NULL CORRELATION BUNDLE AND SECANT BUNDLES

By

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KRISHANU

To

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Contents

SYNOPSIS

This thesis is divided into two parts. In the first part, we consider an involution of $\mathbb{P}^3$ which induces a finite morphism $\phi : \mathbb{P}^3 \to \mathbb{P}^5$ such that the image of ϕ is contained in the Grassmannian variety $Gr(2, \mathbb{C}^4)$ in $\mathbb{P}^5$. We consider the pull back of the universal quotient bundle on the Grassmannian $Gr(2, \mathbb{C}^4)$ under the morphism ϕ. We show that the resulting rank 2 bundle on $\mathbb{P}^3$ is a null correlation bundle. In the second part, we consider the second symmetric power $S^2(C)$ of a smooth irreducible projective curve C of genus g over the field of complex numbers. Given a vector bundle E over C of rank r, there is a naturally associated vector bundle of rank $2r$ over $S^2(C)$. We study the semi-stability of this vector bundle with respect to the ample divisor $x + C$ on $S^2(C)$ when E is a rank two stable bundle on C. We also study the semi-stability of the restriction of this vector bundle on the curves of the form $x + C$. In the subsequent sections we will state those results in details.

Let X be a non-singular irreducible projective variety of dimension n over $\mathbb{C}$. Fix an ample divisor H on X. For a coherent torsion free sheaf $\mathcal{V}$ on X, the first Chern class of $\mathcal{V}$ is defined as $c_1(\mathcal{V}) := c_1(\det(\mathcal{V}))$. Given a coherent torsion free sheaf $\mathcal{V}$ on X, we define the *slope* of $\mathcal{V}$ with respect to the ample divisor class H as $\mu_H(\mathcal{V}) := \frac{c_1(\mathcal{V}) \cdot H^{n-1}}{\mathrm{rank}(\mathcal{V})}$. We say that $\mathcal{V}$ is μ_H-*stable* (respectively, μ_H-*semi-stable*) if for every coherent subsheaf $\mathcal{W}$ of $\mathcal{V}$ with $0 < \mathrm{rank}(\mathcal{W}) < \mathrm{rank}(\mathcal{V})$ we have $\mu_H(\mathcal{W}) < \mu_H(\mathcal{V})$ (respectively, $\mu_H(\mathcal{W}) \leq \mu_H(\mathcal{V})$). We will simply write *stable* (respectively, *semi-stable*) to denote μ_H-*stable* (respectively, μ_H- *semi-stable*) when there is no confusion regarding the ample divisor H.

The study of stable and semistable bundles are very fundamental in algebraic geometry, specially when we want to construct moduli space of vector bundles. The rank and the Chern classes are the most natural numerical invariants of a vector bundle. If we try to describe the space of all (isomorphic classes) of vector bundles of fixed rank and Chern classes, we see that these invariants are not sufficient to describe such space as it may be too “big” (for an example see [Har80, Theorem 2.5]). We need some more conditions and thus stability and semistability comes into picture by the works of Seshadri, Narasimhan, Mumford, Takemoto, Gieseker, Maruyama ...

Null Correlation Bundle on Projective Three Space

Let $\mathbb{P}^n$ denotes the projective n-space over $\mathbb{C}$, the field of complex numbers. Let $\mathcal{V}$ be a stable rank two vector bundles on $\mathbb{P}^3$ and let $H \subset \mathbb{P}^3$ be hyperplane. One particular interest is to study the stability property of the restriction $\mathcal{V}|_H$ on H. Maruyama showed that if $\mathcal{V}$ is a semistable rank two vector bundle on $\mathbb{P}^3$, then for a general hyperplane $H \subset \mathbb{P}^3$, the restriction $\mathcal{V}|_H$ is semistable. W. Barth proved a more stronger result. He showed that if $\mathcal{V}$ is a stable rank two bundle on $\mathbb{P}^n, n \geq 4$ then for a general hyperplane $H \subset \mathbb{P}^n$, the restriction $\mathcal{V}|_H$ is stable. For rank two stable bundles on $\mathbb{P}^3$, the same conclusion holds except in one case. W. Barth called the bundle on $\mathbb{P}^3$ with this exceptional property a "null correlation bundle". More precisely, he proved the following result:

Theorem 0.1. *([Barth77]) Let $\mathcal{V}$ be some rank two μ-stable bundle on $\mathbb{P}^n$, the complex projective n-space. If $n \geq 4$, there is a non-empty open set $U \subset \mathbb{P}^{n*}$, the dual projective space, such that the restriction of $\mathcal{V}$ to all hyperplanes parametrized by U is again stable. The same holds for $n = 3$, unless $\mathcal{V}$ is a null correlation bundle.*

A null correlation bundle is unique up to translation by automorphisms of $\mathbb{P}^3$ and by tensoring by line bundles. In [Barth77], Barth gives a construction of this bundle and showed that the Chern classes of this bundle are $c_1 = 0, c_2 = 1$. It turns out that any stable rank two vector bundle on $\mathbb{P}^3$ with Chern classes $c_1 = 0, c_2 = 1$ is a null correlation bundle ([O-S-S, Chapter 2, Lemma 4.3.2]). So the moduli space $\mathcal{M}_{\mathbb{P}^3}(0, 1)$ of stable rank two vector bundles with Chern classes $c_1 = 0, c_2 = 1$ is the space of null correlation bundles on $\mathbb{P}^3$. The moduli space $\mathcal{M}_{\mathbb{P}^3}(0, 1)$ for the null correlation bundles on $\mathbb{P}^3$ is isomorphic to the complement $\mathbb{P}^5 \setminus Gr(2, \mathbb{C}^4)$ in $\mathbb{P}^5$ of the Grassmannian variety of two dimensional quotients of $\mathbb{C}^4$ ([O-S-S, Chapter 2, Theorem 4.3.4]). Since the stability and semistability preserves with respect to tensoring by line bundles, any vector bundle which arises from a null correlation bundle by tensoring by line bundles, will also be called a null correlation bundle.

Let $[x_1, x_2, x_3, x_4]$ be the homogeneous co-ordinates of $\mathbb{P}^3$ and let $\sigma : \mathbb{P}^3 \to \mathbb{P}^3$ be an involution defined by $[x_1, x_2, x_3, x_4] \mapsto [-x_1, -x_2, x_3, x_4]$. The automorphism σ lifts to an automorphism of the vector space $H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(2))$. Let V be the σ-invariant subspace of $H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(2))$, i.e. $V = \{v \in H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(2)) : \sigma(v) = v\}$. Then $V = \langle X_1^2, X_2^2, X_3^2, X_4^2, X_1X_2 + X_3X_4, X_1X_2 - X_3X_4\rangle$. The vector space V defines a base point free linear system on $\mathbb{P}^3$ and hence induces a finite morphism

$$\phi : \mathbb{P}^3 \to \mathbb{P}^5$$

given by

$$[x_1, x_2, x_3, x_4] \mapsto [x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2 + x_3x_4, x_1x_2 - x_3x_4].$$

The image of ϕ is contained in the quadric $Z_2Z_3 - Z_0Z_1 + Z_4Z_5$, where $[Z_0, Z_1, Z_2, Z_3, Z_4, Z_5]$ is the homogeneous co-ordinates of $\mathbb{P}^5$. This quadric defines the Grassmannian variety, $Gr(2, \mathbb{C}^4)$ – of two dimensional quotient spaces of $\mathbb{C}^4$, in $\mathbb{P}^5$ under the Plücker embedding. So we can think of ϕ as a morphism $\phi : \mathbb{P}^3 \to Gr(2, \mathbb{C}^4)$. Let Q_u denotes the universal quotient bundle on the Grassmannian $Gr(2, \mathbb{C}^4)$ and define $Q := \phi^*(Q_u)$. Then Q is a globally generated rank two vector bundle on $\mathbb{P}^3$. We use the geometry of Grassmannian variety to show that the Chern classes of this bundle are $c_1(Q) = 2, c_2(Q) = 2$. We obtain the following result:

Theorem 0.2. *The rank two vector bundle Q is a null correlation bundle on $\mathbb{P}^3$.*

This gives an alternative construction of null correlation bundle on $\mathbb{P}^3$. Now consider an automorphism $g : \mathbb{P}^3 \to \mathbb{P}^3$ of $\mathbb{P}^3$. If we consider the morphism $\tau : \mathbb{P}^3 \to \mathbb{P}^3$ defined by $\tau = g\sigma g^{-1}$, then τ is also an involution on $\mathbb{P}^3$. If we replace σ with τ, then the vector bundle we get is $g^*(Q)$. Since null correlation bundle is unique up to translation by automorphisms of $\mathbb{P}^3$, the bundle $g^*(Q)$ is also a null correlation bundle.

Semi-stability of Secant Bundle on Second Symmetric Power of a Curve

Let C be a smooth irreducible projective curve of genus g over the field of complex numbers $\mathbb{C}$. On the space $C \times C$, consider the following involution $C \times C \longrightarrow C \times C, (x, y) \mapsto (y, x)$. The resulting quotient space is denoted by $S^2(C)$, called the *second symmetric power* of C. It is a smooth irreducible projective surface over $\mathbb{C}$. A point in $S^2(C)$ can be thought as an effective divisor of degree 2 on C and $S^2(C)$ can be regarded as the set of all effective divisors of degree 2 on C. With respect to this identification, we can write an element of $S^2(C)$ as $x + y, x, y \in C$. Set $\Delta_2 := \{(D, p) \in S^2(C) \times C : D = p + q$, for some $q \in C\}$. Then Δ_2 is a divisor in $S^2(C) \times C$, called *the universal divisor of degree* 2 *on* C. Let q_1 and q_2 be the projections from $S^2(C) \times C$ onto the first and second factors respectively. Then the restriction of the first projection to Δ_2 induces a morphism $q : \Delta_2 \longrightarrow S^2(C)$, which is a two sheeted ramified covering. For any vector bundle E of rank r on C we construct a bundle $\mathcal{F}_2(E) := (q)_*(q_2^*(E) \mid_{\Delta_2})$ of rank $2r$ over $S^2(C)$. Note that the map $C \times C \to \Delta_2, (x, y) \mapsto (x + y, y)$ is an isomorphism. Let $p_i : C \times C \to C$ be the i-th coordinate projection, $i = 1, 2$, and $\pi : C \times C \to S^2(C)$ be the quotient map. Then $\mathcal{F}_2(E) = \pi_* p_2^* E$. This bundle is called the *secant bundle* on $S^2(C)$. This was first introduced by R. Schwarzenberger ([S64]).

Assume $g \geq 2$. Let $\mathrm{Pic}^2(C)$ denotes the variety parameterizing all degree 2 line bundles on C and let $\nu : S^2(C) \to \mathrm{Pic}^2(C)$ be map defined by $\{x, y\} \mapsto x + y$. (Here we are considering the unordered collection $\{x, y\}$ of points $x, y \in C$ as an element of $S^2(C)$ and we are identifying the divisor $x + y$ on C and the corresponding line bundle on C.) Let $\theta \in H^2(S^2(C), \mathbb{Z})$ be the pull back of the class of theta divisor in $H^2(\mathrm{Pic}^2(C), \mathbb{Z})$ under the map ν. Using Künneth formula we can write $H^2(S^2(C) \times C, \mathbb{Z}) = (H^2(S^2(C), \mathbb{Z}) \otimes H^0(C, \mathbb{Z})) \oplus (H^1(S^2(C), \mathbb{Z}) \otimes H^1(C, \mathbb{Z})) \oplus (H^0(S^2(C), \mathbb{Z}) \otimes H^2(C, \mathbb{Z}))$. Let δ be the class of the universal divisor Δ_2. Then using above we can decompose δ as $\delta = \delta^{2,0} + \delta^{1,1} + \delta^{0,2}$, where $\delta^{i,j} \in H^i(S^2(C), \mathbb{Z}) \otimes H^j(C, \mathbb{Z})$. Let's denote the Künneth component $\delta^{2,0}$ by x.

