ON POLYNOMIAL ALGEBRAS AND RELATIVISTIC WAVE EQUATIONS

THESIS

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CONTENTS

Preface

This thesis comprises the work done by the author during the period 1967-1972 under the supervision of Professor Alladi Ramakrishnan, Director, Matscience, The Institute of Mathematical Sciences, Madras.

The thesis deals with the field of polynomial algebras and applications to relativistic wave equations. It consists of three parts the first part dealing with the polynomial algebras and some applications to the higher spin theories of relativistic wave equations. Part II deals with general involutional matrices and their representations. In Part III we discuss relativistic equations for a spin 1/2 particle inequivalent to the Dirac equations and the algebra involved.

Five papers which form part of the subject matter of the thesis have been published or are in the course of publication in established journals. Collaboration with some of colleagues particularly with Dr. T. S. Santhanam and Dr. I. V. V. Raghavacharyulu was necessitated by the nature and range of the problems dealt with and due acknowledgement is made in the relevant chapters.

It is with great pleasure that I record my deep gratitude to Professor Alladi Ramakrishnan for his constant encouragement, help and guidance at every stage of my endeavours in the field of research. I am thankful to the Council of Scientific and Industrial Research, India, for the award of a Junior Fellowship during the period October 1968 to October 1971, and to the Institute of Mathematical Sciences, Madras, for financial support during the remaining period.

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INTRODUCTION

In recent years, there has been considerable interest in the algebra of matrices which obey restricted polynomial equations. The simplest example is the Dirac Clifford algebra where the polynomial is just quadratic. Another example is the generalized Clifford algebra, the general mathematical formulation of which has been made by Morinaga and Nono, Yamazaki, and Morris, while its relation to physics through a study of the specific representations has been made systematically under the title L-matrix theory by Alladi Ramakrishnan and his collaborators[1].

This thesis deals with generalizations of the L-matrix theory on the one hand to more general polynomial algebras and on the other to problems relating to higher spins. It is divided into five chapters as indicated in the following.

Chapter I is concerned with what are called 'Polynomial algebras'[2] as an extension of the work of Ramakrishnan and his colleagues on the algebra of matrices

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1) Proceedings of the Conference on Clifford Algebra, its generalizations and applications (1971) edited by Alladi Ramakrishnan.

2) I.V.V. Raghavacharyulu and Nalini B. Menon, J. Math. Phys. 11, 3055 (1970).

satisfying conditions like $L^m = I$ and $L^m = L^k\ (k < l)$. We consider a generalization of this by requiring L to satisfy a polynomial equation. When these matrices show very special properties, we call the algebra satisfied by them as polynomial algebras. We show that some very important algebras which physicists have found useful can be obtained by various restrictions on the polynomial, such as both ordinary and generalized Clifford and Grassman algebras.

In Chapter II, we show that this sort of generalization of L-matrix approach is useful in deriving in a simple way the generating relations for spin and parafield algebras[3]. One interesting feature is that in the course of the derivation, we make use of a set of permutation identities, which are directly verified, and are true for any set of associative operators.

In Chapter III[4], we study general involutional matrices, i.e., matrices which are such that their m^{th} power is a multiple of the unit matrix. A 2 x 2 involutional matrix $A^{(2)}$ will involve three independent parameters. If this is regarded as an element of the general linear group

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3) I.V.V.Raghavacharyulu and Nalini B. Menon, Proceedings of the First Mastech Conference (Bangalore, 1969).

4) T.S.Santhanam, P.S.Chandrasekaran and Nalini B. Menon, J. Math. Phys. **12**, 377 (1971)

in two dimensions, its matrix representation as a transformation on a basis set of homogeneous polynomials of q^{th} degree in two variables will yield a $(q+1) \times (q+1)$ involutional matrix with three parameters. This is just the q^{th} induced matrix of $A^{(2)}$ and since induced matrices are a special class of invariant matrices, the property of involution is carried through for an arbitrary $n \times n$ matrix. We have set up the generating equation for the q^{th} induced matrix of an arbitrary $n \times n$ matrix and discussed the case $n = 3$ in some detail. It is shown that a 3×3 involutional matrix $A^{(3)}$ satisfying $[A^{(3)}]^3 = 1$ can be expanded in the basis of the generalized Clifford algebra C_2^3 with coefficients which are the generalized hyperbolic functions. We have also calculated the eigenvalues of the matrix belonging to $GL(n)$ obtained through induction, and specialized it to the case of involutional matrices.

Chapter IV deals with a generalization of Clifford algebra relating to a hierarchy of linear relativistic wave equations for spin 1/2, which are inequivalent to the Dirac equation[5]. This hierarchy has been introduced by Capri. We reexamine the work of Umezawa and Viscouti on

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5) P.S.Chandrasekaran, Nalini B. Menon and T.S.Santhanam Prog. Theoret. Phys. **47**, 671 (1972).

general linear relativistic wave equations and show that the condition on β_0 (matrix coefficient of the fourth component of momentum in the linear equation) given by them can be relaxed to include wave equations of the type given by Capri. We consider a particular case of the hierarchy and show that there are two possible algebras which the β -matrices can satisfy. The particular β -matrices given by Capri satisfy one of these algebras. The other is a new algebra. Both however describe a spin 1/2 particle.

In Chapter V, we consider the spin 1/2 wave equation involving matrices satisfying the second algebra and derive the solutions of this equation in the absence of any interaction[6)]. Then the equation with a minimal electromagnetic interaction put in is studied and the magnetic moment calculated.

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6) Nalini B. Menon (to be published)

CHAPTER I

POLYNOMIAL ALGEBRAS *

I. Introduction:

In a series of contributions[1),2)], Ramakrishnan and his colleagues have initiated and studied the matrix algebras obtained by imposing restrictive polynomial conditions like

$$L^m = I \quad \text{and} \quad L^m = L^k \tag{1.1.1}$$

This entire theory of 'L-matrices' had its starting point when Ramakrishnan[3)] devised a method of building a hierarchy of matrices which have the property

$$L^2_{2n+1} = (\lambda_1^2 + \lambda_2^2 + \text{---} + \lambda^2_{2n+1})I = \lambda_n^2 I \tag{1.1.2}$$

where the matrix L_{2n+1} contains (2n+1) parameters $\lambda_1, \lambda_2, \text{---} \lambda_{2n+1}$ and I is a unit matrix of the same dimension as L_{2n+1}. The structure of this hierarchy of matrices was studied in great detail. Later, it was realized[4)] that many considerations of L-matrix theory are applicable even to matrices obeying a generalized Clifford

* I.V.V.Raghavacharyulu and Nalini B. Menon, J. Math. Phys. **11**, 3055 (1970).

1) Alladi Ramakrishnan, R.Vasudevan, N.R.Ranganathan and P.S.Chandrasekaran, J. Math. Anal. Appl. 23, 10 (1968).

2) Alladi Ramakrishnan and R.Vasudevan, J. Math. Anal. Appl. 32, 141 (1970).

3) Alladi Ramakrishnan, J. Math. Anal. Appl. 20, 9 (1967).

condition, which is defined as follows. If we call Eq.(1.1.2) as the Clifford condition, then it is possible to generalize this condition on the L-matrices by requiring that the m-th power of the L-matrix is a product of a unit matrix and a number, i.e.,

$$L_{2n+1}^{m} = (\lambda_1^{m} + \lambda_2^{m} + \cdots \lambda_{2n+1}^{m}) I = \lambda_n^{m} I \qquad (1.1.3)$$

This has been done by Morris[4] in a recent contribution.

The work presented in this chapter and the next extends these studies by imposing more general polynomial conditions, leading to what we shall call polynomial algebras.

Let L_m be an m-dimensional linear space over a field F. We generate a class of associative algebras called polynomial algebras $A[\alpha_1, \alpha_2, \cdots \alpha_m]$ with $\{\alpha_i \mid i = 1, 2, \cdots m\}$ as generating elements by requiring that every element

$$L(\lambda) = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_m\alpha_m \qquad (1.1.4)$$

belonging to L_m satisfy a polynomial equation

$$P[\lambda; L] \equiv L^{n} + p_1 L^{n-1} + \cdots + p_n I = 0 \qquad (1.1.5)$$

where n is independent of m. We show that some very important algebras in physics such as Clifford and Grassman

4) A. O. Morris, Quart. J. Math. (Oxford) 18, 7 (1967).

algebras (ordinary and generalized) and spin and parafield algebras are indeed polynomial algebras.

2. The Clifford conditions:

Suppose we take the set $\{\alpha_i\}$ to be the set of basis elements of the Clifford algebra, which is the algebra satisfied by the matrices occurring in the Dirac equation[5] for spin 1/2 particles. In this case, the α_i satisfy the Clifford commutation relation

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \tag{1.2.1}$$

It can be seen that this condition is really a consistency relation. The L-matrix associated with the algebra defined by the α_i is then just given by

$$L = \Lambda_1 \alpha_1 + \Lambda_2 \alpha_2 + - - \Lambda_m \alpha_m \tag{1.2.2}$$

It can be seen that

$$L^2 = (\Lambda_1^2 + \Lambda_2^2 + - - \Lambda_m^2) I \tag{1.2.3}$$

due to the fact that the α_i satisfy (1.2.1). Since

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5) P.A.M.Dirac, Proc. Roy. Soc. A117, 610 (1928).

the minimal matrix equation satisfied by the L-matrix is the same as that satisfied by a basis element of the Clifford algebra if $\Sigma \lambda_i^2 = 1$. When viewed from this angle, the Clifford commutation relation is nothing but a consistency relation satisfied by the basis elements such that the L-matrix satisfies the minimal equation of a basis element. This minimal equation is a quadratic equation in the case of spin 1/2 particles. When extending this to higher spins and internal symmetries of particles, as discussed in detail in Chapter II, the minimal equation becomes a polynomial equation, but the L-matrix as such will not be able to furnish the generalized commutation relations unless some additional conditions are imposed.

3. <u>Polynomial commutation relations</u>:

Let $\alpha_1, \alpha_2, \ldots \alpha_m$ be the basis elements of an algebra defined over either a real or complex field F. Let $\lambda_1, \ldots \lambda_m$ be m numbers from the field. Following Ramakrishnan, we construct the L-matrix as

$$L = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \ldots + \lambda_m\alpha_m$$

Now let

$$L^n + p_1 L^{n-1} + p_2 L^{n-2} + \ldots + p_n I = 0 \tag{1.3.1}$$

be the minimal polynomial equation satisfied by L , where the P_i are symmetric homogeneous polynomials of degree i in $\lambda_1, \ldots, \lambda_m$, and are given by

$$
\begin{aligned}
P_1 &= a_{11} \Sigma \lambda_i \\
P_2 &= a_{21} \Sigma' \lambda_i \lambda_j + a_{22} \Sigma \lambda_i^2 \\
P_3 &= a_{31} \Sigma' \lambda_i \lambda_j \lambda_k + a_{32} \Sigma' \lambda_i^2 \lambda_j + a_{33} \Sigma \lambda_i^3 \qquad (1.3.2) \\
&\ldots\ldots \\
P_n &= a_{n1} \Sigma' \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n} + \cdots + a_{nn} \Sigma \lambda_i^n
\end{aligned}
$$

where the prime on the summation sign indicates that only terms with inequal indices are to be taken in the summation. Obviously, if we take $\lambda_i = 1$ and $\lambda_j = 0$ for $j \neq i$, , Eq.(1.3.1) reduces to

$$\alpha_i^n + a_{11} \alpha_i^{n-1} + \cdots a_{nn} = 0 \qquad (1.3.3)$$

Substituting (1.3.2) in (1.3.1) and collecting the coefficients of $\lambda_{i_1} \cdots \lambda_{i_n}$ we obtain the most general commutation relations satisfied by the α's , which we call 'polynomial commutation relations.' The commutation relations of the algebra $\{\alpha_1, \ldots, \alpha_n\}$ should be compatible with these polynomial commutation relations. In the general case, of course, these polynomial relations are too complicated to be interesting. In the following we shall therefore consider some special cases which lead to certain well known algebras.

When n = 2, Eq. (1.3.1) reduces to

$$L^2 + P_1 L + P_2 I = 0, \tag{1.3.4}$$

which, when written out in full after substituting for L, P_1 and P_2 is as follows:

$$\sum_{i,j} \Lambda_i \Lambda_j \alpha_i \alpha_j + a_{11} \sum_i \Lambda_i \sum_j \Lambda_j \alpha_j + a_{21} \sum_{i,j}{}' \Lambda_i \Lambda_j + a_{22} \sum \Lambda_i^2 =$$

which becomes

$$\frac{1}{2} \sum_{i,j} \Lambda_i \Lambda_j \, (\alpha_i \alpha_j + \alpha_j \alpha_i) + \frac{1}{2} a_{11} \sum_{i,j} \Lambda_i \Lambda_j \, (\alpha_i + \alpha_j)$$

$$+ \frac{1}{2} a_{21} \sum_{i,j} \Lambda_i \Lambda_j \, (1 - \delta_{ij}) + a_{22} \sum_{i,j} \Lambda_i \Lambda_j \, \delta_{ij} = 0$$

when we rewrite it as a symmetric expression in i and j, with every term a complete summation over i,j, so that now we can compare coefficients of $\Lambda_i \Lambda_j$. . Doing so we obtain the polynomial commutation relation in this case as

$$(\alpha_i \alpha_j + \alpha_j \alpha_i) + a_{11} (\alpha_i + \alpha_j) \tag{1.3.5}$$

$$+ a_{21} (1 - \delta_{ij}) + 2 a_{22} \delta_{ij} = 0$$

For $n = 3$, we confine ourselves to the case when $a_{11} = a_{21} = a_{31} = a_{32} = 0$. Then L satisfies

$$L^3 + a_{22} \Sigma \lambda_i^2 L + a_{33} \Sigma \lambda_i^3 = 0$$

which when written in symmetric form becomes

$$\frac{1}{6} \Sigma \lambda_i \lambda_j \lambda_k (\alpha_i \alpha_j \alpha_k + \alpha_i \alpha_k \alpha_j + \alpha_j \alpha_i \alpha_k + \alpha_j \alpha_k \alpha_i + \alpha_k \alpha_i \alpha_j + \alpha_k \alpha_j \alpha_i)$$

$$+ \frac{1}{3} a_{22} \Sigma \lambda_i \lambda_j \lambda_k (\alpha_i \delta_{jk} + \alpha_j \delta_{ik} + \alpha_k \delta_{ij}) + a_{33} \Sigma \lambda_i \lambda_j \lambda_k \delta_{ij} \delta_{jk} \delta_{ki} = 0$$

Comparing coefficients of $\lambda_i \lambda_j \lambda_k$, we get

$$S_{\alpha_i \alpha_j \alpha_k} + 2(\alpha_i \delta_{jk} + \alpha_j \delta_{ki} + \alpha_k \delta_{ij}) a_{22} + 6 \delta_{ij} \delta_{kj} \delta_{ki} a_{33} = 0 \qquad (1.3.6)$$

where the symbol S stands for all the terms obtained by permuting the suffixes. If $a_{22} = -1$ and $a_{33} = 0$ the polynomial equation satisfied by L reduces to

$$L^3 = (\lambda_1^2 + \lambda_2^2 + \;-\;-\; + \lambda_m^2) L$$

which is the matrix equation considered by Bhabha[6)] and Ramakrishnan and Vasudevan[7)]. It is interesting to note

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6) H.J. Bhabha, Rev. Mod. Phys. **17**, 200 (1945).

7) Alladi Ramakrishnan and R. Vasudevan, Symposia in Theoretical Physics and Mathematics, Vol.9 (Plenum Press) New York.

that the Clifford condition satisfied in (1.3.6) is not identical with the Clifford condition satisfied by the Duffin-Kemmer algebra, even after putting $a_{22} = -1$ and $a_{33} = 0$. This case is to be contrasted with the case when L satisfies a polynomial equation of the second degree, where putting $a_{11} = a_{21} = 0$), $a_{22} = -1$ to get $L^2 = \Sigma \lambda_i^2$ automatically reduces (1.3.5) to the Clifford commutation relation (1.2.1). In fact, Eq.(1.3.6) and similar equations correspond to an infinite algebra generated by a finite number of elements if $n \geqslant 3$. To make this algebra finite, we have to impose some more conditions which recover for us the spin algebras on the internal symmetry algebras etc.

4. Polynomial algebras for n = 2:

Let us consider the algebras when the L-matrix satisfies a quadratic equation of the form

$$L^2 + p_2 I = 0 \tag{1.4.1}$$

that is, we have put $a_{11} = 0$. Now the commutation relations of this algebra are obtained from Eq.(1.3.5), which in this case reduces to

$$\alpha_i^2 = - a_{22} I$$
$$\alpha_i \alpha_j + \alpha_j \alpha_i = - a_{21} I, \quad i \neq j . \tag{1.4.2}$$

In contrast to the general case $n > 2$, polynomial algebras for $n = 2$ are finite algebras. All special algebras with different values of a_{22} and a_{21} have both mathematical and physical significance. We shall now consider the algebras that are obtained for different values of a_{21} and a_{22}.

When $a_{21} = a_{22} = 0$, then the Eqs. (1.4.2) reduce to

$$\alpha_i^2 = 0$$

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0, \quad i \neq j \tag{1.4.3}$$

and the algebra having this commutation relation is isomorphic with the Grassman algebra of differential forms[8]. When $a_{21} = 0$ and $a_{22} \neq 0$, then, without loss of generality, a_{22} may be put equal to -1, when the Eqs. (1.4.2) becomes

$$\alpha_i^2 = I$$

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0, \quad i \neq j \tag{1.4.4}$$

In this case, then, the α_i give rise to the Clifford algebra of order n, which is of importance in the study of spinor representations of orthogonal groups.

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8) M. Schonberg, Anales Acad. Cienc. (Brazil) 28, 11 (1957)
H. Flanders, Differential Forms (Academic Press, New York 1963).

