Spectral multiplicity for Random Operators with projection valued randomness

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MATH10201104001

The Institute of Mathematical Sciences, Chennai

A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

For the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

August, 2016

Homi Bhabha National Institute

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LIST OF PUBLICATIONS ARISING FROM THE THESIS

Journal

1. *Jakšić-Last Theorem for Higher Rank Perturbations*, Anish Mallick, Mathematische Nachrichten, 2015, DOI: 10.1002/mana.201400423.

Anish Mallick

This thesis is dedicated to my parents for their constant support and encouragement. Without them I could never have reached this stage.

ACKNOWLEDGEMENTS

A PhD dissertation is a summary of the knowledge accumulated over the period of five years. So there is no way to write everything that I have learnt over the period. There are many experiences gained over the years and too many people involved in some way or another, and I thank them for their presence.

First of all I will express my thanks to my advisor, Professor Krishna Maddaly for his guidance and helpful advice. Without his help, it would be very difficult to transverse the vast literature of disordered systems. I should also point out to all the discussions with him, which not only helped me to correct the mistakes I made, but also helped in increasing my conceptual understanding of the subject. Beside my advisor, I thank Professor Partha Sarathi Chakraborty for helpful discussions and many suggestions he provided to better my understanding.

Next I thank my friends and office mates for their presence and constant source joy. Without them, I would not have made it this far. I am very much grateful to members of IMSc for giving me an opportunity and support during my work here. I thank my family for their constant support and having belief in me. Finally I thank the referees for spending their valuable time to read the thesis and provide their valuable advice.

Anish Mallick

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Synopsis

The development of Quantum mechanics lead to explanation of many phenomena and discovery of many new effects. One of its immediate application was in explaining conduction in metals. When semiconductor and doping started developing, conduction and insulation were hard to explain. Anderson [6] developed a quantum mechanical theory to explain spin waves in doped silicon. It was extended to explain metal insulator transition in disordered media. Since then a lot of work has been done in this field to show localization as well as transmission.

The Quantum theory is a Hilbert space theory with self adjoint operators representing observables. Because of this representation, it is natural to ask question about the spectrum. Spectral theorem is a natural theorem that is useful. Since multiplicity of spectrum is part of the spectral theorem and occurs owing to symmetries in the system, it also throws light on presence of symmetries. For example hydrogen spectrum has non-trivial multiplicity (the problem is spherically symmetric), but presence of magnetic or electric field breaks it. So magnetic field can be computed using the spectrum itself. This phenomenon is called Zeeman effect (for magnetic field) and is used by astrophysicist to get an estimate of magnetic field for stars.

It is in general believed that, because of randomness, the point spectrum is simple, i.e the multiplicity of spectrum is one. This was proved by Barry Simon [84] for one dimension and related to Poisson statistics for energy statistics in region of localization for Anderson tight binding model. But simplicity is not proved in case of continuum random Schrödinger operator. The content of this thesis is a step in that direction. In the case of continuum model, each of the perturbations are infinite rank operators, and most of the time are very hard to handle. For example if one consider periodic potential over lattice, then any eigenvalue for which corresponding eigenfunction has bounded support, has infinite multiplicity. Here we handle the case when the perturbation is only finite rank projections.

The class of random operator that is handled here is of the form $A^\omega = A + \sum_{n\in\mathcal{N}} \omega_n P_n$, where A is a bounded self adjoint operator on the separable Hilbert space $\mathcal{H}$, $\mathcal{N}$ is a countable set, $\{P_n\}_{n\in\mathcal{N}}$ are rank N projections and $\{\omega_n\}_{n\in\mathcal{N}}$ are independent real random variables with absolutely continuous distribution. Anderson tight-binding model is an example of this type of random operator for the case $N = 1$, and random dimer model is another ($N = 2$). For tight binding model presence of localized regime is known in many setting and in case of Bethe lattice presence of absolute continuous spectrum is known for low disorder. It is proved by Jakšić-Last [43] that the spectral measure associated with $P_n = |\delta_n\rangle\langle\delta_n|$ when $\{\delta_n\}_{n\in\mathcal{N}}$ is a basis of $\mathscr{H}$ (i.e the rank of perturbation is one), for Anderson type Hamiltonian are equivalent and singular spectrum is simple. Here similar type of results are shown for higher rank cases.

Let E^ω be the spectral projection for the operator A^ω and E^ω_{ac} (similarly E^ω_{sing}) denotes the projection associated to absolutely continuous part (singular part) of the spectrum. Set

$$\Omega_{n,m} = \{\omega \in \Omega : Q^\omega_n P_m \text{ has full rank}\},$$

where Q^ω_n is the canonical projection from $\mathscr{H}$ onto $\mathscr{H}^\omega_n$, which is the minimal closed A^ω-invariant subspace containing the vector space $P_n\mathscr{H}$, and $\sigma^\omega_n(\cdot) = tr(P_n E^\omega(\cdot)P_n)$ is the trace measure. The set $\mathcal{M}$ is maximal subset of $\mathcal{N}$ such that for $n \in \mathcal{M}$, the measure σ^ω_n is not equivalent to Lebesgue measure. The main result proved here is the following theorem:

Theorem : *Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{N}$ be a countable set, $(\Omega, \mathscr{B}, \mathbb{P})$ be a probability space and $N \in \mathbb{N}$ be given. Let $\{P_n\}_{n\in\mathcal{N}}$ be a collection of rank N projections satisfying $\sum_{n\in\mathcal{N}} P_n = I$ and $\{\omega_n\}_{n\in\mathcal{N}}$ are independent real bounded random variables on $(\Omega, \mathscr{B}, \mathbb{P})$ with absolutely continuous distribution. Let $\{A^\omega\}_{\omega\in\Omega}$ be a family of operators defined by $A^\omega = A + \sum_{n\in\mathcal{N}} \omega_n P_n$, then*

1. *For $n, m \in \mathcal{M}$, we have $\mathbb{P}(\Omega_{n,m}) \in \{0, 1\}$.*

2. *Let $n, m \in \mathcal{M}$ such that $\mathbb{P}(\Omega_{n,m} \cap \Omega_{m,n}) = 1$, then for almost all $\omega \in \Omega$ the restrictions onto absolutely continuous part $E^{\omega}_{ac}A^{\omega}|_{\mathcal{H}^{\omega}_n}$ and $E^{\omega}_{ac}A^{\omega}|_{\mathcal{H}^{\omega}_m}$ are equivalent.*

3. *Let $n, m \in \mathcal{M}$ such that $\mathbb{P}(\Omega_{n,m} \cap \Omega_{m,n}) = 1$, then for almost all $\omega \in \Omega$ the trace measures σ^{ω}_n and σ^{ω}_m are equivalent as Borel measures.*

4. *Let $\mathbb{P}(\Omega_{n,m} \cap \Omega_{m,n}) = 1$ for any $n, m \in \mathcal{M}$, then for almost all $\omega \in \Omega$, $E^{\omega}_{sing}\mathcal{H} = E^{\omega}_{sing}\mathcal{H}^{\omega}_n$ for any $n \in \mathcal{M}$.*

The thesis is divided into four chapters:

1. In the first chapter, preliminaries on measure spaces, probability spaces, Hilbert spaces and spectral theory for self adjoint operators is provided. The chapter is designed to be self-contained. Most of the theorems stated are fairly standard and their proofs can be found in [11, 21, 22, 28, 69, 73, 75, 94, 95].

2. In the second chapter, techniques related to identifying spectrum are kept. Most of the work are done using Herglotz functions and their generalisation. So Holomorphic functional calculus is an important tool. Results about zeros of holomorphic function are needed for statements involving uniqueness and classification of the spectral measures. Most of the results presented are from various sources, such as [8, 9, 10, 12, 16, 37, 44, 68, 74, 78, 82].

3. In the third chapter, an introduction is given to the Anderson Model and important results related to spectral structure for certain class of random operators. Some important examples and results pertaining to a class of operators that we are interested in are given. In addition some necessary conditions are also given.

4. In the fourth chapter, the theorem stated earlier is proved. The essential part of the proof are:

- Set of $\omega \in \Omega$ where the analysis may not work is measure zero set.
- Next step is to set up a condition which will state when the two spectral measures (say $P_n E^\omega(\cdot) P_n$ and $P_m E^\omega(\cdot) P_m$) can be compared. The event $\Omega_{n,m}$ gives this criteria, hence first part of the theorem is important. The proof is done by showing that the event $\Omega_{n,m}$ is independent of any other perturbations.
- To show the equivalence of the absolute continuous part of measure, looking at perturbations by two projection is enough. Since the perturbation involved are finite ranked, the problem involve matrices only.
- In case of trace measure, the problem is handled by solving for two perturbations.
- For the multiplicity results, equivalence of trace measure is used, and cyclic vector for each of the Hilbert subspace for singular part of measures are identified.

List of Symbols

$(\Omega, \mathcal{B}, \mathbb{P})$ Probability space. 24

A^ω The main class of operator in consideration. The operator has form $A + \sum_n \omega_n P_n$ where A is self adjoint and $\{P_n\}_n$ are finite rank projections. 53

E^T_{sing} Spectral projection onto the singular part of spectrum for the operator T.

E^T Spectral projection for the self adjoint operator T. 33

E^ω_{ac} The spectral projection onto the absolutely continuous part of the spectrum of the self adjoint operator A^ω. 57

E^ω_{sing} The spectral projection onto the singular part of the spectrum of the self adjoint operator A^ω. 57

E^ω The spectral measure for the self adjoint operator A^ω. 57

E_{ac} Projection onto the absolutely continuous part of the spectral measure E. 35

E_{pp} Projection onto the pure point part of the spectral measure E. 35

E_{sc} Projection onto the singular continuous part of the spectral measure E. 35

$F_\mu(z)$ Borel transform $\int \frac{d\mu(x)}{x-z}$ of the measure μ. 40

$G^\omega_{ij}(z)$ Green's function for the operator A^ω associated with the projection i and j.

$G^{\omega,\mu,p}_{nm}(z)$ Green's function for the operator $A^\omega + \mu P_p$ associated with the projection n and m. 57

$L^2(\mathbb{R}, \nu, V)$ Hilbert space of V-valued function which are L^2 integrable w.r.t ν. 34

Q^ω_n The canonical projection from $\mathscr{H}$ into $\mathscr{H}^\omega_n$. 57

$S^\perp$ Linear subspace containing vectors orthogonal to any vectors of S.

Δ discrete Laplacian on $\mathbb{Z}^d$. 47

$\mathscr{H}^\omega_n$ closed A^ω-invariant subspace of $\mathscr{H}$ containing $P_n\mathscr{H}$. 56

$\mathscr{H}$ Separable Hilbert space. 27

$\Im T$ Defined as $\frac{1}{2\imath}(T - T^*)$. 31

$\Omega_{n,m}$ Set of $\omega \in \Omega$ such that $rank(Q^\omega_n P_m) = rank(P_m)$. 57

$\Re T$ Defined as $\frac{1}{2}(T + T^*)$. 31

$\Sigma^\omega_n(\cdot)$ The measure $P_n E^\omega(\cdot) P_n$. 57

δ_n Standard basis element of the Hilbert space $\ell^2(\mathbb{Z}^d)$. 48

$\int f d\mu$ Integration of the function f with respect to the measure μ. 21

$\mathcal{M}$ Set of indices $n \in \mathcal{N}$ such that the trace measure σ^ω_n is not equivalent to Lebesgue measure for almost all ω. 57

$\mathcal{N}$ Indexing set of projection. Here it is always countable.

$\mathscr{B}_\mathbb{R}$ The Borel σ-algebra over $\mathbb{R}$. 18

$\mu \perp \nu$ The measure μ and ν are mutually singular to each other. 23

μ_{ac} Absolutely continuous component of μ w.r.t Lebesgue measure. 24

μ_{sing} Singular component of μ w.r.t Lebesgue measure. 24

$\nu \ll \mu$ The measure ν is absolutely continuous with respect to μ. 23

$|\phi\rangle\langle\phi|$ The orthogonal projection onto the subspace $\mathbb{C}\phi$. 30

$\rho(T)$ Resolvent set for the operator T. 32

$\sigma(T)$ Spectrum for the operator T. 32

$\sigma_n^\omega(\cdot)$ The measure $tr(P_n E^\omega(\cdot) P_n)$. 57

$f(x + \iota 0)$ The limit $\lim_{\epsilon \downarrow 0} f(x + \iota\epsilon)$ at $x \in \mathbb{R}$ of the holomorphic function f defined on $\mathbb{C}^+$. 38

$f \perp g$ The vectors f and g are orthogonal to each other in the Hilbert space. 28

$f d\mu$ The measure $\mathcal{B} \ni E \mapsto \int_E f d\mu$. 23

$ker(T)$ Kernel of the operator T. 30

$range(T)$ Range of the operator T. 30

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Chapter 1

Preliminaries

In this chapter most of the basics are covered. This chapter has the definitions and some results related to measure theory, probability theory and Hilbert space theory. In the last section Spectral theorem for an unbounded self-adjoint operator is stated and continuous functional calculus is defined. Most of these can be found in [11, 21, 22, 28, 69, 73, 75, 94, 95].

1.1 Measure theory

In this section σ-algebra and basic measure theory are presented. Some examples of measures and the terminology which will be used later are stated. Since a probability space is a finite measure space, many concepts which are used in the case of probability are given in this section itself, but used as part of probability space.

1.1.1 σ-algebra

An *algebra* of sets of X is a non-empty collection $\mathcal{G}$ of subsets of X which is closed under finite union and complements, i.e if $A_1, \cdots, A_n \in \mathcal{G}$, then so is $\cup_{i=1}^{n} A_i$ and $\cap_{i=1}^{n} A_i^c$. A σ-algebra is an algebra which is closed under countable union also. The *σ-algebra generated by* Ω (a collection of sets) is the smallest σ-algebra $\sigma(\Omega)$ containing Ω.

The *Borel σ-algebra* $\mathscr{B}_{\mathbb{R}}$ on $\mathbb{R}$ is the σ-algebra generated by open sets. Here we will only deal with σ-algebra related to Borel σ-algebra of $\mathbb{R}$.

The *product σ-algebra* on the set $\prod_{i\in I} X_i$ where $\{X_i\}_{i\in I}$ is an indexed collection of sets and $\mathcal{M}_i$ are σ-algebra for each X_i, is the σ-algebra generated by

$$\{p_i^{-1}(A_i) : A_i \in \mathcal{M}_i, i \in I\},$$

where $p_j : \prod_{i\in I} X_i \to X_j$ is the projection map onto j^{th} coordinate. The indexing set can be uncountable (but we will deal with countable set only). Only the product Borel σ-algebra on the space of real sequences are needed, i.e the space is $\mathbb{R}^{\mathcal{N}} := \{\{x_i\}_{i\in\mathcal{N}} : x_i \in \mathbb{R}\}$ and the σ-algebra $\mathcal{B}(\mathbb{R}^{\mathcal{N}})$(also denoted as $\otimes_{n\in\mathcal{N}}\mathscr{B}_{\mathbb{R}}$) is the product σ-algebra generated by Borel σ-algebra $\mathscr{B}_{\mathbb{R}}$.

1.1.2 Measures

Definition 1.1.1. *Let $\mathcal{M}$ be a σ-algebra on a set X. A* **measure** *on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0,\infty]$ such that*

1. $\mu(\phi) = 0$
2. *If $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty}\mu(M_i)$*

Examples 1.1.2. (Dirac measure) *One the space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$, the Dirac measure $\delta_x : \mathscr{B}_{\mathbb{R}} \to \{0,1\}$ at $x \in \mathbb{R}$, is given by*

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & otherwise \end{cases}.$$

Some other example are $\sum_{n\in\mathbb{Z}}\delta_n$ and $\sum_{\substack{n\in\mathbb{N}\\0\le j\le 2^n}} 2^{-(n+j)}\delta_{\frac{j}{2^n}}$. One crucial difference between these two example is that, in the first case the total measure is infinite, but the set supporting the *measure is discrete, and in the second case the total measure is finite, but the set supporting the measure is dense in* $[0, 1]$.

A *measure space* is a triple $(X, \mathcal{M}, \mu)$ where μ is a measure over a σ-algebra $\mathcal{M}$ for the set X. If $\mu(X) < \infty$, then μ is finite measure. If $X = \cup_{i=1}^{\infty} X_i$ where $X_i \in \mathcal{M}$ and $\mu(X_i) < \infty$, then μ is σ-finite measure. A set $E \in \mathcal{M}$ such that $\mu(E) = 0$ is called *null set*. If a statement about points $x \in X$ holds except for x in some null set, then that statement is true *almost everywhere* or *almost all x* (when the measure needs to be specified *μ-almost everywhere*). A measure space is *complete* if the σ-algebra contains all the subsets of null sets, i.e if $N \in \mathcal{M}$ such that $\mu(N) = 0$, then $F \in \mathcal{M}$ for all $F \in \mathcal{P}(N)$, where the notation $\mathcal{P}(N)$ denotes the power set of N.

Definition 1.1.3. *An* **outer measure** *on a non-empty set X is a function* $\mu^* : \mathcal{P}(X) \to [0, \infty]$ *satisfying*

1. $\mu^*(\phi) = 0$
2. $\mu^*(A) \leq \mu^*(B)$ *if* $A \subset B$
3. $\mu^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

If μ^* is an outer measure on X, a set $A \subset X$ is μ^*-measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad \forall\ E \in \mathcal{P}(X).$$

Theorem 1.1.4. [Carathéodary's Theorem] *If* μ^* *is an outer measure on X, the collection* $\mathcal{M}$ *of* μ^**-measurable sets is a* σ*-algebra and the restriction of* μ^* *to* $\mathcal{M}$ *is a complete measure.*

If μ^* is an outer measure on X and $\mathcal{M}$ is the σ-algebra of μ^*-measurable sets, then denote $\mu = \mu^*|_{\mathcal{M}}$, and the measure space is $(X, \mathcal{M}, \mu)$. Some measures arising as a consequence of this theorem are:

Examples 1.1.5. (Lebesgue measure) *The outer measure is defined by*

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : A \subset \bigcup_{i=1}^{\infty} (a_i, b_i), \;\; a_i < b_i \ \forall i \right\}$$

and Borel sets are m^*-measurable.

Examples 1.1.6. (Hausdorff measure) *Given $0 < \alpha \leq 1$, one can define the outer measure*

$$h_\alpha^*(A) = \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i)^\alpha : A \subset \bigcup_{i=1}^{\infty}(a_i, b_i),\ \ a_i < b_i\ \forall i\right\}$$

here also Borel sets are h_α^*-measurable.

The outer measures h_1^* and m^* are the same. The case $\alpha = 0$ is defined as counting measure. A larger class of measures arises by taking $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that f is increasing and $f(0) = 0$, then defining the outer measure by

$$h_f^*(A) = \inf\left\{\sum_{i=1}^{\infty} f(b_i - a_i) : A \subset \bigcup_{i=1}^{\infty}(a_i, b_i),\ \ a_i < b_i\ \forall i\right\}.$$

So there exists a uncountable family of measure spaces $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, h_f)$.

Borel Measures on $\mathbb{R}$ are those measures whose domain is $\mathscr{B}_{\mathbb{R}}$. So Dirac measure, Lebesgue measure and Hausdorff measures are example of Borel measures. A large class of Borel measures can be constructed by:

Theorem 1.1.7. *If $F : \mathbb{R} \to \mathbb{R}$ is an increasing, right continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all a, b. Let G is another such function, then $\mu_F = \mu_G$ if and only if $F - G$ is constant. Conversely if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel set, define*

$$F(x) = \begin{cases} \mu((0, x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x, 0]) & x < 0 \end{cases}$$

then F is increasing, right continuous and $\mu = \mu_F$.

In case of finite measure the theorem gives an one-to-one correspondence with bounded right continuous increasing function which is zero at 0. For Lebesgue measure m the function is

$F(x) = x$, and for Dirac measure δ_x it is:

$$F_x(t) = \begin{cases} 1 & t \geq x \\ 0 & t < x \end{cases}.$$

The Hausdorff measure space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, h_\alpha)$ for $\alpha < 1$ are not σ-finite, so such function does not exist, but the result holds when the measure is restricted onto a subset with finite measure.

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces. Define the outer measure $(\mu \otimes \nu)^*$ on the set $X \times Y$ by

$$(\mu \otimes \nu)^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu(E_i)\nu(F_i) : A \subset \bigcup_{i=1}^{\infty} E_i \times F_i, E_i \in \mathcal{M}, F_i \in \mathcal{N}\ \forall i \right\}.$$

This set of $(\mu \otimes \nu)^*$-measurable sets contains the σ-algebra $\mathcal{M} \otimes \mathcal{N}$ and so define the *product measure space* $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \otimes \nu)$.

