On some Group Theoretic, Ergodic Theoretic and Operator Algebraic Aspects of Compact Quantum Groups

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ISSAN PATRI

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DECLARATION

I, hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree/ diploma at this or any other Institution/University.

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Contents

SYNOPSIS

The theory of quantum groups has its roots in the work of G.I. Kac, in his attempt to extend Pontryagin Duality to the case of non-commutative groups. However, it really came to the fore in the 1980's in the pathbreaking work of Drinfeld, Jimbo and Woronowicz, done independently, at roughly around the same time. While Drinfeld [32] and Jimbo [46] constructed deformations of the universal enveloping algebra of simple lie algebras, the approach of Woronowicz was different. Inspired by Gelfand Duality for commutative C^*-algebras, Woronowicz, in a series of seminal papers [93][94][95], initiated the study of what are now called compact quantum groups.

The point of view of Woronowicz of quantum groups has now evolved into the succesful theory of operator algebraic quantum groups, with connections to a wide range of subjects. Quantum groups, being generalizations of groups, can be studied from a group theoetic perspective, which includes, study of representation theory [6][8], the study of subgroups[75][91], etc. On the other hand, quantum groups, being defined as certain operator algebras with some additional properties, can hence be studied from an operator algebraic point of view (often in tandem with the group theoretic perspective). This includes study of various operator algebraic properties and approximation properties like Property T [34], Amenability [82][11], exactness[84], etc. At the same time, the existence of the Haar state for compact quantum groups, which is always preserved under quantum automorphisms (see definition 1.4.1), means a study of group actions on compact quantum groups by quantum automorphisms, from the point of view of non-commutative dynamical systems, is viable.

In this thesis, we make a study on three aspects of quantum groups. More precisely, we make a study of certain group theoretic aspects of compact quantum groups, viz. definition of inner automorphisms, relation with Wang's notion of normal subgroups and definition of center of a compact quantum group. We also study discrete group actions on compact quantum groups by quantum automorphisms and obtain combinatorial conditions for when such actions are ergodic, weak mixing, mixing, compact, topologically transitive, etc. We give a structure theorem for such actions for a special class of compact quantum groups, showing the existence of maximal ergodic normal subgroup. Finally, we make a thorough and comprehensive study of the representation theory and various approximation properties, viz. Haagerup property, Rapid Decay, Weak Amenability, etc of (a class of) the bicrossed product quantum groups and crossed product quantum groups. We also present the first non-trivial examples of discrete quantum groups with Property T.

Let us now briefly explain the layout of this thesis.

This thesis comprises of five chapters, including the present one. The second chapter, titled Preliminaries, gives a brief introduction to various aspects of operator algebraic quantum groups that will be needed subsequently. We also give here accounts of various approximation properties and a quick introduction to the notion of quantum automorphisms of compact quantum groups.

The third chapter, titled "Normal subgroups, center and inner automorphisms of compact quantum groups", deals with some group theoretic aspects of compact quantum groups. In particular, we define and study inner automorphisms, obtain structure theorems for the group of inner automorphisms. We also study the relation of these inner automorphisms to Wang's notion of normal subgroups of compact quantum groups. In the latter half of this chapter, we define and study the notion of centers of compact quantum groups and compute it for several examples, using a result that shows that centers of compact quantum groups with identical fusion rules are isomorphic. The original part of this chapter is culled from the paper [72].

The fourth chapter, titled "Automorphisms of compact quantum groups", deals with the study of group actions on compact quantum groups by quantum automorphisms. We present combinatorial conditions for various spectral properties like Ergodicity, Weak Mixing, Compactness of such actions. We give several examples of such dynamical systems. This chapter is based on the paper [67].

The fifth, and final, chapter, deals with the bicrossed product and the crossed product quantum groups. We make an exhaustive study of special classes of the bicrossed product quantum group and the crossed product quantum group, explaining their representation theory and permanence properties of various approximation properties, like Haagerup Property, Rapid Decay, etc. We provide a large number of examples, exhibiting various approximation properties in these particular cases and also present an infinite family of non-trivial discrete quantum groups possessing the Property T. This chapter is based on the paper [35].

Each chapter starts with a brief introduction and outline of results.

Notations. In this thesis, the inner products of Hilbert spaces are assumed to be linear in the first variable. The same symbol $\otimes$ will denote the tensor product of Hilbert spaces, the minimal tensor product of C*-algebras and as well as the tensor product of von Neumann algebras.

Chapter 1

Preliminaries

In this section, we present a brief account of operator algebraic quantum groups and various associated approximation properties, that will be needed in the subsequent chapters. We also study the notion of quantum automorphisms of compact quantum groups in various guises and prove results that will be crucial in the sequel.

1.1 Compact and Discrete Quantum Groups

Definition 1.1.1. *A compact quantum group $G = (A, \Delta)$ is a unital C^*-algebra A together with a comultiplication Δ which is a coassociative $*$-homomorphism:*

$$\Delta : A \to A \otimes A$$

such that the sets $(A \otimes 1)\Delta(A)$ and $(1 \otimes A)\Delta(A)$ are total in $A \otimes A$.

Now, let $\mathcal{H}$ be a finite dimensional hilbert space with an orthonormal basis given by $\{e_1, e_2, \ldots, e_n\}$ and with e_{ij} the corresponding system of matrix units in $B(\mathcal{H})$. A unitary element $u = \sum_{i,j} e_{ij} \otimes u_{ij}$ in $B(\mathcal{H}) \otimes A$ is said to be a finite dimensional representation for the compact quantum group $G = (A, \Delta)$ if $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all $i, j \in \{1, 2, ..., n\}$. A finite dimensional representation is said to be irreducible if it has no invariant subspace, see for example [62] and [96], and the book [81] for a detailed introduction to compact quantum groups and their representation theory.

For two finite dimensional representations u and v of G, we denote by $\mathrm{Mor}(u, v)$ the space of intertwiners from u to v and by $u \otimes v$ their tensor product representation, which is simply the representation $u_{13}v_{23}$. Let us note that several authors use the notation $u \oplus v$ to denote the tensor product representation. The trivial representation is denoted by 1. We also denote by $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible unitary representations of G. For $x \in \mathrm{Irr}(G)$, we choose a representative $u^x \in \mathcal{B}(H_x) \otimes C(G)$, where u^x is a irreducible representation on the Hilbert space H_x. We denote by $\mathrm{Pol}(G)$ the linear span of the coefficients of $\{u^x : x \in \mathrm{Irr}(G)\}$. This is a unital $*$-subalgebra of A, and it can be shown that this is dense subalgebra of A. Further, $\mathrm{Pol}(G)$ is equipped with a co-unit $\epsilon_G : \mathrm{Pol}(G) \to \mathbb{C}$, which is a $*$-homomorphism satisfying, for all $a \in \mathrm{Pol}(G)$,

$$(\epsilon_G \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \epsilon_G)\Delta(a) = a$$

It also has an antipode, which is an anti-multiplicative map $S_G : \mathrm{Pol}(G) \to \mathrm{Pol}(G)$ satisfying for all $a \in \mathrm{Pol}(G)$

$$m((S_G \otimes \mathrm{id})\Delta(a)) = m((\mathrm{id} \otimes S_G)\Delta(a)) = \epsilon_G(a) \cdot 1$$

where $m : \mathrm{Pol}(G) \otimes \mathrm{Pol}(G) \to \mathrm{Pol}(G)$ denotes the multiplication map, $m(a \otimes b) = ab$

It is well known that there is a natural involution $x \mapsto \overline{x}$ such that $u^{\overline{x}}$ is the unique (up to equivalence) irreducible representation of G such that $\mathrm{Mor}(1, x \otimes \overline{x}) \neq \{0\} \neq \mathrm{Mor}(\overline{x} \otimes x, 1)$. For any $x \in \mathrm{Irr}(G)$, take a non-zero element $E_x \in \mathrm{Mor}(1, x \otimes \overline{x})$ and define an anti-linear map $J_x : H_x \to H_{\overline{x}}$ by letting $\xi \mapsto (\xi^* \otimes 1)E_x$. Define $Q_x = J_x J_x^* \in \mathcal{B}(H_x)$. We normalize E_x in such a way that $\mathrm{Tr}_x(Q_x) = \mathrm{Tr}_x(Q_x^{-1})$, where Tr_x is the unique trace on $\mathcal{B}(H_x)$ such that $\mathrm{Tr}_x(1) = \dim(x)$. This uniquely determines Q_x and fixes E_x up to a complex number of modulus 1. The number $\dim_q(x) := \mathrm{Tr}_x(Q_x) = \mathrm{Tr}_x(Q_x^{-1})$ is called the *quantum dimension of* x. Let $u^x_{cc} = (\mathrm{id} \otimes S_G^2)(u^x)$, where S_G denotes the antipode of G. It can be shown that (see e.g. section 5 of [94]) that Q_x is also uniquely determined by the fact that $Q_x \in \mathrm{Mor}(u^x, u^x_{cc})$ and that Q_x is invertible and $\mathrm{Tr}_x(Q_x) = \mathrm{Tr}_x(Q_x^{-1}) > 0$.

Definition 1.1.2. *A compact matrix quantum group $G = (A, \Delta, u)$ is a triple such that $G = (A, \Delta)$ is a compact quantum group and u is a finite dimensional representation of G such that any irreducible representation is a sub-representation of some tensor power of u.*

A very fundamental fact is the existence of the Haar state associated to a compact quantum group G, which we denote as h or h_G if the context is not clear. This is a unique state on the C^*-algebra A, which is left and right invariant, i.e. it satisfies the equations

$$(h \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes h)\Delta(a) = h(a)1$$

for any $a \in A$. It is also well known that h is faithful on $\mathrm{Pol}(G)$.

Associated also to any compact quantum group G are the maximal C^*-algebra, which we shall denote by $C_m(G)$, which can be defined as the enveloping C^*-algebra of $\mathrm{Pol}(G)$. Similarly, we have the reduced C^*-algebra, denoted as $C(G)$, generated by the GNS construction of h (i.e. as the image of the GNS representation of h). Hence we view $C(G) \subset \mathcal{B}(\mathrm{L}^2(G))$, where $\mathrm{L}^2(G)$ is the GNS space of h. The von-Neumann algebra thus generated by $C(G)$ will be denoted as $\mathrm{L}^\infty(G)$. Let us note that the comultiplication is well defined on all these algebras, and the reader is cautioned that it is always denoted as Δ (or as Δ_G if the context is not clear) for all these algebras. Also, we denote by λ (or λ_G) the canonical surjection from $C_m(G)$ to $C(G)$. Let us note that there can exist "exotic" C^*-completions of $\mathrm{Pol}(G)$, which are not isomorphic with either $C_m(G)$ or $C(G)$. Interesting examples of such completions are constructed in [57], where Property (T) of the associated discrete quantum group is used crucially. Let us note that in this thesis we construct the first non-trivial examples of Property (T) discrete quantum groups (see Example 4.5.4 and Example 4.5.10).

Let us now turn to the case of discrete quantum groups. Associated to any compact quantum group G, is a discrete quantum group $\widehat{G}$ and the correspondence is one-to-one. The associated operator algebras of the discrete dual $\widehat{G}$ of G are denoted by

$$\ell^\infty(\widehat{G}) = \bigoplus_{x \in \mathrm{Irr}(G)}^{\ell^\infty} \mathcal{B}(H_x) \quad \text{and} \quad c_0(\widehat{G}) = \bigoplus_{x \in \mathrm{Irr}(G)}^{c_0} \mathcal{B}(H_x).$$

We denote by $V_G = \bigoplus_{x \in \mathrm{Irr}(G)} u^x \in M(c_0(\widehat{G}) \otimes C_m(G))$ the *maximal multiplicative unitary*. Let p_x be the minimal central projection of $\ell^\infty(\widehat{G})$ corresponding to the block $\mathcal{B}(H_x)$. We say that $a \in \ell^\infty(\widehat{G})$ has *finite support* if $ap_x = 0$ for all but finitely many

$x \in \mathrm{Irr}(G)$. The set of finitely supported elements of $\ell^\infty(\widehat{G})$ is dense in $c_0(\widehat{G})$ and the latter is equal to the algebraic direct sum $c_c(\widehat{G}) = \bigoplus_{x\in\mathrm{Irr}(G)}^{alg} \mathcal{B}(H_x)$.

The (left-invariant) Haar weight on $\widehat{G}$ is the n.s.f. weight on $\ell^\infty(\widehat{G})$ defined by

$$h_{\widehat{G}}(a) = \sum_{x\in\mathrm{Irr}(G)} \mathrm{Tr}_x(Q_x)\mathrm{Tr}_x(Q_x a p_x),$$

whenever the formula makes sense. It is known that the GNS representation of $h_{\widehat{G}}$ is of the form $(\widehat{\lambda}_G, \mathrm{L}^2(G), \Lambda_{\widehat{G}})$, where $\Lambda_{\widehat{G}} : c_c(\widehat{G}) \to \mathrm{L}^2(G)$ is linear with dense range and $\widehat{\lambda}_G$: $\ell^\infty(\widehat{G}) \to \mathcal{B}(\mathrm{L}^2(G))$ is a unital normal $*$-homomorphism such that $\Delta_G(x) = W_G(x\otimes 1)W_G^*$ for all $x \in C(G)$, where $W_G = (\widehat{\lambda}_G \otimes \lambda_G)(V_G)$. We call W_G the *reduced multiplicative unitary.*

1.2 Subgroups and Normal Subgroups

We now move to the notion of subgroups introduced by Podles [75] and normal subgroups introduced by Wang [87][91]. We note that the term subgroup will mean quantum subgroup and will be used interchangeably with it. We also refer to [30] for a comphrehensive study of subgroups of locally compact quantum groups.

Definition 1.2.1. *A compact quantum group H is said to be a quantum subgroup of G if there exists a surjective $*$-homomorphism $\rho : C_m(G) \to C_m(H)$ such that*

$$(\rho \otimes \rho)\Delta_G = \Delta_H \circ \rho$$

This ρ will be called the corresponding surjection.

Associated with a quantum subgroup H of G are the left coset space and the right coset space given by:

$$C_m(G/H) := \{a \in C_m(G) \mid (\mathrm{id} \otimes \rho)\Delta_G(a) = a \otimes 1\}$$
$$C_m(H\backslash G) := \{a \in C_m(G) \mid (\rho \otimes \mathrm{id})\Delta_G(a) = 1 \otimes a\}$$

respectively.

These spaces have natural conditional expectation onto them given by:

$$E_{G/H} := (\mathrm{id} \otimes h_H \circ \rho)\Delta_G$$

and

$$E_{H\backslash G} := (h_H \circ \rho \otimes \mathrm{id})\Delta_G$$

respectively.

Definition 1.2.2. *A quantum subgroup H of G is said to be finite index subgroup if $C_m(G/H)$ is finite dimensional.*

Definition 1.2.3. *Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group. We say then that A is a Woronowicz C^*-algebra. If $A_0 \subseteq A$ is a sub-C^*-algebra such that the tuple $(A_0, \Delta_{|A_0})$ is compact quantum group, then we say that A_0 is a Woronowicz sub-C^*-algebra of A.*

Definition 1.2.4. *Given a CQG $\mathbb{G} = (A, \Delta)$, with unitary dual Irr($\mathbb{G}$), we say that a subset $\mathcal{T} \subseteq$ Irr($\mathbb{G}$) is a sub-object of Irr($\mathbb{G}$) if $e \in \mathcal{T}$ (e denotes the class of the trivial representation), if $\beta \in \mathcal{T}$ then $\overline{\beta} \in \mathcal{T}$ and if β_1 and $\beta_2 \in \mathcal{T}$, then for any $\gamma \subset \beta_1 \otimes \beta_2$, $\gamma \in \mathcal{T}$. So, in other words, closure of the linear span of the matrix coefficients of representatives of elements in $\mathcal{T}$ is a Woronowicz sub-C^*-algebra of A.*

Definition 1.2.5. *A quantum subgroup N of a compact quantum group G is said to be normal if $C_m(G/N) = C_m(N\backslash G)$. In this case, $G/N = (C_m(G/N), \Delta_{G/N})$ is a compact quantum group itself.*

It is pertinent to note here that the notation used in the previous definition is in fact consistent, in the sense of the following theorem, for which we refer to [2] for a proof.

Theorem 1.2.6. *Let G be a compact quantum group. Let $A_0 \subset C_m(G)$ be a Woronowicz sub-C^*-algebra of $C_m(G)$, with the associated compact quantum group denoted by G_0. Then we have that $C_m(G_0) = A_0$.*

In other words, the previous theorem says that, Woronowicz sub-C^*-algebras of Woronowicz C^*-algebra which are maximal C^*-algebras associated to some compact quantum group, are themselves the maximal. This result is well known for group C^*-algebras which corresponds to the cocommutative case (see for example Proposition 2.5.8 of [19]).

The following theorem is proved by Wang in [91].

Theorem 1.2.7. *A quantum subgroup N of G is normal if and only if any of the following equivalent conditions hold –*

1. $\Delta_G(C_m(G/N)) \subseteq C_m(G/N) \otimes C_m(G/N)$
2. $\Delta_G(C_m(N\backslash G) \subseteq C_m(N\backslash G) \otimes C_m(N\backslash G)$
3. *The multiplicity of 1_N, the trivial representation of N in $u_{|N}$, where u is any irreducible representation of G, is either 0 or d_u, the dimension of u.*

1.3 Approximation properties

In this section we recall the definition of the Coamenability, Haagerup property, weak amenability and Cowling-Haagerup constants for discrete quantum groups. We also show some basic facts we could not find in the literature: for example, permanence of the (co)-Haagerup property and (co)-weak amenability from a quantum subgroup of finite index to the ambient compact quantum group.

Let G be a compact quantum group. We say G is coamenable if the haar state on $C_m(G)$ is faithful (equivalence of this with the original definition given in [11] follows easily from Theorem 2.2 of [11]).

Proposition 1.3.1. *If G has a coamenable normal subgroup N such that G/N is coamenable, then G is coamenable.*

Proof. Let $\rho : C_m(G) \to C_m(N)$ be the corresponding surjection. It follows from the uniqueness of the co-unit ϵ_G that $\epsilon_G = \epsilon_N \circ \rho$. Also note that the co-unit is norm bounded on $\mathrm{Pol}(G)$.

To show that the Haar measure h_G is faithful, we first note that the conditional expectation onto $N\backslash G$ given by

$$E_{N\backslash G} = (h_H \circ \rho \otimes \mathrm{id})\Delta$$

is faithful since $(\rho \otimes \mathrm{id})\Delta$ is injective as

$$(\epsilon_N \circ \rho \otimes \mathrm{id})\Delta(a) = (\epsilon_G \otimes \mathrm{id})\Delta(a) = a$$

and h_H is faithful. But then as $E_{N\backslash G}$ is invariant under h_G, it follows that h_G is faithful. Hence, by [11], we are done. □

Proposition 1.3.2. *Let G be a coamenable compact quantum group. Then, any subgroup H of G is coamenable as well.*

Proof. We use a little bit of machinery for this. It is shown in [56] that a compact quantum group is coamenable if and only if its fusion algebra is amenable in the sense of [43]. The proposition now follows from Proposition 7.4(2) of [43]. □

Now, for G a compact quantum group and $\omega \in C_m(G)^*$, define its Fourier transform $\widehat{\omega} = (\mathrm{id}\otimes\omega)(V) \in M(c_0(\widehat{G}))$, where $V = \bigoplus_{x\in\mathrm{Irr}(G)} u^x \in M(c_0(\widehat{G})\otimes C_m(G))$ is the maximal multiplicative unitary. Observe that $\omega \mapsto \widehat{\omega}$ is linear and $\|\widehat{\omega}\|_{\mathcal{B}(\mathrm{L}^2(G))} \leq \|\omega\|_{C_m(G)^*}$ for all $\omega \in C_m(G)^*$.

When G is a classical compact group with Haar measure μ and ν is a complex Borel measure on G, then the Fourier transform $\widehat{\nu} \in M(C_r^*(G))$ is the operator $\widehat{\nu} = \int_G \lambda_g d\nu(g) \in M(C_r^*(G)) \subset \mathcal{B}(\mathrm{L}^2(G))$.

Following [29], we say that $\widehat{G}$ *has the Haagerup property* if there exists a sequence of states $\omega_n \in C_m(G)^*$ such that $\omega_n \to \varepsilon_G$ in the weak* topology and $\widehat{\omega}_n \in c_0(\widehat{G})$ for all $n \in \mathbb{N}$.

For $a \in \ell^\infty(\widehat{G})$ with finite support, we define a finite rank map $m_a : C(G) \to C(G)$ by $(\mathrm{id} \otimes m_a)(u^x) = u^x(ap_x \otimes 1)$. We say that a sequence $a_i \in \ell^\infty(\widehat{G})$ *converges pointwise to* 1, if $\|a_i p_x - p_x\|_{\mathcal{B}(H_x)} \to 0$ for all $x \in \mathrm{Irr}(G)$.

Recall that $\widehat{G}$ is said to be *weakly amenable* if there exists a sequence of finitely supported $a_i \in \ell^\infty(\widehat{G})$ converging pointwise to 1 and such that $C = \sup_i \|m_{a_i}\|_{cb} < \infty$. The infimum of those C, called the Haagerup constant of $\widehat{G}$, is denoted by $\Lambda_{cb}(\widehat{G})$ (and is, by definition, infinite if $\widehat{G}$ is not weakly amenable). Similarly, associated to any C^*-algebra A and to any von-Neumann algebra M, we have a corresponding Haagerup constant, which is denoted by $\Lambda_{cb}(A)$ and $\Lambda_{cb}(M)$ respectively (see Definition 12.3.9 of [19]). It was proved in [55] that, when G is Kac, we have $\Lambda_{cb}(\widehat{G}) = \Lambda_{cb}(C(G)) = \Lambda_{cb}(\mathrm{L}^\infty(G))$.

Theorem 1.3.3. *Let H be a finite index quantum subgroup of G. Then the following holds.*

1. *If $\widehat{H}$ has the Haagerup property, then $\widehat{G}$ has the Haagerup property.*

2. $\Lambda_{cb}(\widehat{G}) \leq \Lambda_{cb}(\widehat{H})$.

Proof. We will need the following Claim.

Claim. *If H is a finite index quantum subgroup of G with surjective morphism $\rho : C_m(G) \to C_m(H)$ then the set $N_y^\rho = \{x \in \mathrm{Irr}(G) : \mathrm{Mor}(v^y, (\mathrm{id} \otimes \rho)(u^x)) \neq \{0\}\}$ is finite for all $y \in \mathrm{Irr}(H)$, where $\{v^y : y \in \mathrm{Irr}(H)\}$ is a complete set of representatives.*

Proof of the Claim. We first show that N_1^ρ is finite. Let $x \in N_1^\rho$ and $\xi \in H_x$ be such that $\|\xi\| = 1$ and $(\mathrm{id} \otimes \rho)(u^x)(\xi \otimes)1 = \xi \otimes 1$. Choose an orthonormal basis $(e_k^x)_k$ of H_x such that $e_1^x = \xi$. Observe that the coefficients of u^x with respect to this orthonormal basis satisfy $\rho(u_{11}^x) = 1$ and $\rho(u_{k1}^x) = 0$ for all $k \neq 1$. It follows that $u_{11}^x \in C_m(G/H)$. Since the coefficients of non-equivalent representations are linearly independent and since $C_m(G/H)$ is finite dimensional, it follows that the set N_1^ρ is finite.

Suppose that there exists $y \in \mathrm{Irr}(H) \setminus \{1\}$ such that N_y^ρ is infinite and let $(x_n)_{n \in \mathbb{N} \cup \{0\}}$ be an infinite sequence of elements in N_y^ρ. Since $(\mathrm{id} \otimes \rho)(u^{\overline{x}_0} \otimes u^{x_i})$ has a sub-representation isomorphic to $v^{\overline{y}} \otimes v^y$, it contains the trivial representation. It follows that, for all $i \geq 1$, there exists $z_i \in N_1^\rho$ such that $z_i \subset \overline{x}_0 \otimes x_i$. Hence, by Proposition 3.2 of [76], $x_i \subset x_0 \otimes z_i$ and the set $\{z_i : i \geq 1\}$ is infinite, a contradiction.

(1). Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of states on $C_m(H)$ such that $\widehat{\mu_n} \in c_0(\widehat{H})$ for all $n \in \mathbb{N}$ and $\mu_n \to \varepsilon_H$ in the weak* topology. Define $\omega_n = \mu_n \circ \rho \in C_m(G)^*$, where $\rho : C_m(G) \to C_m(H)$ is the subgroup surjection. Since $\varepsilon_G = \varepsilon_H \circ \rho$, we have $\omega_n \to \varepsilon_G$ in the weak* topology. Let $n \in \mathbb{N}$ and $\epsilon > 0$. We need to show that the set $G_{n,\epsilon} = \{x \in \mathrm{Irr}(G) : ||(\mathrm{id} \otimes \omega_n)(u^x)|| \geq \epsilon\}$ is finite. Since $\widehat{\mu_n} \in c_0(\widehat{H})$, the set $H_{n,\epsilon} = \{y \in \mathrm{Irr}(H) : ||(id \otimes \mu_n)(v^y)|| \geq \epsilon\}$ is finite, and since $G_{n,\epsilon} = \cup_{y \in H_{n,\epsilon}} N_y^\rho$, by the previous claim we are done.

(2). We may and will suppose that $\widehat{H}$ is weakly amenable. Let $\epsilon > 0$ and $a_i \in \ell^\infty(\widehat{H})$ be a sequence of finitely supported elements that converges to 1 pointwise and such that $\sup_i \|m_{a_i}\|_{cb} \leq \Lambda_{cb}(\widehat{H}) + \epsilon$.

We consider the dual morphism $\widehat{\rho} : c_0(\widehat{H}) \to M(c_0(\widehat{G}))$, which is the unique non-degenerate $*$-homomorphism satisfying $(\mathrm{id} \otimes \rho)(V_G) = (\widehat{\rho} \otimes \mathrm{id})(V_H)$.

We first show that $\widehat{\rho}(a_i) \in \ell^\infty(\widehat{G})$ is finitely supported for all i and the sequence $(\widehat{\rho}(a_i))_i$ converges to 1 pointwise. Consider the functional $\omega_{a_i} \in C_m(H)^*$ defined by $(\mathrm{id} \otimes \omega_{a_i})(v^y) = a_i p_y$ for all $y \in \mathrm{Irr}(H)$ so $(\mathrm{id} \otimes \omega_{a_i})(V_H) = a_i$ and, by definition of the dual morphism $\widehat{\rho}(a_i) = (\mathrm{id} \otimes \omega_{a_i} \circ \rho)(V_G)$, we have $\widehat{\rho}(a_i)p_x = (\mathrm{id} \otimes \omega_{a_i} \circ \rho)(u^x)$ and $\{x \in \mathrm{Irr}(G) : \widehat{\rho}(a_i)p_x \neq 0\} = \cup_{y \in \mathrm{Irr}(H), a_i p_y \neq 0} N_y^\rho$. Hence, $\widehat{\rho}(a_i)$ is finitely supported for all i. Moreover, for all $x \in \mathrm{Irr}(G)$,

$$\begin{aligned}
\|\widehat{\rho}(a_i)p_x - p_x\| &= \|(\mathrm{id} \otimes \omega_{a_i} \circ \rho)(u^x) - p_x\| = \sup_{y \in \mathrm{Irr}(H) \text{ and } x \in N_y^\rho} \|(\mathrm{id} \otimes \omega_{a_i})(v^y) - p_y\| \\
&= \sup_{y \in \mathrm{Irr}(H) \text{ and } x \in N_y^\rho} \|a_i p_y - p_y\| \to_i 0.
\end{aligned}$$

We now show that $\sup_i \|m_{\widehat{\rho}(a_i)}\|_{cb} < \Lambda_{cb}(\widehat{H}) + \epsilon$. First let us note that, by Fell's

Absorption Principle, we have $(W_G)_{12}(V_G)_{13} = (V_G)_{23}(W_G)_{12}(V_G)^*_{23}$. Thus, there exists a $*$-homomorphism $\widetilde{\Delta}_G : C(G) \to C(G) \otimes C_m(G)$ which extends the comultiplication Δ_G on $\mathrm{Pol}(G)$. We now define a unital $*$-homomorphism $\pi : C(G) \to C(G) \otimes C(H)$ such that

$$\pi(x) = (\mathrm{id} \otimes \lambda_H \circ \rho) \circ \widetilde{\Delta}_G$$

where $\lambda_H : C_m(H) \to C(H)$ denotes the canonical surjection given by the GNS-representation with respect to the Haar state of H. Clearly, π extends the map $(\mathrm{id} \otimes \rho) \circ \Delta_G$ on $\mathrm{Pol}(G)$. Now it not hard to see that the map π is a *right quantum homomorphism* (see section 1 of [65]); in other words π satisfies the equations -

$$(\Delta_G \otimes \mathrm{id}) \circ \pi = (\mathrm{id} \otimes \pi) \circ \Delta_G,$$

$$(\mathrm{id} \otimes \Delta_H) \circ \pi = (\pi \otimes \mathrm{id}) \circ \pi.$$

Both of the above equations follow easily from the coassociativity condition of the comultiplication of G and H and from the fact that $\pi = (\mathrm{id} \otimes \rho) \circ \Delta_G$ and $(\rho \otimes \rho) \circ \Delta_G = \Delta_H \circ \rho$ on $\mathrm{Pol}(G)$. This together with Theorem 5.3 of [65] implies that there exists a unitary operator $V_\rho \in \mathcal{B}(\mathrm{L}^2(G)) \otimes C(H)$ such that

$$\pi(x) = V_\rho (x \otimes 1) V_\rho^*.$$

Hence, it follows that, π is isometric.

It is now not hard to see that $(\mathrm{id} \otimes m_{a_i})\pi = \pi \circ m_{\widehat{\rho}(a_i)}$ for all i. Indeed, since $m_{a_i}(x) = (\mathrm{id} \otimes \omega_{a_i})\Delta_H(x)$ and $m_{\widehat{\rho}(a_i)}(x) = (\mathrm{id} \otimes \omega_{a_i} \circ \rho)\Delta_G(x)$ for all $x \in \mathrm{Pol}(G)$, we find that for $x \in \mathrm{Pol}(G)$,

$$\begin{aligned}
(\mathrm{id} \otimes m_{a_i})\pi(x) &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega_{a_i})(\mathrm{id} \otimes \Delta_H) \circ \pi(x) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega_{a_i})(\pi \otimes \rho) \circ \Delta_G(x) \\
&= \pi\left((\mathrm{id} \otimes \omega_{a_i} \circ \rho)(\Delta_G(x)\right) = \pi \circ m_{\widehat{\rho}(a_i)}(x).
\end{aligned}$$

Since π is isometric, we have $\|m_{\widehat{\rho}(a_i)}\|_{cb} \leq \|m_{a_i}\| \leq \Lambda_{cb}(\widehat{H}) + \epsilon$ for all i. Hence, $\Lambda_{cb}(\widehat{G}) \leq \Lambda_{cb}(\widehat{H}) + \epsilon$. Since ϵ is arbitrary the proof is complete. $\square$

1.4 Automorphisms of Compact Quantum Groups

In this chapter, we study the notion of quantum automorphisms of compact quantum groups.

Definition 1.4.1. *Let G be a compact quantum group. A quantum automorphism is a C^*-algebraic automorphism $\alpha : C_m(G) \to C_m(G)$ such that $(\alpha \otimes \alpha)\Delta = \Delta \circ \alpha$.*

We denote the group of all quantum automorphisms of G by $\mathrm{Aut}(G)$. It is not hard to see that $\mathrm{Aut}(G)$ is a closed subgroup of $\mathrm{Aut}(C_m(G))$, the group of all C^*-algebraic automorphisms, given the pointwise norm topology. Hence, as $\mathrm{Aut}(C_m(G))$ is a Polish group, we have that $\mathrm{Aut}(G)$ is also Polish.

We record some properties of quantum automorphisms in the following proposition.

Proposition 1.4.2. *Let $\alpha \in Aut(G)$. Then,*

1. *$\alpha(Pol(G)) = Pol(G)$;*

2. *$\epsilon_G \circ \alpha = \epsilon_G$;*

3. *$S_G \circ \alpha = \alpha \circ S_G$;*

4. *$h_G \circ \alpha = h_G$;*

5. *If $((u_{ij})) \in M_n(C_m(G))$ is a finite dimensional irreducible representation of G, then so is $((\alpha(u_{ij}))) \in M_n(C_m(G))$.*

Proof.

1. Let $((u_{ij})) \in M_n(C_m(G))$ be a unitary representation of G. Then, it is easy to see that $((\alpha(u_{ij}))) \in M_n(C_m(G))$ is also a representation. It then follows that $\alpha(\mathrm{Pol}(G)) \subseteq \mathrm{Pol}(G)$. But, it is also easily checked that α^{-1} is an automorphism of G and hence $\alpha(\mathrm{Pol}(G)) \subseteq \mathrm{Pol}(G)$.

2. Follows from the uniqueness of the counit ϵ_G under the condition that $(\epsilon_G \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \epsilon_G)\Delta = \mathrm{id}$.

3. Follows from the uniqueness of the antipode under the condition that $m(S_G \otimes \mathrm{id})\Delta(\cdot) = m(\mathrm{id} \otimes S_G)\Delta(\cdot) = \epsilon_G(\cdot)1$.

4. Follows from the uniqueness of Haar measure h_G under the condition that $(h_G \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes h_G)\Delta(a) = h_G(a)1$

5. Let $u = ((u_{ij})) \in M_n(C_m(G))$ be an finite dimensional representation. Then, $\chi_u = \sum u_{ii}$ is the character of the representation u. It is shown in [94] that u is irreducible if and only if $h_G(\chi_u^* \chi_u) = 1$. The result now follows immediately from part (4).

□

Observe that it follows from the last proposition that each $\alpha \in \mathrm{Aut}(G)$ induces a bijection $\alpha \in S(\mathrm{Irr}(G))$, the group of bijections of the set $\mathrm{Irr}(G)$. Indeed, for $x \in \mathrm{Irr}(G)$, $\alpha(x)$ is the equivalence class of the irreducible unitary representation $(\mathrm{id} \otimes \alpha)(u^x)$. By construction, the map $\mathrm{Aut}(G) \to S(\mathrm{Irr}(G))$ is a group homomorphism.

Proposition 1.4.3. *The map* $\mathrm{Aut}(G) \to S(\mathrm{Irr}(G))$ *is continuous.*

Proof. We shall need the following well known lemma which is of independent interest. We include a proof for the convenience of the reader.

Lemma 1.4.4. *Let* $u, v \in \mathcal{B}(H) \otimes C_m(G)$ *be two unitary representations of* G *on the same finite dimensional Hilbert space* H*. If* $\|u - v\| < 1$*, then* u *and* v *are equivalent.*

Proof. Define $x = (\mathrm{id} \otimes h)(v^* u) \in \mathcal{B}(H)$. Since u and v are unitary representations, h being the Haar state forces $(x \otimes 1)u = v(x \otimes 1)$. We have $u^*(x^* x \otimes 1)u = x^* x \otimes 1$. Hence, $u^*(|x| \otimes 1)u = |x| \otimes 1$. Now observe that $\|1 - x\| = \|(\mathrm{id} \otimes h)(1 - v^* u)\| \leq \|1 - v^* u\| = \|v - u\| < 1$. Hence x is invertible, and in the polar decomposition $x = w|x|$, the polar part w is a unitary. Consequently, $v^*(w|x| \otimes 1)u = v^*(w \otimes 1)u(|x| \otimes 1) = (w \otimes 1)(|x| \otimes 1)$.

By uniqueness of the polar decomposition of $x \otimes 1$, we deduce that $v^*(w \otimes 1)u = w \otimes 1$. Hence, u and v are equivalent. $\square$

We can now prove the proposition. Let $(\alpha_n)_n$ be a sequence in $\mathrm{Aut}(G)$ which converges to $\alpha \in \mathrm{Aut}(G)$. Let $F \subset \mathrm{Irr}(G)$ be a finite subset and let $N \in \mathbb{N}$ be such that for all $n \geq N$

$$\|(\mathrm{id} \otimes \alpha_n)(u^x) - (\mathrm{id} \otimes \alpha)(u^x)\| < \frac{1}{2} \quad \text{for all } x \in F.$$

It follows from Lemma 1.4.4 that $(\mathrm{id} \otimes \alpha_n)(u^x)$ and $(\mathrm{id} \otimes \alpha)(u^x)$ are equivalent for all $n \geq N$ and for all $x \in F$. This means that $\alpha_n(x) = \alpha(x)$ for all $x \in F$ whenever $n \geq N$. This establishes the continuity. $\square$

Remark 1.4.5. *We can also define* $\mathrm{Aut}_r(G) = \{\alpha \in \mathrm{Aut}(C(G)) : \Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta\}$ *which is again a Polish group as it is a closed subgroup of the Polish group* $\mathrm{Aut}(C(G))$*. Since any* $\alpha \in \mathrm{Aut}(G)$ *preserves the Haar state, it defines a unique element in* $\mathrm{Aut}_r(G)$*. Hence, we have a canonical map* $\mathrm{Aut}(G) \to \mathrm{Aut}_r(G)$ *which is obviously a group homomorphism. Moreover, it is actually bijective. The inverse bijection is constructed in the following way. Since any* $\alpha \in \mathrm{Aut}_r(G)$ *restrict to an automorphism of* $\mathrm{Pol}(G)$*, it extends uniquely by the universal property to an automorphism in* $\mathrm{Aut}(G)$*. It is also easy to check that the map* $\mathrm{Aut}(G) \to \mathrm{Aut}_r(G)$ *is continuous.*

Also, since any automorphism of $C(G)$ *intertwining* Δ *has a unique normal extension to* $\mathrm{L}^\infty(G)$*, it induces a map* $\mathrm{Aut}_r(G) \to \mathrm{Aut}_\infty(G)$*, where* $\mathrm{Aut}_\infty(G) = \{\alpha \in \mathrm{Aut}(\mathrm{L}^\infty(G)) : \Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta\}$*. As before, this map is a bijective group homomorphism and is continuous (the topology on* $\mathrm{Aut}(\mathrm{L}^\infty(G))$ *being governed by the pointwise* $\|\cdot\|_{2,h}$ *convergence).*

1.5 Compact group action on countable sets

We end this chapter by recording some facts regarding actions of compact groups on countable sets. This will be necessary in Chapter 4 where we will be studying the bicrossed product construction for compact matched pairs of groups.

Let X be a countable infinite set and let $S(X)$ be the group of bijections of X. It is a Polish group equipped with the topology of pointwise convergence which is the topology generated by the sets $S_{x,y} = \{\alpha \in S(X) : \alpha(x) = y\}$ for $x, y \in X$. Since $S^c_{x,y} = \cup_{z\in X\setminus\{y\}} S_{x,z}$, these sets are clopen in $S(X)$. Moreover, for any compact subset $K \subset S(X)$ and for any $x \in X$, the orbit $K \cdot x \subset X$ is finite. Indeed, from the open cover $K \subset \cup_{y\in X} S_{x,y}$, we find $y_1, \cdots, y_n \in X$ such that $K \subset \cup_{i=1}^n S_{x,y_i}$, which implies that $K \cdot x \subset \{y_1, \cdots, y_n\}$.

Let $\beta : G \to S(X)$ be a continuous right action of G on X. To simplify the notations, we write $x \cdot g = \beta_g(x)$ for $g \in G$ and $x \in X$.

Observe that, since β is continuous and G is compact, every β-orbit in X is finite. Fix $r, s \in X$ and denote by $A_{r,s}$ the set

$$A_{r,s} = \{g \in G : r \cdot g = s\} = \beta^{-1}(S_{r,s}).$$

Note that, since β is continuous, $A_{r,s}$ is open and closed in G for all $r, s \in X$. Hence, $1_{A_{r,s}} \in C(G)$. Moreover, $1_{A_{r,s}} \neq 0$ if and only if r and s are in the same orbit and we have the following relations:

1. $1_{A_{s,r}} 1_{A_{t,r}} = \delta_{t,s} 1_{A_{s,r}}$ for all $r, s, t \in X$.

2. $1_{A_{s,r}} 1_{A_{s,t}} = \delta_{r,t} 1_{A_{s,r}}$ for all $r, s, t \in X$.

3. $\sum_{s\in X} 1_{A_{r,s}} = \sum_{s\in r\cdot G} 1_{A_{r,s}} = 1$ for all $r \in X$.

4. $\sum_{r\in X} 1_{A_{r,s}} = \sum_{r\in s\cdot G} 1_{A_{r,s}} = 1$ for all $r \in X$.

5. If $r \cdot G = s \cdot G$, then $\Delta_G(1_{A_{s,r}}) = \sum_{t\in s.G} 1_{A_{s,t}} \otimes 1_{A_{t,r}}$,

where Δ_G is the usual comultiplication on $C(G)$. In other words, for every orbit $x \cdot G$, the matrix $(1_{A_{r,s}})_{r,s\in x\cdot G} \in M_{|x\cdot G|}(\mathbb{C}) \otimes C(G)$ is a magic unitary and a unitary representation of G. We note also that formally, equality also holds in the case $r \cdot G \neq s \cdot G$, as is easily checked.

Chapter 2

Normal Subgroups, Center and Inner automorphisms of Compact Quantum Groups

In this chapter, we define and study a notion of “inner“ automorphisms of compact quantum groups and its relation to Wang's notion of normal subgroups of compact quantum groups. We also define the center of a compact quantum group and compute the center for several examples.

2.1 Inner Automorphisms

In the classical case, when G is a compact group, a special class of automorphisms are the inner automorphisms, the automorphisms of the form $\alpha_s : G \to G$ where

$$\alpha_s(g) = sgs^{-1}, \quad s, g \in G$$

Let's note that the inner automorphisms preserve the class of any unitary representation of the compact group. In other words, the fixed point algebra of $C(G)$ under the action of G on it by inner automorphisms contains the characters of all irreducible representations of the group, and in fact, it follows from Peter-Weyl theorem that the linear span of characters of all irreducible representations is dense in this algebra.

Let's consider the more general class of automorphisms that preserve the representation class of each irreducible representation. Denoting this subset of the automorphism group by $\mathrm{Aut}_\chi(G)$, it is easily seen that this gives a normal subgroup of the group $\mathrm{Aut}(G)$. It is known for compact connected groups that $\mathrm{Aut}_\chi(G) = \mathrm{Inn}(G)$, where $\mathrm{Inn}(G)$ denotes the inner automorphisms of G [64]. But there are several examples of finite groups for which $\mathrm{Inn}(G)$ is a proper subgroup of $\mathrm{Aut}_\chi(G)$ (for more on this we refer to the survey [97] and to references therein).

Given a compact quantum group G, let G_{char} denote the set of characters of irreducible representations of G. We want to consider the group of automorphisms

$$\mathrm{Aut}_\chi(G) = \{\alpha \in \mathrm{Aut}(G) \mid \alpha(\chi_{u^a}) = \chi_{u^a} \ \forall \ \chi_{u^a} \in G_{\mathrm{char}}\}$$

It is straightforward to see, using Proposition 1.4.2, that $\mathrm{Aut}_\chi(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

We topologise $\mathrm{Aut}(G)$ by taking as neighbourhood of identity, sets of the form:

$$u(a_1, \ldots, a_n \in A \mid \epsilon > 0) := \{\alpha \in \mathrm{Aut}(G) \mid \|a_i - \alpha(a_i)\| < \epsilon \; \forall i \in \{1, 2, \ldots, n\}\}$$

$\mathrm{Aut}_\chi(G)$ is easily seen to be a closed normal subgroup. For the next theorem, we assume that $G = (A, \Phi, u)$ is compact matrix quantum group. Before proceeding further, let us recall that a unital subspace of a C^*-algebra is said to be an operator system if it is closed under the involution. One can then study completely positive maps between operator systems, which is the appropriate morphism in the category of operator systems. Another notion that we need is that of a multiplicative domain of a unital completely positive (UCP) map. Given A, B C^*-algebras and a UCP $T : A \to B$, we define the multiplicative domain $M_T = \{a \in A : T(a^*a) = T(a)^*T(a) \text{and} T(aa^*) = T(a)T(a)^*\}$. It follows easily from the Cauchy-Schwarz inequality for CP maps that for any $a \in M_t$ and $b \in A$, $T(ab) = T(a)T(b)$ and $T(ba) = T(b)T(a)$. We refer to the book [73] for more on operator systems, CP maps and multiplicative domains.

Theorem 2.1.1. $\mathrm{Aut}_\chi(G)$ *is a compact group.*

Proof. Viewing u as an element of $M_n(C_m(G)) = M_n(\mathbb{C}) \otimes C_m(G)$, let $u = ((u_{ij}))$. Let $\alpha \in \mathrm{Aut}_\chi(G)$, then we have:

$$\sum u_{ii} = \sum \alpha(u_{ii})$$

so that there exists a $u_\alpha \in U(n)$, the group of $n \times n$ unitary (scalar) matrices. such that:

$$(u_\alpha \otimes 1)u(u_\alpha^* \otimes 1) = \alpha(u)$$

so, this gives us an anti-homomorphism,

$$\gamma : \mathrm{Aut}_\chi(G) \to U(n)$$

$$\alpha \mapsto u_\alpha$$

where $\alpha(u) = (u_\alpha \otimes 1)u(u_\alpha^* \otimes 1)$. Clearly γ is injective.

To show $\mathrm{Aut}_\chi(G)$ is compact, we want to show that the image of γ in $U(n)$, denoted by $\Im(\gamma)$, is closed in $U(n)$ and the map $\gamma^{-1} : \Im(\gamma) \to \mathrm{Aut}_\chi(G)$ is continuous. We have a lemma:

Lemma 2.1.2. *Let $\{\alpha_i\}_{i\in I}$ be a net of automorphisms of a C^*-algebra A which is generated as a C^*-algebra by the set $\{s_1, ..., s_n\}_{n\in\mathbb{N}}$. If $\alpha_i(s_k) \to s_k$ for all $k \in \{1, \ldots, n\}$ in norm, then, $\alpha_i(a) \to a$ for all $a \in A$ in norm.*

Proof. This is straightforward. □

It now follows from the previous lemma that γ^{-1} is continuous since $C_m(G)$ is generated by u_{ij}'s, the matrix entries of the representation $u = ((u_{ij}))$.

To show that $\Im(\gamma)$ is closed, we need the following lemma, a matrix version of a lemma of Pisier [74].

Lemma 2.1.3. *Let A be a unital C^*-algebra and $u = ((u_{ij})) \in M_n(A)$ be an unitary element, such that A is generated as a C^*-algebra by $u_{ij}, i, j \in \{1, 2, ..., n\}$ and suppose $T : A \to B$ is a unital completely positive map into some C^*-algebra B such that $((T(u_{ij}))) \in M_n(B)$ is also a unitary element. Then, T is a $*$-homomorphism.*

Proof. The proof is by a multiplicative domain argument. We want to show that u_{ij}'s are in the multiplication of domain of T from which the result will follow.

Let us take $u_{11} \in A$. Then, we know that

$$\sum u_{1j}u_{1j}^* = 1.$$

Now, by Cauchy-Schwarz inequality,

$$T(u_{11}u_{11}^*) \geqslant T(u_{11})T(u_{11})^*$$

But, we also have,

$$
\begin{aligned}
T(u_{1j}u_{1j}^*) &\geqslant T(u_{1j})T(u_{1j})^* \\
\Rightarrow \sum_{j=2}^{n} T(u_{1j}u_{1j}^*) &\geqslant \sum_{j=2}^{n} T(u_{1j})T(u_{1j})^* \\
\Rightarrow T(1-u_{11}u_{11}^*) &\geqslant 1-T(u_{11})T(u_{11}^*) \\
\Rightarrow T(u_{11}u_{11}^*) &= T(u_{11})T(u_{11})^*
\end{aligned}
$$

Similarly, this can be proved for all u_{ij}'s. □

We continue with the proof of the theorem. Let $\{t_i\}_{i\in I}$ be a net of unitary matrices in $U(n)$ such that $t_i \to t$ in $U(n)$, with $\gamma(\alpha_i) = t_i\ \forall i \in I$. Consider the finite dimensional operator system S generated by $\{u_{ij}\}, i, j \in \{1, 2, ..., n\}$, the matrix entries of $u \in M_n(C_m(G))$. Since α_i are automorphisms such that:

$$\alpha_i(u) = (t_i \otimes 1)u(t_i^* \otimes 1)$$

the map on S defined by

$$1 \mapsto 1$$

and

$$u \mapsto (t_i \otimes 1)u(t_i^* \otimes 1)$$

gives us an unital completely positive map $\phi_i : S \to S$ for all $i \in I$. Now, consider the map $\phi : S \to S$ given by

$$1 \mapsto 1$$

$$u \mapsto (t \otimes 1)u(t \otimes 1)^*$$

Since $t_i \to t$ in norm, it follows that $\phi_i \to \phi$ in point norm topology. This implies $\|\phi\|_{cb} = 1$ and since ϕ is unital, we have that ϕ is unital completely positive.

But by Arvenson Extension theorem and the previous lemma, ϕ extends uniquely to an automorphism of the quantum group and so range of γ is closed and we are done. $\square$

We now revert back to our original assumption of G being a compact quantum group.

Theorem 2.1.4. *The group* $\mathrm{Out}_\chi(G) = \mathrm{Aut}(G)/\mathrm{Aut}_\chi(G)$ *is totally disconnected.*

Proof. Let u be an irreducible representation of G and χ_u be its character, $\chi_u \in C_m(G)$. Let

$$K_{[u]} := \{\alpha \in \mathrm{Aut}(G) \mid \alpha(\chi_u) = \chi_u\}$$

Then we have,

$$\mathrm{Aut}_\chi(G) = \bigcap_{[u]\in \mathrm{Irr}(G)} K_{[u]}$$

where $[u]$ the equivalence class corresponding to the irreducible representation u of G. We shall show that $K_{[u]}$ is an open subgroup of $\mathrm{Aut}(G)$ which will imply that $\mathrm{Aut}(G)/K_{[u]}$ is discrete and so it will follow that $\mathrm{Aut}(G)/\mathrm{Aut}_\chi(G)$ is totally disconnected.

To show that $K_{[u]}$ is open, consider the open neighbourhood:

$$u(\chi_u, 1) := \{\alpha \in \mathrm{Aut}(G) \mid \|\chi_u - \alpha(\chi_u)\| < 1\}$$

Now, $h(\chi_u^*\chi_u) = 1$ and

$$h(\alpha(\chi_u)^*\chi_u) = \begin{cases} 1, & \text{if } \chi_u = \alpha(\chi_u) \\ 0, & \text{otherwise} \end{cases}$$

So, for any $\alpha \in u(\chi_u, 1)$, we have,

$$|h(\chi_u^*\chi_u) - h(\alpha(\chi_u)^*\chi_u)| \leqslant \|\chi_u - \alpha(\chi_u)\| < 1$$

and hence

$$|1 - h(\alpha(\chi_u)^*\chi_u)| < 1 \Rightarrow \alpha(\chi_u) = \chi_u \Rightarrow \alpha \in K_{[u]}$$

Since $u(\chi_u, 1)$ is an open neighbourhood in $K_{[u]}$ we indeed have that $K_{[u]}$ is an open subgroup of $\mathrm{Aut}(G)$. $\square$

In fact, much more is true for compact quantum groups which have fusion rules identical to those of connected compact simple Lie Groups.

Proposition 2.1.5. *Let G be a compact quantum group having fusion rules identical to those of a connected compact Lie group. Then the group $\mathrm{Out}_\chi(G) = \mathrm{Aut}(G)/\mathrm{Aut}_\chi(G)$ has finite order. In particular, if G is a $q-$deformation of some simply connected simple compact Lie group, then $\mathrm{Out}_\chi(G)$ has order $1, 2, 3$ or 6.*

Proof. Any automorphism α of G induces an order isomorphism of its representation ring $\mathbb{Z}[\widehat{G}]$. And clearly, if $\alpha \in \mathrm{Aut}_\chi(G)$, then α induces the trivial isomorphism of its representation ring. So, $\mathrm{Out}_\chi(G)$ is easily seen to be a subgroup of the group of order isomorphisms of the representation ring $\mathbb{Z}[\widehat{G}]$.

But, by results of Mcmullen [64] and Handelman [40], it follows that for connected compact groups, the group of order isomorphisms of the representation ring of the group and its outer automorphism group are isomorphic. The proposition now follows from the facts that for connected compact Lie groups, the outer automorphism group is finite and that for simple compact Lie groups, it can only have order $1, 2$, or 6. $\square$

Definition 2.1.6. *A compact group G is said to be Hopfian if every surjective homomorphism $f : G \to G$ is an isomorphism. In other words, there exists no proper normal subgroup N of G such that $G/N \cong G$.*

Analogously, one can define Hopfian compact quantum groups.

