MEASURE THEORETIC ASPECTS OF ERROR TERMS

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I, Kamalakshya Mahatab, hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.

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List of Publications

Journal

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2. Kamalakshya Mahatab, Number of Prime Factors of an Integer. Mathematics News Letter, Ramanujan Mathematical Society, volume 24 (2013).

Others

1. Kamalakshya Mahatab and Anirban Mukhopadhyay, Measure Theoretic Aspects of Oscillations of Error Terms. arXiv:1512.03144v1 (2015).
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Kamalakshya Mahatab

CONTENTS

Synopsis

This thesis studies fluctuation of error terms that appears in various asymptotic formulas and size of the sets where these fluctuations occur. As a consequence, this approach replaces Landau's criterion on oscillation of error terms.

General Theory

Consider a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ having Dirichlet series

$$D(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s},$$

which is convergent in some half-plane. As in Perron summation formula [37, II.2.1], we write

$$\sum_{n\leq x}^{*} a_n = \mathcal{M}(x)+\Delta(x),$$

where $\mathcal{M}(x)$ is the main term, $\Delta(x)$ is the error term and $\sum^{*}$ is defined as

$$\sum_{n\leq x}^{*} a_n = \begin{cases} \sum_{n\leq x} a_n & \text{if } x\notin\mathbb{N}, \\ \sum_{n<x} a_n+\frac{1}{2}a_x & \text{if } x\in\mathbb{N}. \end{cases}$$

In this thesis, we obtain Ω and $\Omega_{\pm}$ estimates for $\Delta(x)$. We shall use the Mellin transform of $\Delta(x)$ (defined below) to obtain such estimates.

Definition. *The Mellin transform of $\Delta(x)$ be $A(s)$, defined as*

$$A(s) = \int_1^\infty \frac{\Delta(x)}{x^{s+1}} \mathrm{d}x.$$

In this direction, under some natural assumptions and for a suitably defined contour $\mathscr{C}$, we shall show that

$$A(s) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{D(\eta)}{\eta(s-\eta)} \mathrm{d}\eta.$$

In the above formula, the poles of $D(s)$ that lie left to $\mathscr{C}$ are all the poles that contributes to the main term $\mathcal{M}(x)$. Landau [26] used the meromorphic continuation of $A(s)$ to obtain $\Omega_\pm$ results for $\Delta(x)$. He proved that if $A(s)$ has a pole at $\sigma_0 + it_0$ for some $t_0 \neq 0$ and has no real pole for $s \geq \sigma_0$, then

$$\Delta(x) = \Omega_\pm(x^{\sigma_0}).$$

We shall show a quantitative version of Landau's theorem, which also generalizes a theorem of Bhowmik, Ramaré and Schlage-Puchta [6]. Below we state this theorem in a simplified way. We introduce the following notations to state these theorems.

Definition. *Let*

$$\mathcal{A}_T^+(x^{\sigma_0}) := \{T \leq x \leq 2T : \Delta(x) > \lambda x^{\sigma_0}\},$$

$$\mathcal{A}_T^-(x^{\sigma_0}) := \{T \leq x \leq 2T : \Delta(x) < -\lambda x^{\sigma_0}\},$$

$$\mathcal{A}_T(x^{\sigma_0}) := \mathcal{A}_T^+(x^{\sigma_0}) \cup \mathcal{A}_T^-(x^{\sigma_0}),$$

for some $\lambda, \sigma_0 > 0$.

Theorem. *Let $\sigma_0 > 0$, and let the following conditions hold:*

(1) $A(s)$ has no real pole for $\Re(s) \geq \sigma_0$,

(2) there is a complex pole $s_0 = \sigma_0 + it_0$, $t_0 \neq 0$, of $A(s)$, and

(3) for positive functions $h^{\pm}(x)$ such that $h \pm (x) \to \infty$ as $x \to \infty$, we have

$$\int_{\mathcal{A}_T^{\pm}(x^{\sigma_0})} \frac{\Delta^2(x)}{x^{2\sigma_0+1}} \mathrm{d}x \ll h^{\pm}(T).$$

Then

$$\mu(\mathcal{A}_T^{\pm}(x^{\sigma_0})) = \Omega\left(\frac{T}{h^{\pm}(T)}\right),$$

where μ denotes the Lebesgue measure.

In the above theorem, Condition 2 is a very strong criterion. In the following theorem, we replace Condition 2 by an Ω-bound of $\mu(\mathcal{A}_T(x^{\sigma_0}))$ and obtain an $\Omega_{\pm}$-result from the given Ω-bound.

Theorem. *Let $\sigma_0 > 0$, and let the following conditions hold:*

(1) $A(s)$ has no real pole for $\Re(s) \geq \sigma_0$, and

(2) $\mu(\mathcal{A}_T(x^{\sigma_0})) = \Omega(T^{1-\delta})$ for $0 < \delta < \sigma_0$.

Then

$$\Delta(x) = \Omega_{\pm}(T^{\sigma_0-\delta'})$$

for any δ' such that $0 < \delta' < \delta$.

The above two theorems are applicable to a wide class of arithmetic functions. Now we mention some results obtained by applying these theorems.

A Twisted Divisor Function

Given $\theta \neq 0$, define

$$\tau(n,\theta) = \sum_{d|n} d^{i\theta}.$$

The Dirichlet series of $|\tau(n,\theta)|^2$ can be expressed in terms of Riemann zeta function as

$$D(s) = \sum_{n=1}^{\infty} \frac{|\tau(n,\theta)|^2}{n^s} = \frac{\zeta^2(s)\zeta(s+i\theta)\zeta(s-i\theta)}{\zeta(2s)} \quad \text{for} \quad \Re(s) > 1.$$

In [14, Theorem 33], Hall and Tenenbaum proved that

$$\sum_{n\leq x}^{*} |\tau(n,\theta)|^2 = \omega_1(\theta)x\log x + \omega_2(\theta)x\cos(\theta\log x) + \omega_3(\theta)x + \Delta(x),$$

where $\omega_i(\theta)$s are explicit constants depending only on θ. They also showed that

$$\Delta(x) = O_\theta(x^{1/2}\log^6 x).$$

Here the main term comes from the residues of $D(s)$ at $s = 1, 1 \pm i\theta$. All other poles of $D(s)$ come from zeros of $\zeta(2s)$. Using a pole on the line $\Re(s) = 1/4$, Landau's method gives

$$\Delta(x) = \Omega_\pm(x^{1/4}).$$

We prove the following bounds for a computable $\lambda(\theta) > 0$ and for any $\epsilon > 0$:

$$\mu\left(\{T \leq x \leq 2T : \Delta(x) > (\lambda(\theta)-\epsilon)x^{1/4}\}\right) = \Omega\left(T^{1/2}(\log T)^{-12}\right),$$
$$\mu\left(\{T \leq x \leq 2T : \Delta(x) < (-\lambda(\theta)+\epsilon)x^{1/4}\}\right) = \Omega\left(T^{1/2}(\log T)^{-12}\right).$$

For a constant $c > 0$, define

$$\alpha(T) = \frac{3}{8} - \frac{c}{(\log T)^{1/8}}.$$

Applying a method due to Balasubramanian, Ramachandra and Subbarao [5], we prove

$$\Delta(T) = \Omega\left(T^{\alpha(T)}\right).$$

In fact, this method gives Ω-estimate for the measure of the sets involved:

$$\mu(\mathcal{A} \cap [T, 2T]) = \Omega\left(T^{2\alpha(T)}\right),$$

where

$$\mathcal{A} = \{x : |\Delta(x)| \geq M x^{\alpha(x)}\}$$

and $M > 0$ is a positive constant. We also show that

$$\text{either } \ \Delta(x) = \Omega\left(x^{\alpha(x)+\delta/2}\right) \ \text{ or } \ \Delta(x) = \Omega_{\pm}\left(x^{3/8-\delta'}\right),$$

for $0 < \delta < \delta' < 1/8$. For any $\epsilon > 0$, this result and the conjecture

$$\Delta(x) = O(x^{3/8+\epsilon})$$

proves that

$$\Delta(x) = \Omega_{\pm}(x^{3/8-\epsilon}).$$

Prime Number Theorem Error

Let a_n be the von Mandoldt function $\Lambda(n)$:

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^r,\ r \geq 1,\ p \text{ prime}, \\ 0 & \text{otherwise.} \end{cases}.$$

Let

$$\sum_{n\leq x}^{*} \Lambda_n = x + \Delta(x).$$

From the Vinogradov's zero free region for Riemann zeta function, one gets [23, Theorem 12.2]

$$\Delta(x) = O\left(x \exp\left(-c(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right)$$

for some constant $c > 0$.

Hardy and Littlewood [16] proved that

$$\Delta(x) = \Omega_{\pm}\left(x^{1/2}\log\log\log x\right).$$

But this result does not say about the measure of the sets, where the above $\Omega_{\pm}$ bounds are attained by $\Delta(x)$. We obtain the following weaker result, but with an Ω-estimates for the measure of the corresponding sets.

Let $\lambda_1 > 0$ denotes a computable constant. For a fixed ϵ, $0 < \epsilon < \lambda_1$, we write

$$\mathcal{A}_1 := \left\{x : \Delta(x) > (\lambda_1 - \epsilon)x^{1/2}\right\},$$
$$\mathcal{A}_2 := \left\{x : \Delta(x) < (-\lambda_1 + \epsilon)x^{1/2}\right\}.$$

Then

$$\mu([T, 2T] \cap \mathcal{A}_j) = \Omega\left(T^{1-\epsilon}\right), \text{ for } j = 1, 2 \text{ and for any } \epsilon > 0.$$

Under Riemann Hypothesis, we have

$$\mu([T, 2T] \cap A_j) = \Omega\left(\frac{T}{(\log T)^4}\right) \text{ for } j = 1, 2.$$

We also show the following unconditional Ω-bounds for the second moment of Δ:

$$\int_{[T,2T]\cap\mathcal{A}_j} \Delta^2(x)\mathrm{d}x = \Omega(T^2) \quad \text{for } j = 1, 2.$$

Non-isomorphic Abelian Groups

Let a_n denote the number of non-isomorphic abelian groups of order n. We write

$$\sum_{n\leq x}^{*} a_n = \sum_{k=1}^{6} b_k x^{1/k} + \Delta(x).$$

In the above formula, we define b_k as

$$b_k := \prod_{j=1, j\neq k}^{\infty} \zeta(j/k).$$

It is an open problem to show that

$$\Delta(x) \ll x^{1/6+\delta} \text{ for any } \delta > 0. \tag{1}$$

The best result on upper bound of $\Delta(x)$ is due to O. Robert and P. Sargos [33], which gives

$$\Delta(x) \ll x^{1/4+\epsilon} \text{ for any } \epsilon > 0.$$

Also Balasubramanian and Ramachandra [4] proved that

$$\Delta(x) = \Omega\left(x^{1/6}\sqrt{\log x}\right).$$

From this result, we may obtain

$$\mu\left(\{T \le x \le 2T : |\Delta(x)| \ge \lambda_2 x^{1/6}(\log x)^{1/2}\}\right) = \Omega(T^{5/6-\epsilon}),$$

for some $\lambda_2 > 0$ and for any $\epsilon > 0$. Sankaranarayanan and Srinivas [35] proved that

$$\Delta(x) = \Omega_{\pm}\left(x^{1/10}\exp\left(c\sqrt{\log x}\right)\right)$$

for some constant $c > 0$, while it has been conjectured that

$$\Delta(x) = \Omega_{\pm}(x^{1/6-\delta}),$$

for any $\delta > 0$. We shall show that either

$$\int_T^{2T} \Delta^4(x)\mathrm{d}x = \Omega(T^{5/3+\delta}) \text{ or } \Delta(x) = \Omega_{\pm}(x^{1/6-\delta}),$$

for any $0 < \delta < 1/42$. The conjectured upper bound (1) of $\Delta(x)$ gives

$$\int_T^{2T} \Delta^4(x)\mathrm{d}x \ll T^{5/3+\delta}.$$

This along with our result implies that

$$\Delta(x) = \Omega_{\pm}(x^{1/6-\delta}) \text{ for any } 0 < \delta < 1/42.$$

Notations

We denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$, the set of real numbers by $\mathbb{R}$, the set of positive real numbers by $\mathbb{R}^+$, and the set of complex numbers by $\mathbb{C}$.

The notaion i stands for $\sqrt{-1}$, the square root of -1 that belongs to the upper half plane in $\mathbb{C}$.

We denote the Lebesgue mesure on the real line $\mathbb{R}$ by μ.

For $z = \sigma + it \in \mathbb{C}$, we denote σ by $\Re(z)$ and t by $\Im(z)$.

Let $f(x)$ be a complex valued function and $g(x)$ be a positive real valued function on $\mathbb{R}^+$. As $x \to \infty$, we write:

$$f(x) = O(g(x)),\ \text{if}\ \lim_{x\to\infty} \left|\tfrac{f(x)}{g(x)}\right| < \infty;$$

$$f(x) = o(g(x)),\ \text{if}\ \lim_{x\to\infty} \left|\tfrac{f(x)}{g(x)}\right| = 0;$$

$$f(x) \ll g(x),\ \text{if}\ f(x) = O(g(x));$$

$$f(x) \gg g(x),\ \text{if}\ g(x) = O(f(x));$$

$$f(x) \sim g(x),\ \text{if}\ \lim_{x\to\infty} \tfrac{f(x)}{g(x)} = 1;$$

$$f(x) \asymp g(x),\ \text{if}\ 0 < \lim_{x\to\infty} \left|\tfrac{f(x)}{g(x)}\right| < \infty.$$

Let $f(x)$ be a complex valued function on $\mathbb{R}^+$, and let $g(x)$ be a positive monotonic function on $\mathbb{R}^+$. As $x \to \infty$, we write

$$f(x) = \Omega(g(x)), \text{ if } \limsup_{x\to\infty} \frac{|f(x)|}{g(x)} > 0;$$

$$f(x) = \Omega_+(g(x)), \text{ if } \limsup_{x\to\infty} \frac{f(x)}{g(x)} > 0;$$

$$f(x) = \Omega_-(g(x)), \text{ if } \liminf_{x\to\infty} \frac{f(x)}{g(x)} < 0;$$

$$f(x) = \Omega_\pm(g(x)), \text{ if } f(x) = \Omega_+(g(x)) \text{ and } f(x) = \Omega_-(g(x)).$$

THEOREM INDEX

[I] INTRODUCTION

In 1896, Jacques Hadamard and Charles Jean de la Vallée-Poussin proved that the number of primes upto x is asymptotic to $x/\log x$. This result is well known as the Prime Number Theorem (PNT). Below we state a version of this theorem (PNT*) in terms of the von-Mangoldt function.

Definition I.1. *For $n \in \mathbb{N}$, the von-Mangoldt function $\Lambda(n)$ is defined as*

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^r,\ r \in \mathbb{N} \text{ and } p \text{ prime}, \\ 0 & \text{otherwise}. \end{cases}$$

Theorem (PNT*)**.** *For a constant $c_1 > 0$, we have*

$$\sideset{}{^*}\sum_{n\leq x} \Lambda(n) = x + O\left(x \exp\left(-c_1(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right),$$

where

$$\sideset{}{^*}\sum_{n\leq x} \Lambda(n) = \begin{cases} \sum_{n\leq x} \Lambda(n) & \text{if } x \notin \mathbb{N}, \\ \sum_{n\leq x} \Lambda(n) - \Lambda(x)/2 & \text{otherwise}. \end{cases}$$

For a proof of the above theorem see [23, Theorem 12.2]. Proof of PNT* uses analytic continuation of the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

defined for $\Re(s) > 1$. The function $\zeta(s)$ is called the 'Riemann zeta function', named after the famous German mathematician Bernhard Riemann. In 1859, Riemann showed that this has a meromorphic continuation to the whole complex plane. He also showed PNT by assuming that the meromorphic continuation of $\zeta(s)$ does not have zeros for $\Re(s) > \frac{1}{2}$. This conjecture of Riemann is popularly known as the 'Riemann Hypothesis' (RH), and is an unsolved problem. Under RH, the upper bound for $\Delta(x)$ in PNT* can be improved as in the following theorem:

Theorem (PNT**)**.** *Let $\Delta(x)$ be defined as in PNT*. Further, if we assume RH, then*

$$\Delta(x) = O\left(x^{\frac{1}{2}} \log^2 x\right).$$

Proof. See [40]. □

In fact, we shall see in Theorem III.3 that PNT** is equivalent to RH. At this point, it is natural to ask the following questions:

- Can we obtain a bound for $\Delta(x)$, better than the bound in PNT**?
- Is $\Delta(x)$ an increasing or a decreasing function?
- Can $\Delta(x)$ be both positive and negative depending on x?
- How large are positive and negative values of $\Delta(x)$?

We shall make an attempt to answer these question by obtaining Ω and $\Omega_{\pm}$ results. The following result was obtained by Hardy and Littlewood [16] in the year 1916:

$$\Delta(x) = \Omega_{\pm}\left(x^{\frac{1}{2}} \log\log\log x\right). \tag{I.1}$$

The above $\Omega_{\pm}$ bound on $\Delta(x)$ gives some answer to our earlier questions. It says that we can not have an upper bound for $\Delta(x)$ which is smaller than $x^{\frac{1}{2}} \log\log\log x$. It also says that $\Delta(x)$ often takes both positive and negative values with magnitude of order $x^{\frac{1}{2}} \log\log\log x$. This suggests, it is important to obtain Ω and $\Omega_{\pm}$ bounds for various other error terms. In this direction, Landau's theorem [26] (see Theorem III.3 below) gives an elegant tool to obtain $\Omega_{\pm}$ results. Applying this theorem, we have

$$\Delta(x) = \Omega_{\pm}\left(x^{\frac{1}{2}}\right).$$

The advantage of Landau's method as compared to Hardy and Littlewood's method is in its applicability to a wide class of error terms of various summatory functions. In Landau's method, the existence of a complex pole with real part $\frac{1}{2}$ serves as a criterion for the existence of above limits. In this thesis, we shall investigate on a quantitative version of Landau's result by obtaining the Lebesgue measure of the sets where $\Delta(x) > \lambda x^{1/2}$ and $\Delta(x) < -\lambda x^{\frac{1}{2}}$, for some $\lambda > 0$. We shall show that the large Lebesgue measure of the set where $|\Delta(x)| > \lambda x^{\frac{1}{2}}$, for some $\lambda > 0$ replaces the criterion of existence of a complex pole in Landau's method. This approach has the advantage of getting $\Omega_{\pm}$ results even when no such complex pole exists. This is evident from some applications which we discuss in this thesis.

I.1 Framework

In this thesis, we consider a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ having Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

that converges in some half-plane. The Perron summation formula (see Theorem II.1) uses analytic properties of $D(s)$ to give

$$\sideset{}{^*}\sum_{n \leq x} a_n = \mathcal{M}(x) + \Delta(x),$$

where $\mathcal{M}(x)$ is the main term, $\Delta(x)$ is the error term (which would be specified later) and $\sum^*$ is defined as

$$\sideset{}{^*}\sum_{n \leq x} a_n = \begin{cases} \sum_{n \leq x} a_n & \text{if } x \notin \mathbb{N} \\ \sum_{n \leq x} a_n - \frac{1}{2} a_x & \text{if } x \in \mathbb{N}. \end{cases}$$

We may define the Mellin transform of $\Delta(x)$ as follows (which is different from the standard definition of the Mellin transform).

Definition I.2. *For a complex variable s, the Mellin transform $A(s)$ of $\Delta(x)$ is defined by*

$$A(s) = \int_1^{\infty} \frac{\Delta(x)}{x^{s+1}} \mathrm{d}x.$$

In general, $A(s)$ is holomorphic in some half plane. We shall discuss a method to obtain a meromorphic continuation of $A(s)$ from the meromorphic continuation of $D(s)$. In particular, we shall prove in Theorem II.3 that under some natural assumptions

$$A(s) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{D(\eta)}{\eta(s-\eta)}\mathrm{d}\eta,$$

where the contour $\mathscr{C}$ is as in Definition II.1 and s lies to the right of $\mathscr{C}$. Later, this result will complement Theorem III.6 and Theorem III.8 in their applications.

In Chapter III, we revisit Landau's method and obtain measure theoretic results. Also we generalize a theorem of Kaczorowski and Szydło [24] and a theorem of Bhowmik, Ramaré and Schlage-Puchta [6] in Theorem III.8.

Let

$$\mathcal{A}(\alpha, T) := \{x : x \in [T, 2T], |\Delta(x)| > x^{\alpha}\},$$

and let μ denotes the Lebesgue measure on $\mathbb{R}$. In Chapter IV, we establish a connection between $\mu(\mathcal{A}(\alpha, T))$ and fluctuations of $\Delta(x)$. In Proposition IV.1, we see that

$$\mu(\mathcal{A}(\alpha, T)) \ll T^{1-\delta} \text{ implies } \Delta(x) = \Omega(x^{\alpha+\delta/2}).$$

However, Theorem IV.3 gives that

$$\mu(\mathcal{A}(\alpha, T)) = \Omega(T^{1-\delta}) \text{ implies } \Delta(x) = \Omega_{\pm}(x^{\alpha-\delta}),$$

provided $A(s)$ does not have a real pole for $\Re(s) \geq \alpha - \delta$. In particular, this says that either we can improve on the Ω result or we can obtain a tight $\Omega_{\pm}$ result for $\Delta(x)$.

In Chapter V we study a twisted divisor function defined as follows:

$$\tau(n, \theta) = \sum_{d|n} d^{i\theta} \text{ for } \theta \neq 0. \tag{I.2}$$

This function is used in [14, Chapter 4] to measure the clustering of divisors. We give a brief note on some applications of $\tau(n,\theta)$ in Section V.V.1. In [14, Theorem 33], Hall and Tenenbaum proved that

$$\sum_{n\leq x}^{*} |\tau(n,\theta)|^2 = \omega_1(\theta)x\log x + \omega_2(\theta)x\cos(\theta\log x) + \omega_3(\theta)x + \Delta(x), \tag{I.3}$$

where $\omega_i(\theta)$s are explicit constants depending only on θ. They also showed that

$$\Delta(x) = O_\theta(x^{1/2}\log^6 x). \tag{I.4}$$

We give a proof of this formula in Theorem V.1. In Theorem V.2, we obtain an Ω bound for the second moment of $\Delta(x)$ by adopting a technique due to Balasubramanian, Ramachandra and Subbarao [5]. Also we derive conditional $\Omega_\pm$ bounds for $\Delta(x)$ in Theorem V.4 using techniques from the previous chapters.

The main theorems of this thesis, except Theorem III.8, are from [28], which is a joint work of the author with A. Mukhopadhyay.

I.2 Applications

We conclude the introduction by mentioning a few applications of the methods given in this thesis.

