ON THE ANALYTIC CONTINUATION OF MULTIPLE DIRICHLET SERIES AND THEIR SINGULARITIES

By

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Biswajyoti Saha

List of Publications arising from the thesis

Journal

1. "An elementary approach to the meromorphic continuation of some classical Dirichlet series", Biswajyoti Saha (to appear in *Proc. Indian Acad. Sci. Math. Sci.*).

2. "Analytic properties of multiple zeta functions and certain weighted variants, an elementary approach", Jay Mehta, Biswajyoti Saha and G.K. Viswanadham, J. Number Theory **168** (2016), 487–508.

Others

1. "Multiple Lerch zeta functions and an idea of Ramanujan", Sanoli Gun and Biswajyoti Saha (*submitted*, arXiv:1510.05835).

2. "Multiple Dirichlet series associated to additive and Dirichlet characters", Biswajyoti Saha (*submitted*).

Biswajyoti Saha

Dedicated to my Teachers

রহিবে বিবেক ! সে শুধু আমার !
বিকাবো না তারে কভু !

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Notations

Symbol	**Description**
$\mathbb{N}$	The set of non-negative integers
$\mathbb{Z}$	The ring of integers
$\mathbb{Z}_{\leq n}$	The set of integers less than or equal to n
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
$\zeta(s)$	The Riemann zeta function
$\zeta_r(s_1, \ldots, s_r)$	The multiple zeta function of depth r
ι	The primitive 4-th root of unity with positive imaginary part
$\Re(s)$	The real part of a complex number s
$\Im(s)$	The imaginary part of a complex number s
I_q	The set $\{k \in \mathbb{N} \mid 0 \leq k \leq q-1\}$ where q is a positive integer
J_q	The set $\{k \in \mathbb{N} \mid k \geq q\}$ where q is a positive integer

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Synopsis

For an integer $r \geq 1$, consider the open subset U_r of $\mathbb{C}^r$ given by

$$U_r := \{(s_1, \ldots, s_r) \in \mathbb{C}^r \ : \ \Re(s_1 + \cdots + s_i) > i \ \text{ for all } \ 1 \leq i \leq r\}.$$

The multiple zeta function of depth r, denoted by $\zeta_r(s_1, \ldots, s_r)$, is a function on U_r defined by

$$\zeta_r(s_1, \ldots, s_r) := \sum_{n_1 > \cdots > n_r > 0} n_1^{-s_1} \cdots n_r^{-s_r}.$$

Note that the above series converges normally on any compact subset of U_r and hence defines an analytic function on U_r. When $r = 1$, this is the classical Riemann zeta function.

Riemann zeta function has been the focus of study for sometime now, playing a central role both in the development of modern number theory as well as arithmetic geometry. However, the study of multiple zeta functions has achieved prominence only in recent times though its origin can be traced back to Euler who studied the case $r = 2$ (see [11] for more details). Euler showed that the Riemann zeta values are inter-connected to the multiple zeta values. In particular, it is easy to see that

$$\zeta_1(s_1) \cdot \zeta_1(s_2) = \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) + \zeta_1(s_1 + s_2). \tag{0.0.1}$$

Thus the question of algebraic independence of Riemann zeta values gets related to the question of linear independence of certain multiple zeta values. There have been some recent developments by Hoffman, Goncharov, Terasoma, Zagier, Kaneko, Brown among others in the context of discovering possible structures in the set of special values of the multiple zeta functions at integral points.

In another direction, one can ask the question of analytic continuation of these multiple zeta functions. The first result in this direction is due to Atkinson who considered this question for $r = 2$. The question for general r was studied by Arakawa and Kaneko [4]. They showed that for a fixed tuple $(k_1, \cdots, k_{r-1})$, the function $\zeta_r(k_1, \cdots, k_{r-1}, s)$ has a meromorphic continuation to the complex plane. The analytic continuation of the multiple zeta functions as a function of several complex variables was first proved by Zhao [33]. He also provided a list of the possible polar singularities. The exact location of these polar singularities was later determined by Akiyama, Egami and Tanigawa [1]. The vanishing of the odd Bernoulli numbers played a crucial role in their work.

In order to establish the analytic continuation of the Riemann zeta function, Ramanujan [28] wrote down an identity involving the translates of the Riemann zeta function. More precisely, Ramanujan proved the following theorem.

Theorem 0.0.1 (S. Ramanujan)**.** *The Riemann zeta function satisfies the following identity:*

$$1 = \sum_{k \geq 0} (s-1)_k \big(\zeta_1(s+k) - 1\big), \quad \Re(s) > 1$$

where for $k \geq 0$

$$(s)_k := \frac{s \cdots (s+k)}{(k+1)!}.$$

Since this identity involves translates of the Riemann zeta function, from now on we refer to it as the translation formula for the Riemann zeta function. In this thesis, we built upon this idea of Ramanujan to study the analytic continuation and singularities of the multiple zeta functions and their various generalisations.

A later work of Ecalle [10] alluded to the possibility of extending the idea of Ramanujan for the multiple zeta functions. In a recent work with Mehta and Viswanadham [24], following the idea of Ecalle, we prove the following theorem.

Theorem 0.0.2. *For each integer $r \geq 2$, the multiple zeta function of depth r extends to a meromorphic function on $\mathbb{C}^r$ satisfying the translation formula*

$$\zeta_{r-1}(s_1 + s_2 - 1, s_3, \ldots, s_r) = \sum_{k \geq 0} (s_1 - 1)_k \zeta_r(s_1 + k, s_2, \ldots, s_r), \tag{0.0.2}$$

where the series of meromorphic functions on the right hand side converges normally on all compact subsets of $\mathbb{C}^r$.

We could also recover the following theorem of Akiyama, Egami and Tanigawa.

Theorem 0.0.3 (Akiyama-Egami-Tanigawa)**.** *The multiple zeta function of depth r is holomorphic in the open set obtained by removing the following hyperplanes from $\mathbb{C}^r$ and it has simple poles at the hyperplanes given by the equations*

$$s_1 = 1;\ s_1 + s_2 = 2, 1, 0, -2, -4, -6, \ldots;$$

$$s_1 + \cdots + s_i = n \ \textit{for all} \ n \in \mathbb{Z}_{\leq i} \ \textit{for all} \ 3 \leq i \leq r.$$

Here $\mathbb{Z}_{\leq i}$ denotes set of all integers less than or equal to i.

In order to prove Theorem 0.0.3, we introduce the method of “matrix formulation" to write down the residues along the possible polar hyperplanes (listed by Zhao) in a computable form. Here we would like to mention that while Zhao had given a formula to calculate the residues along the possible polar hyperplanes, the non-vanishing of these residues could not be concluded from that expression.

As a natural generalisation of the multiple zeta functions, Akiyama and Ishikawa [2] introduced the notion of multiple Hurwitz zeta function.

Definition 0.0.4. Let $r \geq 1$ be an integer and $\alpha_1, \ldots, \alpha_r \in [0,1)$. The multiple Hurwitz zeta function of depth r is denoted by $\zeta_r(s_1, \ldots, s_r;\ \alpha_1, \ldots, \alpha_r)$ and defined by the following convergent series in U_r:

$$\zeta_r(s_1, \ldots, s_r;\ \alpha_1, \ldots, \alpha_r) := \sum_{n_1 > \cdots > n_r > 0} (n_1 + \alpha_1)^{-s_1} \cdots (n_r + \alpha_r)^{-s_r}.$$

The analytic continuation of this function was established by Akiyama and Ishikawa. In the same paper [2], they also provided a list of possible singularities and were able to determine the exact set of singularities in some special cases. Here is their theorem.

Theorem 0.0.5 (Akiyama-Ishikawa)**.** *The multiple Hurwitz zeta function of depth r can be extended as a meromorphic function to $\mathbb{C}^r$ with possible simple poles at the hyperplanes given by the equations*

$$s_1 = 1;\ s_1 + \cdots + s_i = n \ \textit{for all} \ n \in \mathbb{Z}_{\leq i} \ \textit{and} \ 2 \leq i \leq r.$$

Later Kelliher and Masri [19] extended Zhao's method to reprove this theorem. They also obtained an expression for the residues along these possible polar hyperplanes but could not isolate the exact set of polar singularities.

We are now able to determine the exact set of singularities of the multiple Hurwitz zeta functions. Just as the vanishing of odd Bernoulli numbers plays a central role in determining the exact location of polar hyperplanes of the multiple zeta functions, in the case of the multiple Hurwitz functions an analogous pivotal role is played by the zeros of the Bernoulli polynomials. The Bernoulli polynomials $B_n(t) \in \mathbb{Q}[t]$ are defined by the following exponential generating function.

$$\sum_{n\geq 0} B_n(t)\frac{x^n}{n!} = \frac{xe^{tx}}{e^x - 1}.$$

In this context we proved the following theorem.

Theorem 0.0.6. *The multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ of depth r can be analytically continued to an open subset V_r of $\mathbb{C}^r$, where the V_r is obtained by removing the hyperplanes given by the equations*

$$s_1 = 1;\ s_1 + \cdots + s_k = n \textit{ for all } n \in \mathbb{Z}_{\leq k}, \textit{ for all } 2 \leq k \leq r$$

from $\mathbb{C}^r$. It has at most simple poles along each of these hyperplanes.

Further suppose that I is the set of $i \in \mathbb{N}$ such that $B_i(\alpha_2 - \alpha_1) = 0$. Let us define a subset J of $\mathbb{Z}_{\leq 2}$ by $J := \{2 - i : i \in I\}$. Then the multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ of depth r can be analytically continued to an open subset W_r of $\mathbb{C}^r$, where W_r is obtained by removing the hyperplanes given by the equations

$$s_1 = 1;\ s_1 + s_2 = n \textit{ for all } n \in \mathbb{Z}_{\leq 2} \setminus J;$$
$$s_1 + \cdots + s_k = n \textit{ for all } n \in \mathbb{Z}_{\leq k}, \textit{ for all } 3 \leq k \leq r$$

from $\mathbb{C}^r$. It has simple poles along each of these hyperplanes.

Akiyama and Ishikawa [2] also considered the multiple Dirichlet L-function.

Definition 0.0.7. Let $r \geq 1$ be an integer and $\chi_1, \ldots, \chi_r$ be Dirichlet characters. The multiple Dirichlet L-function of depth r is denoted by $L_r(s_1, \ldots, s_r;\ \chi_1, \ldots, \chi_r)$ and defined by the following convergent series in U_r:

$$L_r(s_1, \ldots, s_r;\ \chi_1, \ldots, \chi_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{\chi_1(n_1) \cdots \chi_r(n_r)}{n_1^{s_1} \cdots n_r^{s_r}}.$$

When $r = 1$, classically the analytic continuation is achieved by writing the function in terms of the classical Hurwitz zeta function. For $r > 1$, one may attempt to do that. In this case some variants of the multiple Hurwitz zeta functions come up. The exact set of singularities of the multiple Dirichlet L-functions are not well understood. For $r = 2$ and specific choices of characters χ_1 and χ_2, Akiyama and Ishikawa provided a complete description of the polar hyperplanes and for general r, they could prove the following theorem.

Theorem 0.0.8 (Akiyama-Ishikawa)**.** *Let $\chi_1, \ldots, \chi_r$ be Dirichlet characters of same conductor. Then the multiple Dirichlet L-function $L_r(s_1, \ldots, s_r;\ \chi_1, \ldots, \chi_r)$ of depth r can be extended as a meromorphic function to $\mathbb{C}^r$ with possible simple poles at the hyperplanes given by the equations*

$$s_1 = 1;\ s_1 + \cdots + s_i = n \ \textit{for all}\ \ n \in \mathbb{Z}_{\leq i} \ \textit{and}\ \ 2 \leq i \leq r.$$

To address this difficult question, we aim to obtain a translation formula satisfied by the multiple Dirichlet L-functions. In this direction, we are able to establish such a translation formula for the classical Dirichlet L-functions and obtain their meromorphic continuation in [31], the proof of which in fact carries over for Dirichlet series associated to periodic arithmetical functions.

Theorem 0.0.9. *Let f be a periodic arithmetical function with period q. Then the associated Dirichlet series $D(s, f) := \sum_{n \geq 1} \frac{f(n)}{n^s}$ satisfies the following translation formula:*

$$\sum_{a=1}^{q} \frac{f(a)}{a^{(s-1)}} = \sum_{k \geq 0} (s-1)_k q^{k+1} \left(D(s+k, f) - \sum_{a=1}^{q} \frac{f(a)}{a^{(s+k)}} \right), \tag{0.0.3}$$

where the infinite series on the right hand side converges normally on every compact subset of $\Re(s) > 1$.

Using Theorem 0.0.9, we can derive the meromorphic continuation of $D(s, f)$.

Theorem 0.0.10. *Let f be as in Theorem 0.0.9. Then, by means of the translation formula (0.0.3), the Dirichlet series $D(s, f)$ can be analytically continued to the entire complex plane except at $s = 1$, where the function has simple pole with residue $\frac{1}{q}\sum_{a=1}^{q} f(a)$. If $\sum_{a=1}^{q} f(a) = 0$, then $D(s, f)$ can be extended to an entire function.*

However, obtaining such a translation formula for multiple Dirichlet L-functions seems harder and we plan to take it up in near future. On the other hand, if we consider additive characters, that is, group homomorphisms $f : \mathbb{Z} \to \mathbb{C}^*$ in place of Dirichlet characters, the problem becomes tractable.

Definition 0.0.11. For a natural number $r \geq 1$ and additive characters $f_1, \ldots, f_r$, the multiple L-function associated to $f_1, \ldots, f_r$ is denoted as $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ and defined by following series:

$$L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{f_1(1)^{n_1} \cdots f_r(1)^{n_r}}{n_1^{s_1} \cdots n_r^{s_r}}.$$

The necessary and sufficient condition for absolute convergence of the above series is given by

$$|g_i(1)| \leq 1 \text{ for all } 1 \leq i \leq r,$$

where $g_i := \prod_{1 \leq j \leq i} f_j$ for all $1 \leq i \leq r$. With these conditions, the above series converges normally on any compact subset of U_r and hence defines an analytic function there. If $f_i(1) = e^{2\pi\iota\lambda_i}$ for some $\lambda_i \in \mathbb{C}$, the conditions $|g_i(1)| \leq 1$ can be rewritten as $\Im(\lambda_1 + \cdots + \lambda_i) \geq 0$. For simplicity, we will assume that $\lambda_i \in \mathbb{R}$ for all $1 \leq i \leq r$ so that the conditions are vacuously true. In this context, we have the following theorem which is valid also for complex λ_i's.

Theorem 0.0.12. *The multiple L-function $L_r(f_1, \ldots, f_r; s_1, \ldots, s_r)$, associated to additive characters $f_1, \ldots, f_r$ has different set of singularities depending on the values of*

$$\mu_i := \sum_{j=1}^{i} \lambda_j, \quad \textit{for } 1 \leq i \leq r.$$

(a) If $\mu_i \notin \mathbb{Z}$ for all $1 \leq i \leq r$, then $L_r(f_1, \ldots, f_r; s_1, \ldots, s_r)$ can be extended analytically to the whole of $\mathbb{C}^r$.

Let $i_1 < \cdots < i_m$ be all the indices such that $\mu_{i_k} \in \mathbb{Z}$ for all $1 \leq k \leq m$. Then the set of all possible singularities of $L_r(f_1, \ldots, f_r; s_1, \ldots, s_r)$ is described in the following two cases.

(b) If $i_1 = 1$, then $L_r(f_1, \ldots, f_r; s_1, \ldots, s_r)$ can be extended analytically to an open subset V_r of $\mathbb{C}^r$, where V_r is obtained by removing the hyperplanes given by the equations

$$s_1 = 1;\ s_1 + \cdots + s_{i_k} = n \text{ for all } n \in \mathbb{Z}_{\leq k}, \text{ for all } 2 \leq k \leq m$$

from $\mathbb{C}^r$. It has at most simple poles along each of these hyperplanes.

(c) If $i_1 \neq 1$, then $L_r(f_1, \ldots, f_r; s_1, \ldots, s_r)$ can be extended analytically to an open subset W_r of $\mathbb{C}^r$, where W_r is obtained by removing the hyperplanes given by the equations

$$s_1 + \cdots + s_{i_k} = n \text{ for all } n \in \mathbb{Z}_{\leq k}, \text{ for all } 1 \leq k \leq m$$

from $\mathbb{C}^r$. It has at most simple poles along each of these hyperplanes.

We further determine the exact set of singularities when $\lambda_i \in \mathbb{R}$ for all i. We call the corresponding additive characters to be real additive characters.

Theorem 0.0.13. *The exact set of singularities of the multiple L-function associated to real additive characters $f_1, \ldots, f_r$ differs from the set of all possible singularities as described in Theorem 0.0.12 only in the following two cases.*

(a) If $i_1 = 1$ and $i_2 = 2$ i.e. both $\lambda_1 = \lambda_2 = 0$, then $L_r(f_1, \ldots, f_r; s_1, \ldots, s_r)$ can be extended analytically to an open subset X_r of $\mathbb{C}^r$, where X_r is obtained by removing the hyperplanes given by the equations

$$s_1 = 1;\ s_1 + s_2 = n \text{ for all } n \in \mathbb{Z}_{\leq 2} \setminus J;$$
$$s_1 + \cdots + s_{i_k} = n \text{ for all } n \in \mathbb{Z}_{\leq k}, \text{ for all } 3 \leq k \leq m$$

from $\mathbb{C}^r$*, where* $J := \{-2n-1 : n \in \mathbb{N}\}$*. It has simple poles along each of these hyperplanes.*

(b) *If* $i_1 = 2$ *and* $\lambda_1 = 1/2$ *i.e. both* $\lambda_1 = \lambda_2 = 1/2$*, then* $L_r(f_1, \ldots, f_r; s_1, \ldots, s_r)$ *can be extended analytically to an open subset* Y_r *of* $\mathbb{C}^r$*, where* Y_r *is obtained by removing the hyperplanes given by the equations*

$$s_1 + s_2 = n \text{ for all } n \in \mathbb{Z}_{\leq 1} \setminus J;$$

$$s_1 + \cdots + s_{i_k} = n \text{ for all } n \in \mathbb{Z}_{\leq k} \text{ for all } 2 \leq k \leq m$$

from $\mathbb{C}^r$*, where* $J := \{-2n-1 : n \in \mathbb{N}\}$*. It has simple poles along each of these hyperplanes.*

In fact, one can unify the previous notions and consider the following general function. For a natural number $r \geq 1$ and two sets of real numbers $\lambda_1, \ldots, \lambda_r$ and $\alpha_1, \ldots, \alpha_r$ in $[0, 1)$, one can consider the complex valued function defined by the following convergent series in U_r:

$$\sum_{n_1 > \cdots > n_r > 0} \frac{e(\lambda_1 n_1 + \cdots + \lambda_r n_r)}{(n_1 + \alpha_1)^{s_1} \cdots (n_r + \alpha_r)^{s_r}},$$

where for a real number a, $e(a)$ means the complex number $e^{2\pi\iota a}$. This function is a generalisation of one variable Lerch zeta function, and hence we call it as the multiple Lerch zeta function of depth r.

We extend the idea of Ecalle to obtain a translation formula for the multiple Lerch zeta functions analogous to the one discovered by Ramanujan for the Riemann zeta function. Using this translation formula, we can then establish the meromorphic continuation of the Lerch zeta function, of which Theorem 0.0.12 and first part of Theorem 0.0.6 are particular cases.

1

Introduction

1.1 Riemann zeta function and its special values

For a complex number s with $\Re(s) > 1$, the Riemann zeta function $\zeta(s)$ is defined by the absolutely convergent series

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}.$$

Before Riemann, this function was considered by Euler for positive integer values, and is therefore also referred to as the Euler-Riemann zeta function. Euler derived many beautiful results about these special values. For instance, he proved that $\zeta(2) = \pi^2/6$, and more generally that

$$\zeta(2n) = \frac{(-1)^{n-1}2^{2n-1}B_{2n}}{(2n)!}\pi^{2n} \quad \text{for all} \quad n \geq 1. \tag{1.1.1}$$

Here B_n denotes the n-th Bernoulli number which is defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}.$$

The relation (1.1.1) together with the fact that π is transcendental, proved by F. Lindemann in 1882, yields that the zeta values at positive even integers are transcendental numbers and furthermore linearly independent over $\overline{\mathbb{Q}}$.

On the other hand the arithmetic nature of the zeta values at odd positive integers are yet to be determined and any relations among these special values are yet to be found. In this context we have the following conjecture which is regarded as a mathematical folklore.

Conjecture 1.1.1. *The numbers $\pi, \zeta(2n+1)$ for all $n \geq 1$ are algebraically independent i.e. there are no polynomial relations among any of these numbers. In particular they are all transcendental.*

While this conjecture is far from being proved, there have been a number of recent developments in this direction. Besides π, the lone number in the above list whose arithmetic nature has been revealed to some extent is $\zeta(3)$. R. Apéry [3] proved it to be irrational in 1978. Later in 2000, K. Ball and T. Rivoal [6] proved the following notable theorem from which it follows that there are infinitely many odd zeta values which are irrational.

Theorem 1.1.2 (Ball-Rivoal)**.** *Given any $\epsilon > 0$, there exists an integer $N = N(\epsilon)$ such that for all $n > N$, the dimension of the $\overline{\mathbb{Q}}$-vector space generated by the numbers*

$$1, \zeta(3), \ldots, \zeta(2n+1)$$

exceeds

$$\frac{1-\epsilon}{1+\log 2} \log n.$$

About the specific values, Zudilin [34] proved that at least one of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational.

As we can see that Conjecture 1.1.1 predicts no algebraic relation among the odd Riemann zeta values, but on the other hand when we consider the set of special values of the so-called multiple zeta functions, which also contains the Riemann zeta values, has a rich structure with a number of known relations and hence is more amenable for investigation. These multiple zeta functions and their various generalisations are the principal objects of study in this thesis and we introduce them in the following sections.

1.2 Multiple zeta functions and their special values

For an integer $r \geq 1$, consider the open subset U_r of $\mathbb{C}^r$:

$$U_r := \{(s_1, \ldots, s_r) \in \mathbb{C}^r \ : \ \Re(s_1 + \cdots + s_i) > i \ \text{ for all } \ 1 \leq i \leq r\}.$$

The multiple zeta function of depth r, denoted by $\zeta_r(s_1, \ldots, s_r)$, is a function on U_r defined by

$$\zeta_r(s_1, \ldots, s_r) := \sum_{n_1 > \cdots > n_r > 0} n_1^{-s_1} \cdots n_r^{-s_r}.$$

The above series converges normally on any compact subset of U_r (see §3.2 of Chapter 3 for a proof) and hence defines an analytic function on U_r. This is a multi-variable generalisation of the classical Riemann zeta function. The origin of these functions can again be traced back to Euler who studied the case $r = 2$. Euler showed that the Riemann zeta values are inter-connected to the multiple zeta values. In particular, it is easy to see that

$$\zeta(s_1) \cdot \zeta(s_2) = \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) + \zeta(s_1 + s_2). \tag{1.2.1}$$

Thus the question of algebraic independence of special values of the Riemann zeta function gets linked to the question of linear independence of some other special values of the multiple zeta functions.

In this context it is convenient to introduce the following terminology. The special values of the multiple zeta function of depth r at the points $(k_1, \ldots, k_r)$ of U_r such that k_i's are positive integers for all $1 \leq i \leq r$, are called the *multi zeta values* of depth r. The number

$(k_1 + \cdots + k_r)$ is called the *weight* of multi zeta value $\zeta_r(k_1, \ldots, k_r)$. Besides the relations coming from (1.2.1), Euler could also prove that

$$\zeta(3) = \zeta_2(2, 1).$$

More generally, he proved the following theorem.

Theorem 1.2.1 (Euler)**.** *For any integer $k \geq 3$, we have*

$$\zeta(k) = \sum_{j=2}^{k-1} \zeta_2(j, k-j).$$

A generalisation of this theorem was later proposed by M.E. Hoffman [16], which is now known as *the sum theorem*. It has been proved independently by A. Granville [14] and D. Zagier.

Theorem 1.2.2 (Granville-Zagier)**.** *For any integers $s \geq 2$ and $r \geq 1$, the identity*

$$\zeta(k) = \sum_{\substack{k_1 > 1, k_2 \geq 1, \ldots, k_r \geq 1 \\ k_1 + \cdots + k_r = k}} \zeta_r(k_1, \ldots, k_r)$$

holds.

The relation $\zeta(3) = \zeta_2(2, 1)$ can also be seen as a special instance of the so called duality relation. To elaborate on this we need some more notations. For any r-tuple of positive integers $\mathbf{k} = (k_1, \ldots, k_r)$, we associate a $(k_1 + \cdots + k_r)$-tuple with entries in the set $\{0, 1\}$ by the following prescription:

$$(k_1, \ldots, k_r) \leftrightarrow (\,\underbrace{0 \ldots 0}_{k_1 - 1} 1 \underbrace{0 \ldots 0}_{k_2 - 1} 1 \ldots \underbrace{0 \ldots 0}_{k_r - 1} 1\,). \tag{1.2.2}$$

We denote this tuple by $\mathbf{w}(\mathbf{k})$. One often writes $\zeta_r(\mathbf{w}(\mathbf{k}))$ to denote $\zeta_r(k_1, \ldots, k_r)$.

We now introduce the dual map τ on the set of tuples with entries in $\{0, 1\}$ to itself as follows. First of all, for $\epsilon \in \{0,1\}$, let $\bar{\epsilon} := 1-\epsilon$. Then for any tuple $\mathbf{w} = (\epsilon_1 \ldots \epsilon_n)$ with entries in $\{0,1\}$, we define its dual $\tau(\mathbf{w}) = \bar{\mathbf{w}}$ to be the tuple $\bar{\mathbf{w}} := (\bar{\epsilon}_n \ldots \bar{\epsilon}_1)$. Note that if $\mathbf{w} = \mathbf{w}(\mathbf{k})$ for some r-tuple of positive integers $\mathbf{k} = (k_1, \ldots, k_r) \in U_r$, then $\bar{\mathbf{w}}$ is also associated to an $(k_1 + \cdots + k_r - r)$-tuple of positive integers in $U_{(k_1+\cdots+k_r-r)}$, say $\bar{\mathbf{k}}$. We call $\bar{\mathbf{k}}$ to be the *dual* of $\mathbf{k}$. In this context, the following theorem, due to Zagier, is known as the *duality theorem*.

Theorem 1.2.3 (Zagier)**.** *Let* $\mathbf{k} = (k_1, \ldots, k_r) \in U_r$ *be an* r*-tuple of positive integers and* $\bar{\mathbf{k}} \in U_{(k_1+\cdots+k_r-r)}$ *denote its dual. Then we have*

$$\zeta_r(\mathbf{k}) = \zeta_{(k_1+\cdots+k_r-r)}(\bar{\mathbf{k}}).$$

For example, note that $(3) \leftrightarrow (001)$ and $\tau(001) = 011$. Further, $(011) \leftrightarrow (2,1)$. Hence, Euler's identity $\zeta(3) = \zeta_2(2,1)$ follows as a special case of this more general theorem.

Further, we also have the *shuffle product formula* of multi zeta values. The formula (1.2.1) can be seen as a special case of the *stuffle product formula* of multiple zeta functions. Below we define the notion of shuffling and stuffling.

Let p and q be two non-negative integers. We define a *stuffling* of p and q to be a pair (A, B) of sets such that $|A| = p,\ |B| = q$ and $A \cup B = \{1, \ldots, r\}$ for some integer r. We then have $\max(p,q) \le r \le p+q$. We call this r to be the *length of the stuffling*. Such a stuffling is called a *shuffling* when A and B are disjoint, i.e. when $r = p+q$. We denote the stuffle product by $\star$ and the shuffle product is denoted by Ш.

Let $(s_1, \ldots, s_p)$ and $(t_1, \ldots, t_q)$ be two sequences of complex numbers and (A, B) be a stuffling of p and q, with $A \cup B = \{1, \ldots, r\}$. Let σ and τ denote the unique increasing bijections from $A \to \{1, \ldots, p\}$ and $B \to \{1, \ldots, q\}$ respectively. Let us define a sequence of complex numbers $(z_1, \ldots, z_r)$ as follows:

$$z_i := \begin{cases} s_{\sigma(i)} & \text{when } i \in A \setminus B, \\ t_{\tau(i)} & \text{when } i \in B \setminus A, \\ s_{\sigma(i)} + t_{\tau(i)} & \text{when } i \in A \cap B. \end{cases}$$

We call it the *sequence deduced from* $(s_1, \ldots, s_p)$ *and* $(t_1, \ldots, t_q)$ *by the stuffling* (A, B). Clearly, if $(s_1, \ldots, s_p) \in U_p$ and $(t_1, \ldots, t_q) \in U_q$, then $(z_1, \ldots, z_r) \in U_r$. With the above notation one has the following theorem, which is known as the stuffle product formula.

Theorem 1.2.4. *Let* $(s_1, \ldots, s_p) \in U_p$ *and* $(t_1, \ldots, t_q) \in U_q$. *Then we have,*

$$\zeta_p(s_1, \ldots, s_p)\, \zeta_q(t_1, \ldots, t_q) = \sum_{(A,B)} \zeta_r(z_1, \ldots, z_r), \tag{1.2.3}$$

where in the summation on the right hand side (A, B) *runs over the stufflings of* p *and* q, *and* $(z_1, \ldots, z_r)$ *denotes the sequence deduced from* $(s_1, \ldots, s_p)$ *and* $(t_1, \ldots, t_q)$ *by this stuffling.*

On the other hand, the shuffle product formula of multi zeta values involves the correspondence given in (1.2.2).

Theorem 1.2.5. *Let* $\mathbf{k} = (k_1, \ldots, k_p) \in U_p$ *be a* p*-tuple of positive integers of weight* m *and* $\mathbf{l} = (l_1, \ldots, l_q) \in U_q$ *be a* q*-tuple of positive integers of weight* n. *Then we have,*

$$\zeta_p(k_1, \ldots, k_p)\, \zeta_q(l_1, \ldots, l_q) = \sum_{(A,B)} \zeta_{p+q}(\epsilon_1 \ldots \epsilon_{m+n}), \tag{1.2.4}$$

where in the summation on the right hand side, (A, B) *runs over the shufflings of* m *and* n, *and* $(\epsilon_1 \ldots \epsilon_{m+n})$ *denotes the sequence deduced from* $\mathbf{w}(\mathbf{k})$ *and* $\mathbf{w}(\mathbf{l})$ *by this shuffling.*

Note that in the above theorem, by $\zeta_{p+q}(\epsilon_1 \ldots \epsilon_{m+n})$ we mean the multi zeta value $\zeta_{p+q}(\mathbf{u})$, where $\mathbf{u} \in U_{p+q}$ is the element corresponding to $(\epsilon_1 \ldots \epsilon_{m+n})$ as per (1.2.2).