Definition 2.1.7. *A compact quantum group $G = (A, \Phi)$ is said to be Hopfian if every injective quantum homomorphism $\phi : C_m(G) \to C_m(G)$ is also surjective.*

Proposition 2.1.8. *Let G be a compact quantum group which has fusion rules identical to those of a compact connected group $\mathcal{G}$. Suppose that $\mathcal{G}$ is a Hopfian compact group. Then, G is Hopfian as well.*

Proof. Suppose not, then there exists a quantum homomorphism $\alpha : C_m(G) \to C_m(G)$ which is injective but not surjective. Then the induced map of the representation ring $\widehat{\alpha} : \mathbb{Z}[\widehat{G}] \to \mathbb{Z}[\widehat{G}]$ is injective and order preserving but not surjective.

However, $\mathbb{Z}[\widehat{G}]$ is also the representation ring of the compact connected group $\mathcal{G}$. Now the range of $\widehat{\alpha}$ corresponds to a proper subobject of $\widehat{\mathcal{G}}$, which by the Galois correspondence between the subobjects and normal subgroups, corresponds to a proper normal subgroup $\mathcal{N}$ of $\mathcal{G}$ and hence, the representation ring of $\mathcal{G}/\mathcal{N}$ and $\mathcal{G}$ are order isomorphic, and so once again by [64] and [40], it follows that $\mathcal{G}$ is isomorphic to $\mathcal{G}/\mathcal{N}$, which is a contradiction. $\square$

2.2 Inner Automorphisms and Normal Subgroups I

In the classical case of compact groups, subgroups are said to be normal if they are stable under all automorphisms of the form $\alpha_s(g) = sgs^{-1}$, $s, g \in G$, a compact group. So, for a compact group G, a subgroup N is, by definition, normal if for every automorphism of the form α_s, $s \in G$, there exists an automorphism $\beta : N \to N$ such that:

$$\begin{tikzcd} G \arrow[r, "\alpha_s"] & G \\ N \arrow[u, hook, "i"] \arrow[r, "\beta"] & N \arrow[u, hook, "i"] \end{tikzcd}$$

commutes.

In fact, more is true: if N is a normal subgroup of G, then it corresponds to a unique subobject Γ_N of $\mathrm{Irr}(G)$ where $\mathrm{Irr}(G)$ denotes the equivalence classes of all irreducible representations of G [42]. So, it follows that N is always stable under any representation class preserving automorphism (the converse of course is trivial as $\mathrm{Inn}(G) \subseteq \mathrm{Aut}_\chi(G)$). In this section, we show that this holds true in the quantum case as well, under certain assumptions.

Essentially, we want to show that, given a compact quantum group G, $\alpha \in \mathrm{Aut}_\chi(G)$ and N a normal subgroup, with $\rho : C_m(G) \to C_m(N)$ as the corresponding surjection, there exists a quantum group automorphism $\beta : C_m(N) \to C_m(N)$ such that:

$$\begin{tikzcd} C_m(G) \arrow[r, "\alpha"] \arrow[d, "\rho"'] & C_m(G) \arrow[d, "\rho"] \\ C_m(N) \arrow[r, "\beta"'] & C_m(N) \end{tikzcd}$$

commutes.

Lemma 2.2.1. *Let G be a compact quantum group with H a subgroup of it and $\rho : C_m(G) \to C_m(H)$ the corresponing surjection. Let $\alpha : C_m(G) \to C_m(G)$ be an automorphism of G and $\beta : C_m(H) \to C_m(H)$ be a C^*-algebraic automorphism such that*

$$\begin{tikzcd} C_m(G) \arrow[r, "\alpha"] \arrow[d, "\rho"] & C_m(G) \arrow[d, "\rho"] \\ C_m(H) \arrow[r, "\beta"] & C_m(H) \end{tikzcd}$$

commutes. Then β is also a quantum group automorphism.

Proof. We want to show that $\Delta_H \circ \beta = (\beta \otimes \beta)\Delta_H$. Since ρ is surjective, given any $b \in C_m(H)$, there exists some $a \in C_m(G)$ such that $\rho(a) = b$. Now,

$$\begin{aligned} (\beta \otimes \beta)\Delta_H(b) &= (\beta \otimes \beta)\Delta_H(\rho(a)) \\ &= (\beta \circ \rho \otimes \beta \circ \rho)\Delta_G(a) \\ &= (\rho \otimes \rho)(\alpha \otimes \alpha)\Delta_G(a) \\ &= \Delta_H(\rho \circ \alpha(a)) \\ &= \Delta_H(\beta \circ \rho(a)) = \Delta_H(\beta(b)) \end{aligned}$$

and so we are done. $\square$

Proposition 2.2.2. *Let G be a compact quantum group and let H be a subgroup of G. Let α be an automorphism of G and let β be an automorphism of H such that*

$$\begin{tikzcd} C_m(G) \arrow[r, "\alpha"] \arrow[d, "\rho"] & C_m(G) \arrow[d, "\rho"] \\ C_m(H) \arrow[r, "\beta"] & C_m(H) \end{tikzcd}$$

commutes. Then $\alpha : C_m(G/H) \to C_m(G/H)$ is a C^*-algebraic automorphism. Similarly, $\alpha : C_m(H\backslash G) \to C_m(H\backslash G)$ is a C^*-algebraic automorphism.

Proof. First we note that

$$\rho \circ \alpha = \beta \circ \rho \iff \rho \circ \alpha^{-1} = \beta^{-1} \circ \rho$$

Now, we just have to show that

$$\alpha(C_m(G/H)) \subseteq C_m(G/H)$$

This is clear as if $a \in C_m(G/H)$, then by definition, $(\rho \otimes \mathrm{id})\Delta_G(a) = 1 \otimes a$. Now,

$$\begin{aligned}(\rho \otimes \mathrm{id})\Delta_G(\alpha(a)) &= (\rho \otimes \mathrm{id})(\alpha \otimes \alpha)\Delta_G(a)\\ &= (\beta \circ \rho \otimes \alpha)\Delta_G(a)\\ &= (\beta \otimes \alpha)(1 \otimes a)\\ &= 1 \otimes \alpha(a)\end{aligned}$$

and so $\alpha(a) \in C_m(G/H)$. One can similarly prove the proposition in the case of $C_m(H\backslash G)$. □

Proposition 2.2.3. *Let G be a compact quantum group and let H be a subgroup of G. Let $\alpha : C_m(G) \to C_m(G)$ be an automorphism of G, such that $\alpha : C_m(G/H) \to C_m(G/H)$ is C^*-algebraic automorphism, then there exists $\beta : C_m(H) \to C_m(H)$ such that β is also an automorphism of H and such that*

$$\begin{tikzcd} C_m(G) \arrow[r, "\alpha"] \arrow[d, "\rho"] & C_m(G) \arrow[d, "\rho"] \\ C_m(H) \arrow[r, "\beta"] & C_m(H) \end{tikzcd}$$

commutes.

Proof. Consider $\mathrm{Pol}(G)$ and $\mathrm{Pol}(H)$. We have that $\rho(\mathrm{Pol}(G)) = \mathrm{Pol}(H)$ and $\alpha(\mathrm{Pol}(G)) = \mathrm{Pol}(G)$. Hence, we will be done if we can find a $\beta_0 : \mathrm{Pol}(H) \to \mathrm{Pol}(H)$, a $*$-algebra automorphism such that

$$\begin{array}{ccc} \mathrm{Pol}(G) & \xrightarrow{\alpha} & \mathrm{Pol}(G) \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ \mathrm{Pol}(H) & \xrightarrow{\beta_0} & \mathrm{Pol}(H) \end{array}$$

commutes. This is because it is then easy to show that β_0 is actually a Hopf algebra automorphism, and if we denote the natural extension of β_0 to $C_m(H)$ by β, then β is a quantum group automorphism such that $\alpha \circ \rho = \rho \circ \beta$. First let us note the following straightforward

Lemma 2.2.4. *Let A, A_1, A_2 be unital $*$-algebras and $\pi_k : A \to A_k$, for $k = 1, 2$ be surjective $*$-algebraic morphisms, with kernels denoted by $I_k, k = 1, 2$. Then the following are equivalent-*

1. *There is an $*$-algebraic surjective morphism $\alpha : A_1 \to A_2$ such that $\pi_2 = \alpha \circ \pi_1$.*

2. *$I_1 \subseteq I_2$*

We also have that, in the previous case, α is an isomorphism if and only if $I_1 = I_2$

Thus, with $C = \mathrm{Pol}(G)$, $C_1 = C_2 = \mathrm{Pol}(H)$, $\pi_1 = \rho$ and $\pi_2 = \rho \circ \alpha$, if we show that $\ker(\rho) = \ker(\rho \circ \alpha)$, then we will have a $*$-isomorphism $\beta_0 : \mathrm{Pol}(H) \to \mathrm{Pol}(H)$ such that $\beta_0 \circ \rho = \rho \circ \alpha$.

Let $\mathrm{Pol}(G)_{G/H} := \mathrm{Pol}(G) \cap C_m(G/H)$. We have that $\alpha(\mathrm{Pol}(G)_{G/H} = \mathrm{Pol}(G)_{G/H}$. Let $a \in \mathrm{Pol}(G)_{G/H}$, then as $\rho(a) = \epsilon_G(a) \cdot 1$, we have that $\rho(a) = 0$ if and only if $\epsilon_G(a) = 0$. But since $\epsilon_G \circ \alpha = \epsilon_G$, we have that $\rho(a) = 0$ if and only if $\rho(\alpha(a)) = 0$ for any $a \in \mathrm{Pol}(G)_{G/\mathbb{H}}$.

Suppose now that for some $a \in \mathrm{Pol}(G)$, $a > 0$, we have that $\rho(a) = 0$ but $\rho(\alpha(a)) > 0$. We then have that $h_h(\rho(\alpha(a))) > 0$. But then $\rho(E_{G/H}(\alpha(a))) = (\rho \otimes h_H \circ \rho)\Phi(a) = (id \otimes h_H)\Psi(\rho(a)) = h_H(\rho(a)) \cdot 1 > 0$.

Before proceeding further, we need the following

Lemma 2.2.5. *Let C be a unital $*$-algebra and $C_0 \subseteq C$ be a unital $*$-subalgebra. Suppose that ϕ is a faithful state on C. Then there can exist at most one linear projection map*

$E_0 : C \to C_0$ such that $E_0(abc) = aE_0(b)c$ for any $a, c \in C_0, b \in C$, $E_0(a)^* = E_0(a^*)$, $E_0(a)^* E_0(a) \leq E_0(a^*a)$ and $\phi \circ E_0 = \phi$.

The proof of this lemma is very similar to the proof of the correspoding C^*-algebraic statement, as can be found in Corollary II.6.10.8 of [16] and is left to the reader.

So, since $E_{G/H}(\mathrm{Pol}(G)) = \mathrm{Pol}(G)_{G/H}$, it is easily checked that the maps $E_{G/H}$ and $\alpha^{-1} \circ E_{G/H} \circ \alpha$ both satisfy the hypothesis of the lemma with $\phi = h_G$, we have now by the lemma that for any $a \in \mathrm{Pol}(G)$, $E_{G/H}(a) = \alpha^{-1} \circ E_{G/H} \circ \alpha(a)$. Hence, $\rho(\alpha(E_{G/H}(a))) = \rho(E_{G/H}(\alpha(a))) > 0$. But since $\rho(a) = 0$, we have that $\rho(E_{G/H}(a)) = 0$ and as $E_{G/H}(a) \in \mathrm{Pol}(G)_{G/H}$, we get a contradiction to the first part of the proof. Hence, we are done. □

Theorem 2.2.6. *Let G be a compact quantum group, $\alpha \in \mathrm{Aut}_\chi(G)$ and N be a normal subgroup of it, with $\rho : C_m(G) \to C_m(N)$ the corresponding surjection. Then there exists a $\beta \in \mathrm{Aut}(N)$ such that*

$$\begin{tikzcd} C_m(G) \arrow[r, "\alpha"] \arrow[d, "\rho"] & C_m(G) \arrow[d, "\rho"] \\ C_m(N) \arrow[r, "\beta"] & C_m(N) \end{tikzcd}$$

commutes.

Proof. This follows from the previous proposition and from the fact that as $\alpha \in \mathrm{Aut}_\chi(G)$, $\alpha : C_m(G/N) \to C_m(G/N)$ is a C^*-algebraic automorphism.

□

2.3 Inner Automorphisms and Normal Subgroups II

We start this section by giving a recipe to produce representation class preserving automorphism.

Definition 2.3.1. *For a locally compact group K, we denote the group of one dimensional representations of K by $Sp(K)$. It is not hard to see that $Sp(K)$ is the Pontryagin dual of the abelianisation $K/[K, K]$ of K, where $[K, K]$ denotes the commutator subgroup of*

K. Similarly, for a C^*-algebra A, we denote the set of one dimensional $*$-representations by $Sp(A)$.

Example 2.3.2. Let G be a compact quantum group and write $\chi(G) := \mathrm{Sp}(C_m(G))$. It is a group with the product defined by $gh = (g \otimes h) \circ \Delta$, for $g, h \in \chi(G)$. The unit of $\chi(G)$ is the counit $\epsilon_G \in C_m(G)^*$ and the inverse of $g \in \chi(G)$ is given by $g \circ S_G$, where S_G is the antipode on $C_m(G)$. Viewing $\chi(G)$ as a closed subset of the unit ball of $C_m(G)^*$, one can consider the weak* topology on $\chi(G)$ which turns $\chi(G)$ to a compact group. It is easy to see that $\chi(G)$ is in fact a subgroup of G, the surjective map is the obvious evaluation map from $C_m(G)$,

$$\rho : C_m(G) \to C(\chi(G))$$

$$a \mapsto e_a$$

where $e_a(f) = f(a)$. It is also not hard to see that $\chi(G)$ is the maximal classical compact subgroup of G.

Let Γ be a discrete group and $C^*(\Gamma)$ be the full group C^*-algebra of Γ. Then for this co-commutative compact quantum group, the maximal classical compact group is $C^*(\Gamma/[\Gamma, \Gamma])$, i.e. the full group C^*-algebra of the abelianisation of Γ. In case of $SU_q(2)$ with $-1 < q < 1$, the maximal classical compact group is S^1, the circle group [75], while in the case of $A_u(n)$, the maximal classical compact group is the unitary group $U(n)$ [87].

Let G be a compact quantum group and define, for all $g \in \chi(G)$, the map $\alpha_g = (g^{-1} \otimes \mathrm{id} \otimes g) \circ \Delta^{(2)}$. It defines a continuous group homomorphism $\chi(G) \ni g \mapsto \alpha_g \in \mathrm{Aut}(G)$. It is not hard to see that the action of $\chi(G)$ on $\mathrm{Irr}(G)$ is trivial. Indeed, for $g \in \chi(G)$ and $x \in \mathrm{Irr}(G)$ a straightforward computation gives $(\mathrm{id} \otimes \alpha_g)(u^x) = (V_g^* \otimes 1)u^x(V_g \otimes 1)$, where $V_g = (\mathrm{id} \otimes g)(u^x)$. This class of automorphisms were first defined by Wang in [88].

We now give two examples for compact quantum groups, for which there exist non-normal subgroups stabilised by each representation class preserving automorphism:

2.3.1 The $SU_q(2)$ case

We show that such is the case for $SU_q(2)$, $-1 < q < 1, q \neq 0$. We refer to the book [81] for more details on the compact quantum group $SU_q(2)$. We note first that since $SU_q(2)$ has a unique irreducible representation class for a given dimension, any quantum group automorphism is in fact in $\mathrm{Aut}_\chi(SU_q(2))$, i.e. $\mathrm{Aut}(SU_q(2)) = \mathrm{Aut}_\chi(SU_q(2))$. For $SU_q(2)$, the generating unitary representation, denoted by u, is the 2×2 matrix

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

Now, let $\tau : C(SU_q(2)) \to C(SU_q(2))$ be an automorphism of $SU_q(2)$. Then, since the irreducible representations u and $\tau(u)$ are in the same representation class, we have for some $((\tau_{ij})) \in U(2)$

$$\tau(u) = (((\tau_{ij})) \otimes 1)u(((\overline{\tau_{ji}})) \otimes 1)$$

which tells us that

$$\alpha \mapsto \tau_{11}\overline{\tau_{11}}\alpha - \tau_{11}\overline{\tau_{12}}q\gamma^* + \tau_{12}\overline{\tau_{11}}\gamma + \tau_{12}\overline{\tau_{12}}\alpha^*$$

$$-q\gamma^* \mapsto \tau_{11}\overline{\tau_{21}}\alpha - \tau_{11}\overline{\tau_{22}}q\gamma^* + \tau_{12}\overline{\tau_{21}}\gamma + \tau_{12}\overline{\tau_{22}}\alpha^*$$

$$\gamma \mapsto \tau_{21}\overline{\tau_{11}}\alpha - \tau_{21}\overline{\tau_{12}}q\gamma^* + \tau_{22}\overline{\tau_{11}}\gamma + \tau_{22}\overline{\tau_{12}}\alpha^*$$

$$\alpha^* \mapsto \tau_{21}\overline{\tau_{21}}\alpha - \tau_{21}\overline{\tau_{22}}q\gamma^* + \tau_{22}\overline{\tau_{21}}\gamma + \tau_{22}\overline{\tau_{22}}\alpha^*$$

Since τ is $*$-preserving and $\alpha, \gamma, -q\gamma^*, \alpha^*$ are linearly independent, we get,

$$|\tau_{11}|^2 = |\tau_{22}|^2$$

$$q^2\overline{\tau_{21}}\tau_{12} = \overline{\tau_{21}}\tau_{12}$$

But as $0 < q^2 < 1$, we have $\tau_{21} = \tau_{12} = 0$ and $\tau_{11}, \tau_{22} \in S^1$, the circle group. So,

$$\alpha \mapsto \alpha$$

$$\gamma \mapsto \tau_{22}\overline{\tau_{11}}\gamma$$

But $C(SU_q(2))$ is the universal C^*-algebra generated α, γ such that

$$\alpha^*\alpha + \gamma^*\gamma = 1,\ \alpha\alpha^* + q^2\gamma\gamma^* = 1$$

$$\alpha\gamma = q\gamma\alpha,\ \alpha\gamma^* = q\gamma^*\alpha,\ \gamma\gamma^* = \gamma^*\gamma$$

We see that α and $\tau_{22}\overline{\tau_{11}}\gamma$ also satisfy these relations, which implies that indeed we have an automorphism

$$\tau : C(SU_q(2)) \to C(SU_q(2))$$

$$\alpha \mapsto \alpha$$

$$\gamma \mapsto \tau_{22}\overline{\tau}_{11}\gamma$$

where $\tau_{11}, \tau_{22} \in S^1$, the circle group. It is easily checked that this automorphism is also a quantum group automorphism by verifying the relation on the generators.

However, since $\tau_{22}\overline{\tau_{11}} \in S^1$, there exists some $\kappa \in S^1$ such that $\overline{\kappa^2} = \tau_{22}\overline{\tau_{11}}$ and so, by [88], we get that the same automorphism is induced by the matrix

$$\begin{bmatrix} \kappa & 0 \\ 0 & \overline{\kappa} \end{bmatrix}$$

and this comes from the induced automorphism of the maximal compact group S^1 of $SU_q(2)$. But, for these automorphisms, as they are induced from the quantum subgroup S^1 of $SU_q(2)$, we then have the following:

Theorem 2.3.3. *For any $\alpha \in \mathrm{Aut}(SU_q(2)) = \mathrm{Aut}_\chi(SU_q(2))$, with $0 < q^2 < 1$, we have* *the following commutative diagram-*

$$\begin{tikzcd} C(SU_q(2)) \arrow[r, "\alpha"] \arrow[d, "\rho"'] & C(SU_q(2)) \arrow[d, "\rho"] \\ C(S^1) \arrow[r, "id"'] & C(S^1) \end{tikzcd}$$

2.3.2 The $A_u(n)$ case

We now consider the case of the compact quantum group $A_u(n)$. Let us first note that the C^*-algebra $C_m(A_u(n))$ is defined as the universal C^*-algebra generated u_{ij}, $i, j \in 1, 2, ..., n$ such that the matrices $u = ((u_{ij}))$ and $\bar{u} = ((u_{ij}^*))$ are both unitary.

Now, in case of $A_u(n)$, we have a surjective homomorphism

$$\phi : U(n) \to \mathrm{Aut}_\chi(A_u(n))$$

$$t \mapsto \phi_t$$

where $\phi_t : C_m(A_u(n)) \to C_m(A_u(n))$ is defined by the property that

$$\phi_t(u) = (t \otimes 1)u(t^* \otimes 1)$$

where $u = ((u_{ij})) \in M_n(C_m(A_u(n)))$ is the fundamental unitary. This, by the universal property of $C_m(A_u(n))$, extends to an automorphism of $C_m(A_u(n))$. Now, for any $\alpha \in \mathrm{Aut}_\chi(A_u(n))$, there exists some $t \in U(n)$ such that $\alpha(u) = (t \otimes 1)u(t^* \otimes 1)$. Hence, $\alpha = \phi_t$, which shows that the map ϕ is surjective.

Theorem 2.3.4. *For any $t \in U(n)$, the following diagram commutes-*

$$\begin{tikzcd} C_m(A_u(n)) \arrow[r, "\phi_t"] \arrow[d, "\rho"'] & C_m(A_u(n)) \arrow[d, "\rho"] \\ C(U(n)) \arrow[r, "\psi_t"'] & C(U(n)) \end{tikzcd}$$

where ρ denotes the canonical surjection onto $C(U(n))$, the algebra of complex-valued functions of the group $U(n)$, which is the maximal compact subgroup of $A_u(n)$ and ψ_t

denotes the automorphism of $C(U(n))$, induced by the inner automorphism

$$\beta_t : U(n) \to U(n), \quad s \mapsto tst^*$$

Proof. This is easily shown by checking the relation on u_{ij}'s, the matrix entries of u, the fundamental unitary of $A_u(n)$. □

We have the following corollary, which follows easily as the diagram of the previous proposition commutes.

Corollary 2.3.5. *For the homomorphism $\phi : U(n) \to \mathrm{Aut}_\chi(A_u(n))$, we have*

$$\ker \phi = \mathrm{Z}(U(n)) = \{\lambda I \ : \ \lambda \in S^1\}$$

Also, each non-trivial ϕ_t is a C^*-algebraic outer automorphism of $C_m(A_u(n))$.

We now want to show that $U(n)$ is not normal in $A_u(n)$.

We first prove the following lemma, which should be well known to experts, but we nonetheless sketch a quick proof.

Lemma 2.3.6. *Let $\mathbb{G} = (A, \Delta)$ be a CQG and $\mathbb{H} = (B, \Psi)$ a quantum subgroup of $\mathbb{G}$, with $\rho : A \to B$ the associated subgroup surjection. Then given any irreducible represenation $v^\gamma = ((v^\gamma_{ij})) \in M_n(B)$ of $\mathbb{H}$, there exists an irreducible representation $u^\beta = ((u^\beta_{kl})) \in M_m(A)$ of $\mathbb{G}$, such that v^γ is a subrepresentation of the representation $((\rho(u^\beta_{kl})))$ (which we denote by $\rho(u^\beta)$) of $\mathbb{H}$.*

Proof. Suppose not, i.e. suppose there exists v^γ, an irreducible representation of $\mathbb{H}$, such that it is not a subrepresentation of $\rho(u^\gamma)$ for any irreducible representation u^β of $\mathbb{G}$. Then, it follows from [94] (see Equation 5.9), that

$$h_{\mathbb{H}}((v^\gamma_{ij})^* \rho(u^\beta_{kl})) = 0$$

for all i, j, k, l and for all $\beta \in \mathrm{Irr}(\mathbb{G})$. But as $\rho(\mathrm{Pol}(\mathbb{G}))$ is dense in B ($\mathrm{Pol}(\mathbb{G})$ denotes the canonical dense hopf-$*$-algebra of $\mathbb{G}$), it is then easy to see that $h_{\mathbb{H}}((v_{ij}^{\gamma})^* v_{ij}^{\gamma}) = 0$, for all i, j. But this is a contradiction as $h_{\mathbb{H}}$ is faithful on $\mathcal{B}$, the canonical dense hopf-$*$-algebra of $\mathbb{H}$. Hence, we are done. □

Lemma 2.3.7. *Let G be a compact quantum group with subgroups N_1 and N_2. Suppose that N_2 is a subgroup of N_1 and that N_2 is normal in G. Then, N_2 is normal in N_1.*

Proof. This follows easily from the previous lemma. □

The following lemma generalises the well known Third Isomorphism theorem for groups.

Lemma 2.3.8. *Let G, N and H be compact quantum groups such that H is a subgroup of N and N is a subgroup of G, so H is also a subgroup of G. Suppose further that N and H are normal in G. Then the compact quantum group G/H has N/H as a normal subgroup with the quotient being G/N.*

Proof. It follows from the previous lemma that H is normal in N. We have the three corresponding surjections

$$\rho_0 : C_m(G) \to C_m(H)$$

$$\rho_1 : C_m(G) \to C_m(N)$$

$$\rho_2 : C_m(N) \to C_m(H)$$

such that $\rho_0 = \rho_2 \circ \rho_1$.

Now, $\rho_1(C_m(G/H)) = C_m(N/H)$ as if $a \in C_m(G/H)$ then

$$\begin{aligned}(\mathrm{id} \otimes \rho_2)\Delta_N(\rho_1(a)) &= (\rho_1 \otimes \rho_2 \circ \rho_1)\Delta_G(a) \\ &= (\rho_1 \otimes \mathrm{id})(\mathrm{id} \otimes \rho_0)\Delta_G(a) \\ &= \rho_1(a) \otimes 1\end{aligned}$$

and so $\rho_1(C_m(G/H)) \subseteq C_m(N/H)$. But then, if $b \in C_m(N/H)$, and if $x \in C_m(G)$ is such that $\rho_1(x) = b$, then for $E_{G/H}(x) = (\mathrm{id} \otimes h_H \circ \rho_0)\Delta_G(a)$, we have

$$\begin{aligned}\rho_1(E_{G/H}(x)) &= (\mathrm{id} \otimes h_H \circ \rho_2)(\rho_1 \otimes \rho_1)\Delta_G(x)\\ &= (\mathrm{id} \otimes h_H \circ \rho_2)\Delta_N(\rho_1(x)) = b\end{aligned}$$

So, we have that N/H is indeed a subgroup of G/H with the surjection being the map ρ_1 restricted to $C_m(G/H)$. As $C_m(G/N) \subseteq C_m(G/H)$, it is easily seen that N/H is normal in G/H, and that $(G/H)/(N/H) = G/N$. □

Theorem 2.3.9. *The subgroup $U(n)$ is not normal in $A_u(n)$.*

Proof. Observe that S^1 is normal in $U(n)$, and as shown in the proof of Proposition 4.5 of [91], S^1 is a normal subgroup of $A_u(n)$. If $U(n)$ is normal in $A_u(n)$, then by the previous lemma, $U(n)/S^1$ is a normal subgroup of $A_u(n)/S^1$. But this contradicts Theorem 1 of [24]. □

Remark 2.3.10. *The previous theorem is also easily shown by direct calculation.*

We end this section by computing the group

$$\mathrm{Out}_\chi(A_u(n)) = \mathrm{Aut}(A_u(n))/\operatorname{Aut}_\chi(A_u(n))$$

Proposition 2.3.11. *$Aut(A_u(n))/Aut_\chi(A_u(n)) = \mathbb{Z}_2$*

Proof. Since $A_u(n)$ has exactly 2 irreducible representation of dimension n, and the automorphism γ defined by

$$u = (u_{ij}) \mapsto \overline{u} = (u_{ij}^*)$$

is a quantum group automorphism, the proposition follows from the fact that any automorphism not in $Aut_\chi(A_u(n))$ will map

$$u \mapsto (t \otimes 1)\overline{u}(t^* \otimes 1), t \in U(n)$$

, which is composition of two quantum group automorphisms

$$u \xrightarrow{\gamma} \overline{u} \xrightarrow{\gamma_t} t\overline{u}t^*$$

But γ_t is in $Aut_\chi(A_u(n))$ and hence the result follows. □

2.4 Central Subgroup and Center

In case of compact groups, we have that $\mathrm{Inn}(G) = G/Z(G)$, where $Z(G)$ denotes the center of G. In this section, we try to identify the center of a given compact quantum group (see also [25] for an alternative approach).

Definition 2.4.1 ([91])**.** *A subgroup H of a compact quantum group G is said to be central if $(\rho\otimes\mathrm{id})\Delta_G = (\rho\otimes\mathrm{id})\sigma\Delta_G$, where $\rho : C_m(G) \to C_m(H)$ denotes the corresponding surjection and σ the flip map on $C_m(G) \otimes C_m(G)$.*

Proposition 2.4.2. *Let H be a central subgroup of compact quantum group G. Then H is co-commutative.*

Proof. We have that to show that $\Delta_H(a) = \sigma\Delta_H(a)$ for all $a \in C_m(H)$.

Let $s \in C_m(G)$ such that $\rho(s) = a$, then we have

$$\begin{aligned}
\Delta_H(\rho(s)) &= (\rho\otimes\rho)\Delta_G(s) \\
&= (\mathrm{id}\otimes\rho)(\rho\otimes\mathrm{id})\Delta_G(s) \\
&= (\mathrm{id}\otimes\rho)(\rho\otimes\mathrm{id})\sigma\Delta_G(s) \\
\sigma(\rho\otimes\rho)\Delta_G(s) &= \sigma\Delta_H(\rho(s))
\end{aligned}$$

Hence, we are done. □

It follows easily from the definitions that central subgroups of compact quantum groups are always normal.

Theorem 2.4.3. *Let H be a subgroup of a compact quantum group G. Then H is a central subgroup of G if and only if given any irreducible representation u^τ of G, there exists a unique 1-dimensional representation λ_n of H such that u^τ restricted to H decomposes as a direct sum of d_τ copies of λ_n, where d_τ denotes the dimension of u_τ. In other words,*

$$(u^\tau_{|H}, \lambda_n) = d_\tau$$

Proof. ($\Leftarrow$) This is easily seen by a straightforward calculation as

$$(\rho \otimes \mathrm{id})\Delta_G(u^\tau_{ij}) = (\rho \otimes \mathrm{id})\sigma\Delta_G(u^\tau_{ij})$$

for matrix elements of all irreducible representations.

($\Rightarrow$) H is central, so by the previous lemma, H is co-commutative. But then any irreducible representation (u^τ) when restricted to H decomposes as a sum of 1-dimensional representations of H. By unitary equivalence, we may assume that $\rho(u^\tau_{ij}) = 0$ if $i \neq j$ and $\rho(u^\tau_{11}) = \lambda_1$ and $\rho(u^\tau_{ii}) = \lambda_i$ for some $i \neq 1$. We want to show that $\lambda_1 = \lambda_i$. Now,

$$\begin{aligned}(\rho \otimes \mathrm{id})\Delta_G(u^\tau_{1i}) &= (\rho \otimes \mathrm{id}) \sum_k u^\tau_{1k} \otimes u^\tau_{ki} \\ &= \sum_k \rho(u^\tau_{1k}) \otimes u^\tau_{ki} = \lambda_1 \otimes u^\tau_{1i}\end{aligned}$$

and

$$\begin{aligned}(\rho \otimes \mathrm{id})\sigma\Delta_G(u^\tau_{1i}) &= (\rho \otimes \mathrm{id}) \sum_k u^\tau_{ki} \otimes u^\tau_{1k} \\ &= \sum_k \rho(u^\tau_{ki}) \otimes u^\tau_{1k} = \lambda_i \otimes u^\tau_{1i}\end{aligned}$$

and hence, $\lambda_1 = \lambda_i$. □

Corollary 2.4.4. *Suppose G is a compact quantum group and H is a central subgroup of it. Suppose $\alpha \in \mathrm{Aut}_\chi(G)$. Then,*

$$\begin{tikzcd}
C_m(G) \arrow[r, "\alpha"] \arrow[d, "\rho"'] & C_m(G) \arrow[d, "\rho"] \\
C_m(H) \arrow[r, "id"'] & C_m(H)
\end{tikzcd}$$

commutes.

Proof. Since $\alpha \in \mathrm{Aut}_\chi(G)$, for any irreducible representation u^τ of G, with dimension d_τ,

$$\alpha(u^\tau) = (t \otimes 1)u(t^* \otimes 1)$$

for some t in $U(d_\tau)$. The result now follows by direct calculation, using the previous theorem and assuming, by unitary equivalence, that given an irreducible representation u^τ:

$$\rho(u^\tau_{ij}) = \begin{cases} 0 & \text{if } i \neq j \\ \lambda^\tau & \text{if } i = j \end{cases}$$

where λ^τ is a 1-dimensional representation of H. $\square$

Proposition 2.4.5. *Let G be a compact quantum group and let $\chi(G)$ be its maximal classical compact subgroup. Let $X_0 := \{s \in \chi(G) : \alpha_s = id$ on $C_m(G)\}$, i.e. the set of 1-dimensional $*-$representations of $C_m(G)$ such that the associated induced inner automorphisms of G is trivial. Then X_0 is a closed subgroup of $\chi(G)$ and is a central subgroup of G.*

Proof. Its easy to check that X_0 is a subgroup of $\chi(G)$.

Now since

$$\begin{aligned}
& \alpha_s = \lambda_s \rho_s \\
\Rightarrow\quad & \alpha_s = \mathrm{id} \Leftrightarrow \lambda_{s^{-1}} = \rho_s \\
\Rightarrow\quad & (s \otimes \mathrm{id})(\rho \otimes \mathrm{id})\Delta_G = (\mathrm{id} \otimes s)(\mathrm{id} \otimes \rho)\Delta_G, \forall s \in X_0 \\
\Rightarrow\quad & (s \otimes \mathrm{id})(\rho \otimes \mathrm{id})\Delta_G = (s \otimes \mathrm{id})(\rho \otimes \mathrm{id})\sigma\Delta_G \\
\Rightarrow\quad & (\rho \otimes \mathrm{id})\Delta_G = (\rho \otimes \mathrm{id})\sigma\Delta_G
\end{aligned}$$

and hence, X_0 is central in G. $\square$

Let G be a compact quantum group, and consider the set $\mathrm{Irr}(G)$. Following [64], we give the following

Definition 2.4.6. *Let $\Sigma \subseteq \mathrm{Irr}(G)$ be a subobject. For $a, b \in \mathrm{Irr}(G)$, we define $a \sim b$ if and only if $a \times \bar{b} \cap \Sigma \neq \phi$. Here $\bar{b}$ denotes the conjugate of b and $a \times b$ denotes the set of all elements of $\mathrm{Irr}(G)$ representatives of which are subrepresentations of $u^a \otimes u^b$ where u^a, u^b are irreducible representations of G and $[u^a] = a$ and $[u^b] = b$.*

Using Propn 3.2 of [76], it is easy to check that this defines an equivalence relation. We call the equivalence classes of this relation as $\Sigma-$cosets.

Given, two $\Sigma-$cosets, $\mathcal{A}$ and $\mathcal{B}$, we define the product set $\mathcal{A} \times \mathcal{B} := \{c \in \mathrm{Irr}(G) : c \subset a \times b\}$. Obviously, we have that the set-wise product of two $\Sigma-$ cosets is a union of $\Sigma-$cosets.

Definition 2.4.7. *We say a subobject $\Sigma \subseteq \mathrm{Irr}(G)$ is a central subobject if the $\Sigma-$cosets form a group. In this case, the product of two $\Sigma-$cosets is itself a $\Sigma-$coset. Σ, which is itself a $\Sigma-$coset, acts as the identity element. The group is denoted as $\mathrm{Irr}(G)/\Sigma$.*

Proposition 2.4.8. *Let G be a compact quantum group and H a normal subgroup. Let Σ denote the subobject of $\mathrm{Irr}(G)$ corresponding to equivalence classes of irreducible representations of G that decompose as direct sum of trivial representation when restricted to H. Then H is central if and only if $\Sigma-$cosets form a group.*

Proof. ($\Rightarrow$) Let H be central. Since H is cocommutative, we know that $\mathrm{Irr}(H)$ is a discrete group. By Theorem 2.4.3, we know that for any irreducible representation u^τ of G, there exists a unique 1-dimensional representation $\phi^\tau \in \mathrm{Irr}(H)$ such that $(u^\tau_{|H}, \phi^\tau) = d_\tau$, where d_τ denotes the dimension of u^τ. We consider the following map-

$$\pi : \mathrm{Irr}(G) \mapsto \mathrm{Irr}(H)$$

$$[u^\tau] \mapsto \phi^\tau$$

where $[u^\tau]$ denotes the equivalence class of u^τ in $\mathrm{Irr}(G)$. The map π is then easily checked to be multiplicative, in the sense that if we take two irreducible representations of G, u^α and u^β, then

$$\sigma \in [u^\alpha] \times [u^\beta] \mapsto \phi^\alpha \cdot \phi^\beta$$

Now, Σ consists of $[u^\tau]$ such that $\pi([u^\tau]) = 1_H$, so this implies that

$$\alpha \sim \beta \Leftrightarrow \pi(u^\alpha) = \pi(u^\beta)$$

and hence, cosets correspond to preimages of elements of the group $\mathrm{Irr}(H)$ and since π is multiplicative, we have that $\mathrm{Irr}(G)/\Sigma$ is indeed a group.
($\Leftarrow$) Let H not be central, then by Theorem 2.4.3, there exists some irreducible representation u^σ of G, with $[u^\sigma] \in \mathrm{Irr}(G)$ denoting its equivalence class, such that $u^\sigma_{|H} = \oplus_{i=1}^n \xi_i$ upto equivalence, where either for some i, $\dim(\xi_i) > 1$ or, in case all ξ_i have dimension 1, for some $i \neq j, \xi_i \neq \xi_j$, where ξ_i's are irreducible representations of H. Then we have that

$$u^\sigma \otimes \bar{u^\sigma}_{|H} = \oplus_{i,j=1}^n (\xi_i \otimes \bar{\xi_j})$$

upto equivalence. But then, in either of the aforementioned cases, 1_H and some other non-trivial representation appear in the decomposition of $(u^\sigma \otimes \bar{u^\sigma})_{|H}$ into irreducible representations of H. And so if we let $[\sigma]$ denote the Σ-coset corresponding to $[u^\sigma]$ and $[\bar{\sigma}]$ the Σ-coset corresponding to $[\bar{u^\sigma}]$, $[\sigma] \times [\bar{\sigma}]$ is a union of more than one coset, which gives a contradiction. $\square$

Proposition 2.4.9. *Let $\mathcal{Z} := \{\Sigma \subseteq \mathrm{Irr}(G) : \Sigma$ is a central subobject $\}$. Let*

$$\widetilde{\Sigma} := \cap_{\Sigma \in \mathcal{Z}} \Sigma$$

Then $\widetilde{\Sigma} \in \mathcal{Z}$

Proof. Let $[a]$ and $[b]$ be two $\widetilde{\Sigma}-$cosets, we want to show that $[a] \times [b]$ is also a $\widetilde{\Sigma}-$coset.

This follows easily from the following two facts -:

(i) If a and a_1 belong to the same $\widetilde{\Sigma}-$coset, then they belong to the same $\Sigma-$coset for all $\Sigma \in Z$, obvious as $\widetilde{\Sigma} \subseteq \Sigma$ for all $\Sigma \in \mathcal{Z}$.

(ii) If a and a_1 belong to the same $\Sigma-$coset for every $\Sigma \in \mathcal{Z}$ then a and a_1 belong to the same $\widetilde{\Sigma}-$coset. This is true because for $\Sigma \in \mathcal{Z}$, the $\Sigma-$cosets form a group. Now, since a and a_1 belong to the same $\Sigma-$coset, we have that

$$a \times \bar{a_1} \cap \Sigma \neq \phi$$

But then as $\Sigma-$cosets form a group, we have that $[a] \times [\bar{a_1}] = \Sigma$, where $[a]$ denotes the Σ-coset corresponding to a. This implies that for any $\sigma \in a \times \bar{a_1}, \sigma \in \Sigma$ for all $\Sigma \in \mathcal{Z}$, so $\sigma \in \widetilde{\Sigma}$ and hence, a and a_1 are in the same $\widetilde{\Sigma}-$coset. $\square$

We call $\widetilde{\Sigma} \subseteq \mathrm{Irr}(G)$ the center subobject of G. This corresponds to the center of the compact quantum group G. The subalgebra in $C_m(G)$ generated by $\widetilde{\Sigma}$ gives us the underlying C^*-algebra of the quotient quantum group $G/Z(G)$.

Let G be a compact quantum group. Let $\Sigma \subseteq \mathrm{Irr}(G)$ be a central object and $H_\Sigma = \mathrm{Irr}(G)/\Sigma$ be the group of Σ-cosets. We then have:

Proposition 2.4.10. *$C^*(H_\Sigma)$ is a central subgroup of G, with Σ being the subobject of $\mathrm{Irr}(G)$ cosisting of equivalence classes of irreducible representations of G that decompose as a sum of trivial representation when restricted to it, so its left and equivalently right, coset space is generated by matrix entries of the representatives of elements of Σ as a subalgebra of $C_m(G)$.*

Proof. We refer to [95] for definitions of unexplained terms.

We have for the compact quantum group G, a triple

$$(R_G, \{\mathcal{H}_\alpha\}_{\alpha \in R_G}, \{\mathrm{Mor}(\alpha, \beta)\}_{\alpha,\beta \in R_G})$$

By Proposition 3.3 of [95] for any model , $(M, \{V_r\}_{r \in R_G})$, there exists a $*$-homomorphism

$$\varphi_M : \mathrm{Pol}(G) \to M$$

which extends to $C_m(G)$ (using universal property of $C_m(G)$).

We want to give a model with $M = C^*(H_\Sigma)$. First, let us take a complete set of irreducible representations $\{\alpha_k\}_{k \in I}$ of G. So, α_k's are representatives of elements of $\mathrm{Irr}(G)$ and belong to R_G. And, they are complete in the sense that, the map

$$\kappa : \{\alpha_k\}_{k \in I} \subseteq R_G \to \mathrm{Irr}(G)$$

$$\alpha_k \mapsto [\alpha_k]$$

is one-one and onto.

For these $\alpha_k \in R_G$ such that $\alpha_k \in B(\mathcal{H}_k) \otimes C_m(G)$, we define

$$V_{\alpha_k} = 1_{\mu_k} \otimes \delta_k \in B(\mathcal{H}_k) \otimes C^*(H_\Sigma)$$

Here, δ_k is the element of the group H_Σ corresponding to the Σ-coset that contains $[\alpha_k]$. Now, for any $r \in R_G$, we know that there exists a finite set $\{\alpha_{k_1}, \dots, \alpha_{k_n}\} \subseteq \{\alpha_k\}_{k \in I}$ such that there exist $P_{k_i} \in \mathrm{Mor}(\alpha_{k_i}, r)$ with the properties that:

$$\sum P_{k_1} P_{k_i}^* = I_{\mathcal{H}_r}$$
$$P_{k_i}^* P_{k_i} = I_{\mathcal{H}_k}$$
$$\text{and } P_{k_i} P_{k_i}^* P_{k_j} P_{k_j}^* = 0 \text{ for all } i \neq j \in \{1, \dots, n\}$$

We then define $V_r \in B(\mathcal{H}_r) \otimes C^*(H_\Sigma)$,

$$V_r := \sum_i P_{k_i} P_{k_i}^* \otimes \delta_{k_i}.$$

It is then straightforward to check that:

$$V_r \otimes V_s = V_{rs}$$

and

$$V_r(t \otimes 1) = (t \otimes 1)V_s$$

for all $r, s \in R_G, t \in \mathrm{Mor}(s, r)$. So, we have a model for

$$\{R_G, \{\mathcal{H}_\xi\}_{\xi \in R_G}, \mathrm{Mor}(\alpha, \beta)_{\alpha,\beta \in R_G}\}$$

and so a $*$-homomorphism:

$$\varphi_\Sigma : C_m(G) \to C^*(H_\Sigma)$$

It follows easily that φ_Σ is a quantum group homomorphism and also that $C^*(H_\Sigma)$ is a central subgroup of G with $\Sigma \subseteq \mathrm{Irr}(G)$ as the corresponding subobject. □

So, associated to any central subobject $\Sigma \subseteq \mathrm{Irr}(G)$, there exists a central subgroup of G, with the algebra generated by representatives of elements of Σ giving its left/right coset space.

Remark 2.4.11. *Let us note that for $\widetilde{\Sigma}$, we have the group $H_{\widetilde{\Sigma}}$, which can be regarded as the center of the compact quantum group G (more precisely, it is the dual of the center of G, but we will continue with this slight abuse, since we often end up calculating $H_{\widetilde{\Sigma}}$).*

Lemma 2.4.12. *Suppose we have compact quantum groups G, N_1, N_2. Let N_1 and N_2 be normal subgroups of G such that $\Sigma_{N_1} \subseteq \Sigma_{N_2}$ where Σ_{N_i} is the subobject of $\mathrm{Irr}(G)$ corresponding to N_i (and hence, consisting of equivalence classes of irreducible representations* *of G that are direct sums of the trivial representation when restricted to N_i), $i = 1, 2$. Then, N_2 is a normal subgroup of N_1. Further, if the corresponding surjections are denoted*

$$\rho_i : C_m(G) \to C_m(N_i),\ i = 1, 2$$

then, the surjection

$$\rho_0 : C_m(N_1) \to C_m(N_2)$$

satisfies:

$$\rho_2 = \rho_0 \circ \rho_1$$

Proof. We have that $\rho_1|_{\mathrm{Pol}(G)} = \mathrm{Pol}(N_1)$ and $\rho_2|_{\mathrm{Pol}(G)} = \mathrm{Pol}(N_2)$. We define:

$$\rho_0 : \mathrm{Pol}(N_1) \to \mathrm{Pol}(N_2)$$

by

$$\rho_0(\rho_1(a)) = \rho_2(a) \text{ for } a \in \mathrm{Pol}(G)$$

This is well-defined, as $\ker(\rho_1) \subseteq \ker(\rho_2)$, which follows from Lemma 4.4 of [91]. It is then easy to check that ρ_0 is a $*$-homomorphism, so can be extended to $C_m(N_1)$. It is also surjective and is in fact a quantum group homomorphism. Normality follows from Lemma 2.3.7. □

So, it follows that any central subgroup of G is in fact a subgroup of the center of that G.

Let now G be compact quantum group. We consider the set of discrete groups

$$\bar{Z}(G) := \{F : C^*(F) \text{ is a central subgroup of G}\}$$

We say that $F_0 \leq F_1$ if there exists a surjective map

$$\rho_0^1 : F_1 \to F_0$$

such that the induced map $\widetilde{\rho_0^1} : C^*(F_1) \to C^*(F_0)$ has the property that

$$\begin{tikzcd} C_m(G) \arrow[r, "\rho_1"] \arrow[dr, "\rho_0"'] & C^*(F_1) \arrow[d, "\widetilde{\rho_0^1}"] \\ & C^*(F_0) \end{tikzcd}$$

commutes.

This then gives us an inverse system of discrete groups. One can directly show this without appealing to the previous proposition. It is easy to see that if we have $F_0 \leq F_1 \leq F_2$, with respective maps,

$$\rho_0^1 : F_1 \to F_0$$

$$\rho_1^2 : F_2 \to F_1$$

and

$$\rho_0^2 : F_2 \to F_1$$

then

$$\rho_0^2 = \rho_0^1 \circ \rho_1^2$$

Now given F_1 and F_2 in $\bar{Z}(G)$, we want to show that there exists F_0 in $\bar{Z}(G)$ such that $F_1 \leq F_0$ and $F_2 \leq F_0$. We have surjections

$$\rho_1 : C_m(G) \to C^*(F_1)$$

$$\rho_2 : C_m(G) \to C^*(F_2)$$

We consider the following map

$$C_m(G) \xrightarrow{\widetilde{\Delta_G}} C_m(G) \otimes_{\max} C_m(G) \xrightarrow{\rho_1 \otimes_{\max} \rho_2} C^*(F_1) \otimes_{\max} C^*(F_2) \cong C^*(F_1 \times F_2)$$

Here $\widetilde{\Delta_G}$ denotes the extension of the map

$$\Delta_G : \mathrm{Pol}(G) \subseteq C_m(G) \to \mathrm{Pol}(G) \otimes_{\mathrm{alg}} \mathrm{Pol}(G) \subseteq C_m(G) \otimes_{\max} C_m(G)$$

We have to check that $(\rho_1 \otimes_{\max} \rho_2)\widetilde{\Delta_G}$ is a quantum group homomorphism and that it is central. This is easy to check as (F_1, ρ_1) and (F_2, ρ_2) are central subgroups, so it is easily checked for matrix entries of irreducible representations of G.

However, the map need not be surjective. But we can consider its range, which is a cocommutative quantum group. Let

$$\mathrm{Ran}(\rho_1 \otimes_{\max} \rho_2) = C^*(F_0)$$

We now wish to show that $F_0 \geq F_1$ and $F_0 \geq F_2$. Let us show that $F_0 \geq F_1$. So, we want to show that

$$\begin{tikzcd} C_m(G) \arrow[r, "\Psi_0"] \arrow[dr, "\rho_1"'] & C^*(F_0) \arrow[d, "\pi_1"] \\ & C^*(F_1) \end{tikzcd}$$

commutes. Here,

$$\Psi_0 = (\rho_1 \otimes_{\max} \rho_2)\widetilde{\Delta_G}$$

and

$$\pi_1 : C^*(F_0) \to C^*(F_1)$$

is the restriction to $C^*(F_0)$ of the map

$$\widetilde{\pi_1} : C^*(F_1 \times F_2) \to C^*(F_1)$$

$$\delta_{g_1} \otimes \delta_{g_2} \mapsto \delta_{g_1}$$

Let (u^τ_{kl}) be matrix entries of some arbitrary irreducible representation of G such that

$$\rho_2(u^\tau_{kl}) = \begin{cases} \delta_g & \text{if } k = l, g \in F_2 \\ 0 & \text{if } k \neq l \end{cases}$$

This is possible as F_2 is a central subgroup of G, invoking Theorem 2.4.3. Then we have

$$\pi_1(\rho_1 \otimes \rho_2)\Delta_G(u^{\tau}_{kl}) = \pi_1(\rho_1 \otimes \rho_2)\sum_j u^{\tau}_{kj} \otimes u^{\tau}_{jl} = \pi_1(\rho_1(u^{\tau}_{kl}) \otimes \delta_g)$$

But since

$$\rho_1(u^{\tau}_{kl}) = \lambda \cdot \delta_{g_1}$$

for some $g_1 \in F_1$ and $\lambda \in \mathbb{C}$, we have that

$$\pi_1(\rho_1(u^{\tau}_{kl}) \otimes \delta_g) = \rho_1(u^{\tau}_{kl})$$

and so we have $F_0 \geq F_1$ and similarly, $F_0 \geq F_2$.

So, we have an inverse directed system and taking inverse limit, we get a discrete group $F = \varprojlim F_i$. This is of course the center of the compact quantum group G, as can be seen using Proposition 2.4.10.

2.5 Center Calculations

We calculate the center of some compact quantum groups. The following theorem is helpful in many cases:

Theorem 2.5.1. *Compact quantum groups having identical fusion rules have isomorphic center.*

Proof. Let G_1 and G_2 be two compact quantum groups with $\mathrm{Irr}(G_1)$ and $\mathrm{Irr}(G_2)$, the set of equivalence classes of irreducible representations respectively. Then, as G_1 and G_2 have identical fusion rules, hence, there exists a bijection:

$$p : \mathrm{Irr}(G_1) \to \mathrm{Irr}(G_2)$$

such that for all $v, s \in \mathrm{Irr}(G_1)$, we have,

$$p(v \times s) = p(v) \times p(s)$$

as subsets of $\mathrm{Irr}(G_2)$. Thus, $\Sigma \subseteq \mathrm{Irr}(G_1)$ is central if and only if $p(\Sigma) \subseteq \mathrm{Irr}(G_2)$ is central, as follows easily from Proposition 2.4.8. So, centers of G_1 and G_2 are isomorphic. □

Thus, we have the following:

1. The center of $SU_q(2)$ is $\mathbb{Z}_2$ for $-1 \leqslant q \leqslant 1$ and $q \neq 0$.
2. $SO_q(3)$ has trivial center, $-1 \leqslant q \leqslant 1$, and $q \neq 0$.
3. $C^*(\Gamma)$, for any discrete subgroup Γ, has as center Γ.
4. $B_u(Q)$ has the same fusion rules as $SU(2)$ [6] and so its center is $\mathbb{Z}_2$, where Q is a $n \times n$ matrix with $Q\overline{Q} = cI_n'$.
5. For B a finite dimensional C^*-algebra with $\dim(B) \geq 4$ and τ the canonical trace on it, the compact quantum group $A_{\mathrm{aut}}(B, \tau)$ [89] has the same fusion rules as $SO(3)$ [8] and so has trivial center.