1.1.3 Integration

Given two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$, a function $f : X \to Y$ is *measurable* if $f^{-1}(E) := \{x \in X : f(x) \in E\} \in \mathcal{M}$ for all $E \in \mathcal{N}$. Now define

$$M^+(X) = \{f : X \to [0, \infty] : f \text{ is measurable}\},$$

here $[0, \infty]$ is equipped with the Borel σ-algebra. For any set $A \in \mathcal{M}$ define the *characteristic function* as

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & otherwise \end{cases}.$$

Define a linear functional Ψ such that for any $A \in \mathcal{M}$

$$\Psi(\chi_A) = \int \chi_A d\mu = \mu(A).$$

This take care of any finite linear combination of characteristic functions (called simple functions). Next for $f \in M^+(X)$ set

$$\int f d\mu = \sup \{\Psi(\phi) : \phi \leq f,\ \phi \text{ is simple function with positive coefficient}\}.$$

A real measurable function f is called *μ-integrable* if $\int |f| d\mu$ is finite and *extended μ-integrable* if at least one of $\int f_{\pm} d\mu$ (here $f_{\pm}(x) = \pm f(x)\chi_{\{x:\pm f(x)>0\}}(x)$) is finite. In either of the cases define the integral by

$$\int f d\mu = \int f_{+} d\mu - \int f_{-} d\mu.$$

Similarly a complex measurable function f is integrable if $\int |f| d\mu$ is finite and define

$$\int f d\mu = \int \Re f d\mu + \iota \int \Im f d\mu.$$

Set of complex integrable functions is denoted by $L^1(X, \mu)$.

Theorem 1.1.8. [Fubini-Tonelli Theorem] *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are two σ-finite measure space.*

1. *(Tonelli) If $f \in M^+(X \times Y)$, then the functions $g(x) = \int f(x, \cdot) d\nu$ and $h(y) = \int f(\cdot, y) d\mu$ are in $M^+(X)$ and $M^+(Y)$ respectively, and*

$$\int f d(\mu \times \nu) = \int g d\mu = \int h d\nu.$$

2. *(Fubini) If $f \in L^1(\mu \times \nu)$, then $f(x, \cdot) \in L^1(\nu)$ for almost every $x \in X$, $f(\cdot, y) \in L^1(\mu)$ for almost every $y \in Y$, the functions $g(x) = \int f(x, \cdot) d\nu$ and $h(y) = \int f(\cdot, y) d\mu$ are defined almost everywhere and belong to $L^1(\mu)$ and $L^1(\nu)$ respectively. Finally*

$$\int f d(\mu \times \nu) = \int \left(\int f(x, y) d\mu(x) \right) d\nu(y) = \int \left(\int f(x, y) d\nu(y) \right) d\mu(x).$$

1.1.4 Measure class

Definition 1.1.9. *Let $\mathcal{M}$ be a σ-algebra on the set X. A* **signed measure** *on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that*

1. *$\nu(\phi) = 0$.*

2. *ν assumes at most one of the values $\pm\infty$.*

3. *If* $\{E_j\}$ *is a sequence of disjoint sets in* $\mathcal{M}$*, then* $\nu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \nu(E_i)$*, where the later sum is absolutely convergent if* $\nu(\cup_{i=1}^{\infty} E_i)$ *is finite.*

The definition stated previously is a special case of this and can be viewed as *positive measure*. Since for signed measure space $(X, \mathcal{M}, \mu)$, the measure can take any value, a set E is *null set* if $\mu(F) = 0$ for all $F \in \mathcal{P}(E)$ such that $F \in \mathcal{M}$.

Two signed measures μ and ν on $(X, \mathcal{M})$ are *mutually singular* (μ is singular with respect to ν and vice-versa), if there exists $E, F \in \mathcal{M}$ such that $E \cap F = \phi$, $E \cup F = X$, E is a null set of μ and F is a null set of ν. This relation is symmetric and will denote by $\mu \perp \nu$. *Jordan decomposition theorem* states that any signed measure ν can be decomposed in terms of two unique positive measures $\nu^{\pm}$ such that $\nu = \nu^{+} - \nu^{-}$ and $\nu^{+} \perp \nu^{-}$. The *total variation measure* denoted as $|\nu|$ is defined by $|\nu| = \nu^{+} + \nu^{-}$.

Each of the Hausdorff measures h_{α} restricted to finite measure subsets are mutually singular with respect to each other. In the case of measures of the form $\sum_i \alpha_i \delta_{x_i}$, two such measures are mutually singular if the set of x_i are disjoint.

Let ν be a signed measure and μ a positive measure on $(X, \mathcal{M})$. The measure ν is *absolutely continuous* with respect to μ if $\nu(E) = 0$ for every $E \in \mathcal{M}$ such that $\mu(E) = 0$. This is denoted as $\nu \ll \mu$.

For a positive measure μ on the measure space $(X, \mathcal{M})$, let $f : X \to \mathbb{R}$ be an extended μ-integrable function and define $\nu(E) = \int_E f d\mu$ for $E \in \mathcal{M}$. This makes ν a signed measure and $\nu \ll \mu$. The notation $f d\mu$ will be used to denote $\nu(E) = \int_E f d\mu$.

Theorem 1.1.10. [Lebesgue-Radon-Nikodym Theorem] *Let* ν *be a finite signed measure and* μ *a* σ*-finite positive measure on* $(X, \mathcal{M})$*. There exist unique* σ*-finite signed measures* λ, ρ *on* $(X, \mathcal{M})$ *such that*

$$\lambda \perp \mu, \ \rho \ll \mu, \ \& \ \nu = \lambda + \rho$$

moreover there exists an extended μ*-integrable function* $f : X \to \mathbb{R}$ *such that* $d\rho = f d\mu$*, and* *any two such functions are equal μ-almost everywhere.*

The decomposition $\nu = \lambda + \rho$ where $\lambda \perp \mu$ and $\rho \ll \mu$ is called *Lebesgue decomposition* of ν with respect of μ. In case $\nu \ll \mu$, the theorem implies $d\nu = f d\mu$ for some extended μ-integrable function. In this case, the function f is called *Radon-Nikodym derivative* of ν with respect to μ and denoted by $\frac{d\nu}{d\mu}$.

Given a signed measure μ on $\mathbb{R}$ we will use the decomposition

$$\mu = \mu_{ac} + \mu_{sing}$$

to denote the Lebesgue-Radon-Nikodym decomposition for μ with respect to Lebesgue measure. The measure μ_{ac} is absolutely continuous with respect to Lebesgue measure and μ_{sing} is the singular with respect to Lebesgue measure.

1.2 Probability theory

In this section the basics of probability theory are recalled. Notion of independence and tail events are defined for series of random variables.

A *probability space* is a measure space $(\Omega, \mathcal{B}, \mathbb{P})$ where the measure $\mathbb{P}$ is positive and $\mathbb{P}[\Omega] = 1$. Ω is called sample space and elements of σ-algebra $\mathcal{B}$ are called *events*.

A *random variable* X on the space $(S, \mathcal{M})$ is a measurable function from probability space Ω to S. In case of real/complex random variable, the σ-algebra on $\mathbb{R}/\mathbb{C}$ will always be Borel σ-algebra. Later random variables are also denoted by X^ω, which will also be used as evaluation at $\omega \in \Omega$ (most of the random variables are almost everywhere defined, so any evaluation is always done in complement of some set of measure zero).

Expectation of a random variable X is the integration with respect to the probability measure and is denoted by

$$\mathbb{E}[X] = \int X d\mathbb{P} \quad \text{or} \quad \mathop{\mathbb{E}}_{\omega}[X^\omega] = \int X^\omega d\mathbb{P}(\omega).$$

Let X be a real random variable on the probability space $(\Omega, \mathcal{B}, \mathbb{P})$, then the measure $\mathbb{P}_X$ defined by

$$P_X(E) = \mathbb{P}(\{\omega : X^\omega \in E\}) \qquad \forall E \in \mathscr{B}_{\mathbb{R}},$$

is a probability measure on $\mathbb{R}$ and is called *distribution* of X. When the distribution is absolutely continuous with respect to Lebesgue measure, then the distribution is said to be *absolutely continuous distribution*. For a sequence of real/complex random variables $\{X_i\}_{i=1}^N$, define the joint distribution by

$$\mathbb{P}_N[E_1 \times \cdots \times E_N] = \mathbb{P}[\{\omega : X_i^\omega \in E_i \ \forall i\}] \qquad \forall E_i \in \mathscr{B}_{\mathbb{R}/\mathbb{C}}.$$

For a real/complex random variable X, the notation $X^{-1}(E) = \{\omega : X^\omega \in E\}$ for $E \in \mathscr{B}_{\mathbb{R}/\mathbb{C}}$, will be used.

Definition 1.2.1. *For a probability space* $(\Omega, \mathcal{B}, \mathbb{P})$

1. *Two events $E_1, E_2 \in \mathcal{B}$ are independent if* $\mathbb{P}[E_1 \cap E_2] = \mathbb{P}[E_1]\mathbb{P}[E_2]$.
2. *Two real/complex random variables X_1, X_2 are independent if for every $E_1, E_2 \in \mathscr{B}_{\mathbb{R}/\mathbb{C}}$, we have*

$$\mathbb{P}[\{\omega : X_1^\omega \in E_1, X_2^\omega \in E_2\}] = \mathbb{P}[X_1^{-1}(E_1)]\mathbb{P}[X_2^{-1}(E_2)].$$

3. *Two sub-σ-algebra $\mathcal{F}$ and $\mathcal{G}$ are independent if for each $F \in \mathcal{F}$ and $G \in \mathcal{F}$, F and G are independent.*

A sequence of real/complex random variables $\{X_i\}_{i=1}^N$ are independent if

$$\mathbb{P}\left[\bigcap_{i=1}^{N} X_i^{-1}(E_i)\right] = \prod_{i=1}^{N} \mathbb{P}_{X_i}[E_i] \qquad \forall\ E_i \in \mathscr{B}_{\mathbb{R}/\mathbb{C}}.$$

A sequence of random variables $\{X_i\}_i$ is said to have *identical* distribution if the probability measures $\mathbb{P}_{X_i}$ are the same.

Theorem 1.2.2. [Kolmogorov Extension theorem] *Let $\mathscr{I}$ be a set (can be uncountable) and let $\mathbb{P}_{\alpha_1,\cdots,\alpha_n}$ be a Borel probability measure on $\mathbb{R}^n$ for each $\alpha_1, \cdots, \alpha_n \in \mathscr{I}$ and $n \in \mathbb{N}$. Assume that this family of measure satisfies:*

1. *If $\pi \in S_n$ be a permutation of set $\{1, \cdots, n\}$ and $f_\pi : \mathbb{R}^n \to \mathbb{R}^n$ defined as $f_\pi(x_1, \cdots, x_n) = (x_{\pi(1)}, \cdots, x_{\pi(n)})$, then*

$$\mathbb{P}_{\alpha_{\pi(1)},\cdots,\alpha_{\pi(n)}}[S] = \mathbb{P}_{\alpha_1,\cdots,\alpha_n}[f_\pi^{-1}(S)]$$

   for all $S \in \mathscr{B}_{\mathbb{R}^n}$.

2. *Let $\sigma_{n+m,n} : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the projection $\sigma_{n+m,n}(x_1, \cdots, x_{n+m}) = (x_1, \cdots, x_n)$, then*

$$\mathbb{P}_{\alpha_1,\cdots,\alpha_{n+m}}[\sigma_{n+m,n}^{-1}(S)] = \mathbb{P}_{\alpha_1,\cdots,\alpha_n}[S]$$

   for all $S \in \mathscr{B}_{\mathbb{R}^n}$.

Then there exists a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and real random variables $\{X_\alpha\}_{\alpha\in\mathscr{I}}$ such that for any finite $(\alpha_1, \cdots, \alpha_n)$,

$$\mathbb{P}_{\alpha_1,\cdots,\alpha_n}[S] = \mathbb{P}\left[\phi_{\alpha_1,\cdots,\alpha_n}^{-1}(S)\right]$$

where $\phi_{\alpha_1,\cdots,\alpha_n} : \Omega \to \mathbb{R}^n$ is the map $\omega \mapsto (X_{\alpha_1}^\omega, \cdots, X_{\alpha_n}^\omega)$.

So given a sequence of probability measures $\{\mu_n\}_{n\in\mathcal{N}}$ on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$, the above theorem gives the existence of a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and a sequence of independent random variables $\{X_n\}_{n\in\mathcal{N}}$, such that $\mathbb{P}_{X_n} = \mu_n$ for each $n \in \mathcal{N}$. Through the proof of the theorem the probability space turns out to be $(\mathbb{R}^\mathcal{N}, \mathcal{B}(\mathbb{R}^\mathcal{N}), \otimes_{n\in\mathcal{N}}\mu_n)$ and so is called *product probability space*.

Given a sequence of events $\{E_n\}_{n\in\mathbb{N}}$ define

$$\limsup_n A_n = \bigcap_{n=1}^{\infty}\bigcup_{m\geq n} A_m \quad and \quad \liminf_n A_n = \bigcup_{n=1}^{\infty}\bigcap_{m\geq n} A_m.$$

Lemma 1.2.3. [Borel-Cantelli] *Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space and $A_n \in \mathcal{B}$ for $n \in \mathbb{N}$ be given. Then*

1. *If* $\sum_n \mathbb{P}[A_n] < \infty$*, then* $\mathbb{P}[\limsup_n A_n] = 0$.

2. *If* $\{A_n\}_n$ *are independent and* $\sum_n \mathbb{P}[A_n] = \infty$*, then* $\mathbb{P}[\limsup_n A_n] = 1$.

Given a sequence of real/complex random variables $\{X_i\}_{i\in\mathcal{N}}$, the σ-algebra generated $\{X_i\}_{i\in I}$ for some $I \subseteq \mathcal{N}$, is the σ-algebra generated by the collection $\{X_i^{-1}(E) : i \in I, E \in \mathscr{B}_{\mathbb{R}/\mathbb{C}}\}$, and it is denoted as $\sigma(\{X_i\}_{i\in I})$. Given a sequence of real/complex random variables $\{X_i\}_{i\in\mathbb{N}}$, $E \in \sigma(\{X_i\}_{i\in\mathbb{N}})$ is a *tail event* if $E \in \sigma(\{X_i\}_{i\geq n})$ for every $n \geq 0$.

Theorem 1.2.4. [Kolmogorov Zero-One law] *A tail event for a sequence of independent random variables has probability either zero or one.*

1.3 Hilbert space

The operators in consideration here are on separable complex Hilbert space. Some of the basic properties are listed here.

For a vector space V over $\mathbb{C}$, an *inner product* $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ is a function satisfying:

1. $\langle u, \alpha v + \beta w\rangle = \alpha\langle u, v\rangle + \beta\langle u, w\rangle$ for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{C}$,

2. $\langle \alpha u + \beta v, w\rangle = \bar{\alpha}\langle u, w\rangle + \bar{\beta}\langle v, w\rangle$ for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{C}$,

3. $\langle u, u\rangle \geq 0$ for $u \in V$ and $\langle u, u\rangle = 0 \implies u = 0$,

4. $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for $u, v \in V$.

The *norm* of $u \in V$ is defined by $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$. The pair $(V, \langle,\rangle)$ is called an inner product space.

Definition 1.3.1. *A* **complex Hilbert space** *is a complex inner product space* $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ *such that the metric* $d(x, y) = \|x - y\|$ *induced by the norm makes* $\mathscr{H}$ *a complete metric space.*

A **separable Hilbert space** *is a Hilbert space which has a countable dense subset.*

Let $\mathcal{S} \subseteq \mathscr{H}$ be a closed linear subspace, then the function $p_{\mathcal{S}}(h) = \inf\{||h - s|| : s \in \mathcal{S}\}$ is continuous and since $\mathcal{S}$ is convex the minimum is obtained for the map $s \mapsto ||h - s||$ for $s \in \mathcal{S}$. Denote the minimum by Ph. The map $h \mapsto Ph$ is linear, continuous and $||Ph|| \leq ||h||$ for every $h \in \mathscr{H}$. The map $P : \mathscr{H} \to \mathscr{H}$ is called *orthogonal projection* of $\mathscr{H}$ onto $\mathcal{S}$. To make the dependence on $\mathcal{S}$ on the projection explicit $P_{\mathcal{S}}$ is used.

Two vectors v, w in an Hilbert space $(\mathscr{H}, \langle , \rangle)$ are *orthogonal* if $\langle v, w \rangle = 0$ and is denoted as $v \perp w$. For $A, B \subseteq \mathscr{H}$, if $f \perp g$ for each $f \in A$ and $g \in B$ then the sets are orthogonal to each other and is denoted as $A \perp B$.

A *orthonormal* subset $\mathcal{O}$ of a Hilbert space $\mathscr{H}$ is a subset with the properties:

1. $||v|| = 1$ for each $v \in \mathcal{O}$,
2. $\langle v, w \rangle = 0$ if $v \neq w$ for $v, w \in \mathcal{O}$.

A *orthonormal basis* is a maximal orthonormal set. For a separable Hilbert space any orthonormal basis is countable.

Theorem 1.3.2. *Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of a separable Hilbert space $\mathscr{H}$, then*

1. *If $h \in \mathscr{H}$ and $h \perp e_n$ for all $n \in \mathbb{N}$, then $h = 0$.*
2. *For any $h \in \mathscr{H}$*
$$h = \sum_{n=1}^{\infty} \langle e_n, h \rangle \, e_n, \tag{1.1}$$
here the convergence is in norm.
3. *For $f, g \in \mathscr{H}$*
$$\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, e_n \rangle \, \langle e_n, g \rangle \tag{1.2}$$

Two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are *isomorphic* if a linear surjection $U : \mathcal{H} \to \mathcal{K}$ exists which satisfies

$$\langle Uf, Ug\rangle_{\mathcal{K}} = \langle f, g\rangle_{\mathcal{H}}$$

for all $f, g \in \mathcal{H}$. The map U is called *isomorphism* from $\mathcal{H}$ to $\mathcal{K}$. In particular U is a *isometry* (norm preserving).

1.3.1 Linear functional

On a complex Hilbert space $\mathscr{H}$, a linear functional $L : \mathscr{H} \to \mathbb{C}$ is continuous if and only if there exists a constant $c > 0$ such that $|L(h)| \leq c\,||h||$ for every $h \in \mathscr{H}$. Since a continuous linear functional follows the bound, it is also called *bounded linear functional* and

$$||L|| = \sup\{|L(h)| : ||h|| \leq 1,\ h \in \mathscr{H}\}.$$

Theorem 1.3.3. [Riesz Representation theorem] *If $L : \mathscr{H} \to \mathbb{C}$ is a bounded linear functional on a complex Hilbert space $\mathscr{H}$, then there exists a unique vector $l \in \mathscr{H}$ such that $L(h) = \langle l, h\rangle$ for every $h \in \mathscr{H}$. Moreover $||L|| = ||l||$.*

So this theorem guarantees that any bounded linear functional can be viewed as inner product with some elements of the Hilbert space. Next theorem is about abundance of linear functionals.

Theorem 1.3.4. [complex Hahn-Banach theorem] *Let $\mathscr{H}$ be a complex Hilbert space, p a real-valued function defined on $\mathscr{H}$ satisfying $p(\alpha u + \beta v) \leq |\alpha|p(u) + |\beta|p(v)$ for all $u, v \in \mathscr{H}$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| = 1$. Let λ be a linear functional defined on a subspace $\mathcal{S}$ of $\mathscr{H}$ satisfying $|\lambda(u)| \leq p(u)$ for $u \in \mathcal{S}$. Then there exists a liner functional Λ defined on $\mathscr{H}$, such that $|\Lambda(u)| \leq p(u)$ for all $u \in \mathscr{H}$ and $\Lambda(u) = \lambda(u)$ for all $u \in \mathcal{S}$.*

1.3.2 Bounded linear operator

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces, then the linear transformation $T : \mathcal{H} \to \mathcal{K}$ is said to be *bounded* if there exists $c > 0$ such that $||Th||_{\mathcal{K}} \leq c\, ||h||_{\mathcal{H}}$ for each $h \in \mathcal{H}$. Given a linear operator T define

$$ker(T) = \{v \in \mathcal{H} : Tv = 0\} \;\; and \;\; range(T) = \{v \in \mathcal{K} : \; v = Tw \; \exists w \in \mathcal{H}\}. \tag{1.3}$$

A linear operator T is continuous if and only if it is bounded and one can define

$$||T|| = \sup\{||Th|| : ||h|| \leq 1, \; h \in \mathcal{H}\}. \tag{1.4}$$

The set of bounded linear operator from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathscr{B}(\mathcal{H}, \mathcal{K})$ (set of bounded linear operators from complex Hilbert space $\mathscr{H}$ to itself is denoted by $\mathscr{B}(\mathscr{H})$). The space $\mathscr{B}(\mathcal{H}, \mathcal{K})$ together with the *operator norm* (1.4) forms a complete metric space. There are two other senses of convergence:

1. Given a sequence of operators $\{T_n\}_{n\in\mathbb{N}}$ in $\mathscr{B}(\mathscr{H})$, T_n is said to converge to T in the *strong operator topology* if $||(T_n - T)h|| \to 0$ for each $h \in \mathscr{H}$.