One can further equate the right hand sides of (1.2.3) and (1.2.4) to obtain more relations. The relations obtain this way are called the *double shuffle relations*. For example,

$$(2) \star (2) = 2(2,2) + (4).$$

Now

$$(2) \leftrightarrow (01) \ \text{ and } \ (01) ⧢ (01) = 4(0011) + 2(0101) \leftrightarrow 4(3,1) + 2(2,2).$$

Thus we get,

$$\zeta_2(3,1) = \frac{\zeta(4)}{4}.$$

The most significant property of all these relations among multi zeta values that we have discussed here, is the 'preservation of weight', i.e. the weights of all the multi zeta values involved in any of the above discussed relations are the same. In view of this observation, the following conjecture seems reasonable.

Conjecture 1.2.6. *There are no non-trivial $\mathbb{Q}$-linear relations among multi zeta values of different weights.*

Here by a non-trivial $\mathbb{Q}$-linear relation we mean that such a relation cannot be further reduced to two or more uniform-weight relations. An example of a trivial relation is the following:

$$\zeta(3) + \zeta(4) = \zeta_2(2,1) + 4\,\zeta_2(3,1).$$

The above conjecture can also be formulated in an abstract setting.

We define a graded $\mathbb{Q}$-vector space Z of multi zeta values where the grading is over the weight of the multi zeta values. Let us set

$$Z := \bigoplus_{n \in \mathbb{N}} Z_n,$$

where $Z_0 := \mathbb{Q}$, $Z_1 := \{0\}$ and for $n \geq 2$

$$Z_n := \mathbb{Q}\langle \zeta(k_1, \ldots, k_r) : r, k_1, \ldots, k_r \text{ are integers } \geq 1 \text{ with } k_1 > 1 \text{ and } k_1 + \cdots + k_r = n \rangle.$$

Then the above conjecture can be reformulated as the following one.

Conjecture 1.2.7. *For all $m, n \in \mathbb{N}$ such that $m \neq n$,*

$$Z_n \cap Z_m = \{0\}$$

There are some other conjectures concerning the structure of Z. For instance, the following conjecture due to Zagier predicts the dimension of each Z_n.

Conjecture 1.2.8. *The dimension d_n of $\mathbb{Q}$-vector space Z_n is given by the recurrence relation*

$$d_n = d_{n-2} + d_{n-3},$$

with the initial data that $d_1 = 0$ and $d_0 = d_2 = 1$. In other words, the generating series of d_n is the following:

$$\sum_{n \geq 0} d_n X^n = \frac{1}{1 - X^2 - X^3}.$$

This conjecture is far from being proved. Though this conjecture predicts exponential growth of d_n, till date we do not have a single example of $d_n \geq 2$. For all $n \in \mathbb{N}$, a set D_n of basis elements of Z_n was predicted by Hoffman. Recently, Brown [8] has showed that D_n in fact generates Z_n for all $n \in \mathbb{N}$.

The theory of multi zeta values has been expanded to a great extent in the past couple of decades by the likes of Hoffman, Zagier, Goncharov, Terasoma, Kaneko, Ohno and more recently, by Brown. Beside this, the analytic theory of the multiple zeta functions has also been a subject of development in these years. In the following section we discuss this briefly.

1.3 Analytic theory of the multiple zeta functions

We begin this section by recalling the analytic properties of $\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}$. Riemann showed that the function defined by the above series on the half plane $\Re(s) > 1$ can be continued analytically to the entire complex plane except at $s = 1$, where it has simple pole with residue 1, i.e.

$$\lim_{s\to 1}(s-1)\zeta(s) = 1.$$

In fact, it is possible to extend the Riemann zeta function to the half plane $\Re(s) > 0$ by just using the Abel's partial summation formula.

For a real number $x > 1$ and complex number s such that $\Re(s) > 1$, we get that

$$\sum_{n\leq x} n^{-s} = \frac{[x]}{x^s} + s\int_1^x \frac{[t]}{t^{s+1}}dt.$$

Then letting $x \to \infty$ we get

$$\zeta(s) = s\int_1^\infty \frac{[t]}{t^{s+1}}dt = \frac{s}{s-1} - \int_1^\infty \frac{\{t\}}{t^{s+1}}dt,$$

where the integral

$$\int_1^\infty \frac{\{t\}}{t^{s+1}}dt$$

converges in $\Re(s) > 0$. Thus using the above expression one can extend the Riemann zeta function to the half plane $\Re(s) > 0$ as a meromorphic function with a simple pole at $s = 1$ with residue 1.

In 1859, Riemann [30] established its meromorphic continuation to the entire complex plane satisfying the functional equation

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s),$$

where Γ denotes the gamma function. This is undoubtedly the most fundamental and the most referred functional equation of the Riemann zeta function. But there is another elegant but not so well-known functional equation of the Riemann zeta function due to Ramanujan [28]. Ramanujan proved that the Riemann zeta function satisfies the following formula:

$$1 = \sum_{k \geq 0} (s-1)_k (\zeta(s+k) - 1),$$

where for $k \geq 0$

$$(s)_k := \frac{s(s+1)\cdots(s+k)}{(k+1)!},$$

and the series on the right hand side converges normally on compact subsets of $\Re(s) > 1$. It is convenient to define $(s)_{-1} := 1$. One can deduce the meromorphic continuation of the Riemann zeta function from the above translation formula. Since this identity involves translates of the Riemann zeta function, from now on we refer to it as the translation formula for the Riemann zeta function.

A similar translation formula was obtained by V. Ramaswami [29] in 1934. He proved that the Riemann zeta function satisfies the following translation formula:

$$(1 - 2^{1-s})\zeta(s) = \sum_{k \geq 0} (s)_k \frac{\zeta(s+k+1)}{2^{s+k+1}},$$

where the series on the right hand side converges normally on compact subsets of $\Re(s) > 0$.

On the contrary, even though the multiple zeta function of depth 2, often called the double zeta function, was known since the time of Euler, its meromorphic continuation was studied much later. In 1949, almost a century after Riemann's fundamental work, F.V. Atkinson [5] addressed the question of meromorphic continuation of the double zeta function while studying the mean-values of the Riemann zeta function.

For general r, initially the meromorphic continuation of the multiple zeta function of depth r was obtained for each variable separately. Such treatment can be found in [4]. As a function of several variable, the analytic continuation was first established by J. Zhao [33] in 1999. He used the theory of generalised functions.

Theorem 1.3.1 (Zhao). *The multiple zeta function of depth r can be extended as a meromorphic function to $\mathbb{C}^r$ with possible simple poles at the hyperplanes given by the equations*

$$s_1 = 1;\ s_1 + \cdots + s_i = n \text{ for all } n \in \mathbb{Z}_{\leq i} \text{ and } 2 \leq i \leq r.$$

Here $\mathbb{Z}_{\leq i}$ denotes set of all integers less than or equal to i.

Around the same time, S. Akiyama, S. Egami and Y. Tanigawa [1] gave a simpler proof of the above fact using the classical Euler-Maclaurin summation formula. Besides, what was even more special in their work is that they could identify the exact set of singularities. The vanishing of the odd Bernoulli numbers played a central role in this context.

Theorem 1.3.2 (Akiyama-Egami-Tanigawa). *The multiple zeta function of depth r is holomorphic in the open set obtained by removing the following hyperplanes from $\mathbb{C}^r$ and it has simple poles at the hyperplanes given by the equations*

$$s_1 = 1;\ s_1 + s_2 = 2, 1, 0, -2, -4, -6, \ldots;$$
$$s_1 + \cdots + s_i = n \text{ for all } n \in \mathbb{Z}_{\leq i} \text{ and } 3 \leq i \leq r.$$

Thereafter, the problem of the meromorphic continuation of the multiple zeta functions received a lot of attention. In this process, a variety of methods evolved to address this problem. For instance, Goncharov [13] obtained the meromorphic continuation using the theory of distributions. Alternate proofs using Mellin-Barnes integrals was given by K. Matsumoto. Later he went on to apply this method to a set of other related problems. His expositions can be found in [22, 23]. Matsumoto's work has further been generalised in [26].

However, the simplest possible approach to this problem was indicated by J. Ecalle [10]. His idea germinated from Ramanujan's translation formula for the Riemann zeta function. In his article [10], he indicated how one could have obtained Ramanujan's identity in an elementary way and extend it for the multiple zeta functions. His idea has recently been penned down explicitly in a joint work [24] with J. Mehta and G.K. Viswanadham, carried out under the supervision of J. Oesterlé.

In this work we further introduce the method of *matrix formulation* to write down the residues along the possible polar hyperplanes (listed by Zhao) in a computable form. Here we would like to mention that Zhao [33] had also given a formula to calculate the residues along the possible polar hyperplanes. But the non-vanishing of these residues could not be concluded from that expression, whereas our expression of residues enabled us to isolate the non-existing polar hyperplanes from his list and recover the above mentioned theorem of Akiyama, Egami and Tanigawa.

Soon after Zhao and Akiyama, Egami and Tanigawa's work, several generalisations of the multiple zeta functions were introduced and their analytic properties were discussed. One important example is the multiple Hurwitz zeta functions. In 2002, Akiyama and Ishikawa [2] introduced the notion of multiple Hurwitz zeta functions.

Definition 1.3.3. Let $r \geq 1$ be an integer and $\alpha_1, \ldots, \alpha_r \in [0, 1)$. The multiple Hurwitz zeta function of depth r is denoted by $\zeta_r(s_1, \ldots, s_r;\ \alpha_1, \ldots, \alpha_r)$ and defined by the following convergent series in U_r:

$$\zeta_r(s_1, \ldots, s_r;\ \alpha_1, \ldots, \alpha_r) := \sum_{n_1 > \cdots > n_r > 0} (n_1 + \alpha_1)^{-s_1} \cdots (n_r + \alpha_r)^{-s_r}.$$

Following the method of [2], they established meromorphic continuations of these functions as well as listed possible polar singularities for them. They were able to determine the exact set of singularities for some specific values of α_i's.

Theorem 1.3.4 (Akiyama-Ishikawa)**.** *The multiple Hurwitz zeta function of depth r can be extended as a meromorphic function to $\mathbb{C}^r$ with possible simple poles at the hyperplanes given* *by the equations*

$$s_1 = 1;\ s_1 + \cdots + s_i = n \textit{ for all } n \in \mathbb{Z}_{\leq i} \textit{ and } 2 \leq i \leq r.$$

Perhaps motivated by the classical relation between the Hurwitz zeta function and Dirichlet L-functions, Akiyama and Ishikawa also considered the following several variable generalisation of the Dirichlet L-functions in [2].

Definition 1.3.5. Let $r \geq 1$ be an integer and $\chi_1, \ldots, \chi_r$ be Dirichlet characters of any modulus. The multiple Dirichlet L-function of depth r is denoted by $L_r(s_1, \ldots, s_r;\ \chi_1, \ldots, \chi_r)$ and defined by the following convergent series in U_r:

$$L_r(s_1, \ldots, s_r;\ \chi_1, \ldots, \chi_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{\chi_1(n_1) \cdots \chi_r(n_r)}{n_1^{s_1} \cdots n_r^{s_r}}.$$

It is classically known that the Dirichlet L-functions can be written as linear combinations of the Hurwitz zeta functions. Thus the meromorphic continuation of Dirichlet L-functions follows from that of the Hurwitz zeta functions. To obtain the meromorphic continuation of the multiple Dirichlet L-functions, they followed this very approach and derived the following theorem.

Theorem 1.3.6 (Akiyama-Ishikawa)**.** *Let $\chi_1, \ldots, \chi_r$ be primitive Dirichlet characters of same conductor. Then the multiple Dirichlet L-function $L_r(s_1, \ldots, s_r;\ \chi_1, \ldots, \chi_r)$ of depth r can be extended as a meromorphic function to $\mathbb{C}^r$ with possible simple poles at the hyperplanes given by the equations*

$$s_1 = 1;\ s_1 + \cdots + s_i = n \textit{ for all } n \in \mathbb{Z}_{\leq i} \textit{ and } 2 \leq i \leq r.$$

But till date, we do not have precise information about the exact set of singularities of the multiple Hurwitz zeta functions and the multiple Dirichlet L-functions. Major part of this thesis is devoted to study these yet to be resolved problems, following the methods developed in [24]. In the next section we give a brief outline of this thesis.

1.4 Arrangement of the thesis

In the second chapter, we derive translation formulas and thereby the meromorphic continuation of certain families of Dirichlet series along the line of Ramanujan. To establish such formulas we follow Ecalle's indication to obtain an elementary proof of Ramanujan's theorem.

In the third chapter we discuss the analytic properties of the multiple zeta functions. We obtain translation formulas for these functions and then write them in terms of infinite matrices to obtain a matrix formulation of these translation formulas. We also deduce the meromorphic continuation of the multiple zeta functions by means of such a translation formula and induction on the depth. We use the matrix formulation to write down an expression for residues along the possible polar hyperplanes and study the non-vanishing of these residues.

In the fourth chapter we consider the multiple Hurwitz zeta functions. Building upon the work on the previous chapter, we derive translation formulas for the multiple Hurwitz zeta functions. We then deduce the meromorphic continuation and derive a list of possible singularities. Using a fundamental property of the zeros of the Bernoulli polynomials we then determine the exact set of singularities of the multiple Hurwitz zeta functions.

In the penultimate chapter, we consider multiple Dirichlet series associated to additive characters and derive their meromorphic continuations as well as their exact list of polar singularities. We next show that the multiple Dirichlet series associated to additive characters are related to the multiple Dirichlet L-functions. Using such relations, we then derive meromorphic continuations and possible list of polar singularities for multiple Dirichlet L-functions.

Our last chapter deals with a weighted variant of the multiple zeta functions. Study of this weighted variant is not esoteric. We show that this weighted variant has some rich arithmetic structures and their location of singularities have an uniform pattern.

2

Dirichlet series and their translation formulas

We begin this chapter by proving the aforementioned theorem of S. Ramanujan following an outline by J. Ecalle [10]. We then extend Ecalle's idea to prove analogous theorems for certain Dirichlet series. To the best of our knowledge, such results were not known before. In later chapters we extend some of these results for certain multiple Dirichlet series.

2.1 Proof of Ramanujan's theorem

We first recall Ramanujan's theorem.

Theorem 2.1.1 (S. Ramanujan)**.** *The Riemann zeta function satisfies the following identity*

$$1 = \sum_{k\geq 0}(s-1)_k\big(\zeta(s+k)-1\big) \textit{ for } \Re(s) > 1, \tag{2.1.1}$$

where the series on the right hand side converges normally on any compact subset of $\Re(s) > 1$ *and for any* $k \geq 0, s \in \mathbb{C}$,

$$(s)_k := \frac{s(s+1)\cdots(s+k)}{(k+1)!}.$$

Since this identity involves translates of the Riemann zeta function, from now on we refer to it as the translation formula for the Riemann zeta function. Proof of this theorem is obtained following an elegant idea of Ecalle [10]. However in order to write down the complete proof, we need couple of short lemmas. We state and prove these lemmas here so that we can refer to them whenever required. The notion of normal convergence is integral to our study. We first recall the definition.

For a complex valued function f on a set X, let $\|f\|_X := \sup_{x \in X} |f(x)|$. We say that a family $(f_i)_{i \in I}$ of complex valued functions on X is *normally summable* if $\|f_i\|_X < \infty$ for all $i \in I$ and the family of real numbers $(\|f_i\|_X)_{i \in I}$ is summable. In this case, we also say that the series $\sum_{i \in I} f_i$ *converges normally* on X.

Lemma 2.1.2. *Let $m \in \mathbb{R}$ and K be a compact subset of $\Re(s) > -(m+1)$. Then the family*

$$\left(\left\| \frac{1}{n^{s+m+2}} \right\|_K \right)_{n \geq 1}$$

is summable. Here for a function $f : K \to \mathbb{C}$,

$$\|f\|_K := \sup_{s \in K} |f(s)|.$$

Proof. We have that K is a compact subset of $\Re(s) > -(m+1)$ and the set

$$L := \{s \in \mathbb{C} : \Re(s) = -(m+1)\}$$

is closed. Hence we get that the distance of L and K is positive, as they are disjoint. More precisely, there exists a $\delta > -(m+1)$ such that $\Re(s) > \delta$ for all $s \in K$. Thus

$$\left\| \frac{1}{n^{s+m+2}} \right\|_K = \sup_{s \in K} \left| \frac{1}{n^{s+m+2}} \right| = \sup_{s \in K} \frac{1}{n^{\Re(s)+m+2}} < \frac{1}{n^{\delta+m+2}}.$$

Since $\delta + m + 2 > 1$, we get the desired result. $\square$

Lemma 2.1.3. *Let $m \in \mathbb{R}$, K be a compact subset of $\Re(s) > -(m+1)$ and $q \geq 1$ be an integer . Then the family*

$$\left((s-1)_k \frac{q^{k+1}}{n^{s+k}}\right)_{n>q,k\geq m+2}$$

is normally summable in K.

Proof. Let $S := \sup_{s\in K} |s-1|$. Then for $n > q$, we have

$$\left\|(s-1)_k \frac{q^{k+1}}{n^{s+k}}\right\|_K \leq q^{m+3}(S)_k \left(\frac{q}{q+1}\right)^{k-(m+2)} \left\|\frac{1}{n^{s+m+2}}\right\|_K .$$

Further note that the series

$$\sum_{k\geq m+2} (S)_k \left(\frac{q}{q+1}\right)^{k-(m+2)}$$

is convergent. This together with Lemma 2.1.2 proves our claim. □

We are now ready to prove Ramanujan's theorem.

Proof of Theorem 2.1.1. Following Ecalle, we start with the following identity which is valid for any $n > 1$ and $s \in \mathbb{C}$:

$$(n-1)^{1-s} - n^{1-s} = \sum_{k\geq 0} (s-1)_k \, n^{-s-k}. \tag{2.1.2}$$

This identity is easily obtained by writing the left hand side as $n^{1-s}\left((1-\frac{1}{n})^{1-s} - 1\right)$ and expanding $(1-\frac{1}{n})^{1-s}$ as a Taylor series in $\frac{1}{n}$. We know by Lemma 2.1.3 (for $m = -2$ and $q = 1$) that the family

$$\left((s-1)_k \frac{1}{n^{s+k}}\right)_{n>1,k\geq 0}$$

is normally summable on every compact subset of $\Re(s) > 1$. Then we sum the left hand side of (2.1.2) for $n > 1$ and $\Re(s) > 1$. Upon interchanging the summations, we get (2.1.1). □

2.2 Ramanujan's theorem for some Dirichlet series

We now generalise Ramanujan's idea in order to derive meromorphic continuation of some classes of Dirichlet series. In this process we recover some classical results in this direction. But the aim of this chapter is to unify some of these proofs and to provide an elementary and simple way to reach the state of the art.

By an arithmetical function we mean a function $f : \mathbb{N} \to \mathbb{C}$. Now for an arithmetical function f and complex parameter s, we define the associated Dirichlet series $D(s, f)$ by

$$D(s, f) := \sum_{n\geq 1} \frac{f(n)}{n^s}.$$

If the function f has polynomial growth then the above series converges in some half plane. More generally, the necessary growth condition on f, so as to make sense of the above definition, can be given in terms of the partial sums $F(x) := \sum_{n\leq x} f(n)$. If $F(x)$ has polynomial growth i.e. $F(x) = O(x^{\delta})$ for some positive real number δ, then the Dirichlet series $D(s, f)$ converges absolutely in the half plane $\Re(s) > \delta$.

Here we mainly consider two types of arithmetical functions. First we consider an arithmetical function f which is periodic i.e. there exists a natural number $q \geq 1$ such that

$$f(n + q) = f(n) \text{ for all } n \in \mathbb{N}.$$

Note that for such an arithmetical function f, the Dirichlet series $D(s, f)$ converges absolutely for $\Re(s) > 1$. Such Dirichlet series are known as periodic Dirichlet series.

Next we consider a non-zero arithmetical function f which satisfies

$$f(m + n) = f(m)f(n) \text{ for all } m, n \in \mathbb{N}.$$

Such an arithmetical function can be extended to $\mathbb{Z}$ so that it becomes a homomorphism from

$\mathbb{Z} \to \mathbb{C}^*$. These homomorphisms are known as additive characters. On the other hand, an additive character $f : \mathbb{Z} \to \mathbb{C}^*$ gives rise to such an arithmetical function. In this case, the function is determined by its value at 1 as $f(n) = f(1)^n$. Hence the sum $\sum_{n\geq 1} \frac{f(n)}{n^s}$ converges only if $|f(1)| \leq 1$ and in that case the Dirichlet series $D(s, f)$ converges absolutely for $\Re(s) > 1$. In fact if $|f(1)| < 1$, then the Dirichlet series $D(s, f)$ converges normally on any compact subset of $\mathbb{C}$, hence defines an entire function. For Dirichlet series associated to these arithmetical functions we prove the following theorems.

Theorem 2.2.1. *Let f be a periodic arithmetical function with period q. Then the associated Dirichlet series $D(s, f)$ satisfies the following translation formula:*

$$\sum_{a=1}^{q} \frac{f(a)}{a^{(s-1)}} = \sum_{k\geq 0} (s-1)_k q^{k+1} \left(D(s+k, f) - \sum_{a=1}^{q} \frac{f(a)}{a^{(s+k)}} \right), \tag{2.2.1}$$

where the infinite series on the right hand side converges normally on every compact subset of $\Re(s) > 1$.

The above theorem includes Ramanujan's theorem as a special case. Using Theorem 2.2.1, we can now derive the meromorphic continuation of $D(s, f)$. Classically it was derived by writing such Dirichlet series as linear combinations of the Hurwitz zeta functions and then using the meromorphic continuation of these Hurwitz zeta functions.

Theorem 2.2.2. *Let f be as in Theorem 2.2.1. Then using the translation formula (2.2.1), the Dirichlet series $D(s, f)$ can be analytically continued to the entire complex plane except at $s = 1$, where the function has simple pole with residue $\frac{1}{q}\sum_{a=1}^{q} f(a)$. If $\sum_{a=1}^{q} f(a) = 0$, then $D(s, f)$ can be extended to an entire function.*

Example 2.2.3. Besides obtaining the meromorphic continuation of the Riemann zeta function, we can also recover the following results from Theorem 2.2.2. Let χ be a non-trivial Dirichlet character mod q. Then it is known that $\sum_{a=1}^{q} \chi(a) = 0$. Hence, the Dirichlet L-

function

$$L(s,\chi) := \sum_{n\geq 1} \frac{\chi(n)}{n^s}$$

can be extended to an entire function. If $\chi = \chi_0$, the trivial Dirichlet character mod q, then $\sum_{a=1}^{q} \chi_0(a) = \varphi(q)$. Hence $L(s,\chi_0)$ has a simple pole at $s = 1$ with residue $\frac{\varphi(q)}{q}$.

Next we consider additive characters. The Dirichlet series associated to the trivial character is the Riemann zeta function. Hence we consider a non-trivial additive character $f : \mathbb{Z} \to \mathbb{C}^*$ such that $|f(1)| \leq 1$. For such functions we prove the following theorems.

Theorem 2.2.4. *Let $f : \mathbb{Z} \to \mathbb{C}^*$ be a non-trivial additive character such that $|f(1)| \leq 1$. Then the associated Dirichlet series $D(s,f)$ satisfies the following translation formula:*

$$f(1) = (1 - f(1))D(s,f) + \sum_{k\geq 0} (s)_k \left(D(s+k+1,f) - f(1)\right), \tag{2.2.2}$$

where the series on the right hand side converges normally on any compact subset of $\Re(s) > 1$.

It is not difficult to see that for a non-trivial additive character $f : \mathbb{Z} \to \mathbb{C}^*$ such that $|f(1)| < 1$, the Dirichlet series $D(s,f)$ itself is an entire function. But for any non-trivial additive character $f : \mathbb{Z} \to \mathbb{C}^*$ such that $|f(1)| \leq 1$, Theorem 2.2.4 enables us to deduce the following theorem in general.

Theorem 2.2.5. *Let f be as in Theorem 2.2.4. Then using the translation formula (2.2.2), the Dirichlet series $D(s,f)$ can be extended to an entire function.*

2.3 Proof of the theorems

Now we give the proofs of the above mentioned theorems.

2.3.1 Proof of Theorem 2.2.1

We start with the following identity which is valid for any $n > q$ and $s \in \mathbb{C}$:

$$(n-q)^{1-s} - n^{1-s} = \sum_{k \geq 0} (s-1)_k \, q^{k+1} \, n^{-s-k}. \tag{2.3.1}$$

This identity is obtained by writing the left hand side as $n^{1-s}\left((1-\frac{q}{n})^{1-s} - 1\right)$ and expanding $(1-\frac{q}{n})^{1-s}$ as a Taylor series in $\frac{q}{n}$. Now by Lemma 2.1.3 (for $m = -2$), the family

$$\left((s-1)_k \frac{q^{k+1}}{n^{s+k}}\right)_{n>q, k\geq 0}$$

is normally summable on compact subsets of $\Re(s) > 1$. Now we multiply $f(n)$ to both the sides of (2.3.1) and sum for $n > q$ and $\Re(s) > 1$. By interchanging the summations we obtain (2.2.1). □

2.3.2 Proof of Theorem 2.2.2

To prove Theorem 2.2.2, we establish the analytic continuation of $(s-1)D(s,f)$ to the entire complex plane which takes the value $\frac{1}{q}\sum_{a=1}^{q} f(a)$ at $s = 1$. This is done recursively.

First we establish the analytic continuation to $\Re(s) > 0$, then to $\Re(s) > -1$ and so on. Since the half planes $\Re(s) > -m$, for $m \in \mathbb{N}$ form an open cover of $\mathbb{C}$, we will obtain the desired analytic continuation. Note that the left hand side of (2.2.1) is entire, and all but finitely many terms on the right hand side of (2.2.1) are holomorphic in any proper half plane of $\mathbb{C}$.

Analytic continuation to $\Re(s) > 0$

If $\Re(s) > 0$, then all the summands corresponding to $k \geq 1$ on the right hand side of (2.2.1) are holomorphic. Next note that by Lemma 2.1.3 (for $m = -1$), the family

$$\left((s-1)_k \frac{q^{k+1}}{n^{s+k}}\right)_{n>q, k\geq 1}$$

is normally summable on every compact subset of $\Re(s) > 0$. Hence the sum

$$\sum_{k\geq 1}(s-1)_k q^{k+1}\left(D(s+k,f) - \sum_{a=1}^{q}\frac{f(a)}{a^{(s+k)}}\right)$$

defines a holomorphic function on $\Re(s) > 0$. Thus by means of translation formula (2.2.1), we can extend $(s-1)D(s,f)$ as a holomorphic function on $\Re(s) > 0$. Note that for all $k \geq 1$, $D(s+k,f)$ is holomorphic in $\Re(s) > 0$. Hence from (2.2.1) we get

$$\lim_{s\to 1}(s-1)D(s,f) = \frac{1}{q}\sum_{a=1}^{q} f(a).$$

Analytic continuation to $\Re(s) > -1$ and so on

Now we establish the analytic continuation of $(s-1)D(s,f)$ to $\Re(s) > -(m+1)$ assuming that it has been analytically continued to $\Re(s) > -m$. Note that if $\Re(s) > -(m+1)$, then all the summands corresponding to $k \geq m+2$ on the right hand side of (2.2.1) are holomorphic. Again by Lemma 2.1.3, the family

$$\left((s-1)_k \frac{q^{k+1}}{n^{s+k}}\right)_{n>q,k\geq m+2}$$

is normally summable on every compact subset of $\Re(s) > -(m+1)$.

Now the analytic continuation of $(s-1)D(s,f)$ to $\Re(s) > -m$ implies the analytic continuation of $(s+k-1)D(s+k,f)$ to $\Re(s) > -(m+1)$ for all $1 \leq k \leq m+1$. Hence the sum

$$\sum_{k\geq 1}(s-1)_k q^{k+1}\left(D(s+k,f) - \sum_{a=1}^{q}\frac{f(a)}{a^{(s+k)}}\right)$$

defines a holomorphic function on $\Re(s) > -(m+1)$. Thus by means of translation formula (2.2.1), we can extend $(s-1)D(s,f)$ as a holomorphic function on $\Re(s) > -(m+1)$. $\square$

2.3.3 Proof of Theorem 2.2.4

To prove the Theorem 2.2.4, we need the following variant of Lemma 2.1.3.

Lemma 2.3.1. *Let $m \in \mathbb{R}$ and K be a compact subset of $\Re(s) > -(m+1)$. Then for integers $k \geq m+1$, the family*

$$\left((s)_k \, n^{-s-k-1}\right)_{n>1, k \geq m+1}$$

is normally summable in K.

Proof. This proof almost follows the argument presented in the proof of Lemma 2.1.3. Let $S := \sup_{s \in K} |s|$. Then for $n > 1$, we have

$$\left\| (s)_k \frac{1}{n^{s+k+1}} \right\|_K \leq \frac{(S)_k}{2^{k-m-1}} \left\| \frac{1}{n^{s+m+2}} \right\|_K .$$

Note that the series

$$\sum_{k \geq m+1} \frac{(S)_k}{2^{k-m-1}}$$

is convergent. This together with Lemma 2.1.2 completes the proof. □

Now we resume the proof of Theorem 2.2.4. Here we work with a variant of (2.1.2). The following identity is valid for any $n > 1$ and $s \in \mathbb{C}$:

$$(n-1)^{-s} - n^{-s} = \sum_{k \geq 0} (s)_k \, n^{-s-k-1}. \tag{2.3.2}$$

By Lemma 2.3.1 (for $m = -1$), we get that the family

$$\left((s)_k \, n^{-s-k-1}\right)_{n>1, k \geq 0}$$

is normally summable on every compact subset of $\Re(s) > 1$. In fact they are normally summable on every compact subset of $\Re(s) > 0$. Now we multiply $f(n)$ to both the sides of (2.3.2), and then sum for $n > 1$ and $\Re(s) > 1$. Since $f(n)$ can be written as $f(1)f(n-1)$, we obtain (2.2.2). □

2.3.4 Proof of Theorem 2.2.5

As in the proof of Theorem 2.2.2, here also we first establish the analytic continuation of $D(s,f)$ to $\Re(s) > 0$, then to $\Re(s) > -1$ and so on.

By Lemma 2.3.1 (for $m = -1$), we know that the family

$$\left((s)_k \, n^{-s-k-1}\right)_{n>1, k\geq 0}$$

is normally summable on every compact subset of $\Re(s) > 0$. Hence the sum

$$\sum_{k\geq 0} (s)_k \left(D(s+k+1, f) - f(1)\right)$$

defines a holomorphic function on the half plane $\Re(s) > 0$. Thus by means of the translation formula (2.2.2), we can extend $D(s,f)$ to $\Re(s) > 0$.