Proposition 2.5.2. *Let G be a compact quantum matrix group with u being its fundamental representation. Assume that u is irreducible, then $Z(G)$, the center of G, is either $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{Z}$ or $\mathbb{Z}$.*

Proof. By Theorem 2.4.3, and the fact that u is the fundamental representation and is irreducible, it follows that if $Z(G) = \Gamma$, then, Γ is generated by δ_g where

$$\rho(u) = \underbrace{\delta_g \oplus \cdots \oplus \delta_g}_{\text{dim(u)-times}}$$

where ρ denotes the surjection from G onto its center. Hence, Γ is a quotient of $\mathbb{Z}$. □

Corollary 2.5.3. *The compact quantum group $A_u(Q)$, $Q \in GL_n(\mathbb{C})$ has center $\mathbb{Z}$.*

Proof. It is shown in [91] that $A_u(Q)$ has $C(S^1) = C^*(\mathbb{Z})$ as a central subgroup. The result now follows from the previous lemma, as u, the fundamental representation of $A_u(Q)$, is irreducible. $\square$

The notion of the chain group $c(G)$ for a given compact group G was defined by Baümgartel and Lledo [10] and was shown by Müger in [66] to be isomorphic to $\widehat{Z(G)}$, the dual group of the center $Z(G)$ of G. We prove that the same is true for compact quantum groups.

Definition 2.5.4. *Given a compact quantum group G, we define the chain group*

$$c(G) := \mathrm{Irr}(G)/\sim_1$$

to be the group of equivalence classes of the equivalence relation $\sim_1$ defined for $X, Y \in$ $\mathrm{Irr}(G)$ as follows :

$$X \sim_1 Y$$

if and only if

$$\exists n \in \mathbb{N} \text{ and } z_1, \ldots, z_n \in \mathrm{Irr}(G)$$

such that

$$X \in z_1 \times \cdots \times z_n \text{ and } Y \in z_1 \times \cdots \times z_n$$

Let

$$\mathcal{E} := \{x \in \mathrm{Irr}(G) : x \sim_1 [1_G]\}$$

where $[1_G] \in \mathrm{Irr}(G)$ denotes the equivalence class of 1_G, the trivial representation of G. It is then, straightforward to see that $\mathcal{E}$ is a subobject of $\mathrm{Irr}(G)$.

Proposition 2.5.5. *$\mathcal{E}$ is a central subobject of $\mathrm{Irr}(G)$, and $\mathrm{Irr}(G)/\mathcal{E} \cong \mathrm{Irr}(G)/\sim_1 \cong c(G)$.*

Proof. This will follow if we can show that, for $a, b \in \mathrm{Irr}(G)$, $a \sim b$ in the sense of Definition 2.4.7, with $\Sigma = \mathcal{E}$ if and only if $a \sim_1 b$.

($\Rightarrow$) Let for $a, b \in \mathrm{Irr}(G)$, $a \sim b$ in the sense of Definition 2.4.7. Then we have $a \times \bar{b} \cap \mathcal{E} \neq \varnothing$, so there exists $z_1, \ldots, z_n \in \mathrm{Irr}(G)$ such that for some $k \in a \times \bar{b}$, we have

$$k \in z_1 \times \cdots \times z_n$$

and

$$[1_G] \in z_1 \times \cdots \times z_n$$

This implies

$$[1_G] \in z_1 \times \cdots \times z_n \times a \times \bar{b}$$

and

$$a \times \bar{b} \subseteq z_1 \times \cdots \times z_n \times a \times \bar{b}$$

So, $a \sim_1 b$ as $a \in z_1 \times \cdots \times z_n \times a \times \bar{b} \times b$ and $b \in z_1 \times \cdots \times z_n \times a \times \bar{b} \times b$

($\Leftarrow$) Let $a \sim_1 b$ for $a, b \in \mathrm{Irr}(G)$. Then $a \in z_1 \times \cdots \times z_n$ and $y \in z_1 \times \cdots \times z_n$ for some $z_1, \ldots, z_n \in \mathrm{Irr}(G)$. But then obviously,

$$1 \in z_1 \times \cdots \times z_n \times \bar{b}$$

and

$$a \times \bar{b} \subseteq z_1 \times \cdots \times z_n \times \bar{b}$$

So, we have that $a \sim b$ in the sense of Definition 2.4.7 with $\Sigma = \mathcal{E}$. $\square$

Now to show that the chain group is indeed isomorphic to the center of the compact quantum group G, we have to show

$$\widetilde{\Sigma} = \mathcal{E}$$

as subsets of $\mathrm{Irr}(G)$, where $\widetilde{\Sigma}$ denotes the center subobject of G. But by the previous proposition we have $\widetilde{\Sigma} \subseteq \mathcal{E}$. So, we want to show that $\mathcal{E} \subseteq \widetilde{\Sigma}$ which is true if and only if $\mathcal{E} \subseteq \Sigma$ for any central subobject Σ. This can be easily shown as if $a \in \mathcal{E}$, then there exist $z_1, ..., z_n$ in $\mathrm{Irr}(G)$ such that

$$1 \in z_1 \times z_2 \times ... \times z_n$$

and

$$a \in z_1 \times z_2 \times ... \times z_n$$

But for any central subobject Σ, the Σ-cosets form a group, denoted by H_Σ, by Proposition 2.4.8. Now consider the product

$$[z_1][z_2]...[z_n]$$

in H_Σ. This is the identity element of H_Σ. But then $a \in \Sigma$. So, we indeed have that the chain group of G is isomorphic to the center of G.

Chapter 3

Group Actions on Compact Quantum Groups

In this chapter, we study group actions on compact quantum groups by quantum automorphisms. We will give characterizations as to when this type of an action is ergodic, weak mixing, mixing, etc. We also study constructions and examples of such group actions. We end this chapter by proving a structure theorem for a specific family of compact quantum groups, by showing the existence of a maximal ergodic normal subgroup of the compact quantum group on which a group is acting. Let us note that we often write CQG for compact quantum group and also that we will be dealing with the reduced C^*-algebra of compact quantum groups in this chapter, unless stated otherwise. In this context, let us remind the reader of Remark 1.4.5, and hence that this is not an unreasonable restriction.

3.1 CQG dynamical systems

This section is divided into three subsections. In these subsections, we study ergodicity, weak mixing, mixing and compactness of quantum automorphisms on CQGs. All throughout this chapter $\mathbb{G}$ will denote a CQG in its reduced avatar and Γ will denote a discrete group acting on $\mathbb{G}$ by quantum automorphisms.

Definition 3.1.1. *We say that the tuple* $(\mathbb{G}, \Gamma)$ *is a CQG dynamical system if* Γ *acts on* $C(\mathbb{G})$ *via quantum automorphims. In other words, there exists an homomorphism* $\alpha : \Gamma \to \mathrm{Aut}(\mathbb{G})$.

Let $(\mathbb{G}, \Gamma)$ be a CQG dynamical system. Then, it follows from Prop. 1.4.2(5) that Γ induces an action on $\mathrm{Irr}(\mathbb{G})$, the unitary dual of $\mathbb{G}$. Let $[u] \in \mathrm{Irr}(\mathbb{G})$ denote the equivalence class of an irreducible representation of $\mathbb{G}$. We denote by $\Gamma[u]$, the orbit of $[u]$ under the action of Γ and elements of the orbit by $\gamma \cdot [u]$, $\gamma \in \Gamma$.

Consider the GNS representation of $C(\mathbb{G})$ with respect to the Haar state $h_{\mathbb{G}}$. Write $L^2(\mathbb{G}) = L^2(C(\mathbb{G}), h_{\mathbb{G}})$ with distinguished cyclic vector $\Omega_{h_{\mathbb{G}}} = \widehat{1}$. As a consequence of Prop. 1.4.2, any automorphism α of $\mathbb{G}$ is implemented by the unitary operator $U_\alpha : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ by $U_\alpha(\widehat{a}) = \widehat{\alpha(a)}$, $a \in C(\mathbb{G})$; and this choice is unique on demanding that the vacuum vector be fixed. In the case of a CQG dynamical system, let U_γ, $\gamma \in \Gamma$, denote the unique unitary in $\mathbf{B}((L^2(C(\mathbb{G})))$ implementing the automorphism α_γ. Also note that the GNS representation is faithful as $h_{\mathbb{G}}$ is faithful.

3.1.1 Ergodicity and Weak Mixing

We make the following definitions.

Definition 3.1.2. *We say that the CQG dynamical system* $(\mathbb{G}, \Gamma)$ *has the infinite orbit condition* (i.o.c. *in the sequel), if the orbit of every non-trivial element in Irr($\mathbb{G}$) under the induced action of* Γ *on Irr($\mathbb{G}$) is infinite.*

Definition 3.1.3. *Let* $(\mathbb{G}, \Gamma)$ *be a CQG dynamical system.*

1. Γ *is a said to act ergodically on* $\mathbb{G}$*, if* $\zeta \in L^2(\mathbb{G})$ *and* $U_\gamma \zeta = \zeta$ *for all* $\gamma \in \Gamma$ *forces that* $\zeta = \lambda \Omega_h$ *for some* $\lambda \in \mathbb{C}$.

2. *The action is said to be topologically transitive, if given two nonzero elements* $a, b \in C(\mathbb{G})$*, there exists* $\gamma \in \Gamma$ *such that* $a\alpha_\gamma(b) \neq 0$.

3. *The action is said to be strongly topologically transitive, if given* $(a_i, b_i) \in C(\mathbb{G}) \times C(\mathbb{G})$*,* $1 \leq i \leq n$*, such that* $\sum_{i=1}^n a_i \otimes b_i \neq 0$*, there exists* $\gamma \in \Gamma$ *such that* $\sum_{i=1}^n a_i \alpha_\gamma(b_i) \neq 0$.

4. *The associated action of* Γ *on* $L^\infty(\mathbb{G})$ *is weakly mixing, if given* $\epsilon > 0$ *and* $a_i \in L^\infty(\mathbb{G})$*,* $1 \leq i \leq n$*, there exists* $\gamma \in \Gamma$ *such that*

$$|h_{\mathbb{G}}(\alpha_\gamma(a_i)a_j) - h_{\mathbb{G}}(a_i)h_{\mathbb{G}}(a_j)| < \epsilon \forall i, j.$$

5. *A nonzero vector* $\zeta \in L^2(\mathbb{G})$ *is weakly wandering for the action of* Γ *on* $\mathbb{G}$*, if there exists a sequence of elements* $\{\gamma_i\} \subseteq \Gamma$ *such that* $\langle U_{\gamma_i}\zeta, \zeta\rangle_{h_{\mathbb{G}}} = 0$ *for all* i*. We*

refer to [54] for more on the notion of weakly wandering vectors in case of classical dynamical systems.

It should be noted that *i.o.c.* is a combinatorial property analogous to the requirement in von Neumann algebras of a group being of infinite conjugacy class. In fact, if G is a countable discrete group then G acts on the co-commutative CQG $C_r^*(G)$ by the obvious quantum automorphisms given by inner automorphims. Then the pair $(C_r^*(G), G)$ has *i.o.c.* if and only if G is an i.c.c. group. Ergodicity and weak mixing are spectral theoretic phenomena. On the other side, (strong) topological transitivity is a C^*-algebraic phenomena. But as we show, when the underlying space is a CQG these notions coincide. A result from [18] will be useful for our purpose. We state it here for convenience. We have assumed a stronger definition of ergodicity than the usual requirement that the fixed point algebra of the action is trivial, but the two notions are in fact equivalent.

Theorem 3.1.4. *Let (A, Γ, α) be a C^* dynamical system, where Γ is a countable discrete group and α is the associated action. Suppose the system has a state ω, such that*
(i) ω is α-invariant;
(ii) if $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ denote the associated GNS triple, then Ω_ω is the unique (modulo scaling) fixed vector of the induced action of Γ on $\mathcal{H}_\omega$.
Further, if π_ω is faithful and Ω_ω is separating for $(\pi_\omega(A))''$, then the action is strongly topologically transitive.

The next theorem is a result about rigidity in Ergodic Hierarchy. The first result of such kind was proved by Halmos [41], where it was proved that an ergodic automorphism of a compact abelian group is mixing, in fact the spectral measure of the action is Lebesgue and the spectral multiplicity is infinite. As far as we are aware, the second result of such flavor was proved by Jaksic and Pillet [45], where they show that an ergodic action of $\mathbb{R}$ on a von Neumann algebra is weakly mixing. It can be deduced from the main result in [44] (also see [9] for an independent proof) that, an ergodic action of a locally compact seperable group on a separably acting von Neumann algebra M preserving a prescribed faithful normal state φ is always weak mixing on $M \ominus M^\varphi$, where M^φ denotes the centralizer of the state φ. Such class of results have value in Statistical Physics, in order to recognize the region in the phase space where a dynamics implemented by symmetries of the Hamiltonian is chaotic. Topological transitivity is a counterpart in C^*-dynamics of the notion of ergodicity in W^*-dynamics. This was first studied by Longo and Peligrad in [61] in the non commutative case and then by Bratteli, Eliott and Robinson in [18]; though in the classical case it was first studied by Berend [13]. We prove that in the context of quantum automorphisms all these notions are equivalent to weak mixing. The noncommutativity of the group forces us to compromise with weak mixing instead of mixing as in [41]. When Γ is (torsion-free) more can be said, but for this we refer to the paper [67] for a study of spectral properties of such actions.

Theorem 3.1.5. *Let* $(\mathbb{G}, \Gamma)$ *be a CQG dynamical system. Then, the following are equivalent:*

(i) $(\mathbb{G}, \Gamma)$ *satisfies i.o.c.;*

(ii) Γ *acts ergodically*[1]*;*

(iii) $L^\infty(\mathbb{G})^\Gamma = \{a \in L^\infty(\mathbb{G}) : \alpha_\gamma(a) = a \ \forall \gamma \in \Gamma\} = \mathbb{C}1$*;*

(iv) *the action is topologically transitive;*

(v) *the action is strongly topologically transitive;*

(vi) *the associated action on* $L^\infty(\mathbb{G})$ *is weak mixing;*

(vii) *the linear span of weakly wandering vectors of the representation* $\gamma \mapsto U_\gamma$, $\gamma \in \Gamma$ *are dense in* $L^2(\mathbb{G}) \ominus \mathbb{C}\Omega_{h_\mathbb{G}}$.

Proof. We will prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$, $(ii) \Rightarrow (v)$, $(iv) \Rightarrow (iii)$, $(i) \Rightarrow (vi)$ and $(vii) \Rightarrow (vi)$. Note that $(v) \Rightarrow (iv)$, $(vi) \Rightarrow (ii)$ and $(i) \Rightarrow (vii)$ are trivial.

Note that $L^2(\mathbb{G}) = \mathbb{C}\Omega_{h_\mathbb{G}} \oplus \oplus_{s \in I} \mathcal{H}_s$, where $\mathcal{H}_s = \text{ span } \{u_{i,j}^{(s)} \Omega_{h_\mathbb{G}}\}$ with $u^{(s)} = ((u_{i,j}^{(s)}))$, $s \in I$, being non trivial irreducible representations of $\mathbb{G}$. Clearly, I is countable. For $\gamma \in \Gamma$ and $s \in I$, note that $u^{(\gamma \cdot s)} = ((\alpha_\gamma(u_{i,j}^{(s)})))$ is a irreducible representation of $\mathbb{G}$. Consequently, Γ acts on I by bijections of I. Thus, $U_\gamma : \mathcal{H}_s \to \mathcal{H}_{\gamma \cdot s}$ for all γ and $s \in I$.

[1] (ii) and (iii) are equivalent for any W^*-dynamical system.

$(i) \Rightarrow (ii)$. Let $\zeta \in L^2(\mathbb{G})$ be such that $U_\gamma \zeta = \zeta$ for all $\gamma \in \Gamma$. Fix $s \in I$. Let P_s denote the orthogonal projection from $L^2(\mathbb{G})$ onto $\mathcal{H}_s$. Then $P_s U_\gamma \zeta = P_s \zeta$ for all $\gamma \in \Gamma$. But $P_s U_\gamma = U_\gamma P_{\gamma^{-1} \cdot s}$ for all $\gamma \in \Gamma$. Therefore,

$$\|P_s \zeta\|_{h_{\mathbb{G}}} = \|P_s U_\gamma \zeta\|_{h_{\mathbb{G}}} = \|U_\gamma P_{\gamma^{-1} \cdot s} \zeta\|_{h_{\mathbb{G}}}, \text{ for all } \gamma \in \Gamma$$

Thus, the norm of $P_{\gamma \cdot s} \zeta$ remains constant on the orbit of s. Consequently, if the orbit of every non trivial irreducible representation is infinite, then $P_s \zeta = 0$ for each $s \in I$. Thus, $\zeta = \lambda \widehat{1}$ for some $\lambda \in \mathbb{C}$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$. Suppose to the contrary $(\mathbb{G}, \Gamma)$ does not satisfy *i.o.c.* Choose a non trivial $[u] \in \mathrm{Irr}(\mathbb{G})$ such that $\Gamma[u]$ is finite. Let $\Gamma[u] = \{\gamma_1 \cdot [u], \cdots, \gamma_n \cdot [u]\}$, where $\gamma_i \in \Gamma$ for $1 \leq i \leq n$. Choose representatives $\gamma_i \cdot u$, $1 \leq i \leq n$ from the equivalence classes. Note that the character of each finite dimensional representation is independent of its class representative. For $1 \leq i \leq n$, let $\chi_{\gamma_i \cdot u}$ denote the character of $\gamma_i \cdot u$. Then, $x = \sum_{i=1}^n \chi_{\gamma_i \cdot u} \in A$. As $\alpha_\gamma(\chi_{\gamma_i \cdot u}) = \chi_{(\gamma\gamma_i) \cdot u}$ for all $\gamma \in \Gamma$, so $\alpha_\gamma(x) = x$ for all $\gamma \in \Gamma$. But $L^\infty(\mathbb{G}) = \mathbb{C}1$. So $(\mathbb{G}, \Gamma)$ must satisfy *i.o.c.*

$(ii) \Rightarrow (v)$. Suppose the action is ergodic. The Haar state $h_{\mathbb{G}}$ is Γ-invariant by Prop. 1.4.2. Clearly, (i) and (ii) of Thm. 3.1.4 is satisfied. By hypothesis the GNS representation of $(C(G), h_{\mathbb{G}})$ is faithful. We now show that $\Omega_{h_{\mathbb{G}}}$ is a separating vector for $L^\infty(\mathbb{G})$. Recall [62] that if $\mathrm{Pol}(\mathbb{G})$ denote the canonical dense Hopf $*$-algebra associated to A, then there exists an (algebraic) automorphism $\sigma : \mathrm{Pol}(\mathbb{G}) \mapsto \mathrm{Pol}(\mathbb{G})$ such that

$$h_{\mathbb{G}}(ab) = h_{\mathbb{G}}(b\sigma(a)), \ \forall a \in \mathrm{Pol}(\mathbb{G}), b \in C(\mathbb{G})$$

Let $c \in L^\infty(\mathbb{G})$. Use Kaplansky density theorem to choose a net $c_i \in C(G)$ with $\|c_i\| \leq \|c\|$ such that $c_i \to c$ in strong operator topology $(s.o.t)$. Then for all $a \in \mathrm{Pol}(\mathbb{G})$, one has $h_{\mathbb{G}}(ac_i) = h_{\mathbb{G}}(c_i \sigma(a))$, for all i. The extension of $h_{\mathbb{G}}$ to $L^\infty(\mathbb{G})$ is a normal state of

$L^\infty(\mathbb{G})$, so

$$h_{\mathbb{G}}(ac) = h_{\mathbb{G}}(c\sigma(a)), \forall\ a \in \mathrm{Pol}(\mathbb{G}), c \in L^\infty(\mathbb{G}).$$

Let $x \in L^\infty(\mathbb{G})$ be such that $x\Omega_{h_{\mathbb{G}}} = 0$. For $y, z \in \mathrm{Pol}(\mathbb{G})$, we have

$$\begin{aligned}
0 &= \langle x\Omega_{h_{\mathbb{G}}}, yz\Omega_{h_{\mathbb{G}}}\rangle_{h_{\mathbb{G}}} \\
&= \langle z^*y^*x\Omega_{h_{\mathbb{G}}}, \Omega_{h_{\mathbb{G}}}\rangle_{h_{\mathbb{G}}} \\
&= h_{\mathbb{G}}(z^*y^*x) \\
&= h_{\mathbb{G}}(y^*x\sigma(z^*)) \\
&= \langle x\sigma(z^*)\Omega_{h_{\mathbb{G}}}, y\Omega_{h_{\mathbb{G}}}\rangle_{h_{\mathbb{G}}}.
\end{aligned}$$

Since the $\mathrm{Pol}(\mathbb{G})$ is dense in $C(\mathbb{G})$, one concludes that $x = 0$. Then by Thm. 3.1.4 (v) follows.

$(iv) \Rightarrow (iii)$. Suppose to the contrary let $L^\infty(\mathbb{G})^\Gamma$ is not trivial. We have already shown that $(iii) \Leftrightarrow (i)$. It then follows from the proof of $(iii) \Rightarrow (i)$ that $C(\mathbb{G})^\Gamma$ is not trivial. Then choose $0 \neq a, b \in C(\mathbb{G})^\Gamma$ such that $ab = 0$. This can be achieved using the Gelfand-Naimark theorem and spectral theorem. Then $a\alpha_\gamma(b) = 0$ for all $\gamma \in \Gamma$, which is a contradiction.

$(v) \Rightarrow (iv)$ follows from Defn. 3.1.3.

$(vi) \Rightarrow (ii)$ is well known classical fact.

$(i) \Rightarrow (vi)$. Fix $s \in I$. If $a_i \in \mathcal{H}_s$, $1 \leq i \leq N$, then $h(a_i) = 0$ for all i, and by *i.o.c.* there is $\gamma \in \Gamma$ (actually infinitely many such elements) such the $\langle \alpha_\gamma(a_i), a_j\rangle_{h_{\mathbb{G}}} = 0$ for all $1 \leq i, j \leq N$.

Similarly one claims that, if $s_1, s_2, \cdots, s_t \in I$ then there is $\gamma \in \Gamma$ such that $\mathcal{H}_{\gamma \cdot s_i} \perp \oplus_{i=1}^t \mathcal{H}_{s_i}$ for all $1 \leq i \leq t$. Indeed, for $s, t \in I$ write $s \sim t$ if and only if there exists $\gamma \in \Gamma$ such that $\gamma \cdot s = t$. Clearly, $\sim$ defines a equivalence relation on I. Write $J = \{s_i : 1 \leq i \leq t\}$. Partition J as J_{i_j}, $1 \leq j \leq k$, where $J_{i_j} = J \cap [s_{i_j}]$, $[\cdot]$ denotes the equivalence classe of $\sim$. If for any $1 \leq j \leq k$, $\gamma \cdot s_{i_j} \in J_{i_j}$ for all $\gamma \in \Gamma$, then *i.o.c.* would be violated for the corresponding element of $\mathrm{Irr}(\mathbb{G})$. This establishes the claim.

Consequently, if $a_i \in \oplus_{i=1}^{t} \mathcal{H}_{s_i}$, $1 \leq i \leq N$ with $h(a_i) = 0$ for all i, then there exists $\gamma \in \Gamma$ such that $\langle \alpha_\gamma(a_i), a_j \rangle_{h_{\mathbb{G}}} = 0$ for all $1 \leq i, j \leq N$.

Finally, a standard approximation argument in the $\|\cdot\|_{h_{\mathbb{G}}}$ shows that the Γ-action is weakly mixing.

$(i) \Rightarrow (vii)$. Follows directly from the defintion of *i.o.c.*

$(vii) \Rightarrow (vi)$. By a standard result in ergodic theory (c.f. Theorem 1.23 [52]), $L^2(\mathbb{G})$ admits a unique decomposition $L^2(\mathbb{G}) = \mathcal{H}_c \oplus \mathcal{H}_{wm}$ into invariant subspaces of the action such that $\Gamma \ni \gamma \mapsto V_\gamma \in \mathcal{U}(\mathbf{B}(\mathcal{H}_c))$ is a compact represenation (i.e. it decomposes as a direct sum of finite dimensional represenations) and $\Gamma \ni \gamma \mapsto U_\gamma \in \mathcal{U}(\mathbf{B}(\mathcal{H}_{wm}))$ is weakly mixing. By definition, if $0 \neq \zeta \in L^2(\mathbb{G})$ is a weakly wandering vector then $\zeta \in \mathcal{H}_{wm}$. This completes the argument. $\square$

Let us recall that a unitary group representation $\pi : \Gamma \to B(H)$ is said to be ergodic if for any $\xi \in H$, $\pi(g)\xi = \xi$ for all $g \in \Gamma$ implies that $\xi = 0$. Then, in the discrete dual case (i.e. the cocommutative case) we have the following:

Corollary 3.1.6. *Let Γ be a countable non abelian group and let Γ_0 be a countable abelian subgroup of Γ. Let us denote the group von-Neumann algebra associated to Γ by $L(\Gamma)$. Consider the action of Γ_0 on $L(\Gamma)$ implemented by inner automorphisms of Γ. Then:*

(i) the action extends to an ergodic representation of Γ_0 on $\ell^2(\Gamma) \ominus \ell^2(\Gamma_0)$ if and only if $L(\Gamma_0)$ is a maximal abelian subalgebra (masa) in $L(\Gamma)$;

(ii) the action is ergodic if and only if $L(\Gamma_0)' \cap L(\Gamma) = \mathbb{C}1$.

Proof. (i) Note that since $\Gamma_0 \leq \Gamma$, so $hgh^{-1} \in \Gamma \setminus \Gamma_0$ for all $g \in \Gamma \setminus \Gamma_0$ and $h \in \Gamma_0$. Since the action of Γ_0 on $\ell^2(\Gamma) \ominus \ell^2(\Gamma_0)$ implemented by inner automorphisms arising from Γ_0 is ergodic, so the orbit $\Gamma_0 g$ for each $g \in \Gamma \setminus \Gamma_0$ is infinite. For otherwise, arguing as in $(iii) \Rightarrow (i)$ of Thm. 3.1.5 one can produce a fixed point of the action of the form $\sum_{g \in F} a_g \delta_g$, where $F \subseteq \Gamma \setminus \Gamma_0$ is a finite set, $0 \neq a_g \in \mathbb{C}$ for all $g \in F$ and δ_g is the standard basis vector for $g \in \Gamma$. Thus, by ergodicity one has $\sum_{g\in F} a_g \delta_g = 0$. By linear independence, $a_g = 0$ for all $g \in F$, which is a contradiciton. Finally, this is equivalent to saying that $L(\Gamma_0) \subseteq L(\Gamma)$ is a masa (c.f. Lemma 3.3.1 [78]).

(ii) This follows easily from the equivalence $(ii) \Leftrightarrow (iii)$ of Theorem 3.1.5. Also, note that in this case, *i.o.c.* is exactly equivalent to the inclusion of $L(\Gamma_0)$ inside $L(\Gamma)$ being irreducible (see Lemma 5.1 [79]). □

3.1.2 Compactness

In this section we study compact actions.

Definition 3.1.7. *Let $(\mathbb{G}, \Gamma)$ be a CQG dynamical system. Let $\|\cdot\|$ denote the C^*-norm on $C(\mathbb{G})$.*

1. *We say that the action is almost periodic if given any $a \in C(\mathbb{G})$, the set $\{\alpha_\gamma(a) : \gamma \in \Gamma\}$ is relatively compact in $C(\mathbb{G})$ with respect to $\|\cdot\|$ [3].*

2. *We say that the action is compact if given any $a \in C(\mathbb{G})$, the set $\{\alpha_\gamma(a)\Omega_{h_\mathbb{G}} : \gamma \in \Gamma\}$ is relatively compact in $C(\mathbb{G})$ with respect to the $\|\cdot\|_{h_\mathbb{G}}$.*

We now prove that these two notions are equivalent in our case.

Theorem 3.1.8. *Let $(\mathbb{G}, \Gamma)$ be a CQG dynamical system. The following are equivalent:*

(i) The action is almost periodic;

(ii) the action is compact;

(iii) the orbit of any irreducible representation in Irr($\mathbb{G}$) is finite.

Proof. $(i) \Rightarrow (ii)$. This follows from the fact that convergence in $\|\cdot\|$ implies convergence in $\|\cdot\|_{h_\mathbb{G}}$.

$(ii) \Rightarrow (iii)$ Suppose to the contrary there exists $[u] \in \mathrm{Irr}(\mathbb{G})$ such that its orbit $\Gamma[u]$, under the Γ-action, is infinite. Choose an infinite enumerated subset $\Gamma_0 = \{\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots\}$

of Γ such that

$$\gamma_i \cdot [u] \neq \gamma_j \cdot [u] \text{ for all } i \neq j. \tag{3.1.1}$$

Let us denote by $\chi_{[u]}$ the character of $[u]$. Then $\chi_{[u]}\Omega_{h_{\mathbb{G}}} \in L^2(\mathbb{G})$. It follows from the hypothesis that $\{\alpha_\gamma(\chi_{[u]})\Omega_{h_{\mathbb{G}}} : \gamma \in \Gamma\}$ is relatively compact in $L^2(\mathbb{G})$. However, from Eq. (3.1.1) it follows that the set $\{\alpha_\gamma(\chi_{[u]})\Omega_{h_{\mathbb{G}}} : \gamma \in \Gamma_0\}$ is orthonormal and hence, this gives us a contradiction.

$(iii) \Rightarrow (i)$ It is easy to see that for any action of a group G on a C^*-algebra B, the set

$$B_c := \{a \in B : G \cdot a \text{ is relatively compact in } B\}$$

is a closed $*$-subalgebra of B.

Since in our case, the closed linear span of the matrix coefficients of irreducible representations of $\mathbb{G}$ is $C(\mathbb{G})$, we will be done if we show that $\Gamma \cdot u^\beta_{ij}$ is relatively compact (with respect to $\|\cdot\|$) for all β, i, j, where u^β_{ij} denote a matrix coefficients of an irreducible representation u^β. But this follows, as by hypothesis, the orbit of any $u^\beta \in \mathrm{Irr}(\mathbb{G})$ is finite, which then implies that the set $\Gamma \cdot u^\beta_{ij}$ is subset of a finite dimensional subspace of $C(\mathbb{G})$. Hence, we are done. $\square$

Remark 3.1.9. *Theorems 3.1.5 and 3.1.8 show that it is not possible that a CQG dynamical system is both compact and ergodic. More generally, this also shows that a separable compact group, say K, cannot act ergodically on a CQG by CQG automorphisms. This follows as let Γ be a countable group dense in K. Then by continuity of the action it follows that the action of Γ is almost periodic and hence compact by Theorem 3.1.8. But if the action of K were ergodic, then by density of Γ in K, the action of Γ is ergodic as well. But this is a contradiction.*

3.1.3 Mixing

We now consider mixing actions. In this case also, there is a characterization of a CQG dynamical system to be mixing in terms of the induced action on the unitary dual of the CQG. The system $(\mathbb{G}, \Gamma)$ is mixing if $h_{\mathbb{G}}(b\alpha_\gamma(a)) \to h_{\mathbb{G}}(a)h_{\mathbb{G}}(b)$ as $\gamma \to \infty \ \forall \ a, b \in C(\mathbb{G})$. Equivalently, we say that the system $(\mathbb{G}, \Gamma)$ is mixing, if for any $x, y \in L^\infty(\mathbb{G})$ such that $h_{\mathbb{G}}(x) = 0 = h_{\mathbb{G}}(y)$, we have that the function

$$\tau_{a,b} : \Gamma \ni \gamma \mapsto h_{\mathbb{G}}(y^* \alpha_\gamma(x))$$

is in $c_0(\Gamma)$.

Note that the next theorem generalizes Prop. 4.4 of [20].

Theorem 3.1.10. *Let* $(\mathbb{G}, \Gamma)$ *be a CQG dynamical system. The following are equivalent:*
(i) *The action is mixing.*
(ii) *Let* $[u] \in$ *Irr*$(\mathbb{G})$ *be any non-trivial element. Then the stabilizer subgroup of* $[u]$

$$\Gamma_{[u]} := \{\gamma \in \Gamma : \gamma \cdot [u] = [u]\}$$

is finite.

Proof. $(i) \Rightarrow (ii)$. Suppose to the contrary that there exists $[u] \in \mathrm{Irr}(\mathbb{G})$ such that $\Gamma_{[u]}$ is infinite. Denoting $\chi_{[u]}$ to be the character of $[u]$, we first note that $h_{\mathbb{G}}(\chi_{[u]}) = 0$. Then we have that the function $\Gamma \ni \gamma \mapsto h_{\mathbb{G}}(\chi_{[u]}^* \alpha_\gamma(\chi_{[u]}))$ does not belong to $c_0(\Gamma)$, as the function takes constant value 1 on the infinite subgroup $\Gamma_{[u]}$. This is a contradiction and hence, we have proved (ii).
$(ii) \Rightarrow (i)$. Let $1 \neq [u] \in \mathrm{Irr}(\mathbb{G})$ with $u = ((u_{ij}))$ a representative for $[u]$. Let us first note that any $x \in L^\infty(\mathbb{G})$ with $h_{\mathbb{G}}(x) = 0$ can be obtained as a weak limit of linear combinations of such $u_{i,j}$, as $[u]$ varies in the set $\mathrm{Irr}(\mathbb{G}) \setminus \{1\}$. Now since the subgroup $\Gamma_{[u]}$ is assumed to be finite, it follows that the function $\Gamma \ni \gamma \mapsto h_{\mathbb{G}}(u_{kl}^* \alpha_\gamma(u_{ij}))$, is in $c_0(\Gamma)$, as it takes non-zero values only on $\Gamma_{[u]}$. But since $[u]$ was arbitrary and $h_{\mathbb{G}}$ is

normal on $L^\infty(\mathbb{G})$, so $\Gamma \ni \gamma \mapsto h_{\mathbb{G}}(y^*\alpha_\gamma(x))$ is in $c_0(\Gamma)$ for all $x, y \in L^\infty(\mathbb{G})$ such that $h_{\mathbb{G}}(x) = 0 = h_{\mathbb{G}}(y)$. The argument is then complete. $\square$

Corollary 3.1.11. *Let $(\mathbb{G}, \Gamma)$ be a CQG dynamical system. Then the following are equivalent:*

(i) The action is mixing;

(ii) For any infinite subgroup Λ of Γ, the induced CQG system $(\mathbb{G}, \Lambda)$ is mixing;

(iii) For any infinite subgroup Λ of Γ, the induced CQG system $(\mathbb{G}, \Lambda)$ is ergodic.

Proof. $(i) \Rightarrow (ii)$. This follows from the definition of mixing actions.

$(ii) \Rightarrow (iii)$. This follows from the fact that every mixing action is ergodic.

$(iii) \Rightarrow (i)$. Suppose the action were not mixing, then by Theorem 3.1.10, there exists nontrivial $[u] \in \widehat{G}$ such that the stabilizer group $\Gamma_{[u]} := \{\gamma : \gamma \cdot [u] = [u]\}$ is infinite. Consequently by Theorem 3.1.5, the system $(\mathbb{G}, \Gamma_{[u]})$ is not ergodic, since the orbit of $[u]$ under $\Gamma_{[u]}$ is $\{[u]\}$. This is contradiction to the hypothesis. Thus (i) holds. $\square$

In the case $\Gamma = \mathbb{Z}^d$, we have:

Corollary 3.1.12. *TFAE:*

(i) $(\mathbb{G}, \Gamma)$ is mixing.

(ii) For any $t \in \Gamma$ of infinite order, the induced $\mathbb{Z}$ action is mixing.

(iii) For any $t \in \Gamma$ of infinite order, the induced $\mathbb{Z}$ action is ergodic.

Proof. This follows easily from the Cor. 3.1.11 and the fact that any non-trivial subgroup of $\mathbb{Z}^d$ is torsion-free, and hence, generated by copies of $\mathbb{Z}$. $\square$

3.2 Permanence Properties and Examples

In this section, we prove some permanence properties of CQG dynamical systems. These can be used to construct examples of CQG dynamical systems with assigned properties. We also give some standalone examples. It should be pointed that although some of these properties follow from general theory, however, we prove them using the *i.o.c.*, to illustrate this property.

It was shown by Wang [87] that given two maximal CQGs $\mathbb{G}_1 = (A_1, \Phi_1)$ and $\mathbb{G}_2 = (A_2, \Phi_2)$, a CQG structure can be given on the full free product $A_1 * A_2$ (we refer to the book [33] for more on free products of C^*-algebras). We shall denote the correspoding CQG by $\mathbb{G}_1 * \mathbb{G}_2$. Now suppose we are given a quantum automorphism α_1 of $\mathbb{G}_1$ and α_2 of $\mathbb{G}_2$, it follows easily from the definition of the co-multiplication in this case (see Theorem 3.4 of [87]) that the automorphism $\alpha_1 * \alpha_2$ of the full free product $A_1 * A_2$ (defined by the universal property for full free products) is a quantum automorphism of $\mathbb{G}_1 * \mathbb{G}_2$. Hence, given an action of a discrete group Γ on the CQGs $\mathbb{G}_1$ and $\mathbb{G}_2$ by quantum automorphisms, we get, in a natural way, an action of Γ on $\mathbb{G}_1 * \mathbb{G}_2$ by quantum automorphisms. Hence, we can construct the CQG dynamical system $((\mathbb{G}_1 * \mathbb{G}_2)_{red}, \Gamma)$.

Proposition 3.2.1. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be maximal compact quantum groups and Γ be a discrete group such that Γ acts on $\mathbb{G}_1$ and $\mathbb{G}_2$ by quantum automorphisms. Suppose now that the CQG dynamical systems $((\mathbb{G}_1)_{red}, \Gamma)$ and $((\mathbb{G}_2)_{red}, \Gamma)$ are both ergodic (resp. mixing, compact). Then the CQG dynamical system $((\mathbb{G}_1 * \mathbb{G}_2)_{red}, \Gamma)$ is also ergodic (resp. mixing, compact).*

Proof. We only prove for the case when the actions are ergodic; the proofs for the cases when the actions are mixing and compact are similar and follow from Thm. 3.1.10 and Thm. 3.1.8. As is shown in Theorem 3.10 of [87], if $U = \{u^\xi\}$ and $V = \{v^\eta\}$ denote, respectively, complete sets of ireducible representations of $(\mathbb{G}_1)_{red}$ and $(\mathbb{G}_2)_{red}$, upto equivalence, then a complete set of mutually inequivalent, irreducible representation of $\mathbb{G} = (\mathbb{G}_1 * \mathbb{G}_2)_{red}$ is given by the set consisting of the trivial representation together with the collection of interior tensor product representations of the form

$$w^{\gamma_1} \otimes_{in} w^{\gamma_2} \otimes_{in} \cdots \otimes_{in} w^{\gamma_n} \tag{3.2.1}$$

where w^{γ_i} belongs either to U or to V and w^{γ_i} and $w^{\gamma_{i+1}}$ belong to different sets.

Now let us take a non-trivial representation of the form (3.2.1). Then there exists an i such that w^{γ_i}, which without loss of generality we may assume belongs to U, is non-trivial. But as the action of Γ on $(\mathbb{G}_1)_{red}$ is ergodic, so the orbit of w^{γ_i} under Γ, denoted by the set

$$W = \{w^{t\cdot\gamma_i} : t \in \Gamma\}$$

is infinite. But as the orbit of any irreducible representation of $\mathbb{G}$ of the form (3.2.1) under Γ is

$$\{w^{t\cdot\gamma_1} \otimes_{in} w^{t\cdot\gamma_2} \otimes_{in} \ldots \otimes_{in} w^{t\cdot\gamma_n} : t \in \Gamma\}$$

it follows that this set must also be infinite. This is because it follows from Theorem 3.10 of [87] for any $t_1, t_2 \in \Gamma$ for which $w^{t_1\cdot\gamma_i}$ and $w^{t_2\cdot\gamma_i}$ are inequivalent, the irreducible representations of $\mathbb{G}$ given by $w^{t_1\cdot\gamma_1} \otimes_{in} w^{t_1\cdot\gamma_2} \otimes_{in} \ldots \otimes_{in} w^{t_1\cdot\gamma_n}$ and $w^{t_2\cdot\gamma_1} \otimes_{in} w^{t_2\cdot\gamma_2} \otimes_{in} \cdots \otimes_{in} w^{t_2\cdot\gamma_n}$ are also inequivalent. □

For notational simplicity, we demonstrated the above result for a free product of two CQG, but it is not hard to see that the same holds if we are given a sequence of CQGs $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$. As should be clear, the above result can be used to construct non-trivial CQG dynamical systems starting from classical dynamical systems, i.e. where a group is acting on a compact group by group automorphisms.

Similarly, if $\mathbb{G}_i = (A_i, \Phi_i)$, $i = 1, 2$, are two maximal CQGs, then as shown by Wang [88], one can give a CQG structure on $A_1 \otimes_{max} A_2$ (the model for $\mathbb{G}_1 \otimes \mathbb{G}_2$ on the spatial tensor product is given in [4]). We denote the corresponding CQG as $\mathbb{G}_1 \otimes \mathbb{G}_2$. Now, suppose α_1 is a quantum automorphism of $\mathbb{G}_1$ and α_2 is a quantum automorphism of $\mathbb{G}_2$. Then, as follows from the definition of the co-multiplication of $\mathbb{G}_1 \otimes \mathbb{G}_2$ (see Theorem 2.2 of [88]), the automorphism $\alpha_1 \otimes_{max} \alpha_2$ of $A_1 \otimes_{max} A_2$ (given by the universal property of maximal tensor products) is in fact a quantum automorphism.

Proposition 3.2.2. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be maximal CQGs such that $((\mathbb{G}_1)_{red}, \Gamma_1)$ and $((\mathbb{G}_2)_{red}, \Gamma_2)$ be CQG dynamical systems which are both ergodic (resp. mixing, resp. compact). Then the CQG dynamical system $((\mathbb{G}_1 \otimes \mathbb{G}_2)_{red}, \Gamma_1 \times \Gamma_2)$ is ergodic (resp. mixing, resp. compact).*

Proof. Once again, we will only show the ergodic case, since the other two are similar.

It follows from Theorem 2.11 of [88] that if $U = \{u^{\xi}\}$ and $V = \{v^{\eta}\}$ denote the sets of mutually inequivalent irreducible representations of $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively, complete upto equivalence, then a complete set of mutually inequivalent irreducible representations is given exterior tensor products of irreducible representations of $\mathbb{G}_1$ and $\mathbb{G}_2$ of the form

$$\{u^{\xi_i} \otimes_{ex} v^{\eta_j}\} \tag{3.2.2}$$

where u^{ξ_i} belongs to U and v^{η_j} belongs to V. Now the orbit of any irreducible represention of $\mathbb{G}_1 \otimes \mathbb{G}_2$ of the form (3.2.2) under the action of $\Gamma_1 \times \Gamma_2$ is given by the set

$$\{u^{t_1 \cdot \alpha_i} \otimes_{ex} v^{t_2 \cdot \beta_j} : (t_1, t_2) \in \Gamma_1 \times \Gamma_2\}$$

But then if we pick a nontrivial irreducible representation of $\mathbb{G}_1 \otimes \mathbb{G}_2$ of the form (3.2.2), then atleast one of u^{ξ_i} and v^{η_j} must be non-trivial. The result now follows from Theorem 2.11 of [88] and a reasoning similar to one used in the proof of Theorem 3.2.1.
□

In this case too, the result holds for an arbitrary sequence of groups $\{\Gamma_n\}_{n \in \mathbb{N}}$ and CQGs $\{\mathbb{G}_i\}$.

We now look at some examples.

Example 3.2.3. Let $\mathbb{G} = (A, \Delta)$ be any CQG. Then (see Example 4.4 of [88]) CQG structures can be given on the C^*-algebras $\mathbb{A}_1 = *_{i=-\infty}^{\infty} A_i$ and $\mathbb{A}_2 = \otimes_{i=-\infty}^{\infty} A_i$, where $A_i = A$ for all $i \in \mathbb{Z}$. One can then show that the natural shift action of $\mathbb{Z}$ on $\mathbb{A}_1$ and $\mathbb{A}_2$ gives us CQG dynamical systems. It is then easy to show that both these dynamical systems are actually ergodic, in fact mixing.

Example 3.2.4. In the last chapter (as also [72]), we studied a class of automorphisms of CQGs which behave like "inner automorphisms" in the group theoretic sense. Following [72], let $\mathbb{G}_{char}$ denote the set of characters of irreducible representations of a CQG $\mathbb{G}$. The group

$$\mathrm{Aut}_\chi(\mathbb{G}) := \{\alpha \in \mathrm{Aut}(\mathbb{G}) : \alpha(\chi_a) = \chi_a, \forall \chi_a \in \mathbb{G}_{char}\}$$

is a closed compact normal subgroup of $\mathrm{Aut}(\mathbb{G})$, the group of all CQG automorphisms of $\mathbb{G}$, where the topology is of pointwise norm convergence (as shown in Theorem 2.1.1 for compact matrix quantum groups). To give explicit examples of such automorphisms, we refer to Example 2.3.2. However, for the convenience of the reader we recall the main steps here. Denote by $\chi(\mathbb{G})$ the set of unital $*$-homomorphisms from $C_m(\mathbb{G})$ to $\mathbb{C}$. It is a group with the product defined by $gh = (g \otimes h) \circ \Delta$, for $g, h \in \chi(\mathbb{G})$. The unit of $\chi(\mathbb{G})$ is the counit $\varepsilon_{\mathbb{G}} \in C_m(\mathbb{G})^*$ and the inverse of $g \in \chi(\mathbb{G})$ is given by $g \circ S$, where S is the antipode on $\mathbb{G}$. Viewing $\chi(\mathbb{G})$ as a closed subset of the unit ball of $C_m(\mathbb{G})^*$, one can consider the weak* topology on $\chi(\mathbb{G})$ which make $\chi(\mathbb{G})$ a compact group. Define, for all $g \in \chi(\mathbb{G})$, the map $\alpha_g = (g^{-1} \otimes \mathrm{id} \otimes g) \circ \Delta^{(2)}$. It defines a continuous group homomorphism $\chi(\mathbb{G}) \ni g \mapsto \alpha_g \in \mathrm{Aut}_\chi(\mathbb{G}) \subseteq \mathrm{Aut}(\mathbb{G})$ as it can be easily checked these automorphisms are quantum automorphisms and that the action of $\chi(\mathbb{G})$ on $\mathrm{Irr}(\mathbb{G})$ is trivial. Indeed, for $g \in \chi(G)$ and $x \in \mathrm{Irr}(G)$ a straightforward computation gives $(\mathrm{id} \otimes \alpha_g)(u^x) = (V^* \otimes 1)u^x(V \otimes 1)$, where $V = (\mathrm{id} \otimes g)(u^x)$.

Now if we have a CQG dynamical system $(\mathbb{G}, \Gamma)$ with the action defined by the homomorphism

$$\alpha : \Gamma \to \mathrm{Aut}(\mathbb{G})$$

Suppose now that $\alpha(\Gamma) \subseteq \mathrm{Aut}_\chi(\mathbb{G})$, then it follows from Theorem 3.1.8 and from the definition of $\mathrm{Aut}_\chi(\mathbb{G})$ that this action must be compact, since the orbit of any irreducible representation of $\mathbb{G}$ under the Γ-action has one element (i.e. itself).

It is shown in [87] that for the free unitary quantum group $A_u(n)$, the maximal compact subgroup is the unitary group $U(n)$ and for the free orthogonal quantum group $A_o(n)$, the maximal compact subgroup is the orthogonal group $O(n)$. Similarly, it follows from [75] that the maximal compact subgroup of $SU_q(2)$ is the circle group $\mathbb{T}$. In each of these cases, taking Γ to be a countable discrete subgroup of $\chi(\mathbb{G})$, we get a compact CQG dynamical system.

In the previous chapter, the group

$$\mathrm{Out}_\chi(\mathbb{G}) = \mathrm{Aut}(\mathbb{G})/\,\mathrm{Aut}_\chi(\mathbb{G})$$

was also studied and e.g. it was shown that for CQG with fusion rules identical to those of simple compact Lie groups, this group has order $1, 2, 3$ or 6 (Proposition 2.1.5).

Proposition 3.2.5. *Let $\mathbb{G}$ be a CQG such that $|\mathrm{Out}_\chi(\mathbb{G})| < \infty$. Then any CQG dynamical system $(\mathbb{G}, \Gamma)$ is compact.*

Proof. Suppose the action is not compact, then it follows from Theorem 3.1.8 that there exists an $\alpha \in \mathrm{Irr}(\mathbb{G})$ such that the orbit $\{\Gamma\alpha\}$ is infinite. Let $g_1, g_2, g_3, \ldots$ be elements in Γ such that

$$g_i\alpha \neq g_j\alpha \mathrm{for} i \neq j$$

By hypothesis, we can find infinitely many such elements. Now we have the homomorphism

$$\gamma : \Gamma \to \mathrm{Aut}(\mathbb{G})$$

which encodes the action of Γ on $\mathbb{G}$ and let

$$\gamma_\chi : \Gamma \to \mathrm{Out}_\chi(\mathbb{G})$$

be the obvious map into the outer automorphism group of $\mathbb{G}$. But then, it follows from the equation above that $\gamma_\chi(g_i) \neq \gamma_\chi(g_j)$ if $i \neq j$. This gives us a contradiction since $\mathrm{Out}_\chi(\mathbb{G})$ is finite.

$\square$

So, it follows that for CQG dynamical system $(\mathbb{G}, \Gamma)$ where $\mathbb{G}$ has identical fusion rules as simple compact lie groups, the action must be compact. Similarly, it was shown in Proposition 2.3.11 that for the free quantum group $A_u(n)$, the outer automorphism group is $\mathbb{Z}_2$, for any $n \in \mathbb{N}$. So, by the previous proposition, we have that for any CQG dynamical system $(\mathbb{G}, \Gamma)$ is compact, when $\mathbb{G} = A_u(n)$.

In fact, for a large class of compact quantum groups, one can show that a compact action by quantum group automorphisms is virtually "inner".

Theorem 3.2.6. *Suppose for the CQG $\mathbb{G}$ $\mathrm{Out}_\chi(\mathbb{G})$ is discrete. Let $(\mathbb{G}, \Gamma)$ be a compact CQG dynamical system. Then the subgroup*

$$\Gamma_\chi := \{\gamma \in \Gamma : \alpha_\gamma \in \mathrm{Aut}_\chi(\mathbb{G})\}$$

of Γ is of finite index.

Proof. Let us first recall that as is shown in Theorem 2.2 of [39], any C^*-dynamical system (A, Γ, α) is almost periodic if and only if the closure of the image of Γ under α, in $\mathrm{Aut}(A)$, is compact in the pointwise norm topology. Now, since $(\mathbb{G}, \Gamma)$ is compact, it follows that $H := \overline{\Gamma} \subseteq \mathrm{Aut}(\mathbb{G}) \subseteq \mathrm{Aut}(C(\mathbb{G}))$ is compact. But we have that $\mathrm{Out}_\chi(\mathbb{G})$ is discrete, so equivalently, $\mathrm{Aut}_\chi(\mathbb{G})$ is an open subgroup of $\mathrm{Aut}(\mathbb{G})$. So, $H_\chi = \mathrm{Aut}_\chi(\mathbb{G}) \cap H$ is open in H. Since H is compact, we have that H_χ is finite index in H. This then implies that Γ_χ is finite index subgroup of Γ and hence, we are done. $\square$

The question now arises as to for which CQG's the above property holds, i.e. $\mathrm{Out}_\chi(\mathbb{G})$ is discrete.

Let us recall that given a compact quantum group $\mathbb{G}$, its fusion ring, denoted by $Z(\mathbb{G})$, is defined to be the additive group of finitely supported functions from $\mathbb{G}_{char}$, the set of characters of irreducible representations of $\mathbb{G}$, into $\mathbb{Z}$ with the multiplication given by the fusion rules of $\mathbb{G}$.

The following was exhibited in the proof of the Theorem 2.1.4.

Lemma 3.2.7. *Let $\beta \in Irr(\mathbb{G})$. The subgroup of* $\mathrm{Aut}(\mathbb{G})$ *defined as*

$$K_\beta := \{\alpha \in \mathrm{Aut}(\mathbb{G}) : \alpha(\chi_\beta) = \chi_\beta\}$$

where χ_β denotes the character corresponing to β, is an open subgroup of $\mathrm{Aut}(\mathbb{G})$.

Proposition 3.2.8. *Suppose the fusion ring $Z(\mathbb{G})$ of $\mathbb{G}$ is finitely generated as a ring. Then* $\mathrm{Out}_\chi(\mathbb{G})$ *is discrete, or equivalently, that* $\mathrm{Aut}_\chi(\mathbb{G})$ *is an open subgroup of* $\mathrm{Aut}(G)$.