2. Given a sequence of operators $\{T_n\}_{n\in\mathbb{N}}$ in $\mathscr{B}(\mathscr{H})$, T_n is said to converge to T in the *weak operator topology* if $\langle g, (T_n - T)h\rangle \to 0$ for each $h, g \in \mathscr{H}$.

The equation (1.1) can be expressed as $\sum_{n=1}^{N} |e_n\rangle \langle e_n| \xrightarrow{N\to\infty} I$ in strong operator topology, where following *Dirac notation* the object $|\phi\rangle \langle\phi|$ is the orthogonal projection onto the subspace $\mathbb{C}\phi$.

Definition 1.3.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be two complex Hilbert spaces, a function $\phi : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ is a* **sesquilinear form** *if*

1. *$\phi(x, \alpha u + \beta v) = \alpha\phi(x, u) + \beta\phi(x, v)$ for all $x \in \mathcal{H}$, $u, v \in \mathcal{K}$ and $\alpha, \beta \in \mathbb{C}$.*

2. *$\phi(\alpha x + \beta y, u) = \bar{\alpha}\phi(x, u) + \bar{\beta}\phi(y, u)$ for all $x, y \in \mathcal{H}$, $u \in \mathcal{K}$ and $\alpha, \beta \in \mathbb{C}$.*

It is **bounded** *if there exists a constant M such that $|u(u, v)| \leq M\, ||u||_{\mathcal{H}}\, ||v||_{\mathcal{K}}$.*

The following theorem ensures that to define a bounded operator, one only needs to define a sesquilinear form.

Theorem 1.3.6. *If $\phi : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ is a bounded sesquilinear form with bound M, then there are unique operator $S \in \mathscr{B}(\mathcal{H}, \mathcal{K})$ and $T \in \mathscr{B}(\mathcal{K}, \mathcal{H})$ such that*

$$\phi(u, v) = \langle S u, v \rangle_{\mathcal{K}} = \langle u, T v \rangle_{\mathcal{H}} \tag{1.5}$$

for all $u \in \mathcal{H}$, $v \in \mathcal{K}$ and $\|S\|, \|T\| \leq M$.

Definition 1.3.7. *If $T \in \mathscr{B}(\mathcal{H}, \mathcal{K})$, then there exists a unique operator $S \in \mathscr{B}(\mathcal{K}, \mathcal{H})$ such that* (1.5) *holds and is called the* **adjoint** *of T. The adjoint of an operator T is denoted by T^*.*

Given $T \in \mathscr{B}(\mathscr{H})$, it can be decomposed as $T = \Re T + \imath \Im T$ where $\Re T = \frac{T+T^*}{2}$ and $\Im T = \frac{T-T^*}{2\imath}$. This decomposition has the property that $(\Re T)^* = \Re T$ and $(\Im T)^* = \Im T$.

Definition 1.3.8. *Let $T \in \mathscr{B}(\mathscr{H})$,*

1. *T is* **self-adjoint operator** *if $T^* = T$.*
2. *T is* **normal operator** *if $T^*T = TT^*$.*
3. *T is* **unitary operator** *if $T^*T = I = TT^*$.*
4. *T is* **idempotent operator** *if $T^2 = T$.*

If $T \in \mathscr{B}(\mathscr{H})$ is a idempotent operator, then it is an orthogonal projection of $\mathscr{H}$ onto $range(T)$.

1.3.3 Unbounded linear operator

Definition 1.3.9. *If $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces, a linear operator $T : \mathcal{H} \to \mathcal{K}$ is a function whose domain of definition is a linear subspace $dom(T) \subset \mathcal{H}$, such that $T(\alpha u + \beta v) = \alpha T u + \beta T v$ for $u, v \in dom(T)$ and $\alpha, \beta \in \mathbb{C}$. T is bounded if there is a constant $c > 0$ such that $\|Tu\| \leq c \|u\|$ for all $u \in dom(T)$.*

A linear operator T is said to be *densely defined* if $dom(T)$ is a dense subset of $\mathcal{H}$. The operator S is called an *extension* of T if $dom(T) \subset dom(S)$ and $Tu = Su$ for all $u \in dom(T)$, it is denoted by $T \subseteq S$. The graph of a linear operator $T : \mathcal{H} \to \mathcal{K}$ is the set:

$$graph(T) = \{(u, Tu) \in \mathcal{H} \times \mathcal{K} : u \in dom(T)\} \tag{1.6}$$

An linear operator $T : \mathcal{H} \to \mathcal{K}$ is *closed* if its graph is closed in $\mathcal{H} \times \mathcal{K}$. T is called *closable* if there exists a closed extension.

Here we will deal with operators of the form $S + T$ where both S and T are densely defined. So, let S, T be two linear operators from $\mathcal{H}$ to $\mathcal{K}$, then $S + T$ is defined on the domain $dom(S + T) = dom(S) \cap dom(T)$.

Definition 1.3.10. *If $T : \mathcal{H} \to \mathcal{K}$ is densely defined, let*

$$dom(T^*) = \{u \in \mathcal{K} : v \mapsto \langle u, Tv\rangle \text{ is a bounded linear functional on } dom(T)\}$$

Since $dom(T)$ is dense, for $u \in dom(T^*)$ there exists a unique $v \in \mathcal{H}$ such that

$$\langle u, Tw\rangle = \langle v, w\rangle \qquad \forall w \in dom(T)$$

and so denote $T^*u = v$.

Definition 1.3.11. *A densely defined operator $T : \mathscr{H} \to \mathscr{H}$ is* **self-adjoint** *if $dom(T) = dom(T^*)$ and $T = T^*$.*

To define inverse one needs to define composition of two linear operators. Let $T : \mathcal{H} \to \mathcal{K}$ and $S : \mathcal{K} \to \mathcal{L}$ be two linear operators, the linear operator $ST : \mathcal{H} \to \mathcal{L}$ is defined on $dom(ST) = T^{-1}dom(S)$ (here T^{-1} is the set theoretic inverse).

Definition 1.3.12. *An linear operator $T : \mathcal{H} \to \mathcal{K}$ is* **boundedly invertible** *if there is a bounded operator $S : \mathcal{K} \to \mathcal{H}$ such that $TS = I$ and $ST \subseteq I$ (I is an extension of ST).*

Definition 1.3.13. *For a linear operator $T : \mathcal{H} \to \mathcal{K}$, the* **resolvent set** *$\rho(T)$ is defined by $\{\lambda \in \mathbb{C} : T - \lambda$ is bounded invertible$\}$. The* **spectrum** *of T is the set $\sigma(T) = \mathbb{C} \setminus \rho(T)$.*

1.4 Functional calculus and spectral theorem

Definition 1.4.1. *A* **projection valued measure** *on a set* X *is a map* $E : \mathcal{M} \to \mathscr{P}(\mathscr{H})$*, where* $\mathcal{M}$ *is a* σ*-algebra on* X *and* $\mathscr{P}(\mathscr{H})$ *is collection of projections on the separable complex Hilbert space* $\mathscr{H}$*, which satisfies:*

1. $E(\phi) = 0$ *and* $E(X) = I$*,*
2. $E(Y \cap Z) = E(Y)E(Z)$ *for* $Y, Z \in \mathcal{M}$*,*
3. *Let* $\{Y_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{M}$ *of pairwise disjoint sets, then*
$$E\left(\bigcup_n Y_n\right) = \sum_n E(Y_n).$$

For a projection valued measure E on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, \mathscr{H})$ and $\phi, \psi \in \mathscr{H}$, the set function

$$E_{\phi,\psi}(W) = \langle \phi, E(W)\psi \rangle \qquad \forall W \in \mathscr{B}_{\mathbb{R}}$$

defines a signed Borel measure on $\mathbb{R}$ with total variation $\leq \|\phi\|\,\|\psi\|$. For a bounded measurable function $f : \mathbb{R} \to \mathbb{C}$, one can define the sesquilinear form $\Psi(\phi, \psi) = \int f dE_{\phi,\psi}$, and so there is an operator $A_f \in \mathscr{B}(\mathscr{H})$ such that

$$\Psi(\phi, \psi) = \left\langle \phi, A_f \psi \right\rangle \qquad \forall \phi, \psi \in \mathscr{H}.$$

The operator A_f is denoted by $\int f dE$.

Theorem 1.4.2. [Spectral theorem for self-adjoint operators] *For any self-adjoint operator* T *on the Hilbert space* $\mathscr{H}$ *there exists exactly one projection valued measure* E *on* $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, \mathscr{H})$ *such that*

1. $T = \int t dE(t)$*,*
2. *If* $A \in \mathscr{B}(\mathscr{H})$ *such that* $AT = TA$*, then* $AE(\Omega) = E(\Omega)A$ *for all* $\Omega \in \mathscr{B}_{\mathbb{R}}$*.*

The measure E *is called* **spectral measure** *for* T *and is denoted by* E^T*.*

In particular any $f \in C(\sigma(T))$ we have

$$f(T) = \int f(t) dE^T(t)$$

which defines the *continuous functional calculus*. One of the function that will keep appearing is the *resolvent* which is

$$G_T(z) = (T - z)^{-1} = \int \frac{1}{t - z} dE^T(t) \qquad \forall z \notin \sigma(T). \tag{1.7}$$

For $\phi, \psi \in \mathscr{H}$ we have

$$G_T(\phi, \psi; z) = \left\langle \phi, (T - z)^{-1} \psi \right\rangle = \int \frac{dE^T_{\phi,\psi}(t)}{t - z} \qquad \forall \Im z \neq 0.$$

For a Borel measure space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, \nu)$ and a Hilbert space $(V, \langle \cdot, \cdot \rangle)$ define

$$L^2(\mathbb{R}, \nu, V) = \left\{ f : \mathbb{R} \to V : \ f \text{ is measurable and } \int \|f(x)\|^2 \, d\nu(x) < \infty \right\}.$$

Next theorem helps in distinguishing the multiplicity from previous theorem.

Theorem 1.4.3. [Hahn-Hellinger Theorem] *Let E be a spectral measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, \mathscr{H})$. Then there exist mutually singular σ-finite measures $\nu_\infty, \nu_1, \nu_2, \cdots$ and an invertible isometry*

$$U : \mathscr{H} \to L^2(\mathbb{R}, \nu_\infty, \ell^2(\mathbb{N})) \oplus \sum_{n=1}^{\infty} L^2(\mathbb{R}, \nu_n, \mathbb{C}^n),$$

such that for all $A \in \mathscr{B}_{\mathbb{R}}$ and $f \in L^2(\mathbb{R}, \nu_\infty, \ell^2(\mathbb{N})) \oplus \sum_{n=1}^{\infty} L^2(\mathbb{R}, \nu_n, \mathbb{C}^n)$,

$$UE(A)U^{-1} f = \chi_A f.$$

If $\nu'_\infty, \nu'_1, \nu'_2, \cdots$ is another sequence of mutually singular measures then for each i, ν_i and ν'_i are absolutely continuous with respect to each other.

We will also need another decomposition of the Hilbert space under the action of a operator.

Definition 1.4.4. *Given a self adjoint operator T on a the separable Hilbert space $\mathscr{H}$, define the linear subspace*

$$\mathscr{H}_{pp} = \overline{\{\phi \in \mathscr{H} : \ \phi \text{ is a eigenvector of } T\}},$$

where the bar denotes the closed linear span of the set in the Hilbert space $\mathscr{H}$, and

$$\mathscr{H}_{ac} = \left\{\phi \in \mathscr{H} : \left\langle \phi, E^T(\cdot)\phi \right\rangle \textit{ is absolutely continuous w.r.t Lebesgue measure}\right\}.$$

The set $\mathscr{H}_{ac}$ is closed subspace of $\mathscr{H}$ (see [46, Chapter 10, Theorem 1.5]). Set $\mathscr{H}_{sc} = (\mathscr{H}_{pp} \oplus \mathscr{H}_{ac})^{\perp}$.

The canonical projections from $\mathscr{H}$ to $\mathscr{H}_{pp}$, $\mathscr{H}_{ac}$ and $\mathscr{H}_{sc}$ will be denoted by E_{pp}, E_{ac} and E_{sc} respectively. These subspaces are closed under the action of T. So the projections E_{ac}, E_{pp} and E_{sc} commutes with the spectral measure E^T itself, which provides the Lebesgue decomposition of the measure $\left\langle \phi, E^T(\cdot)\phi \right\rangle$ as

$$\left\langle \phi, E^T(\cdot)\phi \right\rangle = \left\langle E_{ac}\phi, E^T(\cdot)E_{ac}\phi \right\rangle + \left\langle E_{sc}\phi, E^T(\cdot)E_{sc}\phi \right\rangle + \left\langle E_{pp}\phi, E^T(\cdot)E_{pp}\phi \right\rangle,$$

where the measure $\left\langle E_{ac}\phi, E^T(\cdot)E_{ac}\phi \right\rangle$ is absolutely continuous with respect to Lebesgue measure, $\left\langle E_{pp}\phi, E^T(\cdot)E_{pp}\phi \right\rangle$ is sum of Dirac measures and $\left\langle E_{sc}\phi, E^T(\cdot)E_{sc}\phi \right\rangle$ is mutually singular to other two measures. The set $\sigma_{pp}(T)$ is spectrum of T restricted to $\mathscr{H}_{pp}$ and is called *pure point spectrum*. Similarly $\sigma_{ac}(T)$ is *absolute continuous spectrum* which is spectrum of T restricted to $\mathscr{H}_{ac}$ and $\sigma_{sc}(T)$ is the *singular continuous spectrum* which is spectrum of T restricted to $\mathscr{H}_{sc}$.

Chapter 2

Borel transform and its properties

In this chapter important properties of the Borel transform are listed. This is the main tool that is used to determine the properties of the spectral measure. We will extract the information about the spectral measures through the linear maps

$$P(A - z)^{-1}P : P\mathscr{H} \to P\mathscr{H}$$

for $z \in \mathbb{C}^+$, where A is a self-adjoint operator and P is a projection on the separable Hilbert space $\mathscr{H}$. These are termed as Matrix valued Herglotz function or Birman-Schwinger operators. Birman-Schwinger principle was developed for compact perturbation in [12, 81] and some notable applications can be found in [16, 51, 82]. Since we will be focusing on the case $rank(P) = N$, we will view them as matrix valued Herglotz functions.

In the first section we will setup the equations arising from single perturbation. These equations are the main reason to look at matrix-valued Herglotz functions. We will be working with Holomorphic functional calculus for self adjoint operators, so some properties of a class of holomorphic functions are needed. These properties are recalled in the second section.

The definition of Borel transform is presented in third section along with all it's properties. Fourth section contains their generalisation to matrix valued measures.

2.1 Perturbation by a single projection

Given the triple $(A, \{P_i\}_{i=1}^3, \mathscr{H})$, where A is a self-adjoint operator on the separable Hilbert space $\mathscr{H}$ and $\{P_i\}_{i=1}^3$ are three rank N projections with the property that $P_iP_j = 0$ if $i \neq j$, we set $A_\lambda = A + \lambda P_1$. We will follow the notation

$$G_{ij}(z) = P_i(A - z)^{-1}P_j \;\; and \;\; G_{ij}^\lambda(z) = P_i(A_\lambda - z)^{-1}P_j \qquad \forall z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.1}$$

So $G_{ij}(z)$ and $G_{ij}^\lambda(z)$ can be viewed as linear maps from $P_j\mathscr{H}$ to $P_i\mathscr{H}$. In this section whenever I appears, it is viewed as identity map on $P_1\mathscr{H}$. For example in case of $I - \lambda G_{11}(z)$, it is used as a linear map from $P_1\mathscr{H}$ to $P_1\mathscr{H}$. It is easy to check

$$\Im G_{11}(z) \geq 0 \;\; and \;\; \|G_{11}(z)\| \leq \frac{1}{\Im z} \qquad for \; \Im z > 0.$$

Using the resolvent equation $B^{-1} - C^{-1} = B^{-1}(C - B)C^{-1}$, we have for $\Im z > 0$

$$G_{11}^\lambda(z) = G_{11}(z)(I + \lambda G_{11}(z))^{-1}, \tag{2.2}$$

and

$$G_{ij}^\lambda(z) = G_{ij}(z) - \lambda G_{i1}(z)G_{1j}(z) + \lambda^2 G_{i1}(z)G_{11}^\lambda(z)G_{1j}(z) \qquad \forall (i, j) \neq (1, 1). \tag{2.3}$$

Another way of writing (2.2) is

$$(I - \lambda G_{11}^\lambda(z))(I + \lambda G_{11}(z)) = I \;\; \Leftrightarrow \;\; (I + \lambda G_{11}(z))(I - \lambda G_{11}^\lambda(z)) = I. \tag{2.4}$$

The equations (2.2),(2.3) and (2.4) will be used later for obtaining all the results related to spectral measures.

2.2 Herglotz functions and uniqueness

In this section we will consider holomorphic functions on the domain $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$. The class of holomorphic functions $f : \mathbb{C}^+ \to \mathbb{C}^+$ are called *Herglotz functions*. One of the important properties for Herglotz functions is their uniqueness upto constant.

The following theorems give such uniqueness for functions which are holomorphic inside the unit disc. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathcal{M}(\mathbb{D})$ (respectively $\mathcal{H}(\mathbb{D})$) denote the set of meromorphic (respectively holomorphic) functions $f : \mathbb{D} \to \mathbb{C}$.

Theorem 2.2.1. **[9, Theorem 1]** *There exists a non-constant function f in $\mathcal{M}(\mathbb{D})$ (respectively $\mathcal{H}(\mathbb{D})$) such that $\lim_{r\to 1} f(rz) = 0$ for $z \in E \subset S^1$ if and only if the outer measure of $E \cap B$ is zero for all open $B \subset S^1$.*

For any Herglotz function f, we can define $g : \mathbb{D} \to \mathbb{C}$ by $g(z) = f(\iota\frac{z-1}{z+1})$ and use the above theorem. So the set

$$A_\alpha = \{x \in \mathbb{R} : \lim_{\epsilon\downarrow 0} f(x + \iota\epsilon) = \alpha\} \qquad \forall \alpha \in \mathbb{C} \cup \{\infty\},$$

has zero Lebesgue measure. Next theorem is a statement about the existence of the limit.

Theorem 2.2.2. **[82, Theorem 11.4]** *For a Herglotz function f, the limit $\lim_{\epsilon\downarrow 0} f(x + \iota\epsilon)$ exists and is finite for almost all x (with respect to Lebesgue measure).*

We will denote

$$f(x + \iota 0) = \lim_{\epsilon\downarrow 0} f(x + \iota\epsilon), \tag{2.5}$$

wherever the limit exists and the above theorem guarantees its existence almost everywhere.

Any Herglotz function $f : \mathbb{C}^+ \to \mathbb{C}^+$ can be extended to $\tilde{f} : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C} \setminus \mathbb{R}$ by defining it as follows

$$\tilde{f}(z) = \begin{cases} f(z) & \Im z > 0 \\ \overline{f(\bar{z})} & \Im z < 0 \end{cases}.$$

2.3 Borel-Stieltjes transform

Since we are going to use matrix valued functions of the form (2.1), we are interested in their relation to the spectral measures. This connection is through Nevanlinna-Reisz-Herglotz representation of measures (see theorem 2.3.5). But first we need to define Borel-Stieltjes transform for a positive measure.

Definition 2.3.1. *Let μ be a positive measure on $\mathbb{R}$ satisfying the condition:*

$$\int \frac{d\mu(x)}{1+x^2} < \infty,$$

then the **Borel transform** *(or Borel-Stieltjes transform) of μ is the function:*

$$F_\mu(z) = \int_{\mathbb{R}} \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu(x) \qquad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

The Borel transform of a measure is an holomorphic function on $\mathbb{C}^\pm$ ($= \{z \in \mathbb{C} : \pm\Im z > 0\}$) and maps each component to itself (i.e $F_\mu : \mathbb{C}^\pm \to \mathbb{C}^\pm$).

The definition does not guarantee uniqueness of Borel transform. The theorem F. and M. Riesz [78] tells us when a Borel transform will be zero (since the map $\mu \mapsto F_\mu$ is linear, we only need to look at the kernel). Here we state the following version of the theorem

Theorem 2.3.2. [79, Theorem 17.13] *If μ is a Borel measure on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and if*

$$\int e^{\imath n\theta} d\mu(\theta) = 0 \qquad \forall n \in \mathbb{N},$$

then μ is absolutely continuous with respect to Lebesgue measure.

The theorem is stated for measures on S^1, but by using a simple transformation it can be used for Borel measures on $\mathbb{R}$. The version that will be used is

Corollary 2.3.3. *The Borel transform of any complex measure which is zero in $\mathbb{C}^+$ has to be absolutely continuous with respect to Lebesgue measure.*

Remark 2.3.4. *One can prove (see [45, Theorem 2.2]) that the total variation measure need to be equivalent to Lebesgue measure.*

Because of this we will work only with measures which are not equivalent to Lebesgue measure and so the Borel transform will be unique.

Theorem 2.3.5. [Herglotz Representation Theorem][65, Theorem 1.4.2] *Let $F : \mathbb{C}^+ \to \mathbb{C}^+$ be a holomorphic function, then there exists a non-negative number a, a real number b and a Borel measure μ satisfying*

$$\int \frac{d\mu(x)}{1+x^2} < \infty$$

such that

$$F(z) = az + b + \int \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu(x).$$

The triple (a, b, μ) is uniquely associated with F.

