ARITHMETIC PROPERTIES OF GENERALISED EULER-BRIGGS CONSTANTS

By

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I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

Journal

1. "Transcendence of generalized Euler-Lehmer constants", Sanoli Gun, Ekata Saha and Sneh Bala Sinha, *J. Number Theory*, **2014**, *145*, 329-339.

2. "A note on generalized Briggs-Lehmer constants", Sanoli Gun and Ekata Saha, *Ramanujan Math. Soc. Lect. Notes Ser.*, **2016**, *23*, 93-104.

3. "Linear and algebraic independence of Generalized Euler-Briggs constants", Sanoli Gun, V. Kumar Murty and Ekata Saha, *J. Number Theory*, **2016**, *166*, 117-136.

Ekata Saha

Dedicated to my Parents & Teachers

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NOTATIONS

Symbol	Description
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The ring of rational integers
$\mathbb{Q}$	The field of rational numbers
$\overline{\mathbb{Q}}$	The field of algebraic numbers
$\mathbb{C}$	The field of complex numbers
$\mathscr{O}_K$	The ring of integers of a number field K
$K[X]$	Polynomial ring in one indeterminate X over a field K
Δ_K	Discriminant of a number field K
Ω	A finite set of prime numbers
P_Ω	The product of all the primes in a finite set of primes Ω
(a,b)	The greatest common divisor of two natural numbers a and b
$a \mid b$	a divides b
ζ_n	A primitive n-th root of unity
$\Phi_n(X)$	The n-th cyclotomic polynomial
φ	The Euler's totient function
μ	The Möbius function
Λ	The von Mangoldt function
Γ	The gamma function
Ψ	The digamma function
$\zeta(s)$	The Riemann zeta function
$\Re(s)$	The real part of a complex number s
$\widehat{G}$	The group of characters of a finite abelian group G

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SYNOPSIS

Introduction

The central object of study in this doctoral thesis is Euler's constant γ and its various generalisations. Our research is devoted to study their arithmetic properties such as their transcendence, linear independence and algebraic independence.

In 1731, Euler introduced γ as the following limit

$$\gamma := \lim_{x\to\infty}\left(\sum_{n\leq x}\frac{1}{n} - \log x\right).$$

Besides proving the existence of the limit, he also expressed this limit as a conditionally convergent sum. More precisely, he showed that

$$\gamma = \sum_{n\geq 2}(-1)^n\frac{\zeta(n)}{n}.$$

Further, γ appears as the constant term in the Laurent series expansion of the Riemann zeta function around $s = 1$.

The nature of γ continues to elude us though it is almost three centuries since its inception. The central theme of our doctoral research is to study the arithmetic nature of families of numbers of which γ is a member, rather than studying γ in isolation. These families owe their origin to analytic as well as arithmetic contexts.

This approach is evident in the works of Murty-Saradha [26] and Rivoal [38]. While Rivoal studied γ along with e and Euler-Gompertz constant $\delta := \int_0^\infty \frac{e^{-w}}{1+w} dw$, Murty-Saradha studied it as a member of a family of constants which were introduced by Briggs [7] and studied extensively by Lehmer [21].

Diamond and Ford [10], trying to relate e^γ with the Riemann hypothesis, defined another family of generalisations of γ, whose arithmetic properties were studied by R. Murty with Zaytseva [27].

In our work, we study a family of numbers which unifies the two seemingly different families of constants introduced by Briggs and Diamond-Ford respectively. We call these unified constants as generalised Euler-Briggs constants. The first part of our thesis revolves around the transcendence of these family of numbers.

Transcendental results

Definition 0.0.1 (Generalised Euler-Briggs constants)**.** For a finite set of primes Ω and natural numbers a, q with $(q, P_\Omega) = 1$, the constants

$$\gamma(\Omega, a, q) := \lim_{x \to \infty} \left(\sum_{\substack{n \le x \\ n \equiv a \bmod q \\ (n, P_\Omega) = 1}} \frac{1}{n} - \delta_\Omega \frac{\log x}{q} \right)$$

are called generalised Euler-Briggs constants. Here $P_\Omega := \prod_{p \in \Omega} p$ and $\delta_\Omega := \prod_{p \in \Omega} \left(1 - \frac{1}{p}\right)$.

For these constants, we along with Gun and Sinha prove the following theorems in [16].

Theorem 0.0.2. *Let a and $q > 1$ be natural numbers with $(a, q) = 1$. Also let*

$$U := \{\, \Omega \mid \Omega \text{ is a finite set of primes},\ (q, P_\Omega) = 1 \}.$$

Then the set $T := \{\gamma(\Omega, a, q) \mid \Omega \in U\}$ has at most one algebraic element.

In our next theorem, we work with a fixed set of finitely many primes.

Theorem 0.0.3. *Let Ω be a finite set of primes and $S = \{q_1, q_2, \cdots\}$ be an infinite set of mutually co-prime natural numbers $q_i > 1$ with $(q_i, P_\Omega) = 1$ for all $i \geq 1$. Then for any $a \in \mathbb{N}$ co-prime to q_i for all i, the set $T := \{\ \gamma(\Omega, a, q_i)\ |\ q_i \in S\}$ has at most one algebraic element.*

The following general transcendental result plays a crucial role in some of our investigations. This we obtain by using Baker's theory of linear forms in logarithm and theory of cyclotomic units, in particular, Ramachandra units.

Theorem 0.0.4. *Let $q_1, q_2 > 1$ be natural numbers with $(q_1, q_2) = 1$ and Ω be a finite set of primes co-prime to $q_1 q_2$. Also let $\alpha_p, \mu_r, \beta_\chi, \beta'_\chi, \eta_\psi$ and η'_ψ be algebraic numbers, where p and r vary over the prime divisors of $q_1 q_2$ and elements of Ω respectively and χ, ψ are non-trivial characters modulo q_1 and q_2 respectively. Then the number*

$$\sum_{p | q_1 q_2} \alpha_p \log p + \sum_{r \in \Omega} \mu_r \log r + \sum_{\substack{\chi\ even \\ \chi \neq \chi_0}} \beta_\chi L(1, \chi) + \sum_{\substack{\psi\ even \\ \psi \neq \psi_0}} \eta_\psi L(1, \psi) + \sum_{\chi\ odd} \beta'_\chi L(1, \chi) + \sum_{\psi\ odd} \eta'_\psi L(1, \psi)$$

is transcendental provided not all $\alpha_p, \mu_r, \beta_\chi, \eta_\psi$'s are zero.

Linear independence results

In the second part of our work, we consider questions of linear independence of these constants over number fields and $\overline{\mathbb{Q}}$. The following theorem proved jointly with Gun [15] constitutes a main ingredient for the study of linear independence of these constants.

Theorem 0.0.5. *Let f be a periodic arithmetic function with period $q \geq 1$ and M be a natural number co-prime to q. Then the series*

$$\sum_{\substack{n \geq 1 \\ (n,M)=1}} \frac{f(n)}{n}$$

converges if and only if $\sum_{a=1}^{q} f(a) = 0$ and in which case, it is equal to $\sum_{a=1}^{q} f(a)\gamma(\Omega, a, q)$. Here Ω is the set of prime divisors of M.

Thus the question of linear independence of generalised Euler-Briggs constants is now reduced to non-vanishing of certain periodic Dirichlet series at the point $s = 1$. The following theorem of Baker, Birch and Wirsing [5], which answers a question of Chowla, therefore enters naturally into our investigation.

Theorem 0.0.6 (Baker, Birch and Wirsing)**.** *Let f be a non-zero algebraic valued periodic arithmetical function with period $q \geq 1$. Also let $f(n) = 0$ whenever $1 < (n,q) < q$ and the q-th cyclotomic polynomial $\Phi_q(X)$ be irreducible over $\mathbb{Q}(f(1), \cdots, f(q))$, then $\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$.*

Building upon these results, we prove the following theorem in [14], which gives a lower bound on the dimension of the vector space generated by these numbers.

Theorem 0.0.7. *Let Ω be a finite set of primes. Consider the $\mathbb{Q}$-vector space*

$$V_{\mathbb{Q},N} := \mathbb{Q}\left\langle \gamma(\Omega, m, n) \mid 1 \leq m \leq n \leq N, \;\; (m,n) = 1 = (n, P_\Omega) \right\rangle.$$

Then for N sufficiently large, we have $N \ll_\Omega \dim_{\mathbb{Q}} V_{\mathbb{Q},N}$. In particular, the dimension of the $\mathbb{Q}$-vector space

$$V_{\mathbb{Q}} := \mathbb{Q}\left\langle \gamma(\Omega, m, n) \mid m, n \in \mathbb{N}, \; (m,n) = 1 = (n, P_\Omega) \right\rangle$$

is infinite.

In fact, we have the following general theorem about linear independence of these constants over any number field.

Theorem 0.0.8. *Let K be a number field with discriminant $d > 1$, Ω be a finite set of primes such that $K \cap \mathbb{Q}(\zeta_{P_\Omega}) = \mathbb{Q}$, where $\zeta_{P_\Omega} := e^{\frac{2\pi i}{P_\Omega}}$. Consider the K-vector space*

$$V_{K,N} := K\,\langle \gamma(\Omega, m, n) \mid 1 \le m \le n \le N,\ (m,n) = 1 = (n, dP_\Omega)\rangle\,.$$

Then for N sufficiently large, we have

$$N \ \ll_{K,\Omega}\ \dim_K V_{K,N}.$$

In particular, the K-vector space

$$V_K := K\,\langle \gamma(\Omega, m, n) \mid m, n \in \mathbb{N},\ (m,n) = 1 = (n, dP_\Omega)\rangle$$

is infinite dimensional.

Note that the trivial upper bounds for $\dim_{\mathbb{Q}} V_{\mathbb{Q},N}$ in Theorem 0.0.7 and for $\dim_K V_{K,N}$ in Theorem 0.0.8 are N^2.

We now study the analogous linear spaces over $\overline{\mathbb{Q}}$. For this, we shall need to introduce some notions.

For a finite set of primes Ω and $a \in \mathbb{N}$, consider $C(a,\Omega) := \{q \in \mathbb{N} \mid\ (a,q) = 1 = (q, P_\Omega)\}$. We define an equivalence relation on the set $X := \{\gamma(\Omega, a, q) : q \in C(a,\Omega)\}$, given by $\gamma(\Omega, a, q_1) \sim \gamma(\Omega, a, q_2)$ if $\gamma(\Omega, a, q_1) = \lambda\gamma(\Omega, a, q_2)$ for some $\lambda \in \overline{\mathbb{Q}}^*$. Then we have the following theorem for subsets of X.

Theorem 0.0.9. *Let Ω be a finite set of primes and $a \in \mathbb{N}$. Let Y be a subset of $C(a,\Omega)$, consisting of co-prime integers. Then in $\{\gamma(\Omega, a, q) : q \in Y\}$, each equivalence class $[\gamma(\Omega, a, q)]$, where the equivalence relation is restricted to Y, has at most two elements.*

In particular, it follows that the following $\overline{\mathbb{Q}}$ linear space

$$V_{\overline{\mathbb{Q}}} := \overline{\mathbb{Q}}\langle \gamma(\Omega, m, n) \mid m, n \in \mathbb{N},\ (m, n) = 1 = (n, P_\Omega)\rangle$$

has dimension at least two.

It is also possible to derive similar theorems for family of generalised Euler-Briggs constants where we vary Ω's with fixed q. For this, define

$$C(q) := \{\Omega \mid \Omega \text{ is a finite set of primes and } (q, P_\Omega) = 1\}.$$

As before, for a fixed a with $(a, q) = 1$, one can define a similar equivalence relation on the set $Z := \{\gamma(\Omega, a, q) : \Omega \in C(q)\}$. In this set-up, we have the following theorem.

Theorem 0.0.10. *The orbit of any element $\gamma(\Omega, a, q) \in Z$ has at most two elements.*

In particular, it follows from above that the following $\overline{\mathbb{Q}}$ linear space

$$W_{\overline{\mathbb{Q}}} := \overline{\mathbb{Q}}\langle \gamma(\Omega, a, q) \mid \Omega \in C(q)\rangle$$

has dimension at least two.

However, we prove the following much stronger result using different tools.

Theorem 0.0.11. *The vector space $W_{\overline{\mathbb{Q}}}$ is infinite dimensional over $\overline{\mathbb{Q}}$.*

This summarises our study of linear independence of generalised Euler-Briggs constants.

Algebraic independence results

In the penultimate part of our work, we study the question of algebraic independence of these constants. Algebraic independence of numbers is rather delicate with very few known results and we shall need to assume the weak Schanuel conjecture which we recall below.

Conjecture 0.0.12 (weak Schanuel)**.** *Let $\alpha_1, \cdots, \alpha_n$ be non-zero algebraic numbers such that the numbers $\log\alpha_1, \cdots, \log\alpha_n$ are $\mathbb{Q}$-linearly independent. Then $\log\alpha_1, \cdots, \log\alpha_n$ are algebraically independent.*

We shall just illustrate one of our results in this direction. For this, let us introduce some more notations. For $a, q \in \mathbb{N}$ with $1 \le a \le q$ and $(a,q) = 1$, define

$$\gamma^*(\Omega, a, q) := \frac{q\gamma(\Omega, a, q)}{\delta_\Omega}.$$

We call a finite set $\{\Omega_1, \cdots, \Omega_n\}$ of sets to be irreducible if $\cup_{i=1}^n \Omega_i \neq \cup_{j\in J}\Omega_j$ for any proper subset $J \subset \{1, \cdots, n\}$. We call an infinite set X of sets to be irreducible if every finite subset of X is irreducible. In this context we prove the following theorem in [14].

Theorem 0.0.13. *Suppose that the weak Schanuel conjecture is true. Let $q \in \mathbb{N}$ and T be an infinite set consisting of finite subsets Ω of primes with $(P_\Omega, q) = 1$. Consider the set*

$$S_1 := \{\gamma^*(\Omega, a, q) - \gamma - \sum_{\chi \neq \chi_0} \alpha^*_{\chi,\Omega,q} L(1,\chi) \mid \Omega \in T\},$$

where χ runs over non-principal Dirichlet character modulo q and

$$\alpha^*_{\chi,\Omega,q} := \overline{\chi}(a) \prod_{p\in\Omega}\left(1 - \frac{\chi(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-1} \prod_{p|q}\left(1 - \frac{1}{p}\right)^{-1}.$$

Then the elements of S_1 are algebraically independent if the infinite set T is irreducible.

Generalised Euler-Briggs constants and certain infinite series

In the final part of our work, we study the non-vanishing of certain series which are variants of the Hurwitz zeta function. In this context, we prove the following theorem.

Theorem 0.0.14. *Let M, q be co-prime natural numbers and f be a periodic arithmetic function with period q. Also let $a < b$ be co-prime natural numbers such that $a \equiv 0 \bmod M$. Then the infinite series*

$$\sum_{\substack{n \geq 0 \\ (n,M)=1}} \frac{f(n)}{n + a/b}$$

converges if and only if $\sum_{t=1}^{q} f(t) = 0$. In that case, the sum is equal to

$$b \sum_{t=0}^{q-1} f(t)\, \gamma(\Omega,\ a + tb,\ bq).$$

Here Ω is the set of prime divisors of M.

We also study sums of the type

$$\sum_{\substack{n \geq 0 \\ (n,M)=1}} \frac{A(n)}{B(n)},$$

where $A(X), B(X) \in \mathbb{Q}[X]$ are non-zero polynomials and link such sums to linear combinations of generalised Euler-Briggs constants.

CHAPTER 1

INTRODUCTION

1.1 Euler's constant

Perhaps the most fundamental object of study in number theory is the Riemann zeta function $\zeta(s)$. For a complex number s with $\Re(s) > 1$, the Riemann zeta function $\zeta(s)$ is defined by the absolutely convergent series

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}.$$

Sometimes it is also called the Euler-Riemann zeta function. Leonhard Euler was the first person to introduce this function as a function of a real variable and studied the special values of this function at positive integral points. More than a century later, in 1859, Bernhard Riemann [37] extended Euler's definition to complex numbers and studied its analytic properties.

While studying the special values of the Riemann zeta function at positive integral points, Euler derived that $\zeta(2) = \pi^2/6$. In fact he could prove a general result about the Riemann zeta values at even arguments. He proved that for all $n \geq 1$,

$$\zeta(2n) = \frac{(-1)^{n-1}2^{2n-1}B_{2n}}{(2n)!}\pi^{2n}.$$

Here B_n denotes the n-th Bernoulli number which is defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}.$$

However Euler could not succeed in obtaining similar results for Riemann zeta values at odd integers. In his quest to assign a value for $\zeta(1)$, he introduced the constant γ as the limit

$$\gamma := \lim_{x\to\infty}\left(\sum_{1\leq n\leq x}\frac{1}{n} - \log x\right).$$

The above constant is now known as Euler's constant. The convergence of the above series can be easily proved by using Abel's partial summation formula which we recall below.

Theorem 1.1.1. *Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers and f is a continuously differentiable function on $[y,x]$ where $0 < y < x$. Let $A(t) := \sum_{n\leq t} a_n$. Then*

$$\sum_{y<n\leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\,dt.$$

For a proof of Theorem 1.1.1, we refer the reader to [2], page 77. Now by Theorem 1.1.1, we can write

$$\begin{aligned}\sum_{n\leq x}\frac{1}{n} &= \frac{[x]}{x} + \int_1^x \frac{[t]}{t^2}dt \\ &= 1 - \frac{\{x\}}{x} + \log x - \int_1^x \frac{\{t\}}{t^2}dt.\end{aligned}$$

Hence

$$\sum_{n\leq x}\frac{1}{n} - \log x = \left(1 - \int_1^\infty \frac{\{t\}}{t^2}dt\right) + \int_x^\infty \frac{\{t\}}{t^2}dt - \frac{\{x\}}{x}.$$

Letting $x \to \infty$, we see that the limit

$$\lim_{x\to\infty}\left(\sum_{n\le x}\frac{1}{n}-\log x\right)$$

exists. Besides proving the existence of this limit, Euler also expressed this limit as the conditionally convergent sum

$$\gamma = \sum_{n\ge 2}(-1)^n\frac{\zeta(n)}{n}.$$

Euler obtained this formula using the expansion

$$\log(1+x) = \sum_{k\ge 1}(-1)^{k+1}\frac{x^k}{k} \quad \text{for} \quad |x|<1.$$

Note that this formula is also valid for $|x| = 1$ except for $x = -1$.

While Euler was trying to find the explicit decimal expansion of γ, one of his key idea was to express γ by means of infinite series involving other known constants. For example, he observed that

$$\gamma = \sum_{n\ge 1}\frac{n}{n+1}(\zeta(n+1)-1)$$

and

$$1-\gamma = \sum_{n\ge 2}\frac{1}{n}(\zeta(n)-1).$$

These two absolutely convergent series expressions are of particular interest, for instance they imply that $0<\gamma<1$. Euler could calculate γ up to 15 decimal places and in particular he obtained

$$\gamma = 0.577215664901532\cdots.$$

As of 2013, γ has been calculated up to 119377958182 decimal digits by A. J. Yee (see http://www.numberworld.org/digits/EulerGamma/ for more details).

1.2 Significance of γ

Euler's constant γ makes its appearance in several themes. To start with, it occurs in the context of the ubiquitous gamma function

$$\Gamma(t) := \int_0^\infty x^{t-1} e^{-x} dx$$

defined for $t > 0$. This is one of the main reasons to denote Euler's constant by the Greek lower case 'γ', though Euler used the notation 'C' to denote this constant.

Euler himself noticed that $\Gamma'(1) = -\gamma$. Thus γ features in the Taylor series expansion of $\Gamma(t)$ around $t = 1$. In addition to that, γ also appears in the Hadamard product expansion of the entire function $\frac{1}{\Gamma(z)}$ which is

$$\frac{1}{\Gamma(z)} := z e^{\gamma z} \prod_{n \geq 1} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}.$$

Euler's constant is also connected to the digamma function which is defined to be the logarithmic derivative of gamma function. More precisely, for any real number $x \neq 0, -1, -2, \ldots$ the digamma function $\Psi(x)$ can be defined as

$$-\Psi(x) := \gamma + \frac{1}{x} + \sum_{n \geq 1} \left(\frac{1}{n+x} - \frac{1}{n}\right).$$

In particular, $-\Psi(1) = \gamma$, as noticed by Euler. Gauss gave an explicit formula for the digamma function at rational numbers involving γ. For the digamma function Euler proved the following theorem.

Theorem 1.2.1 (Euler, 1765)**.** *The digamma function has Taylor series expansion given by*

$$\Psi(z+1) = -\gamma + \sum_{k \geq 1} (-1)^{k+1} \zeta(k+1) z^k,$$

where the series on the right hand side converges absolutely for $|z| < 1$.

The constant γ further appears in the Laurent series expansion of the Riemann zeta function. This fact was first proved by Stieltjes.