Next we establish the analytic continuation of $D(s,f)$ to $\Re(s) > -(m+1)$ assuming that it has been analytically continued to $\Re(s) > -m$, for $m \in \mathbb{N}$. Note that if $\Re(s) > -(m+1)$, then all the summands corresponding to $k \geq m+1$ on the right hand side of (2.2.2) are holomorphic. Further using Lemma 2.3.1, we see that the family

$$\left((s)_k \, n^{-s-k-1}\right)_{n>1, k\geq m+1}$$

is normally summable on every compact subset of $\Re(s) > -(m+1)$.

Now the analytic continuation of $D(s,f)$ to $\Re(s) > -m$ implies the analytic continuation of $D(s+k+1,f)$ to $\Re(s) > -(m+1)$ for all $0 \le k \le m$. Hence the sum

$$\sum_{k\ge 0}(s)_k\left(D(s+k+1,f)-f(1)\right)$$

defines a holomorphic function on the half plane $\Re(s) > -(m+1)$. Thus by means of the translation formula (2.2.2), we can extend $D(s,f)$ to $\Re(s) > -(m+1)$. This completes the proof, as the half planes of the form $\Re(s) > -m$ for $m \in \mathbb{N}$ cover $\mathbb{C}$. $\square$

2.4 Hurwitz zeta function and shifted Dirichlet series

One considers Hurwitz zeta function as another natural genralisation of the Riemann zeta function. For a complex parameter s and a real number $x \in (0,1]$, the Hurwitz zeta function is denoted by $\zeta(s,x)$ and defined by the following absolutely convergent sum for $\Re(s) > 1$:

$$\zeta(s,x) := \sum_{n\ge 0}\frac{1}{(n+x)^s}.$$

A. Hurwitz [17] proved that the above function can be extended analytically to the entire complex plane except at $s=1$, where it has simple pole with residue 1.

Now for an arithmetical function f such that its partial sums $F(y) := \sum_{n\le y} f(n)$ have polynomial growth i.e. $F(y) = O(y^\delta)$ for some positive real number δ, we can define the following analogue of Dirichlet series, which we denote by $D(s,x,f)$ and call it the shifted Dirichlet series associated to f. For $\Re(s) > \delta$ and $x \in (0,1]$,

$$D(s,x,f) := \sum_{n\ge 0}\frac{f(n)}{(n+x)^s}.$$

For the function $f(n) = 1$ for all n, we get back the Hurwitz zeta function.

For the shifted Dirichlet series associated to the arithmetical functions we have considered in this chapter before, we can prove the following analogous theorems. The proofs of these theorems are omitted, as they can be obtained by imitating the proofs of the theorems in the case of Dirichlet series. However, we indicate how to obtain the relevant translation formulas.

Theorem 2.4.1. *Let f be as in Theorem 2.2.1. Then the associated shifted Dirichlet series $D(s,x,f)$ satisfies the following translation formula:*

$$\sum_{a=0}^{q-1} \frac{f(a)}{(a+x)^{(s-1)}} = \sum_{k\geq 0}(s-1)_k q^{k+1}\left(D(s+k,x,f) - \sum_{a=0}^{q-1}\frac{f(a)}{(a+x)^{(s+k)}}\right), \tag{2.4.1}$$

where the infinite series on the right hand side converges normally on every compact subset of $\Re(s) > 1$.

To obtain this theorem we follow the proof of Theorem 2.2.1 starting with the following identity which is valid for any $n \geq q$, any $x \in (0,1]$ and $s \in \mathbb{C}$:

$$(n+x-q)^{1-s} - (n+x)^{1-s} = \sum_{k\geq 0}(s-1)_k\, q^{k+1}\,(n+x)^{-s-k}. \tag{2.4.2}$$

As a particular case of the above theorem, we get a translation formula for the Hurwitz zeta function. The meromorphic continuation of the Hurwitz zeta function and the shifted Dirichlet series $D(s,x,f)$ for periodic arithmetical function f, follows from Theorem 2.4.1.

Theorem 2.4.2. *Let f be as in Theorem 2.2.1. Then using the translation formula (2.4.1), the shifted Dirichlet series $D(s,x,f)$ can be analytically continued to the entire complex plane except at $s=1$, where the function has simple pole with residue $\frac{1}{q}\sum_{a=0}^{q-1} f(a)$. If $\sum_{a=0}^{q-1} f(a) = 0$, then $D(s,x,f)$ can be extended to an entire function.*

The shifted Dirichlet series $D(s,x,f)$ associated to an additive character $f : \mathbb{Z} \to \mathbb{C}^*$ is a specialization of the famous Lerch transcendent. The Lerch transcendent is denoted by $\Phi(z,s,x)$ and defined by the following convergent series

$$\Phi(z,s,x) := \sum_{n\geq 0}\frac{z^n}{(n+x)^s},$$

for $x \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and $|z| < 1$ with $s \in \mathbb{C}$ or $|z| = 1$ with $\Re(s) > 1$. Now for this type of shifted Dirichlet series $D(s, x, f)$ we can prove the following theorems.

Theorem 2.4.3. *Let f be as in Theorem 2.2.4. Then the associated shifted Dirichlet series $D(s, x, f)$ satisfies the following translation formula:*

$$\frac{1}{x^s} = (1 - f(1))D(s, x, f) + \sum_{k \geq 0} (s)_k \left(D(s + k + 1, x, f) - \frac{1}{x^{(s+k+1)}} \right), \tag{2.4.3}$$

where the series on the right hand side converges normally on any compact subset of $\Re(s) > 1$.

To obtain this theorem we follow the proof of Theorem 2.2.4 and we work with the following variant of (2.3.2). The identity is valid for any $n \geq 1, x \in (0, 1]$ and $s \in \mathbb{C}$:

$$(n + x - 1)^{-s} - (n + x)^{-s} = \sum_{k \geq 0} (s)_k \, (n + x)^{-s-k-1}. \tag{2.4.4}$$

This theorem enables us to deduce the following theorem about the analytic continuation of such shifted Dirichlet series $D(s, x, f)$.

Theorem 2.4.4. *Let f be as in Theorem 2.2.4. Then using the translation formula (2.4.3), the shifted Dirichlet series $D(s, x, f)$ can be extended to an entire function.*

Example 2.4.5. A prototypical shifted Dirichlet series of this kind is the Lerch zeta function, which is also a genralisation of the Hurwitz zeta function. These zeta functions were first considered by M. Lerch [21]. For real numbers $\lambda, \alpha \in (0, 1]$, the Lerch zeta function is denoted by $L(\lambda, \alpha, s)$ and defined by the following convergent sum in $\Re(s) > 1$:

$$L(\lambda, \alpha, s) := \sum_{n \geq 0} \frac{e(\lambda n)}{(n + \alpha)^s},$$

where for a real number a, $e(a)$ denotes the uni-modulus complex number $e^{2\pi \iota a}$. Lerch [21] showed that the meromorphic continuation of the Lerch zeta function $L(\lambda, \alpha, s)$ depends on the values of the parameter λ. If $\lambda = 1$, the Lerch zeta function has an analytic continuation to the whole complex plane except at $s = 1$, where it has simple pole with residue 1. If $\lambda \neq 1$, the Lerch zeta function can be extended to an entire function. These two assertions follow from Theorem 2.4.2 and Theorem 2.4.4 respectively.

3

Multiple zeta functions

3.1 Introduction

We begin this chapter by recalling the definition of the multiple zeta functions.

Definition 3.1.1. For an integer $r \geq 1$, the multiple zeta function of depth r, denoted by $\zeta_r(s_1, \ldots, s_r)$, is a function on U_r defined by

$$\zeta_r(s_1, \ldots, s_r) := \sum_{n_1 > \cdots > n_r > 0} n_1^{-s_1} \cdots n_r^{-s_r},$$

where

$$U_r := \{(s_1, \ldots, s_r) \in \mathbb{C}^r \ : \ \Re(s_1 + \cdots + s_i) > i \ \text{ for all } \ 1 \leq i \leq r\}.$$

When $r = 1$, the multiple zeta function of depth 1 is nothing but the Riemann zeta function, which is generally denoted by $\zeta(s)$, in place of $\zeta_1(s)$. The series

$$\sum_{n_1 > \cdots > n_r > 0} n_1^{-s_1} \cdots n_r^{-s_r}$$

converges normally on any compact subset of U_r and hence the function $(s_1, \ldots, s_r) \mapsto \zeta_r(s_1, \ldots, s_r)$ is holomorphic on U_r.

For any $r \geq 1$, the multiple zeta function of depth r can be extended meromorphically to $\mathbb{C}^r$. This was proved by J. Zhao [33] in 1999, using the theory of generalised functions. Later in 2001, a simpler proof was given by Akiyama, Egami and Tanigawa [1], where the coveted meromorphic continuation was obtained by applying the classical Euler-Maclaurin summation formula to the first index of the summation n_1. An alternate proof using Mellin-Barnes integrals was given by K. Matsumoto. In fact he applied this method to a number of variants of multiple zeta functions. A brief summary of his works can be found in [22, 23]. The most recent contribution to this topic is due to T. Onozuka [27] in 2013.

In this chapter we explain, to the best of our knowledge, the simplest proof of the meromorphic continuation of the multiple zeta functions. In fact, expanding on a remark of J. Ecalle [10], in [24] we prove that the meromorphic continuation of the multiple zeta functions follows from the following identity.

Theorem 3.1.2. *For any integer $r \geq 2$ and any $(s_1, \ldots, s_r) \in U_r$, we have*

$$\zeta_{r-1}(s_1 + s_2 - 1, s_3, \ldots, s_r) = \sum_{k \geq 0} (s_1 - 1)_k \, \zeta_r(s_1 + k, s_2, \ldots, s_r), \tag{3.1.1}$$

where the series on the right hand side converges normally on any compact subset of U_r and for any $k \geq 0$ and $s \in \mathbb{C}$,

$$(s)_k := \frac{s(s+1)\cdots(s+k)}{(k+1)!}.$$

We call (3.1.1) the translation formula for the multiple zeta function of depth $r \geq 2$. This is the several variable generalisation of the identity (2.1.1) proved by Ramanujan for the Riemann zeta function.

Note that the left hand side of (3.1.1), only involves the multiple zeta function of depth $(r-1)$, whereas the multiple zeta functions appearing on the right hand side are all translates by non-negative integers in the first variable of the multiple zeta function of depth r. This feature of this formula allows us, by induction on r, to meromorphically extend the multiple zeta function of depth r from U_r to $\mathbb{C}^r$ essentially in the same way as we did in case of the Riemann zeta function and its variants in Chapter 2.

It is very useful to view the translation formula (3.1.1) as the first of an infinite family of relations, each obtained successively by applying the translation $s_1 \mapsto s_1 + n$ for $n \geq 0$, to both the sides of (3.1.1). We express this infinite family of relations in terms of infinite matrices. This method allows us to write down explicitly the residues along the possible polar hyperplanes as certain matrix coefficients which in turn helps us to examine the non-vanishing of these residues.

3.2 Normal convergence of the multiple zeta functions

For the sake of completeness, we include the properties of normal convergence of the multiple zeta functions.

For a complex valued function f on a set X, let

$$\|f\|_X := \sup_{x \in X} |f(x)|.$$

Recall that we say that a family $(f_i)_{i \in I}$ of complex valued functions on X is normally summable if $\|f_i\|_X < \infty$ for all $i \in I$ and the family of real numbers $(\|f_i\|_X)_{i \in I}$ is summable. In this case, we also say that the series $\sum_{i \in I} f_i$ converges normally on X.

As a consequence of *Weierstrass M-test*, it is easy to show that normal convergence implies uniform convergence. Thus if X is an open subset of $\mathbb{C}^r$ and all the f_i's are holomorphic on X, then their sum is also holomorphic on X.

The normal convergence of the multiple zeta functions follows from the following proposition as an easy consequence.

Proposition 3.2.1. *Let $r \geq 1$ be an integer and $(\sigma_1, \ldots, \sigma_r)$ be an r-tuple of real numbers in U_r. Then the family of functions*

$$(n_1^{-s_1} \cdots n_r^{-s_r})_{n_1 > \cdots > n_r > 0} \tag{3.2.1}$$

is normally summable on $D(\sigma_1, \ldots, \sigma_r) := \{(s_1, \ldots, s_r) \in \mathbb{C}^r : \Re(s_i) > \sigma_i \text{ for } 1 \leq i \leq r\}$.

Proof. Note that for sequence of integers $n_1 > \cdots > n_r > 0$, we have

$$\|n_1^{-s_1} \cdots n_r^{-s_r}\|_{D(\sigma_1, \ldots, \sigma_r)} = n_1^{-\sigma_1} \cdots n_r^{-\sigma_r}. \tag{3.2.2}$$

Thus we have to prove that the family of real numbers

$$(n_1^{-\sigma_1} \cdots n_r^{-\sigma_r})_{n_1 > \cdots > n_r > 0}$$

is summable. We have $\sigma_1 > 1$ by definition of U_r. Hence our assertion is true for $r = 1$. If $r \geq 2$, then note that for every integer $n_2 \geq 1$, we have

$$\sum_{n_1 \geq n_2 + 1} n_1^{-\sigma_1} \leq \int_{n_2}^{\infty} x^{-\sigma_1} dx = \frac{1}{\sigma_1 - 1} n_2^{1 - \sigma_1}. \tag{3.2.3}$$

Since $(\sigma_1, \ldots, \sigma_r) \in U_r$, we get that $(\sigma_1 + \sigma_2 - 1, \sigma_3, \ldots, \sigma_r) \in U_{r-1}$. Thus our claim follows from induction on r. $\square$

We conclude this section with the following corollary.

Corollary 3.2.2. *For $r \geq 1$ is an integer, the series*

$$\sum_{n_1 > \cdots > n_r > 0} n_1^{-s_1} \cdots n_r^{-s_r},$$

converges normally on any compact subset of U_r.

Proof. Any point of U_r has a neighbourhood of the form $D(\sigma_1,\ldots,\sigma_r)$ with $(\sigma_1,\ldots,\sigma_r)$ in $U_r\cap\mathbb{R}^r$. To see this, let $(s_1,\ldots,s_r)\in U_r$. Set $\tau_i:=\Re(s_i)$ for all i. Since $(s_1,\ldots,s_r)\in U_r$, we have $\tau_1+\cdots+\tau_i>i$ for all $1\le i\le r$.

Hence it is possible to choose $\epsilon_1,\ldots,\epsilon_r$ such that

$$0<\epsilon_i<(\tau_1+\cdots+\tau_i)-i-(\epsilon_1+\cdots+\epsilon_{i-1})$$

for all $1\le i\le r$. For these choices of ϵ_i's, let us set $\sigma_i:=\tau_i-\epsilon_i$ for all $1\le i\le r$. Then $(\sigma_1,\ldots,\sigma_r)\in U_r\cap\mathbb{R}^r$ and $(s_1,\ldots,s_r)\in D(\sigma_1,\ldots,\sigma_r)$.

Now any compact subset K of U_r can be covered by finitely many open sets of the form $D(\sigma_1,\ldots,\sigma_r)$ and hence applying Proposition 3.2.1, we get the corollary. $\square$

3.3 The translation formula

In this section, we establish the translation formula (3.1.1) for the multiple zeta functions. As in the case of Riemann zeta function, we deduce it from the following identity which is valid for any integer $n\ge 2$ and any $s\in\mathbb{C}$:

$$(n-1)^{1-s}-n^{1-s}=\sum_{k\ge 0}(s-1)_k\,n^{-s-k}. \tag{3.3.1}$$

The following proposition is key to the interchange of summations involved in the proof of Theorem 3.1.2.

Proposition 3.3.1. *Let $r\ge 2$ be an integer. The family of functions*

$$\left((s_1-1)_k n_1^{-s_1-k}n_2^{-s_2}\cdots n_r^{-s_r}\right)_{n_1>\cdots>n_r>0,\ k\ge 0} \tag{3.3.2}$$

is normally summable on any compact subset of U_r.

Proof. Let K be a compact subset of U_r and set

$$S := \sup_{(s_1,\dots,s_r)\in K} |s_1 - 1|.$$

Since $r \geq 2$, for any strictly decreasing sequence $n_1, \dots, n_r$ of r positive integers, we have $n_1 \geq 2$. Hence for $k \geq 0$, we get

$$\left\|(s_1 - 1)_k \, n_1^{-s_1-k} n_2^{-s_2} \cdots n_r^{-s_r}\right\|_K \;\leq\; \frac{(S)_k}{2^k} \|n_1^{-s_1} n_2^{-s_2} \cdots n_r^{-s_r}\|_K.$$

By Corollary 3.2.2, we know that the family $(\|n_1^{-s_1} \cdots n_r^{-s_r}\|_K)_{n_1>\cdots>n_r>0}$ is summable. Further, by ratio test one can see that for any real number a, the series

$$\sum_{k\geq 0} \frac{(a)_k}{2^k}$$

is convergent. This completes the proof of Proposition 3.3.1. □

We are now ready to prove Theorem 3.1.2.

3.3.1 Proof of Theorem 3.1.2

If we sum the family (3.3.2) successively with respect to the variables $k, n_1, \dots, n_r$ and use identity (3.3.1) with (n, s) replaced by (n_1, s_1), we get the left hand side of (3.1.1). On the other hand, if we sum it successively with respect to the variables $n_1, \dots, n_r, k$, we get the right hand side of (3.1.1). As this interchange of summation is justified by Proposition 3.3.1, we obtain (3.1.1).

From Proposition 3.3.1, we also get that the series on the right hand side of (3.1.1) converges normally on any compact subset of U_r. This completes the proof of Theorem 3.1.2. □

3.4 Meromorphic continuation

Let $(f_i)_{i\in I}$ be a family of meromorphic functions on an open subset U of $\mathbb{C}^r$. We say that the series $\sum_{i\in I} f_i$ is *normally convergent on all compact subsets of* U if for any compact subset K of U, there exists a finite subset J of I such that each function f_i for $i \in I\setminus J$ is holomorphic in an open neighbourhood of K, and the family of functions $(f_i|K)_{i\in I\setminus J}$ is normally summable on K. The sum of the family $(f_i)_{i\in I}$ is then a well defined meromorphic function on U. This definition agrees with the one given before in case of holomorphic functions.

We now establish the meromorphic continuation of the multiple zeta function to $\mathbb{C}^r$.

Theorem 3.4.1. *For each integer $r \geq 2$, the multiple zeta function of depth r extends to a meromorphic function on $\mathbb{C}^r$ satisfying the translation formula*

$$\zeta_{r-1}(s_1+s_2-1, s_3, \ldots, s_r) = \sum_{k\geq 0} (s_1-1)_k\, \zeta_r(s_1+k, s_2, \ldots, s_r), \tag{3.4.1}$$

where the series of meromorphic functions on the right hand side converges normally on all compact subsets of $\mathbb{C}^r$.

Proof. We argue by induction on r. When $r = 2$, the left hand side of (3.4.1) is the Riemann zeta function and hence has a meromorphic continuation. For $r \geq 3$, the left hand side of (3.4.1) can be extended to a meromorphic function on $\mathbb{C}^r$ by induction hypothesis. Now for each integer $m \geq 0$, let

$$U_r(m) := \{(s_1, \ldots, s_r) \in \mathbb{C}^r \;:\; \Re(s_1 + \cdots + s_i) > i - m \;\text{ for }\; 1 \leq i \leq r\}.$$

We shall prove by induction on m that the multiple zeta function of depth r extends to a meromorphic function on $U_r(m)$ satisfying (3.4.1) and the series of meromorphic functions on the right hand side of (3.4.1) converges normally on all compact subsets of $U_r(m)$. Since $\{U_r(m)\}_{m\geq 0}$ is an open covering of $\mathbb{C}^r$, Theorem 3.4.1 will follow.

The case $m = 0$ is nothing but Theorem 3.1.2. Now suppose that $m \geq 1$ and our assertion is true for $(m-1)$. Then all terms of the series on the right hand side of (3.4.1), except possibly the first one, are meromorphic on $U_r(m)$. In fact, those corresponding to indices $k \geq m$ are even holomorphic on $U_r(m)$. It is therefore sufficient to prove that the series

$$\sum_{k\geq m} (s_1 - 1)_k \, \zeta_r(s_1 + k, s_2, \dots, s_r)$$

is normally convergent on any compact subset K of $U_r(m)$.

Here we follow the argument as in the proof of Proposition 3.3.1. We just have to note that if S is the supremum of $|s_1 - 1|$ in K, then for any strictly decreasing sequence $n_1, \dots, n_r$ of r positive integers and any integer $k \geq m$,

$$\left\| (s_1 - 1)_k \, n_1^{-s_1-k} n_2^{-s_2} \cdots n_r^{-s_r} \right\|_K$$

is bounded above by

$$\frac{(S)_k}{2^{k-m}} \left\| n_1^{-s_1-m} n_2^{-s_2} \cdots n_r^{-s_r} \right\|_K .$$

Now as $(s_1, \dots, s_r)$ varies over K, $(s_1 + m, s_2 \dots, s_r)$ varies over the compact subset of U_r which is obtained by translating K by $(m, 0, \dots, 0)$. Hence the family

$$\left(\left\| n_1^{-s_1-m} n_2^{-s_2} \cdots n_r^{-s_r} \right\|_K \right)_{n_1 > \cdots > n_r > 0}$$

is summable by Proposition 3.2.1. Finally, we note that the series $\sum_{k\geq m} \frac{(S)_k}{2^{k-m}}$ is convergent. This completes the proof of Theorem 3.4.1. $\square$

3.5 Matrix formulation of the translation formula

In this section we deal with infinite upper triangular matrices. Let R be a ring and $\mathbf{T}(R)$ denote the set of upper triangular matrices of type $\mathbb{N} \times \mathbb{N}$ with coefficients in R. Adding or multiplying such matrices involves only finite sums. Hence $\mathbf{T}(R)$ has a ring structure. The group of invertible elements of $\mathbf{T}(R)$ consists of the matrices whose diagonal elements are invertible. With the topology induced by the product topology on $R^{\mathbb{N}\times\mathbb{N}}$, where each factor is considered as a discrete space, $\mathbf{T}(R)$ becomes a topological ring. Now if M is a matrix in $\mathbf{T}(R)$ with all diagonal elements equal to 0, and $f = \sum_{n\geq 0} a_n x^n \in R[[x]]$ is a formal power series, then the series $\sum_{n\geq 0} a_n M^n$ converges in $\mathbf{T}(R)$ and its sum is denoted by $f(M)$. For our purpose we take R to be the field of rational fractions $\mathbb{Q}(t)$ in one indeterminate t over $\mathbb{Q}$.

As indicated before, we shall now give a matrix formulation of the translation formula (3.4.1). For this we set up some further notations. Let $M = (m_{ij})$ be a matrix of type $\mathbb{N}\times\mathbb{N}$, where for each $(i,j) \in \mathbb{N}\times\mathbb{N}$, m_{ij} is a meromorphic function on $\mathbb{C}^r$. Also let $u = (u_i)$, $v = (v_i)$ be column vectors with entries indexed by $\mathbb{N}$, where each u_i, v_i is a meromorphic function on $\mathbb{C}^r$. Then we write $Mu = v$ if for each $i \in \mathbb{N}$, the series of meromorphic functions $\sum_{j\in\ N} m_{ij}u_j$ converges normally on compact subsets of $\mathbb{C}^r$, and its sum is equal to v_i.

With these notations, the translation formula (3.4.1), together with the relations obtained by applying successively the change of variables $s_1 \mapsto s_1 + n$ for $n \geq 0$ in (3.4.1), are represented on $\mathbb{C}^r$ as follows:

$$\mathbf{V}_{r-1}(s_1+s_2-1, s_3, \ldots, s_r) = \mathbf{A_1}(s_1-1)\mathbf{V}_r(s_1,\ldots,s_r), \tag{3.5.1}$$

where $\mathbf{V}_r(s_1,\ldots,s_r)$ denotes the infinite column vector

$$\mathbf{V}_r(s_1,\ldots,s_r) := \begin{pmatrix} \zeta_r(s_1,s_2,\ldots,s_r) \\ \zeta_r(s_1+1,s_2,\ldots,s_r) \\ \zeta_r(s_1+2,s_2,\ldots,s_r) \\ \vdots \end{pmatrix}. \tag{3.5.2}$$

and for an indeterminate t, $\mathbf{A_1}(t)$ denotes the matrix in $\mathbf{T}(\mathbb{Q}(t))$ defined by

$$\mathbf{A_1}(t) := \begin{pmatrix} t & \frac{t(t+1)}{2!} & \frac{t(t+1)(t+2)}{3!} & \cdots \\ 0 & t+1 & \frac{(t+1)(t+2)}{2!} & \cdots \\ 0 & 0 & t+2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.5.3}$$

This matrix $\mathbf{A_1}(t)$ is invertible in $\mathbf{T}(\mathbb{Q}(t))$. To see this note that $\mathbf{A_1}(t)$ can be written as

$$\mathbf{A_1}(t) = \mathbf{U}(t)\boldsymbol{\Delta}(t) = \boldsymbol{\Delta}(t)\mathbf{U}(t+1), \tag{3.5.4}$$

where

$$\boldsymbol{\Delta}(t) = \begin{pmatrix} t & 0 & 0 & \cdots \\ 0 & t+1 & 0 & \cdots \\ 0 & 0 & t+2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \mathbf{U}(t) = \begin{pmatrix} 1 & \frac{t}{2!} & \frac{t(t+1)}{3!} & \cdots \\ 0 & 1 & \frac{t+1}{2!} & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.5.5}$$

Hence the matrix $\mathbf{A_1}(t)$ is invertible in $\mathbf{T}(\mathbb{Q}(t))$ and its inverse matrix $\mathbf{B_1}(t)$ is given by

$$\mathbf{B_1}(t) = \boldsymbol{\Delta}(t)^{-1}\mathbf{U}(t)^{-1} = \mathbf{U}(t+1)^{-1}\boldsymbol{\Delta}(t)^{-1}.$$

Next we note that the matrix $\mathbf{U}(t)$ can be written as

$$\mathbf{U}(t) = f(\mathbf{M}(t)),$$

where f is the formal power series

$$\frac{e^x - 1}{x} = \sum_{n \geq 0} \frac{x^n}{(n+1)!}$$

and

$$\mathbf{M}(t) = \begin{pmatrix} 0 & t & 0 & \cdots \\ 0 & 0 & t+1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.5.6}$$

Thus

$$\mathbf{U}(t)^{-1} = g(\mathbf{M}(t)),$$

where g is the exponential generating series of Bernoulli numbers

$$\frac{x}{e^x - 1} = \sum_{n \geq 0} \frac{B_n}{n!} x^n.$$

Thus we have

$$\mathbf{B_1}(t) = \mathbf{\Delta}(t)^{-1} g(\mathbf{M}(t)) = g(\mathbf{M}(t+1))\mathbf{\Delta}(t)^{-1},$$

or equivalently, we have

$$\mathbf{B_1}(t) = \begin{pmatrix} \frac{1}{t} & \frac{B_1}{1!} & \frac{(t+1)B_2}{2!} & \frac{(t+1)(t+2)B_3}{3!} & \cdots \\ 0 & \frac{1}{t+1} & \frac{B_1}{1!} & \frac{(t+2)B_2}{2!} & \cdots \\ 0 & 0 & \frac{1}{t+2} & \frac{B_1}{1!} & \cdots \\ 0 & 0 & 0 & \frac{1}{t+3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.5.7}$$

This suggests that one may attempt to express the column vector $\mathbf{V}_r(s_1, \ldots, s_r)$ in terms of $\mathbf{V}_{r-1}(s_1 + s_2 - 1, s_3, \ldots, s_r)$, so as to obtain an expression of the multiple zeta function of depth r in terms of the translates of the multiple zeta function of depth $(r-1)$, by multiplying both sides of (3.5.1) by $\mathbf{B_1}(s_1 - 1)$. However this is not allowed. The reason is that the product of infinite matrices $\mathbf{B_1}(s_1 - 1)\mathbf{V}_{r-1}(s_1 + s_2 - 1, s_3, \ldots, s_r)$ is not defined, as the entries of the formal product of the matrix $\mathbf{B_1}(s_1 - 1)$ with the column vector $\mathbf{V}_{r-1}(s_1 + s_2 - 1, s_3, \ldots, s_r)$

are not convergent series.

We get around this difficulty by writing (3.5.1) in the form

$$\mathbf{\Delta}(s_1-1)^{-1}\mathbf{V}_{r-1}(s_1+s_2-1,s_3,\ldots,s_r)=\mathbf{U}(s_1)\mathbf{V}_r(s_1,\ldots,s_r). \tag{3.5.8}$$

We then choose an integer $q\geq 1$ and define

$$I=I_q:=\{k\in\mathbb{N}:0\leq k\leq q-1\}\ \text{ and }\ J=J_q:=\{k\in\mathbb{N}:k\geq q\}.$$

This allows us to write the previous matrices as block matrices such as, for example

$$\mathbf{U}(s_1)=\begin{pmatrix}\mathbf{U}^{II}(s_1) & \mathbf{U}^{IJ}(s_1)\\ \mathbf{0}^{JI} & \mathbf{U}^{JJ}(s_1)\end{pmatrix}.$$

From (3.5.8) we then deduce that

$$\begin{aligned}&\mathbf{\Delta}^{II}(s_1-1)^{-1}\mathbf{V}^I_{r-1}(s_1+s_2-1,s_3,\ldots,s_r)\\ &=\mathbf{U}^{II}(s_1)\mathbf{V}^I_r(s_1,\ldots,s_r)+\mathbf{U}^{IJ}(s_1)\mathbf{V}^J_r(s_1,\ldots,s_r).\end{aligned} \tag{3.5.9}$$

Now $\mathbf{U}^{II}(s_1)$ is a finite square invertible matrix and we have

$$\mathbf{U}^{II}(s_1)^{-1}\mathbf{\Delta}^{II}(s_1-1)^{-1}=\mathbf{B_1}^{II}(s_1-1).$$

Hence from (3.5.9) we get that

$$\mathbf{V}^I_r(s_1,\ldots,s_r)=\mathbf{B_1}^{II}(s_1-1)\mathbf{V}^I_{r-1}(s_1+s_2-1,s_3,\ldots,s_r)+\mathbf{W}^I(s_1,\ldots,s_r), \tag{3.5.10}$$

where

$$\mathbf{W}^I(s_1,\ldots,s_r)=-\mathbf{U}^{II}(s_1)^{-1}\mathbf{U}^{IJ}(s_1)\mathbf{V}^J_r(s_1,\ldots,s_r). \tag{3.5.11}$$

All the series of meromorphic functions involved in the products of matrices in formulas (3.5.10) and (3.5.11) converge normally on all compact subsets of $\mathbb{C}^r$. Moreover, all entries of the matrices on the right hand side of (3.5.11) are holomorphic on the open set $U_r(q)$, translate of U_r by $(-q, 0, \dots, 0)$. Therefore the entries of $\mathbf{W}^I(s_1, \dots, s_r)$ are also holomorphic in $U_r(q)$.