Next theorem provides some of the important properties of Borel transform:

Theorem 2.3.6. [82, Theorem 11.6] *Let F be a Borel transform of a measure μ. Then*

1. $\frac{1}{\pi}\Im F(x+\iota\epsilon)dx \to d\mu(x)$ *weakly, in the sense that*

$$\lim_{\epsilon\downarrow 0} \frac{1}{\pi} \int f(x) \Im F(x+\iota\epsilon)dx = \int f d\mu \qquad \forall f \in C_c(\mathbb{R}).$$

2. μ_{sing} *is supported on* $\{x : \lim_{\epsilon\downarrow 0} F(x+\iota\epsilon) = \infty\}$.

3. $d\mu_{ac}(x) = \frac{1}{\pi}\Im F(x+\iota 0)dx$.

Above results give a way to extract results about the absolute continuous part of the measure and provide the set where singular part of the measure lies. The only thing left is to extract the type of singular measure. For that Poltoratskii's theorem[74] is used. For the Borel measure μ satisfying $\int \frac{d\mu(x)}{1+|x|} < \infty$, we will use the notation

$$F_\mu(z) = \int \frac{d\mu(x)}{x-z} \qquad \forall z \in \mathbb{C} \setminus \mathbb{R},$$

that is a and b are zero in the representation obtained through the theorem 2.3.5.

Theorem 2.3.7. [Poltoratskii's theorem] [44, Theorem 1.1] *For any complex valued Borel measure μ on $\mathbb{R}$ and $f \in L^1(\mathbb{R}, d\mu)$,*

$$\lim_{\epsilon\downarrow 0} \frac{F_{f\mu}(x+\iota\epsilon)}{F_\mu(x+\iota\epsilon)} = f(x)$$

for almost all x with respect to μ_{sing}.

2.4 Matrix valued Herglotz functions

A *Matrix valued Herglotz function* $M : \mathbb{C}^+ \to M_n(\mathbb{C})$ is a function with each of the entries being holomorphic on the domain and $\Im(M(z)) \geq 0$ for $z \in \mathbb{C}^+$.

Analogous version of theorem 2.3.5 and 2.3.6 can be stated as follows:

Theorem 2.4.1. **[37, Theorem 5.4]** *Let $M : \mathbb{C}^+ \to M_n(\mathbb{C})$ be a matrix-valued Herglotz function, then*

1. *$M(z)$ has finite normal limits, i.e $M(x + \iota 0) = \lim_{\epsilon \downarrow 0} M(x + \iota\epsilon)$ exists for a.e $x \in \mathbb{R}$ (with respect to Lebesgue measure).*

2. *If each diagonal element $M_{ii}(z)$, $1 \leq i \leq n$, of $M(z)$ has zero normal limit on a fixed subset of $\mathbb{R}$ which has positive Lebesgue measure, then $M(z) = C_0$ where C_0 is a constant self-adjoint $n \times n$ matrix with 0 on the diagonal.*

3. *There exists a matrix-valued measure Σ on bounded Borel set of $\mathbb{R}$ satisfying*
$$\int \frac{\langle v, d\Sigma(x)v\rangle}{1 + x^2} < \infty \qquad \forall v \in \mathbb{C}^n,$$
such that the Nevanlinna-Reisz-Herglotz representation
$$M(z) = C + Dz + \int_{\mathbb{R}} \left(\frac{1}{x - z} - \frac{x}{1 + x^2} \right) d\Sigma(x) \qquad \forall z \in \mathbb{C}^+,$$
holds where
$$C = M(\iota) \ \text{ and } \ D = \lim_{\eta \to \infty} \frac{1}{\iota\eta} M(\iota\eta).$$

4. *The Stieltjes inversion formula for Σ is*
$$\lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_a^b \Im(M(x + \iota\epsilon))dx = \frac{1}{2}\left(\Sigma(\{b\}) + \Sigma(\{a\})\right) + \Sigma((a, b)).$$

5. *The absolutely continuous part of the measure is given by*

$$d\Sigma_{ac}(x) = \frac{1}{\pi}\Im(M(x+\iota 0))dx$$

6. *Any poles of* $M(z)$ *are simple and are located on the real axis.*

Finally using the fact that Σ is absolutely continuous with respect to the trace measure $\sigma(= tr(\Sigma))$, and using theorem 1.1.10 (Lebesgue-Radon-Nikodym theorem) we observe that there exists $M_\Sigma \in L^1(\mathbb{R}, \sigma; M_n(\mathbb{C}))$ such that

$$d\Sigma(x) = M_\Sigma(x)d\sigma(x). \tag{2.6}$$

Using theorem 2.3.7 for each of the entries of Σ, we get

$$\lim_{\epsilon\downarrow 0} \frac{1}{F_\sigma(x+\iota\epsilon)} F_\Sigma(x+\iota\epsilon) = M_\Sigma(x) \tag{2.7}$$

for almost all x w.r.t σ_{sing}. Here F_Σ denotes the Borel transform of Σ. Since we are working with non-negative measures, i.e the measures $\langle u, \Sigma(\cdot)u\rangle$ are non-negative for all $u \in \mathbb{C}^n$, we also have $M_\Sigma(x) \geq 0$ for almost all x with respect to σ.

The only transformation that will be used, as seen in (2.2), is analogous to linear fractional transform. For $A_{ij} \in M_n(\mathbb{C})$, such that $A_{21}^*A_{11} = A_{11}^*A_{21}$, $A_{22}^*A_{12} = A_{12}^*A_{22}$ and $A_{11}^*A_{22} - A_{21}^*A_{12} = I$, define the transformation

$$\tau(M) = (A_{11} - A_{12}M)(A_{21} - A_{22}M)^{-1},$$

for $M \in M_n(\mathbb{C})$ such that $\Im M \geq 0$. This transformation is important because

$$\Im\tau(M) = \left((A_{21} - A_{22}M)^{-1}\right)^* \Im M \left((A_{21} - A_{22}M)^{-1}\right),$$

hence positivity of the imaginary part is preserved, so if $M : \mathbb{C}^+ \to M_n(\mathbb{C})$ is a matrix valued Herglotz function, then so is $\tau(M(z))$. One other property that will be used is:

Lemma 2.4.2. [66, Lemma A.1] *Let* $A_{ij} \in M_n(\mathbb{C})$ $i, j = 1, 2$, *such that*

$$\Im\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix} \geq 0. \tag{2.8}$$

Then for $u, v \in \mathbb{C}^n$

$$\left|\left\langle u, \frac{A_{12} - A_{21}^*}{2\iota} v \right\rangle\right|^2 \leq \langle u, (\Im A_{11})u\rangle \langle v, (\Im A_{22})v\rangle . \tag{2.9}$$

as a consequence of (2.9)*, let* $v \in \mathbb{C}^n$ *be such that* $(\Im A_{22})v = 0$ *then*

$$A_{12}v = A_{21}^* v, \tag{2.10}$$

and $u \in \mathbb{C}^n$ *be such that* $(\Im A_{11})u = 0$ *then*

$$A_{21}u = A_{12}^* u. \tag{2.11}$$

So if $tr(\Im A_{22}) = 0$ *then* $A_{12} = A_{21}^*$ *and if* $tr(\Im A_{11}) = 0$ *then* $A_{21} = A_{12}^*$.

Proof. For any $u, v \in \mathbb{C}^n$. using (2.8) we have

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \Im A_{11} & \frac{A_{12}-A_{21}^*}{2\iota} \\ \frac{A_{21}-A_{12}^*}{2\iota} & \Im A_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle \geq 0$$

$$\Rightarrow \qquad 0 \leq \langle u, (\Im A_{11})u\rangle + \langle v, (\Im A_{22})v\rangle + 2\Re \left\langle u, \frac{A_{12} - A_{21}^*}{2\iota} v \right\rangle$$

Since we can choose $v = 0$ where as $u \in \mathbb{C}^n$ (similarly other way around) we require

$$\langle u, (\Im A_{11})u\rangle \geq 0 \ \& \ \langle v, (\Im A_{22})v\rangle \geq 0 \qquad \forall u, v \in \mathbb{C}^n.$$

This implies $\Im A_{11} \geq 0$ and $\Im A_{22} \geq 0$. Next replacing u by tu for $t \in \mathbb{R}$, we obtain

$$\langle u, (\Im A_{11})u\rangle t^2 + \langle v, (\Im A_{22})v\rangle + 2t\Re \left\langle u, \frac{A_{12} - A_{21}^*}{2\iota} v \right\rangle \geq 0.$$

Since this is valid for all t, we have

$$4\left(\Re \left\langle u, \frac{A_{12} - A_{21}^*}{2\iota} v \right\rangle\right)^2 - 4 \langle v, (\Im A_{22})v\rangle \langle u, (\Im A_{11})u\rangle \leq 0$$

giving us

$$\left|\Re \left\langle u, \frac{A_{12} - A_{21}^*}{2\iota} v \right\rangle\right| \leq \sqrt{\langle v, (\Im A_{22})v\rangle \langle u, (\Im A_{11})u\rangle}. \tag{2.12}$$

So in case $\left\langle u, \frac{A_{12}-A_{21}^*}{2\iota} v \right\rangle \neq 0$, choosing $\alpha = \frac{\left\langle u, \frac{A_{12}-A_{21}^*}{2\iota} v \right\rangle}{\left|\left\langle u, \frac{A_{12}-A_{21}^*}{2\iota} v \right\rangle\right|}$ and replacing u by αu we get (2.9).

For proving (2.10) let $v \in \mathbb{C}^n$ to be such that $(\Im A_{22})v = 0$ then by (2.9), for any $u \in \mathbb{C}^n$

$$\left\langle u, \frac{A_{12} - A_{21}^*}{2\iota} v \right\rangle = 0 \quad \Rightarrow \quad (A_{12} - A_{21}^*)v = 0.$$

Similarly for proving (2.11) assuming $u \in \mathbb{C}^n$ be such that $(\Im A_{11})u = 0$, using (2.9) for any $v \in \mathbb{C}^n$,

$$\left\langle u, \frac{A_{12} - A_{21}^*}{2\iota} v \right\rangle = 0 \quad \Rightarrow \quad (A_{12}^* - A_{21})u = 0.$$

Finally if $tr(\Im A_{11}) = 0$ then $\Im A_{11} = 0$ because $\Im A_{11} \geq 0$. So using (2.11) for any $u \in \mathbb{C}^n$

$$(A_{21} - A_{12}^*)u = 0.$$

Similarly if $tr(\Im A_{22}) = 0$ we have $(A_{12} - A_{21}^*)v = 0$ for any $v \in \mathbb{C}^n$. □

2.5 Spectral projection results

The spectral theorem stated in previous chapter is the most general version but a restricted case is only needed.

Theorem 2.5.1. [66, Theorem A.3] *Let T be a self-adjoint operator on a separable Hilbert space $\mathscr{H}$, and P be a rank N projection. Let $\{\delta_n\}_{n=1}^N$ be a basis of the vector space $P\mathscr{H}$ and $\mathscr{H}_P$ denotes the closed subspace generated by T containing $P\mathscr{H}$. Let E^T be the spectral projection associated to T through theorem 1.4.2. Then the map*

$$U : L^2(\mathbb{R}, PE^T P, \mathbb{C}^N) \to \mathscr{H}_P$$

defined by

$$U(f_1, \cdots, f_N) \mapsto \sum_{i=1}^{N} f_i(T)\delta_i.$$

is unitary and

$$UId = TU,$$

where Id is multiplication by identity on $L^2(\mathbb{R}, PE^T P, \mathbb{C}^N)$.

Proof. This map U is an injection because

$$0 = \|U(f_1, \cdots, f_n)\|_2^2 = \sum_{i,j=1}^{N} \left\langle f_i(T)\delta_i, f_j(T)\delta_j \right\rangle = \sum_{i,j=1}^{N} \int \bar{f}_i(x) f_j(x) d\mu_{ij}(x),$$

where $\mu_{ij}(\cdot)$ are the measures $\left\langle \delta_i, E^T(\cdot)\delta_j \right\rangle$, so the previous equation is giving us

$$\int \left\langle f(x), d(PE^T P)(x) f(x) \right\rangle = 0 \;\Rightarrow\; \|f\|_2^2 = 0,$$

where $f(x) = (f_1(x), \cdots, f_N(x))$, and $(PE^T(\cdot)P)_{ij} = \left\langle \delta_i, E^T(\cdot)\delta_j \right\rangle$. The map U is isometry, because (for $f = (f_1, \cdots, f_N)$, $g = (g_1, \cdots, g_N)$)

$$\begin{aligned} \langle Uf, Ug \rangle_{\mathscr{H}_P} &= \sum_{i,j=1}^{N} \left\langle f_i(T)\delta_i, g_j(T)\delta_j \right\rangle = \sum_{i,j=1}^{N} \int \bar{f}_i(x) g_j(x) d\mu_{ij}(x) \\ &= \int \left\langle f(x), d(PE^T P)(x) g(x) \right\rangle = \langle f, g \rangle_{L^2(\mathbb{R}, PE^T P, \mathbb{C}^N)} . \end{aligned}$$

Next we will prove that U is a surjection. Let $\phi \in \mathscr{H}_P$, then there exists a sequence $\{(f_{1m, \cdots, f_{Nm}})\}_{m=1}^{\infty}$ where $f_{im} \in C_c(\mathbb{R})$ such that

$$\sum_{i=1}^{N} f_{im}(T)\delta_i \xrightarrow{m\to\infty} \phi$$

in norm, so

$$\lim_{m\to\infty} \|\phi - U(f_{1m}, \cdots, f_{Nm})\|_2 = 0.$$

Finally

$$U(Idf) = \sum_{i=1}^{N} (T f_i(T))\delta_i = T \sum_{i=1}^{N} f_i(T)\delta_i = T(Uf),$$

giving us the identity $UId = TU$. □

Another result that will be used is the spectral averaging result.

Lemma 2.5.2. [Spectral Averaging][19, Corollary 4.2] *Let $E_\lambda(\cdot)$ be the spectral projection for the operator $A_\lambda = A + \lambda P$, where A is a self-adjoint operator and P is a rank N projection. Then for $M \subset \mathbb{R}$ such that $|M| = 0$ (lebesgue measure), we have $PE_\lambda(M)P = 0$ for a.e λ w.r.t Lebesgue measure.*

As a consequence of this we can leave any fixed (Lebesgue) measure zero set from the analysis and the results will still hold almost everywhere.

Chapter 3

Random operators for certain disordered systems

Disorder is part of almost every physical system. Every model developed to understand physical system is some kind of idealisation. Sometime just studying the idealised model is not enough to make prediction. So understanding the role of the disorder is an important topic in science. Given the very meaning of the term disorder, it is important to verify if disorder fundamentally changes the nature of solution from an idealised scenario. Whenever the solution does not change drastically, it is enough to look at the idealised model and only correction are needed to be estimated.

Any real life problem always has some amount of external noise and as part of modelling that noise is ignored. But often this creates a significant difference between the predicted results from the model and the observed behaviour. One such case is the problem of explaining conduction and insulation for materials. In a seminal work by P. W. Anderson [6] to explain characteristic of spin waves over doped silicon, he proposed a quantum mechanical model and showed that at high disorder the wave functions are exponentially localized at all energies. A consequence of localized state is its inability to carry any kind of current over macroscopic distances. Thus, for complete description in such systems, one has to take into account of the disorder.

3.1 Anderson model

The Anderson model is a simplified model to describe movement of a single electron through a lattice of nuclei. Physically speaking, we are looking at some crystal, so we have a periodic background and as a simplification we assume that only one electron is moving. The disorder arises because of doping, which randomly replaces some nuclei of the lattice with some other nuclei of different charges.

In case of usual lattice $\mathbb{Z}^d$, there is a potential at each lattice point $\{\omega_n\}_{n\in\mathbb{Z}^d}$ which is how the disorder is introduced, and there is an interaction $\mathcal{I} : \mathbb{R} \to \mathbb{R}$ for each nucleus and the electron, which only depends upon the distance, that is assumed to be constant (though in some cases this can also be random). Evolution of the wave function $\{\psi(x,t)\}_{x\in\mathbb{Z}^d}$ is governed by the equation

$$\iota\frac{\partial\psi}{\partial t}(x,t) = \omega_x\psi(x,t) + \sum_{y\in\mathbb{Z}^d}\mathcal{I}(|x-y|)\psi(y,t) \qquad \forall x \in \mathbb{Z}^d. \tag{3.1}$$

To understand the solution of above equation, it is important to study the operator

$$(H^\omega u)(x) = \sum_{y\in\mathbb{Z}^d}\mathcal{I}(|x-y|)u(y) + \omega_x u(x) \qquad \forall x \in \mathbb{Z}^d,\ |supp(u)| < \infty.$$

A further simplification can be done by taking $\mathcal{I}(1) = 1$ and rest to be zero. This is the case when the interaction effects only nearest neighbour. Then operator is of the form

$$(H^\omega u)(x) = \sum_{|x-y|=1} u(y) + \omega_x u(x) \qquad \forall x \in \mathbb{Z}^d,\ |supp(u)| < \infty. \tag{3.2}$$

The operator H^ω can be written as $\Delta + V^\omega$ where

$$(\Delta u)(x) = \sum_{|x-y|=1} u(y) \ \ and\ \ (V^\omega u)(x) = \omega_x u(x) \qquad \forall x \in \mathbb{Z}^d.$$

The spectral properties of Δ (*discrete Laplacian*) is well understood. Its spectrum is $[-2d, 2d]$ and the spectral measure is absolutely continuous. The operator V^ω has pure point spectrum and is given by $\{\omega_x : x \in \mathbb{Z}^d\}$, the eigenvectors are $\{\delta_x\}_{x\in\mathbb{Z}^d}$ (Kronecker delta function)

$$\delta_x(y) = \begin{cases} 1 & if\ x = y \\ 0 & otherwise \end{cases}.$$

The set $\{\delta_n\}_{n\in\mathbb{Z}^d}$ is also the canonical basis of $\ell^2(\mathbb{Z}^d)$.

3.1.1 Anderson tight-binding model

For *tight-binding Hamiltonian*, the potential $\{\omega_x\}_{x\in\mathbb{Z}^d}$ are taken to be independent identically distributed real random variables. Hence H^ω is not a single operator but a family of operators. This is because $\{\omega_x\}_{x\in\mathbb{Z}^d}$ can be viewed as identically distributed independent random variables over some probability space $(\Omega, \mathcal{B}, \mathbb{P})$ (by using theorem 1.2.2) and so we have the map

$$H^{\cdot} : \Omega \to \mathcal{S}(\ell^2(\mathbb{Z}^d)),$$

given by $\omega \mapsto H^\omega$, where $\mathcal{S}(\ell^2(\mathbb{Z}^d))$ is set of essentially self-adjoint operators. The operator H^ω is unbounded whenever V^ω is unbounded, which is the case when the distribution of ω_x has unbounded support. In case H^ω is unbounded, the domain of definition always contains all $u \in \ell^2(\mathbb{Z}^d)$ with finite support. All the statements made for H^ω are statements which holds almost surely.

To study the effect of disorder an extra parameter is introduced and the Anderson Hamiltonian is usually defined by

$$H^\omega_\lambda = \Delta + \lambda V^\omega. \tag{3.3}$$

This way of defining it can be extended to case of graphs where the Laplacian is replaced by adjacency operator for the graph.

Early work by Pasture [71, 72] showed that the spectrum of these operators are almost surely constant and is given by $\sigma(\Delta) + \lambda supp(\mu)$ where μ is the distribution of the random variables $\{\omega_x\}$.

One of the main reason for developing this model is because the Green's function are exponentially decaying for high disorder (λ being large in (3.3)). Based on initial estimates by Fröhlich-Spencer[31], Multi-scale analysis was developed by Fröhlich-Martinelli-Scoppola-Spencer[30], Simon-Taylor-Wolff[87] and Delyon-Levy-Souillard [24] (see also Stollmann [89] and Germinet-Klein[34] for Bootstrap multi-scale analysis) to give a rigorous proof of the exponential decay of Green's function. Carmona-Klein-Martinelli[13] extended the method for singular single site distribution. Later Aizenman-Molchanov [2] developed fractional moment method.

For more comprehensive details see [35, 90, 91]. But as of yet no proof of absolute continuous spectrum for Anderson tight-binding model on $\mathbb{Z}^d$ exist. Abel Klein in [52] proved existence of absolutely continuous spectrum for tight-binding model on Bethe lattice at low disorder (see also Froese-Hasler-Spitzer[29]). recently Aizenman-Warzel[4] showed resonant delocalisation on Bethe lattice, which implies the absence of point spectrum in the region. Another important property is that the point spectrum is simple (i.e for almost ω any eigenvalue has unique eigenfunction), this was shown by Simon [84] and later Klein-Molchanov [55]. This is also proved in more general setup by Jakšić-Last [45].

