{aligned} F_{2,0}^{(2)} = &\ \frac{23}{128} - \frac{2a_2^2 + m_1^2 + m_2^2 + 2m_{12}^2}{256 a_1^2} \\ &+ \frac{a_2^4 + 2a_2^2\big((m_1 - m_2)^2 - m_{12}^2\big) + m_1^2 m_2^2 + 2m_{12}^2 (m_1 + m_2)^2 + m_{12}^4}{64 a_1^4} \\ &- \frac{15\Big[a_2^4(m_1^2 + m_2^4) + 2a_2^2\big(m_1^2 m_2^2 - m_{12}^2(m_1^2 + m_2^2)\big) + 2m_1^2 m_2^2 m_{12}^2 + (m_1^2 + m_2^2) m_{12}^4\Big]}{256 a_2^6} \\ &+ \frac{21 m_1^2 m_2^2 (a_2^4 - m_{12}^2 a_2^2 + m_{12}^4)}{128 a_1^8} \ , \end{aligned} \tag{A.19b}$$

$$\begin{aligned} F_{1,1}^{(2)} = &\ \frac{7}{16} + \frac{m_1 m_2 m_3 m_4 (a_1^4 + a_1^2 a_2^2 + a_2^4)}{16 a_1^4 a_2^4} + \frac{m_1 m_2 (a_2^2 - m_{12}^2)}{16 a_1^4} + \frac{m_3 m_4 (a_1^2 - m_{12}^2)}{16 a_2^4} \\ &+ \frac{m_1 m_2 m_3 m_4 m_{12}^2 (a_1^2 + a_2^2)}{16 a_1^4 a_2^4} \ . \end{aligned} \tag{A.19c}$$

The other prepotential terms $F_{k,\ell}$ can be obtained from $F_{\ell,k}$ by the operations

$$a_1 \leftrightarrow a_2 \ , \qquad (m_1, m_2) \leftrightarrow (m_3, m_4) \ , \qquad m_{12} \leftrightarrow -m_{12} \ . \tag{A.20}$$

An important check of these results is that $F_{k,0}$ with $a_2 = 0$ matches exactly the k-instanton prepotential of the conformal SU(2) gauge theory with $N_f = 4$ if we choose to label the Coulomb parameter of the gauge group by a_1 and take the four masses to be given by

$$\big(m_1, m_2, m_{12}, m_{12}\big) \tag{A.21}$$

(see for example [29], taking into account that $m_i^{\text{here}} = \sqrt{2} m_i^{\text{there}}$). These calculations can be extended to higher instanton numbers without any problem.

We conclude by recalling the structure of the perturbative part of the prepotential for the quiver theory. The basic ingredient is the double-Gamma function

$$\gamma_{\epsilon_1,\epsilon_2}(x) := \log \Gamma_2(x|\epsilon_1, \epsilon_2) = \frac{d}{ds}\left[\frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} \frac{t^s \mathrm{e}^{-tx}}{(1 - \mathrm{e}^{-\epsilon_1 t})(1 - \mathrm{e}^{-\epsilon_2 t})}\right]_{s=0} \tag{A.22}$$

where Λ is an arbitrary mass scale. For large values of x, the function $\gamma_{\epsilon_1,\epsilon_2}$ has a series expansion of the form

$$\begin{aligned}\gamma_{\epsilon_1,\epsilon_2}(x) = \frac{x^2}{4}\left(3-\log\frac{x^2}{\Lambda^2}\right)b_0 - x\left(1-\frac{1}{2}\log\frac{x^2}{\Lambda^2}\right)b_1 - \frac{1}{4}\log\frac{x^2}{\Lambda^2}\,b_2 \\ + \sum_{n\geq 3}\frac{x^{2-n}}{n(n-1)(n-2)}\,b_n\end{aligned} \tag{A.23}$$

where the coefficients b_n's are defined by

$$\frac{1}{(1-\mathrm{e}^{-\epsilon_1 t})(1-\mathrm{e}^{-\epsilon_2 t})} = \sum_{n=0}^{\infty}\frac{b_n}{n!}t^{n-2}\ . \tag{A.24}$$

For the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ quiver the perturbative part of the prepotential is

$$\begin{aligned}F_{\text{pert}} = \epsilon_1\epsilon_2\Bigg[&\gamma_{\epsilon_1,\epsilon_2}(2a_1)+\gamma_{\epsilon_1,\epsilon_2}(-2a_1)+\gamma_{\epsilon_1,\epsilon_2}(2a_2)+\gamma_{\epsilon_1,\epsilon_2}(-2a_2)\\ &-\sum_{f=1,2}\Big(\gamma_{\epsilon_1,\epsilon_2}(a_1+\widehat{m}_f)+\gamma_{\epsilon_1,\epsilon_2}(-a_1+\widehat{m}_f)\Big)\\ &-\sum_{f=3,4}\Big(\gamma_{\epsilon_1,\epsilon_2}(a_2+\widehat{m}_f)+\gamma_{\epsilon_1,\epsilon_2}(-a_2+\widehat{m}_f)\Big)\\ &-\gamma_{\epsilon_1,\epsilon_2}(a_1+a_2-\widehat{m}_{12}+\epsilon)-\gamma_{\epsilon_1,\epsilon_2}(-a_1+a_2-\widehat{m}_{12}+\epsilon)\\ &-\gamma_{\epsilon_1,\epsilon_2}(a_1-a_2-\widehat{m}_{12}+\epsilon)-\gamma_{\epsilon_1,\epsilon_2}(-a_1-a_2-\widehat{m}_{12}+\epsilon)\Bigg]\end{aligned} \tag{A.25}$$

where we recall that $\widehat{m}$ stands for $m+\frac{1}{2}\epsilon$, with ϵ defined in (A.12) . The first line in the above formula represents the contribution of the two adjoint vector multiplets, the second and third lines represent the contributions of the fundamental hypermultiplets of the two gauge groups, while the last two lines are the contribution of the bi-fundamental matter.

This perturbative potential can be expanded for small ϵ_1 and ϵ_2 using (A.23). Up to order four in the masses and up to order two in the ϵ's we get

$$
\begin{aligned}
F_{\text{pert}} = &- \left(a_1^2 + a_2^2 + \frac{1}{12}\big(\epsilon^2 + \epsilon_1\epsilon_2\big)\right) \log 16 \\
&- \left(a_1^2 - \frac{1}{2}\big(m_1^2 + m_2^2\big) + \frac{1}{12}\big(2\epsilon^2 - \epsilon_1\epsilon_2\big)\right) \log \frac{a_1^2}{\Lambda^2} \\
&- \left(a_2^2 - \frac{1}{2}\big(m_3^2 + m_4^2\big) + \frac{1}{12}\big(2\epsilon^2 - \epsilon_1\epsilon_2\big)\right) \log \frac{a_2^2}{\Lambda^2} \\
&+ \left(\frac{1}{2}\big(a_1 + a_2\big)^2 + \frac{1}{2}m_{12}^2 - \frac{1}{24}\big(\epsilon^2 - 2\epsilon_1\epsilon_2\big)\right) \log \frac{\big(a_1 + a_2\big)^2}{\Lambda^2} \\
&+ \left(\frac{1}{2}\big(a_1 - a_2\big)^2 + \frac{1}{2}m_{12}^2 - \frac{1}{24}\big(\epsilon^2 - 2\epsilon_1\epsilon_2\big)\right) \log \frac{(a_1 - a_2)^2}{\Lambda^2} \\
&- \frac{2\big(m_1^4 + m_2^4\big) - \big(\epsilon^2 - 2\epsilon_1\epsilon_2\big)\big(m_1^2 + m_2^2\big)}{24a_1^2} - \frac{2\big(m_3^4 + m_4^4\big) - \big(\epsilon^2 - 2\epsilon_1\epsilon_2\big)\big(m_3^2 + m_4^2\big)}{24a_2^2} \\
&+ \frac{m_{12}^2\big(\epsilon^2 - 2\epsilon_1\epsilon_2\big) - 2m_{12}^4}{24(a_1 + a_2)^2} + \frac{m_{12}^2\big(\epsilon^2 - 2\epsilon_1\epsilon_2\big) - 2m_{12}^4}{24(a_1 - a_2)^2} \\
&+ \frac{\big(m_1^4 + m_2^4\big)\big(\epsilon^2 - 2\epsilon_1\epsilon_2\big)}{48a_1^4} + \frac{\big(m_3^4 + m_4^4\big)\big(\epsilon^2 - 2\epsilon_1\epsilon_2\big)}{48a_2^4} \\
&+ \frac{m_{12}^4\big(\epsilon^2 - 2\epsilon_1\epsilon_2\big)}{48(a_1 + a_2)^2} + \frac{m_{12}^4\big(\epsilon^2 - 2\epsilon_1\epsilon_2\big)}{48(a_1 - a_2)^2} + \ldots
\end{aligned}
\tag{A.26}
$$

It is easy to check that in the limit $\epsilon_1,\epsilon_2 \to 0$ we recover the expected expression of the 1-loop prepotential for the linear quiver we have considered. Notice that only in the massless undeformed theory the dependence on the arbitrary scale Λ drops out, in agreement with conformal invariance.

Appendix B

Polynomials appearing in the Seiberg-Witten curves

The fourth-order polynomial $\mathcal{P}_4$ appearing in the numerator of the Seiberg-Witten curve (2.71) for the SU(2) $N_f = 4$ theory is

$$\mathcal{P}_4(t) = \sum_{\ell=0}^{4} C_\ell \, t^\ell \tag{B.1}$$

where

$$\begin{aligned}
C_0 &= \frac{q^2}{4}(m_1 - m_2)^2 \ , \\
C_1 &= -qu + qm_1m_2 - \frac{q^2}{2}\big[(m_1+m_2)(m_3+m_4) + m_1^2 + m_2^2\big] \ , \\
C_2 &= u + qu + \frac{q}{2}\big[(m_1+m_2)(m_3+m_4) - 2m_1m_2 - 2m_3m_4\big] + \frac{q^2}{4}\Big(\sum_{f=1}^{4} m_f\Big)^2 \ , \\
C_3 &= -u + m_3m_4 - \frac{q}{2}\big[(m_1+m_2)(m_3+m_4) + m_3^2 + m_4^2\big] \ , \\
C_4 &= \frac{1}{4}(m_3 - m_4)^2 \ .
\end{aligned} \tag{B.2}$$