It is very interesting to note that the matrix representation of these algebras can be set up very easily, by forming the direct products of the Pauli triplet of matrices $\sigma_x, \sigma_y, \sigma_z$ where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (1.4.5)$$

which satisfy the following commutation relations

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I,$$

$$\sigma_x\sigma_y + \sigma_y\sigma_x = \sigma_y\sigma_z + \sigma_z\sigma_y = \sigma_z\sigma_x + \sigma_x\sigma_z = 0, \qquad (1.4.6)$$

$$\sigma_x\sigma_y - \sigma_y\sigma_x = \epsilon_{xyz}\sigma_z, \quad (x, y, z, \text{ cyclic})$$

We form $\sigma^{\pm} = \sigma_x \pm i\sigma_y$ and note that they satisfy

$$\sigma^{\pm}\sigma_z + \sigma_z\sigma^{\pm} = 0 \qquad (1.4.7)$$

$$(\sigma^{\pm})^2 = 0$$

Consider now the Grassman algebra G . If the basis elements of this algebra are denoted by $g_1, -- g_n$, then these elements satisfy the commutation relation

$$g_i g_j + g_j g_i = 0 \qquad (1.4.8)$$

The algebra generated by these basis elements then consists of $2^n - 1$ elements, which are given by the distinct products of the basis elements. The algebraic structure of the Grassman algebra being well known, we may write down the matrix representation of this algebra. When n = 1, the only element $g_1 = \sigma^+$ or σ^-. For $n=2$ n = 2, the basis elements of G are given by

$$\begin{aligned} g_1 &= \sigma^{\pm} \otimes I \\ g_2 &= \pm\sigma_z \otimes \sigma^{\pm} \end{aligned} \tag{1.4.9}$$

Obviously

$$g_1^2 = (\sigma^{\pm} \otimes I)^2 = (\sigma^{\pm})^2 \otimes I = 0$$

$$g_2^2 = (\pm\sigma_z \otimes \sigma^{\pm})^2 = \sigma_z^2 \otimes (\sigma^{\pm})^2 = 0$$

$$\begin{aligned} g_1 g_2 + g_2 g_1 &= \pm(\sigma^{\pm}\sigma_z \otimes \sigma^{\pm} + \sigma_z\sigma^{\pm} \otimes \sigma^{\pm}) \\ &= \pm(\sigma^{\pm}\sigma_z + \sigma_z\sigma^{\pm}) \otimes \sigma^{\pm} = 0 \end{aligned}$$

using Eqs.(1.4.7). Extending the procedure for the case when there are n basis elements we obtain

$$\begin{aligned} g_1 &= \sigma^{\pm} \otimes \underbrace{I \otimes I \otimes \cdots \otimes I}_{n-1 \text{ times}} \\ g_2 &= \pm\sigma_z \otimes \sigma^{\pm} \otimes I \otimes \cdots \otimes I \\ g_3 &= \pm\sigma_z \otimes \pm\sigma_z \otimes \sigma^{\pm} \otimes I \otimes \cdots \otimes I \\ &\cdots\cdots \\ g_i &= \underbrace{\pm\sigma_z \otimes \pm\sigma_z \otimes \cdots \otimes \pm\sigma_z}_{i-1 \text{ times}} \otimes \sigma^{\pm} \otimes I \otimes \cdots \otimes I \\ &\cdots\cdots \\ g_n &= \pm\sigma_z \otimes \pm\sigma_z \otimes \cdots \otimes \pm\sigma_z \otimes \sigma^{\pm} \end{aligned} \tag{1.4.10}$$

From these the representative matrices for the other elements of the Grassman algebra can be obtained directly. Notice that since we can take either σ^+ or σ^- in each of the n basis elements, there are altogether 2^n distinct representations. It is interesting to note that any of these 2^n distinct representations can be transformed into any other by taking its transform by a suitable permutation matrix obtained by taking direct products of

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with P in suitable places. For example, in the case n = 2, the four possible representations for $\{g_1, g_2\}$ are

(i) $\{\sigma^+ \otimes I, \pm\sigma_z \otimes \sigma^+\}$; (ii) $\{\sigma^+ \otimes I, \pm\sigma_z \otimes \sigma^-\}$;

(iii) $\{\sigma^- \otimes I, \pm\sigma_z \otimes \sigma^+\}$; (iv) $\{\sigma^- \otimes I, \pm\sigma_z \otimes \sigma^-\}$; and

(i) is transformed into (ii) by the matrix $I \otimes P$, or (ii) into (iii) by $P \otimes I$, since $P\sigma^+P = \sigma^-$ and $P\sigma^-P = \sigma^+$, as can be easily verified.

The method of getting representations of the Clifford algebra in terms of direct product of the Pauli matrices has already been discussed extensively by Ramakrishnan and his group.

5. Generalized Grassman and Clifford algebras:

Let the minimal polynomial equation satisfied by the L-matrix be of degree three. Suppose we put $a_{11} = 0$ so as to make L satisfy

$$L^3 + p_2 L + p_3 I = 0 \tag{1.5.1}$$

We shall only consider some simple specific cases of this which lead to well known algebras. When $a_{22} = -1$ and the rest of the a_{ij} are zero, this polynomial algebra corresponds to the algebra of spin 1 particles used by Bhabha to obtain the commutation relations of spin 1 particles. In this case Eq.(1.5.1) becomes

$$L^3 = (\Lambda_1^2 + \Lambda_2^2 + \cdots \Lambda_m^2) L . \tag{1.5.2}$$

This equation has been also considered by Ramakrishnan and Vasudevan when the extended the σ -operation to the spin 1 algebra[7].

If we put all the a_{ij} equal to zero except a_{33}, Eqs.(1.5.1) reduces to

$$L^3 + a_{33} \Sigma \Lambda_i^3 = 0$$

which in a symmetric form is just

$$\Sigma \lambda_i \lambda_j \lambda_k S_{\alpha_i \alpha_j \alpha_k} + a_{33} \Sigma \lambda_i \lambda_j \lambda_k \delta_{ij} \delta_{jk} \delta_{ki} = 0$$

so that we get the α_i to satisfy the condition

$$S_{\alpha_i \alpha_j \alpha_k} = - a_{33} \delta_{ij} \delta_{jk} \delta_{ki} \qquad (1.5.3)$$

If we put $a_{33} = -1$, this just becomes

$$S_{\alpha_i \alpha_j \alpha_k} = \delta_{ij} \delta_{jk} \delta_{ki} , \qquad (1.5.4)$$

or $S_{\alpha_i \alpha_j \alpha_k} = 0$ unless i = j = k . Obviously this will be satisfied if the set of basis elements α_i satisfy the commutation relation of the generalized Clifford algebra, viz.,

$$\alpha_i \alpha_j = \omega \alpha_j \alpha_i , \quad i < j , \qquad (1.5.5)$$

where ω is a primitive cube root of unity. On the other hand if we put $a_{33} = 0$ we obtain the relation

$$S_{\alpha_i \alpha_j \alpha_k} = 0 , \quad i \neq j \neq k , \qquad (1.5.6)$$

which is satisfied when the α_i obey the relations

$$\begin{aligned} \alpha_i^3 &= 0 , \\ \alpha_i \alpha_j &= \omega \alpha_j \alpha_i , \quad i < j \end{aligned} \qquad (1.5.7)$$

These are just the commutation relations of the generalized Grassman algebra of the third order.

Now let an L-matrix satisfy an n^{th} order minimal equation. As usual we can write down the most general polynomial equation in this case also. However, if a set of basis elements of the algebra satisfy the commutation relations

$$\alpha_i \alpha_j = \omega \alpha_j \alpha_i \quad (i < j) \tag{1.5.8}$$

where ω is a primitive n^{th} root of unity, then $S\, \alpha_{i_1} - \alpha_{i_n}$, where S denotes the sum over all permutations over the suffixes, is zero unless $i_1 = \cdots i_n = i$, in which case this term reduces to α_i^n . In this case obviously we must set all the a's other than a_{nn} to be equal to zero, when the minimal equation satisfied by L is of the form

$$L^n = a_{nn} (\Lambda_1^n + \Lambda_2^n + - - \Lambda_m^n) \tag{1.5.9}$$

If $a_{nn} \neq 0$, we can take $a_{nn} = -1$ without loss of generality, when the basis elements of this algebra satisfy the commutation relations

$$\alpha_i^n = 1, \tag{1.5.10}$$

$$\alpha_i \alpha_j = \omega \alpha_j \alpha_i$$

This is the generalized Clifford algebra. Consider now the algebra obtained when $a_{nn} = 0$. Then the generating elements satisfy commutation relations

$$\begin{aligned} \alpha_i^{\,n} &= 0, \\ \alpha_i\alpha_j &= w\alpha_j\alpha_i \end{aligned} \qquad (1.5.11)$$

This algebra is called the generalized Grassman algebra. It has $m^n - 1$ basis elements excluding the identity. All these elements are obtained by taking products of the form

$$\alpha_1^{p_1}\alpha_2^{p_2} - - \alpha_n^{p_n}$$

where $p_i = 0, - \cdot\, n-1.$

CHAPTER II

A NEW DERIVATION OF THE GENERATING RELATIONS OF SPIN AND PARAFIELD ALGEBRAS *

1. Introduction:

In a classic paper[1], Madhava Rao has obtained the generating relations of algebras of higher spins. This method has been extended by Kamefuchi and Takahashi to parafield algebras[2]. Madhava Rao's method consists in imposing the Lie algebraic condition

$$[\alpha_i, \alpha_j\alpha_k - \alpha_k\alpha_j] = \delta_{ij}\alpha_k - \delta_{ik}\alpha_j \qquad (2.1.1)$$

On the sequence of polynomial conditions

$$\begin{aligned}(\alpha_i - 1/2)(\alpha_i + 1/2) &= 0, \\ (\alpha_i - 1)\alpha_i(\alpha_i + 1) &= 0, \qquad (2.1.2) \\ (\alpha_i - 3/2)(\alpha_i - 1/2)(\alpha_i + 1/2)(\alpha_i + 3/2) &= 0,\end{aligned}$$

satisfied by each basis element α_i of the algebra. We now demonstrate that the generating relations can be obtained

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* I.V.V.Raghavacharyulu and Nalini B. Menon, Proceedings of the First Mastech Conference, (Bangalore, 1969), p.211.

1) B.S.Madhava Rao, Proc. Indian Acad. Sci. 15, 139 (1942).

2) S.Kamefuchi and Y.Takahashi, Nucl. Phys. 36, 177 (1962).

in a much simpler manner if we adopt the spirit of the L-matrix approach first introduced by Ramakrishnan[3] which consists of imposing on linear combinations of matrices conditions similar to those satisfied by the individual matrices.

2. Spin Algebras:

The L-matrix associated with the algebra defined by the α_i is defined as

$$L = (\Lambda_1\alpha_1 + \Lambda_2\alpha_2 + \text{-----} + \Lambda_m\alpha_m) \qquad (2.2.1)$$

~~The~~ Thus corresponding to Eqs.(2.1.2) we take the polynomial conditions[4] satisfied by L as

$$(L - \Lambda/2)(L + \Lambda/2) = 0 \qquad (2.2.2a)$$

$$(L - \Lambda)\,L\,(L + \Lambda) = 0 \qquad (2.2.2b)$$

$$(L - 3\Lambda/2)(L - \Lambda/2)(L + \Lambda/2)(L + \tfrac{3\Lambda}{2}) = 0 \qquad (2.2.2c)$$

where $\Lambda^2 = \Lambda_1^2 + \Lambda_2^2 + \text{----} + \Lambda_m^2$

(2.2.3)

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3) Alladi Ramakrishnan, J. Math. Anal. Appl. 20, 9 (1967).

4) I. V. V. Raghavacharyulu and Nalini B. Menon, J. Math Phys. 11, 3055 (1970)

In order to compare coefficients of products of Λ's in Eqs.(2.2.2) we expand these equations in terms of L and Λ and rearrange in symmetric form as follows:

$$\frac{1}{2} \sum \Lambda_i \Lambda_j S_{\alpha_i \alpha_j} = \frac{1}{4} \sum \Lambda_i \Lambda_j \delta_{ij}$$

$$\frac{1}{6} \sum \Lambda_i \Lambda_j \Lambda_k S_{\alpha_i \alpha_j \alpha_k} = \frac{1}{3} \sum \Lambda_i \Lambda_j \Lambda_k (\alpha_k \delta_{ij} + \alpha_j \delta_{ik} + \alpha_i \delta_{jk})$$

$$\begin{aligned} \frac{1}{24} \sum \Lambda_i \Lambda_j \Lambda_k \Lambda_l S_{\alpha_i \alpha_j \alpha_k \alpha_l} = {} & \frac{5}{24} \sum \Lambda_i \Lambda_j \Lambda_k \Lambda_l (\{\alpha_i, \alpha_j\} \delta_{kl} \\ & + \{\alpha_i, \alpha_k\} \delta_{jl} + \{\alpha_i, \alpha_l\} \delta_{jk} + \{\alpha_j, \alpha_k\} \delta_{il} \\ & + \{\alpha_j, \alpha_l\} \delta_{ik} + \{\alpha_k, \alpha_l\} \delta_{ij}) \\ & - \frac{3}{16} \sum \Lambda_i \Lambda_j \Lambda_k \Lambda_l (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \end{aligned}$$

where $\{\alpha_i, \alpha_j\} \equiv \alpha_i \alpha_j + \alpha_j \alpha_i$. Comparing the coefficients of $\Lambda_i \Lambda_j$, $\Lambda_i \Lambda_j \Lambda_k$ and $\Lambda_i \Lambda_j \Lambda_k \Lambda_l$ in the first, second and third of the above equations respectively, we obtain

$$S_{\alpha_i \alpha_j} = \frac{1}{2} \delta_{ij} \tag{2.2.4a}$$

$$S_{\alpha_i \alpha_j \alpha_k} = 2 (\alpha_i \delta_{jk} + \alpha_j \delta_{ik} + \alpha_k \delta_{ij}) \tag{2.2.4b}$$

$$\begin{aligned} S_{\alpha_i \alpha_j \alpha_k \alpha_l} = {} & 5 (\{\alpha_k, \alpha_l\} \delta_{ij} + \{\alpha_j, \alpha_l\} \delta_{ik} \\ & + \{\alpha_j, \alpha_k\} \delta_{il} + \{\alpha_i, \alpha_l\} \delta_{jk} \\ & + \{\alpha_i, \alpha_k\} \delta_{jl} + \{\alpha_i, \alpha_j\} \delta_{kl}) \\ & - \frac{9}{2} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \end{aligned} \tag{2.2.4c}$$

These relations which are the direct consequences of the conditions on the L-matrix simplify the process for obtaining the generating relations. Eqs. (2.2.4) are simplified using Eq. (2.1.1) along with the following <u>permutation identities</u>, which are directly verified, and are true for any set of associative operators:

$$3(\gamma_i\gamma_j\gamma_k + \gamma_k\gamma_j\gamma_i) - S_{\gamma_i}\gamma_j\gamma_k \equiv [\gamma_i, L_{jk}] + [\gamma_k, L_{ji}], \tag{2.2.5}$$

$$\begin{aligned} &6(\gamma_i\gamma_j\gamma_k\gamma_l + \gamma_i\gamma_l\gamma_k\gamma_j + \gamma_j\gamma_k\gamma_l\gamma_i + \gamma_l\gamma_k\gamma_j\gamma_i) - S\gamma_i\gamma_j\gamma_k\gamma_l \\ &\equiv 2\{\gamma_i, [\gamma_j, L_{kl}]\} - 2\{\gamma_i, [\gamma_l, L_{kj}]\} + 2\{\gamma_l, [\gamma_j, L_{ki}]\} \\ &\quad + 2\{\gamma_j, [\gamma_l, L_{ki}]\} - \{\gamma_j, [\gamma_i, L_{kl}]\} - \{\gamma_l, [\gamma_i, L_{kj}]\} \\ &\quad - \{\gamma_k, [\gamma_j, L_{il}]\} - \{\gamma_k, [\gamma_l, L_{ij}]\} \end{aligned} \tag{2.2.6}$$

where $[\gamma_i, L_{jk}] \equiv \gamma_i L_{jk} - L_{jk}\gamma_i$ and

$$L_{ij} \equiv [\gamma_i, \gamma_j] = \gamma_i\gamma_j - \gamma_j\gamma_i \tag{2.2.7}$$

Eq. (2.2.4a) directly gives the generating relation of the spin 1/2 algebra. Eqs. (2.2.4b) and (2.2.4c) lead to the generating relations of spin 1 and spin 3/2 algebras after simplifying using Eqs. (2.2.5) and (2.2.6) respectively. We shall give the steps involved in arriving at the generating relations in the case of spin 1 only. An exactly similar method applies for spin 3/2 also.