There are many important properties that are not listed here but can be found in surveys, such as [5, 14, 48, 49, 92].

3.1.2 Multi-particle Anderson model

In recent years, study of *multi-particle Anderson Hamiltonian* has gained importance. The N-particle Anderson Hamiltonian on $\mathbb{Z}^d$ can be described as follows. The Hilbert space in consideration is $\otimes^N \ell^2(\mathbb{Z}^d)$ (which is same as $\ell^2(\mathbb{Z}^{dN})$) and the operator H^ω is described by

$$(H^\omega u)(x) = \left(\sum_{n=1}^{N} [(\Delta u_n)(x_n) + \omega_{x_n} u_n(x_n)] \prod_{m\neq n} u_m(x_m)\right) + \sum_{n<m} w(|x_n - x_m|)u(x) \qquad \forall x \in (\mathbb{Z}^d)^N,$$

for functions of the form $u(x) = \prod_{n=1}^{N} u_n(x_n)$ where each $u_n : \mathbb{Z}^d \to \mathbb{C}$ are finite supported. Here $\{\omega_x\}_{x\in\mathbb{Z}^d}$ are i.i.d real random variables and $w : \mathbb{R} \to \mathbb{R}$ is the interaction between the electrons. The functions of form $\prod_{n=1}^{N} u_n(x_n)$ are dense in $\ell^2(\mathbb{Z}^{dN})$ and so above operator is densely defined operator. One can modify above equation and write

$$(H^\omega u)(x) = \left[\left(\sum_{n=1}^{N}(\Delta u_n)(x_n)\prod_{m\neq n} u_m(x_m)\right) + \sum_{n\neq m} w(|x_n - x_m|)u(x)\right] + \left(\sum_{n=1}^{N}\omega_{x_n}\right)u(x) \qquad x \in (\mathbb{Z}^d)^N.$$

In these models, the meaning of localization is not entirely clear. But exponential decay of Green's function are proved by many, for example by Chulaevsky-Boutet De Monvel-Suhov [15], Aizenman-Warzel [3] and Klein-Nguyen [56]. Not much is known in these models and lots of questions are still to be answered.

3.1.3 Non-Ergodic random operators

In all of the previous examples, there is a $\mathbb{Z}^d$ action T on probability space $(\Omega, \mathcal{B}, \mathbb{P})$ defined by $T_m(\{\omega_x\}_{x\in\mathbb{Z}^d}) = \{\omega_{x+m}\}_{x\in\mathbb{Z}^d}$ for any $m \in \mathbb{Z}^d$ (this action is measure preserving for above examples), and on the Hilbert space the action is defined through translation, for example on $\ell^2(\mathbb{Z}^d)$ and $L^2(\mathbb{R}^d)$ the action is $(U_m u)(x) = u(x+m)$. The operators in previous examples follows

$$U_m H^\omega U_m^* = H^{T(\omega)} \qquad \forall m \in \mathbb{Z}^d.$$

There are few models developed to study certain aspect of random operators which are not ergodic, for example on $\ell^2(\mathbb{Z}^d)$ one can define the random operator

$$H_\alpha^\omega = \Delta + \sum_{n\in\mathbb{Z}^d}(1 + \|n\|_1)^\alpha \omega_n \, |\delta_n\rangle\langle\delta_n|$$

for $\alpha \in \mathbb{R}$. As before $\{\omega_n\}_n$ are iid real random variables. Element $u \in \ell^2(\mathbb{Z}^d)$ with finite support lies in the domain of these operator and so is densely defined operator.

In the case $\alpha > 0$, the operator is called unbounded random Schrödinger operator. The spectral theory for this was studied by Gordon-Molchanov-Tsagani [38] for one-dimension and Gordon-Jakšić-Molchanov-Simon [39] for higher dimensions. They showed that the spectrum is almost surely point spectrum and eigenfunctions are exponentially decaying.

When $\alpha < 0$, the operator has much richer structure. In case of dimension one, Delyon-Simon-Souillard [25] showed that for $-\frac{1}{2} < \alpha < 0$, the spectrum is pure point and for $\alpha < -\frac{1}{2}$ the spectrum in $[-2, 2]$ is continuous (when distribution of random variable has bounded support). For higher dimension, Kirsch-Krishna-Obermeit [47] showed that $[-2d, 2d]$ has absolutely continuous spectrum for $\alpha < -1$ (when second or higher moment exists for the distribution function of randomness). Jakšić-Last[42] showed purity of absolute continuous spectrum in these models.

Another class of random models is the sparse potential. Given $\phi : \mathbb{Z}^d \to \mathbb{R}$, and a set $S \subset \mathbb{Z}^d$ with the property

$$\lim_{R\to\infty} \frac{|S \cap \{x \in \mathbb{Z}^d : \|x\| < R\}|}{R^d} = 0,$$

and

$$|\phi(x)| \leq \frac{C_0}{(1 + \|x\|)^{d+\epsilon}},$$

for $C_0, \epsilon > 0$, define the operator

$$(H^\omega u)(x) = (\Delta u)(x) + \sum_{p\in S} \omega_p \phi(x - p)u(x) \qquad x \in \mathbb{Z}^d, |supp(u)| < \infty.$$

These kind of operators are densely defined and are also non-ergodic. Then under certain conditions Krishna [62] showed the presence of absolutely continuous spectrum. Other result includes work by Molchanov-Vainberg[67], Simon-Stolz[86] and Remling[76, 77].

3.2 Other results

There are many result concerning the effect of perturbation on singular spectrum. Some examples of work involving rank one perturbations are Simon-Wolff[88], Donoghue[27], Rio-Jitomirskaya-Last-Simon[23] and Gesztesy-Simon[36].

The main result of this thesis is similar to results by Jakšić-Last from [43] and [45]. They worked with Anderson type operators where the perturbations are rank one. When the spectral subspace generated by perturbing vectors have non-trivial intersection, then their work showed that the spectral measure associated with the perturbing vectors are absolutely continuous with respect to each other, they also showed that the singular subspaces are equal. This result along with other results from previous works shows simplicity of the singular part of the spectrum.

There are some work on higher rank perturbation like Naboko-Nichols-Stolz[68], and Sadel-Schulz-Baldes[80]. In [68] the authors proved simplicity of point spectrum for some special class of perturbing projections. In [80] the authors showed that based on dimension of the underlying space, multiplicity of the spectrum can change for quasi one-dimensional Dirac operators with matrix valued perturbations.

These results implies the possible limitation of any result that can be obtained in general scenario. With these models in mind, we will restrict to certain class of random operators described in next section.

3.3 Model in consideration

All of the above operators have the form:

$$A^{\omega} = A + \sum_{n\in\mathcal{N}} \omega_n C_n, \tag{3.4}$$

where A is self adjoint operator (or essentially self adjoint operator) on some separable Hilbert space $\mathscr{H}$, $\mathcal{N}$ is countable a countable set, $\{C_n\}_{n\in\mathcal{N}}$ is a countable collection of bounded operators and $\{\omega_n\}_{n\in\mathcal{N}}$ are independent real random variables. In case of tight binding Hamiltonian C_n's are rank one projection and in case of multi-particle Anderson Hamiltonian, they are infinite rank projections. In case of continuum random Schrödinger operator C_n are compact relative to the operator A.

In this thesis, we are interested in class of operator A^ω on the separable Hilbert space $\mathscr{H}$ of the form

$$A^\omega = A + \sum_{n \in \mathcal{N}} \omega_n P_n, \tag{3.5}$$

where A is a bounded self-adjoint operator, $\mathcal{N}$ is a countable set, $\{P_n\}_{n\in\mathcal{N}}$ are rank N projections with the property $\sum_{n\in\mathcal{N}} P_n = I$, and $\{\omega_n\}_{n\in\mathcal{N}}$ are independent real random variables with absolutely continuous distribution over the probability space $(\Omega, \mathcal{B}, \mathbb{P})$. Following examples will help in establishing some of the conditions for setting up the theorem.

Examples 3.3.1. *Let $N \in \mathbb{N}$ be fixed. The Hilbert space in consideration is $\ell^2(\mathbb{Z})$, and the random operator is of the form*

$$H^\omega = \Delta + \sum_{n\in\mathbb{Z}} \omega_n P_n,$$

where

$$P_n = \sum_{k=0}^{N-1} |\delta_{nN+k}\rangle \langle \delta_{nN+k}| \,.$$

When $N = 1$ and $\{\omega_n\}_{n\in\mathbb{Z}}$, this is one-dimensional Anderson tight binding model and for $N = 2$ (or higher) is called dimer (polymer respectively) model. The action of H^ω can be described by

$$(H^\omega u)(x) = u(x+1) + u(x-1) + \omega_{\lfloor \frac{x}{N} \rfloor} u(x) \qquad \forall x \in \mathbb{Z}$$

for any $u \in \ell^2(\mathbb{Z})$. We use the notation $\lfloor x \rfloor$ to denote the greatest integer less than x.

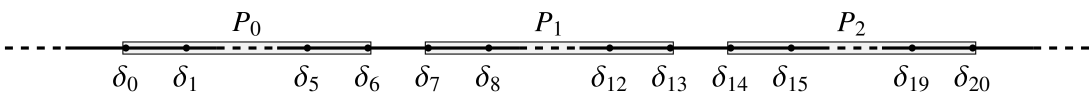

Figure 3.1: Representation of action of the operator H^ω on $\mathbb{Z}$ for the case $N = 7$. The lattice $\mathbb{Z}$ is viewed as a graph where the lines indicating the edges between neighbours, and Δ acts as adjacency operator on this graph. The support of the projections P_i are indicated by the shaded rectangles.

The figure 3.1 gives a representation for the operator H^ω acting over the Hilbert space of the graph $\mathbb{Z}$. Note that for any $n, m \in \mathbb{Z}$, we have $\left\langle \delta_n, (H^\omega)^{|n-m|}\delta_m \right\rangle = \left\langle \delta_n, \Delta^{|n-m|}\delta_m \right\rangle \neq 0$, so the spectral measure for any element δ_n will be influenced because of perturbation P_m for any m.

Examples 3.3.2. *Let $N \in \mathbb{N}$ be fixed. On the Hilbert space $\ell^2(\mathbb{Z} \times \{1, \cdots, N\})$ consider the operator H^ω defined by*

$$(H^\omega u)(x, n) = u(x+1, n) + u(x-1, n) + \omega_{\pi_n(x)} u(x, n) \qquad \forall (x, n) \in \mathbb{Z} \times \{1, \cdots, N\}.$$

for all u such that $|supp(u)| < \infty$. Here $\{\omega_x\}_{x\in\mathbb{Z}}$ are independent random variables and $\pi_i : \mathbb{Z} \to \mathbb{Z}$ are bijections. Simplest case is when π_i are shift (i.e they are defined by $x \mapsto x + m$), then above operator is a collection of N identical Anderson Hamiltonian and so the spectrum has multiplicity N.

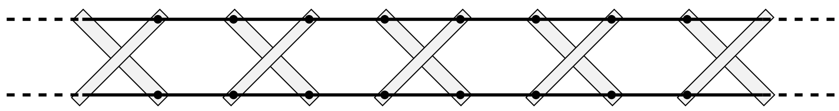

Figure 3.2: Here $N = 2$ with $\pi_1(x) = x$ and π_2 switches odd and even numbers, i.e $\pi_2(2n) = 2n + 1$, $\pi_2(2n + 1) = 2n$. The line represent the edges between the vertices $\{(x, n)\}_{x\in\mathbb{Z}, n\in\{1,2\}}$ giving a graph structure on $\mathbb{Z} \times \{1, 2\}$. The action of the constant part of the operator is same as adjacency operator over the graph. Like figure 3.1, the support of projections are represented by shaded rectangles.

The perturbation P_n can be written as

$$P_n = \sum_{i=1}^{N} \left|\delta_{(\pi_i^{-1}(n), i)}\right\rangle \left\langle \delta_{(\pi_i^{-1}(n), i)}\right|.$$

In this case for any basis vector $\delta_{x,n}$, only basis vectors $\delta_{y,n}$ can be reached, i.e $\left\langle \delta_{x,n}, (H^\omega)^k \delta_{y,m}\right\rangle = 0$ for any k if $n \neq m$. This in turn tells us that there are multiple cyclic subspaces (though if the spectral measure are singular for each subspace, then we can write a single cyclic vector). So to get complete information about spectral measure associated with some vector, say $\delta_{x,n}$, we need to focus on only the linear subspace generated by $\overline{\langle \delta_{x,n} : x \in \mathbb{Z}\rangle}$. Hence to get the spectral information for the entire operator, looking at $\{\delta_{\pi_n^{-1}(0), n}\}_{n=1}^{N}$ is enough, which is associated to the spectral measure associated with the projection P_0.

These two example has something in common, the linear maps $P_n(H^\omega - z)^{-1} P_m$ are invertible for all n, m (follows from the proof of lemma 4.3.1 in next chapter). The next example is a mix of both and gives us cases that cannot be handled easily.

Examples 3.3.3. *Let $N \in \mathbb{N}$ be fixed and consider the Hilbert space $\ell^2(\mathbb{Z} \times \{1, \cdots, N\})$. Set the projections*

$$P_{n(N+1)+m} = \begin{cases} \sum_{i=1}^{N} \left|\delta_{n(N+1),i}\right\rangle \left\langle\delta_{n(N+1),i}\right| & m = 0 \\ \sum_{i=1}^{N} \left|\delta_{n(N+1)+i,m}\right\rangle \left\langle\delta_{n(N+1)+i,m}\right| & m \neq 0 \end{cases},$$

and define the operator H^ω as

$$(H^\omega u)(x,n) = u(x+1,n) + u(x-1,n) + \sum_{m \in \mathbb{Z}} \omega_m (P_m u)(x,n) \qquad \forall (x,n) \in \mathbb{Z} \times \{1, \cdots, N\},$$

for u with finite support. The action of the operator H^ω can be visualised by figure 3.3.

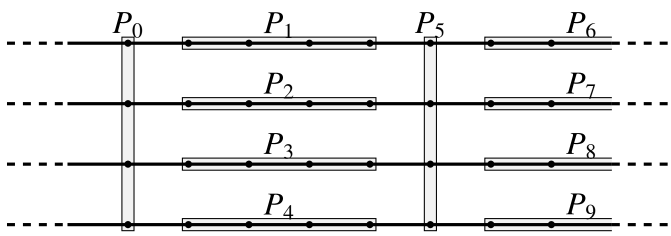

Figure 3.3: Here $N = 4$, we follow convention of figure 3.2. It can be seen that $P_0, P_5, \cdots$ behaves like example 3.3.2 and $P_1, P_6, \cdots$ (similarly $P_2, P_7, \cdots$ and others) behaves like example 3.3.1.

Here $P_n(H^\omega - z)^{-1} P_m$ is invertible if and only if $n \equiv m \mod (N+1)$. Since the linear subspaces $\overline{\langle \delta_{x,n} : x \in \mathbb{Z} \rangle}$ are closed under the action of H^ω for each n, it is clear that to get the spectral measure one only need to look at $\{\delta_{0,n}\}_{n=1}^{N}$. This is associated with the spectral measure (through theorem 2.5.1) of P_0.

As seen in previous example, even though spectral measure can be computed by looking at P_0, there is no way of making sure that it is enough. There can be exceptional cases. Next example is one such case

Examples 3.3.4. *Consider the Hilbert space $\ell^2(\mathbb{N}^2)$, with the self adjoint operator*

$$(\tilde{\Delta} u)(x,y) = \begin{cases} u(x+1,y) + u(x-1,y) & x > 1, y \in \mathbb{N} \\ u(2,y) & x = 1, y \in \mathbb{N} \end{cases}$$

and sequence of rank 2 projections $P_{n,m,j}$ by

$$P_{n,m,j} = \left|\delta_{(n,2nm+j)}\right\rangle \left\langle\delta_{(n,2nm+j)}\right| + \left|\delta_{(n,2nm+j+n)}\right\rangle \left\langle\delta_{(n,2nm+j+n)}\right|.$$

Let $\{\omega_{n,m,j}\}$ be real iid random variables and define the operator

$$H^\omega = \tilde{\Delta} + \sum_{n,m,j} \omega_{n,m,j} P_{n,m,j}.$$

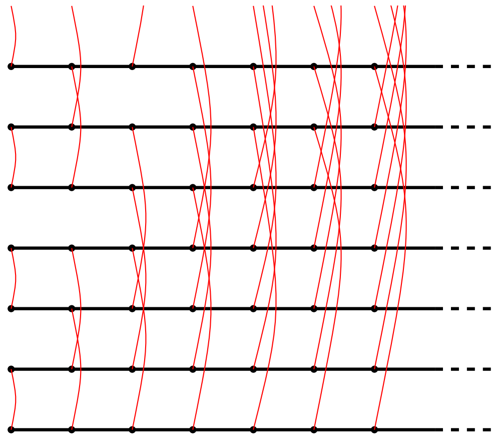

Figure 3.4: The operator described above is visualised here. The operator $\tilde{\Delta}$ is the adjacency operator over the graph $\mathbb{N}^2$ where the edges are denoted by the black lines. The red lines indicates the support of the projections.

In this case none of the matrix $P_{p,q,r}(H^\omega - z)^{-1}P_{m,n,o}$ are invertible if $(p,q,r) \neq (m,n,o)$. So here looking at these matrices doesn't help in getting the spectral measure and one has to focus on spectral measures for each $\delta_{n,m}$ separately (even though the subspace $\overline{\langle \delta_{n,m} : n \in \mathbb{Z} \rangle}$ are closed under action of H^ω for each m)

3.3.1 Notation

For the next chapter, we will set up the notation here itself. As stated in the beginning of the section, we have a separable Hilbert space $\mathscr{H}$ and a probability space $(\Omega, \mathcal{B}, \mathbb{P})$. We have a class of essentially self adjoint operator $A^{\cdot} : \Omega \to \mathcal{S}(\mathscr{H})$ given by (3.5). For $n \in \mathcal{N}$ and $\omega \in \Omega$, define $\mathscr{H}_n^\omega$ to be the closed A^ω-invariant subspace containing $P_n\mathscr{H}$, i.e

$$\mathscr{H}_n^\omega = \overline{\{f(A^\omega)\phi : \phi \in C_c(\mathbb{R}), \phi \in P_n\mathscr{H}\}},$$

where the bar denotes the closed linear span in $\mathscr{H}$. Set $Q_n^\omega : \mathscr{H} \to \mathscr{H}_n^\omega$ to be the canonical projection onto the subspace $\mathscr{H}_n^\omega$. Let E^ω denote the spectral measure E^{A^ω} (obtained through spectral theorem 1.4.2), set $\Sigma_n^\omega(\cdot) = P_n E^\omega(\cdot) P_n$ and $\sigma_n^\omega(\cdot) = tr(\Sigma_n^\omega(\cdot))$ as the trace measures. Let E_{ac}^ω (similarly E_{sing}^ω) to be the orthogonal projection onto the absolutely continuous (respectively singular) spectral subspace of A^ω. For $n, m \in \mathcal{N}$, define

$$\Omega_{n,m} = \{\omega \in \Omega | Q_n^\omega P_m \text{ has same rank as } P_m\}. \tag{3.6}$$

We will be focusing on the set

$$\mathcal{M} = \{n \in \mathcal{N} | \sigma_n^\omega \text{ is not equivalent to Lebesgue measure for a.a } \omega\}.$$

This is because of F. and M. Riesz theorem (the result used here is corollary 2.3.3). Since we will be working with Borel transform, on the set of indices $\mathcal{M}$, the Borel transform will be non-zero. Finally we will denote

$$G_{nm}^\omega(z) = P_n(A^\omega - z)^{-1} P_m \qquad \forall n, m \in \mathcal{N}, z \in \mathbb{C}^+.$$

Let $A_p^{\omega,\mu} = A^\omega + \mu P_p$ for some $p \in \mathcal{N}$, and set

$$G_{nm}^{\omega,\mu,p}(z) = P_n(A_p^{\omega,\mu} - z)^{-1} P_m \qquad \forall n, m \in \mathcal{N}, z \in \mathbb{C}^+.$$

Observe that as a consequence of theorem 2.4.1 (5)

$$d\Sigma_{n,ac}^\omega(x) = \frac{1}{\pi} G_{nn}^\omega(x + \iota 0) dx.$$

Finally for examples 3.3.1 and 3.3.2 we have $\mathbb{P}(\Omega_{n,m}) = 1$ for any $n, m \in \mathcal{N}$. For example 3.3.4 we have $\mathbb{P}(\Omega_{n,m}) = 0$ if $n \neq m$, and for example 3.3.3 we have $\mathbb{P}(\Omega_{n,m}) = 1$ if and only if $n \equiv m \mod (N + 1)$ otherwise is zero.