I.2.1 Twisted Divisors

Consider the twisted divisor function $\tau(n,\theta)$ defined in the previous section. The Dirichlet series of $|\tau(n,\theta)|^2$ can be expressed in terms of the Riemann zeta function as:

$$D(s) = \sum_{n=1}^{\infty} \frac{|\tau(n,\theta)|^2}{n^s} = \frac{\zeta^2(s)\zeta(s+i\theta)\zeta(s-i\theta)}{\zeta(2s)} \qquad \text{for} \quad \Re(s) > 1. \tag{I.5}$$

In Theorem V.1, we shall show

$$\sum_{n\leq x}^{*} |\tau(n,\theta)|^2 = \omega_1(\theta)x\log x + \omega_2(\theta)x\cos(\theta\log x) + \omega_3(\theta)x + \Delta(x),$$

where $\omega_i(\theta)$s are explicit constants depending only on θ and

$$\Delta(x) = O_\theta(x^{1/2}\log^6 x).$$

The Dirichlet series $D(s)$ has poles at $s = 1, 1 \pm i\theta$ and at the zeros of $\zeta(2s)$. Using a complex pole on the line $\Re(s) = 1/4$, Landau's method gives

$$\Delta(x) = \Omega_\pm(x^{1/4}).$$

In order to apply the method of Bhowmik, Ramaré and Schlage-Puchta [6], we need

$$\int_T^{2T} \Delta^2(x)\mathrm{d}x \ll T^{2\sigma_0+1+\epsilon}$$

for any $\epsilon > 0$ and $\sigma_0 = 1/4$; such an estimate is not possible due to Corollary V.1. Generalization of this method in Theorem III.6 can be applied to get

$$\mu\left(\mathcal{A}_j \cap [T, 2T]\right) = \Omega\left(T^{1/2}(\log T)^{-12}\right) \quad \text{for } j = 1, 2,$$

and here $\mathcal{A}_j$s' for $\Delta(x)$ are defined as

$$\mathcal{A}_1 = \left\{x : \Delta(x) > (\lambda(\theta) - \epsilon)x^{1/4}\right\} \text{ and } \mathcal{A}_2 = \left\{x : \Delta(x) < (-\lambda(\theta) + \epsilon)x^{1/4}\right\},$$

for any $\epsilon > 0$ and $\lambda(\theta) > 0$ as in (V.3). But under Riemann Hypothesis, we show in (V.5) that the above Ω bounds can be improved to

$$\mu\left(\mathcal{A}_j\right) = \Omega\left(T^{3/4-\epsilon}\right), \text{ for } j = 1, 2 \text{ and for any } \epsilon > 0.$$

Fix a constant $c_2 > 0$ and define

$$\alpha(T) = \frac{3}{8} - \frac{c_2}{(\log T)^{1/8}}.$$

In Corollary V.2, we prove that

$$\Delta(T) = \Omega\left(T^{\alpha(T)}\right).$$

In Proposition V.3, we give an Ω estimate for the measure of the sets involved in the above bound:

$$\mu(\mathcal{A} \cap [T, 2T]) = \Omega\left(T^{2\alpha(T)}\right),$$

where

$$\mathcal{A} = \left\{x : |\Delta(x)| \geq M x^{\alpha(x)}\right\}$$

for a positive constant $M > 0$. In Theorem V.4, we show that

$$\text{either } \Delta(x) = \Omega\left(x^{\alpha(x)+\delta/2}\right) \text{ or } \Delta(x) = \Omega_{\pm}\left(x^{3/8-\delta'}\right),$$

for $0 < \delta < \delta' < 1/8$. We may conjecture that

$$\Delta(x) = O(x^{3/8+\epsilon}) \text{ for any } \epsilon > 0.$$

Theorem V.4 and this conjecture imply that

$$\Delta(x) = \Omega_{\pm}(x^{3/8-\epsilon}) \text{ for any } \epsilon > 0.$$

I.2.2 Square Free Divisors

Let $\Delta(x)$ be the error term in the asymptotic formula for partial sums of the square free divisors:

$$\Delta(x) = \sum_{n\leq x}^{*} 2^{\omega(n)} - \frac{x \log x}{\zeta(2)} + \left(-\frac{2\zeta'(2)}{\zeta^2(2)} + \frac{2\gamma-1}{\zeta(2)}\right) x,$$

where $\omega(n)$ denotes the number of distinct prime divisors of n. It is known that $\Delta(x) \ll x^{1/2}$ (see [12]). Let $\lambda_1 > 0$ and the sets $\mathcal{A}_j$ for $j = 1, 2$ be defined as in Section III.4.1:

$$\mathcal{A}_1 = \left\{x : \Delta(x) > (\lambda_1 - \epsilon)x^{1/4}\right\}, \text{ and } \mathcal{A}_2 = \left\{x : \Delta(x) < (-\lambda_1 + \epsilon)x^{1/4}\right\}.$$

In (III.14), we show that

$$\mu\left(\mathcal{A}_j \cap [T, 2T]\right) = \Omega\left(T^{1/2}\right) \text{ for } j = 1, 2.$$

But under Riemann Hypothesis, we prove the following Ω bounds in (III.15):

$$\mu\left(\mathcal{A}_j \cap [T, 2T]\right) = \Omega\left(T^{1-\epsilon}\right), \text{ for } j = 1, 2 \text{ and for any } \epsilon > 0.$$

I.2.3 Divisors

Let $d(n)$ denotes the number of divisors of n:

$$d(n) = \sum_{d|n} 1.$$

Dirichlet [18, Theorem 320] showed that

$$\sum_{n\le x}^{*} d(n) = x \log x + (2\gamma - 1)x + \Delta(x),$$

where γ is the Euler constant and

$$\Delta(x) = O(\sqrt{x}).$$

Latest result on $\Delta(x)$ is due to Huxley [20], which is

$$\Delta(x) = O(x^{131/416}).$$

On the other hand, Hardy [15] showed that

$$\Delta(x) = \Omega_+((x\log x)^{1/4}\log\log x),$$
$$= \Omega_-(x^{1/4}).$$

There are many improvements on Hardy's result due to K. Corrádi and I. Kátai [7], J. L. Hafner [13] and K. Sounderarajan [36]. As a consequence of Theorem IV.3, we shall show in Chapter IV that for all sufficiently large T and for a constant $c_3 > 0$, there exist $x_1, x_2 \in [T, 2T]$ such that

$$\Delta(x_1) > c_3 x_1 \ \text{ and } \ \Delta(x_2) < -c_3 x_2.$$

In particular, we get

$$\Delta(x) = \Omega_\pm(x^{1/4}).$$

I.2.4 Error Term in the Prime Number Theorem

Let $\Delta(x)$ be the error term in the Prime Number Theorem:

$$\Delta(x) = \sum_{n\le x}^{*} \Lambda(n) - x.$$

We know from Landau's theorem [26] that

$$\Delta(x) = \Omega_\pm\left(x^{1/2}\right)$$

and from the theorem of Hardy and Littlewood [16] that

$$\Delta(x) = \Omega_{\pm}\left(x^{1/2}\log\log x\right).$$

We define

$$\mathcal{A}_1 = \left\{x : \Delta(x) > (\lambda_2 - \epsilon)x^{1/2}\right\} \text{ and } \mathcal{A}_2 = \left\{x : \Delta(x) < (-\lambda_2 + \epsilon)x^{1/2}\right\},$$

where $\lambda_2 > 0$ be as in Section III.4.2. If we assume Riemann Hypothesis, then the theorem of Bhowmik, Ramaré and Schlage-Puchta (see Theorem III.5 below) along with PNT** gives

$$\mu\left(\mathcal{A}_j \cap [T, 2T]\right) = \Omega\left(\frac{T}{\log^4 T}\right) \text{ for } j = 1, 2.$$

However, as an application of Corollary III.1 of Theorem III.6, we prove the following weaker bound unconditionally:

$$\mu\left(\mathcal{A}_j \cap [T, 2T]\right) = \Omega\left(T^{1-\epsilon}\right), \text{ for } j = 1, 2 \text{ and for any } \epsilon > 0.$$

I.2.5 Non-isomorphic Abelian Groups

Let a_n be the number of non-isomorphic abelian groups of order n, and the corresponding Dirichlet series is given by

$$\sum_{n=1}^{\infty}\frac{a(n)}{n^s} = \prod_{k=1}^{\infty}\zeta(ks) \text{ for } \Re(s) > 1.$$

Let $\Delta(x)$ be defined as

$$\Delta(x) = \sum_{n\leq x}^{*} a_n - \sum_{k=1}^{6} \Big(\prod_{j\neq k} \zeta(j/k)\Big) x^{1/k}.$$

It is an open problem to show that

$$\Delta(x) \ll x^{1/6+\epsilon} \;\text{ for any } \epsilon > 0. \tag{I.6}$$

The best result on upper bound of $\Delta(x)$ is due to O. Robert and P. Sargos [33], which gives

$$\Delta(x) \ll x^{1/4+\epsilon} \;\text{ for any } \epsilon > 0.$$

Balasubramanian and Ramachandra [4] proved that

$$\int_T^{2T} \Delta^2(x)\mathrm{d}x = \Omega(T^{4/3}\log T).$$

Following the proof of Proposition V.3, we get

$$\mu\left(\{T \leq x \leq 2T : |\Delta(x)| \geq \lambda_3 x^{1/6} (\log x)^{1/2}\}\right) = \Omega(T^{5/6-\epsilon}),$$

for some $\lambda_2 > 0$ and for any $\epsilon > 0$. Sankaranarayanan and Srinivas [35] proved that

$$\Delta(x) = \Omega_{\pm}\left(x^{1/10} \exp\left(c\sqrt{\log x}\right)\right)$$

for some constant $c > 0$. It has been conjectured that

$$\Delta(x) = \Omega_{\pm}(x^{1/6-\delta}) \;\text{ for any } \delta > 0.$$

In Theorem IV.1, we prove that either

$$\int_T^{2T} \Delta^4(x)\mathrm{d}x = \Omega(T^{5/3+\delta}) \text{ or } \Delta(x) = \Omega_\pm(x^{1/6-\delta}),$$

for any $0 < \delta < 1/42$. The conjectured upper bound (I.6) of $\Delta(x)$ gives

$$\int_T^{2T} \Delta^4(x)\mathrm{d}x \ll T^{5/3+\delta}.$$

This along with Theorem IV.1 implies that

$$\Delta(x) = \Omega_\pm(x^{1/6-\delta}) \text{ for any } 0 < \delta < 1/42.$$

[II] Analytic Continuation Of The Mellin Transform

In this chapter, we express the error term $\Delta(x)$ as a contour integral using the Perron's formula. This allows us to obtain a meromorphic continuation of $A(s)$ (see Definition I.2) in terms of the meromorphic continuation of $D(s)$, which is the main theorem of this chapter (Theorem II.3). This theorem will be used in the next chapter to obtain $\Omega_\pm$ results for $\Delta(x)$.

II.1 Perron's Formula

Recall that we have a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$, with its Dirichlet series $D(s)$. The Perron summation formula approximates the partial sums of a_n by expressing it as a contour integral involving $D(s)$.

Theorem II.1 (Perron's Formula, Theorem II.2.1 [37]). *Let $D(s)$ be absolutely convergent for $\Re(s) > \sigma_c$, and let $\kappa > \max(0, \sigma_c)$. Then for $x \geq 1$, we have*

$$\sum_{n\leq x}^{*} a_n = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{D(s)x^s}{s}\mathrm{d}s.$$

But in practice, we use the following effective version of the Perron's formula.

Theorem II.2 (Effective Perron's Formula, Theorem II.2.1 [37]). *Let $\{a_n\}_{n=1}^{\infty}$, $D(s)$ and κ be defined as in Theorem II.1. Then for $T \geq 1$ and $x \geq 1$, we have*

$$\sideset{}{^*}\sum_{n\leq x} a_n = \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} \frac{D(s)x^s}{s}\mathrm{d}s + O\left(x^\kappa \sum_{n=1}^{\infty} \frac{|a_n|}{n^\kappa(1+T|\log(x/n)|)}\right).$$

The above formulas are used by shifting the line of integration, and thus by collecting the residues of $D(s)x^s/s$ at its poles lying to the right of the shifted contour. The residues contribute to the main term $\mathcal{M}(x)$, leaving an expression for $\Delta(x)$ as a contour integral. So we write

$$\sideset{}{^*}\sum_{n\leq x} a_n = \mathcal{M}(x) + \Delta(x),$$

where $\mathcal{M}(x)$ is the main term and $\Delta(x)$ is the error term. We make the following natural assumptions on $D(s)$, $\mathcal{M}(x)$ and $\Delta(x)$.

Assumptions II.1. *Suppose there exist real numbers T_0, σ_1, σ_2 satisfying $0 < \sigma_1 < \sigma_2$, and $T_0 > 0$ such that*

(i) $D(s)$ is absolutely convergent for $\Re(s) > \sigma_2$.

(ii) $D(s)$ can be meromorphically continued to the half plane $\Re(s) > \sigma_1$ and is analytic on the following line segments

$$\{\sigma + it : \sigma_1 \leq \sigma \leq \sigma_2, t = \pm T_0\}$$

$$\{\sigma + it : \sigma = \sigma_1, -T_0 \leq t \leq T_0\}.$$

(iii) For $\mathcal{P}$ define as

$$\mathcal{P} = \{\sigma + it : \sigma + it \text{ is a pole of } D(s), \sigma > \sigma_1, |t| < T_0\},$$

the main term $\mathcal{M}(x)$ is sum of residues of $\frac{D(s)x^s}{s}$ at poles in $\mathcal{P}$:

$$\mathcal{M}(x) = \sum_{\rho \in \mathcal{P}} Res_{s=\rho} \left(\frac{D(s)x^s}{s} \right).$$

We may note that $\mathcal{P}$ is a finite set.

The above assumptions also imply:

Note II.1. *We may also observe:*

(i) For any $\epsilon > 0$, we have

$$|a_n|, |\mathcal{M}(x)|, |\Delta(x)|, \left| \sum_{n \le x} a_n \right| \ll x^{\sigma_2 + \epsilon}.$$

(ii) The main term $\mathcal{M}(x)$ is a polynomial in x, and $\log x$:

$$\mathcal{M}(x) = \sum_{j \in \mathscr{J}} \nu_{1,j} x^{\nu_{2,j}} (\log x)^{\nu_{3,j}},$$

where $\nu_{1,j}$ are complex numbers, $\nu_{2,j}$ are real numbers with $\sigma_1 < \nu_{2,j} \le \sigma_2$, $\nu_{3,j}$ are positive integers, and $\mathscr{J}$ is a finite index set.

To express $\Delta(x)$ in terms of a contour integration, we define the following contour.

Definition II.1. *Let σ_1, σ_2 be as defined in Assumptions II.1. Choose a positive real number σ_3 such that $\sigma_3 > \sigma_2$. We define the contour $\mathscr{C}$, as in Figure II.1, as the union of the following five line segments:*

$$\mathscr{C} = L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5,$$

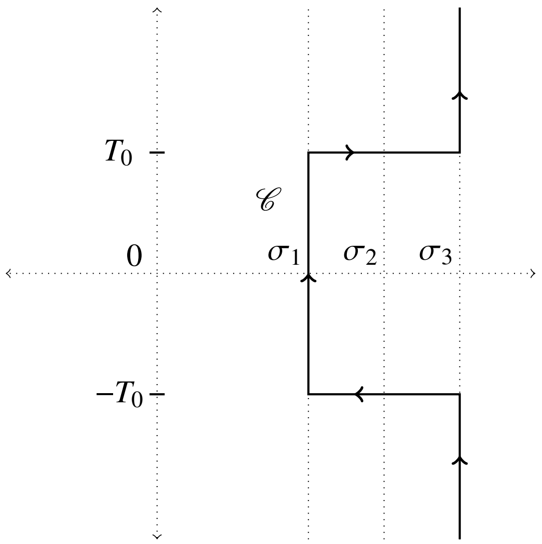

Figure II.1: Contour $\mathscr{C}$

where

$$L_1 = \{\sigma_3 + iv : T_0 \le v < \infty\}, \qquad L_2 = \{u + iT_0 : \sigma_1 \le u \le \sigma_3\},$$

$$L_3 = \{\sigma_1 + iv : -T_0 \le v \le T_0\}, \qquad L_4 = \{u - iT_0 : \sigma_1 \le u \le \sigma_3\},$$

$$L_5 = \{\sigma_3 + iv : -\infty < v \le -T_0\}.$$

Now, we write $\Delta(x)$ as an integration over $\mathscr{C}$ in the following lemma.

Lemma II.1. *Under Assumptions II.1, the error term $\Delta(x)$ can be expressed as:*

$$\Delta(x) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{D(\eta)x^{\eta}}{\eta}\mathrm{d}\eta.$$

Proof. Follows from Theorem II.1. □

II.2 Analytic continuation of $A(s)$

Now, we shall discuss a method to obtain a meromorphic continuation of $A(s)$, which will serve as an important tool to obtain $\Omega_{\pm}$ results for $\Delta(x)$ in the following chapter.
Below we present the main theorem of this chapter.

Theorem II.3. *Under Assumptions-II.1, we have*

$$A(s) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{D(\eta)}{\eta(s-\eta)}\mathrm{d}\eta,$$

when s lies on the right-hand side of the contour $\mathscr{C}$ (Figure II.1).

II.2.1 Preparatory Lemmas

We shall need the following preparatory lemmas to prove the above theorem.

From Lemma II.1, we have:

$$A(s) = \frac{1}{2\pi i}\int_{1}^{\infty}\int_{\mathscr{C}} \frac{D(\eta)x^{\eta}}{\eta}\mathrm{d}\eta\frac{\mathrm{d}x}{x^{s+1}}. \tag{II.1}$$

To prove Theorem II.3, we need to justify the interchange of the integrals of η and x in (II.1).

Definition II.2. *Define the following complex valued function $B(s)$:*

$$\begin{aligned} B(s) &:= \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{D(\eta)}{\eta}\int_{1}^{\infty}\frac{\mathrm{d}x}{x^{s-\eta+1}}\mathrm{d}\eta \\ &= \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{D(\eta)\mathrm{d}\eta}{(s-\eta)\eta} \qquad \textit{for } \Re(s) > \Re(\eta). \end{aligned}$$

The integral defining $B(s)$ being absolutely convergent, we have $B(s)$ is well defined and analytic.

Definition II.3. *For a positive integer N, define the contour $\mathscr{C}(N)$ as:*

$$\mathscr{C}(N) = \{\eta \in \mathscr{C} : |\Im(\eta)| \leq N\}.$$

Definition II.4. *Integrating the integrals of η and x, define $B_N(s)$ as:*

$$\begin{aligned} B_N(s) &= \frac{1}{2\pi i} \int_{\mathscr{C}(N)} \frac{D(\eta)\mathrm{d}\eta}{\eta} \int_1^\infty \frac{\mathrm{d}x}{x^{s-\eta+1}} \\ &= \frac{1}{2\pi i} \int_{\mathscr{C}(N)} \frac{D(\eta)\mathrm{d}\eta}{(s-\eta)\eta} \qquad \textit{for } \Re(s) > \Re(\eta). \end{aligned}$$

With above definitions we prove:

Lemma II.2. *The functions B and B_N satisfy the following identities:*

$$B(s) = \lim_{N\to\infty} B_N(s) \tag{II.2}$$

$$= \lim_{N\to\infty} \frac{1}{2\pi i} \int_1^\infty \int_{\mathscr{C}(N)} \frac{D(\eta)x^\eta}{\eta} \mathrm{d}\eta \frac{\mathrm{d}x}{x^{s+1}}. \tag{II.3}$$

Proof. Assume $N > T_0$. To show (II.2), note:

$$\begin{aligned} |B(s) - B_N(s)| &\leq \left| \frac{1}{2\pi i} \int_{\mathscr{C}-\mathscr{C}(N)} \frac{D(\eta)\mathrm{d}\eta}{(s-\eta)\eta} \right| \\ &\ll \left| \int_{\sigma_3+iN}^{\sigma_3+i\infty} \frac{D(\eta)\mathrm{d}\eta}{(s-\eta)\eta} + \int_{\sigma_3-i\infty}^{\sigma_3-iN} \frac{D(\eta)\mathrm{d}\eta}{(s-\eta)\eta} \right| \\ &\ll \int_N^\infty \frac{\mathrm{d}v}{v^2} \ll \frac{1}{N}. \quad (\text{ substituting } \eta = \sigma_3 + iv) \end{aligned}$$

This completes proof of (II.2).

We shall prove (II.3) using a theorem of Fubini and Tonelli [8, Theorem B.3.1, (b)]. To show that the integrals commute, we need to show that one of the iterated integrals in (II.3) converges absolutely. We note:

$$\int_{\mathscr{C}(N)}\int_1^\infty \left|\frac{D(\eta)}{\eta x^{s-\eta+1}}\right| \mathrm{d}x|\mathrm{d}\eta|$$
$$\ll \int_{\mathscr{C}(N)} \left|\frac{D(\eta)}{\eta(\Re(s)-\Re(\eta))}\right| |\mathrm{d}\eta| < \infty.$$

This implies (II.3). □

Let

$$B'_N(s) := \frac{1}{2\pi i}\int_1^\infty \int_{\mathscr{C}(N)} \frac{D(\eta)x^\eta}{\eta}\mathrm{d}\eta \frac{\mathrm{d}x}{x^{s+1}}. \tag{II.4}$$

We re-write (II.3) of Lemma II.2 as:

$$\lim_{N\to\infty} B'_N(s) = B(s).$$

Observe that $A(s) = B(s)$, if

$$\lim_{N\to\infty}\int_1^\infty \int_{\mathscr{C}-\mathscr{C}(N)} \frac{D(\eta)x^\eta}{\eta}\mathrm{d}\eta \frac{\mathrm{d}x}{x^{s+1}} = 0;$$

can be shown by interchanging the integral of x with the limit. For this, we need the uniform convergence of the integrand, which we do not have. It is easy to see from Theorem II.2 that the problem arises when x is an integer. To handle this problem, we shall divide the integral in two parts, with one part having neighborhoods of integers.

Definition II.5. *For* $\delta = \frac{1}{\sqrt{N}}$ *(where* $N \geq 2$ *), we construct the following set as a neigh-*

borhood of integers:

$$\mathcal{S}(\delta) := [1, 1+\delta] \cup (\cup_{m\geq 2}[m-\delta, m+\delta]).$$

Write

$$A(s) - B'_N(s) = \frac{1}{2\pi i}(J_{1,N}(s) + J_{2,N}(s) - J_{3,N}(s)), \tag{II.5}$$

where

$$J_{1,N}(s) = \int_{\mathcal{S}(\delta)^c} \int_{\mathscr{C}-\mathscr{C}_N} \frac{D(\eta)x^\eta}{\eta} \mathrm{d}\eta \frac{\mathrm{d}x}{x^{s+1}},$$
$$J_{2,N}(s) = \int_{\mathcal{S}(\delta)} \int_{\sigma_3 - i\infty}^{\sigma_3 + i\infty} \frac{D(\eta)x^\eta}{\eta} \mathrm{d}\eta \frac{\mathrm{d}x}{x^{s+1}},$$
$$J_{3,N}(s) = \int_{\mathcal{S}(\delta)} \int_{\sigma_3 - iN}^{\sigma_3 + iN} \frac{D(\eta)x^\eta}{\eta} \mathrm{d}\eta \frac{\mathrm{d}x}{x^{s+1}}.$$

In the next three lemmas, we shall show that each of $J_{i,N}(s) \to 0$ as $N \to \infty$.