(i) Spin 1/2:

The commutation relation is just Eqs.(2.2.4a) which is

$$\alpha_i \alpha_j + \alpha_j \alpha_i = \tfrac{1}{2}\delta_{ij} \qquad (2.2.8)$$

(ii) Spin 1: Substituting α_i in the place of γ_i in the permutation identity (2.2.5) we have

$$3(\alpha_i\alpha_j\alpha_k + \alpha_k\alpha_j\alpha_i) - S_{\alpha_i\alpha_j\alpha_k}$$
$$= [\alpha_i, \alpha_j\alpha_k - \alpha_k\alpha_j] + [\alpha_k, \alpha_j\alpha_i - \alpha_i\alpha_j]$$

which reduces to

$$3(\alpha_i\alpha_j\alpha_k + \alpha_k\alpha_j\alpha_i) - S_{\alpha_i\alpha_j\alpha_k}$$
$$= \delta_{ij}\alpha_k - \delta_{ik}\alpha_j + \delta_{jk}\alpha_i - \delta_{ik}\alpha_j$$
$$= \delta_{ij}\alpha_k + \delta_{jk}\alpha_i - 2\delta_{ik}\alpha_j$$

on using Eqs.(2.1.1). Now adding this equation to Eqs.(2.2.4b) directly gives the generating relation for the spin 1 algebra,

$$(\alpha_i\alpha_j\alpha_k + \alpha_k\alpha_j\alpha_i) = \alpha_i\delta_{jk} + \alpha_k\delta_{ij} \qquad (2.2.9)$$

(iii) Spin 3/2:

Here again the permutation identity (2.2.6) put in terms of α_i is simplified using Eq.(2.1.1) and then added to Eq.(2.2.4c) to get the following generating relation:

$$\alpha_i(\alpha_j\alpha_k\alpha_l + \alpha_l\alpha_k\alpha_j) + (\alpha_j\alpha_k\alpha_l + \alpha_l\alpha_k\alpha_j)\alpha_i$$
$$= \tfrac{1}{2}\{\alpha_k, \alpha_l\}\delta_{ij} + \tfrac{1}{2}\{\alpha_j, \alpha_l\}\delta_{ik} + \tfrac{1}{2}\{\alpha_j, \alpha_k\}\delta_{il}$$
$$+ \tfrac{1}{2}\{\alpha_i, \alpha_k\}\delta_{jl} + \tfrac{3}{2}\{\alpha_i, \alpha_l\}\delta_{jk} + \tfrac{3}{2}\{\alpha_i, \alpha_j\}\delta_{kl} \qquad (2.2.10)$$
$$- \tfrac{3}{4}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

3. Parafermi algebras:

By an exactly similar prodedure as for spin algebras, the complete set of generating relations for parafermi algebras can also be obtained. The generating elements of the parafield algebra are given by

$$a_i = (\alpha_{2i-1} + i\alpha_{2i})/\sqrt{2}$$

$$a_i^+ = (\alpha_{2i-1} - i\alpha_{2i})/\sqrt{2}, \quad i = 1, --, k. \qquad (2.3.1)$$

and the Lie algebraic conditions became generalized and are

$$[a_i, a_j a_k - a_k a_j] = 0 \qquad (2.3.2a)$$

$$[a_i, a_j^+ a_k - a_k a_j^+] = 2\delta_{ij} a_k \qquad (2.3.2b)$$

$$[a_i^+, a_j a_k - a_k a_j] = 2\delta_{ij} a_k - 2\delta_{ik} a_j \qquad (2.3.2c)$$

and their complex conjugates. The L-matrix written in terms of the a_i and a_i^+ now looks like

$$L = (\mu_1 a_1 + \bar{\mu}_1 a_1^+ + \cdots + \mu_k a_k + \bar{\mu}_k a_k^+) \tag{2.3.3}$$

where

$$\mu_i = (\Lambda_{2i-1} - i\Lambda_{2i})/\sqrt{2} \tag{2.3.4}$$

We require L to satisfy the same equations (2.2.2). Obviously, in the case of parafields n should be even and equal to 2k. Λ^2 expressed in terms of μ and $\bar{\mu}$ is given by

$$\Lambda^2 = 2(\mu_1\bar{\mu}_1 + \mu_2\bar{\mu}_2 + \cdots \quad \mu_k\bar{\mu}_k) \tag{2.3.5}$$

(i) <u>Parafield of Order 1</u>: In this case L satisfies the minimal equation $L^2 = \frac{\Lambda^2}{4} I$. Written in terms of a_i and a_i^+ and rearranged in symmetric form this equation becomes

$$\frac{1}{2}\sum \mu_i\mu_j(a_i a_j + a_j a_i) + \sum \bar{\mu}_i \mu_j (a_i^+ a_j + a_j a_i^+)$$
$$+ \frac{1}{2}\sum \bar{\mu}_i \bar{\mu}_j (a_i^+ a_j^+ + a_j^+ a_i^+) = \frac{1}{2}\sum \bar{\mu}_i \mu_j \delta_{ij}$$

Comparing coefficients of $\mu_i\mu_j$, $\bar{\mu}_i\mu_j$ and $\bar{\mu}_i\bar{\mu}_j$ in turn, we directly get the commutation relations

$$\begin{aligned} &a_i a_j + a_j a_i = 0 \quad , \\ &a_i^+ a_j + a_j a_i^+ = \tfrac{1}{2}\delta_{ij} \quad , \\ &a_i^+ a_j^+ + a_j^+ a_i^+ = 0 \; , \end{aligned} \tag{2.3.6}$$

of the first order parafermi algebra.

(ii) <u>Parafield of Order 2:</u>

We rewrite Eqs.(2.2.2b) in terms of and and as

$$\Sigma(\mu_i \mu_j \mu_k \tfrac{1}{6} S a_i a_j a_k + \mu_i \mu_j \bar{\mu}_k S a_i a_j a_k^+$$

$$+ \mu_i \bar{\mu}_j \bar{\mu}_k S a_i a_j^+ a_k^+ + \bar{\mu}_i \bar{\mu}_j \cdot \bar{\mu}_k \tfrac{1}{6} S a_i^+ a_j^+ a_k^+$$

$$= \Sigma \mu_i \mu_j \bar{\mu}_k (a_i \delta_{jk} + a_j \delta_{ik})$$

$$+ \Sigma \mu_i \bar{\mu}_j \bar{\mu}_k (a_j^+ \delta_{ik} + a_k^+ \delta_{ij}) .$$

Compare coefficients as before to get the following equations:

$$S a_i a_j a_k = 0, \tag{2.3.6a}$$

$$S a_i a_j a_k^+ = a_i \delta_{jk} + a_j \delta_{ik}, \tag{2.3.6b}$$

$$S a_i a_j^+ a_k^+ = a_j^+ \delta_{ik} + a_k^+ \delta_{ij}, \tag{2.3.6c}$$

$$S a_i^+ a_j^+ a_k^+ = 0 \tag{2.3.6d}$$

Note that Eqs.(2.3.6c and d) are really not independent, since they are just the conjugates of (2.3.6a and b) respectively. Substituting a_i or a_i^+ as necessary in the permutation identity (2.2.5) we can arrive at the following equations:

$$3(a_i a_j a_k + a_k a_j a_i) - 5 a_i a_j a_k = 0, \tag{2.3.7a}$$

$$\begin{aligned} 3(a_i a_k^+ a_j + a_j a_k^+ a_i) &- 5 a_i a_j a_k^+ \\ &\equiv [a_i, a_k^+ a_j - a_j a_k^+] + [a_j, a_k^+ a_i - a_i a_k^+] \\ &= 2\delta_{ik} a_j + 2\delta_{jk} a_i , \end{aligned} \tag{2.3.7b}$$

$$\begin{aligned} 3[a_i a_j a_k^+ + a_k^+ a_j a_i] &- 5 a_i a_j a_k^+ \\ &\equiv [a_i, a_j a_k^+ - a_k^+ a_j] + [a_k^+, a_j a_i - a_i a_j] \\ &= 2\delta_{jk} a_i - 4\delta_{ik} a_j . \end{aligned} \tag{2.3.7c}$$

Now we add (2.3.6a) to (2.3.7a), and (2.3.6b) to (2.3.7b) and (2.3.7c) in turn to arrive at the following commutation relations of the second order parafield:

$$\begin{aligned} a_i a_j a_k + a_k a_j a_i &= 0 , \\ a_i a_k^+ a_j + a_j a_k^+ a_i &= a_j \delta_{ik} + a_i \delta_{jk} , \\ a_i a_j a_k^+ + a_k^+ a_j a_i &= a_i \delta_{jk} - a_j \delta_{ik} . \end{aligned} \tag{2.3.8}$$

<u>Parafield of Order 3:</u>

The minimal polynomial equation satisfied by the L-matrix in this case is just

$$L^4 - \frac{5}{2}\Lambda^2 L^2 + \frac{9}{16}\Lambda^4 I = 0$$

As before this equation is written in terms of the a_i and a_i^+ and coefficients of (a) $\mu_i \mu_j \mu_k \mu_l$, (b) $\mu_i \mu_j \bar{\mu}_k \mu_l$ (c) $\bar{\mu}_i \bar{\mu}_j \mu_k \mu_l$ are compared. This gives the following equations

$$S_{a_i a_j a_k a_l} = 0 \quad , \tag{2.3.9a}$$

$$S_{a_i a_j a_k^+ a_l} = 10\,[\delta_{ik}\{a_j, a_l\} + \delta_{jk}\{a_i, a_l\} + \delta_{kl}\{a_i, a_j\}] \,, \tag{2.3.9b}$$

$$S_{a_i^+ a_j a_k^+ a_l} = 10\,[\delta_{ij}\{a_k^+, a_l\} + \delta_{il}\{a_j, a_k^+\} + \delta_{jk}\{a_i^+, a_l\} + \delta_{kl}\{a_i^+, a_j\}] - 18\,[\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}] \,. \tag{2.3.9c}$$

We now write the permutation identity (2.2.6) with the identification $\gamma_i \equiv a_i$, $\gamma_j \equiv a_j$, $\gamma_k \equiv a_k$ and $\gamma_l \equiv a_l$ which immediately gives

$$6(a_i a_j a_k a_l + a_i a_l a_k a_j + a_j a_k a_l a_i + a_l a_k a_j a_i) - S_{a_i a_j a_k a_l} = 0,$$

On making use of the first of Eqs.(2.3.8). Adding this to Eqs.(2.3.9a), we get the first of the generating relations of the third order parafermi field

$$a_i a_j a_k a_l + a_i a_l a_k a_j + a_j a_k a_l a_i + a_l a_k a_j a_i = 0 . \qquad (2.3.10a)$$

To get the second generating relation, we identify $(\gamma_i, \gamma_j, \gamma_k, \gamma_l) \equiv (a_i, a_j, a_k^+, a_l)$ in the permutation identity. After simplifying using Eqs.(2.3.8) we get

$$6(a_i a_j a_k^+ a_l + a_i a_l a_k^+ a_j + a_j a_k^+ a_l a_i + a_l a_k^+ a_j a_i) - 5 a_i a_j a_k^+ a_l$$

$$= 8\delta_{jk}\{a_i, a_l\} + 8\delta_{lk}\{a_i, a_j\} - 4\delta_{ik}\{a_j, a_l\} .$$

Adding this to Eqs.(2.3.9b) we get the generating relation

$$a_i(a_j a_k^+ a_l + a_l a_k^+ a_j) + (a_j a_k^+ a_l + a_l a_k^+ a_j) a_i$$

$$= \delta_{ik}\{a_j, a_l\} + 3\delta_{jk}\{a_i, a_l\} + 3\delta_{kl}\{a_i, a_j\} \qquad (2.3.10b)$$

In a similar manner, with suitable rearrangements of the a's and a^+'s in Eqs.(2.2.6) and (2.3.9) we can obtain the rest of the generating relations:

$$a_i(a_k^+ a_j a_l + a_l a_j a_k^+) + (a_j a_k^+ a_l + a_l a_k^+ a_j) a_i = \delta_{ik}\{a_j, a_l\} + 3\delta_{jk}\{a_i, a_l\} + 3\delta_{kl} a_i a_j + \delta_{kl} a_j a_i, \tag{2.3.10c}$$

$$a_k^+(a_i a_j a_l + a_l a_j a_i) + (a_j a_i a_l + a_l a_i a_j) a_k^+ = \delta_{ik}\{a_j, a_l\} + \delta_{jk}\{a_i, a_l\} + \delta_{kl}\{a_i, a_j\}, \tag{2.3.10d}$$

$$a_i^+(a_j a_k^+ a_l + a_l a_k^+ a_j) + (a_l a_k^+ a_j + a_j a_k^+ a_l) a_i^+ = 3\{a_i^+, a_l\}\delta_{jk} + 3\{a_i^+, a_j\}\delta_{kl} + \{a_k^+, a_l\}\delta_{ij} + 3\{a_k^+, a_j\}\delta_{il} - 3(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}), \tag{2.3.10e}$$

$$a_l(a_k^+ a_j a_i^+ + a_i^+ a_j a_k^+) + (a_k^+ a_i^+ a_j + a_j a_i^+ a_k^+) a_l = 3\{a_k^+, a_l\}\delta_{ij} + \{a_i^+, a_j\}\delta_{kl} + \{a_k^+, a_j\}\delta_{il} + a_i^+ a_l \delta_{jk} + 3 a_l a_i^+ \delta_{jk} - 3(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}), \tag{2.3.10f}$$

$$a_i^+(a_j a_l a_k^+ + a_k^+ a_l a_j) + (a_j a_k^+ a_l + a_l a_k^+ a_j) a_i^+ = 3\{a_i^+, a_j\}\delta_{kl} + \{a_k^+, a_l\}\delta_{ij} + \{a_k^+, a_j\}\delta_{il} + \{a_i^+, a_l\}\delta_{jk} + 2 a_l a_i^+ \delta_{jk} - 3(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \tag{2.3.10g}$$

$$a_\ell(a_i^+ a_j a_k^+ + a_k^+ a_j a_i) + (a_k^+ a_i^+ a_j + a_j a_i^+ a_k^+) a_\ell = 3\{a_k^+, a_\ell\}\delta_{ij} + 3\{a_i^+, a_\ell\}\delta_{jk} + \{a_k^+, a_j\}\delta_{i\ell} + \{a_i^+, a_j\}\delta_{k\ell} - 2a_i^+ a_\ell \delta_{jk} - 3(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{jk}) \quad (2.3.10h)$$

We notice that the Eqs.(2.3.10a-d) are the same as those given by Kamefuchi and Takahashi. However, the Eqs.(2.3.10e-h) which we have obtained represent a simplification over the equations of Kamefuchi and Takahashi, in that they have obtained only the sum of Eqs.(2.3.10e) and (2.3.10f), and that of Eqs.(2.3.10g) and (2.3.10h).

It is to be noted that the four equations (2.3.10e-h) are not distinct and one could be deduced from the other by making use of the Lie algebraic identity and some trivial changes in the order of the indices. Further in deriving some of these generating relations of the third order parafield, some slight algebraic modifications making use of Eqs.(2.3.8) have been made. This was done only to get the equations in exactly the same form as those given by Kamefuchi and Takahashi.

CHAPTER III

GENERAL INVOLUTIONAL TRANSFORMATIONS AND THE REPRESENTATION OF GL(n)*

1. Introduction:

General involutional transformations which include homographic projective transformations (apart from sign) have wide applications in physics[1]. These are matrices satisfying the relation $A^m = kI$, k = constant, of which a particular case is the set of Pauli matrices.

The case when the set of matrices A obey the generalized Clifford algebra C_n^m (GCA) defined by

$$e_i e_j = w e_j e_i, \quad i < j, \quad i, j = 1 - - n, \tag{3.1.1}$$

$$e_i^m = 1, \tag{3.1.2}$$

where w is a primitive m^{th} root of unity, has been studied exhaustively. The general mathematical formulation has been made by Morinaga and Nono[2], Yamazaki[3] and

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* T. S. Santhanam, P. S. Chandrasekaran and Nalini B. Menon J. Math. Phys. 12, 377 (1971).

1) Alladi Ramakrishnan et. al., J. Math. Anal. Appl. 27, 164 (1969);
L. A. Pipes, J. Franklin Inst. 287, 285 (1969);
S. K. Kim, J. Math. Phys. 9, 1705 (1968);
M. E. Fisher, Phys. Rev. 113, 969 (1959);
See also A. Deepak et. al., Intern. J. Quant. Chem. 3 445 (1969).

2) K. Morinaga and T. Nono, J. Sci. Hiroshima Univ. A6, 13 (1952)

3) K. Yamazaki, J. Fac. Sci. Univ. Tokyo, Sec 1, 10, 147 (1964).

Morris[4], while its relation to physics through the study of their specific representations has been made systematically by Ramakrishnan[5] and collaborators[1]. The present investigation, however, is on involutional matrices which satisfy Eq.(3.1.2) and may or may not satisfy Eqs.(3.1.1). In this sense, Eqs.(3.1.2) alone envelops a wider class of matrices than those implied by both the Eqs.(3.1.1) and (3.1.2). The case when $m = 2$ has been studied in detail by Kim[6]. In this chapter, we shall study general involutional matrices.

When $m = 2$, an involutional matrix has the general form (except for trivial constant matrices)

$$A^{(2)} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \qquad (3.1.3)$$

where a, b and c are arbitrary parameters. If this is regarded as an element of the general linear group in two dimensions, the matrix representation of $A^{(2)}$ as a transformation on a basis set of homogeneous polynomials of q^{th} degree in two variables will yield a $(q+1) \times (q+1)$

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4) A.O.Morris, Quart. J. Math. (Oxford) (2) 18, 7 (1967); 19, 289 (1968).

5) Alladi Ramakrishnan, J. Math. Anal. Appl. 20, 9 (1967).

6) S.K.Kim, J. Math. Phys. 10, 1225 (1969).

involutional matrix with three arbitrary parameters. This is just the q^{th} induced representation of $A^{(2)}$[7]. Since the above procedure can be recognised as a very simple method of induction and since induced matrices are a special class of invariant matrices, the property of involution is carried through for an arbitrary $n \times n$ matrix[8].

In this chapter the following are dealt with:

(1) We show that the conditions on the 2×2 matrix $A^{(2)}$ such that $[A^{(2)}]^m = kI$ are sufficient to make the q^{th} induced matrix of $A^{(2)}$ obey the equation

$$[A_q^{(2)}]^m = k^q I \tag{3.1.4}$$

(2) We set up generating equations for the q^{th} induced matrix of a 3×3 matrix $A^{(3)}$. In particular, if the matrix $A^{(3)}$ is involutional in the sense $[A^{(3)}]^m = kI$, the q^{th} induced matrix $A_q^{(3)}$ satisfies the equation

$$[A_q^{(3)}]^m = k^q I \tag{3.1.5}$$

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7) D.E. Littlewood, The Theory of Group Characters and Matrix Representations of Groups (Oxford University Press, Oxford, 1958), p.178.

8) It has been pointed out to us by Professor Alladi Ramakrishnan that the method of induction can be related to taking the direct product of helicity matrices defined by him. See Alladi Ramakrishnan, J. Math. Anal. Appl. **26**, 88 (1969).

(3) It is now quite clear how to write down the generating equation for the q^{th} induced matrix of an arbitrary $n \times n$ matrix $A^{(n)}$. A particular case of interest is when $A^{(n)}$ is involutional.

(4) The special case of a 3×3 matrix $A^{(3)}$ satisfying $[A^{(3)}]^3 = 1$ is discussed in detail. It is shown that it can be expanded in the basis of the generalized Clifford algebra C_2^3 with coefficients which are the generalized hyperbolic functions.

(5) We calculate the eigenvalues of the matrix belonging to $GL(n)$ obtained through induction, and specialize it to the case of involutional matrices.

2. <u>Involutional transformations of</u> $GL(2)$:

The complete set of q^{th} degree polynomials in two variables x and y,

$$F_v(r) = x^{q-v} y^{v} \qquad (3.2.1)$$

$$r = (x, y), \quad v = 0, 1 - - q$$

is taken as the basis set. An element $R^{(2)}$ of $GL(2)$ is given by the general 2×2 (nonsingular) matrix

$$R^{(2)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2), \qquad (3.2.2)$$

$$ad - bc \neq 0,$$

when a, b, c and d are arbitrary parameters. The (q+1)-dimensional representation $R_q^{(2)}$ is furnished by the q^{th} induced matrix of $R^{(2)}$ and is given by[6)]

$$F_v(R^{(2)}r) = (ax+by)^{q-v}(cx+dy)^v$$

$$= \sum_{\mu=0}^{q} [R_q^{(2)}]_{v\mu} F_\mu(r). \tag{3.2.3}$$

The explicit form of $R_q^{(2)}$ is obtained by developing Eqs.(3.2.3) in power series, and one gets

$$[R_q^{(2)}]_{v\mu} = a^{q-\mu-v} b^\mu c^v \sum_k \binom{v}{k}\binom{q-v}{\mu-k}\left(\frac{ad}{bc}\right)^k \tag{3.2.4}$$

Now, an invariant matrix A_q of a matrix A is defined by the relation[7)]

$$A_q B_q = (AB)_q, \tag{3.2.5}$$

where A_q is the matrix whose entries are polynomials in the elements of the matrix A, from which it easily follows that

$$[A_q]^m = [A^m]_q = k^q 1, \tag{3.2.6}$$

if

$$A^m = k1 \tag{3.2.7}$$

where A_q is of dimension (q+1).

