

Demazure flags, Chebyshev polynomials and mock
theta functions

by

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Rekha Biswal

List of publications

- **Published**

- (1) Rekha Biswal and Ghislain Fourier, Minuscule schubert varieties: poset polytopes, PBW-degenerated Demazure modules, and Kogan faces
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- (2) Rekha Biswal, Vyjayanthi Chari, Lisa Schneider and Sankaran Viswanath, Demazure Flags, Chebyshev polynomials, Partial and Mock theta functions
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Dedicated to my brother
Nigamananda Biswal

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SYNOPSIS

The mathematical objects known as current algebras are closely related to affine Lie algebras which are some of the most interesting examples of Kac-Moody algebras. In this thesis, we investigate an important class of representations of the current algebras, the so-called Demazure modules. The work of several mathematicians over the past decades has thrown much light on the structure of these modules. But, in very recent work (2013), Vyjayanthi Chari, of the University of California at Riverside, and her collaborators introduced a rather novel point of view which we may refer to as the “Demazure flags approach”. Roughly speaking, this approach is a way of decomposing a Demazure module into simpler Demazure modules, but of higher “level”. In this document, we carry out a deeper analysis of Demazure flags.

It was observed in [23] that one could use the results of [17] and [21] to show the following: for all integers $m \geq \ell > 0$ and any non-negative integer s , the Demazure module denoted by $D(\ell, s)$, admits a Demazure flag of level m , i.e., there exists a decreasing sequence of graded submodules of $D(\ell, s)$ such that the successive quotients are isomorphic to $\tau_p^* D(m, n)$ where $p \geq 0$, $0 \leq n \leq s$ and $s - n$ is even. The proof of the statement requires taking the $q = 1$ limit of a result proved by A. Joseph [17] in the quantum case using the theory of canonical bases. In [9], the authors gave a direct proof of this result. Their methods also showed the existence of a level m Demazure flag in a much wider class of modules for $\mathfrak{sl}_2[t]$. The number of times a particular level m -Demazure module appears as a quotient in a level m -flag is independent of the choice

of the flag. We define a polynomial in an indeterminate q by,

$$[D(\ell, s) : D(m, n)]_q = \sum_{p \geq 0} [D(\ell, s) : \tau_p^* D(m, n)] q^p,$$

where $[D(\ell, s) : \tau_p^* D(m, n)]$ is the multiplicity of $\tau_p^* D(m, n)$ in a level m -Demazure flag of $D(\ell, s)$. The polynomial $[D(\ell, s) : D(m, n)]_q$ is called the graded multiplicity of $D(m, n)$ in the level m -Demazure flag of $D(\ell, s)$ and the polynomial $[D(\ell, s) : D(m, n)]_q$ evaluated at $q = 1$ is called the numerical multiplicity of $D(m, n)$ in the level m -Demazure flag of $D(\ell, s)$. It is known that

$$[D(\ell, s) : D(m, s)]_q = 1, \quad [D(\ell, s) : D(m, n)]_q = 0 \quad s - n \notin 2\mathbb{Z}_+.$$

Moreover, for $m \geq \ell' \geq \ell$ we have

$$[D(\ell, s) : D(m, n)]_q = \sum_{p \in \mathbb{Z}_{\geq 0}} [D(\ell, s) : D(\ell', p)]_q [D(\ell', p) : D(m, n)]_q. \quad (0.0.1)$$

In [9], explicit recurrence relations were given for the multiplicity of a level $(\ell + 1)$ -Demazure module occurring in a filtration of $D(\ell, n)$. A closed form solution of these recurrences was however only obtained in some special cases: the numerical multiplicities (the $q = 1$ case) were computed for $\ell = 2, m = 3$, and the graded multiplicities for $\ell = 1, m = 2$. The authors also showed that the graded multiplicities of level 3 Demazure modules in level 2 Demazure module are related to partial theta series. In this thesis we undertake a deeper study of the polynomials $[D(\ell, s) : D(m, n)]_q$ and the associated

generating series: given $\ell, m \in \mathbb{N}$ with $m \geq \ell$, set

$$A_n^{\ell \rightarrow m}(x, q) = \sum_{k \geq 0} [D(\ell, n + 2k) : D(m, n)]_q x^k, \quad n \geq 0.$$

We extend the results of [9], obtaining closed form expression of those generating series in more general cases. We also relate the generating series to mock theta functions in some special cases. This thesis consists of 5 chapters which we briefly describe below:

- In chapter 1, we give a brief introduction to the problems.
- In chapter 2, we recall some preliminaries that will help us present the results in this thesis.
- In chapter 3, we establish a recursive formula for the polynomials $[D(\ell, s) : D(m, n)]_q$.
- In chapter 4, we study the series $A_n^{\ell \rightarrow m}(x, 1)$ and give explicit formulae for $A_n^{1 \rightarrow m}(x, 1)$ and $A_n^{2 \rightarrow m}(x, 1)$ involving Chebyshev polynomials.
- In chapter 5, we are concerned about the q -multiplicities when $\ell = 1$ and $m = 3$. We prove that in this case, the generating series $A_n^{1 \rightarrow 3}(x, q)$ when appropriately specialized reduce to expressions involving the fifth order mock theta functions $\phi_0, \phi_1, \psi_0, \psi_1$ of Ramanujan.

Chapter 1

Introduction

Let $\mathfrak{sl}_2[t] = \mathfrak{sl}_2 \otimes \mathbb{C}[t]$ be the Lie algebra of two by two matrices of trace zero with entries in the algebra $\mathbb{C}[t]$ of polynomials with complex coefficients in an indeterminate t . The degree grading of $\mathbb{C}[t]$ defines a natural grading on $\mathfrak{sl}_2[t]$. Let $D(\ell, s)$ be the graded $\mathfrak{sl}_2[t]$ -module generated by an element v_s with defining relations:

$$(x \otimes \mathbb{C}[t])v_s = 0, \quad (h \otimes f(t))v_s = sf(0)v_s, \quad (y \otimes 1)^{s+1}v_s = 0, \quad (1.0.2)$$

$$(y \otimes t^{s_1+1})v_s = 0, \quad (y \otimes t^{s_1})^{s_0+1}v_s = 0, \quad \text{if } s_0 < \ell. \quad (1.0.3)$$

Here, x, h, y is the standard basis of \mathfrak{sl}_2 and $s_0 \in \mathbb{N}$ and $s_1 \in \mathbb{Z}$ with $s_1 \geq -1$ and $s_0 \leq \ell$ are such that $s = \ell s_1 + s_0$. The $D(\ell, s)$ are called *Demazure modules*; they are finite dimensional and ℓ is called the level of the Demazure module. If V is a graded $\mathfrak{sl}_2[t]$ -module, let τ_r^*V denote the graded $\mathfrak{sl}_2[t]$ -module with the graded pieces shifted uniformly by r and the action of $\mathfrak{sl}_2[t]$ unchanged. Let \mathbb{Z}_+ denote the set of non-negative integers. For $n \in \mathbb{Z}_+$, the local Weyl module $W_{\text{loc}}(n)$ is the $\mathfrak{sl}_2[t]$ -module generated by an element

w_n with following defining relations:

$$(x \otimes \mathbb{C}[t])w_n = 0, \quad (h \otimes t^s)w_n = n\delta_{s,0}w_n, \quad (y \otimes 1)^{n+1}w_n = 0. \quad (1.0.4)$$

1.1. RECURSIVE FORMULAE FOR $[D(\ell, s) : D(m, n)]_q$

In chapter 3 of the thesis, we give a recursive formula for the polynomials $[D(\ell, s) : D(m, n)]_q$, which could be viewed as giving the definition of these polynomials. This recursive formula plays a critical role in studying $A_n^{1 \rightarrow m}(x, 1)$ and relating $A_n^{1 \rightarrow 3}(x, q)$ to mock theta functions.

Given integers $m \geq \ell > 0$ and integers s, n , set

$$[D(\ell, s) : D(m, n)]_q = 0, \quad \text{if } s < 0 \text{ or } n < 0. \quad (1.1.1)$$

We have

$$[D(\ell, 0) : D(m, n)]_q = \delta_{n,0}, \quad n \in \mathbb{Z}_+, \quad (1.1.2)$$

where $\delta_{j,k}$ is the Kronecker delta function. More generally,

$$[D(\ell, s) : D(m, n)]_q = 0, \quad \text{if } s - n \notin 2\mathbb{Z}_+, \text{ and} \quad (1.1.3)$$

$$[D(\ell, s) : D(m, s)]_q = 1, \quad s \in \mathbb{Z}_+. \quad (1.1.4)$$

Given a non-negative integer n and a positive integer m let $0 \leq r(n, m) < m$ be the unique integer such that $n = m \lfloor \frac{n}{m} \rfloor + r(n, m)$. We prove the following theorem:

Theorem 1.1.1. *Let ℓ, m be positive integers with $m \geq \ell$. For all $s, n \in \mathbb{Z}_+$, we have*

$$\begin{aligned}
[D(\ell, s+1) : D(m, n)]_q &= [D(\ell, s) : D(m, n-1)]_q + (1 - \delta_{r(n+1, m), 0})[D(\ell, s) : D(m, n+1)]_q \\
&- (1 - \delta_{r(s, \ell), 0})[D(\ell, s-1) : D(m, n)]_q - q^{\lfloor \frac{s}{\ell} \rfloor r(s, \ell)} (1 - q^{\lfloor \frac{s}{\ell} \rfloor}) [D(\ell, s - 2r(s, \ell) - 1) : D(m, n)]_q \\
&+ q^{\lfloor \frac{n}{m} \rfloor + 1} (m - r(n, m) - 1) (1 - q^{\lfloor \frac{n}{m} \rfloor + 1}) [D(\ell, s) : D(m, n + 2m - 2r(n, m) - 1)]_q.
\end{aligned}$$

This theorem can be viewed as giving a recursive definition of the polynomials $[D(\ell, s) : D(m, n)]_q$. Thus, (1.1.1) and (1.1.2) define $[D(\ell, s) : D(m, n)]_q$ for all $s \leq 0$ and $n \in \mathbb{Z}$. For $s \geq 0$, assume that we have defined $[D(\ell, s') : D(m, n)]_q$ for all $s' \leq s$ and all $n \in \mathbb{Z}$. The right hand side in Theorem 1.1.1 only involves $[D(\ell, s') : D(m, n')]_q$ with $s' \leq s, n' \in \mathbb{Z}$ and hence shows that $[D(\ell, s+1) : D(m, n)]_q$ is defined for all $n \in \mathbb{Z}_+$, and hence, by (1.1.1), for all $n \in \mathbb{Z}$. To prove the above theorem, we study the tensor product $D(\ell, s) \otimes D(\ell, 1)$ and write the graded character of the tensor product explicitly as a linear combination of graded characters of level ℓ -Demazure modules. If $m > \ell$, this result allows us to write the graded character of $D(\ell, s) \otimes D(\ell, 1)$ as linear combination of graded characters of level m -Demazure modules in two different ways. A comparison of the coefficients then proves the theorem. We briefly explain the strategy of the proof below. First we prove the following proposition:

Proposition 1.1.2. *Let ℓ be a positive integer and let $s \in \mathbb{Z}_+$. Write $s = \ell s_1 + s_0$ with $s_1, s_0 \in \mathbb{Z}$, $s_1 \geq -1$ and $0 < s_0 \leq \ell$. We have,*

$$\begin{aligned} \text{ch}_{\text{gr}} D(\ell, s) \text{ch}_{\text{gr}} D(\ell, 1) &= \text{ch}_{\text{gr}} D(\ell, s+1) + (1 - \delta_{s_0, \ell}) \text{ch}_{\text{gr}} D(\ell, s-1) \\ &+ q^{s_1(s_0 - \ell \delta_{s_0, \ell})} (1 - q^{s_1 + \delta_{s_0, \ell}}) \text{ch}_{\text{gr}} D(\ell, s - 2(s_0 - \ell \delta_{s_0, \ell}) - 1). \end{aligned}$$

It is easy to see that if V is a finite dimensional graded $\mathfrak{sl}_2[t]$ -module and admits a Demazure flag of level m , then

$$\text{ch}_{\text{gr}} V = \sum_{s \in \mathbb{Z}} [V : D(m, s)]_q \text{ch}_{\text{gr}} D(m, s). \quad (1.1.5)$$

In particular, if $D(\ell, s)$ admits a Demazure flag of level m , then we can write,

$$\text{ch}_{\text{gr}} D(\ell, s) = \sum_{p \geq 0} [D(\ell, s) : D(m, p)]_q \text{ch}_{\text{gr}} D(m, p),$$

where $m \in \mathbb{Z}_+$ with $m \geq \ell$. Multiplying both sides of the equation by $\text{ch}_{\text{gr}} D(\ell, 1)$ gives,

$$\text{ch}_{\text{gr}} D(\ell, s) \text{ch}_{\text{gr}} D(\ell, 1) = \sum_{p \geq 0} [D(\ell, s) : D(m, p)]_q \text{ch}_{\text{gr}} D(m, p) \text{ch}_{\text{gr}} D(m, 1). \quad (1.1.6)$$

Here, we have used the fact that $D(\ell, 1) \cong D(m, 1)$ as $\mathfrak{sl}_2[t]$ -modules. Now, we know that the product of graded characters is the graded character of the tensor product. We can therefore apply Proposition 1.1.2 to both sides of the preceding equation. Applying it to the right hand side gives us a linear combination of the graded characters of level m -Demazure modules. Applying it to the left hand side, gives a linear combination of graded characters of level ℓ -Demazure modules. These can be further expressed as

a combination of the graded characters of level m -Demazure modules. Equating the coefficients of a level m -Demazure module on both sides will prove Theorem 1.1.1.

1.2. NUMERICAL MULTIPLICITIES AND CHEBYSHEV POLYNOMIALS

In chapter 4 of the thesis, we greatly extend the results for the numerical multiplicities in [9] by using the recursive formula in Theorem 1.1.1. We prove that the generating function for the numerical multiplicity when $\ell = 1$ is a rational function involving the Chebyshev polynomials. A level one Demazure module is isomorphic to a *local Weyl module* [7] and hence our result completely determines the numerical multiplicities of a level m flag of a local Weyl module for any given $m \geq 1$. We prove the following recursion on the generating series $A_n^{1 \rightarrow m}(x, 1)$ using Theorem 1.1.1.

Theorem 1.2.1. *For $n \geq -1$ and $m \geq 1$, the power series $A_n^{1 \rightarrow m}(x, 1)$ satisfies the recurrence,*

$$A_n^{1 \rightarrow m}(x, 1) = \begin{cases} A_{n+1}^{1 \rightarrow m}(x, 1) - x A_{n+2}^{1 \rightarrow m}(x, 1) & \text{if } m \nmid n + 2. \\ A_{n+1}^{1 \rightarrow m}(x, 1) & \text{if } m \mid n + 2. \end{cases} \quad (1.2.1)$$

Then we use the above theorem to give a closed form for $A_n^{1 \rightarrow m}(x, 1)$. We first recall some relevant facts about Chebyshev polynomials. For $n \geq 0$, the Chebyshev polynomial $U_n(x)$ of the second kind, of degree n , is given by the recurrence relation:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \text{ for } n \geq 1, \quad U_0(x) = 1, \quad U_1(x) = 2x.$$

We define

$$P_n(x^2) = x^n U_n((2x)^{-1})$$

and then it is easy to see that $P_n(x)$'s satisfy the following recurrence.

$$P_0 = P_1 = 1 \text{ and } P_{n+1}(x) = P_n(x) - xP_{n-1}(x) \text{ for } n \geq 1. \quad (1.2.2)$$

We now establish the following corollary of Theorem 1.2.1 which gives the closed form of $A_n^{1 \rightarrow m}(x, 1)$.