Lemma II.3. *For $\Re(s) = \sigma > \sigma_3 + 1$, we have the limit*

$$\lim_{N\to\infty} J_{1,N}(s) = 0.$$

Proof. Using Theorem II.2 with $x \in \mathcal{S}(\delta)^c$, we have

$$\begin{aligned}\left|\int_{\mathscr{C}-\mathscr{C}_N} \frac{D(\eta)x^\eta}{\eta}\mathrm{d}\eta\right| &\ll x^{\sigma_3} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_3}(1 + N|\log(x/n)|)} \\ &\ll \frac{x^{\sigma_3}}{N} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_3}} + \frac{1}{N} \sum_{x/2\leq n\leq 2x} \frac{x|a_n|}{|x-n|} \left(\frac{x}{n}\right)^{\sigma_3}\end{aligned}$$

$$\ll \frac{x^{\sigma_3}}{N} + \frac{x^{\sigma_3+1+\epsilon}}{\delta N} \ll \frac{x^{\sigma_3+1+\epsilon}}{\sqrt{N}} \quad (\text{ as } \delta = N^{-\frac{1}{2}}).$$

From the above calculation, we see that

$$|J_{1,N}| \ll \frac{1}{\sqrt{N}} \int_1^\infty x^{\sigma_3-\sigma+\epsilon} dx \ll \frac{1}{\sqrt{N}}$$

for $\sigma = \Re(s) > \sigma_3 + 1 + \epsilon$. This proves our required result. □

Lemma II.4. *For $\Re(s) = \sigma > \sigma_3$,*

$$\lim_{N\to\infty} J_{2,N}(s) = 0.$$

Proof. Recall that

$$\sum_{n\leq x}^{*} a_n = \begin{cases} \sum_{n<x} a_n + a_x/2 & \text{if } x \in \mathbb{N}, \\ \sum_{n\leq x} a_n & \text{if } x \notin \mathbb{N}. \end{cases}$$

By Note II.1,

$$\sum_{n\leq x}^{*} a_n \ll x^{\sigma_3}.$$

Using this bound, we calculate an upper bound for $J_{2,N}$ as follows:

$$\left| \int_{S(\delta)} \int_{\sigma_3-i\infty}^{\sigma_3+i\infty} \frac{D(\eta)x^\eta}{\eta} \mathrm{d}\eta \frac{\mathrm{d}x}{x^{s+1}} \right| \leq \int_{S(\delta)} \frac{\left|\sum_{n\leq x}^{*} a_n\right|}{x^{\sigma+1}} \mathrm{d}x$$
$$\ll \int_{S(\delta)} x^{\sigma_3-\sigma-1} \mathrm{d}x \ll \int_1^{1+\delta} x^{\sigma_3-\sigma-1} + \sum_{m=2}^{\infty} \int_{m-\delta}^{m+\delta} x^{\sigma_3-\sigma-1} \mathrm{d}x.$$

This gives

$$|J_{2,N}(s)| \ll \delta + \sum_{m\geq 2} \left(\frac{1}{(m-\delta)^{\sigma-\sigma_3}} - \frac{1}{(m+\delta)^{\sigma-\sigma_3}} \right).$$

Using the mean value theorem, for all $m \geq 2$ there exists a real number $\overline{m} \in [m-\delta, m+\delta]$

such that

$$|J_{2,N}(s)| \ll \delta + \sum_{m\geq 2} \frac{\delta}{m^{\sigma-\sigma_3+1}} \ll \delta = \frac{1}{\sqrt{N}} \quad \text{by choosing } \sigma > \sigma_3.$$

This implies that $J_{2,N} \to 0$ as $N \to \infty$. □

Lemma II.5. *For $\sigma > \sigma_3$, we have*

$$\lim_{N\to\infty} J_{3,N}(s) = 0.$$

Proof. Consider

$$J_{3,N}(s) = \int_{\mathcal{S}(\delta)} \int_{\sigma_3-iN}^{\sigma_3+iN} \frac{D(\eta)x^\eta}{\eta} \mathrm{d}\eta \frac{\mathrm{d}x}{x^{s+1}}.$$

This double integral is absolutely convergent for $\Re(s) > \sigma_3$. Using the Theorem of Fubini and Tonelli [8, Theorem B.3.1, (b)], we can interchange the integrals:

$$\begin{aligned} J_{3,N}(s) &= \int_{\sigma_3-iN}^{\sigma_3+iN} \frac{D(\eta)}{\eta} \int_{\mathcal{S}(\delta)} x^{\eta-s-1}\mathrm{d}x \ \mathrm{d}\eta \\ &= \int_{\sigma_3-iN}^{\sigma_3+iN} \frac{D(\eta)}{\eta} \left\{ \int_1^{1+\delta} \frac{x^\eta}{x^{s+1}}\mathrm{d}x + \sum_{m\geq 2} \int_{m-\delta}^{m+\delta} \frac{x^\eta}{x^{s+1}}\mathrm{d}x \right\} \mathrm{d}\eta. \end{aligned}$$

For any θ_1, θ_2 such that $0 < \theta_1 < \theta_2 < \infty$, we have

$$\int_{\theta_1}^{\theta_2} x^{\eta-s-1}\mathrm{d}x = \frac{1}{s-\eta} \left\{ \frac{1}{\theta_1^{s-\eta}} - \frac{1}{\theta_2^{s-\eta}} \right\} = \frac{\theta_2-\theta_1}{\overline{\theta}^{s-\eta+1}},$$

for some $\overline{\theta} \in [\theta_1, \theta_2]$. Applying the above formula to $J_{3,N}(s)$, we get

$$J_{3,N}(s) = \int_{\sigma_3 - iN}^{\sigma_3 + iN} \frac{D(\eta)}{\eta} \sum_{m \geq 1} \frac{2\delta}{\overline{m}^{s-\eta+1}} \mathrm{d}\eta = 2\delta \sum_{m \geq 1} \int_{\sigma_3 - iN}^{\sigma_3 + iN} \frac{D(\eta)}{\overline{m}^{s-\eta+1}\eta} \mathrm{d}\eta,$$

where $\overline{1/2} \in [1, 1+\delta]$ and $\overline{m} \in [m-\delta, m+\delta]$ for all integers $m \geq 2$. In the above calculation, we can interchange the series and the integral as the series is absolutely convergent. So we have

$$\begin{aligned} J_{3,N}(s) &\ll \delta \sum_{m \geq 1} \int_{-N}^{N} \frac{1}{(1+|v|)\overline{m}^{\sigma-\sigma_3+1}} \mathrm{d}v \quad (\text{ substituting } \eta = \sigma_3 + iv \text{ }) \\ &\ll \delta \log N \sum_{m \geq 1} \frac{1}{\overline{m}^{\sigma-\sigma_3+1}} \ll \frac{\log N}{\sqrt{N}}. \end{aligned}$$

Here we used the fact that for $\sigma > \sigma_3$, the series

$$\sum_{m \geq 1} \frac{1}{\overline{m}^{s-\eta+1}}$$

is absolutely convergent. This proves our required result. □

II.2.2 Proof of Theorem II.3

Proof. From equation (II.5) and Lemma II.3, II.4 and II.5, we get

$$A(s) = \lim_{N \to \infty} B'_N(s)$$

for $\Re(s) > \sigma_3 + 1$, and where $B'_N(s)$ is defined by (II.4). From Lemma II.2, we have

$$B(s) = \lim_{N \to \infty} B'_N(s).$$

This gives $A(s)$ and $B(s)$ are equal for $\Re(s) > \sigma_3 + 1$. By analytic continuation, $A(s)$ and $B(s)$ are equal for any s that lies right to $\mathscr{C}$. $\square$

In this chapter, we shall use the meromorphic continuation of $A(s)$ derived in Theorem II.3 to obtain mesure theoretic $\Omega_\pm$ results for $\Delta(x)$.

II.3 Alternative Approches

Theorem II.3 gives a way for meromorphic continuation of $A(s)$ by formulating it as a contour integral. This theorem has its significance in terms of elegance and generality. However, there are alternative and easier ways in many cases. Below we give an example.

Note that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = -\int_1^{\infty} \Big(\sum_{n\leq x} a_n\Big) \mathrm{d}x^{-s} \quad \text{for } \Re(s) > \sigma_2.$$

This gives

$$\frac{D(s)}{s} = \int_1^{\infty} \Big(\sum_{n\leq x} a_n\Big) x^{-s-1} \mathrm{d}x \quad \text{for } \Re(s) > \sigma_2.$$

So we can express $A(s)$ as

$$A(s) = \frac{D(s)}{s} - \int_1^{\infty} \mathcal{M}(x) x^{-s-1} \mathrm{d}x \quad \text{for } \Re(s) > \sigma_2. \tag{II.6}$$

The above formula reduces the problem of meromorphically continuing $A(s)$ to that of

$$\int_1^{\infty} \mathcal{M}(x) x^{-s-1} \mathrm{d}x.$$

To demonstrate this method, we consider the case when $D(\eta)$ has a pole at $\eta = 1$ and residue at this pole gives the main term $\mathcal{M}(x)$, i.e $\mathcal{P} = \{1\}$. The following meromorphic functions may serve as examples of $D(\eta)$ in this situation:

$$\frac{\zeta(s)}{\zeta(2s)}, \zeta^2(s), \frac{\zeta^2(s)}{\zeta(2s)}, -\frac{\zeta'(s)}{\zeta(s)}, \ldots .$$

For a small positive real number r, we can write $\mathcal{M}(x)$ as

$$\mathcal{M}(x) = \frac{1}{2\pi i}\int_{|\eta-1|=r} \frac{D(\eta)x^\eta}{\eta}\mathrm{d}\eta.$$

Thus

$$\begin{aligned}
\int_1^\infty \frac{\mathcal{M}(x)}{x^{s+1}}\mathrm{d}x &= \int_1^\infty \frac{1}{2\pi i}\int_{|\eta-1|=r} \frac{D(\eta)x^\eta}{\eta} d\eta \frac{\mathrm{d}x}{x^{s+1}} \\
&= \frac{1}{2\pi i}\int_{|\eta-1|=r} \frac{D(\eta)}{\eta}\left(\int_1^\infty \frac{\mathrm{d}x}{x^{s-\eta+1}}\right)\mathrm{d}\eta \\
&\quad \text{(using [8, Theorem B.3.1, (b)])} \\
&= \frac{1}{2\pi i}\int_{|\eta-1|=r} \frac{D(\eta)}{\eta(s-\eta)}\mathrm{d}\eta. \qquad \text{(II.7)}
\end{aligned}$$

Let the Laurent series expansion of $D(\eta)$ at $\eta = 1$ be

$$\frac{D(\eta)}{\eta} = \sum_{n\le N} \frac{b_n}{(\eta-1)^n} + H(\eta),$$

where $H(\eta)$ is holomorphic for $\Re(\eta) > \sigma_1$. Plugging in this expression for $D(\eta)$ in (II.7), we get

$$\int_1^\infty \frac{\mathcal{M}(x)}{x^{s+1}}\mathrm{d}x = \sum_{n\le N} b_n \frac{1}{2\pi i}\int_{|\eta-1|=r} \frac{\mathrm{d}\eta}{(\eta-1)^n(s-\eta)}. \qquad \text{(II.8)}$$

Let $\Re(s) \ge 1 + 2r$, then

$$\frac{|\eta-1|}{|s-1|} \le \frac{1}{2} \quad \text{for } |\eta-1| = r.$$

This gives

$$\frac{1}{s-\eta} = \sum_{n=0}^{\infty} \frac{(\eta-1)^n}{(s-1)^{n+1}}$$

is an absolutely convergent series. Using the above expansion of $(s-\eta)^{-1}$ in (II.8), we have

$$\begin{aligned}\int_1^{\infty} \frac{\mathcal{M}(x)}{x^{s+1}} \mathrm{d}x &= \sum_{n\leq N} b_n \frac{1}{2\pi i} \int_{|\eta-1|=r} \left\{ \sum_{m=0}^{\infty} \frac{(\eta-1)^m}{(s-1)^{m+1}} \right\} \frac{\mathrm{d}\eta}{(\eta-1)^n} \\ &= \sum_{n\leq N} \frac{b_n}{(s-1)^n} \quad \text{(by [34, Theorem 6.1])} \\ &= \frac{D(s)}{s} - H(s).\end{aligned}$$

Thus we got

$$A(s) = H(s) \text{ for } \Re(s) \geq 1+2r.$$

But the right hand side is holomorphic for $\Re(s) > \sigma_1$ hence the formula gives analytic continuation of $A(s)$ in the half plane $\Re(s) > \sigma_1$.

Similar calculations can be done when the main term $\mathcal{M}(x)$ is more complicated.

[III] Landau's Oscillation Theorem

In this chapter, we revisit a result due to Landau and obtain $\Omega_{\pm}$ results for $\Delta(x)$ using certain singularities of $D(s)$. Also we shall measure the fluctuations of $\Delta(x)$ in terms of Ω bounds, which generalizes a result of Kaczorowski and Szydło [24], and a result of Bhowmik, Ramaré and Schlage-Puchta [6].

III.1 Landau's Criterion for Sign Change

We begin with a result on real valued functions that do not change sign. This appears in a paper of Landau [26], attributed to Phragmén and stated without a proof. Here we present a proof of this result following [37, II.1.3, Theorem 6].

Theorem III.1 (Phragmén-Landau)**.** *Let $f(x)$ be a real valued piecewise continuous function defined for $x \geq 1$. Let $F(s)$ be its Mellin transform:*

$$F(s) = \int_1^{\infty} \frac{f(x)}{x^{s+1}} \mathrm{d}x,$$

converging absolutely in some complex right half plane. Also assume that $f(x)$ does not change sign for $x \geq x_0$, for some $x_0 \geq 1$. If $F(s)$ diverges for some real s, then there exist a real number σ_0 satisfying the following properties:

(1) *the integral defining* $F(s)$ *is divergent for* $s < \sigma_0$ *and convergent for* $s > \sigma_0$*,*

(2) $s = \sigma_0$ *is a singularity of* $F(s)$*,*

(3) *and* $F(s)$ *is analytic for* $\Re(s) > \sigma_0$*.*

Proof. Let σ_0 be:

$$\sigma_0 = \inf\{\sigma \in \mathbb{R} : F(\sigma) \text{ converges}\}.$$

We shall show that σ_0 satisfies the properties given in the theorem.

As $f(x)$ does not change sign for $x \geq x_0$, convergence of $F(\sigma)$ implies the absolute convergence of $F(s)$ for $\Re(s) \geq \sigma$. This proves (1) and (3). To prove (2), we proceed by method of contradiction. Assume that $s = \sigma_0$ is not a singularity of $F(s)$. Then there exist $\sigma_0' > \sigma_0$ and $r > \sigma_0' - \sigma_0$ such that $F(s)$ has the following Taylor series expansion:

$$F(s) = \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(\sigma_0')(s - \sigma_0')^k,$$

for all s satisfying $|s - \sigma_0'| < r$.

Claim (1)**.** *For* σ_0' *as above, we have*

$$F(s) = \sum_{k=0}^{\infty} \frac{1}{k!}(s - \sigma_0')^k \int_1^{\infty} (-\log x)^k \frac{f(x)}{x^{\sigma_0'+1}} \mathrm{d}x.$$

Proof of Claim (1). By Cauchy's integral formula, we can write

$$F^{(k)}(\sigma_0') = \frac{k!}{2\pi i} \int_C \frac{F(z)}{(z - \sigma_0')^{k+1}} \mathrm{d}z,$$

where C is a circle with a small enough radius having its center at σ_0'. So we have

$$F(s) = \sum_{k=0}^{\infty} \frac{(s-\sigma_0')^k}{2\pi i} \int_C \frac{1}{(z-\sigma_0')^{k+1}} \int_1^{\infty} \frac{f(x)}{x^{z+1}} \mathrm{d}x \, \mathrm{d}z.$$

Suppose we can exchange the integrals of x and z, then

$$\begin{aligned} F(s) &= \sum_{k=0}^{\infty} \frac{(s-\sigma_0')^k}{k!} \int_1^{\infty} \frac{f(x)}{x} \frac{k!}{2\pi i} \int_C \frac{x^{-z}\mathrm{d}z}{(z-\sigma_0')^{k+1}} \mathrm{d}x \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}(s-\sigma_0')^k \int_1^{\infty} (-\log x)^k \frac{f(x)}{x^{\sigma_0'+1}} \mathrm{d}x, \end{aligned}$$

which proves Claim 1 conditionally. The only thing remains is to show that we can exchange integrals of x and z. If we choose C with a small enough radius, then

$$\int_1^{\infty} \frac{f(x)}{x^{\Re(z)+1}} \mathrm{d}x$$

is absolutely convergent and so is the double integral

$$\int_C \frac{1}{(z-\sigma_0')^{k+1}} \int_1^{\infty} \frac{f(x)}{x^{z+1}} \mathrm{d}x \, \mathrm{d}z.$$

By the theorem of Fubinni and Tonelli [8, Theorem B.3.1, (b)], we can exchange these two iterated integrals. This completes the proof of Claim 1.

Claim (2)**.** *For $|s-\sigma_0'| < r$, the integral*

$$F(s) = \int_1^{\infty} \frac{f(x)}{x^{s+1}} \mathrm{d}x$$

converges.

Proof of Claim (2). We shall simplify $F(s)$ using Claim 1. We write

$$F(s) = \sum_{k=0}^{\infty} \frac{(\sigma_0' - s)^k}{k!} \int_1^{\infty} \frac{(\log x)^k f(x)}{x^{\sigma_0'+1}} \mathrm{d}x.$$

In the above identity, we can exchange the series and the integral as the series is absolutely convergent. So we have

$$\begin{aligned} &\int_1^{\infty} \frac{f(x)}{x^{\sigma_0'+1}} \left(\sum_{k=0}^{\infty} \frac{(\sigma_0' - s)^k}{k!} (\log x)^k \right) \mathrm{d}x \\ &= \int_1^{\infty} \frac{f(x)}{x^{\sigma_0'+1}} \exp((\sigma_0' - s)\log x) \mathrm{d}x = \int_1^{\infty} \frac{f(x)}{x^{s+1}} \mathrm{d}x. \end{aligned}$$

This completes the proof of Claim 2.

But Claim 2 implies that we have a real number smaller than σ_0, say σ_0'', such that the integral of $F(\sigma_0'')$ converges. This is a contradiction to the definition of σ_0. So σ_0 is a singularity of $F(s)$, which proves (2). □

The following theorem appears in [1, Section 2] without a proof and is attributed to Landau. We shall prove this theorem using Theorem III.1.

Theorem III.2 (Phragmén-Landau-Anderson-Stark)**.** *Let $f(x)$ be a real valued piecewise continuous function defined on $[1, \infty)$, and does not change sign when $x > x_0$ for some $1 < x_0 < \infty$. Define*

$$F(s) := \int_1^{\infty} \frac{f(x)}{x^{s+1}} \mathrm{d}x,$$

and assume that the above integral is absolutely convergent in some half plane. Further, assume that we have an analytic continuation of $F(s)$ in a region containing a part of the real line

$$l(\sigma_0, \infty) := \{\sigma + i0 : \sigma > \sigma_0\}.$$

Then the integral representing $F(s)$ is absolutely convergent for $\Re(s) > \sigma_0$, and hence $F(s)$ is an analytic function in this region.

Proof. By Theorem III.1, if

$$\int_1^\infty \frac{f(x)}{x^{\sigma'+1}}\mathrm{d}x$$

diverges for some $\sigma' > \sigma_0$, then there exist a real number $\sigma_0' \geq \sigma' > \sigma_0$ such that F is not analytic at σ_0'. But this contradicts our assumption that F is analytic on $l(\sigma_0, \infty)$. So the integral

$$\int_1^\infty \frac{f(x)}{x^{\sigma'+1}}\mathrm{d}x \ \text{ converges } \ \forall\, \sigma' > \sigma_0,$$

and since f does not change sign for $x \geq x_0$, $F(s)$ converges absolutely for $\Re(s) > \sigma_0$. This also gives that $F(s)$ is analytic for $\Re(s) > \sigma_0$. □

The above two theorems give some criteria when a function does not change sign. In the next section we will use these results to show the sign changes of $\Delta(x)$.

III.2 $\Omega_\pm$ Results

Consider the Mellin transform $A(s)$ of $\Delta(x)$. We need the following assumptions to apply Theorem III.2.

Assumptions III.1. *Suppose there exists a real number σ_0, $0 < \sigma_0 < \sigma_1$, such that $A(s)$ has the following properties.*

(i) There exists $t_0 \neq 0$ such that

$$\lambda := \limsup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0)|A(\sigma + it_0)| > 0.$$

(ii) We have

$$l_s := \limsup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0) A(\sigma) < \infty,$$

$$l_i := \liminf_{\sigma \searrow \sigma_0} (\sigma - \sigma_0) A(\sigma) > -\infty.$$

(iii) The limits l_i, l_s and λ satisfy

$$l_i + \lambda > 0 \quad \textit{and} \quad l_s - \lambda < 0.$$

(iv) We can analytically continue $A(s)$ in a region containing the real line $l(\sigma_0, \infty)$.

Remark III.1. *Assumptions III.1 (i) implies that $\sigma_0 + it_0$ is a singularity of $A(s)$.*

Now we construct the following sets for further use.

Definition III.1. *With l_s, l_i and λ as in Assumptions III.1, and for an ϵ such that $0 < \epsilon < \min(l_i + \lambda, \lambda - l_s)$, define*

$$\mathcal{A}_1 := \{x : x \in [1, \infty), \Delta(x) > (l_i + \lambda - \epsilon) x^{\sigma_0}\}$$

$$\textit{and} \quad \mathcal{A}_2 := \{x : x \in [1, \infty), \Delta(x) < (l_s - \lambda + \epsilon) x^{\sigma_0}\}.$$

Under Assumptions III.1 and using methods from [24], we can derive the following measure theoretic theorem.

Theorem III.3. *Let the conditions in Assumptions III.1 hold. Then for any real number $M > 1$, we have*

$$\mu(\mathcal{A}_1 \cap [M, \infty]) > 0,$$

$$\textit{and} \quad \mu(\mathcal{A}_2 \cap [M, \infty]) > 0.$$

This implies

$$\Delta(x) = \Omega_{\pm}(x^{\sigma_0}).$$

Proof. We prove the theorem only for $\mathcal{A}_1$ as the other part is similar.

Now define the following integrals whenever they are absolutely convergent:

$$\begin{aligned}
g(x) &:= \Delta(x) - (l_i + \lambda - \epsilon)x^{\sigma_0}, & G(s) &:= \int_1^\infty \frac{g(x)}{x^{s+1}} \mathrm{d}x; \\
g^+(x) &:= \max(g(x), 0), & G^+(s) &:= \int_1^\infty \frac{g^+(x)}{x^{s+1}} \mathrm{d}x; \\
g^-(x) &:= \max(-g(x), 0), & G^-(s) &:= \int_1^\infty \frac{g^-(x)}{x^{s+1}} \mathrm{d}x.
\end{aligned}$$

With the above notations, we have

$$\begin{aligned}
& g(x) = g^+(x) - g^-(x) \\
\text{and} \quad & G(s) = G^+(s) - G^-(s).
\end{aligned}$$

Note that

$$\begin{aligned}
G(s) &= A(s) - \int_1^\infty (l_i + \lambda - \epsilon) x^{\sigma_0 - s - 1} \mathrm{d}x \\
&= A(s) + \frac{l_i + \lambda - \epsilon}{\sigma_0 - s} \quad \text{for } \Re(s) > \sigma_0,
\end{aligned}$$

where ϵ is fixed as in definition III.1. So $G(s)$ is analytic wherever $A(s)$ is, except possibly for a pole at σ_0. This gives

$$\limsup_{\sigma\searrow\sigma_0}(\sigma-\sigma_0)|G(\sigma+it_0)| = \limsup_{\sigma\searrow\sigma_0}(\sigma-\sigma_0)|A(\sigma+it_0)| = \lambda. \qquad \text{(III.1)}$$

We shall use the above limit to prove our theorem. We proceed by method of contradiction. Assume that there exists an $M > 1$ such that

$$\mu(\mathcal{A}_1 \cap [M,\infty)) = 0.$$

This implies

$$G^+(s) = \int_1^\infty \frac{g^+(x)}{x^{s+1}}\mathrm{d}x = \int_1^M \frac{g^+(x)}{x^{s+1}}\mathrm{d}x$$

is bounded for any s, and so is an entire function. By Assumptions III.1, $A(s)$ and $G(s)$ can be analytically continued on the line $l(\sigma_0,\infty)$. As $G(s)$ and $G^+(s)$ are analytic on $l(\sigma_0,\infty)$, $G^-(s)$ is also analytic on $l(\sigma_0,\infty)$. The integral for $G^-(s)$ is absolutely convergent for $\Re(s) > \sigma_3 + 1$, and $g^-(x)$ is a piecewise continuous function. This suggests that we can apply Theorem III.2 to $G^-(s)$, and conclude that

$$G^-(s) = \int_1^\infty \frac{g^-(x)}{x^{s+1}}\mathrm{d}x$$

is absolutely convergent for $\Re(s) > \sigma_0$.

From the above discussion, we summarize that the Mellin transforms of g, g^+ and g^- converge absolutely for $\Re(s) > \sigma_0$. As a consequence, we see that $G(\sigma), G^+(\sigma)$ and $G^-(\sigma)$ are finite real numbers for $\sigma > \sigma_0$. We note that for any $t \in \mathbb{R}$

$$|G^+(\sigma+it)| \le \int_1^M \frac{g^+(x)}{x^{\sigma+1}}\mathrm{d}x = O(1).$$

Thus

$$(\sigma - \sigma_0)|G^{+}(\sigma + it)| \longrightarrow 0 \text{ as } \sigma \longrightarrow \sigma_0 + .$$

Observe that

$$\begin{aligned}
(\sigma - \sigma_0)|G(\sigma + it_0)| &\leq (\sigma - \sigma_0)G^{+}(\sigma) + (\sigma - \sigma_0)G^{-}(\sigma) \\
&\leq 2(\sigma - \sigma_0)G^{+}(\sigma) - (\sigma - \sigma_0)G(\sigma) \\
&\leq 2(\sigma - \sigma_0)G^{+}(\sigma) - (\sigma - \sigma_0)A(\sigma) + l_i + \lambda - \epsilon.
\end{aligned}$$

So we have

$$\limsup_{\sigma \searrow \sigma_0}(\sigma - \sigma_0)|G(\sigma + it_0)| \leq -\liminf_{\sigma \searrow \sigma_0}(\sigma - \sigma_0)A(\sigma) + l_i + \lambda - \epsilon = \lambda - \epsilon.$$

This contradicts (III.1). Thus $\mu(\mathcal{A}_1 \cap [M, \infty)) > 0$ for any $M > 1$, which completes the proof. □

III.3 Measure Theoretic $\Omega_{\pm}$ Results

Now we know that $\mathcal{A}_1$ and $\mathcal{A}_2$ are unbounded. But we do not know how the size of these sets grow. An answer to this question was given by Kaczorowski and Szydło in [24, Theorem 4].