Proof. Let us denote the generators of $Z(\mathbb{G})$ by $\lambda_1, \lambda_2, ..., \lambda_n$. Since characters of irreducible representations of $\mathbb{G}$ form a basis of $Z(\mathbb{G})$, we can define the set

$$N := \{\chi_\beta : \chi_\beta \text{ appears in the linear decomposition of some } \lambda_i, 1 \leq i \leq n\}$$

N is then a finite subset of the set of characters of irreducible representations of $\mathbb{G}$. Now, we claim that $\alpha \in \mathrm{Aut}(\mathbb{G})$ is an element of $\mathrm{Aut}_\chi(\mathbb{G})$ if and only if $\alpha(\chi_\beta) = \chi_\beta$ for all $\chi_\beta \in N$. Of course, by definition of $\mathrm{Aut}_\chi(\mathbb{G})$, it is clear that $\alpha(\chi_\beta) = \chi_\beta$ if $\alpha \in \mathrm{Aut}_\chi(\mathbb{G})$. For the other implication, we note that if $\alpha(\chi_\beta) = \chi_\beta$ for all $\chi_\beta \in N$, we also have that $\alpha(\lambda_i) = \lambda_i$ for all $1 \leq i \leq n$. But since λ_i generate $Z(\mathbb{G})$, we have that $\alpha(a) = a$ for all $a \in Z(\mathbb{G})$ and hence, as characters of all irreducible representations of $\mathbb{G}$ are elements of $Z(\mathbb{G})$, we are done. Now, to compelete the proof of the proposition, let us once again define

$$K_\beta := \{\alpha \in \mathrm{Aut}(\mathbb{G}) : \alpha(\chi_\beta) = \chi_\beta\}$$

and note that, in our case, we have that

$$\mathrm{Aut}_\chi(\mathbb{G}) = \cap_{\{\chi_\beta \in N\}} K_{\chi_\beta}$$

Since N is a finite set, the proof now follows from the previous lemma. $\square$

Remark 3.2.9. *A large class of compact quantum groups have finitely generated representation rings. For example, duals of finitely generated discrete groups, $A_u(n)$ [7], $A_o(n)$ (since it has same fusion ring as $SU(2)$)[6]. In fact, it was shown by Segal in [77] that the representation ring of any compact lie group is finitely generated, hence, any compact quantum group with fusion rules isomorphic to that of a compact lie group will also have finitely generated representation ring.*

Remark 3.2.10. *It is to be expected that the fusion ring of compact matrix quantum groups should be finitely generated, since these correspond, in the classical case, to compact lie groups. But the proof is unclear to us. In light of Lemma 3.2.7, a direct way of showing that inner automorphism groups of compact matrix quantum groups is open is showing that given an automorphism of a compact quantum group, the sub-category of representations which under the action of this automorphism, get transformed to a unitarily equivalent representation is in fact a full sub-category. But this is not true as can be seen in the case of the compact group $SU(3)$. In this case, the outer automorphism group is also of order 2 and in fact, modulo the inner automorphism group, this outer automorphism is uniquely given by the complex conjugation automorphism, i.e. the automorphism of $SU(3)$ that sends any matrix to its complex conjugate. It can now be checked that this automorphism fixes the adjoint representation of $SU(3)$, which is irreducible of dimension 8 (and in fact is the unique irreducible representation of dimension 8 for $SU(3)$). But tensoring the adjoint representation to itself, we get a 10 dimensional irreducible representation of $SU(3)$, which under the complex conjugation automorphism, gets sent to its contragradient representation.*

Example 3.2.11. Let Λ be a countable discrete group. Let $A = C_r^*(\Lambda)$ be the reduced group C^*-algebra. We consider the standard CQG structure on A, which is co-commutative, which we denote $\mathbb{G}$. Let now $\alpha : \Gamma \to \mathrm{Aut}(\Lambda)$ be a homomorphism of Γ into the automorphism group of Λ. This induces a CQG dynamical system $(\mathbb{G}, \Gamma)$. Let us now consider the CQG dynamical system $(\mathbb{G}, \Lambda)$ which is induced from the action of Λ on itself by inner automorphisms. It is not hard now to see that this action in ergodic if and only if Λ is an ICC group. To see this, note that in this case all irreducible representations of $\mathbb{G}$ are one-dimensional and are given by group unitaries $\lambda_g, g \in \Lambda$. Now, the orbit of λ_g under the Λ-action, is given by

$$\{\lambda_{sgs^{-1}} | s \in \Lambda\}$$

which is infinite for all $g \neq e$ if and only if Λ is ICC. In general though, this action need not be mixing. In fact, it follows from Theorem 3.1.10 that this action will be mixing if and only if the centralizer subgroup of any given nontrivial element is finite. So, in particular, ICC groups which have torsion free elements, like free groups, fail this property. But, if Λ is a Tarski Monster [70], then it is known that centralizer of any nontrivial element is indeed finite, and hence this action is mixing.

It can also be shown (assuming that Λ is finitely generated) that the Λ-action on $\mathbb{G}$ by inner automorphisms is compact (i.e. all conjugacy classes are finite) if and only if the center $Z(\Lambda)$ has finite index in Λ. While the backwards implication is obvious, to see the forward implication, we use Theorem 3.2.6. In this case, it follows from the definitions that $\mathrm{Aut}_\chi(\mathbb{G})$ is trivial and hence, $\mathrm{Out}_\chi(\mathbb{G}) = \mathrm{Aut}(\Lambda)$. But as Λ is finitely generated, $\mathrm{Aut}(\Lambda)$ is discrete. Now invoking Theorem 3.2.6, we conclude that the center $Z(\Lambda)$ is a finite index subgroup of Λ.

Example 3.2.12. Let $\mathbb{G}$ be a compact quantum group and $\mathcal{U}(C_m(G))$ denote the group of unitaries in $C_m(G)$. We then define the intrinsic group of $\mathbb{G}$ as

$$\mathrm{Int}(\mathbb{G}) = \{u \in \mathcal{U}(C_m(G)) : \Delta(u) = u \otimes u\}$$

It is then easy to check that the automorphism $\mathrm{Ad}(u) : a \mapsto uau^*$ of $C_m(G)$ is in fact a quantum automorphism. Hence, for any countable discrete subgroup Γ of $\mathrm{Int}(\mathbb{G})$, we get a CQG dynamical system $(\mathbb{G}, \Gamma)$ (by inducing this quantum automorphism to $C(G)$).

Let now G be a compact quantum group and let (G, Γ) be a CQG dynamical system. It was then shown in Theorem 1.5 of [88] that the full crossed product $C_m(G) \rtimes \Gamma$ can be given a compact quantum group structure. Let $\mathbb{G}$ denote this compact quantum group, with the comultiplication denoted by $\Delta_{\mathbb{G}}$. It was then shown in [35] (Theorem 6.1(3)) (and as will be exhibited in the following chapter, see 4.4.1) that $C(\mathbb{G}) = C(G) \rtimes_r \Gamma$. It was also shown in [35](Proposition 6.5)(see also 4.4.5) that $\mathrm{Int}(\mathbb{G})$ $=\mathrm{Int}(G) \rtimes \Gamma$. Hence, for any subgroup Λ of Γ, we have the CQG dynamical system $(\mathbb{G}, \Lambda)$. We then have the following

Theorem 3.2.13. *The following holds-*

1. *Suppose the CQG dynamical system (G, Λ) is compact and suppose that for the natural Λ-action on Γ by inner automophisms, the orbit of any element is finite. Then the CQG dynamical system $(\mathbb{G}, \Lambda)$ is also compact.*

2. *Suppose the CQG dynamical system (G, Λ) is ergodic and suppose that for the natural Λ-action on Γ by inner automophisms, the orbit of any non-trivial element is infinite. Then the CQG dynamical system $(\mathbb{G}, \Lambda)$ is ergodic.*

Proof. 1. If we denote the natural embedding of $C(G)$ in $C(G) \rtimes_r \Gamma$ by $\alpha(C(G))$ and the group unitaries corresponding to Γ by u_γ for any $\gamma \in \Gamma$ (hence, $C(G) \rtimes_r \Gamma$ is the closed linear span of elements of the form $\alpha(a)u_\gamma$, with $a \in C(G)$ and $\gamma \in \Gamma$), then as shown in Theorem 6.1(2) of [35] (as also 4.4.1), the set $\{u^x_\gamma : \gamma \in \Gamma, x \in \mathrm{Irr}(G)\}$ gives a complete set of irreducible representations of $\mathbb{G}$, where, given $x \in\mathrm{Irr}(G)$, with a representative $u^x \in B(H_x) \otimes A$, we define $u^x_\gamma = (1 \otimes u_\gamma)(id \otimes \alpha)(u^x) \in B(H_x) \otimes (A \rtimes_r \Gamma)$. Thus, any element of $\mathrm{Irr}(\mathbb{G})$ can be uniquely identified with a tuple (x, γ), with $x \in\mathrm{Irr}(G)$ and $\gamma \in \Gamma$, with u^x_γ as a representative.

Now since the CQG dynamical system (G, Λ) is compact, we have from Theorem 3.1.8 that for any $x \in$Irr(G), the set $\{\gamma \cdot x : \gamma \in \Lambda\}$ is finite. Similarly, we have from the hypothesis for any γ_0 that the set $\{\gamma\gamma_0\gamma^{-1} : \gamma \in \Lambda\}$ is finite. It then follows that the set $\{(\gamma \cdot x, \gamma\gamma_0\gamma^{-1}) : \gamma \in \Lambda\}$ for any $x \in$Irr(G) and $\gamma_0 \in \Gamma$ (which is in fact the orbit of (x, γ_0) under the Λ-action on Irr$(\mathbb{G})$) is finite. Hence, it follows from Theorem 3.1.8 that the CQG dynamical system $(\mathbb{G}, \Lambda)$ is compact.

2. We have from the hypothesis that for $x \in$Irr(G), which doesnot correspond to the trivial representation and for any $\gamma_0 \neq e \in \Gamma$, the sets $\{\gamma \cdot x : \gamma \in \Lambda\}$ and $\{\gamma\gamma_0\gamma^{-1} : \gamma \in \Lambda\}$ are infinite. Then it follows easily that the set $(\gamma \cdot x, \gamma\gamma_0\gamma^{-1})$ is infinite, except when both x and γ_0 are trivial. Hence, we have from Theorem 3.1.5 that the CQG dynamical system $(\mathbb{G}, \Lambda)$ is ergodic.

$\square$

3.3 Maximal Ergodic Normal Subgroup

In this section, we study the structure theory of general actions of groups on CQGs by quantum automorphisms and for a large class of CQGs, demonstrate the existence of a maximal ergodic invariant normal subgroup. We will refer to 1.2 for the basics and notations of subgroups and normal subgroups. In this section, given a compact quantum group G, we will denote by G_{red} the compact quantum group $(C(G), \Delta)$.

We first demonstrate a characterization of non-ergodic actions paralleling the classical characterisation given in Lemma 2.2 of [53].

Theorem 3.3.1. *Let the discrete group Γ act on a compact quantum group $\mathbb{G} = (A, \Delta)$ by quantum automorphisms, with the action denoted by α. Then the following are equivalent-*

1. *The induced action of Γ on $\mathbb{G}_{red}$ is non-ergodic*

2. *There exists a non-trivial $\beta \in Irr(\mathbb{G})$, with character denoted by χ_β, such that the*

subgroup of Γ*, defined as*

$$\Gamma_\beta := \{g \in \Gamma : \alpha_g(\chi_\beta) = \chi_\beta\}$$

is a finite index subgroup.

3. *There exists a non-trivial Compact Matrix Quantum Group* $\mathbb{G}_0 = (A_0, \Delta_0)$ *such that* $A_0 \subseteq A$ *is a Woronowicz sub-*C^**-algebra of* A*, and such that* A_0 *is invariant under the* Γ*-action and the induced action on* $(\mathbb{G}_0)_{red}$ *is compact.*

Proof. First let us note that by Remark 1.4.2, the action of Γ on $\mathbb{G}$ induces an action of Γ on $\mathbb{G}_{red}$.
(1) $\Rightarrow$ (2) Since the action of Γ on $\mathbb{G}_{red}$ is non-ergodic, by Theorem 3.1.5, we have that there must exist $\beta \in \mathrm{Irr}(\mathbb{G})$ such that the orbit of β under the induced Γ action on $\mathrm{Irr}(\mathbb{G})$, is finite. Γ_β is in fact the stabilizer of β and the proof now follows from the orbit-stabilizer theorem.
(2) $\Rightarrow$ (3) Applying the orbit-stabilizer theorem again, we deduce that the orbit of β is finite. Let $\mathcal{T} = \Gamma \cdot \beta \subseteq \mathrm{Irr}(\mathbb{G})$ denote the (finite) orbit of β. Let us denote the Woronowicz sub-C^*-algebra generated by matrix coefficients of representatives of elements of $\mathcal{T}$ by A_0. This gives us a CMQG, which we denote by $\mathbb{G}_0$. First we show that A_0 is invariant under Γ-action. For this, it is enough to show that $\mathrm{Irr}(\mathbb{G}_0) \subseteq \mathrm{Irr}(\mathbb{G})$ is invariant under the induced Γ-action. So, let $\gamma \in \mathrm{Irr}(\mathbb{G}_0)$. Since $\mathcal{T}$ is a generating set for $\mathrm{Irr}(\mathbb{G}_0)$, we have that (abusing definitions slightly) γ must be a sub-representation of some representation which is a tensor product of representations coming from $\mathcal{T}$. Our task is to show that for any g, the irreducible representation representated by $g \cdot \gamma$ is also a subrepresentation of some representation which is a tensor product of representations coming from $\mathcal{T}$. This is easy to see, since if

$$\gamma \subseteq t_1 \otimes t_2 \otimes ... \otimes t_n$$

for $t_1, t_2, ..., t_n \in \mathcal{T}$, we have

$$g \cdot \gamma \subseteq g \cdot t_1 \otimes g \cdot t_2 \otimes ... \otimes g \cdot t_n$$

. But of course, $g \cdot t_i$ is in $\mathcal{T}$ for all $1 \leq i \leq n$ as $\mathcal{T}$ is the orbit of β and hence is invariant under the Γ-action. Now, we show that the induced action of Γ on $(\mathbb{G}_0)_{red}$ is compact. So, by Theorem 3.1.8, we have to show that for any $\gamma \in \mathrm{Irr}(\mathbb{G}_0)$, the orbit $\Gamma \cdot \gamma$ is finite. Let $\gamma \subseteq t_1 \otimes t_2 \otimes ... \otimes t_n$ for $t_1, t_2, ..., t_n \in \mathcal{T}$. Since all t_i have finite orbits, so upto equivalence, the set of representations of $\mathbb{G}_0$ represented by $g \cdot t_1 \otimes g \cdot t_2 \otimes ... \otimes g \cdot t_n$ for $g \in \Gamma$, is finite, and hence, upto equivalence, the set of (irreducible) representations represented by $g \cdot \gamma$, for $g \in \Gamma$, is also finite, since for all $g \in \Gamma$, we have

$$g \cdot \gamma \subseteq g \cdot t_1 \otimes g \cdot t_2 \otimes ... \otimes g \cdot t_n$$

In other words, the orbit of γ under the Γ-action is finite. Hence, we are done.
(3) $\Rightarrow$ (1) Since the induced action of Γ on $(\mathbb{G}_0)_{red}$ is compact, the orbit of any $\gamma \in \mathrm{Irr}(\mathbb{G}_0)$ is finite. Since $\mathrm{Irr}(\mathbb{G}_0) \subseteq \mathrm{Irr}(\mathbb{G})$, taking any non-trivial $\gamma \in \mathrm{Irr}(\mathbb{G}_0) \subseteq \mathrm{Irr}(\mathbb{G})$, we see that the action is non-ergodic, as the orbit of γ is finite. □

Remark 3.3.2. *In case the CMQG is such that its fusion ring is finitely generated as a ring, then it follows from Proposition 3.2.8 and Theorem 3.2.6 that the action is virtually inner. This is of course the case for compact lie groups and this is exactly the statement of Lemma 2.2 of [53].*

Implicit in the proof of the previous theorem is the following

Proposition 3.3.3. *Suppose Γ acts on $\mathbb{G}$ by quantum automorphisms. Suppose β_1 and β_2 are elements of Irr($\mathbb{G}$), and suppose further that the orbits of β_1 and β_2 under the Γ-action are finite. Then the following holds:*

1. *For any $\beta \subset \beta_1 \otimes \beta_2$, the orbit of β is finite.*

2. *Orbit of $\overline{\beta_1}$ (as also $\overline{\beta_2}$) is finite*

Proof. 1. For any $g \in \Gamma$, we have that for the induced action on $\mathrm{Irr}(\mathbb{G})$, $g \cdot \beta \subset g \cdot \beta_1 \otimes g \cdot \beta_2$. But as both β_1 and β_2 have finite orbits, this implies that upto equivalence, the set of representations of $\mathbb{G}$ which are isomorphic to a representation represented by $g \cdot \beta_1 \otimes g \cdot \beta_2$ for some $g \in \Gamma$ is finite, which forces that the set of (irreducible) representations, represented by $g \cdot \beta$ for some $g \in \Gamma$, is upto equivalence, finite. Hence, the orbit of β is finite.

2. This is easy as $\overline{g \cdot \beta} = g \cdot \overline{\beta}$ for any $\beta \in \mathrm{Irr}(\mathbb{G})$.

□

Remark 3.3.4. *In other words, the previous proposition says that the "compact part" of any given action of a group Γ on a CQG $\mathbb{G}$ by quantum automorphisms, is a Woronowicz sub-C^*-algebra of $\mathbb{G}$, and so, is a CQG itself. The irreducible representations of this quotient CQG are exactly the irreducible representations whose orbit under the Γ-action is finite. We use this to construct the maximal invariant ergodic normal subgroup. This is similar in spirit to Theorem 2.3 of [53].*

The following class of CQG have been defined and studied by Wang in [91] and [92].

Definition 3.3.5. *A compact quantum group $\mathbb{G}$ is said to have Property F if each Woronowicz sub-C^*-algebra of $C_m(\mathbb{G})$ is of the form $C_m(\mathbb{G}/\mathbb{H})$ for some normal subgroup $\mathbb{H}$ of $\mathbb{G}$.*

It has been shown in [91] that all compact quantum groups obtained by deformation of compact Lie groups, such as the compact real forms of Drinfeld-Jimbo quantum groups and Rieffel's deformation of compact Lie groups, and several universal quantum groups like $A_o(n)$ and $A_S(n)$ have Property F. It has also been shown by Wang in [92] that Property F passes to quotients and to subgroups (under suitable conditions). So, there is an abundance of examples of compact quantum groups having Property F.

Proposition 2.2.3 will be crucial for our construction of the maximal normal ergodic subgroup, hence we recall the statement here.

Proposition 3.3.6. *Let $\mathbb{G}$ be a compact quantum group and let $\mathbb{H}$ be a subgroup of G. Let $\alpha : C_m(\mathbb{G}) \to C_m(\mathbb{G})$ be an automorphism of $\mathbb{G}$, such that $\alpha : C_m(\mathbb{G}/\mathbb{H}) \to C_m(\mathbb{G}/\mathbb{H})$ is C^*-algebraic automorphism, then there exists $\beta : C_m(\mathbb{H}) \to C_m(\mathbb{H})$ such that β is also an automorphism of $\mathbb{H}$ and such that*

$$\begin{tikzcd} C_m(\mathbb{G}) \arrow[r, "\alpha"] \arrow[d, "\rho"] & C_m(\mathbb{G}) \arrow[d, "\rho"] \\ C_m(\mathbb{H}) \arrow[r, "\beta"] & C_m(\mathbb{H}) \end{tikzcd}$$

commutes.

The following lemma now follows easily.

Lemma 3.3.7. *Suppose $\mathbb{G}$ is a compact quantum group, and suppose $\mathbb{H}$ is a normal subgroup of $\mathbb{G}$. Suppose now that a group Γ acts on $\mathbb{G}$ by quantum automorphisms, with action denoted by α, and that for the set Irr($\mathbb{G}/\mathbb{H}$) $\subseteq$ Irr($\mathbb{G}$), for any $g \in \Gamma$, we have that α_g(Irr($\mathbb{G}/\mathbb{H}$)) $=$ Irr($\mathbb{G}/\mathbb{H}$). Then, for each α_g, there we can construct a quantum automorphism γ_g such that the diagram*

$$\begin{tikzcd} C_m(\mathbb{G}) \arrow[r, "\alpha_g"] \arrow[d, "\rho"] & C_m(\mathbb{G}) \arrow[d, "\rho"] \\ C_m(\mathbb{H}) \arrow[r, "\gamma_g"] & C_m(\mathbb{H}) \end{tikzcd}$$

commutes. So, we have an action of Γ on $\mathbb{H}$ by quantum automorphisms. In other words, $\mathbb{H}$ is an invariant subgroup of $\mathbb{G}$ under the Γ-action.

Proof. To invoke the previous proposition, we need to show that the Woronowicz sub-C^*-algebra $C_m(\mathbb{G}/\mathbb{H})$ is invariant under the Γ-action. But $C_m(\mathbb{G}/\mathbb{H})$ is generated as a C^*-algebra by matrix coefficients of irreducible representations of $\mathbb{G}/\mathbb{H}$. But since for any $g \in \Gamma$, we have that $g\cdot\mathrm{Irr}(\mathbb{G}/\mathbb{H}) = \mathrm{Irr}(\mathbb{G}/\mathbb{H})$, it follows that $\alpha_g(C_m(\mathbb{G}/\mathbb{H})) = C_m(\mathbb{G}/\mathbb{H})$ and hence, we are done. □

For the next two lemmas, we need a preliminary result. The following lemma was proved in [91], see Lemma 4.4.

Lemma 3.3.8. *Let $\mathbb{H}$ be a normal subgroup of $\mathbb{G}$, with the subgroup surjection ρ. Let $\widehat{\rho}$ denote the associated surjection from* $\mathrm{Pol}(\mathbb{G})$ *to* $\mathrm{Pol}(\mathbb{H})$. *Then*

$$ker(\widehat{\rho}) = \mathrm{Pol}(\mathbb{G})^+_{\mathbb{G}/\mathbb{H}}\mathrm{Pol}(\mathbb{G}) = \mathrm{Pol}(\mathbb{G})\mathrm{Pol}(\mathbb{G})^+_{\mathbb{G}/\mathbb{H}} = \mathrm{Pol}(\mathbb{G})\mathrm{Pol}(\mathbb{G})^+_{\mathbb{G}/\mathbb{H}}\mathrm{Pol}(\mathbb{G})$$

Here, $\mathrm{Pol}(\mathbb{G})_{\mathbb{G}/\mathbb{H}}$ *denotes the dense Hopf-$*$-algebra associated with the CQG $\mathbb{G}/\mathbb{H}$ and* $\mathrm{Pol}(\mathbb{G})^+_{\mathbb{G}/\mathbb{H}}$ *denotes its augmentation ideal (i.e. kernel of its counit).*

Lemma 3.3.9. *Let $\mathbb{G}$ be a compact quantum group and $\mathbb{H}_1$ and $\mathbb{H}_2$ be normal subgroups of $\mathbb{G}$ such that $C_m(\mathbb{G}/\mathbb{H}_1) = C_m(\mathbb{G}/\mathbb{H}_2)$. There is a quantum automorphism $\alpha : C_m(\mathbb{H}_1) \to C_m(\mathbb{H}_2)$ such that $\rho_2 = \alpha \circ \rho_1$, where for $k = 1, 2$, ρ_k denotes the subgroup surjection corresponding to $\mathbb{H}_k$.*

Proof. We follow the notation as explained in Lemma 3.3.8. Since $C_m(\mathbb{G}/\mathbb{H}_1) = C_m(\mathbb{G}/\mathbb{H}_2)$, we have that $\mathrm{Pol}(\mathbb{G})_{\mathbb{G}/\mathbb{H}_1} = \mathrm{Pol}(\mathbb{G})_{\mathbb{G}/\mathbb{H}_2}$ and hence $\mathrm{Pol}(\mathbb{G})^+_{\mathbb{G}/\mathbb{H}_1} = \mathrm{Pol}(\mathbb{G})^+_{\mathbb{G}/\mathbb{H}_2}$. So, we have, by Lemma 3.3.8, that $\ker(\widehat{\rho}_1) = \ker(\widehat{\rho}_2)$. But then, it now follows from Lemma 2.2.4 that there exists an isomorphism $\widehat{\alpha} : \mathrm{Pol}(\mathbb{H}_1) \to \mathrm{Pol}(\mathbb{H}_2)$ such that $\widehat{\rho}_2 = \widehat{\alpha} \circ \widehat{\rho}_1$. We can now extend $\widehat{\alpha}$ to give us a C^*-algebraic isomorphism $\alpha : C_m(\mathbb{H}_1) \to C_m(\mathbb{H}_2)$. It is now easy to check that α is a quantum automorphism (see Lemma 2.2.1) and that $\rho_2 = \alpha \circ \rho_1$. □

The following lemma is not hard to deduce by methods similar to the previous one. Once again the notation is same as above.

Lemma 3.3.10. *Let $\mathbb{G}$ be a compact quantum group and $\mathbb{H}_1$ and $\mathbb{H}_2$ be normal subgroups of $\mathbb{G}$. Then the following are equivalent-*

1. *There exists a quantum group surjective morphism $\rho_0 : C_m(\mathbb{H}_1) \to C_m(\mathbb{H}_2)$ such that the following diagram*

$$\begin{tikzcd} C_m(\mathbb{G}) \arrow[r, "\rho_1"] \arrow[dr, "\rho_2"'] & C_m(\mathbb{H}_1) \arrow[d, "\rho_0"] \\ & C_m(\mathbb{H}_2) \end{tikzcd}$$

commutes. Here ρ_k for $k = 1, 2$ denote the subgroup surjections corresponding to $\mathbb{H}_k$. So, in particular $\mathbb{H}_2$ is a (normal) subgroup of $\mathbb{H}_1$.

2. $C_m(\mathbb{G}/\mathbb{H}_1) \subseteq C_m(\mathbb{G}/\mathbb{H}_2)$

Proof. (1) $\Rightarrow$ (2) This follows from the definitions of $C_m(\mathbb{G}/\mathbb{H}_k), k = 1, 2$.

(2) $\Rightarrow$ (1) Since $C_m(\mathbb{G}/\mathbb{H}_1) \subseteq C_m(\mathbb{G}/\mathbb{H}_2)$, so $\mathrm{Pol}(\mathbb{G})^+_{\mathbb{G}/\mathbb{H}_1} \subseteq \mathrm{Pol}(\mathbb{G})^+_{\mathbb{G}/\mathbb{H}_2}$ and hence, by Lemma 3.3.8, we have that $\ker(\widehat{\rho}_1) \subseteq \ker(\widehat{\rho}_2)$. So, invoking Lemma 2.2.4, we get a surjective map $\widehat{\rho}_0 : \mathrm{Pol}(\mathbb{H}_1) \to \mathrm{Pol}(\mathbb{H}_2)$, from which we can get the quantum group morphism ρ_0, by extending $\widehat{\rho}_0$ to $C_m(\mathbb{H}_1)$, which satisfies all the required properties.

$\square$

We are now in a position to state and prove the main theorem of this section.

Theorem 3.3.11. *Let $\mathbb{G}$ be a compact quantum group, possessing Property F. We further assume that $C_m(\mathbb{G})$ is separable. Suppose Γ acts on $\mathbb{G}$ by quantum automorphisms, with action denoted by α. Then there exists a countable ordinal ω and a family of normal subgroups of $\mathbb{G}$, given by*

$$\{\mathbb{H}_\xi : \xi \leq \omega\}$$

such that

1. *$\mathbb{H}_{\xi_1}$ is a subgroup of $\mathbb{H}_{\xi_0}$ for $\xi_0 < \xi_1$, with $\mathbb{H}_0 = \mathbb{G}$. In other words, we have a decreasing family of subgroups.*

2. *For all ξ, $\mathbb{H}_\xi$ is an invariant subgroup of $\mathbb{G}$ under the Γ-action and the induced Γ-action on $(\mathbb{H}_\xi)_{red}$ is non-ergodic for all $\xi < \omega$*

3. *$\mathbb{H}_\omega$ is the maximal unique normal subgroup of $\mathbb{G}$ which is invariant under the Γ-action and such that the induced action on $(\mathbb{H}_\omega)_{red}$ is ergodic.*

Proof. To minimise notation, we make an abuse, a statement of the form "Γ-action on a compact quantum group $\mathbb{G}$ is ergodic (resp. non-ergodic, compact, etc)" will mean that the induced Γ-action on $\mathbb{G}_{red}$ is ergodic (resp. non-ergodic, compact, etc).

We proceed by transfinite induction (separability of $C_m(\mathbb{G})$ ensures that $\mathrm{Irr}(\mathbb{G})$ is countable, hence we are dealing with countable transfinite induction). Since we have assumed that $\mathbb{G}$ possesses Property F, so, we have to construct, inductively, an increasing sequence of Woronowicz sub-C^*-algebras of $C_m(\mathbb{G})$, which are invariant under the Γ-action, so, by Lemma 3.3.7, the corresponding normal subgroup will be invariant as well. But then it suffices to construct, inductively, subobjects of $\mathrm{Irr}(\mathbb{G})$ which are invariant under the induced Γ-action. To this end, we construct the first subobject in the following manner. If the action is ergodic, we have nothing more to do. Hence, let the action be non-ergodic. We define the subobject of $\mathrm{Irr}(\mathbb{G})$

$$\mathcal{T}_1 := \{\beta \in \mathrm{Irr}(\mathbb{G}) : \text{orbit of } \beta \text{ is finite}\}$$

That it is a subobject follows from Proposition 3.3.3. We call it the compact sub-object of $\mathbb{G}$ under the Γ-action. It is also straightforward to see that for any $\beta \in \mathcal{T}_1$ and any $g \in \Gamma$, $g \cdot \beta \in \mathcal{T}_1$. So, the corresponding normal subgroup $\mathbb{H}_1$ is an invariant subgroup of $\mathbb{G}$ under the Γ-action. If the induced Γ-action on $\mathbb{H}_1$ is ergodic, we stop. Else, we construct the subobject of $\mathrm{Irr}(\mathbb{G})$ in the following manner- Since the induced action of Γ on $\mathbb{H}_1$ is non-ergodic, the compact subobject $\mathcal{S}_1 \subseteq \mathrm{Irr}(\mathbb{H}_1)$ under Γ-action is non-trivial. Let now $\beta \in \mathrm{Irr}(\mathbb{G})$, with a representative u^β which is an irreducible representation of $\mathbb{G}$. We then have a representation of $\mathbb{H}_1$ given by $v^\beta = (id \otimes \rho_1)(u^\beta)$, where ρ_1 denotes the subgroup surjection corresponding to $\mathbb{H}_1$. For $\gamma \in \mathrm{Irr}(\mathbb{H}_1)$, we write $\gamma \preccurlyeq v^\beta$, if there exist some representative v^γ of γ, which is a subrepresentation of v^β. We then define

$$S_\beta := \{\gamma \in \mathrm{Irr}(\mathbb{H}_1) : \gamma \preccurlyeq v^\beta\}$$

We then define the set $T_2 \subset \mathrm{Irr}(\mathbb{G})$ as follows

$$T_2 := \{\beta \in \mathrm{Irr}(\mathbb{G}) : S_\beta \cap \mathcal{S}_1 \neq \emptyset\}$$

Since $\mathcal{S}_1$ is non-trivial, it now follows from Lemma 2.3.6 that T_2 is also non-trivial. So, we now define

$$\mathcal{T}_2 := \{\text{subobject of } \mathrm{Irr}(\mathbb{G}) \text{ generated by } \mathcal{T}_1 \text{ and } T_2\}$$

In other words, $\mathcal{T}_2$ is the smallest subobject of $\mathrm{Irr}(\mathbb{G})$ containing $\mathcal{T}_1$ and T_2. Let us now show that $\mathcal{T}_2$ is invariant under the induced Γ-action. We first note that, since $\mathcal{S}_1$ is invariant under the induced Γ-action on $\mathrm{Irr}(\mathbb{H}_1)$, hence T_2 is also invariant under the Γ-action on $\mathrm{Irr}(\mathbb{G})$. We note also that T_2 is closed under taking conjugates. Now, suppose $\beta \in \mathcal{T}_2$ and

$$\beta \subset t_1 \otimes t_2 \otimes ... \otimes t_n$$

where t_i's comes either from $\mathcal{T}_1$ or T_2. Since for any $g \in \Gamma$ we have

$$g \cdot \beta \subset g \cdot t_1 \otimes g \cdot t_2 \otimes ... \otimes g \cdot t_n$$

and since both $\mathcal{T}_1$ and T_2 are invariant under the Γ-action, we have that $g \cdot \beta \in \mathcal{T}_2$ for any $g \in \Gamma$. Hence, corresponding to $\mathcal{T}_2$, we have an normal subgroup $\mathbb{H}_2$ of $\mathbb{G}$ which is invariant under the induced Γ-action. So, we proceed in this manner. If the ordinal is of successor type, i.e. the ordinal is of the form $\xi + 1$ for an ordinal ξ, then having constructed $\mathbb{H}_\xi$, normal subgroup of $\mathbb{G}$ invariant under the Γ-action, we construct $\mathbb{H}_{\xi+1}$ by constructing the subobject $\mathcal{T}_{\xi+1}$ of $\mathrm{Irr}(\mathbb{G})$ as the subobject generated by $\mathcal{T}_\xi$ and the set $T_{\xi+1}$. If the ordinal, say ξ, is a limit ordinal, i.e. for any $\eta_0 < \xi$, there exists an ordinal η_1 such that $\eta_0 < \eta_1 < \xi$, we define the subobject T_ξ as the subobject generated by all the subobjects T_η such that $\eta < \xi$. The smallest ordinal κ for which the induced action on the normal subgroup $\mathbb{H}_\kappa$ is ergodic will be our ω. So, we have constructed our family of normal subgroups

$$\{\mathbb{H}_\xi : \xi \leq \omega\}$$

The first two properties follow from the construction and from Lemma 3.3.10. It also follows from the construction that the normal subgroup $\mathbb{H}_\omega$ is invariant under the Γ-action and the induced action is ergodic.

Now we exhibit maximality of $\mathbb{H}_\omega$ as the normal subgroup on which the induced Γ-action is ergodic. In light of Lemma 3.3.10, it is enough to show that there exist no Γ-invariant subobject $\mathcal{T}$ of $\mathrm{Irr}(\mathbb{G})$ (with associated normal subgroup denoted by $\mathbb{H}$) such that $\mathcal{T} \subset \mathcal{T}_\omega$. To this end, first note that for such a subobject $\mathcal{T}$, one must have $\mathcal{T}_\xi \subset \mathcal{T} \subset \mathcal{T}_{\xi+1}$, with $\xi+1 = \omega$, since we are assuming that the action of Γ on $\mathbb{H}$ is ergodic. The inclusion $\mathcal{T}_\xi \subset \mathcal{T}$ is proper since the action of Γ on $\mathbb{H}_\xi$ is non-ergodic. From the inductive construction above, it follows that there exists $[u^s] \in T_{\xi+1}$ such that u^s when restricted to $\mathbb{H}$, decomposes as a sum of irreducible representations of $\mathbb{H}$, each of which are non trivial. Since $[u^s] \in T_{\xi+1}$, the restriction of u^s to $\mathbb{H}_\xi$ has as a subrepresentation an irreducible representation v^t of $\mathbb{H}_\xi$ whose orbit is finite under the Γ-action. Since, $\mathbb{H}$ is a normal subgroup of $\mathbb{H}_\xi$, the restriction of v^t to $\mathbb{H}$ decomposes as a sum of non trivial irreducible representations of $\mathbb{H}$, because this is the case for u^s. But, as the orbit of v^t is finite, it follows that the orbit of any irreducible representation of $\mathbb{H}$ which is a subrepresentation of the restriction of v^t to $\mathbb{H}$, is also finite. This is a contradiction to the ergodicity of the induced Γ-action on $\mathbb{H}$.

To exhibit uniqueness, we will show that if $\mathbb{H}$ is a normal subgroup of $\mathbb{G}$ which is invariant under the Γ-action and such that this induced Γ-action is ergodic, then the subobject $\mathrm{Irr}(\mathbb{G}/\mathbb{H}) \subseteq \mathrm{Irr}(\mathbb{G})$ contains $\mathcal{T}_\omega$, i.e. $\mathcal{T}_\omega \subseteq \mathrm{Irr}(\mathbb{G}/\mathbb{H})$. Uniqueness (upto quantum automorphism), will then follow from Lemma 3.3.9 and Lemma 3.3.10. To show this, we first prove the following claim

Claim: Let $\mathbb{G}$ be a CQG and suppose Γ acts on $\mathbb{G}$ by quantum automorphisms. Suppose $\mathbb{K}_0, \mathbb{K}_1, \mathbb{K}_2$ are Γ-invariant normal subgroups of $\mathbb{G}$ such that $C_m(\mathbb{G}/\mathbb{K}_1) \cap C_m(\mathbb{G}/\mathbb{K}_2) = C_m(\mathbb{G}/\mathbb{K}_0)$. Suppose further that the CQG dynamical systems $(\mathbb{K}_1, \Gamma)$ and $(\mathbb{K}_2, \Gamma)$ are ergodic. Then the CQG dynamical system $(\mathbb{K}_0, \Gamma)$ is also ergodic.

Proof of the Claim: Let us note first that by Lemma 3.3.10, $\mathbb{K}_1$ and $\mathbb{K}_2$ are normal subgroups of $\mathbb{K}_0$. Consider now the CQG dynamical system $(\mathbb{K}_0, \Gamma)$. Suppose there exists $t \in \mathrm{Irr}(\mathbb{K}_0)$ such that $\{\Gamma \cdot t\}$ (i.e. the Γ-orbit of t) is finite. We want to show t is trivial.

Note that $\mathrm{Irr}(\mathbb{G}/\mathbb{K}_0) = \mathrm{Irr}(\mathbb{G}/\mathbb{K}_1) \cap \mathrm{Irr}(\mathbb{G}/\mathbb{K}_2)$. By Lemma 2.3.6, we can find $\beta \in \mathrm{Irr}(\mathbb{G})$ such that $t \in S_\beta$. We will be done if we show that $\beta \in \mathrm{Irr}(\mathbb{G}/\mathbb{K}_0)$. To this end, we first note that since Γ-orbit of t is finite and the induced action of Γ on $\mathbb{K}_1$ and $\mathbb{K}_2$ is ergodic, we will have by Theorem 3.1.5 that, $S_t^1 \subset \mathrm{Irr}(\mathbb{K}_1)$ and $S_t^2 \subset \mathrm{Irr}(\mathbb{K}_2)$ are both trivial (i.e. just consist of e, the element corresponding to the trivial representation), where

$$S_t^k := \{\gamma \in \mathrm{Irr}(\mathbb{K}_k) : \gamma \preccurlyeq v^t\}$$

for $k = 1, 2$. But then, since $\mathbb{K}_1$ and $\mathbb{K}_2$ are normal in $\mathbb{G}$, it easily follows from Theorem 1.2.7 that $\beta \in \mathrm{Irr}(\mathbb{G}/\mathbb{K}_1)$ and $\beta \in \mathrm{Irr}(\mathbb{G}/\mathbb{K}_2)$, and hence, we have that $\beta \in \mathrm{Irr}(\mathbb{G}/\mathbb{K}_0)$ and so, we are done.

Now, uniqueness follows from the previous claim as suppose that there is a Γ-invariant normal subgroup of $\mathbb{G}$, $\mathbb{H}$ such that the Γ-action is ergodic. Now, since the subobjects $\mathrm{Irr}(\mathbb{G}/\mathbb{H})$ and $\mathcal{T}_\omega$ are Γ-invariant, so is the subobject $\mathcal{S}=\mathrm{Irr}(\mathbb{G}/\mathbb{H}) \cap \mathcal{T}_\omega$. By the previous claim, the Γ-action on the normal subgroup corresponding to $\mathcal{S}$ is ergodic. But then this contradicts maximality of $\mathcal{T}_\omega$, and hence $\mathcal{S} = \mathcal{T}_\omega$. So, we have that $\mathcal{T}_\omega \subseteq \mathrm{Irr}(\mathbb{G}/\mathbb{H})$ and we are done.

□

Remark 3.3.12. *For a CQG $\mathbb{G}$, its fusion ring $Z(\mathbb{G})$ is said to be of Lie type if all increasing sequences*

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \cdots$$

of sub-fusion rings of $Z(\mathbb{G})$ stabilize, i.e. $A_m = A_n$ for all $m \geq n$ for some n (see definition 5.2 of [26]). So, if in addition to the hypothesis of the previous theorem, the fusion ring is of Lie type, then we can get to the Maximal Normal Ergodic Subgroup in finitely many steps.

Remark 3.3.13. *Often, depending on the CQG dynamical system, we can compute the maximal ergodic normal subgroup, without requiring that the compact quantum group have Property F, as we shall see in the next example.*

Example 3.3.14. Given a Woronowicz C^*-algebra A, the notion of free wreath product $A *_w C(A_s(n))$ was defined and studied by Bichon in [15], which is also a Woronowicz C^*-algebra. So given a compact quantum group $\mathbb{G}$, we have an associated compact quantum group $\mathbb{G} \wr_* A_s(n) = (C_m(\mathbb{G}) *_w C(A_s(n)), \Delta_*)$. Now let Λ be a discrete group, and $\mathbb{G} = (C^*(\Lambda), \Delta)$ be the associated compact quantum group. In this case, we denote $\mathbb{G} \wr_* A_s(n)$ by $\Lambda \wr_* A_s(n)$. This class of compact quantum groups were extensively studied in [60]. Let e denote the neutral element of Λ and let $n \geq 4$. Let $A_n(\Lambda)$ denote the universal C^*-algebra with generators $a_{ij}(g)$ for $1 \leq i, j \leq n$ and $g \in \Lambda$ together with the relations:

$$a_{ij}(g)a_{ik}(h) = \delta_{jk}a_{ij}(gh) \; ; \; a_{ji}(g)a_{ki}(h) = \delta_{jk}a_{ji}(gh)$$

$$\sum_{l=1}^{n} a_{il}(e) = 1 = \sum_{l=1}^{n} a_{li}(e)$$

with involution given by

$$a_{ij}(g)^* = a_{ij}(g^{-1})$$

Then $\mathbb{G}_{\Lambda,n} = (A_n(\Lambda), \Delta)$ is a compact quantum group, where we have

$$\Delta(a_{ij}(g)) = \sum_{k=1}^{n} a_{ik}(g) \otimes a_{kj}(g)$$

In fact, we have that $\mathbb{G}_{\Lambda,n}$ is isomorphic as a compact quantum group to $\Lambda \wr_* A_s(n)$ (see Example 2.5 of [15]).

Let now Γ act on the group Λ by group automorphisms. We can now induce an action of Γ on $\mathbb{G}_{\Lambda,n}$ in the following way: given any $\gamma \in \Gamma$, it follows from the definition of $A_n(\Lambda)$ that the map

$$\alpha_\gamma(a_{ij}(g)) = a_{ij}(\gamma \cdot g)$$

defines an automorphism of $A_n(\Lambda)$. It is also easy to check that in fact this is a quantum automorphim. So, we have a CQG dynamical system $((\mathbb{G}_{\Lambda,n})_{red}, \Gamma)$ with the action denoted by α. We now proceed to study some properties of this action, for which we first elucidate the representation theory of $\mathbb{G}_{\Lambda,n}$. It was shown in [60] that there is a family of n-dimensional representations of $\mathbb{G}_{\Lambda,n}$, $\{a(g) : g \in \Lambda\}$ satisfying $(a(g))_{ij} = a_{ij}(g)$ and $\overline{a(g)} = a(g^{-1})$. For all $g \neq e$, the representation $a(g)$ are irreducible while $a(e) = 1 \oplus w(e)$, where $w(e)$ is an irreducible representation, and further, $a(g)$ for all $g \neq e$ and $w(e)$ are pairwise inequivalent. Let $M = \langle \Lambda \rangle$ denote the monoid formed by words over Λ endowed with the following operations- Involution: $(g_1, ..., g_k)^- = (g_k^{-1}, ..., g_1^{-1})$, Concatenation: $(g_1, ..., g_k) * (h_1, ...h_l) = (g_1, ..., g_{k-1}, g_k, h_1, ..., h_l)$ and Fusion: $(g_1, ..., g_k) \cdot (h_1, ..., h_l) = (g_1, ..., g_{k-1}, g_k h_1, h_2, ..., h_l)$. We then have the following theorem (see Theorem 2.25 of [60])

Theorem 3.3.15. *The irreducible representations of $\mathbb{G}_{\Lambda,n}$ can be labelled by $w(x)$ with $x \in M$ and involution $\overline{w(x)} = w(\overline{x})$ and the fusion rules:*

$$w(x) \otimes w(y) = \sum_{x=u*t, y=\bar{t}*v} w(u*v) \oplus \sum_{x=u*t, y=\bar{t}*v; u\neq\emptyset, v\neq\emptyset} w(u \cdot v)$$

Further, we have for all $g \in \Lambda$, $w(g) = a(g) \ominus \delta_{g,e} 1$

We can now prove the following

Theorem 3.3.16. *Let Γ act on the group Λ by group automorphisms. Consider the CQG dynamical system $((\mathbb{G}_{\Lambda,n})_{red}, \Gamma)$. The following then holds-*

1. *Suppose that Γ action on Λ is such that the orbit of any element is finite. Then the CQG dynamical system $((\mathbb{G}_{\Lambda,n})_{red}, \Gamma)$ is compact.*

2. *Consider the map $\rho : A_n(\Lambda) \to C^*(\Lambda^{*n}), a_{ij}(g) \mapsto \delta_{ij}\lambda_{v_i(g)}$ for all $1 \leq i, j \leq n$ and $g \in \Lambda$, where Λ^{*n} denotes the n-fold free product of Λ and v_i denotes the canonical homomorphism from Λ to Λ^{*n}, sending Λ to the ith copy of Λ in Λ^{*n}. We have that ρ defines a $*$-surjection which makes the compact quantum group $\mathbb{H} = C^*(\Lambda^{*n})$ into a normal subgroup of $\mathbb{G} = \mathbb{G}_{\Lambda,n}$. In this case, we have that the compact quantum group $\mathbb{G}/\mathbb{H}$ is isomorphic as quantum group to the compact quantum group $A_s(n)$.*

3. *Suppose that the Γ action on Λ is such that the orbit of any non-trivial element is infinite. Then the invariant normal subgroup $C^*(\Lambda^{*n})$ is the maximal ergodic normal subgroup of $\mathbb{G}_{\Lambda,n}$ under the induced Γ-action.*

Proof. 1. In view of Proposition 3.3.3 (and Remark 3.3.4), Theorem 3.1.8 and Theorem 3.3.15, we will be done if we can show that the orbit of the irreducible representation $a(g)$ of $\mathbb{G}_{\Lambda,n}$ under the Γ-action is finite for all $g \in \Lambda$. But it follows from the definition of the Γ-action α on $\mathbb{G}_{\Lambda,n}$ that the orbit of $a(g)$ is the set $\{a(\gamma \cdot g) : \gamma \in \Gamma\}$. The result now follows from the hypothesis.

2. It follows from the universal property of $A_n(\Lambda)$ that the map ρ is in fact a $*$-surjection. It is also straightforward to show that ρ is in fact a subgroup surjection. Hence, $\mathbb{H}$ is a quantum subgroup of $\mathbb{G}_{\Lambda,n}$. To show that it is normal, we first need the following

Lemma 3.3.17. *Let $\mathbb{G}_0, \mathbb{H}_0$ and $\mathbb{N}_0$ denote compact quantum groups such that $\mathbb{H}_0$ is a subgroup of $\mathbb{G}_0$, with the subgroup surjection denoted by ρ_1 and $\mathbb{N}_0$ is a normal subgroup of $\mathbb{G}$ with subgroup surjection denoted by ρ_2. Suppose further that $\mathbb{N}_0$ is a subgroup of $\mathbb{H}_0$ with subgroup surjection ρ such that $\rho \circ \rho_1 = \rho_2$. We then have that $\mathbb{N}_0$ is a normal subgroup of $\mathbb{H}_0$ and that $\rho_1(C_m(\mathbb{G}_0/\mathbb{N}_0)) = C_m(\mathbb{H}_0/\mathbb{N}_0)$.*

The proof of this lemma is straightforward. Indeed, suppose $\mathbb{N}_0$ is not normal subgroup of $\mathbb{H}_0$. Then, by Theorem 1.2.7 (3), we know that there exists some irreducible representation u^a of $\mathbb{H}_0$ such that $0 < [1_{\mathbb{N}_0}, u^a_{|\mathbb{N}_0}] <dim(u^a)$, where $[1_{\mathbb{N}_0}, u^a_{|\mathbb{N}_0}]$ denotes the multiplicity of the trivial representation as a sub-represenation of $u^a_{|\mathbb{N}_0}$. But then since $\mathbb{N}_0$ is a normal subgroup of $\mathbb{G}_0$, using Lemma 2.3.6, it is easy to derive a contradiction. Also, since $C_m(\mathbb{H}_0/\mathbb{N}_0)$ is spanned by the matrix coefficients of those irreducible representations u^a of $\mathbb{H}_0$ for which $[1_{\mathbb{N}_0}, u^a_{|\mathbb{N}_0}] =dim(u^a)$ and a similar result holds for $C_m(\mathbb{G}_0/\mathbb{N}_0)$, a simple application of Lemma 2.3.6 gives us that $\rho_1(C_m(\mathbb{G}_0/\mathbb{N}_0)) = C_m(\mathbb{H}_0/\mathbb{N}_0)$. This proves the lemma.

Let us now consider the free product compact quantum group $\mathbb{H} * A_s(n)$, as defined in [87]. It follows from Example 2.5 of [15] that that $\mathbb{G}_{\Lambda,n}$ is a quantum subgroup of $\mathbb{H} * A_s(n)$, with the subgroup surjection ρ_s defined by $\rho_s(\lambda_{v_i(g)}x_{ij}) = a_{ij}(g)$, where $x = ((x_{ij}))_{1\leq i,j\leq n}$ is the defining magic unitary of $A_s(n)$. It also follows from Theorem 3.4 of [87] that $\mathbb{H}$ is a normal subgroup of $\mathbb{H} * A_s(n)$ with the subgroup surjection ρ_0 defined as $\rho_0(\lambda_{v_i(g)}x_{ij}) = \lambda_{v_i(g)} \cdot \epsilon_{A_s(n)}(x_{ij}) = \delta_{ij}\lambda_{v_i(g)}$. It is then easy to see that $\rho \circ \rho_s = \rho_0$. It then follows from the previous lemma that $\mathbb{H}$ is normal subgroup of $\mathbb{G}_{\Lambda,n}$.

We now define $p_s : A_n(\Lambda) \to C_m(A_s(n)), p_s(a_ij(g)) = x_{ij}$. It easily follows from the universal property of $A_n(\Gamma)$ that p_s defines a $*$-surjection. It is then easy to check that $A_s(n)$ is quantum subgroup of $\mathbb{G}_{\Lambda,n}$ with subgroup surjection p_s. Further, we have the $*$-embedding $i_s : C_m(A_s(n)) \to C^*(\Lambda^{*n}) * C_m(A_s(n))$. We then have $p_s \circ \rho \circ i_s = id$. Now since it follows from Theorem 3.4 of [87] that the compact quantum group $\mathbb{H} * A_s(n)/\mathbb{H}$ is isomorphic to $A_s(n)$ with the isomorphism given by the map i_s, and using the fact that i_s, ρ and p_s are all quantum group homomorphisms, it follows easily that the compact quantum group $\mathbb{G}/\mathbb{H}$ is isomorphic as a quantum group to $A_s(n)$.

3. We consider the action of Γ on $\mathbb{G}_{\Lambda,n}$ as defined above. We then have an induced action of Γ on $\mathrm{Irr}(\mathbb{G}_{\Lambda,n})$. For an irreducible representation labelled by $w(g_1, g_2, ..., g_m)$, we have that for any $\gamma \in \Gamma$, $\gamma \cdot w(g_1, ..., g_m) = w(\gamma \cdot g_1, ..., \gamma \cdot g_m)$. This follows easily from Theorem 3.3.15 using induction. Indeed, we have that $\gamma \cdot w(g) = w(\gamma \cdot g)$, now suppose it is true for any sequence of m elements of Λ, $g_1, ..., g_m$ that $\gamma \cdot w(g_1, ..., g_m) = w(\gamma \cdot g_1, ..., \gamma \cdot g_m)$, we want to shown that the same is true for any sequence of $(m + 1)$ elements of Λ, $t_1, ..., t_m, t_{m+1}$. To do this, let us first assume that $t_m t_{m+1} \neq 1$. In this case, we have that $w(t_1, ..., t_m) \otimes w(t_{m+1}) =$

$w(t_1, ..., t_m, t_{m+1}) \oplus w(t_1, ..., t_m t_{m+1})$. Hence, for any $\gamma \in \Gamma$, we have that

$$\gamma \cdot (w(t_1, ..., t_m) \otimes w(t_{m+1})) = \gamma \cdot w(t_1, ...t_m, t_{m+1}) \oplus \gamma \cdot w(t_1, ..., t_m t_{m+1})$$

$$\Rightarrow \gamma \cdot w(t_1, ..., t_m) \otimes \gamma \cdot w(t_{m+1}) = \gamma \cdot w(t_1, ...t_m, t_{m+1}) \oplus \gamma \cdot w(t_1, ..., t_m t_{m+1})$$

$$\Rightarrow \gamma \cdot w(t_1, ..., t_m) \otimes \gamma \cdot w(t_{m+1}) = \gamma \cdot w(t_1, ..., t_m, t_{m+1}) \oplus w(\gamma t_1, ..., \gamma t_m t_{m+1})$$

$$\Rightarrow w(\gamma \cdot t_1, ..., \gamma \cdot t_m) \otimes w(\gamma \cdot t_{m+1}) = \gamma \cdot w(t_1, ..., t_m, t_{m+1}) \oplus w(\gamma t_1, ..., \gamma t_m t_{m+1})$$

$$\Rightarrow w(\gamma \cdot t_1, ..., \gamma \cdot t_m, \gamma \cdot t_{m+1}) \oplus w(\gamma \cdot t_1, ..., \gamma \cdot (t_m t_{m+1}))$$
$$= \gamma \cdot w(t_1, ..., t_m, t_{m+1}) \oplus w(\gamma t_1, ..., \gamma t_m t_{m+1})$$

$$\Rightarrow w(\gamma \cdot t_1, ..., \gamma \cdot t_m, \gamma \cdot t_{m+1}) = \gamma \cdot w(t_1, ..., t_m, t_{m+1})$$

The case when $t_m t_{m+1} = 1$ can be proved similarly. Thus, it is now easy to see that since the orbit of any non-trivial element in Λ under the Γ-action is infinite, we have that an irreducible representation of $\mathbb{G}_{\Lambda,n}$ labelled by $w(g_1, ...g_k)$ has a finite orbit under the induced Γ-action if and only if $g_1 = g_2 = ... = g_k = e$. Consider now the subgroup surjection $\rho : A_n(\Lambda) \to C^*(\Lambda^{*n})$. Let $\mathbb{G} = \mathbb{G}_{\Lambda,n}$ and $\mathbb{H} = C^*(\Lambda^{*n})$. By induction method as above, it can be shown that an irreducible representation labelled by $w(g_1, ..., g_k)$ with $g_1 = g_2 = ... = g_k = e$ lies in the coset space $\mathbb{G}/\mathbb{H}$. But, as is shown in the proof of Proposition 3.2 of [60], the closed subspace of $A_s(\Gamma)$ spanned by matrix coefficients of irreducible represenations labelled by $w(g_1, .., g_k)$ for any $k \in \mathbb{N}$, with $g_1 = ... = g_k = e$, is isomorphic as a C^*-algebra to $C_m(A_s(n))$. So, we have, using (2) of this Theorem, that the compact part (see Remark 3.3.4) of the Γ-action on $\mathbb{G}$ is the compact quantum group $\mathbb{G}/\mathbb{H}$.