If we write $\xi_q(s_1, \dots, s_r)$ to be the first entry of the column vector $\mathbf{W}^I(s_1, \dots, s_r)$, we then get from (3.5.10) that

$$\begin{aligned}\zeta_r(s_1, \dots, s_r) = {} & \frac{1}{s_1 - 1}\, \zeta_{r-1}(s_1 + s_2 - 1, s_3, \dots, s_r) \\ & + \sum_{k=0}^{q-2} \frac{s_1 \cdots (s_1 + k - 1)}{(k+1)!} B_{k+1}\, \zeta_{r-1}(s_1 + s_2 + k, s_3, \dots, s_r) \\ & + \xi_q(s_1, \dots, s_r),\end{aligned} \tag{3.5.12}$$

and ξ_q is holomorphic on the open set $U_r(q)$. In fact, this formula can also be obtained by using the Euler-Maclaurin summation formula. This has been done in [1]. Note that in (3.5.12), we get a more explicit remainder term.

3.6 Poles and residues

In this section, we shall recover the exact list of polar hyperplanes of the multiple zeta functions and write down the residues explicitly along these polar hyperplanes as certain matrix coefficients. We shall proceed by induction on r. When $r = 1$, it is well known that the Riemann zeta function has meromorphic continuation to $\mathbb{C}$ with simple pole at $s = 1$ with residue 1. So from now on we fix the depth $r \geq 2$ and we shall prove Theorem 3.6.1, Theorem 3.6.2 and Theorem 3.6.3 below by assuming that they hold for multiple zeta functions of smaller depths. For $1 \leq i \leq r$ and $k \geq 0$, we denote by $H_{i,k}$ the hyperplane of $\mathbb{C}^r$ defined by the equation $s_1 + \cdots + s_i = i - k$. It is disjoint from $U_r(q)$ when $q \leq k$.

3.6.1 Set of all possible singularities

In the following theorem, we give a tentative list of polar hyperplanes. This theorem was proved by Zhao [33] in 1999.

Theorem 3.6.1. *The multiple zeta function of depth r is holomorphic outside the union of the hyperplanes $H_{1,0}$ and $H_{i,k}$, where $2 \leq i \leq r$ and $k \geq 0$. It can have at most simple poles along these hyperplanes.*

Proof. Let $q \geq 1$ be an integer. We adopt the notations from previous section and in particular denote by I and J the sets $\{k \in \mathbb{N} : 0 \leq k \leq q-1\}$ and $\{k \in \mathbb{N} : k \geq q\}$ respectively. We will make use of equation (3.5.10) for our proof.

The entries of the first row of the matrix $\mathbf{B_1}^{II}(s_1 - 1)$ are holomorphic outside the hyperplane $H_{1,0}$ and have at most simple pole along this hyperplane. By the induction hypothesis, the entries of the column vector $\mathbf{V}^I_{r-1}(s_1 + s_2 - 1, s_3, \dots, s_r)$ are holomorphic outside the union of the hyperplanes $H_{i,k}$, where $2 \leq i \leq r$ and $k \geq 0$ and have at most simple poles along these hyperplanes. Finally, the entries of the column vector $\mathbf{W}^I(s_1, \dots, s_r)$ are holomorphic in $U_r(q)$. Since $\mathbb{C}^r$ is covered by the open sets $U_r(q)$ for $q \geq 1$, Theorem 3.6.1 follows. $\square$

3.6.2 Expression for residues

To check if each $H_{i,k}$ is indeed a polar hyperplane, we compute the residue of the multiple zeta function of depth r along this hyperplane. We define this residue to be the restriction of the meromorphic function $(s_1 + \cdots + s_i - i + k)\,\zeta_r(s_1, \dots, s_r)$ to $H_{i,k}$. This definition, while somewhat ad hoc, is the one generally used in the literature on multiple zeta functions.

Theorem 3.6.2. *The residue of the multiple zeta function of depth r along the hyperplane $H_{1,0}$ is the restriction of $\zeta_{r-1}(s_2, \dots, s_r)$ to $H_{1,0}$ and its residue along the hyperplane $H_{i,k}$, where*

$2 \le i \le r$ and $k \ge 0$, is the restriction to $H_{i,k}$ of the product of $\zeta_{r-i}(s_{i+1}, \dots, s_r)$ with the $(0,k)^{th}$ entry of the matrix $\prod_{d=1}^{i-1} \mathbf{B_1}(s_1 + \cdots + s_d - d)$.

Proof. Let $q \ge 1$ be an integer. As in the proof of Theorem 3.6.1, we deduce from (3.5.10) (or (3.5.12)) that

$$\zeta_r(s_1, \dots, s_r) - \frac{1}{s_1 - 1}\, \zeta_{r-1}(s_1 + s_2 - 1, s_3, \dots, s_r)$$

has no pole along $H_{1,0}$ inside the open set $U_r(q)$. These open sets cover $\mathbb{C}^r$. Hence the residue of $\zeta_r(s_1, \dots, s_r)$ along $H_{1,0}$ is the restriction to $H_{1,0}$ of the meromorphic function $\zeta_{r-1}(s_1 + s_2 - 1, s_3, \dots, s_r)$ or equivalently of $\zeta_{r-1}(s_2, \dots, s_r)$. This proves the first part of Theorem 3.6.2.

Now let i and k be integers with $2 \le i \le r$ and $k \ge 0$. Also let $q \in \mathbb{N}$ be such that $q > k$. Now if one iterates $(i-1)$ times the formula (3.5.10), one gets

$$\begin{aligned}\mathbf{V}_r^I(s_1, \dots, s_r) = &\left(\prod_{d=1}^{i-1} \mathbf{B_1}^{II}(s_1 + \cdots + s_d - d)\right) \mathbf{V}_{r-i+1}^I(s_1 + \cdots + s_i - i + 1, s_{i+1}, \dots, s_r) \\ &+ \mathbf{W}^{i,I}(s_1, \dots, s_r),\end{aligned}$$

where $\mathbf{W}^{i,I}(s_1, \dots, s_r)$ is a column matrix whose entries are finite sums of products of rational functions in $s_1, \dots, s_{i-1}$ with meromorphic functions which are holomorphic in $U_r(q)$. These entries therefore have no pole along the hyperplane $H_{i,k}$ in $U_r(q)$. The entries of

$$\prod_{d=1}^{i-1} \mathbf{B_1}^{II}(s_1 + \cdots + s_d - d)$$

are rational functions in $s_1, \dots, s_{i-1}$ and hence have no poles along $H_{i,k}$. It now follows from the induction hypothesis that the only entry of $\mathbf{V}_{r-i+1}^I(s_1 + \cdots + s_i - i + 1, s_{i+1}, \dots, s_r)$ that can possibly have a pole along $H_{i,k}$ in $U_r(q)$ is the one of index k, which is

$$\zeta_{r-i+1}(s_1 + \ldots + s_i - i + k + 1, s_{i+1}, \ldots, s_r).$$

Its residue is the restriction of $\zeta_{r-i}(s_{i+1}, \ldots, s_r)$ to $H_{i,k} \cap U_r(q)$. Since the open sets $U_r(q)$ for $q > k$ cover $\mathbb{C}^r$, the residue of $\zeta_r(s_1, \ldots, s_r)$ along $H_{i,k}$ is the restriction to $H_{i,k}$ of the product of the $(0,k)^{\text{th}}$ entry of the matrix $\prod_{d=1}^{i-1} \mathbf{B_1}(s_1 + \cdots + s_d - d)$ with $\zeta_{r-i}(s_{i+1}, \ldots, s_r)$. This proves the last part of Theorem 3.6.2. □

3.6.3 Exact set of singularities

We shall now deduce the exact list of poles from Theorem 3.6.2. The exact set of poles of the multiple zeta function of depth r (with a proof for $r = 2$) were mentioned by Akiyama, Egami and Tanigawa (see [1], Theorem 1 for details). But the residues were not determine explicitly in their work.

Theorem 3.6.3. *The multiple zeta function of depth r has simple pole along the hyperplane $H_{1,0}$. It also has simple poles along the hyperplanes $H_{i,k}$, for $2 \leq i \leq r$ and $k \geq 0$, except when $i = 2$ and $k \geq 3$ is an odd integer.*

Proof. When $1 \leq i \leq r$ and $k \geq 0$, the restriction to $H_{i,k}$ of $\zeta_{r-i}(s_{i+1}, \ldots, s_r)$ is a non-zero meromorphic function. Hence in order to prove Theorem 3.6.3 we need to show that when $2 \leq i \leq r$ and $k \geq 0$, the $(0,k)^{\text{th}}$ entry of the matrix $\prod_{d=1}^{i-1} \mathbf{B_1}(s_1 + \cdots + s_d - d)$ is identically zero if and only if $i = 2$, $k \geq 3$ is odd. By changing co-ordinates, the above statement is equivalent to say that when $t_1, \ldots, t_{i-1}$ are indeterminate, the $(0,k)^{\text{th}}$ entry of the matrix $\prod_{d=1}^{i-1} \mathbf{B_1}(t_d)$ is non-zero in $\mathbb{Q}(t_1, \ldots, t_{i-1})$ except when $i = 2$ and $k \geq 3$ is an odd integer.

We complete the proof by induction on i. For $i = 2$, our matrix is $\mathbf{B_1}(t_1)$ and hence our assertion follows from the fact that the Bernoulli numbers B_k are non-zero except when $k \geq 3$ is an odd integer.

Now assume that $i \geq 3$. The entries of the first row of the matrix $\prod_{d=1}^{i-2} \mathbf{B_1}(t_d)$ belong to $\mathbb{Q}(t_1, \dots, t_{i-2})$. The first two of them are not equal to zero, by above discussion when $i = 3$ and by the induction hypothesis when $i \geq 4$.

The entries of the k-th column of $\mathbf{B_1}(t_{i-1})$ belong to $\mathbb{Q}(t_{i-1})$ and the non-zero entries are linearly independent over $\mathbb{Q}$, as can be seen on formula (3.5.7), hence also over $\mathbb{Q}(t_1, \dots, t_{i-2})$. At least one of the first two entries in this column is not equal to zero. This implies that the $(0,k)^{\text{th}}$ entry of $\prod_{d=1}^{i-1} \mathbf{B_1}(t_d)$ is a non-zero element of $\mathbb{Q}(t_1, \dots, t_{i-1})$. This completes the proof of Theorem 3.6.3. □

Remark 3.6.4. Theorem 3.6.2 implies that, when $1 \leq i \leq r$ and $k \geq 0$, the meromorphic function $\zeta_r(s_1, \dots, s_r) - \zeta_i(s_1, \dots, s_i)\, \zeta_{r-i}(s_{i+1}, \dots, s_r)$ has no pole along the hyperplane $H_{i,k}$. This in fact follows from Theorem 3.6.1. The function $\zeta_i(s_1, \dots, s_i)\, \zeta_{r-i}(s_{i+1}, \dots, s_r)$ can be expressed as the sum, over all possible stufflings of i and $(r-i)$, of $\zeta_l(\mathbf{z}(s_1, \dots, s_r))$, where $\mathbf{z}(s_1, \dots, s_r)$ is the sequence of complex numbers deduced from $(s_1, \dots, s_i)$ and $(s_{i+1}, \dots, s_r)$ by the chosen stuffling and l is the length of the stuffling. Moreover, Theorem 3.6.1 implies that the meromorphic function $\zeta_l(\mathbf{z}(s_1, \dots, s_r))$ has no pole along the hyperplane $H_{i,k}$, except for the unique stuffling for which $\mathbf{z}(s_1, \dots, s_r) = (s_1, \dots, s_r)$.

Remark 3.6.5. By Theorem 3.6.2 we get that, for $r \geq 1$ and $k \geq 0$, the residue of $\zeta_r(s_1, \dots, s_r)$ along the hyperplane $H_{r,k}$ is a rational function of the variables $s_1, \dots, s_{r-1}$. More precisely, this rational function can be written as

$$\sum_{\substack{k_1, \dots, k_{r-1} \geq 0 \\ k_1 + \cdots + k_{r-1} = k}} F_{k_1, \dots, k_{r-1}}(s_1, \dots, s_{r-1}) \frac{B_{k_1}}{k_1!} \cdots \frac{B_{k_{r-1}}}{k_{r-1}!}, \tag{3.6.1}$$

where the B_i's are the Bernoulli numbers, the rational functions $F_{k_1, \dots, k_{r-1}}$ are defined by

$$F_{k_1, \dots, k_{r-1}}(s_1, \dots, s_{r-1}) := \prod_{1 \leq i \leq r-1} G_{k_i}(s_1 + \cdots + s_i - i + k_1 + \cdots + k_{i-1}), \tag{3.6.2}$$

and the rational fractions $G_j(t)$ in one indeterminate t are defined by

$$G_0(t) = t^{-1} \text{ and } G_j(t) = (t+1)\cdots(t+j-1) \text{ if } j \geq 1. \tag{3.6.3}$$

The rational fractions $G_j(t)$, for $j \geq 0$, are linearly independent over $\mathbb{Q}$. Hence it follows from (3.6.2) that the rational functions $F_{k_1,\ldots,k_{r-1}}$ in the $(r-1)$ variables $s_1, \ldots, s_{r-1}$, where $k_1, \ldots, k_{r-1} \geq 0$, are linearly independent over $\mathbb{Q}$. Hence, Theorem 3.6.3 can also be deduced from Remark 3.6.4, formula (3.6.1) and the following observations :

1. When $r \geq 3$, any integer $k \geq 0$ can be written as $k_1 + \cdots + k_{r-1}$, where $k_i \geq 0$ and $B_{k_i} \neq 0$ for $1 \leq i \leq r-1$.

2. When $r = 2$, the same is true except when k is an odd integer ≥ 3.

4

Multiple Hurwitz zeta functions

4.1 Introduction

In this chapter, we discuss the analytic properties of the multiple Hurwitz zeta functions which are several variable generalisations of the classical Hurwitz zeta functions. These functions were introduced by S. Akiyama and H. Ishikawa [2] in 2002. Some of the notations that have been introduced earlier will be used here as well.

Definition 4.1.1. Let $r \geq 1$ be an integer and $\alpha_1, \ldots, \alpha_r \in [0, 1)$. The multiple Hurwitz zeta function of depth r is denoted by $\zeta_r(s_1, \ldots, s_r;\ \alpha_1, \ldots, \alpha_r)$ and defined by the following convergent series in U_r:

$$\zeta_r(s_1, \ldots, s_r;\ \alpha_1, \ldots, \alpha_r) := \sum_{n_1 > \cdots > n_r > 0} (n_1 + \alpha_1)^{-s_1} \cdots (n_r + \alpha_r)^{-s_r}.$$

Normal convergence of the above series follows from the normal convergence of the multiple zeta function of depth r and we have the following proposition.

Proposition 4.1.2. *Let $r \geq 1$ be an integer and $\alpha_1, \ldots, \alpha_r$ be non-negative real numbers.*

Then the family of functions

$$\big((n_1+\alpha_1)^{-s_1}\cdots(n_r+\alpha_r)^{-s_r}\big)_{n_1>\cdots>n_r>0}$$

is normally summable on compact subsets of U_r*.*

Hence $(s_1,\ldots,s_r)\mapsto\zeta_r(s_1,\ldots,s_r;\ \alpha_1,\ldots,\alpha_r)$ defines a holomorphic function on U_r. Following the method described in [1], Akiyama and Ishikawa [2] established its meromorphic continuation to $\mathbb{C}^r$. They also obtained a list of all possible singularities and determined the exact set of singularities in some specific cases. Later J.P. Kelliher and R. Masri [19], following the work of Zhao [33], wrote down expressions for residues along these possible polar hyperplanes but could not conclude their non-vanishing. Another proof for the meromorphic continuation of these multiple zeta functions, using the binomial theorem and Hartogs' theorem, can be found in the work of M. R. Murty and K. Sinha [25].

In this chapter, we determine the exact set of singularities of the multiple Hurwitz zeta functions. In this process, we also obtain their meromorphic continuation following the method of Ramanujan and Ecalle.

In [24], we exhibited how one can obtain a translation formula for the multiple Hurwitz zeta function. Following that indication and the proof in case of the multiple zeta functions it is not difficult to prove the following theorem.

Theorem 4.1.3. *For any integer* $r\geq 2$, $\alpha_1,\ldots,\alpha_r\in[0,1)$ *and all* $(s_1,\ldots,s_r)$ *in the open set* U_r*, we have*

$$\begin{aligned}&\zeta_{r-1}(s_1+s_2-1,s_3,\ldots,s_r;\alpha_2,\alpha_3,\ldots,\alpha_r)\\&=\sum_{k\geq 0}(s_1-1)_k\big((1+\alpha_1-\alpha_2)^{k+1}-(\alpha_1-\alpha_2)^{k+1}\big)\zeta_r(s_1+k,s_2,\ldots,s_r;\alpha_1,\alpha_2,\ldots,\alpha_r),\end{aligned}$$

where the series on the right hand side converges normally on any compact subset of U_r*.*

But for our purpose we prove the following translation formula for the multiple Hurwitz zeta functions. This particular translation formula has the advantage that its matrix formulation is somewhat similar to the one for the multiple zeta functions.

Theorem 4.1.4. *For any integer $r \geq 2$ and $\alpha_1, \ldots, \alpha_r \in [0, 1)$, the multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ satisfies the following translation formula in U_r:*

$$\begin{aligned}&\sum_{k \geq -1} (s_1 - 1)_k \, (\alpha_2 - \alpha_1)^{k+1} \, \zeta_{r-1}(s_1 + s_2 + k, s_3, \ldots, s_r; \alpha_2, \alpha_3, \ldots, \alpha_r) \\ &= \sum_{k \geq 0} (s_1 - 1)_k \, \zeta_r(s_1 + k, s_2, \ldots, s_r; \alpha_1, \alpha_2, \ldots, \alpha_r),\end{aligned} \tag{4.1.1}$$

where the series on both the sides of (4.1.1) *converge normally on any compact subset of U_r.*

From now on, we will call (4.1.1) the translation formula for the multiple Hurwitz zeta function of depth $r \geq 2$. The left hand side of (4.1.1) involves the multiple Hurwitz zeta function of depth $(r - 1)$ and its translates by integers $k \geq -1$ in the first co-ordinate, and the right hand side of (4.1.1) comprises of the translates of the multiple Hurwitz zeta function of depth r by non-negative integers in the first variable. As in the case of the multiple zeta functions, here also we use induction on r and the translation formula (4.1.1) to extend the multiple Hurwitz zeta function of depth r to $\mathbb{C}^r$ meromorphically.

Next we obtain a matrix formulation of the translation formula (4.1.1) so that we can write down an expression for residues along the possible polar hyperplanes, from which we are going to determine the exact set of singularities of the multiple Hurwitz zeta functions.

4.2 The translation formula

In this section we prove the translation formula (4.1.1) for the multiple Hurwitz zeta functions. To begin with we need the following two identities. For any integer $n \geq 2$, $\alpha \in \mathbb{R}$ with $\alpha \geq 0$

and $s \in \mathbb{C}$, one has

$$(n+\alpha-1)^{1-s} - (n+\alpha)^{1-s} = \sum_{k\geq 0}(s-1)_k(n+\alpha)^{-s-k}. \tag{4.2.1}$$

We prove this identity by writing

$$(n+\alpha-1)^{1-s} - (n+\alpha)^{1-s} = (n+\alpha)^{1-s}\left(\left(1-\frac{1}{n+\alpha}\right)^{1-s} - 1\right)$$

and then expanding $(1-\frac{1}{n+\alpha})^{1-s}$ as a Taylor series in $\frac{1}{n+\alpha}$.

The next identity is valid for any natural number $n \geq 1$, $s \in \mathbb{C}$ $\alpha, \beta \in \mathbb{R}$ with $\alpha, \beta \geq 0$ and $|\alpha - \beta| < 1$. We have

$$(n+\alpha)^{1-s} = \sum_{k\geq -1}(s-1)_k(\beta-\alpha)^{k+1}(n+\beta)^{-s-k}. \tag{4.2.2}$$

The proof follows by writing the left hand side as $(n+\beta)^{1-s}\left(1-\frac{\beta-\alpha}{n+\beta}\right)^{1-s}$ and then expanding $\left(1-\frac{\beta-\alpha}{n+\beta}\right)^{1-s}$ as a Taylor series in $\frac{\beta-\alpha}{n+\beta}$.

We now prove the following propositions which are needed to justify the interchange of summations involved in the proof of Theorem 4.1.4.

Proposition 4.2.1. *Let $r \geq 2$ be an integer and $\alpha_1, \ldots, \alpha_r$ be non-negative real numbers. Then the family of functions*

$$\left((s_1-1)_k(n_1+\alpha_1)^{-s_1-k}(n_2+\alpha_2)^{-s_2}\cdots(n_r+\alpha_r)^{-s_r}\right)_{n_1>\cdots>n_r>0,\ k\geq 0}$$

is normally summable on any compact subset of U_r.

Proof. This proposition is an immediate consequence of Proposition 3.3.1. □

Proposition 4.2.2. *Let $r \geq 2$ be an integer and $\alpha_1, \ldots, \alpha_r$ be non-negative real numbers such* *that* $|\alpha_1 - \alpha_2| < 1$*. Then the family of functions*

$$\left((s_1-1)_k(\alpha_2-\alpha_1)^{k+1}(n_2+\alpha_2)^{-s_1-s_2-k}(n_3+\alpha_3)^{-s_3}\cdots(n_r+\alpha_r)^{-s_r}\right)_{n_2>\cdots>n_r>0,\ k\geq -1}$$

is normally summable on any compact subset of U_r*.*

Proof. Let K be a compact subset of U_r and

$$S := \sup_{(s_1,\ldots,s_r)\in K} |s_1 - 1|.$$

Then for $k \geq -1$,

$$\begin{aligned}&\left\|(s_1-1)_k(\alpha_2-\alpha_1)^{k+1}(n_2+\alpha_2)^{-s_1-s_2-k}(n_3+\alpha_3)^{-s_3}\cdots(n_r+\alpha_r)^{-s_r}\right\|_K\\ &\leq (S)_k\,|\alpha_2-\alpha_1|^{k+1}\,\|n_2^{-s_1-s_2-k}n_3^{-s_3}\cdots n_r^{-s_r}\|_K\end{aligned}$$

Now since $|\alpha_1 - \alpha_2| < 1$, using normal convergence of the multiple zeta function of depth $(r-1)$, we get the desired result. □

We are now ready to prove Theorem 4.1.4.

4.2.1 Proof of Theorem 4.1.4

We first replace n, α, s by n_1, α_1, s_1 in (4.2.1) and then multiply $(n_2+\alpha_2)^{-s_2}\cdots(n_r+\alpha_r)^{-s_r}$ to both the sides of (4.2.1) and obtain that

$$\begin{aligned}&\left((n_1+\alpha_1-1)^{1-s_1}-(n_1+\alpha_1)^{1-s_1}\right)(n_2+\alpha_2)^{-s_2}\cdots(n_r+\alpha_r)^{-s_r}\\ &=\sum_{k\geq 0}(s_1-1)_k(n_1+\alpha_1)^{-s_1-k}(n_2+\alpha_2)^{-s_2}\cdots(n_r+\alpha_r)^{-s_r}.\end{aligned} \tag{4.2.3}$$

Now we sum both the sides of (4.2.3) for $n_1 > \cdots > n_r > 0$. Using Proposition 4.2.1, we get

$$\begin{aligned} &\sum_{n_2 > \cdots > n_r > 0} (n_2 + \alpha_1)^{1-s_1} (n_2 + \alpha_2)^{-s_2} \cdots (n_r + \alpha_r)^{-s_r} \\ &= \sum_{k \geq 0} (s_1 - 1)_k \, \zeta_r(s_1 + k, s_2, \ldots, s_r; \alpha_1, \alpha_2, \ldots, \alpha_r). \end{aligned} \tag{4.2.4}$$

To evaluate the left hand side of the above expression, we use (4.2.2) with n, α, β, s replaced by $n_2, \alpha_1, \alpha_2, s_1$ respectively and then appeal to Proposition 4.2.2. With the interchange of summations being justified by Proposition 4.2.2, we obtain

$$\begin{aligned} &\sum_{k \geq -1} (s_1 - 1)_k \, (\alpha_2 - \alpha_1)^{k+1} \, \zeta_{r-1}(s_1 + s_2 + k, s_3, \ldots, s_r; \alpha_2, \alpha_3, \ldots, \alpha_r) \\ &= \sum_{k \geq 0} (s_1 - 1)_k \, \zeta_r(s_1 + k, s_2, \ldots, s_r; \alpha_1, \alpha_2, \ldots, \alpha_r). \end{aligned}$$

This together with Proposition 4.2.1 and Proposition 4.2.2, completes the proof of Theorem 4.1.4. □

4.3 Meromorphic continuation

In this section, we establish the meromorphic continuation of multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ to $\mathbb{C}^r$.

Theorem 4.3.1. *For integer $r \geq 2$, the multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ extends to a meromorphic function on $\mathbb{C}^r$ satisfying the translation formula*

$$\begin{aligned} &\sum_{k \geq -1} (s_1 - 1)_k \, (\alpha_2 - \alpha_1)^{k+1} \, \zeta_{r-1}(s_1 + s_2 + k, s_3, \ldots, s_r; \alpha_2, \alpha_3, \ldots, \alpha_r) \\ &= \sum_{k \geq 0} (s_1 - 1)_k \, \zeta_r(s_1 + k, s_2, \ldots, s_r; \alpha_1, \alpha_2, \ldots, \alpha_r), \end{aligned} \tag{4.3.1}$$

where both the above series of meromorphic functions converge normally on all compact sub-

sets of $\mathbb{C}^r$.

To complete the proof, we need the following propositions which can be viewed as extensions of Proposition 4.2.1 and Proposition 4.2.2 respectively. We start by recalling the following notation. For an integer $m \geq 0$,

$$U_r(m) := \{(s_1, \ldots, s_r) \in \mathbb{C}^r \ : \ \Re(s_1 + \cdots + s_i) > i - m \ \text{ for all } \ 1 \leq i \leq r\}.$$

Proposition 4.3.2. *Let $r \geq 2$ be an integer and $\alpha_1, \ldots, \alpha_r$ be non-negative real numbers. Then the family of functions*

$$\left((s_1 - 1)_k (n_1 + \alpha_1)^{-s_1-k} (n_2 + \alpha_2)^{-s_2} \cdots (n_r + \alpha_r)^{-s_r}\right)_{n_1 > \cdots > n_r > 0,\ k \geq m}$$

is normally summable on any compact subset of $U_r(m)$.

Proof. Note that if $(s_1, \ldots, s_r) \in U_r(m)$, then $(s_1 + m, s_2, \ldots, s_r) \in U_r$. Now the proof follows from the proof of Proposition 4.2.1 (or Proposition 3.3.1). □

Proposition 4.3.3. *Let $r \geq 2$ be an integer and $\alpha_1, \ldots, \alpha_r$ be non-negative real numbers such that $|\alpha_1 - \alpha_2| < 1$. Then the family of functions*

$$\left((s_1 - 1)_k (\alpha_2 - \alpha_1)^{k+1} (n_2 + \alpha_2)^{-s_1-s_2-k} (n_3 + \alpha_3)^{-s_3} \cdots (n_r + \alpha_r)^{-s_r}\right)_{n_2 > \cdots > n_r > 0,\ k \geq m-1}$$

is normally summable on any compact subset of $U_r(m)$.

Proof. Note that if $(s_1, \ldots, s_r) \in U_r(m)$, then $(s_1 + s_2 - 1, s_3, \ldots, s_r) \in U_{r-1}(m)$ and hence $(s_1 + s_2 + m - 1, s_3 \ldots, s_r) \in U_{r-1}$. Now the proof can be completed following the proof of Proposition 4.2.2. □

4.3.1 Proof of Theorem 4.3.1

We first show by induction on depth r that $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ can be extended meromorphically to $U_r(1)$ satisfying (4.3.1). We then extend it to $U_r(m)$ for all $m \geq 1$, by induction. This will complete the proof as $\{U_r(m) : m \geq 1\}$ is an open cover of $\mathbb{C}^r$.

When $r = 2$, all the terms on the left hand side of (4.3.1) are translates of the Hurwitz zeta function and hence have a meromorphic continuation to $\mathbb{C}^2$. If $r \geq 3$, then by induction hypothesis all the terms in the left hand side of (4.3.1) have a meromorphic continuation to $\mathbb{C}^r$.

In fact, for any integer $r \geq 2$ and $m \geq 0$, all the summands for $k \geq m - 1$ on the left hand side of (4.3.1) are holomorphic in $U_r(m)$. Hence by applying Proposition 4.3.3, we can now extend the left hand side of (4.3.1) as a meromorphic function to $U_r(m)$. Since $U_r(m)$ for $m \geq 0$ form an open cover of $\mathbb{C}^r$, we see that the left hand side of (4.3.1) extends to a meromorphic function on $\mathbb{C}^r$.

Now by Theorem 4.1.4, the identity (4.3.1) is valid on U_r. We have already proved that the left hand side of (4.3.1) extends to a meromorphic function on $\mathbb{C}^r$. Applying Proposition 4.3.2, we see that the series

$$\sum_{k \geq 1} (s_1 - 1)_k \, \zeta_r(s_1 + k, s_2, \ldots, s_r; \alpha_1, \alpha_2, \ldots, \alpha_r)$$

defines a holomorphic function on $U_r(1)$. Hence $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ can be extended meromorphically to $U_r(1)$, by means of (4.3.1).

We now assume that for integers $m \geq 2$, the function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ has been extended meromorphically to $U_r(m-1)$ and it satisfies (4.3.1). Again by Proposition 4.3.2, the series

$$\sum_{k \geq m} (s_1 - 1)_k \, \zeta_r(s_1 + k, s_2, \ldots, s_r; \alpha_1, \alpha_2, \ldots, \alpha_r)$$

defines a holomorphic function on $U_r(m)$. As $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ has been extended meromorphically to $U_r(m-1)$, we can extend $\zeta_r(s_1 + k, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ meromorphically to $U_r(m)$ for all $1 \le k \le m-1$.