Theorem III.4 (Kaczorowski and Szydło [24])**.** *Let the conditions in Assumptions III.1 hold. Also assume that for a non-decreasing positive continuous function h satisfying*

$$h(x) \ll x^{\epsilon},$$

we have
$$\int_T^{2T} \Delta^2(x)\mathrm{d}x \ll T^{2\sigma_0+1}h(T). \tag{III.2}$$

Then as $T \to \infty$,
$$\mu\left(\mathcal{A}_j \cap [1,T]\right) = \Omega\Big(\frac{T}{h(T)}\Big) \quad \textit{for } j = 1,2.$$

In [24], Kaczorowski and Szydło applied this theorem to the error term appearing in the asymptotic formula for the fourth power moment of Riemann zeta function. We write this error term as $E_2(x)$:

$$\int_0^x \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t = xP(\log x) + E_2(x),$$

where P is a polynomial of degree 4. Motohashi [31] proved that

$$E_2(x) \ll x^{2/3+\epsilon},$$

and further in [32] he showed that

$$E_2(x) = \Omega_\pm(\sqrt{x}).$$

Theorem of Kaczorowski and Szydło (Theorem III.5) gives that there exist $\lambda_0, \nu > 0$ such that

$$\mu\{1 \le x \le T : E_2(x) > \lambda_0\sqrt{x}\} = \Omega(T/(\log T)^\nu)$$

and

$$\mu\{1 \le x \le T : E_2(x) < -\lambda_0\sqrt{x}\} = \Omega(T/(\log T)^\nu)$$

as $T \to \infty$. These results not only prove $\Omega_\pm$-results, but also give quantitative estimates for the occurrences of such fluctuations. The above theorem of Kaczorowski and Szydło has been generalized by Bhowmik, Ramaré and Schlage-Puchta by localizing the fluctuations of $\Delta(x)$ to $[T, 2T]$. Proof of this theorem follows from [6, Theorem 2] (also see Theorem III.7 below).

Theorem III.5 (Bhowmik, Ramaré and Schlage-Puchta [6])**.** *Let the assumptions in Theorem III.4 hold. Then as $T \to \infty$,*

$$\mu\left(\mathcal{A}_j \cap [T, 2T]\right) = \Omega\Big(\frac{T}{h(T)}\Big) \quad \textit{for } j = 1, 2.$$

An application of the above theorem to Goldbach's problem is given in [6]. Let

$$\sum_{n\le x} G_k(n) = \frac{x^k}{k!} - k\sum_{\rho} \frac{x^{k-1+\rho}}{\rho(1+\rho)\cdots(k-1+\rho)} + \Delta_k(x),$$

where the Goldbach numbers $G_k(n)$ are defined as

$$G_k(n) = \sum_{\substack{n_1,\ldots n_k \\ n_1+\cdots+n_k=n}} \Lambda(n_1)\cdots\Lambda(n_k),$$

and ρ runs over nontrivial zeros of the Riemann zeta function $\zeta(s)$. Bhowmik, Ramaré and Schlage-Puchta proved that under Riemann Hypothesis

$$\mu\{T \le x \le 2T : \Delta_k(x) > (\mathfrak{c}_k + \mathfrak{c}'_k)x^{k-1}\} = \Omega(T/(\log T)^6)$$

$$\text{and} \qquad \mu\{T \le x \le 2T : \Delta_k(x) < (\mathfrak{c}_k - \mathfrak{c}'_k)x^{k-1}\} = \Omega(T/(\log T)^6) \ \text{ as } T \to \infty,$$

where $k \ge 2$ and $\mathfrak{c}_k, \mathfrak{c}'_k$ are well defined real number depending on k with $\mathfrak{c}'_k > 0$.

Note that Theorem III.4 implies Theorem III.5, but both the theorems are applicable to the same set of examples. The main obstacle in applicability of these theorems is the condition (III.2). For example, if $\Delta(x)$ is the error term in approximating $\sum_{n\leq x}|\tau(n,\theta)|^2$, we can not apply Theorem III.4 and Theorem III.5. However, the following theorem due to the author and A. Mukhopadhyay [28, Theorem 3] overcomes this obstacle by replacing the condition (III.2).

Theorem III.6. *Let the conditions in Assumptions III.1 hold. Assume that there is an analytic continuation of $A(s)$ in a region containing the real line $l(\sigma_0,\infty)$. Let h_1 and h_2 be two positive monotonic functions with polynomial growth* [1] *such that*

$$\int_{[T,2T]\cap\mathcal{A}_j}\frac{\Delta^2(x)}{x^{2\sigma_0+1}}\mathrm{d}x \ll h_j(T) \quad \text{for } j=1,2. \tag{III.3}$$

Then as $T\longrightarrow\infty$,

$$\mu(\mathcal{A}_j\cap[T,2T])=\Omega\Big(\frac{T}{h_j(T)}\Big) \quad \text{for } j=1,2. \tag{III.4}$$

Below we state an integral version of Theorem III.5 as in [6].

Theorem III.7 (Bhowmik, Ramaré and Schlage-Puchta [6])**.** *Suppose the conditions in Assumptions III.1 hold, and let $h(x)$ be as in Theorem III.5. Then as $\delta\to 0^+$,*

$$\int_1^\infty\frac{\mu(\mathcal{A}_j\cap[x,2x])h(4x)}{x^{2+\delta}}\mathrm{d}x=\Omega\Big(\frac{1}{\delta}\Big), \quad \text{for } j=1,2.$$

The following lemma shows that Theorem III.7 implies Theorem III.5 and Theorem III.8 (below) implies Theorem III.6.

[1] $h_1^{\pm1}(x),\ h_2^{\pm1}(x)\ll x^k$ for some $k\in\mathbb{N}$.

Lemma III.1. *Let f be a real valued function defined on $\mathbb{R}_{\geq 1}$ such that f is bounded and measurable on $[1, x]$ for all $x > 1$, and $f(x) \to 0$ as $x \to \infty$. Then as $\delta \to 0$, we have*

$$\int_1^\infty \frac{f(x)}{x^{\delta+1}} \mathrm{d}x = o\left(\frac{1}{\delta}\right).$$

Proof. As $f(x) \to 0$ when $x \to \infty$, for any $\epsilon > 0$ there exists $x_0 \geq 1$ such that

$$|f(x)| < \epsilon \text{ for all } x \geq x_0.$$

Also $f(x)$ is bounded by some positive constant c:

$$|f(x)| < c \text{ for } x \geq 1.$$

So, we may write

$$\int_1^{x_0} \frac{f(x)}{x^{1+\delta}} \mathrm{d}x \leq \int_1^{x_0} \frac{|f(x)|}{x} \mathrm{d}x \leq c \log x_0 \leq M(\epsilon),$$

where we can choose $M(\epsilon)$ as a positive monotonic function of ϵ mapping $0 < \epsilon < 1$ onto $\mathbb{R}_{\geq 1}$, and

$$M(\epsilon) \to \infty \Leftrightarrow \epsilon \to 0.$$

From the above inequalities we get

$$\int_1^\infty \frac{f(x)}{x^{1+\delta}} \mathrm{d}x \leq \int_1^{x_0} \frac{|f(x)|}{x^{1+\delta}} \mathrm{d}x + \int_{x_0}^\infty \frac{|f(x)|}{x^{1+\delta}} \mathrm{d}x \leq M(\epsilon) + \frac{\epsilon}{\delta T^\delta}.$$

We choose $M(\epsilon) = \delta^{-\frac{1}{2}}$. Then as $\delta \to 0$, $M(\epsilon) \to \infty$, and so $\epsilon \to 0$. Thus

$$\lim_{\delta \to 0} \delta \int_1^\infty \frac{f(x)}{x^{1+\delta}} \mathrm{d}x = 0.$$

□

In our next theorem, we generalize Theorem III.4, III.5, III.6 and III.7.

Theorem III.8. *Let the conditions in Theorem III.6 hold. Then as* $\delta \to 0^+$,

$$\int_1^\infty \frac{\mu(\mathcal{A}_j \cap [x, 2x])h_j(x)}{x^{2+\delta}} \mathrm{d}x = \Omega\Big(\frac{1}{\delta}\Big) \quad \textit{for } j = 1, 2. \tag{III.5}$$

Proof. We shall prove the theorem for $j = 1$; the proof is similar for $j = 2$. We define g, g^+, g^-, G, G^+ and G^-, as in Theorem III.3. Let

$$m^\#(x) := h_1(x)\mu(\mathcal{A}_1 \cap [x, 2x])x^{-1}.$$

First, we shall show:

Claim. *As* $\delta \to 0$,

$$\sum_{k \geq 0} \frac{m^\#(2^k)}{2^{k\delta}} = \Omega\left(\frac{1}{\delta}\right).$$

Assume that

$$\sum_{k \geq 0} \frac{m^\#(2^k)}{2^{k\delta}} = o\left(\frac{1}{\delta}\right). \tag{III.6}$$

From the above assumption, we may obtain an upper bound for $G^+(\sigma)$ as follows:

$$\int_{\mathcal{A}_1} \frac{g^+(x)\mathrm{d}x}{x^{\sigma+1}} \leq \sum_{k \geq 0} \int_{\mathcal{A}_1 \cap [2^k, 2^{k+1}]} \frac{\Delta(x)\mathrm{d}x}{x^{\sigma+1}} \quad (\text{as } \Delta(x) > g(x) \text{ on } \mathcal{A}_1)$$

$$\leq \sum_{k\geq 0}\left(\int_{\mathcal{A}_1\cap[2^k,2^{k+1}]}\frac{\Delta^2(x)\mathrm{d}x}{x^{2\sigma_0+1}}\right)^{\frac{1}{2}}\left(\frac{\mu(\mathcal{A}_1\cap[2^k,2^{k+1}])}{2^{k(2\delta+1)}}\right)^{\frac{1}{2}} \text{ (where } \sigma-\sigma_0=\delta>0)$$
$$\leq c_3\sum_{k\geq 0}\left(\frac{h_1(2^k)\mu(\mathcal{A}_1\cap[2^k,2^{k+1}])}{2^{k(2\delta+1)}}\right)^{\frac{1}{2}} \leq c_3\sum_{k\geq 0}\left(\frac{m^{\#}(2^k)}{2^{2k\delta}}\right)^{\frac{1}{2}}.$$

From the above inequality, we get

$$\delta G^+(\sigma) \ll \delta\left(\sum_{k\geq 0}\frac{1}{2^{k\delta}}\right)^{\frac{1}{2}}\left(\sum_{k\geq 0}\frac{m^{\#}(2^k)}{2^{k(\sigma-\sigma_0)}}\right)^{\frac{1}{2}} = o(1) \tag{III.7}$$

as $\delta\to 0^+$. In particular, $G^+(\sigma)$ is bounded for every $\sigma>\sigma_0$. Therefore

$$G^+(s) = \int_1^\infty \frac{g^+(x)\mathrm{d}x}{x^{s+1}}$$

is absolutely convergent for $\Re(s)>\sigma_0$, and so it is analytic in this region. But

$$G^-(s) = G(s) - G^+(s),$$

and G is analytic on $l(\sigma_0,\infty)$. So G^- is also analytic on $l(\sigma_0,\infty)$. Using Theorem III.2, we get

$$G^-(s) = \int_1^\infty \frac{g^-(x)\mathrm{d}x}{x^{s+1}}$$

is absolutely convergent for $\Re(s)>\sigma_0$. As a consequence, we get $G(\sigma), G^+(\sigma)$, and $G^-(\sigma)$ are finite real numbers for $\sigma>\sigma_0$.

Now observe that

$$\begin{aligned}(\sigma-\sigma_0)|G(\sigma+it_0)| &\leq 2(\sigma-\sigma_0)G^+(\sigma) - (\sigma-\sigma_0)G(\sigma)\\ &\leq 2(\sigma-\sigma_0)G^+(\sigma) - (\sigma-\sigma_0)A(\sigma) + l_i + \lambda - \epsilon.\end{aligned}$$

Using (III.7), we get

$$\limsup_{\sigma\searrow\sigma_0}(\sigma-\sigma_0)|G(\sigma+it_0)| \le -\liminf_{\sigma\searrow\sigma_0}(\sigma-\sigma_0)A(\sigma)+l_i+\lambda-\epsilon=\lambda-\epsilon.$$

This is a contradiction to (III.1), and so (III.6) is wrong. This proves our Claim.

Now we are ready to prove the theorem. For $k \ge 1$, observe that

$$\int_{k-1}^{k}\frac{m^{\#}(2^x)}{2^{\delta x}}\mathrm{d}x=\int_{k-1}^{k}\frac{h_1(2^x)\mu(\mathcal{A}_1\cap[2^x,2^{x+1}])}{2^{x(\delta+1)}}\mathrm{d}x=\int_{k-1}^{k}\int_{2^x}^{2^{x+1}}\frac{h_1(2^x)}{2^{\delta x+x}}\chi_{\mathcal{A}_1}(t)\mathrm{d}t\mathrm{d}x$$

(where $\chi_{\mathcal{A}_1}(t)$ is the indicator function of $\mathcal{A}_1$)

$$=\int_{k-1}^{k}\int_{2^x}^{2^k}\frac{h_1(2^x)}{2^{\delta x+x}}\chi_{\mathcal{A}_1}(t)\mathrm{d}t\mathrm{d}x+\int_{k-1}^{k}\int_{2^k}^{2^{x+1}}\frac{h_1(2^x)}{2^{\delta x+x}}\chi_{\mathcal{A}_1}(t)\mathrm{d}t\mathrm{d}x$$

$$=\int_{2^{k-1}}^{2^k}\int_{k-1}^{\frac{\log t}{\log 2}}\frac{h_1(2^x)}{2^{x(1+\delta)}}\chi_{\mathcal{A}_1}(t)\mathrm{d}x\mathrm{d}t+\int_{2^k}^{2^{k+1}}\int_{\frac{\log t}{\log 2}-1}^{k}\frac{h_1(2^x)}{2^{x(1+\delta)}}\chi_{\mathcal{A}_1}(t)\mathrm{d}x\mathrm{d}t.$$

From the above identity, we have

$$\int_{k-1}^{k}\frac{m^{\#}(2^x)}{2^{\delta x}}\mathrm{d}x\ge\int_{2^k}^{2^{k+1}}\int_{\frac{\log t}{\log 2}-1}^{k}\frac{h_1(2^x)}{2^{x(1+\delta)}}\chi_{\mathcal{A}_1}(t)\mathrm{d}x\mathrm{d}t$$

and

$$\int_{k}^{k+1}\frac{m^{\#}(2^x)}{2^{\delta x}}\mathrm{d}x\ge\int_{2^k}^{2^{k+1}}\int_{k}^{\frac{\log t}{\log 2}}\frac{h_1(2^x)}{2^{x(1+\delta)}}\chi_{\mathcal{A}_1}(t)\mathrm{d}x\mathrm{d}t.$$

So we get

$$\int_{k-1}^{k+1}\frac{m^{\#}(2^x)}{2^{\delta x}}\mathrm{d}x\ge\int_{2^k}^{2^{k+1}}\int_{\frac{\log t}{\log 2}-1}^{k}\frac{h_1(2^x)}{2^{x(1+\delta)}}\chi_{\mathcal{A}_1}(t)\mathrm{d}x\mathrm{d}t+\int_{2^k}^{2^{k+1}}\int_{k}^{\frac{\log t}{\log 2}}\frac{h_1(2^x)}{2^{x(1+\delta)}}\chi_{\mathcal{A}_1}(t)\mathrm{d}x\mathrm{d}t$$

$$=\int_{2^k}^{2^{k+1}}\int_{\frac{\log t}{\log 2}-1}^{\frac{\log t}{\log 2}}\frac{h_1(2^x)}{2^{x(1+\delta)}}\chi_{\mathcal{A}_1}(t)\mathrm{d}x\mathrm{d}t.$$

Now, we may use the fact that h_1 is a monotonic function having polynomial growth, and simplify the above calculation as follows:

$$\begin{aligned}\int_{k-1}^{k+1}\frac{m^{\#}(2^x)}{2^{\delta x}}\mathrm{d}x &\gg h_1(2^k)\int_{2^k}^{2^{k+1}}\int_{\frac{\log t}{\log 2}-1}^{\frac{\log t}{\log 2}}\frac{\mathrm{d}x}{2^{x(1+\delta)}}\chi_{\mathcal{A}_1}(t)\mathrm{d}t\\
&= \frac{h_1(2^k)}{\log 2}\int_{2^k}^{2^{k+1}}\left(2^{-\left(\frac{\log t}{\log 2}-1\right)(1+\delta)}-2^{-\frac{\log t}{\log 2}(1+\delta)}\right)\chi_{\mathcal{A}_1}(t)\mathrm{d}t\\
&= \frac{h_1(2^k)}{\log 2}\int_{2^k}^{2^{k+1}}\frac{2^{1+\delta}-1}{t^{1+\delta}}\chi_{\mathcal{A}_1}(t)\mathrm{d}t \geq \frac{h_1(2^k)}{2^{(k+1)(\delta+1)}}\mu(\mathcal{A}_1\cap[2^k,2^{k+1}]) \geq \frac{1}{4}\frac{m^{\#}(2^k)}{2^{k\delta}}. \quad \text{(III.8)}\end{aligned}$$

Now using the Claim and (III.8), we get

$$\int_0^{\infty}\frac{m^{\#}(2^x)}{2^{\delta x}}\mathrm{d}x \gg \sum_{k=1}^{\infty}\frac{m^{\#}(2^k)}{2^{k\delta}} = \Omega\left(\frac{1}{\delta}\right).$$

Changing the variable x to $u = 2^x$ in the above inequality gives

$$\frac{1}{\log 2}\int_1^{\infty}\frac{m^{\#}(u)}{u^{1+\delta}}\mathrm{d}u = \Omega\left(\frac{1}{\delta}\right),$$

or

$$\int_1^{\infty}\frac{\mu(\mathcal{A}_j\cap[u,2u])h_j(u)}{u^{2+\delta}}\mathrm{d}u = \Omega\left(\frac{1}{\delta}\right).$$

This proves the theorem. □

Corollary III.1. *Let the conditions given in Theorem III.6 hold. Suppose we have a monotonically increasing positive function h such that*

$$\Delta(x) = O(h(x)), \quad \text{(III.9)}$$

then

$$\mu(\mathcal{A}_j\cap[T,2T]) = \Omega\left(\frac{T^{1+2\sigma_0}}{h^2(T)}\right) \quad \textit{for } j = 1,2. \quad \text{(III.10)}$$

Corollary III.2. *Similar to Corollary III.1, we assume that the conditions in Theorem III.6* *hold. Then we have*

$$\int_{[T,2T]\cap\mathcal{A}_j} \Delta^2(x)\mathrm{d}x = \Omega(T^{2\sigma_0+1}) \quad \textit{for } j = 1,2. \tag{III.11}$$

Proof. This Corollary follows from the proof of Theorem III.8. We shall prove this Corollary for $\mathcal{A}_1$, and the proof for $\mathcal{A}_2$ is similar. Note that in the proof of Theorem III.8, we showed that the integral for $G^+(s)$ is absolutely convergent for $\Re(s) > \sigma_0$ by assuming (III.6). Then we got a contradiction which proves Claim (1) of Theorem III.8. Now we proceed in a similar manner by assuming (III.11) is false. So we have

$$\int_{[T,2T]\cap\mathcal{A}_1} \Delta^2(x)\mathrm{d}x = o(T^{2\sigma_0+1}).$$

So for an arbitrarily small constant ε, we have

$$\begin{aligned}
|G^+(s)| &\leq \int_{\mathcal{A}_1} \frac{g^+(x)\mathrm{d}x}{x^{\sigma+1}} \leq \sum_{k\geq 0} \int_{\mathcal{A}_1\cap[2^k,2^{k+1}]} \frac{\Delta(x)\mathrm{d}x}{x^{\sigma+1}} \\
&\leq \sum_{k\geq 0} \frac{1}{2^{k(\sigma-\sigma_0)}} \left(\int_{\mathcal{A}_1\cap[2^k,2^{k+1}]} \frac{\Delta^2(x)\mathrm{d}x}{x^{2\sigma_0+1}} \right)^{1/2} \\
&\leq c_4(\varepsilon) + \varepsilon \sum_{k\geq k(\varepsilon)} \frac{1}{2^{k(\sigma-\sigma_0)}},
\end{aligned}$$

where $c_4(\varepsilon)$ is a positive constant depending on ε. From this we obtain that $G^+(s)$ is absolutely convergent for $\Re(s) > \sigma_0$. Now onwards the proof is same as that of Theorem III.8. □

III.4 Applications

Now we demonstrate applications of our theorems in the previous section to error terms appearing in two well known asymptotic formulas.

III.4.1 Square Free Divisors

Let $a_n = 2^{\omega(n)}$, where $\omega(n)$ denotes the number of distinct prime factors of n; equivalently, a_n denotes the number of square free divisors of n. We write

$$\sum_{n \le x}^{*} 2^{\omega(n)} = \mathcal{M}(x) + \Delta(x),$$

where

$$\mathcal{M}(x) = \frac{x \log x}{\zeta(2)} + \left(-\frac{2\zeta'(2)}{\zeta^2(2)} + \frac{2\gamma - 1}{\zeta(2)} \right) x,$$

and by a theorem of Gioia and Vaidya [12]

$$\Delta(x) \ll x^{1/2}. \tag{III.12}$$

Under Riemann Hypothesis, Baker [2] has improved the above upper bound to

$$\Delta(x) \ll x^{4/11}.$$

It is easy to see that the Dirichlet series $D(s)$ has the following form:

$$D(s) = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}.$$

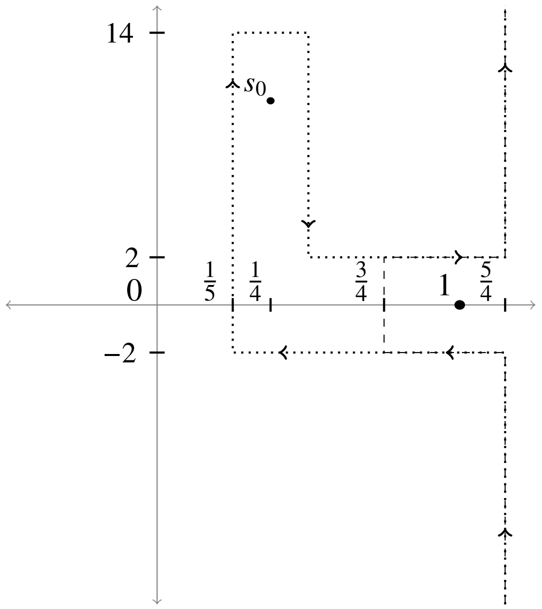

Figure III.1: Contours for square-free divisors.

Let $A(s)$ be the Mellin transform of $\Delta(x)$ at s, and let s_0 be the zero of $\zeta(2s)$ with least positive imaginary part:

$$2s_0 = \frac{1}{2} + i14.134\ldots. \tag{III.13}$$

We define a contour $\mathscr{C}^{(1)}$ as union of the following five lines:

$$\mathscr{C}^{(1)} := \left(\frac{5}{4} - i\infty,\ \frac{5}{4} - i2\right] \cup \left[\frac{5}{4} - i2,\ \frac{3}{4} - i2\right] \cup \left[\frac{3}{4} - i2,\ \frac{3}{4} + i2\right] \cup \left[\frac{3}{4} + i2,\ \frac{5}{4} + i2\right] \cup \left[\frac{5}{4} + i2,\ \frac{5}{4} + i\infty\right)$$

The contour $\mathscr{C}^{(1)}$ is represented by 'dashed' lines in Figure III.1. By Theorem II.3, we have

$$A(s) = \int_1^\infty \frac{\Delta(x)}{x^{s+1}}\mathrm{d}x = \frac{1}{2\pi i}\int_{\mathscr{C}^{(1)}} \frac{D(\eta)}{\eta(s-\eta)}\mathrm{d}\eta.$$

Now, we shift the contour $\mathscr{C}^{(1)}$ to form a new contour $\mathscr{C}^{(2)}$, so that

$$1,\ s_0,\ l\left(\frac{1}{4},\infty\right)$$

lie to the right of $\mathscr{C}^{(2)}$ and no other pole of $D(s)$ lie to the right of this contour. We have represented the contour $\mathscr{C}^{(2)}$ by dotted lines in Figure III.1.

Since s_0 is a pole of $D(s)$ and is on the right side of $\mathscr{C}^{(1)}$, we have

$$A(s) = \frac{1}{2\pi i}\int_{\mathscr{C}^{(2)}} \frac{D(\eta)}{\eta(s-\eta)}\mathrm{d}\eta + \operatorname*{Res}_{\eta=s_0}\left(\frac{D(\eta)}{\eta(s-\eta)}\right).$$

From the above formula, we may compute the following limits:

$$\lambda_1 := \lim_{\sigma\searrow 0}\sigma|A(\sigma+s_0)| = |s_0|^{-1}\left|\operatorname*{Res}_{\eta=s_0}D(\eta)\right| > 0$$

and

$$\lim_{\sigma\searrow 0}\sigma A(\sigma+1/4) = 0.$$

For a fixed $\epsilon_0 > 0$,

$$\mathcal{A}_1 = \left\{x : \Delta(x) > (\lambda_1-\epsilon_0)x^{1/4}\right\}$$
$$\text{and}\quad \mathcal{A}_2 = \left\{x : \Delta(x) < (-\lambda_1+\epsilon_0)x^{1/4}\right\}.$$

Using Corollary III.1 and (III.12), we get

$$\mu\left(\mathcal{A}_j\cap[T,2T]\right) = \Omega\left(T^{1/2}\right)\ \text{for } j=1,2. \tag{III.14}$$

Under Riemann Hypothesis, we may argue similarly as in Proposition V.4 and show that

$$\int_T^{2T} \Delta^2(x) \ll T^{3/2+\epsilon} \text{ for any } \epsilon > 0.$$

The above upper bound and Theorem III.6 give

$$\mu\left(\mathcal{A}_j \cap [T, 2T]\right) = \Omega\left(T^{1-\epsilon}\right), \text{ for } j = 1, 2 \text{ and for any } \epsilon > 0. \qquad \text{(III.15)}$$

III.4.2 The Prime Number Theorem Error

Consider the error term in the Prime Number Theorem:

$$\Delta(x) = \sum_{n \le x}^{*} \Lambda(n) - x.$$

Let

$$\lambda_2 = |2s_0|^{-1},$$

where $2s_0$ is the first nontrivial zero of $\zeta(s)$ as in (III.13). We shall apply Corollary III.1 to prove the following proposition.