Corollary 1.2.2. *For $n \in \mathbb{Z}_+$, let r, s be the unique non-negative integers such that $n = ms + r$ with $0 \leq r < m$. Then*

$$A_n^{1 \rightarrow m}(x, 1) = \frac{P_{m-r-1}(x)}{P_m(x)^{s+1}}.$$

Finally, we consider the general case, i.e., the multiplicities of level m Demazure modules in level ℓ Demazure modules for any $m \geq \ell$. For $n \geq 0$, define

$$\tilde{A}_n^{\ell \rightarrow m}(x, q) = \sum_{s \geq 0} [D(\ell, s) : D(m, n)]_q x^s.$$

Since the coefficient of x^s is zero unless $s - n$ is a non-negative even integer, we have $\tilde{A}_n^{\ell \rightarrow m}(x, q) = x^n \tilde{A}_n^{\ell \rightarrow m}(x^2, q)$. We prove the following proposition which gives us a way of computing $\tilde{A}_n^{\ell \rightarrow m}(x, 1)$ implicitly.

Proposition 1.2.3. *Let $1 \leq \ell \leq m$ and $n \geq 0$. Let $\beta_r(x) \in \mathbb{C}[[x]]$, $0 \leq r < \ell$, be the unique power series such that*

$$\tilde{A}_n^{\ell \rightarrow m}(x, 1) = \sum_{r=0}^{\ell-1} x^r \beta_r(x^\ell).$$

Then we have

$$\tilde{A}_n^{1 \rightarrow m}(x, 1) = \sum_{r=0}^{\ell-1} \tilde{A}_r^{1 \rightarrow \ell}(x, 1) \beta_r(y^\ell),$$

where $y = x/P_\ell(x^2)^{\frac{1}{\ell}}$.

Using the above proposition, we obtain the following corollary which gives an explicit expression for $A_n^{2 \rightarrow m}(x, 1)$.

Corollary 1.2.4. *Let $m \geq 2, n \geq 0$. Then*

$$A_n^{2 \rightarrow m}(x, 1) = \left(\frac{1}{1+x} \right)^{\lfloor \frac{n}{2} \rfloor + 1} A_n^{1 \rightarrow m} \left(\frac{x}{1+x}, 1 \right).$$

1.3. GRADED MULTIPLICITIES AND MOCK THETA FUNCTIONS

In chapter 5 of the thesis, we focus on graded multiplicities. Our next main result concerns the graded multiplicities when $\ell = 1$ and $m = 3$. In this case, we show that the generating series when appropriately specialized reduce to expressions involving the following fifth order mock theta functions $\phi_0, \phi_1, \psi_0, \psi_1$ of Ramanujan.

$$\phi_0(q) = \sum_{n=0}^{\infty} q^{n^2} (-q; q^2)_n \qquad \phi_1(q) = \sum_{n=0}^{\infty} q^{(n+1)^2} (-q; q^2)_n \qquad (1.3.1)$$

$$\psi_0(q) = \sum_{n=0}^{\infty} q^{\frac{(n+1)(n+2)}{2}} (-q; q)_n \qquad \psi_1(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} (-q; q)_n \qquad (1.3.2)$$

The appearance of Ramanujan's mock theta functions in this set-up is quite unexpected and intriguing. Certain *Hecke type double sums*, which are closely related objects, have previously appeared in Kac-Peterson's work [19] on characters of integrable representations of $\widehat{\mathfrak{sl}}_2$. Further, mock theta functions (in the modern sense, following Zwegers [30]) appear in Kac-Wakimoto's theory of affine superalgebras and their characters [20]. Given any power series f in the indeterminate q , we define

$$f^+(q) = \sum_{n \geq 0} c_{2n} q^n = \frac{f(q^{\frac{1}{2}}) + f(-q^{\frac{1}{2}})}{2}, \quad f^-(q) = \sum_{n \geq 0} c_{2n+1} q^n = \frac{f(q^{\frac{1}{2}}) - f(-q^{\frac{1}{2}})}{2q^{\frac{1}{2}}},$$

so that $f(q) = f^+(q^2) + q f^-(q^2)$. Then we prove the following results:

Theorem 1.3.1.

$$\begin{aligned} A_0^{1 \rightarrow 3}(1, q) &= \phi_0^+(q) & A_0^{1 \rightarrow 3}(q, q) &= \phi_1^-(q) \\ A_1^{1 \rightarrow 3}(1, q) &= \psi_1(q) & A_1^{1 \rightarrow 3}(q, q) &= \psi_0(q)/q \\ A_2^{1 \rightarrow 3}(1, q) &= \phi_0^-(q) & A_2^{1 \rightarrow 3}(q, q) &= \phi_1^+(q)/q^2 \end{aligned}$$

We now consider the specializations $A_n^{1 \rightarrow 3}(q^k, q)$ for arbitrary $k \in \mathbb{Z}$ and $0 \leq n \leq 2$. We show that these are in fact linear combinations of the mock theta functions with coefficients in $\mathbb{Z}[q, q^{-1}]$. More precisely, we have

Theorem 1.3.2. *Let $k \in \mathbb{Z}$. Then:*

(1)

$$A_1^{1 \rightarrow 3}(q^k, q) = a_{k,0}(q) \psi_0(q) + a_{k,1}(q) \psi_1(q) + b_k(q),$$

for some $a_{k,0}, a_{k,1}, b_k \in \mathbb{Z}[q, q^{-1}]$.

(2)

$$A_0^{1 \rightarrow 3}(q^k, q) = c_{k,0}(q) \phi_0^\pm(q) + c_{k,1}(q) \phi_1^\pm(q) + d_k(q),$$

for some $c_{k,0}, c_{k,1}, d_k \in \mathbb{Z}[q, q^{-1}]$. The choice of signs (\pm) on the right hand side is made as follows: both signs are (+) if k is even, and both are (-) if k is odd.

(3)

$$A_2^{1 \rightarrow 3}(q^k, q) = e_{k,0}(q) \phi_0^\pm(q) + e_{k,1}(q) \phi_1^\pm(q) + f_k(q),$$

for some $e_{k,0}, e_{k,1}, f_k \in \mathbb{Z}[q, q^{-1}]$. The choice of signs (\pm) on the right hand side is now opposite to that above, with both signs (-) if k is even, and (+) if k is odd.

Finally, we turn to $A_n^{1 \rightarrow 3}(x, q)$ for arbitrary $n \geq 0$. Let us define

$$F_n(x, q) = A_n^{1 \rightarrow 3}(x, q) \prod_{i=1}^{\lfloor \frac{n}{3} \rfloor} (1 - q^i),$$

with $F_{-1}(x, q) = 0$. Let $\mathbb{Z}((q))$ denote the ring of Laurent series with integer coefficients.

We then have the following:

Proposition 1.3.3. *Let $R \subset \mathbb{Z}((q))$ denote the $\mathbb{Z}[q, q^{-1}]$ -span of $\{1, \phi_0^\pm, \phi_1^\pm, \psi_0, \psi_1\}$. Let $n \geq 0, k \in \mathbb{Z}$. Then $F_n(q^k, q) \in R$.*

Chapter 2

Preliminaries

In this chapter, we recall certain well-known definitions and results which will be used in this thesis. We begin with brief history. The theory of semisimple Lie algebras and their representations lies at the heart of modern mathematics. The finite-dimensional simple Lie algebras over the field of complex numbers were classified in the works of Élie Cartan and Wilhelm Killing in the 1930's. There are four infinite series A_r ($r \geq 1$); B_r, C_r ($r \geq 2$); D_r ($r \geq 4$) which are called the classical Lie algebras, and five exceptional Lie algebras E_6, E_7, E_8, F_4, G_2 . The Lie algebras of type A, D , and E are called of type simply laced. The structure of these Lie algebras is uniformly described in terms of certain finite sets of vectors in a Euclidean space called root systems. The theory of finite-dimensional representations of semisimple Lie algebras is largely reduced to the study of their irreducible representations, due to Weyl's complete reducibility theorem. The irreducibles are parametrized by their highest weights. In the late 1960's, Victor Kac and Robert Moody built on this work and independently defined and studied a class of Lie algebras, now called the Kac-Moody Lie algebras. These are generalizations of

the finite-dimensional simple Lie algebras. In the four decades since their discovery, the theory of Kac-Moody Lie algebras and their representations has emerged as a field that has deep and intriguing connections to diverse fields of mathematics and mathematical physics, such as invariant theory, combinatorics, topology, modular forms and theta functions, singularities, finite simple groups, Hamiltonian mechanics, soliton equations, and quantum field theory. The reader is referred to, e.g., the books of Bourbaki [3], Carter [4], Dixmier [12], Humphreys [15], or Kac [18] for a detailed exposition of the theory. Throughout the thesis, \mathbb{C} denotes the field of complex numbers, \mathbb{Z} the set of integers, \mathbb{Z}_+ the set of non-negative integers, \mathbb{N} the set of positive integers, $\mathbb{C}[t]$ the polynomial ring in an indeterminate t , $\mathbb{C}[t, t^{-1}]$ the ring of Laurent polynomials, and $\mathbf{U}(\mathfrak{a})$ the universal enveloping algebra corresponding to a complex Lie algebra \mathfrak{a} .

2.4. THE SIMPLE LIE ALGEBRA \mathfrak{g}

Let \mathfrak{g} be a finite-dimensional simple Lie algebra over \mathbb{C} of rank r , with Cartan subalgebra \mathfrak{h} . Set $I = \{1, 2, \dots, r\}$. Let R (resp. R^+) be the set of roots (resp. positive roots) of \mathfrak{g} with respect to \mathfrak{h} and let $\theta \in R^+$ be the highest root in R . Let $(\cdot | \cdot)$ be a non-degenerate, symmetric, invariant bilinear form on \mathfrak{h}^* normalized so that the square length of a long root is two. For $\alpha \in R$, let $\alpha^\vee \in \mathfrak{h}$ be the corresponding co-root and let \mathfrak{g}_α be the corresponding root space of \mathfrak{g} . It is well-known that $\dim \mathfrak{g}_\alpha = 1, \forall \alpha \in R$. For each $\alpha \in R^+$, we fix non-zero elements $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$ such that $[x_\alpha^+, x_\alpha^-] = \alpha^\vee$. We set $\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}$. The weight lattice P (resp. the set of dominant weights P^+) is the \mathbb{Z} -span (resp. \mathbb{Z}_+ -span) of the fundamental weights $\omega_i, i \in I$ of \mathfrak{g} . The root lattice Q is the \mathbb{Z} -span of the simple roots $\alpha_i, i \in I$ of \mathfrak{g} . The dominant root lattice

$Q^+ = \sum_{i \in I} \mathbb{Z}_+ \alpha_i$. Set $d_i = 2/(\alpha_i | \alpha_i)$, $\forall i \in I$. We define $L = \sum_{i \in I} \mathbb{Z} d_i \omega_i$, a sub lattice of P , and $M = \sum_{i \in I} \mathbb{Z} d_i \alpha_i$, a sub lattice of Q . We note that L and M are the images of the co-weight and co-root lattices respectively under the identification of \mathfrak{h} and \mathfrak{h}^* induced by the form $(. | .)$.

2.5. THE WEYL GROUP OF \mathfrak{g}

For each $i \in I$, the fundamental reflection s_{α_i} (or s_i) is given by

$$s_{\alpha_i}(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \forall \lambda \in \mathfrak{h}^*.$$

The subgroup W of $GL(\mathfrak{h}^*)$ generated by all fundamental reflections s_i , $i \in I$ is called the Weyl group of \mathfrak{g} . Given $w \in W$, let $\ell(w)$ be the length of a reduced expression for w . Let w_0 be the longest element in W .

2.6. THE FINITE-DIMENSIONAL IRREDUCIBLE \mathfrak{g} -MODULES

It is well-known that the finite-dimensional irreducible \mathfrak{g} -modules (up to isomorphism) are indexed by the elements of P^+ . For $\lambda \in P^+$, the corresponding finite-dimensional irreducible \mathfrak{g} -module $V(\lambda)$ is the cyclic \mathfrak{g} -module generated by an element v_λ with the following defining relations:

$$x_\alpha^+ v_\lambda = 0, \quad h v_\lambda = \langle \lambda, h \rangle v_\lambda, \quad (x_\alpha^-)^{\langle \lambda, \alpha^\vee \rangle + 1} v_\lambda = 0, \quad \forall \alpha \in R^+, h \in \mathfrak{h}.$$

2.7. THE AFFINE LIE ALGEBRA $\widehat{\mathfrak{g}}$

Let \mathfrak{g} be a finite-dimensional simple Lie algebra over \mathbb{C} as in §2.4. Let $\widehat{\mathfrak{g}}$ be the corresponding (untwisted) affine Lie algebra defined by

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where c is central and the other Lie brackets are given by

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\delta_{m,-n}(x|y)c,$$

$$[d, x \otimes t^m] = m(x \otimes t^m),$$

for all $x, y \in \mathfrak{g}$ and integers m, n . The Lie subalgebras $\widehat{\mathfrak{h}}$, $\widehat{\mathfrak{n}}^+$, and $\widehat{\mathfrak{b}}$ of $\widehat{\mathfrak{g}}$ are defined as follows:

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \widehat{\mathfrak{n}}^+ = \mathfrak{n}^+ \otimes \mathbb{C}[t] \oplus (\mathfrak{n}^- \oplus \mathbb{C}) \otimes t\mathbb{C}[t], \quad \widehat{\mathfrak{b}} = \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}^+.$$

We regard \mathfrak{h}^* as a subspace of $\widehat{\mathfrak{h}}^*$ by setting $\langle \lambda, c \rangle = \langle \lambda, d \rangle = 0$ for $\lambda \in \mathfrak{h}^*$. For $\xi \in \widehat{\mathfrak{h}}^*$, let $\xi|_{\mathfrak{h}}$ be the element of \mathfrak{h}^* obtained by restricting ξ to \mathfrak{h} . Let $\delta, \Lambda_0 \in \widehat{\mathfrak{h}}^*$ be given by

$$\langle \delta, \mathfrak{h} + \mathbb{C}c \rangle = 0, \quad \langle \delta, d \rangle = 1, \quad \langle \Lambda_0, \mathfrak{h} + \mathbb{C}d \rangle = 0, \quad \langle \Lambda_0, c \rangle = 1.$$

Extend the non-degenerate form on \mathfrak{h}^* to a non-degenerate symmetric bilinear form on $\widehat{\mathfrak{h}}^*$ by setting,

$$(\mathfrak{h}^*|\mathbb{C}\delta + \mathbb{C}\Lambda_0) = (\delta|\delta) = (\Lambda_0|\Lambda_0) = 0 \quad \text{and} \quad (\delta|\Lambda_0) = 1.$$