Thus we can now extend $\zeta_r(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r)$ meromorphically to $U_r(m)$ by means of (4.3.1). This completes the proof as the open sets $U_r(m)$ for $m \ge 0$ form an open cover of $\mathbb{C}^r$. $\square$

4.4 Matrix formulation of the translation formula

As in Chapter 3, we now write down the matrix formulation of the translation formula (4.3.1). The translation formula (4.3.1) together with the set of relations obtained by applying successively the change of variable $s_1 \mapsto s_1+n$ for $n \ge 1$ to (4.3.1), can be written as

$$\begin{aligned}&\mathbf{A_2}(\alpha_2-\alpha_1;s_1-1)\mathbf{V}_{r-1}(s_1+s_2-1,s_3,\dots,s_r;\alpha_2,\dots,\alpha_r)\\&=\mathbf{A_1}(s_1-1)\mathbf{V}_r(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r).\end{aligned}\tag{4.4.1}$$

Here for an indeterminate t, we have

$$\mathbf{A_1}(t) := \begin{pmatrix} t & \frac{t(t+1)}{2!} & \frac{t(t+1)(t+2)}{3!} & \cdots \\ 0 & t+1 & \frac{(t+1)(t+2)}{2!} & \cdots \\ 0 & 0 & t+2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4.4.2}$$

$$\mathbf{A_2}(\alpha_2-\alpha_1;t) := \begin{pmatrix} 1 & t(\alpha_2-\alpha_1) & \frac{t(t+1)}{2!}(\alpha_2-\alpha_1)^2 & \cdots \\ 0 & 1 & (t+1)(\alpha_2-\alpha_1) & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.4.3}$$

and

$$\mathbf{V}_r(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r) := \begin{pmatrix} \zeta_r(s_1,s_2,\ldots,s_r;\alpha_1,\ldots,\alpha_r) \\ \zeta_r(s_1+1,s_2,\ldots,s_r;\alpha_1,\ldots,\alpha_r) \\ \zeta_r(s_1+2,s_3\ldots,s_r;\alpha_1,\ldots,\alpha_r) \\ \vdots \end{pmatrix}. \tag{4.4.4}$$

Note that the matrix $\mathbf{A_1}(t)$ has also featured in the previous chapter. Thus we have that

$$\mathbf{A_1}(t) = \mathbf{\Delta}(t) f(\mathbf{M}(t+1)),$$

where f is the formal power series

$$f(x) := \frac{e^x - 1}{x} = \sum_{n\geq 0} \frac{x^n}{(n+1)!},$$

and $\mathbf{\Delta}(t), \mathbf{M}(t)$ are as in Chapter 3, i.e.

$$\mathbf{\Delta}(t) = \begin{pmatrix} t & 0 & 0 & \cdots \\ 0 & t+1 & 0 & \cdots \\ 0 & 0 & t+2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \mathbf{M}(t) = \begin{pmatrix} 0 & t & 0 & \cdots \\ 0 & 0 & t+1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to see that $\mathbf{\Delta}(t), \mathbf{M}(t)$ satisfy the following commuting relation:

$$\mathbf{\Delta}(t)\mathbf{M}(t+1) = \mathbf{M}(t)\mathbf{\Delta}(t). \tag{4.4.5}$$

Thus using (4.4.5), we have

$$\mathbf{A_1}(t) = f(\mathbf{M}(t))\mathbf{\Delta}(t).$$

Further, it is also possible to write that

$$\mathbf{A_2}(\alpha_2-\alpha_1;t) = h(\mathbf{M}(t)),$$

where h denotes the power series

$$e^{(\alpha_2-\alpha_1)x} = \sum_{n\geq 0}(\alpha_2-\alpha_1)^n\frac{x^n}{n!}.$$

Clearly the matrix $\mathbf{A_2}(\alpha_2-\alpha_1;t)$ is invertible and we see that

$$\mathbf{A_2}(\alpha_2-\alpha_1;t)^{-1}\mathbf{A_1}(t) = \frac{f}{h}(\mathbf{M}(t))\,\mathbf{\Delta}(t) = \mathbf{\Delta}(t)\,\frac{f}{h}(\mathbf{M}(t+1)).$$

Hence the inverse of the matrix $\mathbf{A_2}(\alpha_2-\alpha_1;t)^{-1}\mathbf{A_1}(t)$ is given by

$$\mathbf{B_2}(\alpha_2-\alpha_1;t) := \mathbf{A_1}(t)^{-1}\mathbf{A_2}(\alpha_2-\alpha_1;t) = \frac{h}{f}(\mathbf{M}(t+1))\,\mathbf{\Delta}(t)^{-1} = \mathbf{\Delta}(t)^{-1}\,\frac{h}{f}(\mathbf{M}(t)),$$

where $\frac{h}{f}$ is the exponential generating series of the Bernoulli polynomials evaluated at the point $(\alpha_2-\alpha_1)$, i.e.

$$\frac{h}{f}(x) = \frac{xe^{(\alpha_2-\alpha_1)x}}{e^x-1} = \sum_{n\geq 0}\frac{B_n(\alpha_2-\alpha_1)}{n!}x^n.$$

More precisely, we have

$$\mathbf{B_2}(\alpha_2-\alpha_1;t) = \begin{pmatrix} \frac{1}{t} & \frac{B_1(\alpha_2-\alpha_1)}{1!} & \frac{(t+1)B_2(\alpha_2-\alpha_1)}{2!} & \frac{(t+1)(t+2)B_3(\alpha_2-\alpha_1)}{3!} & \cdots \\ 0 & \frac{1}{t+1} & \frac{B_1(\alpha_2-\alpha_1)}{1!} & \frac{(t+2)B_2(\alpha_2-\alpha_1)}{2!} & \cdots \\ 0 & 0 & \frac{1}{t+2} & \frac{B_1(\alpha_2-\alpha_1)}{1!} & \cdots \\ 0 & 0 & 0 & \frac{1}{t+3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \qquad (4.4.6)$$

As in the case of the multiple zeta functions, here also we can not express the column vector $\mathbf{V}_r(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r)$ as the product of the matrix $\mathbf{B_2}(\alpha_2-\alpha_1;s_1-1)$ and the column vector $\mathbf{V}_{r-1}(s_1+s_2-1,s_3,\dots,s_r;\alpha_2,\dots,\alpha_r)$. To get around this difficulty we essentially repeat what we did in the case of the multiple zeta functions.

We first rewrite (4.4.1) in the form

$$\begin{aligned}&\mathbf{\Delta}(s_1-1)^{-1}\mathbf{V}_{r-1}(s_1+s_2-1,s_3,\dots,s_r;\alpha_2,\dots,\alpha_r)\\ &=\frac{f}{h}(\mathbf{M}(s_1))\mathbf{V}_r(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r).\end{aligned} \tag{4.4.7}$$

For notational convenience, let us denote $\frac{f}{h}(\mathbf{M}(s_1))$ by $\mathbf{X}(s_1)$. We then choose an integer $q\geq 1$ and define

$$I:=\{k\mid 0\leq k\leq q-1\}\quad\text{and}\quad J:=\{k\mid k\geq q\}.$$

Then we write our matrices as block matrices, for example

$$\mathbf{X}(s_1)=\begin{pmatrix}\mathbf{X}^{II}(s_1) & \mathbf{X}^{IJ}(s_1)\\ \mathbf{0}^{JI} & \mathbf{X}^{JJ}(s_1)\end{pmatrix}.$$

Hence from (4.4.7) we get that

$$\begin{aligned}&\mathbf{\Delta}^{II}(s_1-1)^{-1}\mathbf{V}^{I}_{r-1}(s_1+s_2-1,s_3,\dots,s_r;\alpha_2,\dots,\alpha_r)\\ &=\mathbf{X}^{II}(s_1)\mathbf{V}^{I}_r(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r)+\mathbf{X}^{IJ}(s_1)\mathbf{V}^{J}_r(s_1,\dots,s_r;\alpha_1,\dots,\alpha_r).\end{aligned} \tag{4.4.8}$$

Since $\mathbf{X}^{II}(s_1)$ is a finite invertible square matrix, we have

$$\mathbf{X}^{II}(s_1)^{-1}\mathbf{\Delta}^{II}(s_1-1)^{-1}=\mathbf{B_2}^{II}(\alpha_2-\alpha_1;s_1-1).$$

Therefore we deduce from (4.4.8) that

$$\begin{aligned}
&\mathbf{V}_r^I(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r)\\
&=\mathbf{B_2}^{II}(\alpha_2-\alpha_1;s_1-1)\mathbf{V}_{r-1}^I(s_1+s_2-1,s_3,\ldots,s_r;\alpha_2,\ldots,\alpha_r)\\
&+\mathbf{Y}^I(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r),
\end{aligned}\tag{4.4.9}$$

where

$$\mathbf{Y}^I(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r)=-\mathbf{X}^{II}(s_1)^{-1}\mathbf{X}^{IJ}(s_1)\mathbf{V}_r^J(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r).\tag{4.4.10}$$

All the series of meromorphic functions involved in the products of matrices in formulas (4.4.9) and (4.4.10) converge normally on all compact subsets of $\mathbb{C}^r$. Moreover, all entries of the matrices on the right hand side of (4.4.10) are holomorphic on the open set $U_r(q)$, translate of U_r by $(-q,0,\ldots,0)$. Therefore the entries of $\mathbf{Y}^I(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r)$ are also holomorphic in $U_r(q)$. Let us denote $\xi_q(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r)$ to be the first entry of the column vector $\mathbf{Y}^I(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r)$. Then we get from (4.4.9) that

$$\begin{aligned}
&\zeta_r(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r)\\
&=\frac{1}{s_1-1}\zeta_{r-1}(s_1+s_2-1,s_3,\ldots,s_r;\alpha_2,\ldots,\alpha_r)\\
&+\sum_{k=0}^{q-2}\frac{s_1\cdots(s_1+k-1)}{(k+1)!}B_{k+1}(\alpha_2-\alpha_1)\,\zeta_{r-1}(s_1+s_2+k,s_3,\ldots,s_r;\alpha_2,\ldots,\alpha_r)\\
&+\xi_q(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r),
\end{aligned}\tag{4.4.11}$$

and $\xi_q(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r)$ is holomorphic in the open set $U_r(q)$. In the above formula, whenever empty products and empty sums appear, they are assumed to be 1 and 0 respectively. Formula (4.4.11) can also be obtained by using the Euler-Maclaurin summation formula which was done in [2].

4.5 Poles and residues

As we have already mentioned in §4.1 that the exact set of singularities of the multiple Hurwitz zeta functions were only known for some specific values of α_i's from the work of Akiyama and Ishikawa [2]. In this section, we shall determine the exact list of polar hyperplanes of the multiple Hurwitz zeta functions for any values of α_i's and write down the residues explicitly along these polar hyperplanes as certain matrix coefficients.

We shall proceed by induction on r. When $r = 1$, it is well known that the Hurwitz zeta function has meromorphic continuation to $\mathbb{C}$ with simple pole at $s = 1$ with residue 1. So from now on, we fix the depth $r \geq 2$ and we shall prove Theorems 4.5.1, 4.5.2 and 4.5.6 below by assuming that they hold for multiple Hurwitz zeta functions of smaller depths. As in Chapter 3, for $1 \leq i \leq r$ and $k \geq 0$, we define the hyperplane $H_{i,k}$ as follows:

$$H_{i,k} := \{(s_1, \cdots, s_r) \in \mathbb{C}^r \mid s_1 + \cdots + s_i = i - k\}.$$

It is disjoint from $U_r(q) := \{(s_1, \cdots, s_r) \in \mathbb{C}^r \mid \Re(s_1 + \cdots + s_i) > i - q\}$ when $q \leq k$.

4.5.1 Set of all possible singularities

Before deriving the exact set of polar hyperplanes, in the following theorem we derive a possible list of polar hyperplanes. This result was proved by Akiyama and Ishikawa [2] and later reproved by Kelliher and Masri [19]. Our proof here is different from the works [2, 19].

Theorem 4.5.1. *The multiple Hurwitz zeta function of depth r is holomorphic outside the union of the hyperplanes $H_{1,0}$ and $H_{i,k}$, where $2 \leq i \leq r$ and $k \geq 0$. It has at most simple poles along each of these hyperplanes.*

Proof. We will make use of equation (4.4.11) for our proof. For $q \geq 1$, consider the open set

$U_r(q)$ of $\mathbb{C}^r$. By induction hypothesis, we know that in $U_r(q)$, the functions

$$\zeta_{r-1}(s_1 + s_2 + m, s_3, \ldots, s_r; \alpha_2, \ldots, \alpha_r) \text{ for all } -1 \leq m \leq q-2$$

are holomorphic outside the union of the hyperplanes $H_{i,k}$, where $2 \leq i \leq r$ and $0 \leq k < q$ and they have at most simple poles along these hyperplanes. Also $\xi_q(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ is holomorphic in $U_r(q)$. Hence from (4.4.11), we see that the only possible polar hyperplanes of $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ in $U_r(q)$ are $H_{1,0}$ and $H_{i,k}$ for $2 \leq i \leq r,\ 0 \leq k < q$. Since $\mathbb{C}^r$ is covered by the open sets $U_r(q)$ for $q \geq 1$, Theorem 4.5.1 follows. □

4.5.2 Expression for residues

To check if each $H_{i,k}$ is indeed a polar hyperplane, we compute the residue of the multiple Hurwitz zeta function of depth r along this hyperplane. Recall that it is defined as the restriction of the meromorphic function $(s_1 + \cdots + s_i - i + k)\, \zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ to $H_{i,k}$.

Theorem 4.5.2. *The residue of the multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ along the hyperplane $H_{1,0}$ is the restriction of $\zeta_{r-1}(s_2, \ldots, s_r; \alpha_2, \ldots, \alpha_r)$ to $H_{1,0}$ and its residue along the hyperplane $H_{i,k}$, where $2 \leq i \leq r$ and $k \geq 0$, is the restriction to $H_{i,k}$ of the product of $\zeta_{r-i}(s_{i+1}, \ldots, s_r; \alpha_{i+1}, \ldots, \alpha_r)$ with the $(0,k)^{th}$ entry of the matrix*

$$\prod_{d=1}^{i-1} \mathbf{B_2}(\alpha_{d+1} - \alpha_d; s_1 + \cdots + s_d - d).$$

Proof. Let $q \geq 1$ be an integer. As in the proof of Theorem 4.5.1, we know from (4.4.11) that

$$\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r) - \frac{1}{s_1 - 1}\, \zeta_{r-1}(s_1 + s_2 - 1, s_3, \ldots, s_r; \alpha_2, \ldots, \alpha_r)$$

has no pole along $H_{1,0}$ inside the open set $U_r(q)$. These open sets cover $\mathbb{C}^r$. Hence the residue of $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ along $H_{1,0}$ is the restriction to $H_{1,0}$ of the meromorphic function

$\zeta_{r-1}(s_1 + s_2 - 1, s_3, \ldots, s_r; \alpha_2, \ldots, \alpha_r)$ or equivalently of $\zeta_{r-1}(s_2, \ldots, s_r; \alpha_2, \ldots, \alpha_r)$. This proves the first part of Theorem 4.5.2.

Now let i, k be integers with $2 \leq i \leq r$ and $0 \leq k < q$. Also let I and J be as in §4.4. Now if one iterates $(i-1)$ times the formula (4.4.9), one gets

$$\begin{aligned} \mathbf{V}_r^I(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r) = &\left(\prod_{d=1}^{i-1} \mathbf{B_2}^{II}(\alpha_{d+1} - \alpha_d; s_1 + \cdots + s_d - d)\right) \\ &\times \mathbf{V}_{r-i+1}^I(s_1 + \cdots + s_i - i + 1, s_{i+1}, \ldots, s_r; \alpha_i, \ldots, \alpha_r) \\ &+ \mathbf{Y}^{i,I}(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r), \end{aligned}$$

where $\mathbf{Y}^{i,I}(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ is a column matrix whose entries are finite sums of products of rational functions in $s_1, \ldots, s_{i-1}$ with meromorphic functions which are holomorphic in $U_r(q)$. These entries therefore have no pole along the hyperplane $H_{i,k}$ in $U_r(q)$. The entries of

$$\prod_{d=1}^{i-1} \mathbf{B_2}^{II}(\alpha_{d+1} - \alpha_d; s_1 + \cdots + s_d - d)$$

are rational functions in $s_1, \ldots, s_{i-1}$ and hence have no poles along $H_{i,k}$. It now follows from the induction hypothesis that the only entry of $\mathbf{V}_{r-i+1}^I(s_1+\cdots+s_i-i+1, s_{i+1}, \ldots, s_r; \alpha_i, \ldots, \alpha_r)$ that can possibly have a pole along $H_{i,k}$ in $U_r(q)$ is the one of index k, which is

$$\zeta_{r-i+1}(s_1 + \ldots + s_i - i + k + 1, s_{i+1}, \ldots, s_r; \alpha_i, \ldots, \alpha_r).$$

Its residue is the restriction of $\zeta_{r-i}(s_{i+1}, \ldots, s_r; \alpha_{i+1}, \ldots, \alpha_r)$ to $H_{i,k} \cap U_r(q)$, where $2 \leq i \leq r$ and $0 \leq k < q$. Since the open sets $U_r(q)$ for $q > k$ cover $\mathbb{C}^r$, the residue of $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ along $H_{i,k}$ is the restriction to $H_{i,k}$ of the product of the $(0,k)^{\text{th}}$ entry of the matrix

$$\prod_{d=1}^{i-1} \mathbf{B_2}(\alpha_{d+1} - \alpha_d; s_1 + \cdots + s_d - d)$$

with $\zeta_{r-i}(s_{i+1}, \ldots, s_r; \alpha_{i+1}, \ldots, \alpha_r)$. This proves the last part of Theorem 4.5.2. $\square$

4.5.3 Exact set of singularities

We shall now deduce the exact list of poles from Theorem 4.5.2. For this we need the following important theorem due to Brillhart and Dilcher.

Theorem 4.5.3 (Brillhart-Dilcher)**.** *Bernoulli polynomials do not have multiple roots.*

This theorem was first proved for the odd Bernoulli polynomials by Brillhart [7] and later extended for the even Bernoulli polynomials by Dilcher [9]. Theorem 4.5.3 amounts to say that the Bernoulli polynomials $B_{n+1}(t)$ and $B_n(t)$ are relatively prime as they satisfy the relation

$$B'_{n+1}(t) = (n+1)B_n(t) \text{ for all } n \geq 1.$$

where $B'_{n+1}(t)$ denotes the derivative of the polynomial $B_{n+1}(t)$. With the theorem of Brillhart and Dilcher in place we can now describe the exact set of singularities of the multiple zeta functions. For that it is convenient to have the following lemmas in place.

Lemma 4.5.4. *Let x, y be two indeterminates. Then all the entries in the first row of the matrix*

$$\mathbf{B_2}(\beta-\alpha; x)\ \mathbf{B_2}(\gamma-\beta; y),$$

where $0 \leq \alpha, \beta, \gamma < 1$, are non-zero rational functions in x, y with coefficients in $\mathbb{R}$.

Proof. Since entries of these matrices are indexed by $\mathbb{N} \times \mathbb{N}$, the entries of the first row are written as $(0, k)^{\text{th}}$ entry for $k \geq 0$. Let us denote the $(0, k)^{\text{th}}$ entry by $a_{0,k}$. Then we have the following formula:

$$x(y+k)\ a_{0,k} = \sum_{i=0}^{k} (x)_{i-1}(y+i+1)_{k-i-1} B_i(\beta-\alpha)\ B_{k-i}(\gamma-\beta)$$

for all $k \geq 0$. As the Bernoulli polynomial $B_0(t)$ is equal to 1, we get $a_{0,0} = \frac{1}{xy}$ and hence non-zero. For $k \geq 1$, we first note that the set of polynomials

$$P := \{(x)_{i-1}(y+i+1)_{k-i-1} : 0 \leq i \leq k\}$$

is linearly independent over $\mathbb{R}$.

Now suppose, $B_1(\beta - \alpha) \neq 0$. We know by Theorem 4.5.3 that at least one of $B_k(\gamma - \beta)$ and $B_{k-1}(\gamma - \beta)$ is non-zero. Hence using the linearly independence of the set of polynomials P, we get $a_{0,k} \neq 0$.

Next suppose, $B_1(\beta - \alpha) = 0$, i.e. $\beta - \alpha = 1/2$. Thus $\gamma - \beta \neq 1/2$ as $0 \leq \alpha, \gamma < 1$. Hence $B_1(\gamma - \beta) \neq 0$. Again by Theorem 4.5.3 we know that at least one of $B_k(\beta - \alpha)$ and $B_{k-1}(\beta - \alpha)$ is non-zero. Hence using the linearly independence of the set of polynomials P, we get $a_{0,k} \neq 0$. This completes the proof of Lemma 4.5.4. □

Lemma 4.5.5. *Let $n \geq 0$ be an integer and $x, x_1, \ldots, x_n$ be indeterminates. Suppose that $\mathbf{D}$ be an infinite square matrix whose entries are indexed by $\mathbb{N} \times \mathbb{N}$ and coming from the ring $\mathbb{R}(x_1, \ldots, x_n)$. Further suppose that all the entries in the first row of $\mathbf{D}$ are non-zero. Then for any $\alpha, \beta \in \mathbb{R}$, all the entries in the first row of $\mathbf{DB_2}(\beta - \alpha; x)$ are non-zero.*

Proof. We first note that each column of $\mathbf{B_2}(\beta - \alpha; x)$ has at least one non-zero entry and the non-zero entries of each of these columns are linearly independent over $\mathbb{R}$ as rational functions in x with coefficients in $\mathbb{R}$. Since all the entries in the first row of $\mathbf{D}$ are non-zero, the proof is complete by the above observation. □

Theorem 4.5.6. *The multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ has simple pole along the hyperplane $H_{1,0}$. It also has simple poles along the hyperplanes $H_{i,k}$, for $2 \leq i \leq r$ and $k \geq 0$, except when $i = 2$ and $k \in J$, where J denotes the set of indices j such that $B_j(\alpha_2 - \alpha_1) = 0$, i.e.*

$$J = \{j \in \mathbb{N} : B_j(\alpha_2 - \alpha_1) = 0\}.$$

Proof. When $1 \leq i \leq r$ and $k \geq 0$, the restriction of $\zeta_{r-i}(s_{i+1}, \ldots, s_r, \alpha_{i+1}, \ldots, \alpha_r)$ to $H_{i,k}$ is a non-zero meromorphic function. Hence in order to prove Theorem 4.5.6, we need to show that when $2 \leq i \leq r$ and $k \geq 0$, the $(0,k)^{\text{th}}$ entry of the matrix

$$\prod_{d=1}^{i-1} \mathbf{B_2}(\alpha_{d+1} - \alpha_d; s_1 + \cdots + s_d - d)$$

is identically zero if and only if $i = 2,\ k \in J$. By changing co-ordinates, the above statement is equivalent to say that when $t_1, \ldots, t_{i-1}$ are indeterminates, the $(0,k)^{\text{th}}$ entry of the matrix

$$\prod_{d=1}^{i-1} \mathbf{B_2}(\alpha_{d+1} - \alpha_d; t_d)$$

is non-zero in $\mathbb{R}(t_1, \ldots, t_{i-1})$ except when $i = 2$ and $k \in J$.

For $i = 2$, our matrix is $\mathbf{B_2}(\alpha_2 - \alpha_1; t_1)$ and hence our assertion follows immediately. Now assume that $i \geq 3$. By Lemma 4.5.4, we know that all the entries in the first row of the matrix

$$\mathbf{B_2}(\alpha_2 - \alpha_1; t_1)\mathbf{B_2}(\alpha_3 - \alpha_2; t_2)$$

is non-zero in $\mathbb{R}(t_1, t_2)$. Hence the theorem follows from Lemma 4.5.4 if $i = 3$ and from repeated application of Lemma 4.5.5 if $i > 3$. $\square$

We have precise knowledge about the rational zeros of the Bernoulli polynomials due to K. Inkeri [18].

Theorem 4.5.7 (Inkeri)**.** *The rational roots of a Bernoulli polynomial $B_n(t)$ can only be $0, 1/2$ and 1. This happens only when n is odd and precisely in the following cases:*

1. $B_n(0) = B_n(1) = 0$ *for all odd* $n \geq 3$*,*

2. $B_n(1/2) = 0$ *for all odd* $n \geq 1$.

Using Theorem 4.5.7, we deduce the following results as an immediate consequence of Theorem 4.5.6.

Corollary 4.5.8. *If $\alpha_2 - \alpha_1 = 0$, then the exact set of singularities of the multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ is given by the hyperplanes*

$$H_{1,0}, H_{2,1}, H_{2,2k} \text{ and } H_{i,k} \text{ for all } k \geq 0 \text{ and } 3 \leq i \leq r.$$

If $\alpha_2 - \alpha_1 = 1/2$, then the exact set of singularities of the multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ is given by the hyperplanes

$$H_{1,0}, H_{2,2k} \text{ and } H_{i,k} \text{ for all } k \geq 0 \text{ and } 3 \leq i \leq r.$$

If $\alpha_2 - \alpha_1$ is a rational number $\neq 0, 1/2$, then the exact set of singularities of the multiple Hurwitz zeta function $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ is given by the hyperplanes

$$H_{1,0} \text{ and } H_{i,k} \text{ for all } k \geq 0 \text{ and } 2 \leq i \leq r.$$

A particular case of this result, namely when $\alpha_i \in \mathbb{Q}$ for all $1 \leq i \leq r$, was proved by Akaiyama and Ishikawa [2]. Corollary 4.5.8 above shows that such condition can be removed.

5

Multiple Dirichlet series with additive characters

5.1 Introduction

Akiyama and Ishikawa [2] introduced the notion of multiple Dirichlet L-functions which are several variable generalisations of the classical Dirichlet L-functions.

Definition 5.1.1. Let $r \geq 1$ be an integer and $\chi_1, \ldots, \chi_r$ be Dirichlet characters of arbitrary modulus. The multiple Dirichlet L-function of depth r associated to the Dirichlet characters $\chi_1, \ldots, \chi_r$ is denoted by $L_r(s_1, \ldots, s_r;\ \chi_1, \ldots, \chi_r)$ and defined by the following normally convergent series in U_r:

$$L_r(s_1, \ldots, s_r;\ \chi_1, \ldots, \chi_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{\chi_1(n_1) \cdots \chi_r(n_r)}{n_1^{s_1} \cdots n_r^{s_r}}.$$

It is worthwhile to mention that Akiyama and Ishikawa defined the multiple Dirichlet L-functions for characters with same conductor, but their definition also makes sense for Dirichlet characters of arbitrary modulus. The normal convergence of the above series follows from the normal convergence of the multiple zeta function of depth r as an immediate consequence and we record it here in the following proposition. Throughout this chapter, whenever we consider a set of characters, they are not necessarily of same modulus unless otherwise stated.

Proposition 5.1.2. *Let $r \geq 1$ be an integer and $\chi_1, \ldots, \chi_r$ be Dirichlet characters. Then the family of functions*

$$\left(\frac{\chi_1(n_1) \cdots \chi_r(n_r)}{n_1^{s_1} \cdots n_r^{s_r}} \right)_{n_1 > \cdots > n_r > 0}$$

is normally summable on compact subsets of U_r.

Hence $(s_1, \ldots, s_r) \mapsto L_r(s_1, \ldots, s_r;\ \chi_1, \ldots, \chi_r)$ defines a holomorphic function on U_r. Akiyama and Ishikawa have also discussed the question of meromorphic continuation of the multiple Dirichlet L-functions. When $r = 1$, classically the meromorphic continuation is achieved by writing the function in terms of the Hurwitz zeta function. For $r > 1$, one may try to mimic this. In this case some variants of the multiple Hurwitz zeta functions come up. This is exactly what they have done and they could prove the following theorem.

Theorem 5.1.3 (Akiyama-Ishikawa)**.** *Let $\chi_1, \ldots, \chi_r$ be primitive Dirichlet characters of the same modulus. Then the multiple Dirichlet L-function $L_r(s_1, \ldots, s_r;\ \chi_1, \ldots, \chi_r)$ of depth r can be extended as a meromorphic function to $\mathbb{C}^r$ with possible simple poles at the hyperplanes given by the equations*

$$s_1 = 1;\ s_1 + \cdots + s_i = n \ \textit{for all} \ n \in \mathbb{Z}_{\leq i} \ \textit{and} \ 2 \leq i \leq r.$$

The exact set of singularities of the multiple Dirichlet L-functions is not well understood. For $r = 2$ and specific choices of the characters χ_1 and χ_2, Akiyama and Ishikawa provided a complete description of the polar hyperplanes. To address this difficult question, one can aim to obtain a translation formula satisfied by the multiple Dirichlet L-functions. However, obtaining such a translation formula for multiple Dirichlet L-functions seems harder and so is the analogue of Theorem 2.2.1.

On the other hand, if we consider additive characters, i.e. group homomorphisms $f : \mathbb{Z} \to \mathbb{C}^*$ in place of Dirichlet characters the problem seems to be more amenable to study and we could derive analogues of Theorem 2.2.4 and Theorem 2.2.5. It is interesting to note that Dirichlet characters are linked with additive characters and so is the Dirichlet L-functions with the Dirichlet series associated to additive characters. Thus studying multiple Dirichlet series associated to additive characters are very much relevant to the study of multiple Dirichlet L-functions. We elaborate below.