Theorem III.9. *We write*

$$\mathcal{A}_1 = \left\{x : \Delta(x) > (\lambda_2 - \epsilon_0)x^{1/2}\right\}$$
$$\text{and} \quad \mathcal{A}_2 = \left\{x : \Delta(x) < (-\lambda_2 + \epsilon_0)x^{1/2}\right\},$$

for a fixed ϵ_0 *such that* $0 < \epsilon_0 < \lambda_2$. *Then*

$$\mu\left(\mathcal{A}_j \cap [T, 2T]\right) = \Omega\left(T^{1-\epsilon}\right), \text{ for } j = 1, 2 \text{ and for any } \epsilon > 0.$$

Proof. Here we apply Corollary III.1 in a similar way as in the previous application, so we shall skip the details.

The Riemann Hypothesis, Theorem III.5 and Theorem PNT** give

$$\mu\left(\mathcal{A}_j \cap [T, 2T]\right) = \Omega\left(\frac{T}{\log^4 T}\right) \text{ for } j = 1, 2;$$

this implies the proposition. But if the Riemann Hypothesis is false, then there exists a constant $\mathfrak{a}$, with $1/2 < \mathfrak{a} \leq 1$, such that

$$\mathfrak{a} = \sup\{\sigma : \zeta(\sigma + it) = 0\}.$$

Using Perron summation formula, we may show that

$$\Delta(x) \ll x^{\mathfrak{a}+\epsilon},$$

for any $\epsilon > 0$. Also for any arbitrarily small δ, we have $\mathfrak{a} - \delta < \sigma' < \mathfrak{a}$ such that $\zeta(\sigma' + it') = 0$ for some real number t'. If $\lambda'' := |\sigma' + it'|^{-1}$, then by Corollary III.1 we get

$$\begin{aligned} &\mu\left(\left\{x \in [T, 2T] : \Delta(x) > (\lambda''/2)x^{\sigma'}\right\}\right) = \Omega\left(T^{1-2\delta-2\epsilon}\right) \\ \text{and} \quad &\mu\left(\left\{x \in [T, 2T] : \Delta(x) < -(\lambda''/2)x^{\sigma'}\right\}\right) = \Omega\left(T^{1-2\delta-2\epsilon}\right). \end{aligned}$$

As δ and ϵ are arbitrarily small and $\sigma' > 1/2$, the above Ω bounds imply the proposition.

□

Remark III.2. *Results similar to Theorem III.9 can be obtained for error terms in asymptotic formulas for partial sums of Mobius function and for partial sums of the indicator* *function of square-free numbers.*

Remark III.3. *In Section III.4.1 and III.4.2, we saw that* $\mu(\mathcal{A}_j)$ *are large. Now suppose that* $\mu(\mathcal{A}_1 \cup \mathcal{A}_2)$ *is large, then what can we say about the individual sizes of* $\mathcal{A}_j$*? We may guess that* $\mu(\mathcal{A}_1)$ *and* $\mu(\mathcal{A}_2)$ *are both large and almost equal. But this may be very difficult to prove. In the next chapter, we shall show that if* $\mu(\mathcal{A}_1 \cup \mathcal{A}_2)$ *is large, then both* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are nonempty.*

[IV] Influence Of Measure

In this chapter, we study the influence of measure of the set where Ω-result holds, on its possible improvements. The following proposition is an interesting application of the main theorem (Theorem IV.3) of this chapter.

Let $\Delta(x)$ denotes the error term appearing in the assymptotic formula for average order of non-isomorphic abelian groups:

$$\Delta(x) = \sum_{n \leq x}^{*} a_n - \sum_{k=1}^{6} \Big(\prod_{j \neq k} \zeta(j/k) \Big) x^{1/k}, \tag{IV.1}$$

where a_n denotes the number of non-isomophic abelian groups of order n. One would expect that

$$\Delta(x) = O\left(x^{1/6+\epsilon}\right) \quad \text{for any } \epsilon > 0$$

(see Section IV.3.2 for more details), so an analogus $\Omega_{\pm}$ bound for $\Delta(x)$ is also expected. The proposition below gives a sufficient condition to obtain such an $\Omega_{\pm}$ bound.

Theorem IV.1. *Let δ be such that $0 < \delta < 1/42$, and $\Delta(x)$ be as in (IV.1). Then either*

$$\int_{T}^{2T} \Delta^4(x) \mathrm{d}x = \Omega(T^{5/3+\delta}),$$

or

$$\Delta(x) = \Omega_{\pm}(x^{1/6-\delta}).$$

It may be conjectured that

$$\int_T^{2T} \Delta^4(x)\mathrm{d}x = O(T^{5/3+\epsilon})$$

for any $\epsilon > 0$. By the above proposition, this conjecture implies

$$\Delta(x) = \Omega_{\pm}(x^{1/6-\epsilon}) \text{ for any } \epsilon > 0.$$

We begin by assuming the conditions and notations given in Assumptions II.1. Further we have the following notations for this chapter.

Notations. *For a real valued and non-negative function f, we denote*

$$\mathcal{A}(f(x)) := \{x \geq 1 : |\Delta(x)| > f(x)\}.$$

IV.1 Refining Omega Result from Measure

We define an **X**-Set as follows:

Definition IV.1. *An infinite discrete subset $\mathcal{S}$ of non-negative real numbers is called an **X**-Set.*

Now we hypothesize a situation when there is a lower bound estimate for the second moment of the error term.

Assumptions IV.1. *Let $\mathcal{S}$ be an* ***X****-Set and let $\alpha(T)$ be a real valued positive bounded function such that*

$$0 \leq \alpha(T) < M < \infty$$

for some constant M. We shall denote $\alpha(T)$ by α throughout this section. Let h_0 be a positive monotonic function. For a fixed T and for a fixed constant $c_5 > 0$, we write

$$\mathcal{A}_T := [T/2, T] \cap \mathcal{A}\left(c_5 x^{\alpha}\right).$$

For all $T \in \mathcal{S}$ and for constants $c_6,\ c_7 > 0$, we assume that the following three conditions hold:

(i)

$$\int_{\mathcal{A}_T} \frac{\Delta^2(x)}{x^{2\alpha+1}} \mathrm{d}x > c_6,$$

(ii)

$$\mu(\mathcal{A}_T) < c_7 h_0(T), \quad \textit{and}$$

(iii) the function

$$x^{\alpha+1/2} h_0^{-1/2}(x)$$

is monotonically increasing for $x \in [T/2, T]$.

Note that the first assumption indicates an Ω-estimate. The next two assumptions indicate that the measure of the set on which the Ω estimate holds is not 'too big'.

Proposition IV.1. *Suppose there exists an* ***X****-Set $\mathcal{S}$ having properties as described in*

Assumptions IV.1. Let the constant c_8 be given by

$$c_8 := \sqrt{\frac{c_6}{2^{2M+1}c_7}}.$$

Then there exists a T_0 such that for all $T > T_0$ and $T \in \mathcal{S}$, we have

$$|\Delta(x)| > c_8 x^{\alpha+1/2} h_0^{-1/2}(x)$$

for some $x \in [T/2, T]$.

In particular

$$\Delta(x) = \Omega(x^{\alpha+1/2} h_0^{-1/2}(x)).$$

Proof. If the statement of the above proposition is not true, then for all $x \in [T/2, T]$ we have

$$\Delta(x) \leq c_8 x^{\alpha+1/2} h_0^{-1/2}(x).$$

From this, we may derive an upper bound for second moment of $\Delta(x)$:

$$\int_{\mathcal{A}_T} \frac{\Delta^2(x)}{x^{2\alpha+1}} \mathrm{d}x \leq \frac{c_8^2 T^{2\alpha+1} \mu(\mathcal{A}_T)}{h_0(T)(T/2)^{2\alpha+1}} \leq c_8^2 2^{2M+1} c_7 \leq c_6.$$

This bound contradicts (i) of Assumptions IV.1, which proves the proposition. □

The above proposition will be used in the next chapter to obtain a result on the error term appearing in the asymptotic formula for $\sum_{n \leq x}^{*} |\tau(n, \theta)|^2$.

IV.2 Omega Plus-Minus Result from Measure

In this section, we prove an $\Omega_\pm$ result for $\Delta(x)$ when $\mu(A_T)$ is big. We formalize the conditions in the following assumptions.

Assumptions IV.2. *Suppose Assumptions II.1 holds. Let l be an integer such that*

$$l > \max(\sigma_2, 1),$$

and let $\alpha_1(u)$ be a monotonic function satisfying

$$0 < \alpha_1(u) \leq \sigma_1.$$

We also assume that $D(s)$ has no pole for $\Re(s) \geq \sigma_1$ except for the poles in $\mathcal{P}$.
Let $\mathcal{S}$ be an $\boldsymbol{X}$-Set such that for all $T \in \mathcal{S}$

$D(\sigma + it)$ is holomorphic for $\alpha_1(T) \leq \sigma \leq \sigma_1$ and $|t| \leq T^{2l}$ and there exists a constant constant $c_9 > 0$ such that

$$|D(\sigma + it)| \leq c_9(|t| + 1)^{l-1}.$$

Assumptions IV.3. *Suppose Assumptions II.1 holds. Let α_1 and l be as in Assumptions IV.2, and $D(s)$ has no pole for $\Re(s) \geq \sigma_1$ except for the poles in $\mathcal{P}$. Let $\mathcal{S}$ be an $\boldsymbol{X}$-Set such that there exist constants $c_{10}, \epsilon > 0$, $0 < \epsilon < \alpha_1(T)$ for all $T \in \mathcal{S}$, such that the following conditional statement holds.*

For all $T \in \mathcal{S}$, if $D(\sigma + it)$ has no pole for $\alpha_1(T) - \epsilon < \sigma \leq \sigma_1$ and $|t| \leq 2T^{2l}$, then

$$|D(\sigma + it)| \leq c_{10}(|t| + 1)^{l-1}$$

when $\alpha_1(T) \le \sigma \le \sigma_1$ *and* $|t| \le T^{2l}$.

Assumptions IV.3 says that if $D(s)$ does not have pole in $\alpha_1(T) - \epsilon < \sigma \le \sigma_1$, then it has polynomial growth in a certain region.

Lemma IV.1. *Under the conditions in Assumptions IV.2, we have*

$$\Delta(x) = \frac{1}{2\pi i}\int_{\alpha_1 - iT^{2l}}^{\alpha_1 + iT^{2l}} \frac{D(\eta)x^\eta}{\eta}\mathrm{d}\eta + O(T^{-1})$$

for all $x \in [T/2, 5T/2]$.

Proof. Follows from Theorem II.2. □

Lemma IV.2 (Balasubramanian and Ramachandra [4])**.** *Let* $T \ge 1$, $\delta_0 > 0$ *and* $f(x)$ *be a real-valued integrable function such that*

$$f(x) \ge 0 \quad \text{for } x \in [T - \delta_0 T,\ 2T + \delta_0 T].$$

Then for any $\delta > 0$ *and for a positive integer* l *satisfying* $\delta l \le \delta_0$, *we have*

$$\int_T^{2T} f(x)\mathrm{d}x \le \frac{1}{(\delta T)^l}\underbrace{\int_0^{\delta T}\cdots\int_0^{\delta T}}_{l \text{ times}}\int_{T - \sum_1^l y_i}^{2T + \sum_1^l y_i} f(x)\mathrm{d}x\, \mathrm{d}y_1 \ldots \mathrm{d}y_l.$$

Proof. For $0 \le y_i \le \delta T$, $i = 1, 2, ..., l$

$$\int_T^{2T} f(x)\mathrm{d}x \le \int_{T - \sum_1^l y_i}^{2T + \sum_1^l y_i} f(x)\mathrm{d}x,$$

as $f(x) \ge 0$ in

$$\left[T - \sum_1^l y_i, 2T + \sum_1^l y_i\right] \subseteq [T - \delta_0 T, 2T + \delta_0 T].$$

This gives

$$\frac{1}{(\delta T)^l}\underbrace{\int_0^{\delta T}\cdots\int_0^{\delta T}}_{l\text{ times}}\int_{T-\sum_1^l y_i}^{2T+\sum_1^l y_i} f(x)\mathrm{d}x\ \mathrm{d}y_1\ldots\mathrm{d}y_l$$
$$\geq \frac{1}{(\delta T)^l}\underbrace{\int_0^{\delta T}\cdots\int_0^{\delta T}}_{l\text{ times}}\int_T^{2T} f(x)\mathrm{d}x\ \mathrm{d}y_1\ldots\mathrm{d}y_l = \int_T^{2T} f(x)\mathrm{d}x.$$

□

The next theorem shows that if $\Delta(x)$ does not change sign then the set on which Ω-estimate holds can not be 'too big'.

Theorem IV.2. *Let $\mathcal{S}$ be an **X**-Set for which Assumptions IV.2 holds. Let $h_1(u)$ be a monotonically increasing function such that $h_1(u) \to \infty$. Let $\alpha_2(u)$ be a bounded positive monotonic function such that*

$$0 < \alpha_1(u) < \alpha_2(u) \leq \sigma_1.$$

Then there exists a T_0 such that for $T \in \mathcal{S}$ and $T \geq T_0$, if $\Delta(x)$ does not change sign on $\mathcal{A}(h_1(x)) \cap [T/2, 5T/2]$, then

$$\mu(\mathcal{A}(x^{\alpha_2}) \cap [T, 2T]) \leq 4h_1(5T/2)T^{1-\alpha_2} + O(1 + T^{1-\alpha_2+\alpha_1}),$$

where α_1 and α_2 denote $\alpha_1(T)$ and $\alpha_2(T)$ respectively.

Proof. We may easily verify that

$$\mu(\mathcal{A}(x^{\alpha_2}) \cap [T, 2T]) \leq \int_T^{2T} \frac{|\Delta(x)|}{x^{\alpha_2}}\mathrm{d}x.$$

Using Lemma IV.2 on the above inequality, we get

$$\mu(\mathcal{A}(x^{\alpha_2}) \cap [T, 2T]) \leq \frac{1}{(\delta T)^l} \underbrace{\int_0^{\delta T} \cdots \int_0^{\delta T}}_{l \text{ times}} \int_{T-\sum_1^l y_i}^{2T+\sum_1^l y_i} \frac{|\Delta(x)|}{x^{\alpha_2}} \mathrm{d}x \, \mathrm{d}y_1 \ldots \mathrm{d}y_l,$$

where $\delta = \frac{1}{2l}$.

Let χ denote the characteristic function of the complement of $\mathcal{A}(h_1(x))$:

$$\chi(x) = \begin{cases} 1 & \text{if } x \notin \mathcal{A}(h_1(x)), \\ 0 & \text{if } x \in \mathcal{A}(h_1(x)). \end{cases}$$

For $T \geq 2T_0$, $\Delta(x)$ does not change sign on

$$\left[T - \sum_1^l y_i, \; 2T + \sum_1^l y_i\right] \cap \mathcal{A}(h_1(x)),$$

as $0 \leq y_i \leq \delta T$ for all $i = 1, ..., l$. So we can write the above inequality as

$$\begin{aligned} \mu(\mathcal{A}(x^{\alpha_2}) \cap [T, 2T]) \leq{}& \frac{2}{(\delta T)^l} \underbrace{\int_0^{\delta T} \cdots \int_0^{\delta T}}_{l \text{ times}} \int_{T-\sum_1^l y_i}^{2T+\sum_1^l y_i} \frac{|\Delta(x)|}{x^{\alpha_2}} \chi(x) \mathrm{d}x \, \mathrm{d}y_1 \ldots \mathrm{d}y_l \\ &+ \frac{1}{(\delta T)^l} \left| \underbrace{\int_0^{\delta T} \cdots \int_0^{\delta T}}_{l \text{ times}} \int_{T-\sum_1^l y_i}^{2T+\sum_1^l y_i} \frac{\Delta(x)}{x^{\alpha_2}} \mathrm{d}x \, \mathrm{d}y_1 \ldots \mathrm{d}y_l \right|. \qquad \text{(IV.2)} \end{aligned}$$

Since $x \notin \mathcal{A}(h_1(x))$ implies $|\Delta(x)| \leq h_1(x)$, we get

$$\begin{aligned} &\frac{2}{(\delta T)^l} \underbrace{\int_0^{\delta T} \cdots \int_0^{\delta T}}_{l \text{ times}} \int_{T-\sum_1^l y_i}^{2T+\sum_1^l y_i} \frac{|\Delta(x)|}{x^{\alpha_2}} \chi(x) \mathrm{d}x \, \mathrm{d}y_1 \ldots \mathrm{d}y_l \\ &\quad \leq 4 h_1(5T/2) T^{1-\alpha_2}. \qquad \text{(IV.3)} \end{aligned}$$

We use the integral expression for $\Delta(x)$ as given in Lemma IV.1, and get

$$
\begin{aligned}
&\frac{1}{(\delta T)^l}\left|\underbrace{\int_0^{\delta T}\cdots\int_0^{\delta T}}_{l\text{ times}}\int_{T-\sum_1^l y_i}^{2T+\sum_1^l y_i}\frac{\Delta(x)}{x^{\alpha_2}}\mathrm{d}x\ \mathrm{d}y_1\ldots\mathrm{d}y_l\right| \\
&\leq \frac{1}{(\delta T)^l}\left|\underbrace{\int_0^{\delta T}\cdots\int_0^{\delta T}}_{l\text{ times}}\int_{T-\sum_1^l y_i}^{2T+\sum_1^l y_i}\int_{\alpha_1-iT^{2l}}^{\alpha_1+iT^{2l}}\frac{D(\eta)x^{\eta-\alpha_2}}{\eta}\mathrm{d}\eta\ \mathrm{d}x\ \mathrm{d}y_1\ldots\mathrm{d}y_l\right| + O(1) \\
&\ll 1+\frac{1}{(\delta T)^l}\left|\int_{\alpha_1-iT^{2l}}^{\alpha_1+iT^{2l}}\frac{D(\eta)}{\eta}\underbrace{\int_0^{\delta T}\cdots\int_0^{\delta T}}_{l\text{ times}}\int_{T-\sum_1^l y_i}^{2T+\sum_1^l y_i}x^{\eta-\alpha_2}\mathrm{d}x\ \mathrm{d}y_1\ldots\mathrm{d}y_l\ \mathrm{d}\eta\right| \\
&\ll 1+\frac{1}{(\delta T)^l}\left|\int_{\alpha_1-iT^{2l}}^{\alpha_1+iT^{2l}}\frac{D(\eta)(2T+l\delta T)^{\eta-\alpha_2+l+1}}{\eta\prod_{j=1}^{l+1}(\eta-\alpha_2+j)}\mathrm{d}\eta\right| \\
&\ll 1+\frac{T^{\alpha_1-\alpha_2+l+1}}{(\delta T)^l}\int_{-T^{2l}}^{T^{2l}}\frac{(1+|t|)^{l-1}}{(1+|t|)^{l+2}}\mathrm{d}t \ll 1+T^{1-\alpha_2+\alpha_1}.
\end{aligned}
\tag{IV.4}
$$

The theorem follows from (IV.2), (IV.3) and (IV.4). □

Theorem IV.3. *Let Assumptions II.1 holds. Let $\alpha_1(u), \alpha_2(u), \sigma_1, h_1(u)$ be as in Theorem IV.2, and*

$$\nu := \lim_{u\to\infty}\alpha_1(u).$$

We also have

$$\frac{h_1(u)}{u^{\alpha_1(u)}} \to \infty \text{ as } u \to \infty.$$

*Further, if there exists an **X**-Set $\mathcal{S}$ such that for all $T \in \mathcal{S}$*

$$\mu(\mathcal{A}(x^{\alpha_2}) \cap [T, 2T]) > 5h_1(5T/2)T^{1-\alpha_2},$$

where $\alpha_2 = \alpha_2(T)$, then the following statements hold.

(i) Suppose $\mathcal{S}$ satisfy Assumptions IV.2. Then there exists a $T_0 > 0$ such that $\Delta(x)$ changes sign in $[T/2, 5T/2] \cap \mathcal{A}(h_1(x))$ for all $T \in \mathcal{S}$ and $T \geq T_0$. In particular,

$$\Delta(x) = \Omega_{\pm}(h_1(x)).$$

(ii) Suppose $\mathcal{S}$ satisfy Assumptions IV.3 and let ϵ be as in that Assumptions. We also assume that $D(s)$ does not have a real pole in $[\alpha_1(T) - \epsilon, \infty) - \mathcal{P}$ for all $T \in \mathcal{S}$ and $\mathcal{P}$ be as in Assumptions II.1. Then for any $\epsilon' > \epsilon$, we have

$$\Delta(x) = \Omega_{\pm}(x^{\nu-\epsilon'}).$$

Proof. If $\mathcal{S}$ satisfy Assumptions IV.2, then (i) follows from Theorem IV.2. So let Assumptions IV.3 holds for $\mathcal{S}$.

If $D(\sigma + it)$ has no pole when $\alpha_1(T) - \epsilon < \sigma \leq \sigma_1, \ |t| \leq 2T^{2l}$ except for finitely many $T \in S$, then we may reconstruct our **X**-Set by removing those finitely many T and apply Theorem IV.2 to get the required result. Otherwise, there are infinitely many $T \in S$ such that $D(\sigma + it)$ has a pole $\sigma_T + it_T$ with $\alpha_1(T) - \epsilon < \sigma_T \leq \sigma_1, \ |t_T| \leq 2T^{2l}$. By our assumptions in (ii), $\sigma_T + it_T$ is not a real pole. So Theorem III.3 gives

$$\Delta(x) = \Omega_{\pm}\left(x^{\alpha_1(T)-\epsilon}\right)$$

for T in an **X**-Set. This in particular implies

$$\Delta(x) = \Omega_{\pm}(x^{\nu-\epsilon'}) \ \text{ for any } \ \epsilon' > \epsilon.$$

□

IV.3 Applications

Now we shall see two examples demonstrating applications of Theorem IV.3.

IV.3.1 Divisors

Let $d(n)$ denote the number of divisors of n:

$$d(n) = \sum_{d|n} 1.$$

Dirichlet [18, Theorem 320] showed that

$$\sum_{n\leq x}^{*} \tau(n) = x\log(x) + (2\gamma - 1)x + \Delta(x),$$

where γ is the Euler constant and

$$\Delta(x) = O(\sqrt{x}).$$

Latest result on $\Delta(x)$ is due to Huxley [20], which is

$$\Delta(x) = O(x^{131/416}).$$

On the other hand, Hardy [15] showed that

$$\begin{aligned}\Delta(x) &= \Omega_+((x\log x)^{1/4}\log\log x),\\ &= \Omega_-(x^{1/4}).\end{aligned}$$

There are many improvements of Hardy's result. Some notable results are due to K. Corrádi and I. Kátai [7], J. L. Hafner [13], and K. Sounderarajan [36]. Below, we shall show that $\Delta(x)$ is $\Omega_{\pm}(x^{1/4})$ as a consequence of Theorem IV.3 and results of Ivić and Tsang (see below). Moreover, we shall how that such fluctuations occur in $[T, 2T]$ for every sufficiently large T.