The sixth-order polynomial $\mathcal{P}_6$ appearing in the numerator of the Seiberg-Witten curve (2.80) for the SU(2) × SU(2) quiver theory is

$$\mathcal{P}_6(t) = \sum_{\ell=0}^{6} C'_\ell \, t^\ell \tag{B.3}$$

where,

$$
\begin{aligned}
C'_0 &= \frac{t_1^2\, t_2^2}{4}\,(m_1 - m_2)^2 \ , \\
C'_1 &= -t_1\, t_2^2\,(u_1 - m_1 m_2) + \frac{t_1^2\, t_2}{4}\Big(m_{12}^2 - 2m_1^2 - 2m_2^2 + 2m_{12}(m_1 + m_2 + m_{12})\Big) \\
&\quad - \frac{t_1^2\, t_2^2}{4}\Big(m_{12}^2 + 2(m_1 + m_2 + m_{12})\sum_{f=1}^{4} m_f - 4m_1 m_2\Big) \ , \\
C'_2 &= \frac{t_1\, t_2}{4}\Big(4(u_1 + u_2) - 7m_{12}^2 - 2m_{12}(m_1 + m_2) - 4m_1 m_2\Big) + t_2^2\, u_1 \\
&\quad + \frac{t_1\, t_2^2}{2}\Big(2u_1 + (m_1 + m_2 + m_{12})(m_3 + m_4 + m_{12}) + m_{12}(m_3 + m_4) - 2m_1 m_2\Big) \\
&\quad + \frac{t_1^2}{4}\,(m_1 + m_2 - m_{12})^2 + \frac{t_1^2\, t_2^2}{4}\Big(m_{12} + m_1 + m_2 + m_3 + m_4\Big)^2 \\
&\quad - \frac{t_1^2\, t_2}{4}\Big(3m_{12}^2 + 2m_{12}(m_1 + m_2 + m_{12}) - 4(m_1 + m_2)\sum_{f=1}^{4} m_f + 4m_1 m_2\Big) \\
C'_3 &= -\frac{t_1}{4}\Big(4u_2 + m_{12}^2 - 2m_{12}(m_1 + m_2)\Big) - t_2\big(u_1 + u_2 - m_{12}^2\big) \\
&\quad - \frac{t_1\, t_2}{2}\Big(2u_1 + 2u_2 - 6m_{12}^2 - m_{12}(m_1 + m_2 - m_3 - m_4) + 2(m_1 + m_2)(m_3 + m_4) \\
&\qquad\qquad - 2m_1 m_2 - 2m_3 m_4\Big) - \frac{t_1^2}{2}\big(m_1 + m_2 - m_{12}\big)\big(m_1 + m_2 + m_3 + m_4 - m_{12}\big) \\
&\quad - \frac{t_2^2}{4}\Big(4u_1 + m_{12}^2 + 2m_{12}(m_3 + m_4)\Big) + \frac{t_1^2\, t_2}{2}\Big(m_{12} - \sum_{f=1}^{4} m_f\Big)\Big(m_{12} - \sum_{f=1}^{4} m_f\Big) \\
&\quad - \frac{t_1\, t_2^2}{2}\big(m_{12} + m_3 + m_4\big)\big(m_{12} + m_1 + m_2 + m_3 + m_4\big) \ , \\
&\quad - \frac{t_1\, t_2}{4}\Big(5m_{12}^2 - 2m_{12}(m_3 + m_4) - 4(m_1 + m_2)(m_3 + m_4) - 4m_3^3 - 4m_3 m_4 - 4m_4^2\Big) \ ,
\end{aligned}
$$

$$
\begin{aligned}
C'_4 &= u_2+\frac{t_1}{2}\Big(2u_2+m_{12}^2-m_{12}(2m_1+2m_2+m_3+m_4)+(m_1+m_2)(m_3+m_4)-2m_3m_4\Big)\\
&\quad+\frac{t_2}{4}\Big(4u_1+4u_2-7m_{12}^2+2m_{12}(m_3+m_4)-4m_3m_4\Big)\\
&\quad+\frac{t_1^2}{4}\Big(m_{12}^2-2m_{12}\sum_{f=1}^{4}m_f+2\sum_{f<f'}m_fm_{f'}+\sum_{f=1}^{4}m_f^2\Big)+\frac{t_2^2}{4}\big(m_{12}+m_3+m_4\big)^2\\
&\quad-\frac{t_1\,t_2}{4}\Big(5m_{12}^2-2m_{12}(m_3+m_4)-4(m_1+m_2)(m_3+m_4)-4m_3^3-4m_3m_4-4m_4^2\Big)\ ,\\
C'_5 &= -u_2+m_3m_4-\frac{t_1}{4}\left(m_{12}^2+2(m_3+m_4-m_{12})\sum_{f=1}^{4}m_f-4m_3m_4\right)\\
&\quad+\frac{t_2}{4}\left(m_{12}^2-2m_3^2-2m_4^2-2(m_3+m_4-m_{12})m_{12}\right)\ ,\\
C'_6 &= \frac{1}{4}(m_3-m_4)^2
\end{aligned}
\tag{B.4}
$$

where $t_1=q_1q_2$ and $t_2=q_2$.

Appendix C

Some useful integrals

The calculation of the periods of the Seiberg-Witten differential λ requires the evaluation of integrals of the following types

$$I_1 = \frac{1}{\pi}\int_0^z \sqrt{\frac{z-t}{t}}\,\frac{f(t)}{q-t}\,dt \qquad \text{for } |q|<1\ , \tag{C.1}$$

and

$$I_2 = \frac{1}{\pi}\int_0^z \sqrt{\frac{z-t}{t}}\,\frac{f(t)}{1-t}\,dt \tag{C.2}$$

where $f(t)$ is a function admitting a Taylor expansion $\sum_n f_n\, t^n$. Using the identities

$$\frac{f(t)}{q-t} = \sum_{n=0}^{\infty}\frac{t^n}{q^{n+1}}\left(f(q) - \sum_{\ell=n+1}^{\infty} f_\ell\, q^\ell\right) \tag{C.3}$$

and

$$\int_0^z \sqrt{\frac{z-t}{t}}\, t^n = (-1)^n\,\pi\binom{1/2}{n+1}\, z^{n+1}\ , \tag{C.4}$$

we can prove that

$$I_1 = f(q) - \sqrt{\frac{q-z}{q}}\, f(q) - \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-1)^n\binom{1/2}{n+1} f_{n+\ell+1}\, z^{n+1}\, q^\ell\ . \tag{C.5}$$

On the other hand, from

$$\frac{f(t)}{1-t} = \sum_{n=0}^{\infty}\sum_{\ell=0}^{n} f_\ell\, t^n \tag{C.6}$$

and (C.4), we have

$$I_2 = \sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^n\binom{1/2}{n+1} f_\ell\, z^{n+1}\,. \tag{C.7}$$

These results can be used to compute the periods of the Seiberg-Witten differential. For example in the SU(2) $N_f = 4$ theory considered in Section 2.6, we can rewrite the last term of (2.106) as

$$J = \frac{\sqrt{C}}{\pi(1-q)}\int_0^{e_2}\sqrt{\frac{e_2-t}{t}}\left(\frac{\sqrt{e_3-t}}{q-t} - \frac{\sqrt{e_3-t}}{1-t}\right)dt = \frac{\sqrt{C}}{1-q}(I_1 - I_2) \tag{C.8}$$

where I_1 and I_2 are as in (C.1) and (C.2) with $z = e_2$ and $f(t) = \sqrt{e_3 - t}\,$. Then, from (C.5) and (C.7) we get

$$\begin{aligned} J = \frac{\sqrt{C}}{1-q}\Bigg(&\sqrt{e_3-q} - \sqrt{\frac{(e_2-q)(q-e_3)}{q}} + \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-1)^\ell\binom{1/2}{n+1}\binom{1/2}{n+\ell+1}\frac{e_2^{n+1}\,q^\ell}{e_3^{n+\ell+1/2}} \\ &- \sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{(n+\ell)}\binom{1/2}{n+1}\binom{1/2}{\ell}\frac{e_2^{n+1}}{e_3^{\ell-1/2}}\Bigg)\,. \end{aligned} \tag{C.9}$$

This is the result used to obtain (2.107) in the main text.

In the quiver theory described in Section 2.7 we had to compute the integral (see (2.141)

$$J' = \frac{1}{\pi}\int_0^{\zeta_1}\sqrt{\frac{\zeta_1-t}{t}}\,\sqrt{\frac{u_2(\widehat{\zeta}-t)}{(1-t)(1-q_2t)}}\,\frac{dt}{q_1-t} \tag{C.10}$$

which is again of the type I_1 with $z = \zeta_1$, $q = q_1$ and

$$f(t) = \sqrt{\frac{u_2(\widehat{\zeta}-t)}{(1-t)(1-q_2t)}}\,. \tag{C.11}$$

Using (C.5) we then find

$$J' = \sqrt{\frac{u_2(\widehat{\zeta} - q_1)}{(1-q_1)(1-q_1 q_2)}} - \sqrt{\frac{u_2(q_1-\zeta_1)(\widehat{\zeta} - q_1)}{q_1(q_1-1)(q_1 q_2 - 1)}} - \sum_{n,\ell=0}^{\infty} (-1)^n \binom{1/2}{n+1} f_{n+\ell+1}\, \zeta_1^{n+1}\, q_1^{\ell} \tag{C.12}$$

where the f_n's are the Taylor expansion coefficients of the function (C.11). This is the result used to obtain (2.145) in the main text.

Appendix D

Conformal Ward identities

The chiral blocks that are relevant for the discussion in Sections 3 and 3.5 are

$$\begin{aligned}
\left\langle T(z)\prod_{i=0}^{n+2}V_{\alpha_i}(z_i)\right\rangle &= \sum_{i=0}^{n+2}\left(\frac{\Delta_{\alpha_i}}{(z-z_i)^2}+\frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right)\left\langle \prod_{i=0}^{n+2}V_{\alpha_i}(z_i)\right\rangle \ , \\
\left\langle :T(z)\Phi_{2,1}(z): \prod_{i=0}^{n+2}V_{\alpha_i}(z_i)\right\rangle &= \sum_{i=0}^{n+2}\left(\frac{\Delta_{\alpha_i}}{(z-z_i)^2}+\frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right)\left\langle \Phi_{2,1}(z)\prod_{i=0}^{n+2}V_{\alpha_i}(z_i)\right\rangle \ .
\end{aligned} \tag{D.1}$$

We can simplify the right hand sides by imposing the constraints that follow from the global conformal invariance of the theory. For an $(n+3)$-point correlator these are:

$$\widehat{\Lambda}_k\left\langle \prod_{i=0}^{n+2}V_{\alpha_i}(z_i)\right\rangle = 0 \qquad \text{for} \quad k=-1,0,1 \ , \tag{D.2}$$

where

$$\widehat{\Lambda}_{-1}=\sum_{i=0}^{n+2}\frac{\partial}{\partial z_i} \ , \qquad \widehat{\Lambda}_{0}=\sum_{i=0}^{n+2}\left(z_i\frac{\partial}{\partial z_i}+\Delta_i\right) \ , \qquad \widehat{\Lambda}_{1}=\sum_{i=0}^{n+2}\left(z_i^2\frac{\partial}{\partial z_i}+2z_i\Delta_i\right) \tag{D.3}$$

are the generators of the global conformal group. The relations (D.2) allow to express the derivatives with respect to, say, z_0, z_{n+1} and z_{n+2} in terms of the derivatives with respect to the remaining n coordinates. If we fix $z_0=0$, $z_{n+1}=1$ and $z_{n+2}=\infty$, we have

$$
\begin{aligned}
\frac{\partial}{\partial z_0} &= -\sum_{i=1}^{n}\left((z_i-1)\frac{\partial}{\partial z_i}+\Delta_{\alpha_i}\right)+\Delta_{\alpha_0}+\Delta_{\alpha_{n+1}}-\Delta_{\alpha_{n+2}}\ , \\
\frac{\partial}{\partial z_{n+1}} &= -\sum_{i=1}^{n}\left(z_i\frac{\partial}{\partial z_i}+\Delta_{\alpha_i}\right)-\Delta_{\alpha_0}-\Delta_{\alpha_{n+1}}+\Delta_{\alpha_{n+2}}\ , \\
\frac{\partial}{\partial z_{n+2}} &= 0\ .
\end{aligned}
\tag{D.4}
$$

Applying these relations to the first correlator in (D.1), we get

$$
\begin{aligned}
\left\langle T(z)\prod_{i=0}^{n+2}V_{\alpha_i}(z_i)\right\rangle = \Bigg[\sum_{i=1}^{n}\left(\frac{\Delta_{\alpha_i}}{(z-z_i)^2}+\frac{z_i(z_i-1)}{z(z-1)(z-z_i)}\frac{\partial}{\partial z_i}\right)+\frac{\Delta_{\alpha_0}}{z^2}+\frac{\Delta_{\alpha_{n+1}}}{(z-1)^2} \\
-\frac{\sum_{i=1}^{n}\Delta_{\alpha_i}+\Delta_{\alpha_0}+\Delta_{\alpha_{n+1}}-\Delta_{\alpha_{n+2}}}{z(z-1)}\Bigg]\left\langle\prod_{i=0}^{n+2}V_{\alpha_i}(z_i)\right\rangle
\end{aligned}
\tag{D.5}
$$

where, both in the left and in the right hand side, it is understood that $z_0=0$, $z_{n+1}=1$ and $z_{n+2}=\infty$.