Set $\widehat{I} = I \cup \{0\}$. The elements $\alpha_i, i \in \widehat{I}$ where $\alpha_0 = \delta - \theta$ are the set of simple roots of $\widehat{\mathfrak{g}}$, and the elements $\alpha_i^\vee, i \in \widehat{I}$ where $\alpha_0^\vee = c - \theta^\vee$ are the corresponding co-roots. The Chevalley generators e_i and f_i ($i \in \widehat{I}$) of $\widehat{\mathfrak{g}}$ are given by following:

$$e_0 = x_\theta^- \otimes t, \quad f_0 = x_\theta^+ \otimes t^{-1}, \quad e_i = x_{\alpha_i}^+ \otimes 1, \quad f_i = x_{\alpha_i}^- \otimes 1, \quad (i \in I).$$

Let \widehat{R}^+ be the set of positive roots,

$$\widehat{R}^+ = \{\alpha + n\delta : \alpha \in R, n \in \mathbb{N}\} \cup R^+ \cup \{n\delta : n \in \mathbb{N}\},$$

and \widehat{R}^- be the set of negative roots,

$$\widehat{R}^- = \{\alpha + n\delta : \alpha \in R, n \in -\mathbb{N}\} \cup R^- \cup \{n\delta : n \in -\mathbb{N}\}.$$

Let $\widehat{R}_{re} = \{\alpha + n\delta : \alpha \in R, n \in \mathbb{Z}\}$ be the set of real roots, and $\widehat{R}_{im} = \{n\delta : n \in \mathbb{Z} \setminus \{0\}\}$ be the set of imaginary roots. The set of roots \widehat{R} of $\widehat{\mathfrak{g}}$ is given by $\widehat{R} = \widehat{R}_{re} \cup \widehat{R}_{im} = \widehat{R}^- \cup \widehat{R}^+$. The root space decomposition of $\widehat{\mathfrak{g}}$ is given by

$$\widehat{\mathfrak{g}} = \bigoplus_{\gamma \in \widehat{R}} \widehat{\mathfrak{g}}_\gamma \oplus \widehat{\mathfrak{h}},$$

where $\widehat{\mathfrak{g}}_\gamma = \{x \in \widehat{\mathfrak{g}} : [h, x] = \langle \gamma, h \rangle x, \forall h \in \widehat{\mathfrak{h}}\}$. It is well-known that $\dim \widehat{\mathfrak{g}}_\gamma = 1, \forall \gamma \in \widehat{R}_{re}$. For each real root $\alpha + n\delta$, we have the Lie subalgebra of $\widehat{\mathfrak{g}}$ generated by $\{x_\alpha^+ \otimes t^n, x_\alpha^- \otimes t^{-n}\}$ which is isomorphic to \mathfrak{sl}_2 . Let $\widehat{Q} = \sum_{i \in \widehat{I}} \mathbb{Z}\alpha_i$ be the root lattice, and $\widehat{Q}^+ = \sum_{i \in \widehat{I}} \mathbb{Z}_+ \alpha_i$. The weight lattice (resp. the set of dominant integral weights) is

defined by

$$\widehat{P} \text{ (resp. } \widehat{P}^+) = \{\lambda \in \widehat{\mathfrak{h}}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ (resp. } \mathbb{Z}_+), \forall i \in \widehat{I}\}.$$

For an element $\lambda \in \widehat{P}$, the integer $\langle \lambda, c \rangle$ is called the *level* of λ .

2.8. THE WEYL GROUP OF $\widehat{\mathfrak{g}}$

For each $i \in \widehat{I}$, the fundamental reflection s_{α_i} (or s_i) is given by

$$s_{\alpha_i}(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \forall \lambda \in \widehat{\mathfrak{h}}^*.$$

The subgroup \widehat{W} of $GL(\widehat{\mathfrak{h}}^*)$ generated by all fundamental reflections $s_i, i \in \widehat{I}$ is called the affine Weyl group or the Weyl group of $\widehat{\mathfrak{g}}$. We regard W naturally as a subgroup of \widehat{W} . Given $\alpha \in \widehat{\mathfrak{h}}^*$, let $t_\alpha \in GL(\widehat{\mathfrak{h}}^*)$ be defined by

$$t_\alpha(\lambda) = \lambda + (\lambda | \delta)\alpha - (\lambda | \alpha)\delta - \frac{1}{2}(\lambda | \delta)(\alpha | \alpha)\delta \quad \text{for } \lambda \in \widehat{\mathfrak{h}}^*.$$

The translation subgroup T_M of \widehat{W} is defined by $T_M = \{t_\alpha \in GL(\widehat{\mathfrak{h}}^*) : \alpha \in M\}$ (where, you may recall the definition of M from §2.4). The following proposition gives the relation between W and \widehat{W} . It is well-known and may be found in [18].

Proposition 2.8.1. [18, Proposition 6.5] $\widehat{W} = W \ltimes T_M$.

The extended affine Weyl group \widetilde{W} is the semi-direct product

$$\widetilde{W} = W \ltimes T_L,$$

where $T_L = \{t_\alpha \in GL(\widehat{\mathfrak{h}}^*) : \alpha \in L\}$. Let $\widehat{C} = \{\Lambda \in \widehat{\mathfrak{h}}^* : \langle \Lambda, \alpha_i^\vee \rangle \geq 0 \forall i \in \widehat{I}\}$ be the fundamental Weyl chamber for $\widehat{\mathfrak{g}}$. Let $\Sigma = \{\sigma \in \widetilde{W} : \sigma(\widehat{C}) = \widehat{C}\}$; it is a subgroup of the group of diagram automorphisms of $\widehat{\mathfrak{g}}$. Then Σ provides a complete system of coset representatives of $\widetilde{W}/\widehat{W}$ and we have $\widetilde{W} = \widehat{W} \rtimes \Sigma$ (see [13], as also [3]). Given $w \in \widehat{W}$, let $\ell(w)$ be the length of a reduced expression for w . The length function ℓ is extended to \widetilde{W} by setting

$$\ell(w\sigma) = \ell(w), \quad (2.8.1)$$

for $w \in \widehat{W}$ and $\sigma \in \Sigma$.

2.9. THE CATEGORY \mathcal{O}

A $\widehat{\mathfrak{g}}$ -module V is called $\widehat{\mathfrak{h}}$ -diagonalizable if it admits a weight space decomposition

$$V = \bigoplus_{\mu \in \widehat{\mathfrak{h}}^*} V_\mu,$$

where $V_\mu = \{v \in V : hv = \langle \mu, h \rangle v, \forall h \in \widehat{\mathfrak{h}}\}$. A non-zero vector of V_μ is called a weight vector of weight μ . Let $P(V) := \{\mu \in \widehat{\mathfrak{h}}^* : V_\mu \neq 0\}$ denote the set of weights of V . For $\Lambda \in \widehat{\mathfrak{h}}^*$, let us denote $D(\Lambda) := \{\mu \in \widehat{\mathfrak{h}}^* : \mu \leq \Lambda\}$. Recall that the partial order \leq on $\widehat{\mathfrak{h}}^*$ is defined by $\mu \leq \Lambda$ iff $\Lambda - \mu \in \widehat{Q}^+$.

Definition 2.9.1. *A $\widehat{\mathfrak{g}}$ -module V is said to be in category \mathcal{O} if*

- (1) *It is $\widehat{\mathfrak{h}}$ -diagonalizable with finite-dimensional weight spaces, and*
- (2) *There exist finitely many elements $\Lambda_1, \dots, \Lambda_m \in \widehat{\mathfrak{h}}^*$ such that $P(V) \subset \cup_{i=1}^m D(\Lambda_i)$.*

The morphisms in \mathcal{O} are homomorphisms of $\widehat{\mathfrak{g}}$ -modules. The category \mathcal{O} is abelian.

2.10. HIGHEST-WEIGHT MODULES

Highest-weight modules are important examples of objects from the category \mathcal{O} .

Definition 2.10.1. *A $\widehat{\mathfrak{g}}$ -module V is said to be a highest-weight module with highest weight $\Lambda \in \widehat{\mathfrak{h}}^*$ if there exists a non-zero vector v_Λ such that*

$$\widehat{\mathfrak{n}}^+ v_\Lambda = 0, \quad h v_\Lambda = \langle \Lambda, h \rangle v_\Lambda, \quad \forall h \in \widehat{\mathfrak{h}}, \quad \text{and} \quad \mathbf{U}(\widehat{\mathfrak{g}}) v_\Lambda = V. \quad (2.10.1)$$

Remark. By condition (2.10.1) it is easy to see that $\mathbf{U}(\widehat{\mathfrak{n}}^-) v_\Lambda = V$, and we have $V = \bigoplus_{\mu \leq \Lambda} V_\mu$, $V_\Lambda = \mathbb{C}v_\Lambda$, $\dim V_\mu < \infty \forall \mu \in \widehat{\mathfrak{h}}^*$. Therefore, a highest-weight module is an object of category \mathcal{O} . Now, we recall an important family of highest-weight modules known as Verma modules.

Definition 2.10.2. *A $\widehat{\mathfrak{g}}$ -module $M(\Lambda)$ with highest weight Λ is called a Verma module if every $\widehat{\mathfrak{g}}$ -module with highest weight Λ is a quotient of $M(\Lambda)$.*

The following proposition justifies the importance of Verma modules.

Proposition 2.10.3. [18, Proposition 9.2]

- (1) *For every $\Lambda \in \widehat{\mathfrak{h}}^*$ there exists a unique (up to isomorphism) Verma module $M(\Lambda)$.*
- (2) *Viewed as a $U(\widehat{\mathfrak{n}}^-)$ -module, $M(\Lambda)$ is a free module of rank 1 generated by the highest weight vector.*
- (3) *$M(\Lambda)$ contains a unique proper maximal submodule $M'(\Lambda)$.*

It follows from part 3 of the above proposition that for $\Lambda \in \widehat{\mathfrak{h}}^*$, there is a unique irreducible module of highest weight Λ which we denote by $L(\Lambda) := M(\Lambda)/M'(\Lambda)$. The

$\widehat{\mathfrak{g}}$ -modules $L(\Lambda)$, for $\Lambda \in \widehat{\mathfrak{h}}^*$, exhaust all irreducible modules of the category \mathcal{O} [18, Proposition 9.3].

2.11. INTEGRABLE MODULES

Definition 2.11.1. *A $\widehat{\mathfrak{g}}$ -module V is said to be integrable if the following holds:*

- *It is $\widehat{\mathfrak{h}}$ -diagonalizable with finite-dimensional weight spaces.*
- *The Chevalley generators e_i and f_i ($i \in \widehat{I}$) are locally nilpotent on V . i.e., given any $v \in V$, there exists $n \geq 0$ such that $e_i^n v = 0 = f_i^n v$.*

We will further restrict our attention to the category $\mathcal{O}^{\text{int}}(\widehat{\mathfrak{g}})$ of integrable modules in category \mathcal{O} . We record the following fact from [18].

Proposition 2.11.2. [18, Lemma 10.1] *The $\widehat{\mathfrak{g}}$ -module $L(\Lambda)$ is integrable if and only if $\Lambda \in \widehat{P}^+$.*

The following Proposition gives the defining relations for the modules $L(\Lambda)$, $\Lambda \in \widehat{P}^+$.

Proposition 2.11.3. [18, Corollary 10.4] *Let $\Lambda \in \widehat{P}^+$. The $\widehat{\mathfrak{g}}$ -module $L(\Lambda)$ is the cyclic module generated by v_Λ , with defining relations*

$$\begin{aligned} h v_\Lambda &= \langle \Lambda, h \rangle v_\Lambda \quad \forall h \in \widehat{\mathfrak{h}}, \\ e_i v_\Lambda &= 0 \quad (i \in \widehat{I}), \\ f_i^{\langle \Lambda, \alpha_i^\vee \rangle + 1} v_\Lambda &= 0 \quad (i \in \widehat{I}). \end{aligned}$$

In particular, an integrable highest-weight module of $\widehat{\mathfrak{g}}$ is automatically irreducible.

The $\widehat{\mathfrak{g}}$ -modules $L(\Lambda)$, for $\Lambda \in \widehat{P}^+$, exhaust all irreducible integrable modules of the category \mathcal{O} [18, Corollary 10.7]. Denote by $P(\Lambda)$ the set of weights of $L(\Lambda)$.

The following proposition may be found in [18]. For $\mathfrak{g} = \mathfrak{sl}_2$, see also [4, Proposition 20.22].

Proposition 2.11.4. [18, Lemma 12.6, Proposition 12.13] *Assume that \mathfrak{g} is simply laced of rank r . Let $\Lambda \in \widehat{P}^+$ be of level 1. Then*

- $P(\Lambda) = \{t_\mu(\Lambda) - n\delta : \mu \in Q, n \in \mathbb{Z}_+\}$,
- $\dim L(\Lambda)_{t_\mu(\Lambda) - n\delta} =$ the number of partitions of n into r colors, $\forall \mu \in Q, n \in \mathbb{Z}_+$.

2.12. DEMAZURE MODULES

Let $L(\Lambda)$ be the irreducible integrable highest-weight module of $\widehat{\mathfrak{g}}$ corresponding to a dominant integral weight Λ . Given an element w of \widehat{W} , define a $\widehat{\mathfrak{b}}$ -submodule $V_w(\Lambda)$ of $L(\Lambda)$ by

$$V_w(\Lambda) = \mathbf{U}(\widehat{\mathfrak{b}})(L(\Lambda)_{w\Lambda}).$$

We call the $\widehat{\mathfrak{b}}$ -module $V_w(\Lambda)$ as a Demazure module. Since $f_i L(\Lambda)_{w\Lambda} = 0$ holds if and only if $\langle w\Lambda, \alpha_i^\vee \rangle \leq 0$, we see that $V_w(\Lambda)$ is \mathfrak{g} -stable if and only if $\langle w\Lambda, \alpha_i^\vee \rangle \leq 0, \forall i \in I$.

The notion of Demazure module associated to an element of \widetilde{W} is defined by setting

$$V_{w\sigma}(\Lambda) = V_w(\sigma\Lambda),$$

for $\sigma \in \Sigma$ and $w \in \widehat{W}$.

2.13. THE CURRENT ALGEBRA $\mathfrak{g}[t]$

The current algebra $\mathfrak{g}[t]$ associated to \mathfrak{g} is defined as $\mathfrak{g} \otimes \mathbb{C}[t]$, with the Lie bracket

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} \quad \forall x, y \in \mathfrak{g}, m, n \in \mathbb{Z}_+.$$

The degree grading on $\mathbb{C}[t]$ gives a natural \mathbb{Z}_+ -grading on $\mathbf{U}(\mathfrak{g}[t])$: the element $(a_1 \otimes t^{r_1}) \cdots (a_k \otimes t^{r_k})$, for $a_i \in \mathfrak{g}, r_i \in \mathbb{Z}_+$, has grade $r_1 + \cdots + r_k$. A graded $\mathfrak{g}[t]$ -module is a \mathbb{Z} -graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V[n]$ such that

$$(\mathfrak{g} \otimes t^m)V[n] \subset V[n+m], \quad \forall m \in \mathbb{Z}_+, n \in \mathbb{Z}.$$

Let $\text{ev}_0 : \mathfrak{g}[t] \rightarrow \mathfrak{g}$ be the morphism of Lie algebras given by setting $t = 0$. The pull back of any \mathfrak{g} -module V by ev_0 defines a graded $\mathfrak{g}[t]$ -module structure on V , and we denote this module by $\text{ev}_0^* V$. We define the morphism of graded $\mathfrak{g}[t]$ -modules as a degree zero morphism of $\mathfrak{g}[t]$ -modules. For $m \in \mathbb{Z}$ and a graded $\mathfrak{g}[t]$ -module V , we let $\tau_m V$ be the m -th graded shift of V , defined by setting $(\tau_m V)[n] = V[n - m]$.