Let E be a line bundle on C of degree d. Then the Chern polynomial of the vector bundle

$\mathcal{F}_2(E)$ over $S^2(C)$ is given by $c_t(\mathcal{F}_2(E)) = (1-xt)^{1-d+g}e^{t\theta/(1-xt)}$ (See [ACGH, Chapter VIII, Lemma 2.5]). From this we get that the first Chern class

$$c_1(\mathcal{F}_2(E)) = (d-g-1)x + \theta$$

and the second Chern class

$$c_2(\mathcal{F}_2(E)) = \binom{d-g}{2}x^2 + (d-g)x\cdot\theta + \frac{\theta^2}{2}.$$

The cohomology group $H^4(S^2(C),\mathbb{Z})$ is canonically isomorphic to $\mathbb{Z}$. Also we have $x^2 = 1, x\cdot\theta = g, \theta^2 = g(g-1)$. Using this we get $c_2(\mathcal{F}_2(E)) = \binom{d}{2}$. (See [ELN11])

Fixing a point $x \in C$, the image of $\{x\}\times C$ in $S^2(C)$ defines an ample divisor H' on $S^2(C)$, which we denote by $x + C$. The stability and semistability of the vector bundle $\mathcal{F}_2(E)$, where E is a line bundle is obtained in [ELN11] by El Mazouni, Laytimi, and Nagaraj. They proved the following result.

Theorem 0.3. *[ELN11] Let E be a very ample line bundle on C.*

(1) The vector bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is semistable with respect to the ample divisor $x+C$.

(2) Assume that genus of C is greater than zero or genus of C is zero and degree(E) is greater than one. Then the vector bundle $\mathcal{F}_2(E)$ is stable with respect to the ample divisor $x + C$.

In [BN13], Biswas and Nagaraj improved the above results significantly by classifying all the stable and semistable secant bundles $\mathcal{F}_2(E)$ for non-trivial line bundles E.

Theorem 0.4. *[BN13] Let C be a smooth irreducible projective curve of genus g on $\mathbb{C}$ and let E be a non-trivial line bundle on C.*

(1) The vector bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is μ-semistable with respect to the ample divisor $x + C$.

(2) If E and E' are two line bundles on C such that $\mathcal{F}_2(E) \cong \mathcal{F}_2(E')$, then $E \cong E'$. In fact, every isomorphism of $\mathcal{F}_2(E)$ and $\mathcal{F}_2(E')$ induces an isomorphism of E and E'.

(3) The vector bundle $\mathcal{F}_2(E)$ is μ-stable with respect to the ample divisor $x + C$, unless $E \cong \mathcal{O}_C(x)$ or $E \cong \mathcal{O}_C(-x)$ for some $x \in C$.

Now let E be a rank two stable vector bundle on C and consider the associated rank 4 vector bundle $\mathcal{F}_2(E)$ on $S^2(C)$. One can ask about the stability and semistability of this bundle with respect to the ample divisor $x + C$. In the second part of this thesis we gave a partial answer of this question.

To find the Chern character of $\mathcal{F}_2(E)$, for any rank r vector bundle E, first choose a filtration of E such that the successive quotients are line bundles and use the fact that $\mathcal{F}_2(\oplus M_k) = \oplus \mathcal{F}_2(M_k)$ where M_k's are line bundles over C. Then the Chern character of $\mathcal{F}_2(E)$ has the following expression ([BL11]):

$$\mathrm{ch}(\mathcal{F}_2(E)) = \mathrm{degree}(E)(1 - \exp(-x)) - r(g-1) + r(1 + g + \theta)\exp(-x).$$

From the above expression one can easily see that $c_1(\mathcal{F}_2(E)) = (d - r(g+1))x + r\theta$, where $d = \mathrm{degree}\ E$.