Let a, b, N be natural numbers with $1 \leq a, b \leq N$ and χ denote a Dirichlet character modulo N. For $\Re(s) > 1$, we consider the following Dirichlet series:

$$L(s;\chi) := \sum_{n \geq 1} \frac{\chi(n)}{n^s},$$
$$\Phi(s;a) := \sum_{\substack{n \geq 1 \\ n \equiv a \bmod N}} \frac{1}{n^s},$$
$$\Psi(s;b) := \sum_{n \geq 1} \frac{e^{2\pi\iota bn/N}}{n^s}.$$

Here $L(s;\chi)$ is the Dirichlet L-function associated to the Dirichlet character χ, $\Phi(s;a)$ is the product of N^{-s} with the Hurwitz zeta function $\zeta(s;a/N)$ and $\Psi(s;b)$ is essentially a constant multiple of the Lerch zeta function $L(\frac{b}{N};1;s)$. Now note that we have the following relations among these Dirichlet series:

$$L(s;\chi) = \sum_{1 \leq a \leq N} \chi(a)\Phi(s;a), \tag{5.1.1}$$

$$\Psi(s;b) = \sum_{1 \leq a \leq N} e^{2\pi\iota ab/N}\Phi(s;a). \tag{5.1.2}$$

Further using (5.1.2), we can deduce that

$$\Phi(s;a) = \frac{1}{N} \sum_{1 \leq b \leq N} e^{-2\pi\iota ab/N}\Psi(s;b). \tag{5.1.3}$$

Thus we get that

$$L(s;\chi) = \frac{1}{N} \sum_{1\le a\le N} \chi(a) \sum_{1\le b\le N} e^{-2\pi\iota ab/N}\Psi(s;b). \tag{5.1.4}$$

In fact (5.1.4) can be generalised for multiple Dirichlet L-functions. Let $r \geq 1$ be a natural number and for each $1 \leq i \leq r$, let a_i, b_i, N_i be natural numbers with $1 \leq a_i, b_i \leq N_i$. Also let χ_i be a Dirichlet character mod N_i. Then the depth-r multiple Dirichlet L-function associated to $\chi_1, \ldots, \chi_r$ can be written as follows:

$$L_r(s_1,\ldots,s_r;\ \chi_1,\ldots,\chi_r) = \sum_{\substack{1\le a_i\le N_i\\ \text{for all } 1\le i\le r}} \chi_1(a_1)\cdots\chi_r(a_r)\Phi_r(s_1,\ldots,s_r;a_1,\ldots,a_r), \tag{5.1.5}$$

where $\Phi_r(s_1,\ldots,s_r;a_1,\ldots,a_r)$ is defined as the following multiple Dirichlet series in U_r:

$$\Phi_r(s_1,\ldots,s_r;a_1,\ldots,a_r) := \sum_{\substack{n_1>\cdots>n_r>0\\ n_i\equiv a_i \bmod N_i\\ \text{for all } 1\le i\le r}} n_1^{-s_1}\cdots n_r^{-s_r}.$$

Next we consider the following multiple Dirichlet series in U_r:

$$\Psi_r(s_1,\ldots,s_r;b_1,\ldots,b_r) := \sum_{n_1>\cdots>n_r>0} \frac{e^{2\pi\iota\left(\frac{b_1n_1}{N_1}+\cdots+\frac{b_rn_r}{N_r}\right)}}{n_1^{s_1}\cdots n_r^{s_r}}.$$

Now we can write down a several variable generalisation of (5.1.2):

$$\Psi_r(s_1,\ldots,s_r;b_1,\ldots,b_r) = \sum_{\substack{1\le a_i\le N_i\\ \text{for all } 1\le i\le r}} e^{2\pi\iota\left(\frac{a_1b_1}{N_1}+\cdots+\frac{a_rb_r}{N_r}\right)}\Phi_r(s_1,\ldots,s_r;a_1,\ldots,a_r). \tag{5.1.6}$$

Further using (5.1.6), we get

$$\begin{aligned}&\Phi_r(s_1,\ldots,s_r;a_1,\ldots,a_r)\\ &= \frac{1}{N_1\cdots N_r} \sum_{\substack{1\le b_i\le N_i\\ \text{for all } 1\le i\le r}} e^{-2\pi\iota\left(\frac{a_1b_1}{N_1}+\cdots+\frac{a_rb_r}{N_r}\right)}\Psi_r(s_1,\ldots,s_r;b_1,\ldots,b_r).\end{aligned} \tag{5.1.7}$$

Using (5.1.5) and (5.1.7) we then obtain that

$$L_r(s_1,\ldots,s_r;\ \chi_1,\ldots,\chi_r) \\ = \frac{1}{N_1\cdots N_r}\sum_{\substack{1\le a_i\le N_i\\ \text{for all } 1\le i\le r}}\chi_1(a_1)\cdots\chi_r(a_r)\sum_{\substack{1\le b_i\le N_i\\ \text{for all } 1\le i\le r}} e^{-2\pi\iota\left(\frac{a_1b_1}{N_1}+\cdots+\frac{a_rb_r}{N_r}\right)}\Psi_r(s_1,\ldots,s_r;b_1,\ldots,b_r). \tag{5.1.8}$$

We now introduce the notion of multiple Dirichlet series associated to additive characters.

Definition 5.1.4. For a natural number $r\ge 1$ and additive characters $f_1,\ldots,f_r$, the multiple L-function associated to $f_1,\ldots,f_r$ is denoted by $L_r(f_1,\ldots,f_r;\ s_1,\ldots,s_r)$ and defined by the following series:

$$L_r(f_1,\ldots,f_r;\ s_1,\ldots,s_r) := \sum_{n_1>\cdots>n_r>0}\frac{f_1(n_1)\cdots f_r(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \sum_{n_1>\cdots>n_r>0}\frac{f_1(1)^{n_1}\cdots f_r(1)^{n_r}}{n_1^{s_1}\cdots n_r^{s_r}}.$$

From now on we refer to this function as *multiple additive L-function of depth r associated to the additive characters $f_1,\ldots,f_r$*. A necessary and sufficient condition for the absolute convergence of the above series in U_r is given in terms of the partial products $g_i := \prod_{1\le j\le i} f_j$ for all $1\le i\le r$. The condition is that

$$|g_i(1)|\le 1 \text{ for all } 1\le i\le r.$$

The sufficiency of this condition is easily established by noting that for any arbitrary r integers $n_1>\cdots>n_r>0$, one has

$$|f_1(1)^{n_1}\cdots f_r(1)^{n_r}| \le \left|g_1(1)^{n_1-n_2}\cdots g_{r-1}(1)^{n_{r-1}-n_r}g_r(1)^{n_r}\right|.$$

It can also be shown that if $|g_i(1)|>1$ for some i, the above series does not converge absolutely in U_r, in fact, anywhere in $\mathbb{C}^r$. To see this let i be one of the index such that $|g_i(1)|>1$. Now if possible let for a complex r-tuple $(s_1, \ldots, s_r)$, the series

$$\sum_{n_1 > \cdots > n_r > 0} \frac{f_1(n_1) \cdots f_r(n_r)}{n_1^{s_1} \cdots n_r^{s_r}}$$

converges absolutely i.e. the series

$$\sum_{n_1 > \cdots > n_r > 0} \frac{|f_1(n_1) \cdots f_r(n_r)|}{n_1^{\sigma_1} \cdots n_r^{\sigma_r}}$$

of non-negative real numbers converges, where σ_i denotes the real part of s_i. Hence the following smaller series

$$\sum_{n > r-i} \frac{|f_1(n+i-1) f_2(n+i-2) \cdots f_i(n) f_{i+1}(r-i) \cdots f_r(1)|}{(n+i-1)^{\sigma_1} (n+i-2)^{\sigma_2} \cdots n^{\sigma_i} (r-i)^{\sigma_{i+1}} \cdots 1^{\sigma_r}}$$

is convergent. Note that the numerator of the summand in the above series is nothing but

$$|g_1(1) \cdots g_{i-1}(1) g_i(1)^n f_{i+1}(r-i) \cdots f_r(1)|$$

and the denominator is smaller than

$$(n+i-1)^{\sigma_1 + \cdots + \sigma_i} (r-i)^{\sigma_{i+1}} \cdots 1^{\sigma_r}.$$

Now we get a contradiction since the series

$$\sum_{n > r-i} \frac{|g_i(1)|^n}{(n+i-1)^{\sigma_1 + \cdots + \sigma_i}}$$

does not converge for any choice of complex r-tuple $(s_1, \ldots, s_r)$ as $|g_i(1)| > 1$.

If we write $f_i(1) = e^{2\pi\iota\lambda_i}$ for some $\lambda_i \in \mathbb{C}$, the condition $|g_i(1)| \leq 1$ for $1 \leq i \leq r$ can be rewritten as

$$\Im(\lambda_1 + \cdots + \lambda_i) \geq 0 \text{ for } 1 \leq i \leq r.$$

For simplicity one can assume that $\lambda_i \in \mathbb{R}$ for $1 \leq i \leq r$ so that the above conditions are vacuously true. If f is an additive character such that $f(1) = e^{2\pi\iota\lambda}$ for some $\lambda \in \mathbb{R}$, then we call such an f a *real additive character*.

With the necessary and sufficient condition for absolute convergence of the multiple additive L-functions in place, we derive the following result as an immediate consequence of the normal convergence of the multiple zeta function of depth r.

Proposition 5.1.5. *Let $r \geq 1$ be an integer and $f_1, \ldots, f_r$ be additive characters such that the partial products $g_i := \prod_{1 \leq j \leq i} f_j$ satisfy the condition $|g_i(1)| \leq 1$ for all $1 \leq i \leq r$. Then the family of functions*

$$\left(\frac{f_1(n_1) \cdots f_r(n_r)}{n_1^{s_1} \cdots n_r^{s_r}} \right)_{n_1 > \cdots > n_r > 0}$$

is normally summable on compact subsets of U_r.

Thus for a natural number $r \geq 1$ and additive characters $f_1, \ldots, f_r$ such that the partial products $g_i := \prod_{1 \leq j \leq i} f_j$ satisfy the condition $|g_i(1)| \leq 1$ for all $1 \leq i \leq r$, the multiple additive L-function $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ defines an analytic function in U_r.

In this chapter, we will discuss the question of meromorphic continuation of such multiple additive L-functions, their translation formulas and the location of their singularities. Major part of this chapter is the reproduction of the work [32].

5.2 The Translation formulas

In this section, we shall derive translation formulas satisfied by the multiple additive L-functions and using these we are going to establish their meromorphic continuation in the subsequent section. As before, we start with a natural number $r \geq 1$ and additive characters $f_1, \ldots, f_r$ such that the partial products $g_i := \prod_{1 \leq j \leq i} f_j$ satisfy the condition $|g_i(1)| \leq 1$ for all $1 \leq i \leq r$. Then the associated multiple additive L-function $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ satisfies the following translation formulas depending on the condition that whether $f_1(1) = 1$ or not.

Theorem 5.2.1. *For any integer $r \geq 2$ and additive characters $f_1, \ldots, f_r$ such that the partial products $g_i := \prod_{1 \leq j \leq i} f_j$ satisfy the condition $|g_i(1)| \leq 1$ for all $1 \leq i \leq r$ with $f_1(1) = 1$, the associated multiple additive L-function $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ satisfies the following translation formula in U_r:*

$$L_{r-1}(f_2, \ldots, f_r;\ s_1 + s_2 - 1, s_3, \ldots, s_r) = \sum_{k \geq 0} (s_1 - 1)_k\, L_r(f_1, \ldots, f_r;\ s_1 + k, s_2, \ldots, s_r), \tag{5.2.1}$$

where the series on the right hand side converges normally on any compact subset of U_r .

Theorem 5.2.2. *Let $r \geq 2$ be an integer and $f_1, \ldots, f_r$ be additive characters such that the partial products $g_i := \prod_{1 \leq j \leq i} f_j$ satisfy the condition $|g_i(1)| \leq 1$ for all $1 \leq i \leq r$ with $f_1(1) \neq 1$. Then the associated multiple additive L-function $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ satisfies the following translation formula in U_r:*

$$\begin{aligned} &f_1(1) L_{r-1}(g_2, f_3, \ldots, f_r; s_1 + s_2, s_3, \ldots, s_r) + (f_1(1) - 1) L_r(f_1, \ldots, f_r; s_1, \ldots, s_r) \\ &= \sum_{k \geq 0} (s_1)_k\, L_r(f_1, \ldots, f_r;\ s_1 + k + 1, s_2, \ldots, s_r), \end{aligned} \tag{5.2.2}$$

where the series on the right side converges normally on any compact subset of U_r .

It is worthwhile to note that Theorem 5.2.2 is the multiple Dirichlet series analogue of Theorem 2.2.4. To prove Theorem 5.2.1, we need the ubiquitous identity which is valid for any integer $n \geq 2$ and any complex number s:

$$(n-1)^{1-s} - n^{1-s} = \sum_{k \geq 0} (s-1)_k\, n^{-s-k}. \tag{5.2.3}$$

Whereas to prove Theorem 5.2.2, we need the following version of (5.2.3), obtained by replacing s with $s + 1$ in (5.2.3):

$$(n-1)^{-s} - n^{-s} = \sum_{k \geq 0} (s)_k\, n^{-s-k-1}. \tag{5.2.4}$$

Besides, we need the following proposition.

Proposition 5.2.3. *Let $r \geq 2$ be an integer and $f_1, \ldots, f_r$ be additive characters such that the partial products $g_i := \prod_{1 \leq j \leq i} f_j$ satisfy the condition $|g_i(1)| \leq 1$ for all $1 \leq i \leq r$. Then the family of functions*

$$\left((s_1 - 1)_k \frac{f_1(n_1) \cdots f_r(n_r)}{n_1^{s_1+k} n_2^{s_2} \cdots n_r^{s_r}} \right)_{n_1 > \cdots > n_r > 0, k \geq 0}$$

is normally summable on compact subsets of U_r.

Proof. This proposition is an immediate consequence of Proposition 3.3.1 as for any r integers $n_1 > \cdots > n_r > 0$, we have

$$|f_1(n_1) \cdots f_r(n_r)| = |f_1(1)^{n_1} \cdots f_r(1)^{n_r}| \leq \left| g_1(1)^{n_1 - n_2} \cdots g_{r-1}(1)^{n_{r-1} - n_r} g_r(1)^{n_r} \right| \leq 1$$

from the hypothesis. □

We are now ready to prove Theorem 5.2.1 and Theorem 5.2.2.

5.2.1 Proof of Theorem 5.2.1

We replace n, s by n_1, s_1 in (5.2.3) and multiply $\frac{f_1(n_1) \cdots f_r(n_r)}{n_2^{s_2} \cdots n_r^{s_r}}$ to both sides of (5.2.3) and obtain that

$$\left(\frac{1}{(n_1 - 1)^{s_1 - 1}} - \frac{1}{n_1^{s_1 - 1}} \right) \frac{f_1(n_1) \cdots f_r(n_r)}{n_2^{s_2} \cdots n_r^{s_r}} = \sum_{k \geq 0} (s_1 - 1)_k \frac{f_1(n_1) \cdots f_r(n_r)}{n_1^{s_1+k} n_2^{s_2} \cdots n_r^{s_r}}.$$

Now we sum both the sides for $n_1 > \cdots > n_r > 0$. Since $f_1(1) = 1$, using Proposition 5.2.3, we get

$$L_{r-1}(f_2, \ldots, f_r;\ s_1 + s_2 - 1, s_3, \ldots, s_r) = \sum_{k \geq 0} (s_1 - 1)_k\, L_r(f_1, \ldots, f_r;\ s_1 + k, s_2, \ldots, s_r).$$

This together with Proposition 5.2.3 completes the proof. □

5.2.2 Proof of Theorem 5.2.2

For this we replace n, s by n_1, s_1 in (5.2.4) and multiply $\frac{f_1(n_1)\cdots f_r(n_r)}{n_2^{s_2}\cdots n_r^{s_r}}$ to both the sides of (5.2.4) and obtain that

$$\left(\frac{1}{(n_1-1)^{s_1}} - \frac{1}{n_1^{s_1}}\right)\frac{f_1(n_1)\cdots f_r(n_r)}{n_2^{s_2}\cdots n_r^{s_r}} = \sum_{k\geq 0}(s_1)_k \frac{f_1(n_1)\cdots f_r(n_r)}{n_1^{s_1+k+1}n_2^{s_2}\cdots n_r^{s_r}}.$$

Now we sum both the sides for $n_1 > \cdots > n_r > 0$ and use Proposition 5.2.3 with s_1 replaced by s_1+1. We then obtain

$$\begin{aligned}
&f_1(1)L_{r-1}(g_2, f_3, \ldots, f_r; s_1+s_2, s_3, \ldots, s_r) + (f_1(1)-1)L_r(f_1, \ldots, f_r; s_1, \ldots, s_r)\\
&= \sum_{k\geq 0}(s_1)_k\, L_r(f_1, \ldots, f_r;\ s_1+k+1, s_2, \ldots, s_r).
\end{aligned}$$

This together with Proposition 5.2.3 completes the proof. □

5.3 Meromorphic continuation

In this section, we establish the meromorphic continuation of the multiple additive L-functions using the translation formulas (5.2.1) and (5.2.2).

Theorem 5.3.1. *Let $r \geq 2$ be an integer and $f_1, \ldots, f_r$ be additive characters such that the partial products $g_i := \prod_{1\leq j\leq i} f_j$ satisfy the condition $|g_i(1)| \leq 1$ for all $1 \leq i \leq r$. Then the associated multiple additive L-function $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ extends to a meromorphic function on $\mathbb{C}^r$ satisfying the following translation formulas on $\mathbb{C}^r$:*

$$\begin{aligned}
&L_{r-1}(f_2, \ldots, f_r;\ s_1+s_2-1, s_3, \ldots, s_r)\\
&= \sum_{k\geq 0}(s_1-1)_k\, L_r(f_1, \ldots, f_r;\ s_1+k, s_2, \ldots, s_r) \quad \textit{if} \quad f_1(1)=1
\end{aligned} \tag{5.3.1}$$

and

$$\begin{aligned} &f_1(1)L_{r-1}(g_2, f_3, \ldots, f_r; s_1+s_2, s_3, \ldots, s_r) + (f_1(1)-1)L_r(f_1, \ldots, f_r; s_1, \ldots, s_r) \\ &= \sum_{k\geq 0}(s_1)_k\, L_r(f_1, \ldots, f_r;\ s_1+k+1, s_2, \ldots, s_r) \quad \textit{if} \quad f_1(1) \neq 1. \end{aligned} \tag{5.3.2}$$

The series of meromorphic functions on the right hand sides of (5.3.1) *and* (5.3.2) *converge normally on every compact subset of* $\mathbb{C}^r$*.*

We prove this theorem by induction on depth r. Assuming induction hypothesis for multiple additive L-functions of depth $(r-1)$, we first extend the multiple additive L-function $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ as a meromorphic function to $U_r(m)$ for each $m \geq 1$, where

$$U_r(m) := \{(s_1, \ldots, s_r) \in \mathbb{C}^r \ :\ \Re(s_1 + \cdots + s_i) > i - m \text{ for all } 1 \leq i \leq r\}.$$

Since open sets of the form $U_r(m)$ for $m \geq 1$ form an open cover of $\mathbb{C}^r$, we get the coveted meromorphic continuation to $\mathbb{C}^r$. Here we need the following variant of Proposition 5.2.3.

Proposition 5.3.2. *Let $r \geq 2$ be an integer and $f_1, \ldots, f_r$ be additive characters such that the partial products $g_i := \prod_{1\leq j\leq i} f_j$ satisfy the condition $|g_i(1)| \leq 1$ for all $1 \leq i \leq r$. Then the family of functions*

$$\left((s_1)_k \frac{f_1(n_1)\cdots f_r(n_r)}{n_1^{s_1+k+1} n_2^{s_2}\cdots n_r^{s_r}}\right)_{n_1>\cdots>n_r>0, k\geq m-1}$$

is normally summable on compact subsets of $U_r(m)$.

Proof. Let K be a compact subset of $U_r(m)$ and $S := \sup_{(s_1,\ldots,s_r)\in K} |s_1|$. Then for $k \geq m-1$,

$$\left\| (s_1)_k \frac{f_1(n_1)\cdots f_r(n_r)}{n_1^{s_1+k+1} n_2^{s_2}\cdots n_r^{s_r}} \right\| \leq \frac{(S)_k}{2^{k-m+1}} \left\| \frac{1}{n_1^{s_1+m} n_2^{s_2}\cdots n_r^{s_r}} \right\|.$$

Now as $(s_1, \ldots, s_r)$ varies over $U_r(m)$, $(s_1+m, \ldots, s_r)$ varies over U_r. Then the proof follows from Corollary 3.2.2 as the series $\sum_{k\geq m-1} \frac{(S)_k}{2^{k-m+1}}$ converges. $\square$

5.3.1 Proof of Theorem 5.3.1

As mentioned before, we prove this theorem by induction on the depth r. If $r = 2$, then the left hand side of (5.3.1) has a meromorphic continuation to $\mathbb{C}^2$ by Theorem 2.2.2 if $f_2(1) = 1$ and by Theorem 2.2.5 if $f_2(1) \neq 1$. Similarly when $r = 2$, the first term on the left hand side of (5.3.2) has a meromorphic continuation to $\mathbb{C}^2$ by Theorem 2.2.2 if $g_2(1) = 1$ and by Theorem 2.2.5 if $g_2(1) \neq 1$. For $r \geq 3$, the left hand side of (5.3.1) and the first term in the left hand side of (5.3.2) have a meromorphic continuation to $\mathbb{C}^r$ by the induction hypothesis.

We now establish the meromorphic continuation of the multiple additive L-function $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ separately for each of the cases $f_1(1) = 1$ and $f_1(1) \neq 1$.

First we consider the case $f_1(1) = 1$. As we have shown that in this case the multiple additive L-function $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ satisfies the translation formula (5.2.1) in U_r. Note that the translation formula (5.2.1) is exactly the same as (3.1.1) which is satisfied by the multiple zeta functions of depth r. Hence in this case the meromorphic continuation follows exactly as in the case of the multiple zeta function.

Next we consider the case $f_1(1) \neq 1$. Now by induction hypothesis we know that

$$f_1(1) L_{r-1}(g_2, f_3, \ldots, f_r; s_1 + s_2, s_3, \ldots, s_r)$$

has a meromorphic continuation to $\mathbb{C}^r$ and by Proposition 5.3.2, (for $m = 1$) we have that

$$\sum_{k \geq 0} (s_1)_k\, L_r(f_1, \ldots, f_r;\ s_1 + k + 1, s_2, \ldots, s_r)$$

is a holomorphic function on $U_r(1)$. Hence we can extend $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ to $U_r(1)$ as a meromorphic function satisfying (5.2.2).

Now we assume that for an integer $m \geq 2$, $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ has been extended meromorphically to $U_r(m-1)$ satisfying (5.2.2). Again by Proposition 5.3.2, the series

$$\sum_{k\geq m-1} (s_1)_k\, L_r(f_1,\ldots,f_r;\ s_1+k+1, s_2,\ldots,s_r)$$

defines a holomorphic function on $U_r(m)$. As $L_r(f_1,\ldots,f_r;\ s_1,\ldots,s_r)$ has been extended meromorphically to $U_r(m-1)$, we can extend $L_r(f_1,\ldots,f_r;\ s_1+k+1,\ldots,s_r)$ meromorphically to $U_r(m)$ for all $0\leq k\leq m-2$.

Thus we can now extend $L_r(f_1,\ldots,f_r;\ s_1,\ldots,s_r)$ meromorphically to $U_r(m)$ by means of (5.2.2). This completes the proof as $\{U_r(m) : m\geq 1\}$ is an open cover of $\mathbb{C}^r$. $\square$

5.4 Matrix formulation of the translation formulas

Since the translation formulas (5.3.1) and (3.4.1) are similar, one can easily notice that their matrix formulation will be identical. We briefly write down the matrix formulation of the translation formula (5.3.1).

The translation formula (5.3.1) together with the other relations obtained by applying successively the change of variables $s_1 \mapsto s_1+n$ to it for each $n\geq 0$ is equivalent to the single relation

$$\mathbf{V}_{r-1}(f_2,f_3,\ldots,f_r;\ s_1+s_2-1,s_3,\ldots,s_r) = \mathbf{A_1}(s_1-1)\mathbf{V}_r(f_1,\ldots,f_r;\ s_1,\ldots,s_r), \tag{5.4.1}$$

where $\mathbf{V}_r(f_1,\ldots,f_r;\ s_1,\ldots,s_r)$ denotes the infinite column vector

$$\mathbf{V}_r(f_1,\ldots,f_r;\ s_1,\ldots,s_r) := \begin{pmatrix} L_r(f_1,\ldots,f_r;\ s_1,s_2,\ldots,s_r) \\ L_r(f_1,\ldots,f_r;\ s_1+1,s_2,\ldots,s_r) \\ L_r(f_1,\ldots,f_r;\ s_1+2,s_2,\ldots,s_r) \\ \vdots \end{pmatrix} \tag{5.4.2}$$

and for an indeterminate t, $\mathbf{A_1}(t)$ is defined by

$$\mathbf{A_1}(t) := \begin{pmatrix} t & \frac{t(t+1)}{2!} & \frac{t(t+1)(t+2)}{3!} & \cdots \\ 0 & t+1 & \frac{(t+1)(t+2)}{2!} & \cdots \\ 0 & 0 & t+2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.4.3}$$

As in Chapter 3, for any integer $q \geq 1$, we write

$$I = I_q = \{k \in \mathbb{N} \mid 0 \leq k \leq q-1\} \quad \text{and} \quad J = J_q = \{k \in \mathbb{N} \mid k \geq q\}.$$

We can then reformulate (5.4.1) to write

$$\begin{aligned} &\mathbf{V}_r^I(f_1, \ldots, f_r;\ s_1, \ldots, s_r) \\ &= \mathbf{B_1}^{II}(s_1 - 1)\mathbf{V}_{r-1}^I(f_2, \ldots, f_r;\ s_1 + s_2 - 1, s_3, \ldots, s_r) + \mathbf{W}^I(f_1, \ldots, f_r;\ s_1, \ldots, s_r), \end{aligned} \tag{5.4.4}$$

where

$$\mathbf{W}^I(f_1, \ldots, f_r;\ s_1, \ldots, s_r) = -\mathbf{U}^{II}(s_1)^{-1}\mathbf{U}^{IJ}(s_1)\mathbf{V}_r^J(f_1, \ldots, f_r;\ s_1, \ldots, s_r), \tag{5.4.5}$$

and for an indeterminate t,

$$\mathbf{B_1}(t) := \begin{pmatrix} \frac{1}{t} & \frac{B_1}{1!} & \frac{(t+1)B_2}{2!} & \frac{(t+1)(t+2)B_3}{3!} & \cdots \\ 0 & \frac{1}{t+1} & \frac{B_1}{1!} & \frac{(t+2)B_2}{2!} & \cdots \\ 0 & 0 & \frac{1}{t+2} & \frac{B_1}{1!} & \cdots \\ 0 & 0 & 0 & \frac{1}{t+3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \mathbf{U}(t) := \begin{pmatrix} 1 & \frac{t}{2!} & \frac{t(t+1)}{3!} & \cdots \\ 0 & 1 & \frac{t+1}{2!} & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

All the series of meromorphic functions involved in the products of matrices in formulas

(5.4.4) and (5.4.5) converge normally on all compact subsets of $\mathbb{C}^r$. Moreover, all the entries of the matrices on the right hand side of (5.4.5) are holomorphic in the open set $U_r(q)$, translate of U_r by $(-q, 0, \ldots, 0)$. Therefore the entries of $\mathbf{W}^I(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ are also holomorphic in $U_r(q)$.

If we write $\xi_q(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ to be the first entry of $\mathbf{W}^I(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$, we then get from (5.4.4) that

$$\begin{aligned} &L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r) \\ &= \frac{1}{s_1 - 1}\, L_{r-1}(f_2, \ldots, f_r;\ s_1 + s_2 - 1, s_3, \ldots, s_r) \\ &\quad + \sum_{k=0}^{q-2} \frac{s_1 \cdots (s_1 + k - 1)}{(k+1)!} B_{k+1}\, L_{r-1}(f_2, \ldots, f_r;\ s_1 + s_2 + k, s_3, \ldots, s_r) \\ &\quad + \xi_q(f_1, \ldots, f_r;\ s_1, \ldots, s_r), \end{aligned} \tag{5.4.6}$$

where $\xi_q(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ is holomorphic in the open set $U_r(q)$.

On the other hand, the translation formula (5.3.2) and the other relations obtained by applying successively the change of variable $s_1 \mapsto s_1 + n$ to it for each $n \geq 0$, is equivalent to

$$f_1(1)\mathbf{V}_{r-1}(g_2, f_3, \ldots, f_r;\ s_1 + s_2, s_3, \ldots, s_r) = \mathbf{A_3}(f_1; s_1)\mathbf{V}_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r), \tag{5.4.7}$$

where for an indeterminate t and a non-trivial group homomorphism $f : \mathbb{Z} \to \mathbb{C}^*$ with $f(1) \neq 1$, the matrix $\mathbf{A_3}(f; t)$ is defined as follows:

$$\mathbf{A_3}(f; t) := \begin{pmatrix} 1 - f(1) & t & \frac{t(t+1)}{2!} & \frac{t(t+1)(t+2)}{3!} & \cdots \\ 0 & 1 - f(1) & t + 1 & \frac{(t+1)(t+2)}{2!} & \cdots \\ 0 & 0 & 1 - f(1) & t + 2 & \cdots \\ 0 & 0 & 0 & 1 - f(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.4.8}$$

Clearly the matrix $\mathbf{A_3}(f;t)$ is invertible in $\mathbf{T}(\mathbb{C}(t))$. To find the inverse of $\mathbf{A_3}(f;t)$ explicitly, we notice that

$$\mathbf{A_3}(f;t) = e(\mathbf{M}(t)) - f(1)\mathbf{I}_{\mathbb{N}\times\mathbb{N}}$$

where $\mathbf{I}_{\mathbb{N}\times\mathbb{N}}$ denotes the identity matrix in $\mathbf{T}(\mathbb{C}(t))$ and as in the previous chapters

$$\mathbf{M}(t) := \begin{pmatrix} 0 & t & 0 & \cdots \\ 0 & 0 & t+1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We would like to invert the power series $e^x - c$ in $\mathbb{C}[[x]]$ for some $c \neq 1$. Now for the sake of convenience, let us replace the variable x by $(c-1)y$, where $c \neq 1$. Now our aim is to invert the power series $e^{(c-1)y} - c$ when $c \neq 1$. For this we recall the generating function for the Eulerian polynomials. The Eulerian polynomials $A_n(t)$'s are defined by the following exponential generating function

$$\sum_{n\geq 0} A_n(t)\frac{y^n}{n!} = \frac{1-t}{e^{(t-1)y} - t}.$$

Thus

$$\frac{1}{e^{(c-1)y} - c} = \frac{1}{1-c}\sum_{n\geq 0} A_n(c)\frac{y^n}{n!} \quad \text{as} \quad c \neq 1.$$

Hence

$$\frac{1}{e^x - c} = \frac{1}{1-c}\sum_{n\geq 0} A_n(c)\frac{x^n}{(c-1)^n n!} \quad \text{when} \quad c \neq 1.$$

This in turn yields that the inverse of $\mathbf{A_3}(f;t)$, which we denote by $\mathbf{B_3}(f;t)$, is given by the formula

$$\mathbf{B_3}(f;t) = \frac{1}{1-f(1)}\sum_{n\geq 0} A_n(f(1))\frac{\mathbf{M}(t)^n}{(f(1)-1)^n n!} \tag{5.4.9}$$

as $f(1) \neq 1$. It can be calculated that $A_0(t) = A_1(t) = 1$. Hence

$$\mathbf{B_3}(f;t) = \frac{1}{1-f(1)} \begin{pmatrix} 1 & \frac{1}{f(1)-1}t & \frac{A_2(f(1))}{(f(1)-1)^2}\frac{t(t+1)}{2!} & \frac{A_3(f(1))}{(f(1)-1)^3}\frac{t(t+1)(t+2)}{3!} & \cdots \\ 0 & 1 & \frac{1}{f(1)-1}(t+1) & \frac{A_2(f(1))}{(f(1)-1)^2}\frac{(t+1)(t+2)}{2!} & \cdots \\ 0 & 0 & 1 & \frac{1}{f(1)-1}(t+2) & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus one may attempt to express the column vector $\mathbf{V}_r(f_1,\ldots,f_r;\ s_1,\ldots,s_r)$ in terms of $\mathbf{V}_{r-1}(g_2,f_3,\ldots,f_r;\ s_1{+}s_2,s_3,\ldots,s_r)$ so as to obtain an expression of the multiple additive L-function of depth r in terms of translates of the multiple additive L-function of depth $(r-1)$ by multiplying both sides of (5.4.7) by $\mathbf{B_3}(f_1;s_1)$. However this is not allowed as the coefficients of the Eulerian polynomials grow very fast. In fact it is known that the sum of the coefficients of $A_n(t)$ is $n!$ for each $n \geq 0$.