Proceeding in a similar way, we can rewrite the second correlator in (D.1) as

$$
\begin{aligned}
\left\langle :T(z)\Phi_{2,1}(z):\prod_{i=0}^{n+2}V_{\alpha_i}(z_i)\right\rangle = \Bigg[\sum_{i=1}^{n}\left(\frac{\Delta_{\alpha_i}}{(z-z_i)^2}+\frac{z_i(z_i-1)}{z(z-1)(z-z_i)}\frac{\partial}{\partial z_i}\right)-\frac{2z-1}{z(z-1)}\frac{\partial}{\partial z} \\
+\frac{\Delta_{\alpha_0}}{z^2}+\frac{\Delta_{\alpha_{n+1}}}{(z-1)^2}-\frac{\sum_{i=1}^{n}\Delta_{\alpha_i}+\Delta_z+\Delta_{\alpha_0}+\Delta_{\alpha_{n+1}}-\Delta_{\alpha_{n+2}}}{z(z-1)}\Bigg]\left\langle\Phi_{2,1}(z)\prod_{i=0}^{n+2}V_{\alpha_i}(z_i)\right\rangle\ .
\end{aligned}
\tag{D.6}
$$

To make contact with the discussion in Sections 3 and 3.5, we should notice that the punctures z_i have been denoted by t_i and that these are related to the gauge couplings according to $q_i=t_i/t_{i+1}$. Using this we can obtain from (D.5) and (D.6) the formulæ (3.24) and (3.37) of the main text.

Appendix E

Useful formulas for modular forms and elliptic functions

In this appendix we collect some formulas about quasi-modular forms and elliptic functions that are useful to check the statements of the main text.

Eisenstein series

We begin with the Eisenstein series E_{2n}, which admit a Fourier expansion in terms of $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$ of the form

$$E_{2n} = 1 + \frac{2}{\zeta(1-2n)} \sum_{k=1}^{\infty} \sigma_{2n-1}(k) q^k \ , \tag{E.1}$$

where $\sigma_p(k)$ is the sum of the p-th powers of the divisors of k. More explicitly we have

$$
\begin{aligned}
E_2 &= 1 - 24\sum_{k=1}^{\infty}\sigma_1(k)q^k = 1 - 24q - 72q^2 - 96q^3 - 168q^4 + \cdots \ , \\
E_4 &= 1 + 240\sum_{k=1}^{\infty}\sigma_3(k)q^k = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \cdots \ , \\
E_6 &= 1 - 504\sum_{k=1}^{\infty}\sigma_5(k)q^k = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \cdots \ .
\end{aligned}
\tag{E.2}
$$

Under a modular transformation $\tau \to \frac{a\tau+b}{c\tau+d}$, with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$, the Eisenstein series transform as

$$
E_2 \to (c\tau+d)^2\, E_2 + \frac{6}{\pi \mathrm{i}}\, c\,(c\tau+d)\ , \quad E_4 \to (c\tau+d)^4\, E_4\ , \quad E_6 \ \to\ (c\tau+d)^6\, E_6\ . \tag{E.3}
$$

In particular, under S-duality we have

$$
\begin{aligned}
E_2(\tau) &\to E_2\Big(-\frac{1}{\tau}\Big) = \tau^2\big(E_2(\tau) + \delta\big)\ , \\
E_4(\tau) &\to E_4\Big(-\frac{1}{\tau}\Big) = \tau^4 E_4(\tau)\ , \\
E_6(\tau) &\to E_6\Big(-\frac{1}{\tau}\Big) = \tau^6\, E_6(\tau)
\end{aligned}
\tag{E.4}
$$

where $\delta = \frac{6}{\pi \mathrm{i}\tau}$.

Elliptic functions

The elliptic functions that are relevant for this thesis can all be obtained from the Jacobi θ-function

$$
\theta_1(z|\tau) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2}\,(-x)^{(n-\frac{1}{2})} \tag{E.5}
$$

where $x = \mathrm{e}^{2\pi \mathrm{i} z}$. From θ_1, we first define the function

$$h_1(z|\tau) = \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial z}\log\theta_1(z|\tau) = x\frac{\partial}{\partial x}\log\theta_1(z|\tau)\ , \tag{E.6}$$

and the Weierstraß $\wp$-function

$$\wp(z|\tau) = -\frac{\partial^2}{\partial z^2}\log\theta_1(z|\tau) - \frac{\pi^2}{3}E_2(\tau)\ . \tag{E.7}$$

In most of our formulas the following rescaled $\wp$-function appears:

$$\widetilde{\wp}(z|\tau) := \frac{\wp(z,\tau)}{4\pi^2} = x\frac{\partial}{\partial x}\Big(x\frac{\partial}{\partial x}\log\theta_1(z|\tau)\Big) - \frac{1}{12}E_2(\tau)\ , \tag{E.8}$$

which we can write also as

$$\widetilde{\wp}(z|\tau) = h_1'(z|\tau) - \frac{1}{12}E_2(\tau)\ . \tag{E.9}$$

Another relevant elliptic function is the derivative of the Weierstraß function, namely

$$\widetilde{\wp}'(z|\tau) := \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial z}\widetilde{\wp}(z|\tau) = x\frac{\partial}{\partial x}\widetilde{\wp}(z|\tau) = h_1''(z|\tau)\ . \tag{E.10}$$

The Weierstraß function and its derivative satisfy the equation of an elliptic curve, given by

$$\widetilde{\wp}'(z|\tau)^2 + 4\,\widetilde{\wp}(z|\tau)^3 - \frac{E_4}{12}\,\widetilde{\wp}(z|\tau) - \frac{E_6}{216} = 0\ . \tag{E.11}$$

By differentiating this equation, we obtain

$$\widetilde{\wp}''(z|\tau) = -6\,\widetilde{\wp}(z|\tau)^2 + \frac{E_4}{24} \tag{E.12}$$

which, using (E.9) and (E.10), we can rewrite as

$$h_1'''(z|\tau) = -6\left(h_1'(z|\tau)\right)^2 + E_2\,h_1'(z|\tau) - \frac{E_2^2 - E_4}{24}\ . \tag{E.13}$$

The function h_1, $\widetilde{\wp}$ and $\widetilde{\wp}'$ have well-known expansions near the point $z = 0$. However, a different expansion is needed for our purposes, namely the expansion for small q and x. To find such an expansion we observe that q and x variables must be rescaled differently, as is clear from the map (4.52) between the gauge theory parameters and the microscopic counting parameters. In particular for $M = 2$ this map reads (see also (4.59))

$$q = q_1 q_2 \quad , \quad x = q_2 \ , \tag{E.14}$$

so that if the microscopic parameters are all scaled equally as $q_i \longrightarrow \lambda q_i$, then the gauge theory parameters scale as

$$q \to \lambda^2 q \qquad x \to \lambda x \ . \tag{E.15}$$

With this in mind, we now expand the elliptic functions for small λ and set $\lambda = 1$ in the end, since this is the relevant expansion needed to compare with the instanton calculations. Proceeding in this way, we find [1]

$$\begin{aligned}
h_1(x|q) &= h_1(\lambda x|\lambda^2 q)\Big|_{\lambda=1} \\
&= \Big[-\frac{1}{2} + \lambda\Big(\frac{q}{x} - x\Big) + \lambda^2\Big(\frac{q^2}{x^2} - x^2\Big) + \lambda^3\Big(\frac{q^3}{x^3} + \frac{q^2}{x} - qx - x^3\Big) \\
&\qquad - \lambda^4 x^4 + \lambda^5 \left(\frac{q^3}{x} - q^2 x - x^5\right) - \lambda^6 \left(q^2 x^2 + x^6\right) + \cdots \Big]_{\lambda=1} \\
&= -\frac{1}{2} - \Big(x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots\Big) + \Big(\frac{1}{x} - x\Big) q \\
&\qquad + \Big(\frac{1}{x^2} + \frac{1}{x} - x - x^2\Big) q^2 + \Big(\frac{1}{x^3} + \frac{1}{x} + \cdots\Big) q^3 + \cdots \ ,
\end{aligned} \tag{E.16}$$

[1] Depending on the context, we denote the arguments of the elliptic functions by either $(z|\tau)$ as we did so far, or by their exponentials $(x|q)$ when the expansions are being used.

$$
\begin{aligned}
\widetilde{\wp}(x|q) &= \widetilde{\wp}(\lambda x|\lambda^2 q)\Big|_{\lambda=1} \\
&= \Big[-\frac{1}{12} - \lambda\Big(\frac{q}{x} + x\Big) + \lambda^2\Big(-\frac{2q^2}{x^2} + 2q - 2x^2\Big) \\
&\quad - \lambda^3\Big(\frac{3q^3}{x^3} + \frac{q^2}{x} + qx + 3x^3\Big) + \lambda^4\big(6q^2 - 4x^4\big) + \cdots\Big]_{\lambda=1} \\
&= -\frac{1}{12} - \Big(x + 2x^2 + 3x^3 + 4x^4 + \cdots\Big) - \Big(\frac{1}{x} - 2 + x\Big)q \\
&\quad - \Big(\frac{2}{x^2} + \frac{1}{x} - 6 + \cdots\Big)q^2 - \frac{3q^3}{x^3} + \cdots \ ,
\end{aligned} \tag{E.17}
$$

$$
\begin{aligned}
\widetilde{\wp}'(x|q) &= \widetilde{\wp}'(\lambda x|\lambda^2 q)\Big|_{\lambda=1} \\
&= \Big[\lambda\Big(\frac{q}{x} - x\Big) + \lambda^2\left(\frac{4q^2}{x^2} - 4x^2\right) \\
&\quad + \lambda^3\Big(\frac{9q^3}{x^3} + \frac{q^2}{x} - qx - 9x^3\Big) - 16\lambda^4 x^4 + \cdots\Big]_{\lambda=1} \\
&= -\Big(x + 4x^2 + 9x^3 + 16x^4 + \cdots\Big) + \Big(\frac{1}{x} - x\Big)q \\
&\quad + \Big(\frac{4}{x^2} + \frac{1}{x} + \cdots\Big)q^2 + \frac{9q^3}{x^3} + \cdots \ .
\end{aligned} \tag{E.18}
$$

As a consistency check, it is possible to verify that, using these expansions and those of the Eisenstein series in (E.2), the elliptic curve equation (E.11) is identically satisfied order by order in λ.