Therefore the conditions on the four parameters of the 2 x 2 matrix, in order that Eqs.(3.2.7) is satisfied, automatically leave $A_q^{(2)}$ involutional. When k = 1, the involutional matrix $A^{(2)}$ involves only two parameters since in this case $a+d=0$ and $bc=1-a^2$, and thus it can be expressed as

$$A^{(2)}(\theta) = \sigma_3 R(\theta), \tag{3.2.8}$$

where

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.2.9}$$

and $R(\theta)$ is the rotation matrix in two dimensions given by

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{3.2.10}$$

with θ defined through

$$(bc)^{1/2} = \sin\theta \tag{3.2.11}$$

3. <u>Generating equations for the general involutional matrices:</u>

By an exactly similar procedure as that utilized for the case of m = 2, we now write down the generating equations for the case m = 3. We define the q^{th} degree homogeneous polynomials in three variables x, y, z as

$$F_{\alpha_1,\alpha_2}(\underline{r}) = x^{q-\alpha_1-\alpha_2}\, y^{\alpha_1} z^{\alpha_2}, \tag{3.3.1}$$

where the nonnegative integers α_1 and α_2 obey

$$\alpha_1 + \alpha_2 \leq q \tag{3.3.2}$$

The linear homogeneous transformation $R^{(3)}$ in three dimensions is given by a 3 x 3 matrix

$$R^{(3)} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \in GL(3), \tag{3.3.3}$$

where the a_{ij} are arbitrary parameters. The q^{th} induced matrix of $R^{(3)}$ involving nine parameters is then obtained from the following equation:

$$F_{(\alpha_1\alpha_2)}(R^{(3)}r) = (a_{00}x + a_{01}y + a_{02}z)^{q-\alpha_1-\alpha_2}$$
$$\times (a_{10}x + a_{11}y + a_{12}z)^{\alpha_1}$$
$$\times (a_{20}x + a_{21}y + a_{22}z)^{\alpha_2}$$

$$= \sum_{(\alpha_1'\alpha_2')} [R_q^{(3)}]_{(\alpha_1\alpha_2)(\alpha_1'\alpha_2')} F_{(\alpha_1'\alpha_2')}(\underline{r})$$

$$\underline{r} = (x, y, z) \qquad (3.3.4)$$

where the matrix $R_q^{(3)}$ is labeled by the different partitions of the non-negative integers (α_1', α_2') and (α_1, α_2) satisfying

$$(\alpha_1' + \alpha_2') \le q, \quad \alpha_1 + \alpha_2 \le q \qquad (3.3.5)$$

Hence the dimension of $R_q^{(3)}$ is simply given by the number of solutions (α_1, α_2) of Eqs. (3.3.5), which in this case is equal to $\binom{q+2}{2}$.

Obviously, $R_q^{(3)}$ reduces to $R^{(3)}$ when $q = 1$. For convenience, we can choose the partitions in decreasing order in α, for a given value of $(\alpha_1 + \alpha_2)$ and increasing order in $(\alpha_1 + \alpha_2)$ for labeling the matrix. To make this clear, we calculate $R_2^{(3)}$ in the following. Since $q = 2$, $\alpha_1, \alpha_2, \alpha_1', \alpha_2'$ can each take any of the values (0,1,2). The partitions (α_1, α_2) are then as follows:

$\alpha_1+\alpha_2$	(α_1, α_2)
0	(0, 0)
1	(1, 0)
	(0, 1)
2	(2, 0)
	(1, 1)
	(0, 2)

Using the first set of values of (α_1, α_2) in Eqs.(3.3.4) gives

$$(a_{00}x + a_{01}y + a_{02}z)^2 = \sum_{(\alpha_1'\alpha_2')} [R_2^{(3)}]_{(00)(\alpha_1'\alpha_2')} x^{2-\alpha_1'-\alpha_2'} y^{\alpha_1'} z^{\alpha_2'}$$

$$= [R_2^{(3)}]_{(00)(00)} x^2 + [R_2^{(3)}]_{(00)(10)} xy + [R_2^{(3)}]_{(00)(01)} xz$$

$$+ [R_2^{(3)}]_{(00)(20)} y^2 + [R_2^{(3)}]_{(00)(11)} yz + [R_2^{(3)}]_{(00)(02)} z^2$$

from which one directly gets

$$[R_2^{(3)}]_{(00)(00)} = a_{00}^{3} \qquad [R_2^{(3)}]_{(00)(20)} = a_{01}^2$$

$$[R_2^{(3)}]_{(00)(10)} = 2a_{00}a_{01} \qquad [R_2^{(3)}]_{(00)(11)} = 2a_{01}a_{02}$$

$$[R_2^{(3)}]_{(00)(01)} = 2a_{00}a_{02} \qquad [R_2^{(3)}]_{(00)(02)} = a_{02}^2$$

Thus the first row of the matrix $[R_{2}^{(3)}]$ is obtained. Similarly the rest of the elements can be easily read off from Eqs. (3.3.4) by successively using the different partitions (α_1, α_2) . Explicitly the matrix $[R_2^{(3)}]$ looks like

$$
\begin{bmatrix}
a_{00}^2 & 2a_{00}a_{01} & 2a_{00}a_{02} & a_{01}^2 & 2a_{01}a_{02} & a_{02}^2 \\
a_{00}a_{10} & a_{00}a_{11} + a_{01}a_{10} & a_{00}a_{12} + a_{02}a_{10} & a_{01}a_{11} & a_{01}a_{12} + a_{02}a_{11} & a_{02}a_{12} \\
a_{00}a_{20} & a_{00}a_{21} + a_{01}a_{20} & a_{00}a_{22} + a_{02}a_{20} & a_{01}a_{21} & a_{01}a_{22} + a_{02}a_{21} & a_{02}a_{22} \\
a_{10}^2 & 2a_{10}a_{11} & 2a_{10}a_{12} & a_{11}^2 & 2a_{11}a_{22} & a_{12}^2 \\
a_{10}a_{20} & a_{10}a_{21} + a_{11}a_{20} & a_{10}a_{22} + a_{12}a_{20} & a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} & a_{12}a_{22} \\
a_{20}^2 & 2a_{20}a_{21} & 2a_{20}a_{22} & a_{21}^2 & 2a_{21}a_{22} & a_{22}^2
\end{bmatrix}
$$

Eqs. (3.3.4) can be inverted to get explicit expressions for the elements $[R_q^{(3)}]$ by simply expanding in power series,

$$(a_{00}x + a_{01}y + a_{02})^{q-\alpha_1-\alpha_2}(a_{10}x + a_{11}y + a_{12}z)^{\alpha_1}(a_{20}x + a_{21}y + a_{22}z$$

$$= \sum_{\substack{\mu_1\mu_2\nu_1\nu_2 \\ \lambda_1\lambda_2}} \binom{q-\alpha_1-\alpha_2}{\mu_1}\binom{\mu_1}{\mu_2}(a_{00}x)^{q-\alpha_1-\alpha_2-\mu_1}(a_{01}y)^{\mu_1-\mu_2}(a_{02}$$

$$\times \binom{\alpha_1}{\nu_1}\binom{\nu_1}{\nu_2}(a_{10}x)^{\alpha_1-\nu_1}(a_{11}y)^{\nu_1-\nu_2}(a_{12}z)^{\nu_2}$$

$$\times \binom{\alpha_2}{\lambda_1}\binom{\lambda_1}{\lambda_2}(a_{20}x)^{\alpha_2-\lambda_1}(a_{21}y)^{\lambda_1-\lambda_2}(a_{22}z)^{\lambda_2}$$

$$= \sum_{\substack{\mu_1\mu_2\\ \nu_1\nu_2\\ \lambda_1\lambda_2}} \binom{q-\alpha_1-\alpha_2}{\mu_1}\binom{\mu_1}{\mu_2}\binom{\alpha_1}{\nu_1}\binom{\nu_1}{\nu_2}\binom{\alpha_2}{\lambda_1}\binom{\lambda_1}{\lambda_2}$$

$$\times (a_{00})^{q-\alpha_1-\alpha_2-\mu_1}(a_{01})^{\mu_1-\mu_2}(a_{02})^{\mu_2}(a_{10})^{\alpha_1-\nu_1} \times (a_{11})^{\nu_1-\nu_2}(a_{12})^{\nu_2}$$

$$\times\ x^{q-(\mu_1+\nu_1+\lambda_1)}\ y^{(\mu_1+\nu_1+\lambda_1)-(\mu_2+\nu_2+\lambda_2)}\ z^{\mu_2+\nu_2+\lambda_2}$$

Now we set $(\mu_1+\nu_1+\lambda_1)-(\mu_2+\nu_2+\lambda_2) = \alpha_1'$ and $\mu_2+\nu_2+\lambda_2 = \alpha_2'$ and comparing with the righthand-side of Eq.(.3.4) we get after some rearrangement

$$[R_q^{(3)}]_{(\alpha_1\alpha_2)(\alpha_1'\alpha_2')} = \left(\frac{a_{21}}{a_{20}}\right)^{\alpha_1'+\alpha_2'}\left(\frac{a_{22}}{a_{21}}\right)^{\alpha_2'}(a_{00})^{q-\alpha_1-\alpha_2}(a_{10})^{\alpha_1}(a_{2}$$

$$\times \sum_{\substack{\mu_1\mu_2\\ \nu_1\nu_2}} \binom{q-\alpha_1-\alpha_2}{\mu_1}\binom{\alpha_1}{\nu_1}\binom{\alpha_2}{\alpha_1'+\alpha_2'-\mu_1-\nu_1}\binom{\mu_1}{\mu_2} \times \binom{\nu_1}{\nu_2}$$

$$\times \binom{\alpha_1'+\alpha_2'-\mu_1-\nu_1}{\alpha_2'-\mu_2-\nu_2} \qquad (3.3.6)$$

$$\times \left(\frac{a_{01}a_{20}}{a_{00}a_{21}}\right)^{\mu_1}\left(\frac{a_{02}a_{21}}{a_{01}a_{22}}\right)^{\mu_2}\left(\frac{a_{11}a_{20}}{a_{10}a_{21}}\right)^{\nu_1}\left(\frac{a_{12}a_{21}}{a_{11}a_{22}}\right)^{\nu_2}$$

This may possibly be related to the Lauricella functions[9].

The above procedure of generating the induced matrix can be easily generalized to the case of an arbitrary $n \times n$ matrix $R^{(n)}$ given by

$$R^{(n)} = \begin{bmatrix} a_{00} & a_{01} & \cdot & a_{0\,n-1} \\ a_{10} & a_{11} & - - & a_{1\,n-1} \\ | & | & - - & - \\ & | & - - & - \\ a_{n-1,0} & a_{n-1,1} & - - & a_{n-1,n-1} \end{bmatrix} \qquad (3.3.7)$$

In this case we define the q^{th} degree polynomials $F_{(\alpha_1,\cdots\alpha_{n-1})}(\underline{x})$ in n variables $x_1, - \cdot x_n$,

$$F_{(\alpha_1,-\alpha_{n-1})}(\underline{x}) = x_1^{q-\sum_{i=1}^{n-1}\alpha_i}\, x_2^{\alpha_1} \cdots x_n^{\alpha_{n-1}},$$

$$\underline{x} = (x_1, x_2, -, x_n) \qquad (3.3.8)$$

with the non-negative integers α_i satisfying the partition equation

$$\sum_{i=1}^{n-1} \alpha_i \leq q \qquad (3.3.9)$$

---

9) J. Slater, Generalized Hypergeometric functions (Cambridge University Press, Cambridge, 1966), p.227

The q^{th} induced representation of $R^{(n)}$ is given by the matrix $R_q^{(n)}$ defined by

$$F_{(\alpha_1, \cdots \alpha_{n-1})}(R^{(n)} r) = (a_{00} x_1 + \cdots + a_{qn-1} x_n)^{q - \sum_{i=1}^{n-1} \alpha_i} \prod_{j=1}^{n-1} \left(\sum_{k=0}^{n-1} a_{jk} x_{k+1} \right)^{\alpha_j}$$

$$= \sum_{(\alpha_1', \cdots \alpha_{n-1}')} [R_q^{(n)}]_{(\alpha_1, \cdots \alpha_{n-1})(\alpha_1' \cdots \alpha_{n-1}')} F_{(\alpha_1' - \alpha_{n-1}')}(\underline{r}),$$

(3.3.10)

where the matrix is labeled by the distinct partitions given by Eqs. (3.3.9). We can choose them in the decreasing order $(\alpha_1, \cdots \alpha_{n-1})$ for a given value of

$$(\alpha_1 + \cdots + \alpha_{n-1}) \leq q$$

The dimension of $R_q^{(n)}$ is just given by the number of solutions to the partitions equation (3.9), which is simply equal to $\binom{n+q-1}{q} = \binom{n+q-1}{n-1}$

Let us know specialize the method of induction to the case of involutional matrices satisfying the equation

$$[R^{(n)}]^m = k I \qquad (3.3.11)$$

As in the case of a (2 x 2) matrix, the conditions on $R^{(n)}$

so that it satisfies Eqs. (3.3.11) make its q^{th} induced representation obey

$$[R_q^{(n)}]^m = k^q I \tag{3.3.12}$$

This follows directly from the property of induced matrices, which form a special case of invariant matrices satisfying Eqs. (3.2.5). The conditions on $R^{(n)}$ implied by Eqs. (3.3.11), when m = n, follow from the characteristic equation of $R^{(n)}$ and are:

$$\mathrm{Tr}\, R^{(n)} = \mathrm{Tr}[R^{(n)}]^2 = \cdots = \mathrm{Tr}[R^{(n)}]^{n-1} = 0, \tag{3.3.13}$$

and

$$\det R^{(n)} = (-1)^n k.$$

Let us consider the special case of a 3 x 3 matrix satisfying the equation

$$[A^{(3)}]^3 = I. \tag{3.3.14}$$

The eigenvalues of $A^{(3)}$ are then given by the cube roots of unity $(1, \omega, \omega^2)$. As in the case of $A^{(2)}$, $A^{(3)}$ can be reduced to the form

$$F^{(3)}(\theta) = VA^{(3)}V^{-1} \tag{3.3.15}$$

$$= \begin{bmatrix} f_1^{(3)} & \omega f_2^{(3)} & f_3^{(3)} \\ f_3^{(3)} & \omega f_1^{(3)} & f_2^{(3)} \\ \omega^2 f_2^{(3)} & f_3^{(3)} & \omega^2 f_1^{(3)} \end{bmatrix},$$

where the $f_i^{(3)}$ are the generalized hyperbolic functions of order three with argument $(\Lambda\theta)$, with $\Lambda = \exp(\frac{1}{3}\pi i) = \omega^{1/2}$, ω, being a primitive cube root of unity. It is inessential to compute the matrix V whose existence can be inferred from the fact that both $A^{(3)}$ and $F^{(3)}$ are nonsingular and satisfy Eqs.(3.3.14). The f's are functions of the entries of the matrix $A^{(3)}$. They satisfy the determinantal condition[10)]

$$\begin{vmatrix} f_1^{(3)} & f_2^{(3)} & f_3^{(3)} \\ f_3^{(3)} & f_1^{(3)} & f_2^{(3)} \\ f_2^{(3)} & f_3^{(3)} & f_1^{(3)} \end{vmatrix} = 1 \qquad (3.3.16)$$

The $f_i^{(3)}$ are related to the trigonometric functions of order three $k_i^{(3)}$ (see Appendix B) through the relation

$$k_i^{(3)}(\theta) = \Lambda^{(1-i)} f_i^{(3)}(\Lambda\theta) \qquad (3.3.17)$$

$F^3(\theta)$ can be expressed as

$$F^{(3)}(\theta) = B^{(3)} R^{(3)}(\theta) \qquad (3.3.18)$$

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19) A.Erdélyi, Higher transcendental functions (McGraw Hill, New York, 1955), Vol. III, pp.212-17.

where
$$B^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \tag{3.3.19}$$

and $R^{(3)}(\theta)$ is the matrix

$$R^{(3)}(\theta) = \begin{bmatrix} f_1 & \omega f_2 & f_3 \\ \omega^2 f_3 & f_1 & \omega^2 f_2 \\ f_2 & \omega f_3 & f_1 \end{bmatrix} \tag{3.3.20}$$

The interesting point is that $R^{(3)}(\theta)$ can be expressed as

$$R^{(3)}(\theta) = \sum_{i=1}^{3} f_i^{(3)}(\Lambda\theta) Q_3^{i-1} \tag{3.3.21}$$

$$= \sum_{i=1}^{3} \Lambda^{i-1} k_i^{(3)}(\theta) Q_3^{i-1} \tag{3.3.22}$$

where the matrix

$$Q_3 = \begin{bmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{bmatrix} \tag{3.3.23}$$

is a base element of the generalized Clifford algebra C_2^{3}

(see Appendix A). The determinantal condition (3.3.16) can also be written as

$$\det \sum_{i=1}^{3} f_i^{(3)}(\lambda\theta)\, P_3^{\,i-1} = \det \sum_{i=1}^{3} k_i^{(3)}(\theta)\, \Lambda^{i-1} P_3^{\,i-1} = 1 \tag{3.3.24}$$

where the matrix

$$P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{3.3.25}$$

is the other base element of $C_3^{\,2}$.