We prove the following:

Theorem 0.5. *Let E be a rank two μ-stable vector bundle of even degree $d \geq 2$ on C such that $\mathcal{F}_2(E)$ is globally generated. Then the bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is $\mu_{H'}$-semistable with respect to the ample class $H' = x + C$.*

Let E be a non-zero vector bundle on C and $k \in \mathbb{Z}$, we denote by $\mu_k(E)$ the rational number $\mu_k(E) := \frac{\mathrm{degree}(E)+k}{\mathrm{rank}(E)}$. We say that the vector bundle E is (k,l)-*stable* (respectively, (k,l)-*semistable*) if, for every proper subbundle F of E we have $\mu_k(F) < \mu_{-l}(E/F)$ (resp. $\mu_k(F) \leq \mu_{-l}(E/F)$). Note that usual μ-stability is equivalent to $(0,0)$-stability. If $g \geq 3$, then there always exists a $(0,1)$-stable bundle and if $g \geq 4$, then the set of $(0,1)$-stable bundles form a dense open subset of the moduli space of stable bundles over C of rank 2 and degree d ([NR78, Section 5]). We obtain the following result:

Theorem 0.6. *Assume the genus of C greater than 2. Let E be a rank two $(0,1)$-stable bundle of odd degree $d \geq 1$ on C such that $\mathcal{F}_2(E)$ is globally generated. Then the bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is $\mu_{H'}$-semistable with respect to the ample class $H' = x + C$.*

Also, regarding the restriction of $\mathcal{F}_2(E)$ on the curves of the form $x + C$ we prove the following:

Theorem 0.7. *Let C be a smooth irreducible projective curve over $\mathbb{C}$ of genus g and let E be a rank to vector bundle on C of degree $d \geq 3$. Then for any $x \in C, \mathcal{F}_2(E)|_{x+C}$ is not semistable.*

1 PRELIMINARIES

In this section we will recall some basic and well known results of stability and semistability of vector bundles on a smooth irreducible projective varieties, which are essential for the rest of the chapters. We will follow mainly [O-S-S], [Friedman], [HL]. For the sake of completeness we will also give proof of some of the stated results. The words "line bundle", "invertible sheaf" and "locally free sheaf of rank one" will be used interchangeably. The same is true for the words "vector bundle" and "locally free sheaf of finite rank".

1.1 Stable and Semistable Sheaves

Let X be a non-singular irreducible projective variety of dimension n over $\mathbb{C}$, the field of complex numbers. Let $\mathcal{V}$ be coherent sheaf on X. We say that $\mathcal{V}$ is *torsion free* if for every $x \in X, \mathcal{V}_x$ is a torsion free $\mathcal{O}_{X,x}$ module, i.e. $fa = 0$ for $a \in \mathcal{V}_x$ and $f \in \mathcal{O}_{X,x}$ always implies $a = 0$ or $f = 0$ The *dual* of $\mathcal{V}$, denoted by $\mathcal{V}^*$, is the coherent sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{V}, \mathcal{O}_X)$. The coherent sheaf $\mathcal{V}^{**} := (\mathcal{V}^*)^*$ is called the *double dual* of $\mathcal{V}$. There is a canonical homomorphism $\sigma : \mathcal{V} \to \mathcal{V}^{**}$. The stalk of $\mathcal{V}$ at the generic point of X is a finite dimensional vector space. Since this vector space is isomorphic to its double dual, the kernel and cokernel of the map σ are supported on proper closed subsets of X. Also $\mathrm{Ker}(\sigma)$ is precisely the torsion subsheaf of $\mathcal{V}$, i.e. for every $x \in X$

$$\mathrm{Ker}(\sigma)_x = \{a \in \mathcal{V}_x : fa = 0, \text{for some} f \in \mathcal{O}_{X,x}, f \neq 0\}.$$

Thus $\mathcal{V}$ is torsion free if and only if σ is a monomorphism.

Definition 1.1. A coherent sheaf $\mathcal{V}$ on X is called *reflexive* if the canonical homomorphism $\sigma : \mathcal{V} \to \mathcal{V}^{**}$ is an isomorphism.