The action of Γ on Λ^{*n}, defined by $\gamma \cdot v_i(g) = v_i(\gamma \cdot g)$ induces an action of Γ on $\mathbb{H}$. Since the action of Γ on Λ is such that any non-trivial element has an infinite orbit, we have using Theorem 3.1.5 and Proposition 3.2.1, that the action of Γ on $\mathbb{H}_{red}$ is ergodic. Also, since the compact part of the Γ-action on $\mathbb{G}$ is $\mathbb{G}/\mathbb{H}$, it can be shown, as in the proof of Theorem 3.3.11, that $\mathbb{H}$ is the maximal ergodic normal subgroup of $\mathbb{G}_{\Lambda,n}$ under the induced Γ-action and hence, we are done.

$\square$

Chapter 4

The Bicrossed Product and the Crossed Product quantum group

In this chapter, we study the bicrossed product and the crossed product compact quantum groups. We study their representation theory and make a detailed study of various approximation properties possessed by these quantum groups, under suitable conditions. The first part of this chapter studies the bicrossed product quantum group of a specific type while in the second part, we discuss the crossed product case. The last section of this chapter is devoted to examples, and a study of approximation properties for these specific examples. We also give examples of discrete quantum groups possessing Property (T).

The theory of quantum groups finds its roots in the work of Kac [49, 50] in the early sixties, and his notion of ring groups in modern terms are known as finite dimensional Kac algebras. In the fundamental work [51] on extensions of finite groups, Kac introduced the notion of *matched pair* of finite groups and developed the bicrossed product construction giving the first examples of semisimple Hopf algebras that are neither commutative nor cocommutative. It was later generalized by Baaj and Skandalis [4] in the context of Kac algebras and then by Vaes and Vainerman [83] in the framework of locally compact [1] (l.c. in the sequel) quantum groups; the latter was introduced by Kustermans and Vaes in [59]. In the classical case, i.e., in the ambience of groups, Baaj and Skandalis concentrated only on the case of *regular* matched pairs of l.c. groups. In [83], the authors extended the study to *semi-regular* matched pairs of l.c. groups. The case of a general matched pair of locally compact groups was settled by Baaj, Skandalis and Vaes in [5].

As a standing assumption, all throughout this chapter, all Hilbert spaces and all C*-algebras are separable, all von Neumann algebras have separable preduals, all discrete groups are countable and all compact groups are Hausdorff and second countable.

[1] All l.c. spaces are assumed to be Hausdorff.

4.1 Representation theory of bicrossed products

This section has two parts. In the first part, we discuss matched pair of groups of which one is compact and show an automatic regularity property of such matched pairs (Proposition 4.1.2). In the second part, we study bicrossed products of compact matched pair of groups and study their representation theory and related concepts.

4.1.1 Matched pairs

Definition 4.1.1 ([5]). *We say that a pair of l.c. groups (G_1, G_2) is matched if both G_1, G_2 are closed subgroups of a l.c. group H satisfying $G_1 \cap G_2 = \{e\}$ and such that the complement of G_1G_2 in H has Haar measure zero.*

From a matched pair (G_1, G_2) as above, one can construct a l.c. quantum group called the bicrossed product and it follows from [83] that the bicrossed product is compact if and only if G_1 is discrete and G_2 is compact. In the next proposition, we show some regularity properties of matched pairs (G_1, G_2) with G_2 being compact.

Proposition 4.1.2. *Let (G_1, G_2) be a matched pair and suppose that G_2 is compact. Then $G_1G_2 = H$, and, for all $(\gamma, g) \in G_1 \times G_2$ there exists unique $(\alpha_\gamma(g), \beta_g(\gamma)) \in G_2 \times G_1$ such that $\gamma g = \alpha_\gamma(g)\beta_g(\gamma)$. Moreover,*

1. *For $g, h \in G_2$ and $r, s \in G_1$, we have*

$$\alpha_r(gh) = \alpha_r(g)\alpha_{\beta_g(r)}(h), \ \ \beta_g(rs) = \beta_{\alpha_s(g)}(r)\beta_g(s) \quad \text{and} \quad \alpha_r(e) = e, \ \ \beta_g(e) = e. \tag{4.1.1}$$

2. *α is a continuous left action of G_1 on the topological space G_2. Moreover, the Haar measure on G_2 is α-invariant whenever G_1 is discrete.*

3. *β is a continuous right action of G_2 on the topological space G_1.*

4. *α is trivial $\Leftrightarrow$ G_1 is normal in H. Also, β is trivial $\Leftrightarrow$ G_2 is normal in H.*

Proof. First observe that, since G_2 is compact, H is Hausdorff and G_1 is closed, the set G_1G_2 is closed. Hence, the complement of G_1G_2 is open and has Haar measure zero. It follows that $G_1G_2 = H = H^{-1} = G_2^{-1}G_1^{-1} = G_2G_1$. Since $G_1 \cap G_2 = \{e\}$, the existence and uniqueness of $\alpha_\gamma(g)$ and $\beta_g(\gamma)$ for all $\gamma \in G_1$ and $g \in G_2$ are obvious. Then, the relations in (1) and the facts that α (resp. β) is a left (resp. right) action as in the statement easily follow from the aforementioned uniqueness.

Now let us check the continuity of these actions. Since the subgroup G_1 is closed in the l.c. group H, so H/G_1 is a l.c. Hausdorff space equipped with the quotient topology and the projection map $\pi : H \to H/G_1$ is continuous. Hence, $\pi_{|G_2} : G_2 \to H/G_1$ is continuous and bijective since $G_1 \cap G_2 = \{e\}$ and $G_1G_2 = H$. Since G_2 is compact, $\pi_{|G_2}$ is an homeomorphism. Let $\rho : H/G_1 \to G_2$ be the inverse of $\pi_{|G_2}$ and observe that the map $\alpha : G_1 \times G_2 \to G_2$, $(\gamma, g) \mapsto \alpha_\gamma(g)$ satisfies $\alpha = \rho \circ \pi \circ \psi$, where $\psi : G_1 \times G_2 \to H$ is the continuous map given by $\psi(\gamma, g) = \gamma g$, for $\gamma \in G_1, g \in G_2$. Consequently, the action α is continuous. Since for all $\gamma \in G_1$ and $g \in G_2$, we have $\beta_g(\gamma) = \alpha_\gamma(g)^{-1}\gamma g$, we deduce the continuity of $\beta : G_1 \times G_2 \to G_1$, $(\gamma, g) \mapsto \beta_g(\gamma)$ from the continuity of α and the continuity of the product and inverse operations in H.

Moreover, suppose that G_1 is discrete. Then G_1 is a co-compact lattice in H and it follows from the general theory (see for example Section 2.6 of [37]) that H is unimodular and hence there exists a unique H-invariant Borel probability measure ν on H/G_1. Consider the homeomorphism $\pi_{|G_2} : G_2 \to H/G_1$ and the Borel probability measure $\mu = (\pi_{|G_2})_*(\nu)$ on G_2. Since, for all $\gamma \in G_1$, the map $\pi_{|G_2}$ intertwines the homeomorphism α_γ of G_2 with the left translation by γ on H/G_1 and since ν is invariant under the left translation by γ, it follows that μ is invariant under α_γ. Also, $\pi_{|G_2}$ intertwines the left translation by h on G_2 with the left translation by h on H/G_1 for all $h \in G_2$. Hence, μ is invariant under the left translation by h for all $h \in G_2$. It follows that μ is the Haar measure.

Suppose that G_1 is normal is H. Then for all $\gamma \in G_1$, $g \in G_2$, we have $g^{-1}\gamma g = g^{-1}\alpha_\gamma(g)\beta_g(\gamma) \in G_1$. Since $g^{-1}\alpha_\gamma(g) \in G_2$ and $G_1 \cap G_2 = \{1\}$, we deduce that $g^{-1}\alpha_\gamma(g) =$

1 for all $\gamma \in G_1$, $g \in G_2$. For the reverse implication in (4), suppose that α is trivial. Then for all $\gamma \in G_1$, $g \in G_2$, we have $\gamma g = g\beta_g(\gamma) \in G_1$. Hence, $g^{-1}G_1 g \subset G_1$ for all $g \in G_2$ and since we trivially have $\gamma^{-1}G_1\gamma \subset G_1$ for all $\gamma \in G_1$ and $H = G_1 G_2$, we deduce that G_1 is normal in H. The proof of the last assertion of the Proposition is analogous. $\square$

In the next proposition, we discuss the well known Zappa-Szép product (also known as the Zappa-Rédei-Szép product, general product or knit product). It is a converse of Proposition 4.1.2. We include a proof for the convenience of the reader.

Proposition 4.1.3. *Suppose that G_1 and G_2 are two l.c. groups with a continuous left action α of G_1 on the topological space G_2 and a continuous right action β of G_2 on the topological space G_1 satisfying the relations (4.1.1). Then there exists a l.c. group H for which G_1, G_2 are closed subgroups satisfying $G_1 \cap G_2 = \{e\}$, $H = G_1 G_2$, and for all $\gamma \in G_1, g \in G_2$, $\gamma g = \alpha_\gamma(g)\beta_g(\gamma)$.*

Proof. Consider the l.c. space $H = G_1 \times G_2$ and define a product on H by the formula:

$$(r, g)(s, h) = (\beta_h(r)s, g\alpha_r(h)) \quad \text{for all } r, s \in G_1,\ g, h \in G_2.$$

It is routine to check that this multiplication turns H into a l.c. group. Moreover, we may identify G_1 with a closed subgroup of H by the map $G_1 \ni r \mapsto (r, 1) \in G_1 \times G_2$ and G_2 with a closed subgroup of H by the map $G_2 \ni g \mapsto (1, g) \in G_1 \times G_2$. After these identifications, we have $H = G_1 G_2$, $G_1 \cap G_2 = \{e\}$, and for all $\gamma \in G_1, g \in G_2$, $\gamma g = \alpha_\gamma(g)\beta_g(\gamma)$. $\square$

4.1.2 Representation theory

We first construct the bicrossed product from a compact matched pair and then study its representation theory. Along the way we prove some straightforward consequences e.g., amenability, K-amenability and Haagerup property of the dual of the bicrossed product.

We also compute the intrinsic group and the spectrum of the maximal C*-algebra of the bicrossed product.

Let (Γ, G) be a matched pair of a countable discrete group Γ and a compact group G. Associated to the continuous action β of the compact group G on the countable infinite set Γ, we have a magic unitary $v^{\gamma \cdot G} = (v_{rs})_{r,s \in \gamma \cdot G} \in M_{|\gamma \cdot G|}(\mathbb{C}) \otimes C(G)$ for every $\gamma \cdot G \in \Gamma / G$, where $v_{rs} = 1_{A_{r,s}}$ and $A_{r,s} = \{g \in G : \beta_g(r) = s\}$ (this is a continuous function on G, as is explained in Section 1.5 of the first chapter).

We define the C*-algebra $A_m = \Gamma\,_{\alpha,f} \ltimes C(G)$ to be the full crossed product and the C*-algebra $A = \Gamma_\alpha \ltimes C(G)$ to be the reduced crossed product. With abuse of notation, we denote by α the canonical injective maps from $C(G)$ to A_m and from $C(G)$ to A. We also denote by u_γ, $\gamma \in \Gamma$, the canonical unitaries viewed in either A_m or A. Observe that A_m is the enveloping C*-algebra of the unital *-algebra

$$\mathcal{A} = \mathrm{Span}\{u_\gamma \alpha(u^x_{ij}) \,:\, \gamma \in \Gamma, x \in \mathrm{Irr}(G), 1 \leq i, j \leq \dim(x)\}.$$

Let $\lambda : A_m \to A$ be the canonical surjection. Since the action α on (G, μ) is μ-preserving and μ is a probability measure, so there exists a unique faithful trace τ on A such that

$$\tau(u_\gamma \alpha(F)) = \delta_{\gamma,e} \int F d\mu, \;\; \gamma \in \Gamma, F \in C(G).$$

Theorem 4.1.4. *There exists a unique unital $*$-homomorphism $\Delta_m : A_m \to A_m \otimes A_m$ such that*

$$\Delta_m \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_G \quad \textit{and} \quad \Delta_m(u_\gamma) = \sum_{r \in \gamma \cdot G} u_\gamma \alpha(v_{\gamma,r}) \otimes u_r, \;\; \forall \gamma \in \Gamma.$$

Moreover, $\mathbb{G} = (A_m, \Delta_m)$ is a compact quantum group and we have:

1. *The Haar state of $\mathbb{G}$ is $h = \tau \circ \lambda$, hence $\mathbb{G}$ is Kac.*

2. *The set of unitary representations of* $\mathbb{G}$ *of the form* $V^{\gamma\cdot G}\otimes v^x$ *for some* $\gamma\cdot G\in\Gamma/G$ *and* $x\in\mathrm{Irr}(G)$*, where* $V^{\gamma\cdot G}=\sum_{r,s\in\gamma\cdot G}e_{r,s}\otimes u_r\alpha(v_{r,s})\in M_{|\gamma\cdot G|}(\mathbb{C})\otimes\mathcal{A}$ *and* $v^x=(\mathrm{id}\otimes\alpha)(u^x)$*, is a complete set of irreducible unitary representations of* $\mathbb{G}$*.*

3. *We have* $C_m(\mathbb{G})=A_m$*,* $C(\mathbb{G})=A$*,* $\mathrm{Pol}(G)=\mathcal{A}$*,* λ *is the canonical surjection and* $\mathrm{L}^\infty(\mathbb{G})$ *is the von Neumann algebraic crossed product.*

4. *The counit* $\varepsilon_{\mathbb{G}}:C_m(\mathbb{G})\to\mathbb{C}$ *is the unique unital* $*$*-homomorphism such that* $\varepsilon_{\mathbb{G}}(\alpha(F))=F(e)$ *for all* $F\in C(G)$ *and* $\varepsilon_{\mathbb{G}}(u_\gamma)=1$ *for all* $\gamma\in\Gamma$*.*

The compact quantum group $\mathbb{G}$ associated to the compact matched pair (Γ,G) in Theorem 4.1.4 is called the *bicrossed product.*

Proof. The uniqueness of Δ_m is obvious. To show the existence, it suffices to check that Δ_m satisfies the universal property of A_m.

Let us check that $\gamma\mapsto\Delta_m(u_\gamma)$ is a unitary representation of Γ. Let $\gamma\in\Gamma$. We first check that $\Delta_m(u_\gamma)$ is unitary. Observe that, for all $g\in G$ and $\gamma\in\Gamma$, we have

$$1=\beta_g(\gamma^{-1}\gamma)=\beta_{\alpha_\gamma(g)}(\gamma^{-1})\beta_g(\gamma).$$

Hence, $(\beta_g(\gamma))^{-1}=\beta_{\alpha_\gamma(g)}(\gamma^{-1})$. From this relation it is easy to check that $\Gamma^{-1}\cdot G=\{r^{-1}:r\in\gamma\cdot G\}$ and $\alpha_\gamma(v_{\gamma,r^{-1}})=v_{\gamma^{-1},r}$ for all $r\in\Gamma$. It follows that

$$\begin{aligned}\Delta_m(u_\gamma)^*&=\sum_{r\in\gamma\cdot G}\alpha(v_{\gamma,r})u_{\gamma^{-1}}\otimes u_{r^{-1}}=\sum_{r\in\gamma\cdot G}u_{\gamma^{-1}}\alpha(\alpha_\gamma(v_{\gamma,r}))\otimes u_{r^{-1}}\\&=\sum_{r\in\gamma^{-1}\cdot G}u_{\gamma^{-1}}\alpha(v_{\gamma^{-1},r})\otimes u_r=\Delta_m(u_{\gamma^{-1}}).\end{aligned}$$

Let $\gamma_1,\gamma_2\in\Gamma$. We have

$$\Delta_m(u_{\gamma_1})\Delta_m(u_{\gamma_2})=\sum_{r\in\gamma_1\cdot G,s\in\gamma_2\cdot G}u_{\gamma_1}\alpha(v_{\gamma_1,r})u_{\gamma_2}\alpha(v_{\gamma_2,s})\otimes u_{rs}=\sum_{r,s}u_{\gamma_1\gamma_2}\alpha\left(\alpha_{\gamma_2^{-1}}(v_{\gamma_1,r})v_{\gamma_2,s}\right)\otimes u_{rs}.$$

Observe that $\alpha_{\gamma_2^{-1}}(v_{\gamma_1,r})v_{\gamma_2,s} = 1_{B_{\gamma_1,\gamma_2,r,s}}$, where

$$B_{\gamma_1,\gamma_2,r,s} = \{g \in G \,:\, \beta_{\alpha_{\gamma_2}(g)}(\gamma_1) = r \text{ and } \beta_g(\gamma_2) = s\} \subset A_{\gamma_1\gamma_2,rs} = \{g \in G \,:\, \beta_g(\gamma_1\gamma_2) = rs\},$$

since $\beta_{\alpha_{\gamma_2}(g)}(\gamma_1)\beta_g(\gamma_2) = \beta_g(\gamma_1\gamma_2)$. In particular, $B_{\gamma_1,\gamma_2,r,s} = \emptyset$ whenever $rs \notin \gamma_1\gamma_2 \cdot G$; hence

$$\Delta_m(u_{\gamma_1})\Delta_m(u_{\gamma_2}) = \sum_{t\in\gamma_1\gamma_2\cdot G, r\in\gamma_1\cdot G} u_{\gamma_1\gamma_2}\alpha\left(1_{B_{\gamma_1,\gamma_2,r,r^{-1}t}}\right) \otimes u_t = \sum_{t\in\gamma_1\gamma_2\cdot G} u_{\gamma_1\gamma_2}\alpha(F_t) \otimes u_t,$$

where $F_t = \sum_r 1_{B_{\gamma_1,\gamma_2,r,r^{-1}t}} = 1_{\sqcup_r B_{\gamma_1,\gamma_2,r,r^{-1}t}} = 1_{A_{\gamma_1\gamma_2,t}}$, and $A_{\gamma_1\gamma_2,t} = \{g \in G \,:\, \gamma_1\gamma_2\cdot g = t\}$. Consequently, $1_{A_{\gamma_1\gamma_2,t}} = v_{\gamma_1\gamma_2,t}$ and $\Delta_m(u_{\gamma_1})\Delta_m(u_{\gamma_2}) = \Delta_m(u_{\gamma_1\gamma_2})$. Since $\Delta_m(u_e) = 1$, it follows that $\gamma \mapsto \Delta_m(u_\gamma)$ is a unitary representation of Γ.

Let us now check that the relations of the crossed product are satisfied. For $\gamma \in \Gamma$ and $F \in \mathrm{Pol}(G)$ we have:

$$\begin{aligned}
\Delta_m(u_\gamma)\Delta_m(\alpha(F))\Delta_m(u_\gamma^*) &= \sum_{r,s}(u_\gamma \otimes u_r)(\alpha\otimes\alpha)\left((v_{\gamma,r}\otimes 1)\Delta_G(F)\right)\left(u_{\gamma^{-1}}\alpha(v_{\gamma^{-1},s})\otimes u_s\right) \\
&= \sum_{r,s}(u_\gamma \otimes u_r)(\alpha\otimes\alpha)\left((v_{\gamma,r}\alpha_{\gamma^{-1}}(v_{\gamma^{-1},s})\otimes 1)\Delta_G(F)\right)\left(u_{\gamma^{-1}}\otimes u_s\right) \\
&= \sum_{r,s}(\alpha\otimes\alpha)\left((\alpha_\gamma(v_{\gamma,r})v_{\gamma^{-1},s}\otimes 1)(\alpha_\gamma\otimes\alpha_r)(\Delta_G(F))\right)(1\otimes u_{rs}) \\
&= \sum_{r,t}(\alpha\otimes\alpha)\left((\alpha_\gamma(v_{\gamma,r})v_{\gamma^{-1},r^{-1}t}\otimes 1)(\alpha_\gamma\otimes\alpha_r)(\Delta_G(F))\right)(1\otimes u_t) \\
&= \sum_t(\alpha\otimes\alpha)(H_t)(1\otimes u_t),
\end{aligned}$$

where $H_t = \sum_r(\alpha_\gamma(v_{\gamma,r})v_{\gamma^{-1},r^{-1}t}\otimes 1)(\alpha_\gamma\otimes\alpha_r)(\Delta_G(F))$.

Observe that $\alpha_\gamma(v_{\gamma,r})v_{\gamma^{-1},r^{-1}t} = 1_{B_{\gamma,r,t}}$, where

$$B_{\gamma,r,t} = \{g \in G \,:\, \beta_{\alpha_{\gamma^{-1}}(g)}(\gamma) = r \text{ and } \beta_g(\gamma^{-1}) = r^{-1}t\}.$$

Since $\beta_{\alpha_{\gamma^{-1}}(g)}(\gamma)\beta_g(\gamma^{-1}) = \beta_g(\gamma\gamma^{-1}) = \beta_g(e) = e$, we deduce that $B_{\gamma,r,t} = \emptyset$ whenever $t \neq e$, and it is easy to see that $\sqcup_{r\in\gamma\cdot G}B_{\gamma,r,e} = G$. Hence, $H_t = 0$ for $t \neq e$. Again for

$g \in B_{\gamma,r,e}$ and $h \in G$, one has $H_e(g,h) = F(\alpha_{\gamma^{-1}}(g)\alpha_{r^{-1}}(h)) = F(\alpha_{\gamma^{-1}}(g)\alpha_{\beta_g(\gamma^{-1})}(h)) = F(\alpha_{\gamma^{-1}}(gh))$. It follows that $H_e = \Delta_G(\alpha_\gamma(F))$. Consequently, $\Delta_m(u_\gamma)\Delta_m(\alpha(F))\Delta_m(u_\gamma^*) = (\alpha \otimes \alpha)(H_e)$. This completes the proof of the existence of Δ_m.

It is clear that v^x (as defined in the statement) is unitary and since $(\alpha\otimes\alpha)\Delta_G = \Delta_m \circ \alpha$, we have $\Delta_m(v^x_{ij}) = \sum_k v^x_{ik} \otimes v^x_{kj}$. Observe that $V^{\gamma\cdot G} = D_\gamma(\mathrm{id}\otimes\alpha)(v^{\gamma\cdot G}) \in M_{|\gamma\cdot G|}(\mathbb{C}) \otimes \mathcal{A}$, where D_γ is the diagonal matrix with entries u_r, $r \in \gamma \cdot G$. Hence, $V^{\gamma\cdot G}$ is unitary. Moreover,

$$\begin{aligned}
\Delta_m(V^{\gamma\cdot G}_{rs}) &= \Delta_m(u_r\alpha(v_{rs})) = \sum_{t\in r\cdot G=\gamma\cdot G} (u_r\alpha(v_{rt})\otimes u_t)(\alpha\otimes\alpha)(\Delta_G(v_{rs})) \\
&= \sum_{t,z\in\gamma\cdot G} u_r\alpha(v_{rt}v_{rz})\otimes u_t\alpha(v_{zs}) = \sum_{t\in\gamma\cdot G} u_r\alpha(v_{rt})\otimes u_t\alpha(v_{ts}) = \sum_{t\in\gamma\cdot G} V^{\gamma\cdot G}_{rt}\otimes V^{\gamma\cdot G}_{ts}.
\end{aligned}$$

It follows from [87, Definition 2.1'] that $\mathbb{G}$ is a compact quantum group and $V^{\gamma\cdot G}$, v^x are unitary representations of $\mathbb{G}$ for all $\gamma\cdot G \in \Gamma/G$ and $x \in \mathrm{Irr}(G)$.

(1). Since $\sum_s V^{\gamma\cdot G}_{rs} = u_r$, the linear span of the coefficients of the representations $V^{\gamma\cdot G}\otimes v^x$ for $\gamma \in \Gamma/G$ and $x \in \mathrm{Irr}(G)$ is equal to $\mathcal{A}$. Hence, it suffices to check the invariance of h on the coefficients of $V^{\gamma\cdot G}\otimes v^x$. We have

$$h(V^{\gamma\cdot G}_{rs}v^x_{ij})) = h(u_r\alpha(v_{rs}v^x_{ij})) = \delta_{r,e}\int_G v_{es}v^x_{ij}d\mu = \delta_{r,e}\delta_{s,e}\int_G v^x_{ij}d\mu = \delta_{r,e}\delta_{s,e}\delta_{x,1},$$

since $v_{es} = \delta_{s,e}1$ and v^x is irreducible. Hence, if $x \neq 1$, we have

$$(\mathrm{id}\otimes h)\Delta_m(V^{\gamma\cdot G}_{rs}v^x_{ij}) = \sum_{t,k} V^{\gamma\cdot G}_{rt}v^x_{ik}h(V^{\gamma\cdot G}_{ts}v^x_{kj}) = 0 = \sum_{t,k} h(V^{\gamma\cdot G}_{rt}v^x_{ik})V^{\gamma\cdot G}_{ts}v^x_{kj} = (h\otimes\mathrm{id})\Delta_m(V^{\gamma\cdot G}_{rs}v^x_{ij}).$$

And, if $x = 1$, we have $(\mathrm{id}\otimes h)\Delta_m(V^{\gamma\cdot G}_{rs}) = \sum_t V^{\gamma\cdot G}_{rt}h(V^{\gamma}_{ts}) = \delta_{\gamma,e}1 = (h\otimes\mathrm{id})\Delta_m(V^{\gamma\cdot G}_{rs})$. It follows that h is the Haar state.

(2). To simplify the notations, we write $\gamma\cdot G\otimes x = V^{\gamma\cdot G}\otimes v^x$ during this proof. For a unitary representation u (of $\mathbb{G}$ or G), we denote by $\chi(u) = \sum_i u_{ii}$ its character. Observe that $\chi(\gamma\cdot G\otimes x) = \chi(V^{\gamma\cdot G})\alpha(\chi(x)) = \sum_{r\in\gamma\cdot G} u_r\alpha(v_{rr})\alpha(\chi(x))$. Hence, for all $\gamma,\gamma' \in \Gamma$, and all $x, y \in \mathrm{Irr}(G)$, we have

$$\begin{aligned}\chi(\overline{\gamma' \cdot G \otimes y} \otimes \gamma \cdot G \otimes x) &= \chi(\gamma' \cdot G \otimes y)^* \chi(\gamma \cdot G \otimes x) = \sum_{s \in \gamma' \cdot G, r \in \gamma \cdot G} \alpha(\chi(\overline{y}) v_{ss}) u_{s^{-1}r} \alpha(v_{rr} \chi(x)) \\ &= \sum_{s \in \gamma' \cdot G, r \in \gamma \cdot G} u_{s^{-1}r} \alpha(\alpha_{r^{-1}s}(\chi(\overline{y}) v_{ss}) v_{rr} \chi(x)).\end{aligned}$$

Hence,

$$h(\chi(\overline{\gamma' \cdot G \otimes y} \otimes \gamma \cdot G \otimes x)) = \delta_{\gamma' \cdot G, \gamma \cdot G} \sum_{s \in \gamma \cdot G} \int_G v_{ss} \chi(\overline{y}) \chi(x) d\mu = \delta_{\gamma' \cdot G, \gamma \cdot G} \int_G \chi(\overline{y} \otimes x) d\mu = \delta_{\gamma' \cdot G, \gamma \cdot G} \delta_{x,y}. \tag{4.1.2}$$

Taking $\gamma' = \gamma$ and $y = x$ this shows that $\dim(\mathrm{Mor}(\gamma \cdot G \otimes x, \gamma \cdot G \otimes x)) = 1$. Hence, such representations are irreducible. Since the linear span of the coefficients of $\gamma \cdot G \otimes x$ is equal to $\mathcal{A}$ and hence dense in A_m, it follows that any irreducible representation of $\mathbb{G}$ is equivalent to some $\gamma \cdot G \otimes x$.

It also follows from Equation (4.1.2) that $\gamma \cdot x \simeq \gamma' \cdot y$ if and only if $\gamma \cdot G = \gamma' \cdot G$ and $x = y$.

(3). We have already shown that $\mathrm{Pol}(\mathbb{G}) = \mathcal{A}$. It follows that $C_m(\mathbb{G}) = A_m$. Since λ is surjective and τ is faithful on A, it follows that $C(\mathbb{G}) = A$ and $\mathrm{L}^\infty(\mathbb{G})$ is the bicommutant of A in $\mathcal{B}(\ell^2(\Gamma) \otimes \mathrm{L}^2(G))$ i.e., it is the von Neumann algebraic crossed product. Finally, since λ is the identity on $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$, it follows that λ is the canonical surjection.

(4). The fact that $\varepsilon_{\mathbb{G}}(\alpha(F)) = F(e)$ for all $F \in C(G)$ is obvious since α intertwines the colmultiplication. Fix $\gamma \in \Gamma$. Since $V^{\gamma \cdot G}$ is irreducible, we have that $(\mathrm{id} \otimes \varepsilon_{\mathbb{G}})(V^{\gamma \cdot G}) = 1$. Hence,

$$1 = \sum_{r,s \in \gamma.G} e_{r,s} \varepsilon_{\mathbb{G}}(u_r) v_{r,s}(e) = \sum_{r \in \gamma.G} e_{r,r} \varepsilon_{\mathbb{G}}(u_r).$$

It follows that $\varepsilon_{\mathbb{G}}(u_\gamma) = 1$. □

Remark 4.1.5. *Let $\mathbb{G}$ be the bicrossed product coming from a compact matched pair (Γ, G) as above. From the definition, it is easy to check that $C_m(\mathbb{G})$ is commutative if and only if the action α is trivial and Γ is abelian. Moreover, $\mathbb{G}$ is cocommutative if and* *only if the action β is trivial and G is abelian.*

Remark 4.1.6. *The following observation is well known. Let $\alpha : \Gamma \curvearrowright A$ be an action of the countable group Γ on the unital C^*-algebra A and let C be the full crossed product which is generated by the unitaries u_γ, $\gamma \in \Gamma$, and by the copy $\alpha(A)$ of the C^*-algebra A. If A has a character $\varepsilon \in A^*$ such that $\varepsilon(\alpha_\gamma(a)) = \varepsilon(a)$ for all $a \in A$ and $\gamma \in \Gamma$, then the C^*-subalgebra $B \subset C$ generated by $\{u_\gamma : \gamma \in \Gamma\}$ is canonically isomorphic to $C^*(\Gamma)$. Indeed, it suffices to check that B satisfies the universal property of $C^*(\Gamma)$. Let $v : \Gamma \to \mathcal{U}(H)$ be a unitary representation of Γ on H. Consider the unital $*$-homomorphism $\pi : A \to \mathcal{B}(H)$ given by $\pi(a) = \varepsilon(a)\mathrm{id}_H$, $a \in A$. We have $v_\gamma \pi(a) v_{\gamma^{-1}} = \varepsilon(a)\mathrm{id}_H = \varepsilon(\alpha_\gamma(a))\mathrm{id}_H = \pi(\alpha_\gamma(a))$. Hence, we obtain a representation of C that we can restrict to B to get the universal property.*

Let (Γ, G) be a matched pair. Since the map $\varepsilon : C(G) \to \mathbb{C}$ defined by $F \mapsto F(e)$ is a α-invariant character, it follows from the preceding observation that the C^*-subalgebra of A_m generated by u_γ, $\gamma \in \Gamma$, is canonically isomorphic to $C^*(\Gamma)$.

We now give some obvious consequences of the preceding result concerning amenability, K-amenability and the Hagerup property. The first assertion of the following corollary is already known [31] but we include an easy proof for the convenience of the reader. We refer to [85] for the definition of K-amenability of discrete quantum groups.

Corollary 4.1.7. *The following holds:*

1. *$\mathbb{G}$ is co-amenable if and only if Γ is amenable.*

2. *If Γ is K-amenable, then $\widehat{\mathbb{G}}$ is K-amenable.*

3. *If $\widehat{\mathbb{G}}$ has the Haagerup property, then Γ has the Haagerup property.*

4. *If the action of Γ on $\mathrm{L}^\infty(G)$ is compact and Γ has the Haagerup property, then $\widehat{\mathbb{G}}$ has the Haagerup property.*

Proof. (1). If Γ is amenable, then we trivially have that λ is an isomorphism; hence, $\mathbb{G}$ is co-amenable. Conversely, if $\mathbb{G}$ is co-amenable, then the Haar state $h = \tau \circ \lambda$ is faithful on A_m. Since $h(u_\gamma) = \delta_{\gamma,e}$, $\gamma \in \Gamma$, we conclude from Remark 4.1.6, that the canonical trace on $C^*(\Gamma)$ has to be faithful. Hence, Γ is amenable.

(2). It is an immediate consequence of [28, Theorem 2.1 (c)].

(3). It follows from [29, Theorem 6.7], since $\mathrm{L}(\Gamma)$ is a von Neumann subalgebra of $\mathrm{L}^\infty(\mathbb{G})$.

(4). This is a direct consequence of [47, Corollary 3.4] and [29, Theorem 6.7]. $\square$

Now we describe the fusion rules of a bicrossed product.

For $r, s \in \Gamma$, let $B_{rs} \subset G$ be the clopen set defined by $B_{r,s} = \{g \in G : \beta_{\alpha_s(g)}(r) = r \text{ and } \beta_g(s) = s\}$. To reduce notation, we denote by $\gamma \cdot G \in \mathrm{Irr}(\mathbb{G})$ the equivalence class of $V^{\gamma \cdot G}$ for $\gamma \cdot G \in \Gamma/G$, and we view $\mathrm{Irr}(G) \subset \mathrm{Irr}(\mathbb{G})$.

Theorem 4.1.8. *The following holds:*

1. *The set of unitary representations of $\mathbb{G}$ of the form $v^x \otimes V^{\gamma \cdot G}$ for $\gamma \cdot G \in \Gamma/G$ and $x \in \mathrm{Irr}(G)$ is a complete set of irreducible unitary representations of $\mathbb{G}$. In particular, for all $\gamma \cdot G \in \Gamma/G$ and all $x \in \mathrm{Irr}(G)$, there exists a unique $\alpha_{\gamma \cdot G}(x) \in \mathrm{Irr}(G)$ and a unique $\beta_x(\gamma \cdot G) \in \Gamma/G$ such that*

$$\gamma \cdot G \otimes x \simeq \alpha_{\gamma \cdot G}(x) \otimes \beta_x(\gamma \cdot G).$$

   Moreover, for all $\gamma \cdot G \in \Gamma/G$ and all $x \in \mathrm{Irr}(G)$, the maps

$$\alpha_{\gamma \cdot G} : \mathrm{Irr}(G) \to \mathrm{Irr}(G) \quad \textit{and} \quad \beta_x : \Gamma/G \to \Gamma/G$$

   are bijections.

2. *For all $r, s, \gamma \in \Gamma$ and $x \in \operatorname{Irr}(G)$ we have*

$$\dim(\operatorname{Mor}(\gamma \cdot G \otimes x, r \cdot G \otimes s \cdot G)) = \sum_{s' \in s \cdot G, r' \in r \cdot G} |\{t \in \gamma \cdot G \,:\, t = r's'\}| \int_{B_{r',s'}} \chi(\overline{x}) d\mu.$$

Proof. (1). The proof of (1) is exactly as the proof of assertion (2) in Theorem 4.1.4. The second assertion is trivial, since the representations $V^{\gamma \cdot G} \otimes v^x$ are irreducible. Finally, the fact that the maps are bijective follows from uniqueness.

(2). For all $\gamma, r, s \in \Gamma$, we have

$$\begin{aligned}
\chi(\overline{\gamma \cdot G \otimes x} \otimes r \cdot G \otimes s \cdot G) &= \sum_{\gamma' \in \gamma \cdot G, r' \in r \cdot G, s' \in s \cdot G} \alpha(\chi(\overline{x}) v_{\gamma'\gamma'}) u_{(\gamma')^{-1} r'} \alpha(v_{r'r'}) u_{s'} \alpha(v_{s's'}) \\
&= \sum_{\gamma' \in \gamma \cdot G, r' \in r \cdot G, s' \in s \cdot G} u_{(\gamma')^{-1} r's'} \alpha(\alpha_{(r's')^{-1}\gamma'}(\chi(\overline{x}) v_{\gamma'\gamma'}) \alpha_{(s')^{-1}}(v_{r'r'}) v_{s's'}).
\end{aligned}$$

It follows that

$$\dim(\operatorname{Mor}(\gamma \cdot G \otimes x, r \cdot G \otimes s \cdot G)) = h(\chi(\overline{\gamma \cdot G \otimes x} \otimes r \cdot G \otimes s \cdot G))$$

$$= \sum_{\gamma' \in \gamma \cdot G, r' \in r \cdot G, s' \in s \cdot G} \delta_{\gamma', r's'} \int_G \chi(\overline{x}) v_{r's', r's'} \alpha_{(s')^{-1}}(v_{r'r'}) v_{s's'} d\mu$$

$$= \sum_{s' \in s \cdot G, r' \in r \cdot G} |\{t \in \gamma \cdot G \,:\, t = r's'\}| \int_G \chi(\overline{x}) v_{r's', r's'} \alpha_{(s')^{-1}}(v_{r'r'}) v_{s's'} d\mu.$$

Observe that $v_{r's', r's'} \alpha_{(s')^{-1}}(v_{r'r'}) v_{s's'} = 1_{D_{r',s'}}$, where $D_{r',s'} = \{g \in G \,:\, \beta_g(r's') = r's'\} \cap B_{r',s'}$. Since $\beta_g(r's') = \beta_{\alpha_{s'}(g)}(r')\beta_g(s')$, it follows that $B_{r',s'} \subset D_{r',s'}$. Hence, $D_{r',s'} = B_{r',s'}$. □

Before proceeding further, we remind the reader of the following

Definition 4.1.9. *Let G be a compact quantum group. Then,* $\operatorname{Int}(G) := \{u \in \mathcal{U}(C_m(G)) \,:\, \Delta_G(u) = u \otimes u\}$, *called the intrinsic group of G. It is the set of all 1-dimensional irreducible unitary representations of G and it is countable when $C_m(G)$ is assumed to be* *separable.*

We end this section with a description of the $\mathrm{Int}(\mathbb{G})$ and $\chi(\mathbb{G})$, the maximal classical compact subgroup of $\mathbb{G}$, in terms of the matched pair (G,Γ). It will be used to distinguish various explicit examples in Section 4.5.

Observe that the relations in Equation (4.1.1) imply that $\Gamma^\beta = \{\gamma \in \Gamma : \beta_g(\gamma) = \gamma \ \forall g \in G\}$ and $G^\alpha = \{g \in G : \alpha_\gamma(g) = g \ \forall \gamma \in \Gamma\}$ are respectively subgroups of Γ and G. Moreover, since β is continuous, G^β is closed, hence compact. Thus, when (Γ, G) is a compact matched pair, the relations in Equation (4.1.1) imply that the associations

$$\gamma \cdot \omega = \omega \circ \alpha_\gamma \quad \text{and} \quad g \cdot \mu = \mu \circ \beta_g, \quad \text{for all } \gamma \in \Gamma, g \in G, \omega \in \mathrm{Sp}(G), \mu \in \mathrm{Sp}(\Gamma),$$

define two actions by group homomorphisms, namely: (i) right action of Γ^β on $\mathrm{Sp}(G)$ that we still denote by α, and (ii) left action of G^α on $\mathrm{Sp}(\Gamma)$ that we still denote by β. Also, β is a continuous action by homeomorphisms.

Proposition 4.1.10. *There are canonical group isomorphisms:*

$$\mathrm{Int}(\mathbb{G}) \simeq \mathrm{Sp}(G) \rtimes_\alpha \Gamma^\beta \quad \textit{and} \quad \chi(\mathbb{G}) \simeq G^\alpha {}_\beta\!\ltimes \mathrm{Sp}(\Gamma).$$

The second isomorphism is moreover a homeomorphism.

Proof. The irreducible representation $V^{\gamma \cdot G}$ of $\mathbb{G}$ is of dimension $1 \Leftrightarrow |\gamma \cdot G| = 1 \Leftrightarrow \gamma \in \Gamma^\beta$. By assertion (2) of Theorem 4.1.4, there is a bijective map

$$\pi : \mathrm{Sp}(G) \rtimes_\alpha \Gamma^\beta \to \mathrm{Int}(\mathbb{G}) : (\omega, \gamma) \mapsto u_\gamma \alpha(\omega) \in C_m(\mathbb{G}), \quad \omega \in \mathrm{Sp}(G), \gamma \in \Gamma^\beta.$$

The relations of the crossed product and the group law in the right semi-direct product imply that π is a group homomorphism.

Let $(g, \mu) \in G^\alpha \times \mathrm{Sp}(\Gamma)$. Since $g \in G^\alpha$, the unital $*$-homomorphism $C(G) \to \mathbb{C}$ given by $F \mapsto F(g)$ and the unitary representation $\mu : \Gamma \to S^1$ give a covariant representation.

Hence, we get a unique $\rho(g,\mu) \in \chi(\mathbb{G})$ such that $\rho(g,\mu)(u_\gamma \alpha(F)) = \mu(\gamma)F(g)$ for all $\gamma \in \Gamma$, $F \in C(G)$. It defines a map $\rho : G^\alpha{}_\beta \ltimes \mathrm{Sp}(\Gamma) \to \chi(\mathbb{G})$ which is obviously injective.

For all $g, h \in G^\alpha$,$\gamma \in \Gamma$ and $F \in C(G)$, one has

$$\begin{aligned}(\rho(g,\omega) \cdot \rho(h,\mu))(u_\gamma \alpha(F)) &= (\rho(g,\omega) \otimes \rho(h,\mu))(\Delta_m(u_\gamma \alpha(F))) = \sum_{r \in \gamma \cdot G} \omega(\gamma) v_{\gamma,r}(g)\mu(r)F(gh) \\ &= \omega(\gamma)\mu(\beta_g(\gamma))F(gh) = (\rho(gh, \omega \cdot \mu \circ \beta_g))(u_\gamma \alpha(F)).\end{aligned}$$

Hence, ρ is a group homomorphism.

Let us check that ρ is surjective. Let $\chi \in \chi(\mathbb{G})$, then $\chi \circ \alpha \in \mathrm{Sp}(C(G))$. Let $g \in G$ be such that $\chi(\alpha(F)) = F(g)$ for all $F \in C(G)$. Actually $g \in G^\alpha$. Indeed, for all $\gamma \in \Gamma$ and all $F \in C(G)$, one has

$$F(\alpha_{\gamma^{-1}}(g)) = \chi(\alpha(\alpha_\gamma(F))) = \chi(u_\gamma \alpha(F) u_\gamma^*) = \chi(\alpha(F)) = F(g);$$

now use the fact that $C(G)$ separates points of G to establish $g \in G^\alpha$. Define $\omega = (\gamma \mapsto \chi(u_\gamma)) \in \mathrm{Sp}(\Gamma)$. Consequently, $\chi = \rho(g,\omega)$ and ρ is surjective.

Finally, the map $\rho^{-1} : \chi(\mathbb{G}) \to G^\alpha{}_\beta \ltimes \mathrm{Sp}(\Gamma)$ is continuous, since $p_1 \circ \rho^{-1} : \chi(\mathbb{G}) \to \mathrm{Sp}(C(G)) = G$ by $\chi \mapsto \chi \circ \alpha$ and $p_2 \circ \rho^{-1} : \chi(\mathbb{G}) \to \mathrm{Sp}(\Gamma)$ by $\chi \mapsto (\gamma \mapsto \chi(u_\gamma))$, are obviously continuous, where p_1 and p_2 are the canonical projections. By compactness, ρ is an homeomorphism. □

4.2 Property T and bicrossed product

This section is dedicated to the relative co-property (T) of the pair $(G, \mathbb{G})$ and Kazhdan property of the dual of the bicrossed product $\mathbb{G}$ constructed in Section 3. The results in this section generalize classical results on relative property (T) for inclusion of groups of the form $(H, \Gamma \ltimes H)$, where H and Γ are discrete groups and H is abelian [27].

4.2.1 Relative property T for compact bicrossed product

Definition 4.2.1. *Let G and $\mathbb{G}$ be two compact quantum groups with an injective unital $*$-homomorphism $\alpha : C_m(G) \to C_m(\mathbb{G})$ such that $\Delta_{\mathbb{G}} \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_G$. We say that the pair $(G, \mathbb{G})$ has the relative co-property (T), if for every representation $\pi : C_m(\mathbb{G}) \to \mathcal{B}(H)$ we have $\varepsilon_{\mathbb{G}} \prec \pi \implies \varepsilon_G \subset \pi \circ \alpha$, where we use $\varepsilon_{\mathbb{G}} \prec \pi$ to denote the existence of a sequence of unit vectors ξ_n in H such that for any $a \in C_m(\mathbb{G})$, we have $\|\pi(a)\xi_n - \varepsilon_{\mathbb{G}}(a)\xi_n\| \to 0$ as $n \to \infty$.*

Observe that, by [58, Proposition 2.3], $\widehat{\mathbb{G}}$ has the property (T) in the sense of [34] if and only if the pair $(\mathbb{G}, \mathbb{G})$ has the relative co-property (T) (with $\alpha = \mathrm{id}$). Also, if Λ, Γ are countable discrete groups and $\Lambda < \Gamma$, then the pair $(\widehat{\Lambda}, \widehat{\Gamma})$ has the relative co-property (T) if and only if the pair (Λ, Γ) has the relative property (T) in the classical sense.

Let (Γ, G) be a matched pair of a countable discrete group Γ and a compact group G. Let $\mathbb{G}$ be the bicrossed product. In the following result, we characterize the relative co-property (T) of the pair $(G, \mathbb{G})$ in terms of the action α of Γ on $C(G)$. This is a non-commutative version of [27, Theorem 1] and the proof is similar. We will use freely the notations and results of Section 4.1.

Theorem 4.2.2. *The following are equivalent:*

1. *The pair $(G, \mathbb{G})$ does not have the relative co-property (T).*
2. *There exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of Borel probability measures on G such that*
   - *(a) $\mu_n(\{e\}) = 0$ for all $n \in \mathbb{N}$;*
   - *(b) $\mu_n \to \delta_e$ weak*;*
   - *(c) $\|\alpha_\gamma(\mu_n) - \mu_n\| \to 0$ for all $\gamma \in \Gamma$.*

Proof. For a representation $\pi : C_m(\mathbb{G}) \to \mathcal{B}(H)$, we have $\varepsilon_G \subset \pi \circ \alpha$ if and only if $K_\pi \neq \{0\}$, where

$$K_\pi = \{\xi \in H : \pi \circ \alpha(F)\xi = F(e)\xi \text{ for all } F \in C(G)\}.$$

Define $\rho = \pi \circ \alpha : C(G) \to \mathcal{B}(H)$, and for all $\xi, \eta \in H$, let $\mu_{\xi,\eta}$ be the unique complex Borel measure on G such that $\int_G F d\mu_{\xi,\eta} = \langle \rho(F)\xi, \eta \rangle$ for all $F \in C(G)$. Let $\mathcal{B}(G)$ be the collection of Borel subsets of G and $E : \mathcal{B}(G) \to \mathcal{B}(H)$ be the projection-valued measure associated to ρ i.e., for all $B \in \mathcal{B}(G)$, the projection $E(B) \in \mathcal{B}(H)$ is the unique operator such that $\langle E(B)\xi, \eta \rangle = \mu_{\xi,\eta}(B)$ for all $\xi, \eta \in H$.

Observe that a vector $\xi \in H$ satisfies $\rho(F)\xi = F(e)\xi$ for all $F \in C(G)$, if and only if $\mu_{\xi,\eta} = \langle \xi, \eta \rangle \delta_e$ for all $\eta \in H$, which in turn is true if and only if $\langle E(\{e\})\xi, \eta \rangle = \langle \xi, \eta \rangle$ for all $\eta \in H$. Hence, $E(\{e\})$ is the orthogonal projection onto K_π.

(1) $\implies$ (2). Suppose that the pair $(G, \mathbb{G})$ *does not* have the relative co-property (T). Let $\pi : C_m(\mathbb{G}) \to \mathcal{B}(H)$ be a representation such that $\varepsilon_{\mathbb{G}} \prec \pi$ and $K_\pi = \{0\}$. Hence, $\mu_{\xi,\eta}(\{e\}) = \langle E(\{e\})\xi, \eta \rangle = 0$ for all $\xi, \eta \in H$.

Since $\varepsilon_{\mathbb{G}} \prec \pi$, let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of unit vectors in H such that $\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| \to 0$ for all $x \in C_m(\mathbb{G})$. Define $\mu_n = \mu_{\xi_n,\xi_n}$. Then, we have $\mu_n(\{e\}) = 0$ for all $n \in \mathbb{N}$. Since μ_n is a probability measure, $|\mu_n(F) - \delta_e(F)| = |\int_G (F - F(e)) d\mu_n| \leq \|F - F(e)\|_{L^1(\mu_n)} \leq \|F - F(e)\|_{L^2(\mu_n)}$, for all $F \in C(G)$. Moreover,

$$\|F - F(e)\|^2_{L^2(\mu_n)} = \|\rho(F - F(e)1)\xi_n\|^2 = \|\pi(\alpha(F))\xi_n - \varepsilon_{\mathbb{G}}(\alpha(F))\xi_n\|^2 \to 0.$$

Hence, $\mu_n \to \delta_e$ weak*. Finally, for all $\gamma \in \Gamma$ and all $F \in C(G)$, we have:

$$\begin{aligned}\int_G F d\alpha_\gamma(\mu_n) = \int_G \alpha_{\gamma^{-1}}(F) d\mu_n &= \langle \rho(\alpha_{\gamma^{-1}}(F))\xi_n, \xi_n \rangle = \langle \pi(u_\gamma)^* \rho(F) \pi(u_\gamma)\xi_n, \xi_n \rangle \\ &= \langle \rho(F)\pi(u_\gamma)\xi_n, \pi(u_\gamma)\xi_n \rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\left| \int_G F d\alpha_\gamma(\mu_n) - \int_G F d\mu_n \right| &= |\langle \rho(F)\pi(u_\gamma)\xi_n, \pi(u_\gamma)\xi_n \rangle - \langle \rho(F)\xi_n, \xi_n \rangle| \\ &\leq |\langle \rho(F)(\pi(u_\gamma)\xi_n - \xi_n), \pi(u_\gamma)\xi_n \rangle| + |\langle \rho(F)\xi_n, \pi(u_\gamma)\xi_n - \xi_n \rangle| \\ &\leq 2\|F\| \, \|\pi(u_\gamma)\xi_n - \xi_n\|, \text{ for all } F \in C(G) \text{ and } \gamma \in \Gamma.\end{aligned}$$

Hence, $\|\alpha_\gamma(\mu_n) - \mu_n\| \leq 2\|\pi(u_\gamma)\xi_n - \xi_n\| = 2\|\pi(u_\gamma)\xi_n - \varepsilon_{\mathbb{G}}(u_\gamma)\xi_n\| \to 0$ (see (4) of Theorem 4.1.4).

(2) $\implies$ (1). We first prove the following claim.