Ivić [21] proved that for a positive constant c_{11},

$$\int_T^{2T} \Delta^2(x)\mathrm{d}x \sim c_{11}T^{3/2}.$$

A similar result for fourth moment of $\Delta(x)$ was proved by Tsang [39]:

$$\int_T^{2T} \Delta^4(x)\mathrm{d}x \sim c_{12}T^2,$$

for a positive constant c_{12}. Let $\mathcal{A}$ denote the following set:

$$\mathcal{A} := \left\{ x : |\Delta(x)| > \frac{c_{11}x^{1/4}}{6} \right\}.$$

For sufficiently large T, using the result of Ivić [21], we get

$$\begin{aligned}
\int_{[T,2T]\cap\mathcal{A}} \frac{\Delta^2(x)}{x^{3/2}}\mathrm{d}x &= \int_T^{2T} \frac{\Delta(x)^2}{x^{3/2}}\mathrm{d}x - \int_{[T,2T]\cap\mathcal{A}^c} \frac{\Delta^2(x)}{x^{3/2}}\mathrm{d}x \\
&\geq \frac{1}{4T^{3/2}} \int_T^{2T} \Delta^2(x)\mathrm{d}x - \frac{c_{11}}{6} \\
&\geq \frac{c_{11}}{5} - \frac{c_{11}}{6} \geq \frac{c_{11}}{30}.
\end{aligned}$$

Using Cauchy-Schwarz inequality and the result due to Tsang [39] we get

$$\begin{aligned}\int_{[T,2T]\cap\mathcal{A}} \frac{\Delta^2(x)}{x^{3/2}}\mathrm{d}x &\le \left(\int_{[T,2T]\cap\mathcal{A}} \frac{\Delta^4(x)}{x^2}\mathrm{d}x\right)^{1/2} \left(\int_{[T,2T]\cap\mathcal{A}} \frac{1}{x}\mathrm{d}x\right)^{1/2}\\ &\le \left(\frac{c_{12}\mu([T,2T]\cap\mathcal{A})}{T}\right)^{1/2}.\end{aligned}$$

The above lower and upper bounds on second moment of Δ gives the following lower bound for measure of $\mathcal{A}$:

$$\mu([T,2T]\cap\mathcal{A}) > \frac{c_{11}^2}{901c_{12}}T,$$

for some $T \ge T_0$. Now, Theorem IV.3 applies with the following choices:

$$\alpha_1(T) = 1/5, \quad \alpha_2(T) = 1/4, \quad h_1(T) = \frac{c_{11}^2}{9000c_{12}}T^{1/4}.$$

Finally using Theorem IV.3, we get that for all $T \ge T_0$ there exists $x_1, x_2 \in [T, 2T]$ such that

$$\Delta(x_1) > h_1(x_1) \text{ and } \Delta(x_2) < -h_1(x_2).$$

In particular we get

$$\Delta(x) = \Omega_{\pm}(x^{1/4}).$$

IV.3.2 Average order of Non-Isomorphic abelian Groups

Let a_n denote the number of non-isomorphic abelian groups of order n. The Dirichlet series $D(s)$ is given by

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{k=1}^{\infty} \zeta(ks), \quad \Re(s) > 1.$$

The meromorphic continuation of $D(s)$ has poles at $1/k$, for all positive integer $k \geq 1$. Let the main term $\mathcal{M}(x)$ be

$$\mathcal{M}(x) = \sum_{k=1}^{6} \Big(\prod_{j \neq k} \zeta(j/k) \Big) x^{1/k},$$

and the error term $\Delta(x)$ be

$$\sum_{n \leq x}^{*} a_n - \mathcal{M}(x).$$

Balasubramanian and Ramachandra [4] proved that

$$\int_T^{2T} \Delta^2(x) \mathrm{d}x = \Omega(T^{4/3} \log T), \text{ and } \Delta(x) = \Omega_{\pm}(x^{92/1221}).$$

Sankaranarayanan and Srinivas [35] improved the $\Omega_{\pm}$ bound to

$$\Delta(x) = \Omega_{\pm} \left(x^{1/10} \exp \left(c \sqrt{\log x} \right) \right)$$

for some constant $c > 0$. An upper bound for the second moment of $\Delta(x)$ was first given by Ivić [22], and then improved by Heath-Brown [19] to

$$\int_T^{2T} \Delta^2(x) \mathrm{d}x \ll T^{4/3} (\log T)^{89}.$$

This bound of Heath-Brown is best possible in terms of power of T. But for the fourth moment, the similar statement

$$\int_T^{2T} \Delta^4(x)\mathrm{d}x \ll T^{5/3}(\log T)^C,$$

which is best possible in terms of power of T, is an open problem. Another open problem is to show that

$$\Delta(x) = \Omega_\pm(x^{1/6-\delta}) \text{ for any } \delta > 0.$$

For $0 < \delta < 1/42$, we have stated in Theorem IV.1 that either

$$\int_T^{2T} \Delta^4(x)\mathrm{d}x = \Omega(T^{5/3+\delta}) \text{ or } \Delta(x) = \Omega_\pm(x^{1/6-\delta}).$$

Below, we present a proof of this proposition.

Proof of Theorem IV.1. If the first statement is false, then we have

$$\int_T^{2T} \Delta^4(x)\mathrm{d}x \le c_{13}T^{5/3+\delta},$$

for some constant c_{13} depending on δ and for all $T \ge T_0$. Let $\mathcal{A}$ be defined by:

$$\mathcal{A} = \{x : |\Delta(x)| > c_{14}x^{1/6}\}, \quad c_{14} > 0.$$

By the result of Balasubramanian and Ramachandra [4], we have an **X**-Set $\mathcal{S}$, such that

$$\int_{[T,2T]\cap\mathcal{A}} \Delta^2(x)\mathrm{d}x \ge c_{15}T^{4/3}(\log T)$$

for $T \in S$. Using Cauchy-Schwartz inequality, we get

$$c_{15}T^{4/3}(\log T) \leq \int_{[T,2T]\cap\mathcal{A}} \Delta^2(x)\mathrm{d}x \leq \left(\int_T^{2T} \Delta^4(x)\mathrm{d}x\right)^{1/2} (\mu(\mathcal{A}\cap[T,2T]))^{1/2}$$
$$\leq c_{13}^{1/2}T^{5/6+\delta/2}(\mu(\mathcal{A}\cap[T,2T]))^{1/2}.$$

This gives, for a suitable positive constant c_{16},

$$\mu(\mathcal{A}\cap[T,2T]) \geq c_{16}T^{1-\delta}(\log T)^2.$$

Now we use Theorem IV.3, (i), with

$$\alpha_2 = \frac{1}{6}, \quad \alpha_1 = \frac{13}{84} - \frac{\delta}{2}, \quad \text{and} \quad h_1(T) = T^{1/6-\delta}.$$

So we get

$$\Delta(x) = \Omega_{\pm}(x^{1/6-\delta}).$$

This completes the proof. □

[V] The Twisted Divisor Function

Recall that in Chapter I, we have defined the twisted divisor function $\tau(n,\theta)$ as follows:

$$\tau(n,\theta) = \sum_{d|n} d^{i\theta}, \quad \text{for } \theta \in \mathbb{R} - \{0\},\ n \in \mathbb{N}.$$

We also have stated the following asymptotic formula:

$$\sideset{}{^*}\sum_{n\le x} |\tau(n,\theta)|^2 = \omega_1(\theta)x\log x + \omega_2(\theta)x\cos(\theta\log x) + \omega_3(\theta)x + \Delta(x),$$

where $\omega_i(\theta)$s are explicit constants depending only on θ and

$$\Delta(x) = O_\theta(x^{1/2}\log^6 x).$$

In this chapter, we give a proof of this formula (see Section V.2, Theorem V.1). In Section V.3, we use Theorem III.6 to obtain some measure theoretic $\Omega_\pm$ results. Further, we obtain an Ω bound for the second moment of $\Delta(x)$ in Section V.4 by adopting a technique due to Balasubramanian, Ramachandra and Subbarao [5]. In the final section, we prove that if the Ω bound obtained in the previous section can not be improved, then

$$\Delta(x) = \Omega(x^{3/8-\epsilon}) \text{ for any } \epsilon > 0.$$

Now we motivate with a brief note on few applications of $\tau(n,\theta)$.

V.1 Applications of $\tau(n,\theta)$

The function $\tau(n,\theta)$ can be used to study various properties related to the distribution of divisors of an integer:

$$\sum_{\substack{d|n \\ a\le \log d\le b}}^{*} 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tau(n,\theta)\frac{e^{-ib\theta}-e^{-ia\theta}}{-i\theta}\mathrm{d}\theta,$$

here $\sum^*$ means that the corresponding contribution to the sum is $\frac{1}{2}$ if $e^a|n$ or $e^b|n$. Below we present two applications.

V.1.1 Clustering of Divisors

The following function measures the clustering of divisors of an integer:

$$W(n,f) := \sum_{d,d'|n} f(\log(d/d')),$$

for some constant $c > 0$ and for a function $f \in L^1(\mathbb{R})$. We assume that f has a Fourier transformation, say $\hat{f}$, and $\hat{f} \in L^1(\mathbb{R})$.

Proposition V.1. *With the above notations:*

$$\sum_{n\le x} W(n,f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\theta)\sum_{n\le x}|\tau(n,\theta)|^2\mathrm{d}\theta.$$

Proof. Note that by the Fourier inversion formula, we get

$$W(n,f) = \sum_{d,d'|n} f(\log(d/d')) = \frac{1}{2\pi}\sum_{d,d'|n}\int_{-\infty}^{\infty}\hat{f}(\theta)\left(\frac{d}{d'}\right)^{i\theta}\mathrm{d}\theta$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\theta)\left(\sum_{d,d'|n}\left(\frac{d}{d'}\right)^{i\theta}\right)\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\theta)|\tau(n,\theta)|^2\mathrm{d}\theta.$$

This implies the proposition. □

Using Proposition V.1 and the formula in (I.3), we may write

$$\sum_{n\le x} W(n,f) = \frac{x\log x}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\theta)\omega_1(\theta)\mathrm{d}\theta + \frac{x}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\theta)(\omega_2(\theta)\cos(\theta\log x) + \omega_3(\theta))\mathrm{d}\theta$$
$$+ \frac{x}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\theta)\Delta(x,\theta)\mathrm{d}\theta.$$

(In the above identity, we denoted $\Delta(x)$ by $\Delta(x,\theta)$.)

This gives that the function $\sum_{n\le x} W(n,f)$ behaves like $x\log x$. Further, if we want to obtain more information on $\sum_{n\le x} W(n,f)$, we may analyzing other terms in the above formula. But now, we skip the details and refer to [14, Chapter 4].

V.1.2 The Multiplication Table Problem

The multiplication table problem asks for an estimate on the order of the growth of $|\mathrm{Mul}(N)|$ as $N \to \infty$, where

$$\mathrm{Mul}(N) := \{1 \le m \le N^2 : m = ab,\ a,b \in \mathbb{Z} \text{ and } 1 \le a,b \le N\}.$$

The initial attempts in this direction are due to Erdős [9]. He used a result of Hardy and Ramanujan [17] (also see [27]) to show

$$|\mathrm{Mul}(N)| \ll \frac{N^2}{(\log N)^{\nu_0}\sqrt{\log\log N}} \text{ as } N \to \infty,$$

and here

$$\nu_0 = 1 - \frac{1+\log\log 2}{\log 2}.$$

Intuitively, the theorem of Hardy and Ramanujan says that most of the positive integers less than x have around $\log\log x$ prime factors; more precisely,

$$\#\left\{n \le x : |\omega(n) - \log\log n| < (\log\log n)^{\frac{1}{2}+\epsilon}\right\} \sim x$$

as $x \to \infty$ and for any $\epsilon > 0$. This gives that most of the positive integers less than N^2 have around $\log\log N$ prime factors, whereas most of the integers in the multiplication table have around $2\omega(n) \approx 2\log\log N$ prime factors. This huristic can be refined to show $|\mathrm{Mul}(N)| = o(N^2)$. Erdős has used this idea to obtain the given upper bound for $|\mathrm{Mul}(N)|$.

The best known bound on the asymptotic growth of $|\mathrm{Mul}(N)|$ is due to Ford [10]:

$$|\mathrm{Mul}(N)| \asymp \frac{N^2}{(\log N)^{\nu_0}(\log\log N)^{3/2}} \text{ as } N \to \infty.$$

To obtain the expected lower bound for $|\mathrm{Mul}(N)|$, Ford first proved that

$$|\mathrm{Mul}(N)| \gg \frac{N^2}{(\log N)^2} \sum_{n \le N^{1/8}} \frac{L(n)}{n}, \text{ where } L(n) := \mu\left(\cup_{d|n}[\log(d/2), \log d]\right).$$

We may also observe that

$$\sum_{n\le N^{1/8}}\frac{L(n)}{n}\ge\frac{\left(\sum_{n\le N^{1/8}}\frac{d(n)}{n}\right)^2}{6\sum_{n\le N^{1/8}}\frac{W(n)}{n}}.$$

Rest of the part in Ford's argument deals with the above sums involving the divisor function $d(n)$ and $W(n) := W\left(n, 1_{[\frac{1}{2},2]}\right)$, where $1_{[\frac{1}{2},2]}$ is the indicator function of the interval $[\frac{1}{2}, 2]$. We skip the details and refer to [11].

V.2 Asymptotic Formula for $\sum^*_{n\le x}|\tau(n,\theta)|^2$

In this section, we shall prove the following asymptotic formula for $\sum^*_{n\le x}|\tau(n,\theta)|^2$.

Theorem V.1 (Theorem 33, [14]). *Let $\theta \neq 0$ be a fixed real number. Then for $x \ge 1$, we have*

$$\sum_{n\le x}^{*}|\tau(n,\theta)|^2=\omega_1(\theta)x\log x+\omega_2(\theta)x\cos(\theta\log x)+\omega_3(\theta)x+O_\theta(x^{1/2}\log^6 x)$$

where $\omega_i(\theta)$s are explicit constants depending only on θ.

Proof. Recall that the corresponding Dirichlet series $D(s)$ has the following meromorphic continuation:

$$D(s)=\sum_{1}^{\infty}\frac{|\tau(n,\theta)|^2}{n^s}=\frac{\zeta^2(s)\zeta(s+i\theta)\zeta(s-i\theta)}{\zeta(2s)},\quad\text{for } s>1.$$

For $x \ge 2$, we denote $\kappa = 1 + \frac{1}{\log x}$ and $T = x + |\theta| + 1$. By Perron's formula

$$\sum_{n\le x}^{*}|\tau(n,\theta)|^2=\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT}D(s)x^s\frac{\mathrm{d}s}{s}+O(x^{\epsilon}).$$

After shifting the line of integration to $\Re(s) = \frac{1}{2}$, we may estimate the contributions from horizontal lines as follows:

$$T^{-1}\int_{\frac{1}{2}}^{1} |D(\sigma \pm iT)| x^{\sigma} \mathrm{d}\sigma \ll T^{-1}\int_{\frac{1}{2}}^{1} T^{1-\sigma+\epsilon} x^{\sigma} \mathrm{d}\sigma \ll x^{\epsilon}.$$

To obtain an asymptotic formula for $\sum_{n\le x}^{*} |\tau(n,\theta)|^2$, we add up the residues from the poles $1, 1 \pm i\theta$ after shifting the line of integration to $\Re(s) = \frac{1}{2}$:

$$\sum_{n\le x}^{*} |\tau(n,\theta)|^2 = \mathcal{M}(x) + O\left(x^{\epsilon} + x^{\frac{1}{2}}\int_{-T}^{T}\left|\frac{\zeta^2(\frac{1}{2}+it)\zeta(\frac{1}{2}+i(t+\theta))\zeta(\frac{1}{2}+i(t-\theta))}{\zeta(1+2it)(\frac{1}{2}+it)}\right| \mathrm{d}t\right),$$

where

$$\mathcal{M}(x) = \omega_1(\theta) x\log x + \omega_2(\theta) x\cos(\theta\log x) + \omega_3(\theta) x.$$

If we write

$$\mathcal{J}(\mathfrak{a},T) := \int_{-T}^{T}\frac{\zeta^4(\frac{1}{2}+i(\mathfrak{a}+t))}{\sqrt{t^2+\frac{1}{4}}}\mathrm{d}t \quad \text{for } \mathfrak{a}, T \in \mathbb{R} \text{ and } T \ge 1,$$

then we have [23, Theorem 5.1]

$$\mathcal{J}(\mathfrak{a},T) \ll_{\mathfrak{a}} \log^5 T. \tag{V.1}$$

To express $\Delta(x)$ in terms of $\mathcal{J}(\mathfrak{a},T)$, observe that

$$\begin{aligned}\Delta(x) &= \sum_{n\le x}^{*} |\tau(n,\theta)|^2 - \mathcal{M}(x)\\ &\ll x^{\epsilon} + x^{\frac{1}{2}}\int_{-T}^{T}\left|\frac{\zeta^2(\frac{1}{2}+it)\zeta(\frac{1}{2}+i(t+\theta))\zeta(\frac{1}{2}+i(t-\theta))}{\zeta(1+i2t)(\frac{1}{2}+it)}\right| \mathrm{d}t\end{aligned}$$

$$\ll x^{\epsilon} + x^{\frac{1}{2}} \log x \int_{-T}^{T} |\zeta^2(\frac{1}{2}+it)\zeta(\frac{1}{2}+i(t+\theta))\zeta(\frac{1}{2}+i(t-\theta))| \frac{\mathrm{d}t}{|\frac{1}{2}+it|}.$$

From (V.1) and using the Cauchy-Schwartz inequality twice, we get

$$\Delta(x) \ll x^{\epsilon} + x^{\frac{1}{2}} \log x \mathcal{J}^{\frac{1}{2}}(0,x)\mathcal{J}^{\frac{1}{4}}(\theta,x)\mathcal{J}^{\frac{1}{4}}(-\theta,x) \ll_{\theta} x^{\frac{1}{2}} \log^6 x,$$

which gives the required result. □

In the following sections, we shall obtain various Ω and $\Omega_{\pm}$ bounds for $\Delta(x)$.

V.3 Oscillations of the Error Term

Here we shall apply results in Chapter III to $\Delta(x)$ and obtain some measure theoretic $\Omega_{\pm}$ results. We begin by defining a contour $\mathscr{C}$ as given in Figure V.1:

$$\begin{aligned}\mathscr{C} = &\left(\frac{5}{4} - i\infty, \frac{5}{4} - i(\theta+1)\right] \cup \left[\frac{5}{4} - i(\theta+1), \frac{3}{4} - i(\theta+1)\right] \\ &\cup \left[\frac{3}{4} - i(\theta+1), \frac{3}{4} + i(\theta+1)\right] \cup \left[\frac{3}{4} + i(\theta+1), \frac{5}{4} + i(\theta+1)\right] \\ &\cup \left[\frac{5}{4} + i(\theta+1), \frac{5}{4} + i\infty\right).\end{aligned}$$

From Theorem II.1, we have

$$\Delta(x) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{D(\eta)x^{\eta}}{\eta} \mathrm{d}\eta.$$

The above identity expresses the Mellin transform $A(s)$ of $\Delta(x)$ as a contour integral involving $D(s)$. Using Theorem II.3, we write

$$A(s) = \int_1^\infty \frac{\Delta(x)}{x^{s+1}}\mathrm{d}x = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{D(\eta)}{\eta(s-\eta)}\mathrm{d}\eta,$$

when s lies right to the contour $\mathscr{C}$. Denote the first nontrivial zero of $\zeta(s)$ with least positive imaginary part by $2s_0$. An approximate value of this point is

$$2s_0 = \frac{1}{2} + i14.134\ldots.$$

Define the contour $\mathscr{C}(s_0)$, as in Figure V.2, such that s_0 and any real number $s \geq 1/4$ lie in the right side of this contour. A meromorphic continuation of $A(s)$ to all s that lies right side of $\mathscr{C}(s_0)$ is given by

$$A(s) = \frac{1}{2\pi i}\int_{\mathscr{C}(s_0)} \frac{D(\eta)x^\eta}{\eta}\mathrm{d}\eta + \frac{\operatorname*{Res}_{\eta=s_0} D(\eta)}{s_0(s-s_0)}. \tag{V.2}$$

From (V.2), we calculate the following two limits:

$$\lambda(\theta) := \lim_{\sigma\searrow 0} \sigma|A(\sigma+s_0)| = |s_0|^{-1}\left|\operatorname*{Res}_{\eta=s_0} D(\eta)\right| > 0 \tag{V.3}$$

and

$$\lim_{\sigma\searrow 0} \sigma A(\sigma + 1/4) = 0.$$

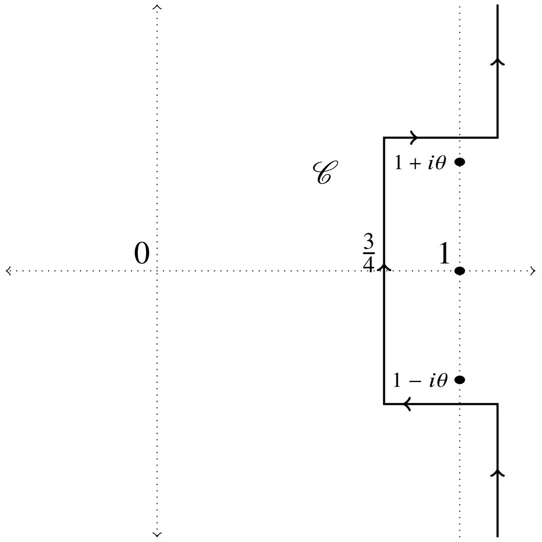

Figure V.1: Contour $\mathscr{C}$, for $D(s) = \sum_{n=1}^{\infty} \frac{|\tau(n,\theta)|^2}{n^s}$.

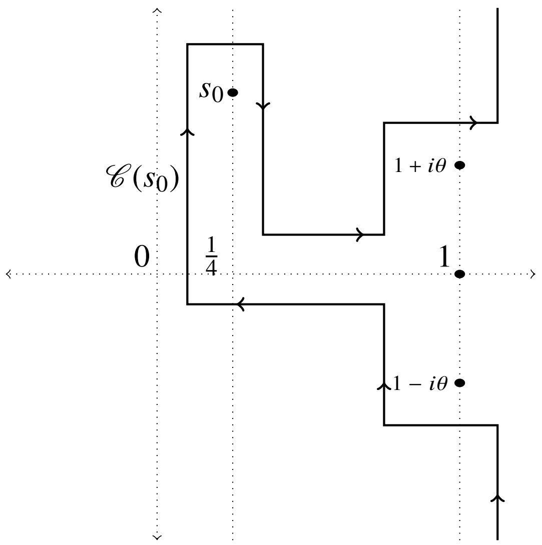

Figure V.2: Contour $\mathscr{C}(s_0)$

For a fixed small enough $\epsilon > 0$, define

$$\mathcal{A}_1 = \left\{x : \Delta(x) > (\lambda(\theta) - \epsilon)x^{1/4}\right\},$$
$$\mathcal{A}_2 = \left\{x : \Delta(x) < (-\lambda(\theta) + \epsilon)x^{1/4}\right\}.$$

Corollary III.1 and Theorem V.1 give

$$\mu\left(\mathcal{A}_j \cap [T, 2T]\right) = \Omega\left(T^{1/2}(\log T)^{-12}\right) \text{ for } j = 1, 2. \tag{V.4}$$

Under Riemann Hypothesis, Theorem III.6 and Proposition V.4 give

$$\mu\left(\mathcal{A}_j \cap [T, 2T]\right) = \Omega\left(T^{3/4-\epsilon}\right) \text{ for } j = 1, 2. \tag{V.5}$$

Note that the above statements in particular show that

$$\Delta(x) = \Omega_{\pm}(x^{1/4}).$$

From Corollary III.2 of Chapter III, we get

$$\int_{\mathcal{A}_j \cap [T, 2T]} \Delta^2(x) \mathrm{d}x = \Omega\left(T^{3/2}\right) \text{ for } j = 1, 2. \tag{V.6}$$

V.4 An Omega Theorem

Recall that (see Theorem V.1)

$$\sum_{n \leq x} |\tau(n, \theta)|^2 = \mathcal{M}(x) + \Delta(x),$$

where the main term

$$\mathcal{M}(x) = \omega_1(\theta)x\log x + \omega_2(\theta)x\cos(\theta\log x) + \omega_3(\theta)x$$

comes from the poles of $D(s)$ at $s = 1, 1+i\theta$ and $s = 1-i\theta$. We may observe from Corollary III.2 that if $D(s)$ has a complex pole at $s_0 = \sigma_0 + it_0$, other than $1+i\theta$ and $1-i\theta$, then

$$\int_T^{2T} \Delta(x)\mathrm{d}x = \Omega(x^{2\sigma_0+1}).$$

By Riemann Hypothesis, the only positive value for σ_0 is $\frac{1}{4}$, which is same as (V.6). In this section, we shall use a technique due to Balasubramanian, Ramachandra and Subbarao [5] to improve this omega bound further. Now we state the main theorem of this section.

Theorem V.2. *For any $c > 0$, there exists $K(c) > 0$ and $T(c) > 0$ such that for all $T \geq T(c)$, we get*

$$\int_T^{\infty} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} e^{-2x/y}\mathrm{d}x \geq K(c)\exp\left(c(\log T)^{7/8}\right), \tag{V.7}$$

where

$$\alpha = \alpha(T) = \frac{3}{8} - \frac{c}{(\log T)^{1/8}} \quad \text{and} \quad y = T^{\mathfrak{b}} \quad \text{for } \mathfrak{b} \geq 80.$$

In particular, this implies

$$\Delta(x) = \Omega(x^{3/8}\exp(-c(\log x)^{7/8}),$$

for some suitable $c > 0$.