Theorem 1.2.2 (Stieltjes, 1885)**.** *The Laurent series expansion of $\zeta(s)$ around $s = 1$ is given by*

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n\geq 1} \frac{(-1)^n}{n!}\gamma_n (s-1)^n,$$

where for $n \geq 1$, $\gamma_n := \lim_{m\to\infty}\left(\sum_{1\leq k\leq m} \frac{(\log k)^n}{k} - \frac{(\log m)^{n+1}}{n+1}\right)$.

Euler's constant is also connected to several formulations of the Riemann hypothesis, which states that all complex zeros of the Riemann zeta function off the real line have real part equal to $\frac{1}{2}$. The first result in this direction seems to be the following one due to Ramanujan, where he linked the Riemann hypothesis with the sum of divisor function

$$\sigma(n) := \sum_{d|n} d.$$

Theorem 1.2.3 (Ramanujan [1, 31, 35, 36])**.** *If Riemann hypothesis is true, then for n_0 large enough*

$$\frac{\sigma(n)}{n \log\log n} < e^\gamma$$

for all $n \geq n_0$.

At this point we also mention the following theorem of Gronwall.

Theorem 1.2.4 (Gronwall [11])**.** *One has*

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n\log\log n} = e^\gamma.$$

In 1984, Robin improved Ramanujan's theorem and proved:

Theorem 1.2.5 (Robin [39])**.** *The Riemann hypothesis is equivalent to the inequality*

$$\frac{\sigma(n)}{n\log\log n} < e^\gamma,$$

for all $n \geq 5041$.

In [9], authors showed that the above inequality holds for squarefree integers greater than 30 and odd integers greater than 9 and thereby reformulated the above equivalent condition of the Riemann hypothesis.

The following equivalent formulations of the Riemann hypothesis are due to Nicolas.

Theorem 1.2.6 (Nicolas [29])**.** *The Riemann hypothesis is equivalent to the inequality*

$$\frac{P_k}{\varphi(P_k)\log\log P_k} > e^{\gamma},$$

for all $k \geq 2$*. Here* P_k *denotes the product of first* k *primes and* φ *denotes the Euler's totient function.*

Theorem 1.2.7 (Nicolas [30])**.** *For all* $n \geq 2$*, define*

$$c_n := \left(\frac{n}{\varphi(n)} - e^{\gamma}\log\log n\right)\sqrt{\log n}.$$

Then the Riemann hypothesis is equivalent to the statement that

$$\limsup_{n\to\infty} c_n = e^{\gamma}(4+\gamma-\log(4\pi)).$$

In fact Euler's constant also appears in the estimation of the divisor function $d(n) := \sum_{d|n} 1$. The following theorem is due to Dirichlet.

Theorem 1.2.8 (Dirichlet, 1849)**.** *For all* $x \geq 1$*, we have*

$$\sum_{n\leq x} d(n) = x\log x + (2\gamma-1)x + O(\sqrt{x}).$$

These are some of the instances of the central roles played by this constant. For an encyclopedic account on Euler's constant, we refer to the beautiful article of J. C. Lagarias [20].

1.3 Arithmetic nature of γ

As far as the arithmetic nature of γ is concerned, we can only say that $\gamma \notin \mathbb{Z}$. Though the arithmetic nature of γ has been in the focus of research since the time of Euler, it has been very difficult to establish any result in that direction, for instance, to answer the question of irrationality of γ.

Continued fraction of a number is directly linked to its irrationality and its algebraicity. We know that a real number has a finite continued fraction if and only if it is rational. Also the continued fraction of a quadratic irrational is eventually periodic. Papanikolaou computed the first few terms of the continued fraction expansion of γ which are $[0;1,1,2,1,2,1,4,3,13,\ldots]$ and he showed that it has at least 470,000 terms without any evident pattern. From this he derived that if γ is rational, then its denominator has to be greater than 10^{242080}.

Without further ado, let us state the following folklore conjecture:

Conjecture 1.3.1. *Euler's constant is irrational.*

This long-standing open problem even has a stronger version due to Kontsevich and Zagier.

Conjecture 1.3.2 (Kontsevich-Zagier [19])**.** *Euler's constant is not a Kontsevich-Zagier period. In particular, it is transcendental.*

A period, as defined by Kontsevich and Zagier [19], is a complex number whose real and imaginary parts are values of absolutely convergent integrals of algebraic functions with algebraic coefficients over domains in $\mathbb{R}^n$ given by polynomial inequalities with algebraic coefficients. Clearly, an algebraic number is a period. The simplest example of a period which is not algebraic is

$$\pi = \int\int_{x^2+y^2\leq 1} dx\, dy.$$

In fact all the Riemann zeta values are examples of periods. Besides these, logarithms of algebraic numbers are also periods. Though the set of periods is countable, till date we do not have an explicit natural example of a non-period. Kontsevich and Zagier have also conjectured e to be a non-period. A special subset of the set of periods is the set of Baker periods.

Definition 1.3.3. A Baker period is a complex number which is a $\overline{\mathbb{Q}}$-linear combination of logarithms of non-zero algebraic numbers.

For more details on Baker periods we refer to [24], page 113. As a consequence of the celebrated theorem of Baker, it follows that every non-zero Baker period is transcendental. So one can ask whether γ is a Baker period. A partial answer to this question is given by M. R. Murty and N. Saradha.

Theorem 1.3.4 (Murty-Saradha [25]). *Let $q > 1$ be a natural number and K be a number field over which the q-th cyclotomic polynomial $\Phi_q(X)$ is irreducible. Then the K-vector space generated by the numbers*

$$\{\gamma, \Psi(a/q) : 1 \leq a \leq q \text{ and } (a,q) = 1\}$$

has dimension at least $\varphi(q)$.

Inspired by the above theorem, they formulated the following conjecture.

Conjecture 1.3.5 (Murty-Saradha [25]). *Let $q > 1$ be a natural number and K be a number field over which the q-th cyclotomic polynomial $\Phi_q(X)$ is irreducible. Then the numbers*

$$\{\Psi(a/q) : 1 \leq a \leq q \text{ and } (a,q) = 1\}$$

are linearly independent over K.

In connection to the question of γ being a Baker period, they have the following theorem.

Theorem 1.3.6 (Murty-Saradha [25]). *At least one of the following statements is true.*

1. *γ is a Baker period.*

2. *The above conjecture is true.*

In [12], the authors showed that the above conjecture is true most of the times. More precisely, they proved the following theorem.

Theorem 1.3.7 (Gun-Murty-Rath [12]). *Let $q, r > 1$ be two co-prime integers. Let K be a number field over which both the q-th and r-th cyclotomic polynomials are irreducible. Then at least one of the following sets of real numbers*

$$\{\Psi(a/q) : 1 \leq a \leq q \text{ and } (a, q) = 1\},$$

$$\{\Psi(b/r) : 1 \leq b \leq r \text{ and } (b, r) = 1\}$$

is linearly independent over K. Thus in particular, there exists an integer $q_0 > 1$ such that for any integer q co-prime to q_0, the set of real numbers

$$\{\Psi(a/q) : 1 \leq a \leq q \text{ and } (a, q) = 1\}$$

is linearly independent over $\mathbb{Q}$.

Now from Theorem 1.3.4, one can conclude that the set

$$\{\gamma, \Psi(a/q) : 1 \leq a \leq q \text{ and } (a, q) = 1\} \cap \mathbb{Q}$$

has cardinality at most one. But they also proved a stronger result about the arithmetic nature of the numbers in that set.

Theorem 1.3.8 (Murty-Saradha [25])**.** *Let $q > 1$ be a natural number. At most one of the $\varphi(q)+1$ numbers in the set*

$$\{\gamma, \Psi(a/q) : 1 \leq a \leq q \text{ and } (a,q) = 1\}$$

is algebraic.

In the above theorem γ is viewed as a special value of the digamma function. This result suggests an alternate approach for the study of the arithmetic nature of γ. It indicates that it might be judicious to study γ as a member of a family of constants rather than studying it in isolation. This idea is further supported by the following result of Rivoal.

Theorem 1.3.9 (Rivoal [38])**.** *At least one of the numbers γ and $\delta := \int_0^\infty \frac{e^{-w}}{1+w} dw$ is transcendental.*

This result was also proved independently by Kh. Hessami Pilehrood and T. Hessami Pilehrood [33] using a different method. In fact the above statement follows immediately from the following stronger theorem of Rivoal.

Theorem 1.3.10 (Rivoal [38])**.** *At least two of the numbers e, γ and δ are algebraically independent.*

Note that the constants e, γ and δ are part of a family of numbers called exponential periods, introduced in [18, 19]. Exponential periods are natural extensions of the set of periods. Here the integrand is the product of an algebraic function and the exponential of an algebraic function. It is known that

$$\gamma = \int_0^1 \int_x^1 \frac{e^{-x}}{y} dy dx - \int_1^\infty \int_1^x \frac{e^{-x}}{y} dy dx,$$

and hence an exponential period (for more details see [18] and [20], page 595).

1.4 Generalisations of Euler's constant

Over the past fifty years, Euler's constant has been generalised and studied by many mathematicians. Perhaps the first such instance is in a work by W. E. Briggs [7], where he considered Euler's constant for arithmetic progression. For a natural number $q > 1$, he considered the constants

$$\gamma(a,q) := \lim_{x\to\infty}\left(\sum_{\substack{n\le x\\ n\equiv a \bmod q}} \frac{1}{n} - \frac{1}{q}\log x\right).$$

Convergence of these sums can again be established using Abel's partial summation formula. From now on we refer to these constants as Euler-Briggs constants. These constants were studied extensively by D. H. Lehmer [21]. We mention one of his result below. The following result allows one to study the arithmetic nature of the family of Euler-Briggs constants.

Theorem 1.4.1 (Lehmer [21]). *For a natural number $q > 1$, one has*

$$q\gamma(a,q) = \gamma + \sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\ne 1}} \zeta_q^{-a}\log(1-\zeta_q).$$

Here μ_q denotes the set of q-th roots of unity and ζ_q denotes a q-th root of unity.

Murty and Saradha derived the following theorem concerning the arithmetic nature of the Euler-Briggs constants.

Theorem 1.4.2 (Murty-Saradha [25]). *Let $q > 1$ be a natural number. At most one of the $\varphi(q)+1$ numbers in the set*

$$\{\gamma, \gamma(a,q) : 1 \le a \le q \text{ and } (a,q) = 1\}$$

is algebraic.

In a follow-up article [26], they strengthened this result and proved the following theorem.

Theorem 1.4.3 (Murty-Saradha [26]). *At most one number in the infinite list of numbers*

$$\{\gamma(a,q) : 1 \le a < q \text{ and } q \ge 2\}$$

is an algebraic number.

It can be verified that $\gamma(2,4) = \frac{\gamma}{4}$. In 2008, a further generalisation of γ was introduced by H. Diamond and K. Ford [10]. They named those numbers as generalised Euler constants. In their work, they connected these numbers to the Riemann hypothesis.

We need to set up some notations to define these constants. Let Ω be a finite set primes. Let

$$P_\Omega := \begin{cases} \prod_{p\in\Omega} p & \text{if } \Omega \ne \phi, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\delta_\Omega := \begin{cases} \prod_{p\in\Omega}(1-\frac{1}{p}) & \text{if } \Omega \ne \phi, \\ 1 & \text{otherwise.} \end{cases}$$

For a finite set of primes Ω, Diamond and Ford [10] denoted the generalised Euler constants by $\gamma(\Omega)$ and defined as the following limit:

$$\gamma(\Omega) := \lim_{x\to\infty}\left(\sum_{\substack{n\le x \\ (n,P_\Omega)=1}} \frac{1}{n} - \delta_\Omega \log x \right).$$

Again, the existence of these constants can be showed using Abel's partial summation formula. Note that, when $\Omega = \phi$, then $\gamma(\Omega) = \gamma$.

Besides defining these generalisations of γ, they derive an expression linking γ with $\gamma(\Omega)$'s. In their article they mainly considered the subfamily consisting of the constants

$\gamma(\Omega_r)$, where Ω_r denotes the set of first r primes. One of their main result is the following theorem.

Theorem 1.4.4 (Diamond-Ford [10]). *The Riemann hypothesis is equivalent to the statement*

$$\gamma(\Omega_r) > e^{-\gamma} \text{ for all } r \geq 0.$$

More recently, the arithmetic nature of these constants was discussed by M. R. Murty and A. Zaytseva.

Theorem 1.4.5 (Murty-Zaytseva [27]). *Let S be the set of numbers $\{\gamma(\Omega)\}$, as Ω ranges over all finite sets of distinct primes. Then all numbers of S are transcendental with at most one exception.*

In our work [16], we introduce the following generalisation of γ, which unifies the families introduced by Briggs and Diamond-Ford. We refer to them as generalised Euler-Briggs constants. Let Ω be a finite subset of primes and a, q be natural numbers such that $(q, P_\Omega) = 1$, where P_Ω is as defined earlier. Then the generalised Euler-Briggs constant $\gamma(\Omega, a, q)$ is defined by the following limit:

$$\gamma(\Omega, a, q) := \lim_{x\to\infty}\left(\sum_{\substack{n\leq x \\ n\equiv a \bmod q \\ (n,P_\Omega)=1}} \frac{1}{n} - \delta_\Omega \frac{\log x}{q}\right).$$

Note that for any finite subset of primes Ω and a natural number q such that $(P_\Omega, q) = 1$, one has

$$\gamma(\Omega, q, q) = \frac{1}{q}(\gamma(\Omega) - \delta_\Omega \log q).$$

This thesis is devoted to study various facets of these constants. In the following section we briefly discuss the main contents of the chapters in this thesis.

1.5 Arrangement of the thesis

In the second chapter, we recall various basic definitions and some of the known results from algebraic, analytic and transcendental number theory which are required for our theorems in the upcoming chapters. At times we indicate briefly the proofs of some of these theorems to keep the exposition self-contained to the extent possible.

In the third chapter, we discuss the possible transcendental nature of the generalised Euler-Briggs constants. Some of the main ingredients for the theorems in this chapter are coming from the theory of linear forms in logarithm as developed by A. Baker [3] and the theory of multiplicatively independent cyclotomic units due to K. Ramachandra [34].

In the fourth chapter, we study the linear independence of the generalised Euler-Briggs constants over $\mathbb{Q}$ as well as over other number fields and $\overline{\mathbb{Q}}$. We also derive a non-trivial lower bound of certain vector spaces generated by these constants. In addition to the ingredients alluded to above, we shall need a theorem of A. Baker, B. J. Birch and E. A. Wirsing.

The penultimate chapter deals with the algebraic independence of these generalised Euler-Briggs constants. The results in this sections are conditional, subject to the weak Schanuel conjecture. Several consequences of Schanuel's conjecture and the weak Schanuel conjecture can be found in [13].

In the last chapter, we explore the connection between the generalised Euler-Briggs constants and certain infinite series. Inspired by a result of Lehmer [21], we derive a necessary and sufficient condition for the existence of periodic Dirichlet series at $s = 1$ twisted by certain principal Dirichlet character. We express this sum as a linear combination of generalised Euler-Briggs constants. We also prove a result about the special values of a shifted periodic Dirichlet series which can be seen as a variant of the Hurwitz zeta function.

CHAPTER 2

PRELIMINARIES

In this chapter we list the basic results from all the branches of number theory required to prove the theorems in the thesis. We include some of the proofs which follow from elementary observations. In the first section we recall basic definitions from algebraic number theory. For the first section we follow [28] and [40]. In the second section we recall the basics from analytic number theory. Here we follow [2, 22]. In the third section we have given the prerequisites from transcendental number theory following [4, 24]. In the last but one we give an exposition on characters of finite abelian group. We end the chapter by discussing the arithmetic properties of Dirichlet L-functions.

2.1 Some requisites from algebraic number theory

We start with the definition of a number field.

Definition 2.1.1. Any finite field extension $K \subset \mathbb{C}$ of $\mathbb{Q}$ is called a number field.

By degree of K, we mean the dimension of the vector space K over $\mathbb{Q}$.

Definition 2.1.2. A complex number α is said to be an algebraic number if there exists a non-zero polynomial $P(X) \in \mathbb{Q}[X]$ such that $P(\alpha) = 0$.

The least degree monic polynomial satisfied by α is unique and called the minimal polynomial of α. One can easily verify that all the elements of a number field K are algebraic numbers.

Definition 2.1.3. A complex number α which satisfies a monic polynomial over $\mathbb{Z}$ is called an algebraic integer. Set of all the elements of a number field K which are algebraic integers is a ring. This ring is called the ring of integers of K and denoted by $\mathcal{O}_K$.

Definition 2.1.4. Let K be a number field of degree n. A set of elements $\alpha_1, \ldots, \alpha_n \in \mathcal{O}_K$ is called an integral basis of K over $\mathbb{Q}$ or a basis of $\mathcal{O}_K$ over $\mathbb{Z}$ if

$$\mathcal{O}_K = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n.$$

Note that any integral basis is always a basis of K over $\mathbb{Q}$. Next we define the trace of an algebraic number in K.

Definition 2.1.5. Let K be a number field and $\alpha \in K$. Consider the linear map

$$T_\alpha : K \to K$$

defined by

$$T_\alpha(x) = \alpha x$$

for all $x \in K$. Then the trace of $\alpha \in K$, denoted by $Tr_{K/\mathbb{Q}}(\alpha)$, is the trace of the linear operator T_α.

It can be deduced that $Tr_{K/\mathbb{Q}}(\alpha) = \sum_{\sigma} \sigma(x)$, where σ varies over all embeddings of K into $\overline{\mathbb{Q}}$.

Definition 2.1.6. Let K be a number field of degree n. Then the discriminant of the number field K is defined by

$$\Delta_K := \det\, [Tr_{K/\mathbb{Q}}(\alpha_i \alpha_j)],$$

where $\alpha_1, \ldots, \alpha_n$ is an integral basis of K over $\mathbb{Q}$.

One can also show that $\Delta_K = \det(\sigma_i \alpha_j)^2$, where σ_i's vary over all the distinct embeddings of K into $\overline{\mathbb{Q}}$. Note that the definition of the discriminant is independent of the integral basis as the determinant of the base change matrix is a unit in $\mathbb{Z}$.

Definition 2.1.7. A noetherian, integrally closed integral domain where every non-zero prime ideal is maximal is called a Dedekind domain.

Theorem 2.1.8. *In a Dedekind domain every non-zero proper ideal factorizes uniquely into product of prime ideals.*

The following theorem gives us important examples of Dedekind domains.

Theorem 2.1.9. *For any number field K, $\mathcal{O}_K$ is a Dedekind domain.*

Let p be a prime in $\mathbb{Z}$ and K be a number field. Consider the ideal $p\mathcal{O}_K$ in $\mathcal{O}_K$. Since $\mathcal{O}_K$ is a Dedekind domain, we can write

$$p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r},$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are distinct prime ideals of $\mathcal{O}_K$. The prime p is said to be ramified in K if at least one of the e_i's is greater than 1.

Now we mention the following theorem which tells us that there are only finitely many primes which ramify in a number field.

Theorem 2.1.10. *Let K be a number field and Δ_K be the discriminant of K. Then a prime p in $\mathbb{Z}$ ramifies in K if and only if $p|\Delta_K$.*

The n-th cyclotomic field is the number field obtained by adjoining a primitive n-th root of unity to $\mathbb{Q}$. We know that the absolute value of the discriminant of the cyclotomic field $\mathbb{Q}(\zeta_{p^n})$ is equal to

$$p^{p^{n-1}(pn-n-1)}.$$

Thus by Theorem 2.1.10, the only prime that ramifies in $\mathbb{Q}(\zeta_{p^n})$ is p. The following theorem about the cyclotomic fields has been used in the subsequent chapters.

Theorem 2.1.11. *Let m,n be two positive integers. Then $\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$ if and only if $(m,n) = 1$.*

To prove the above theorem, we shall need the following result.

Theorem 2.1.12. *Let n be a positive integer. Then a prime $p \in \mathbb{Z}$ ramifies in $\mathbb{Q}(\zeta_n)$ if and only if $p|n$.*

Proof. Let $p|n$. So $\mathbb{Q}(\zeta_p) \subseteq \mathbb{Q}(\zeta_n)$. Now by Theorem 2.1.10, p ramifies in $\mathbb{Q}(\zeta_p)$ and hence in $\mathbb{Q}(\zeta_n)$. Next let p ramifies in $\mathbb{Q}(\zeta_n)$ but p does not divide $n = \prod_{i=1}^{k} p_i^{n_i}$. Now p does not ramify in the field $\mathbb{Q}(\zeta_{p_i^{n_i}})$ for all $1 \le i \le k$. Hence p does not ramify in the compositum, which is $\mathbb{Q}(\zeta_n)$. This fact follows from Theorem 2.1.10 and the following proposition. ■

Proposition 2.1.13. *Let K,L be two number fields of degree m,n and with discriminant c,d respectively. Further suppose such that both the extensions $L/\mathbb{Q}$ and $K/\mathbb{Q}$ are Galois and $K \cap L = \mathbb{Q}$. Then the compositum KL has discriminant $c^n d^m$.*

For a proof of a more general version of the above proposition see page 13 of [28]. We also need the following theorem of Minkowski.