A reflexive sheaf is necessarily torsion free. If $\mathcal{V}$ is locally free then obviously it is reflexive. But there are example of reflexive sheaves which are not locally free ([O-S-S, Chapter 2, Example 1.1.13]). Also dual of any coherent sheaf is always reflexive ([Har80, Corollary 1.2]).

Let $\mathcal{V}$ be a locally free sheaf on X of rank r. The *determinant* of $\mathcal{V}$, denoted by $\det(\mathcal{V})$, is defined as $\det(\mathcal{V}) := (\Lambda^r \mathcal{V})^{**}$. Now assume $\mathcal{V}$ is a coherent torsion free sheaf on X. Then there is an open dense subset U of X such that the restriction $\mathcal{V}|_U$ is locally free. The rank of $\mathcal{V}$ is defined to be the rank of the locally free sheaf $\mathcal{V}|_U$. Let the rank of $\mathcal{V}$ be r. we define the determinant of $\mathcal{V}$ to be $\det(\mathcal{V}) := (\Lambda^r \mathcal{V})^{**}$. The determinant of a coherent torsion free sheaf $\mathcal{V}$ is a reflexive sheaf of rank one. This is actually a line bundle. This follows from the following result.

Lemma 1.2. *Let $\mathcal{E}$ be a rank one reflexive sheaf on X. Then $\mathcal{E}$ is locally free.*

Proof. See [O-S-S, Chapter 2, Lemma 1.1.15]. □

Remark 1.3. *Let $\mathcal{V}$ be a coherent sheaf on X. Then it has a finite locally free resolution $0 \to \mathcal{V}_m \to \mathcal{V}_{m-1} \to \cdots \to \mathcal{V}_0 \to \mathcal{V} \to 0$. The determinant of $\mathcal{V}$ is defined by $det(\mathcal{V}) := \bigotimes det(\mathcal{V}_i)^{(-1)^i}$. This definition does not depend on the locally free resolutions ([HL, 1.1.17], [Har, Chapter III, Ex. 6.8, 6.9]).*

Definition 1.4. Let $\mathcal{V}$ be a coherent sheaf on X. We say that $\mathcal{V}$ is *normal* if for every open subset $U \subseteq X$ and every closed subset $Y \subseteq U$ of codimension at least 2, the restriction map $\Gamma(U, \mathcal{V}) \to \Gamma(U - Y, \mathcal{V})$ is surjective. ([Barth77, Page 128])

Reflexive sheaves on X has the following characterization:

Lemma 1.5. *Let $\mathcal{V}$ be a coherent sheaf on X. Then $\mathcal{V}$ is reflexive if and only if it is torsion free and normal.*

Proof. See [O-S-S, Chapter 2, Lemma 1.1.12]. □

The following results gives a criterion for a coherent sheaf to be normal.

Lemma 1.6. *Let $\mathcal{V}$ be a coherent sheaf on X and assume that there is a short exact sequence $0 \to \mathcal{V} \to \mathcal{E} \to \mathcal{Q} \to 0$ of sheaves with $\mathcal{E}$ reflexive. If $\mathcal{Q}$ is torsion free then $\mathcal{V}$ is normal.*

Proof. ([O-S-S, Chapter 2, Lemma 1.1.16]) Let U ba an open subset of X and let Y be a closed subset of U of codimension at least 2. Since $\mathcal{V}, \mathcal{Q}$ are torsion free, the restrictions $\mathcal{V}(U) \to \mathcal{V}(U - Y)$ and $\mathcal{Q}(U) \to \mathcal{Q}(U - Y)$ are injective. Now consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{V}(U) & \longrightarrow & \mathcal{E}(U) & \longrightarrow & \mathcal{Q}(U) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{V}(U-Y) & \longrightarrow & \mathcal{E}(U-Y) & \longrightarrow & \mathcal{Q}(U-Y) \end{array}$$

where the horizontal rows are exact, the leftmost and rightmost arrows are injective. Also the middle arrow is an isomorphism since $\mathcal{E}$ being reflexive is both torsion free and normal. From this diagram it follows that the restriction $\mathcal{V}(U) \to \mathcal{V}(U - Y)$ is an isomorphism. □