As we have seen in Section 4.2, the modular group acts on $(z|\tau)$ as follows:

$$
(z|\tau) \to \Big(\frac{z}{c\tau+d}\,\Big|\,\frac{a\tau+b}{c\tau+d}\Big) \tag{E.19}
$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Under such transformations the Weierstraß function and its derivative have, respectively, weight 2 and 3, namely

$$
\begin{aligned}
\wp(z|\tau) &\to \wp\Big(\frac{z}{c\tau+d}\,\Big|\,\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^2\,\wp(z|\tau) \ , \\
\wp'(z|\tau) &\to \wp'\Big(\frac{z}{c\tau+d}\,\Big|\,\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^3\,\wp'(z|\tau) \ .
\end{aligned} \tag{E.20}
$$

Of course, similar relations hold for the rescaled functions $\widetilde{\wp}$ and $\widetilde{\wp}'$. In particular, under S-duality we have

$$\begin{aligned} \widetilde{\wp}(z|\tau) &\to \widetilde{\wp}\Big(-\frac{z}{\tau}\,\Big|-\frac{1}{\tau}\Big) = \tau^2\,\widetilde{\wp}(z|\tau)\ , \\ \widetilde{\wp}'(z|\tau) &\to \widetilde{\wp}'\Big(-\frac{z}{\tau}\,\Big|-\frac{1}{\tau}\Big) = -\tau^3\,\widetilde{\wp}'(z|\tau)\ . \end{aligned} \tag{E.21}$$

Appendix F

Generalized instanton number in the presence of fluxes

In this Appendix we calculate the second Chern class of the gauge field in the presence of a surface operator for a generic Lie algebra $\mathfrak{g}$.

Surface operator Ansatz

A surface operator creates a singularity in the gauge field A. As discussed in the main text, we parametrize the space-time $\mathbb{R}^4 \simeq \mathbb{C}^2$ by two complex variables ($z_1 = \rho\, \mathrm{e}^{\mathrm{i}\phi}$, $z_2 = r\, \mathrm{e}^{\mathrm{i}\theta}$), and consider a two-dimensional defect D located at $z_2 = 0$ and filling the z_1-plane. In this set-up, we make the following Ansatz [43]:

$$A = \widehat{A} + g(r)\, d\theta\ , \tag{F.1}$$

where $\widehat{A}$ is regular all over $\mathbb{R}^4$, and $g(r)$ is a $\mathfrak{g}$-valued function which is regular when $r \to \infty$. The corresponding field strength is then,

$$F := dA - \mathrm{i}A \wedge A = \widehat{F} + d(g(r)d\theta) - \mathrm{i}d\theta \wedge [g(r), \widehat{A}] \ . \tag{F.2}$$

From this expression we obtain,

$$\begin{aligned}\mathrm{Tr}\ F \wedge F &= \mathrm{Tr}\widehat{F} \wedge \widehat{F} + 2\ \mathrm{Tr}\ \left(d(g(r)d\theta) \wedge \widehat{F}\right) - 2\ \mathrm{i}\ \mathrm{Tr}\ \left(d\theta \wedge [g(r), \widehat{A}] \wedge \widehat{F}\right) \\ &= \mathrm{Tr}\widehat{F} \wedge \widehat{F} + 2\ \mathrm{Tr}\ d(g(r)d\theta \wedge \widehat{F}) + 2\ \mathrm{Tr}\ \left(g(r)d\theta \wedge \left(d\widehat{F} - \mathrm{i}\ \widehat{A} \wedge \widehat{F} - \mathrm{i}\ \widehat{F} \wedge \widehat{A}\right)\right)\end{aligned} \tag{F.3}$$

The last term vanishes due to the Bianchi identity, and we are left with,

$$\mathrm{Tr}\ F \wedge F = \mathrm{Tr}\ \widehat{F} \wedge \widehat{F} + 2\ \mathrm{Tr}\ d(g(r)d\theta \wedge \widehat{F}) \tag{F.4}$$

We now assume that the function $g(r)$ has components only along the Cartan directions of $\mathfrak{g}$, labeled by an index i, such that,

$$\lim_{r\to 0} g_i(r) = -\gamma_i \quad \text{and} \quad \lim_{r\to\infty} g_i(r) = 0 \tag{F.5}$$

This means that near the defect the gauge connection behaves as,

$$A = A_\mu\, dx^\mu \ \simeq\ -\,\mathrm{diag}\left(\gamma_1, \cdots, \gamma_{\mathrm{rank}(\mathfrak{g})}\right)\ d\theta \tag{F.6}$$

for $r \to 0$. Using this in (F.4), we have

$$\mathrm{Tr}F \wedge F = \mathrm{Tr}\ \widehat{F} \wedge \widehat{F} + 2\sum_i d(g_i(r)d\theta \wedge \widehat{F}_i) \tag{F.7}$$

Notice that in the last term we can replace $\widehat{F}_i$ with F_i because the difference lies entirely in the transverse directions of the surface operator, and thus does not contribute in the wedge product with $d\theta$. Since the defect D effectively acts as a boundary in $\mathbb{R}^4$ located at $r=0$, integrating (F.7) over $\mathbb{R}^4$ we have,

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{Tr}\,F\wedge F=\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{Tr}\,\widehat{F}\wedge\widehat{F}+\sum_i\frac{\gamma_i}{2\pi}\int_D F_i=k+\sum_i\gamma_i\,m_i\ . \tag{F.8}$$

Here we have denoted by k the instanton number of the smooth connection $\widehat{A}$ and taken into account a factor of 2π originating from the integration over θ. Finally, we have defined

$$m_i=\frac{1}{2\pi}\int_D F_i\ . \tag{F.9}$$

These quantities, which we call fluxes, must satisfy a quantization condition that can be understood as follows. All fields of the gauge theory are organized in representations of $\mathfrak{g}$ and, in particular, can be chosen to be eigenstates of the Cartan generators H_i with eigenvalues λ_i. These eigenvalues define a vector $\vec{\lambda}=\{\lambda_i\}$, which is an element of the weight lattice Λ_W of $\mathfrak{g}$. Let us now consider a gauge transformation in the Cartan subgroup with parameters $\vec{\omega}=\{\omega_i\}$. On a field with weight $\vec{\lambda}$, this transformation simply acts by a phase factor $\exp\big(\mathrm{i}\,\vec{\omega}\cdot\vec{\lambda}\big)$. From the point of view of the two-dimensional theory on the defect, the Cartan gauge fields A_i must approach a pure-gauge configuration at infinity so that

$$A_i\sim d\omega_i\quad\text{for }\rho\to\infty\ , \tag{F.10}$$

with ω_i being a function of ϕ, the polar angle in the z_1-plane. In this situation, for the corresponding gauge transformation to be single-valued, one finds

$$\vec{\omega}(\phi+2\pi)\cdot\vec{\lambda}=\vec{\omega}(\phi)\cdot\vec{\lambda}+2\pi n \tag{F.11}$$

with integer n. In other words, $\vec{\omega} \cdot \vec{\lambda}$ must be a map from the circle at infinity S_1^∞ into S_1 with integer winding number n. Given this, we have

$$2\pi m_i = \int_D F_i = \oint_{S_1^\infty} d\omega_i = \omega_i(\phi + 2\pi) - \omega_i(\phi) \ . \tag{F.12}$$

Then, using (F.11), we immediately deduce that

$$\vec{m} \cdot \vec{\lambda} \in \mathbb{Z} \ , \tag{F.13}$$

which amounts to saying that the flux vector $\vec{m}$ must belong to the dual of the weight lattice of $\mathfrak{g}$:

$$m \in (\Lambda_W)^* \ . \tag{F.14}$$

The SU(N) case

For U(N) the Cartan generators H_i can be taken as the diagonal $(N \times N)$ matrices with just a single non-zero entry equal to 1 in the i-th place ($i = 1, \cdots, N$). The restriction to SU(N) can be obtained by choosing a basis of $(N-1)$ traceless generators, for instance $(H_i - H_{i+1})/\sqrt{2}$. In terms of the standard orthonormal basis $\{\vec{e}_i\}$ of $\mathbb{R}^N$, the $(N-1)$ simple roots of SU(N) are then $\{(\vec{e}_1 - \vec{e}_2), (\vec{e}_2 - \vec{e}_3), \cdots\}$ and the root lattice Λ_R is the $\mathbb{Z}$-span of these simple roots. Note that Λ_R lies in a codimension-1 subspace orthogonal to $\sum_i \vec{e}_i$, and that the integrality condition for the weights is simply $\vec{\alpha} \cdot \vec{\lambda} \in \mathbb{Z}$ for any root $\vec{\alpha}$. This shows that the weight lattice is the dual of the root lattice, or equivalently that the dual of the weight lattice is the root lattice: $(\Lambda_W)^* = \Lambda_R$. Therefore, the condition (F.14) implies that the flux vector $\vec{m}$ must be of the form

$$\vec{m} = n_1(\vec{e}_1 - \vec{e}_2) + n_2(\vec{e}_2 - \vec{e}_3) + \cdots + n_{N-1}(\vec{e}_{N-1} - \vec{e}_N) \quad \text{with } \ n_i \in \mathbb{Z} \ . \tag{F.15}$$

This simply corresponds to

$$\vec{m} = \sum_i m_i \, \vec{e}_i \quad \text{with } \; m_i \in \mathbb{Z} \quad \text{and } \; \sum_i m_i = 0 \; . \tag{F.16}$$

The fact that the fluxes m_i are integers (adding up to zero) has been used in the main text.

Generic surface operator

The case in which all the γ_i's defined in (F.5) are distinct, corresponds to the surface operator of type $[1, 1, \dots, 1]$, also called full surface operator. If instead some of the γ_i's coincide, the surface operator has a more generic form. Let us consider for example the case in which the SU(N) gauge field at the defect takes the form (see (4.8)):

$$A = A_\mu \, dx^\mu \; \simeq \; -\operatorname{diag}\Big(\underbrace{\gamma_1, \cdots, \gamma_1}_{n_1}, \underbrace{\gamma_2, \cdots, \gamma_2}_{n_2}, \cdots, \underbrace{\gamma_M, \cdots, \gamma_M}_{n_M} \Big) \, d\theta \; , \tag{F.17}$$

for $r \to 0$, which corresponds to splitting the gauge group according to

$$\mathrm{SU}(N) \to \mathrm{S}\big[\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \cdots \times \mathrm{U}(n_M)\big] \; . \tag{F.18}$$

The calculation of the second Chern class (F.8) proceeds as before, but the result can be written as follows

$$\frac{1}{8\pi^2} \int_M \mathrm{Tr}\, F \wedge F = k + \sum_{I=1}^{M} \gamma_I \, m_I \tag{F.19}$$

with

$$m_I = \sum_{i=1}^{n_I} m_i = \frac{1}{2\pi} \int_D \sum_{i=1}^{n_I} F_i = \frac{1}{2\pi} \int_D \mathrm{Tr}\, F_{\mathrm{U}(n_I)} \; . \tag{F.20}$$

Here we see that it is the magnetic flux associated with the U(1) factor in each subgroup U(n_I) that appears in the expression for the generalized instanton number in the presence of magnetic fluxes.

Appendix G

Ramified instanton moduli and their properties

In this appendix we describe the instanton moduli in the various sectors. Our results are summarized in Tab.G.1.

Let us first consider the neutral states of the strings stretching between two D-instantons.