Theorem 2.1.14 (Minkowski)**.** *The discriminant of a number field K of degree n satisfies*

$$|\Delta_K| \geq \frac{n^{2n}}{(n!)^2}\left(\frac{\pi}{4}\right)^n.$$

For a proof of the above theorem we refer to [28], page 204. As a consequence we get that the discriminant of a number field K of degree $n > 1$ is not equal to ± 1. Now we give the proof of Theorem 2.1.11.

Proof of Theorem 2.1.11 Clearly $\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$ implies $(m,n) = 1$ as otherwise if $(m,n) = d > 1$, then $\mathbb{Q} \subsetneq \mathbb{Q}(\zeta_d) \subseteq \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n)$. Now suppose that $(m,n) = 1$. Let $K = \mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n)$. If $K \neq \mathbb{Q}$, then by Theorem 2.1.14 and Theorem 2.1.10, there exists a prime $p \in \mathbb{Z}$ such that p divides the discriminant of K, hence p ramifies in K. So we have p ramifies in both the fields $\mathbb{Q}(\zeta_m)$ and $\mathbb{Q}(\zeta_n)$. By Theorem 2.1.12, $p|(m,n)$, which is a contradiction. ■

2.2 Some definitions and tools from analytic number theory

By an arithmetical function we mean a function $f : \mathbb{N} \to \mathbb{C}$. An arithmetical function f is said to be multiplicative if $f(mn) = f(m)f(n)$ for all $(m,n) = 1$. To give an example, consider the Möbius function

$$\mu(n) := \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \cdots p_k \text{ where } p_i\text{'s are distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

Another important example is the Euler's totient function

$$\varphi(n) := \sum_{\substack{1\le m\le n\\(m,n)=1}} 1.$$

With the two functions in place we mention the following propositions.

Proposition 2.2.1. *For $n \ge 1$ we have*

$$\sum_{d|n} \mu(d) = \left[\frac{1}{n}\right] = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise}. \end{cases}$$

The above result is known as the fundamental property of Möbius function and can easily be derived from the definition using the fact that μ is multiplicative.

Proposition 2.2.2. *For $n \ge 1$, we have*

$$\frac{\varphi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}.$$

Proof. We have

$$\varphi(n) = \sum_{1\le m\le n} \left[\frac{1}{(n,m)}\right].$$

By Proposition 2.2.1 we can write

$$\varphi(n) = \sum_{1\le m\le n} \sum_{d|(m,n)} \mu(d).$$

So,

$$\begin{aligned} \varphi(n) &= \sum_{d|n} \sum_{\substack{1\le m\le n\\ d|m}} \mu(d) \\ &= \sum_{d|n} \mu(d) \sum_{1\le k\le n/d} 1 \\ &= n \sum_{d|n} \frac{\mu(d)}{d}. \end{aligned}$$

■

Next proposition gives us an expression of $\varphi(n)$ in terms of the prime divisors of n.

Proposition 2.2.3. *For $n \geq 1$, one has*

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Proof. Since φ is multiplicative, it is enough to prove the formula for prime powers. Let $n = p^k$ for some prime p and $k \in \mathbb{N}$. The numbers that are smaller than or equal to n and share a common factor greater than 1 with n, are the multiples of p up to p^k. Now there are p^{k-1} many multiples of p which are smaller than or equal to n. Hence,

$$\varphi(n) = p^k - p^{k-1} = n\left(1 - \frac{1}{p}\right).$$

■

Before we prove the next identity we recall the definition of the von Mangoldt function and the Möbius inversion formula.

Definition 2.2.4. The von Mangoldt function Λ is defined by

$$\Lambda(n) = \begin{cases} 1 & \text{if } n = 1, \\ \log p & \text{if } n = p^k, k \geq 1, \\ 0 & \text{otherwise }. \end{cases}$$

Theorem 2.2.5 (Möbius inversion formula)**.** *Let f, g be two arithmetical functions. Then*

$$f(n) = \sum_{d|n} g(d),$$

if and only if

$$g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$$

for all n.

From the definition of Λ, one can deduce that

$$\log n = \sum_{d|n} \Lambda(d).$$

Hence by the Möbius inversion formula, we have

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d.$$

Now we give a proof of the following identity which we shall need later.

Proposition 2.2.6. *Let n be a natural number. Then*

$$-\sum_{d|n} \frac{\mu(d)\log d}{d} = \prod_{p|n}\left(1-\frac{1}{p}\right)\sum_{p|n}\frac{\log p}{p-1}$$

Proof. First we prove that for a square-free number n, one has

$$-\sum_{d|n} \frac{\mu(d)\log d}{d} = \prod_{p|n}\left(1-\frac{1}{p}\right)\sum_{p|n}\frac{\log p}{p-1}.$$

Since

$$\Lambda(d) = -\sum_{t|d} \mu(t) \log t,$$

by Möbius inversion formula, one can write

$$-\mu(d)\log d = \sum_{t|d} \mu(t)\Lambda(\frac{d}{t}).$$

So

$$
\begin{aligned}
-\sum_{d|n}\frac{\mu(d)\log d}{d} &= \sum_{d|n}\sum_{t|d}\frac{\mu(t)\Lambda(\frac{d}{t})}{d}\\
&= \sum_{ts|n}\frac{\mu(t)\Lambda(s)}{ts}\\
&= \sum_{p|n}\frac{\log p}{p}\sum_{t|\frac{n}{p}}\frac{\mu(t)}{t}\\
&= \sum_{p|n}\frac{\log p}{p}\frac{\varphi(n/p)}{n/p}\\
&= \sum_{p|n}\frac{\log p}{p}\prod_{p'|\frac{n}{p}}\left(1-\frac{1}{p'}\right)\\
&= \prod_{p|n}\left(1-\frac{1}{p}\right)\sum_{p|n}\frac{\log p}{p-1}.
\end{aligned}
$$

Note that for any $n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, $\alpha_i \geq 0$ for $1 \leq i \leq r$, one has

$$-\sum_{d|n}\frac{\mu(d)\log d}{d} = -\sum_{d|p_1\cdots p_r}\frac{\mu(d)\log d}{d}.$$

■

2.3 Inputs from transcendental number theory

A complex number α is transcendental if it is not algebraic. We begin this section with the famous theorem of Baker concerning linear forms in logarithms of non-zero algebraic numbers.

Theorem 2.3.1 (Baker [3]). *If $\alpha_1,\ldots,\alpha_n$ are non-zero algebraic numbers such that the numbers $\log\alpha_1,\ldots,\log\alpha_n$ are linearly independent over $\mathbb{Q}$, then $1,\log\alpha_1,\ldots,\log\alpha_n$ are linearly independent over $\overline{\mathbb{Q}}$.*

Another equivalent formulation of the Baker's theorem is the following theorem, which we use frequently in the subsequent chapters .

Theorem 2.3.2 (Baker [4])**.** *Let $\alpha_1, \cdots, \alpha_n \in \overline{\mathbb{Q}} \setminus \{0\}$ and $\beta_1, \cdots, \beta_n \in \overline{\mathbb{Q}}$, then*

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental. The latter case arises if $\log \alpha_1, \cdots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$ and not all $\beta_1, \cdots, \beta_n$ are zero.

For more details on Baker's theorem, we refer to Chapter 19-20 of [24]. The following theorem of Baker, Birch and Wirsing, which displays a very important application of Baker's theorem, plays a central role in our investigations in Chapter 4.

Theorem 2.3.3 (Baker, Birch and Wirsing [5])**.** *Let f be a non-zero algebraic valued periodic arithmetical function with period $q \geq 1$. Also let $f(n) = 0$ whenever $1 < (n, q) < q$ and the q-th cyclotomic polynomial $\Phi_q(X)$ be irreducible over $\mathbb{Q}(f(1), \cdots, f(q))$, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

The following theorem from the theory linear forms in logarithms of algebraic numbers is also used multiple times in this thesis.

Theorem 2.3.4 (Murty-Murty, Murty-Saradha [23, 26])**.** *Let $\alpha_1, \cdots, \alpha_n$ be positive algebraic numbers. If $\beta_0, \cdots, \beta_n$ are algebraic numbers with $\beta_0 \neq 0$, then*

$$\beta_0 \pi \ + \sum_{j=1}^{n} \beta_j \log \alpha_j$$

is a transcendental number and hence non-zero.

Proof. Denote

$$\alpha := \beta_0 \pi \ + \sum_{j=1}^{n} \beta_j \log \alpha_j.$$

We choose a maximal $\mathbb{Q}$-linearly independent subset J of the set $\{\log\alpha_1,\ldots,\log\alpha_n\}$. Without loss of generality, let $J=\{\log\alpha_1,\ldots,\log\alpha_s\}$. We then rewrite

$$i\alpha = c_0\log(-1) \ + \ \sum_{j=1}^{s} c_j\log\alpha_j,$$

where we have used the fact that $i\pi=\log(-1)$. Note that $\beta_0\neq 0$ implies $c_0\neq 0$. Now by Theorem 2.3.2, $i\alpha$ is either zero or transcendental. If $i\alpha$ is transcendental, then we get α is transcendental. So we assume $i\alpha=0$. Hence $\{\log(-1),\log\alpha_j : 1\leq j\leq s\}$ is not $\overline{\mathbb{Q}}$-linearly independent. Using Theorem 2.3.1 we get that there exists integers $a_0,\ldots,a_s$, not all zero such that

$$a_0\log(-1)=\sum_{j=1}^{s} a_j\log\alpha_j. \tag{2.3.1}$$

This gives us

$$1=\prod_{j=1}^{s}\alpha_j^{2a_j}.$$

Now $\{\alpha_j : 1\leq j\leq s\}$ is multiplicatively independent. So we get $a_j=0$ for all $1\leq j\leq s$. Putting the values of a_j for all $1\leq j\leq s$ in (2.3.1) we get $a_0=0$. Hence a contradiction to the fact that not all a_j for $0\leq j\leq s$ are zero. ■

2.4 Dirichlet characters

In this section, we recall the definition and certain properties of Dirichlet characters. For the sake of completeness, we begin with the basic theory of characters defined on an arbitrary abelian group G of finite order.

Definition 2.4.1. Let G be a finite abelian group. By a character on G, we mean a group homomorphism $f: G\to\mathbb{C}^*$.

The set of all such characters on G is denoted by $\widehat{G}$. It is easy to see that $\widehat{G}$ has an abelian group structure with the identity element $\mathbf{1}_G : G \to \mathbb{C}^*$ for which $g \mapsto 1$ for all $g \in G$. For the group $\widehat{G}$, we have the following propositions.

Proposition 2.4.2. *Let G be a finite abelian group of order n. Then* $|\widehat{G}| = n$.

Proof. We prove this by induction on the order of the group. Clearly, this is true if G is singleton. Now we assume the induction hypothesis. Let H be a maximal subgroup of G with $|H| = m < n$ and $a \in G \setminus H$. Then the group generated by H and a is the full group G. Further let k be the least positive integer such that $a^k \in H$. Then every element of G is uniquely written as $a^i x$ for some $x \in H$ and $1 \le i \le k$ i.e.

$$G = \{a^i x : x \in H, 1 \le i \le k\}.$$

Hence, $n = |G| = k|H| = km$. Note that a character on G restricts to a character on H. By induction hypothesis we know $|\widehat{H}| = m$. Now we prove that a character on H can be extended to a character on G only in k distinct ways.

Let $f : H \to \mathbb{C}^*$ be a character. Since, $a^k \in H$, $f(a^k)$ is defined. Let $f(a^k) = c \in \mathbb{C}^*$. Now we can extend f to G by setting $f(a)$ to be a k-root of c. Thus f can be extended to a character on G at least in k ways. Now suppose that $\tilde{f} : G \to \mathbb{C}^*$ is an extension of the character $f : H \to \mathbb{C}^*$. Then, $\tilde{f}(a^k) = f(a^k)$ and thus $\tilde{f}(a)^k = f(a^k) = c$. Hence $\tilde{f}(a)$ is a k-root of c. This proves that $|\widehat{G}| = km = n$. ■

Proposition 2.4.3. *Let G be a finite abelian group. Then* $G \simeq \widehat{G}$.

Proof. We first prove this for cyclic groups. Let G be a cyclic group of order n and $a \in G$ be a generator of G. Let $f : G \to \mathbb{C}^*$ be a character. Then $f(a^n) = 1$ and hence $f(a)^n = 1$. This implies that $f(a)$ is a n-th root of unity. If $f(a)$ is a primitive n-th root of unity, then f generates the character group $\widehat{G}$. This proves $G \simeq \widehat{G}$ for any cyclic group G.

Now for a finite abelian group G we appeal to the fundamental theorem of finite abelian groups and write it as product of cyclic groups. By virtue of this we are now reduced to prove that if $G_1,\dots,G_r$ are finite cyclic groups then

$$\widehat{H} \simeq \widehat{G_1} \times \cdots \times \widehat{G_r},$$

where $H := G_1 \times \cdots \times G_r$. Clearly both these groups are of same order. Now we provide an injective homomorphism from $\widehat{G_1} \times \cdots \times \widehat{G_r}$ to $\widehat{H}$. Define $f : \widehat{G_1} \times \cdots \times \widehat{G_r} \to \widehat{H}$ by $(f_1,\dots,f_r) \mapsto f_1 \cdots f_r$, where $f_1 \cdots f_r : H \to \mathbb{C}^*$ denotes the map for which $(g_1,\dots,g_r) \mapsto f_1(g_1)\cdots f_r(g_r)$ for all $(g_1,\dots,g_r) \in G_1 \times \cdots \times G_r$. It is easy to see that f defines an injective homomorphism from $\widehat{G_1} \times \cdots \times \widehat{G_r}$ to $\widehat{H}$. This completes the proof. ■

We now prove the orthogonality relations satisfied by the characters.

Proposition 2.4.4. *Let G be a finite abelian group of order n. Then*

$$\sum_{g \in G} f(g) = \begin{cases} n & \text{if } f = \mathbf{1}_G, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{f \in \widehat{G}} f(g) = \begin{cases} n & \text{if } g = e_G, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\mathbf{1}_G : G \to \mathbb{C}^*$ denotes the trivial homomorphism and e_G denotes the identity element of the group G.

Proof. Since $\mathbf{1}_G(g) = 1$ for all $g \in G$ we obtain $\sum_{g \in G} \mathbf{1}_G(g) = n$. Now if $f \neq \mathbf{1}_G$, then there exists $h \in G$ such that $f(h) \neq 1$. Hence,

$$f(h) \sum_{g \in G} f(g) = \sum_{g \in G} f(gh) = \sum_{g \in G} f(g).$$

Since $f(h) \neq 1$, we get the desired result.

For the second identity, note that $f(e_G) = 1$ for all $f \in \widehat{G}$. Hence $\sum_{f\in\widehat{G}} f(e_G) = n$. Now if $g \neq e_G$, then there exists $\phi \in \widehat{G}$ such that $\phi(g) \neq 1$. To see this we start with a non-trivial character on the cyclic group H generated by g. This character on H then can be extended to G following the method described in the proof of Proposition 2.4.2. Now for such an ϕ on G,

$$\phi(g)\sum_{f\in\widehat{G}} f(g) = \sum_{f\in\widehat{G}} \phi f(g) = \sum_{f\in\widehat{G}} f(g).$$

Since $\phi(g) \neq 1$, we get the desired result. ■

In fact the second identity can also be derived from the first one using the canonical isomorphism $G \simeq \widehat{\widehat{G}}$. A further extension of the orthogonality relations is given by the following proposition, which is in fact an easy consequence of Proposition 2.4.4.

Proposition 2.4.5. *Let G be a finite abelian group of order n. Then*

$$\sum_{g\in G} f_1(g)f_2^{-1}(g) = \begin{cases} n & \textit{if } f_1 = f_2, \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\sum_{f\in\widehat{G}} f(gh^{-1}) = \begin{cases} n & \textit{if } g = h, \\ 0 & \textit{otherwise.} \end{cases}$$

Now we define Dirichlet characters.

Definition 2.4.6. Let $q \geq 1$ be an integer and $f : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ be a character. This f can be extended to an arithmetical function χ by defining

$$\chi(n) := \begin{cases} f(\bar{n}) & \text{if } (n,q) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{n}$ denotes the residue class $n \bmod q$. Such a χ is called a Dirichlet character modulo q. If f is the trivial homomorphism then the corresponding Dirichlet character is called the principal Dirichlet character modulo q and denoted by χ_0.

There are $\varphi(q)$ Dirichlet characters modulo q. Since -1 is of order 2 in $(\mathbb{Z}/q\mathbb{Z})^*$, $q > 2$, the possible values of $\chi(-1)$ are ± 1. We say that a Dirichlet character χ is even if $\chi(-1) = 1$ and odd if $\chi(-1) = -1$. Let χ be a Dirichlet character corresponding to a character $f : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$. Then the Dirichlet character corresponding to the character $f^{-1} : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ is denoted by $\overline{\chi}$ and called the conjugate of χ. The orthogonality relations satisfied by the Dirichlet characters are given in the following proposition.

Proposition 2.4.7. *Let a, b, q be positive integers and χ_1, χ_2 be two Dirichlet characters modulo q. Then*

$$\sum_{1 \le a \le q} \chi_1(a)\overline{\chi}_2(a) = \begin{cases} \varphi(q) & \text{if } \chi_1 = \chi_2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{\chi \bmod q} \chi(a)\overline{\chi}(b) = \begin{cases} \varphi(q) & \text{if } a \equiv b \bmod q, \\ 0 & \text{otherwise.} \end{cases}$$

2.5 Dirichlet L-functions

Let $q \ge 1$ be an integer and χ be a Dirichlet character modulo q. By the Dirichlet L-function associated to the character χ, we mean the Dirichlet series

$$\sum_{n \ge 0} \frac{\chi(n)}{n^s},$$

defined initially for $\Re(s) > 1$. This function is denoted by $L(s, \chi)$. The above series converges absolutely and uniformly on compact subsets of $\Re(s) > 1$, hence defines an analytic function there. The meromorphic continuation of the Dirichlet L-function $L(s,\chi)$ is given by the following general theorem which is valid for any periodic Dirichlet series.

Theorem 2.5.1. *Let f be a periodic arithmetical function with period q and let*

$$D(s,f) := \sum_{n\geq 1} \frac{f(n)}{n^s}$$

be the associated Dirichlet series. If $\sum_{a=1}^{q} f(a) = 0$, then $D(s,f)$ can be extended to an entire function. Otherwise it can be continued analytically except at $s = 1$ where it has a simple pole with residue $\frac{1}{q}\sum_{a=1}^{q} f(a)$.

For a periodic arithmetical function f of period q, we define

$$D(1,f) := \lim_{s\to 1+} D(s,f), \text{ when it exists.}$$

So, from the above theorem we get that $D(1,f)$ exists if and only if $\sum_{a=1}^{q} f(a) = 0$. On the other hand if we assume $\sum_{a=1}^{q} f(a) = 0$, using Abel's partial summation formula it can also be shown that $\sum_{n\geq 1}\frac{f(n)}{n}$ exists. Therefore, a natural question arises whether

$$D(1,f) = \sum_{n\geq 1} \frac{f(n)}{n}.$$

This question is answered in the following proposition which is valid for more general Dirichlet series.

Proposition 2.5.2. *Let f be an arithmetical function such that the associated Dirichlet series*

$$D(s,f) := \sum_{n\geq 1} \frac{f(n)}{n^s}$$

converges absolutely for $\Re(s) > 1$. Also suppose that $\sum_{n\geq 1}\frac{f(n)}{n}$ exists. Then for a real parameter σ we have,

$$\lim_{\sigma\to 1+} D(\sigma,f) = \sum_{n\geq 1} \frac{f(n)}{n}.$$

Proof. For a positive integer N, let

$$F(N) := \sum_{1 \le n \le N} \frac{f(n)}{n} \quad \text{and} \quad \lim_{N \to \infty} F(N) = \alpha.$$

We write $F(t) = E(t) + \alpha$. Using Abel's partial summation formula, for $\Re(s) > 1$, we get

$$\begin{aligned} \sum_{1 \le n \le N} \frac{f(n)}{n^s} &= \frac{F(N)}{N^{s-1}} + (s-1) \int_1^N \frac{F(t)}{t^s} dt \\ &= \frac{F(N)}{N^{s-1}} + (s-1)\alpha \int_1^N \frac{dt}{t^s} + (s-1) \int_1^N \frac{E(t)}{t^s} dt \\ &= \frac{F(N) - \alpha}{N^{s-1}} + \alpha + (s-1) \int_1^N \frac{E(t)}{t^s} dt. \end{aligned}$$

Letting $N \to \infty$, for $\Re(s) > 1$, we get

$$D(s, f) = \alpha + (s-1) \int_1^\infty \frac{E(t)}{t^s} dt.$$

Hence for a real parameter $\sigma > 1$ we have,

$$D(\sigma, f) = \alpha + (\sigma - 1) \int_1^x \frac{E(t)}{t^\sigma} dt + (\sigma - 1) \int_x^\infty \frac{E(t)}{t^\sigma} dt,$$

which holds for all $x \ge 1$. Let $\epsilon > 0$ be given. Since $E(t) \to 0$ as $t \to \infty$, we can choose x such that $|E(t)| < \epsilon$ for all $t \ge x$. Hence,

$$|D(\sigma, f) - \alpha| \le (\sigma - 1) \int_1^x \frac{|E(t)|}{t^\sigma} dt + \epsilon(\sigma - 1) \int_x^\infty \frac{1}{t^\sigma} dt = c_x(\sigma - 1) + \frac{\epsilon}{x^{\sigma - 1}},$$

for some positive real number c_x. Thus letting $\sigma \to 1$, we obtain

$$\lim_{\sigma \to 1+} D(\sigma, f) = \sum_{n \ge 1} \frac{f(n)}{n}.$$

■

With this proposition in place, it is now justified to write

$$D(1, f) = \sum_{n \ge 1} \frac{f(n)}{n},$$

for a periodic arithmetical function f of period q with $\sum_{a=1}^{q} f(a) = 0$. From Proposition 2.4.7, we know that for a non-principal Dirichlet character χ modulo q, one has

$$\sum_{a=1}^{q} \chi(a) = 0.$$

Hence $L(1,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n}$ exists for $\chi \neq \chi_0$. Dirichlet himself proved that $L(1,\chi) \neq 0$. For a proof see page 118 of [8]. Using Baker's theorem, we give a proof of the fact that for non-principal Dirichlet character χ modulo q, $L(1,\chi)$ is transcendental.