The above discussion can now be carried for an arbitrary (n x n) involutional matrix

$$[A^{(n)}]^n = I \tag{3.3.26}$$

which can be transformed to the form

$$V^{(n)} A^{(n)} [V^{(n)}]^{-1} = F^{(n)}(\theta) = B^{(n)} R^{(n)}(\theta) \tag{3.3.27}$$

where

$$B^{(n)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & & \omega^{n-1} \end{bmatrix} \tag{3.3.28}$$

and[11])

$$R^{(n)}(\theta) = \sum_{i=1}^{n} f_i^{(n)}(\Lambda\theta)\, \beta_n^{\,i-1} \tag{3.3.29}$$

$$= \sum_{i=1}^{n} \Lambda^{i-1} k_i^{(n)}(\theta)\, \beta_n^{\,i-1}, \quad \Lambda = \exp(n^{-1}\pi i)$$

where the matrix

$$\beta_n = \begin{bmatrix} 0 & \omega & 0 & - & - & 0 \\ 0 & 0 & \omega^2 & - & - & 0 \\ - & - & - & & - & \\ 0 & 0 & 0 & - & - & \omega^{n-1} \\ 1 & 0 & 0 & - & - & 0 \end{bmatrix} \tag{3.3.30}$$

$= 1$ for n odd

$= \Lambda$ for n even,

is an element of the generalized Clifford algebra $C_2^{\,n}$. The determinantal condition on the hyperbolic and trigonometric functions of order n is simply given by

$$\det \sum_{i=1}^{n} f_i^{(n)}(\Lambda\theta)\beta_n^{\,i-1} = \det \sum_{i=1}^{n} \Lambda^{i-1} k_i^{(n)}(\theta)\beta_n^{\,i-1} = 1, \tag{3.3.31}$$

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11) It has been noted by Professor Alladi Ramakrishnan (Private communication) that if a matrix T(x) has the form $T(x) = Me^{Nx}$, with the matrices M and N satisfying the relation $MN = NM$, $M^m = N^m = I$, then it follows that $[T(x)]^m = I$.

where the matrix

$$P_n = \begin{bmatrix} 0 & 1 & 0 & - & - & 0 \\ 0 & 0 & 1 & - & - & 0 \\ - & - & - & & & - \\ 0 & 0 & 0 & - & - & 1 \\ 1 & 0 & 0 & - & - & 0 \end{bmatrix} \qquad (3.3.32)$$

is the other base element of C_2^n. The f's are functions of the entries of the matrix $A^{(n)}$, and the explicit relation is of little concern to us. The existence of $V^{(n)}$ is again guaranteed by the fact that $A^{(n)}$ and $F^{(n)}(\theta)$ are both nonsingular and satisfy Eqs. (3.3.26).

4. <u>Eigenvalues of</u> $R_q^{(n)}$ <u>and</u> $A_q^{(n)}$:

In this section we first calculate the eigenvalues of the q^{th} induced matrix $R_q^{(n)}$ of the matrix $R^{(n)}$ given by Eqs. (3.3.10) and specialize it to the case when $R_q^{(n)}$ is involutional. The calculation is based on the simple theorem that if the matrix $R^{(n)}$ is triangular, then its induced matrix $R_q^{(n)}$ is also triangular in shape similar to $R^{(n)}$. This theorem has been proved by Kim[6] for $n = 2$, and it is true even in the general case. Consider for example the case of $n = 3$. If $R^{(3)}$ has the form

$$R^{(3)t} = \begin{bmatrix} a_{00} & 0 & 0 \\ a_{10} & a_{11} & 0 \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \tag{3.4.1}$$

Eqs.(3.3.4) has the form

$$(a_{00}x)^{q-\alpha_1-\alpha_2}(a_{10}x + a_{11}y)^{\alpha_1}(a_{20}x + a_{21}y + a_{22}z)^{\alpha_2}$$

$$= \sum_{(\alpha_1'\alpha_2')} [R_q^{(3)t}]_{(\alpha_1\alpha_2)(\alpha_1'\alpha_2')} x^{q-\alpha_1'-\alpha_2'} y^{\alpha_1'} z^{\alpha_2'}$$

and it follows that

$$[R_q^{(3)t}]_{\alpha\alpha'} = 0$$

unless $\alpha_2' \leq \alpha_2$ and $\alpha_1' + \alpha_2' \leq \alpha_1 + \alpha_2$ where $\alpha = (\alpha_1, \alpha_2)$ and $\alpha' = (\alpha_1', \alpha_2')$. These are simply the conditions for the matrix $R_q^{(3)}$ to be triangular in shape similar to $R^{(3)}$. It is not hard to prove the same result for any n.

In fact it follows directly from Eqs.(3.3.10) that if the matrix R^n is triangular, then, since $a_{ij} = 0$, $i < j$

we have $[R_q^{(n)}]_{\alpha\alpha'} = 0$ unless

$$\sum_{i=k}^{n-1} \alpha_i \geq \sum_{i=k}^{n-1} \alpha_i', \quad k = 1 - - n-1 \tag{3.4.2}$$

These are just the conditions for $R_q^{(n)t}$ to be triangular and similar in shape to $R/^{(n)t}$, Eqs.(3.4.2) incidentally suggests a more convenient labeling of $R_q^{(n)}$ by $(\beta_1, - - \beta_{n-1})$ satisfying

$$\begin{aligned} &0 \leq \beta_{n-1} \leq - - \leq \beta_1 \leq q, \\ &\beta_1 + \beta_2 + - - + \beta_{n-1} \leq (n-1)q \end{aligned} \tag{3.4.3}$$

where

$$\beta_j = \sum_{i=j}^{n-1} \alpha_i, \quad j = 1 - - n-1$$

The generating equation (3.3.10) for the induced matrix then simply becomes

$$\prod_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} a_{jk} x_{k+1} \right)^{\beta_j - \beta_{j+1}} = \sum_{\beta'} [R_q^{(n)}]_{\beta\beta'} F_{\beta'}(\underline{x}) \tag{3.4.4}$$

with

$$\beta_0 = q, \quad \beta_n = 0 \tag{3.4.5}$$

and β, β' denote $(\beta_1 \cdots \beta_{n-1})$, $(\beta_1' \cdots \beta_{n-1}')$ respectively. Equations (3.3.10) and (3.4.4) are completely equivalent.

Now it is always possible to transform the matrix $R^{(3)}$ into the triangular matrix $R^{(3)T}$,

$$R^{(3)T} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ \xi_1 & \epsilon_2 & 0 \\ \xi_2 & \xi_3 & 0 \end{bmatrix} \qquad (3.4.6)$$

through a suitable unitary transformation. Here the ξ's are constants, and the ϵ's are the eigenvalues of $R^{(3)}$. Substituting Eqs. (3.4.6) in Eqs. (3.3.6), we obtain

$$\begin{aligned}
[R_q^{(3)T}] &= \epsilon_1^{\,q-\alpha_1-\alpha_2}\, \epsilon_3^{\,\alpha_2} \\
&\quad \times \sum_v \binom{\alpha_1}{v} \binom{\alpha_2}{\alpha_1'+\alpha_2'-v} \binom{\alpha_1'+\alpha_2'-v}{\alpha_2'} \\
&\quad \times (\epsilon_2)^v (\xi_1)^{\alpha_1-v} (\xi_2)^{\alpha_2-\alpha_1'-\alpha_2'+v} (\xi_3)^{\alpha_1'-v}, \\
&\qquad \xi_1,\ \xi_2,\ \xi_3 \neq 0 \qquad (3.4.7) \\
&= (\epsilon_1)^{q-\alpha_1-\alpha_2} (\epsilon_2)^{\alpha_1} (\epsilon_3)^{\alpha_2}\, \delta_{\alpha_1\alpha_1'}\, \delta_{\alpha_2\alpha_2'} \\
&\qquad \xi_1 = \xi_2 = \xi_3 = 0
\end{aligned}$$

The eigenvalues of $R_q^{(3)}$ are then given by

$$\epsilon_1^{\,q-\alpha_1-\alpha_2}\,\epsilon_2^{\alpha_1}\,\epsilon_3^{\alpha_2}, \qquad \alpha_1+\alpha_2 \le q \tag{3.4.8}$$

The determinant of $R_q^{(3)}$ is given by

$$\det R_q^{(3)} = \prod_{\substack{\alpha_1,\alpha_2\\ \alpha_1+\alpha_2\le q}} \epsilon_1^{\,q-\alpha_1-\alpha_2}\,\epsilon_2^{\alpha_1}\,\epsilon_3^{\alpha_2}$$

$$= \prod_{\alpha_1} \epsilon_1^{(q-\alpha_1)+(q-\alpha_1-1)+\cdots 1+0} \left(\epsilon_2^{\alpha_1}\right)^{q-\alpha_1+1} \times \epsilon_3^{0+1+\cdots+(q-\alpha_1-1)+(q-}$$

$$= (\epsilon_1\,\epsilon_2\,\epsilon_3)^{\frac{q(q+1)(q+2)}{6}}$$

therefore,

$$\det R_q^{(3)} = (\epsilon_1\,\epsilon_2\,\epsilon_3)^{\binom{q+2}{3}} = \Delta^{\binom{q+2}{3}} \tag{3.4.9}$$

where Δ denotes the determinant of $R^{(3)}$. The trace of $R_q^{(3)}$ is given by

$$\mathrm{Tr}\, R_q^{(3)} = \sum_{\alpha_1, \alpha_2} \epsilon_1^{q-\alpha_1-\alpha_2} \epsilon_2^{\alpha_1} \epsilon_3^{\alpha_2}$$

$$= \sum_{\alpha_1} \epsilon_2^{\alpha_1} \left(\frac{\epsilon_1^{(q-\alpha_1+1)} - \epsilon_3^{q-\alpha_1+1}}{\epsilon_1 - \epsilon_3} \right)$$

$$= \frac{1}{\epsilon_1 - \epsilon_3} \left[\epsilon_1^{q+1} \sum_{\alpha_1=0}^{q} \left(\frac{\epsilon_2}{\epsilon_1} \right)^{\alpha_1} - \epsilon_3^{q+1} \sum_{\alpha_1} \left(\frac{\epsilon_2}{\epsilon_3} \right)^{\alpha_1} \right]$$

$$= \frac{1}{\epsilon_1 - \epsilon_3} \left[\frac{\epsilon_1 (\epsilon_1^{q+1} - \epsilon_2^{q+1})}{\epsilon_1 - \epsilon_2} - \frac{\epsilon_3 (\epsilon_2^{q+1} - \epsilon_3^{q+1})}{\epsilon_2 - \epsilon_3} \right]$$

which after simplifying and rearranging gives

$$\mathrm{Tr}\, R_q^{(3)} = \frac{1}{(\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)(\epsilon_1 - \epsilon_3)} \times$$

$$\times \left[\epsilon_1 \epsilon_2 (\epsilon_1^{q+1} - \epsilon_2^{q+1}) + \epsilon_2 \epsilon_3 (\epsilon_2^{q+1} - \epsilon_3^{q+1}) + \epsilon_3 \epsilon_1 (\epsilon_3^{q+1} - \epsilon_1^{q+1}) \right] \quad (3.4.10)$$

If $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon$, we just have

$$\mathrm{Tr}\, R_q^{(3)} = \sum_{\alpha_1,\alpha_2} (\epsilon)^q = \binom{q+2}{q} \epsilon^q \tag{3.4.11}$$

since $\binom{q+2}{2}$ is the number of partitions (α_1, α_2) having $0 \le \alpha_1, \alpha_2 \le q$ and $\alpha_1 + \alpha_2 \le q$.

The above formulae can be immediately generalized to yield

$$\det R_q^{(n)} = (\Delta)^{\binom{q+n-1}{n}} \tag{3.4.12}$$

where

$$\Delta = \epsilon_1 \epsilon_2 \cdots \epsilon_n \tag{3.4.13}$$

is the determinant of $R^{(n)}$. Further we have

$$\mathrm{Tr}\, R_q^{(n)} = \prod_{\substack{i<j \\ i,j=1,\ldots,n}} (\epsilon_i - \epsilon_j)^{-1} \sum_{\substack{k \neq l \\ k,l \text{ cyclic} \\ k,l=1,\ldots,n}} \epsilon_k \epsilon_l [\epsilon_k^{q+1} - \epsilon_l^{q+1}], \quad \epsilon_1 \neq \epsilon_2 \neq \cdots \neq \epsilon_n,$$

$$= \binom{q+n-1}{n-1} \epsilon^q, \quad \epsilon_1 = \epsilon_2 = \cdots = \epsilon_n = \epsilon \tag{3.4.14}$$

The eigenvalues of $R_q^{(n)}$ are given by

$$\epsilon_1^{\,q - \sum_{i=1}^{n-1} \alpha_i} \epsilon_2^{\alpha_1} \epsilon_3^{\alpha_2} \cdots \epsilon_n^{\alpha_{n-1}}, \quad \sum_{i=1}^{n-1} \alpha_i \le q \tag{3.4.15}$$

Another interesting property of $R_q^{(n)}$ which can be easily derived from Eqs. (3.3.10) is that

$$\sum_{i,j\neq 0}^{n} a_{ij} \frac{\delta R_q^{(n)}}{\delta a_{ij}} = q R_q^{(n)} \qquad (3.4.16)$$

This directly follows on partially differentiating Eqs. (3.3.10) with respect to each a_{ij}, multiplying the resulting equation by a_{ij} and then adding up all the equations for the different a_{ij}.

All that has been discussed in this section can be specialized to the case of the general involutional n x n matrix $A^{(n)}$. The eigenvalues of $A^{(n)}$ are given by

$$\epsilon_1 = \epsilon, \quad \epsilon_2 = \omega\epsilon, \quad \epsilon_3 = \omega^2\epsilon_2, \; - - \;, \epsilon_n = \omega^{n-1}\epsilon,$$
$$\omega^n = 1 \qquad (3.4.17)$$

In this case we have

$$\mathrm{Tr}\, A_q^{(n)} = \frac{(-1)^n \epsilon^q}{n} (1 + \omega^q + \omega^{2q} + - - \omega^{(n-1)q}) \qquad (3.4.18)$$

so that

$$\mathrm{Tr}\, A_q^{(n)} = 0 \qquad \text{for } q \neq 0 \bmod n$$
$$= (-1)^n \epsilon^q \qquad \text{for } q \neq 0 \bmod n. \qquad (3.4.19)$$

The determinant of $A_q^{(n)}$ is given by

$$\det A_q^{(n)} = [\epsilon^n \omega^{\binom{n}{2}}]^{\binom{q+n-1}{n}} \qquad (3.4.20)$$

<u>Appendix A:</u>

We summarize here the relevant details of the generalized Clifford algebra[2-4]. The equation

$$\sum_{i=1}^{n} x_i^m = \left(\sum_{i=1}^{n} \alpha_i x_i\right)^m \qquad (3.A.1)$$

is satisfied if α's obey the relations

$$\alpha_i \alpha_j = \omega \alpha_j \alpha_i, \quad i < j, \quad i, j = 1, \ldots, n, \qquad (3.A.2)$$
$$\omega^m = 1$$

The set of elements defined by

$$\prod_{i=1}^{m} \alpha_i^{p_i} \qquad (3.A.3)$$

where the integers p_i satisfy

$$0 \leq p_i \leq m-1, \qquad (3.A.4)$$

is linearly independent. They are m^n in number. Obviously they form a vector space of dimension m^n, and with the product defined by Eqs.(3.A.2) they form an associative algebra called the generalized Clifford algebra C_n^m . The case when m = 2 can be realized to be the Dirac Clifford algebra. The matrix representation of α's has been obtained by using the Dirac procedure by Morinaga and Nono[2] and has also been obtained by Ramakrishnan, Santhanam and Chandrasekaran[12] by using vector space methods. The results are: C_n^m for $n = 2v$ has a faithful representation by the matrix ring $m^v \times m^v$; when n is odd, it has again the matrix representation in terms of $(m^v \times m^v)$ -dimensional matrices, which, however, breaks up into m sets of inequivalent matrix rings $m^v \times m^v$. That is, if the set $\{\beta\}$ furnishes a representation of dimension $m^v \times m^v$, then

$\omega^i\{\beta\}$, $i = 1, \ldots, m-1$ also furnish inequivalent representations of the same dimension, ω being a primitive m^{th} root of unity. The case when m = 2 is, of course, very well known[13].

Appendix B:

We assumarize here some general properties of the trigonometric and hyperbolic functions of order n[10]. The n functions

$$f_i = \frac{1}{n}\sum_{m=1}^{n} \omega^{(l-i)m} \exp(\omega^m x), \quad i = 1, \ldots n, \qquad (3.B.1)$$

$$\omega = \exp\left(\frac{2\pi i}{n}\right),$$

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12) Alladi Ramakrishnan, T.S. Santhanam and P.S. Chandrasekaran, J. Math. Phys. Sci. (Madras) 3, 307 (1969).

13) See, for example, H. Boerner, Representations of Groups (North-Holland, Amsterdam, 1963), Chap. 8.

are called the hyperbolic functions of order n .

The f_i satisfy the differential equation

$$\left(\frac{d^n}{dx^n} - 1\right) y = 0 \qquad \text{(3.B.2)}$$

$f_1, \ldots, f_n$ from a linearly independent set of solutions of Eqs.(3.B.2) and their Wronskian is equal to unity. From the definition of f_i it follows that

$$\exp(w^m x) = \sum_{i=1}^{n} w^{(1-i)m} f_i(x,n), \; m \text{ integer} \qquad \text{(3.B.3)}$$

From (3.B.3) it follows that

$$\prod_{m=1}^{n} \left(\sum_{i=1}^{n} w^{(1-i)m} f_i(x,n) \right) = 1. \qquad \text{(3.B.4)}$$

Eqs.(3.B.4) can also be written as

$$\det \sum_{i=1}^{n} f_i P^{i-1} = 1, \qquad \text{(3.B.5)}$$

where the permutation matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \text{(3.B.6)}$$

is a base element of C_2^n .

The functions

$$k_i(x,n) = \lambda^{1-i} f_i(\lambda x, n), \quad i=1, \cdots, n, \qquad \lambda = \exp(\pi i/n) \tag{3.B.7}$$

are called the trigonometric functions of order n . They are the solutions of the differential equation

$$\left(\frac{d^n}{dx^n} + 1\right) y = 0 \tag{3.B.8}$$

From (3.B.7) it is clear that

$$k_i(x,n) = \frac{1}{n} \sum_{m=1}^{n} \lambda^{(1-i)(2m+1)} \exp(\lambda^{2m+1} x) \tag{3.B.9}$$

and

$$\prod_{m=1}^{n} \left(\sum_{i=1}^{n} \lambda^{(1-i)(2m+1)} K_i(x) \right) = \det \sum_{i=1}^{n} \lambda^{(i-1)} K_i \beta^{i-1} = 1 \tag{3.B.10}$$

<u>Appendix C:</u>

We demonstrate the use of the expansion of a matrix in the basis of the roots of the unit matrix to find its arbitrary power. The problem of finding the arbitrary power of an n x n matrix has already been considered before[14]. We believe that the expansion of a matrix in terms of the roots of the unit matrix will simplify the problem very much. Since we have the results only for the case of a 2 x 2 matrix, which is very well known, we content ourselves by just giving the results. Any 2 x 2 matrix X can be uniquely expanded as

$$X = l_0 1 + \underline{l}\,\underline{\sigma} \;, \tag{3.C.1}$$

where the σ's are the Pauli matrices forming the algebra C_2^2 along with the unit matrix. If the matrix X has the form

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$l_0 = \frac{1}{2}(a+p) \;, \quad l_3 = \frac{1}{2}(a-d)$$
$$l_1 = \frac{1}{2}(b+c) \;, \quad l_2 = \frac{1}{2}[i(b-c)] \tag{3.C.2}$$

Then it is easy to see that

$$X^m = \frac{1}{2}[(l_0+l)^m + (l_0-l)^m] + \frac{1}{2}\frac{(\underline{l}\cdot\underline{\sigma})}{l}[(l_0+l)^m - (l_0-l)^m], \tag{3.C.3}$$

where $l = (l_1^2 + l_2^2 + l_3^2)^{1/2}$.