In order to prove the theorem, we need several lemmas, which form the content of this section. We begin with a fixed $\delta_0 \in (0, 1/16]$ for which we would choose a numerical value at the end of this section.

Definition V.1. *For $T > 1$, let $Z(T)$ be the set of all γ such that*

1. $T \leq \gamma \leq 2T$,

2. *either* $\zeta(\beta_1 + i\gamma) = 0$ *for some* $\beta_1 \geq \frac{1}{2} + \delta_0$
   or $\zeta(\beta_2 + i2\gamma) = 0$ *for some* $\beta_2 \geq \frac{1}{2} + \delta_0$.

Let

$$I_{\gamma,k} = \{T \leq t \leq 2T : |t - \gamma| \leq k \log^2 T\} \text{ for } k = 1, 2.$$

We finally define

$$J_k(T) = [T, 2T] \setminus \cup_{\gamma \in Z(T)} I_{\gamma,k}.$$

Lemma V.1. *With the above definition, we have for $k = 1, 2$*

$$\mu(J_k(T)) = T + O\left(T^{1-\delta_0/4} \log^3 T\right).$$

Proof. We shall use an estimate on the function $N(\sigma, T)$, which is defined as

$$N(\sigma, T) := \left|\{\sigma' + it : \sigma' \geq \sigma,\ 0 < t \leq T,\ \zeta(\sigma' + it) = 0\}\right|.$$

Selberg [38, Page 237] proved that

$$N(\sigma, T) \ll T^{1-\frac{1}{4}(\sigma - \frac{1}{2})} \log T, \quad \text{for } \sigma > 1/2.$$

Now the lemma follows from the above upper bound on $N(\sigma, t)$, and the observation that

$$\mu\left(\cup_{\gamma \in Z(T)} I_{\gamma,k}\right) \ll N\left(\frac{1}{2} + \delta_0, T\right) \log^2 T.$$

□

The next lemma closely follows Theorem 14.2 of [38], but does not depend on Riemann Hypothesis.

Lemma V.2. *For* $t \in J_1(T)$ *and* $\sigma = 1/2 + \delta$ *with* $\delta_0 < \delta < 1/4 - \delta_0/2$*, we have*

$$|\zeta(\sigma + it)|^{\pm 1} \ll \exp\left(\log\log t \left(\frac{\log t}{\delta_0}\right)^{\frac{1-2\delta}{1-2\delta_0}}\right)$$

and

$$|\zeta(\sigma + 2it)|^{\pm 1} \ll \exp\left(\log\log t \left(\frac{\log t}{\delta_0}\right)^{\frac{1-2\delta}{1-2\delta_0}}\right).$$

Proof. We provide a proof of the first statement, and the second statement can be similarly proved.

Let $1 < \sigma' \le \log t$. We consider two concentric circles centered at $\sigma' + it$, with radius $\sigma' - 1/2 - \delta_0/2$ and $\sigma' - 1/2 - \delta_0$. Since $t \in J_1(T)$ and the radius of the circle is $\ll \log t$, we conclude that

$$\zeta(z) \ne 0 \ \text{ for } \ |z - \sigma' - it| \le \sigma' - \frac{1}{2} - \frac{\delta_0}{2}$$

and also $\zeta(z)$ has polynomial growth in this region. Thus on the larger circle, $\log|\zeta(z)| \le c_{17} \log t$, for some constant $c_{17} > 0$. By Borel-Carathéodory theorem,

$$|z - \sigma' - it| \le \sigma' - \frac{1}{2} - \delta_0 \ \text{ implies } \ |\log\zeta(z)| \le \frac{c_{18}\sigma'}{\delta_0}\log t,$$

for some $c_{18} > 0$. Let $1/2 + \delta_0 < \sigma < 1$, and $\xi > 0$ be such that $1 + \xi < \sigma'$. We consider three concentric circles centered at $\sigma' + it$ with radius $r_1 = \sigma' - 1 - \xi$, $r_2 = \sigma' - \sigma$ and

$r_3 = \sigma' - 1/2 - \delta_0$, and call them C_1, C_2 and C_3 respectively. Let

$$M_i = \sup_{z \in C_i} |\log \zeta(z)|.$$

From the above bound on $|\log \zeta(z)|$, we get

$$M_3 \leq \frac{c_{18}\sigma'}{\delta_0} \log t.$$

Suitably enlarging c_{18}, we see that

$$M_1 \leq \frac{c_{18}}{\xi}.$$

Hence we can apply the Hadamard's three circle theorem to conclude that

$$M_2 \leq M_1^{1-\nu} M_3^{\nu}, \quad \text{for } \nu = \frac{\log(r_2/r_1)}{\log(r_3/r_1)}.$$

Thus

$$M_2 \leq \left(\frac{c_{18}}{\xi}\right)^{1-\nu} \left(\frac{c_{18}\sigma' \log t}{\delta_0}\right)^{\nu}.$$

It is easy to see that

$$\nu = 2 - 2\sigma + \frac{4\delta_0(1-\sigma)}{1 + 2\xi - 2\delta_0} + O(\xi) + O\left(\frac{1}{\sigma'}\right).$$

Now we put

$$\xi = \frac{1}{\sigma'} = \frac{1}{\log \log t}.$$

Hence

$$M_2 \leq \frac{c_{18} \log^{\nu} t \log\log t}{\delta_0^{\nu}} = \frac{c_{19} \log\log t}{\delta_0^{\nu}} (\log t)^{2-2\sigma+\frac{4\delta_0(1-\sigma)}{1+2\xi-2\delta_0}},$$

for some $c_{19} > 0$. We observe that

$$2 - 2\sigma + \frac{4\delta_0(1-\sigma)}{1+2\xi-2\delta_0} < 2 - 2\sigma + \frac{4\delta_0(1-\sigma)}{1-2\delta_0} = \frac{1-2\delta}{1-2\delta_0}.$$

So we get

$$|\log\zeta(\sigma+it)| \leq c_{19} \log\log t \left(\frac{\log t}{\delta_0}\right)^{\frac{1-2\delta}{1-2\delta_0}},$$

and hence the lemma. □

We put $y = T^{\mathfrak{b}}$, for a constant $\mathfrak{b} \geq 80$. Now suppose that

$$\int_T^{\infty} \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-u/y} du \geq \log^2 T,$$

for sufficiently large T. Then clearly

$$\Delta(u) = \Omega(u^{\alpha}).$$

Our next result explores the situation when such an inequality does not hold.

Proposition V.2. *Let* $\delta_0 < \delta < \frac{1}{4} - \frac{\delta_0}{2}$. *For* $1/4 + \delta < \alpha < 1/2$, *suppose that*

$$\int_T^{\infty} \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-u/y} du \leq \log^2 T \tag{V.8}$$

for any sufficiently large T. Then we have

$$\int_{\substack{Re(s)=\alpha \\ t\in J_2(T)}} \frac{|D(s)|^2}{|s|^2} \ll 1 + \int_T^\infty \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-2u/y} du.$$

Before embarking on a proof, we need the following technical lemmas.

Lemma V.3. *For $0 \le \Re(z) \le 1$ and $|Im(z)| \ge \log^2 T$, we have*

$$\int_T^\infty e^{-u/y} u^{-z} du = \frac{T^{1-z}}{1-z} + O(T^{-\mathfrak{b}'}) \tag{V.9}$$

and

$$\int_T^\infty e^{-u/y} u^{-z} \log u \, du = \frac{T^{1-z}}{1-z} \log T + O(T^{-\mathfrak{b}'}), \tag{V.10}$$

where $\mathfrak{b}' > 0$ depends only on $\mathfrak{b}$.

Proof. By changing variable by $v = u/y$, we get

$$\int_T^\infty \frac{e^{-u/y}}{u^z} du = y^{1-z} \int_{T/y}^\infty e^{-v} v^{-z} dv.$$

Integrating the right hand side by parts

$$\int_{T/y}^\infty e^{-v} v^{-z} dv = \frac{e^{-T/y}}{1-z} \left(\frac{T}{y}\right)^{1-z} + \frac{1}{1-z} \int_{T/y}^\infty e^{-v} v^{1-z} dv$$

It is easy to see that

$$\int_{T/y}^\infty e^{-v} v^{1-z} dv = \Gamma(2-z) + O\left(\left(\frac{T}{y}\right)^{2-Re(z)}\right).$$

Hence (V.9) follows using $e^{-T/y} = 1 + O(T/y)$ and Stirling's formula along with the assumption that $|Im(z)| \ge \log^2 T$.

Proof of (V.10) proceeds in the same line and uses the fact that

$$\int_{T/y}^{\infty} e^{-v} v^{1-z} \log v \, dv = \Gamma'(2-z) + O\left(\left(\frac{T}{y}\right)^{2-Re(z)} \log T\right).$$

Then we apply Stirling's formula for $\Gamma'(s)$ instead of $\Gamma(s)$. □

Lemma V.4. *Under the assumption (V.8), there exists T_0 with $T \leq T_0 \leq 2T$ such that*

$$\frac{\Delta(T_0) e^{-T_0/y}}{T_0^{\alpha}} \ll \log^2 T,$$

$$\text{and} \quad \frac{1}{y} \int_{T_0}^{\infty} \frac{\Delta(u) e^{-u/y}}{u^{\alpha}} du \ll \log T.$$

Proof. The assumption (V.8) implies that

$$\begin{aligned} \log^2 T &\geq \int_{T}^{2T} \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-u/y} du \\ &= \int_{T}^{2T} \frac{|\Delta(u)|^2}{u^{2\alpha}} e^{-2u/y} \frac{e^{u/y}}{u} du \\ &\geq \min_{T \leq u \leq 2T} \left(\frac{|\Delta(u)|}{u^{\alpha}} e^{-u/y}\right)^2, \end{aligned}$$

which proves the first assertion. To prove the second assertion, we use the previous assertion and Cauchy- Schwartz inequality along with assumption (V.8) to get

$$\begin{aligned} \left(\int_{T_0}^{\infty} \frac{\Delta(u)}{u^{\alpha}} e^{-u/y} du\right)^2 &\leq \left(\int_{T_0}^{\infty} \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-u/y} du\right)\left(\int_{T_0}^{\infty} u e^{-u/y} du\right) \\ &\ll y^2 \log^2 T. \end{aligned}$$

This completes the proof of this lemma. □

We now recall a mean value theorem due to Montgomery and Vaughan [30].

Notation. *For a real number θ, let $\|\theta\| := \min_{n\in\mathbb{Z}} |\theta - n|$.*

Theorem V.3 (Montgomery and Vaughan [30])**.** *Let $a_1, \cdots, a_N$ be arbitrary complex numbers, and let $\lambda_1, \cdots, \lambda_N$ be distinct real numbers such that*

$$\delta = \min_{m,n} \|\lambda_m - \lambda_n\| > 0.$$

Then

$$\int_0^T \left| \sum_{n\leq N} a_n \exp(i\lambda_n t) \right|^2 dt = \left(T + O\left(\frac{1}{\delta}\right)\right) \sum_{n\leq N} |a_n|^2.$$

Lemma V.5. *For $T \leq T_0 \leq 2T$ and $\Re(s) = \alpha$, we have*

$$\int_T^{2T} \left| \sum_{n\leq T_0} \frac{|\tau(n,\theta)|^2}{n^s} e^{-n/y} \right|^2 t^{-2} dt \ll 1.$$

Proof. Using theorem V.3, we get

$$\int_T^{2T} \left| \sum_{n\leq T_0} \frac{|\tau(n,\theta)|^2}{n^s} e^{-n/y} \right|^2 t^{-2}\mathrm{d}t \quad\leq\quad \frac{1}{T^2}\left(T\sum_{n\leq T_0} |b(n)|^2 + O\left(\sum_{n\leq T_0} n|b(n)|^2 \right)\right),$$

$$\text{where} \quad b(n) = \frac{|\tau(n,\theta)|^2}{n^\alpha} e^{-n/y}.$$

Thus

$$\sum_{n\leq T_0} |b(n)|^2 \leq \sum_{n\leq T_0} \frac{d(n)^4}{n^{2\alpha}} \ll T_0^{1-2\alpha+\epsilon} \text{ and } \sum_{n\leq T_0} n|b(n)|^2 \leq \sum_{n\leq T_0} \frac{d(n)^4}{n^{2\alpha-1}} \ll T_0^{2-2\alpha+\epsilon}$$

for any $\epsilon > 0$, since the divisor function $d(n) \ll n^\epsilon$. As we have $\alpha > 0$, this completes the proof. □

Lemma V.6. *For $\Re(s) = \alpha$ and $T \leq T_0 \leq 2T$, we have*

$$\int_T^{2T} \left| \sum_{n\geq 0} \int_0^1 \frac{\Delta(n+x+T_0)e^{-(n+x+T_0)/y}}{(n+x+T_0)^{s+1}} \mathrm{d}x \right|^2 \mathrm{d}t \ll \int_T^{\infty} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} e^{-2x/y} \mathrm{d}x.$$

Proof. Using Cauchy- Schwarz inequality, we get

$$\begin{aligned}
&\left| \sum_{n\geq 0} \int_0^1 \frac{\Delta(n+x+T_0)}{(n+x+T_0)^{s+1}} e^{-(n+x+T_0)/y} \mathrm{d}x \right|^2 \\
\leq &\int_0^1 \left| \sum_{n\geq 0} \frac{\Delta(n+x+T_0)}{(n+x+T_0)^{s+1}} e^{-(n+x+T_0)/y} \right|^2 \mathrm{d}x.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_T^{2T} \left| \int_0^1 \sum_{n\geq 0} \frac{\Delta(n+x+T_0)e^{-(n+x+T_0)/y}}{(n+x+T_0)^{s+1}} \mathrm{d}x \right|^2 \mathrm{d}t \\
\leq &\int_T^{2T} \int_0^1 \left| \sum_{n\geq 0} \frac{\Delta(n+x+T_0)}{(n+x+T_0)^{s+1}} e^{-(n+x+T_0)/y} \right|^2 \mathrm{d}x\mathrm{d}t \\
= &\int_0^1 \int_T^{2T} \left| \sum_{n\geq 0} \frac{\Delta(n+x+T_0)}{(n+x+T_0)^{s+1}} e^{-(n+x+T_0)/y} \right|^2 \mathrm{d}t\mathrm{d}x.
\end{aligned}$$

From Theorem V.3, we can get

$$\begin{aligned}
&\int_T^{2T} \left| \sum_{n\geq 0} \frac{\Delta(n+x+T_0)}{(n+x+T_0)^{s+1}} e^{-(n+x+T_0)/y} \right|^2 \mathrm{d}t \\
= \quad &T \sum_{n\geq 0} \frac{|\Delta(n+x+T_0)|^2}{(n+x+T_0)^{2\alpha+2}} e^{-2(n+x+T_0)/y} + O\left(\sum_{n\geq 0} \frac{|\Delta(n+x+T_0)|^2}{(n+x+T_0)^{2\alpha+1}} e^{-2(n+x+T_0)/y} \right) \\
\ll \quad &\sum_{n\geq 0} \frac{|\Delta(n+x+T_0)|^2}{(n+x+T_0)^{2\alpha+1}} e^{-2(n+x+T_0)/y}.
\end{aligned}$$

Hence

$$\int_T^{2T} \left| \sum_{n\geq 0} \int_0^1 \frac{\Delta(n+x+T_0)e^{-(n+x+T_0)/T}}{(n+x+T_0)^{s+1}} \mathrm{d}x \right|^2 \mathrm{d}t$$
$$\ll \int_0^1 \sum_{n\geq 0} \frac{|\Delta(n+x+T_0)|^2}{(n+x+T_0)^{2\alpha+1}} e^{-2(n+x+T_0)/y} \mathrm{d}x \ll \int_T^\infty \frac{|\Delta(x)|^2}{x^{2\alpha+1}} e^{-2x/y} \mathrm{d}x,$$

completing the proof. □

Proof of Proposition V.2. For $s = \alpha + it$ with $1/4 + \delta < \alpha < 1/2$ and $t \in J_2(T)$, we have

$$\begin{aligned} \sum_{n=1}^\infty \frac{|\tau(n,\theta)|^2}{n^s} e^{-n/y} &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} D(s+w)\Gamma(w) y^w \mathrm{d}w \\ &= \frac{1}{2\pi i} \int_{2-i\log^2 T}^{2+i\log^2 T} + O\left(y^2 \int_{\log^2 T}^\infty |D(s+2+iv)||\Gamma(2+iv)| \mathrm{d}v \right). \end{aligned}$$

The above error term is estimated to be $o(1)$. We move the integral to

$$\left[\frac{1}{4} + \frac{\delta}{2} - \alpha - i\log^2 T,\ \frac{1}{4} + \frac{\delta}{2} - \alpha + i\log^2 T \right].$$

Let $\delta' = 1/4 + \delta/2 - \alpha$. In the region to the right side of this line, $\Re(2s+2w) \geq 1/2 + \delta$. Writing $w = u + iv$ we observe that $t + v \in J_1(T)$ since $t \in J_2(T)$. So we can apply Lemma V.2 to conclude that

$$\zeta(2s+2w) \gg T^{-1}.$$

On the above line, we have $\Re(s+w) = 1/4 + \delta/2$, Thus

$$\zeta^2(s+w)\zeta(s+w+i\theta)\zeta(s+w-i\theta) \ll T^{3/2-\delta} \log^4 T$$

where we use the fact that $\zeta(z) \ll \Im(z)^{(1-\Re(z))/2} \log(\Im(z))$ if $0 \leq \Re(z) \leq 1$. Hence by convexity, we see that $\zeta^2(s+w)\zeta(s+w+i\theta)\zeta(s+w-i\theta)$ has polynomial growth on the horizontal lines of integration. Therefore the horizontal integrals are $o(1)$ by exponential decay of Γ-function. Since the only pole inside this contour is at $w=0$, we get

$$\sum_{n=1}^{\infty} \frac{|\tau(n,\theta)|^2}{n^s} e^{-n/y} = D(s) + \frac{1}{2\pi i} \int_{\delta'-i\log^2 T}^{\delta'+i\log^2 T} D(s+w)\Gamma(w)y^w \mathrm{d}w + o(1).$$

For the integral on the right hand side, we have

$$D(s+w)y^w \ll T^{5/2-\delta(\mathfrak{b}/2+1)}$$

where the exponent of T is negative by our choice of $\mathfrak{b}$ and δ. Therefore this integral is also $o(1)$.

Using T_0 as in Lemma V.4, we now divide the sum into two parts:

$$D(s) = \sum_{n\le T_0} \frac{|\tau(n,\theta)|^2}{n^s} e^{-n/y} + \sum_{n>T_0} \frac{|\tau(n,\theta)|^2}{n^s} e^{-n/y} + o(1).$$

To estimate the second sum, we write

$$\begin{aligned} \sum_{n>T_0} \frac{|\tau(n,\theta)|^2}{n^s} e^{-n/y} &= \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} \mathrm{d}\left(\sum_{n\le x} |\tau(n,\theta)|^2\right) \\ &= \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} \mathrm{d}(\mathcal{M}(x)+\Delta(x)) \\ &= \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} \mathcal{M}'(x)\mathrm{d}x + \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} \mathrm{d}(\Delta(x)). \end{aligned}$$

Recall that

$$\mathcal{M}(x) = \omega_1(\theta)x\log x + \omega_2(\theta)x\cos(\theta\log x) + \omega_3(\theta)x,$$

thus

$$\mathcal{M}'(x) = \omega_1(\theta)\log x + \omega_2(\theta)\cos(\theta\log x) - \theta\omega_2(\theta)\sin(\theta\log x) + \omega_1(\theta) + \omega_3(\theta).$$

Observe that

$$\int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s}\cos(\theta\log x)\mathrm{d}x = \frac{1}{2}\int_{T_0}^{\infty}\frac{e^{-x/y}}{x^{s+i\theta}}\mathrm{d}x + \frac{1}{2}\int_{T_0}^{\infty}\frac{e^{-x/y}}{x^{s-i\theta}}\mathrm{d}x.$$

Applying Lemma V.3, we conclude that

$$\int_{T_0}^{\infty}\frac{e^{-x/y}}{x^s}\mathcal{M}'(x)\mathrm{d}x = o(1).$$

Integrating the second integral by parts:

$$\begin{aligned}\int_{T_0}^{\infty}\frac{e^{-x/y}}{x^s}\mathrm{d}(\Delta(x)) &= \frac{e^{-T_0/y}\Delta(T_0)}{T_0^s} \\ &+ \frac{1}{y}\int_{T_0}^{\infty}\frac{e^{-x/y}}{x^s}\Delta(x)\mathrm{d}x - s\int_{T_0}^{\infty}\frac{e^{-x/y}}{x^{s+1}}\Delta(x)\mathrm{d}x.\end{aligned}$$

Applying Lemma V.4, we get

$$\begin{aligned}\sum_{n>T_0}\frac{|\tau(n,\theta)|^2}{n^s}e^{-n/y} &= s\int_{T_0}^{\infty}\frac{\Delta(x)e^{-x/y}}{x^{s+1}}\mathrm{d}x + O(\log T) \\ &= s\sum_{n\geq 0}\int_0^1\frac{\Delta(n+x+T_0)e^{-(n+x+T_0)/y}}{(n+x+T_0)^{s+1}}\mathrm{d}x + O(\log T).\end{aligned}$$

Hence we have

$$D(s) = \sum_{n\leq T_0}\frac{|\tau(n,\theta)|^2}{n^s}e^{-n/y} + s\sum_{n\geq 0}\int_0^1\frac{\Delta(n+x+T_0)e^{-(n+x+T_0)/y}}{(n+x+T_0)^{s+1}}\mathrm{d}x + O(\log T).$$

Squaring both sides, and then integrating on $J_2(T)$, we get

$$\int_{J_2(T)} \frac{|D(\alpha+it)|^2}{|\alpha+it|^2}\mathrm{d}t \ll \int_T^{2T} \left| \sum_{n\le T_0} \frac{|\tau(n,\theta)|^2}{n^s} e^{-n/y} \right|^2 \frac{\mathrm{d}t}{t^2}$$
$$+ \int_T^{2T} \left| \sum_{n\ge 0} \int_0^1 \frac{\Delta(n+x+T_0)e^{-(n+x+T_0)/y}}{(n+x+T_0)^{s+1}} \mathrm{d}x \right|^2 \mathrm{d}t.$$

The proposition now follows using Lemma V.5 and Lemma V.6. □

We are now ready to prove our main theorem of this section.

Proof of Theorem V.2. We prove by contradiction. Suppose that (V.7) does not hold. Then there exists a constant $c > 0$ such that given any $N_0 > 1$, there exists $T > N_0$ for which

$$\int_T^\infty \frac{|\Delta(x)|^2}{x^{2\alpha+1}} e^{-2x/y}\mathrm{d}x \ll \exp\left(c(\log T)^{7/8}\right),$$

for all $c > 0$. Note that the above statement is weaker than the contrapositive of the statement of theorem. This gives

$$\int_T^\infty \frac{|\Delta(x)|^2}{x^{2\beta+1}} e^{-2x/y}\mathrm{d}x \ll 1,$$

where

$$\beta = \frac{3}{8} - \frac{c}{2(\log T)^{1/8}}.$$

We apply Proposition V.2 to get

$$\int_{J_2(T)} \frac{|D(\beta+it)|^2}{|\beta+it|^2}\mathrm{d}t \ll 1. \tag{V.11}$$

Now we compute a lower bound for the last integral over $J_2(T)$. Write the functional equation for $\zeta(s)$ as

$$\zeta(s) = \pi^{1/2-s}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\zeta(1-s).$$

Using the Stirling's formula for Γ function, we get

$$|\zeta(s)| = \pi^{1/2-\sigma}t^{1/2-\sigma}|\zeta(1-s)|\left(1+O\left(\frac{1}{T}\right)\right),$$

for $s = \sigma + it$. This implies

$$|D(\beta+it)| = t^{2-4\beta}\frac{|\zeta(1-\beta+it)^2\zeta(1-\beta-it-i\theta)\zeta(1-\beta-it+i\theta)|}{|\zeta(2\beta+i2t)|}.$$

Let $\delta_0 = 1/16$, and

$$\beta = \frac{3}{8} - \frac{c}{2(\log T)^{1/8}} = \frac{1}{2} - \delta$$

with

$$\delta = \frac{1}{8} + \frac{c}{2(\log T)^{1/8}}.$$

Then using Lemma V.2, we get

$$|\zeta(1-\beta+it)| = \left|\zeta\left(\frac{1}{2}+\delta+it\right)\right| \gg \exp\left(\log\log t\left(\frac{\log t}{\delta_0}\right)^{\frac{1-2\delta}{1-2\delta_0}}\right).$$

For $t \in J_2(T)$ we observe that $t\pm\theta \in J_1(T)$, and so the same bounds hold for $\zeta(1-\beta+it+i\theta)$ and $\zeta(1-\beta+it-i\theta)$. Further

$$|\zeta(2\beta+i2t)| = \left|\zeta\left(\frac{1}{2}+\left(\frac{1}{2}-2\delta\right)+i2t\right)\right| \ll \exp\left(\log\log t\left(\frac{\log t}{\delta_0}\right)^{\frac{4\delta}{1-2\delta_0}}\right).$$

Combining these bounds, we get

$$|D(\beta + it)| \gg t^{2-4\beta} \exp\left(-5 \log\log t \left(\frac{\log t}{\delta_0}\right)^{\frac{1-2\delta}{1-2\delta_0}}\right).$$

Therefore

$$\begin{aligned}\int_{J_2(T)} |D(\beta + it)|^2 \mathrm{d}t &\gg T^{4-8\beta} \exp\left(-10 \log\log T \left(\frac{\log T}{\delta_0}\right)^{\frac{1-2\delta}{1-2\delta_0}}\right) \mu(J_2(T)) \\ &\gg T^{5-8\beta} \exp\left(-10 \log\log T \left(\frac{\log T}{\delta_0}\right)^{\frac{1-2\delta}{1-2\delta_0}}\right),\end{aligned}$$

where we use Lemma V.1 to show that $\mu(J_2(T)) \gg T$. Now putting the values of δ and δ_0 as chosen above, we get

$$\int_{J_2(T)} \frac{|D(\beta + it)|^2}{|\beta + it|^2} dt \gg \exp\left(3c(\log T)^{7/8}\right),$$

since $\frac{1-2\delta}{1-2\delta_0} < 7/8$. This contradicts (V.11), and hence the theorem follows. □

The following two corollaries are immediate.