We consider the Fourier transform of the periodic function χ. Let

$$\hat{\chi}(n) := \frac{1}{q} \sum_{1\leq a\leq q} \chi(a) e^{-2\pi i a n/q}$$

denote the Fourier transform of χ. Using orthogonality, we get

$$\chi(n) = \sum_{1\leq a\leq q} \hat{\chi}(a) e^{2\pi i a n/q}.$$

The condition $\sum_{a=1}^{q} \chi(a) = 0$ implies that $\hat{\chi}(q) = 0$. Thus we obtain

$$\begin{aligned} L(1,\chi) &= \sum_{n\geq 1} \frac{\chi(n)}{n} \\ &= \sum_{n\geq 1} \frac{1}{n} \sum_{1\leq a\leq q-1} \hat{\chi}(a) e^{2\pi i a n/q} \\ &= -\sum_{1\leq a\leq q-1} \hat{\chi}(a) \log(1 - e^{2\pi i a/q}). \end{aligned}$$

We know that $L(1,\chi) \neq 0$ and this is a linear form in logarithms of algebraic numbers with algebraic co-efficients. So by Baker's theorem we conclude that $L(1,\chi)$ is transcendental.

If χ is even, then $\hat{\chi}$ is also even. So $\hat{\chi}(a) = \hat{\chi}(q-a)$. Thus it follows from the above expression of $L(1,\chi)$ that if χ is a non-principal even Dirichlet character modulo q then

$$-L(1,\chi) = \begin{cases} \sum\limits_{1\leq a<q/2} 2\hat{\chi}(a) \log|1 - e^{2\pi i a/q}| & \text{when } q \text{ is odd,} \\ \sum\limits_{1\leq a<q/2} 2\hat{\chi}(a) \log|1 - e^{2\pi i a/q}| + \hat{\chi}(q/2)\log 2 & \text{when } q \text{ is even.} \end{cases} \tag{2.5.1}$$

Here we have used the fact that for the principal branch of logarithm

$$\log z + \log \bar{z} = \log |z|^2.$$

In case of odd Dirichlet characters, we can express $L(1,\chi)$ as an algebraic multiple of π.

Proposition 2.5.3. *Let χ be an odd Dirichlet character modulo q, then $L(1,\chi)$ is an algebraic multiple of π.*

Proof. We use the following cotangent expression to give a proof of this fact. For a real number x which is not an integer, we have

$$\pi \cot(\pi x) = \sum_{n\in\mathbb{Z}} \frac{1}{n+x}.$$

In the above expression the infinite series $\sum_{n\in\mathbb{Z}} \frac{1}{n+x}$ denotes the limit

$$\lim_{N\to\infty} \sum_{-N\le n\le N} \frac{1}{n+x}.$$

Since χ is odd, we have

$$\sum_{n\in\mathbb{Z}, n\neq 0} \frac{\chi(n)}{n} = 2\sum_{n\ge 1} \frac{\chi(n)}{n}.$$

Now we write

$$\begin{aligned}
\sum_{n\in\mathbb{Z}, n\neq 0} \frac{\chi(n)}{n} &= \sum_{1\le a\le q} \chi(a) \sum_{\substack{n\equiv a \bmod q \\ n\neq 0}} \frac{1}{n} \\
&= \frac{1}{q} \sum_{1\le a\le q-1} \chi(a) \sum_{n\in\mathbb{Z}} \frac{1}{n+a/q} \\
&= \frac{\pi}{q} \sum_{1\le a\le q-1} \chi(a) \cot(\pi a/q).
\end{aligned}$$

Hence we obtain that

$$L(1,\chi) = \frac{\pi}{2q} \sum_{1\le a\le q-1} \chi(a) \cot(\pi a/q).$$

Now we know that $\cot(\pi a/q)$ is algebraic for all $1\le a\le q-1$. This completes the proof. ■

Let us denote $V_{\text{even},q}$ and $V_{\text{odd},q}$ to be the $\overline{\mathbb{Q}}$ vector spaces generated by the values $L(1,\chi)$ as χ varies over non-principal even and odd Dirichlet characters modulo q respectively. Therefore in view of Theorem 2.3.4, Proposition 2.5.3 and formula (2.5.1), we obtain that

$$V_{\text{even},q} \cap V_{\text{odd},q} = \{0\}.$$

From Proposition 2.5.3 we also get that

$$\dim(V_{\text{odd},q}) = 1.$$

On the other hand, T. Okada [32] proved the following theorem about $V_{\text{even},q}$.

Theorem 2.5.4 (Okada [32])**.** *The values $L(1,\chi)$ and $\log p$ as χ varies over non-principal even Dirichlet characters modulo q and p runs through all the prime factors of q, are linearly independent over $\overline{\mathbb{Q}}$. In particular, one has*

$$\dim(V_{even,q}) = \frac{\varphi(q)}{2} - 1.$$

Hence from the above discussion we obtain the following theorem as an immediate consequence.

Theorem 2.5.5. *The $\overline{\mathbb{Q}}$ vector space generated by the values $L(1,\chi)$ as χ varies over non-principal Dirichlet characters modulo q is of dimension $\frac{\varphi(q)}{2}$.*

Now we end this chapter by recalling real multiplicatively independent units of a cyclotomic field which are known as Ramachandra units. Ramachandra [34] discovered a set of real multiplicatively independent units in cyclotomic field $\mathbb{Q}(\zeta_q)$ for any arbitrary natural number q. Following the notation used in [40], we denote them as ξ_a (with $1 < a < q/2$ and $(a,q) = 1$) where the ambient cyclotomic field is $\mathbb{Q}(\zeta_q)$. One of the fundamental property of these units which is very relevant in our context is that for any non-principal even Dirichlet character χ modulo q,

$$L(1,\chi) = A_\chi \sum_{\substack{1<a<q/2 \\ (a,q)=1}} \bar{\chi}(a) \log \xi_a,$$

where A_χ's are non-zero algebraic number and $\xi_a = \zeta_q^{d_a} \eta_a$. Here

$$\eta_a := \prod_{\substack{d|q, d\neq q \\ (d,q/d)=1}} \frac{1-\zeta_q^{ad}}{1-\zeta_q^{d}} \quad \text{and} \quad d_a := \frac{1}{2}(1-a) \sum_{\substack{d|q, d\neq q \\ (d,q/d)=1}} d.$$

This particular expression of $L(1,\chi)$ is the key to the proof of Theorem 2.5.4.

CHAPTER 3

TRANSCENDENTAL RESULTS

We begin this chapter by recalling that a complex number α is said to be algebraic if there exists a non-zero polynomial $P(x) \in \mathbb{Q}[x]$ such that $P(\alpha) = 0$. Otherwise, it is called transcendental. In the first section of this chapter, we state the results related to the transcendental nature of the generalised Euler-Briggs constants. In the second section, we record all the intermediate results, along with their proofs, which are required to prove the main theorems of this chapter. In the last section, we provide the proofs of the theorems mentioned in the first section. These results appear in [16] which generalise as well as unify the works in [26] and [27].

3.1 Statement of the theorems

It is easy to derive from the definition of $\gamma(\Omega,a,q)$ that if $(a,q)=d$ for some $d \geq 1$, then

$$\gamma(\Omega,a,q)=\frac{1}{d}\gamma(\Omega,a/d,q/d)-\frac{\delta_\Omega}{q}\log d.$$

For our purpose, we work with the condition $(a,q)=1$.

Theorem 3.1.1. *Let Ω be a non-empty finite set of primes and $a,q \geq 1$ be natural numbers with $(a,q)=1$. Then*

$$\gamma(\Omega,a,q)\ -\ \delta_\Omega\frac{\gamma}{q}$$

is transcendental.

The above theorem does not shed any light about the transcendence of the generalised Euler-Briggs constants (except in the very unlikely scenario that γ is algebraic). To study the transcendental nature of the generalised Euler-Briggs constants, we consider a family of such constants and derive the following theorem.

Theorem 3.1.2. *Let a and $q \geq 1$ be natural numbers with $(a,q)=1$. Let*

$$U:=\left\{\ \Omega:\Omega \textit{ is a finite set of primes and } (P_\Omega,q)=1\right\}.$$

Then the set

$$T(a,q):=\left\{\gamma(\Omega,a,q)\mid\Omega\in U\right\}$$

is infinite and has at most one algebraic element.

In our next theorem, we fix a finite set of primes Ω and vary q over a collection of mutually co-prime integers.

Theorem 3.1.3. *Let Ω be a finite set of primes and $S = \{q_1, q_2, \cdots\}$ be an infinite set of mutually co-prime natural numbers $q_i \geq 1$ for all $i \in \mathbb{N}$. Further suppose that each q_i is co-prime to P_Ω. Let a be a natural number with $(a, q_i) = 1$ for all i. Then the set*

$$T(\Omega, a) := \{\, \gamma(\Omega, a, q_i) \mid q_i \in S\}$$

has at most one algebraic element.

3.2 Intermediate results

We need a number of intermediate results to prove the above mentioned theorems. We begin with a suitable expression for the generalised Euler-Briggs constants.

Lemma 3.2.1. *Let a and $q \geq 1$ be natural numbers such that $(a, q) = 1$. Also, let Ω be a finite set of primes such that $(P_\Omega, q) = 1$. Then $\gamma(\Omega, a, q)$ exists and*

$$\gamma(\Omega, a, q) \;=\; \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \overline{\chi}(a) L(1, \chi) \prod_{p \in \Omega} \left(1 - \frac{\chi(p)}{p}\right) \;+\; \frac{\delta_\Omega}{q} \left(\gamma + \sum_{p \mid q} \frac{\log p}{p-1} + \sum_{p \in \Omega} \frac{\log p}{p-1}\right).$$

Proof. When $q = 1$, the above identity reduces to

$$\gamma(\Omega) \;=\; \delta_\Omega \left(\gamma + \sum_{p \in \Omega} \frac{\log p}{p-1}\right),$$

which has been proved by Diamond and Ford in [10]. Hence we assume that $q > 1$ and write

$$\begin{aligned}
\sum_{\substack{n \leq x \\ n \equiv a \bmod q \\ (n, P_\Omega) = 1}} \frac{1}{n} \;&=\; \frac{1}{\varphi(q)} \sum_{\substack{n \leq x \\ (n, P_\Omega) = 1}} \frac{1}{n} \sum_{\chi \bmod q} \overline{\chi}(a) \chi(n) \\
&=\; \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \sum_{n \leq x} \frac{\chi(n)}{n} \sum_{d \mid (n, P_\Omega)} \mu(d) \\
&=\; \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \sum_{d \mid P_\Omega} \frac{\mu(d)\chi(d)}{d} \sum_{m \leq x/d} \frac{\chi(m)}{m} \\
&=\; A \,+\, B,
\end{aligned}$$

where

$$A := \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \overline{\chi}(a) \sum_{d \mid P_\Omega} \frac{\mu(d)\chi(d)}{d} \sum_{m \le x/d} \frac{\chi(m)}{m}$$

and

$$B := \frac{1}{\varphi(q)} \sum_{d \mid P_\Omega} \frac{\mu(d)}{d} \sum_{m \le x/d} \frac{\chi_0(m)}{m}.$$

We deduce that

$$\begin{aligned}
A &= \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \overline{\chi}(a) \sum_{d \mid P_\Omega} \frac{\mu(d)\chi(d)}{d} \sum_{m \le x/d} \frac{\chi(m)}{m} \\
&= \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \overline{\chi}(a) \sum_{d \mid P_\Omega} \frac{\mu(d)\chi(d)}{d} \left(L(1,\chi) + O\left(\frac{d}{x}\right)\right) \\
&= \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \overline{\chi}(a) L(1,\chi) \prod_{p \in \Omega} \left(1 - \frac{\chi(p)}{p}\right) + O\left(\frac{1}{x}\right),
\end{aligned}$$

and

$$\begin{aligned}
B &= \frac{1}{\varphi(q)} \sum_{d \mid P_\Omega} \frac{\mu(d)}{d} \sum_{m \le x/d} \frac{\chi_0(m)}{m} \\
&= \frac{1}{\varphi(q)} \sum_{d \mid P_\Omega} \frac{\mu(d)}{d} \sum_{\substack{m \le x/d \\ (m,q)=1}} \frac{1}{m} \\
&= \frac{1}{\varphi(q)} \sum_{d \mid P_\Omega} \frac{\mu(d)}{d} \sum_{m \le x/d} \frac{1}{m} \sum_{d_1 \mid (m,q)} \mu(d_1) \\
&= \frac{1}{\varphi(q)} \sum_{d \mid P_\Omega} \frac{\mu(d)}{d} \sum_{d_1 \mid q} \frac{\mu(d_1)}{d_1} \sum_{t \le x/dd_1} \frac{1}{t} \\
&= \frac{1}{\varphi(q)} \sum_{d \mid P_\Omega} \frac{\mu(d)}{d} \sum_{d_1 \mid q} \frac{\mu(d_1)}{d_1} \left(\log \frac{x}{dd_1} + \gamma + O\left(\frac{dd_1}{x}\right)\right) \\
&= \frac{\delta_\Omega}{q} \left(\log x + \gamma + \sum_{p \mid q} \frac{\log p}{p-1} + \sum_{p \in \Omega} \frac{\log p}{p-1} \right) + O\left(\frac{1}{x}\right).
\end{aligned}$$

Here we use the identities

$$-\sum_{d \mid q} \frac{\mu(d) \log d}{d} = \left(\prod_{p \mid q} \frac{p-1}{p} \right) \sum_{p \mid q} \frac{\log p}{p-1}$$

and

$$-\frac{1}{q}\sum_{d|P_\Omega}\frac{\mu(d)\log d}{d}=\frac{\delta_\Omega}{q}\sum_{p\in\Omega}\frac{\log p}{p-1}.$$

Thus

$$\begin{aligned}\sum_{\substack{n\le x\\ n\equiv a \bmod q\\ (n,P_\Omega)=1}}\frac{1}{n} &= \frac{1}{\varphi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\overline{\chi}(a)L(1,\chi)\prod_{p\in\Omega}\left(1-\frac{\chi(p)}{p}\right)\\ &+ \frac{\delta_\Omega}{q}\left(\log x+\gamma+\sum_{p|q}\frac{\log p}{p-1}+\sum_{p\in\Omega}\frac{\log p}{p-1}\right)+O\left(\frac{1}{x}\right).\end{aligned}$$

Hence we obtain

$$\gamma(\Omega,a,q)=\frac{1}{\varphi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\overline{\chi}(a)L(1,\chi)\prod_{p\in\Omega}\left(1-\frac{\chi(p)}{p}\right)+\frac{\delta_\Omega}{q}\left(\gamma+\sum_{p|q}\frac{\log p}{p-1}+\sum_{p\in\Omega}\frac{\log p}{p-1}\right).$$

This completes the proof. ■

The next set of propositions, apart from being central to the proofs of the theorems in this chapter, are also of independent interest. Baker's theory of linear forms in logarithms and theory of Ramachandra units constitute the crucial ingredients in their proof.

Proposition 3.2.2. *Let $q>1$ be a natural number and Ω be a finite set of primes co-prime to q. Then the number*

$$\sum_{p|q}\alpha_p\log p+\sum_{p\in\Omega}\eta_p\log p+\sum_{\substack{\chi\ even\\ \chi\neq\chi_0}}b_\chi\, L(1,\chi)+\sum_{\chi\ odd}d_\chi\, L(1,\chi) \tag{3.2.1}$$

is transcendental. Here α_p, η_p, b_χ are algebraic numbers, not all of them zero. Further, d_χ's are arbitrary algebraic numbers.

Proof. We prove it by contradiction. We know by (2.5.1) that for any even Dirichlet character $\chi\neq\chi_0$, one can write $L(1,\chi)$ as follows:

$$-L(1,\chi)=\begin{cases}\sum\limits_{1\le a<q/2}2\hat{\chi}(a)\log|1-e^{2\pi ia/q}| & \text{when } q \text{ is odd,}\\ \sum\limits_{1\le a<q/2}2\hat{\chi}(a)\log|1-e^{2\pi ia/q}|+\hat{\chi}(q/2)\log 2 & \text{when } q \text{ is even.}\end{cases}$$

Now for any odd Dirichlet character χ, we know by Proposition 2.5.3 that $L(1,\chi)$ is a non-zero algebraic multiple of π. Hence Using these two facts one can rewrite (3.2.1) as

$$\sum_{p|q} \alpha_p \log p \ + \sum_{p\in\Omega} \eta_p \log p \ + \sum_{a=1}^{q} \beta_a \ \log \alpha_a + \beta\,\pi,$$

where β, β_a's are algebraic numbers and α_a's are positive algebraic numbers.

Now if β is non-zero, then by Theorem 2.3.4, we get that the above sum is necessarily transcendental. Next we assume that $\beta = 0$ and hence that (3.2.1) is of the form

$$\sum_{p|q} \alpha_p \log p \ + \sum_{p\in\Omega} \eta_p \log p \ + \sum_{\substack{\chi \text{ even} \\ \chi\neq\chi_0}} b_\chi \, L(1,\chi).$$

Now by Theorem 2.3.2, this sum is either zero or transcendental. Suppose that

$$\sum_{p|q} \alpha_p \log p \ + \sum_{p\in\Omega} \eta_p \log p \ + \sum_{\substack{\chi \text{ even} \\ \chi\neq\chi_0}} b_\chi \, L(1,\chi) = 0. \tag{3.2.2}$$

As mentioned in Chapter 2, for a non-principal even Dirichlet character χ, we have

$$L(1,\chi) = W_\chi \sum_{\substack{1<a<q/2 \\ (a,q)=1}} \overline{\chi}(a) \log \xi_a,$$

where ξ_a's are real multiplicatively independent units in the cyclotomic field $\mathbb{Q}(\zeta_q)$ and W_χ is a non-zero algebraic number. Thus (3.2.2) can be written as

$$\sum_{p|q} \alpha_p \log p \ + \sum_{p\in\Omega} \eta_p \log p \ + \sum_{\substack{1<a<q/2 \\ (a,q)=1}} \lambda_a \log \xi_a = 0,$$

where λ_a's are algebraic numbers. Note that by Theorem 2.5.4,

$$\sum_{\substack{\chi \text{ even} \\ \chi\neq\chi_0}} b_\chi \, L(1,\chi) = 0 \iff b_\chi = 0 \text{ for all } \chi,$$

and by Theorem 2.3.2,

$$\sum_{\substack{1<a<q/2 \\ (a,q)=1}} \lambda_a \log \xi_a = 0 \iff \lambda_a = 0 \text{ for all } a.$$

Hence we conclude that $\lambda_a = 0$ for all a if and only if $b_\chi = 0$ for all even non-principal χ.

Now by Theorem 2.3.2, there exist integers d_p for $p|q$, e_p for $p \in \Omega$ and c_a for $1 < a < q/2$ with $(a,q) = 1$, not all zero, such that

$$\sum_{p|q} d_p \log p \;+\; \sum_{p\in\Omega} e_p \log p \;-\; \sum_{\substack{1<a<q/2\\(a,q)=1}} c_a \log \xi_a \;=\; 0.$$

The above expression gives

$$\prod_{p|q} p^{d_p} \prod_{p\in\Omega} p^{e_p} = \prod_{\substack{1<a<q/2\\(a,q)=1}} \xi_a^{c_a}.$$

Now taking norm on both sides, we see that

$$\prod_{p|q} p^{\varphi(q)d_p} \prod_{p\in\Omega} p^{\varphi(q)e_p} = 1.$$

This forces $e_p = 0 = d_p$ for all $p \in \Omega$ and $p|q$. Further, ξ_a's are multiplicatively independent implies that c_a's are zero for all $1 < a < q/2$ with $(a,q) = 1$. This proves the proposition. ■

In the following proposition, we deal with Dirichlet characters of different moduli.