14) A. Herpin, Compt. Rend. Acad. Sci. (Paris), 225, 17 (1947). Recently this problem has been studied by R. Bakarat, J. Math. and Phys. 43, 332 (1964); R. Bakarat and E. Baumann, J. Math. Phys. 10, 1474 (1969).

Equation (3.C.3) can also be written as

$$X^m = U_m(p,q)\,X + q\,U_{m-1}(p,q)\,\mathbb{1},$$

$$p = (a+d) = \mathrm{Tr}\,X \quad \text{and} \quad q = (ad-bc) = \det X, \tag{3.C.4}$$

where the U's are the Lucas polynomials given by

$$U_m(p,q) = \frac{1}{2^m (p^2-4q)^{1/2}} \left[[p + (p^2-4q)^{1/2}]^m - [p - (p^2-4q)^{1/2}]^m \right]$$

Of course there are methods of Sylvester using the explicit eigenvalues of X and the method of using the characteristic equation of X. But we hope that the expansion in terms of the roots of the unit matrix can be much simpler, as in the case of $m = 2$ demonstrated above. The simple reason is that the (nontrivial) roots of the unit matrix are traceless matrices, and hence their characteristic is much simpler.

C H A P T E R IV

A CLASS OF LINEAR RELATIVISTIC WAVE EQUATIONS DESCRIBING PARTICLES WITH SPIN 1/2 *

1. Introduction:

In the year 1928 Dirac discovered the relativistic equation which now bears his name[1], describing particles with spin 1/2. In 1936, Dirac extended his relativistic theory of electron to the case of general spin[2]. This theory was investigated in detail by Fierz and Pauli in 1939[3]. Since that time, several types of generalizations of the Dirac equations have been attempted. The generalized equations of Dirac, Fierz and Pauli mentioned above can be written in a linear form only if additional subsidiary conditions are imposed. The existence of three subsidiary conditions has always been a difficulty of the Dirac-Fierz-Pauli formulation. This is particularly marked if we introduce an interaction, say with the electromagnetic field in the conventional way, when the subsidiary conditions become inconsistent with the original

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* P.S.Chandrasekaran, Nalini B. Menon and T.S.Santhanam Prog. Theoret. Phys. 47, 671 (1972).

1) P.A.M.Dirac, Proc. Roy. Soc. A117, 610 (1928)
P.A.M.Dirac, Proc. Roy. Soc. A118, 351 (1928).

2) P.A.M.Dirac, Proc. Roy. Soc. A155, 447 (1936)

3) M.Fierz and W.Pauli, Proc. Roy. Soc. A173, 211 (1939).

equations. To get over this difficulty Bhabha[4] formulated a set of relativistic wave equations. His requirements were that these fundamental equations must be first - order equations and that all properties of the particles described by them must be derivable from the equations themselves without the use of any further subsidiary conditions. However this resulted in multimass solutions for spins greater than 1, that is, the particle has states of higher rest mass which are simple rational multiples of the lowest value of the rest mass. Harischandra[5] tried a linear equation of the type

$$(\beta_\mu \delta_\mu + m)\chi = 0 \qquad (4.1.1)$$

which has no subsidiary conditions, but still describes a particle of unique mass m . He derived minimal β_μ conditions on the matrices β_μ entering into the linear equation (4.1.1) , in order that the equations is relativistically invariant. His condition is simply that

$$\beta_0^{n+1} = \beta_0^{n-1}, \quad n \geq 2. \qquad (4.1.2)$$

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4) H.J. Bhabha, Rev. Mod. Phys. **17**, 200 (1945).

5) Harish Chandra, Phys. Rev. **71**, 793 (1947).

Umezawa and Viscouti[6] showed that n must be equal to 2f where f is the maximum spin contained in the field function. Such an analysis makes it almost obvious that a particle with spin 1/2 is described uniquely by the Dirac equation.

Recently, Capri[7a,b] has obtained an equation for spin 1/2 particles different from and inequivalent to the Dirac equation. He shows that there can exist first-order differential equations other than the Dirac equation that are form invariant under Lorentz transformations, irreducible and derivable from a Lagrangian, and whose solutions correspond to mass m and spin 1/2. The only additional condition satisfied by the Dirac equation is that β_0 is diagonalizable. Capri's argument is that there does not seem to be any sufficiently strong reason why β_0 should be diagonalizable, because if such a condition is imposed, it automatically excludes all equations of the type (4.1.1) for all spins except spin 0, 1/2 and 1. He drops the requirement that β_0 be diagonalizable and obtains a hierarchy of spin 1/2 equations, of which he discusses a particular case in some detail,

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6) H. Umezawa and A. Viscouti, Nucl. Phys. 1, 248 (1956).

7) A. Z. Capri, a) Phys. Rev. 178, 2427 (1969).
b) Phys. Rev. 187, 1811 (1969).

giving an explicit representation for the β -matrices. For this example, he gets four solutions for β_0, of which he throws out two, since the equation satisfied by β_0 in these cases is not minimal. A close look shows that the other two solutions also do not satisfy the minimal condition $\beta_0{}^4 = \beta_0$, but actually obey $\beta_0{}^3 = \beta_0$ and hence is diagonalizable. In fact there are also two more solutions where $\beta_0{}^2 = 1$ and $\beta_0{}^2 = 0$ respectively. From the general considerations of Umezawa and Viscouti[6], it should therefore follow that the equation written by Capri cannot describe a particle of spin 1/2.

However, a re-examination of the work of Umezawa and Viscouti shows that the condition n = 2f is only a special case of the more general inequality

$$2f \geq n \geq 2s,$$

when s = f. Hence we can admit a class of equations to describe a particle of spin 1/2.

As a particular example, we examine the work of Umezawa and Viscouti for the case when f = 3/2 and s = 1/2. In this case, as will be seen later in the actual calculation, the condition $\alpha_{\mu\nu\lambda} = 0$, where $\alpha_{\mu\nu\lambda}$ is the third-order coefficient matrix in the Klein-Gordon divisor[8]

8) H. Umezawa, Quantum Field Theory (North Holland Publishing Company, Amsterdam 1956) Chap. 5, pp. 80-81.

yields three distinct algebras satisfied by the β matrices. One is the Duffin-Kemmer-Petiau algebra[9] describing particles of spin 0 and spin 1. The second we realize to be simply the algebra obeyed by the matrices occurring in the equation given by Capri. There is a third distinct new algebra* again describing a particle of spin 1/2. If, however, we require the existence of a hermitianizing matrix η such that $\eta\beta\eta^{-1} = \beta^{+}$, the latter two algebras coincide and yield a trivial extension of the Dirac algebra as the only admissible solution for a spin 1/2 particle, since a representation of the new algebra is furnished by simply the adjoints of the matrices given by Capri.

2. Algebra of Capri:

Form-invariance of Eqs.(4.1.1) under a homogeneous Lorentz transformation $D(\Lambda)$ with ψ transforming as

$$\psi(x) \longrightarrow \psi'(x') = D(\Lambda)\,\psi(\Lambda^{-1}x') \tag{4.2.1}$$

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* That these two algebras are inequivalent may also be inferred from the fact that a hermitianizing matrix does not exist in this case. See for more details A.R.Tekumalla and T.S.Santhanam (Matscience Preprint).

9) R. J. Duffin, Phys. Rev. 54, 1114 (1938)
N. Kemmer, Proc. Roy. Soc. (London) A173, 91 (1939)
G. Petiau, Thesis, Paris (1936)

requires that the β^μ transform as

$$D(\Lambda)^{-1}\beta^\mu D(\Lambda) = \Lambda^\mu{}_\nu \beta^\nu \qquad (4.2.2)$$

Eqs.(4.2.2) written in terms of the generators of the Lorentz group is just

$$[\beta^\mu, \ I^{\rho\sigma}] = g^{\mu\sigma}\beta^\rho - g^{\mu\rho}\beta^\sigma \qquad (4.2.3)$$

This resolves the problem of finding all equations invariant under homogeneous Lorentz Transformations into the problem of finding all β^μ satisfying the Eqs.(4.2.3). Bhabha[4] has written all the solutions of Eqs.(4.2.3) in terms of the spinor-matrices $u^\alpha(k)$, $v^\beta(k)$, which were first given by Dirac[2] and later studied by Fierz[3]. $u^\alpha(k)$ is a rectangular matrix of dimension $(2k+1)\times 2k$ and $v^\beta(k)$ is of dimension $2k\times(2k+1)$. They were introduced by Dirac in connection with the direct product ~~of~~ $D(\frac{1}{2},\frac{1}{2})\otimes D(k,l)$. A short account of the procedure adopted by Bhabha to get the solutions for β^μ is given in the appendix to this chapter. An explicit representation of $u(k)$ and $v(k)$ is as follows: For k an integer,

$$u_1(k)_{r,s} = (r)^{1/2}\,\delta_{r-1,s}, \qquad v^1(k)_{r,s} = (r+1)^{1/2}\delta_{r+1,s},$$

$$u_2(k)_{r,s} = (2k-r)^{1/2}\delta_{r,s}, \qquad v^2(k)_{r,s} = (2k-r)^{1/2}\delta_{r,s}.$$

For k a half-odd integer,

$$u^1(k)_{r,s} = (r)^{1/2}\delta_{r-1,s}, \qquad v_1(k)_{r,s} = (r+1)^{1/2}\delta_{r+1,s},$$

$$u^2(k)_{r,s} = (2k-r)^{1/2}\delta_{r,s}, \qquad v_2(k)_{r,s} = (2k-r)^{1/2}\delta_{r,s}.$$

Bhabha has obtained the spinor components $\beta^{\alpha\beta}$ of all solutions of (4.2.3) in terms of these spinor matrices. These are of the form

$$\langle (k,l)_s|\beta^{\alpha\beta}|(k+\tfrac{1}{2}, l+\tfrac{1}{2})_t\rangle = c_{st}\, v^{\alpha}(k+\tfrac{1}{2}) \otimes v^{\beta}(l+\tfrac{1}{2})$$

$$\langle (k,l)_s|\beta^{\alpha\beta}|(k+\tfrac{1}{2}, l-\tfrac{1}{2})_t\rangle = c_{st}\, v^{\alpha}(k+\tfrac{1}{2}) \otimes u^{\beta}(l)$$

$$\langle (k,l)_s|\beta^{\alpha\beta}|(k-\tfrac{1}{2}, l+\tfrac{1}{2})_t\rangle = c_{st}\, u^{\alpha}(k) \otimes v^{\beta}(l+\tfrac{1}{2})$$

$$\langle (k,l)_s|\beta^{\alpha\beta}|(k-\tfrac{1}{2}, l-\tfrac{1}{2})_t\rangle = c_{st}\, u^{\alpha}(k) \otimes u^{\beta}(l)$$

(4.2.4)

where the c_{st} are arbitrary coefficients. The indices (k,l) here refer to the indices in $D^{(k,l)}$ labelling the irreducible representations. If one works in a basis in which J^2 is diagonal, then as Wild[10] has

10) E. Wild, Proc. Roy. Soc. A191, 253 (1947).

shown the components of β_0 are given by

$$\langle (k,\ell)_s | \beta_0 | (k-\tfrac{1}{2}, \ell+\tfrac{1}{2})_t \rangle = c_{st} (k+j-\ell)^{1/2} (j+\ell+1-k)^{1/2} \delta_{jj'}, \tag{4.2.5a}$$

$$\langle (k,\ell)_s | \beta_0 | (k-\tfrac{1}{2}, \ell-\tfrac{1}{2})_t \rangle = c_{st} (-1)^{k+\ell+j} (k+\ell+j)^{1/2} (k+\ell+j+1)^{1/2} \delta_{jj'} \tag{4.2.5b}$$

$$\langle (k, k+\tfrac{1}{2})_s | \beta_0 | (k+\tfrac{1}{2}, k)_t \rangle = c_{st} (-1)^{\{j\}+1} (j+\tfrac{1}{2}) \delta_{jj'} \tag{4.2.5c}$$

where $\ell \neq k$ and $\ell' \neq k'$,

$\{j\}$ = integral part of j ,

$$|k-\ell| \leq j' \leq |k+\ell|$$

For $\ell = k$ or $\ell' = k'$ we have

$$\langle (k-\tfrac{1}{2}, k+\tfrac{1}{2})_s | \beta_0 | (k, k')_t \rangle = c_{st} (-1)^{2k+j+1} (j)^{1/2} (j+1)^{1/2} \delta_{jj'} \tag{4.2.5d}$$

All the other components of β_0 are obtained from

$$\langle k, \ell | \beta_0 | k', \ell' \rangle = (-1)^{2k+2} \langle k', \ell' | \beta_0 | k, \ell \rangle \tag{4.2.5e}$$

and

$$\langle k, \ell | \beta_0 | k', \ell' \rangle = -\langle \ell, k | \beta_0 | \ell', k' \rangle . \qquad (4.2.5f)$$

In order to obtain the solutions for any half-odd-integral spin Capri[7a] uses a prepresentation of the homogeneous Lorentz transformation which contains as its highest spin the value of the required spin and then eliminates the lower values of the spin by suitably imposing the necessary conditions on the matrix β_0 . On the other hand, in order to arrive at a hierarchy of linear equations inequivalent to the Dirac equation, he reverses this procedure; that is, using a representation containing a maximum spin $\geqslant$ 1/2 and then eliminates the higher components of spin. If the maximum spin is 1/2 what one obtains is just the Dirac equation. The next possibility is when the highest spin contained in the representation is 3/2. In this case D has the form

$$D = (1, \tfrac{1}{2}) \oplus (0, \tfrac{1}{2}) \oplus (\tfrac{1}{2}, 0) \oplus (\tfrac{1}{2}, 1) \qquad (4.2.6)$$

7a) A.Z.Capri, Phys. Rev. 178, 2427 (1969).

If we want to construct the β matrices occurring in Eqs.(4.1.1) for a particle with spin 3/2, β_0 should satisfy the following conditions:

$$\beta_0{}^4 = \beta_0{}^2 \tag{4.2.7a}$$

$$J^2 \beta_0^2 = 3/2\,(3/2+1)\,\beta_0{}^2 \tag{4.2.7b}$$

where J^2 is the square of the generator of rotations in three dimensions. The explicit representation of β_0 has been given by Capri[7a]. Condition (4.2.7b) eliminates the spin 1/2 component from the mixture of spins 3/2 and 1/2.

On the other hand to get a class of linear relativistic wave equations for spin 1/2 particles, Capri eliminates the spin 3/2 component by requiring β_0 to satisfy

$$J^2 \beta_0{}^2 = 1/2\,(1/2+1)\,\beta_0{}^2 \tag{4.2.8}$$

In addition, of course, β_0 should satisfy the minimal condition

$$\beta_0{}^4 = \beta_0{}^2 \tag{4.2.9}$$

Because of the choice of the representation D as in Eqs.(4.2.6) β_0^2 appears in block-diagonal form with the different blocks being labelled by the values of J^2. Here the block corresponding to spin 3/2 is made nilpotent and that corresponding to spin 1/2 is required to satisfy the minimal equation (4.2.9). This imposes some conditions on the coefficients c_{st} occurring in β_0. Four possible solutions are possible, of which two are rejected since they do not satisfy (4.2.9) minimally. Actually these two solutions are trivial, since for one of them, β_0 becomes equal to the null matrix and for the other, β_0 is just the unit matrix. It is however found that the other two solutions again do not satisfy Eqs.(4.2.9) minimally. Indeed we realize that the matrices constructed by Capri obey the following equation:

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\mu \beta_\lambda \beta_\nu = 2 g_{\nu\lambda} \beta_\mu ; \quad \mu, \nu, \lambda = 1,2,3,4, \tag{4.2.10}$$

and hence

$$\beta_\mu^3 = g_{\mu\mu} \beta_\mu ; \quad g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{4.2.11a}$$

$$\beta_\mu \beta_\lambda^2 = g_{\lambda\lambda} \beta_\mu , \quad \lambda \neq \mu ; \tag{4.2.11b}$$

$$\beta_\mu \beta_\nu \beta_\lambda = - \beta_\mu \beta_\lambda \beta_\nu , \quad \mu \neq \nu \neq \lambda \tag{4.2.11c}$$

and

$$\beta_\mu \beta_\nu \beta_\mu = - \beta_\mu^{\,2} \beta_\nu \quad , \quad \mu \neq \nu \tag{4.2.11d}$$

In these four equations (4.2.11a-d), there is no summation over repeated indices. These equations are easily obtained by using the explicit representation of the β_μ . These matrices are of dimension 16 x 16. We give below the explicit form of these matrices as obtained by Capri:

$$\beta_0 = \left[\begin{array}{cccccccc|cc|cccccc}
 & & & & & & & & 0 & 0 & & & & & & \\
 & & & & & & & & \sqrt{2}c & 0 & & & & & & \\
 & & & & & & & & -c & 0 & & & & & & \\
 & & & & & & & & 0 & c & & & & & & \\
 & & & & & & & & 0 & -\sqrt{2}c & & & & & & \\
 & & & & & & & & 0 & 0 & & & & & & \\
\hline
 & & & & & & & & -1 & 0 & & & & & & \\
 & & & & & & & & 0 & -1 & & & & & & \\
\hline
 & & & & & & -1 & 0 & & & & & & & & \\
 & & & & & & 0 & -1 & & & & & & & & \\
\hline
 & & & & & & 0 & 0 & & & & & & & & \\
 & & & & & & -c & 0 & & & & & & & & \\
 & & & & & & 0 & -\sqrt{2}c & & & & & & & & \\
 & & & & & & \sqrt{2}c & 0 & & & & & & & & \\
 & & & & & & 0 & c & & & & & & & & \\
 & & & & & & 0 & 0 & & & & & & & &
\end{array}\right] \tag{4.2.12a}$$

$$\beta_1 = \left[\begin{array}{cc} & \begin{array}{cc}\sqrt{2}c & 0\\ 0 & 0\\ 0 & c\\ -c & 0\\ 0 & 0\\ 0 & -\sqrt{2}c\\ \hline 0 & -1\\ -1 & 0\end{array} \\ \hline \begin{array}{cc} 0 & 1\\ 1 & 0\\ \hline -\sqrt{2}c & 0\\ 0 & -c\\ 0 & 0\\ 0 & 0\\ c & 0\\ 0 & \sqrt{2}c\end{array} & \end{array}\right] \tag{4.2.12b}$$

$$\beta_2 = \left[\begin{array}{cc} & \begin{array}{cc}\sqrt{2}ic & 0\\ 0 & 0\\ 0 & ic\\ ic & 0\\ 0 & 0\\ 0 & \sqrt{2}ic\\ \hline 0 & -i\\ i & 0\end{array} \\ \hline \begin{array}{cc} 0 & i\\ -i & 0\\ \hline -\sqrt{2}ic & 0\\ 0 & -ic\\ 0 & 0\\ 0 & 0\\ -ic & 0\\ 0 & -\sqrt{2}ic\end{array} & \end{array}\right] \tag{4.2.12c}$$

$$\beta_3 = \left[\begin{array}{c|c} & \begin{array}{cc} a & 0 \\ -\sqrt{2}c & 0 \\ -c & 0 \\ 0 & -c \\ 0 & -\sqrt{2}c \\ 0 & 0 \\ \hline -1 & 0 \\ 0 & -1 \end{array} \\ \hline \begin{array}{cc} -1 & 0 \\ 0 & -1 \\ \hline 0 & 0 \\ c & 0 \\ 0 & \sqrt{2}c \\ \sqrt{2}c & 0 \\ 0 & c \\ 0 & 0 \end{array} & \end{array}\right] \tag{4.2.12d}$$

From the Eqs. (4.2.11) it is therefore obvious that if the condition of Umezawa and Viscouti is strictly followed then the solutions obtained for β_μ cannot admit an equation describing a particle of spin 1/2, contrary to what Capri has envisaged. On the other hand, we show in the next section that the Umezawa-Viscouti condition is only a special case of a more general inequality in the spirit of wave equations constructed by Capri.