Chapter 4

Main Result

4.1 Statement

Most of the content of this chapter is from the work [66]. The main result of this thesis can be summarised by the following theorem:

Theorem 4.1.1. *Let $\mathcal{H}$ be a separable Hilbert space, $(\Omega, \mathscr{B}, \mathbb{P})$ be a probability space, $\mathcal{N}$ be a countable set and $N \in \mathbb{N}$ be given. Let $\{P_n\}_{n\in\mathcal{N}}$ be a collection of rank N projections satisfying $\sum_{n\in\mathcal{N}} P_n = I$ and $\{\omega_n\}_{n\in\mathcal{N}}$ are independent real random variables on $(\Omega, \mathscr{B}, \mathbb{P})$ with absolutely continuous distribution. Let $\{A^\omega\}_{\omega\in\Omega}$ be a family of operators defined by $A^\omega = A + \sum_{n\in\mathcal{N}} \omega_n P_n$, then*

1. *For $n, m \in \mathcal{M}$, we have $\mathbb{P}(\Omega_{n,m}) \in \{0, 1\}$.*
2. *Let $n, m \in \mathcal{M}$ such that $\mathbb{P}(\Omega_{n,m} \cap \Omega_{m,n}) = 1$, then for almost all $\omega \in \Omega$ the restrictions onto absolutely continuous part $E^\omega_{ac}A^\omega|_{\mathcal{H}^\omega_n}$ and $E^\omega_{ac}A^\omega|_{\mathcal{H}^\omega_m}$ are equivalent.*
3. *Let $n, m \in \mathcal{M}$ such that $\mathbb{P}(\Omega_{n,m} \cap \Omega_{m,n}) = 1$, then for almost all $\omega \in \Omega$ the trace measures σ^ω_n and σ^ω_m are equivalent as Borel measures.*

4. *Let* $\mathbb{P}(\Omega_{n,m} \cap \Omega_{m,n}) = 1$ *for any* $n, m \in \mathcal{M}$*, then* $E^{\omega}_{sing}\mathcal{H} = E^{\omega}_{sing}\mathcal{H}^{\omega}_{n}$ *for any* $n \in \mathcal{M}$ *for almost all* $\omega \in \Omega$*.*

Except for part (4) of the theorem, rest is same as in [66, Theorem 1.1].

Second and third part of the theorem 4.1.1 is consequence of perturbations by two projections. For the first part, the event $\Omega_{n,m}$ is shown to be independent of any finite collection of perturbations, then the result follows through Kolmogorov 0-1 law. For the last part, individual cyclic subspaces for the singular part of the operator are identified and then by the help of the third part the equality of the cyclic subspaces are established. Lemma 4.3.5 is the primary step for the first part of the theorem. It tells us that the event $\Omega_{n,m}$ ($Q^{\omega}_{n}P_{m}$ *has same rank as* P_{m}), is independent of any other perturbation, whence Kolmogorov 0-1 law applies. For the second part, whenever the condition is satisfied, we have to show that for $x \in \mathbb{R}$ in a full Lebesgue measure set, density of the measure has same rank for both indices; this is done in corollary 4.3.7. For the third part, the second part of the theorem 4.1.1 helps by asserting that absolute continuous parts are equivalent. As for the singular part we only need to consider the lowest (Hausdorff) dimensional part. This is the case because all the singular measures are singular with respect to each other. Hence showing absolute continuity for each singular measure is enough, which is done using Poltoratskii's theorem [74]. This works because lowest (Hausdorff) dimensional part of the spectrum contributes the maximum rate of growth to the Herglotz function as its argument approaches the boundary of $\mathbb{C}^{+}$. Corollary 4.3.9 provides the equivalence for the lowest dimensional parts of the measure. For the last part, first we show that if $\mathbb{P}(\Omega_{n,m} \cap \Omega_{m,n}) = 1$, then $E^{\omega}_{sing}\mathscr{H}^{\omega}_{n} = E^{\omega}_{sing}\mathscr{H}^{\omega}_{m}$, which is done in corollary 4.3.12, then the result follows.

Before proving it one more result needed. This lemma helps in the proof of the main theorem by ensuring that for almost all perturbation the functions in consideration does not vanish on positive (Lebesgue) measure set (or else the analysis will fail), and so we can ignore because of Spectral Averaging result (see lemma 2.5.2).

4.2 Measure of zero set of certain polynomial

Following lemma is a result concerning the zero sets of polynomials. This is stated in some generality, we only need it on reals with Lebesgue measure.

Lemma 4.2.1. [66, Lemma 2.1] *For a σ-finite positive measure space $(X, \mathscr{B}, m)$, and a collection of measurable functions $a_i : X \to \mathbb{C}$, define the function $f(\lambda, x) = 1 + \sum_{n=1}^{N} \lambda^n a_n(x)$. The set defined by*

$$\Lambda_f = \{\lambda \in \mathbb{C} | m\{x \in X | f(\lambda, x) = 0\} > 0\} \tag{4.1}$$

is countable.

Proof. The proof is by induction on degree of f (as a polynomial of λ). We will use the notation:

$$S_\lambda = \{x \in X | f(\lambda, x) = 0\} \tag{4.2}$$

By definition the sets S_λ are measurable.

Base case of induction is $N = 1$, so $f(\lambda, x) = 1 + \lambda a_1(x)$. Clearly for $\lambda_1 \neq \lambda_2 \in \mathbb{C}$ we have $S_{\lambda_1} \cap S_{\lambda_2} = \phi$. Since, if $x \in S_{\lambda_1} \cap S_{\lambda_2}$ then

$$\begin{aligned} & 1 + \lambda_1 a_1(x) = 0 \text{ and } 1 + \lambda_2 a_1(x) = 0 \\ \Rightarrow \quad & \frac{1}{\lambda_1} = -a_1(x) = \frac{1}{\lambda_2} \\ \Rightarrow \quad & \lambda_1 = \lambda_2 \end{aligned}$$

but we assumed $\lambda_1 \neq \lambda_2$. Since (X, m) is σ-finite, we have a countable collection $\{X_i\}_{i \in \mathbb{N}}$ such that $\cup_i X_i = X$ and for each i we have $m(X_i) < \infty$. Now for each $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ define $S_{\lambda,n} = S_\lambda \cap X_n$, so we have $\cup_n S_{\lambda,n} = S_\lambda$, and $\cup_{\lambda \in \Lambda_f} S_{\lambda,n} \subset X_n$. We have

$$\sum_{\lambda \in \Lambda_f} m(S_{\lambda,n}) = m(\cup_{\lambda \in \Lambda_f} S_{\lambda,n}) \leq m(X_n) < \infty,$$

so only for countably many $\lambda \in \Lambda_f$ we have $m(S_{\lambda,n}) \neq 0$. Set $\Lambda_n = \{\lambda \in \Lambda_f | m(S_{\lambda,n}) > 0\}$, we have $\Lambda_f = \cup_{n \in \mathbb{N}} \Lambda_n$, but since countable union of countable set is countable, we get Λ_f countable. This completes base case.

Now assume the induction hypothesis, i.e for measurable functions $a_i : X \to \mathbb{C}$, and $f(\lambda, x) = 1 + \sum_{n=1}^{N} \lambda^n a_n(x)$, the set Λ_f is countable.

We have to show for $f(\lambda, x) = 1 + \sum_{n=1}^{N+1} \lambda^n a_n(x)$, the set Λ_f is countable. First we define the relation $\sim$ for elements of Λ_f; for $\mu, \nu \in \Lambda_f$ we define $\mu \sim \nu$ if there exists $\{\lambda_i\}_{i=1}^{k}$ such that $\lambda_1 = \mu$, $\lambda_k = \nu$ and $m(S_{\lambda_i} \cap S_{\lambda_{i+1}}) > 0$ for $i = 1, \cdots, k-1$. For $\mu \in \Lambda_f$ we have $\mu \sim \mu$ because $m(S_\mu) > 0$ hence $\sim$ is reflexive. If $\mu \sim \nu$ for $\mu, \nu \in \Lambda_f$, then we have a sequence $\{\lambda_i\}_{i=1}^{k}$ such that $\lambda_1 = \mu$ and $\lambda_k = \nu$ and $m(S_{\lambda_i} \cap S_{\lambda_i+1}) > 0$, hence choosing $\tilde{\lambda}_i = \lambda_{k-i+1}$ we get $\nu \sim \mu$ and so $\sim$ is symmetric. If $\mu \sim \nu$ and $\nu \sim \eta$, then we have sequences $\{\alpha_i\}_{i=1}^{p}$ and $\{\beta_i\}_{i=1}^{q}$ such that $\alpha_1 = \mu$, $\alpha_p = \beta_1 = \nu$ and $\beta_q = \eta$, so defining the sequence $\{\lambda_i\}_{i=1}^{p+q}$ defined as $\lambda_i = \alpha_i$ for $i \leq p$ and $\lambda_i = \beta_{i-p}$ for $i > p$ we get $\mu \sim \eta$ giving transitivity of $\sim$. So $\sim$ is a equivalence relation on Λ_f, and can break the set Λ_f into equivalence classes indexed by $\tilde{\Lambda} = \Lambda_f / \sim$, where we view $[\lambda] \in \tilde{\Lambda}$ as $[\lambda] = \{\mu \in \Lambda_f | \mu \sim \lambda\}$ and define $S_{[\lambda]} = \cup_{\mu \in [\lambda]} S_\mu$.

First we will show for any $[\lambda] \in \tilde{\Lambda}$, the set $[\lambda]$ is countable. Let $\lambda \in \Lambda_f$, so we have the $m(S_\lambda) \neq 0$. We will restrict to subspace S_λ, on this space $f(\nu, x)$ can be written as $f(\nu, x) = \frac{1}{\lambda}(\lambda - \nu)\left(1 + \sum_{n=1}^{N} \tilde{a}_n(x)\nu^n\right)$ (since λ is a solution). So we have the new function $\tilde{f}(\nu, x) = 1 + \sum_{n=1}^{N} \tilde{a}_n(x)\nu^n$, and by our assumption (induction hypothesis) we get $\Lambda_{\tilde{f}}$ is countable. For any $\nu \in \Lambda_f$ with $m(S_\lambda \cap S_\nu) \neq 0$ implies $\nu \in \Lambda_{\tilde{f}}$, so for fixed $\lambda \in \Lambda_f$ the set of $\nu \in \Lambda_f$ such that $m(S_\lambda \cap S_\nu) \neq 0$ is countable.

Next choose $\lambda \in \Lambda_f$, and set $A_0 = \{\lambda\}$, and define

$$A_i = \cup_{\beta \in A_{i-1}} \{\nu \in \Lambda_f | m(S_\nu \cap S_\beta) \neq 0\} \qquad \forall i \in \mathbb{N}$$

by previous step each A_i are countable. So $\cup_{i=0}^{\infty} A_i$ is countable. By definition of $\sim$ we have $[\lambda] = \cup_{i=0}^{\infty} A_i$.

Now we will prove $\tilde{\Lambda}$ is countable. By definition $m(S_{[\lambda]}) > 0$ for $[\lambda] \in \tilde{\Lambda}$, and for $[\lambda] \neq [\mu] \in \tilde{\Lambda}$ we have $m(S_{[\lambda]} \cap S_{[\nu]}) = 0$. For $n \in \mathbb{N}$ define $S_{[\lambda],n} = S_{[\lambda]} \cap X_n$, then we have

$$\sum_{n \in \tilde{\Lambda}} m(S_{[\lambda],n}) = m(\cup_{[\lambda] \in \tilde{\Lambda}} S_{[\lambda],n}) \leq m(X_i) < \infty$$

From last step only countably many $[\lambda]$ can have $m(S_{[\lambda],n}) > 0$. Call $\tilde{\Lambda}_n = \{[\lambda] \in \tilde{\Lambda} | m(S_{[\lambda],n}) > 0\}$ (which are countable); for any $[\lambda] \in \tilde{\Lambda}$ we have

$$0 < m(S_{[\lambda]}) \leq \sum_{n \in \mathbb{N}} m(S_{[\lambda],n})$$

So $[\lambda] \in \tilde{\Lambda}$ for some $n \in \mathbb{N}$ we have $m(S_{[\lambda],n}) > 0$, hence $\tilde{\Lambda} = \cup_{n \in \mathbb{N}} \tilde{\Lambda}_n$; giving us $\tilde{\Lambda}$ is countable.

Since $\Lambda_f = \cup_{[\lambda] \in \tilde{\Lambda}} [\lambda]$ and both the sets are countable we get the countability of Λ_f.

□

Remark 4.2.2. *It should be clear that above result holds for function of the type* $f(\lambda, x) = \sum_{n=0}^{N} a_n(x) \lambda^n$ *on the set* $\{x \in X | a_0(x) \neq 0\}$. *It should be noted that one cannot extend the result for whole of* X.

We can view $f(\lambda, x) = \lambda^N \left(\sum_{n=0}^{N} a_{N-n}(x) \left(\frac{1}{\lambda} \right)^n \right)$, *and so the result also holds on the set* $\{x \in X | a_N(x) \neq 0\}$.

Corollary 4.2.3. **[66, Corollary 2.3]** *For a* σ*-finite positive measure space* $(X, \mathscr{B}, m)$ *and a collection of functions* $a_i : X \to \mathbb{C}$, $b_i : X \to \mathbb{C}$, *define the function* $f(\lambda, x) = \frac{1 + \sum_{i=1}^{N} a_i(x) \lambda^i}{1 + \sum_{i=1}^{N} b_i(x) \lambda^i}$, *then the set*

$$\Lambda_f = \{\lambda \in \mathbb{C} | m\{x \in X | f(\lambda, x) = 0\} \neq 0\} \tag{4.3}$$

is countable

Proof. Set $g(\lambda, x) = 1 + \sum_{n=1}^{N} a_n(x) \lambda^n$, then $\{(x, \mu) \in X \times \mathbb{C} | f(\lambda, x) = 0\} \subseteq \{(x, \mu) \in X \times \mathbb{C} | g(\lambda, x) = 0\}$. So by lemma 4.2.1 we get the desired result.

□

4.3 Proof of main theorem

In this section we will be working with $(H, \mathscr{H}, \{P_i\}_{i=1}^3)$, where H is a self adjoint operator on the Hilbert space $\mathscr{H}$, and $\{P_i\}_{i=1}^3$ are three rank N projections. We will work with the case that the measures $tr(P_i E^H(\cdot)P_i)$ are not equivalent to Lebesgue measure (hence as consequence of theorem 2.3.3, the Borel transform of these measures are non-zero on the upper half plane). Define $H_\mu = H + \mu P_1$, $G_{ij}(z) = P_i(H - z)^{-1}P_j$ and $G^\mu_{ij}(z) = P_i(H_\mu - z)^{-1}P_j$ for $i, j = 1, 2, 3$ and $z \in \mathbb{C}^+$, and will use the notation

$$g(x + \iota 0) := \lim_{\epsilon \downarrow 0} g(x + \iota\epsilon)$$

for $x \in \mathbb{R}$ (whenever the limit exists). We recall the equations (2.2),(2.3) and (2.4) here:

$$G^\mu_{11}(z) = G_{11}(z)(I + \mu G_{11}(z))^{-1}, \tag{4.4}$$

$$(I + \mu G_{11}(z))(I - \mu G^\mu_{11}(z)) = I, \tag{4.5}$$

$$G^\mu_{ij}(z) = G_{ij}(z) - \mu G_{i1}(z)(I + \mu G_{11}(z))^{-1}G_{1j}(z) \qquad (i, j) \neq (1, 1). \tag{4.6}$$

For any $x \in \mathbb{R}$ such that $G_{11}(x+\iota 0)$ exists and finite and $f : (0, \infty) \to \mathbb{C}$ be such that $\lim_{\epsilon\downarrow 0} f(\epsilon) = 0$, using equation (4.5) observe

$$\begin{aligned} & \lim_{\epsilon\downarrow 0} f(\epsilon)(I - \mu G^\mu_{11}(x + \iota\epsilon))(I + \mu G_{11}(x + \iota\epsilon)) - f(\epsilon)I = 0, \\ \Rightarrow \qquad & (I + \mu G_{11}(x + \iota 0))\left(\lim_{\epsilon\downarrow 0} f(\epsilon)G^\mu_{11}(x + \iota\epsilon)\right) = 0. \end{aligned}$$

So

$$range\left(\lim_{\epsilon\downarrow 0} f(\epsilon)G^\mu_{11}(x + \iota\epsilon)\right) \subseteq ker(I + \mu G_{11}(x + \iota 0)) \subseteq ker(\Im G_{11}(x + \iota 0)), \tag{4.7}$$

where left hand side can possibly be empty. The last inclusion comes because of the fact that $\Im G_{11}(x + \iota 0) \geq 0$.

Since $\Im G_{11}(x+\iota 0) \geq 0$ it decomposes the space $P_1\mathscr{H} = ker(\Im G_{11}(x+\iota 0)) \oplus ker(\Im G_{11}(x+\iota 0))^\perp$ with $range(\Im G_{11}(x+\iota 0)) = ker(\Im G_{11}(x+\iota 0))^\perp$, so on $ker(\Im G_{11}(x+\iota 0))^\perp$ we have $\Im G_{ii}(x+\iota 0) > 0$. This fact will be used in identifying appropriate subspaces.

4.3.1 Proof of part (1)

The Following lemma relates the invertibility of the matrices $G^{\mu}_{12}(z)$ with the ranks of Q_1P_2 and P_2.

Lemma 4.3.1. [66, Lemma 3.1] *Let H be a self-adjoint operator on the Hilbert space $\mathscr{H}$ and P_1 and P_2 be two projections of rank N. Let $\mathscr{H}_i$ denote the cyclic subspace generated by H and $P_i\mathscr{H}$ and $Q_i : \mathscr{H} \to \mathscr{H}_i$ be the canonical projection onto that subspace, for $i = 1, 2$. If Q_1P_2 has same rank as P_2, then $P_1(H - z)^{-1}P_2$ is invertible for a.e $z \in \mathbb{C}^+$.*

Proof. Let $\phi \in P_2\mathscr{H} \setminus \{0\}$. Since Q_1P_2 has same rank as P_2, we have $0 \neq Q_1\phi \in \mathscr{H}_1$ (if it is zero, then $ker(Q_1) \cap P_2\mathscr{H} \neq \{0\}$ and so $rank(Q_1P_2) < rank(P_2)$), so there is $\psi \in P_1\mathscr{H}$ and $f \in L^2(\mathbb{R}, d\mu_\psi)$ such that $Q_1\phi = f(H)\psi$. So

$$0 \neq \langle Q_1\phi, Q_1\phi\rangle = \langle\psi, f^*(H)Q_1\phi\rangle = \langle\psi, f^*(H)\phi\rangle = \int \bar{f}(x)d\mu_{\psi,\phi}(x)$$

since Q_1 commutes with any functions of H. So the measure $\mu_{\psi,\phi}$ is non-zero, hence the Borel transform

$$\int \frac{d\mu_{\psi,\phi}(x)}{x - z} = \left\langle \psi, (H - z)^{-1}\phi \right\rangle,$$

is almost surely non-zero on $\mathbb{C}^+$.

So for each vector $\phi \in P_2\mathscr{H}$ there exists a $\psi \in P_1\mathscr{H}$ such that $\left\langle \psi, (H - z)^{-1}\phi \right\rangle$ is non-zero, in other words $P_1(H - z)^{-1}P_2$ is an injection, and since $P_1(H - z)^{-1}P_2$ is an $n \times n$ matrix we get invertibility.

□

Remark 4.3.2. *By above lemma the holomorphic function* $\det(P_1(H-z)^{-1}P_2)$ *is not zero on* $\mathbb{C}^+$. *So using theorem 2.2.1 the normal limit* $\lim_{\epsilon\downarrow 0} \det(P_1(H-x-\iota\epsilon)^{-1}P_2)$ *cannot be zero on a set of positive Lebesgue measure. So $P_1(H - x - \iota 0)^{-1}P_2$ is invertible for almost all x w.r.t. Lebesgue measure.*

For some $z \in \mathbb{C}^+$, the invertibility of $P_1(H-z)^{-1}P_2$ give us Q_1P_2 has same rank as P_2. This is the case because if $rank(Q_1P_2) < rank(P_2)$ then there exists $\phi \in P_2\mathscr{H}$ such that $Q_1\phi = 0$, which implies $P_1(H-z)^{-1}\phi = 0$ for any z.

So by looking at $\det(G_{mn}(z))$ we can obtain a statement about non-orthogonality of the subspace $\{\mathscr{H}_i\}_{i=1,2}$.

Choose a basis of $P_i\mathscr{H}$, then $G_{ij}(z)$ is a matrix in the basis. We can write

$$S = \{x \in \mathbb{R} | \text{ Entries of } G_{ij}(x + \iota 0) \text{ exists and are finite } \forall i, j = 1, 2, 3\} \tag{4.8}$$

Then by theorem 2.2.2 we know that S has full measure. Define

$$S_{ij} = \{x \in S | G_{ij}(x + \iota 0) \text{ is invertible}\} \qquad \forall i, j = 1, 2, 3 \tag{4.9}$$

By lemma 4.3.1, S_{ij} has full measure whenever Q_iP_j has same rank as P_j.