Proposition 3.2.3. *Let $q_1, q_2 > 1$ be natural numbers with $(q_1, q_2) = 1$. Also let $\alpha_p, \beta_\chi, \beta'_\chi, \eta_\psi$ and η'_ψ be algebraic numbers, where p varies over the prime divisors of q_1q_2, $\chi \neq \chi_0, \psi \neq \psi_0$ are characters modulo q_1 and q_2 respectively. Then the number*

$$\sum_{p|q_1q_2} \alpha_p \log p \;+\; \sum_{\substack{\chi\ even\\ \chi\neq\chi_0}} \beta_\chi L(1,\chi) \;+\; \sum_{\substack{\psi\ even\\ \psi\neq\psi_0}} \eta_\psi L(1,\psi) \;+\; \sum_{\chi\ odd} \beta'_\chi L(1,\chi) \;+\; \sum_{\psi\ odd} \eta'_\psi L(1,\psi)$$

is transcendental provided not all $\alpha_p, \beta_\chi, \eta_\psi$'s are zero.

Proof. As in the case of proof of Proposition 3.2.2, Theorem 2.3.4 allows us to ignore the odd characters. We show that

$$\sum_{p|q_1q_2} \alpha_p \log p \;+\; \sum_{\substack{\chi \text{ even}\\ \chi\neq\chi_0}} \beta_\chi L(1,\chi) \;+\; \sum_{\substack{\psi \text{ even}\\ \psi\neq\psi_0}} \eta_\psi L(1,\psi)$$

is transcendental where α_p for $p|q_1q_2$ and β_χ, η_ψ where χ, ψ varies over non-principal even Dirichlet characters modulo q_1 and q_2 respectively, are algebraic numbers, not all zero.

As in Proposition 3.2.2, we can write the above expression as

$$\sum_{p|q_1q_2} \alpha_p \log p + \sum_{\substack{1<a<q_1/2 \\ (a,q_1)=1}} \delta_a \log \xi_a + \sum_{\substack{1<b<q_2/2 \\ (b,q_2)=1}} \theta_b \log \xi_b,$$

where ξ_a, ξ_b's are multiplicatively independent units in $\mathbb{Q}(\zeta_{q_1})$ and $\mathbb{Q}(\zeta_{q_2})$ respectively.

Note that using Theorem 2.5.4 and Theorem 2.3.2 it is easy to deduce that $\delta_a = 0$ for all a if and only if $\beta_\chi = 0$ for all even non-trivial χ modulo q_1. Similarly $\theta_b = 0$ for all b if and only if $\eta_\psi = 0$ for all even non-trivial ψ modulo q_2.

Now by Theorem 2.3.2, the above expression is either zero or transcendental. Suppose that it is zero i.e. the numbers appearing in the above sum are not $\overline{\mathbb{Q}}$-linearly independent and thus again by Baker's theorem they are $\mathbb{Q}$-linearly dependent. Hence there exist integers c_p, d_a, e_b, not all zero, such that

$$\prod_{p|q_1q_2} p^{c_p} = \prod_{\substack{1<a<q_1/2 \\ (a,q_1)=1}} \xi_a^{d_a} \prod_{\substack{1<b<q_2/2 \\ (b,q_2)=1}} \xi_b^{e_b}, \tag{3.2.3}$$

By taking norm on both sides of (3.2.3), we get $c_p = 0$ for all p. Hence

$$\prod_{\substack{1<a<q_1/2 \\ (a,q_1)=1}} \xi_a^{d_a} = \prod_{\substack{1<b<q_2/2 \\ (b,q_2)=1}} \xi_b^{-e_b}. \tag{3.2.4}$$

Since $(q_1, q_2) = 1$, we know that $\mathbb{Q}(\zeta_{q_1}) \cap \mathbb{Q}(\zeta_{q_2}) = \mathbb{Q}$ and hence we get that both sides of (3.2.4) are rational numbers. Further they are algebraic integers and units, thus they have to be equal to ± 1. Now squaring both sides, we get

$$\prod_{\substack{1<a<q_1/2 \\ (a,q_1)=1}} \xi_a^{2d_a} = \prod_{\substack{1<b<q_2/2 \\ (b,q_2)=1}} \xi_b^{-2e_b} = 1. \tag{3.2.5}$$

This forces that $d_a = 0$ for all a's and $e_b = 0$ for all b since ξ_a's and ξ_b's are multiplicatively independent. This completes the proof of the proposition. ■

3.3 Proof of the theorems

3.3.1 Proof of Theorem 3.1.1

The proof is now straightforward. From Lemma 3.2.1 we obtain the following expression:

$$\gamma(\Omega,a,q) - \delta_\Omega\frac{\gamma}{q} = \frac{1}{\varphi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} \overline{\chi}(a)L(1,\chi)\prod_{p\in\Omega}\left(1-\frac{\chi(p)}{p}\right) + \frac{\delta_\Omega}{q}\left(\sum_{p|q}\frac{\log p}{p-1}+\sum_{p\in\Omega}\frac{\log p}{p-1}\right).$$

Now the theorem follows as an immediate corollary of Proposition 3.2.2. ■

3.3.2 Proof of Theorem 3.1.2

First we prove that the set $T(a,q)$ is infinite. In fact we show that the list of numbers $\gamma(\Omega,a,q)$, where Ω varies over U can have at most one pair of repetitions, i.e. there is at most one unordered pair (Ω_1,Ω_2) of distinct elements in U such that $\gamma(\Omega_1,a,q)=\gamma(\Omega_2,a,q)$. Suppose not, then there exist two unordered pairs $(\Omega_1,\Omega_2),(\Omega_3,\Omega_4)$ of distinct elements in U such that

$$\gamma(\Omega_1,a,q)=\gamma(\Omega_2,a,q) \ \text{ and } \ \gamma(\Omega_3,a,q)=\gamma(\Omega_4,a,q),$$

with $(\Omega_1,\Omega_2)\neq(\Omega_3,\Omega_4)$. Now we rewrite Lemma 3.2.1 as

$$\gamma(\Omega,a,q)=\alpha(\Omega,a,q)+\frac{\delta_\Omega}{q}\gamma+\beta(\Omega,q),$$

where

$$\alpha(\Omega,a,q)=\frac{1}{\varphi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} \overline{\chi}(a)L(1,\chi)\prod_{p\in\Omega}\left(1-\frac{\chi(p)}{p}\right)$$

and

$$\beta(\Omega,q)=\frac{\delta_\Omega}{q}\left(\sum_{p|q}\frac{\log p}{p-1}+\sum_{p\in\Omega}\frac{\log p}{p-1}\right).$$

Thus the equation $\gamma(\Omega_1,a,q) = \gamma(\Omega_2,a,q)$ is equivalent to the equation

$$\gamma = \frac{q}{\delta_{\Omega_1} - \delta_{\Omega_2}}(\alpha(\Omega_2,a,q) + \beta(\Omega_2,q) - \alpha(\Omega_1,a,q) - \beta(\Omega_1,q)).$$

Similarly the equation $\gamma(\Omega_3,a,q) = \gamma(\Omega_4,a,q)$ is equivalent to the equation

$$\gamma = \frac{q}{\delta_{\Omega_3} - \delta_{\Omega_4}}(\alpha(\Omega_4,a,q) + \beta(\Omega_4,q) - \alpha(\Omega_3,a,q) - \beta(\Omega_3,q)).$$

Now we equating the right hand side of the above two equations we get a contradiction by Proposition 3.2.2.

Next we prove that the set $T(a,q)$ can have at most one algebraic element. We prove it by contradiction. Suppose that $\gamma(\Omega_1,a,q)$ and $\gamma(\Omega_2,a,q)$ are algebraic, where $\Omega_1,\Omega_2 \in U$ with $\Omega_1 \neq \Omega_2$. Then by Lemma 3.2.1, we have

$$\delta_{\Omega_2}\gamma(\Omega_1,a,q) - \delta_{\Omega_1}\gamma(\Omega_2,a,q) = M + \frac{\delta_{\Omega_1}\delta_{\Omega_2}}{q}\left(\sum_{p\in\Omega_1}\frac{\log p}{p-1} - \sum_{p\in\Omega_2}\frac{\log p}{p-1}\right) \in \overline{\mathbb{Q}}, \tag{3.3.1}$$

where

$$M := \frac{1}{\varphi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\overline{\chi}(a)L(1,\chi)\left(\delta_{\Omega_2}\prod_{p\in\Omega_1}\left(1-\frac{\chi(p)}{p}\right) - \delta_{\Omega_1}\prod_{p\in\Omega_2}\left(1-\frac{\chi(p)}{p}\right)\right).$$

But by Proposition 3.2.2, the number in the right hand side of (3.3.1) is transcendental, which is a contradiction. This completes the proof of the theorem. ■

3.3.3 Proof of Theorem 3.1.3

The proof is carried out along the same line. Suppose that the theorem is not true and there exist natural numbers q_1,q_2 such that $\gamma(\Omega,a,q_1)$ and $\gamma(\Omega,a,q_2)$ are algebraic. Then by Lemma 3.2.1, we have

$$\frac{1}{q_2}\gamma(\Omega,a,q_1) - \frac{1}{q_1}\gamma(\Omega,a,q_2) = N + \frac{\delta_\Omega}{q_1q_2}\left(\sum_{\substack{p\mid q_1,\\ p\text{ prime}}}\frac{\log p}{p-1} - \sum_{\substack{p\mid q_2,\\ p\text{ prime}}}\frac{\log p}{p-1}\right) \in \overline{\mathbb{Q}}, \tag{3.3.2}$$

where

$$N := \sum_{\substack{\chi \bmod q_1 \\ \chi \neq \chi_0}} \frac{\overline{\chi}(a)\, L(1,\chi)}{q_2 \varphi(q_1)} \prod_{p \in \Omega} \left(1 - \frac{\chi(p)}{p}\right) - \sum_{\substack{\chi \bmod q_2 \\ \chi \neq \chi_0}} \frac{\overline{\chi}(a)\, L(1,\chi)}{q_1 \varphi(q_2)} \prod_{p \in \Omega} \left(1 - \frac{\chi(p)}{p}\right).$$

But by Proposition 3.2.3, we know that the right hand side of (3.3.2) is transcendental, which is a contradiction. This proves the theorem. ■

CHAPTER 4

LINEAR INDEPENDENCE RESULTS

4.1 Introduction

A complex number α is irrational if the numbers 1 and α are linearly independent over $\mathbb{Q}$. It is transcendental if for any natural number d, the numbers $1, \alpha, \alpha^2 \cdots \alpha^d$ are linearly independent over $\mathbb{Q}$. The study of possible linear relations among any family of interesting numbers constitutes an interesting theme in transcendence theory. For instance, any non-zero period ω of an elliptic curve $y^2 = 4x^3 - g_2x - g_3$ defined over $\overline{\mathbb{Q}}$ and π are linearly independent over $\overline{\mathbb{Q}}$. Note that this neither implies, nor follows from the transcendence of π and ω and hence is of independent interest.

To cite another instance, we can consider the family of Riemann zeta values at positive odd integers. It is known that zeta value at non-positive integers are given by the formula

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \quad \text{for all} \quad n \geq 0,$$

where B_{n+1} denotes the $(n+1)$-th Bernoulli number and hence the above zeta values are rational numbers. For zeta values at positive even integers, Euler proved that

$$\zeta(2n) \in \pi^{2n}\mathbb{Q}^* \quad \text{for all} \quad n \geq 1.$$

It is known due to Lindemann that π is transcendental. So the zeta values at positive even integers are transcendental numbers and furthermore are linearly independent over $\overline{\mathbb{Q}}$. But the arithmetic nature of the zeta values at odd positive integers is still shrouded in mystery. We have the following folklore conjecture for the odd zeta values.

Conjecture 4.1.1. *The numbers $\pi, \zeta(2n+1)$ for all $n \geq 1$ are algebraically independent.*

For the definition of algebraic independence see Chapter 5. In particular, all the odd zeta values are expected to be transcendental. But, till date we have no example of a single odd zeta value which is shown to be transcendental. The only example of an odd zeta value known to be irrational is $\zeta(3)$. This was proved by R. Apéry in 1978. Later in 2000, K. Ball and T. Rivoal [6] proved the following remarkable theorem in this direction.

Theorem 4.1.2 (Ball-Rivoal)**.** *Given any $\epsilon > 0$, there exists an integer $N = N(\epsilon)$ such that for all $n > N$, the dimension of the $\overline{\mathbb{Q}}$-vector space generated by the numbers*

$$1,\ \zeta(3),\ \cdots,\zeta(2n-1),\ \zeta(2n+1)$$

exceeds

$$\frac{1-\epsilon}{1+\log 2}\log n.$$

In particular, this implies that there are infinitely many odd zeta values which are irrational.

Let us now consider another family of numbers, namely for a given integer $q > 1$:

$$\{L(1,\chi) : \chi \text{ is a non-principal Dirichlet character mod } q\}.$$

We know that all the numbers in the above family are non-zero. Further, for any odd Dirichlet character χ modulo q, $L(1,\chi) \in \pi\,\overline{\mathbb{Q}}^*$. For even Dirichlet characters, by Theorem 2.3.2 these special values are transcendental. It follows from Theorem 2.5.4 that the dimension of the $\overline{\mathbb{Q}}$-vector space generated by all these constants is $\frac{\varphi(q)}{2}$. But the dimension of the $\mathbb{Q}$-vector space generated by these constants is conjectured to be $\varphi(q)-1$ by Baker and it is yet to be proved.

Against the backdrop of these, we now consider the family of generalised Euler-Briggs constants and study the question of linear independence over various number fields as well as over $\overline{\mathbb{Q}}$.

4.2 Linear independence over $\mathbb{Q}$

In this section we prove the following result.

Theorem 4.2.1. *Let Ω be a finite set of primes. Consider the $\mathbb{Q}$-vector space*

$$V_{\mathbb{Q},N} := \mathbb{Q}\,\langle \gamma(\Omega,m,n) \mid 1 \le m \le n \le N,\ \ (m,n) = 1 = (n,P_\Omega)\rangle\,.$$

Then for N sufficiently large, we have $N \ll_\Omega \dim_{\mathbb{Q}} V_{\mathbb{Q},N}$. In particular, the dimension of the $\mathbb{Q}$-vector space

$$V_{\mathbb{Q}} := \mathbb{Q}\,\langle \gamma(\Omega,m,n) \mid m,n \in \mathbb{N},\ (m,n) = 1 = (n,P_\Omega)\rangle$$

is infinite.

Before giving a proof of this theorem, we first mention the following result proved in [15] which plays an important role in our proof.

Theorem 4.2.2. *Let f be a periodic arithmetic function with period $q \geq 1$ and M be a natural number co-prime to q. Then*

$$\sum_{\substack{n\geq 1\\(n,M)=1}} \frac{f(n)}{n}$$

exists if and only if $\sum_{a=1}^{q} f(a) = 0$. Moreover, whenever the above sum exists, we have

$$\sum_{\substack{n\geq 1\\(n,M)=1}} \frac{f(n)}{n} = \sum_{a=1}^{q} f(a)\gamma(\Omega, a, q),$$

where Ω is the set of prime divisors of M.

We will give a proof of this theorem in Chapter 6.

4.2.1 Proof of Theorem 4.2.1

For any finite subset of primes Ω, let us define

$$S_\Omega := \{u \in \mathbb{N} \mid (u, P_\Omega) = 1\},$$

and for $u \in S_\Omega$,

$$\Gamma_{\Omega,u} := \{\gamma(\Omega, v, u) \mid 1 \leq v \leq u,\ (v,u) = 1\}.$$

Note that for a fixed u the cardinality of $\Gamma_{\Omega,u}$ is $\varphi(u)$. We claim that for any two relatively prime natural numbers s,t such that $(s,P_\Omega) = (t,P_\Omega) = 1$, at least one of the sets $\Gamma_{\Omega,s}, \Gamma_{\Omega,t}$ is $\mathbb{Q}$-linearly independent.

Suppose our claim is not true. Then for all $1 \leq a \leq s$ with $(a,s) = 1$ and for all $1 \leq b \leq t$ with $(b,t) = 1$, there exists rational numbers α_a (not all zero), β_b (not all zero) such that

$$\sum_{\substack{1\leq a<s\\(a,s)=1}} \alpha_a \gamma(\Omega, a, s) = 0, \quad \text{and} \quad \sum_{\substack{1\leq b<t\\(b,t)=1}} \beta_b \gamma(\Omega, b, t) = 0. \tag{4.2.1}$$

Now we define the following two arithmetic functions:

$$f(n) := \begin{cases} \alpha_a & \text{if } n \equiv a \bmod s,\ (a,s)=1 \\ -\sum_{\substack{1\le a\le s\\(a,s)=1}} \alpha_a & \text{if } n \equiv 0 \bmod s, \\ 0 & \text{otherwise,} \end{cases}$$

$$g(n) := \begin{cases} \beta_b & \text{if } n \equiv b \bmod t,\ (b,t)=1 \\ -\sum_{\substack{1\le b\le r\\(b,t)=1}} \beta_b & \text{if } n \equiv 0 \bmod t, \\ 0 & \text{otherwise.} \end{cases}$$

We can see that f and g are periodic functions with periods s and t respectively. Further,

$$\sum_{1\le a\le s} f(a) = 0, \quad \text{and} \quad \sum_{1\le b\le t} g(b) = 0.$$

Hence by Theorem 4.2.2 and equation (4.2.1), we have

$$\sum_{\substack{n\ge1\\(n,P_\Omega)=1}} \frac{f(n)}{n} = \sum_{a=1}^{s} f(a)\gamma(\Omega,a,s) = f(s)\gamma(\Omega,s,s) = \frac{f(s)}{s}\big(\gamma(\Omega)-\delta_\Omega \log s\big) \tag{4.2.2}$$

$$\text{and} \quad \sum_{\substack{m\ge1\\(m,P_\Omega)=1}} \frac{g(m)}{m} = \sum_{b=1}^{t} g(b)\gamma(\Omega,b,t) = g(t)\gamma(\Omega,t,t) = \frac{g(t)}{t}\big(\gamma(\Omega)-\delta_\Omega \log t\big).$$

Next we show that both $f(s)$ and $g(t)$ are non-zero. Suppose, $f(s)=0$. Then by above equation we get that

$$\sum_{\substack{n\ge1\\(n,P_\Omega)=1}} \frac{f(n)}{n} = 0.$$

We rewrite,

$$\sum_{\substack{n\ge1\\(n,P_\Omega)=1}} \frac{f(n)}{n} = \sum_{n\ge1} \frac{f\chi_0(n)}{n},$$

where χ_0 denotes the trivial character mod P_Ω. Now $f\chi_0$ is a rational valued sP_Ω periodic function for which we have

$$f\chi_0(n) = 0 \quad \text{for all} \quad 1 < (n, sP_\Omega) < sP_\Omega.$$

Now applying the theorem of Baker, Birch and Wirsing [5], we get

$$\sum_{\substack{n\geq 1\\(n,P_\Omega)=1}} \frac{f(n)}{n} \neq 0,$$

a contradiction and hence $f(s)\neq 0$. Similarly we get that $g(t)\neq 0$. So we have the following two expressions for $\gamma(\Omega)$ from each of the equations of (4.2.2) i.e.

$$\gamma(\Omega) = \frac{s}{f(s)}\sum_{a=1}^{s} f(a)\gamma(\Omega,a,s)+\delta_\Omega \log s = \frac{t}{g(t)}\sum_{b=1}^{t} g(b)\gamma(\Omega,b,t)+\delta_\Omega \log t.$$

Now using the definition of f and g we have

$$\frac{s}{f(s)}\sum_{\substack{1\leq a\leq s\\(a,s)=1}} f(a)\gamma(\Omega,a,s) + s\gamma(\Omega,s,s) + \delta_\Omega \log s = \frac{t}{g(t)}\sum_{\substack{1\leq b\leq t\\(b,t)=1}} g(b)\gamma(\Omega,b,t) + t\gamma(\Omega,t,t) + \delta_\Omega \log t.$$

i.e.

$$\frac{s}{f(s)}\sum_{\substack{1\leq a\leq s\\(a,s)=1}} f(a)\gamma(\Omega,a,s) - \frac{t}{g(t)}\sum_{\substack{1\leq b\leq t\\(b,t)=1}} g(b)\gamma(\Omega,b,t) = 0.$$

Since

$$f(s) = -\sum_{\substack{1\leq a\leq s\\(a,s)=1}} f(a) \text{ and } g(t) = -\sum_{\substack{1\leq b\leq t\\(b,t)=1}} g(b),$$

using Lemma 3.2.1 we get

$$\begin{aligned}
&\frac{s}{f(s)\varphi(s)}\sum_{\substack{\chi \bmod s\\ \chi\neq\chi_0}} L(1,\chi)\prod_{p\in\Omega}\left(1-\frac{\chi(p)}{p}\right)\sum_{\substack{1\leq a\leq s\\(a,s)=1}} f(a)\overline{\chi}(a) \;-\; \delta_\Omega\sum_{p\mid s}\frac{\log p}{p-1}\\
&-\frac{t}{g(t)\varphi(t)}\sum_{\substack{\psi \bmod t\\ \psi\neq\psi_0}} L(1,\psi)\prod_{p\in\Omega}\left(1-\frac{\psi(p)}{p}\right)\sum_{\substack{1\leq b\leq t\\(b,t)=1}} g(b)\overline{\psi}(b) \;+\; \delta_\Omega\sum_{p\mid t}\frac{\log p}{p-1} = 0,
\end{aligned} \tag{4.2.3}$$

which leads to a contradiction to Proposition 3.2.3. This proves our claim.