Claim. *If (2) holds, then there exists a sequence $(\nu_n)_{n\in\mathbb{N}}$ of Borel probability measures on G satifying (a), (b) and (c) and such that $\alpha_\gamma(\nu_n) \sim \nu_n$ for all $\gamma \in \Gamma$, $n \in \mathbb{N}$.*

Proof of the claim. Denote by $\ell^1(\Gamma)_{1,+}$ the set of positive ℓ^1 functions on Γ with $\|f\|_1 = 1$. For μ a Borel probability measure on G and $f \in \ell^1(\Gamma)_{1,+}$, define the Borel probability measure $f * \mu$ on G by the convex combination

$$f * \mu = \sum_{\gamma\in\Gamma} f(\gamma)\alpha_\gamma(\mu).$$

Observe that for all $\gamma \in \Gamma$, we have $\delta_\gamma * \mu = \alpha_\gamma(\mu)$ and $\alpha_\gamma(f * \mu) = f_\gamma * \mu$, where $f_\gamma \in \ell^1(\Gamma)_{1,+}$ is defined by $f_\gamma(r) = f(\gamma^{-1}r)$, $r \in \Gamma$.

Moreover, if $f \in \ell^1(\Gamma)_{1,+}$ is such that $f(\gamma) > 0$ for all $\gamma \in \Gamma$, then since $(f * \mu)(E) = \sum_\gamma f(\gamma)\mu(\alpha_{\gamma^{-1}}(E))$ (E is Borel subset of G), so we have that $(f * \mu)(E) = 0$ if and only if $\mu(\alpha_\gamma(E)) = 0$ for all $\gamma \in \Gamma$. This last condition does not depend on f. Hence, if $f \in \ell^1(\Gamma)_{1,+}$ is such that $f > 0$, then since $f_\gamma(r) > 0$ for all $\gamma, r \in \Gamma$, it follows that $f * \mu \sim \alpha_\gamma(f * \mu) = f_\gamma * \mu$ for all $\gamma \in \Gamma$ as they have the same null sets: the Borel subsets E of G such that $\mu(\alpha_s(E)) = 0$ for all $s \in \Gamma$.

Therefore, since $\alpha_\gamma(e) = e$ for all $\gamma \in \Gamma$, so

$$(f * \mu)(\{e\}) = \sum_\gamma f(\gamma)\mu(\alpha_{\gamma^{-1}}(\{e\})) = \sum_\gamma f(\gamma)\mu(\{e\}) = \mu(\{e\}), \text{ for all } f \in \ell^1(\Gamma)_{1,+}.$$

Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of Borel probability on G satisfying (a), (b) and (c). For all $f \in \ell^1(\Gamma)_{1,+}$ with finite support we have,

$$\|f * \mu_n - \mu_n\| \leq \sum_\gamma f(\gamma)\|\delta_\gamma * \mu_n - \mu_n\| = \sum_\gamma f(\gamma)\|\alpha_\gamma(\mu_n) - \mu_n\| \to 0. \tag{4.2.1}$$

Since such functions are dense in $\ell^1(\Gamma)_{1,+}$ (in the ℓ^1-norm), it follows that $\|f*\mu_n-\mu_n\| \to 0$

for all $f \in \ell^1(\Gamma)_{1,+}$.

Let $\xi \in \ell^1(\Gamma)_{1,+}$ be any function such that $\xi > 0$ and define $\nu_n = \xi * \mu_n$. By the preceding discussion, we know that $\alpha_\gamma(\nu_n) \sim \nu_n$ for all $\gamma \in \Gamma$ and $\nu_n(\{e\}) = \mu_n(\{e\}) = 0$ for all $n \in \mathbb{N}$. Moreover, by Equation (4.2.1),

$$\|\alpha_\gamma(\nu_n) - \nu_n\| = \|\xi_\gamma * \mu_n - \xi * \mu_n\| \leq \|\xi_\gamma * \mu_n - \mu_n\| + \|\mu_n - \xi * \mu_n\| \to 0, \text{ for all } \gamma \in \Gamma.$$

Finally, since $\mu_n \to \delta_e$ weak* and $\alpha_\gamma(e) = e$, one has $|\mu_n(F \circ \alpha_\gamma) - F(e)| \to 0$ for all $\gamma \in \Gamma$ and for all $F \in C(G)$. Hence, for all $F \in C(G)$, the dominated convergence theorem implies that

$$|\nu_n(F) - \delta_e(F)| = \left|\sum_\gamma f(\gamma)(\mu_n(F \circ \alpha_\gamma) - F(e))\right| \leq \sum_\gamma f(\gamma)|\mu_n(F \circ \alpha_\gamma) - F(e)| \to 0.$$

It follows that $\nu_n \to \delta_e$ weak* and this finishes the proof of the claim. □

We now finish the proof of the Theorem. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of Borel probability measures on G as prescribed in the Claim. For $n \in \mathbb{N}$ and $\gamma \in \Gamma$, let $h_n(\gamma) = \frac{d\alpha_\gamma(\mu_n)}{d\mu_n}$; then, by uniqueness of the Radon-Nikodym derivatives and since α is an action, we have for all $n \in \mathbb{N}$, $h_n(\gamma, g)h_n(\gamma^{-1}, \alpha_{\gamma^{-1}}(g)) = 1$, μ_n a.e. $g \in G$, and for all $\gamma \in \Gamma$. Define $H_n = \mathrm{L}^2(G, \mu_n)$ and let $u_n : \Gamma \to \mathcal{U}(H_n)$ be the unitary representations defined by $(u_n(\gamma)\xi)(g) = \xi(\alpha_{\gamma^{-1}}(g))h_n(\gamma, g)^{\frac{1}{2}}$ for $\gamma \in \Gamma$, $g \in G$, $\xi \in H_n$. Also consider the representations $\rho_n : C(G) \to \mathcal{B}(H_n)$, defined by $\rho_n(F)\xi(g) = F(g)\xi(g)$, for $\xi \in H_n$, $g \in G$ and $F \in C(G)$. Observe that the projection valued measure associated to ρ_n is given by $(E_n(B)\xi)(g) = 1_B(g)\xi(g)$ for all $B \in \mathcal{B}(G)$, $\xi \in H_n$ and $g \in G$. Using the identity $h_n(\gamma, \cdot)h_n(\gamma^{-1}, \alpha_{\gamma^{-1}}(\cdot)) = 1$, we find $u_n(\gamma)\rho_n(F)u_n(\gamma^{-1}) = \rho_n(\alpha_\gamma(F))$ for all $\gamma \in \Gamma$, $F \in C(G)$, $g \in G$. Therefore, by the universal property of A_m, for each $n \in \mathbb{N}$ there is a unital $*$-homomorphism $\pi_n : A_m \to \mathcal{B}(H_n)$ such that $\pi_n(u_\gamma) = u_n(\gamma)$ and $\pi_n \circ \alpha = \rho_n$ for all $n \in \mathbb{N}$. Since $\mu_n(\{e\}) = 0$, we have $E_n(\{e\}) = 0$ for all $n \in \mathbb{N}$. Hence, $K_{\pi_n} = \{0\}$ for all $n \in \mathbb{N}$. Consequently, on defining $H = \oplus_n H_n$ and $\pi = \oplus_n \pi_n : C_m(\mathbb{G}) \to \mathcal{B}(H)$, it follows that $K_\pi = \{0\}$ as well. Hence, it suffices to show that $\varepsilon_{\mathbb{G}} \prec \pi$.

Define the unit vectors $\xi_n = 1 \in \mathrm{L}^2(G, \mu_n) \subset H$, $n \in \mathbb{N}$. Observe that $(\mu_n - \alpha_\gamma(\mu_n))(F) = \int_G F(1 - h_n(\gamma))d\mu_n$ for all $F \in C(G)$. Hence, $\|\mu_n - \alpha_\gamma(\mu_n)\| = \|1 - h_n(\gamma)\|_{L^1(G,\mu_n)} \to 0$ for all $\gamma \in \Gamma$. Moreover, as $0 \leq 1 - \sqrt{t} \leq \sqrt{1-t}$ for all $0 \leq t \leq 1$ and for $t > 1$, we have $\sqrt{t} - 1 \leq 1$, it follows that

$$\|\pi(u_\gamma)\xi_n - \xi_n\|_H^2 = \|u_n(\gamma)1 - 1\|_{H_n}^2 = \int_G (1 - h_n(\gamma)^{\frac{1}{2}})^2 d\mu_n$$

$$\leq \int_G |1 - h_n(\gamma)| d\mu_n = \|1 - h_n(\gamma)\|_{L^1(G,\mu_n)} \to 0$$

for all $\gamma \in \Gamma$. Since $\mu_n \to \delta_e$ weak*, for all $F \in C(G)$, we also have that,

$$\|\pi(\alpha(F))\xi_n - F(e)\xi_n\|_H^2 = \|\rho_n(F)1 - F(e)1\|_{H_n}^2 = \int_G |F(g) - F(e)|^2 d\mu_n \to 0.$$

Consequently, for all $x = u_\gamma \alpha(F) \in C_m(\mathbb{G})$, we have

$$\begin{aligned}
\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| &= \|\pi(u_\gamma)\pi(\alpha(F))\xi_n - F(e)\xi_n\| \\
&\leq \|\pi(u_\gamma)(\pi(\alpha(F))\xi_n - F(e)\xi_n)\| + |F(e)|\,\|\pi(u_\gamma)\xi_n - \xi_n\| \\
&\leq \|\pi(\alpha(F))\xi_n - F(e)\xi_n\| + |F(e)|\,\|\pi(u_\gamma)\xi_n - \xi_n\| \to 0.
\end{aligned}$$

By linearity and the triangle inequality, we have $\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| \to 0$ for all $x \in \mathcal{A}$. The proof is complete by density of $\mathcal{A}$ in $C_m(\mathbb{G})$. □

4.2.2 Property (T)

Now we discuss property (T) of $\mathbb{G}$. Let G^α be the set of fixed points in G under the action α of Γ. It is a closed subset of G, and, by the relations in Equation (4.1.1) it is also a subgroup of G.

Theorem 4.2.3. *The following holds:*

1. *If $\widehat{\mathbb{G}}$ has property (T), then Γ has property (T) and G^α is finite.*

2. *If $\widehat{\mathbb{G}}$ has property (T) and α is compact[2] then Γ has Property (T) and G is finite.*

3. *If Γ has property (T) and G is finite, then $\widehat{\mathbb{G}}$ has property (T).*

Proof. (1). Let $\rho : C(G) \to C^*(\Gamma)$ be the unital $*$-homomorphism defined by $\rho(F) = F(e)1$ and consider the canonical unitary representation of Γ given by $\Gamma \ni \gamma \mapsto U_\gamma \in C^*(\Gamma)$. For all $\gamma \in \Gamma$ and $F \in C(G)$, we have $\rho(\alpha_\gamma(F)) = \alpha_\gamma(F)(e)1 = F(\alpha_{\gamma^{-1}}(e))1 = F(e)1 = U_\gamma \rho(F) U_\gamma^*$. Hence, there exists a unique unital $*$-homomorphism $\pi : C_m(\mathbb{G}) \to C^*(\Gamma)$ such that $\pi \circ \alpha = \rho$ and $\pi(u_\gamma) = U_\gamma$ for all $\gamma \in \Gamma$. Observe that π is surjective and, for all $F \in C(G)$,

$$(\pi \otimes \pi)\Delta_{\mathbb{G}}(\alpha(F)) = (\rho \otimes \rho)(\Delta_G(F)) = \Delta_G(F)(e,e)1 \otimes 1 = F(e)1 \otimes 1 = \Delta_{\widehat{\Gamma}}(\pi(\alpha(F))).$$

Moreover, since for all $\gamma, r \in \Gamma$ one has $1_{A_{\gamma,r}}(e) = \delta_{\gamma,r}$, we find, for all $\gamma \in \Gamma$,

$$(\pi \otimes \pi)\Delta_{\mathbb{G}}(u_\gamma) = \sum_{r \in \gamma \cdot G} \pi(u_\gamma \alpha(v^\gamma_{\gamma,r})) \otimes \pi(u_r) = \sum_{r \in \gamma \cdot G} U_\gamma 1_{A_{\gamma,r}}(e) \otimes U_r = U_\gamma \otimes U_\gamma = \Delta_{\widehat{\Gamma}}(\pi(u_\gamma)).$$

So π intertwines the comultiplications and property (T) for Γ follows from [34, Proposition 6].

To show that G^α is finite it suffices, since G^α is closed in G hence compact, to show that G^α is discrete. Let (g_n) be any sequence in G^α such that $g_n \to e$. Consider the unital $*$-homomorphism $\rho : C(G) \to \mathcal{B}(\ell^2(\mathbb{N}))$ defined by $(\rho(F)\xi)(n) = F(g_n)\xi(n)$, for all $\xi \in \ell^2(\mathbb{N})$, and the trivial representation of Γ on $\ell^2(\mathbb{N})$. Since $g_n \in G^\alpha$ for all $n \in \mathbb{N}$ this pair gives a covariant representation. Hence, there exists a unital $*$-homomorphism $\pi : C_m(\mathbb{G}) \to \mathcal{B}(\ell^2(\mathbb{N}))$ such that $\pi(u_\gamma \alpha(F)) = \rho(F)$ for all $\gamma \in \Gamma$ and $F \in C(G)$. Define $\xi_n = \delta_n \in \ell^2(\mathbb{N})$. One has $\|\pi(u_\gamma \alpha(F))\xi_n - \varepsilon_{\mathbb{G}}(u_\gamma \alpha(F))\xi_n\| = |F(g_n) - F(e)| \to 0$ for all $F \in C(G)$. Hence, π has almost invariant vectors. By property (T), π has a non-zero

[2] We only need to assume that the closure of the image of Γ in the group of homeomorphisms of G is compact for some Hausdorff group topology for which the evaluation map at e is continuous.

invariant vector and for such a vector $\xi \in \ell^2(\mathbb{N})$ we have $F(g_n)\xi(n) = F(e)\xi(n)$ for all $F \in C(G)$ and all $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ for which $\xi(n_0) \neq 0$. We have $F(g_{n_0}) = F(e)$ for all $F \in C(G)$, which implies that $g_{n_0} = e$ and shows that G^α must be discrete.

(2). It suffices to show that G is finite. The proof is similar to (1). Let $g_n \in G$ be any sequence such that $g_n \to e$. We view α as a group homomorphism $\alpha : \Gamma \to H(G)$, $\gamma \mapsto \alpha_\gamma$, where $H(G)$ is the group of homeomorphisms of G and we write $K = \overline{\alpha(\Gamma)} \subset H(G)$. By assumptions, K is a compact group and we denote by ν the Haar probability on K. Note that, since $\alpha_\gamma(e) = e$ for all $\gamma \in \Gamma$, by continuity of the evaluation at e and density, we also have $x(e) = e$ for all $x \in K$. We define a covariant representation (ρ, v), $\rho : C(G) \to \mathcal{B}(\mathrm{L}^2(K \times \mathbb{N}))$ and $v : \Gamma \to \mathcal{U}(\mathrm{L}^2(K \times \mathbb{N}))$ by $(\rho(F)\xi)(x,n) = F(x(g_n))\xi(x,n)$ and $(v_\gamma\xi)(x,n) = \xi(\alpha_{\gamma^{-1}}x, n)$. By the universal property of $C_m(\mathbb{G})$, we get a unital $*$-homomorphism $\pi : C_m(\mathbb{G}) \to \mathcal{B}(\mathrm{L}^2(K \times \mathbb{N}))$ such that $\pi(u_\gamma\alpha(F)) = v_\gamma\rho(F)$ for all $\gamma \in \Gamma$ and $F \in C(G)$. Define, for $k \in \mathbb{N}$, the vector $\xi_k(x,n) = \delta_{k,n}$. Since ν is a probability it follows that ξ_k is a unit vector in $\mathrm{L}^2(K \times \mathbb{N})$. Moreover, for all $\gamma \in \Gamma$ and $F \in C(G)$,

$$\|\pi(u_\gamma\alpha(F))\xi_k - \varepsilon_{\mathbb{G}}(u_\gamma\alpha(F))\xi_k\|^2 = \int_K |F(\alpha_{\gamma^{-1}}x(g_k)) - F(e)|^2 d\nu(x) \to 0,$$

where the convergence follows from the dominated convergence Theorem since, by continuity, we have $F(\alpha_{\gamma^{-1}})x(g_k)) \to F(e)$ for all $\gamma \in \Gamma$, $x \in K$ and $F \in C(G)$ and the domination is obvious since ν is a probability. By property (T), there exists a non-zero $\xi \in \mathrm{L}^2(K \times \mathbb{N}))$ such that $F(e)\xi = \varepsilon_{\mathbb{G}}(\alpha(F))\xi = \pi(\alpha(F))\xi = \rho(F)\xi$ for all $F \in C(G)$. Define $Y := \{x \in K : \sum_{n\in\mathbb{N}} |\xi(x,n)|^2 > 0\}$ and, for $F \in C(G)$, $X_F := \{x \in K : \sum_{n\in\mathbb{N}} |F(x(g_n))\xi(x,n) - F(e)\xi(x,n)|^2 \neq 0\}$. The condition on ξ means that $\nu(Y) > 0$ and, for all $F \in C(G)$, $\nu(X_F) = 0$. Let $F_k \in C(G)$ be a dense sequence and $X = \cup_{k\in\mathbb{N}} X_{F_k}$ then $\nu(X) = 0$ so $\nu(Y \setminus X) > 0$. Hence, $Y \setminus X \neq \emptyset$. Let $x \in Y \setminus X$, we have $\sum_n |\xi(x,n|^2 > 0$ and, for all $k, n \in \mathbb{N}$, $F_k(x(g_n))\xi(x,n) = F_k(e)\xi(x,n)$. By density and continuity, $F(x(g_n))\xi(x,n) = F(e)\xi(x,n)$ for all $n \in \mathbb{N}$ and $F \in C(G)$. Since $\sum_n |\xi(x,n|^2 > 0$, there exists $n_0 \in \mathbb{N}$ such that $\xi(x,n_0) \neq 0$ which implies that

$F(x(g_{n_0})) = F(e)$ for all $F \in C(G)$. Hence, $x(g_{n_0}) = e$ which implies that $g_{n_0} = e$. Hence G must be discrete and, by compactness, G is finite.

(3). Let $\pi : C_m(\mathbb{G}) = \Gamma_{\alpha,f} \ltimes C(G) \to \mathcal{B}(H)$ be a unital $*$-homomorphism and K be the closed subspace H given by $C(G)$-invariant vectors i.e. $K = \{\xi \in H : \pi \circ \alpha(F)\xi = F(e)\xi \text{ for all } F \in C(G)\}$. Then $P = \pi(\alpha(\delta_e))$ is the orthogonal projection onto K which is an invariant subspace of the unitary representation $\gamma \mapsto \pi(u_\gamma)$ since $\pi(u_\gamma)P\pi(u_\gamma)^* = \pi(\alpha(\delta_{\alpha_g(e)})) = \pi(\alpha(\delta_e)) = P$ for all $\gamma \in \Gamma$. Let $\gamma \mapsto v_\gamma$ be the unitary representation of Γ on K obtained by restriction.

Suppose that $\varepsilon_{\mathbb{G}} \prec \pi$ and let $\xi_n \in H$ be a sequence of unit vectors such that $\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| \to 0$ for all $x \in C_m(\mathbb{G})$. Since G is finite (hence $\widehat{G}$ has property (T)), so $K \neq \{0\}$. Moreover, since $|\,\|P\xi_n\| - 1| \leq \|P\xi_n - \xi_n\|$, we have $\|P\xi_n\| \to 1$ and hence we may and will assume that $P\xi_n \neq 0$ for all n. Let $\eta_n = \frac{P\xi_n}{\|P\xi_n\|} \in K$. We have $\|v_\gamma\eta_n - \eta_n\| = \frac{1}{\|P\xi_n\|}\|P(v_\gamma\xi_n - \xi_n)\| \leq \frac{\|\pi(u_\gamma)\xi_n - \xi_n\|}{\|P\xi_n\|} \to 0$. Hence, $\gamma \mapsto v_\gamma$ has almost invariant vectors. Since Γ has property (T), let $\xi \in K$ be a non-zero invariant vector. Then, for all $x \in C_m(\mathbb{G})$ of the form $x = u_\gamma\alpha(F)$, we have $\pi(x)\xi = F(e)\pi(u_\gamma)\xi = F(e)\xi = \varepsilon_{\mathbb{G}}(x)\xi$. By linearity, continuity, and density of $\mathcal{A}$ in $C_m(\mathbb{G})$, we have $\pi(x)\xi = \varepsilon_{\mathbb{G}}(x)\xi$ for all $x \in C_m(\mathbb{G})$. $\square$

We mention that the third assertion of the previous theorem appears in [23] when β is supposed to be the trivial action.

Remark 4.2.4. *The compactness assumption on α in assertion* 2 *of the preceding Corollary can not be removed. Indeed, for $n \geq 3$, the semi-direct product $H = \mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ (for the linear action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{Z}^n$) has property (T) and H may be viewed as the dual of the bicrossed product associated to the matched pair $(\mathrm{SL}_n(\mathbb{Z}), \mathbb{T}^n)$ with the non-compact action $\alpha : \mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{T}^n$ given by viewing $\mathbb{T}^n = \widehat{\mathbb{Z}^n}$ and dualizing the linear action $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{Z}^n$ and the action β being trivial. In this example, the compact group $G = \mathbb{T}^n$ is infinite.*

4.3 Relative Haagerup property and bicrossed product

In this section, we study the relative co-Haagerup property of the pair $(G, \mathbb{G})$ constructed in Section 3. The main result in this section also generalizes one direction of the characterization of relative Haagerup property of the pair $(H, \Gamma \ltimes H)$, where H and Γ are discrete groups and H is abelian [27]. It is not clear to us as to how to show the other direction of the equivalence. We refer to Section 1.3 for the definitions of the Fourier transform and the Haagerup property.

Definition 4.3.1. *Let G and $\mathbb{G}$ be two compact quantum groups with an injective unital $*$-homomorphism $\alpha : C_m(G) \to C_m(\mathbb{G})$ such that $\Delta_{\mathbb{G}} \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_G$. We say that the pair $(G, \mathbb{G})$ has the relative co-Haagerup property, if there exists a sequence of states $\omega_n \in C_m(\mathbb{G})^*$ such that $\omega_n \to \varepsilon_{\mathbb{G}}$ in the weak* topology and $\widehat{\omega_n \circ \alpha} \in c_0(\widehat{G})$ for all $n \in \mathbb{N}$.*

Observe that, for any compact quantum group G, the dual $\widehat{G}$ has the Haagerup property if and only if the pair (G, G) has the co-Haagerup property. Moreover, it is clear that if Λ, Γ are discrete groups with $\Lambda < \Gamma$, then the pair $(\widehat{\Lambda}, \widehat{\Gamma})$ has the relative co-Haagerup property if and only if the pair (Λ, Γ) has the relative Haagerup property in the classical sense, where we denote the compact quantum group dual to the discrete group Λ and Γ by $\widehat{\Lambda}$ and $\widehat{\Delta}$ respectively.

Let (Γ, G) be a matched pair of a discrete group Γ and a compact group G. Let $\mathbb{G}$ be the bicrossed product. In the following theorem, we give a necessary condition for the relative co-Haagerup property of the pair $(G, \mathbb{G})$ in terms of the action α of Γ on $C(G)$. This is a non commutative version (one direction of) of [27, Theorem 4] and the proof is similar in spirit. However, one of the arguments of the classical case does not work in our context since α_γ is not a group homomorphism and substitutive ideas are required. Actually, for a general automorphism $\pi \in \mathrm{Aut}(C(G))$ and $\nu \in$Prob(G), there is no guarantee that $\widehat{\nu} \in C_r^*(G) \Rightarrow \widehat{\pi(\nu)} \in C_r^*(G)$. It is not clear to us, as to how to address this issue. We will freely use the notations and results of Section 4.1.

Theorem 4.3.2. *Suppose that the pair $(G, \mathbb{G})$ has the relative co-Haagerup property. Then we have that there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of Borel probability measures on G such that*

1. $\widehat{\mu}_n \in C_r^*(G)$ *for all* $n \in \mathbb{N}$*;*
2. $\mu_n \to \delta_e$ *weak*;*
3. $\|\alpha_\gamma(\mu_n) - \mu_n\| \to 0$ *for all* $\gamma \in \Gamma$*.*

Proof. Let $\omega_n \in C_m(\mathbb{G})^*$ be a sequence of states such that $\omega_n \to \varepsilon_{\mathbb{G}}$ in the weak* topology and $\widehat{\omega_n \circ \alpha} \in C_r^*(G)$. For each n view $\omega_n \circ \alpha \in C(G)^*$ as a Borel probability measure μ_n on G. By hypothesis, $\widehat{\mu}_n \in C_r^*(G)$ for all $n \in \mathbb{N}$ and $\mu_n \to \delta_e$ in the weak* topology. Writing (H_n, π_n, ξ_n) the GNS construction of ω_n and doing the same computation as in the proof of (1) $\implies$ (2) of Theorem 4.2.2, we find $\left|\int_G F d\alpha_\gamma(\mu_n) - \int_G F d\mu_n\right| \leq \|F\| \, \|\pi_n(u_\gamma)\xi_n - \xi_n\| = \|F\|\sqrt{2(1 - \mathrm{Re}(\omega_n(u_\gamma))}$. Hence, $\|\alpha_\gamma(\mu_n) - \mu_n\| \leq \sqrt{2(1 - \mathrm{Re}(\omega_n(u_\gamma))} \to \sqrt{2(1 - \mathrm{Re}(\varepsilon_{\mathbb{G}}(u_\gamma))} = 0$.

□

4.4 Crossed product quantum group

This section deals with a matched pair of a discrete group and a compact quantum group that arises in a crossed product, where the discrete group acts on the compact quantum group via quantum automorphisms. This section is longer and has four subsections. First, we analyze the quantum group structure and the representation theory of such crossed products which was initially studied by Wang in [88], but unlike Wang we do not rely on free products which allows us to shorten the proofs. We also obtain some obvious consequences related to amenability and K-amenability and the computation of the intrinsic group and the spectrum of the full C*-algebra of a crossed product quantum group. The subsections deal with weak amenability, rapid decay, (relative) property (T) and (relative) Haagerup property.

Let G be a compact quantum group, Γ a discrete group acting on G i.e., $\alpha : \Gamma \curvearrowright G$ be an action by quantum automorphisms. We will denote by the same symbol α the action of Γ on $C_m(G)$ or $C(G)$. Let $A_m = \Gamma_{\alpha,m} \ltimes C_m(G)$ be the full crossed product and $A = \Gamma_\alpha \ltimes C(G)$ be the reduced crossed product. By abuse of notation, we still denote by α the canonical injective map from $C_m(G)$ to A_m and from $C(G)$ to A. We also denote by u_γ, for $\gamma \in \Gamma$, the canonical unitaries viewed in either A_m or A. This will be clear from the context and cause no confusion.

By the universal property of the full crossed product, we have a unique surjective unital $*$-homomorphism $\lambda : A_m \to A$ such that $\lambda(u_\gamma) = u_\gamma$ and $\lambda(\alpha(a)) = \alpha(\lambda_G(a))$ for all $\gamma \in \Gamma$ and for all $a \in C_m(G)$. Finally, we denote by $\omega \in A^*$, the dual state of h_G i.e., ω is the unique (faithful) state such that

$$\omega(u_\gamma \alpha(a)) = \delta_{e,\gamma} h_G(a) \quad \text{for all } a \in C(G), \gamma \in \Gamma.$$

Again by the universal property of the full crossed product, there exists a unique unital $*$-homomorphism $\Delta_m : A_m \to A_m \otimes A_m$ such that $\Delta_m(u_\gamma) = u_\gamma \otimes u_\gamma$ and $\Delta_m \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_G$.

The following theorem is due to Wang [88]. We include a short proof.

Theorem 4.4.1. *$\mathbb{G} = (A_m, \Delta_m)$ is a compact quantum group and the following holds.*

1. *The Haar state of $\mathbb{G}$ is $h = \omega \circ \lambda$, hence, $\mathbb{G}$ is Kac if and only if G is Kac.*
2. *For all $\gamma \in \Gamma$ and all $x \in \mathrm{Irr}(G)$, $u^x_\gamma = (1 \otimes u_\gamma)(\mathrm{id} \otimes \alpha)(u^x) \in \mathcal{B}(H_x) \otimes A_m$ is an irreducible representation of $\mathbb{G}$ and the set $\{u^x_\gamma : \gamma \in \Gamma, x \in \mathrm{Irr}(G)\}$ is a complete set of irreducible representations of $\mathbb{G}$.*
3. *One has $C_m(\mathbb{G}) = A_m$, $C(\mathbb{G}) = A$, $\mathrm{Pol}(\mathbb{G}) = \mathrm{Span}\{u_\gamma \alpha(a) : \gamma \in \Gamma, a \in \mathrm{Pol}(G)\}$, λ is the canonical surjection from $C_m(\mathbb{G})$ to $C(\mathbb{G})$ and $\mathrm{L}^\infty(\mathbb{G})$ is the von Neumann algebraic crossed product.*

Proof. (1). Write $\mathcal{A} = \mathrm{Span}\{u_\gamma \alpha(a) : \gamma \in \Gamma, a \in \mathrm{Pol}(G)\}$. Since, by definition of A_m, $\mathcal{A}$ is dense in A_m it suffices to show the invariance of h on $\mathcal{A}$ and one has

$$\begin{aligned}(\mathrm{id} \otimes h)(\Delta_m(u_\gamma \alpha(u^x_{ij}))) &= \sum_k u_\gamma \alpha(u^x_{ik}) h(u_\gamma \alpha(u^x_{kj})) = \delta_{\gamma,e}\delta_{x,1} \\ &= h(u_\gamma \alpha(u^x_{ij})) = (h \otimes \mathrm{id})(\Delta_m(u_\gamma \alpha(u^x_{ij}))),\ \gamma \in \Gamma, x \in \mathrm{Irr}(G).\end{aligned}$$

(2). By the definition of Δ_m, it is obvious that u^x_γ is a unitary representation of $\mathbb{G}$ for all $\gamma \in \Gamma$ and $x \in \mathrm{Irr}(G)$. The representations u^x_γ, for $\gamma \in \Gamma$ and $x \in \mathrm{Irr}(G)$, are irreducible and pairwise non-equivalent since

$$\begin{aligned}h(\chi(u^x_r)^*\chi(u^y_s)) &= h(\alpha(\chi(\overline{x}))u_{r^{-1}s}\alpha(\chi(y))) = h(u_{r^{-1}s}\alpha(\alpha_{r^{-1}s}(\chi(\overline{x}))\chi(y))) = \delta_{r,s}h_G(\chi(\overline{x})\chi(y)) \\ &= \delta_{r,s}\delta_{x,y}.\end{aligned}$$

Finally, $\{u^x_\gamma : \gamma \in \Gamma,\ x \in \mathrm{Irr}(G)\}$ is a complete set of irreducibles since the linear span of the coefficients of the u^x_γ is $\mathcal{A}$, which is dense in $C_m(G)$.

(3). We established in (2) that $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$. Since, by definition, A_m is the enveloping C*-algebra of $\mathcal{A}$, we have $C_m(\mathbb{G}) = A_m$. Since $\lambda : A_m \to A$ is surjective and ω is faithful on A, we have $C(\mathbb{G}) = A$. Moreover, since λ is identity on $\mathcal{A} = \mathrm{Pol}(G)$, it follows that λ is the canonical surjection. Finally, $\mathrm{L}^\infty(\mathbb{G})$ is, by definition, the bicommutant of $C(\mathbb{G}) = A$ which is also the von Neumann algebraic crossed product. $\square$

Remark 4.4.2. *Observe that the counit satisfies $\varepsilon_{\mathbb{G}}(u_\gamma \alpha(a)) = \varepsilon_G(a)$ for any $\gamma \in \Gamma$ and $a \in$ Pol(G). This follows from the uniqueness of the counit with respect to the equation $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$ and also the fact that $\varepsilon_G \circ \alpha_\gamma(a) = \varepsilon_G(a)$, for any $\gamma \in \Gamma$ and $a \in$ Pol(G). Similarly, $S_{\mathbb{G}}(u_\gamma \alpha(a)) = u_{\gamma^{-1}}\alpha(S_G(\alpha_{\gamma^{-1}}(a)))$. Hence, for any $\gamma \in \Gamma$, we have $\alpha_\gamma \circ S_G = S_G \circ \alpha_\gamma$.*

Remark 4.4.3. *Notice that we have a group homomorphism $\Gamma \to S(\mathrm{Irr}(G))$, $\gamma \mapsto \alpha_\gamma$, where $\alpha_\gamma(x)$, for $x \in \mathrm{Irr}(G)$, is the class of the irreducible representation $(\mathrm{id} \otimes \alpha_\gamma)(u^x)$. Let $\gamma \cdot x \in \mathrm{Irr}(\mathbb{G})$ be the class of u^x_γ. Observe that, we have $\gamma \otimes x \otimes \gamma^{-1} = \alpha_\gamma(x)$ and*

$\gamma \cdot x = \gamma \otimes x$, *by viewing* $\Gamma \subset \mathrm{Irr}(\mathbb{G})$ *and* $\mathrm{Irr}(G) \subset \mathrm{Irr}(\mathbb{G})$. *Hence, the fusion rules of* $\mathbb{G}$ *are described as follows:*

$$r \cdot x \otimes s \cdot y = rs \cdot \alpha_{s^{-1}}(x) \otimes y = \bigoplus_{\substack{t \in \mathrm{Irr}(G) \\ t \subset \alpha_{s^{-1}}(x) \otimes y}} rs \cdot t, \qquad \text{for all } r, s \in \Gamma,\ x, y \in \mathrm{Irr}(G).$$

Moreover, we have $\overline{\gamma \cdot x} = \gamma^{-1} \cdot \alpha_\gamma(\overline{x})$ *for all* $\gamma \in \Gamma$ *and* $x \in \mathrm{Irr}(G)$.

Corollary 4.4.4. *The following hold.*

1. $\mathbb{G}$ *is co-amenable if and only if* G *is co-amenable and* Γ *is amenable.*

2. *If* G *is co-amenable and* Γ *is K-amenable, then* $\widehat{\mathbb{G}}$ *is K-amenable.*

Proof. (1). Let G be co-amenable and Γ be amenable. Then as $C_m(G) = C(G)$ and since the full and the reduced crossed products are the same for actions of amenable groups, it follows from the previous theorem that $\mathbb{G}$ is co-amenable. Now, if $\mathbb{G}$ is co-amenable, its Haar state is faithful on A_m. In particular, $h \circ \lambda \circ \alpha = h_G \circ \lambda_G$ must be be faithful on $C_m(G)$ which implies that G is co-amenable. Since $h(u_\gamma) = \delta_{\gamma,e}$, $\gamma \in \Gamma$, we conclude, from Remark 4.1.6 (since the counit ε_G is an α invariant character on $C_m(G)$), that the canonical trace on $C^*(\Gamma)$ has to be faithful. Hence, Γ is amenable. Note that this direction can also be shown using the continuity of the co-unit on the reduced algebras.

(2). Follows from [28, Theorem 2.1 (c)] since $C_m(G) = C(G)$. □

Note that, from the action $\alpha : \Gamma \curvearrowright C_m(G)$ by quantum automorphisms, we have a natural action, still denoted α, of Γ on $\chi(G)$ by group automorphisms and homeomorphisms. The set of fixed points $\chi(G)^\alpha = \{\chi \in \chi(G), :\ \chi \circ \alpha_\gamma = \chi \text{ for all } \gamma \in \Gamma\}$ is a closed subgroup. Also note that we have a natural action by group automorphisms, still denoted α, of Γ on $\mathrm{Int}(G)$.

Proposition 4.4.5. *There are canonical group isomorphisms:*

$$\mathrm{Int}(\mathbb{G}) \simeq \Gamma\,{}_\alpha\!\ltimes \mathrm{Int}(G) \quad \textit{and} \quad \chi(\mathbb{G}) \simeq \chi(G)^\alpha \times \mathrm{Sp}(\Gamma).$$

The second one is moreover a homeomorphism.

Proof. The proof is the same as the proof of Proposition 4.1.10. The dimension of the irreducible representation $(\mathrm{id} \otimes \alpha)(u^x)(1 \otimes u_\gamma)$ is equal to the dimension of x and such representations, for $x \in \mathrm{Irr}(G)$ and $\gamma \in \Gamma$, form a complete set of irreducibles of $\mathbb{G}$. Hence we get a bijection

$$\pi \,:\, \Gamma\,{}_\alpha \ltimes \mathrm{Int}(G) \to \mathrm{Int}(\mathbb{G}) \,:\, (\gamma, u) \mapsto \alpha(u)u_\gamma \in C_m(\mathbb{G}).$$

Moreover, the relations in the crossed product and the group law in the semi-direct product imply that it is a group homomorphism.

Let $(\chi, \mu) \in \chi(G)^\alpha \times \mathrm{Sp}(\Gamma)$. Since $\chi \circ \alpha_\gamma = \chi$ for all $\gamma \in \Gamma$, the pair (χ, μ) gives a covariant representation in $\mathbb{C}$, hence a unique character $\rho(\chi, \mu) \in \chi(\mathbb{G})$ such that $\rho(\chi, \mu)(u_\gamma\alpha(a)) = \mu(\gamma)\chi(a)$ for all $\gamma \in \Gamma$, $a \in C_m(G)$. It defines a map $\rho \,:\, \chi(G)^\alpha \times \mathrm{Sp}(\Gamma) \to \chi(\mathbb{G})$ which is obviously injective. A direct computation shows that ρ is a group homomorphism. Let us show that ρ is surjective. Let $\omega \in \chi(\mathbb{G})$, then $\chi := \omega \circ \alpha \in \chi(G)$ and, for all $a \in C_m(G)$, $\chi \circ \alpha_\gamma(a) = \omega(u_\gamma\alpha(a)u_\gamma^*) = \omega(u_\gamma)\omega(\alpha(a))\omega(u_\gamma^*) = \chi(a)$. Hence, $\chi \in \chi(G)^\alpha$ and we have $\omega = \rho(\chi, \mu)$, where $\mu = (\gamma \mapsto \omega(u_\gamma))$. Moreover, as in the proof of Proposition 4.1.10, it is easy to see that the map ρ^{-1} is continuous, hence ρ also, by compactness.

□

4.4.1 Weak amenability

This subsection deals with weak amenability of $\widehat{\mathbb{G}}$ constructed in Section 4.4. We first prove an intermediate technical result to construct finite rank u.c.p. maps from $C(G)$ to itself using compactness of the action and elements of $\ell^\infty(\widehat{G})$ of finite support. Using this construction, we estimate the Cowling-Haagerup constant of $C(\mathbb{G})$ and show that $C(\mathbb{G})$ is weakly amenable when both Γ and $\widehat{G}$ are weakly amenable and when the action is compact. This enables us to compute Cowling-Haagerup constants in some explicit examples given in Section 4.5. We freely use the notations and definitions of Section 1.3.

Lemma 4.4.6. *Suppose that the action $\alpha : \Gamma \curvearrowright G$ is compact. Denote by $H < \mathrm{Aut}(G)$ the compact group obtained by taking the closure of the image of Γ in $\mathrm{Aut}(G)$. If $a \in \ell^\infty(\widehat{G})$ has finite support, then the linear map $\Psi : C(G) \to C(G)$, defined by $\Psi(z) = \int_H (h^{-1} \circ m_a \circ h)(z) dh$ has finite rank and $\|\Psi\|_{cb} \leq \|m_a\|_{cb}$, where dh denotes integration with respect to the normalized Haar measure on H.*

Proof. First observe that Ψ is well defined since, for all $z \in C(G)$, the map $H \ni h \mapsto (h^{-1} \circ m_a \circ h)(z) \in C(G)$ is continuous. Moreover, the linearity of Ψ is obvious. Since a has finite support, the map m_a is of the form $m_a(\cdot) = \omega_1(\cdot)y_1 + \cdots + \omega_n(\cdot)y_n$, where $\omega_i \in C(G)^*$ and $y_i \in \mathrm{Pol}(G)$. Hence, to show that Ψ has finite rank, it suffices to show that the map $\Psi_1(z) = \int_H (h^{-1} \circ \varphi \circ h)(z) dh$, $z \in C(G)$, has finite rank when $\varphi(\cdot) = \omega(\cdot)y$, with $\omega \in C(G)^*$ and $y \in \mathrm{Pol}(G)$.

In this case, we have $\Psi_1(z) = \int_H \omega(h(z)) h^{-1}(y) dh$, $z \in C(G)$. Write y as a finite sum $y = \sum_{i=1}^N \sum_{k,l} \lambda_{i,k,l} u_{kl}^{x_i}$, where $F = \{x_1, \cdots, x_N\} \subset \mathrm{Irr}(G)$. Since H is compact, the action of H on $\mathrm{Irr}(G)$ has finite orbits. Writing $h \cdot x$ for the action of $h \in H$ on $x \in \mathrm{Irr}(G)$, the set $H \cdot F = \{h \cdot x : h \in H, x \in F\} \subset \mathrm{Irr}(G)$ is finite and, for all $h \in H$, $h^{-1}(y) \in \mathcal{F}$, where $\mathcal{F}$ is the finite subspace of $C(G)$ generated by the coefficients of the irreducible representations $x \in H \cdot F$. Hence, the map $h \mapsto \omega(h(z)) h^{-1}(y)$ takes values in $\mathcal{F}$, for all $z \in C(G)$. It follows that $\Psi_1(z) = \int_H \omega(h(z)) h^{-1}(y) dh \in \mathcal{F}$ for all $z \in C(G)$. Hence, Ψ has finite rank.

Now we proceed to show that $\|\Psi\|_{cb} \leq \|m_a\|_{cb}$. For $n \in \mathbb{N}$, denote by Ψ_n the map

$$\Psi_n = \mathrm{id} \otimes \Psi : M_n(\mathbb{C}) \otimes C(G) \to M_n(\mathbb{C}) \otimes C(G).$$

Observe that $\Psi_n(X) = \int_H (\mathrm{id} \otimes (h^{-1} \circ m_a \circ h))(X) dh$ for all $X \in M_n(\mathbb{C}) \otimes C(G)$. Hence, for $n \in \mathbb{N}$, one has

$$\|\Psi_n(X)\| \leq \int_H \|(\mathrm{id} \otimes (h^{-1} \circ m_a \circ h))(X)\| dh \leq \|X\| \int_H \|(h^{-1} \circ m_a \circ h)\|_{cb} dh \leq \|X\| \, \|m_a\|_{cb}.$$

It follows that $\|\Psi\|_{cb} \leq \|m_a\|_{cb}$. $\square$

Theorem 4.4.7. *We have* $\max(\Lambda_{cb}(\Gamma), \Lambda_{cb}(C(G))) \leq \Lambda_{cb}(C(\mathbb{G}))$. *Moreover, if the action* $\Gamma \curvearrowright G$ *is compact, then* $\Lambda_{cb}(C(\mathbb{G})) \leq \Lambda_{cb}(\Gamma)\Lambda_{cb}(\widehat{G})$.

Proof. The first inequality is obvious by the existence of conditional expectations from $C(\mathbb{G})$ to $C_r^*(\Gamma)$ and from $C(\mathbb{G})$ to $C(G)$. Let us prove the second inequality. We may and will assume that Γ and $\widehat{G}$ are weakly amenable. Fix $\epsilon > 0$.

Let $a_i \in \ell^\infty(\widehat{G})$ be a sequence of finitely supported elements such that $\sup_i \|m_{a_i}\|_{cb} \leq \Lambda_{cb}(\widehat{G}) + \epsilon$ and m_{a_i} converges pointwise in norm to identity. Consider the maps Ψ_i associated to a_i as in Lemma 4.4.6. Observe that the sequence Ψ_i converges pointwise in norm to identity. Indeed, for $x \in C(G)$,

$$\begin{aligned}\|\Psi_i(x) - x\| &= \|\int_H ((h^{-1} \circ m_{a_i} \circ h)(x) - x)dh\| = \|\int_H (h^{-1}(m_{a_i}(h(x)) - h(x))dh\| \\ &\leq \int_H \|m_{a_i}(h(x)) - h(x)\|dh.\end{aligned}$$

Now the right hand side of the above expression is converging to 0 for all $x \in C(G)$ by the dominated convergence theorem, since $\|m_{a_i}(h(x)) - h(x)\| \to_i 0$ for all $x \in C(G)$ and all $h \in H$, and

$$\|m_{a_i}(h(x)) - h(x)\| \leq (\|m_{a_i}\|_{cb} + 1)\|x\| \leq (\Lambda_{cb}(\widehat{G}) + \epsilon + 1)\|x\| \quad \text{for all } i \text{ and all } x \in C(G).$$

By definition, the maps Ψ_i are Γ-equivariant i.e., $\Psi_i \circ \alpha_\gamma = \alpha_\gamma \circ \Psi_i$. Hence, for all i, there is a unique linear extension $\widetilde{\Psi}_i : C(\mathbb{G}) \to C(\mathbb{G})$ such that $\widetilde{\Psi}_i(u_\gamma \alpha(x)) = u_\gamma \alpha(\Psi_i(x))$ for all $x \in C(G)$ and all $\gamma \in \Gamma$. Moreover, $\|\widetilde{\Psi}_i\|_{cb} \leq \|\Psi_i\|_{cb} \leq \|m_{a_i}\|_{cb} \leq \Lambda_{cb}(\widehat{G}) + \epsilon$.

Consider a sequence of finitely supported maps $\psi_j : \Gamma \to \mathbb{C}$ going pointwise to 1 and such that $\sup \|m_{\psi_j}\|_{cb} \leq (\Lambda_{cb}(\Gamma) + \epsilon)$, and denote by $\widetilde{\psi_j} : C(\mathbb{G}) \to C(\mathbb{G})$ the unique linear extension such that $\widetilde{\psi_j}(u_\gamma \alpha(x)) = \psi_j(\gamma) u_\gamma \alpha(x)$. Then, we have $\|\widetilde{\psi_j}\|_{cb} \leq \|m_{\psi_j}\|_{cb} \leq \Lambda_{cb}(\Gamma) + \epsilon$.

Define the maps $\varphi_{i,j} = \widetilde{\psi}_j \circ \widetilde{\Psi}_i : C(\mathbb{G}) \to C(\mathbb{G})$. Then for all i, j we have $\|\varphi_{i,j}\|_{cb} \leq$

$(\Lambda_{cb}(\Gamma)+\epsilon)(\Lambda_{cb}(\widehat{G})+\epsilon)$. Since $\varphi_{i,j}(u_\gamma\alpha(x))=\psi_j(\gamma)u_\gamma\alpha(\Psi_i(x))$, it is clear that $\varphi_{i,j}$ has finite rank, and $(\varphi_{i,j})_{i,j}$ is going pointwise in norm to identity. Since ϵ was arbitrary, the proof is complete. □

4.4.2 Rapid Decay

In this subsection we study property (RD) for crossed products. We use the notion of property (RD) developed in [14] and recall the definition below. Since for a discrete quantum subgroup $\widehat{G}<\widehat{\mathbb{G}}$, i.e. such that there exists a faithful unital $*$-homomorphism $C_m(G)\to C_m(\mathbb{G})$ which intertwines the comultiplications, property (RD) for $\widehat{\mathbb{G}}$ implies property (RD) for $\widehat{G}$ and, since for a crossed product $\widehat{\mathbb{G}}$ coming from an action $\Gamma\curvearrowright G$ of a discrete group Γ on a compact quantum group G, both Γ and $\widehat{G}$ are discrete quantum subgroups of $\widehat{\mathbb{G}}$, it follows that property (RD) for $\widehat{\mathbb{G}}$ implies property (RD) for Γ and $\widehat{G}$. Hence, we will only concentrate on proving the converse.

For a compact quantum group G and $a\in C_c(\widehat{G})$ we define its Fourier transform (see [76]) as:

$$\mathcal{F}_G(a)=(h_{\widehat{G}}\otimes 1)(V(a\otimes 1))=\sum_{x\in\mathrm{Irr}(G)}\dim_q(x)(\mathrm{Tr}_x\otimes\mathrm{id})((Q_x\otimes 1)u^x(ap_x\otimes 1))\in\mathrm{Pol}(G),$$

and its "Sobolev 0-norm" by $\|a\|_{G,0}^2=\sum_{x\in\mathrm{Irr}(G)}\frac{\dim_q(x)^2}{\dim(x)}\mathrm{Tr}_x(Q_x^*(a^*a)p_xQ_x)$.

Let $\alpha:\Gamma\curvearrowright G$ be an action by quantum automorphisms and denote by $\mathbb{G}$ the crossed product. Recall that $\mathrm{Irr}(\mathbb{G})=\{\gamma\cdot x\,:\,\gamma\in\Gamma \text{ and } x\in\mathrm{Irr}(G)\}$, where $\gamma\cdot x$ is the equivalence class of

$$u_\gamma^x=(1\otimes u_\gamma)(\mathrm{id}\otimes\alpha)(u^x)\in\mathcal{B}(H_x)\otimes C(\mathbb{G}).$$

Let $V_{\gamma\cdot x}:H_{\gamma\cdot x}\to H_x$ be the unique unitary such that $u^{\gamma\cdot x}=(V_{\gamma\cdot x}^*\otimes 1)u_\gamma^x(V_{\gamma\cdot x}\otimes 1)$.

Lemma 4.4.8. *For any $\gamma\in\Gamma$ and $x\in\mathrm{Irr}(G)$, one has $Q_{\gamma\cdot x}=V_{\gamma\cdot x}^*Q_xV_{\gamma\cdot x}$ and $\dim_q(\gamma\cdot x)=\dim_q(x)$.*

Proof. Since $V_{\gamma\cdot x}$ is unitary, it suffices to show the first assertion. Recall that $Q_{\gamma\cdot x}$ is uniquely determined by the properties that it is invertible, $\mathrm{Tr}_{\gamma\cdot x}(Q_{\gamma\cdot x}) = \mathrm{Tr}_{\gamma\cdot x}(Q_{\gamma\cdot x}^{-1}) > 0$ and that $Q_{\gamma\cdot x} \in \mathrm{Mor}(u^{\gamma\cdot x}, u_{cc}^{\gamma\cdot x})$, where $u_{cc}^{\gamma\cdot x} = (\mathrm{id} \otimes S_{\mathbb{G}}^2)(u^{\gamma\cdot x})$. It is obvious that $Q := V_{\gamma\cdot x}^* Q_x V_{\gamma\cdot x}$ is invertible and that $\mathrm{Tr}_{\gamma\cdot x}(Q) = \mathrm{Tr}_{\gamma\cdot x}(Q^{-1}) > 0$. Hence, we will be done once we show that $Q \in \mathrm{Mor}(u^{\gamma\cdot x}, u_{cc}^{\gamma\cdot x})$. To this end, we first note that we have, by Remark 4.4.2, for any $\gamma \in \Gamma$ and $a \in \mathrm{Pol}(G)$, $S_{\mathbb{G}}^2(u_\gamma \alpha(a)) = u_\gamma \alpha(S_G^2(a))$. Thus, $(\mathrm{id} \otimes S_{\mathbb{G}}^2)(u_\gamma^x) = (1 \otimes u_\gamma)(\mathrm{id} \otimes \alpha)((\mathrm{id} \otimes S_G^2)(u^x))$. It follows that $Q_x \in \mathrm{Mor}(u_\gamma^x, (u_\gamma^x)_{cc})$ hence $Q \in \mathrm{Mor}(u^{\gamma\cdot x}, u_{cc}^{\gamma\cdot x})$. □

Lemma 4.4.9. *Let $a \in C_c(\widehat{\mathbb{G}})$ and write $a = \sum_{\gamma\in S, x\in T} a p_{\gamma\cdot x}$, where $S \subset \Gamma$ and $T \subset \mathrm{Irr}(G)$ are finite subsets. For $\gamma \in S$, define $a_\gamma \in C_c(\widehat{G})$ by $a_\gamma = \sum_{x\in T} V_{\gamma\cdot x} a p_{\gamma\cdot x} V_{\gamma\cdot x}^* p_x$. The following statements hold.*

1. $\mathcal{F}_{\mathbb{G}}(a) = \sum_{\gamma\in S} u_\gamma \alpha(\mathcal{F}_G(a_\gamma))$.

2. $\|a\|_{\mathbb{G},0}^2 = \sum_{\gamma\in S} \|a_\gamma\|_{G,0}^2$.