Remark 4.3.3. *On the set S, the limit $G_{11}(x + \iota 0)$ exists and since $\det(I + \mu G_{11}(x + \iota 0)) = 1 + \sum_{i=1}^{N} a_i(x)\mu^i$, using lemma 4.2.1 for almost all μ the matrix $I + \mu G_{11}(x + \iota 0)$ is invertible for μ in a set of full Lebesgue measure.*

Remark 4.3.4. *By using lemma 2.5.2 we can conclude that $P_1E^{H_\mu}(\mathbb{R} \setminus S)P_1 = 0$ for almost all μ (with respect to Lebesgue measure), so we need to focus our analysis on the set S only.*

Lemma 4.3.5. [66, Lemma 3.4] *Let H be self adjoint operator on the Hilbert space $\mathscr{H}$ and $\{P_i\}_{i=1}^3$ be rank N projections. Define $H_\mu = H + \mu P_1$, $G_{ij}(z) = P_i(H-z)^{-1}P_j$ and $G^\mu_{ij}(z) = P_i(H_\mu - z)^{-1}P_j$. If $G_{23}(x + \iota 0)$ is invertible for almost all x (with respect to Lebesgue measure), then $G^\mu_{23}(x + \iota 0)$ is also invertible for a.e (x, μ) (with respect to Lebesgue measure).*

Proof. From equations (4.4) and (4.6) and remark 4.3.3 we get for x in a set of full Lebesgue measure

$$G^\mu_{23}(x + \iota 0) = G_{23}(x + \iota 0) - \mu G_{21}(x + \iota 0)(I + \mu G_{11}(x + \iota 0))^{-1}G_{13}(x + \iota 0).$$

Since we are only looking for invertibility, looking at determinant is enough. So

$$\det(G_{23}^{\mu}(x+\iota 0)) = \frac{\det(G_{23}(x+\iota 0)) + \sum_{n=1}^{N} a_n(x)\mu^n}{\det(I+\mu G_{11}(x+\iota 0))}.$$

Again by corollary 4.2.3 we get that for almost all μ the matrix $G_{23}(x+\iota 0)$ is invertible on a set of full Lebesgue measure.

□

Proof of part (1) of main theorem [66]

For $n, m \in \mathscr{M}$, let $\omega \in \Omega_{n,m}$, using lemma 4.3.1 we get $G^{\omega}_{nm}(z)$ is almost surely invertible. For any $p \in \mathcal{N}$, we have $H^{\omega}_{\mu,p}$ and using lemma 4.3.5 we get $G^{\omega,\mu,p}_{nm}(z)$ is also almost surely invertible for almost all μ (with respect to Lebesgue measure). So we get, if $\omega \in \Omega_{n,m}$ then so is $\tilde{\omega} \in \Omega_{n,m}$ ($\tilde{\omega}$ is defined by $\omega_n = \tilde{\omega}_n \; \forall n \in \mathscr{M} \setminus \{p\}$) or in other words the event $\Omega_{n,m}$ is independent of the ω_p for any $p \in \mathscr{N}$. We can repeat the procedure and show that $\Omega_{n,m}$ is independent of $\{\omega_{p_i}\}_{i=1}^{K}$ for any finite collection of $p_i \in \mathscr{N}$. So we can use Kolmogorov 0-1 law (see theorem 1.2.4) to conclude that $\mathbb{P}(\Omega_{n,m}) \in \{0, 1\}$.

4.3.2 Proof of part (2)

Next lemma provide the relation between the absolute continuous component of the measures.

Lemma 4.3.6. [66, Lemma 3.5] *On the Hilbert space $\mathscr{H}$ we have two rank N projections P_1, P_2 and a self adjoint operator H. Set $H_\mu = H + \mu P_1$, $G_{ij}(z) = P_i(H-z)^{-1}P_j$ and $G^{\mu}_{ij}(z) = P_i(H_\mu - z)^{-1}P_j$; set S and S_{12} as (4.8),(4.9). Define*

$$V^{\mu}_{x,i} = ker(\Im G^{\mu}_{ii}(x+\iota 0))^{\perp}$$

for each $x \in S \cap \{x \in \mathbb{R} | \lim_{\epsilon \downarrow 0} G^{\mu}_{11}(x+\iota\epsilon)$ exists and finite$\}$. Assume S_{12} has full measure. Then for almost all μ

$$(G_{12}(x+\iota 0))^{-1} : V^{\mu}_{x,1} \to V^{\mu}_{x,2}$$

is injective and

$$(I + \mu G_{11}(x + \iota 0)) : V^0_{x,1} \to V^\mu_{x,1}$$

is isomorphism.

Proof. From the equation (4.6) and (4.5) we get

$$G^\mu_{22}(z) = G_{22}(z) - \mu G_{21}(z)G_{12}(z) + \mu^2 G_{21}(z)G^\mu_{11}(z)G_{12}(z)$$

For $x \in S \cap \{y \in \mathbb{R} | \lim_{\epsilon\downarrow 0} G^\mu_{11}(y + \iota\epsilon)$ exists and finite$\}$, let $v \in V^\mu_{x,1}$, and set $\phi = (G_{12}(x + \iota 0))^{-1}v$, observe (every quantity in RHS below exists and finite so limit can be taken)

$$\lim_{\epsilon\downarrow 0} \left\langle \phi, (\Im G^\mu_{22}(x + \iota\epsilon))\phi \right\rangle = \lim_{\epsilon\downarrow 0} [\langle \phi, (\Im G_{22}(x + \iota\epsilon))\phi\rangle - \mu \langle \phi, \Im(G_{21}(x + \iota\epsilon)G_{12}(x + \iota\epsilon))\phi\rangle$$
$$+ \mu^2 \left\langle \phi, (\Im G_{21}(x + \iota\epsilon)G^\mu_{11}(x + \iota\epsilon)G_{12}(x + \iota\epsilon))\phi \right\rangle]$$

Since $\Im G^\mu_{22}(x + \iota 0)$ is positive matrix, looking at $\left\langle \phi, (\Im G^\mu_{22}(x + \iota 0))\phi \right\rangle$ is enough.

If $\langle \phi, (\Im G_{22}(x + \iota 0))\phi\rangle = 0$ which implies $(\Im G_{22}(x + \iota 0))\phi = 0$ so using (2.10) we have $G_{12}(x + \iota 0)\phi = G^*_{21}(x + \iota 0)\phi$, so

$$\lim_{\epsilon\downarrow 0} \left\langle \phi, (\Im G^\mu_{22}(x + \iota\epsilon))\phi \right\rangle = \mu^2 \left\langle G_{12}(x + \iota 0)\phi, (\Im G^\mu_{11}(x + \iota 0))G_{12}(x + \iota 0)\phi \right\rangle$$
$$- \mu \langle \phi, \Im(G_{21}(x + \iota 0)G_{12}(x + \iota 0))\phi\rangle$$
$$= \mu^2 \left\langle v, (\Im G^\mu_{11}(x + \iota 0))v \right\rangle$$

So $\phi \in V^\mu_{E,2}$ and hence $G_{12}(x + \iota 0)^{-1}$ gives the injection.

For the other assertion, let $v \in V^0_{x,1}$ observe

$$\langle v, (I + \mu G_{11}(x + \iota 0))v\rangle = \|v\|^2_2 + \mu(\langle v, \Re G_{11}(x + \iota 0)v\rangle + \iota \langle v, \Im G_{11}(x + \iota 0)v\rangle)$$

since $\langle v, \Im G_{11}(x + \iota 0)v\rangle \neq 0$, so the above equation cannot be zero for any $\mu \in \mathbb{R}$. So on $V^0_{x,1}$ the operator $(I + \mu G_{11}(x + \iota 0))$ is invertible. Set $\phi = (I + \mu G_{11}(x + \iota 0))v$, observe

$$\lim_{\epsilon\to 0} \left\langle \phi, (\Im G^\mu_{11}(x + \iota\epsilon))\phi \right\rangle = \lim_{\epsilon\to 0} \left\langle \phi, \Im(G_{11}(x + \iota\epsilon)(I + \mu G_{11}(x + \iota\epsilon))^{-1})\phi \right\rangle$$
$$= \left\langle (I + \mu G_{11}(x + \iota 0))^{-1}\phi, (\Im G_{11}(x + \iota 0))(I + \mu G_{11}(x + \iota 0))^{-1}\phi \right\rangle$$

$$= \langle v, (\Im G_{11}(x + \iota 0))v \rangle \neq 0$$

This gives the isomorphism $(I + \mu G_{11}(x + \iota 0)) : V^0_{x,1} \to V^\mu_{x,1}$.

□

This only gives the injection between the absolutely continuous spectral subspaces. One cannot expect more from this setting. By a second perturbation we obtain an isomorphism, which is attained in the next corollary.

Corollary 4.3.7. [66, Corollary 3.6] *Let H be self adjoint operator on the Hilbert space $\mathscr{H}$, and P_1, P_2 are two rank N projections. Set $H_\mu = H + \mu_1 P_1 + \mu_2 P_2$ and $G_{ij}(z) = P_i(H - z)^{-1} P_j$, $G^{\mu_1,\mu_2}_{ij}(z) = P_i(H_{\mu_1,\mu_2} - z)^{-1} P_j$ for $i, j = 1, 2$ and define the vector space*

$$V^{\mu_1,\mu_2}_{x,i} = ker(\Im G^{\mu_1,\mu_2}_{ii}(x + \iota 0))^\perp$$

for each $x \in S \cap \{y \in \mathbb{R} | \lim_{\epsilon \downarrow 0} G^{\mu_1,\mu_2}_{ii}(y + \iota\epsilon)$ exists and finite for $i = 1, 2\}$. Assume S_{12}, S_{21} have full measure. Then for a.e μ_1, μ_2 the two vector space $V^{\mu_1,\mu_2}_{x,1}$ and $V^{\mu_1,\mu_2}_{x,2}$ are isomorphic.

Proof. This is just application of lemma 4.3.6. For x in full Lebesgue measure set we have

$$V^{\mu_1,\mu_2}_{x,2} \hookrightarrow V^{\mu_1,\mu_2}_{x,1}$$

where the map is $(G^{\mu_1,0}_{21}(x+\iota 0))^{-1}$. Lemma 4.3.5 tells us $G^{\mu_1,0}_{21}(x+\iota 0)$ is also invertible for almost all μ_1 (with respect to Lebesgue measure). Now we can do the same thing other way around:

$$V^{\mu_1,\mu_2}_{x,1} \hookrightarrow V^{\mu_1,\mu_2}_{x,2}$$

Since we are working in finite dimensional spaces ($V^{\mu_1,\mu_2}_{x,i}$ are finite dimensional), injection in both direction provides the isomorphism.

□

Proof of part (2) of main theorem [66]

For any $n \in \mathscr{M}$, we have $(A^\omega, \mathscr{H}_n^\omega)$ is unitary equivalent to $(M_{id}, L^2(\mathbb{R}, \Sigma_n^\omega, \mathbb{C}^N))$ (see theorem 2.5.1). For $m \in \mathscr{M}$ such that $\mathbb{P}(\Omega_{n,m} \cap \Omega_{m,n}) = 1$, we have to show $(\Sigma_n^\omega)_{ac}$ is equivalent to $(\Sigma_m^\omega)_{ac}$. Using (5) of theorem 2.4.1 we have

$$d(\Sigma_n^\omega)_{ac}(x) = \frac{1}{\pi} \Im G_{nn}^\omega(x + \iota 0) dE.$$

For $\omega \in \Omega_{n,m}$, we can write the operator $A^{\tilde{\omega}} = A^\omega + \mu_1 P_n + \mu_2 P_m$, and using corollary 4.3.7 we get $V_n^{\tilde{\omega}}$ are isomorphic to $V_m^{\tilde{\omega}}$, where

$$V_i^{\tilde{\omega}} = ker\left(P_i(A^{\tilde{\omega}} - x - \iota 0)^{-1} P_i\right)^{\perp}$$

Since $\Im G_{nn}^\omega(x+\iota 0) = \Im\left(P_n(A^\omega - x - \iota 0)^{-1} P_n\right)$, the isomorphism gives the equivalence. By proof of part (1), we know $\Omega_{n,m}$ is independent of ω_n and ω_m, so the result holds for almost all ω.

4.3.3 Proof of part (3)

The next lemma is similar to lemma 4.3.6 but for the singular part. The conclusion is for subspaces where growth of the Herglotz function is maximum or equivalently, its associated measure has lowest (Hausdorff) dimension. We will use the fact that a matrix valued measure $\Sigma_n(\cdot) = P_n E^H(\cdot) P_n$ is absolutely continuous with respect to the trace measure $\sigma_n(\cdot) = tr(\Sigma_n(\cdot))$ and so $\lim_{\epsilon\downarrow 0} \frac{1}{\sigma_n(x+\iota\epsilon)} \Sigma_n(x + \iota\epsilon) = M(x)$ is L^1 w.r.t σ_n-singular ($\sigma_n(z), \Sigma_n(z)$ are Borel transforms of the measures σ_n and Σ_n respectively).

Lemma 4.3.8. [66, Lemma 3.7] *On the Hilbert space $\mathscr{H}$ we have two rank N projections P_1, P_2 and a self adjoint operator H. Set $H_\mu = H + \mu P_1$, $G_{ij}(z) = P_i(H - z)^{-1} P_j$ and $G_{ij}^\mu(z) = P_i(H_\mu - z)^{-1} P_j$. Set $f_x(\epsilon) = tr(G_{11}^\mu(x + \iota\epsilon))^{-1}$ and $x \in \mathbb{R}$ be such that $f_x(\epsilon) \xrightarrow{\epsilon\downarrow 0} 0$, define*

$$\tilde{V}_{x,i}^\mu = ker\left(\lim_{\epsilon\downarrow 0} f_x(\epsilon) G_{ii}^\mu(x + \iota\epsilon)\right)^{\perp}$$

Assume S_{12} defined as (4.9) has full measure, then for $x \in S$ such that $f_x(\epsilon) \xrightarrow{\epsilon \downarrow 0} 0$ defined as in (4.8) the map

$$(G_{12}(x + \iota 0))^{-1} : \tilde{V}^{\mu}_{x,1} \to \tilde{V}^{\mu}_{x,2}$$

is injective. So the measure σ_2^{μ} (where $\sigma_i^{\mu}(\cdot) = tr\left(P_i E^{H_\mu}(\cdot) P_i\right)$) is absolutely continuous with respect to σ_1^{μ}-singular.

Proof. Using $i, j = 2$ in the equation (4.6), we have

$$G^{\mu}_{22}(z) = G_{22}(z) - \mu G_{21}(z) G_{12}(z) + \mu^2 G_{21}(z) G^{\mu}_{11}(z) G_{12}(z)$$

Since we are working with $x \in S$, the limits for $G_{ij}(x + \iota 0)$ exists for $i, j = 1, 2$. For $\phi, \psi \in P_2 \mathscr{H}$ we have

$$\begin{aligned}
\left\langle \psi, G^{\mu}_{22}(x + \iota\epsilon)\phi \right\rangle &= \langle \psi, G_{22}(x + \iota\epsilon)\phi \rangle - \mu \langle \psi, G_{21}(x + \iota\epsilon) G_{12}(x + \iota\epsilon)\phi \rangle \\
&\quad + \mu^2 \left\langle \psi, G_{21}(x + \iota\epsilon) G^{\mu}_{11}(x + \iota\epsilon) G_{12}(x + \iota\epsilon)\phi \right\rangle \\
\lim_{\epsilon \downarrow 0} f_x(\epsilon) \left\langle \psi, G^{\mu}_{22}(x + \iota\epsilon)\phi \right\rangle &= \mu^2 \lim_{\epsilon \downarrow 0} f_x(\epsilon) \left\langle \psi, G_{21}(x + \iota\epsilon) G^{\mu}_{11}(x + \iota\epsilon) G_{12}(x + \iota\epsilon)\phi \right\rangle \\
&= \mu^2 \left\langle \psi, G_{21}(x + \iota 0) \left(\lim_{\epsilon \downarrow 0} f_x(\epsilon) G^{\mu}_{11}(x + \iota\epsilon) \right) G_{12}(x + \iota 0)\phi \right\rangle
\end{aligned}$$

And now using (4.7) and (2.11) we have

$$\begin{aligned}
&\left\langle \psi, G_{21}(x + \iota 0) \left(\lim_{\epsilon \downarrow 0} f_x(\epsilon) G^{\mu}_{11}(x + \iota\epsilon) \right) G_{12}(x + \iota 0)\phi \right\rangle \\
&= \left\langle \psi, G_{12}(x + \iota 0)^* \left(\lim_{\epsilon \downarrow 0} f_x(\epsilon) G^{\mu}_{11}(x + \iota\epsilon) \right) G_{12}(x + \iota 0)\phi \right\rangle
\end{aligned}$$

From above if $\phi = G_{12}(x + \iota 0)^{-1} v$ for $v \in \tilde{V}^{\mu}_{x,1}$, then $\phi \in \tilde{V}^{\mu}_{x,2}$, giving us that the map $G_{12}(x + \iota 0)^{-1}$ is injection.

Finally

$$\lim_{\epsilon \downarrow 0} \frac{tr\left(G^{\mu}_{22}(x + \iota\epsilon)\right)}{tr\left(G^{\mu}_{11}(x + \iota\epsilon)\right)} = tr\left(G_{12}(x + \iota 0)^* \left(\lim_{\epsilon \downarrow 0} f_x(\epsilon) G^{\mu}_{11}(x + \iota\epsilon) \right) G_{12}(x + \iota 0) \right)$$

where RHS is L^1 for σ_1^{μ}-singular by lemma 2.3.7 (Poltoratskii's theorem).

□

Next lemma makes the injection to isomorphism by taking second perturbation in account.

Corollary 4.3.9. **[66, Corollary 3.8]** *Let H be self adjoint operator on the Hilbert space $\mathscr{H}$, and P_1, P_2 are two rank N projections. Set $H_\mu = H + \mu_1 P_1 + \mu_2 P_2$, $G_{ij}(z) = P_i(H - z)^{-1}P_j$ and $G_{ij}^{\mu_1,\mu_2}(z) = P_i(H_{\mu_1,\mu_2} - z)^{-1}P_j$ for $i, j = 1, 2$. Let $x \in S_{12} \cap S_{21}$ (defined as in (4.9)) and $tr(G_{ii}^{\mu_1,\mu_2}(x + \iota\epsilon))^{-1} \xrightarrow{\epsilon\downarrow 0} 0$ for either $i = 1, 2$, then*

$$\tilde{V}_{x,i}^{\mu_1,\mu_2} = ker\left(\lim_{\epsilon\downarrow 0} tr(G_{ii}^{\mu_1,\mu_2}(x + \iota\epsilon))^{-1} G_{ii}^{\mu_1,\mu_2}(x + \iota\epsilon)\right)^{\perp} \qquad i = 1, 2$$

are isomorphic. In particular the singular part of trace measure associated with $G_{ii}^{\mu_1,\mu_2}$ are equivalent to each other.

Proof. Define

$$\tilde{V}_{x,i,j}^{\mu_1,\mu_2} = ker\left(\lim_{\epsilon\downarrow 0} tr(G_{jj}^{\mu_1,\mu_2}(x + \iota\epsilon))^{-1} G_{ii}^{\mu_1,\mu_2}(x + \iota\epsilon)\right)^{\perp}$$

This is exactly like corollary 4.3.7. By action of lemma 4.3.8 we have

$$V_{x,1,1}^{\mu_1,\mu_2} \hookrightarrow V_{x,2,1}^{\mu_1,\mu_2} \textit{ and } V_{x,2,2}^{\mu_1,\mu_2} \hookrightarrow V_{x,1,2}^{\mu_1,\mu_2}$$

where first is given by $G_{12}^{0,\mu_2}(x + \iota 0)^{-1}$ and second is given by $G_{21}^{\mu_1,0}(x + \iota 0)^{-1}$ which are a.e (with respect to perturbation μ_1, μ_2) invertible because of lemma 4.3.5. Because of the second conclusion of the previous lemma 4.3.8 we have

$$\lim_{\epsilon\downarrow 0} \frac{tr\left(G_{11}^{\mu}(x + \iota\epsilon)\right)}{tr\left(G_{22}^{\mu}(x + \iota\epsilon)\right)} \text{ exists for almost all } x \text{ w.r.t } tr(P_2 E^{H_\mu}(\cdot) P_2)\text{-singular,}$$

$$\lim_{\epsilon\downarrow 0} \frac{tr\left(G_{22}^{\mu}(x + \iota\epsilon)\right)}{tr\left(G_{11}^{\mu}(x + \iota\epsilon)\right)} \text{ exists for almost all } x \text{ w.r.t } tr(P_1 E^{H_\mu}(\cdot) P_1)\text{-singular.}$$

So as a vector space $V_{x,i,j}^{\mu_1,\mu_2} = V_{x,i,i}^{\mu_1,\mu_2} = V_{x,i}^{\mu_1,\mu_2}$ for a.e $tr(P_i E^{H_\mu}(\cdot) P_i)$-singular. Since we have the injection both direction and finite dimensionality of the spaces involved, we get the isomorphism.