• $(-1)/(-1)$ **strings of type I-I:** All moduli of this type transform in the adjoint representation $(\mathbf{d}_I, \bar{\mathbf{d}}_I)$ of $\mathrm{U}(d_I)$. A special role is played by the bosonic states created in the Neveu-Schwarz (NS) sector of such strings by the complex oscillator ψ^v in the last complex space-time direction, which is neutral with respect to the orbifold. We denote them by χ_I. They are characterized by a $\mathrm{U}(1)^4$ weight $\{0,0,0,0\}$ and a charge $(+1)$ with respect to the last $\mathrm{U}(1)$. The complex conjugate moduli $\bar{\chi}_I$, with weight $\{0,0,0,0\}$ and charge (-1), are paired in a Q-doublet with the fermionic moduli $\bar{\eta}_I$ coming from the ground state of the Ramond (R) sector with weight $\left\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\}$ and charge $(-\frac{1}{2})$. All other moduli in this sector are arranged in

Q-doublets. One doublet is $(A_I^{z_1}, M_I^{z_1})$, where $A_I^{z_1}$ is from the ψ^{z_1} oscillator in the NS sector with weight $\{+1,0,0,0\}$ and charge 0, and $M_I^{z_1}$ is from the R ground state $\{+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$ with charge $(+\frac{1}{2})$. Another doublet is $(A_I^{z_4}, M_I^{z_4})$, where $A_I^{z_4}$ is from the ψ^{z_4} oscillator in the NS sector with weight $\{0,0,0,+1\}$ and charge 0, and $M_I^{z_4}$ is from the R ground state with weight $\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2}\}$ and charge $(+\frac{1}{2})$. Also the complex conjugate doublets are present. Finally, there is a (real) doublet (λ_I, D_I) where λ_I is from the R ground state with weight $\{+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2}\}$ and charge $(-\frac{1}{2})$, and D_I is an auxiliary field, and a complex doublet $(\lambda_I^{z_1}, D_I^{z_1})$ with $\lambda_I^{z_1}$ associated to the R ground state with weight $\{+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2}\}$ and charge $(-\frac{1}{2})$, and $D_I^{z_1}$ an auxiliary field.

- $(-1)/(-1)$ **strings of type I-$(I+1)$:** In this sector the moduli transform in the bi-fundamental representation $(\mathbf{d}_I, \bar{\mathbf{d}}_{I+1})$ of $\mathrm{U}(\mathbf{d}_I)\times\mathrm{U}(\mathbf{d}_{I+1})$. In order to cancel the phase ω^{-1} due to the different representations on the Chan-Paton indices at the two endpoints, the weights under spacetime rotations of the operators creating the states in this sector must be such that $l_2 - l_3 = 1$. In this way they can survive the $\mathbb{Z}_M$-orbifold projection. Applying this requirement, we find a doublet $(A_I^{z_2}, M_I^{z_2})$, $A_I^{z_2}$ is from the ψ^{z_2} oscillator in the NS sector with weight $\{0,+1,0,0\}$ and charge 0, and $M_I^{z_2}$ is from the R ground state $\{-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$ with charge $(+\frac{1}{2})$. Another doublet is $(\bar{A}_I^{z_3}, \bar{M}_I^{z_3})$ where $\bar{A}_I^{z_3}$ is from the $\bar{\psi}^{z_3}$ oscillator in the NS sector with weight $\{0,0,-1,0\}$ and charge 0, and $\bar{M}_I^{z_3}$ is from the R ground state $\{+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2}\}$ with charge $(+\frac{1}{2})$ [1]. Furthermore, we find two other complex Q-doublets, $(\lambda_I^{z_2}, D_I^{z_2})$ and $(\lambda_I^{z_3}, D_I^{z_3})$ where $\lambda_I^{z_2}$ and $\lambda_I^{z_3}$ are associated to the R ground states with weights $\{-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2}\}$ and $\{+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$ and charges $(-\frac{1}{2})$, while $D_I^{z_2}$ and $D_I^{z_3}$ are auxiliary fields. Also the complex conjugate doublets are present in the $\mathbb{Z}_M$-invariant spectrum, and arise from strings with the opposite orientation.

[1] Notice that this last doublet is actually the complex conjugate of a Q-doublet of type $(I+1)$-I, which is made of $(A_I^{z_3}, M_I^{z_3})$ with $A_I^{z_3}$ corresponding to the weight $\{0,0,1,0\}$ and $M_I^{z_3}$ corresponding to the weight $\{-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2}\}$.

• $3/(-1)$ **strings of type** I-I**:** These open strings have mixed Neumann-Dirichlet boundary conditions along the (z_1, z_2)-directions and thus the corresponding states are characterized by the action of a twist operator Δ [78]. We assign an orbifold charge $\omega^{-\frac{1}{2}}$ to this twist operator, so that the states which survive the $\mathbb{Z}_M$-projection are those with weights such that $l_2 - l_3 = 1/2$. The moduli in this sector belong to the bi-fundamental representation $(\mathbf{n}_I \times \bar{\mathbf{d}}_I)$ of the gauge and ADHM groups, and form two complex doublets. One is (w_I, μ_I) where the NS component w_I has weight $\{+\frac{1}{2}, +\frac{1}{2}, 0, 0\}$ and charge 0, and the R component μ_I has weight $\{0, 0, -\frac{1}{2}, -\frac{1}{2}\}$ and charge $(+\frac{1}{2})$. The other doublet is (μ'_I, h'_I) where μ'_I is associated to the R ground state with weight $\{0, 0, -\frac{1}{2}, +\frac{1}{2}\}$ and charge $(-\frac{1}{2})$, while h'_I is an auxiliary field. Also the complex conjugate doublets, associated to the $(-1)/3$ strings of type I-I, are present in the spectrum.

• $(-1)/3$ **strings of type** I-$(I+1)$**:** These open strings have mixed Dirichlet-Neumann boundary conditions along the (z_1, z_2)-directions and transform in the bi-fundamental representation $(\mathbf{d}_I \times \bar{\mathbf{n}}_{I+1})$ of the gauge and ADHM groups. As compared to the previous case, the states in this sector are characterized by the action of an anti-twist operator $\bar{\Delta}$ which carries an orbifold parity $\omega^{+\frac{1}{2}}$. Thus the $\mathbb{Z}_M$-invariant configurations must have again weights with $l_2 - l_3 = \frac{1}{2}$ in order to compensate for the ω^{-1} factor carried by the Chan-Paton indices. Taking this into account, we find two complex doublets: $(\hat{w}_I, \hat{\mu}_I)$ where the NS component $\hat{w}_I$ has weight $\{+\frac{1}{2}, +\frac{1}{2}, 0, 0\}$ and charge 0, and the R component $\hat{\mu}_I$ has weight $\{0, 0, -\frac{1}{2}, -\frac{1}{2}\}$ and charge $(+\frac{1}{2})$, and $(\hat{\mu}'_I, \hat{h}'_I)$ where $\hat{\mu}'_I$ is associated to the R ground state with weight $\{0, 0, -\frac{1}{2}, +\frac{1}{2}\}$ and charge $(-\frac{1}{2})$, while $\hat{h}'_I$ is an auxiliary field. Also the complex conjugate doublets, associated to the $3/(-1)$ strings of type $(I+1)$-I, are present in the spectrum.

Notice that no states from the $3/(-1)$ strings of type I-$(I+1)$ or from the $(-1)/3$

Doublet	$(-)^{F_\alpha}$	Chan-Paton	$\mathrm{U}(1)^4$charge	Q^2-eigenvalue λ_α	Character
$(\bar{\chi}_I, \bar{\eta}_I)$	$+$	$(\mathbf{d}_I, \bar{\mathbf{d}}_I)$	$\{0,0,0,0\}$	$\chi_{I,\sigma} - \chi_{I,\tau}$	
$(A_I^{z_1}, M_I^{z_1})$	$+$	$(\mathbf{d}_I, \bar{\mathbf{d}}_I)$	$\{+1,0,0,0\}$	$\chi_{I,\sigma} - \chi_{I,\tau} + \epsilon_1$	$V_I^* V_I T_1$
$(A_I^{z_4}, M_I^{z_4})$	$+$	$(\mathbf{d}_I, \bar{\mathbf{d}}_I)$	$\{0,0,0,+1\}$	$\chi_{I,\sigma} - \chi_{I,\tau} + \epsilon_4$	$V_I^* V_I T_4$
(λ_I, D_I)	$-$	$(\mathbf{d}_I, \bar{\mathbf{d}}_I)$	$\{+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2}\}$	$\chi_{I,\sigma} - \chi_{I,\tau}$	
$(\lambda_I^{z_1}, D_I^{z_1})$	$-$	$(\mathbf{d}_I, \bar{\mathbf{d}}_I)$	$\{+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2}\}$	$\chi_{I,\sigma} - \chi_{I,\tau} + \epsilon_1 + \epsilon_4$	$-V_I^* V_I T_1 T_4$
$(A_I^{z_2}, M_I^{z_2})$	$+$	$(\mathbf{d}_I, \bar{\mathbf{d}}_{I+1})$	$\{0,+1,0,0\}$	$\chi_{I,\sigma} - \chi_{I+1,\rho} + \hat{\epsilon}_2$	$V_{I+1}^* V_I T_2$
$(\lambda_I^{z_2}, D_I^{z_2})$	$-$	$(\mathbf{d}_I, \bar{\mathbf{d}}_{I+1})$	$\{-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2}\}$	$\chi_{I,\sigma} - \chi_{I+1,\rho} + \hat{\epsilon}_2 + \epsilon_4$	$-V_{I+1}^* V_I T_2 T_4$
$(\bar{A}_I^{z_3}, \bar{M}_I^{z_3})$	$+$	$(\mathbf{d}_I, \bar{\mathbf{d}}_{I+1})$	$\{0,0,-1,0\}$	$\chi_{I,\sigma} - \chi_{I+1,\rho} - \hat{\epsilon}_3$	$V_{I+1}^* V_I T_1 T_2 T_4$
$(\lambda_I^{z_3}, D_I^{z_3})$	$-$	$(\mathbf{d}_I, \bar{\mathbf{d}}_{I+1})$	$\{+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$	$\chi_{I,\sigma} - \chi_{I+1,\rho} + \epsilon_1 + \hat{\epsilon}_2$	$-V_{I+1}^* V_I T_1 T_2$
(w_I, μ_I)	$+$	$(\mathbf{n}_I, \bar{\mathbf{d}}_I)$	$\{+\frac{1}{2},+\frac{1}{2},0,0\}$	$a_{I,s} - \chi_{I,\sigma} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)$	$V_I^* W_I$
(μ_I', h_I')	$-$	$(\mathbf{n}_I, \bar{\mathbf{d}}_I)$	$\{0,0,-\frac{1}{2},+\frac{1}{2}\}$	$a_{I,s} - \chi_{I,\sigma} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2) + \epsilon_4$	$-V_I^* W_I T_4$
$(\hat{w}_I, \hat{\mu}_I)$	$+$	$(\mathbf{d}_I, \bar{\mathbf{n}}_{I+1})$	$\{+\frac{1}{2},+\frac{1}{2},0,0\}$	$\chi_{I,\sigma} - a_{I+1,t} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)$	$W_{I+1}^* V_I T_1 T_2$
$(\hat{\mu}_I', \hat{h}_I')$	$-$	$(\mathbf{d}_I, \bar{\mathbf{n}}_{I+1})$	$\{0,0,-\frac{1}{2},+\frac{1}{2}\}$	$\chi_{I,\sigma} - a_{I+1,t} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2) + \epsilon_4$	$-W_{I+1}^* V_I T_1 T_2 T_4$

Table G.1: The spectrum of moduli, organized in doublets of the BRST charge Q (or its conjugate $\bar{Q}$). For each of them, we display their statistics $(-)^{F_\alpha}$, the representation of the color and ADHM groups in which they transform, their charge vector with respect to the $\mathrm{U}(1)^4$ symmetry, the eigenvalue λ_α of Q^2 and the corresponding contribution to the character. The neutral moduli carrying a superscript z_1, z_2, z_3 or z_4, and the colored moduli in this table are complex.

strings of type $(I+1)$-I survive the orbifold projection. Indeed, in the first case the phases $\omega^{-\frac{1}{2}}$ and ω^{-1} from the twist operator Δ and the Chan-Paton factors cannot be compensated by the NS or R weights; while in the second case the phases $\omega^{+\frac{1}{2}}$ and ω^{+1} from the anti-twist operator $\bar{\Delta}$ and the Chan-Paton factors cannot be canceled. All the above results are summarized in Tab. G.1 which contains also other relevant information about the moduli.