2.14. THE WEYL MODULES OF $\mathfrak{g}[t]$

In [8], Chari and Pressley introduced the notion of local Weyl modules for the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. In [14], a more general case was considered by replacing the Laurent polynomial ring with the co-ordinate ring of an algebraic variety. Later in [5], a functorial approach is used to study local Weyl modules associated with the Lie algebra $\mathfrak{g} \otimes A$, where A is a commutative \mathbb{C} -algebra with unit.

Definition 2.14.1. Given $\lambda \in P^+$, the local Weyl module $W(\lambda)$ is the cyclic $\mathfrak{g}[t]$ -module generated by an element w_λ , with following defining relations:

$$\begin{aligned} (\mathfrak{n}^+ \otimes \mathbb{C}[t]) w_\lambda = 0, \quad (\mathfrak{h} \otimes t\mathbb{C}[t]) w_\lambda = 0, \quad \text{and} \quad h w_\lambda = \langle \lambda, h \rangle w_\lambda, \quad \forall h \in \mathfrak{h}, \\ (x_\alpha^- \otimes 1)^{\langle \lambda, \alpha^\vee \rangle + 1} w_\lambda = 0, \quad \forall \alpha \in R^+. \end{aligned} \quad (2.14.1)$$

We set the grade of w_λ to be zero. Since the defining relations of $W(\lambda)$ are graded, it inherits a \mathbb{Z}_+ -grading from the grading on $\mathbf{U}(\mathfrak{g}[t])$. For $s \in \mathbb{N}$, the subspace of grade s is given by

$$W(\lambda)[s] = \text{span} \{ (a_1 \otimes t^{r_1}) \cdots (a_k \otimes t^{r_k}) w_\lambda : k \geq 1, a_i \in \mathfrak{g}, r_i \in \mathbb{Z}_+, \sum r_i = s \},$$

and the subspace of grade zero is given by

$$W(\lambda)[0] = \mathbf{U}(\mathfrak{g}) w_\lambda.$$

The following proposition is well-known and the proof is analogous to that in [8, §§1-2].

Proposition 2.14.2. [8] For $\lambda \in P^+$, we have the following:

- $W(\lambda)$ has a unique finite-dimensional graded irreducible quotient, which is isomorphic to $\text{ev}_0^* V(\lambda)$. In particular, $W(\lambda) \neq \{0\}$.
- The zeroth graded piece $W(\lambda)[0]$ of $W(\lambda)$ is isomorphic to $V(\lambda)$.

- $W(\lambda)$ is finite-dimensional. Moreover, any finite-dimensional $\mathfrak{g}[t]$ -module V generated by an element $v \in V$ satisfying the relations

$$(\mathfrak{n}^+ \otimes \mathbb{C}[t])v = 0, \quad (\mathfrak{h} \otimes t\mathbb{C}[t])v = 0, \quad \text{and} \quad hv = \langle \lambda, h \rangle v, \quad \forall h \in \mathfrak{h}, \quad (2.14.2)$$

is a quotient of $W(\lambda)$.

Definition 2.14.3. For $\lambda \in P^+$, an element $v \neq 0$ of a $\mathfrak{g}[t]$ -module satisfying the relations (2.14.2) is said to be a highest weight vector of weight λ .

2.15. THE GRADED CHARACTER OF LOCAL WEYL MODULES

For $s \geq 0$, the subspace $W(\lambda)[s]$ of grade s of the local Weyl module $W(\lambda)$ is a \mathfrak{g} -submodule, and we have the following weight space decomposition for $W(\lambda)$:

$$W(\lambda) = \bigoplus_{(\mu,s) \in P \times \mathbb{Z}_+} W(\lambda)_{\mu,s},$$

where $W(\lambda)_{\mu,s} := \{w \in W(\lambda)[s] : hw = \langle \mu, h \rangle w, \forall h \in \mathfrak{h}\}$. For $\mu \in P$, let

$$W(\lambda)_\mu := \bigoplus_{s \geq 0} W(\lambda)_{\mu,s} = \{w \in W(\lambda) : hw = \langle \mu, h \rangle w, \forall h \in \mathfrak{h}\}.$$

The μ for which $W(\lambda)_\mu \neq 0$ are the *weights* of $W(\lambda)$. The graded character $\text{ch}_q W(\lambda)$ of $W(\lambda)$ is defined as,

$$\text{ch}_q W(\lambda) := \sum_{(\mu,s) \in P \times \mathbb{Z}_+} \dim W(\lambda)_{\mu,s} q^s e^\mu \in \mathbb{Z}[P][q]. \quad (2.15.1)$$

2.16. LOCAL WEYL MODULES AS LEVEL ONE DEMAZURE MODULES

The following theorem gives the connection of local Weyl modules with Demazure modules. For $\mathfrak{g} = \mathfrak{sl}_2$, it follows from a result in [8]. For $\mathfrak{g} = \mathfrak{sl}_{r+1}$, it is proved in [7] by using the result in [8]. For \mathfrak{g} simply laced, it is proved in [13] also by using the result in [8].

Theorem 2.16.1. [13, Theorem 7] *Assume that \mathfrak{g} is simply laced. Given $\lambda \in P^+$, let $w \in \widehat{W}$, $\sigma \in \Sigma$ and $\Lambda \in \widehat{P}^+$ such that*

$$w\sigma\Lambda \equiv w_0\lambda + \Lambda_0 \pmod{\mathbb{Z}\delta}.$$

Then we have the following isomorphism of $\mathfrak{g}[t]$ -modules,

$$W(\lambda) \cong V_{w\sigma}(\Lambda).$$

Definition 2.16.2. *Let $\ell \in \mathbb{N}$, $s \in \mathbb{Z}_+$ and write $s = \ell s_1 + s_0$ with $s_1 \geq -1$ and $s_0 \in \mathbb{N}$ with $s_0 \leq \ell$. Then $D(\ell, s)$ is defined as a graded $\mathfrak{sl}_2[t]$ -module generated by an element v_s (which lies in the zeroth grade piece of $D(\ell, s)$) with the following defining relations:*

$$(x \otimes \mathbb{C}[t])v_s = 0, \quad (h \otimes f)v_s = sf(0)v_s, \quad (y \otimes 1)^{s+1}v_s = 0. \quad (2.16.1)$$

$$(y \otimes t^{s_1+1})v_s = 0, \quad (y \otimes t^{s_1})^{s_0+1}v_s = 0, \quad \text{if } s_0 < \ell. \quad (2.16.2)$$

*Let $\tau_r^*D(\ell, s)$ be the graded $\mathfrak{sl}_2[t]$ -module obtained defining the grade of the element v_s to be r .*

The following result is a special case of a result established in [11, Theorem 2, Proposition 6.7] for $s > 0$.

Proposition 2.16.3. *Let Λ be a dominant integral weight for $\widehat{\mathfrak{h}}$ and let $w \in \widehat{W}$ be such that*

$$\Lambda(c) = \ell, \quad w\Lambda(h) = -s, \quad w\Lambda(d) = r.$$

We have an isomorphism of graded $\mathfrak{sl}_2[t]$ -modules

$$V_w(\Lambda) \cong \tau_r^* D(\ell, s).$$

□

Chapter 3

Recursive Formula

The goal in the first part of this chapter is to collect together the relevant definitions and results that we shall need to prove our main Theorem 1.1.1. We begin this chapter by briefly reminding the reader the definition of a Demazure module occurring in a highest weight integrable irreducible representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. We are interested only in stable Demazure modules and we recall several results from [11] about this family.

3.17. THE AFFINE LIE ALGEBRA $\widehat{\mathfrak{sl}}_2$

Recall that \mathfrak{sl}_2 is the complex simple Lie algebra of two by two matrices of trace zero and that $\{x, h, y\}$ is the standard basis of \mathfrak{sl}_2 , with $[h, x] = 2x$, $[h, y] = -2y$ and $[x, y] = h$. The associated affine Lie algebra $\widehat{\mathfrak{sl}}_2$ with canonical central element c and scaling operator d can be realized as follows: as vector spaces we have

$$\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $\mathbb{C}[t, t^{-1}]$ is the Laurent polynomial ring in an indeterminate t , and the commutator is given by

$$[a \otimes t^r, b \otimes t^s] = [a, b] \otimes t^{r+s} + r\delta_{r,-s}(a, b)c, \quad [d, a \otimes f] = a \otimes tdf/dt, \quad [c, \widehat{\mathfrak{sl}}_2] = 0 = [d, d],$$

where $(\ , \) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ is a nondegenerate invariant symmetric bilinear form which is normalized so the root has squared length 2. The action of d can also be regarded as defining a \mathbb{Z} -grading on $\widehat{\mathfrak{sl}}_2$ where we declare the grade of d and c to be zero and the grade of $a \otimes t^r$ to be r for $a \in \mathfrak{sl}_2$. Let $\widehat{\mathfrak{h}} = \mathbb{C}h \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the Cartan subalgebra and define the Borel and the standard maximal parabolic subalgebras by

$$\widehat{\mathfrak{b}} = \mathfrak{sl}_2 \otimes t\mathbb{C}[t] \oplus \mathbb{C}x \oplus \widehat{\mathfrak{h}}, \quad \widehat{\mathfrak{p}} = \widehat{\mathfrak{b}} \oplus \mathbb{C}y = \mathfrak{sl}_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Notice that $\widehat{\mathfrak{b}}$ and $\widehat{\mathfrak{p}}$ are \mathbb{Z}_+ -graded subalgebras of $\widehat{\mathfrak{g}}$. We identify \mathfrak{sl}_2 with the grade zero subalgebra $\mathfrak{sl}_2 \otimes 1$ of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$. Define $\delta \in \widehat{\mathfrak{h}}^*$ by: $\delta(d) = 1, \delta(\mathfrak{h} \oplus \mathbb{C}c) = 0$. Let \widehat{W} be the affine Weyl group associated to $\widehat{\mathfrak{g}}$ and recall that it acts on $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}^*$ and leaves c and δ fixed.

3.18. DEMAZURE MODULES OF $\widehat{\mathfrak{sl}}_2$

Suppose that $\Lambda \in \widehat{\mathfrak{h}}^*$ is dominant integral: i.e., $\Lambda(h), \Lambda(c-h) \in \mathbb{Z}_+$ and we assume that $\Lambda(d) \in \mathbb{Z}$. Let $V(\Lambda)$ be the irreducible integrable highest weight $\widehat{\mathfrak{g}}$ -module generated by a highest weight vector v_Λ . The action of $\widehat{\mathfrak{h}}$ on $V(\Lambda)$ is diagonalizable and the central element c acts via the scalar $\Lambda(c)$ on $V(\Lambda)$. The non-negative integer $\Lambda(c)$ is called the level of $V(\Lambda)$. For all $w \in \widehat{W}$ the element $w\Lambda$ is also an eigenvalue for the action of $\widehat{\mathfrak{h}}$

on $V(\Lambda)$ with corresponding eigenspace $V(\Lambda)_{w\Lambda}$. The Demazure module associated to w and Λ is defined to be

$$V_w(\Lambda) = \mathbf{U}(\widehat{\mathfrak{b}})V(\Lambda)_{w\Lambda}.$$

The Demazure modules are finite-dimensional and if $w\Lambda(h) \leq 0$, then $V_w(\Lambda)$ is a module for $\widehat{\mathfrak{p}}$. *From now on, we shall only be interested in such Demazure modules.* Notice that these Demazure modules are indexed by the integers

$$-s = w\Lambda(h) \leq 0, \quad \ell = \Lambda(c), \quad p = w\Lambda(d),$$

The action of d on the Demazure modules defines a \mathbb{Z} -grading on them compatible with \mathbb{Z}_+ -grading on $\mathfrak{sl}_2[t]$. Moreover, since $w(\Lambda + p\delta) = w\Lambda + p\delta$ and $(\Lambda + p\delta)(\mathfrak{h} \oplus \mathbb{C}c) = \Lambda(\mathfrak{h} \oplus \mathbb{C}c)$, it follows that for a fixed ℓ and s the modules are just grade shifts. If $s = 0$ then $D(\ell, 0)$ is the trivial $\mathfrak{sl}_2[t]$ -module.

3.19. GRADED CHARACTER OF $\mathfrak{sl}_2[t]$ -MODULES

As the discussion in Section 3.18 shows, the proper setting for our study is the category of finite-dimensional \mathbb{Z} -graded $\mathfrak{sl}_2[t]$ -modules. We recall briefly some of the elementary definitions and properties of this category. A finite-dimensional \mathbb{Z} -graded $\mathfrak{sl}_2[t]$ -module is a \mathbb{Z} -graded vector space admitting a compatible graded action of $\mathfrak{sl}_2[t]$:

$$V = \bigoplus_{k \in \mathbb{Z}} V[k], \quad (a \otimes t^r)V[k] \subset V[k+r] \quad a \in \mathfrak{sl}_2, \quad r \in \mathbb{Z}_+.$$

In particular, $V[r]$ is a module for the subalgebra \mathfrak{sl}_2 of $\mathfrak{sl}_2[t]$ and hence the action of \mathfrak{h} on $V[r]$ is semisimple, i.e.,

$$V[r] = \bigoplus_{m \in \mathbb{Z}} V[r]_m, \quad V[r]_m = \{v \in V[r] : hv = mv\}.$$

The *graded character* of V is the Laurent polynomial in two variables e, q given by

$$\text{ch}_{\text{gr}} V = \sum_{m, r \in \mathbb{Z}} \dim V[r]_m e^m q^r.$$

A map of graded $\mathfrak{sl}_2[t]$ -modules is a degree zero map of $\mathfrak{sl}_2[t]$ -modules. If V_1 and V_2 are graded $\mathfrak{sl}_2[t]$ -modules, then the direct sum and tensor product are again graded $\mathfrak{sl}_2[t]$ -modules, with grading,

$$(V_1 \oplus V_2)[k] = V_1[k] \oplus V_2[k], \quad (V_1 \otimes V_2)[k] = \bigoplus_{s \in \mathbb{Z}} (V_1[s] \otimes V_2[k - s]).$$

The graded character is additive on short exact sequences and multiplicative on tensor products. Given a \mathbb{Z} -graded vector space V , we let $\tau_p^* V$ be the graded vector space whose r -th graded piece is $V[r + p]$. Clearly, a graded action of $\mathfrak{sl}_2[t]$ on V also makes $\tau_p^* V$ into a graded $\mathfrak{sl}_2[t]$ -module. It is now easy to prove (see [6] for instance) that an irreducible object of this category must be of the form $\tau_p^* \text{ev}_0^* V(n)$ where $V(n)$ is the unique (up to isomorphism) irreducible module for \mathfrak{sl}_2 of dimension $(n + 1)$. It follows that if V is an arbitrary finite-dimensional graded $\mathfrak{sl}_2[t]$ -module, then $\text{ch}_{\text{gr}} V$ can be written uniquely as a non-negative integer linear combination of $q^p \text{ch}_{\text{gr}} \tau_0^* V(n)$, $p \in \mathbb{Z}$, $n \in \mathbb{Z}_+$.