To get around this difficulty we do what we have been doing in last few chapters. For an integer $q \geq 1$, let $I = I_q$, $J = J_q$ be as before. Then we rewrite (5.4.7) as follows:

$$\begin{aligned} &f_1(1)\mathbf{V}^I_{r-1}(g_2,f_3,\ldots,f_r;\ s_1+s_2,s_3,\ldots,s_r) \\ &= \mathbf{A_3}^{II}(f_1;s_1)\mathbf{V}^I_r(f_1,\ldots,f_r;\ s_1,\ldots,s_r) + \mathbf{A_3}^{IJ}(f_1;s_1)\mathbf{V}^J_r(f_1,\ldots,f_r;\ s_1,\ldots,s_r). \end{aligned}$$

Now inverting $\mathbf{A_3}^{II}(f_1;s_1)$, we get

$$\begin{aligned} &\mathbf{V}^I_r(f_1,\ldots,f_r;\ s_1,\ldots,s_r) \\ &= f_1(1)\mathbf{B_3}^{II}(f_1;s_1)\mathbf{V}^I_{r-1}(g_2,f_3,\ldots,f_r;\ s_1+s_2,s_3,\ldots,s_r) + \mathbf{Z}^I(f_1,\ldots,f_r;\ s_1,\ldots,s_r), \end{aligned} \tag{5.4.10}$$

where

$$\mathbf{Z}^I(f_1,\ldots,f_r;\ s_1,\ldots,s_r) = -\mathbf{B_3}^{II}(f_1;s_1)\mathbf{A_3}^{IJ}(f_1;s_1)\mathbf{V}^J_r(f_1,\ldots,f_r;\ s_1,\ldots,s_r). \tag{5.4.11}$$

All the series of meromorphic functions involved in the products of matrices in formulas (5.4.10) and (5.4.11) converge normally on all compact subsets of $\mathbb{C}^r$. Moreover, all the entries of the matrices on the right hand side of (5.4.11) are holomorphic on the open set $U_r(q)$. Therefore the entries of $\mathbf{Z}^I(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ are also holomorphic in $U_r(q)$.

If we write $\pi_q(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ to be the first entry of $\mathbf{Z}^I(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$, we then get from (5.4.10) that

$$\begin{aligned} &L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r) \\ &= f_1(1)\, L_{r-1}(g_2, f_3, \ldots, f_r;\ s_1 + s_2, s_3, \ldots, s_r) \\ &\quad + f_1(1) \sum_{k=1}^{q-1} (s_1)_k \frac{A_k(f(1))}{(f(1)-1)^k}\, L_{r-1}(g_2, f_3, \ldots, f_r;\ s_1 + s_2 + k, s_3, \ldots, s_r) \\ &\quad + \pi_q(f_1, \ldots, f_r;\ s_1, \ldots, s_r), \end{aligned} \tag{5.4.12}$$

where $\pi_q(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ is holomorphic in the open set $U_r(q)$.

5.5 Poles and residues

In this section, using the formulas (5.4.4), (5.4.10) and the induction on depth r, we obtain a list of possible singularities of the multiple additive L-functions. When $r = 1$, from Theorem 2.2.2, we know that if $f(1) = 1$ then $L_1(f; s) = D(s, f)$ can be extended to a meromorphic function with simple pole at $s = 1$ with residue 1 and from Theorem 2.2.5, we know that if $f(1) \neq 1$ then $L_1(f; s) = D(s, f)$ can be extended to an entire function.

First we obtain an expression for the residues along the possible polar hyperplane of the multiple additive L-functions and then we deduce the exact set of singularities of these functions. A theorem of G. Frobenius [12] about the zeros of Eulerian polynomials plays a crucial role in this analysis.

5.5.1 Set of all possible singularities

The following theorem gives a description of possible singularities of the multiple additive L-functions.

Theorem 5.5.1. *The multiple additive L-function $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ has different set of singularities depending on the values of $g_i(1)$ for $1 \leq i \leq r$.*

(a) If $g_i(1) \neq 1$ for all $1 \leq i \leq r$, then $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ is a holomorphic function on $\mathbb{C}^r$.

Now let $i_1 < \cdots < i_m$ be all the indices such that $g_{i_j}(1) = 1$ for all $1 \leq j \leq m$. Then the set of all possible singularities of $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ is described as follows:

(b) If $i_1 = 1$, then $L_r(f_1, \ldots, f_r; s_1, \ldots, s_r)$ is holomorphic outside the union of hyperplanes given by the equations

$$s_1 = 1;\ s_1 + \cdots + s_{i_j} = n \text{ for all } n \in \mathbb{Z}_{\leq j} \text{ and } 2 \leq j \leq m.$$

It has at most simple poles along each of these hyperplanes.

(c) If $i_1 \neq 1$, then $L_r(f_1, \ldots, f_r; s_1, \ldots, s_r)$ is holomorphic outside the union of hyperplanes given by the equations

$$s_1 + \cdots + s_{i_j} = n \text{ for all } n \in \mathbb{Z}_{\leq j} \text{ and } 1 \leq j \leq m.$$

It has at most simple poles along each of these hyperplanes.

Proof. We prove the assertions (a), (b) and (c) separately.

Proof of (a): From Theorem 2.2.5, we know that the assertion (a) is true for depth 1 multiple additive L-function. Now let $q \geq 1$ be an integer and $I = I_q,\ J = J_q$ be as before. We will make use of equation (5.4.10) for our proof.

The entries of the first row of the matrix $\mathbf{B_3}^{II}(f_1; s_1)$ are holomorphic on $\mathbb{C}^r$ and by the induction hypothesis, the entries of the column matrix $\mathbf{V}^I_{r-1}(g_2, f_3, \ldots, f_r;\ s_1+s_2, s_3, \ldots, s_r)$ are holomorphic on $\mathbb{C}^r$. Further the entries of the column vector $\mathbf{Z}^I(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ are holomorphic in $U_r(q)$. Since the open sets $U_r(q)$ for $q \geq 1$ cover $\mathbb{C}^r$, the assertion (a) follows.

Proof of (b)*:* For depth 1 multiple additive L-function, the assertion (b) follows from Theorem 2.2.2. Again let $q \geq 1$ be an integer and $I = I_q,\ J = J_q$ be as defined earlier. We now complete the proof of assertion (b) by making use of the equation (5.4.4).

The entries of the first row of the matrix $\mathbf{B_1}^{II}(s_1 - 1)$ are holomorphic outside the hyperplane given by the equation $s_1 = 1$ and have at most simple pole along this hyperplane. By the induction hypothesis, the entries of the column vector $\mathbf{V}^I_{r-1}(f_2, \ldots, f_r;\ s_1+s_2-1, s_3, \ldots, s_r)$ are holomorphic outside the union of the hyperplanes given by the equations

$$s_1 + \cdots + s_{i_j} = n \ \text{ for all } n \in \mathbb{Z}_{\leq j}, \ \text{ for all } 2 \leq j \leq m.$$

and have at most simple poles along these hyperplanes. Finally the entries of the column vector $\mathbf{W}^I(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ are holomorphic in $U_r(q)$. Since $\mathbb{C}^r$ is covered by open sets of the form $U_r(q)$ for $q \geq 1$, the assertion (b) follows.

Proof of (c)*:* The proof of this assertion follows along the lines of the assertion (a). The only difference is the induction hypothesis. Here the induction hypothesis implies that the entries of the column matrix $\mathbf{V}^I_{r-1}(g_2, f_3, \ldots, f_r;\ s_1 + s_2, s_3, \ldots, s_r)$ are holomorphic outside the union of the hyperplanes given by the equations

$$s_1 + \cdots + s_{i_j} = n \ \text{ for all } n \in \mathbb{Z}_{\leq j}, \ \text{ for all } 1 \leq j \leq m.$$

Now the proof of the assertion (c) follows mutatis mutandis the proof of the assertion (a). This completes the proof of Theorem 5.5.1. $\square$

5.5.2 Expression for residues

Here we compute the residues of the multiple additive L-function of depth r along the possible polar hyperplanes.

Theorem 5.5.2. *The residues of the multiple additive L-function $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ are described below. Let $i_1 < \cdots < i_m$ be the indices such that $g_{i_j}(1) = 1$ for all $1 \leq j \leq m$. If $i_1 = 1$, then $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ has a polar singularity along the hyperplane given by the equation $s_1 = 1$ and the residue is the restriction of $L_{r-1}(f_2, \ldots, f_r;\ s_2, \ldots, s_r)$ to this hyperplane.*

In general, the residue along the hyperplane given by the equation

$$s_1 + \cdots + s_{i_j} = n \text{ for } n \in \mathbb{Z}_{\leq j} \text{ and } 1 \leq j \leq m$$

is the restriction of the product of $L_{r-i_j}(f_{i_j+1}, \ldots, f_r;\ s_{i_j+1}, \ldots, s_r)$ with $(0, j-n)$-th entry of

$$\mathbf{C}_j := \left(\prod_{i=1}^{i_1-1} g_i(1)\mathbf{B_3}(g_i; s_1 + \cdots + s_i) \right) \times \prod_{k=1}^{j-1} \left(\mathbf{B_1}(s_1 + \cdots + s_{i_k} - k) \prod_{i=i_k+1}^{i_{k+1}-1} g_i(1)\mathbf{B_3}(g_i; s_1 + \cdots + s_i - k) \right)$$

to the above hyperplane.

Proof. First suppose that $i_1 = 1$. Then for any integer $q \geq 1$, we deduce from (5.4.6) and Theorem 5.5.1 that

$$L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r) - \frac{1}{s_1 - 1} L_{r-1}(f_2, \ldots, f_r;\ s_1 + s_2 - 1, s_3, \ldots, s_r)$$

has no pole inside the open set $U_r(q)$ along the hyperplane given by the equation $s_1 = 1$. These open sets cover $\mathbb{C}^r$ and hence the residue of $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ along the hyperplane given by the equation $s_1 = 1$ is the restriction of the meromorphic function $L_{r-1}(f_2, \ldots, f_r;\ s_2, \ldots, s_r)$

to the hyperplane given by the equation $s_1 = 1$. This proves the first part of Theorem 5.5.2.

Next let $q \geq j - n + 1$ be an integer and $I = I_q,\ J = J_q$ be as before. Now to determine the residue along the hyperplane

$$s_1 + \cdots + s_{i_j} = n \ \text{ for } n \in \mathbb{Z}_{\leq j} \ \text{ and } 1 \leq j \leq m,$$

we iterate the formulas (5.4.4) and (5.4.10) according to the applicable case and obtain that

$$\begin{aligned} &\mathbf{V}_r^I(f_1, \ldots, f_r;\ s_1, \ldots, s_r) \\ &= \mathbf{C}_j^{II} \mathbf{V}_{r-i_j+1}^I(g_{i_j}, f_{i_j+1}, \ldots, f_r;\ s_1 + \cdots + s_{i_j} - (j-1), s_{i_j+1}, \ldots, s_r) \\ &\quad + \mathbf{Z}^{j,I}(f_1, \ldots, f_r;\ s_1, \ldots, s_r). \end{aligned} \tag{5.5.1}$$

Here $\mathbf{Z}^{j,I}(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ is a column matrix whose entries are finite sums of products of rational functions in $s_1, \ldots, s_{i_j-1}$ with meromorphic functions which are holomorphic in $U_r(q)$. These entries therefore have no singularity along the hyperplane given by the equation $s_1 + \cdots + s_{i_j} = n$ in $U_r(q)$. The entries of $\mathbf{C}_j^{II}$ are rational functions in $s_1, \ldots, s_{i_j-1}$ and hence again have no singularity along the above hyperplane. It now follows from the first part of Theorem 5.5.2 that the only entry of

$$\mathbf{V}_{r-i_j+1}^I(g_{i_j}, f_{i_j+1}, \ldots, f_r;\ s_1 + \cdots + s_{i_j} - (j-1), s_{i_j+1}, \ldots, s_r)$$

that can possibly have a pole in $U_r(q)$ along the hyperplane given by the equation $s_1 + \cdots + s_{i_j} = n$ is the one of index $j - n$, which is

$$L_{r-i_j+1}(g_{i_j}, f_{i_j+1}, \ldots, f_r;\ s_1 + \cdots + s_{i_j} - n + 1, s_{i_j+1}, \ldots, s_r).$$

Again by the first part of Theorem 5.5.2, the residue of this function along the hyperplane given by the equation $s_1 + \cdots + s_{i_j} = n$ is the restriction of $L_{r-i_j}(f_{i_j+1}, \ldots, f_r;\ s_{i_j+1}, \ldots, s_r)$

to this hyperplane. Thus the residue of $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ along the hyperplane given by the equation $s_1 + \cdots + s_{i_j} = n$ is the product of $L_{r-i_j}(f_{i_j+1}, \ldots, f_r;\ s_{i_j+1}, \ldots, s_r)$ with $(0, j-n)$-th entry of $\mathbf{C}_j^{II}$. Since open sets of the form $U_r(q)$ cover $\mathbb{C}^r$, this completes the proof of Theorem 5.5.2. □

5.5.3 Exact set of singularities

Now we will deduce the exact set of singularities of the multiple additive L-functions. As mentioned earlier, a theorem of Frobenius [12] about the zeros of Eulerian polynomials helps us in determining the exact set of singularities. We recall that the Eulerian polynomials $A_n(t)$'s are defined by the following exponential generating function

$$\sum_{n \geq 0} A_n(t) \frac{y^n}{n!} = \frac{1-t}{e^{(t-1)y} - t}.$$

These polynomials satisfy the following recurrence relation

$$A_0(t) = 1 \ \text{ and } \ A_n(t) = t(1-t)A'_{n-1}(t) + A_{n-1}(t)(1 + (n-1)t) \ \text{ for all } \ n \geq 1.$$

Frobenius [12] proved the following theorem about the zeros of the Eulerian polynomials $A_n(t)$.

Theorem 5.5.3 (Frobenius). *All the zeros of the Eulerian polynomials $A_n(t)$ are real, negative and simple.*

From this theorem and the above recurrence formula, we can now deduce the following corollary.

Corollary 5.5.4. *For any $n \geq 0$, $A_{n+1}(t)$ and $A_n(t)$ do not have a common zero.*

Proof. To see this, let a be a zero of $A_n(t)$. Then by Theorem 5.5.3, $A'_n(a) \neq 0$. Again by Theorem 5.5.3, $a \neq 0, 1$. Using the above recurrence formula, we deduce that $A_{n+1}(a) \neq 0$. □

Recall that for an indeterminate t and a non-trivial additive character $f : \mathbb{Z} \to \mathbb{C}^*$, we have

$$\mathbf{B_1}(t) = \begin{pmatrix} \frac{1}{t} & \frac{B_1}{1!} & \frac{(t+1)B_2}{2!} & \frac{(t+1)(t+2)B_3}{3!} & \cdots \\ 0 & \frac{1}{t+1} & \frac{B_1}{1!} & \frac{(t+2)B_2}{2!} & \cdots \\ 0 & 0 & \frac{1}{t+2} & \frac{B_1}{1!} & \cdots \\ 0 & 0 & 0 & \frac{1}{t+3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\mathbf{B_3}(f;t) = \frac{1}{1-f(1)} \begin{pmatrix} 1 & \frac{1}{f(1)-1}t & \frac{A_2(f(1))}{(f(1)-1)^2}\frac{t(t+1)}{2!} & \frac{A_3(f(1))}{(f(1)-1)^3}\frac{t(t+1)(t+2)}{3!} & \cdots \\ 0 & 1 & \frac{1}{f(1)-1}(t+1) & \frac{A_2(f(1))}{(f(1)-1)^2}\frac{(t+1)(t+2)}{2!} & \cdots \\ 0 & 0 & 1 & \frac{1}{f(1)-1}(t+2) & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Clearly non-zero elements of each row and each column of these matrices are linearly independent as elements of $\mathbb{C}(t)$. Also the first two entries of the first row of these matrices are non-zero. Since we know that the Bernoulli numbers

$$B_n = 0 \iff n \text{ is odd and } n \geq 3, \tag{5.5.2}$$

we get that at least one of the first two entries in every column of $\mathbf{B_1}(t)$ is non-zero. On the other hand, Corollary 5.5.4 implies that at least one of the first two entries in every column of $\mathbf{B_3}(f;t)$ is non-zero. Hence we know that all entries of the first row of the matrices $\mathbf{B_1}(t_1)\,\mathbf{B_1}(t_2)$, $\mathbf{B_1}(t_1)\,\mathbf{B_3}(f;t_2)$, $\mathbf{B_3}(f;t_1)\,\mathbf{B_1}(t_2)$ and $\mathbf{B_3}(f;t_1)\,\mathbf{B_3}(f;t_2)$ are non-zero. Here t_1 and t_2 are two indeterminates.

With this observation in place, we are now ready to determine the exact set of singularities of the multiple additive L-functions.

Theorem 5.5.5. *The exact set of polar singularities of the multiple additive L-function $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ of depth r differs from the set of all possible singularities (as listed in Theorem 5.5.1) only in the following two cases. Here we keep the notations of Theorem 5.5.1.*

a) If $i_1 = 1$ and $i_2 = 2$ i.e. $f_1(1) = f_2(1) = 1$, then $L_r(f_1, \ldots, f_r; s_1, \ldots, s_r)$ is holomorphic outside the union of hyperplanes given by the equations

$$s_1 = 1;\ s_1 + s_2 = n \textit{ for all } n \in \mathbb{Z}_{\leq 2} \setminus J;$$
$$s_1 + \cdots + s_{i_j} = n \textit{ for all } n \in \mathbb{Z}_{\leq j},\ 3 \leq j \leq m,$$

where $J := \{-2n - 1 : n \in \mathbb{N}\}$. It has simple poles along each of these hyperplanes.

b) If $i_1 = 2$, then $L_r(f_1, \ldots, f_r; s_1, \ldots, s_r)$ is holomorphic outside the union of hyperplanes given by the equations

$$s_1 + s_2 = n \textit{ for all } n \in \mathbb{Z}_{\leq 1} \setminus J;$$
$$s_1 + \cdots + s_{i_j} = n \textit{ for all } n \in \mathbb{Z}_{\leq j},\ 2 \leq j \leq m,$$

where $J := \{1 - n : n \in I\}$ and $I := \{n \in \mathbb{N} : A_n(f_1(1)) = 0\}$. It has simple poles along each of these hyperplanes.

Proof. When $1 \leq j \leq m$ and $n \in \mathbb{Z}_{\leq j}$, the restriction of $L_{r-i_j}(f_{i_j+1}, \ldots, f_r;\ s_{i_j+1}, \ldots, s_r)$ to the hyperplane given by the equation $s_1 + \cdots + s_{i_j} = n$ is a non-zero meromorphic function.

First suppose that $i_1 = 1$ and $i_2 = 2$. Then by (5.5.2), we deduce that only non-zero entries in the first row of $\mathbf{C}_2$ are of index $(0, 1)$ and of index $(0, 2n)$ for $n \in \mathbb{N}$. Also we know that all the entries in the first row of $\mathbf{C}_j$ for $3 \leq j \leq m$ are non-zero. We now conclude from Theorem 5.5.2 that the exact set of singularities in this case consists of the hyperplanes given by the equations

$$s_1 = 1; \ s_1 + s_2 = n \text{ for all } n \in \mathbb{Z}_{\leq 2} \setminus J;$$
$$s_1 + \cdots + s_{i_j} = n \text{ for all } n \in \mathbb{Z}_{\leq j}, \text{ for all } 3 \leq j \leq m,$$

where $J := \{-2n - 1 : n \in \mathbb{N}\}$. This completes the proof of assertion (a).

Next let $i_1 = 2$. Clearly the entries in the first row of $\mathbf{C}_1$ that are zero are of index $(0, n)$ for all $n \in I$. Also we know that all entries in the first row of $\mathbf{C}_j$ for $2 \leq j \leq m$ are non-zero. Hence using Theorem 5.5.2, we have that the exact set of singularities in this case consists of the hyperplanes given by the equations

$$s_1 + s_2 = n \text{ for all } n \in \mathbb{Z}_{\leq 1} \setminus J;$$
$$s_1 + \cdots + s_{i_j} = n \text{ for all } n \in \mathbb{Z}_{\leq j}, \text{ for all } 2 \leq j \leq m,$$

where $J := \{1 - n : n \in I\}$.

Now in all other cases, applying Theorem 5.5.1, we see that the hyperplanes given by the equations of the form $s_1 + s_2 = n$ are not singularities of $L_r(f_1, \ldots, f_r; \ s_1, \ldots, s_r)$. Using Theorem 5.5.2, we know that the expression for residues along the possible polar hyperplanes given by the equations of the form $s_1 + \cdots + s_{i_j} = n$ involves product of at least two matrices of the form $\mathbf{B_1}(x)$ and $\mathbf{B_3}(f; y)$ where x and y are two different indeterminates. Hence such an expression is non-zero. This completes the proof of Theorem 5.5.5. □

Example 5.5.6. We know that $t = -1$ is a zero for the Eulerian polynomials $A_n(t)$ only when n is even. Suppose that we are in a case when $f_1(1) = -1$ and $i_1 = 2$ i.e. $f_1(1) = f_2(1) = -1$. Now Theorem 5.5.5 implies that the exact set of singularities of $L_r(f_1, \ldots, f_r; \ s_1, \ldots, s_r)$

consists of the hyperplanes given by the equation

$$s_1 + s_2 = n \text{ for all } n \in \mathbb{Z}_{\leq 1} \setminus J;$$
$$s_1 + \cdots + s_{i_j} = n \text{ for all } n \in \mathbb{Z}_{\leq j},\ 2 \leq j \leq m,$$

where $J := \{-2n - 1 : n \in \mathbb{N}\}$. Further $L_r(f_1, \ldots, f_r;\ s_1, \ldots, s_r)$ has simple poles along these hyperplanes.

5.6 Multiple Dirichlet L-functions

Though at this moment it seems difficult to determine the exact set of singularities of the multiple Dirichlet L-functions, we can still extend the theorem of Akiyama and Ishikawa (Theorem 5.1.3) for Dirichlet characters, not necessarily of same modulus. As an immediate consequence of (5.1.8) and Theorem 5.5.1, we derive the following theorem.

Theorem 5.6.1. *Let $r \geq 1$ be an integer and $\chi_1, \ldots, \chi_r$ be Dirichlet characters of arbitrary modulus. Then the multiple Dirichlet L-function $L_r(s_1, \ldots, s_r;\ \chi_1, \ldots, \chi_r)$ of depth r can be extended as a meromorphic function to $\mathbb{C}^r$ with possible simple poles at the hyperplanes given by the equations*

$$s_1 = 1;\ s_1 + \cdots + s_i = n \ \textit{for all}\ \ n \in \mathbb{Z}_{\leq i},\ 2 \leq i \leq r.$$

5.7 Multiple Lerch zeta functions

We recall that the Lerch zeta function $L(\lambda, \alpha, s)$ is defined by the following convergent sum in $\Re(s) > 1$:

$$L(\lambda, \alpha, s) := \sum_{n \geq 0} \frac{e(\lambda n)}{(n + \alpha)^s},$$

where for a real number a, $e(a)$ denotes the uni-modulus complex number $e^{2\pi\iota a}$ and $\lambda, \alpha \in (0,1]$. This is an example of a shifted Dirichlet series associated to additive characters. Now we can consider multi-variable generalisations of the Lerch zeta function, and more generally of the shifted Dirichlet series associated to additive characters. Such functions, which we studied in [15], can be viewed as a unification of the multiple Hurwitz zeta functions and the multiple additive L-functions.

Definition 5.7.1. Let $r \geq 1$ be an integer and $\lambda_1, \ldots, \lambda_r, \alpha_1, \ldots, \alpha_r \in [0,1)$. Then the multiple Lerch zeta function of depth r is denoted by $L_r(\lambda_1, \ldots, \lambda_r; \alpha_1, \ldots, \alpha_r; s_1, \ldots, s_r)$ and defined by the following convergent series in U_r:

$$L_r(\lambda_1, \ldots, \lambda_r; \alpha_1, \ldots, \alpha_r; s_1, \ldots, s_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{e(\lambda_1 n_1) \cdots e(\lambda_r n_r)}{(n_1 + \alpha_1)^{s_1} \cdots (n_r + \alpha_r)^{s_r}}.$$

Normal convergence of the above series follows from the normal convergence of the multiple Hurwitz zeta function of depth r. Hence $L_r(\lambda_1, \ldots, \lambda_r; \alpha_1, \ldots, \alpha_r; s_1, \ldots, s_r)$ defines an analytic function in the open set U_r. Next we define the shifted multiple additive L-functions, which further generalise the multiple Lerch zeta functions.

Definition 5.7.2. Let $r \geq 1$ be an integer and $\alpha_1, \ldots, \alpha_r \in [0,1)$. Further suppose $f_1, \ldots, f_r$ are additive characters such that the partial products $g_i := \prod_{1 \leq j \leq i} f_j$ satisfy the condition

$$|g_i(1)| \leq 1 \text{ for all } 1 \leq i \leq r.$$

Then the shifted multiple additive L-function of depth r associated to $f_1, \ldots, f_r$ and $\alpha_1, \ldots, \alpha_r$ is denoted by $L_r(f_1, \ldots, f_r;\ \alpha_1, \ldots, \alpha_r;\ s_1, \ldots, s_r)$ and defined by the following convergent series n U_r:

$$L_r(f_1, \ldots, f_r;\ \alpha_1, \ldots, \alpha_r;\ s_1, \ldots, s_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{f_1(n_1) \cdots f_r(n_r)}{(n_1 + \alpha_1)^{s_1} \cdots (n_r + \alpha_r)^{s_r}}.$$

Again as a consequence of the normal convergence of the multiple Hurwitz zeta function of depth r, we get that the above series is normally summable on compact subsets of U_r, and hence defines an analytic function in the open set U_r.

Now the meromorphic continuation and the location of possible polar singularities of the shifted multiple additive L-function $L_r(f_1, \ldots, f_r;\ \alpha_1, \ldots, \alpha_r;\ s_1, \ldots, s_r)$ are described in the following anticipated theorem, whose proof we omit for brevity.

Theorem 5.7.3. *The set of all possible singularities of the shifted multiple additive L-function $L_r(f_1, \ldots, f_r;\ \alpha_1, \ldots, \alpha_r;\ s_1, \ldots, s_r)$ depends on the values of $g_i(1)$ for $1 \leq i \leq r$.*

a) If $g_i(1) \neq 1$ for all $1 \leq i \leq r$, then $L_r(f_1, \ldots, f_r;\ \alpha_1, \ldots, \alpha_r;\ s_1, \ldots, s_r)$ can be extended analytically to the whole of $\mathbb{C}^r$.

Now let $i_1 < \cdots < i_m$ be all the indices such that $g_{i_j}(1) = 1$ for all $1 \leq j \leq m$, then the set of all possible singularities of $L_r(f_1, \ldots, f_r;\ \alpha_1, \ldots, \alpha_r;\ s_1, \ldots, s_r)$ is described in the following two cases.

b) If $i_1 = 1$, then $L_r(f_1, \ldots, f_r;\ \alpha_1, \ldots, \alpha_r;\ s_1, \ldots, s_r)$ can be extended analytically to an open subset V_r of $\mathbb{C}^r$, where V_r is obtained by removing the hyperplanes given by the equations

$$s_1 = 1; s_1 + \cdots + s_{i_j} = n \text{ for all } n \in \mathbb{Z}_{\leq j}, \text{ for all } 2 \leq j \leq m$$

from $\mathbb{C}^r$. It has at most simple poles along each of these hyperplanes.

c) If $i_1 \neq 1$, then $L_r(f_1, \ldots, f_r;\ \alpha_1, \ldots, \alpha_r;\ s_1, \ldots, s_r)$ can be extended analytically to an open subset W_r of $\mathbb{C}^r$, where W_r is obtained by removing the hyperplanes given by the equations

$$s_1 + \cdots + s_{i_j} = n \text{ for all } n \in \mathbb{Z}_{\leq j}, \text{ for all } 1 \leq j \leq m$$

from $\mathbb{C}^r$. It has at most simple poles along each of these hyperplanes.

Here we would like to mention that an intricate multi-variable generalisation of the Lerch zeta function was studied by Y. Komori [20]. He derived the meromorphic continuation of these functions through their integral representations. Such an integral representation is analogous to one such integral representation of the Riemann zeta function. He also obtained a 'rough estimation' of its possible singularities (see [20], §3.6). It seems Theorem 5.7.3 above provides a more precise information about the singularities of the multiple Lerch zeta functions.

6

Weighted multiple zeta functions

6.1 Introduction

Euler [11] considered (for $r = 2$) a variant of the multiple zeta function $\zeta(s_1, \ldots, s_r)$, where the summation over the sequences $(n_1, \ldots, n_r)$ is replaced by $n_1 \geq \cdots \geq n_r \geq 1$ in place of $n_1 > \cdots > n_r > 0$. For an integer $r \geq 1$,

$$\zeta_r^{\text{Euler}}(s_1, \ldots, s_r) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_r \geq 1} n_1^{-s_1} \cdots n_r^{-s_r}. \tag{6.1.1}$$

These functions are closely related to the multiple zeta functions. It is easy to see that

$$\zeta_2^{\text{Euler}}(s_1, s_2) = \zeta_2(s_1, s_2) + \zeta(s_1 + s_2).$$

More generally, these functions can be expressed as linear combinations of the usual multiple zeta functions of various depths. For this, we need the following notation. We say that a partition $(A_1, \ldots, A_t)$ of $\{1, \ldots, r\}$ is admissible if each subset A_i is non empty, and the elements of A_i are smaller than the elements of A_j when $i < j$. Let $\mathcal{P}_r$ denote the set of all admissible partitions of $\{1, \ldots, r\}$. We then have

$$\zeta_r^{\text{Euler}}(s_1, \ldots, s_r) = \sum_{(A_1, \ldots, A_t) \in \mathcal{P}_r} \zeta_t \left(\sum_{i \in A_1} s_i, \ldots, \sum_{i \in A_t} s_i \right). \tag{6.1.2}$$

The above formula holds on the open subset U_r of $\mathbb{C}^r$. All functions involved in the right hand side of (6.1.2) are holomorphic on U_r. Hence $\zeta_r^{\text{Euler}}(s_1, \ldots, s_r)$ also defines a holomorphic function on U_r. Further, it is also possible to extend it to a meromorphic function on $\mathbb{C}^r$ using the meromorphic continuation of the multiple zeta functions and (6.1.2).