As an illustrative example, we now consider in detail the SU(2) theory.

G.1 SU(2)

In this case we have $M = 2$, and thus necessarily $n_1 = n_2 = 1$. Therefore, in the SU(2) theory we have only simple surface operators. Furthermore, since the index s takes only one value, we can simplify the notation and suppress this index in the following.

Each pair $Y = (Y_1, Y_2)$ of Young tableaux contributes to the instanton partition function with a weight $q_1^{d_1}\, q_2^{d_2}$ where d_1 and d_2 are given by (4.42), which in this case take the simple form [43]

$$d_1 = \sum_j \left(Y_1^{2j+1} + Y_2^{2j+1}\right) \ , \qquad d_2 = \sum_j \left(Y_1^{2j+2} + Y_2^{2j+2}\right) \ . \tag{G.1}$$

with Y_I^k representing the length of the kth column of the tableau Y_I.

Let us begin by considering the case of pairs of Young tableaux with a single box. There are two such pairs that can contribute. One is $Y = (\square, \bullet)$ corresponding to $d_1 = 1$ and $d_2 = 0$. Using these values in (4.38), we find

$$z_{\{1,0\}} = \frac{(\epsilon_1+\epsilon_4)\left(a_1-\chi_{1,1}+\frac{1}{2}(\epsilon_1+\hat{\epsilon}_2)+\epsilon_4\right)\left(\chi_{1,1}-a_2+\frac{1}{2}(\epsilon_1+\hat{\epsilon}_2)+\epsilon_4\right)}{\epsilon_1\,\epsilon_4\,\left(a_1-\chi_{1,1}+\frac{1}{2}(\epsilon_1+\hat{\epsilon}_2)\right)\left(\chi_{1,1}-a_2+\frac{1}{2}(\epsilon_1+\hat{\epsilon}_2)\right)} \tag{G.2}$$

Due to the prescription (4.39), only the pole at

$$\chi_{1,1} = a_1 + \frac{1}{2}(\epsilon_1+\hat{\epsilon}_2) \tag{G.3}$$

contributes to the contour integral over $\chi_{1,1}$, yielding

$$Z_{(\square,\bullet)} = \frac{(\epsilon_1+\epsilon_4)\,(a_{12}+\epsilon_1+\hat{\epsilon}_2+\epsilon_4)}{\epsilon_1\,(a_{12}+\epsilon_1+\hat{\epsilon}_2)} = \frac{(\epsilon_1+\epsilon_4)\,(4a+2\epsilon_1+\epsilon_2+2\epsilon_4)}{\epsilon_1\,(4a+2\epsilon_1+\epsilon_2)} \tag{G.4}$$

where in the last step we used the notation $a_{12} = a_1 - a_2 = 2a$ and reintroduced

Y	weight	poles	Z_Y
$(\square, \square)$	$q_1 q_2$	$\chi_{1,1} = a_1 + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)$ $\chi_{2,1} = a_2 + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)$	$\frac{(\epsilon_1+\epsilon_4)^2(4a+\epsilon_2+2\epsilon_4)(-4a+\epsilon_2+2\epsilon_4)}{\epsilon_1^2(4a+\epsilon_2)(-4a+\epsilon_2)}$
$(\square\square, \bullet)$	$q_1 q_2$	$\chi_{1,1} = a_1 + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)$ $\chi_{2,1} = \chi_{1,1} + \hat{\epsilon}_2$	$\frac{(\epsilon_1+\epsilon_4)(\epsilon_2+\epsilon_4)(4a+\epsilon_2-2\epsilon_4)(4a+2\epsilon_1+\epsilon_2+2\epsilon_4)}{\epsilon_1\epsilon_2(4a+\epsilon_2)(4a+2\epsilon_1+\epsilon_2)}$
$(\bullet, \square\square)$	$q_1 q_2$	$\chi_{2,1} = a_2 + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)$ $\chi_{1,1} = \chi_{2,1} + \hat{\epsilon}_2$	$\frac{(\epsilon_1+\epsilon_4)(\epsilon_2+\epsilon_4)(-4a+\epsilon_2-2\epsilon_4)(-4a+2\epsilon_1+\epsilon_2+2\epsilon_4)}{\epsilon_1\epsilon_2(-4a+\epsilon_2)(-4a+2\epsilon_1+\epsilon_2)}$
$\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \bullet\right)$	q_1^2	$\chi_{1,1} = a_1 + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)$ $\chi_{1,2} = \chi_{1,1} + \epsilon_1$	$\frac{(\epsilon_1+\epsilon_4)(2\epsilon_1+\epsilon_4)(4a+2\epsilon_1+\epsilon_2+2\epsilon_4)(4a+4\epsilon_1+\epsilon_2+2\epsilon_4)}{2\epsilon_1^2(4a+2\epsilon_1+\epsilon_2)(4a+4\epsilon_1+\epsilon_2)}$
$\left(\bullet, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right)$	q_2^2	$\chi_{2,1} = a_2 + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)$ $\chi_{2,2} = \chi_{2,1} + \epsilon_1$	$\frac{(\epsilon_1+\epsilon_4)(2\epsilon_1+\epsilon_4)(-4a+2\epsilon_1+\epsilon_2+2\epsilon_4)(-4a+4\epsilon_1+\epsilon_2+2\epsilon_4)}{2\epsilon_1^2(-4a+2\epsilon_1+\epsilon_2)(-4a+4\epsilon_1+\epsilon_2)}$

Table G.2: We list the tableaux, the weight factors, the pole structure and the contribution to the partition function in all five cases with two boxes for the SU(2) theory.

$\epsilon_2 = 2\hat{\epsilon}_2$. A similar analysis can be done for the second pair of tableaux with one box that contributes, namely $Y = (\bullet, \square)$ corresponding to $d_1 = 0$ and $d_2 = 1$. In this case we find

$$Z_{(\bullet,\square)} = \frac{(\epsilon_1 + \epsilon_4)(-4a + 2\epsilon_1 + \epsilon_2 + 2\epsilon_4)}{\epsilon_1 (-4a + 2\epsilon_1 + \epsilon_2)} \,. \tag{G.5}$$

In the case of two boxes, we have five different pairs of tableaux that can contribute. They are: $Y = (\square, \square)$, $Y = (\square\square, \bullet)$, $Y = (\bullet, \square\square)$, $Y = \left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \bullet\right)$ and $Y = \left(\bullet, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right)$. The contributions of these five diagrams are listed below in Tab. G.2.

Multiplying all contributions with the appropriate weight factor and summing over them, we obtain the instanton partition function for the SU(2) gauge theory in the presence of the surface operator:

$$
\begin{aligned}
Z_{\text{inst}}[1,1] = 1 &+ q_1 \frac{(\epsilon_1+\epsilon_4)\,(4a+2\epsilon_1+\epsilon_2+2\epsilon_4)}{\epsilon_1\,(4a+2\epsilon_1+\epsilon_2)} + q_2 \frac{(\epsilon_1+\epsilon_4)\,(-4a+2\epsilon_1+\epsilon_2+2\epsilon_4)}{\epsilon_1\,(-4a+2\epsilon_1+\epsilon_2)} \\
&+ q_1^2\,\frac{(\epsilon_1+\epsilon_4)\,(2\epsilon_1+\epsilon_4)\,(4a+2\epsilon_1+\epsilon_2+2\epsilon_4)\,(4a+4\epsilon_1+\epsilon_2+2\epsilon_4)}{2\epsilon_1^2\,(4a+2\epsilon_1+\epsilon_2)\,(4a+4\epsilon_1+\epsilon_2)} \\
&+ q_2^2\,\frac{(\epsilon_1+\epsilon_4)\,(2\epsilon_1+\epsilon_4)\,(-4a+2\epsilon_1+\epsilon_2+2\epsilon_4)\,(-4a+4\epsilon_1+\epsilon_2+2\epsilon_4)}{2\epsilon_1^2\,(-4a+2\epsilon_1+\epsilon_2)\,(-4a+4\epsilon_1+\epsilon_2)} \\
&+ q_1 q_2 \left(\frac{(\epsilon_1+\epsilon_4)(\epsilon_2+\epsilon_4)(4a+\epsilon_2-2\epsilon_4)(4a+2\epsilon_1+\epsilon_2+2\epsilon_4)}{\epsilon_1\epsilon_2(4a+\epsilon_2)(4a+2\epsilon_1+\epsilon_2)} \right. \\
&\qquad + \frac{(\epsilon_1+\epsilon_4)(\epsilon_2+\epsilon_4)(-4a+\epsilon_2-2\epsilon_4)(-4a+2\epsilon_1+\epsilon_2+2\epsilon_4)}{\epsilon_1\epsilon_2(-4a+\epsilon_2)(-4a+2\epsilon_1+\epsilon_2)} \\
&\qquad\qquad \left. + \frac{(\epsilon_1+\epsilon_4)^2(4a+\epsilon_2+2\epsilon_4)(-4a+\epsilon_2+2\epsilon_4)}{\epsilon_1^2(4a+\epsilon_2)(-4a+\epsilon_2)} \right) + \cdots
\end{aligned}
\tag{G.6}
$$

where the ellipses stand for the contributions originating from tableaux with higher number of boxes, which can be easily generated with a computer program. We have explicitly computed these terms up six boxes, but we do not write them here since the raw expressions are very long and not particularly illuminating. To the extent it is possible to make comparisons, we observe that the above result agrees with the instanton partition function reported in eq. (B.6) of [43] under the following change of notation

$$
q_1 \to y\ , \quad q_2 \to x\ , \quad \epsilon_4 \to -m\ , \quad 2a \to 2a + \tfrac{\epsilon_2}{2}\ . \tag{G.7}
$$

Note then that the mass m appearing in [43] is the equivariant mass of the hypermultiplet [102], which differs by ϵ-corrections from the mass we have used (see (4.55)).