Proof. Observe that, since $V_{\gamma\cdot x}$ is unitary, $\mathrm{Tr}_{\gamma\cdot x}(V_{\gamma\cdot x}^* A V_{\gamma\cdot x} B) = \mathrm{Tr}_x(A V_{\gamma\cdot x} B V_{\gamma\cdot x}^*)$ for all $\gamma \in \Gamma$, all $x \in \mathrm{Irr}(G)$ and all $A \in \mathcal{B}(H_x)$, $B \in \mathcal{B}(H_{\gamma\cdot x})$. Hence,

$$\begin{aligned}
\mathcal{F}_{\mathbb{G}}(a) &= \sum_{\gamma\in S, x\in T} \dim_q(\gamma\cdot x)(\mathrm{Tr}_{\gamma\cdot x} \otimes \mathrm{id})((Q_{\gamma\cdot x} \otimes 1)u^{\gamma\cdot x}(a p_{\gamma\cdot x} \otimes 1)) \\
&= \sum_{\gamma\in S, x\in T} \dim_q(x)(\mathrm{Tr}_{\gamma\cdot x} \otimes \mathrm{id})((V_{\gamma\cdot x}^* \otimes 1)(Q_x \otimes 1)(V_{\gamma\cdot x} \otimes 1)(V_{\gamma\cdot x}^* \otimes 1)u_\gamma^x(V_{\gamma\cdot x} \otimes 1)(a p_{\gamma\cdot x} \otimes 1)) \\
&= \sum_{\gamma\in S, x\in T} \dim_q(x)(\mathrm{Tr}_x \otimes \mathrm{id})((Q_x \otimes 1)u_\gamma^x(V_{\gamma\cdot x} a p_{\gamma\cdot x} V_{\gamma\cdot x}^* \otimes 1)) \\
&= \sum_{\gamma\in S} u_\gamma \alpha\left(\sum_{x\in T} \dim_q(x)(\mathrm{Tr}_x \otimes \mathrm{id})((Q_x \otimes 1)u^x(V_{\gamma\cdot x} a p_{\gamma\cdot x} V_{\gamma\cdot x}^* \otimes 1))\right) \\
&= \sum_{\gamma\in S} u_\gamma \alpha(\mathcal{F}_G(a_\gamma)).
\end{aligned}$$

This shows assertion 1. Assertion 2 follows from a similar computation using Lemma 4.4.8. □

A function $l : \mathrm{Irr}(G) \to [0, \infty)$ is called a *length function on* $\mathrm{Irr}(G)$ if $l(1) = 0$, $l(\overline{x}) = l(x)$ and that $l(x) \leq l(y) + l(z)$ whenever $x \subset y \otimes z$.

Lemma 4.4.10. *Let $\alpha : \Gamma \curvearrowright G$ be an action of Γ on G by quantum automorphisms and let l be a length function on $\mathrm{Irr}(G)$ which is α-invariant, i.e., $l(x) = l(\alpha_\gamma(x))$ for all $\gamma \in \Gamma$ and $x \in \mathrm{Irr}(G)$. Let l_Γ be a length function on Γ. Let $\mathbb{G}$ be the crossed product. The function $l_0 : \mathrm{Irr}(\mathbb{G}) \to [0, \infty)$, defined by $l_0(\gamma \cdot x) = l_\Gamma(\gamma) + l(x)$ is a length function on $\mathrm{Irr}(\mathbb{G})$.*

Proof. We have $l_0(1) = l_\Gamma(e) + l(1) = 0$ and, by Remark 4.4.3,

$$l_0(\overline{\gamma \cdot x}) = l_0(\gamma^{-1} \cdot \alpha_\gamma(\overline{x})) = l_\Gamma(\gamma^{-1}) + l(\alpha_\gamma(\overline{x})) = l_\Gamma(\gamma) + l(\overline{x}) = l_0(\gamma \cdot x).$$

Again, from Remark 4.4.3, $\gamma \cdot x \subset r \cdot y \otimes s \cdot z$ if and only if $\gamma = rs$ and $x \subset \alpha_{\gamma^{-1}}(y) \otimes z$. Hence,

$$\begin{aligned} l_0(\gamma \cdot x) &= l_\Gamma(\gamma) + l(x) \leq l_\Gamma(r) + l_\Gamma(s) + l(\alpha_{\gamma^{-1}}(y)) + l(z) \\ &= l_\Gamma(r) + l(y) + l_\Gamma(s) + l(z) = l_0(r \cdot y) + l_0(s \cdot z). \end{aligned}$$

□

Given a length function $l : \mathrm{Irr}(G) \to [0, \infty)$, consider the element $L = \sum_{x \in \mathrm{Irr}(G)} l(x) p_x$ which is affiliated to $c_0(\widehat{G})$. Let q_n denote the spectral projections of L associated to the interval $[n, n+1)$. We say that $(\widehat{G}, l)$ *has property* (RD), if there exists a polynomial $P \in \mathbb{R}[X]$ such that for every $k \in \mathbb{N}$ and $a \in q_k c_c(\widehat{G})$, we have $\|\mathcal{F}(a)\|_{C(G)} \leq P(k)\|a\|_{G,0}$. Finally, $\widehat{G}$ *is said to have Property* (RD) if there exists a length function l on $\mathrm{Irr}(G)$ such that $(\widehat{G}, l)$ has property (RD).

We prove property (RD) for the dual of a crossed product in the following Theorem. In case the action of the group is trivial, i.e., when the crossed product reduces to a tensor product, this result is proved in [21, Lemma 4.5]. For semi-direct products of classical groups, this result is due to Jolissaint [48].

Theorem 4.4.11. *Let $\alpha : \Gamma \curvearrowright G$ be an action by quantum automorphisms. Let l be a α-invariant length function on $\mathrm{Irr}(G)$. If $(\widehat{G}, l)$ has property (RD) and Γ has property (RD), then $(\widehat{\mathbb{G}}, l_0)$ has property (RD), where $\mathbb{G}$ is the crossed product and l_0 is as in Lemma 4.4.10.*

Proof. Let l_Γ be any length function on Γ for which (Γ, l_Γ) has property (RD) and let l_0 be the length function on $\mathrm{Irr}(\mathbb{G})$ defined by $l_0(\gamma \cdot x) = l_\Gamma(\gamma) + l(x)$, for $\gamma \in \Gamma$ and $x \in \mathrm{Irr}(G)$. Let $L_0 = \sum_{\gamma\in\Gamma, x\in \mathrm{Irr}(G)} l_0(\gamma\cdot x)p_{\gamma\cdot x} = \sum_{\gamma\in\Gamma, x\in\mathrm{Irr}(G)}(l_\Gamma(\gamma) + l(x))p_{\gamma\cdot x}$ and $L = \sum_{x\in\mathrm{Irr}(G)} l(x)p_x$. Finally, let p_n and q_n be the spectral projections of respectively L_0 and L associated to the interval $[n, n+1)$. Let $a \in c_c(\widehat{\mathbb{G}})$ and write $a = \sum_{\gamma\in S, x\in T} ap_{\gamma\cdot x}$, where $S \subset \Gamma$ and $T \subset \mathrm{Irr}(G)$ are finite subsets. Now suppose that $a \in p_k c_c(\widehat{\mathbb{G}})$. Since

$$p_k = \sum_{\gamma\in\Gamma, x\in\mathrm{Irr}(G), k\le l_\Gamma(\gamma)+l(x)<k+1} p_{\gamma\cdot x}$$

we must have

$$S \subset \{\gamma \in \Gamma \,:\, l_\Gamma(\gamma) < k+1\} \quad \text{and} \quad T \subset \{x \in \mathrm{Irr}(G) \,:\, l(x) < k+1\}.$$

It follows that, for all $\gamma \in S$, the element a_γ defined in Lemma 4.4.9 is in $q_K c_c(\widehat{G})$, where $q_K = \sum_{j=0}^{k} q_j$.

Let P_1 and P_2 be polynomials witnessing (RD) respectively for $(\widehat{G}, l)$ and (Γ, l_Γ). Let, for $i = 1, 2$, $C_i \in \mathbb{R}_+$ and $N_i \in \mathbb{N}$ be such that $P_i(k) \le C_i(k+1)^{N_i}$ for all $k \in \mathbb{N}$. Then, for all $b \in q_K c_c(\widehat{G})$,

$$\begin{aligned} \|\mathcal{F}_G(b)\| &\le \sum_{j\le k} \|\mathcal{F}_G(bq_j)\| \le \sum_{j\le k} P_1(j)\|bq_j\|_{G,0} \le \sum_{j\le k} C_1(j+1)^{N_1}\|bq_j\|_{G,0} \\ &\le C_1(k+1)^{N_1} \sum_{j\le k} \|bq_j\|_{G,0} = C_1(k+1)^{N_1+1}\|b\|_{G,0}. \end{aligned}$$

Similarly, $\|\psi * \phi\|_{\ell^2(\Gamma)} \le C_2(k+1)^{N_2+1}\|\psi\|_{\ell^2(\Gamma)}\|\phi\|_{\ell^2(\Gamma)}$ for all ϕ in $\ell^2(\Gamma)$ and all functions ψ on Γ (finitely) supported on words of l_Γ-length less than equal to k.

Let y be a finite sum $y = \sum_s u_s\alpha(b_s) \in \mathrm{Pol}(\mathbb{G})$. We have $\|y\|_{2,h_{\mathbb{G}}}^2 = \sum_s \|b_s\|_{2,h_G}^2$ and, by Lemma 4.4.9 and the preceding discussion,

$$\begin{aligned}
\|\mathcal{F}_{\mathbb{G}}(a)y\|_{2,h_{\mathbb{G}}}^2 &= \|\sum_{\gamma\in S,s} u_{\gamma s}\alpha(\alpha_{s^{-1}}(\mathcal{F}_G(a_\gamma))b_s)\|_{2,h_{\mathbb{G}}}^2 = \|\sum_{\gamma\in S,t} u_t\alpha(\alpha_{t^{-1}\gamma}(\mathcal{F}_G(a_\gamma))b_{\gamma^{-1}t})\|_{2,h_{\mathbb{G}}}^2 \\
&= \sum_t \|\sum_{\gamma\in S}\alpha_{t^{-1}\gamma}(\mathcal{F}_G(a_\gamma))b_{\gamma^{-1}t}\|_{2,h_G}^2 \leq \sum_t\left(\sum_{\gamma\in S}\|\alpha_{t^{-1}\gamma}(\mathcal{F}_G(a_\gamma))b_{\gamma^{-1}t}\|_{2,h_G}\right)^2 \\
&\leq C_1^2(k+1)^{2(N_1+1)}\sum_t\left(\sum_{\gamma\in S}\|a_\gamma\|_{G,0}\|b_{\gamma^{-1}t}\|_{2,h_G}\right)^2 = C_1^2(k+1)^{2(N_1+1)}\|\psi*\phi\|_{l^2(\Gamma)}^2,
\end{aligned}$$

where $\psi, \phi \in \ell^2(\Gamma)$ are defined by $\psi(\gamma) = \|a_\gamma\|_{G,0}$ and $\phi(s) = \|b_s\|_{2,h_G}$ where $\gamma, s \in \Gamma$. We note that $\|\psi\|_{\ell^2(\Gamma)}^2 = \sum_{\gamma\in S}\|a_\gamma\|_{G,0}^2 = \|a\|_{\mathbb{G},0}^2$ and $\|\phi\|_{\ell^2(\Gamma)}^2 = \sum_s \|b_s\|_{2,h_G}^2 = \|y\|_{2,h_{\mathbb{G}}}^2$. But since ψ is supported on S i.e., on elements of Γ of length less than equal to k, we have

$$\|\mathcal{F}_{\mathbb{G}}(a)y\|_{2,h_{\mathbb{G}}}^2 \leq (C_1C_2)^2(k+1)^{2(N_1+N_2+2)}\|\psi\|_{l^2(\Gamma)}^2\|\phi\|_{\ell^2(\Gamma)}^2 = P(k)^2\|a\|_{\mathbb{G},0}^2\|y\|_{2,h_{\mathbb{G}}}^2,$$

where $P(x) = C_1C_2(x+1)^{N_1+N_2+2}$. As y is arbitrary, the proof is complete. □

Remark 4.4.12. *There may not exist an α-invariant length function on* $\mathrm{Irr}(G)$*. However, if $\Gamma \curvearrowright G$ is compact, then the action $\alpha : \Gamma \curvearrowright \mathrm{Irr}(G)$ has finite orbits. Hence, for any length function l on $\mathrm{Irr}(G)$, the length function l_α defined by $l_\alpha(x) = \sup_{\gamma\in\Gamma} l(\alpha_\gamma(x))$, for $x \in \mathrm{Irr}(G)$, is α-invariant. Hence, $\widehat{\mathbb{G}}$ has (RD) whenever Γ and $\widehat{G}$ have (RD). We refer to the last section and to [67] for several examples of compact group actions on compact quantum groups.*

4.4.3 Property (T)

We characterize relative co-property (T) of the pair $(G, \mathbb{G})$ in a similar way we did characterize relative co-property (T) for bicrossed product. We study the property (T)

for $\widehat{\mathbb{G}}$.

When $\pi : A \to \mathcal{B}(H)$ is a unital $*$-homomorphism from a unital C*-algebra A, we denote by $\widetilde{\pi} : A^{**} \to \mathcal{B}(H)$ its unique normal extension. Also, we view any state $\omega \in A^*$ as a normal state on A^{**}. Observe that if (H, π, ξ) is the GNS construction for the state ω on A, then $(H, \widetilde{\pi}, \xi)$ is the GNS construction for the normal state ω on A^{**}.

Let $M = C_m(G)^{**}$ and $p_0 \in M$ be the unique central projection such that $p_0 x p_0 = \widetilde{\varepsilon}_G(x) p_0$ for all $x \in M$.

In the following theorem, we characterize the relative co-property (T) of the pair $(G, \mathbb{G})$ in terms of the action α of Γ on G. The proof is similar to the proof of Theorem 4.2.2 but technically more involved.

Theorem 4.4.13. *The following are equivalent:*

1. *The pair $(G, \mathbb{G})$ does not have the relative co-property (T).*

2. *There exists a sequence $(\omega_n)_{n \in \mathbb{N}}$ of states on $C_m(G)$ such that*

   (a) $\omega_n(p_0) = 0$ for all $n \in \mathbb{N}$;

   (b) $\omega_n \to \varepsilon_G$ weak;*

   (c) $\|\alpha_\gamma(\omega_n) - \omega_n\| \to 0$ for all $\gamma \in \Gamma$.

Proof. For a representation $\pi : C_m(\mathbb{G}) \to \mathcal{B}(H)$, we have $\varepsilon_G \subset \pi \circ \alpha$ if and only if $K_\pi \neq \{0\}$, where

$$K_\pi = \{\xi \in H : \pi \circ \alpha(a)\xi = \varepsilon_G(a)\xi \text{ for all } a \in C_m(G)\}.$$

Let $\rho = \pi \circ \alpha : C_m(G) \to \mathcal{B}(H)$ and observe that the orthogonal projection onto K_π is the projection $\widetilde{\rho}(p_0)$. Indeed, for all $\xi \in H$, $a \in C_m(G)$, we have $\pi \circ \alpha(a)\widetilde{\rho}(p_0)\xi = \widetilde{\rho}(ap_0)\xi = \varepsilon_G(a)\widetilde{\rho}(p_0)\xi$, which implies that $\mathrm{Im}(\widetilde{\rho}(p_0)) \subset K_\pi$. Moreover, if $\xi \in K_\pi$, we have $\widetilde{\rho}(a)\xi = \widetilde{\varepsilon}_G(a)\xi$ for all $a \in C_m(G)$. Since $C_m(G)$ is σ-weakly dense in M and the representations $\widetilde{\rho}$ and $\widetilde{\varepsilon}_G$ are normal, it follows that the equation $\widetilde{\rho}(a)\xi = \widetilde{\varepsilon}_G(a)\xi$ is valid for all $a \in M$. Hence, for $a = p_0$ we get $\widetilde{\rho}(p_0)\xi = \widetilde{\varepsilon}_G(p_0)\xi = \xi$, which in turn implies that $K_\pi \subset \mathrm{Im}(\widetilde{\rho}(p_0))$.

(1) $\implies$ (2). Suppose that the pair $(G, \mathbb{G})$ does not have the relative co-property (T). Let $\pi : C_m(\mathbb{G}) \to \mathcal{B}(H)$ be a representation such that $\varepsilon_{\mathbb{G}} \prec \pi$ and $K_\pi = \{0\}$. Denote by $\omega_{\xi,\eta} \in C_m(G)^*$ the functional given by $\omega_{\xi,\eta}(a) = \langle \pi \circ \alpha(a)\xi, \eta\rangle$. Hence, $\omega_{\xi,\eta}(p_0) = \langle \widetilde{\rho}(p_0)\xi, \eta\rangle = 0$ for all $\xi, \eta \in H$.

Since $\varepsilon_{\mathbb{G}} \prec \pi$, let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of unit vectors in H such that $\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| \to 0$ for all $x \in C_m(\mathbb{G})$. Define $\omega_n = \omega_{\xi_n,\xi_n}$. Then, we have $\omega_n(p_0) = 0$ for all $n \in \mathbb{N}$. For all $a \in C_m(G)$ we have, $|\omega_n(a) - \varepsilon_G(a)| = |\langle \pi(\alpha(a))\xi_n - \varepsilon_G(a)\xi_n, \xi_n\rangle| \leq \|\pi(\alpha(a))\xi_n - \varepsilon_{\mathbb{G}}(\alpha(a))\xi_n\| \to 0$. Moreover, exactly as in the proof of Theorem 4.2.2, we find $\|\alpha_\gamma(\omega_n) - \omega_n\| \leq 2\|\pi(u_\gamma)\xi_n - \xi_n\| = \|\pi(u_\gamma)\xi_n - \varepsilon_{\mathbb{G}}(u_\gamma)\xi_n\| \to 0$.

(2) $\implies$ (1). For a state $\omega \in C_m(G)^* = M_*$ we denote by $s(\omega) \in M$ its support. Recall that $s(\omega) \in M$ is the unique projection in M such that $N_\omega = M(1 - s(\omega))$, where N_ω is the σ-weakly closed left ideal defined by $N_\omega = \{x \in M : \omega(x^*x) = 0\}$ and note that ω is faithful on $s(\omega)Ms(\omega)$. In the sequel, we still denote by α_γ the unique $*$-isomorphism of M which extends $\alpha_\gamma \in \mathrm{Aut}(C_m(G))$. We first prove the following claim.

Claim. *If* (2) *holds, then there exists a sequence* $(\omega_n)_{n\in\mathbb{N}}$ *of states on* $C_m(G)$ *satifying* (a), (b) *and* (c) *and such that* $\alpha_\gamma(s(\omega_n)) = s(\omega_n)$ *for all* $\gamma \in \Gamma$, $n \in \mathbb{N}$.

Proof of the claim. Denote by $\ell^1(\Gamma)_{1,+}$ the set of positive ℓ^1 functions f on Γ with $\|f\|_1 = 1$. For a state $\omega \in C_m(G)^* = M_*$ and $f \in \ell^1(\Gamma)_{1,+}$, define the state $f*\omega \in C_m(G)^*$ by the convex combination

$$f * \omega = \sum_{\gamma\in\Gamma} f(\gamma)\alpha_\gamma(\omega).$$

Observe that, for all $\gamma \in \Gamma$ we have $\delta_\gamma * \omega = \alpha_\gamma(\omega)$ and $\alpha_\gamma(f * \omega) = f_\gamma * \omega$, where $f_\gamma \in \ell^1(\Gamma)_{1,+}$ is defined by $f_\gamma(r) = f(\gamma^{-1}r)$, $r \in \Gamma$. Moreover, if $f \in \ell^1(\Gamma)_{1,+}$ is such that $f(\gamma) > 0$ for all $\gamma \in \Gamma$, then since $(f * \omega)(x^*x) = \sum_\gamma f(\gamma)\omega(\alpha_{\gamma^{-1}}(x^*x))$, we have that

$(f*\omega)(x^*x)=0$ if and only if $\omega(\alpha_{\gamma^{-1}}(x^*x))=0$ for all $\gamma\in\Gamma$. It follows that

$$N_{f*\omega}=\cap_{\gamma\in\Gamma}\alpha_\gamma(N_\omega)=M\left(\wedge_{\gamma\in\Gamma}(1-\alpha_\gamma(s(\omega)))\right).$$

Hence, $s(f*\omega)=1-\wedge_{\gamma\in\Gamma}(1-\alpha_\gamma(s(\omega)))=\vee_{\gamma\in\Gamma}\alpha_\gamma(s(\omega))$. Hence, we have $\alpha_\gamma(s(f*\omega))=s(f*\omega)$ for all $\gamma\in\Gamma$. Finally, since $\varepsilon_G\circ\alpha_\gamma=\varepsilon_G$, we deduce that, for all $\gamma\in\Gamma$, $\alpha_\gamma(p_0)$ is a central projection of M satisfying $a\alpha_\gamma(p_0)=\alpha_\gamma(\alpha_{\gamma^{-1}}(a)p_0)=\varepsilon_G(\alpha_{\gamma^{-1}}(a))\alpha_\gamma(p_0)=\varepsilon_G(a)\alpha_\gamma(p_0)$, $\gamma\in\Gamma$. By uniqueness of such a projection, we find $\alpha_\gamma(p_0)=p_0$ for all $\gamma\in\Gamma$. Hence, for all $f\in\ell^1(\Gamma)_{1,+}$,

$$(f*\omega)(p_0)=\sum_\gamma f(\gamma)\omega(\alpha_{\gamma^{-1}}(p_0))=\sum_\gamma f(\gamma)\omega(p_0)=\omega(p_0).$$

Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of states on $C_m(G)$ satisfying (a), (b) and (c). We have, for all $f\in\ell^1(\Gamma)_{1,+}$ with finite support

$$\|f*\omega_n-\omega_n\|\leq\sum_\gamma f(\gamma)\|\delta_\gamma*\omega_n-\omega_n\|=\sum_\gamma f(\gamma)\|\alpha_\gamma(\omega_n)-\omega_n\|\to 0. \tag{4.4.1}$$

Since such functions f are dense in $\ell^1(\Gamma)_{1,+}$ (in the ℓ^1-norm), it follows that $\|f*\omega_n-\omega_n\|\to 0$ for all $f\in\ell^1(\Gamma)_{1,+}$.

Let $\xi\in\ell^1(\Gamma)_{1,+}$ be any function such that $\xi>0$ and define $\nu_n=\xi*\omega_n$. By the preceding discussion, we know that $\alpha_\gamma(s(\nu_n))=s(\nu_n)$ for all $\gamma\in\Gamma$ and $\nu_n(p_0)=\omega_n(p_0)=0$ for all $n\in\mathbb{N}$. Moreover, by Equation (4.4.1), we have $\|\alpha_\gamma(\nu_n)-\nu_n\|=\|\xi_\gamma*\omega_n-\xi*\omega_n\|\leq\|\xi_\gamma*\omega_n-\omega_n\|+\|\omega_n-\xi*\omega_n\|\to 0$ for all $\gamma\in\Gamma$. Since $\omega_n\to\varepsilon_G$ in the weak* topology and $\varepsilon_G\circ\alpha_\gamma=\varepsilon_G$, we have, $|\omega_n(\alpha_\gamma(a))-\varepsilon_G(a)|\to 0$ for all $a\in C_m(G)$ and all $\gamma\in\Gamma$. Hence, the Lebesgue dominated convergence theorem implies that, for all $a\in C_m(G)$,

$$|\nu_n(a)-\varepsilon_G(a)|=\left|\sum_\gamma f(\gamma)(\omega_n(\alpha_{\gamma^{-1}}(a))-\varepsilon_G(a))\right|\leq\sum_\gamma f(\gamma)|\omega_n(\alpha_{\gamma^{-1}}(a))-\varepsilon_G(a))|\to 0.$$

It follows that $\nu_n \to \varepsilon_G$ in the weak* topology and this completes the proof of the claim. $\square$

We can now finish the proof of the Theorem. Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of states on $C_m(G)$ as in the Claim. Let $M_n = s(\omega_n)Ms(\omega_n)$ and, since ω_n is faithful on M_n, view $M_n \subset \mathcal{B}(H_n)$ where (H_n, ξ_n) is the GNS construction of the f.n.s. ω_n on M_n. Define $\rho_n : C_m(G) \subset M \to M_n \subset \mathcal{B}(H_n)$ by $a \mapsto s(\omega_n)as(\omega_n)$. By definition, the unique normal extension of ρ_n is the map $\widetilde{\rho}_n : M \to M_n$, defined by $x \mapsto s(\omega_n)xs(\omega_n)$. Since $\alpha_\gamma(s(\omega_n)) = s(\omega_n)$, the action α restricts to an action, still denoted by α of Γ on M_n. Since $M_n \subset \mathcal{B}(H_n)$ is in standard form, we may consider the standard implementation (see Definition 1.6 of [80]) of the action of Γ on M_n to get a unitary representation $u_n : \Gamma \to \mathcal{U}(H_n)$ such that $\alpha_\gamma(x) = u_n(\gamma)xu_n(\gamma^{-1})$ for all $x \in M_n$ and $\gamma \in \Gamma$.

By the universal property of A_m, for $n \in \mathbb{N}$ there exists a unique unital $*$-homomorphism

$$\pi_n : A_m \to \mathcal{B}(H_n) \quad \text{such that} \quad \pi_n(u_\gamma) = u_n(\gamma) \quad \text{and} \quad \pi_n \circ \alpha = \rho_n.$$

Since $\omega_n(p_0) = 0$, we have $s(\omega_n)p_0s(\omega_n) = 0$. Hence, $\widetilde{\rho}_n(p_0) = 0$ and $K_{\pi_n} = \{0\}$ $\forall n \in \mathbb{N}$. It follows that, if we define $H = \oplus_n H_n$ and $\pi = \oplus_n \pi_n : C_m(\mathbb{G}) \to \mathcal{B}(H)$, then $K_\pi = \{0\}$ as well. Hence, it suffices to show that $\varepsilon_{\mathbb{G}} \prec \pi$. Since ξ_n is in the self-dual cone of ω_n and $u_n(\gamma)$ is the standard implementation of α_γ, it follows from [80, Theorem 1.14] that $u_n(\gamma)\xi_n$ is also in the self-dual cone of ω_n for all $n \in \mathbb{N}$. Hence, we may apply [80, Theorem 1.2] to get $\|u_n(\gamma)\xi_n - \xi_n\|^2 \leq \|\omega_{u_n(\gamma)\xi_n} - \omega_{\xi_n}\|$ for all $n \in \mathbb{N}$, $\gamma \in \Gamma$. Observe that $\omega_{u_n(\gamma)\xi_n}(x) = \alpha_\gamma(\omega_n)(x)$ and $\omega_{\xi_n}(x) = \omega_n(x)$ for all $x \in M$. Hence,

$$\|u_n(\gamma)\xi_n - \varepsilon_{\mathbb{G}}(u_\gamma)\xi_n\| = \|u_n(\gamma)\xi_n - \xi_n\| \leq \|\alpha_\gamma(\omega_n) - \omega_n\|^{\frac{1}{2}} \to 0.$$

Since $\omega_n \to \varepsilon_G$ in the weak* topology, it follows that for all $x = u_\gamma\alpha(a) \in C_m(\mathbb{G})$, we have

$$\begin{aligned}
\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| &= \|\pi(u_\gamma)\pi(\alpha(a))\xi_n - \varepsilon_G(a)\xi_n\| \\
&\leq \|\pi(u_\gamma)(\pi(\alpha(a))\xi_n - \varepsilon_G(a)\xi_n)\| + |\varepsilon_G(a)|\,\|\pi(u_\gamma)\xi_n - \xi_n\| \\
&\leq \|\pi(\alpha(a))\xi_n - \varepsilon_G(a)\xi_n\| + |\varepsilon_G(a)|\,\|u_n(\gamma)\xi_n - \xi_n\| \to 0.
\end{aligned}$$

By linearity and the triangle inequality, we have $\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| \to 0$ for all $x \in \mathcal{A}$. We conclude the proof using the density of $\mathcal{A}$ in $C_m(\mathbb{G})$. □

We now turn to Property (T).

Theorem 4.4.14. *The following holds:*

1. *If $\widehat{\mathbb{G}}$ has property (T), then Γ has property (T) and $\chi(G)^\alpha$ is finite.*

2. *If $\widehat{\mathbb{G}}$ has property (T) and α is compact then $\widehat{G}$ have property (T).*

3. *If $\widehat{G}$ has property (T) and Γ has property (T), then $\widehat{\mathbb{G}}$ has property (T).*

Proof. (1). This is the same proof as of assertion 1 of Theorem 4.2.3. First, we use the counit on $C_m(G)$ and the universal property of $C_m(\mathbb{G})$ to construct a surjective $*$-homomorphism $C_m(\mathbb{G}) \to C^*(\Gamma)$ which intertwines the comultiplications. We then use [34, Proposition 6] to conclude that Γ has property (T). To end the proof of (1), we show that $\chi(G)^\alpha$ is discrete. Let $\chi_n \in \chi(G)^\alpha$ be any sequence such that $\chi_n \to \varepsilon_G$ weak* in $C_m(G)^*$. We define a unital $*$-homomorphism $\chi : C_m(G) \to \mathcal{B}(l^2(\mathbb{N}))$ by $(\chi(a)\xi)(n) = \chi_n(a)\xi(n)$ for all $a \in C_m(G)$ and $\xi \in l^2(\mathbb{N})$. Since $\chi_n \in \mathrm{Sp}(C_m(G))^\alpha$ we have $\chi \circ \alpha_\gamma = \chi$ for all $\gamma \in \Gamma$. Hence, considering the trivial representation of Γ on $l^2(\mathbb{N})$ we obtain a covariant representation so there exists a unique unital $*$-homomorphism $\pi : C_m(\mathbb{G}) \to \mathcal{B}(l^2(\mathbb{N}))$ such that $\pi(u_\gamma\alpha(a)) = \chi(a)$ for all $a \in C_m(G)$ and all $\gamma \in \Gamma$. Since $\chi_n \to \varepsilon_G$ weak* the sequence of unit vectors defined by $\xi_n = \delta_n \in l^2(\mathbb{N})$ is a sequence of almost invariant vectors. By property (T) we have $\varepsilon_{\mathbb{G}} \subset \pi$ which easily implies that, for some $n \in \mathbb{N}$, $\chi_n = \varepsilon_G$.

(2). We repeat again the proof of assertion 2 of Theorem 4.2.3. By (1), it suffices to show that $\widehat{G}$ has Property (T). Let $\rho : C_m(G) \to \mathcal{B}(H)$ with $\varepsilon_G \prec \pi$ and define the compact group $K = \overline{\alpha(\Gamma)} \subset \mathrm{Aut}(G)$ with its Haar probability ν. Note that any $x \in \mathrm{Aut}(G)$, in particular any $x \in K$, satisfies $\varepsilon_G \circ x = \varepsilon_G$. Define the covariant representation (ρ_α, v), $\rho_\alpha : C_m(G) \to \mathcal{B}(\mathrm{L}^2(K,H))$ and $v : \Gamma \to \mathcal{U}(\mathrm{L}^2(K,H))$ by $(\rho_\alpha(a)\xi)(x) = \rho(x^{-1}(a))\xi(x)$ and $(v_\gamma\xi)(x) = \xi(\alpha_{\gamma^{-1}}x)$. By the universal property of $C_m(\mathbb{G})$ we get a unital $*$-homomorphism $\pi : C_m(\mathbb{G}) \to \mathcal{B}(\mathrm{L}^2(K,H))$ such that $\pi(u_\gamma\alpha(a)) = v_\gamma\rho_\alpha(a)$. Let $\xi_n \in H$ be a sequence of unit vectors such that $\|\rho(a)\xi_n - \varepsilon_G(a)\xi_n\| \to 0$ for all $a \in C_m(G)$ and define the vectors $\eta_n(x) = \xi_n$ for all $x \in K$, $n \in \mathbb{N}$. Since ν is a probability, η_n is a unit vector in $\mathrm{L}^2(K,H)$ for all $n \in \mathbb{N}$. Moreover, for all $a \in C_m(G)$ and $\gamma \in \Gamma$,

$$\|\pi(u_\gamma\alpha(a))\eta_n - \varepsilon_{\mathbb{G}}(u_\gamma\alpha(a))\xi_n\|^2 = \int_K \|\rho(x^{-1}(\alpha_\gamma(a)))\xi_n - \varepsilon_G(a)\xi_n\|^2 d\nu(x) \to 0,$$

where the convergence follows from the dominated convergence Theorem, since

$$\|\rho(x^{-1}(\alpha_\gamma(a)))\xi_n - \varepsilon_G(a)\xi_n\| = \|\rho(x^{-1}(\alpha_\gamma(a)))\xi_n - \varepsilon_G(x^{-1}(\alpha_\gamma(a)))\xi_n\| \to 0,$$

for all $a \in C_m(G)$, $x \in K$ and $\gamma \in \Gamma$ and the domination hypothesis is obvious since ν is a probability. Hence, $\varepsilon_{\mathbb{G}} \prec \pi$ and it follows from Property (T) that there exists a non-zero π-invariant vector $\xi \in \mathrm{L}^2(G,H)$. In particular, for all $a \in C_m(G)$, $\pi(\alpha(a)\xi = \varepsilon_G(a)\xi$. Hence, $\nu(Y) > 0$ where $Y = \{x \in K : \|\xi(x)\| > 0\}$ and, for all $a \in C_m(G)$, $\nu(X_a) = 0$ where $X_a = \{x \in K : \rho(x^{-1}(a))\xi(x) \neq \varepsilon_G(a)\xi(x)\}$. As in the proof of assertion 2 of Theorem 4.2.3, we deduce from the separability of $C_m(G)$ that there exists $x \in K$ for which $\xi(x) \neq 0$ and $\rho(x^{-1}(a))\xi(x) = \varepsilon_G(a)\xi(x)$ for all $a \in C_m(G)$. It follows that the vector $\eta := \xi(x) \in H$ is a non-zero ρ-invariant vector.

(3). We use the notations introduced in the proof of Theorem 4.4.13. Let $\pi : C_m(\mathbb{G}) \to \mathcal{B}(H)$ be a representation and consider the representation $\rho = \pi \circ \alpha : C_m(G) \to$

$\mathcal{B}(H)$ and the unitary representation $v_\gamma = \pi(u_\gamma)$ of Γ on H. Let $K_\pi = \{\xi \in H : \rho(a)\xi = \varepsilon_G(a)\xi$ for all $a \in C_m(G)\}$ and recall that the orthogonal projection onto K_π is $P = \widetilde{\rho}(p_0)$ and that $\alpha_\gamma(p_0) = p_0$ for all $\gamma \in \Gamma$. Hence, $v_\gamma P v_{\gamma^{-1}} = \widetilde{\rho}(\alpha_\gamma(p_0)) = P$ for all $\gamma \in \Gamma$, and it follows that K_π is an invariant subspace of $\gamma \mapsto v_\gamma$. Suppose that $\varepsilon_{\mathbb{G}} \prec \pi$. By property (T) of $\widehat{G}$, the space K_π is non-zero and we can argue exactly as in the proof of Theorem 4.2.3 to conclude the result. □

Remark 4.4.15. *It follows from the proof of the first assertion of the previous theorem that $C^*(\Gamma)$ is a compact quantum subgroup of the compact quantum group $\mathbb{G}$. Now, an irreducible representation of $\mathbb{G}$ of the form u^x_γ (with dimension say m), when restricted to the subgroup $C^*(\Gamma)$, decomposes as a direct sum of m copies of γ. It now follows from [72, Theorem 6.3] that $C^*(\Gamma)$ is a central subgroup (see [72, Definition 6.1]). Furthermore, Γ induces an action on the chain group $c(G)$ [72, Definition 7.4] of G and it follows from Remark 4.4.3 that the chain group (and hence the center, see [72, Section 7]) of $\mathbb{G}$ is the semidirect product group $c(G) \rtimes \Gamma$.*

Remark 4.4.16. ***(Kazhdan Pair for $\mathbb{G}$)*** *Let (E_1, δ_1) be a Kazhdan pair for G and (E_2, δ_2) be a Kazhdan Pair for Γ. Then it is not hard to show that $E = (E_1 \cup E_2) \subset Irr(\mathbb{G})$ and $\delta = min(\delta_1, \delta_2)$ is a Kazhdan pair for $\mathbb{G}$. Indeed, let $\pi : C_m(\mathbb{G}) \to \mathcal{B}(H)$ be a $*$-representation having a (E, δ)-invariant (unit) vector ξ. Then restricting to the subalgebra $C_m(G)$ (and denoting the corresponding representation by π_G), we get an (E_1, δ_1) invariant vector and hence, there is an invariant vector $\eta \in H$. We may assume $\|\xi - \eta\| < 1$ (this follows from a quantum group version of Proposition 1.1.9 of [12], which can be proved in an exactly similar fashion). Now, by restricting π to Γ, denoting the corresponding representation by u, we have that the closed linear u-invariant subspace generated by $u_\gamma\eta, \gamma \in \Gamma$ (which we denote by H_η), is a subspace of the space of π_G-invariant vectors (as $u_\gamma \pi_G(a) u_\gamma^{-1} = \pi_G(\alpha_\gamma(a))$). Let P_{H_η} denote the orthogonal projection onto H_η. Now, the vector $P_{H_\eta}\xi$, which is non-zero, as $\|\xi - \eta\| < 1$, is an (E_2, δ_2)-invariant vector for the representation u, restricted to H_η. So, there exists an u-invariant vector $\eta_0 \in H_\eta$. This vector is, of course then, π-invariant and hence, we are done.*

4.4.4 Haagerup property

In this section, we study the relative co-Haagerup property of the pair $(G, \mathbb{G})$ given by a crossed product and provide a characterization analogous to the bicrossed product case. We also extend a result of Jolissaint on Haagerup property for finite von Neumann algebra crossed product to a non-finite setting. Thus, we can decide whether $\mathrm{L}^\infty(\mathbb{G})$ has the Haagerup property. Finally, we provide sufficient conditions for $\widehat{\mathbb{G}}$ to posses the Haagerup property.

For the relative Haagerup property of crossed product, we obtain the following result similar to Theorem 4.3.2. The proof is even simpler in the crossed product case, since α is an action by quantum automorphisms.

Theorem 4.4.17. *The following are equivalent:*

1. *The pair $(G, \mathbb{G})$ has the relative co-Haagerup property.*
2. *There exists a sequence $(\omega_n)_{n\in\mathbb{N}}$ of states on $C_m(G)$ such that*
    - *(a) $\widehat{\omega}_n \in c_0(\widehat{G})$ for all $n \in \mathbb{N}$;*
    - *(b) $\omega_n \to \varepsilon_G$ weak*;*
    - *(c) $\|\alpha_\gamma(\omega_n) - \omega_n\| \to 0$ for all $\gamma \in \Gamma$.*

Proof. (1) $\Rightarrow$ (2). The argument is exactly the same as the proof of (1) $\implies$ (2) of Theorem 4.3.2.

(2) $\Rightarrow$ (1). We first prove the following claim.

Claim. *If* (2) *holds, then there exists a sequence $(\nu_n)_{n\in\mathbb{N}}$ of states on $C_m(G)$ satifying* (a), (b) *and* (c) *and such that $\alpha_\gamma(s(\nu_n)) = s(\nu_n)$ for all $\gamma \in \Gamma$, $n \in \mathbb{N}$.*

Proof of the claim. By the proof of the claim in Theorem 4.4.13, it suffices to check that, whenever ν is a state on $C_m(G)$ and $f \in \ell^1(\Gamma)$, we have $\widehat{\nu} \in c_0(\widehat{G}) \Rightarrow \widehat{f * \nu} \in c_0(\widehat{G})$.

We first show that $\widehat{\nu} \in c_0(\widehat{G}) \Rightarrow \widehat{\alpha_\gamma(\nu)} \in c_0(\widehat{G})$. Note that we still denote by α the action of Γ on $\mathrm{Irr}(G)$ (see Remark 4.4.3). Now let ν be a state on $C_m(G)$ such that $\widehat{\nu} \in$

$c_0(\widehat{G})$ and let $\epsilon > 0$. By assumptions, the set $F = \{x \in \mathrm{Irr}(G) : \|(\mathrm{id} \otimes \nu)(u^x)\|_{\mathcal{B}(H_x)} \geq \epsilon\}$ is finite. Hence, the set

$$\{x \in \mathrm{Irr}(G) : \|(\mathrm{id} \otimes \nu)(u^{\alpha_{\gamma^{-1}}(x)})\|_{\mathcal{B}(H_x)} \geq \epsilon\} = \{x \in \mathrm{Irr}(G) : \alpha_{\gamma^{-1}}(x) \in F\} = \alpha_\gamma(F)$$

is also finite. Since $\widehat{\alpha_\gamma(\nu)} = \Big((\mathrm{id} \otimes \nu)(u^{\alpha_{\gamma^{-1}}(x)})\Big)_{x \in \mathrm{Irr}(G)}$, it follows that $\widehat{\alpha_\gamma(\nu)} \in c_0(\widehat{G})$.

From this we can now conclude that for all $f \in \ell^1(\Gamma)$, we have $\widehat{\nu} \in c_0(\widehat{G}) \Rightarrow \widehat{f * \nu} \in c_0(\widehat{G})$ as in the proof of the Claim in Theorem 4.3.2. □

We can now finish the proof of the Theorem. Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of states on $C_m(G)^*$ as in the Claim. As in the proof of Theorem 4.4.13, we construct a representation $\pi : C_m(\mathbb{G}) \to \mathcal{B}(H)$ with a sequence of unit vectors $\xi_n \in H$ such that $\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| \to 0$ for all $x \in C_m(\mathbb{G})$ and $\nu_n = \omega_{\xi_n} \circ \pi \circ \alpha$. It follows that the sequence of states $\omega_n = \omega_{\xi_n} \circ \pi \in C_m(\mathbb{G})^*$, satisfies $\omega_n \to \varepsilon_{\mathbb{G}}$ in the weak* topology and $\widehat{\omega_n \circ \alpha} = \widehat{\nu}_n \in c_0(\widehat{G})$ for all $n \in \mathbb{N}$. □

We now turn to the Haagerup property. We will need the following result which is of independent interest. This is the non-tracial version of [47, Corollary 3.4] and the proof is similar. We include a proof for the convenience of the reader. We refer to [22, 69] for the Haagerup property for arbitrary von Neumann algebras. We will say that a UCP map $\psi : M \to M$ is ν-dominated for a state ν of M if we have that $\nu \circ \psi \leq \nu$.

Proposition 4.4.18. *Let (M, ν) be a von Neumann algebra with a f.n.s. ν and let $\alpha : \Gamma \curvearrowright M$ be an action which leaves ν invariant. If α is compact, Γ and M have the Haagerup property, then $\Gamma \ltimes M$ has the Haagerup property.*

Proof. Let $H < \mathrm{Aut}(M)$ be the closure of the image of Γ in $\mathrm{Aut}(M)$. By assumption H is compact. Let $\mathrm{L}^2(M)$ denote the GNS space of ν.

Let $\psi : M \to M$ be a ucp, normal and ν-dominated map and suppose that T_ψ, the L^2 extension of ψ, is compact. Then it is easy to see that for all $x \in M$, the map $H \ni h \mapsto h^{-1} \circ \psi \circ h(x) \in M$ is σ-weakly continuous. Hence, we can define $\Psi(x) =$

$\int_H h^{-1} \circ \psi \circ h(x) dh$, where dh is the normalized Haar measure on H. By construction, the map $\Psi : M \to M$ is ucp, ν-dominated, Γ-equivariant and normal. Moreover, for all $\xi \in \mathrm{L}^2(M)$, the map $H \ni h \mapsto T_{h^{-1}} \circ T_\psi \circ T_h \xi \in \mathrm{L}^2(M)$, where T_h denotes the L^2-extension of h, is norm continuous. Consequently, $\int_H T_{h^{-1}} \circ T_\psi \circ T_h dh \in \mathcal{B}(\mathrm{L}^2(M))$ and by definition of Ψ we have that the L^2-extension of Ψ is given by $T_\Psi = \int_H T_{h^{-1}} \circ T_\psi \circ T_h dh \in \mathcal{B}(\mathrm{L}^2(M))$. Let $\mathcal{B}$ denote the unit ball of $\mathrm{L}^2(M)$. Consider the set $A = \{h \mapsto T_{h^{-1}} \circ T_\psi \circ T_h \xi : \xi \in \mathcal{B}\} \subset C(H, \mathcal{B})$. It is easy to check that A is equicontinuous and, since T_ψ is compact, the set $A(h) = \{f(h) : f \in A\}$ is precompact for all $h \in H$. By Ascoli's Theorem, A is precompact in $C(H, \mathcal{B})$. Since the map $H \times C(H, \mathcal{B}) \to \mathcal{B}$, defined by $(h, f) \mapsto f(h)$ is continuous, the image of $H \times \overline{A}$ is compact and contains $\mathcal{B}_\psi = \{T_{h^{-1}} \circ T_\psi \circ T_h(\mathcal{B}), h \in H\}$. Since the image of $\mathcal{B}$ under T_Ψ is contained in the closed convex hull of $\overline{\mathcal{B}_\psi}$, it follows that T_Ψ is compact.

We use the standard notations $N = \Gamma \ltimes M = \{u_\gamma x : \gamma \in \Gamma, x \in M\}'' \subset \mathcal{B}(\ell^2(\Gamma) \otimes \mathrm{L}^2(M))$. We write $\widetilde{\nu}$ for the dual state of ν on N. Let ψ_i be a sequence of normal, ucp, ν-dominated and L^2-compact maps on M which converge pointwise in $\|\cdot\|_{2,\nu}$ to identity. Consider the sequence of ν-dominated, ucp, normal, L^2-compact and Γ-equivariant maps Ψ_i given by $\Psi_i(x) = \int_H h^{-1} \circ \psi \circ h(x) dh$ for all $x \in M$. Note that $(\Psi_i)_i$ is still converging pointwise in $\|\cdot\|_{2,\nu}$ to identity since, by the dominated convergence Theorem we have,

$$\|\Psi_i(x) - x\|_{2,\nu} = \left\| \int_H h^{-1}(\psi_i(h(x)) - h(x)) dh \right\|_{2,\nu} \leq \int_H \|\psi_i(h(x)) - h(x)\|_{2,\nu} dh \to 0.$$

By the Γ-equivariance, we can consider the normal ucp $\widetilde{\nu}$-dominated maps on N given by $\widetilde{\Psi_i}(u_\gamma x) = u_\gamma \Psi_i(x)$. Observe that the sequence $(\widetilde{\Psi}_i)$ is still converging pointwise in $\|.\|_{2,\widetilde{\nu}}$ to identity and the L^2-extension of $\widetilde{\Psi}_i$ is given by $T_{\widetilde{\Psi}_i} = 1 \otimes T_{\Psi_i} \in \mathcal{B}(\ell^2(\Gamma) \otimes \mathrm{L}^2(M))$.

Let ϕ_i be a sequence of positive definite and c_0 functions on Γ converging to 1 pointwise and consider the normal ucp $\widetilde{\nu}$-preserving maps on N given by $\widetilde{\phi}_i(u_\gamma x) = \phi_i(\gamma) u_\gamma x$. Observe that the sequence $(\widetilde{\phi}_i)$ is converging pointwise in $\|\cdot\|_{2,\widetilde{\nu}}$ to identity and the L^2-extension of $\widetilde{\phi}_i$ is given by $T_{\widetilde{\phi}_i} = T_{\phi_i} \otimes 1 \in \mathcal{B}(\ell^2(\Gamma) \otimes \mathrm{L}^2(M))$, where $T_{\phi_i}(\delta_\gamma) = \phi_i(\gamma)\delta_\gamma$

is a compact operator on $\ell^2(\Gamma)$.

Hence, if we define the sequence of normal, ucp, $\widetilde{\nu}$-dominated maps on N by $\varphi_{i,j} = \widetilde{\phi}_j \circ \widetilde{\Psi}_i$, we have $\varphi_{i,j}(u_\gamma x) = \phi_j(\gamma) u_\gamma \Psi_i(x)$; the sequence $(\varphi_{i,j})$ is converging pointwise in $\|\cdot\|_{2,\widetilde{\nu}}$ to identity and the L^2-extension of $\varphi_{i,j}$ is given by $T_{\varphi_{i,j}} = T_{\phi_j} \otimes T_{\Psi_i} \in \mathcal{B}(\ell^2(\Gamma) \otimes \mathrm{L}^2(M))$ is compact. □

Corollary 4.4.19. *The following holds.*

1. *If $\mathrm{L}^\infty(\mathbb{G})$ has the Haagerup property, then $\mathrm{L}^\infty(G)$ and Γ both have the Haagerup property.*

2. *If $\mathrm{L}^\infty(G)$ has the Haagerup property, $\alpha : \Gamma \curvearrowright \mathrm{L}^\infty(G)$ is compact and Γ has the Haagerup property, then $\mathrm{L}^\infty(\mathbb{G})$ has the Haagerup property. .*

Proof. (1). Follows from the fact that there exist normal, faithful, Haar-state preserving conditional expectations from $\mathrm{L}(\mathbb{G})$ to $\mathrm{L}(\Gamma)$ and to $\mathrm{L}^\infty(G)$. The former is given by $u_\gamma a \mapsto h_G(a) u_\gamma$ and the latter is given by $u_\gamma a \mapsto \delta_{\gamma,e} a$, $a \in \mathrm{L}^\infty(G)$ and $\gamma \in \Gamma$.

(2). It is an immediate consequence of Proposition 4.4.18. □

Theorem 4.4.20. *Suppose $\widehat{G}$ has the Haagerup property and Γ has the Haagerup property, and further suppose that the action of Γ on G is compact. Then $\widehat{\mathbb{G}}$ has the Haagerup property.*

Proof. Since $\widehat{G}$ has the Haagerup property, this assures the existence of states $(\mu_n)_{n\in\mathbb{N}}$ on $C_m(G)$ such that (1) $\widehat{\mu_n} \in c_0(\widehat{G})$ for all $n \in \mathbb{N}$ and (2) $\mu_n \to \varepsilon_G$ weak*. Our first task is to construct a sequence of α-invariant states on $C_m(G)$ satisfying (1) and (2) above. This is similar to our arguments before (while dealing with property (T) and Haagerup property). Since the action of Γ is compact, the closure of Γ in $\mathrm{Aut}(G)$ is compact, and we denote this subgroup by H. Letting dh denote the normalized Haar measure on H, we define states $\nu_n \in C_m(G)^*$ by $\nu_n(a) = \int_H \mu_n(h^{-1}(a))dh$, for all $a \in C_m(G)$. It is easily seen that ν_n is invariant under the action of Γ for each n. Now, since the action is compact, all orbits of the induced action on $\mathrm{Irr}(G)$ are finite. We need this to show that μ_n satisfy (1) above. So, let $\epsilon > 0$. As μ_n satisfied (1), the set $L = \{x \in \mathrm{Irr}(G) : \|(\mathrm{id} \otimes \mu_n)(u^x)\| \geq \frac{\epsilon}{2}\}$ is finite and the set $K = H \cdot L \subset \mathrm{Irr}(G)$ is also finite, as all the orbits are finite. For $h \in H \subset \mathrm{Aut}(G)$ and $x \in \mathrm{Irr}(G)$ write $V_{h,x} \in \mathcal{B}(H_x)$ to be the unique unitary such that $(\mathrm{id} \otimes h^{-1})(u^x) = (V_{h,x}^* \otimes 1)(\mathrm{id} \otimes u^{h^{-1}(x)})(V_{h,x} \otimes 1)$. If $x \notin K$ then, for all $h \in H$, $h^{-1}(x) \notin L$. Hence, $\|(\mathrm{id} \otimes \mu_n)(u^{h^{-1}(x)})\| < \frac{\epsilon}{2}$ for all $h \in H$ and it follows that

$$\begin{aligned}
\|(\mathrm{id} \otimes \nu_n)(u^x)\| &= \left\| \int_H (\mathrm{id} \otimes \mu_n)((\mathrm{id} \otimes h^{-1})(u^x)) dh \right\| \leq \int_H \|V_{h,x}^* (\mathrm{id} \otimes \mu_n)(u^{h^{-1}(x)}) V_{h,x}\| dh \\
&\leq \int_H \|(\mathrm{id} \otimes \mu_n)(u^{h^{-1}(x)})\| dh \leq \frac{\epsilon}{2} < \epsilon \quad \text{for all } x \notin K.
\end{aligned}$$

Hence, $\{x \in \mathrm{Irr}(G) : \|(\mathrm{id} \otimes \nu_n)(u^x)\| \geq \epsilon\} \subset K$ is a finite set and (1) holds for ν_n. To show that (2) holds, we first note that given any $a \in C_m(G)$, one has $\mu_n(h^{-1}(a)) \to \varepsilon_G(h^{-1}(a)) = \varepsilon_G(a)$ for all $h \in H$ (since H acts on G by quantum automorphisms). By the dominated convergence Theorem, we see that (2) holds for ν_n. Now, since Γ has the Haagerup property, we can construct states τ_n on $C^*(\Gamma)$ satisfying (1) and (2) above. And since the states μ_n on $C_m(G)$ are α-invariant, we can construct the crossed product states $\phi_n = \tau_n \ltimes \mu_n$ on $C_m(\mathbb{G})$ (see [88, Proposition and Definition 3.4] and also [19, Exercise 4.1.4] for the case of c.c.p. maps). The straightforward computations that need to be done to see that the sequence of states $(\phi_n)_{n\in\mathbb{N}}$ satisfy (1) and (2) above, are left to the reader. This then shows that $\widehat{\mathbb{G}}$ has the Haagerup property. $\square$

Remark 4.4.21. *We note that in case G is Kac, the above theorem already follows from Corollary 4.4.19(2) and Theorem 6.7 of [29].*

4.5 Examples

For coherent reading, we have dedicated this section only to examples arising from both matched pairs and crossed products. It is to be noted that it is not hard to come up with examples of compact matched pairs of groups for which only one of the actions α or β is non-trivial which means that the other is an action by group homomorphisms.