Now we proceed to calculate a lower bound of $\dim V_{\mathbb{Q},N}$. We have already shown that for two natural numbers s,t such that $(s,P_\Omega)=(t,P_\Omega)=1$, at least one of the sets $\Gamma_{\Omega,s},\Gamma_{\Omega,t}$ is $\mathbb{Q}$-linearly independent. To get a lower bound of the dimension of $V_{\mathbb{Q},N}$, we find a pair s,t of prime numbers in terms of N.

Let l be the number of primes in Ω. Now using Bertrand's postulate for large enough N, we get that there are at least $l+2$ primes between $\frac{N}{2^{l+2}}$ and N. Thus we can get two primes $s, t \geq \frac{N}{2^{l+2}}$ such that they are co-prime to P_Ω. Hence

$$\dim V_{\mathbb{Q},N} \geq \min\{\varphi(s), \varphi(t)\} = \min\{s-1, t-1\} \geq \frac{N}{2^{l+2}} - 1 \gg_\Omega N.$$

■

Remark 4.2.3. The trivial upper bound of $\dim V_{\mathbb{Q},N}$ is $O(N^2)$. This can be observed by counting the cardinality of the generating set

$$\{\gamma(\Omega, m, n) \mid 1 \leq m \leq n \leq N \in \mathbb{N},\ (m,n) = 1 = (n, P_\Omega)\}.$$

which is

$$\sum_{\substack{n \leq N \\ (n, P_\Omega)=1}} \varphi(n) = O(N^2).$$

4.3 Linear independence over number fields

In this section, we prove result for number fields, analogous to the result in §2 of this chapter. As will be evident, we need to impose certain natural restrictions on the number fields under consideration.

Theorem 4.3.1. *Let Ω be a finite set of primes and K be a number field such that $K \cap \mathbb{Q}(\zeta_{P_\Omega}) = \mathbb{Q}$, where $\zeta_{P_\Omega} := e^{\frac{2\pi i}{P_\Omega}}$. Consider the K-vector space*

$$V_{K,N} := K \left\langle \gamma(\Omega, m, n) \mid 1 \leq m \leq n \leq N,\ \ (m,n) = 1 = (n, P_\Omega) \right\rangle.$$

Then for N sufficiently large, we have

$$N \ll_{K,\Omega} \dim_K V_{K,N},$$

where the implied constant depend on Ω and K. In particular, the K-vector space

$$V_K := K\,\langle \gamma(\Omega, m, n) \mid m, n \in \mathbb{N},\ (m,n) = 1 = (n, P_\Omega)\rangle$$

is infinite dimensional.

The proof of this theorem follows mutatis-mutandis the proof of Theorem 4.2.1. We just indicate the necessary modifications.

1. In order to use the theorem of Baker, Birch and Wirsing, we replace the set S_Ω by S'_Ω, where

   $$S'_\Omega := \{u \in \mathbb{N} \mid (u, P_\Omega) = 1,\ \Phi_{uP_\Omega}(X) \text{ is irreducible over } K\}.$$

   Here we note that the condition $K \cap \mathbb{Q}(\zeta_{P_\Omega}) = \mathbb{Q}$ is equivalent to the condition that $\Phi_{P_\Omega}(X)$ is irreducible over K, and hence S'_Ω is non-empty. Here is a proof of this for the sake of completeness.

   Lemma 4.3.2. *For any number field L, $L \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$ if and only if $\Phi_n(X)$ is irreducible over L, where ζ_n denotes a primitive n-th root of unity.*

   Proof. First suppose that $L \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$. We show that $\Phi_n(X)$ is irreducible over L. Suppose not, then we can write $\Phi_n(X) = f(X)g(X)$ where f, g are polynomials over L with $\deg f, \deg g \geq 1$. Now the co-efficients of f, g are symmetric polynomials of primitive n-th roots of unity. Since $L \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$, we get that $f(X), g(X) \in \mathbb{Q}[X]$. We get a contradiction to the fact that $\Phi_n(X)$ is irreducible over $\mathbb{Q}$.

   Now we prove the converse part. From hypothesis, we have $[L(\zeta_n) : L] = \varphi(n)$. Thus if $[L : \mathbb{Q}] = m$, we have that $[L(\zeta_n) : \mathbb{Q}(\zeta_n)] = m$. Now we show that $L \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$. Suppose not, then there exists $\alpha \in L \cap \mathbb{Q}(\zeta_n)$ such that $\alpha \notin \mathbb{Q}$. Consider the following tower of field extensions.

$$\begin{tikzcd}
& L(\zeta_n) \arrow[dl, dash] \arrow[dr, dash] & \\
L \arrow[dr, dash] & & \mathbb{Q}(\zeta_n) \arrow[dl, dash] \\
& L \cap \mathbb{Q}(\zeta_n) \arrow[d, dash] & \\
& \mathbb{Q}(\alpha) \arrow[d, dash] & \\
& \mathbb{Q} &
\end{tikzcd}$$

Now suppose $[L : \mathbb{Q}(\alpha)] = l$. Then $l \leq m$ and $[L(\zeta_n) : \mathbb{Q}(\zeta_n, \alpha)] \leq l$. But $\mathbb{Q}(\zeta_n, \alpha) = \mathbb{Q}(\zeta_n)$. Hence, $m \leq l$. This in turn gives us

$$[L : \mathbb{Q}] = m = l = [L : \mathbb{Q}(\alpha)],$$

i.e. $\mathbb{Q} = \mathbb{Q}(\alpha)$.

■

2. To conclude the theorem, we need the set S'_Ω to be infinite. This can be proved using the following lemma for $L = K(\zeta_{P_\Omega})$ and the fact that only finitely many primes ramify in a number field.

Lemma 4.3.3. *Let L be a number field and p be a rational prime which does not ramify in L. Then $\Phi_p(X)$ is irreducible over L.*

Proof. We show that $L \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$. Consider the following tower of field extensions.

$$\begin{tikzcd}
 & L(\zeta_p) \arrow[dl, dash] \arrow[dr, dash] & \\
L \arrow[dr, dash] & & \mathbb{Q}(\zeta_p) \arrow[dl, dash] \\
 & L \cap \mathbb{Q}(\zeta_p) \arrow[d, dash] & \\
 & \mathbb{Q} &
\end{tikzcd}$$

For the field extension $L/\mathbb{Q}$, p is a prime which does not ramify and for the field extension $\mathbb{Q}(\zeta_p)/\mathbb{Q}$, p is the only prime that ramifies. Hence, if $L \cap \mathbb{Q}(\zeta_p) \neq \mathbb{Q}$, by considering a prime q which ramifies in $L \cap \mathbb{Q}(\zeta_p)/\mathbb{Q}$, we arrive at a contradiction. ■

To get a lower bound on $\dim_K V_{K,N}$, let m be the number of primes which are ramified in the extension $\mathbb{Q} \subseteq K(\zeta_{P_\Omega})$ and l be the number of primes in Ω. Then again by Bertrand's postulate for large enough N, we get that there are at least $(m+l+2)$ primes between $\frac{N}{2^{m+l+2}}$ and N. Thus we can get two primes $s, t \geq \frac{N}{2^{m+l+2}}$ such that they are co-prime to P_Ω and do not ramify in $K(\zeta_{P_\Omega})$. Hence

$$\dim V_{K,N} \geq \min\{\varphi(s), \varphi(t)\} = \min\{s-1, t-1\} \geq \frac{N}{2^{m+l+2}} - 1 \gg_{\Omega,K} N.$$

This completes the proof. ■

Remark 4.3.4. As earlier, the trivial upper bound of $\dim_K V_{K,N}$ is again $O(N^2)$.

4.4 Linear independence over $\overline{\mathbb{Q}}$

In this section we will discuss the case of $\overline{\mathbb{Q}}$-linear independence of generalised Euler-Briggs constants. First notice that in the case of $\overline{\mathbb{Q}}$ with a fixed finite set of primes Ω, we can no longer use the theorem of Baker, Birch and Wirsing as $\mathbb{Q}(\zeta_{P_\Omega}) \cap \overline{\mathbb{Q}} \neq \mathbb{Q}$.

To state the theorems we begin with the following notations: For a finite set of primes Ω and $a \in \mathbb{N}$, consider $C(a,\Omega) := \{q \in \mathbb{N} \mid (a,q) = 1 = (q, P_\Omega)\}$. We define an equivalence relation on the set $X := \{\gamma(\Omega,a,q) : q \in C(a,\Omega)\}$, given by $\gamma(\Omega,a,q_1) \sim \gamma(\Omega,a,q_2)$ if $\gamma(\Omega,a,q_1) = \lambda\gamma(\Omega,a,q_2)$ for some $\lambda \in \overline{\mathbb{Q}}^*$. Then we have the following theorem for a particular subset Y of X.

Theorem 4.4.1. *Let Ω be a finite set of primes and $a \in \mathbb{N}$. Let Y be a subset of $C(a,\Omega)$, consisting of co-prime integers. Then in $\{\gamma(\Omega,a,q) : q \in Y\}$, each equivalence class $[\gamma(\Omega,a,q)]$, where the equivalence relation is restricted to Y, has at most two elements.*

For the proof of Theorem 4.4.1, we shall need the following theorem.

Theorem 4.4.2. *Let $q_1,q_2,q_3 > 1$ be mutually co-prime natural numbers. Then for any algebraic numbers $\alpha_p, \beta_\chi, \beta_\phi, \beta_\psi$, the number*

$$\sum_{p \mid q_1q_2q_3} \alpha_p \log p + \sum_{\substack{\chi \bmod q_1 \\ \chi \neq \chi_0}} \beta_\chi L(1,\chi) + \sum_{\substack{\phi \bmod q_2 \\ \phi \neq \phi_0}} \beta_\phi L(1,\phi) + \sum_{\substack{\psi \bmod q_3 \\ \psi \neq \psi_0}} \beta_\chi L(1,\psi)$$

is transcendental provided not all $\alpha_p, \beta_\chi, \beta_\phi, \beta_\psi$ for even characters χ,ϕ,ψ are zero.

Proof. We will prove this theorem by contradiction. We know that for any even Dirichlet character $\chi \neq \chi_0$, one can write $L(1,\chi)$ as a non-zero algebraic multiple of

$$\sum_{\substack{1<a<q/2 \\ (a,q)=1}} \overline{\chi}(a) \log \xi_a, \tag{4.4.1}$$

where ξ_a's are real multiplicatively independent units in the cyclotomic field $\mathbb{Q}(\zeta_q)$, known as Ramachandra units and for any odd Dirichlet character χ, we know that $L(1,\chi)$ is a non-zero algebraic multiple of π.

Using these results and Theorem 2.3.4, we can therefore ignore the odd characters. In order to complete the proof of the theorem, we will now show that

1. $\log p$: for all primes $p|q_1q_2q_3$
2. $L(1,\chi)$: for all even non-principal characters χ modulo q_1
3. $L(1,\phi)$: for all even non-principal characters ϕ modulo q_2
4. $L(1,\psi)$: for all even non-principal characters ψ modulo q_3

are linearly independent over $\overline{\mathbb{Q}}$. Suppose not. Then there exists algebraic numbers α_p for $p|q_1q_2q_3$ and $\beta_\chi, \beta_\phi, \beta_\psi$, where χ, ϕ, ψ vary over non-principal even Dirichlet characters modulo q_1, q_2 and q_3 respectively, not all zero, such that

$$\sum_{p|q_1q_2q_3} \alpha_p \log p + \sum_{\substack{\chi \text{ even} \\ \chi \neq \chi_0}} \beta_\chi L(1,\chi) + \sum_{\substack{\phi \text{ even} \\ \phi \neq \phi_0}} \beta_\phi L(1,\phi) + \sum_{\substack{\psi \text{ even} \\ \psi \neq \psi_0}} \beta_\psi L(1,\psi) = 0.$$

We can rewrite the above expression as

$$\sum_{p|q_1q_2q_3} \alpha_p \log p + \sum_{\substack{1<a<q_1/2 \\ (a,q_1)=1}} \delta_a \log \xi_a + \sum_{\substack{1<b<q_2/2 \\ (b,q_2)=1}} \delta_b \log \xi_b + \sum_{\substack{1<c<q_3/2 \\ (c,q_3)=1}} \delta_c \log \xi_c = 0,$$

where ξ_a, ξ_b, ξ_c 's are multiplicatively independent units in $\mathbb{Q}(\zeta_{q_1}), \mathbb{Q}(\zeta_{q_2})$ and $\mathbb{Q}(\zeta_{q_3})$ respectively. Now by Theorem 2.3.2, we have

$$\prod_{p|q_1q_2} p^{c_p} = \prod_{\substack{1<a<q_1/2 \\ (a,q_1)=1}} \xi_a^{d_a} \prod_{\substack{1<b<q_2/2 \\ (b,q_2)=1}} \xi_b^{e_b} \prod_{\substack{1<c<q_3/2 \\ (c,q_3)=1}} \xi_c^{f_c} \tag{4.4.2}$$

where c_p, d_a, e_b, f_c's are integers. By taking norms on both sides of (4.4.2), we get $c_p = 0$ for all p. Hence

$$\prod_{\substack{1<a<q_1/2 \\ (a,q_1)=1}} \xi_a^{d_a} = \prod_{\substack{1<b<q_2/2 \\ (b,q_2)=1}} \xi_b^{-e_b} \prod_{\substack{1<c<q_3/2 \\ (c,q_3)=1}} \xi_c^{-f_c} \tag{4.4.3}$$

Since q_1, q_2, q_3 are mutually co-prime, $\mathbb{Q}(\zeta_{q_1}) \cap \mathbb{Q}(\zeta_{q_2q_3}) = \mathbb{Q}$. So we see that both sides of (4.4.3) are rational numbers and hence equal to ± 1. Now squaring both sides, we get

$$\prod_{\substack{1<a<q_1/2 \\ (a,q_1)=1}} \xi_a^{2d_a} = \prod_{\substack{1<b<q_2/2 \\ (b,q_2)=1}} \xi_b^{-2e_b} \prod_{\substack{1<c<q_3/2 \\ (c,q_3)=1}} \xi_c^{-2f_c} = 1. \tag{4.4.4}$$

This forces that $d_a = 0$ for all a since ξ_a's are multiplicatively independent. Again going back to (4.4.3) and following the same argument, we get $e_b = 0, f_c = 0$ for all b, c. This completes the proof. ■

Now we are ready to prove Theorem 4.4.1:

4.4.1 Proof of Theorem 4.4.1

Suppose that $\gamma(\Omega, a, q_2)$, $\gamma(\Omega, a, q_3) \in [\gamma(\Omega, a, q_1)]$, where q_1, q_2, q_3 are distinct elements in Y. Then there exist non-zero algebraic numbers β, λ such that

$$\gamma(\Omega,a,q_1) = \beta\gamma(\Omega,a,q_2), \quad \text{and} \quad \gamma(\Omega,a,q_1) = \lambda\gamma(\Omega,a,q_3). \tag{4.4.5}$$

Write

$$\begin{aligned} \alpha_{\Omega,q_i} &:= \frac{\delta_\Omega}{q_i} \neq 0, \quad \gamma_1 := \gamma + \sum_{p|\Omega} \frac{\log p}{p-1} \\ \text{and} \quad \alpha_{\Omega,\chi,q_i} &:= \frac{\overline{\chi}(a)}{\varphi(q_i)} \prod_{p\in\Omega}\left(1 - \frac{\chi(p)}{p}\right). \end{aligned}$$

Using Lemma 3.2.1 and (4.4.5), we get

$$\begin{aligned} &\gamma_1(\alpha_{\Omega,q_1} - \beta\alpha_{\Omega,q_2}) + \alpha_{\Omega,q_1}\sum_{p|q_1}\frac{\log p}{p-1} - \beta\alpha_{\Omega,q_2}\sum_{p|q_2}\frac{\log p}{p-1} \\ &+ \sum_{\substack{\chi \bmod q_1 \\ \chi\neq\chi_0}} \alpha_{\Omega,\chi,q_1}L(1,\chi) - \beta \sum_{\substack{\chi \bmod q_2 \\ \chi\neq\chi_0}} \alpha_{\Omega,\chi,q_2}L(1,\chi) = 0. \end{aligned} \tag{4.4.6}$$

Similarly, we have

$$\begin{aligned} &\gamma_1(\alpha_{\Omega,q_1} - \lambda\alpha_{\Omega,q_3}) + \alpha_{\Omega,q_1}\sum_{p|q_1}\frac{\log p}{p-1} - \lambda\alpha_{\Omega,q_3}\sum_{p|q_3}\frac{\log p}{p-1} \\ &+ \sum_{\substack{\chi \bmod q_1 \\ \chi\neq\chi_0}} \alpha_{\Omega,\chi,q_1}L(1,\chi) - \lambda \sum_{\substack{\chi \bmod q_3 \\ \chi\neq\chi_0}} \alpha_{\Omega,\chi,q_3}L(1,\chi) = 0. \end{aligned} \tag{4.4.7}$$

Since q_1, q_2 and q_3 are mutually co-prime natural numbers, applying Proposition 3.2.3 to equations (4.4.6) and (4.4.7), we get

$$a_{\Omega,q_1} - \beta a_{\Omega,q_2} \neq 0, \quad a_{\Omega,q_1} - \lambda a_{\Omega,q_3} \neq 0.$$

Similar reasoning shows that

$$\beta a_{\Omega,q_2} - \lambda a_{\Omega,q_3} \neq 0.$$

Hence

$$\begin{aligned} & C a_{\Omega,q_1} \sum_{p|q_1} \frac{\log p}{p-1} - \frac{\beta a_{\Omega,q_2}}{(a_{\Omega,q_1} - \beta a_{\Omega,q_2})} \sum_{p|q_2} \frac{\log p}{p-1} + \frac{\lambda a_{\Omega,q_3}}{(a_{q_1} - \lambda a_{\Omega,q_3})} \sum_{p|q_3} \frac{\log p}{p-1} \\ & + C \sum_{\substack{\chi \bmod q_1 \\ \chi \neq \chi_0}} a_{\Omega,\chi,q_1} L(1,\chi) + \frac{\lambda}{(a_{\Omega,q_1} - \lambda a_{\Omega,q_3})} \sum_{\substack{\chi \bmod q_3 \\ \chi \neq \chi_0}} a_{\Omega,\chi,q_3} L(1,\chi) \\ & - \frac{\beta}{(a_{\Omega,q_1} - \beta a_{\Omega,q_2})} \sum_{\substack{\chi \bmod q_2 \\ \chi \neq \chi_0}} a_{\Omega,\chi,q_2} L(1,\chi) = 0, \end{aligned} \tag{4.4.8}$$

where

$$C := \frac{\beta a_{\Omega,q_2} - \lambda a_{\Omega,q_3}}{(a_{\Omega,q_1} - \beta a_{\Omega,q_2})(a_{\Omega,q_1} - \lambda a_{\Omega,q_3})} \neq 0,$$

a contradiction to Theorem 4.4.2. ■

Remark 4.4.3. It follows from the above theorem that the following $\overline{\mathbb{Q}}$ linear space

$$V_{\overline{\mathbb{Q}}} := \overline{\mathbb{Q}} \langle \gamma(\Omega, m, n) \mid m, n \in \mathbb{N},\ (m,n) = 1 = (n, P_\Omega) \rangle$$

has dimension at least two.

It is also possible to derive similar theorems for family of generalised Euler-Briggs constants where we vary Ω's with fixed q. For this, define

$$C(q) := \{\Omega \mid \Omega \text{ is a finite set of primes and } (q, P_\Omega) = 1\}.$$

As before, for a fixed a with $(a,q) = 1$, one can define a similar equivalence relation on the set $Z := \{\gamma(\Omega, a, q) : \Omega \in C(q)\}$. In this set-up, we have the following theorem.

Theorem 4.4.4. *The orbit of any element $\gamma(\Omega,a,q)\in Z$ has at most two elements.*

Proof. Suppose that $\gamma(\Omega_2,a,q)$, $\gamma(\Omega_3,a,q)\in[\gamma(\Omega_1,a,q)]$, where Ω_1,Ω_2 and Ω_3 are distinct elements in $C(q)$. Then there exist non-zero algebraic numbers β,λ such that

$$\gamma(\Omega_1,a,q) = \beta\gamma(\Omega_2,a,q) \quad\text{and}\quad \gamma(\Omega_1,a,q) = \lambda\gamma(\Omega_3,a,q). \tag{4.4.9}$$

For a Dirichlet character χ modulo q and a finite set Ω consisting of primes co-prime to q, we have the numbers $a_{\Omega,q}$, γ_1 and $\alpha_{\Omega,\chi,q}$ as defined in the proof of Theorem 4.4.1.