3. The Umezawa-Viscouti condition.

Let us now briefly go through the proof given by Umezawa and Viscouti to show that $n = 2f$. The Klein-Gordon divisor $d(\partial)$ is defined as that operator which when made to operate on a linear equation of the type (4.1.1) reduces it to the Klein-Gordon equation. $d(\partial)$ is assumed to be a polynomial in the derivative operators such that

$$d(\partial)\,\Lambda(\partial) = (\Box - m^2)\,I \tag{4.3.1}$$

where

$$\Lambda(\partial) = -(\beta_\mu \partial_\mu + m), \tag{4.3.2}$$

and $d(\partial)$ is defined as

$$\begin{aligned} d(\partial) &= \alpha_0 + \alpha_\mu \partial_\mu + \alpha_{\mu\nu}\partial_\mu \partial_\nu + \cdots \\ &= \sum_{l=0}^{L} \alpha_{\mu_1 \cdots \mu_l}\, \partial_{\mu_1 \cdots \mu_l} \end{aligned} \tag{4.3.3}$$

It can be assumed, without any loss of generality, that the $\alpha_{\mu_1 \cdots \mu_l}$ are symmetrical with respect to the exchange of their indices. $d(\partial)$ transforms like $\psi \otimes \psi$ where ψ is the field function which satisfies the

Klein-Gordon equation, and hence it contains spins 2f, 2f-1, ... , where f is the maximum spin contained in the field function. Eqs. (4.3.3) can be written as

$$d(\partial) = \sum_{\ell=0}^{2f} \alpha_{\mu_1 \ldots \mu_\ell} \partial_{\mu_1 \ldots \mu_\ell} + \sum_{\ell'=2f+1}^{L} \alpha_{\mu_1 \ldots \mu_{\ell'}} \partial_{\mu_1} \ldots \partial_{\mu_{\ell'}}$$

The terms for which $\ell > 2f$ can be regrouped as

$$\cancel{d(\partial) = \Sigma}$$

$$\alpha'_{\mu_1 \ldots \mu_\ell} \partial_{\mu_1} \ldots \partial_{\mu_{2f}} (\Box)^{(\ell-2f)/2} = \alpha_{\mu_1 \ldots \mu_\ell} \partial_{\mu_1 \ldots \mu_\ell} ; \quad \ell > 2f,$$

since 2f is the maximum spin in $d(\partial)$ and the rest of the terms can only contribute to a power of $\Box$. Obviously, $\alpha_{\mu_1 - \mu_\ell}$ must be zero for $\ell - 2f$ odd. From Eqs. (4.3.1), it follows that

$$\alpha_0 m = m^2 I ,$$

$$(\alpha_0 \beta_\mu + m \alpha_\mu) = 0,$$

$$(\alpha_\mu \beta_\nu + \alpha_\nu \beta_\mu) + 2m\alpha_{\mu\nu} = -2g_{\mu\nu}, \qquad (4.3.4)$$

$$S(\beta_{\mu_1} \alpha_{\mu_2 \ldots \mu_\ell} - m\alpha_{\mu_1 \ldots \mu_\ell}) = 0 \text{ for } \ell > 2$$

where S denotes the summation over terms given by taking all possible permutations over the suffixes. The solutions of the equations (4.3.4) can be worked out to give

$$\begin{aligned} \alpha_0 &= m I \\ \alpha_\mu &= -\beta_\mu \\ \alpha_{\mu\nu} &= \frac{1}{m} g_{\mu\nu} - \frac{1}{2m} S \beta_\mu \beta_\nu \\ &\cdots\cdots\cdots \\ \alpha_{\mu_1 \cdots \mu_\ell} &= \frac{1}{m^{\ell-1}} \cdot \frac{1}{\ell!} S \beta_{\mu_1} \cdots \beta_{\mu_{\ell-2}} \left(\delta_{\mu_{\ell-1},\mu_\ell} - \beta_{\mu_{\ell-1}} \beta_{\mu_\ell} \right) \end{aligned} \tag{4.3.5}$$

Thus we have

$$d(\partial) = m + \beta_\mu \partial_\mu + \frac{1}{m}\left(\Box - \beta_\mu \beta_\nu \partial_\mu \partial_\nu \right) + \cdots \tag{4.3.6}$$

Using these equations, it follows that

$$\alpha_{\mu_1 \cdots \mu_\ell} = 0 \; ; \quad \ell > 2f \tag{4.3.7}$$

even when $(\ell - 2f)$ is even. Hence, the polynomial $d(\partial)$ should terminate at $L \leq 2f$. If, in addition, we require the field function to contain the maximum spin f, $\alpha_{\mu_1 \cdots \mu_{2f}} = 0$. Hence $L = 2f$. This is the proof of Umezawa and Visconti. On the other hand, if we project spin $s < f$ contained in a field function with maximum spin f, we have the inequality

$$2s \leq L \leq 2f \tag{4.3.8}$$

The Harish Chandra condition on β_0 becomes

$$\beta_0^{L+1} = \beta_0^{L-1} \tag{4.3.9}$$

In the spirit of the above discussion it follows therefore that the approach of Capri is fully justified.

In the next section, we shall discuss the implications of Eqs. (4.3.8) in the light of the structure of the Klein-Gordon division.

4. The new algebra:

Let us choose the sequence of representations $(1,\tfrac{1}{2})\oplus(0,\tfrac{1}{2})\oplus(\tfrac{1}{2},0)\oplus(\tfrac{1}{2},1)$ which contains a maximum spin of 3/2 in addition to a spin 1/2 components. We have in this case

$$d(\partial) = m + \beta_\mu \partial_\mu + \alpha_{\mu\nu}\partial_\mu\partial_\nu + \alpha_{\mu\nu\lambda}\partial_\mu\partial_\nu\partial_\lambda \qquad (4.4.1)$$

as the other terms vanish in view of Eqs. (4.3.7). If the field function should have spin 3/2, then $\alpha_{\mu\nu\lambda} \neq 0$. On the other hand if $\alpha_{\mu\nu\lambda} = 0$, we find from Eqs. (4.3.4) that

$$S(\beta_\mu \alpha_{\nu\lambda} - m\,\alpha_{\mu\nu\lambda}) = 0 \qquad (4.4.2)$$

and hence from the third of Eqs. (4.3.4)

$$S\,\beta_\mu \alpha_{\nu\lambda} = -\frac{1}{2m}\, S\,\beta_\mu\,(2g_{\nu\lambda} - \{\beta_\nu, \beta_\lambda\}_+) = 0 .$$

Hence

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\mu \beta_\lambda \beta_\nu + \beta_\nu \beta_\mu \beta_\lambda + \beta_\nu \beta_\lambda \beta_\mu + \beta_\lambda \beta_\mu \beta_\nu + \beta_\lambda \beta_\nu \beta_\mu = 2g_{\mu\nu}\beta_\lambda + 2g_{\mu\lambda}\beta_\nu + 2g_{\nu\lambda}\beta_\mu \tag{4.4.3}$$

Equation (4.4.3) admits three distinct algebras of the Duffin-Kemmer-Petiau type obeyed by the β -matrices. It has been pointed out by Harish Chandra[5]) that, by itself, the commutation relation (4.4.3) will not generate a finite algebra. In order to make the algebra finite, a stronger condition must be imposed on the β_μ . There are open to us three possible ways of imposing such a restrictive condition which is at the same time consistent with Eqs. (4.4.3). The first leads to the Duffin-Kemmer-Petiau algebra,

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu \beta_\mu = g_{\mu\nu}\beta_\lambda + g_{\lambda\nu}\beta_\mu , \tag{4.4.4}$$

describing particles with spin 1 and 0. The second is the algebra obeyed by the matrices of Capri.

$$\beta_\mu (\beta_\lambda \beta_\nu + \beta_\nu \beta_\lambda) = 2g_{\lambda\nu}\beta_\mu$$
$$\beta_\lambda \beta_\nu + \beta_\nu \beta_\lambda \neq 2g_{\lambda\nu} \tag{4.4.5}$$

We find that a third new algebra is also possible:

$$\begin{aligned} (\beta_\lambda\beta_\nu + \beta_\nu\beta_\lambda)\beta_\mu &= 2g_{\lambda\nu}\beta_\mu, \\ \beta_\lambda\beta_\nu + \beta_\nu\beta_\lambda &\neq 2g_{\lambda\nu} \end{aligned} \tag{4.4.6}$$

Both (4.4.5) and (4.4.6) describe particles with spin 1/2. This can be demonstrated by constructing the J^2 operator as Capri has done and showing that $J^2\beta^2 = \frac{1}{2}(\frac{1}{2}+1)\beta^2$ where J^2 is the square of the generator of rotations in three dimensions. That these are the three algebras of the DKP type can be seen as follows. Since the highest non-vanishing term in $\alpha(\partial)$, namely $\alpha_{\mu\nu}\partial_\mu\partial_\nu$, transforms like a spin 2 object $(\partial_\mu \sim (\frac{1}{2},\frac{1}{2}))$ and since $\alpha(\partial)$ transforms like $\psi \otimes \psi$ where ψ is the field function, ψ can either be a combination of spins 1 and 0, which yields the Duffin-Kemmer-Petiau algebra (4.4.4) or it can be a combination of spins 3/2 and 1/2, which yields the Capri and the new algebra. Of course Eqs. (4.4.3) itself generates an (infinite) algebra if there are no subalgebras of the Duffin-Kemmer-Petiau type. A representation of the new algebra (4.4.6) is furnished by the hermitian adjoints of the matrices given by Capri.

The procedure discussed here is not altogether new, since we are used to a spin 0 particle described by a Duffin-Kemmer-Petiau algebra with β_μ and $\alpha_{\mu\nu} \neq 0$. In conclusion it thus looks remarkable that the algebra of β -matrices for a particle with spin s seems to remember the parentage of maximum spin in the choice of the representation. In fact by choosing representations with higher spins, for instance with $\alpha_{\mu\nu\lambda\tau\sigma} = 0$ and $\alpha_{\mu\nu\lambda\tau} \neq 0$, we can get other equations, still describing a particle with spin 1/2. As Capri has already envisaged, all these equations describing a spin 1/2 particle (except Dirac's) lead to non-renormalizable electrodynamics and therefore are inequivalent to the Dirac equation in the presence of an interaction. These different equations can possibily be used to describe the electron and the muon, whose difference is very mysterious.

In the next chapter, we shall calculate the magnetic moment of a particle described by the new equation.

Appendix A:

We indicate briefly the procedure followed by Bhabha to obtain the solutions of Eqs. (4.2.3), i.e. the equation

$$[\beta^\mu, M^{\rho\sigma}] = g^{\mu\sigma}\beta^\rho - g^{\mu\rho}\beta^\sigma \qquad (4.A.1)$$

Bhabha starts by noting that with each matrix β^μ can be connected a spinor $\beta^{\alpha\dot\beta}$ and vice versa through the equations

$$\beta^{\alpha\dot\beta} = \beta^\mu \sigma_\mu{}^{\alpha\dot\beta}, \quad \beta^\mu = \tfrac{1}{2}\sigma^\mu{}_{\dot\beta\alpha}\,\beta^{\alpha\dot\beta} \tag{4.A.2}$$

where $\{\sigma_0{}^{\alpha\dot\beta}, \sigma_1{}^{\alpha\dot\beta}, \sigma_2{}^{\alpha\dot\beta}, \sigma_3{}^{\alpha\dot\beta}\}$ is just the set of the 2 x 2 unit matrix and the three Pauli matrices, the rows and columns of which have been labelled by the spinor indices α and $\dot\beta$. Spinor indices take on only the values 1 and 2. Raising and lowering of spinor indices is carried out by the antisymmetric spinors $\epsilon_{\mu\nu}$ and $\epsilon^{\mu\nu}$ according to

$$b_\mu = \epsilon_{\mu\nu} b^\nu, \quad b^\nu = b_\mu \epsilon^{\mu\nu} \tag{4.A.3}$$

where $\epsilon_{12} = -\epsilon_{21} = 1, \quad \epsilon_{11} = \epsilon_{22} = 0, \quad \epsilon^{12} = -\epsilon^{21} = 1, \quad \epsilon^{11} = \epsilon^{22} = 0.$

Antisymmetric spinors $\epsilon_{\dot\mu\dot\nu}$ and $\epsilon^{\dot\mu\dot\nu}$ for raising and lowering dotted indices are defined similarly. Now, an antisymmetric spinor $M^{\rho\sigma}$ can be connected with two symmetric spinors $K^{\mu\nu}$ and $L^{\dot\mu\dot\nu}$ by

$$4K_\nu^\mu = -M_{\rho\sigma}\sigma^{\rho\mu\dot\lambda}\sigma_{\dot\lambda\nu}{}^\sigma \tag{4.A.4a}$$

$$4L_{\dot\mu}{}^{\dot\lambda} = M_{\rho\sigma}\sigma_{\dot\mu\nu}{}^\rho \sigma^{\sigma\nu\dot\lambda} \tag{4.A.4b}$$

Expanding Eqs. (4.A.4) leads to the following two sets of equations

$$K_1^{\;1} = -K_2^{\;2} = K_z, \quad K_2^{\;1} = K_x - iK_y, \quad K_1^{\;2} = K_x + iK_y, \tag{4.A.5a}$$

$$L_{\dot{1}}^{\;\dot{1}} = -L_{\dot{2}}^{\;\dot{2}} = -L_z, \quad L_{\dot{2}}^{\;\dot{1}} = -L_x - iL_y, \quad L_{\dot{1}}^{\;\dot{2}} = -L_x + iL_y, \tag{4.A.5b}$$

where K_x, K_y, K_z and L_x, L_y, L_z have been defined as

$$K_x = \tfrac{1}{2}(iM^{23} + M^{01}), \quad K_y = \tfrac{1}{2}(iM^{31} + M^{02}), \quad K_z = \tfrac{1}{2}(iM^{12} + M^{03}) \tag{4.A.6a}$$

$$L_x = \tfrac{1}{2}(iM^{23} - M^{01}), \quad L_y = \tfrac{1}{2}(iM^{31} - M^{02}), \quad L_z = \tfrac{1}{2}(iM^{12} - M^{03}). \tag{4.A.6b}$$

When the M's are the infinitesimal transformations of a representation of the Lorentz group, the K's and L's form two sets of matrices of which the matrices of one set commute with those of the other. Also, the members of each set obey among themselves the commutation rules of angular momentum operators, that is

$$[K_x, K_y] = iK_z, \quad [K_y, K_z] = iK_x, \quad [K_z, K_x] = iK_y \tag{4.A.7}$$

and similarly for (L_x, L_y, L_z) . $K^2 = K_x^2 + K_y^2 + K_z^2$ and $L^2 = L_x^2 + L_y^2 + L_z^2$ commute with all the six M's and have values k(k+1) and $\ell(\ell+1)$ respectively, if we consider the irreducible representation $M^{\rho\sigma}(k,\ell)$ of the generators corresponding to the representation $D(k,\ell)$ of the homogeneous Lorentz transformation. The K's and L's have representations of degree (2k+1) and $(2\ell+1)$, and their eigenvalues run from $k, k-1, \ldots, -k+1, -k,$ and $\ell, \ell-1, \ldots, -\ell+1, -\ell,$ respectively. Choosing K_z to be diagonal and labelling the rows and columns by m, where m takes values from k to -k, the matrix elements of the k's are given by

$$
\begin{aligned}
(m|K_z|m) &= m \\
(m+1|K_x + iK_y|m) &= [(k-m)(k+m+1)]^{1/2} \\
(m-1|K_x - iK_y|m) &= [(k+m)(k-m+1)]^{1/2}
\end{aligned}
\tag{4.A.8}
$$

Now, the matrices $u^{\rho}(k)$ and $v^{\rho}(k)$ satisfy

$$
\begin{aligned}
-u_\mu(k+\tfrac{1}{2})\, v^\mu(k+\tfrac{1}{2}) &= v_\mu(k)\, u^\mu(k) = 2k+1, \\
v_\mu(k)\, v^\mu(k+\tfrac{1}{2}) &= u_\mu(k+\tfrac{1}{2})\, u^\mu(k) = 0, \\
-v^\mu(k+\tfrac{1}{2})\, u_\nu(k+\tfrac{1}{2}) &= K_\nu{}^\mu(k) + (k+1)\,\delta_\nu^\mu, \\
-u^\mu(k)\, v_\nu(k) &= K_\nu{}^\mu(k) - k\,\delta_\nu^\mu
\end{aligned}
\tag{4.A.9}
$$

Correspondingly, one has similar equations for $u^{\dot\mu}(\ell)$ and $v^{\dot\mu}(\ell)$ with K(k) replaced by $L(\ell)$. From (4.A.9) the following two equations can be deduced

$$u^{\rho}(k)\, K^{\mu\nu}(k-\tfrac{1}{2}) - K^{\mu\nu}(k)\, u^{\rho}(k) = \frac{1}{2}\epsilon^{\rho\mu} u^{\nu}(k) + \frac{1}{2}\epsilon^{\rho\nu} u^{\mu}(k)$$

(4.A.10)

$$v^{\rho}(k)\, K^{\mu\nu}(k) - K^{\mu\nu}(k-\tfrac{1}{2})\, u^{\rho}(k) = \tfrac{1}{2}\epsilon^{\rho\mu} v^{\nu}(k) + \frac{1}{2}\epsilon^{\rho\nu} v^{\mu}$$

Now consider Eqs.(4.A.1). With some algebraic manipulations and using Eqs.(4.A.2) and (4.A.4) we can deduce from this equation the following relations:

$$2[\beta^{\beta\dot\lambda}, K^{\nu\rho}] = \epsilon^{\beta\nu}\beta^{\rho\dot\lambda} + \epsilon^{\beta\rho}\beta^{\nu\dot\lambda}$$

$$2[\beta^{\beta\dot\lambda}, L^{\dot\mu\dot\nu}] = \epsilon^{\dot\lambda\dot\mu}\beta^{\beta\dot\nu} + \epsilon^{\dot\lambda\dot\nu}\beta^{\beta\dot\mu} \qquad (4.A.11)$$

The matrices $M^{\rho\sigma}$ can be written in the form

$$\begin{bmatrix} M^{\rho\sigma}(k_1,\ell_1) & & & \\ & M^{\rho\sigma}(k_2,\ell_2) & & \\ & & M^{\rho\sigma}(k_3,\ell_3) & \\ & & & \ddots \end{bmatrix} \qquad (4.A.12)$$

Corresponding to this representation of $M^{\rho\sigma}$, both $K^{\nu\rho}$ and $L^{\dot\mu\dot\nu}$ take the same form. The matrices $\beta^{\beta\dot\lambda}$ are also divided into blocks corresponding to the blocks in $M^{\rho\sigma}$, these blocks being labelled as $(k_s, l_s | \beta^{\beta\dot{\lambda}} | k_t, l_t)$.