3.20. THE $\mathfrak{sl}_2[t]$ -STABLE DEMAZURE MODULES $\tau_r^*D(\ell, s)$

We recall for the reader's convenience, the graded $\mathfrak{sl}_2[t]$ module $\tau_r^*D(\ell, s)$ defined in chapter 1. Let $\ell \in \mathbb{N}$, $s \in \mathbb{Z}_+$ and write $s = \ell s_1 + s_0$ with $s_1 \geq -1$ and $s_0 \in \mathbb{N}$ with $s_0 \leq \ell$. Then $D(\ell, s)$ is generated by an element v_s and defining relations:

$$(x \otimes \mathbb{C}[t])v_s = 0, \quad (h \otimes f)v_s = sf(0)v_s, \quad (y \otimes 1)^{s+1}v_s = 0. \quad (3.20.1)$$

$$(y \otimes t^{s_1+1})v_s = 0, \quad (y \otimes t^{s_1})^{s_0+1}v_s = 0, \quad \text{if } s_0 < \ell. \quad (3.20.2)$$

Let $\tau_r^*D(\ell, s)$ be the graded $\mathfrak{sl}_2[t]$ -module obtained by defining the grade of the element v_s to be r . The following result is a special case of a result established in [11, Theorem 2, Proposition 6.7] for $s > 0$.

Theorem 3.20.1. *Let Λ be a dominant integral weight for $\widehat{\mathfrak{h}}$ and let $w \in \widehat{W}$ be such that*

$$\Lambda(c) = \ell, \quad w\Lambda(h) = -s, \quad w\Lambda(d) = r.$$

We have an isomorphism of graded $\mathfrak{sl}_2[t]$ -modules

$$V_w(\Lambda) \cong \tau_r^*D(\ell, s).$$

□

Remark 3.20.2. *A few remarks are in order here. In the case when $s_0 = \ell$ the second relation in equation (3.20.2) is a consequence of the other relations. A presentation of all Demazure modules was given in [17], [22] in the case of simple and Kac–Moody*

algebras respectively. However, it was shown in [11, Theorem 2] that in the case of the \mathfrak{sl}_2 -stable Demazure modules the relations given in [17], [22] are all consequences of the ones stated in the proposition.

We isolate further results from [11, Section 6] that will be needed for our study.

Proposition 3.20.3. *Let $\ell, s \in \mathbb{Z}_+$ and write $s = \ell s_1 + s_0$ with $s_1 \geq -1$ and $s_0 \in \mathbb{N}$ with $s_0 \leq \ell$.*

(i) *For $0 \leq s \leq \ell$ we have*

$$D(\ell, s) \cong \tau_0^* V(s), \quad \text{i.e.} \quad (\mathfrak{sl}_2 \otimes t\mathbb{C}[t]) D(\ell, s) = 0.$$

(ii) *For $s > 0$, we have $\dim D(\ell, s) = (\ell + 1)^{s_1} (s_0 + 1)$.*

(iii) *The $\mathfrak{sl}_2[t]$ -submodule of $D(\ell, s)$ generated by the element $(y \otimes t^{s_1})^{s_0} v_s$ is isomorphic to $\tau_{s_1 s_0}^* D(\ell, s - 2s_0)$. In particular, the quotient $D(\ell, s) / \tau_{s_1 s_0}^* D(\ell, s - 2s_0)$ is generated by an element \bar{v}_s with defining relations, (3.20.1) and,*

$$(y \otimes t^{s_1+1}) \bar{v}_s = 0, \quad (y \otimes t^{s_1})^{s_0} \bar{v}_s = 0. \quad (3.20.3)$$

□

We note that Proposition 3.20.3(iii) is a reformulation of [11, Theorem 5(i) and Proposition 6.4]. The following is a straightforward application of the Poincaré–Birkhoff–Witt theorem.

Lemma 3.20.4. *Let $\ell \in \mathbb{N}$ and $s \in \mathbb{Z}_+$. The module $\tau_0^*V(s)$ is the unique irreducible quotient of $D(\ell, s)$ and occurs with multiplicity one in the Jordan–Holder series of $D(\ell, s)$. Moreover, if $\tau_p^*V(m)$, $m \neq s$ is a Jordan–Holder constituent of $D(\ell, s)$ then $p \in \mathbb{N}$ and $s - m \in 2\mathbb{N}$. \square*

Let $\ell \in \mathbb{N}$. It follows from the Lemma that if V is a graded finite–dimensional module for $\mathfrak{sl}_2[t]$, then $\text{ch}_{\text{gr}} V$ can be written uniquely as a $\mathbb{Z}[q, q^{-1}]$ linear combination of $\text{ch}_{\text{gr}} D(\ell, s)$, $s \in \mathbb{Z}_+$.

3.21. DEMAZURE FLAG OF $\mathfrak{sl}_2[t]$ -STABLE DEMAZURE MODULES

Let V be a finite–dimensional graded $\mathfrak{sl}_2[t]$ –module. We say that a decreasing sequence

$$\mathcal{F}(V) = \{V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_k \supseteq V_{k+1} = 0\}$$

of graded $\mathfrak{sl}_2[t]$ –submodules of V is a Demazure flag of level m , if

$$V_i/V_{i+1} \cong \tau_{p_i}^*D(m, n_i), \quad (n_i, p_i) \in \mathbb{Z}_+ \times \mathbb{Z}, \quad 0 \leq i \leq k.$$

Given a flag $\mathcal{F}(V)$ we say that the multiplicity of $\tau_p^*D(m, n)$ in $\mathcal{F}(V)$ is the cardinality of the set $\{j : V_j/V_{j+1} \cong \tau_p^*D(m, n)\}$. It is not hard to show that the cardinality of this set is independent of the choice of the Demazure flag (see for instance [9, Lemma 2.1]) of V and we denote this number by $[V : \tau_p^*D(m, n)]$. Define

$$[V : D(m, n)]_q = \sum_{p \in \mathbb{Z}} [V : \tau_p^*D(m, n)] q^p, \quad n \geq 0, \quad [V : D(m, n)]_q = 0, \quad n < 0.$$

It follows from the discussion in Section 3.19 and Section 3.20 that if V admits a Demazure flag of level m , then

$$\mathrm{ch}_{\mathrm{gr}} V = \sum_{s \in \mathbb{Z}} [V : D(m, s)]_q \mathrm{ch}_{\mathrm{gr}} D(m, s). \quad (3.21.1)$$

The following result was first proved in [23] for Demazure modules for arbitrary simply-laced simple algebras using the theory of canonical basis. An alternate more constructive and self contained proof was given in [9] for $\mathfrak{sl}_2[t]$.

Proposition 3.21.1. *Let ℓ be a positive integer. For all non-negative integers s and m with $m \geq \ell$, the module $D(\ell, s)$ has a Demazure flag of level m . \square*

This proposition along with Lemma 3.20 proves that the initial condition given in (1.1.2) are satisfied.

3.22. PROOF OF THEOREM 1.1.1

To prove the Theorem 1.1.1, we study the tensor product $D(\ell, s) \otimes D(\ell, 1)$ and write the graded character of the tensor product explicitly as a linear combination of the graded character of level ℓ -Demazure modules. If $m > \ell$, this results allows us to write the graded character of $D(\ell, s) \otimes D(\ell, 1)$ as a linear combination of the graded character of level m Demazure modules in two different ways. A comparison of coefficients then gives Theorem 1.1.1. The proof of the Proposition 1.1.2 can be found in Section 3.23.

Remark 3.22.1. *Let s be as in the proposition. If we let $r(s, \ell)$ be the unique integer with $0 \leq r(s, \ell) < \ell$ such that $s = \ell \lfloor \frac{s}{\ell} \rfloor + r(s, \ell)$, we have*

$$\delta_{s_0, \ell} = \delta_{r(s, \ell), 0} \quad , \quad r(s, \ell) = s_0 - \ell \delta_{s_0, \ell} \quad , \quad \left\lfloor \frac{s}{\ell} \right\rfloor = s_1 + \delta_{s_0, \ell}.$$

In particular, this means $r(s, \ell) \delta_{s_0, \ell} = 0$ and hence $r(s, \ell) \lfloor \frac{s}{\ell} \rfloor = r(s, \ell) s_1$. Using these relations, Proposition 1.1.2 can be reformulated in terms of $\lfloor \frac{s}{\ell} \rfloor$ and $r(s, \ell)$ in place of s_1, s_0 .

We now prove Theorem 1.1.1. We first explain the strategy of the proof. Using equation (3.21.1) and Proposition 3.21.1, we can write,

$$\text{ch}_{\text{gr}} D(\ell, s) = \sum_{p \geq 0} [D(\ell, s) : D(m, p)]_q \text{ch}_{\text{gr}} D(m, p),$$

where $m \in \mathbb{Z}_+$ with $m \geq \ell$. Multiplying both sides of the equation by $\text{ch}_{\text{gr}} D(\ell, 1)$ gives,

$$\text{ch}_{\text{gr}} D(\ell, s) \text{ch}_{\text{gr}} D(\ell, 1) = \sum_{p \geq 0} [D(\ell, s) : D(m, p)]_q \text{ch}_{\text{gr}} D(m, p) \text{ch}_{\text{gr}} D(m, 1). \quad (3.22.1)$$

Here, we have used the fact that $D(\ell, 1) \cong D(m, 1)$ (see Proposition 3.20.3(i)) as $\mathfrak{sl}_2[t]$ -modules. Now, recall that the product of graded characters is the graded character of the tensor product. We can therefore apply Proposition 1.1.2 to both sides of the preceding equation. Applying it to the right hand side gives us a linear combination of the graded characters of level m -Demazure modules. Applying it to the left hand side, gives a linear combination of graded characters of level ℓ -Demazure modules. These can be further expressed as a combination of the graded characters of level m -Demazure

modules. Equating the coefficients of a level m -Demazure module on both sides will prove Theorem 1.1.1. In this subsection, it will be more convenient to work with the notation suggested by Remark 3.22.1. Let us collect the coefficients of $\text{ch}_{\text{gr}} D(m, n)$ which occur on the right hand side of equation (3.22.1) after applying Proposition 1.1.2. It can occur with non-zero coefficients only in the products: $\text{ch}_{\text{gr}} D(m, n \pm 1) \text{ch}_{\text{gr}} D(m, 1)$ and in $\text{ch}_{\text{gr}} D(m, p) \text{ch}_{\text{gr}} D(m, 1)$, where

$$p - 2r(p, m) - 1 = n.$$

We claim that this implies

$$p = 2m + n - 2r(n, m) - 1. \tag{3.22.2}$$

To prove this, we consider $x = p + n + 1$. Since $x = 2(p - r(p, m))$, it is clearly a multiple of $2m$. Further, since $p = n + 1 + 2r(p, m)$, we have

$$n + 1 \leq p \leq n + 1 + 2(m - 1).$$

This implies

$$2n + 2 \leq x \leq 2n + 2m.$$

Thus, we deduce that x is the unique multiple of $2m$ that lies within these bounds; it is given by

$$x = 2m \left(\left\lfloor \frac{2n + 2m}{2m} \right\rfloor \right) = 2m \left(\left\lfloor \frac{n}{m} \right\rfloor + 1 \right),$$

or equivalently by

$$x = 2m + 2n - r(2m + 2n, 2m) = 2m + 2n - 2r(n, m).$$

Thus, $p = x - n - 1$ is given by the required expression. Summarizing (and using Remark 3.22.1 again), we find that the coefficient of $\text{ch}_{\text{gr}} D(m, n)$ on the right hand side is:

$$\begin{aligned} & [D(\ell, s) : D(m, n - 1)]_q + (1 - \delta_{r(n+1, m), 0}) [D(\ell, s) : D(m, n + 1)]_q \\ & + q^{r(p, m)} \lfloor \frac{p}{m} \rfloor (1 - q^{\lfloor \frac{p}{m} \rfloor}) [D(\ell, s) : D(m, p)]_q, \end{aligned} \quad (3.22.3)$$

where p is as in (3.22.2). We note from (3.22.2) that

$$r(p, m) = m - r(n, m) - 1 \text{ and } \lfloor \frac{p}{m} \rfloor = \frac{p - r(p, m)}{m} = 1 + \lfloor \frac{n}{m} \rfloor. \quad (3.22.4)$$

Now, we apply Proposition 1.1.2 to the left hand side of equation (3.22.1). This gives us a linear combination of graded characters of level ℓ -Demazure modules which we can then rewrite using (3.21.1). We find then that the resulting coefficient of $\text{ch}_{\text{gr}} D(m, n)$ is:

$$\begin{aligned} & [D(\ell, s + 1) : D(m, n)]_q + (1 - \delta_{r(s, \ell), 0}) [D(\ell, s - 1) : D(m, n)]_q \\ & + q^{r(s, \ell)} \lfloor \frac{s}{\ell} \rfloor (1 - q^{\lfloor \frac{s}{\ell} \rfloor}) [D(\ell, s - 2r(s, \ell) - 1) : D(m, n)]_q. \end{aligned} \quad (3.22.5)$$

Setting (3.22.3) and (3.22.5) equal to each other and using (3.22.4), we obtain Theorem 1.1.1. □

3.23. PROOF OF PROPOSITION 1.1.2

The rest of the chapter is devoted to the proof of Proposition 1.1.2. If $s = 0$, then $D(\ell, 0)$ is the trivial module and the proposition is trivially true. So, from now on we assume that $s > 0$. For the proof we consider three mutually exclusive cases and it is helpful to write down the equality of characters according to these cases:

(i) If $0 < s = s_0 < \ell$, then

$$\mathrm{ch}_{\mathrm{gr}} D(\ell, s) \otimes D(\ell, 1) = \mathrm{ch}_{\mathrm{gr}} D(\ell, s + 1) + \mathrm{ch}_{\mathrm{gr}} D(\ell, s - 1). \quad (3.23.1)$$

(ii) If $s_0 = \ell$ (in particular if $\ell = 1$), then

$$\mathrm{ch}_{\mathrm{gr}}(D(\ell, s) \otimes D(\ell, 1)) = \mathrm{ch}_{\mathrm{gr}} D(\ell, s + 1) + (1 - q^{s_1+1}) \mathrm{ch}_{\mathrm{gr}} D(\ell, s - 1). \quad (3.23.2)$$

(iii) If $s > \ell > s_0$, then

$$\mathrm{ch}_{\mathrm{gr}}(D(\ell, s) \otimes D(\ell, 1)) = \mathrm{ch}_{\mathrm{gr}} D(\ell, s + 1) + \mathrm{ch}_{\mathrm{gr}} D(\ell, s - 1) + q^{s_1 s_0} (1 - q^{s_1}) \mathrm{ch}_{\mathrm{gr}} D(\ell, s - 2s_0 - 1). \quad (3.23.3)$$

By Proposition 3.20.3(i) we know that $D(\ell, 1) \cong \tau_0^* V(1)$ for all $\ell \in \mathbb{Z}_+$. In particular, the elements v_1, yv_1 are a basis of $D(\ell, 1)$ where we have identified the element $y \in \mathfrak{sl}_2$ with $y \otimes 1$ in $\mathfrak{sl}_2[t]$. From now on for ease of notation, we set

$$U_0 = D(\ell, s) \otimes D(\ell, 1).$$

Lemma 3.23.1. *We have $U_0 \cong \mathbf{U}(\mathfrak{sl}_2[t])(v_s \otimes yv_1)$.*

Proof. Since $y^2v_1 = 0$ we have

$$(y \otimes t^k)(v_s \otimes yv_1) = (y \otimes t^k)v_s \otimes yv_1, \quad k \geq 0.$$