Appendix H

Prepotential coefficients for the SU(N) gauge theory

The prepotential $\mathcal{F}$ of the $\mathcal{N} = 2^\star$ SU(N) gauge theory has been determined in terms of quasi-modular forms in [19, 65]. Expanding $\mathcal{F}$ as in (4.78), the first few non-zero coefficients f_ℓ in the Nekrasov-Shatashvili limit turn out to be

$$f_2 = \frac{1}{4}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \sum_{u\neq v} \log\Big(\frac{a_u - a_v}{\Lambda}\Big)^2 + N\Big(m^2 - \frac{\epsilon_1^2}{4}\Big) \log\widehat{\eta}\ , \tag{H.1}$$

$$f_4 = -\frac{1}{24}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)^2 E_2\, C_2\ , \tag{H.2}$$

$$\begin{aligned} f_6 = -\frac{1}{288}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)^2 \Bigg\{ &\Big[\frac{2}{5}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\big(5E_2^2 + E_4\big) - 6\,\epsilon_1^2\, E_4\Big] C_4 \\ &+ \frac{1}{2}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\big(E_2^2 - E_4\big)\, C_{2;1,1} \Bigg\}\ , \end{aligned} \tag{H.3}$$

$$\begin{aligned} f_8 = -\frac{1}{1728}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)^2 \Bigg\{ &\Big[\frac{2}{105}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)^2 \big(175E_2^3 + 84E_2E_4 + 11E_6\big) \\ &- \frac{24\,\epsilon^2}{35}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\big(7E_2E_4 + 3E_6\big) + \frac{24\,\epsilon^4}{7}\, E_6\Big] C_6 \\ &- \frac{1}{5}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\big(5E_2^3 - 3E_2E_4 - 2E_6\big) - 6\,\epsilon^2\big(E_2E_4 - E_6\big)\Big] C_{4;2} \\ &- \frac{1}{5}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\Big[\frac{1}{12}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)\big(5E_2^3 - 3E_2E_4 - 2E_6\big) - 3\,\epsilon^2\big(E_2E_4 - E_6\big)\Big] C_{3;3} \\ &+ \frac{1}{24}\Big(m^2 - \frac{\epsilon_1^2}{4}\Big)^2 \big(E_2^3 - 3E_2E_4 + 2E_6\big)\, C_{2;1,1,1,1} \Bigg\}\ . \end{aligned} \tag{H.4}$$

Here E_2, E_4 and E_6 are the Eisenstein series and

$$\log\widehat{\eta} = -\sum_{k=1}^{\infty} \frac{\sigma_1(k)}{k}\, q^k = -\frac{1}{24}\log q + \log\eta \tag{H.5}$$

with η being the Dedekind η-function. Finally, the root lattice sums are defined by

$$C_{n;m_1,m_2,\cdots,m_k} = \sum_{\vec{\alpha}\in\Phi}\ \sum_{\vec{\beta}_1\neq\vec{\beta}_2\neq\cdots\neq\vec{\beta}_k\in\Phi(\vec{\alpha})} \frac{1}{(\vec{\alpha}\cdot\vec{a})^n(\vec{\beta}_1\cdot\vec{a})^{m_1}(\vec{\beta}_2\cdot\vec{a})^{m_1}\cdots(\vec{\beta}_k\cdot\vec{a})^{m_k}} \tag{H.6}$$

where Φ is the root system of $\mathrm{SU}(N)$ and

$$\Phi(\vec{\alpha}) = \{\vec{\beta}\in\Phi \ \big|\ \vec{\alpha}\cdot\vec{\beta} = 1\}\ . \tag{H.7}$$

We refer to [19] for the details and the derivation of these results. Notice, however, that we have slightly changed our notation, since $f_{2\ell}^{\text{here}} = f_{\ell}^{\text{there}}$. By expanding the modular functions in powers of q and selecting SU(2) as gauge group, it is easy to show that the above formulas reproduce both the perturbative part and the instanton contributions, reported respectively in (4.76a) and (4.62) of the main text.

Bibliography

[1] N. Seiberg and E. Witten, *Monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*, Nucl. Phys. **B426** (1994) 19–52, arXiv:hep-th/9407087.

[2] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*, Nucl. Phys. **B431** (1994) 484–550, arXiv:hep-th/9408099 [hep-th].

[3] N. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831–864, arXiv:hep-th/0206161.

[4] L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, Lett. Math. Phys. **91** (2010) 167–197, arXiv:0906.3219 [hep-th].

[5] D. Gaiotto, *N=2 dualities*, *JHEP* **1208** (2012) 034, arXiv:0904.2715 [hep-th].

[6] A. Klemm, W. Lerche, S. Yankielowicz, and S. Theisen, *Simple singularities and N=2 supersymmetric Yang-Mills theory*, Phys. Lett. **B344** (1995) 169–175, arXiv:hep-th/9411048 [hep-th].

[7] P. C. Argyres and A. E. Faraggi, *The vacuum structure and spectrum of N=2 supersymmetric SU(n) gauge theory*, Phys. Rev. Lett. **74** (1995) 3931–3934, arXiv:hep-th/9411057 [hep-th].

[8] A. Klemm, W. Lerche, and S. Theisen, *Nonperturbative effective actions of N=2 supersymmetric gauge theories*, *Int. J. Mod. Phys.* **A11** (1996) 1929–1974, `arXiv:hep-th/9505150 [hep-th]`.

[9] S. K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, R. R. John, and A. Lerda, *Non-perturbative studies of N=2 conformal quiver gauge theories*, *Fortsch. Phys.* **63** (2015) 259–293, `arXiv:1502.05581 [hep-th]`.

[10] E. Witten, *Solutions of four-dimensional field theories via M theory*, *Nucl.Phys.* **B500** (1997) 3–42, `arXiv:hep-th/9703166 [hep-th]`.

[11] M. Matone, *Instantons and recursion relations in N=2 SUSY gauge theory*, *Phys. Lett.* **B357** (1995) 342–348, `arXiv:hep-th/9506102`.

[12] A. Marshakov, *Tau-functions for Quiver Gauge Theories*, *JHEP* **1307** (2013) 068, `arXiv:1303.0753 [hep-th]`.

[13] P. Gavrylenko and A. Marshakov, *Residue Formulas for Prepotentials, Instanton Expansions and Conformal Blocks*, *JHEP* **1405** (2014) 097, `arXiv:1312.6382 [hep-th]`.

[14] N. Nekrasov and S. Shatashvili, *Quantization of Integrable Systems and Four Dimensional Gauge Theories*, `arXiv:0908.4052 [hep-th]`.

[15] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, *Nucl. Phys.* **B241** (1984) 333–380.

[16] S. Gukov and E. Witten, *Rigid Surface Operators*, *Adv. Theor. Math. Phys.* **14** no. 1, (2010) 87–178, `arXiv:0804.1561 [hep-th]`.

[17] E. Witten, *Phases of N=2 theories in two-dimensions*, *Nucl. Phys.* **B403** (1993) 159–222, `arXiv:hep-th/9301042 [hep-th]`.

[18] A. Hanany and K. Hori, *Branes and N=2 theories in two-dimensions*, *Nucl. Phys.* **B513** (1998) 119–174, arXiv:hep-th/9707192 [hep-th].

[19] M. Billò, M. Frau, F. Fucito, A. Lerda, and J. F. Morales, *S-duality and the prepotential in $\mathcal{N} = 2^{\star}$ theories (I): the ADE algebras*, *JHEP* **11** (2015) 024, arXiv:1507.07709 [hep-th].

[20] S. K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, R. R. John, and A. Lerda, *Modular and duality properties of surface operators in N=2* gauge theories*, arXiv:1702.02833 [hep-th].

[21] D. Gaiotto, S. Gukov, and N. Seiberg, *Surface Defects and Resolvents*, *JHEP* **09** (2013) 070, arXiv:1307.2578 [hep-th].

[22] D. Gaiotto, *Surface Operators in N = 2 4d Gauge Theories*, *JHEP* **11** (2012) 090, arXiv:0911.1316 [hep-th].

[23] L. Alvarez-Gaume and S. F. Hassan, *Introduction to S duality in N=2 supersymmetric gauge theories: A Pedagogical review of the work of Seiberg and Witten*, *Fortsch. Phys.* **45** (1997) 159–236, arXiv:hep-th/9701069 [hep-th].

[24] N. Seiberg, *Supersymmetry and Nonperturbative beta Functions*, *Phys. Lett.* **B206** (1988) 75–80.

[25] D. Finnell and P. Pouliot, *Instanton calculations versus exact results in four-dimensional SUSY gauge theories*, *Nucl. Phys.* **B453** (1995) 225–239, arXiv:hep-th/9503115 [hep-th].

[26] L. Bao, E. Pomoni, M. Taki, and F. Yagi, *M5-Branes, Toric Diagrams and Gauge Theory Duality*, *JHEP* **1204** (2012) 105, arXiv:1112.5228 [hep-th].

[27] T. Eguchi and K. Maruyoshi, *Penner Type Matrix Model and Seiberg-Witten Theory*, *JHEP* **1002** (2010) 022, arXiv:0911.4797 [hep-th].

[28] T. Eguchi and K. Maruyoshi, *Seiberg-Witten theory, matrix model and AGT relation*, *JHEP* **1007** (2010) 081, arXiv:1006.0828 [hep-th].

[29] M. Billò, L. Gallot, A. Lerda, and I. Pesando, *F-theoretic vs microscopic description of a conformal N=2 SYM theory*, *JHEP* **11** (2010) 041, arXiv:1008.5240 [hep-th].

[30] T. W. Grimm, A. Klemm, M. Marino, and M. Weiss, *Direct Integration of the Topological String*, *JHEP* **0708** (2007) 058, arXiv:hep-th/0702187 [HEP-TH].

[31] M. Billò, M. Frau, L. Gallot, and A. Lerda, *The exact 8d chiral ring from 4d recursion relations*, *JHEP* **1111** (2011) 077, arXiv:1107.3691 [hep-th].

[32] J. Thomae, *Beitrag zur Bestimmung von $\theta(0, 0, \ldots, 0)$ durch die Klassenmodulus algebraischer Functionen*, *Journ. reine angew. Math.* **71** (1870) 201–222.

[33] V. Z. Enolskii and P. Richter, *Periods of hyperelliptic integrals expressed in terms of θ-constants by means of Thomae formulae*, *Phil. Trans. R. Soc. A* **366** (2008) 1005–1024.

[34] L. Martucci, J. F. Morales, and D. R. Pacifici, *Branes, U-folds and hyperelliptic fibrations*, *JHEP* **1301** (2013) 145, arXiv:1207.6120 [hep-th].

[35] N. Dorey, V. V. Khoze, and M. P. Mattis, *A Two instanton test of the exact solution of N=2 supersymmetric QCD*, *Phys. Lett.* **B388** (1996) 324–330, arXiv:hep-th/9607066 [hep-th].

[36] V. Alba and A. Morozov, *Check of AGT Relation for Conformal Blocks on Sphere*, *Nucl.Phys.* **B840** (2010) 441–468, arXiv:0912.2535 [hep-th].

[37] J. Teschner, *Exact results on N=2 supersymmetric gauge theories*, arXiv:1412.7145 [hep-th].

[38] L. F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa, and H. Verlinde, *Loop and surface operators in N=2 gauge theory and Liouville modular geometry*, *JHEP* **1001** (2010) 113, arXiv:0909.0945 [hep-th].

[39] A.-K. Kashani-Poor and J. Troost, *The toroidal block and the genus expansion*, *JHEP* **1303** (2013) 133, arXiv:1212.0722 [hep-th].

[40] A.-K. Kashani-Poor and J. Troost, *Transformations of Spherical Blocks*, arXiv:1305.7408 [hep-th].

[41] S. Gukov, *Surface Operators*, arXiv:1412.7127 [hep-th].

[42] S. Gukov and E. Witten, *Gauge Theory, Ramification, And The Geometric Langlands Program*, arXiv:hep-th/0612073 [hep-th].

[43] L. F. Alday and Y. Tachikawa, *Affine SL(2) conformal blocks from 4d gauge theories*, *Lett. Math. Phys.* **94** (2010) 87–114, arXiv:1005.4469 [hep-th].

[44] C. Kozcaz, S. Pasquetti, F. Passerini, and N. Wyllard, *Affine sl(N) conformal blocks from N=2 SU(N) gauge theories*, *JHEP* **01** (2011) 045, arXiv:1008.1412 [hep-th].

[45] M. Taki, *On AGT Conjecture for Pure Super Yang-Mills and W-algebra*, *JHEP* **05** (2011) 038, arXiv:0912.4789 [hep-th].

[46] A. Marshakov, A. Mironov, and A. Morozov, *On AGT Relations with Surface Operator Insertion and Stationary Limit of Beta-Ensembles*, *J.Geom.Phys.* **61** (2011) 1203–1222, arXiv:1011.4491 [hep-th].

[47] C. Kozcaz, S. Pasquetti, and N. Wyllard, *A & B model approaches to surface operators and Toda theories*, *JHEP* **08** (2010) 042, arXiv:1004.2025 [hep-th].