□

Proof of part (3) of main theorem [66]

For $n, m \in \mathscr{M}$ such that $\mathbb{P}(\Omega_{n,m} \cap \Omega_{m,n}) = 1$. Let $\omega \in \Omega_{n,m}$, define $A^{\tilde{\omega}} = A^{\omega} + \mu_n P_n + \mu_m P_m$, then corollary 4.3.9 gives the equivalence of the trace measure for singular part. As for absolute continuous part, second part of the theorem gives the equivalence.

4.3.4 Proof of part (4)

Till now there was no need for specifying any basis for the $P_i\mathscr{H}$ except for defining the sets S and S_{ij}. But for the following lemma we will work with a fixed basis. Though the result of the lemma is presented in a basis independent form.

Lemma 4.3.10. *On the Hilbert space $\mathscr{H}$ we have two rank N projections P_1, P_2 and a self adjoint operator H. Set $H_\mu = H + \mu P_1$, $G_{ij}(z) = P_i(H - z)^{-1}P_j$ and $G^{\mu}_{ij}(z) = P_i(H_\mu - z)^{-1}P_j$; set S and S_{12} as (4.8),(4.9). Let E^{μ}_{sing} denote the orthogonal projection onto the singular part of spectral measure for H_μ and set $\mathscr{H}^{\mu}_{i,sing}$ denote the closed $E^{\mu}_{sing}H_\mu$-invariant linear subspace containing $P_i\mathscr{H}$. If S_{12} has full Lebesgue measure, then $\mathscr{H}^{\mu}_{2,sing} \subseteq \mathscr{H}^{\mu}_{1,sing}$ for almost all μ (with respect to Lebesgue measure).*

Proof. Let $\{e_{ij}\}_{j=1}^N$ be a basis of $P_i\mathscr{H}$ for $i = 1, 2$. In this basis the linear operators $G^{\mu}_{ij}(z)$ and $G_{ij}(z)$ are matrices. Using Poltoratskii's theorem for the matrix case (see (2.7)) we have

$$\lim_{\epsilon \downarrow 0} \frac{1}{tr(G^{\mu}_{11}(x + \iota\epsilon))} G^{\mu}_{11}(x + \iota\epsilon) = M^{\mu}_1(x),$$

for almost all x w.r.t. σ^{μ}_1-singular (here σ^{μ}_i denotes the trace measure $tr(P_i E^{H_\mu}(\cdot)P_i)$ and set $\sigma^{\mu}_{1,sing}$ to be singular part of the measure). Using non-negativity of the spectral measure we have $M^{\mu}_1(x) \geq 0$ for almost all x with respect to $\sigma^{\mu}_{1,sing}$. Using lemma 4.3.8 and following its proof, we get

$$\lim_{\epsilon \downarrow 0} \frac{1}{tr(G^{\mu}_{11}(x + \iota\epsilon))} G^{\mu}_{ii}(x + \iota\epsilon) = M^{\mu}_i(x) \geq 0$$

for almost all x w.r.t. $\sigma^{\mu}_{1,sing}$. Let $U^{\mu}_{i}(x)$ be the unitary matrix such that $U^{\mu}_{i}(x)M^{\mu}_{i}(x)U^{\mu}_{i}(x)^*$ is diagonal with entries $f^{\mu}_{i1}(x), \cdots, f^{\mu}_{iN}(x)$ for x in support of $\sigma^{\mu}_{1,sing}$ (by using Hahn-Hellinger Theorem 1.4.3, one can choose the $U^{\mu}_{i}(\cdot)$ to be Borel measurable function). For x not in the support of $\sigma^{\mu}_{1,sing}$ set $U^{\mu}_{ij}(x) = 0$ and define $\psi^{\mu}_{ij} = U^{\mu}_{ij}(H_\mu)^* e_{ij}$.

We observe that

$$\begin{aligned}
\left\langle \psi^{\mu}_{ij}, (H_\mu - z)^{-1}\psi^{\mu}_{kl}\right\rangle &= \int \frac{1}{x-z}\left\langle \psi^{\mu}_{ij}, E^{H_\mu}(dx)\psi^{\mu}_{kl}\right\rangle \\
&= \int \frac{1}{x-z}\left\langle U^{\mu}_{i}(x)^* e_{ij}, E^{H_\mu}(dx)U^{\mu}_{k}(x)^* e_{kl}\right\rangle \\
&= \int \frac{1}{x-z}\sum_{p,q}\left\langle e_{ij}, U^{\mu}_{i}(x)e_{ip}\right\rangle\left\langle e_{kq}, U^{\mu}_{k}(x)^* e_{kl}\right\rangle\left\langle e_{ip}, E^{H_\mu}(dx)e_{kq}\right\rangle \\
&= \int \frac{1}{x-z}\sum_{p,q}\left\langle e_{ij}, U^{\mu}_{i}(x)e_{ip}\right\rangle\overline{\left\langle e_{kl}, U^{\mu}_{k}(x)e_{kq}\right\rangle}\left\langle e_{ip}, E^{H_\mu}(dx)e_{kq}\right\rangle.
\end{aligned}$$

So as a consequence of Poltoratskii's theorem

$$\lim_{\epsilon\downarrow 0}\frac{\left\langle \psi^{\mu}_{ij}, (H_\mu - x - \iota\epsilon)^{-1}\psi^{\mu}_{kl}\right\rangle}{tr(G^{\mu}_{11}(x+\iota\epsilon))} = \sum_{p,q}\left\langle e_{ij}, U^{\mu}_{i}(x)e_{ip}\right\rangle\overline{\left\langle e_{kl}, U^{\mu}_{k}(x)e_{kq}\right\rangle}\left(\lim_{\epsilon\downarrow 0}\frac{\left\langle e_{ip}, (H_\mu - x - \iota\epsilon)^{-1}e_{kq}\right\rangle}{tr(G^{\mu}_{11}(x+\iota\epsilon))}\right)$$

Therefore for $j \neq k$ we have $\left\langle \psi^{\mu}_{ij}, (H_\mu - z)^{-1}\psi^{\mu}_{ik}\right\rangle = 0$, because the normal limit to $\mathbb{R}$ is zero for all x. But the measure $\left\langle \psi^{\mu}_{ij}, E^{H_\mu}(\cdot)\psi^{\mu}_{ik}\right\rangle$ cannot have any absolutely continuous component, because by construction of $\{\psi^{\mu}_{pq}\}$, the measure $\left\langle \psi^{\mu}_{pq}, E^{H_\mu}(\cdot)\psi^{\mu}_{pq}\right\rangle$ is supported on the support of $\sigma^{\mu}_{1,sing}$ which is a zero Lebesgue measure set. So as consequence of F. and M. Riesz theorem (theorem 2.3.2) the Hilbert subspace $\mathscr{H}^{\mu}_{\psi^{\mu}_{ij}}$ is orthogonal to $\mathscr{H}^{\mu}_{\psi^{\mu}_{ik}}$ for $j \neq k$, where $\mathscr{H}^{\mu}_{\phi}$ denotes the minimal closed H_μ-invariant subspace containing ϕ.

Using the steps of proof of lemma 4.3.8 we have

$$M^{\mu}_{2}(x) = \lim_{\epsilon\downarrow 0}\frac{1}{tr(G^{\mu}_{11}(x+\iota\epsilon))}G^{\mu}_{22}(x+\iota\epsilon) = \mu^2 G_{12}(x+\iota 0)^* M^{\mu}_{1}(x)G_{12}(x+\iota 0)$$

for almost all x w.r.t. $\sigma^{\mu}_{1,sing}$, hence giving us

$$f^{\mu}_{2i}(x) = \lambda^2\sum_{j=1}^{N}\left|\left\langle \psi^{\mu}_{1j}, G_{12}(x+\iota 0)\psi^{\mu}_{2i}\right\rangle\right|^2 f_{1j}(x)$$

for a.e x wrt $\sigma^\mu_{1,sing}$. This is important because

$$\begin{aligned}\left\langle \psi^\mu_{2i}, g(H_\mu)\psi^\mu_{2i}\right\rangle &= \lim_{\epsilon\downarrow 0}\int g(x)\left\langle \psi^\mu_{2i}, (H_\mu - x - \iota\epsilon)^{-1}\psi^\mu_{2i}\right\rangle dx \qquad\qquad \forall g\in C_c(\mathbb{R})\\ &= \int g(x) f^\mu_{2i}(x) d\sigma^\mu_{1,sing}(x)\\ &= \lambda^2\sum_{i=1}^N \int g(x)\left|\left\langle \psi^\mu_{2i}, G_{12}(x+\iota 0)\psi^\mu_{2i}\right\rangle\right|^2 f_{1j}(x) d\sigma^\mu_{1,sing}(x)\end{aligned}$$

for all $1\leq i\leq N$. Using the equality

$$\lim_{\epsilon\downarrow 0}\frac{1}{tr(G^\mu_{11}(x+\iota\epsilon))}G^\mu_{12}(x+\iota\epsilon) = -\mu M^\mu_1(x)G_{12}(x+\iota 0),$$

for almost all x w.r.t. $\sigma^\mu_{1,sing}$, we have,

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\frac{\left\langle \psi^\mu_{1j}, (H_\mu - x - \iota\epsilon)^{-1}\psi^\mu_{2i}\right\rangle}{tr(G^\mu_{11}(x+\iota\epsilon))}\\ &\quad= \sum_{k,l}\left\langle e_{1j}, U^\mu_1(x)e_{1k}\right\rangle \overline{\left\langle e_{2i}, U^\mu_2(x)e_{2l}\right\rangle}\left\langle e_{1k}, \left(\lim_{\epsilon\downarrow 0}\frac{G^\mu_{12}(x+\iota\epsilon)}{tr(G^\mu_{11}(x+\iota\epsilon))}\right)e_{2l}\right\rangle\\ &\quad= -\mu\sum_{k,l}\left\langle e_{1j}, U^\mu_1(x)e_{1k}\right\rangle \overline{\left\langle e_{2i}, U^\mu_2(x)e_{2l}\right\rangle}\left\langle e_{1k}, M^\mu_1(x)G_{12}(x+\iota 0)e_{2l}\right\rangle\\ &\quad= -\mu\left\langle e_{1j}, U^\mu_1(x)M^\mu_1(x)G_{12}(x+\iota 0)U^\mu_2(x)e_{2i}\right\rangle = -\mu f^\mu_{1j}(x)\left\langle \psi^\mu_{1j}, G_{12}(x+\iota 0)\psi^\mu_{2i}\right\rangle\end{aligned}$$

for almost all x w.r.t $\sigma^\mu_{1,sing}$. On the support of $f^\mu_{1j}\sigma^\mu_{1,sing}$ set

$$\lim_{\epsilon\downarrow 0}\frac{\left\langle \psi^\mu_{1j}, (H_\mu - x - \iota\epsilon)^{-1}\psi^\mu_{2i}\right\rangle}{\left\langle \psi^\mu_{1j}, (H_\mu - x - \iota\epsilon)^{-1}\psi^\mu_{1j}\right\rangle} = p_{ij}(x).$$

Because of Poltoratskii's theorem, the vector $p_{ij}(H_\mu)\psi^\mu_{1j}$ is the projection of ψ^μ_{2i} onto $E^\mu_{sing}\mathscr{H}^\mu_{\psi^\mu_{1j}}$.

Finally for almost all x w.r.t. $f^\mu_{1j}d\sigma^\mu_{1,sing}$ we have

$$\begin{aligned}p_{ij}(x) &= \lim_{\epsilon\downarrow 0}\frac{\left\langle \psi^\mu_{1j}, (H_\mu - x - \iota\epsilon)^{-1}\psi^\mu_{2i}\right\rangle}{\left\langle \psi^\mu_{1j}, (H_\mu - x - \iota\epsilon)^{-1}\psi^\mu_{1j}\right\rangle}\\ &= \lim_{\epsilon\downarrow 0}\frac{\left\langle \psi^\mu_{1j}, (H_\mu - x - \iota\epsilon)^{-1}\psi^\mu_{2i}\right\rangle}{tr(G^\mu_{11}(x+\iota\epsilon))}\frac{tr(G^\mu_{11}(x+\iota\epsilon))}{\left\langle \psi^\mu_{1j}, (H_\mu - x - \iota\epsilon)^{-1}\psi^\mu_{1j}\right\rangle}\\ &= -\mu\left\langle \psi^\mu_{1j}, G_{12}(x+\iota 0)\psi^\mu_{2i}\right\rangle.\end{aligned}$$

Giving us

$$f_{2i}^{\mu}(x) = \sum_{j=1}^{N} |p_{ij}(x)|^2 f_{1j}^{\mu}(x)$$

for almost all x w.r.t. $\sigma_{1,sing}^{\mu}$. So multiplication by p_{ij} is not only projection but also an isometry from $E_{sing}^{\mu}\mathscr{H}_{2i}^{\mu}$ to $\mathscr{H}_{1,sing}^{\mu}$. Since this is valid for all ψ_{2j}^{μ}, we get

$$\mathscr{H}_{2,sing}^{\mu} \subseteq \mathscr{H}_{1,sing}^{\mu}$$

for almost all μ (with respect to Lebesgue measure).

$\square$

Remark 4.3.11. *Since $\left\langle \psi_{ip}^{\mu}, E^{H_\mu}(\cdot)\psi_{iq}^{\mu} \right\rangle \equiv 0$ for $p \neq q$, we have $p_{ip}(x)p_{iq}(x) = 0$ for a.a x w.r.t $\sigma_{1,sing}^{\mu}$. So re-define*

$$\tilde{\psi}_{1i}^{\mu} = \sum_{j=1}^{N} \chi_{\{x:p_{ij}(x)\neq 0\}}(H_\mu)\psi_{1j}^{\mu}$$

and get $f_{2i}^{\mu}(x) = |\tilde{p}_i(x)|^2 f_{1i}(x)$, where $\tilde{p}_i$ is the projection defined using $\tilde{\psi}_{1i}^{\mu}$. So $E_{sing}^{\mu}\mathscr{H}_{\tilde{\psi}_{2i}}^{\mu}$ is contained in $E_{sing}^{\mu}\mathscr{H}_{\tilde{\psi}_{1i}^{\mu}}^{\mu}$.

Using a second perturbation we get the equality of the two Hilbert subspace. This is the statement of the next corollary.

Corollary 4.3.12. *On Hilbert space $\mathscr{H}$ we have two rank N projections P_1, P_2 and a self adjoint operator H. Set $H_{\mu_1,\mu_2} = H + \mu_1 P_1 + \mu_2 P_2$, $G_{ij}(z) = P_i(H - z)^{-1}P_j$ and $G_{ij}^{\mu_1,\mu_2}(z) = P_i(H_{\mu_1,\mu_2} - z)^{-1}P_j$; set S and S_{12}, S_{21} as (4.8),(4.9). Let E_{sing}^{μ} denote the orthogonal projection to the singular part of spectral measure for H_{μ_1,μ_2} and set $\mathscr{H}_{i,sing}^{\mu}$ denote the minimal closed $P_{sing}^{\mu}H_{\mu_1,\mu_2}$-invariant subspace containing $P_i\mathscr{H}$. If S_{12} and S_{21} have full Lebesgue measure, then $\mathscr{H}_{2,sing}^{\mu} = \mathscr{H}_{1,sing}^{\mu}$ for almost all (μ_1, μ_2) (with respect to Lebesgue measure).*

Proof. Viewing A_{μ_1,μ_2} as perturbation of P_1 (i.e $A_{\mu_1,\mu_2} = A_{0,\mu_2} + \mu_1 P_1$) gives

$$\mathscr{H}_{2,sing}^{\mu} \subseteq \mathscr{H}_{1,sing}^{\mu}.$$

Similarly considering A_{μ_1,μ_2} as perturbation of P_2 gives

$$\mathscr{H}^{\mu}_{1,sing} \subseteq \mathscr{H}^{\mu}_{2,sing}.$$

Combining both of them give us the desired result.

□

Proof of (4) of main theorem

Using corollary 4.3.12 we have $\mathscr{H}^{\omega}_{n,sing} = \mathscr{H}^{\omega}_{m,sing}$ for any any n, m such that $\mathbb{P}(\Omega_{n,m} \cap \Omega_{m,n}) = 1$. So using $\mathbb{P}(\Omega_{n,m} \cap \Omega_{m,n}) = 1$ for all $n, m \in \mathcal{M}$ we get

$$E^{\omega}_{sing}\mathscr{H} = \cup_{n\in\mathcal{M}}\mathscr{H}^{\omega}_{n,sing} = \mathscr{H}^{\omega}_{m,sing}$$

for any $m \in \mathcal{M}$.

4.4 Summary and future directions

The result of the corollaries 4.3.7, 4.3.9 and 4.3.12 can be boiled down to the following Venn diagrams.

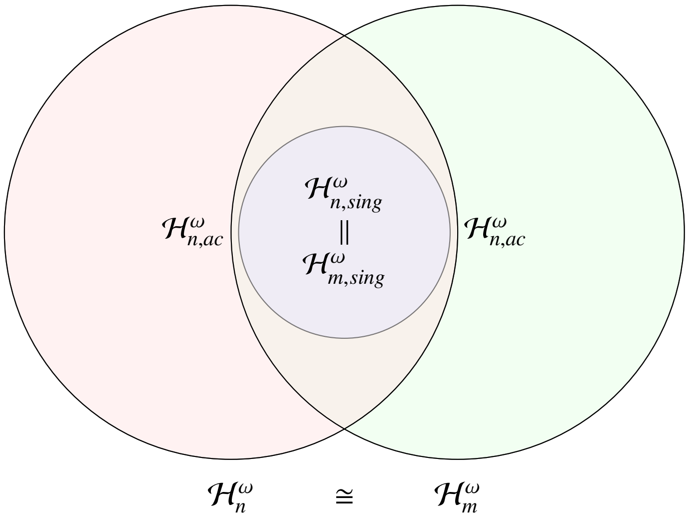

Figure 4.1: When $\mathbb{P}(\Omega_{n,m} \cap \Omega_{m,n}) = 1$, we are able to show that the singular subspace $\mathscr{H}^{\omega}_{n,sing}$ and $\mathscr{H}^{\omega}_{m,sing}$ are equal, but we can only prove the isomorphism for $\mathscr{H}^{\omega}_{n,ac}$ and $\mathscr{H}^{\omega}_{m,ac}$.

The event $\Omega_{n,m}$ provides the information about the event $\{\omega : \mathscr{H}^{\omega}_{n} \cap \mathscr{H}^{\omega}_{m} \neq \phi\}$. In case of rank one, this condition boils down to the fact that the associated Green's function is non-zero.

Definition of $\Omega_{n,m}$ is independent of rank of P_n. But to prove $\mathbb{P}(\Omega_{n,m}) \in \{0, 1\}$, we looked at $\det(G^{\omega}_{nm}(z))$ which can be defined for the case $rank(P_n) = rank(P_m)$ only. Then we showed that the polynomial $\det(G^{\omega,\mu,p}_{nm}(z))$ is almost surely non-zero for almost all (μ, z) w.r.t Lebesgue measure. And the result follows through Kolmogorov 0-1 law. But if $rank(P_n) \neq rank(P_m)$, then also the definition 3.6 is valid. In fact in the lemmas 4.3.1 and 4.3.5, "invertibility" can be modified to "full rank". The main problem arises in lemmas 4.3.6 and 4.3.8, where we used invertibility of $G_{12}(x + \iota 0)$. If those statements could be stated without the inverse, then possibly the theorem can be proved for the case $rank(P_n) < \infty$ only (i.e we allow $rank(P_n) \neq rank(P_m)$). This is probably true because of the way invertibility of $G_{12}(x + \iota 0)$ is used. Hence trying to prove the theorem without the assumption $rank(P_n) = rank(P_m)$ is a possible extension.

Definition of $\Omega_{n,m}$ is too strong, and it cannot give any extra result in cases like example 3.3.4. For the operators of the form (3.5) where we do not assume $rank(P_n) = rank(P_m)$, we can prove that there exists a basis for $P_n\mathscr{H}$ and $P_m\mathscr{H}$ such that $G^{\omega}_{nm}(z)$ can be written down as $S_{n,m} \times S_{m,n}$ sub-matrix with rest of the entries being zero. Even more $S_{i,j}$ are independent of ω and z, and only depends on i and j. While proving the preceding statement, one can get a projection $P_{i,j}$ ($\leq P_i$) with $rank(P_{i,j}) = S_{i,j}$ such that each entries of $P_{i,j}(A^{\omega} - z)^{-1}P_{j,i}$ (in some fixed basis) is non-zero for almost every z. One can also show that the matrix $G^{\omega}_{nn}(z)$ has a block diagonal form (with the block being $P_{i,j}(A^{\omega} - z)^{-1}P_{i,j}$ for any j). So $P_{i,j}$ could be used to replace the set $\Omega_{i,j}$ in theorem 4.1.1 in certain way. And all the results for the spectral measure should be stated for the closed Hilbert subspace generated by A^{ω} and $P_{i,j}\mathscr{H}$. If possible this kind of statement has possibility of classifying all random operator of the form (3.5) whenever the rank of projections are finite.

The next possible question that could be asked is if similar statement holds when the perturbations are compact. Keeping the theme of finite rank situation, the next possibility is replacing the projections P_n with self-adjoint finite ranked operators.

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