However, it is harder to come up with examples for which both α and β are non-trivial. We called such matched pairs *non-trivial*. Starting out with a compact matched pair for which either α or β is trivial, we describe a process to deform the original matched pair by what we call a *crossed homomorphism* in such a way that we manufacture a new compact matched pair for which both actions are non-trivial. For pedagogical reasons, we have made two subsections dealing with matched pairs: the first one (Section 4.5.1.1), in which we describe how to perturb β when it is trivial, followed by Section 4.5.1.2 in which we construe how to perturb α when it is trivial. It has to be noted that it is indeed possible to formalize our process of deformation in a unified way but, since such a formulation would increase the technicalities and would not produce any new explicit examples, we have chosen to separate the presentation in the two basic deformations described above. Our deformations are chosen carefully so as to ensure that the geometric group theoretic properties (that we have studied in detail throughout this chapter) passes from the initial bicrossed product to the one obtained after the deformation very naturally. Such deformations also allow us to keep track of the invariants $\chi(\cdot)$ and $\mathrm{Int}(\cdot)$ of the associated compact quantum groups. These explicit constructions allow us to exhibit: (i) a pair of non-isomorphic non-trivial compact bicrossed products each of which has relative property (T) but the duals do not have property (T), (ii) an infinite family of pairwise non-isomorphic non-trivial compact bicrossed products whose duals are non-amenable with the Haagerup property, (iii) an infinite family of pairwise non-isomorphic non-trivial compact bicrossed products whose duals have property (T).

We also provide non-trivial examples of crossed products of a discrete group on a non-trivial compact quantum group in Section 4.5.2. The action is coming from the conjugation action of a countable subgroup of $\chi(G)$ on the compact quantum group G. In this situation we completely understand weak amenability, (RD), Haagerup property and property (T) in terms of G and Γ and we also discuss explicit examples involving the free orthogonal and free unitary quantum groups.

4.5.1 Examples of bicrossed products

In this section, we focus on deformation of actions in matched pairs when one of them is trivial. The analysis involved helps to construct non-trivial examples.

4.5.1.1 From matched pairs with trivial β

Let α be any action of a discrete group Γ on a compact group G by group homomorphisms. Taking β to be the trivial action of G on Γ, the relations in Equation (4.1.1) are satisfied and we get a compact matched pair. It is possible to upgrade this example in order to obtain a new compact matched pair $(\Gamma, \widetilde{G})$ for which the associated actions $\widetilde{\alpha}$ and $\widetilde{\beta}$ are both non-trivial.

Indeed, given an action α of the discrete group Γ on the compact group G and a continuous map $\chi : G \to \Gamma$, we define a continuous map

$$G \times G \to G \text{ by } (g,h) \mapsto g * h, \quad \text{where} \quad g * h = g\alpha_{\chi(g)}(h) \quad \text{for all } g,h \in G.$$

Observe that $e * g = g * e = g$ for all $g \in G$ if and only if $\chi(e) \in \mathrm{Ker}(\alpha)$. Moreover, it is easy to check that the map $(g,h) \mapsto g * h$ is associative if and only if $\chi(gh)^{-1}\chi(g)\chi(\alpha_{\chi(g)^{-1}}(h)) \in \mathrm{Ker}(\alpha)$ for all $g,h \in G$. Finally, under the preceding hypothesis, the map $(g,h) \mapsto g * h$ turns G into a compact group since the inverse of $g \in G$ exists and is given by $\alpha_{\chi(g)^{-1}}(g^{-1})$ and this inversion is a continuous map from G to itself.

Hence it is natural to define a *crossed homomorphism* as a continuous map $\chi : G \to \Gamma$ such that $\chi(e) = e$ and $\chi(gh) = \chi(g)\chi(\alpha_{\chi(g)^{-1}}(h))$ for all $g,h \in G$. Observe that the continuity of χ, the compactness of G and the discreteness of Γ all together imply that the image of χ is finite. By the preceding discussion, any crossed homomorphism χ gives rise to a new compact group structure on G. We denote this compact group by G_χ. Observe that, since the Haar measure on G is invariant under α, so the Haar measure on G_χ is equal to the Haar measure on G. Hence we have $G_\chi = G$ as probability spaces.

The group G_χ can also be defined as the graph of χ in the semi-direct product $H =$

$\Gamma_\alpha \ltimes G$. Indeed, it is easy to check that the graph $\mathrm{Gr}(\chi) = \{(\chi(g), g) : g \in G\}$ of a continuous map $\chi : G \to \Gamma$, which is a closed subset of H, is a subgroup of H if and only if χ is a crossed homomorphism. Moreover, the map $G_\chi \to \mathrm{Gr}(\chi)$, $g \mapsto (\chi(g), g)$, $g \in G$, is an isomorphism of compact groups.

Since $G_\chi = G$ as topological spaces, α still defines an action of Γ on the compact space G_χ by homeomorphisms. However, α may not be an action by group homomorphisms anymore. Actually, for $\gamma \in \Gamma$, α_γ is a group homomorphism of G_χ if and only if $\chi(g)^{-1}\gamma^{-1}\chi(\alpha_\gamma(g))\gamma \in \mathrm{Ker}(\alpha)$ for all $g \in G$ which happens for example if χ satisfies $\chi \circ \alpha_\gamma = \gamma\chi(\cdot)\gamma^{-1}$.

We define a continuous right action of G_χ on the discrete space Γ by $\beta_g(\gamma) = \chi(\alpha_\gamma(g))^{-1}\gamma\chi(g)$ for all $\gamma \in \Gamma, g \in G$. It is an easy exercise to check that α and β satisfy the relations in Equation (4.1.1), hence, by Proposition 4.1.3 we get a new compact matched pair (Γ, G_χ) with possibly non-trivial actions α and β. To see that the pair (Γ, G_χ) is matched without using Proposition 4.1.3, it suffices to view Γ and G_χ as closed subgroups of $H = \Gamma_\alpha \ltimes G$ via the identification explained before and check that $\Gamma G_\chi = H$ and $\Gamma \cap G_\chi = \{e\}$. It is easy to check that the actions α and β obtained by this explicit matching are the ones we did define.

Let $\mathbb{G}_\chi$ denote the bicrossed product associated with the matched pair (Γ, G_χ).

Proposition 4.5.1. *If the action $\alpha : \Gamma \curvearrowright \mathrm{Irr}(G)$ has all orbits finite and the group Γ has the Haagrup property, then $\widehat{\mathbb{G}}_\chi$ has the Haagerup property for all crossed homomorphisms $\chi : G \to \Gamma$.*

Proof. Recall that if $\alpha : \Gamma \curvearrowright G$ is an action by compact group automorphisms, then the action $\alpha : \Gamma \curvearrowright \mathrm{L}^\infty(G)$ is compact if and only if the image of Γ in $\mathrm{Aut}(G)$ is precompact which in turn is equivalent to the associated action of Γ on $\mathrm{Irr}(G)$ to have all orbits finite. Now let $\chi : G \to \Gamma$ be a crossed homomorphism. Since $G_\chi = G$ as compact spaces and as probability spaces, the action $\alpha : \Gamma \curvearrowright \mathrm{L}^\infty(G)$ is compact if and only if the action $\Gamma \curvearrowright \mathrm{L}^\infty(G_\chi)$ is compact and the former is equivalent to the action $\Gamma \curvearrowright \mathrm{Irr}(G)$ to have all orbits finite. Hence, the proof follows from assertion 4 of Corollary 4.1.7. $\square$

Observe that a continuous group homomorphism $\chi : G \to \Gamma$ is a crossed homomorphism if and only if $\chi \circ \alpha_\gamma = \chi$ for all $\gamma \in \mathrm{Im}(\chi)$.

Now we give a systematic way to construct explicit non-trivial examples of the situation considered in the first part of this section. So, consider a non-trivial action α of a countable discrete group Γ on a compact group G by group homomorphisms and let $\Lambda < \Gamma$ be a finite subgroup. Define the action α^Λ of Γ on $G^\Lambda = \Lambda \times G$ by $\alpha^\Lambda_\gamma(r, g) = (r, \alpha_\gamma(g))$ and the α^Λ-invariant group homomorphism $\chi : G^\Lambda \to \Gamma$ by $\chi(r, g) = r$, $r \in \Lambda, g \in G, \gamma \in \Gamma$. Thus, we get a compact matched pair (Γ, G^Λ_χ) where $G^\Lambda_\chi = \Lambda \times G$ as a compact space and the group law is given by $(r, g) \cdot (s, h) = (r, g)\alpha_{\chi(r,g)}(s, h) = (rs, g\alpha_r(h))$, $r, s \in \Lambda$ and $g, h \in G$. Hence, $G^\Lambda_\chi = \Lambda\,{}_\alpha\!\ltimes G$ as a compact group and the action β of G^Λ_χ on Γ is given by $\beta_{(r,g)}(\gamma) = r^{-1}\gamma r$, $r \in \Lambda, g \in G, \gamma \in \Gamma$. Hence, β is non-trivial if and only if Λ is not in the center of Γ.

One has $\left(G^\Lambda_\chi\right)^\alpha = \Lambda \times G^\alpha$ and, since the action β of $\left(G^\Lambda_\chi\right)^\alpha$ on Γ is by inner automorphisms, the associated action on $\mathrm{Sp}(\Gamma)$ is trivial. Hence, if we denote by $\mathbb{G}_\Lambda$ the associated bicrossed product, then Proposition 4.1.10 implies that $\chi(\mathbb{G}_\Lambda) \simeq \Lambda \times G^\alpha \times \mathrm{Sp}(\Gamma)$. We claim that there is a canonical group isomorphism $\pi : \mathrm{Sp}(G^\Lambda_\chi) \to \mathrm{Sp}(\Lambda) \times \mathrm{Sp}_\Lambda(G)$, where $\mathrm{Sp}_\Lambda(G) = \{\omega \in \mathrm{Sp}(G) : \omega \circ \alpha_r = \omega \text{ for all } r \in \Lambda\}$ is a subgroup of $\mathrm{Sp}(G)$. Indeed, denoting by $\iota_G : G \to G^\Lambda_\chi$, $g \mapsto (1, g)$ and $\iota_\Lambda : \Lambda \to G^\Lambda_\chi$, $r \mapsto (r, 1)$ the two canonical injective (and continuous) group homomorphisms, we may define $\pi(\omega) = (\omega \circ \iota_\Lambda, \omega \circ \iota_G)$. Using the relations in the semi-direct product and the fact that ω is invariant on conjugacy classes, we see that $\omega \circ \iota_G \in \mathrm{Sp}_\Lambda(G)$. Since G^Λ_χ is generated by $\iota_\Lambda(\Lambda)$ and $\iota_G(G)$, so π is injective. The surjectivity of π follows from the universal property of semi-direct products.

Observe that $\Gamma^\beta = C_\Gamma(\Lambda)$ is the centralizer of Λ in Γ. Since, $\alpha_\gamma(\mathrm{Sp}_\Lambda(G)) = \mathrm{Sp}_\Lambda(G)$ for every $\gamma \in C_\Gamma(\Lambda)$, so α induces a right action of $C_\Gamma(\Lambda)$ on $\mathrm{Sp}_\Lambda(G)$ and we have, by Proposition 4.1.10, $\mathrm{Int}(\mathbb{G}_\Lambda) \simeq \mathrm{Sp}(\Lambda) \times (\mathrm{Sp}_\Lambda(G) \rtimes_\alpha C_\Gamma(\Lambda))$.

We will write $\mathbb{G} = \mathbb{G}_{\{1\}}$. We have thus proved the first assertion of the following theorem.

Theorem 4.5.2. *Let $\Lambda < \Gamma$ be any finite subgroup. Then the following holds.*

1. *$\chi(\mathbb{G}_\Lambda) \simeq \Lambda \times G^\alpha \times \mathrm{Sp}(\Gamma)$ and $\mathrm{Int}(\mathbb{G}_\Lambda) \simeq \mathrm{Sp}(\Lambda) \times (\mathrm{Sp}_\Lambda(G) \rtimes_\alpha C_\Gamma(\Lambda))$.*

2. *The following conditions are equivalent.*

   - *$(G, \mathbb{G})$ has the relative property (T).*

   - *$(G^\Lambda_\chi, \mathbb{G}_\Lambda)$ has the relative property (T).*

3. *If the action $\Gamma \curvearrowright \mathrm{Irr}(G)$ has all orbits finite and Γ has the Haagerup property, then $\widehat{\mathbb{G}}_\Lambda$ has the Haagerup property.*

4. *If the action $\Gamma \curvearrowright \mathrm{Irr}(G)$ has all orbits finite and Γ is weakly amenable, then $\widehat{\mathbb{G}}_\Lambda$ is weakly amenable and $\Lambda_{cb}(\widehat{\mathbb{G}}_\Lambda) \leq \Lambda_{cb}(\Gamma)$.*

Proof. (2). ($\Downarrow$) Suppose that the pair $(G^\Lambda_\chi, \mathbb{G}_\Lambda)$ does not have the relative property (T). Let (μ_n) be a sequence of Borel probability measures on $\Lambda \times G$ satisfying the conditions of assertion 2 of Theorem 4.2.2. Since $\{e\} \times G$ is open and closed in $\Lambda \times G$, we have $1_{\{e\}\times G} \in C(\Lambda \times G)$, and since $\mu_n \to \delta_{(e,e)}$ in the weak* topology we deduce that $\mu_n(\{e\} \times G) \to 1$. Hence, we may and will assume that $\mu_n(\{e\} \times G) \neq 0$ for all $n \in \mathbb{N}$. Define a sequence (ν_n) of Borel probability measures on G by $\nu_n(A) = \frac{\mu_n(\{e\}\times A)}{\mu_n(\{e\}\times G)}$, where $A \subset G$ is Borel. Then $\nu_n(\{e\}) = \mu_n(\{(e, e)\}) = 0$ for all $n \in \mathbb{N}$ and it is easy to check that, for all $F \in C(G)$, $1_{\{e\}} \otimes F \in C(\Lambda \times G)$ and $\int_G F d\nu_n = \frac{1}{\mu_n(\{e\}\times G)} \int_{\Lambda\times G} 1_{\{e\}} \otimes F d\mu_n$. It follows from this formula and the fact that $\mu_n \to \delta_{(e,e)}$ in the weak* topology that we also have $\nu_n \to \delta_e$ in the weak* topology. Finally, the previous formula also implies that, for all $F \in C(G)$,

$$\begin{aligned} |\alpha_\gamma(\nu_n)(F) - \nu_n(F)| &= \frac{1}{\mu_n(\{e\} \times G)} |\alpha^\Lambda_\gamma(\mu_n)(1_{\{e\}} \otimes F) - \mu_n(1_{\{e\}} \otimes F)| \\ &\leq \frac{\|1_{\{e\}} \otimes F\|}{\mu_n(\{e\} \times G)} \|\alpha^\Lambda_\gamma(\mu_n) - \mu_n\| \leq \frac{\|F\|}{\mu_n(\{e\} \times G)} \|\alpha^\Lambda_\gamma(\mu_n) - \mu_n\|. \end{aligned}$$

Hence, $\|\alpha_\gamma(\nu_n) - \nu_n\| \leq \frac{\|\alpha^\Lambda_\gamma(\mu_n)-\mu_n\|}{\mu_n(\{e\}\times G)} \to 0$ and thus $(G, \mathbb{G})$ does not have the relative property (T).

($\Uparrow$) Now suppose that the pair $(G, \mathbb{G})$ does not have the relative property (T). Let (μ_n) be a sequence of Borel probability measures on G satisfying the conditions of assertion 2 of Theorem 4.2.2. For each n define the probability measure ν_n on $G_\chi^\Lambda = \Lambda_\alpha \ltimes G$ by $\nu_n = \delta_e \otimes \mu_n$. We have $\nu_n(\{e, e\}) = \mu_n(\{e\}) = 0$ and $\int_{G_\chi^\Lambda} F d\nu_n = \int_G F(e, g) d\mu_n(g)$ for all $F \in C(G_\chi^\Lambda)$. Hence $\nu_n \to \delta_e$ in the weak* topology. Moreover, since for all $F \in C(G_\chi^\Lambda)$, we have

$$\begin{aligned}|\alpha_\gamma^\Lambda(\nu_n)(F) - \nu_n(F)| &= \left|\int_G F(e, \alpha_\gamma(g)) d\mu_n(g) - \int_G F(e, g) d\mu_n\right| = |\alpha_\gamma(\mu_n)(F_e) - \mu_n(F_e)| \\ &\leq \|F_e\| \, \|\alpha_\gamma(\mu_n) - \mu_n\| \leq \|F\| \, \|\alpha_\gamma(\mu_n) - \mu_n\|,\end{aligned}$$

where $F_e = F(e, \cdot) \in C(G)$, we have $\|\alpha_\gamma^\Lambda(\nu_n) - \nu_n\| \leq \|\alpha_\gamma(\mu_n) - \mu_n\| \to 0$.

$\square$

(4). It is easy to check that, if $\alpha : \Gamma \curvearrowright G$ is compact then $\alpha^\Lambda = \mathrm{id} \otimes \alpha : \Gamma \curvearrowright \Lambda \times G$ is compact, for all finite group Λ. Hence, the proof follows from Proposition 4.5.1.

(5). Observe that, for a general compact matched pair (Γ, G) with associated actions α and β, the continuity of β forces each stabilizer subgroup G_γ, for $\gamma \in \Gamma$, to be open, hence finite index by compactness of G. Consider the closed normal subgroup $G_0 = \cap_{\gamma \in \Gamma} G_\gamma = \mathrm{Ker}(\beta) < G$. Equation 4.1.1 implies that G_0 is globally invariant under α and the α-action of Γ on G_0 is by group automorphisms. Hence, we may consider the crossed product quantum group $\mathbb{G}_0$, with $C_m(\mathbb{G}_0) = \Gamma_{\alpha,f} \ltimes C(G_0)$, which is a quantum subgroup (in fact normal subgroup in the sense of Wang [91]) of the bicrossed product quantum group $\mathbb{G}$ with $C_m(\mathbb{G}) = \Gamma_{\alpha,f} \ltimes C(G)$. This is because G_0 is globally invariant under the action α of Γ and hence, by the universal property, we have a surjective unital $*$-homomorphism $\rho : \Gamma_{\alpha,f} \ltimes C(G) \to \Gamma_{\alpha,f} \ltimes C(G_0)$ which is easily seen to intertwines the comultiplications. Since ρ acts as identity on $C_m(\Gamma)$, it follows using Theorem 4.1.4(2) that $C_m(\mathbb{G}/\mathbb{G}_0) = \alpha(C_m(G/G_0))$ (see Definition 1.2.2). Hence, if we assume that G_0 is a finite index subgroup of G, then $\mathbb{G}_0$ is a finite index subgroup of $\mathbb{G}$. If we further assume that Γ is weakly amenable and the action α of Γ on G is compact then the action α of Γ

restricted to G_0 is also compact and Theorem 4.4.7 (with the fact that $\mathbb{G}_0$ is Kac) implies that $\widehat{\mathbb{G}}_0$ is weakly amenable with $\Lambda_{cb}(\widehat{\mathbb{G}}_0) \leq \Lambda_{cb}(\Gamma)$. Using part (2) of Theorem 1.3.3, we conclude that $\widehat{\mathbb{G}}$ is weakly amenable and $\Lambda_{cb}(\widehat{\mathbb{G}}) \leq \Lambda_{cb}(\Gamma)$. In our case, with $G = G_\chi^\Lambda$, the finiteness of Λ forces G_0 to be always of finite index in G. Since, by assumption, the action of Γ on $\mathrm{Irr}(G)$ has all orbits finite, we conclude, as in the proof of Proposition 4.5.1, that the action α is compact. $\square$

Before proceeding further, we prove a result which can be useful to get examples of a pair having the relative Haagerup property.

Proposition 4.5.3. *Let $\alpha : \Lambda \curvearrowright G$ be an action of a finite group Λ on a compact group G by group automorphisms and define the compact group $H = \Lambda \,{}_\alpha\!\ltimes G$. The following holds.*

(a) Let μ be a Borel probability measure on G and define the Borel probability measure ν on H by $\nu = \delta_e \otimes \mu$. If $\widehat{\mu} \in C_r^*(G)$ then $\widehat{\nu} \in C_r^*(H)$.

(b) Let μ be a Borel probability on H such that $\mu(\{e\} \times G) \neq 0$ and define the Borel probability ν on G by $\nu(A) = \frac{\mu(\{e\}\times A)}{\mu(\{e\}\times G)}$ for all $A \in \mathcal{B}(G)$. If $\widehat{\mu} \in C_r^*(H)$ then $\widehat{\nu} \in C_r^*(G)$.

Proof. Let λ^G and λ^H denote the left regular representations of G and H respectively. For $F \in C(G)$ (resp. $F \in C(H)$), write $\lambda^G(F)$ (resp. $\lambda^H(F)$) the convolution operator by F on $\mathrm{L}^2(G, \mu_G)$ (resp. $\mathrm{L}^2(H, \mu_H)$), where μ_G (resp. μ_H), is the Haar probability on G (resp. H). Observe that $\mu_H = m \otimes \mu_G$, where m is the normalized counting measure on Λ.

(a). Recall that, for all $F \in C(H)$, $\int_H F d\nu = \int_G F(e,g) d\mu(g)$. Moreover, using the definition of the group law in H, we find that $\lambda^H_{(e,g)} = 1 \otimes \lambda^G_g \in \mathcal{B}(l^2(\Lambda) \otimes \mathrm{L}^2(G))$, for all $g \in G$. It follows that

$$\widehat{\nu} = \int_G \lambda^H_{(e,g)} d\mu(g) = \int_G (1 \otimes \lambda^G_g) d\mu(g) = 1 \otimes \widehat{\mu} \in M(C_r^*(H)) \subset \mathcal{B}(l^2(\Lambda) \otimes \mathrm{L}^2(G)).$$

Note that for all $F \in C(G)$, $1_{\{e\}} \otimes F \in C(H)$, since Λ is finite. We claim that $\lambda^H(1_{\{e\}} \otimes F) = \frac{1}{|\Lambda|}(1 \otimes \lambda^G(F))$. Indeed,

$$\begin{aligned}
\lambda^H(1_{\{e\}} \otimes F) &= \int_H \delta_{r,e} F(g) \lambda^H_{(e,g)} d\mu_H(r,g) = \int_H \delta_{r,e} F(g)(1 \otimes \lambda^G_g) d\mu_H(r,g) \\
&= \int_G \left(\frac{1}{|\Lambda|} \sum_{r \in \Lambda} \delta_{r,e} F(g)(1 \otimes \lambda^G_g) \right) d\mu_G(g) = \frac{1}{|\Lambda|}(1 \otimes \lambda^G(F)).
\end{aligned}$$

Suppose that $\widehat{\mu} \in C^*_r(G)$ and let $F_n \in C(G)$ be a sequence such that $\|\widehat{\mu} - \lambda^G(F_n)\| \to 0$. Hence, $1 \otimes \lambda^G(F_n) \to \widehat{\nu}$. Since $1 \otimes \lambda^G(F_n) = |\Lambda| \lambda^H(1_{\{e\}} \otimes F_n) \in C^*_r(H)$ $\forall n \in \mathbb{N}$, we have $\widehat{\nu} \in C^*_r(H)$.

(b). Recall that, for all $F \in C(G)$, $1_{\{e\}} \otimes F \in C(\Lambda \times G) = C(H)$ and $\int_G F d\nu = \frac{1}{\mu(\{e\} \times G)} \int_{\Lambda \times G} 1_{\{e\}} \otimes F d\mu$. Using the definition of the group law in H, an easy computation shows that for all $r \in \Lambda$, $\xi \in \mathrm{L}^2(G)$, $\lambda^H_{(r,g)}(\delta_e \otimes \xi) = \delta_r \otimes \lambda^G_g(\xi \circ \alpha_{r^{-1}})$. It follows that,

$$\begin{aligned}
\langle \widehat{\nu} \xi, \eta \rangle &= \int_G \langle \lambda^G_g \xi, \eta \rangle d\nu(g) = \frac{1}{\mu(\{e\} \times G)} \int_{\Lambda \times G} \delta_{e,r} \langle \lambda^G_g \xi, \eta \rangle d\mu(r,g) \\
&= \frac{1}{\mu(\{e\} \times G)} \int_{\Lambda \times G} \langle \lambda^H_{(r,g)} \delta_e \otimes \xi, \delta_e \otimes \eta \rangle d\mu(r,g) \quad \text{for all } \xi, \eta \in \mathrm{L}^2(G).
\end{aligned}$$

Hence, $\widehat{\nu} = \frac{1}{\mu(\{e\} \times G)} V^* \widehat{\mu} V$, where $V : \mathrm{L}^2(G) \to l^2(\Lambda) \otimes \mathrm{L}^2(G) = \mathrm{L}^2(H)$ is the isometry defined by $V\xi = \delta_e \otimes \xi$, $\xi \in \mathrm{L}^2(G)$. To end the proof it suffices to show that $V^* C^*_r(H) V \subset C^*_r(G)$.

Let $F \in C(H)$ and define $F_e \in C(G)$ by $F_e(g) = F(e,g)$, $g \in G$. We will actually show that $V^* \lambda^H(F) V = \frac{1}{|\Lambda|} \lambda^G(F_e)$ and this will finish the argument. For $\xi, \eta \in \mathrm{L}^2(G)$, we have

$$\begin{aligned}
\langle V^* \lambda^H(F) V \xi, \eta \rangle &= \langle \lambda^H(F) \delta_e \otimes \xi, \delta_e \otimes \eta \rangle = \int_H F(r,g) \langle \lambda^H_{(r,g)} \delta_e \otimes \xi, \delta_e \otimes \eta \rangle d\mu_H(r,g) \\
&= \int_H \delta_{r,e} F(e,g) \langle \lambda^G_g \xi, \eta \rangle d\mu_H(r,g) = \int_G \frac{1}{|\Lambda|} \sum_{r \in \Lambda} \delta_{r,e} F(e,g) \langle \lambda^G_g \xi, \eta \rangle d\mu_G(g) \\
&= \frac{1}{|\Lambda|} \int_G F(e,g) \langle \lambda^G_g \xi, \eta \rangle d\mu_G(g) = \frac{1}{|\Lambda|} \langle \lambda^G(F_e) \xi, \eta \rangle.
\end{aligned}$$

and hence, we are done. $\square$

Example 4.5.4. (**Relative Property (T)**) Take $n \in \mathbb{N}$, $n \geq 2$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, $G = \mathbb{T}^n$ and α the canonical action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{T}^n = \mathrm{Sp}(\mathbb{Z}^n)$ coming from the linear action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{Z}^n$. Taking a finite subgroup $\Lambda < \mathrm{SL}_n(\mathbb{Z})$, we manufacture a compact bicrossed product $\mathbb{G}_\Lambda$ with non-trivial actions α and β (described in the beginning of this section) whenever Λ is a non-central subgroup. Note that $(\mathbb{T}^n)^{\mathrm{SL}_n(\mathbb{Z})} = \{e\}$ hence $\chi(\mathbb{G}_\Lambda) \simeq \Lambda \times \mathrm{Sp}(\mathrm{SL}_n(\mathbb{Z}))$.

Suppose $n \geq 3$. In this case, $D(\mathrm{SL}_n(\mathbb{Z})) = \mathrm{SL}_n(\mathbb{Z})$, where $D(F)$ denotes the derived subgroup of a group F. Since every element of $\mathrm{Sp}(\mathrm{SL}_n(\mathbb{Z}))$ is trivial on commutators, we have $\mathrm{Sp}(\mathrm{SL}_n(\mathbb{Z})) = \{1\}$, for all $n \geq 3$. It follows that $\chi(\mathbb{G}_\Lambda) \simeq \Lambda$. Hence, for all $n, m \geq 3$ and all finite subgroups $\Lambda < \mathrm{SL}_n(\mathbb{Z})$, $\Lambda' < \mathrm{SL}_m(\mathbb{Z})$, we have $\mathbb{G}_\Lambda \simeq \mathbb{G}_{\Lambda'}$ implies $\Lambda \simeq \Lambda'$.

However, for $n = 2$, the group $\mathrm{Sp}(\mathrm{SL}_2(\mathbb{Z}))$ is non-trivial. Actually, we have

$$\mathrm{Sp}(\mathrm{SL}_2(\mathbb{Z})) \simeq \{(k, l) \in \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \,:\, k \equiv l \bmod 2\}, \tag{4.5.1}$$

which is a finite group of order 12. Indeed, by the well known isomorphism $\mathrm{SL}_2(\mathbb{Z}) \simeq \mathbb{Z}/4\mathbb{Z} \underset{\mathbb{Z}/2\mathbb{Z}}{*} \mathbb{Z}/6\mathbb{Z}$, it suffices to compute the group of 1-dimensional unitary representations of an amalgamated free product $\Gamma_1 \underset{\Sigma}{*} \Gamma_2$. It is easy to check that the map $\psi : \mathrm{Sp}(\Gamma_1 \underset{\Sigma}{*} \Gamma_2) \to T$ defined by $\psi(\omega) = (\omega|_{\Gamma_1}, \omega|_{\Gamma_2})$, where T is the subgroup of $\mathrm{Sp}(\Gamma_1) \times \mathrm{Sp}(\Gamma_2)$ defined by $T = \{(\omega, \mu) \in \mathrm{Sp}(\Gamma_1) \times \mathrm{Sp}(\Gamma_2) \,:\, \omega|_\Sigma = \mu|_\Sigma\}$, is an isomorphism of compact groups. Hence, using the canonical identification $\mathrm{Sp}(\mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/m\mathbb{Z}$, we obtain the isomorphism in Equation (4.5.1).

Since the pair $(\mathbb{Z}^2, \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2)$ has the relative property (T), we deduce from Theorem 4.5.2 that, for any finite subgroup $\Lambda < \mathrm{SL}_2(\mathbb{Z})$, the pair $(G_\chi^\Lambda, \mathbb{G}_\Lambda)$ has the relative property (T). Identifying $\mathrm{SL}_2(\mathbb{Z})$ with $\mathbb{Z}/4\mathbb{Z} \underset{\mathbb{Z}/2\mathbb{Z}}{*} \mathbb{Z}/6\mathbb{Z}$, one finds that every finite subgroup is conjugated to $\{1\}$ or $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$. The only non-central subgroups are conjugated to $\Lambda_1 = \mathbb{Z}/4\mathbb{Z}$ or $\Lambda_2 = \mathbb{Z}/6\mathbb{Z}$. Hence, we get two non-trivial compact bicrossed products $\mathbb{G}_{\Lambda_i}$, $i = 1, 2$, such that $(G_\chi^{\Lambda_i}, \mathbb{G}_{\Lambda_i})$ has the relative property (T) and

$\widehat{\mathbb{G}_{\Lambda_i}}$ does not have property (T) since $\mathrm{SL}_2(\mathbb{Z})$ has the Haagerup property. Moreover, $\mathbb{G}_{\Lambda_1}$ and $\mathbb{G}_{\Lambda_2}$ are not isomorphic since $|\Lambda_1| \neq |\Lambda_2|$.

Remark 4.5.5. ***(Haagerup Property and Weak Amenability)*** *We depict here a procedure to construct compact bicrossed products with the Haagerup property and Weak Amenability. Suppose that* Γ *is a countable subgroup of a compact group* G *and consider the action* $\alpha : \Gamma \curvearrowright G$ *by inner automorphisms i.e.* $\alpha_\gamma(g) = \gamma g \gamma^{-1}$ *for all* $\gamma \in \Gamma$, $g \in G$. *Let* $\Lambda < \Gamma$ *be any finite subgroup and consider the matched pair* (Γ, G_χ^Λ) *introduced earlier in this section. Let* $\mathbb{G}_\Lambda$ *be the bicrossed product. Observe that, since the action* α *is inner, the associated action on* $\mathrm{Irr}(G)$ *is trivial. Indeed, for any unitary representation* π *of* G, *the unitary* $\pi(\gamma)$ *is an intertwiner between* $\alpha_\gamma(\pi)$ *and* π *for all* $\gamma \in \Gamma$. *Hence, if* Γ *has the Haagerup property, then for any finite subgroup* $\Lambda < \Gamma$ *the bicrossed product* $\widehat{\mathbb{G}}_\Lambda$ *has the Haagerup property. Similarly, if* Γ *is weakly amenable, then for any finite subgroup* $\Lambda < \Gamma$ *the bicrossed product* $\widehat{\mathbb{G}}_\Lambda$ *is weakly amenable and* $\Lambda_{cb}(\widehat{\mathbb{G}}_\Lambda) \leq \Lambda_{cb}(\Gamma)$.

4.5.1.2 From matched pair with trivial α

In this section, we consider the dual situation, i.e., starting with a matched pair with α being trivial and modifying it to some non-trivial action for a different matched pair with α non-trivial.

Let β be any continuous right action of the compact group G on the discrete group Γ by group automorphisms. Taking α to be the trivial action of Γ on G, the relations in Equation (4.1.1) are satisfied and we get a matched pair.

Remark 4.5.6. *Note that if the group* Γ *is finitely generated then the right semi-direct product group* $H = \Gamma \rtimes_\beta G$ *is virtually a direct product. In other words, there is a finite index subgroup of* H *which is a direct product of a subgroup of* G *(which acts trivially on* Γ*) and* Γ.

Indeed, since Γ *is discrete and* β *is continuous, the stabilizer subgroup* $G_\gamma := \{g \in G : \gamma \cdot g = \gamma\}$ *is open in* G *for all* $\gamma \in \Gamma$. *Since* G *is compact,* G_γ *has finite index in* G.

Now consider the subgroup $G_\beta = \cap_{\gamma\in\Gamma} G_\gamma$, which acts trivially on Γ. In case Γ is finitely generated, it follows that G_β is also finite index in G and thus the direct product $\Gamma \times G_\beta$ is a finite index subgroup of H.

However, if the discrete group is not finitely generated then this need not be the case. For instance, let a compact group K act on a finite group F non-trivially. Let $K_n = K$ for $n \in \mathbb{N}$. One can then induce, in the natural way, an action of the compact group $G = \prod_{n\in\mathbb{N}} K_n$ on the discrete group $\Gamma = \oplus_{n\in\mathbb{N}} F_n$, where $F_n = F$ for all n. In this case, it is easy to see that the subgroup G_β is not of finite index.

Getting back to the process of modifying α, we call a map $\chi : \Gamma \to G$ a *crossed homomorphism* if $\chi(e) = e$ and $\chi(rs) = \chi(\beta_{\chi(s)^{-1}}(r))\chi(s)$ for all $r, s \in \Gamma$. Given a crossed homomorphism, we define a new discrete group Γ_χ which is equal to Γ as a set and the group multiplication is given by $r * s = \beta_{\chi(s)}(r)s$ for all $r, s \in \Gamma$. As before, Γ_χ is canonically isomorphic to the graph $\mathrm{Gr}(\chi) = \{(\gamma, \chi(\gamma)) : \gamma \in \Gamma\}$ of χ, which is a subgroup of the right semi-direct product $H = \Gamma \rtimes_\beta G$ (since χ is a crossed homomorphism).

Observe that β still defines a continuous right action of G on the countable set Γ_χ and for $g \in G$, β_g is a group homomorphism of Γ_χ if and only if $g^{-1}\chi(\gamma)^{-1}g\chi(\beta_g(\gamma)) \in \mathrm{Ker}(\beta_g)$ for all $\gamma \in \Gamma$, which happens for example if $\chi \circ \beta_g = g^{-1}\chi(\cdot)g$. Moreover, the formula $\alpha_\gamma(g) = \chi(\gamma)g\chi(\beta_g(\gamma))^{-1}$, for all $\gamma \in \Gamma, g \in G$, defines an action of Γ_χ on the compact space G by homeomorphisms and in addition α and β satisfy the relations in Equation (4.1.1). Consequently, we get a new matched pair (Γ_χ, G) with possibly non-trivial actions α and β. As before, one can describe this new matched pair explicitly by viewing Γ_χ and G as closed subgroups of the right semi-direct product $H = \Gamma \rtimes_\beta G$.

Observe that a group homomorphism $\chi : \Gamma \to G$ is a crossed homomorphism if and only if $\chi = \chi \circ \beta_g$ for all $g \in \mathrm{Im}(\chi)$.

Remark 4.5.7. *Suppose that the crossed homomorphism satisfies $\chi \circ \beta_g = \chi$ for all $g \in$ $\mathrm{Im}(\chi)$ and let ${}_\chi\mathbb{G}$ be the associated bicrossed product. Then the following are equivalent.*

1. *Γ has the Haagerup property.*

2. $\widehat{{}_\chi\mathbb{G}}$ *has the Haagerup property.*

Indeed, by Corollary 4.1.7, it suffices to show that the action α *of* Γ_χ *on* G *is compact when viewed as an action of* Γ_χ *on* $\mathrm{L}^\infty(G)$*. Since* $\alpha_\gamma(g) = \chi(\gamma)g\chi(\gamma)^{-1}$ *for* $g \in G$ *and* $\gamma \in \Gamma_\chi$*,* α *is an action by inner automorphisms, thus it is always compact since it is trivial on* $\mathrm{Irr}(G)$*. Indeed, for any unitary representation* u *of* G*, the unitary* $u(\chi(\gamma))$ *is an intertwiner between* $\alpha_\gamma(u)$ *and* u *for* $\gamma \in \Gamma_\chi$*.*

A systematic way to construct explicit examples using the deformation above is to consider any countable discrete group Γ_0 which has a finite non-abelian quotient G and take $\Gamma = \Gamma_0 \times G$ with the right action of G on Γ given by $\beta_g(\gamma, h) = (\gamma, g^{-1}hg)$, $g, h \in G$ and $\gamma \in \Gamma_0$. Since G is non-abelian, β is non-trivial. Let $q : \Gamma_0 \to G$ be the quotient map and define the morphism $\chi : \Gamma \to G$ by $\chi(\gamma, h) = q(\gamma)$, $\gamma \in \Gamma_0$, $h \in G$. Then, we obviously have $\chi \circ \beta_g = \chi$ for all $g \in G$. Therefore, χ is a crossed homomorphism and the action α of Γ_χ on G is given by $\alpha_{(\gamma,h)}(g) = q(\gamma)gq(\gamma^{-1})$, $\gamma \in \Gamma_0$, $h, g \in G$, which is also non-trivial since G is non-abelian. Thus (Γ_χ, G) is a compact matched pair. Let ${}_\chi\mathbb{G}$ denote the bicrossed product.

Proposition 4.5.8. *We have* $\chi({}_\chi\mathbb{G}) \simeq Z(G) \times \mathrm{Sp}(\Gamma_0) \times \mathrm{Sp}(G)$ *and* $\mathrm{Int}({}_\chi\mathbb{G}) = \mathrm{Sp}(G) \times \Gamma_0 \times Z(G)$*.*

Proof. Note that $\Gamma_\chi = \Gamma_0 \times G$ as a set and the group law is given by $(r, g)(s, h) = (rs, q(s)^{-1}gq(s)h)$ for all $r, s \in \Gamma_0$ and $g, h \in G$. Since the action β of G on Γ_χ is given by $\beta_g(s, h) = (s, g^{-1}hg)$, $s \in \Gamma_0, g, h \in G$, we have $\Gamma_\chi^\beta = \Gamma_0 \times Z(G)$ and the action of $Z(G)$ on Γ_χ is trivial. Since the action α of Γ_χ^β on G is given by $\alpha_{(r,g)}(h) = q(r)hq(r)^{-1}$, $r \in \Gamma_0, g, h \in G$, we find $G^\alpha = Z(G)$. Again, since the action α is by inner automorphisms, the associated action on $\mathrm{Sp}(G)$ is trivial. It follows from Proposition 4.1.10 that $\chi({}_\chi\mathbb{G}) \simeq Z(G) \times \mathrm{Sp}(\Gamma_\chi)$ and $\mathrm{Int}({}_\chi\mathbb{G}) = \mathrm{Sp}(G) \times \Gamma_0 \times Z(G)$. Let $\iota_{\Gamma_0} : \Gamma_0 \to \Gamma_\chi$, $r \mapsto (r, 1)$ and $\iota_G : G \to \Gamma_\chi$, $g \mapsto (1, g)$. Observe that ι_{Γ_0} and ι_G are group homomorphisms. To finish the proof, we claim that the map $\psi : \mathrm{Sp}(\Gamma_\chi) \to \mathrm{Sp}(\Gamma_0) \times \mathrm{Sp}(G)$, defined by $\omega \mapsto (\omega \circ \iota_{\Gamma_0}, \omega \circ \iota_G)$, $\omega \in \mathrm{Sp}(\Gamma_\chi)$, is a group isomorphism. Indeed, it is obviously a group homomorphism. Since Γ_χ is generated by $\iota_{\Gamma_0}(\Gamma_0)$ and $\iota_G(G)$, so ψ is injective. Let $\omega_1 \in \mathrm{Sp}(\Gamma_0)$ and $\omega_2 \in \mathrm{Sp}(G)$. Define the continuous map $\omega : \Gamma_\chi \to S^1$ by $\omega(r, g) = \omega_1(r)\omega_2(g)$, $r \in \Gamma_0, g \in G$. Then, for all $r, s \in \Gamma_0$, $g, h \in G$,

$$\begin{aligned} \omega((r,g)\cdot(s,h)) &= \omega(rs, q(s)^{-1}gq(s)h) = \omega_1(r)\omega_1(s)\omega_2(q(s)^{-1})\omega_2(g)\omega_2(q(s))\omega_2(h) \\ &= \omega_1(r)\omega_2(g)\omega_1(s)\omega_2(h) = \omega(r,g)\omega(s,h). \end{aligned}$$

Hence, $\omega \in \mathrm{Sp}(\Gamma_\chi)$ and $\psi(\omega) = (\omega_1, \omega_2)$, so ψ is surjective. □

Example 4.5.9. (**Haagerup Property**) Observe that any finite non-abelian group G provides an example with $\Gamma_0 = \mathbb{F}_n$, where n is bigger than the number of generators of G, so that G is a quotient of Γ_0 in the obvious way. All bicrossed products obtained in this way are not co-amenable but their duals do have the Haagerup property by Remark 4.5.7.

To get explicit examples we take, for $n \geq 4$, $G = A_n$ the alternating group which is simple, has only one irreducible representation of dimension 1 (the trivial representation) so that $Z(G) = \{1\}$ and $\mathrm{Sp}(G) = \{1\}$. Moreover, viewing A_n generated by the $n-2$ 3-cycles, we have a surjection $\Gamma_0 = \mathbb{F}_{n-2} \to A_n = G$. Associated to this data, we get a non-trivial compact bicrossed product $\mathbb{G}_n$ non co-amenable and whose dual has the Haagerup property and such that $\chi(\mathbb{G}_n) \simeq \mathrm{Sp}(\mathbb{F}_{n-2}) = \mathbb{T}^{n-2}$. In particular $\mathbb{G}_n$ and $\mathbb{G}_m$ are not isomorphic for $n \neq m$. It shows the existence of an infinite family of pairwise non-isomorphic non-trivial compact bicrossed product whose dual are non amenable with the Haagerup property.

We now consider more explicit examples on property (T).

Example 4.5.10. (**Property** (T)) Let $n \geq 3$ be a natural number and $p \geq 3$ be a prime number. Let $\mathbb{F}_p$ denote the finite field of order p. Define $\Gamma_0 = \mathrm{SL}_n(\mathbb{Z})$, $G = \mathrm{SL}_n(\mathbb{F}_p)$ and let $q : \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{F}_p)$ be the canonical quotient map. We get a matched pair (Γ_χ, G) with both actions α and β non-trivial and we denote the bicrossed product by $\mathbb{G}_{n,p}$. Since for $n, p \geq 3$, we have $D(\mathrm{SL}_n(\mathbb{Z})) = \mathrm{SL}_n(\mathbb{Z})$ and $D(\mathrm{SL}_n(\mathbb{F}_p)) = \mathrm{SL}_n(\mathbb{F}_p)$, we deduce as in Example 4.5.4 that $\mathrm{Sp}(\mathrm{SL}_n(\mathbb{F}_p)) = \{1\} = \mathrm{Sp}(\mathrm{SL}_n(\mathbb{Z}))$. It follows from Proposition 4.5.8 that

$$\mathrm{Int}(\mathbb{G}_{n,p}) \simeq \mathrm{SL}_n(\mathbb{Z}) \times Z(\mathrm{SL}_n(\mathbb{F}_p)) \simeq \mathrm{SL}_n(\mathbb{Z}) \times \mathbb{Z}/d\mathbb{Z} \text{ and } \chi(\mathbb{G}_{n,p}) = Z(\mathrm{SL}_n(\mathbb{F}_p)) \simeq \mathbb{Z}/d\mathbb{Z},$$

where $d = \gcd(n, p-1)$. In particular, the quantum groups $\mathbb{G}_p = \mathbb{G}_{p,p}$ for p prime and $p \geq 3$, are pairwise non-isomorphic. They are non-commutative and non-cocommutative by Remark 4.1.5. Moreover, assertion 2 of Theorem 4.2.3 implies that $\widehat{\mathbb{G}}_p$ have property (T). We record this in the form of a theorem.

Theorem 4.5.11. *There exists an infinite family of pairwise non isomorphic non-trivial compact bicrossed products whose duals have property* (T).

These are the first explicit examples of non-trivial discrete quantum groups with property (T).

One can also consider a similar but easier family of examples with β being trivial. We still take a natural number $n \geq 3$ and a prime number $p \geq 3$. But we consider $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ and $G = \mathrm{SL}_n(\mathbb{F}_p)$ with the action α being given by $\alpha_\gamma(g) = [\gamma]g[\gamma]^{-1}$, $\gamma \in \Gamma, g \in G$, and β being the trivial action. Let $\mathbb{H}_{n,p}$ denote the bicrossed product associated to the matched pair (Γ, G). One can check, as before, that $\mathrm{Int}(\mathbb{H}_{n,p}) \simeq \mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{Sp}(C_m(\mathbb{H}_{n,p})) \simeq \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(n, p-1)$. Hence, the quantum groups $\mathbb{H}_p = \mathbb{H}_{p,p}$ for p prime and $p \geq 3$, are pairwise non-isomorphic. They arise from matched pairs for which the β action is trivial but still they are non-commutative and non-cocommutative since Γ and G are both non-abelian. Also, their duals have property (T).

4.5.2 Examples of crossed products

In this section, we provide non-trivial examples of crossed products. Our examples are of the type considered in [88]. Let G be a compact quantum group and define, for all $g \in \chi(G)$, the map $\alpha_g = (g^{-1} \otimes \mathrm{id} \otimes g) \circ \Delta^{(2)}$. It defines a continuous group homomorphism

$\chi(G) \ni g \mapsto \alpha_g \in \mathrm{Aut}(G)$. Since $\chi(G)$ is compact, it follows that the action $\Gamma \curvearrowright G$ is always compact, for any countable subgroup $\Gamma < \chi(G)$. Actually, the action of $\chi(G)$ on $\mathrm{Irr}(G)$ is trivial since, for $g \in \chi(G)$ and $x \in \mathrm{Irr}(G)$ a straightforward computation gives $(\mathrm{id} \otimes \alpha_g)(u^x) = (V_g^* \otimes 1)u^x(V_g \otimes 1)$, where $V_g = (\mathrm{id} \otimes g)(u^x)$. Let G_Γ denote the crossed product. For a subgroup $\Sigma < H$, we denote by $C_H(\Sigma)$ the centralizer of Σ in H. Applying our results on crossed products to G_Γ we get the following Corollary.

Corollary 4.5.12. *The following holds.*

1. $\mathrm{Int}(G_\Gamma) \simeq \mathrm{Int}(G) \times \Gamma$ *and* $\chi(G_\Gamma) \simeq C_{\chi(G)}(\Gamma) \times \mathrm{Sp}(\Gamma)$.

2. $\max(\Lambda_{\mathrm{cb}}(C(G)), \Lambda_{\mathrm{cb}}(\Gamma)) \leq \Lambda_{\mathrm{cb}}(C(G_\Gamma)) \leq \Lambda_{\mathrm{cb}}(\widehat{G})\Lambda_{\mathrm{cb}}(\Gamma)$.

3. $\widehat{G}$ *and* Γ *have (RD) if and only if* $\widehat{G}_\Gamma$ *has (RD).*

4. $\widehat{G_\Gamma}$ *has the Haagerup property if and only if* $\widehat{G}$ *and* Γ *have the Haagerup property.*

5. $\widehat{G}_\Gamma$ *has property (T) if and only if* $\widehat{G}$ *and* Γ *have property (T).*

Proof. All the statements directly follow from the results of section 6 and the discussion preceding the statement of the Corollary except assertion 1 for which there is something to check: the action of $\chi(G)$ on $\mathrm{Int}(G)$ associated to the action α is trivial indeed, for all unitary $u \in C_m(G)$ for which $\Delta(u) = u \otimes u$, one has $\alpha_g(u) = g(u)ug(u^*) = u$. Moreover, the action of $\chi(G)$ on itself associated to the action α is, by definition, the action by conjugation. Hence assertion 1 directly follows from Proposition 4.4.5. □

Example 4.5.13. We consider examples with $G = U_N^+$, the free unitary quantum group or $G = O_N^+$, the free orthogonal quantum group. It is well known that $\chi(U_N^+) = U(N)$ and $\chi(O_N^+) = O(N)$ and that $\mathrm{Int}(U_N^+) = \mathrm{Int}(O_N)^+ = \{1\}$. It is also known that the Cowling-Haagerup constant for O_N^+ and U_N^+ are both 1 [38], and $\widehat{O_N^+}$ and $\widehat{U_N^+}$ have (RD) [86] and the Haagerup property [17]. Hence, for any $N \geq 2$ and any subgroups $\Sigma < O(N)$ and $\Gamma < U(N)$ the following holds.

- $\mathrm{Int}((O_N^+)_\Sigma) \simeq \Sigma$ and $\mathrm{Int}((U_N^+)_\Gamma) \simeq \Gamma$.

- $\chi((O_N^+)_\Sigma) \simeq C_{O(N)}(\Sigma) \times \mathrm{Sp}(\Sigma)$ and $\chi((U_N^+)_\Gamma) \simeq C_{U(N)}(\Gamma) \times \mathrm{Sp}(\Gamma)$.

- $\Lambda_{\mathrm{cb}}(\widehat{(O_N^+)_\Sigma}) = \Lambda_{\mathrm{cb}}(\Sigma)$ and $\Lambda_{\mathrm{cb}}(\widehat{(U_N^+)_\Gamma}) = \Lambda_{\mathrm{cb}}(\Gamma)$.

- $\widehat{(O_N^+)_\Sigma}$ (resp. $(\widehat{(U_N^+)_\Gamma})$ has (RD) if and only if Σ (resp. Γ) has (RD).

- $\widehat{(O_N^+)_\Sigma}$ (resp. $(\widehat{(U_N^+)_\Gamma})$ has the Haagerup property if and only if Σ (resp. Γ) has the Haagerup property.

- $\widehat{(O_N^+)_\Sigma}$ and $(\widehat{(U_N^+)_\Gamma}$ do not have Property (T).

Example 4.5.14. (**Relative Haagerup Property**) Since the action of $\chi(G)$ on $C_m(G)$ is given by $(\mathrm{id} \otimes \alpha_g)(u^x) = (V_g^* \otimes 1)u^x(V_g \otimes 1)$, where $V_g = (\mathrm{id} \otimes g)(u^x)$, we have,

$$\alpha_g(\omega)(u_{ij}^x) = \sum_{r,s} g(u_{ir}^x)\omega(u_{rs}^x)g((u_{js}^x)^*), \text{ for all } \omega \in C_m(G)^*. \tag{4.5.2}$$

Define the sequence of dilated Chebyshev polynomials of second kind by the initial conditions $P_0(X) = 1$, $P_1(X) = X$ and the recursion relation $XP_k(X) = P_{k+1}(X) + P_{k-1}(X)$, $k \geq 1$. It is proved in [17] (see also [36]) that the net of states $\omega_t \in C_m(O_N^+)^*$ defined by $\omega_t(u_{ij}^k) = \frac{P_k(t)}{P_k(N)}\delta_{i,j}$, for $k \in \mathrm{Irr}(O_N^+) = \mathbb{N}$ and $t \in (0,1)$ realize the co-Haagerup property for O_N^+, i.e., $\widehat{\omega}_t \in c_0(\widehat{O_N^+})$ for t close to 1 and $\omega_t \to \varepsilon_{O_N^+}$ in the weak* topology when $t \to 1$. Now let $g \in \chi(O_N^+)$. By Equation (4.5.2), we have $\alpha_g(\omega_t)(u_{ij}^k) = \frac{P_k(t)}{P_k(N)} \sum_r g(u_{ir}^k)g((u_{jr}^k)^*) = \frac{P_k(t)}{P_k(N)}\delta_{i,j} = \omega_t(u_{ij}^k)$. Hence, $\alpha_g(\omega_t) = \omega_t$ for all $g \in \chi(G)$ and all $t \in (0,1)$. It follows that for any $N \geq 2$ and any subgroup $\Gamma < O(N)$, the pair $(O_N^+, (O_N^+)_\Gamma)$ has the relative co-Haagerup property however, the dual of $(O_N^+)_\Gamma)$ does not have the Haagerup property whenever Γ does not have the Haagerup property.

Chapter 5

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