Then Eqs. (4.A.11) just become

$$(k_s, l_s | \beta^{\beta\dot{\lambda}} | k_t, l_t) K^{\gamma\rho}(k_t) - K^{\gamma\rho}(k_s)(k_s, l_s | \beta^{\beta\dot{\lambda}} | k_t, l_t)$$

$$= \frac{1}{2} \epsilon^{\beta\gamma} (k_s, l_s | \beta^{\rho\dot{\lambda}} | k_t, l_t) + \frac{1}{2} \epsilon^{\beta\rho} (k_s, l_s | \beta^{\gamma\dot{\lambda}} | k_t, l_t) \quad (4.A.13)$$

and

$$(k_s, l_s | \beta^{\beta\dot{\lambda}} | k_t, l_t) L^{\dot{\mu}\dot{\nu}}(l_t) - L^{\dot{\mu}\dot{\nu}}(l_s)(k_s, l_s | \beta^{\beta\dot{\lambda}} | k_t, l_t)$$

$$= \frac{1}{2} \epsilon^{\dot{\lambda}\dot{\mu}} (k_s, l_s | \beta^{\beta\dot{\nu}} | k_t, l_t) + \frac{1}{2} \epsilon^{\dot{\lambda}\dot{\nu}} (k_s, l_s | \beta^{\beta\dot{\mu}} | k_t, l_t)$$

(4.A.14)

Because the index λ of $\beta^{\beta\dot{\lambda}}$ remains unaffected in (4.A.13), while index β remains unaffected in (4.A.14), the general solution may be written as a direct product of two rectangular matrices of dimensions $(2k_s+1) \times (2k_t+1)$ and $(2l_s+1) \times (2l_t+1)$ respectively, i.e., we may write

$$(k_s, l_s | \beta^{\beta\dot{\lambda}} | k_t, l_t) = (k_s | \beta^{\beta} | k_t)(l_s | \beta^{\dot{\lambda}} | l_t) \quad (4.A.15)$$

Therefore (4.A.13) reduces to

$$(k_s | \beta^{\beta} | k_t) K^{\gamma\rho}(k_t) - K^{\gamma\rho}(k_s)(k_s | A^{\beta} | k_t)$$

$$= \frac{1}{2} \epsilon^{\beta\gamma} (k_s | A^{\rho} | k_t) \quad (4.A.16)$$

$$+ \frac{1}{2} \epsilon^{\beta\rho} (l_s | A^{\dot{\lambda}} | l_t)$$

and a similar equation holds for $(l_s \,|\, A^{\dot{\lambda}} \,|\, l_t)$.

Comparing this with Eqs. (4.A.10) it is clear that they are exactly of the same form if $k_t = k_s \pm 1/2$. It can be shown that $(k_s | \beta^{\beta} | k_t) = 0$ unless $k_t = k_s \pm 1/2$. A similar result holds for $(l_s | \beta^{\dot{\lambda}} | l_t)$. This indicates that the matrix elements of the spinor components $\beta^{\beta\dot{\lambda}}$ may be written as proportional to direct products of the u's and v's; they are therefore given by

$$(k, l \,|\, \beta^{\beta\dot{\lambda}} \,|\, k+1/2, l-1/2) = c\, v^{\beta}(k+1/2)\, u^{\dot{\lambda}}(l),$$

$$(k+1/2, l-1/2 \,|\, \beta^{\beta\dot{\lambda}} \,|\, k, l) = d\, u^{\beta}(k+1/2)\, v^{\dot{\lambda}}(l),$$

$$(k, l \,|\, \beta^{\beta\dot{\lambda}} \,|\, k+1/2, l+1/2) = c'\, v^{\beta}(k+1/2)\, v^{\dot{\lambda}}(l+1/2),$$

$$(k+1/2, l+1/2 \,|\, \beta^{\beta\dot{\lambda}} \,|\, k, l) = d'\, u^{\beta}(k+1/2)\, u^{\dot{\lambda}}(l+1/2).$$

CHAPTER V

SPIN 1/2 PARTICLE IN ELECTROMAGNETIC FIELD *

1. Introduction:

In the last chapter we studied one particular case of a hierarchy of linear spin-1/2 equations inequivalent to the Dirac equation, which has been obtained by Capri[1] and we found that there exist two possible algebras which the matrices β_μ entering into the equation can obey. One of these is the algebra obeyed by the matrices given by Capri himself. The other, we saw, was a new algebra which is distinct from the first so long as the matrices are not hermitian. In this chapter we consider the spin-1/2 wave equation involving matrices satisfying the second algebra and derive the solutions of this equation in the absence of any interaction. Then the equation with a minimal electromagnetic interaction put in is studied and the magnetic moment calculated. It is shown that the expectation value of the magnetic moment operator is exactly the same as in the Dirac case, in other words, just $e\hbar/2mc$.

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* Nalini B. Menon (to be published)

1) A.Z.Capri, Phys. Rev. 187,(1969), 1811.

2. Free-field solutions:

The equation under consideration is of the form

$$(\beta_\mu p_\mu + m)\psi = 0, \qquad (5.2.1)$$

where the β's satisfy

$$\begin{aligned}(\beta_\lambda\beta_\nu + \beta_\nu\beta_\lambda)\beta_\mu &= 2g_{\lambda\nu}\beta_\mu, \\ (\beta_\lambda\beta_\nu + \beta_\nu\beta_\lambda) &\neq 2g_{\lambda\nu}\end{aligned} \qquad (5.2.2)$$

The matrix representation for the β's is just given by the hermitian conjugates of the matrices given by Capri (see Chapter IV for the explicit form) and these matrices are of dimension 16 x 16. The Klein-Gordon divisor[2] in this case is given by

$$\Lambda(p) = [m - \beta_\mu p_\mu + \frac{1}{2m}(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu - 2g_{\mu\nu})p_\mu p_\nu \qquad (5.2.3)$$

as is easily seen from Eqs.(4.3.3) and (4.3.5). We now operate with $\Lambda(p)$ on Eqs.(5.2.1) to get the Klein-Gordon equation

$$[m - \beta_\mu p_\mu + \frac{1}{2m}(\beta_\mu\beta_\nu + \beta_\mu\beta_\nu - 2g_{\mu\nu})p_\mu p_\nu][\beta_\lambda p_\lambda + m]\psi = 0$$

2) H. Umezawa, Quantum Field Theory (North Holland Publishing Company, Amsterdam, 1956) Chap. 5, pp. 80-81.
Y. Takahashi, An introduction to Field Quantization (Pergamon Press, 1969) pp. 92-95.

i.e.

$$\Big[m^2 - \beta_\mu \beta_\lambda p_\mu p_\lambda + \frac{1}{2m}(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu - 2g_{\mu\nu})\beta_\lambda p_\mu p_\nu p_\lambda$$

$$+ \frac{1}{2}(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu - 2g_{\mu\nu})p_\mu p_\nu\Big]\psi = 0$$

which reduces to just

$$[m^2 - g_{\mu\nu}p_\mu p_\nu]\psi = 0 \tag{5.2.4}$$

on making use of Eqs. (5.2.2) and noting that $\frac{1}{2}(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu)p_\mu p_\nu$ is just equal to $\beta_\mu\beta_\nu p_\mu p_\nu$ . . Therefore we have

$$m^2 - p_0^2 + \not{p}^2 = 0$$

where

$$\not{p}^2 = p_1^2 + p_2^2 + p_3^2, \text{ or}$$

$$p_0 = \pm\sqrt{m^2 + \not{p}^2} \equiv \pm E \tag{5.2.5}$$

To obtain the solutions of the equations, we use the actual forms of the matrices and write Eq. (5.2.1) in full as follows

$$\begin{array}{c} \\ 6 \\ 4 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccc} 6 & 4 & 6 \end{array} \\ \left[\begin{array}{c|c|c} mI & 0 & 0 \\ \hline C_1 & D & C_2 \\ \hline 0 & 0 & mI \end{array}\right] \end{array} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{15} \\ \psi_{16} \end{bmatrix} = 0 \tag{5.2.6}$$

where the dimensions of the different blocks are as indicated and C_1, D, C_2 are given by

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}p_- & \sqrt{2}E_- & -cE_+ & -cE_+ & 0 & 0 \\ 0 & 0 & 0 & cE_- & -\sqrt{2}cE_+ & -\sqrt{2}cp_+ \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -\sqrt{2}cp_- & -cE_- & 0 & \sqrt{2}cE_+ & cp_+ & 0 \\ 0 & -cp_- & -\sqrt{2}cE_- & 0 & cE_+ & \sqrt{2}cp_+ \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

and

$$D = \begin{bmatrix} m & 0 & -E_- & p_+ \\ 0 & m & p_- & -E_+ \\ -E_+ & -p_+ & m & 0 \\ -p_- & -E_- & 0 & m \end{bmatrix}$$

Here the notation $p_\pm = p_1 \pm i p_2$, $E_\pm = p_0 \pm p_3$ has been used. Provided $m \neq 0$, Eq.(5.2.6) implies that $\psi_i = 0$ for $i = 1, 2, \ldots, 6, 11, 12, \ldots 16$.

Therefore Eq.(5.2.6) essentially reduces to the following equation for just four relevant components

$$\begin{bmatrix} m & 0 & -E_- & p_+ \\ 0 & m & p_- & -E_+ \\ -E_+ & -p_+ & m & 0 \\ -p_- & -E_- & 0 & m \end{bmatrix} \begin{bmatrix} \psi_7 \\ \psi_8 \\ \psi_9 \\ \psi_{10} \end{bmatrix} = 0 \qquad (5.2.7)$$

The matrix block operating on $\begin{bmatrix} \psi_7 \\ \psi_8 \\ \psi_9 \\ \psi_{10} \end{bmatrix}$ is seen to be just the Dirac Hamiltonian. Eq. (5.2.7) consists of four equations of which only two are independent. We choose ψ_7 and ψ_8 as the two independent solutions and it is easily calculated that

$$\psi_9 = \frac{E+p_3}{m}\psi_7 + \frac{p_1+ip_2}{m}\psi_8 ,$$

$$\psi_{10} = \frac{p_1-ip_2}{m}\psi_7 + \frac{E-p_3}{m}\psi_8 .$$

Thus we get the solutions for positive energy $(p_0 = +E)$ after normalizing to $u^+u = 1$ as

$$u_I = \frac{1}{\sqrt{2E(E+p_3)}} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m \\ 0 \\ E+p_3 \\ p_1-ip_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ; \qquad u_{II} = \frac{1}{\sqrt{2E(E-p_3)}} \times \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ m \\ p_1+ip_2 \\ E-p_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ; \qquad (5.2.8)$$

where $\begin{pmatrix}\chi_7\\ \psi_8\end{pmatrix}$ has been chosen to be $\begin{pmatrix}1\\0\end{pmatrix}$ in u_I and $u_I\begin{pmatrix}0\\1\end{pmatrix}$ in u_{II}. The solutions for negative energy are similar with E replaced by -E.

3. Magnetic Moment:

The wave equation of a particle in an electromagnetic field is obtained from the free particle equation by replacing p_0 by $p_0 + \frac{e}{c}\Phi$ and $\underline{p}$ by $\underline{p} + \frac{e}{c}\underline{A}$, where Φ is the scalar and $\underline{A}$ the vector potential. In short we make the replacement

$$p_\mu \longrightarrow p_\mu + \frac{e}{c}A_\mu \equiv D_\mu \qquad (5.3.1)$$

where $A_0 \equiv \Phi$. Therefore the equation now looks like

$$(\beta_\mu D_\mu + m)\psi = 0 \qquad (5.3.2)$$

As is done in the case of the Dirac equation (see for example, Reference 3)), we operate on this equation with the Klein-Gordon diviser given by Eq.(5.2.3) wherein again p_μ is replaced by D_μ. There we have

$$\left[m - \beta_\mu D_\mu + \frac{1}{2m}(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu - 2g_{\mu\nu})D_\mu D_\nu\right][\beta_\lambda D_\lambda + m]\psi = 0$$

3) M.E.Rose, Relativistic Electron Theory (John Wiley and Sons. Inc., 1961).pp.116-123.

Due to the rather special algebra satisfied by the β's (Eq. (5.2.2)), the term $(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu - 2g_{\mu\nu})\beta_\lambda D_\mu D_\nu D_\lambda$ vanishes, since each factor in this summation is separately zero and we are left with the following:

$$[m^2 - \beta_\mu\beta_\lambda D_\mu D_\lambda + \tfrac{1}{2}(\beta_\mu\beta_\nu D_\mu D_\nu + \beta_\mu\beta_\nu D_\nu D_\mu) - g_{\mu\nu} D_\mu D_\nu]\psi = 0$$

The third term in the above reduces to

$$\beta_\mu\beta_\nu D_\mu D_\nu + \frac{ie}{2\hbar c}\beta_\mu\beta_\nu F_{\mu\nu}$$

on using the relation

$$[D_\mu, D_\nu] = \frac{ie}{\hbar c} F_{\mu\nu}, \qquad (5.3.3)$$

where $F_{\mu\nu}$ is the electromagnetic field tensor. So we finally get

$$[g_{\mu\nu} D_\mu D_\nu - m^2 + \frac{ie}{2\hbar c}\beta_\mu\beta_\nu F_{\mu\nu}]\psi = 0 \qquad (5.3.4)$$

The first two terms give the Klein-Gordon equation with the replacement of p_μ by D_μ. The last term represents the interaction with the field. The interaction energy is then given by

$$H_{int} = \frac{-\hbar^2}{2m}\left(\frac{ie}{2\hbar c}\beta_\mu \beta_\nu F_{\mu\nu}\right)$$

$$= \frac{-ie\hbar}{2mc}\left(\beta_\mu \beta_\nu F_{\mu\nu}\right) \qquad (5.3.5)$$

To get the coupling with the magnetic field, we use the explicit form of $F_{\mu\nu}$:

$$F_{\mu\nu} : \begin{bmatrix} 0 & \mathcal{H}_3 & -\mathcal{H}_2 & -i\mathcal{E}_1 \\ \mathcal{H}_3 & 0 & \mathcal{H}_1 & -i\mathcal{E}_2 \\ \mathcal{H}_2 & -\mathcal{H}_1 & 0 & -i\mathcal{E}_3 \\ i\mathcal{E}_1 & i\mathcal{E}_2 & i\mathcal{E}_3 & 0 \end{bmatrix}. \qquad \mu,\nu = 1,2,3,0 \qquad (5.3.6)$$

$\mathcal{E}$ and $\underline{\mathcal{H}}$ being the electric and magnetic field vectors.

Then

$$H_{int} = \frac{e\hbar}{2mc}\left[\underline{\Sigma}\cdot\underline{\mathcal{H}} - \underline{\Gamma}\cdot\mathcal{E}\right] \qquad (5.3.7)$$

where

$$\Sigma_i = -i\left(\beta_j \beta_k - \beta_k \beta_j\right), \quad i,j,k, \text{ cyclic}$$

and

$$\Gamma_i = \left(\beta_i \beta_0 - \beta_0 \beta_i\right)$$

The magnetic moment operator is therefore given by

$$\underline{\mu} = -\frac{e\hbar}{2mc}\underline{\Sigma} \tag{5.3.8}$$

Making use of the solutions as given by Eq. (5.2.8) and the matrix representation of β_μ as given in Chapter IV, we can now calculate the expectation value of the z-component of $\underline{\mu}$ when the motion of the particle is in the z-direction. We see that

$$\mu_3 = -\frac{e\hbar}{2mc}\Sigma_3$$

where Σ_3 is given by

$$\Sigma_3 = \left[\begin{array}{cccccc|cccc|cccccc}
 & & 0 & & & & & 0 & & & & & 0 & & & \\
\hline
0 & 0 & -c & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -c & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & -c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -c & 0 \\
\hline
 & & 0 & & & & & 0 & & & & & 0 & & &
\end{array}\right]$$

From the form of the solutions and of Σ_3, it is easily calculated that the expectation value of μ_3 is just $\pm \frac{e\hbar}{2mc}$, which is the same as for a particle described by the Dirac equation. This is as expected. The intrinsic difference between the equation considered here and the Dirac equation should however come out when one considers interactions with other fields, such as a weak interaction. This remains to be investigated.

Incidentally it should be made clear that though the central blocks of the matrices β_μ are identical with the Dirac matrices, the β_μ are in no way equivalent to the Dirac matrices and satisfy an entirely different algebra. The matrices Σ_i cannot here be expressed as commutators of the generators of the Lorentz group and hence do not represent spin operators (unlike the Dirac case where $\underline{\mu} = -(e\hbar/2mc)\underline{\sigma}$, ($\underline{\sigma}$ being the spin matrices).