Definition 1.7. Let $\mathcal{V}$ be be a coherent torsion free sheaf on X. Fix an ample divisor H on X. The first Chern class of $\mathcal{V}$ is defined as $c_1(\mathcal{V}) := c_1(\det(\mathcal{V}))$. The *slope* of $\mathcal{V}$ with respect to the ample divisor H is defined as

$$\mu_H(\mathcal{V}) := \frac{\mathrm{degree}_H(\mathcal{V})}{\mathrm{rank}(\mathcal{V})}$$

where $\mathrm{degree}_H(\mathcal{V}) := c_1(\mathcal{V}) \cdot H^{n-1}$. We say that a coherent torsion free sheaf $\mathcal{V}$ on X is $\mu_H - stable$ (respectively, $\mu_H - semistable$) if for every coherent subsheaf $\mathcal{W}$ of $\mathcal{V}$ with $0 < \mathrm{rank}(\mathcal{W}) < \mathrm{rank}(\mathcal{V})$ we have

$$\mu_H(\mathcal{W}) < \mu_H(\mathcal{V}) \text{ (respectively, } \mu_H(\mathcal{W}) \leq \mu_H(\mathcal{V})).$$

We will write μ-stable (respectively, μ-semistable) when there is no confusion regarding the ample divisor H. Sometimes we will also simply write "stable" (respectively, "semistable") instead of μ-stable (respectively, μ-semistable). μ-stable sheaves are also called *slope stable* or *Mumford stable*. Similarly for μ-semistable sheaves.

Lemma 1.8. *Suppose*

$$0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$$

be an exact sequence of non-zero coherent torsion free sheaves on X and let $\mu := \mu_H$. Then

$$min(\mu(\mathcal{V}'), \mu(\mathcal{V}'')) \leq \mu(\mathcal{V}) \leq max(\mu(\mathcal{V}'), \mu(\mathcal{V}'')),$$

and equality holds at either end if and only if $\mu(\mathcal{V}') = \mu(\mathcal{V}) = \mu(\mathcal{V}'')$.

Proof. ([Friedman, Chapter 4, Lemma 2]) Let $r' = \text{rank}(\mathcal{V}')$, $r'' = \text{rank}(\mathcal{V}'')$. Since both rank and degree are additive, we have

$$\mu(\mathcal{V}) = \frac{r'}{r' + r''}\mu(\mathcal{V}') + \frac{r''}{r' + r''}\mu(\mathcal{V}'').$$

So $\mu(\mathcal{V}) = \lambda\mu(\mathcal{V}') + (1 - \lambda)\mu(\mathcal{V}'')$, for some $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$. From this the conclusion follows. □

Remark 1.9. *More generally, if $\mathcal{V}$ is a vector bundle and $\mathcal{V} = \mathcal{V}_l \supset \mathcal{V}_{l-1} \supset \mathcal{V}_{l-2} \supset \cdots \supset \mathcal{V}_1 \supset (0)$ be a flag of subbundles of $\mathcal{V}$ then $\mu(V) \leq max(\mu(\mathcal{V}_{i+1}/\mathcal{V}_i))$ and $\mu(V) \geq min(\mu(\mathcal{V}_{i+1}/\mathcal{V}_i))$. And equality holds in one case if and only if equality holds for the other one if and only if $\mu(\mathcal{V}_{i+1}/\mathcal{V}_i) = \mu(\mathcal{V}_{j+1}/\mathcal{V}_j)$ for all i and j ([Shatz77, Section 2]).*

Now we state some equivalent condition for a coherent torsion free sheaf on X to be stable (respectively, semistable).

Lemma 1.10. *Let $\mathcal{V}$ be a coherent torsion free sheaf on X and let $\mu := \mu_H$. Then the following conditions are equivalent:*

(i) $\mathcal{V}$ is stable (respectively, semistable).