Repeating this argument we get that the $\mathfrak{sl}_2[t]$ -submodule generated by $v_s \otimes yv_1$ contains the subspace $D(\ell, s) \otimes yv_1$. Since $x(D(\ell, s) \otimes yv_1) = D(\ell, s) \otimes v_1 + (xD(\ell, s)) \otimes yv_1$, the Lemma is established. \square

Set $U_2 = \mathbf{U}(\mathfrak{sl}_2[t])(v_s \otimes v_1)$. It is trivial to check that for all $f \in \mathbb{C}[t]$, we have

$$(x \otimes f)(v_s \otimes v_1) = 0, \quad (h \otimes f)(v_s \otimes v_1) = f(0)(s+1)(v_s \otimes v_1), \quad (y \otimes 1)^{s+2}(v_s \otimes v_1) = 0, \quad (3.23.4)$$

and also that

$$(x \otimes f)(v_s \otimes yv_1) \in U_2, \quad (h \otimes f)(v_s \otimes yv_1) = f(0)(s-1)(v_s \otimes yv_1), \quad (y \otimes 1)^s(v_s \otimes yv_1) \in U_2. \quad (3.23.5)$$

We now prove that equation (3.23.1) is satisfied. Since $s = s_0 < \ell$, we see by using Proposition 3.20.3(i) that

$$(\mathfrak{sl}_2 \otimes t\mathbb{C}[t])(v_s \otimes v_1) = 0, \quad U_2 \cong \tau_0^*V(s+1) \cong D(\ell, s+1).$$

Since the graded character is additive on short exact sequences, it suffices now to prove that $U_0/U_2 \cong D(\ell, s-1)$. Equation (3.23.5) and the fact that $(\mathfrak{sl}_2 \otimes t\mathbb{C}[t])(v_s \otimes yv_1) = 0$ shows that that the image of $v_s \otimes yv_1$ in U_0/U_2 satisfies the relations of $D(\ell, s-1)$ given in

Section 3.20. Since $D(\ell, s-1) \cong \tau_0^*V(s-1)$ is irreducible we see that $U_0/U_2 \cong D(\ell, s-1)$ and (3.23.1) follows. To prove the remaining two cases, we need the following result established in [11, Lemma 2.3, Equation (2.10)]. For any $m \in \mathbb{Z}_+$ and $a \in \mathbf{U}(\mathfrak{sl}_2[t])$ let $a^{(m)} = a^m/m!$. Given a positive integer r and a non-negative integer p , define elements $\mathbf{y}(r, p) \in \mathbf{U}(\mathfrak{sl}_2[t])$ by

$$\mathbf{y}(r, p) = \sum (y \otimes 1)^{(b_0)} \cdots (y \otimes t^p)^{(b_p)}$$

where the sum is over all p -tuples (b_0, \dots, b_p) such that $r = \sum_j b_j$, $p = \sum_j j b_j$.

Proposition 3.23.2. *Let ℓ be a positive integer and $s = \ell s_1 + s_0$ with $s_1, s_0 \in \mathbb{Z}_+$ and $0 < s_0 \leq \ell$. Then $D(\ell, s)$ is the $\mathfrak{sl}_2[t]$ -module generated by an element v_s with the relations given in (3.20.1) and the relation*

$$\mathbf{y}(r, p)v_s = 0$$

for all $r, p \in \mathbb{Z}_+$ satisfying, $p \geq r s_1 + 1$ or $r + p \geq 1 + r k + \ell(s_1 - k) + s_0$ for some $0 \leq k \leq s_1$. □

We now consider the case when $s_0 = \ell$, i.e., $s = \ell(s_1 + 1)$. We shall prove that there exists surjective maps of graded $\mathfrak{sl}_2[t]$ -modules

$$\varphi_1 : D(\ell, s+1)/\tau_{s_1+1}^*D(\ell, s-1) \rightarrow U_2 \rightarrow 0, \quad \varphi_2 : D(\ell, s-1) \rightarrow U_0/U_2 \rightarrow 0.$$

Once this is done, the proof of (3.23.2) is completed as follows. By Proposition 3.20.3(ii), we have

$$\dim D(\ell, s+1) = 2(\ell+1)^{s_1+1} = \dim U_0 = \dim U_0/U_2 + \dim U_2,$$

and hence φ_1 and φ_2 must be isomorphisms. Using the additivity of ch_{gr} gives (3.23.2).

To prove the existence of φ_1 , use Theorem 3.20.1 and Proposition 3.20.3(iii) with s replaced by $s+1 = \ell(s_1+1) + 1$. In view of (3.23.4) it suffices to prove that $(y \otimes t^{s_1+1})(v_s \otimes v_1) = 0$. But this is obvious since $(y \otimes t^{s_1+1})v_s = 0 = (y \otimes t^{s_1+1})v_1$. To prove the existence of φ_2 , note that $s-1 = \ell s_1 + \ell - 1$. In view of (3.23.5) we see that we only have to prove that

$$(y \otimes t^{s_1+1})(v_s \otimes yv_1) \in U_2, \quad \ell > 1, \quad (y \otimes t^{s_1})^\ell(v_s \otimes yv_1) \in U_2, \quad \ell \geq 1.$$

The idea in both cases is the same: namely for all $p \geq 0$ and $r \geq 1$, we have $y^2v_1 = 0$ and hence we can write

$$(y \otimes t^p)^r(v_s \otimes yv_1) = (y \otimes t^p)^r y(v_s \otimes v_1) - C((y \otimes t^p)^r yv_s) \otimes v_1,$$

for some $C \in \mathbb{C}$. Since the first term on the right hand side is in U_2 the left hand side will be in U_2 iff the second term on the right hand side is also in U_2 . In other words, we must prove that

$$(((y \otimes t^{s_1+1})yv_s) \otimes v_1) \in U_2, \quad \ell > 1, \quad (((y \otimes t^{s_1})^\ell yv_s) \otimes v_1) \in U_2, \quad \ell \geq 1. \quad (3.23.6)$$

If $\ell > 1$, then $((y \otimes t^{s_1+1})yv_s) \otimes v_1 = 0$ since $(y \otimes t^{s_1+1})v_s = 0$ and the first assertion of (3.23.6) is established. To prove the second assertion suppose first that $s_1 = 0$, i.e., $s = \ell$. Then equation (3.20.1) gives $(y \otimes 1)^\ell yv_\ell = y^{\ell+1}v_\ell = 0$ and we are done. If $s_1 > 0$, take $r = \ell + 1$, $p = \ell s_1$ and $k = 0$ in Proposition 3.23.2 and observe that

$$\mathbf{y}(\ell + 1, \ell s_1)v_s = 0.$$

Suppose that $b_0, \dots, b_{\ell s_1}$ are such that $\sum_{j=0}^{\ell s_1} b_j = \ell + 1$ and $\sum_{j=1}^{\ell s_1} j b_j = \ell s_1$. If $b_m > 0$ for any $m \geq s_1 + 1$ then $(y \otimes t^m)v_s = 0$ and so

$$(y \otimes 1)^{(b_0)} \dots (y \otimes t^{\ell s_1})^{(b_{\ell s_1})}v_s = 0.$$

Suppose now that $b_j = 0$ for all $j > s_1$ and $b_0 > 1$. Then, we have

$$\sum_{j=1}^{s_1} b_j < \ell, \quad \ell s_1 = \sum_{j=1}^{s_1} j b_j \leq s_1 \sum_{j=1}^{s_1} b_j < \ell s_1,$$

which is absurd. Hence $b_0 \leq 1$. If $b_0 = 1$ and $b_m > 0$ for $0 < m < s_1$, then we again have

$$\ell s_1 = \sum_{j=1}^{s_1} j b_j \leq s_1 \left(\sum_{j \neq m} b_j \right) + m b_m < s_1 \sum_{j=1}^{s_1} b_j = \ell s_1,$$

which is again absurd. Hence we find that

$$0 = \mathbf{y}(\ell + 1, \ell s_1)v_s = (y \otimes 1)(y \otimes t^{s_1})^\ell v_s + Xv_s$$

where $X \in \mathbf{U}(\mathfrak{sl}_2 \otimes t\mathbb{C}[t])$ is an element of grade $\ell s_1 > 0$. This gives,

$$((y \otimes 1)(y \otimes t^{s_1})^\ell v_s) \otimes v_1 = -X v_s \otimes v_1 = -X(v_s \otimes v_1) \in U_2$$

and the proof of (3.23.6) is complete. For the final case of $s > \ell > s_0$, we need an additional submodule,

$$U_1 = U_2 + \mathbf{U}(\mathfrak{sl}_2[t])(y \otimes t^{s_1})^{s_0}(v_s \otimes yv_1) = U_2 + \mathbf{U}(\mathfrak{sl}_2[t])((y \otimes t^{s_1})^{s_0}v_s) \otimes yv_1.$$

We will show the existence of three surjective morphisms of graded $\mathfrak{sl}_2[t]$ -modules:

$$\psi_1 : D(\ell, s+1)/\tau_{s_1(s_0+1)}^* D(\ell, s-2s_0-1) \rightarrow U_2 \rightarrow 0,$$

$$\psi_2 : \tau_{s_1 s_0}^* D(\ell, s-2s_0-1) \rightarrow U_1/U_2 \rightarrow 0, \quad \psi_3 : D(\ell, s-1) \rightarrow U_0/U_1 \rightarrow 0.$$

The proof is then completed as in the preceding case: a dimension count shows that the maps ψ_j , $j = 1, 2, 3$ must be isomorphisms and the equality of graded characters follows. The proof of the existence of the maps is also very similar to the proofs given for φ_j , $j = 1, 2$, and we provide the details only in the case of the module U_1/U_2 which is more complicated. Thus, for ψ_2 to exist we must prove that

$$(x \otimes \mathbb{C}[t])((y \otimes t^{s_1})^{s_0}v_s) \otimes yv_1 \in U_2, \quad ((h \otimes t\mathbb{C}[t])(y \otimes t^{s_1})^{s_0}v_s) \otimes yv_1 = 0, \quad (3.23.7)$$

as well as: if $s_0 < \ell - 1$,

$$(y \otimes t^{s_1})(y \otimes t^{s_1})^{s_0}(v_s \otimes yv_1) \in U_2, \quad (y \otimes t^{s_1-1})^{\ell-s_0}(y \otimes t^{s_1})^{s_0}(v_s \otimes yv_1) \in U_2 \quad (3.23.8)$$

and if $s_0 = \ell - 1$,

$$(y \otimes t^{s_1-1})(y \otimes t^{s_1})^{s_0}(v_s \otimes yv_1) \in U_2. \quad (3.23.9)$$

For (3.23.7), it is enough to note that $xyv_1 = v_1$ and that Proposition 3.20.3(iii) implies that

$$(x \otimes \mathbb{C}[t])(y \otimes t^{s_1})^{s_0}v_s = 0 = (h \otimes t\mathbb{C}[t])v_s.$$

Since $s_1 \geq 1$ we have,

$$(y \otimes t^{s_1})(y \otimes t^{s_1})^{s_0}(v_s \otimes yv_1) = (y \otimes t^{s_1})^{s_0+1}v_s \otimes yv_1 = 0,$$

where the last equality is from (3.20.2). This proves the first assertion in (3.23.8). To prove the second assertion in (3.23.8) and (3.23.9), we argue as in the proof of the existence of map φ_2 that

$$(y \otimes t^{s_1-1})^{\ell-s_0}(y \otimes t^{s_1})^{s_0}(v_s \otimes yv_1) \in U_2 \iff ((y \otimes t^{s_1-1})^{\ell-s_0}(y \otimes t^{s_1})^{s_0}yv_s) \otimes v_1 \in U_2.$$

Taking $r = \ell + 1$, $p = s - \ell$ and $k = 0$ we see by using Proposition 3.23.2 that

$$\mathbf{y}(\ell + 1, s - \ell)v_s = 0.$$

Suppose that $((y \otimes 1)^{(b_0)} \dots (y \otimes t^{s-\ell})^{(b_{s-\ell})})$, is an expression occurring in $\mathbf{y}(\ell + 1, s - \ell)$.

Then its action on v_s is zero if $b_j > 0$ for some $j \geq s_1 + 1$. Moreover, by Proposition 3.20.3(iii), we have

$$(y \otimes t^{s_1})^{s_0+1}v_s = 0, \quad (y \otimes t^{s_1-1})^{\ell-s_0+1}(y \otimes t^{s_1})^{s_0}v_s = 0,$$

it follows that we may assume

$$b_{s_1} \leq s_0, \quad \text{and} \quad b_{s_1} = s_0 \Rightarrow b_{s_1-1} \leq \ell - s_0. \quad (3.23.10)$$

The case $s_1 = 1$ will not arise, because this forces $b_1 = s_0$ and $b_0 = \ell + 1 - s_0$ which violates equation (3.23.10). Suppose that $s_1 > 1$ and $b_0 > 0$. Then $\sum_{j=1}^{s_1} b_j \leq \ell$ and we get the following inequalities,

$$\begin{aligned} s - \ell &= \sum_{j=1}^{s_1} j b_j \leq \left((s_1 - 2) \sum_{j=1}^{s_1} b_j \right) + b_{s_1-1} + 2b_{s_1} \leq \ell(s_1 - 2) + b_{s_1-1} + 2b_{s_1}, \\ s - \ell &= \sum_{j=1}^{s_1} j b_j \leq \left((s_1 - 1) \sum_{j=1}^{s_1} b_j \right) + b_{s_1} \leq \ell(s_1 - 1) + b_{s_1}. \end{aligned}$$

The first inequality implies that $b_{s_1-1} + 2b_{s_1} \geq \ell + s_0$, while the second inequality implies that $b_{s_1} \geq s_0$. It follows from (3.23.10) that we must have $b_{s_1} = s_0$ and $b_{s_1-1} = \ell - s_0$. Hence $b_0 = 1$ and $b_m = 0$ if $m \notin \{0, s_1 - 1, s_1\}$. This proves that,

$$0 = \mathbf{y}(\ell + 1, s - \ell)v_s = ((y \otimes t^{s_1-1})^{\ell-s_0}(y \otimes t^{s_1})^{s_0}y)v_s + Xv_s$$

where $X \in \mathbf{U}(\mathfrak{sl}_2 \otimes t\mathbb{C}[t])$. Since $Xv_s \otimes v_1 = X(v_s \otimes v_1)$ it follows that

$$((y \otimes t^{s_1-1})^{\ell-s_0}(y \otimes t^{s_1})^{s_0}y)v_s \otimes v_1 \in U_2.$$

Chapter 4

Numerical Multiplicities

Given $n \in \mathbb{Z}_+$ and $m \in \mathbb{Z}$, set

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(1 - q^n) \dots (1 - q^{n-m+1})}{(1 - q) \dots (1 - q^m)}, \quad m > 0,$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1, \quad \begin{bmatrix} n \\ m \end{bmatrix}_q = 0, \quad m < 0.$$