[48] T. Dimofte, S. Gukov, and L. Hollands, *Vortex Counting and Lagrangian 3-manifolds*, Lett. Math. Phys. **98** (2011) 225–287, arXiv:1006.0977 [hep-th].

[49] K. Maruyoshi and M. Taki, *Deformed Prepotential, Quantum Integrable System and Liouville Field Theory*, Nucl. Phys. **B841** (2010) 388–425, arXiv:1006.4505 [hep-th].

[50] M. Taki, *Surface Operator, Bubbling Calabi-Yau and AGT Relation*, JHEP **07** (2011) 047, arXiv:1007.2524 [hep-th].

[51] M. Bullimore, H.-C. Kim, and P. Koroteev, *Defects and Quantum Seiberg-Witten Geometry*, JHEP **05** (2015) 095, arXiv:1412.6081 [hep-th].

[52] B. Assel and S. Schäfer-Nameki, *Six-dimensional origin of $\mathcal{N}=4$ SYM with duality defects*, JHEP **12** (2016) 058, arXiv:1610.03663 [hep-th].

[53] M. Billò, M. Frau, L. Gallot, A. Lerda, and I. Pesando, *Deformed N=2 theories, generalized recursion relations and S-duality*, JHEP **1304** (2013) 039, arXiv:1302.0686 [hep-th].

[54] M. Billò, M. Frau, L. Gallot, A. Lerda, and I. Pesando, *Modular anomaly equation, heat kernel and S-duality in $N=2$ theories*, JHEP **1311** (2013) 123, arXiv:1307.6648 [hep-th].

[55] J. Minahan, D. Nemeschansky, and N. Warner, *Instanton expansions for mass deformed N=4 superYang-Mills theories*, Nucl.Phys. **B528** (1998) 109–132, arXiv:hep-th/9710146 [hep-th].

[56] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Holomorphic anomalies in topological field theories*, Nucl.Phys. **B405** (1993) 279–304, arXiv:hep-th/9302103 [hep-th].

[57] E. Witten, *Quantum background independence in string theory*, arXiv:hep-th/9306122 [hep-th].

[58] M. Aganagic, V. Bouchard, and A. Klemm, *Topological Strings and (Almost) Modular Forms*, *Commun.Math.Phys.* **277** (2008) 771–819, arXiv:hep-th/0607100 [hep-th].

[59] M. Gunaydin, A. Neitzke, and B. Pioline, *Topological wave functions and heat equations*, *JHEP* **0612** (2006) 070, arXiv:hep-th/0607200 [hep-th].

[60] M.-x. Huang and A. Klemm, *Holomorphic Anomaly in Gauge Theories and Matrix Models*, *JHEP* **0709** (2007) 054, arXiv:hep-th/0605195 [hep-th].

[61] M.-x. Huang and A. Klemm, *Holomorphicity and Modularity in Seiberg-Witten Theories with Matter*, *JHEP* **1007** (2010) 083, arXiv:0902.1325 [hep-th].

[62] M.-x. Huang and A. Klemm, *Direct integration for general Ω backgrounds*, arXiv:1009.1126 [hep-th].

[63] D. Galakhov, A. Mironov, and A. Morozov, *S-duality as a beta-deformed Fourier transform*, *JHEP* **1208** (2012) 067, arXiv:1205.4998 [hep-th].

[64] N. Nemkov, *S-duality as Fourier transform for arbitrary ϵ_1, ϵ_2*, arXiv:1307.0773 [hep-th].

[65] M. Billò, M. Frau, F. Fucito, A. Lerda, J. Morales, *et al.*, *Modular anomaly equations in $\mathcal{N} = 2^*$ theories and their large-N limit*, *JHEP* **1410** (2014) 131, arXiv:1406.7255 [hep-th].

[66] A.-K. Kashani-Poor and J. Troost, *Quantum geometry from the toroidal block*, *JHEP* **1408** (2014) 117, arXiv:1404.7378 [hep-th].

[67] S. K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, A. Lerda, and M. Raman, *Modular anomaly equations and S-duality in $\mathcal{N}=2$ conformal SQCD*, JHEP **10** (2015) 091, arXiv:1507.07476 [hep-th].

[68] S. K. Ashok, E. Dell'Aquila, A. Lerda, and M. Raman, *S-duality, triangle groups and modular anomalies in $\mathcal{N}=2$ SQCD*, JHEP **04** (2016) 118, arXiv:1601.01827 [hep-th].

[69] M. Beccaria, A. Fachechi, G. Macorini, and L. Martina, *Exact partition functions for deformed $\mathcal{N}=2$ theories with $N_f=4$ flavours*, JHEP **12** (2016) 029, arXiv:1609.01189 [hep-th].

[70] M. Billó, M. Frau, F. Fucito, A. Lerda, and J. F. Morales, *S-duality and the prepotential of $\mathcal{N}=2^\star$ theories (II): the non-simply laced algebras*, JHEP **11** (2015) 026, arXiv:1507.08027 [hep-th].

[71] M. Beccaria and G. Macorini, *Exact partition functions for the Îl'-deformed $\mathcal{N}=2^*$ SU(2) gauge theory*, JHEP **07** (2016) 066, arXiv:1606.00179 [hep-th].

[72] S. K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, A. Lerda, M. Moskovic, and M. Raman, *Chiral observables and S-duality in N = 2* U(N) gauge theories*, JHEP **11** (2016) 020, arXiv:1607.08327 [hep-th].

[73] H. Kanno and Y. Tachikawa, *Instanton counting with a surface operator and the chain-saw quiver*, JHEP **06** (2011) 119, arXiv:1105.0357 [hep-th].

[74] V. Mehta and C. Seshadri, *Moduli of vector bundles on curves with parabolic structures*, *Mathematische Annalen* **248** 205.

[75] I. Biswas, *Parabolic bundles as orbifold bundles*, Duke Math. J. no. 2, 06305–325.

[76] B. Feigin, M. Finkelberg, A. Negut, and L. Rybnikov, *Yangians and cohomology rings of laumon spaces*, Selecta Mathematica **17** no. 3, (2011) 573–607. http://dx.doi.org/10.1007/s00029-011-0059-x.

[77] M. R. Douglas, *Branes within branes*, arXiv:9512077 [hep-th].

[78] M. Billò, M. Frau, I. Pesando, F. Fucito, A. Lerda, and A. Liccardo, *Classical gauge instantons from open strings*, *JHEP* **02** (2003) 045, arXiv:hep-th/0211250.

[79] E. Frenkel, S. Gukov, and J. Teschner, *Surface Operators and Separation of Variables*, *JHEP* **01** (2016) 179, arXiv:1506.07508 [hep-th].

[80] G. W. Moore, N. Nekrasov, and S. Shatashvili, *D-particle bound states and generalized instantons*, *Commun. Math. Phys.* **209** (2000) 77–95, arXiv:hep-th/9803265.

[81] G. Bonelli, A. Sciarappa, A. Tanzini, and P. Vasko, *Vortex partition functions, wall crossing and equivariant Gromov-Witten invariants*, *Commun. Math. Phys.* **333** no. 2, (2015) 717–760, arXiv:1307.5997 [hep-th].

[82] S. Nawata, *Givental J-functions, Quantum integrable systems, AGT relation with surface operator*, *Adv. Theor. Math. Phys.* **19** (2015) 1277–1338, arXiv:1408.4132 [hep-th].

[83] N. Doroud, J. Gomis, B. Le Floch, and S. Lee, *Exact Results in D=2 Supersymmetric Gauge Theories*, *JHEP* **05** (2013) 093, arXiv:1206.2606 [hep-th].

[84] F. Benini and S. Cremonesi, *Partition Functions of $\mathcal{N} = (2,2)$ Gauge Theories on S^2 and Vortices*, *Commun. Math. Phys.* **334** no. 3, (2015) 1483–1527, arXiv:1206.2356 [hep-th].

[85] M. Billò, M. Frau, F. Fucito, and A. Lerda, *Instanton calculus in R-R background and the topological string*, *JHEP* **11** (2006) 012, arXiv:hep-th/0606013.

[86] N. Wyllard, *Instanton partition functions in N=2 SU(N) gauge theories with a general surface operator, and their W-algebra duals*, *JHEP* **02** (2011) 114, arXiv:1012.1355 [hep-th].

[87] H. Awata, H. Fuji, H. Kanno, M. Manabe, and Y. Yamada, *Localization with a Surface Operator, Irregular Conformal Blocks and Open Topological String*, *Adv. Theor. Math. Phys.* **16** no. 3, (2012) 725–804, arXiv:1008.0574 [hep-th].

[88] N. Wyllard, *W-algebras and surface operators in N=2 gauge theories*, *J. Phys.* **A44** (2011) 155401, arXiv:1011.0289 [hep-th].

[89] E. D'Hoker, I. M. Krichever, and D. H. Phong, *The effective prepotential of $N = 2$ supersymmetric SU(N(c)) gauge theories*, *Nucl. Phys.* **B489** (1997) 179–210, arXiv:hep-th/9609041.

[90] S. G. Naculich, H. J. Schnitzer, and N. Wyllard, *The $N = 2$ U(N) gauge theory prepotential and periods from a perturbative matrix model calculation*, *Nucl. Phys.* **B651** (2003) 106–124, arXiv:hep-th/0211123 [hep-th].

[91] N. Nekrasov and V. Pestun, *Seiberg-Witten geometry of four dimensional N=2 quiver gauge theories*, arXiv:1211.2240 [hep-th].

[92] F. Fucito, J. Morales, D. R. Pacifici, and R. Poghossian, *Gauge theories on Ω-backgrounds from non commutative Seiberg-Witten curves*, *JHEP* **1105** (2011) 098, arXiv:1103.4495 [hep-th].

[93] F. Fucito, J. F. Morales, and D. R. Pacifici, *Deformed Seiberg-Witten Curves for ADE Quivers*, *JHEP* **1301** (2013) 091, arXiv:1210.3580 [hep-th].

[94] M. Beccaria, A. Fachechi, and G. Macorini, *Chiral trace relations in Ω-deformed $\mathcal{N} = 2$ theories*, arXiv:1702.01254 [hep-th].

[95] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, Nucl. Phys. **B460** (1996) 299–334, arXiv:hep-th/9510101 [hep-th].

[96] E. D'Hoker and D. H. Phong, *Calogero-Moser systems in SU(N) Seiberg-Witten theory*, Nucl. Phys. **B513** (1998) 405–444, arXiv:hep-th/9709053 [hep-th].

[97] D. Gaiotto, G. W. Moore, and A. Neitzke, *Wall-Crossing in Coupled 2d-4d Systems*, JHEP **12** (2012) 082, arXiv:1103.2598 [hep-th].

[98] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, arXiv:hep-th/0306238.

[99] M. Billò, M. Frau, F. Fucito, L. Giacone, A. Lerda, J. F. Morales, and D. Ricci-Pacifici, *Non-perturbative gauge/gravity correspondence in N=2 theories*, JHEP **1208** (2012) 166, arXiv:1206.3914 [hep-th].

[100] U. Bruzzo, F. Fucito, J. F. Morales, and A. Tanzini, *Multi-instanton calculus and equivariant cohomology*, JHEP **05** (2003) 054, arXiv:hep-th/0211108.

[101] F. Fucito, J. F. Morales, and R. Poghossian, *Instantons on quivers and orientifolds*, JHEP **0410** (2004) 037, arXiv:hep-th/0408090 [hep-th].

[102] T. Okuda and V. Pestun, *On the instantons and the hypermultiplet mass of N=2* super Yang-Mills on S^4*, JHEP **1203** (2012) 017, arXiv:1004.1222 [hep-th].