Corollary V.1. *For any $c > 0$ and for all sufficiently large T depending on c, there exists an*

$$X \in \left[T, \frac{T^{\mathfrak{b}}}{2} \log^2 T\right]$$

for which we have

$$\int_X^{2X} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} dx \geq \exp\left((c - \epsilon)(\log X)^{7/8}\right),$$

with α as in Theorem V.2 and for any $\epsilon > 0$.

Corollary V.2. *For any* $c > 0$ *and for all sufficiently large* T *depending on* c*, there exists an*

$$x \in \left[T, \frac{T^{\flat}}{2} \log^2 T\right]$$

for which we have

$$\Delta(x) \geq x^{3/8} \exp\left(-c(\log x)^{7/8}\right).$$

We can now prove a "measure version" of the above result.

Proposition V.3. *For any* $c > 0$*, let*

$$\alpha(x) = \frac{3}{8} - \frac{c}{(\log x)^{1/8}}$$

and $\mathcal{A} = \{x : |\Delta(x)| \gg x^{\alpha(x)}\}$*. Then for every sufficiently large* X *depending on* c*, we have*

$$\mu(\mathcal{A} \cap [X, 2X]) = \Omega(X^{2\alpha(X)}).$$

Proof. Suppose that the conclusion does not hold, hence

$$\mu(\mathcal{A} \cap [X, 2X]) \ll X^{2\alpha(X)}.$$

Thus for every sufficiently large X, we get

$$\int_{\mathcal{A} \cap [X,2X]} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} dx \ll X^{2\alpha} \frac{M(X)}{X^{2\alpha+1}} = \frac{M(X)}{X},$$

where $\alpha = \alpha(X)$ and $M(X) = \sup_{X \leq x \leq 2X} |\Delta(x)|^2$. Using dyadic partition, we can prove

$$\int_{\mathcal{A} \cap [T,y]} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} dx \ll \frac{M_0(T)}{T} \log T, \quad \text{where } M_0(T) = \sup_{T \leq x \leq y} |\Delta(x)|^2$$

and $y = T^{\mathfrak{b}}$ for some $\mathfrak{b} > 0$ and T sufficiently large. This gives

$$\int_T^\infty \frac{|\Delta(x)|^2}{x^{2\alpha+1}} e^{-2x/y} dx \ll \frac{M_0(T)}{T} \log T.$$

Along with (V.7), this implies

$$M_0(T) \gg T \exp\left(\frac{c}{2}(\log T)^{7/8}\right).$$

Thus

$$|\Delta(x)| \gg x^{\frac{1}{2}} \exp\left(\frac{c}{4}(\log x)^{7/8}\right),$$

for some $x \in [T, y]$. This contradicts the fact that $|\Delta(x)| \ll x^{\frac{1}{2}}(\log x)^6$. □

V.4.1 Optimality of the Omega Bound

The following proposition shows the optimality of the omega bound in Corollary V.1.

Proposition V.4. *Under Riemann Hypothesis (RH), we have*

$$\int_X^{2X} \Delta^2(x) \mathrm{d}x \ll X^{7/4+\epsilon},$$

for any $\epsilon > 0$.

Proof. Theorem II.2 (Perron's formula) gives

$$\Delta(x) = \frac{1}{2\pi i} \int_{-T}^{T} \frac{D(3/8 + it) x^{3/8+it}}{3/8 + it} \mathrm{d}t + O(x^\epsilon),$$

for any $\epsilon > 0$ and for $T = X^2$ with $x \in [X, 2X]$. Using this expression for $\Delta(x)$, we write its second moment as

$$\int_X^{2X} \Delta^2(x)\mathrm{d}x = \frac{1}{(2\pi)^2}\int_X^{2X}\int_{-T}^{T}\int_{-T}^{T}\frac{D(3/8+it_1)D(3/8-it_2)}{(3/8+it_1)(3/8-it_2)}x^{3/4+i(t_1-t_2)}\mathrm{d}x\ \mathrm{d}t_1\mathrm{d}t_2$$
$$+ O\left(X^{1+\epsilon}(1+|\Delta(x)|)\right)$$
$$\ll X^{7/4}\int_{-T}^{T}\int_{-T}^{T}\left|\frac{D(3/8+it_1)D(3/8-it_2)}{(3/8+it_1)(3/8-it_2)(7/4+i(t_1-t_2))}\right|\mathrm{d}t_1\mathrm{d}t_2 + O(X^{3/2+\epsilon}).$$

In the above calculation, we have used the fact that $\Delta(x) \ll x^{\frac{1}{2}+\epsilon}$ as in (I.4). Also note that for complex numbers a, b, we have $|ab| \leq \frac{1}{2}(|a|^2+|b|^2)$. We use this inequality with

$$a = \frac{|D(3/8+it_1)|}{|3/8+it_1|\sqrt{|7/4+i(t_1-t_2)|}} \ \text{ and } \ b = \frac{|D(3/8-it_2)|}{|3/8-it_2|\sqrt{|7/4+i(t_1-t_2)|}},$$

to get

$$\int_X^{2X} \Delta^2(x)\mathrm{d}x \ll X^{7/4}\int_{-T}^{T}\int_{-T}^{T}\left|\frac{D(3/8-it_2)}{(3/8-it_2)}\right|^2\frac{\mathrm{d}t_1}{|7/4+i(t_1-t_2)|}\mathrm{d}t_2 + O(X^{3/2+\epsilon})$$
$$\ll X^{7/4}\log X\int_{-T}^{T}\left|\frac{D(3/8-it_2)}{(3/8-it_2)}\right|^2\mathrm{d}t_2 + O(X^{3/2+\epsilon}).$$

Under RH, convexity bound gives $\zeta(\sigma+it) \ll t^{1/2-\sigma}$ for $0 \leq \sigma \leq 1/2$, hence $|D(3/8-it_2)| \ll |t_2|^{\frac{1}{2}+\epsilon}$. So we have

$$\int_X^{2X} \Delta^2(x)\mathrm{d}x \ll X^{7/4+\epsilon} \ \text{ for any } \ \epsilon > 0.$$

□

Note. The method we have used in Theorem V.2 has its origin from the Plancherel's formula in Fourier analysis. For instance, we may observe from Theorem II.1 that under Riemann Hypothesis and other suitable conditions

$$\frac{\Delta(e^u)}{e^{u\sigma}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{D(\sigma+it)e^{iut}}{\sigma+it}\mathrm{d}t \ \text{ for } \frac{1}{4} < \sigma \le \frac{1}{2}.$$

So $\frac{\Delta(e^u)}{e^{u\sigma}}$ is the Fourier transform of $\frac{D(\sigma+it)}{\sigma+it}$. By Plancherel's formula

$$\int_{-\infty}^{\infty} \frac{|\Delta(e^u)|^2}{e^{2u\sigma}}\mathrm{d}u = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \left|\frac{D(\sigma+it)}{\sigma+it}\right|^2 \mathrm{d}t.$$

Now we change the variable u to $\log x$ and use the functional equation for $\zeta(s)$ to get

$$\int_1^{\infty} \frac{\Delta^2(x)}{x^{2\sigma+1}}\mathrm{d}x \asymp \int_1^{\infty} \left|\frac{D(\sigma+it)}{\sigma+it}\right|^2 \mathrm{d}t \gg \int_1^{\infty} t^{2-8\sigma-10\epsilon}dt$$

for any $\epsilon > 0$ where the last inequality uses Riemann Hypothesis. Now we choose $\sigma = \frac{3}{8} - 2\epsilon$, then the above integral on the right hand side diverges. Now suppose $\Delta(x) \ll x^{\frac{3}{8}-3\epsilon}$, then the integral in the left hand side converges. This contradiction shows

$$\Delta(x) = \Omega(x^{\frac{3}{8}-3\epsilon}).$$

In [3] and [4], Balasubramanian and Ramachandra used this insight to obtain Ω bounds for the error terms in asymptotic formulas for partial sums of square-free divisors and counting function for non-isomorphic abelian groups. This method requires the Riemann Hypothesis to be assumed in certain cases. Later Balasubramanian, Ramachandra and Subbarao [5] modified this technique to apply on error term in the asymptotic formula for the counting function of k-full numbers without assuming Riemann Hypothesis. This method has been used by several authors including [25] and [35].

V.5 Influence of Measure on $\Omega_\pm$ Results

In this section, we shall show that for any $\epsilon > 0$,

$$\text{if } \Delta(x) \ll x^{3/8+\epsilon}, \text{ then } \Delta(x) = \Omega_\pm\left(x^{3/8-\epsilon}\right).$$

This conditionally improves our earlier result, which says that $\Delta(x)$ is $\Omega_\pm\left(x^{1/4}\right)$. Now, we state the main theorem of this section.

Theorem V.4. *Let $\Delta(x)$ be the error term of the summatory function of the twisted divisor function as in Theorem V.1. For $c > 0$, let*

$$\alpha(x) = \frac{3}{8} - \frac{c}{(\log x)^{1/8}}.$$

Let δ and δ' be such that

$$0 < \delta < \delta' < \frac{1}{8}.$$

Then either

$$\Delta(x) = \Omega\left(x^{\alpha(x)+\frac{\delta}{2}}\right) \quad \text{or} \quad \Delta(x) = \Omega_\pm\left(x^{\frac{3}{8}-\delta'}\right).$$

To prove the above theorem, we estimate the growth of the Dirichlet series $D(\sigma + it)$ by assuming that it does not have poles in a certain region.

Lemma V.7. *Let δ and σ be such that*

$$0 < \delta < \frac{1}{8}, \quad \text{and} \quad \frac{3}{8} - \delta \le \sigma < \frac{1}{2}.$$

If $D(\sigma + it)$ does not have a pole in the above mentioned range of σ, then for

$$\frac{3}{8} - \delta + \frac{\delta}{2(1 + \log\log(3 + |t|))} < \sigma < \frac{1}{2},$$

we have

$$D(\sigma + it) \ll_{\delta,\theta} |t|^{2-2\sigma+\epsilon}$$

for any positive constant ϵ.

Proof. Let $s = \sigma + it$ with $3/8 - \delta \le \sigma < 1/2$. Recall that

$$D(s) = \frac{\zeta^2(s)\zeta(s + i\theta)\zeta(s - i\theta)}{\zeta(2s)}.$$

Using functional equation, we write

$$D(s) = \mathcal{X}(s)\frac{\zeta^2(1 - s)\zeta(1 - s - i\theta)\zeta(1 - s + i\theta)}{\zeta(2s)}, \tag{V.12}$$

where

$$\mathcal{X}(s) = \pi^{4s-2}\frac{\Gamma^2\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1-s-i\theta}{2}\right)\Gamma\left(\frac{1-s+i\theta}{2}\right)}{\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{s+i\theta}{2}\right)\Gamma\left(\frac{s-i\theta}{2}\right)}.$$

From Stirling's formula for Γ, we get

$$\mathcal{X}(\sigma + it) \asymp t^{2-4\sigma}. \tag{V.13}$$

Using Stirling's formula and Phragmén-Lindelöf principle, we get

$$|\zeta(1 - s)| \ll |t|^{\sigma/2}\log t.$$

So we get

$$|\zeta^2(1-s)\zeta(1-s-i\theta)\zeta(1-s+i\theta)| \ll t^{2\sigma}(\log t)^4. \tag{V.14}$$

Now we shall compute an upper bound for $|\zeta(2s)|^{-1}$. This can be obtained in a similar way as in Lemma V.2. We choose $t \geq 100$. Similar computation can be done when t is negative.

Consider two concentric circles $C_{1,1}$ and $C_{1,2}$, centered at $2+it$ with radii

$$\frac{5}{4}+2\delta \quad \text{and} \quad \frac{5}{4}+2\delta-\frac{\delta}{1+\log\log(|t|+3)}.$$

The circle $C_{1,1}$ passes through $3/4-2\delta+i2t$ and $C_{1,2}$ passes through $3/4-2\delta+\delta(1+\log\log(|t|+3))^{-1}+i2t$. By our assumption, $\zeta(z)$ does not have any zero for $|z-2-it| \leq 5/4+2\delta$. This implies $\log\zeta(z)$ is a holomorphic function in this region. It is easy to see that on the larger circle $C_{1,1}$, we have $\log|\zeta(z)| < \sigma'\log t$ for some positive constant σ'. We apply Borel-Carathéodory theorem to get an upper bound for $\log\zeta(z)$ on $C_{1,2}$:

$$\begin{aligned}|\log\zeta(z)| &\leq 3\delta^{-1}(1+\log\log(t+3))\left(\sigma'\log t+|\log\zeta(2+it)|\right)\\ &\leq 10\delta^{-1}\sigma'(\log\log t)\log t \qquad \text{for } z\in C_{1,2}.\end{aligned}$$

We may also note that if $\Re(z-3/4-2\delta) > \delta(\log\log t)^{-1}$ and $\Im(z) \leq t/2$, then similar arguments give

$$|\log\zeta(z)| < \delta^{-1}\sigma'(\log\log t)\log t,$$

for some positive constant σ' that has changed appropriately.

Now we consider three concentric circles $C_{2,1}, C_{2,2}, C_{2,3}$, centered at $\sigma''+i2t$ and with radii $r_1 = \sigma'' - 1 - \eta, r_2 = \sigma'' - 2\sigma$ and $r_3 = \sigma'' - \delta_0$ respectively. Here

$$\delta_0 = \frac{3}{4} - 2\delta + \frac{\delta}{1 + \log\log(t+3)}.$$

We shall choose $\sigma'' = \eta^{-1} = \log\log t$. Let M_1, M_2, M_3 denote the supremums of $|\log\zeta(z)|$ on $C_{2,1}, C_{2,2}, C_{2,3}$ respectively. We have already calculated that

$$M_3 \leq \delta^{-1}\sigma'(\log\log t)\log t.$$

It is easy to show that

$$M_1 \leq \sigma' \log\log t,$$

where σ' is again appropriately adjusted. Applying the three circle theorem we conclude

$$M_2 \leq \sigma'(\log\log t)\delta^{-a}\log^a t,$$

where

$$\begin{aligned} a = \frac{\log(r_2/r_1)}{\log(r_3/r_1)} &= \frac{1 - 2\sigma + \eta}{1 - \delta_0 + \eta} + O\left(\frac{1}{\sigma''}\right) \\ &= \frac{4(1-2\sigma)}{1+8\delta} + O_\delta\left(\frac{1}{\log\log t}\right). \end{aligned}$$

This gives

$$|\zeta(2s)|^{-1} \ll \exp\left(c(\log\log t)(\log t)^{\frac{4(1-2\sigma)}{1+8\delta}}\right), \tag{V.15}$$

for a suitable constant $c > 0$ depending on δ. The bound in the lemma follows from (V.12), (V.13), (V.14) and (V.15). □

Now we complete the proof of Theorem V.4.

Proof of Theorem V.4. Let M be any large positive constant, and define

$$\mathcal{A} := \mathcal{A}(Mx^{\alpha(x)}).$$

Then from Corollary V.1, we have

$$\int_{[T,2T]\cap\mathcal{A}} \frac{\Delta^2(x)}{x^{2\alpha(T)}+1} \mathrm{d}x \gg \exp\left(c(\log T)^{7/8}\right).$$

Assuming

$$\mu([T,2T]\cap\mathcal{A}) \leq T^{1-\delta} \quad \text{for } T > T_0, \tag{V.16}$$

Proposition IV.1 gives

$$\Delta(x) = \Omega(x^{\alpha(x)+\delta/2})$$

as $h_0(T) = T^{1-\delta}$, which is the first part of the theorem. But if (V.16) does not hold, then we have

$$\mu([T,2T]\cap\mathcal{A}) > T^{1-\delta}$$

for T in an **X**-Set . We choose

$$h_1(T) = T^{\frac{3}{8}-\frac{2c}{(\log T)^{1/8}}-\delta},\ \alpha_1(T) = \frac{3}{8} - \frac{3c}{(\log T)^{1/8}} - \delta,\ \alpha_2(T) = \alpha(T).$$

Let δ'' be such that $\delta < \delta'' < \delta'$. If $D(\sigma + it)$ does not have pole for $\sigma > 3/8 - \delta''$ then by Lemma V.7, $D(\alpha_1(T) + it)$ has polynomial growth. So Assumptions IV.3 is valid. Since

$$T^{1-\delta} > 5h_1(5T/2)T^{1-\alpha_2(T)},$$

by case (ii) of Theorem IV.3 we have

$$\Delta(T) = \Omega_{\pm}\left(T^{\frac{3}{8}-\delta''}\right).$$

□

Bibliography

[1] R. J. Anderson and H. M. Stark. Oscillation theorems. In *Analytic number theory (Philadelphia, Pa., 1980)*, volume 899 of *Lecture Notes in Math.*, pages 79–106. Springer, Berlin-New York, 1981.

[2] R. C. Baker. The square-free divisor problem. II. *Quart. J. Math. Oxford Ser. (2)*, 47(186):133–146, 1996.

[3] R. Balasubramanian and K. Ramachandra. Effective and noneffective results on certain arithmetical functions. *J. Number Theory*, 12(1):10–19, 1980.

[4] R. Balasubramanian and K. Ramachandra. Some problems of analytic number theory. III. *Hardy-Ramanujan J.*, 4:13–40, 1981.

[5] R. Balasubramanian, K. Ramachandra, and M. V. Subbarao. On the error function in the asymptotic formula for the counting function of k-full numbers. *Acta Arith.*, 50(2):107–118, 1988.

[6] G. Bhowmik, J.-C. Schlage-Puchta, and O. Ramaré. Tauberian oscillation theorems and the distribution of goldbach numbers. *J. Théor. Nombres Bordeaux*. To appear.

[7] K. Corrádi and I. Kátai. A comment on K. S. Gangadharan's paper entitled "Two classical lattice point problems". *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.*, 17:89–97, 1967.

[8] A. Deitmar and S. Echterhoff. *Principles of harmonic analysis*. Universitext. Springer, New York, 2009.

[9] P. Èrdeš. An asymptotic inequality in the theory of numbers. *Vestnik Leningrad. Univ.*, 15(13):41–49, 1960.

[10] K. Ford. The distribution of integers with a divisor in a given interval. *Ann. of Math. (2)*, 168(2):367–433, 2008.

[11] K. Ford. Integers with a divisor in $(y, 2y]$. In *Anatomy of integers*, volume 46 of *CRM Proc. Lecture Notes*, pages 65–80. Amer. Math. Soc., Providence, RI, 2008.

[12] A. A. Gioia and A. M. Vaidya. The number of squarefree divisors of an integer. *Duke Math. J.*, 33:797–799, 1966.

[13] J. L. Hafner. New omega theorems for two classical lattice point problems. *Invent. Math.*, 63(2):181–186, 1981.

[14] R. R. Hall and G. Tenenbaum. *Divisors*, volume 90 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1988.

[15] G. H. Hardy. On Dirichlet's Divisor Problem. *Proc. London Math. Soc.*, S2-15(1):1–g.

[16] G. H. Hardy and J. E. Littlewood. Contributions to the theory of the riemann zeta-function and the theory of the distribution of primes. *Acta Math.*, 41(1):119–196, 1916.

[17] G. H. Hardy and S. Ramanujan. The normal number of prime factors of a number n [Quart. J. Math. **48** (1917), 76–92]. In *Collected papers of Srinivasa Ramanujan*, pages 262–275. AMS Chelsea Publ., Providence, RI, 2000.

[18] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Oxford University Press, Oxford, sixth edition, 2008. Revised by D. R. Heath-Brown and J. H. Silverman, With a foreword by Andrew Wiles.

[19] D. R. Heath-Brown. The number of abelian groups of order at most x. *Astérisque*, (198-200):153–163 (1992), 1991. Journées Arithmétiques, 1989 (Luminy, 1989).

[20] M. N. Huxley. Exponential sums and lattice points. III. *Proc. London Math. Soc. (3)*, 87(3):591–609, 2003.

[21] A. Ivić. Large values of the error term in the divisor problem. *Invent. Math.*, 71(3):513–520, 1983.

[22] A. Ivić. The number of finite nonisomorphic abelian groups in mean square. *Hardy-Ramanujan J.*, 9:17–23, 1986.

[23] A. Ivić. *The Riemann zeta-function*. Dover Publications, Inc., Mineola, NY, 2003. Theory and applications, Reprint of the 1985 original [Wiley, New York; MR0792089 (87d:11062)].

[24] J. Kaczorowski and B. Szydło. Some Ω-results related to the fourth power moment of the Riemann zeta-function and to the additive divisor problem. 9(1):41–50, 1997.

[25] M. Kühleitner and W. G. Nowak. On a question of A. Schinzel: Omega estimates for a special type of arithmetic functions. *Cent. Eur. J. Math.*, 11(3):477–486, 2013.

[26] E. Landau. Über einen Satz von Tschebyschef. *Math. Ann.*, 61(4):527–550, 1906.

[27] K. Mahatab. Number of prime factors of an integer. *Math. Newsl.*, 24(1):7–10, 2013.

[28] K. Mahatab and A. Mukhopadhyay. Measure theoretic aspects of oscillations of error terms. *arXiv:1512.03144*, 2016.

[29] K. Mahatab and K. Sampath. Chinese remainder theorem for cyclotomic polynomials in $\mathbf{Z}[X]$. *J. Algebra*, 435:223–262, 2015.

[30] H. L. Montgomery and R. C. Vaughan. Hilbert's inequality. *J. London Math. Soc. (2)*, 8:73–82, 1974.

[31] Y. Motohashi. An explicit formula for the fourth power mean of the Riemann zeta-function. *Acta Math.*, 170(2):181–220, 1993.

[32] Y. Motohashi. A relation between the Riemann zeta-function and the hyperbolic Laplacian. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 22(2):299–313, 1995.

[33] O. Robert and P. Sargos. Three-dimensional exponential sums with monomials. *J. Reine Angew. Math.*, 591:1–20, 2006.

[34] R. E. Rodríguez, I. Kra, and J. P. Gilman. *Complex analysis*, volume 245 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[35] A. Sankaranarayanan and K. Srinivas. On a method of Balasubramanian and Ramachandra (on the abelian group problem). *Rend. Sem. Mat. Univ. Padova*, 97:135–161, 1997.

[36] K. Soundararajan. Omega results for the divisor and circle problems. *Int. Math. Res. Not.*, (36):1987–1998, 2003.

[37] G. Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 46 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Translated from the second French edition (1995) by C. B. Thomas.

[38] E. C. Titchmarsh. *The theory of the Riemann zeta-function*. The Clarendon Press, Oxford University Press, New York, second edition, 1986. Edited and with a preface by D. R. Heath-Brown.

[39] K. M. Tsang. Higher-power moments of $\Delta(x)$, $E(t)$ and $P(x)$. *Proc. London Math. Soc. (3)*, 65(1):65–84, 1992.

[40] H. von Koch. Sur la distribution des nombres premiers. *Acta Math.*, 24(1):159–182, 1901.