4.24. DEMAZURE FLAGS AND GENERATING SERIES

Let $\mathfrak{sl}_2[t] \cong \mathfrak{sl}_2 \otimes \mathbb{C}[t]$ be the Lie algebra of two by two matrices of trace zero with entries in the algebra $\mathbb{C}[t]$ of polynomials with complex coefficients in an indeterminate t . The degree grading of $\mathbb{C}[t]$ defines a natural grading on $\mathfrak{sl}_2[t]$. Let $D(\ell, s)$ be the $\mathfrak{sl}_2[t]$ -module generated by an element v_s with defining relations:

$$(x \otimes \mathbb{C}[t])v_s = 0, \quad (h \otimes f)v_s = sf(0)v_s, \quad (y \otimes 1)^{s+1}v_s = 0, \quad (4.24.1)$$

$$(y \otimes t^{s_1+1})v_s = 0, \quad (y \otimes t^{s_1})^{s_0+1}v_s = 0, \quad \text{if } s_0 < \ell. \quad (4.24.2)$$

Here, x, h, y is the standard basis of \mathfrak{sl}_2 and $s_0 \in \mathbb{N}$ and $s_1 \in \mathbb{Z}$ with $s_1 \geq -1$ and $s_0 \leq \ell$ are such that $s = \ell s_1 + s_0$. These modules are finite-dimensional and ℓ is called the level of the Demazure module. We refer the reader to chapter 2 for the connection with the more traditional definition of the Demazure modules. It was observed in [23] that one could use the results of [17] and [21] to show the following: for all integers $m \geq \ell > 0$ and any non-negative integer s , the module $D(\ell, s)$ admits a Demazure flag of level m , i.e., there exists a decreasing sequence of graded submodules of $D(\ell, s)$ such that the successive quotients of the flag are isomorphic to $\tau_p^* D(m, n)$ where $p \geq 0$, $0 \leq n \leq s$ and $s - n$ is even. The number of times a particular level m -Demazure module appears as a quotient in a level m -flag is independent of the choice of the flag and we define a polynomial in an indeterminate q by,

$$[D(\ell, s) : D(m, n)]_q = \sum_{p \geq 0} [D(\ell, s) : \tau_p^* D(m, n)] q^p,$$

where $[D(\ell, s) : \tau_p^* D(m, n)]$ is the multiplicity of $\tau_p^* D(m, n)$ in a level m -Demazure flag of $D(\ell, s)$. It is known that

$$[D(\ell, s) : D(m, s)]_q = 1, \quad [D(\ell, s) : D(m, n)]_q = 0 \quad s - n \notin 2\mathbb{Z}_+.$$

Moreover, for $m \geq \ell' \geq \ell$ we have

$$[D(\ell, s) : D(m, n)]_q = \sum_{p \in \mathbb{Z}_{\geq 0}} [D(\ell, s) : D(\ell', p)]_q [D(\ell', p) : D(m, n)]_q. \quad (4.24.3)$$

Our primary goal in this thesis is to understand both the polynomials $[D(\ell, s) : D(m, n)]_q$ and the associated generating series: given $\ell, m \in \mathbb{N}$ with $m \geq \ell$, set

$$A_n^{\ell \rightarrow m}(x, q) = \sum_{k \geq 0} [D(\ell, n + 2k) : D(m, n)]_q x^k, \quad n \geq 0.$$

It will be convenient to set $A_{-1}^{1 \rightarrow m}(x, 1) = 1$.

4.25. NUMERICAL MULTIPLICITY AND CHEBYSHEV POLYNOMIALS

Preliminary work using [27] assisted in the formulation of the results in this chapter. Theorem 1.2.1 is proved in Section 4.27. We now discuss how to use the theorem to give a closed form for $A_n^{1 \rightarrow m}(x, 1)$. We first recall some relevant facts about Chebyshev polynomials. For $n \geq 0$, the Chebyshev polynomial $U_n(x)$ of the second kind, of degree n , is defined as follows:

$$U_0(x) = 1, \quad U_1(x) = 2x$$

and

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad \text{for } n \geq 1.$$

It is known that the polynomials

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} x^k$$

satisfy

$$P_n(x^2) = x^n U_n((2x)^{-1}) = \prod_{k=1}^n \left(1 - 2x \cos \frac{k\pi}{n+1}\right),$$

and also

$$P_0 = P_1 = 1 \text{ and } P_{n+1}(x) = P_n(x) - xP_{n-1}(x) \text{ for } n \geq 1. \quad (4.25.1)$$

We now establish the following corollary of Theorem 1.2.1 which gives the closed form of $A_n^{1 \rightarrow m}(x, 1)$.

Corollary 4.25.1. *For $n \in \mathbb{Z}_+$, let r, s be the unique non-negative integers such that $n = ms + r$ with $0 \leq r < m$. Then*

$$A_n^{1 \rightarrow m}(x, 1) = \frac{P_{m-r-1}(x)}{P_m(x)^{s+1}}.$$

Proof. Set $F_k = A_k^{1 \rightarrow m}(x, 1)$ for $k \geq -1$. The corollary follows if we prove that for all $k \geq 0$ and $0 \leq p < m$, we have

$$(a) F_{mk+p} = P_{m-p-1}(x) F_{mk+m-1}, \quad (b) F_{mk+m-1} = \frac{1}{P_m(x)^{k+1}}.$$

We first prove (a). If $p = m - 1$ this is immediate from the fact that $P_0(x) = 1$, and if $p = m - 2$ it follows from the second case in (1.2.1). Assume now that we have proved the equality for all $0 \leq p' < m$ with $p' > p$ and $p < m - 2$. To prove the equality for p note that $m \nmid n + 2$ and hence the first case of (1.2.1) applies. Together with the induction hypothesis and (4.25.1), we get

$$F_{mk+p} = F_{mk+p+1} - xF_{mk+p+2} = (P_{m-p-2}(x) - xP_{m-p-3}(x)) F_{mk+m-1} = P_{m-p-1}(x) F_{mk+m-1},$$

and the claim is established. To prove (b), observe that the first case of (1.2.1) again, gives

$$F_{m(k-1)+m-1} = F_{mk} - xF_{mk+1} = (P_{m-1}(x) - xP_{m-2}(x))F_{mk+m-1} = P_m(x)F_{mk+m-1}, \quad k \geq 0.$$

Since $F_{-1} = 1$ we get $P_m^{k+1}(x)F_{mk+m-1} = 1$ and the proof of the corollary is complete. □

4.26. THE FUNCTIONS $A_n^{\ell \rightarrow m}(x, 1)$

In this remaining part of this chapter we use Theorem 1.1.1 to analyze the functions $A_n^{\ell \rightarrow m}(x, 1)$. Thus, we first prove Theorem 1.2.1. Finally, we discuss the general case of $A_n^{\ell \rightarrow m}(x, 1)$.

4.27. PROOF OF 1.2.1

To prove Theorem 1.2.1 we use Theorem 1.1.1 with $\ell = 1$ and $q = 1$. Since $r(p, 1) = 0$ for all $p \geq 0$, the recursion takes the following simpler form: for $n \geq -1$ and $k \geq 1$,

$$\begin{aligned} [D(1, n+1+2k) : D(m, n+1)]_{q=1} &= [D(1, n+2k) : D(m, n)]_{q=1} \\ &+ (1 - \delta_{r(n+2, m)}) [D(1, n+2k) : D(m, n+2)]_{q=1}. \end{aligned}$$

Since $r(n+1, m) = m-1 \iff m \mid n+2$, we get

$$[D(1, n+1+2k) : D(m, n+1)]_{q=1} = \begin{cases} [D(1, n+2k) : D(m, n)]_{q=1} + [D(1, n+2k) : D(m, n+2)]_{q=1} & \text{if } m \nmid n+2, \\ [D(1, n+2k) : D(m, n)]_{q=1} & \text{if } m \mid n+2. \end{cases}$$

Multiply both sides of the equation by x^k , sum over $k \geq 1$ and add one to both sides of the resulting equality of power series. Recalling from (1.1.1) and (1.1.4) that $[D(1, p) : D(m, p)]_q = 1$ and $[D(1, p) : D(m, -1)]_q = 0$ for all $p \geq 0$ now proves Theorem 1.2.1.

4.28. NUMERICAL MULTIPLICITIES FOR GENERAL CASE

Finally, we consider the general case, i.e., the multiplicities of level m Demazure modules in level ℓ Demazure modules for any $m \geq \ell$. For $n \geq 0$, define

$$\tilde{A}_n^{\ell \rightarrow m}(x, q) = \sum_{s \geq 0} [D(\ell, s) : D(m, n)]_q x^s.$$

Since the coefficient of x^s is zero unless $s-n$ is a non-negative even integer, we have

$$\tilde{A}_n^{\ell \rightarrow m}(x, q) = x^n A_n^{\ell \rightarrow m}(x^2, q).$$

Proposition 4.28.1. *Let $1 \leq \ell \leq m$ and $n \geq 0$. Let $\beta_r(x) \in \mathbb{C}[[x]]$, $0 \leq r < \ell$, be the unique power series such that*

$$\tilde{A}_n^{\ell \rightarrow m}(x, 1) = \sum_{r=0}^{\ell-1} x^r \beta_r(x^\ell).$$

Then we have

$$\tilde{A}_n^{1 \rightarrow m}(x, 1) = \sum_{r=0}^{\ell-1} \tilde{A}_r^{1 \rightarrow \ell}(x, 1) \beta_r(y^\ell),$$

where $y = x/P_\ell(x^2)^{\frac{1}{\ell}}$.

Proof. Let $\tilde{A}_n^{\ell \rightarrow m}(x, 1) = \sum_{k=0}^{\infty} c_k x^k$. For $k \geq 0$, letting $a(k), b(k)$ denote the unique integers such that $k = \ell a(k) + b(k)$ with $0 \leq b(k) < \ell$, we obtain

$$\beta_r(x) = \sum_{\{k: b(k)=r\}} c_k x^{a(k)} \quad (4.28.1)$$

We now have

$$\begin{aligned} \tilde{A}_n^{1 \rightarrow m}(x, 1) &= \sum_{s \geq 0} [D(1, s) : D(m, n)]_{q=1} x^s = \sum_{s \geq 0} \sum_{u \geq 0} [D(1, s) : D(\ell, u)]_{q=1} [D(\ell, u) : D(m, n)]_{q=1} x^s \\ &= \sum_{u \geq 0} c_u \tilde{A}_u^{1 \rightarrow \ell}(x, 1) \end{aligned} \quad (4.28.2)$$

Corollary 4.25.1 implies that $\tilde{A}_u^{1 \rightarrow \ell}(x, 1) = \tilde{A}_{b(u)}^{1 \rightarrow \ell}(x, 1) \left[\frac{x^\ell}{P_\ell(x^2)} \right]^{a(u)}$. Substituting this into equation (4.28.2):

$$\tilde{A}_n^{1 \rightarrow m}(x, 1) = \sum_{r=0}^{\ell-1} \tilde{A}_r^{1 \rightarrow \ell}(x, 1) \left(\sum_{\substack{u \geq 0 \\ b(u)=r}} c_u \left[\frac{x^\ell}{P_\ell(x^2)} \right]^{a(u)} \right).$$

From equation (4.28.1), the inner sum is just $\beta_r(y^\ell)$ with $y = x/P_\ell(x^2)^{\frac{1}{\ell}}$, and the proof is complete. \square

Corollary 4.28.2. *Let $m \geq 2, n \geq 0$. Then*

$$A_n^{2 \rightarrow m}(x, 1) = \left(\frac{1}{1+x} \right)^{\lfloor \frac{n}{2} \rfloor + 1} A_n^{1 \rightarrow m} \left(\frac{x}{1+x}, 1 \right).$$

Proof. This follows by taking $\ell = 2$ in Proposition 4.28.1, and rewriting everything in terms of the A_n . If n is even, then $\tilde{A}_n^{2 \rightarrow m}(x, 1) = \beta_0(x^2)$ which implies that

$$\tilde{A}_n^{1 \rightarrow m}(x, 1) = \tilde{A}_0^{1 \rightarrow 2}(x, 1) \beta_0 \left(\frac{x^2}{1-x^2} \right) = \left(\frac{1}{1-x^2} \right) \beta_0 \left(\frac{x^2}{1+x^2} \right). \quad (4.28.3)$$

Rewriting this equation in terms of A_n , we have

$$x^n A_n^{1 \rightarrow m} \left(\frac{x^2}{1+x^2}, 1 \right) = (1+x^2)^{\frac{n}{2}+1} \beta_0(x^2). \quad (4.28.4)$$

Substituting $x^{1/2}$ for x , we obtain

$$\left(\frac{1}{1+x} \right)^{\frac{n}{2}+1} A_n^{1 \rightarrow m} \left(\frac{x}{1+x}, 1 \right) = x^{\frac{-n}{2}} \beta_0(x) = A_n^{2 \rightarrow m}(x, 1), \quad (4.28.5)$$

which completes the proof when n is even. When n is odd, it can be proved in the similar way. □

Remark 4.28.3. *Fix $\ell \geq 1$. Let R denote the \mathbb{C} -algebra $\mathbb{C}[[x]]$, and S be the subalgebra $\mathbb{C}[[x^\ell]]$. Then, R is a free S -module of rank ℓ . Further, for any units $u_0, u_1, \dots, u_{\ell-1}$ in R , the set $\{u_r x^r : 0 \leq r < \ell\}$ is an S -basis of R . Consider the following two choices of basis:*

$$\mathcal{B}_1 = \{x^r : 0 \leq r < \ell\}; \quad \mathcal{B}_2 = \{\tilde{A}_r^{1 \rightarrow \ell}(x, 1) : 0 \leq r < \ell\}.$$

The latter forms a basis since $\tilde{A}_r^{1 \rightarrow \ell}(x, 1) = x^r A_r^{1 \rightarrow \ell}(x^2, 1)$ and $A_r^{1 \rightarrow \ell}(x^2, 1)$ is a unit in R since its constant term is 1. Now, the map

$$\phi : \mathbb{C}[[y]] \rightarrow \mathbb{C}[[x]] \text{ defined by } y \mapsto \frac{x}{P_\ell(x^2)^{\frac{1}{\ell}}}$$

is an isomorphism of algebras. Since $\phi^{-1}(x) = uy$ for some unit $u \in \mathbb{C}[[y]]$, it clear that the pull-back $\mathcal{B}'_2 = \{\phi^{-1}(b) : b \in \mathcal{B}_2\}$ of \mathcal{B}_2 is of the form $\{u_r y^r : 0 \leq r < \ell\}$ for some units u_r in $\mathbb{C}[[y]]$. Hence \mathcal{B}'_2 is a basis of $R' = \mathbb{C}[[y]]$ over $S' = \mathbb{C}[[y^\ell]]$. Now, suppose we are given $m \geq \ell$ and $n \geq 0$. To obtain the generating series $\tilde{A}_n^{\ell \rightarrow m}(x, 1) \in R$, it is enough to obtain its coordinates $\beta_r(x^\ell) \in S$ with respect to the basis \mathcal{B}_1 . Proposition 4.28.1 gives us a way of determining the β_r (in principle). Consider $F = \tilde{A}_n^{1 \rightarrow m}(x, 1) \in R$; this is known in closed form by Theorem 1.2.1. The coordinates of $F' = \phi^{-1}(F) \in R'$ with respect to the basis \mathcal{B}'_2 are precisely the $\beta_r(y^\ell)$.