In this chapter, we study the following weighted variant of the functions considered by Euler. For an integer $r \geq 1$,

$$\zeta_r^{\text{weight}}(s_1, \ldots, s_r) := \sum_{n_1 \geq \cdots \geq n_r \geq 1} \frac{{n_1}^{-s_1} \cdots {n_r}^{-s_r}}{w(n_1, \ldots, n_r)}. \tag{6.1.3}$$

Here for a sequence of integer $n_1 \geq \cdots \geq n_r$, $w(n_1, \ldots, n_r)$ denotes the number of permutations σ of $\{1, \ldots, r\}$ such that $n_{\sigma(i)} = n_i$ for all $1 \leq i \leq r$. In other words, $w(n_r, \ldots, n_r)$ denotes the order of the stabilizer of $(n_1, \ldots, n_r)$ in the symmetric group S_r, where the group action is the permutation of the co-ordinates. We call these functions as weighted multiple zeta functions. These functions can also be expressed as linear combinations of the usual multiple zeta functions of various depths and we have

$$\zeta_r^{\text{weight}}(s_1, \ldots, s_r) = \sum_{(A_1, \ldots, A_t) \in \mathcal{P}_r} \frac{1}{|A_1|! \cdots |A_t|!} \zeta_t \left(\sum_{i \in A_1} s_i, \ldots, \sum_{i \in A_t} s_i \right). \tag{6.1.4}$$

These weighted multiple zeta functions have some special properties, which are comparable to the ones satisfied by the multiple zeta functions. We give couple of instances here.

(a) Product formulas: It is known that, when $(s_1, \ldots, s_p) \in U_p$ and $(t_1, \ldots, t_q) \in U_q$, then

$$\zeta_p(s_1, \ldots, s_p)\, \zeta_q(t_1, \ldots, t_q) = \sum \zeta_r(z_1, \ldots, z_r)$$

holds, where $(z_1, \ldots, z_r)$ runs over the family of finite sequences of complex numbers deduced from $(s_1, \ldots, s_p)$ and $(t_1, \ldots, t_q)$ by a stuffling (see §6.3 for the precise definition of this term). We shall prove in §6.3 that a similar formula holds for weighted multiple zeta functions, but with stufflings replaced by shufflings. For example,

$$\zeta(s_1)\zeta_2(s_2, s_3) = \zeta_3(s_1, s_2, s_3)+\zeta_3(s_2, s_1, s_3)+\zeta_3(s_2, s_3, s_1)+\zeta_2(s_1+s_2, s_3)+\zeta_2(s_2, s_1+s_3),$$

whereas

$$\zeta_1^{\text{weight}}(s_1)\zeta_2^{\text{weight}}(s_2, s_3) = \zeta_3^{\text{weight}}(s_1, s_2, s_3) + \zeta_3^{\text{weight}}(s_2, s_1, s_3) + \zeta_3^{\text{weight}}(s_2, s_3, s_1).$$

(b) Location of poles: Due to some cancellations of residues, the set of polar hyperplanes of our weighted multiple zeta functions is smaller than that of the usual ones. More precisely, we shall prove in §6.4 that the meromorphic extension of $\zeta_r^{\text{weight}}(s_1, \ldots, s_r)$ to $\mathbb{C}^r$ is in fact holomorphic outside the hyperplanes given by the equations

$$s_1 = 1;\ s_1 + \cdots + s_i = i - 2k \ \text{ for all } \ 2 \le i \le r \ \text{ and } \ k \ge 0 \ \text{ an integer},$$

and it has simple poles along each of these hyperplanes. The location of the poles in this case has an uniform pattern, which was not the case for the usual multiple zeta functions. Here we use some of the notations from Chapter 3.

6.2 Inversion formula

Interestingly, the formulas (6.1.2) and (6.1.4) can be inverted to express the usual multiple zeta functions in terms of the functions $\zeta_r^{\text{Euler}}(s_1, \ldots, s_r)$ and the weighted multiple zeta functions respectively. To do this, it is more useful to see these formulas in a more general setup. We elaborate below.

Let $f = \sum_{n\geq 1} a_n x^n$ be a formal power series with complex coefficients without constant term. We define

$$\zeta^{(f)}(s_1,\dots,s_r) := \sum_{(A_1,\dots,A_t)\in\mathcal{P}_r} a_{|A_1|}\cdots a_{|A_t|}\,\zeta_t\left(\sum_{i\in A_1} s_i,\dots,\sum_{i\in A_t} s_i\right). \tag{6.2.1}$$

According to the above definition $\zeta_r^{\text{Euler}} = \zeta^{(f)}$ where $f = \frac{x}{1-x}$ and $\zeta_r^{\text{weight}} = \zeta^{(f)}$ where $f = e^x - 1$.

Let $g = \sum_{n\geq 1} b_n x^n$ be another formal power series with complex coefficients and without constant term. We then define

$$(\zeta^{(f)})^{(g)}(s_1,\dots,s_r) := \sum_{(A_1,\dots,A_t)\in\mathcal{P}_r} b_{|A_1|}\cdots b_{|A_t|}\,\zeta^{(f)}\left(\sum_{i\in A_1} s_i,\dots,\sum_{i\in A_t} s_i\right). \tag{6.2.2}$$

With these notations we now prove the following theorem.

Theorem 6.2.1. *Let $f = \sum_{n\geq 1} a_n x^n$, $g = \sum_{n\geq 1} b_n x^n$ be two formal power series with complex coefficients without constant term. Then we have $(\zeta^{(f)})^{(g)} = \zeta^{(f\circ g)}$.*

Proof. By formulas (6.2.1) and (6.2.2), we get that $(\zeta^{(f)})^{(g)}(s_1,\dots,s_r)$ is equal to

$$\sum_{(B_1,\dots,B_s)\in\mathcal{P}_r} b_{|B_1|}\cdots b_{|B_s|} \sum_{(A_1,\dots,A_t)\in\mathcal{P}_s} a_{|A_1|}\cdots a_{|A_t|}\,\zeta_t\left(\sum_{i\in A_1}\sum_{j\in B_i} s_j,\dots,\sum_{i\in A_t}\sum_{j\in B_i} s_j\right).$$

When $(B_1,\dots,B_s)\in\mathcal{P}_r$ and $(A_1,\dots,A_t)\in\mathcal{P}_s$, then $(C_1,\dots,C_t)$ where $C_k = \bigcup_{i\in A_k} B_i$, is an admissible partition of $\{1,\dots,r\}$ and we have

$$\sum_{i\in A_k}\sum_{j\in B_i} s_j = \sum_{i\in C_k} s_i$$

for all $1\leq k\leq t$. Thus $(\zeta^{(f)})^{(g)}(s_1,\dots,s_r)$ is equal to

$$\sum_{(C_1,\dots,C_t)\in\mathcal{P}_r} \zeta_r\left(\sum_{i\in C_1} s_i,\dots,\sum_{i\in C_t} s_i\right)\prod_{k=1}^{t}\left(\sum_{i\geq 1} a_i \sum_{j_1+\cdots+j_i=|C_k|} b_{j_1}\cdots b_{j_i}\right).$$

If $\sum_{n\geq 1} c_n x^n$ denotes the formal power series $f \circ g$, then we have

$$c_{|C_k|} = \sum_{i\geq 1} a_i \sum_{j_1+\cdots+j_i=|C_k|} b_{j_1}\cdots b_{j_i}.$$

This completes the proof of Theorem 6.2.1. □

As an immediate consequence we get the following corollary.

Corollary 6.2.2. *Let $f = \sum_{n\geq 1} a_n x^n$ be a formal power series and $g = \sum_{n\geq 1} b_n x^n$ be the formal power series such that $f \circ g = x$. Then for an integer $r \geq 1$, we have*

$$\zeta_r(s_1,\ldots,s_r) = \sum_{(A_1,\ldots,A_t)\in\mathcal{P}_r} b_{|A_1|}\cdots b_{|A_t|}\, \zeta^{(f)}\left(\sum_{i\in A_1} s_i,\ldots,\sum_{i\in A_t} s_i\right).$$

Applying Corollary 6.2.2 for $f = \frac{x}{1-x}$ and $g = \frac{x}{1+x} = \sum_{n\geq 1}(-1)^{n-1}x^n$, we get

$$\zeta_r(s_1,\ldots,s_r) = \sum_{(A_1,\ldots,A_t)\in\mathcal{P}_r} (-1)^{r-t}\zeta_t^{\text{Euler}}\left(\sum_{i\in A_1} s_i,\ldots,\sum_{i\in A_t} s_i\right). \tag{6.2.3}$$

For $f = e^x - 1$ and $g = \log(1+x) = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}x^n$ we get

$$\zeta_r(s_1,\ldots,s_r) = \sum_{(A_1,\ldots,A_t)\in\mathcal{P}_r} \frac{(-1)^{r-t}}{|A_1|\cdots|A_t|}\zeta_t^{\text{weight}}\left(\sum_{i\in A_1} s_i,\ldots,\sum_{i\in A_t} s_i\right). \tag{6.2.4}$$

It follows from Theorem 3.4.1, Theorem 3.6.1 and the definition of $\zeta^{(f)}(s_1,\ldots,s_r)$ that it can be extended meromorphically to $\mathbb{C}^r$ and has possible simple poles along the hyperplanes $H_{1,0}$ and $H_{i,k}$, where $2 \leq i \leq r$ and $k \geq 0$. We end this section by the following analogue of Remark 3.6.4.

Theorem 6.2.3. *Let $1 \leq i \leq r$ and $k \geq 0$. The meromorphic function*

$$\zeta^{(f)}(s_1,\ldots,s_r) - \zeta^{(f)}(s_1,\ldots,s_i)\zeta^{(f)}(s_{i+1},\ldots,s_r) \tag{6.2.5}$$

has no pole along the hyperplane $H_{i,k}$ of $\mathbb{C}^r$ defined by equation $s_1+\cdots+s_i=i-k$. In other words, the residue of $\zeta^{(f)}(s_1,\dots,s_r)$ along this hyperplane is the product of $\zeta^{(f)}(s_{i+1},\dots,s_r)$ with the residue of $\zeta^{(f)}(s_1,\dots,s_i)$ along the hyperplane of $\mathbb{C}^i$ defined by the same equation.

Proof. By replacing each $\zeta^{(f)}$ by its expression (6.2.1), we see that the meromorphic function (6.2.5) is the sum over all $(A_1,\dots,A_t)$ in $\mathcal{P}_r$, of $a_{|A_1|}\cdots a_{|A_t|}$ times the meromorphic function

$$\zeta_t\left(\sum_{j\in A_1}s_j,\dots,\sum_{j\in A_t}s_j\right)-\zeta_p\left(\sum_{j\in A_1}s_j,\dots,\sum_{j\in A_p}s_j\right)\zeta_{t-p}\left(\sum_{j\in A_{p+1}}s_j,\dots,\sum_{j\in A_t}s_j\right), \quad (6.2.6)$$

if i is the largest element of one of the subsets A_p, and times the meromorphic function

$$\zeta_t\left(\sum_{j\in A_1}s_j,\dots,\sum_{j\in A_t}s_j\right) \quad (6.2.7)$$

otherwise. But the meromorphic functions occurring in (6.2.6) have no singularity along $H_{i,k}$ by Remark 3.6.4, and those occurring in (6.2.7) have no singularity along $H_{i,k}$ by Theorem 3.6.1. This completes the proof. □

6.3 Product of weighted multiple zeta functions

We first recall the notion of shuffling and stuffling. Let p and q be two non-negative integers. We define a *stuffling* of p and q to be a pair (A,B) of sets such that $|A|=p$, $|B|=q$ and $A\cup B=\{1,\dots,r\}$ for some integer r. We then have $\max(p,q)\le r\le p+q$. We call this r to be the *length of the stuffling*. Such a stuffling is called a *shuffling* when A and B are disjoint, i.e. when $r=p+q$.

Let $(s_1,\dots,s_p)$ and $(t_1,\dots,t_q)$ be two sequences of complex numbers and (A,B) be a stuffling of p and q, with $A\cup B=\{1,\dots,r\}$. Let σ and τ denote the unique increasing bijections from $A\to\{1,\dots,p\}$ and $B\to\{1,\dots,q\}$ respectively. Let us define a sequence of complex numbers $(z_1, \ldots, z_r)$ as follows:

$$z_i := \begin{cases} s_{\sigma(i)} & \text{when } i \in A \setminus B; \\ t_{\tau(i)} & \text{when } i \in B \setminus A; \\ s_{\sigma(i)} + t_{\tau(i)} & \text{when } i \in A \cap B. \end{cases}$$

We call it the *sequence deduced from* $(s_1, \ldots, s_p)$ *and* $(t_1, \ldots, t_q)$ *by the stuffling* (A, B). Clearly, if $(s_1, \ldots, s_p)$ belongs to the open set U_p and $(t_1, \ldots, t_q)$ to U_q, then $(z_1, \ldots, z_r)$ belongs to U_r.

It is well known that, for $(s_1, \ldots, s_p) \in U_p$ and $(t_1, \ldots, t_q) \in U_q$, we have

$$\zeta_p(s_1, \ldots, s_p)\, \zeta_q(t_1, \ldots, t_q) = \sum_{(A,B)} \zeta_r(z_1, \ldots, z_r), \tag{6.3.1}$$

where the index of the summation on the right hand side runs over the stufflings of p and q, and $(z_1, \ldots, z_r)$ denotes the sequence deduced from $(s_1, \ldots, s_p)$ and $(t_1, \ldots, t_q)$ by this stuffling.

Our purpose in this section is to prove that the weighted multiple zeta functions have similar properties, where stufflings are replaced by shufflings. More precisely:

Theorem 6.3.1. *For* $(s_1, \ldots, s_p) \in U_p$ *and* $(t_1, \ldots, t_q) \in U_q$, *we have*

$$\zeta_p^{\textit{weight}}(s_1, \ldots, s_p)\, \zeta_q^{\textit{weight}}(t_1, \ldots, t_q) = \sum_{(A,B)} \zeta_{p+q}^{\textit{weight}}(z_1, \ldots, z_{p+q}), \tag{6.3.2}$$

where in the summation on the right hand side, (A, B) *runs over the shufflings of* p *and* q, *and* $(z_1, \ldots, z_{p+q})$ *denotes the sequence deduced from* $(s_1, \ldots, s_p)$ *and* $(t_1, \ldots, t_q)$ *by this shuffling.*

Proof. By formula (6.1.4), the right hand side of (6.3.2) is equal to

$$\sum_{(A,B)} \sum_{(C_1,\ldots,C_t)\in\mathcal{P}_{p+q}} \frac{1}{|C_1|!\cdots|C_t|!}\, \zeta_t\left(\sum_{i\in C_1} z_i, \ldots, \sum_{i\in C_t} z_i\right), \tag{6.3.3}$$

where (A, B) runs over the shufflings of p and q, and $(z_1, \dots, z_{p+q})$ denotes the sequence deduced from $(s_1, \dots, s_p)$ and $(t_1, \dots, t_q)$ by this shuffling.

For a given choice of (A, B) and $(C_1, \dots, C_t)$, let σ denote the unique increasing bijection from $A \to \{1, \dots, p\}$ and τ denote the unique increasing bijection from $B \to \{1, \dots, q\}$. Let

$$A' := \{i : 1 \leq i \leq t \text{ and } C_i \cap A \neq \emptyset\}$$

and

$$B' := \{i : 1 \leq i \leq t \text{ and } C_i \cap B \neq \emptyset\}.$$

Suppose that p' and q' denotes the cardinalities of A' and B' respectively. Then $A' \cup B' = \{1, \dots, t\}$ and (A', B') is a stuffling of p' and q'. Let σ' denote the unique increasing bijection from $\{1, \dots, p'\} \to A'$ and τ' denote the unique increasing bijection from $\{1, \dots, q'\} \to B'$. Then the sequence $(A_1, \dots, A_{p'})$, where $A_i = \sigma(A \cap C_{\sigma'(i)})$, is an admissible partition of $\{1, \dots, p\}$ and $(B_1, \dots, B_{q'})$, where $B_i = \tau(B \cap C_{\tau'(i)})$, is an admissible partition of $\{1, \dots, q\}$. Moreover, $\left(\sum_{i \in C_1} z_i, \dots, \sum_{i \in C_t} z_i\right)$ is the sequence deduced from the sequences $\left(\sum_{i \in A_1} s_i, \dots, \sum_{i \in A_{p'}} s_i\right)$ and $\left(\sum_{i \in B_1} t_i, \dots, \sum_{i \in B_{q'}} t_i\right)$ by the stuffling (A', B') of p' and q'.

Note that on the other hand, when an admissible partition $(A_1, \dots, A_{p'})$ of $\{1, \dots, p\}$, an admissible partition $(B_1, \dots, B_{q'})$ of $\{1, \dots, q\}$ and a stuffling (A', B') of p' and q' are given, they fully determine $(C_1, \dots, C_t)$, but correspond to

$$\prod_{i=1}^{t} \frac{|C_i|!}{|A \cap C_i|!|B \cap C_i|!} = \frac{|C_1|! \cdots |C_t|!}{|A_1|! \cdots |A_{p'}|!|B_1|! \cdots |B_{q'}|!}$$

many different choices of the pair (A, B). Hence expression (6.3.3) can be written as

$$\sum_{(A_1,\ldots,A_{p'})\in\mathcal{P}_p}\ \sum_{(B_1,\ldots,B_{q'})\in\mathcal{P}_q} \frac{1}{|A_1|!\cdots|A_{p'}|!|B_1|!\cdots|B_{q'}|!} \sum_{(A',B')} \zeta_t\left(\sum_{i\in C_1} z_i, \ldots, \sum_{i\in C_t} z_i\right)$$
$$= \sum_{(A_1,\ldots,A_{p'})\in\mathcal{P}_p}\ \sum_{(B_1,\ldots,B_{q'})\in\mathcal{P}_q} \frac{\zeta_{p'}\left(\sum_{i\in A_1} s_i, \ldots, \sum_{i\in A_{p'}} s_i\right)\zeta_{q'}\left(\sum_{i\in B_1} t_i, \ldots, \sum_{i\in B_{q'}} t_i\right)}{|A_1|!\cdots|A_{p'}|!|B_1|!\cdots|B_{q'}|!}$$
$$= \zeta_p^{\text{weight}}(s_1, \ldots, s_p)\, \zeta_q^{\text{weight}}(t_1, \ldots, t_q).$$

□

6.4 Singularities of weighted multiple zeta functions

We have already seen in §6.2 that the weighted multiple zeta function of depth r extends to a meromorphic function on $\mathbb{C}^r$, which is holomorphic outside the hyperplanes $H_{1,0}$ and $H_{i,k}$, where $2 \leq i \leq r$ and $k \geq 0$, and has at most simple poles along these hyperplanes. Our goal in this section is to show that, due to some residue cancellations, not all of them are polar hyperplanes. The precise list of polar hyperplanes has already been mentioned in the introduction of this chapter.

For our purpose it is convenient to introduce another variant of the multiple zeta functions. We define

$$\tilde{\zeta}_r(s_1, \ldots, s_r) := \sum_{n_1\geq\cdots\geq n_r\geq 1} 2^{t-r} n_1^{-s_1}\cdots n_r^{-s_r}, \tag{6.4.1}$$

where t denotes the number of distinct terms in the sequence $(n_1, \ldots, n_r)$. We can write $\tilde{\zeta}$ as $\zeta^{(f)}$ with the notations of §6.2, where f is the formal power series $\sum_{n\geq 1} 2^{1-n}x^n = \frac{2x}{2-x}$. Hence as per our discussion in §6.2, we know that $\tilde{\zeta}_r(s_1, \ldots, s_r)$ can be extended meromorphically to $\mathbb{C}^r$ and has possible simple poles along the hyperplanes $H_{1,0}$ and $H_{i,k}$, where $2 \leq i \leq r$ and $k \geq 0$. The following theorem about the singularities of $\tilde{\zeta}_r(s_1, \ldots, s_r)$ is used to determine the singularities of the weighted multiple zeta functions.

For this we need to set up some notations. Let us define the modified Bernoulli numbers $\widetilde{B}_n$ for $n \geq 0$ as follows:

$$\widetilde{B}_n := \begin{cases} B_n & \text{if } n \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$\widetilde{B}_n := \begin{cases} B_n & \text{if } n \text{ is even}, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding generating series is

$$\sum_{n \geq 0} \frac{\widetilde{B}_n}{n!} x^n = \frac{x}{e^x - 1} + \frac{x}{2} = \frac{x}{2 \tanh \frac{x}{2}}.$$

Let t be an indeterminate and $\widetilde{\mathbf{B_1}}(t)$ denote the infinite $\mathbb{N} \times \mathbb{N}$ upper triangular matrix with coefficients in $\mathbb{Q}(t)$, defined in the same way as the matrix $\mathbf{B_1}(t)$ in formula (3.5.7), except that we replace the Bernoulli numbers B_n in $\mathbf{B_1}(t)$ by their modified counterparts $\widetilde{B}_n$.

With the above notations, we prove:

Theorem 6.4.1. *Let $r \geq 1$ and $k \geq 0$. The residue of the meromorphic function $\tilde{\zeta}_r(s_1, \ldots, s_r)$ along the hyperplane $H_{r,k}$ is the $(0, k)^{th}$ entry of the matrix $\prod_{d=1}^{r-1} \widetilde{\mathbf{B_1}}(s_1 + \cdots + s_d - d)$. In other words, with the notations of Remark 3.6.5, it is given by the formula*

$$\sum_{\substack{k_1, \ldots, k_{r-1} \geq 0 \\ k_1 + \cdots + k_{r-1} = k}} F_{k_1, \ldots, k_{r-1}}(s_1, \ldots, s_{r-1}) \frac{\widetilde{B}_{k_1}}{k_1!} \cdots \frac{\widetilde{B}_{k_{r-1}}}{k_{r-1}!}. \tag{6.4.2}$$

Moreover, when $r \geq 2$, this residue is 0 if and only if k is odd.

Proof. The meromorphic function $\tilde{\zeta}(s_1, \ldots, s_r)$ is the sum of the meromorphic functions

$$2^{t-r} \zeta_t \left(\sum_{i \in A_1} s_i, \ldots, \sum_{i \in A_t} s_i \right), \tag{6.4.3}$$

indexed over all $(A_1,\ldots,A_t)\in\mathcal{P}_r$. Note that the equation $s_1+\cdots+s_r=r-k$ of the hyperplane $H_{r,k}$ can be written as

$$\sum_{i\in A_1} s_i+\cdots+\sum_{i\in A_t} s_i = t-(k+t-r).$$

Therefore by Remark 3.6.5, the residue of the meromorphic function (6.4.3) along this hyperplane is

$$2^{t-r}\sum_{\substack{\ell_1,\ldots,\ell_{t-1}\geq 0\\ \ell_1+\cdots+\ell_{t-1}=k+t-r}} F_{\ell_1,\ldots,\ell_{t-1}}\left(\sum_{i\in A_1}s_i,\ldots,\sum_{i\in A_{t-1}}s_i\right)\frac{B_{\ell_1}}{\ell_1!}\cdots\frac{B_{\ell_{t-1}}}{\ell_{t-1}!}.$$

For a given sequence $(\ell_1,\ldots,\ell_{t-1})$ of non-negative integers with sum $k+t-r$, by formulas (3.6.2) and (3.6.3), we have

$$F_{\ell_1,\ldots,\ell_{t-1}}\left(\sum_{i\in A_1}s_i,\ldots,\sum_{i\in A_{t-1}}s_i\right)=F_{k_1,\ldots,k_{r-1}}(s_1,\ldots,s_{r-1}), \tag{6.4.4}$$

where k_i is equal to ℓ_j when i is the largest element of the subset A_j for some index j such that $1\leq j\leq t-1$, and k_i is equal to 1 otherwise. We then have $k_1+\cdots+k_{r-1}=k$. This observation allows to write the residue of the meromorphic function (6.4.3) along $H_{r,k}$ as

$$\sum_{\substack{k_1,\ldots,k_{r-1}\geq 0\\ k_i=1 \text{ if } i\notin J\\ k_1+\cdots+k_{r-1}=k}} F_{k_1,\ldots,k_{r-1}}(s_1,\ldots,s_{r-1})\left(\frac{1}{2}\right)^{r-1-|J|}\prod_{i\in J}\frac{B_{k_i}}{k_i!},$$

where J denotes the set consisting of the largest elements of the subsets $A_1,\ldots,A_{t-1}$. The map $(A_1,\ldots,A_t)\to J$ is a bijection from the set $\mathcal{P}_r$ of admissible partition of $\{1,\ldots,r\}$ onto the set of subsets of $\{1,\dots,r-1\}$. Hence the residue of $\tilde{\zeta}(s_1,\dots,s_r)$ along $H_{r,k}$ is

$$\sum_{J\subset\{1,\dots,r-1\}}\ \sum_{\substack{k_1,\dots,k_{r-1}\geq 0\\ k_i=1 \text{ if } i\notin J\\ k_1+\cdots+k_{r-1}=k}} F_{k_1,\dots,k_{r-1}}(s_1,\dots,s_{r-1})\left(\frac{1}{2}\right)^{r-1-|J|}\prod_{i\in J}\frac{B_{k_i}}{k_i!}$$

$$=\sum_{\substack{k_1,\dots,k_{r-1}\geq 0\\ k_1+\cdots+k_{r-1}=k}} F_{k_1,\dots,k_{r-1}}(s_1,\dots,s_{r-1})\sum_{J\subset\{i\in\{1,\dots,r-1\}:k_i=1\}}\left(\frac{1}{2}\right)^{r-1-|J|}\prod_{i\in J}\frac{B_{k_i}}{k_i!}$$

$$=\sum_{\substack{k_1,\dots,k_{r-1}\geq 0\\ k_1+\cdots+k_{r-1}=k}} F_{k_1,\dots,k_{r-1}}(s_1,\dots,s_{r-1})\frac{\widetilde{B}_{k_1}}{k_1!}\cdots\frac{\widetilde{B}_{k_{r-1}}}{k_{r-1}!}.$$

Here the last equality follows from the fact that $\frac{\widetilde{B}_n}{n!}$ is equal to $\frac{1}{2}+B_1$ when $n=1$ and $\frac{B_n}{n!}$ when $n\neq 1$. This completes the proof of formula (6.4.2). Now since $\widetilde{B}_n=0$ for odd n, $\widetilde{B}_n\neq 0$ for even n, and the rational functions $F_{k_1,\dots,k_{r-1}}$ in the $(r-1)$ variables $s_1,\dots,s_{r-1}$ are linearly independent over $\mathbb{Q}$ by Remark 3.6.5, the last assertion of the lemma follows. □

We are now ready to determine the singularities of the weighted multiple zeta functions.

Theorem 6.4.2. *The weighted multiple zeta function of depth r is holomorphic outside the union of the hyperplanes $H_{1,0}$ and $H_{i,k}$, where $2\leq i\leq r$, $k\geq 0$ and k is even. It has simple poles along each of these hyperplanes.*

Proof. Theorem 6.2.3 tells us that the residue of $\zeta_r^{\text{weight}}(s_1,\dots,s_r)$ along the hyperplane $H_{1,0}$ is $\zeta_{r-1}^{\text{weight}}(s_2,\dots,s_r)$ and its residue along the hyperplane $H_{i,k}$, where $2\leq i\leq r$ and $k\geq 0$, is the product of $\zeta_{r-i}^{\text{weight}}(s_{i+1},\dots,s_r)$ with the residue of $\zeta_i^{\text{weight}}(s_1,\dots,s_i)$ along the hyperplane of $\mathbb{C}^i$ defined by the equation $s_1+\cdots+s_i=i-k$.

Therefore to prove Theorem 6.4.2, we are reduced to prove that, when $r\geq 2$ and $k\geq 0$, the residue of $\zeta_r^{\text{weight}}(s_1,\dots,s_r)$ along the hyperplane $H_{r,k}$ is different from 0 if and only if k is even.

Recall that the weighted multiple zeta function ζ^{weight} is equal to $\zeta^{(h)}$, where h is the formal power series $\sum_{n\geq 1}\frac{x^n}{n!}=e^x-1$. Note that $h=f\circ g$, where f is the formal power series

$\sum_{n\geq 1} 2^{1-n}x^n = \frac{2x}{2-x}$ and $g = \sum_{n\geq 1} b_n x^n$ is the formal power series expansion of $2\tanh\frac{t}{2}$. Hence we have $\zeta^{\text{weight}} = \tilde{\zeta}^{(g)}$ by Theorem 6.2.1. In other words, we have for $(s_1,\ldots,s_r)$ in U_r,

$$\zeta_r^{\text{weight}}(s_1,\ldots,s_r) = \sum_{(A_1,\ldots,A_t)\in\mathcal{P}_r} b_{|A_1|}\cdots b_{|A_t|}\tilde{\zeta}_t\left(\sum_{i\in A_1} s_i,\ldots,\sum_{i\in A_t} s_i\right). \tag{6.4.5}$$

Here, like before, $\mathcal{P}_r$ denotes the set of all admissible partitions of $\{1,\ldots,r\}$.

We now assume $r \geq 2$ and $k \geq 0$ and investigate the residue of the weighted multiple zeta function $\zeta^{\text{weight}}(s_1,\ldots,s_r)$ along the hyperplane $H_{r,k}$. By formula (6.4.5), this residue is the sum, over all $(A_1,\ldots,A_t)\in\mathcal{P}_r$, of $b_{|A_1|}\cdots b_{|A_t|}$ times the residue along this hyperplane of $\tilde{\zeta}_t\left(\sum_{i\in A_1} s_i,\ldots,\sum_{i\in A_t} s_i\right)$. Here the numbers b_n are the coefficients of the generating series $2\tanh\frac{t}{2} = \sum_{n\geq 1} b_n x^n$. Note that $b_n = 0$ for even n. Therefore, we only have to sum over all $(A_1,\ldots,A_t)\in\mathcal{P}_r$ such that each subset A_j has odd number of elements. For such a partition, the residue of $\tilde{\zeta}\left(\sum_{i\in A_1} s_i,\ldots,\sum_{i\in A_t} s_i\right)$ along $H_{r,k}$ vanishes when k is odd, by Theorem 6.4.1. Hence the residue of $\zeta^{\text{weight}}(s_1,\ldots,s_r)$ along $H_{r,k}$ is zero when k is odd.

Next assume k to be even. Then Theorem 6.4.1 tells us that the residue of $\tilde{\zeta}(s_1,\ldots,s_r)$ along $H_{r,k}$ is a non-zero linear combination of the rational functions $F_{k_1,\ldots,k_{r-1}}(s_1,\ldots,s_{r-1})$, where all the k_i's are even. On the other hand, for any $(A_1,\ldots,A_t)\in\mathcal{P}_r$ where at least one of the sets A_j is not a singleton, Theorem 6.4.1 and formula (6.4.4) show that the residue of $\tilde{\zeta}\left(\sum_{i\in A_1} s_i,\ldots,\sum_{i\in A_t} s_i\right)$ along $H_{r,k}$ is a linear combination of the rational functions $F_{k_1,\ldots,k_{r-1}}(s_1,\ldots,s_{r-1})$, where at least one of the k_i's is equal to 1. By the linear independence of the rational functions $F_{k_1,\ldots,k_{r-1}}(s_1,\ldots,s_{r-1})$ (see Remark 3.6.5), we deduce that the residue of $\zeta_r^{\text{weight}}(s_1,\ldots,s_r)$ along $H_{r,k}$ is not equal to 0. This completes the proof of Theorem 6.4.2. □

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