Using Lemma 3.2.1 and (4.4.9), we get

$$\begin{aligned}&\gamma_1(a_{\Omega_1,q}-\beta a_{\Omega_2,q})+a_{\Omega_1,q}\sum_{p\in\Omega_1}\frac{\log p}{p-1}-\beta a_{\Omega_2,q}\sum_{p\in\Omega_2}\frac{\log p}{p-1}\\&+\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}L(1,\chi)(\alpha_{\Omega_1,\chi,q}-\beta\alpha_{\Omega_2,\chi,q}) = 0\end{aligned} \tag{4.4.10}$$

and

$$\begin{aligned}&\gamma_1(a_{\Omega_1,q}-\lambda a_{\Omega_3,q})+a_{\Omega_1,q}\sum_{p\in\Omega_1}\frac{\log p}{p-1}-\lambda a_{\Omega_3,q}\sum_{p\in\Omega_3}\frac{\log p}{p-1}\\&+\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}L(1,\chi)(\alpha_{\Omega_1,\chi,q}-\lambda\alpha_{\Omega_3,\chi,q}) = 0.\end{aligned} \tag{4.4.11}$$

Since $\Omega_1,\Omega_2,\Omega_3$ are distinct set of primes, applying Proposition 3.2.2 to equations (4.4.10) and (4.4.11), we get

$$a_{\Omega_1,q}-\beta a_{\Omega_2,q}\neq 0,\quad a_{\Omega_1,q}-\lambda a_{\Omega_3,q}\neq 0.$$

Similarly we deduce that

$$a_{\Omega_3}-\frac{\beta}{\lambda}a_{\Omega_2}\neq 0.$$

Now from (4.4.10) and (4.4.11), it follows that

$$\begin{aligned}&\frac{a_{\Omega_1,q}(\beta a_{\Omega_2,q}-\lambda a_{\Omega_3,q})}{(a_{\Omega_1,q}-\beta a_{\Omega_2,q})(a_{\Omega_1,q}-\lambda a_{\Omega_3,q})}\sum_{p\in\Omega_1}\frac{\log p}{p-1}-\frac{\beta a_{\Omega_2,q}}{(a_{\Omega_1,q}-\beta a_{\Omega_2,q})}\sum_{p\in\Omega_2}\frac{\log p}{p-1}\\&+\frac{\lambda a_{\Omega_3,q}}{(a_{\Omega_1,q}-\lambda a_{\Omega_3,q})}\sum_{p\in\Omega_3}\frac{\log p}{p-1}+\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}L(1,\chi)A(\chi) = 0,\end{aligned}$$

where

$$A(\chi) = \frac{(\alpha_{\Omega_1,\chi,q} - \beta\alpha_{\Omega_2,\chi,q})}{(a_{\Omega_1,q} - \beta a_{\Omega_2,q})} - \frac{(\alpha_{\Omega_1,\chi,q} - \lambda\alpha_{\Omega_3,\chi,q})}{(a_{\Omega_1,q} - \lambda a_{\Omega_3,q})}.$$

Since $\Omega_1, \Omega_2, \Omega_3$ are distinct set of primes, without loss of generality, one can assume that there exists a prime $p_1 \in \Omega_1$ such that either $p_1 \notin \Omega_2 \cup \Omega_3$ or $p_1 \in \Omega_2$ but not in Ω_3. The coefficient of $\log p_1$ in the first case is

$$\frac{a_{\Omega_1,q}(\beta a_{\Omega_2,q} - \lambda a_{\Omega_3,q})}{(a_{\Omega_1,q} - \beta a_{\Omega_2,q})(a_{\Omega_1,q} - \lambda a_{\Omega_3,q})(p_1 - 1)} \neq 0$$

and in the second case is

$$\frac{\lambda a_{\Omega_3,q}}{(\lambda a_{\Omega_3,q} - a_{\Omega_1,q})(p_1 - 1)} \neq 0.$$

Hence in both cases we arrive at a contradiction by Proposition 3.2.2. ■

From Theorem 4.4.4, we can conclude that the following $\overline{\mathbb{Q}}$ linear space

$$V_{\overline{\mathbb{Q}},a,q} := \overline{\mathbb{Q}}\langle \gamma(\Omega, a, q) \mid \Omega \in C(q)\rangle$$

has dimension at least two. But we prove something stronger in the following theorem.

Theorem 4.4.5. *Let a, q be natural numbers with $(a, q) = 1$. Then the dimension of the $\overline{\mathbb{Q}}$-vector space*

$$V_{\overline{\mathbb{Q}},a,q} := \overline{\mathbb{Q}}\langle \gamma(\Omega, a, q) \mid \Omega \in C(q)\rangle$$

is infinite.

Proof. It is sufficient to show that given any natural number n, there exist disjoint subsets $\Omega_1, \ldots, \Omega_n \in C(q)$ such that $\gamma(\Omega_1, a, q), \ldots, \gamma(\Omega_n, a, q)$ are linearly independent over $\overline{\mathbb{Q}}$. Suppose that our claim is not true. Then there exists an $n \in \mathbb{N}$ such that for any disjoint sets $\Omega_1, \ldots, \Omega_n \in C(q)$ and $\Omega'_1, \ldots, \Omega'_n \in C(q)$, we can find $\alpha_i, \beta_j \in \overline{\mathbb{Q}}$, $1 \le i, j \le n$, not all zero such that

$$\alpha_1\gamma(\Omega_1, a, q) + \cdots + \alpha_n\gamma(\Omega_n, a, q) = 0 \quad \text{and} \quad \beta_1\gamma(\Omega'_1, a, q) + \cdots + \beta_n\gamma(\Omega'_n, a, q) = 0.$$

Further assume that Ω_i's are disjoint from Ω'_j's for all $1 \le i, j \le n$. Then by Lemma 3.2.1, we have

$$\begin{aligned} \gamma \sum_{i=1}^{n} \alpha_i \delta_{\Omega_i} = \frac{-q}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \overline{\chi}(a) L(1,\chi) \sum_{i=1}^{n} \alpha_i \prod_{p \in \Omega_i} \left(1 - \frac{\chi(p)}{p}\right) \\ - \sum_{p|q} \frac{\log p}{p-1} \sum_{i=1}^{n} \alpha_i \delta_{\Omega_i} - \sum_{i=1}^{n} \alpha_i \delta_{\Omega_i} \sum_{p \in \Omega_i} \frac{\log p}{p-1} \end{aligned} \tag{4.4.12}$$

and

$$\begin{aligned} \gamma \sum_{j=1}^{n} \beta_j \delta_{\Omega'_j} = \frac{-q}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \overline{\chi}(a) L(1,\chi) \sum_{j=1}^{n} \beta_j \prod_{p \in \Omega'_j} \left(1 - \frac{\chi(p)}{p}\right) \\ - \sum_{p|q} \frac{\log p}{p-1} \sum_{j=1}^{n} \beta_j \delta_{\Omega'_j} - \sum_{j=1}^{n} \beta_j \delta_{\Omega'_j} \sum_{p \in \Omega'_j} \frac{\log p}{p-1}. \end{aligned} \tag{4.4.13}$$

Applying Proposition 3.2.2, we see that $A := \sum_{i=1}^{n} \alpha_i \delta_{\Omega_i} \neq 0$ and $B := \sum_{j=1}^{n} \beta_j \delta_{\Omega'_j} \neq 0$. Hence from (4.4.12) and (4.4.13), we get

$$\begin{aligned} \frac{q}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \overline{\chi}(a) L(1,\chi) \left(\sum_{i=1}^{n} \frac{\alpha_i}{A} \prod_{p \in \Omega_i} \left(1 - \frac{\chi(p)}{p}\right) - \sum_{j=1}^{n} \frac{\beta_j}{B} \prod_{p \in \Omega'_j} \left(1 - \frac{\chi(p)}{p}\right) \right) \\ + \sum_{i=1}^{n} \frac{\alpha_i \delta_{\Omega_i}}{A} \sum_{p \in \Omega_i} \frac{\log p}{p-1} - \sum_{j=1}^{n} \frac{\beta_j \delta_{\Omega'_j}}{B} \sum_{p \in \Omega'_j} \frac{\log p}{p-1} = 0, \end{aligned}$$

a contradiction to Proposition 3.2.2. ■

CHAPTER 5

ALGEBRAIC INDEPENDENCE RESULTS

5.1 Introduction

Let $\alpha_1,\dots,\alpha_n$ be complex numbers. They are said to be algebraically independent if $P(\alpha_1,\dots,\alpha_n) \neq 0$ for any non-zero polynomial $P(x_1,\dots,x_n) \in \overline{\mathbb{Q}}[x_1,\dots,x_n]$. Otherwise, the numbers $\alpha_1,\dots,\alpha_n$ are said to be algebraically dependent. An infinite set of complex numbers is called algebraically independent if every finite subset of it is algebraically independent.

We begin with the following remarkable theorem of Lindemann-Weierstrass.

Theorem 5.1.1 (Lindemann-Weierstrass)**.** *If $\alpha_1,\dots,\alpha_n$ are algebraic numbers which are linearly independent over $\mathbb{Q}$, then*

$$e^{\alpha_1},\dots,e^{\alpha_n}$$

are algebraically independent.

The above theorem is a special case of a far reaching conjecture, proposed by Stephen Schanuel in the 1960s.

Conjecture 5.1.2 (Schanuel)**.** *Let $\alpha_1, \ldots, \alpha_n$ be complex numbers which are linearly independent over $\mathbb{Q}$. Then among the following numbers*

$$\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n},$$

at least n are algebraically independent.

Remark 5.1.3. Note that the $\mathbb{Q}$-linear independence of the numbers $\alpha_1, \ldots, \alpha_n$ is a necessary condition as can be seen by taking $\alpha_1 = i\pi$ and $\alpha_2 = 2i\pi$.

The following consequence of Schanuel's conjecture, which is known as the weak Schanuel conjecture, generalises Baker's theorem. See [13, 24] for more details.

Conjecture 5.1.4 (weak Schanuel conjecture)**.** *Let $\alpha_1, \cdots, \alpha_n$ be non-zero algebraic numbers such that the numbers $\log \alpha_1, \cdots, \log \alpha_n$ are $\mathbb{Q}$-linearly independent. Then $\log \alpha_1, \cdots, \log \alpha_n$ are algebraically independent.*

For a proof of Theorem 5.1.1 and consequences of Conjecture 5.1.2 and Conjecture 5.1.4, we refer to [13, 24].

In this chapter, we present some results dealing with the algebraic independence of generalised Euler-Briggs constants. The question of algebraic independence of numbers is generally a delicate one with very few explicit results. Theorems in this chapter are conditional subject to the weak Schanuel conjecture.

5.2 Some Definitions and Notations

For $a, q \in \mathbb{N}$ and $(q, P_\Omega) = 1$, we denote

$$\gamma^*(\Omega, a, q) := \frac{q\gamma(\Omega, a, q)}{\delta_\Omega}.$$

Definition 5.2.1. We call a finite set $\{\Omega_1, \cdots, \Omega_n\}$ of sets to be irreducible if

$$\bigcup_{i=1}^{n} \Omega_i \neq \bigcup_{j \in J} \Omega_j$$

for any proper subset $J \subset \{1, \cdots, n\}$. We call an infinite set X of sets to be irreducible if every finite subset of X is irreducible.

To give an example, let

$$p_1 < p_2 < \cdots$$

be a sequence of distinct prime numbers and $\Omega_i := \{p_i\}$, then $\{\Omega_i : i \in \mathbb{N}\}$ is an irreducible set. On the other hand, the set of sets

$$\{p_1\}, \{p_2\}, \{p_1, p_2\}$$

where p_i's are distinct prime numbers is not irreducible.

In the same spirit, we define the following:

Definition 5.2.2. A finite subset I of $\mathbb{N}$ is called irreducible if and only if

$$P(I) \neq \bigcup_{J \subsetneq I} P(J),$$

where for a subset J of $\mathbb{N}$, $P(J)$ denotes the set of all prime divisors of the elements of J. An infinite subset $T \subseteq \mathbb{N}$ is called irreducible if all finite subsets of T are irreducible.

For example, a subset of primes is irreducible, whereas a set consisting of a prime and its powers is not irreducible.

5.3 Statements of the theorems

With the above notations in place, we prove the following theorems in [14].

Theorem 5.3.1. *Suppose that the weak Schanuel conjecture is true. Let $q \in \mathbb{N}$ and T_1 be an infinite set consisting of finite subsets Ω of primes with $(P_\Omega, q) = 1$. For $a \in \mathbb{N}$ with $1 \le a \le q$ and $(a,q) = 1$, consider the set*

$$S_1 := \{\gamma^*(\Omega, a, q) - \gamma - \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \alpha^*_{\chi,\Omega,q} L(1,\chi) \mid \Omega \in T_1\},$$

where χ runs over non-principal Dirichlet character modulo q and

$$\alpha^*_{\chi,\Omega,q} := \overline{\chi}(a) \prod_{p \in \Omega} \left(1 - \frac{\chi(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-1} \prod_{p \mid q} \left(1 - \frac{1}{p}\right)^{-1}.$$

Then the elements of S_1 are algebraically independent if the infinite set T_1 is irreducible.

In our next theorem, we fix an Ω and vary q in an irreducible subset T of $\mathbb{N}$.

Theorem 5.3.2. *Suppose that the weak Schanuel conjecture is true. Let Ω be a finite set of primes. Assume that T_2 is an infinite irreducible subset of natural numbers consisting of integers which are co-prime to the primes in Ω. Also fix $a \in \mathbb{N}$ such that $(a,q) = 1$ for all $q \in T_2$. Then the elements of the set*

$$S_2 := \{\gamma^*(\Omega, a, q) - \gamma - \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \alpha^*_{\chi,\Omega,q} L(1,\chi) \mid q \in T_2\},$$

where $\alpha^*_{\chi,\Omega,q}$ is as in Theorem 5.3.1 are algebraically independent.

5.4 Proof of Theorem 5.3.1

Let T_1 be an infinite set consisting of finite subsets Ω of primes with $(P_\Omega, q) = 1$. We recall the following expression for generalised Euler-Briggs constants proved in Chapter 3.

For a finite set of primes Ω and natural numbers a,q such that $(a,q)=1=(P_\Omega,q)=1$, we have

$$\gamma(\Omega,a,q)=\frac{1}{\varphi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\overline{\chi}(a)L(1,\chi)\prod_{p\in\Omega}\left(1-\frac{\chi(p)}{p}\right)\;+\;\frac{\delta_\Omega}{q}\left(\gamma+\sum_{p|q}\frac{\log p}{p-1}+\sum_{p\in\Omega}\frac{\log p}{p-1}\right).$$

Hence for an Ω in T_1 and natural numbers a,q with $(a,q)=1$, we get that

$$A_\Omega:=\gamma^*(\Omega,a,q)-\gamma-\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}}a^*_{\chi,\Omega,q}\,L(1,\chi)=\sum_{p\in\Omega}\frac{\log p}{p-1}+\sum_{p|q}\frac{\log p}{p-1}. \tag{5.4.1}$$

Hence by Conjecture 5.1.4, it is sufficient to show that the elements A_Ω's for $\Omega\in T_1$ are linearly independent over $\mathbb{Q}$. We prove it by contradiction. Suppose there exists a finite subset $T_1'=\{\Omega_1,\ldots,\Omega_k\}$ of T_1 and integers $m_1,\ldots,m_k$, not all zero, such that

$$m_1A_{\Omega_1}\;+\;\cdots\;+\;m_kA_{\Omega_k}=0. \tag{5.4.2}$$

Let us set $\Omega:=\cup_{i=1}^k\Omega_{n_i}$. Then applying (5.4.1) in (5.4.2), we get

$$\sum_{p\in\Omega}t_p\log p\;+\;\sum_{\ell|q}r_\ell\log\ell\;=\;0, \tag{5.4.3}$$

where $t_p,r_\ell\in\mathbb{Q}$ and $p\in\Omega$ with $(p,q)=1$. Being a subset of T_1, the set T_1' is also an irreducible set. Also note that not all the m_i's are zero. Hence it follows that not all the t_p's are zero. This is a contradiction to (5.4.3) as the set of primes is multiplicatively independent. ■

5.5 Proof of Theorem 5.3.2

In this section, as in the case of Theorem 5.3.1, it is sufficient to show that the elements of S_2 are linearly independent over $\mathbb{Q}$. Suppose not, then there exists a finite subset $\{q_1,\ldots,q_n\}$

of T_2 and integers $m_1,\dots,m_n$, not all 0, such that

$$\sum_{i=1}^{n} m_i \left(\gamma^*(\Omega, a, q_i) - \gamma - \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q_i}} \alpha^*_{\chi,\Omega,q_i} L(1,\chi) \right) = 0,$$

i.e.

$$\sum_{p\in\Omega} \frac{\log p}{p-1} \sum_{i=1}^{n} m_i \quad + \quad \sum_{i=1}^{n} m_i \sum_{p\mid q_i} \frac{\log p}{p-1} = 0.$$

Without loss of generality, let m_1 be non-zero. Since T_2 is irreducible, by definition all finite subsets of T_2 are irreducible. From definition, we get that $P(\{q_1, q_2, \dots, q_n\}) \neq P(\{q_2, \dots, q_n\})$ and hence there exists a prime p such that $p \mid q_1$ but $p \nmid q_j$ for all $j \neq 1$. This implies that the coefficient of $\log p$ is $\frac{m_1}{p-1} \neq 0$, a contradiction to the fact that primes are multiplicatively independent. ■

CHAPTER 6

GENERALISED EULER-BRIGGS CONSTANTS AND INFINITE SERIES

In this chapter, we will revisit a theorem of Lehmer [21], where he established an identity involving Euler's totient function φ, γ and the Euler's constants in arithmetic progressions considered by Briggs [7]. We extend this theorem for the class of generalised Euler-Briggs constants which in turn gives us an alternate proof of the theorem of Lehmer. Our first section consists of all the basic lemmas required to prove the theorems of this chapter. In the penultimate section, following the prototypical result of Lehmer [21] about the existence of periodic Dirichlet series at $s = 1$, we furnish a necessary and sufficient condition for the existence of a periodic Dirichlet series at $s = 1$ with period q, twisted by the principal Dirichlet character modulo M where $(q, M) = 1$. We also express this sum as a linear combination of generalised Euler-Briggs constants. In the last section, we prove a result about the special values of a shifted periodic Dirichlet series which also can be seen as a variant of Hurwitz zeta function. We end the chapter by proving a theorem connecting some rational functions with the generalised Euler-Briggs constants under certain conditions.

6.1 Basic lemmas

In this section we provide all the results that are required to prove our theorems. From now on, by ζ_n we denote a primitive n-th root of unity, $\Phi_n(X)$ denotes the n-th cyclotomic polynomial i.e. the minimal polynomial of ζ_n over $\mathbb{Q}$ and we will always consider the principal branch of logarithm.

Lemma 6.1.1. *For the n-th ($n > 1$) cyclotomic polynomial*

$$\Phi_n(X) := \prod_{\substack{(a,n)=1 \\ 1 \le a \le n-1}} (X - \zeta_n^a),$$

$\Phi_n(1) \in \mathbb{N}$.

Proof. Clearly $\Phi_n(1) \in \mathbb{Z}$. We only have to show that it is positive. We prove it by induction on n.

Note that $\Phi_2(1) = 2$. Also,

$$\Phi_n(X) = \frac{X^n - 1}{\prod_{\substack{d|n \\ d<n}} \Phi_d(X)} = \frac{X^{n-1} + \cdots + 1}{\prod_{\substack{d|n \\ 1<d<n}} \Phi_d(X)}.$$

Thus by induction hypothesis $\Phi_n(1) > 0$. ■

Lemma 6.1.2. *Let $n > 1$ be an integer having at least two prime divisors, then $\Phi_n(1) = 1$.*

Proof. We first note that for any $n \ge 1$,

$$X^n - 1 = \prod_{i=1}^{n} (X - \zeta_n^i).$$

i.e.,

$$X^{n-1} + \cdots + 1 = \prod_{i=1}^{n-1} (X - \zeta_n^i).$$

Putting $X = 1$, we get that for all $n \geq 1$,

$$n = \prod_{i=1}^{n-1} (1 - \zeta_n^i). \tag{6.1.1}$$

Now suppose that $n = p_1^{a_1} \cdots p_r^{a_r}$, with $r \geq 2$. Using (6.1.1), we get that

$$\prod_{j=1}^{n-1} (1 - \zeta_n^j) = \prod_{i=1}^{r} \prod_{j=1}^{p_i^{a_i}-1} (1 - \zeta_{p_i^{a_i}}^j). \tag{6.1.2}$$

Note that all the terms in the right hand side of (6.1.2) appear in the left hand side of (6.1.2). Now since $r \geq 2$, $\zeta_{p_i^{a_i}}$ is not a primitive n-th root of unity, hence by cancellation we obtain

$$\alpha \prod_{\substack{1 \leq j \leq n-1 \\ (j,n)=1}} (1 - \zeta_n^j) = 1,$$

for some $\alpha \in \mathbb{Z}[\zeta_n]$. Hence it follows that $(1 - \zeta_n)$ is a unit and thus its norm

$$\prod_{\substack{1 \leq j \leq n-1 \\ (j,n)=1}} (1 - \zeta_n^j) = \pm 1,$$

i.e. $\Phi_n(1) = \pm 1$. Then by Lemma 6.1.1, we obtain that $\Phi_n(1) = 1$. ■

Lemma 6.1.3. *For $n \geq 1$,*

$$\sum_{b=1}^{n-1} \log(1 - \zeta_n^b) = \log \left(\prod_{b=1}^{n-1} (1 - \zeta_n^b) \right).$$

Proof. For the principal branch of logarithm, we have $\log z = \log |z| + i\theta$, where $z = |z| e^{i\theta}$ and $-\pi < \theta \leq \pi$. Hence $\log \bar{z} = \log |z| - i\theta$. Thus

$$\log z + \log \bar{z} = \log |z|^2 = \log(z\bar{z}).$$

We know that the only self conjugate roots of unity are ± 1.