Chapter 5

Graded multiplicities

5.29. A CLOSED FORM FOR $A_n^{1 \rightarrow 3}(x, q)$

The following closed form formula was obtained in [26, 2].

$$A_{3s+r}^{1 \rightarrow 3}(x, q) = \sum_{n=0}^{\infty} \sum_{p=0}^n \sum_{\substack{j=0 \\ j \equiv p \\ (\text{mod } 2)}}^p x^n q^{\frac{1}{2} \gamma(n,p,j)} \begin{bmatrix} n + \lfloor \frac{3s+r}{2} \rfloor \\ n-p \end{bmatrix}_q \begin{bmatrix} \frac{p-j}{2} + s \\ s \end{bmatrix}_q \begin{bmatrix} s' \\ j \end{bmatrix}_q \quad (5.29.1)$$

where $\gamma(n, p, j) = (n^2 + (n-p)^2 + j^2) + n(2s+r) + (n-p)(2\lceil \frac{s-r}{2} \rceil + r) + j(-2\lceil \frac{r}{2} \rceil + r)$.

In this chapter, We discuss the relationship between certain specializations of the series

$A_n^{1 \rightarrow 3}(x, q)$ and the following fifth order mock theta functions of Ramanujan [25, 29]:

$$\phi_0(q) = \sum_{n=0}^{\infty} q^{n^2} (-q; q^2)_n, \quad (5.29.2)$$

$$\phi_1(q) = \sum_{n=0}^{\infty} q^{(n+1)^2} (-q; q^2)_n, \quad (5.29.3)$$

$$\psi_0(q) = \sum_{n=0}^{\infty} q^{\frac{(n+1)(n+2)}{2}} (-q; q)_n, \quad (5.29.4)$$

$$\psi_1(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} (-q; q)_n. \quad (5.29.5)$$

where the q -Pochhammer symbols $(a; q)_n$ are defined by

$$(a; q)_n = \prod_{i=1}^n (1 - aq^{i-1}), \quad n > 0, \quad (a; q)_0 = 1$$

Given any power series f in the indeterminate q , we define

$$f^+(q) = \sum_{n \geq 0} c_{2n} q^n = \frac{f(q^{\frac{1}{2}}) + f(-q^{\frac{1}{2}})}{2}, \quad f^-(q) = \sum_{n \geq 0} c_{2n+1} q^n = \frac{f(q^{\frac{1}{2}}) - f(-q^{\frac{1}{2}})}{2q^{\frac{1}{2}}}, \quad (5.29.6)$$

so that $f(q) = f^+(q^2) + q f^-(q^2)$. We shall prove the following theorem.

Theorem 5.29.1.

$$\begin{aligned} A_0^{1 \rightarrow 3}(1, q) &= \phi_0^+(q) & A_0^{1 \rightarrow 3}(q, q) &= \phi_1^-(q) \\ A_1^{1 \rightarrow 3}(1, q) &= \psi_1(q) & A_1^{1 \rightarrow 3}(q, q) &= \psi_0(q)/q \\ A_2^{1 \rightarrow 3}(1, q) &= \phi_0^-(q) & A_2^{1 \rightarrow 3}(q, q) &= \phi_1^+(q)/q^2 \end{aligned}$$

Moreover, for all $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have $(q; q)_{\lfloor \frac{n}{3} \rfloor} A_n^{1 \rightarrow 3}(q^k, q)$ is in the $\mathbb{Z}[q, q^{-1}]$ -span of $\{1, \phi_0^\pm, \phi_1^\pm, \psi_0, \psi_1\}$.

The proof of Theorem 5.29.1 is contained in Corollary 5.32.1, Theorem 5.32.2 and Proposition 5.32.3.

5.30. FORMULAE OF $A_n^{1 \rightarrow 3}(x, q)$ FOR $n = 0, 1, 2$

The following formulae were obtained in [26, 2].

$$A_0^{1 \rightarrow 3}(x, q) = \sum_{n=0}^{\infty} x^n q^{n^2/2} \sum_{\substack{p=0 \\ p \equiv n \\ (\text{mod } 2)}}^n q^{p^2/2} \begin{bmatrix} n \\ p \end{bmatrix}_q. \quad (5.30.1)$$

$$A_2^{1 \rightarrow 3}(x, q) = (xq^{\frac{1}{2}})^{-1} \sum_{n=1}^{\infty} x^n q^{\frac{n^2}{2}} \sum_{\substack{p=0 \\ p \not\equiv n \\ (\text{mod } 2)}}^n q^{\frac{p^2}{2}} \begin{bmatrix} n \\ p \end{bmatrix}_q. \quad (5.30.2)$$

$$A_1^{1 \rightarrow 3}(x, q) = \sum_{n=0}^{\infty} x^n q^{\frac{n(n+1)}{2}} \sum_{p=0}^n q^{\frac{p(p+1)}{2}} \begin{bmatrix} n \\ p \end{bmatrix}_q. \quad (5.30.3)$$

5.31. SPECIALIZATIONS OF $A_n^{1 \rightarrow 3}(x, q)$

For the rest of the chapter we shall be interested in the specializations $A_n^{1 \rightarrow 3}(q^k, q)$ for $k \in \mathbb{Z}$, $n \in \mathbb{Z}_+$. For this, it is convenient to define

$$\Phi(x, q) = \sum_{n=0}^{\infty} x^n q^{n^2} (-q; q^2)_n. \quad (5.31.1)$$

$$\Psi(x, q) = \sum_{n=0}^{\infty} x^n q^{\frac{n(n+1)}{2}} (-q; q)_n. \quad (5.31.2)$$

The following Lemma will be useful.

Lemma 5.31.1.

$$\Psi(x, q) = xq^2 \Psi(xq^2, q) + xq \Psi(xq, q) + 1. \quad (5.31.3)$$

$$\Phi(x, q^{\frac{1}{2}}) = xq \Phi(xq^2, q^{\frac{1}{2}}) + xq^{\frac{1}{2}} \Phi(xq, q^{\frac{1}{2}}) + 1. \quad (5.31.4)$$

Proof. We will only prove (5.31.3), since (5.31.4) is similar. From (5.31.2), it follows that the right hand side of (5.31.3) is the following sum:

$$1 + \sum_{n=0}^{\infty} x^{n+1} (-q; q)_n q^{n(n+1)/2} (q^{2n+2} + q^{n+1}).$$

Reindexing this sum with $n' = n + 1$, it is clear that it equals $\Psi(x, q)$. \square

Proposition 5.31.2.

$$A_0^{1 \rightarrow 3}(x, q) = \frac{1}{2} \left(\Phi(x, q^{\frac{1}{2}}) + \Phi(x, -q^{\frac{1}{2}}) \right) \quad (5.31.5)$$

$$A_2^{1 \rightarrow 3}(x, q) = \frac{1}{2xq^{\frac{1}{2}}} \left(\Phi(x, q^{\frac{1}{2}}) - \Phi(x, -q^{\frac{1}{2}}) \right) \quad (5.31.6)$$

$$A_1^{1 \rightarrow 3}(x, q) = \Psi(x, q) \quad (5.31.7)$$

Proof. For $A_0^{1 \rightarrow 3}(x, q)$, we first use the following q -binomial theorem

$$\sum_{p=0}^n q^{p(p-1)/2} \begin{bmatrix} n \\ p \end{bmatrix}_q x^p = (-x; q)_n, \quad (5.31.8)$$

to obtain

$$\sum_{p=0}^n q^{p^2/2} \begin{bmatrix} n \\ p \end{bmatrix}_q z^{n-p} = z^n \left(-z^{-1}q^{\frac{1}{2}}; q \right)_n. \quad (5.31.9)$$

We obtain a second equation by replacing z by $-z$ in (5.31.9). Adding these two equations together and setting $z = 1$, we have

$$\sum_{\substack{p=0 \\ p \equiv n \\ \pmod{2}}}^n q^{p^2/2} \begin{bmatrix} n \\ p \end{bmatrix}_q = \frac{1}{2} \left(-q^{\frac{1}{2}}; q\right)_n + \frac{(-1)^n}{2} \left(q^{\frac{1}{2}}; q\right)_n.$$

Replacing this summation in (5.30.1), we obtain (5.31.5). The proof for $A_2^{1 \rightarrow 3}(x, q)$ is similar. For $A_1^{1 \rightarrow 3}(x, q)$, apply the q -binomial theorem to the inner sum in (5.30.3) to obtain (5.31.7). \square

5.32. MOCK THETA FUNCTIONS

We are now able to make the connection with mock theta functions and prove the first assertions of Theorem 5.29.1.

Corollary 5.32.1. *For the specializations $x = 1$ and $x = q$, we have*

$$\begin{aligned} A_0^{1 \rightarrow 3}(1, q) &= \phi_0^+(q) & A_0^{1 \rightarrow 3}(q, q) &= \phi_1^-(q) \\ A_1^{1 \rightarrow 3}(1, q) &= \psi_1(q) & A_1^{1 \rightarrow 3}(q, q) &= \psi_0(q)/q \\ A_2^{1 \rightarrow 3}(1, q) &= \phi_0^-(q) & A_2^{1 \rightarrow 3}(q, q) &= \phi_1^+(q)/q^2 \end{aligned}$$

Proof. We note from equations (5.31.1), (5.31.2), (5.29.2)-(5.29.5) that trivial calculations give

$$\Psi(1, q) = \psi_1(q), \quad \Psi(q, q) = \psi_0(q)/q, \quad \Phi(1, q) = \phi_0(q), \quad \Phi(q^2, q) = \phi_1(q)/q. \tag{5.32.1}$$

Since $A_1^{1 \rightarrow 3}(x, q) = \Psi(x, q)$ from (5.31.7), we easily obtain the equalities

$$A_1^{1 \rightarrow 3}(1, q) = \psi_1(q) \quad \text{and} \quad A_1^{1 \rightarrow 3}(q, q) = \psi_0(q)/q.$$

Now, consider (5.31.5) with the equation for $\Phi(1, q^{1/2})$ from (5.32.1) above to obtain

$$A_0^{1 \rightarrow 3}(1, q) = \frac{1}{2}(\phi_0(q^{1/2}) + \phi_0(-q^{1/2})).$$

Thus by (5.29.6), we obtain

$$A_0^{1 \rightarrow 3}(1, q) = \phi_0^+(q).$$

Similar calculations give

$$A_0^{1 \rightarrow 3}(q, q) = \phi_1^-(q).$$

For the last two equalities, we use Equations (5.31.6) and (5.32.1) and proceed as above. □

We now consider the specializations $A_n^{1 \rightarrow 3}(q^k, q)$ for arbitrary $k \in \mathbb{Z}$ and $0 \leq n \leq 2$.

We show that these are in fact linear combinations of the mock theta functions with coefficients in $\mathbb{Z}[q, q^{-1}]$. More precisely, we have

Theorem 5.32.2. *Let $k \in \mathbb{Z}$. Then:*

(1)

$$A_1^{1 \rightarrow 3}(q^k, q) = a_{k,0}(q) \psi_0(q) + a_{k,1}(q) \psi_1(q) + b_k(q),$$

for some $a_{k,0}, a_{k,1}, b_k \in \mathbb{Z}[q, q^{-1}]$.

(2)

$$A_0^{1 \rightarrow 3}(q^k, q) = c_{k,0}(q) \phi_0^\pm(q) + c_{k,1}(q) \phi_1^\pm(q) + d_k(q),$$

for some $c_{k,0}, c_{k,1}, d_k \in \mathbb{Z}[q, q^{-1}]$. The choice of signs (\pm) on the right hand side is made as follows: both signs are $(+)$ if k is even, and both are $(-)$ if k is odd.

(3)

$$A_2^{1 \rightarrow 3}(q^k, q) = e_{k,0}(q) \phi_0^\pm(q) + e_{k,1}(q) \phi_1^\pm(q) + f_k(q),$$

for some $e_{k,0}, e_{k,1}, f_k \in \mathbb{Z}[q, q^{-1}]$. The choice of signs (\pm) on the right hand side is now opposite to that above, with both signs $(-)$ if k is even, and $(+)$ if k is odd.

Proof. All three assertions hold for $k = 0, 1$ by Proposition 5.32.1. We first prove (1).

Let $k \in \mathbb{Z}$; equations (5.31.7) and (5.31.3) imply:

$$1 - A_1^{1 \rightarrow 3}(q^k, q) + q^{k+1} A_1^{1 \rightarrow 3}(q^{k+1}, q) + q^{k+2} A_1^{1 \rightarrow 3}(q^{k+2}, q) = 0. \quad (5.32.2)$$

Consider $A_1^{1 \rightarrow 3}(q^j, q)$ for $j \in \{k, k+1, k+2\}$; equation (5.32.2) shows that if the assertion of the theorem holds for any two of these values of j , then it also holds for the third. Since, as observed earlier, the assertion is true for $k = 0, 1$, it holds for all $k \in \mathbb{Z}$ by induction. To prove (2) and (3), we observe that equations (5.31.4), (5.31.5) and (5.31.6) imply:

$$A_0^{1 \rightarrow 3}(x, q) = xq A_0^{1 \rightarrow 3}(xq^2, q) + x^2 q^2 A_2^{1 \rightarrow 3}(xq, q) + 1. \quad (5.32.3)$$

$$A_2^{1 \rightarrow 3}(x, q) = xq^3 A_2^{1 \rightarrow 3}(xq^2, q) + A_0^{1 \rightarrow 3}(xq, q). \quad (5.32.4)$$

The proof now follows by setting $x = q^k$, and arguing by induction as in (1). \square

Finally, we turn to $A_n^{1 \rightarrow 3}(x, q)$ for arbitrary $n \geq 0$. Let us define

$$F_n(x, q) = A_n^{1 \rightarrow 3}(x, q) \prod_{i=1}^{\lfloor \frac{n}{3} \rfloor} (1 - q^i),$$

with $F_{-1}(x, q) = 0$. Let $\mathbb{Z}((q))$ denote the ring of Laurent series with integer coefficients.

We then have the following:

Proposition 5.32.3. *Let $R \subset \mathbb{Z}((q))$ denote the $\mathbb{Z}[q, q^{-1}]$ -span of $\{1, \phi_0^\pm, \phi_1^\pm, \psi_0, \psi_1\}$.*

Let $n \geq 0, k \in \mathbb{Z}$. Then $F_n(q^k, q) \in R$.

Proof. It is easy to check that the recursion for graded multiplicities obtained in Theorem 1.1.1 translates into the following relation for the generating series (taking $n = 3p - (r+1)$ and $s = n + 2k$), valid for all $p \geq 1, r \in \{0, 1, 2\}$:

$$q^{pr} x^{r+1} F_{3p+r}(x, q) = (1+x)F_{3p-r-1}(x, q) - F_{3p-r-2}(x, q) - xq^{3p-r} F_{3p-r-1}(xq^2, q) + E_{3p+r}(x, q),$$

where

$$E_{3p+r}(x, q) = \begin{cases} 0 & \text{if } r = 0 \\ -xF_{3p-1}(x, q) & \text{if } r = 1 \\ -xF_{3p-2}(x, q) + q^{p-1}F_{3p-4}(x, q) - \delta_{p,1} & \text{if } r = 2. \end{cases}$$

Set $x = q^k$ for $k \in \mathbb{Z}$, and let $n \geq 3$; it is clear from these equations that $F_n(q^k, q)$ lies in the $\mathbb{Z}[q, q^{-1}]$ -span of 1 and the $F_m(q^p, q)$ for $p \in \mathbb{Z}, 0 \leq m < n$. Since by Theorem 5.32.2, we have that $F_m(q^p, q) \in R$ for $p \in \mathbb{Z}, 0 \leq m \leq 2$, our proposition now follows by induction. \square

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