Let $\zeta_n^b \neq \pm 1$. Then $\bar{\zeta_n^b}$ is also a n-th root of unity and not equal to ζ_n^b. Hence

$$\log(1-\zeta_n^b)+\log(1-\bar{\zeta_n^b}) = \log((1-\zeta_n^b)(1-\bar{\zeta_n^b})),$$

which is the logarithm of a positive real number.

Now if $\zeta_n^b = -1$, then $\log(1-\zeta_n^b) = \log 2$. Thus $\sum_{b=1}^{n-1}\log(1-\zeta_n^b)$ can be rewritten as a sum of logarithms of positive real numbers and hence can be written as in Lemma 6.1.3. ∎

6.2 An identity of Lehmer

We begin this section with the following definition:

Definition 6.2.1. For natural numbers $a, q \geq 1$, define

$$\Phi(q) := \sum_{\substack{1\leq a\leq q \\ (a,q)=1}} \gamma(a,q).$$

In 1975, Lehmer proved the following identity about $\Phi(q)$ while studying Euler's constants in arithmetic progressions.

Theorem 6.2.2 (Lehmer [21])**.** *There exists a natural number N_q such that*

$$q\Phi(q) = \varphi(q)\gamma + \log N_q,$$

where φ denotes Euler's totient function.

More precisely, Lehmer showed that N_q is equal to $\prod_{p|q} p^{\varphi(q)/(p-1)}$. In order to prove this, Lehmer used properties of Möbius function and Möbius inversion formula.

To extend the above theorem for generalised Euler-Briggs constants, we first have the following definition.

Definition 6.2.3. For a finite set of primes Ω and a natural number q co-prime to P_Ω,

$$\Phi_\Omega(q) := \sum_{\substack{1\le a\le q\\(a,q)=1}} \gamma(\Omega,a,q).$$

With this definition we prove the following theorem in [15]:

Theorem 6.2.4. *Let Ω be a finite set of primes and q be a natural number co-prime to P_Ω. Then*

$$q\Phi_\Omega(q) = \varphi(q)\gamma(\Omega) + \delta_\Omega \log N_q,$$

where φ denotes Euler's totient function and as before $N_q = \prod_{p|q} p^{\varphi(q)/(p-1)}$.

To prove the result we will require the following theorem proved in [17].

Theorem 6.2.5. *Let Ω be a finite set of primes and $q \ge 1$ such that $(q, P_\Omega) = 1$. Then*

$$q\gamma(\Omega,a,q) - \delta_\Omega\gamma \;=\; \delta_\Omega \sum_{p\in\Omega} \frac{\log p}{p-1} - \sum_{\Omega'\subseteq\Omega} \frac{(-1)^{Card(\Omega')}}{P_{\Omega'}} \sum_{b=1}^{q-1} \zeta_q^{-ab} \log(1-\zeta_q^{bP_{\Omega'}}),$$

where $Card(\Omega')$ denotes the cardinality of the set Ω'.

Proof of Theorem 6.2.4 By using Theorem 6.2.5, we write,

$$\begin{aligned} q\Phi_\Omega(q) &= \sum_{(a,q)=1}\left(\delta_\Omega\gamma \;+\; \delta_\Omega \sum_{p\in\Omega}\frac{\log p}{p-1} - \sum_{\Omega'\subseteq\Omega}\frac{(-1)^{Card(\Omega')}}{P_{\Omega'}}\sum_{b=1}^{q-1}\zeta_q^{-ab}\log(1-\zeta_q^{bP_{\Omega'}})\right)\\ &= \delta_\Omega\varphi(q)\gamma \;+\; \delta_\Omega\varphi(q)\sum_{p\in\Omega}\frac{\log p}{p-1} - \sum_{\Omega'\subseteq\Omega}\frac{(-1)^{Card(\Omega')}}{P_{\Omega'}}\sum_{b=1}^{q-1} n_b\log(1-\zeta_q^{bP_{\Omega'}}), \end{aligned}$$

where $n_b := Tr_{\mathbb{Q}(\zeta_q)/\mathbb{Q}}(\zeta_q^{-b}) \in \mathbb{Z}$. Putting $a = q = 1$ in Theorem 6.2.5, we get

$$q\Phi_\Omega(q) = \varphi(q)\gamma(\Omega) - \sum_{\Omega'\subseteq\Omega}\frac{(-1)^{Card(\Omega')}}{P_{\Omega'}}\sum_{b=1}^{q-1} n_b \log(1-\zeta_q^{bP_{\Omega'}}).$$

Now $n_b = n_{bP_{\Omega'}}$ as $(P_{\Omega'}, q) = 1$. Therefore

$$\sum_{b=1}^{q-1} n_b\log(1-\zeta_q^{bP_{\Omega'}}) = \sum_{b=1}^{q-1} n_{bP_{\Omega'}}\log(1-\zeta_q^{bP_{\Omega'}}) = \sum_{b=1}^{q-1} n_b\log(1-\zeta_q^{b}).$$

Also,

$$\delta_\Omega = \prod_{p\in\Omega}(1-\frac{1}{p}) = \frac{\varphi(P_\Omega)}{P_\Omega} = \sum_{d|P_\Omega}\frac{\mu(d)}{d} = \sum_{\Omega'\subseteq\Omega}\frac{(-1)^{Card(\Omega')}}{P_{\Omega'}}.$$

Thus we obtain

$$q\Phi_\Omega(q) = \varphi(q)\gamma(\Omega) - \delta_\Omega\sum_{b=1}^{q-1} n_b \log(1-\zeta_q^b).$$

Let us set

$$N_q := \prod_{b=1}^{q-1}(1-\zeta_q^b)^{-n_b}.$$

We write

$$N_q = \prod_{d|q}\prod_{\substack{(b,q)=d\\ 1\le b\le q-1}}(1-\zeta_q^b)^{-n_b}.$$

Now if $(b_1,q)=(b_2,q)$, then $\zeta_q^{-b_1}$ and $\zeta_q^{-b_2}$ are conjugate to each other and hence $n_{b_1}=n_{b_2}$. So, for all b such that $(b,q)=d$, $n_b=n_d$. Also by Lemma 6.1.1, one has

$$\prod_{\substack{(b,q)=d\\ 1\le b\le q-1}}(1-\zeta_q^b) = \Phi_{q/d}(1)\in\mathbb{N},$$

where $\Phi_{q/d}$ is the q/d-th cyclotomic polynomial. Hence

$$N_q = \prod_{d|q}\Phi_{q/d}(1)^{-n_d}.$$

By Lemma 6.1.2, we know that $\Phi_{q/d}(1)\neq 1$ only if $q/d=p^k$ for some prime p and $k\ge 1$. Now for $q/d=p^k$, we find out n_d explicitly. In fact

$$\begin{aligned} n_d &= Tr_{\mathbb{Q}(\zeta_q)/\mathbb{Q}}(\zeta_q^{-d}) \\ &= Tr_{\mathbb{Q}(\zeta_q)/\mathbb{Q}}(\zeta_{q/d}) \\ &= [\mathbb{Q}(\zeta_q):\mathbb{Q}(\zeta_{p^k})]\; Tr_{\mathbb{Q}(\zeta_{p^k})/\mathbb{Q}}(\zeta_{p^k}) \end{aligned}$$

We know that the minimal polynomial of ζ_{p^k} is

$$X^{p^{k-1}(p-1)} + X^{p^{k-1}(p-2)} + \cdots + X^{p^{k-1}} + 1.$$

For $k \geq 2$, the coefficient of $X^{p^{k-1}(p-1)-1}$ is 0, i.e. $Tr_{\mathbb{Q}(\zeta_{p^k})/\mathbb{Q}}(\zeta_{p^k}) = 0$ and for $k = 1$, $Tr_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(\zeta_p) = -1$. Hence

$$n_d = \begin{cases} 0 & \text{for } k \geq 2, \\ -\varphi(q)/p - 1 & \text{for } k = 1. \end{cases}$$

This yields

$$N_q = \prod_{p|q} p^{\varphi(q)/p-1}.$$

Now following the proof of Lemma 6.1.3, we get

$$q\Phi_\Omega(q) = \varphi(q)\gamma(\Omega) + \delta_\Omega \log N_q.$$

■

Remark 6.2.6. Besides giving a new proof of Lehmer's theorem, our work gives a natural explanation for the exponent $\varphi(q)/(p-1)$ appearing in the product.

Now we state and prove our next identity:

Theorem 6.2.7. *Let q, M be natural numbers such that $a \equiv 0 \bmod M$ and $(q, M) = 1$. Then*

$$\gamma(\Omega, a, q) = \frac{\gamma(\Omega)}{q} - \frac{\delta_\Omega \log q}{q} - \frac{\Psi_\Omega(a/q)}{q},$$

where Ω is the set of prime divisors of M and

$$\Psi_\Omega(x) =: x \sum_{\substack{n \geq 1 \\ (n,M)=1}} \frac{1}{n(n+x)} \quad \textit{for } x \geq 0.$$

Proof. Let χ_0 be the principal character modulo M, Ω be the set of prime divisors of M and δ_Ω be as in the introduction. We have

$$\begin{aligned}
\gamma(\Omega,a,q) &= \lim_{N\to\infty}\left(\sum_{\substack{m\le a+Nq\\ m\equiv a \bmod q}}\frac{\chi_0(m)}{m} - \frac{\delta_\Omega}{q}\log(a+Nq)\right)\\
&= \lim_{N\to\infty}\left(\sum_{n=0}^{N}\frac{\chi_0(a+nq)}{a+nq} - \frac{\delta_\Omega}{q}\log(a+Nq)\right)\\
&= \frac{\chi_0(a)}{a} + \lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\frac{\chi_0(nq)}{nq}-\frac{\chi_0(nq)}{nq}+\frac{\chi_0(a+nq)}{a+nq}\right) - \frac{\delta_\Omega}{q}\log(a+Nq)\right)\\
&= \frac{\gamma(\Omega)}{q} + \lim_{N\to\infty}\left(-\frac{\delta_\Omega}{q}\log\frac{a+Nq}{N} - \sum_{n=1}^{N}\left(\frac{\chi_0(nq)}{nq}-\frac{\chi_0(a+nq)}{a+nq}\right)\right).
\end{aligned}$$

Since $(q,M)=1$ and $a\equiv 0 \bmod M$, we have

$$(n,M)=1 \iff (a+nq,M)=1.$$

Hence

$$\begin{aligned}
\gamma(\Omega,a,q) &= \frac{\gamma(\Omega)}{q} - \frac{\delta_\Omega}{q}\log q - \lim_{N\to\infty}\left(\sum_{\substack{1\le n\le N\\ (n,M)=1}}\left(\frac{1}{nq}-\frac{1}{a+nq}\right)\right)\\
&= \frac{\gamma(\Omega)}{q} - \frac{\delta_\Omega}{q}\log q - \frac{\Psi_\Omega(a/q)}{q}.
\end{aligned}$$

■

6.3 Existence of periodic Dirichlet series at $s=1$

In this section, we give an equivalent condition for convergence of a periodic Dirichlet series. This result is in the spirit of the following well-known result due to Lehmer.

Theorem 6.3.1 (Lehmer [21]). *Let f be a periodic arithmetic function with period $q\ge 1$. Then a necessary and sufficient condition for the convergence of $\sum_{n\ge 1}\frac{f(n)}{n}$ is $\sum_{a=1}^{q} f(a)=0$. In that case the sum is equal to $\sum_{a=1}^{q} f(a)\gamma(a,q)$.*

In [15], we prove:

Theorem 6.3.2. *Let f be a periodic arithmetic function with period $q \geq 1$ and M be a natural number co-prime to q. Then*

$$\sum_{\substack{n\geq 1\\(n,M)=1}} \frac{f(n)}{n}$$

converges if and only if $\sum_{a=1}^{q} f(a) = 0$. Moreover, whenever the above series converges, we have

$$\sum_{\substack{n\geq 1\\(n,M)=1}} \frac{f(n)}{n} = \sum_{a=1}^{q} f(a)\gamma(\Omega, a, q),$$

where Ω is the set of prime divisors of M.

Proof. Let Ω be the set of prime divisors of M and χ_0 be the principal character modulo M. We write

$$\begin{aligned}
\sum_{\substack{n\leq x\\(n,M)=1}} \frac{f(n)}{n} &= \sum_{n\leq x} \frac{f(n)\chi_0(n)}{n} \\
&= \sum_{a=1}^{q} \sum_{\substack{n\leq x\\ n\equiv a \bmod q}} \frac{f(n)\chi_0(n)}{n} \\
&= \sum_{a=1}^{q} f(a) \sum_{\substack{n\leq x\\ n\equiv a \bmod q}} \frac{\chi_0(n)}{n} \\
&= \sum_{a=1}^{q} f(a) \left(\sum_{\substack{n\leq x\\ n\equiv a \bmod q}} \frac{\chi_0(n)}{n} - \frac{\delta_\Omega \log x}{q} \right) + \frac{\delta_\Omega \log x}{q} \sum_{a=1}^{q} f(a).
\end{aligned}$$

Taking $x \to \infty$, we see that

$$\sum_{\substack{n\geq 1\\(n,M)=1}} \frac{f(n)}{n} = \sum_{a=1}^{q} f(a)\gamma(\Omega, a, q)$$

if and only if $\sum_{a=1}^{q} f(a) = 0$. ■

Remark 6.3.3. Let q, M be co-prime natural numbers and χ_0 be the principal character modulo M. Then by Theorem 6.3.1, a necessary and sufficient condition for the series

$$\sum_{\substack{n\geq 1\\(n,M)=1}} \frac{f(n)}{n} = \sum_{n\geq 1} \frac{f(n)\chi_0(n)}{n}$$

to converge is that $\sum_{a=1}^{qM} f(a)\chi_0(a) = 0$. Note that in Theorem 6.3.2, we obtain a condition involving a sum over smaller set of numbers. However it can be checked that

$$\sum_{a=1}^{qM} f(a)\chi_0(a) = 0 \iff \sum_{a=1}^{q} f(a) = 0.$$

To see this, note that

$$\sum_{a=1}^{qM} f(a)\chi_0(a) = \sum_{a=1}^{q} f(a)\left(\sum_{n=0}^{M-1} \chi_0(a+nq)\right)$$

Now for any $1 \leq a \leq q$, we claim that

$$\{a + nq : 0 \leq n \leq M-1\}$$

is a complete set of residue classes modulo M. Clearly there are M numbers in the above set. Hence if we show that they are distinct modulo M, then we are done. If

$$(a + n_1 q) \equiv (a + n_2 q) \bmod M,$$

for $0 \leq n_1 < n_2 \leq M-1$, then M divides $(n_2 - n_1)q$, a contradiction as $(q, M) = 1$. Thus the inner sum in the right hand side of the above equation is $\varphi(M)$. Therefore we have

$$\sum_{a=1}^{qM} f(a)\chi_0(a) = \varphi(M) \sum_{a=1}^{q} f(a).$$

6.4 An application of Theorem 6.3.2

Theorem 6.4.1. *Let M, q be co-prime natural numbers and f be a periodic arithmetic function with period q. Also let $a \leq b$ be co-prime natural numbers such that $a \equiv 0 \bmod M$.*

Then

$$\sum_{\substack{n\geq 0\\(n,M)=1}} \frac{f(n)}{n+a/b} = b\sum_{t=0}^{q-1} f(t)\,\gamma(\Omega,\ a+tb,\ bq),$$

if and only if $\sum_{t=1}^{q} f(t)=0$. *Here* Ω *is the set of prime divisors of* M.

Proof. Since $(a,b)=1$ and $a\equiv 0 \bmod M$, we have

$$(n,M)=1 \iff (a+nb,M)=1.$$

Hence we can write

$$\sum_{\substack{n\geq 0\\(n,M)=1}} \frac{f(n)}{n+a/b} = \sum_{\substack{m\geq 1\\(m,M)=1}} \frac{g(m)}{m},$$

where

$$g(m)=\begin{cases} bf(n) & \text{for } m=a+nb,\\ 0 & \text{otherwise.}\end{cases}$$

Since g is a periodic function of period bq, using Theorem 6.3.2, we get

$$\sum_{\substack{m\geq 1\\(m,M)=1}} \frac{g(m)}{m} = \sum_{r=1}^{bq} g(r)\gamma(\Omega,r,bq) = b\sum_{t=0}^{q-1} f(t)\gamma(\Omega,a+tb,bq)$$

if and only if

$$\sum_{r=1}^{bq} g(r)=0 \iff \sum_{t=1}^{q} f(t)=0.$$

∎

6.5 Infinite series involving rational functions

In the following theorem we express a series involving rational functions in terms of finite linear combination of generalised Euler-Briggs constants and logarithm of natural numbers.

Theorem 6.5.1. *Let $A(X), B(X) \in \mathbb{Q}[X]$ be non-zero polynomials with $\deg A < \deg B$. Suppose that $B(X)$ has distinct rational roots with the factorisation*

$$B(X) = c \prod_{j=1}^{r} (X + a_j/b_j),$$

where $a_j \leq b_j$ are co-prime natural numbers. Let M be a natural number such that $a_j \equiv 0 \bmod M$ for all j. Then the following sum

$$\sum_{\substack{n \geq 0 \\ (n,M)=1}} \frac{A(n)}{B(n)}$$

converges if and only if $\deg A < \deg B - 1$. In this case, we have

$$\sum_{\substack{n \geq 0 \\ (n,M)=1}} \frac{A(n)}{B(n)} = \sum_{j=1}^{r} c_j \big(\gamma(\Omega, a_j, b_j) \; + \; \frac{\delta_\Omega}{b_j} \log b_j \big)$$

for some explicitly determined constants $c_j \in \mathbb{Q}$. Here Ω is the set of prime divisors of M.

Proof. Since $B(X)$ has simple rational roots with $\deg A < \deg B$, using partial fractions, we can write

$$\frac{A(X)}{B(X)} = \sum_{j=1}^{r} \frac{c_j}{a_j + b_j X}, \text{ where } c_j \in \mathbb{Q}.$$

Hence

$$\sum_{\substack{0 \leq n \leq x \\ (n,M)=1}} \frac{A(n)}{B(n)} \quad = \quad \sum_{j=1}^{r} c_j \sum_{\substack{0 \leq n \leq x \\ (n,M)=1}} \frac{1}{a_j + b_j n}.$$

Also by the hypothesis of the theorem, we have

$$(n, M) = 1 \iff (a_j + b_j n, M) = 1.$$

Thus

$$
\begin{aligned}
\sum_{\substack{n\geq 0\\(n,M)=1}} \frac{A(n)}{B(n)} &= \lim_{x\to\infty}\sum_{j=1}^{r} c_j \sum_{\substack{1\leq m\leq (x+1)b_j\\ m\equiv a_j \bmod b_j\\ (m,M)=1}} \frac{1}{m} \\
&= \lim_{x\to\infty}\sum_{j=1}^{r} c_j \left(\sum_{\substack{m\leq (x+1)b_j\\ m\equiv a_j \bmod b_j\\ (m,M)=1}} \frac{1}{m} - \frac{\delta_\Omega}{b_j}\log(x+1)b_j + \frac{\delta_\Omega}{b_j}\log(x+1)b_j \right).
\end{aligned}
$$

Note that the condition $\deg A(X) \leq \deg B(X) - 2$ is equivalent to the condition

$$\sum_{j=1}^{r} \frac{c_j}{b_j} = 0$$

which in fact is a necessary as well as sufficient condition for the convergence of the series

$$\sum_{\substack{n\geq 0\\(n,M)=1}} \frac{A(n)}{B(n)}.$$

Therefore we have

$$\sum_{\substack{n\geq 0\\(n,M)=1}} \frac{A(n)}{B(n)} = \sum_{j=1}^{r} c_j \left(\gamma(\Omega, a_j, b_j) + \frac{\delta_\Omega}{b_j}\log b_j\right).$$

$\blacksquare$

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