(ii) For every coherent subsheaf $\mathcal{W}$ of $\mathcal{V}$ with $0 < rank(\mathcal{W}) < rank(\mathcal{V})$ such that $\mathcal{V}/\mathcal{W}$ is torsion free we have $\mu(\mathcal{W}) < \mu(\mathcal{V})$ (respectively, $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$).

(iii) For every torsion free quotient Q of $\mathcal{V}$ with $0 < rank(Q) < rank(\mathcal{V})$ we have $\mu(Q) > \mu(\mathcal{V})$ (respectively, $\mu(Q) \geq \mu(\mathcal{V})$).

Proof. ([Friedman, Chapter 4, Lemma 5]) The equivalence follows from Lemma 1.8. □

Example 1.11. Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and fix an ample line bundle H.

(1) Any line bundle L on X is stable.

(2) Let L, M be two line bundles on X and consider the vector bundle $E = L \oplus M$. From Lemma 1.8, it follows that E is semistable if and only if $\text{degree}_H(L) = \text{degree}_H(M)$. Also E is never stable.

(3) Let L, M be two line bundles on X of degrees $0, 1$ respectively. Suppose we have a non-trivial extension $0 \to L \to E \to M \to 0$ on X. Then E is stable. Since degree is additive, we have $\mu(E) = \frac{1}{2}$. Let F be a rank one coherent torsion free sheaf on X which is a subsheaf of E. There are two possibilities. First, suppose there is a non-zero map $F \to L$. In this case, $\mu(F) < \mu(L) = 0 < \mu(E)$. Second, suppose there is a non-zero map $F \to M$. Then $\mu(F) < \mu(M) = 1$. Since $\mu(F)$ is an integer, $\mu(F) \leq 0 < \mu(E)$.

Lemma 1.12. *Let $\mathcal{V}$ be a coherent torsion free sheaf on X. Then*

(i) $\mathcal{V}$ is stable if and only if for any line bundle L on X, $\mathcal{V} \otimes L$ is stable.

(ii) $\mathcal{V}$ is stable if and only if $\mathcal{V}^*$, the dual of $\mathcal{V}$, is stable.

Proof. See [Takemoto72, Proposition 1.4]. □

The example 1.11(3) is a particular case of the following general result.

Lemma 1.13. *Let*

$$0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$$

be an exact sequence of non-zero coherent torsion free sheaves on X with $\mu(\mathcal{V}') = \mu(\mathcal{V}) = \mu(\mathcal{V}'')$ where $\mu := \mu_H$. Then $\mathcal{V}$ is semistable if and only if both $\mathcal{V}'$ and $\mathcal{V}''$ are semistable, and $\mathcal{V}$ is never stable. In particular, if both $\mathcal{V}'$ and $\mathcal{V}''$ are rank 1, then $\mathcal{V}$ is semistable.

Proof. ([Friedman, Chapter 4, Lemma 6]) Let $\mathcal{V}$ be semistable. Suppose $\mathcal{W}'$ be a coherent proper subsheaf of $\mathcal{V}'$. Then $\mathcal{W}'$ is also a coherent proper subsheaf of $\mathcal{V}$ and so $\mu(\mathcal{W}') \leq \mu(\mathcal{V}) = \mu(\mathcal{V}')$. Now suppose $\mathcal{W}''$ be a coherent proper subsheaf of $\mathcal{V}''$ and let $\mathcal{W}$ be the inverse image of $\mathcal{W}''$ in $\mathcal{V}$. Then we have the following exact sequence of coherent torsion free sheaves on X :

$$0 \to \mathcal{V}' \to \mathcal{W} \to \mathcal{W}'' \to 0.$$

Now using Lemma 1.8 and the fact that $\mu(\mathcal{V}') = \mu(\mathcal{V}) = \mu(\mathcal{V}'')$ we see that $\mathcal{V}''$ is also semistable.

Conversely, assume that both $\mathcal{V}'$ and $\mathcal{V}''$ are semistable. Let $\mathcal{W}$ be a proper coherent subsheaf of $\mathcal{V}$. Then we have the following commutative diagram of torsion free sheaves on X :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V}' & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{V}'' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{W}' & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{W}'' & \longrightarrow & 0 \end{array}$$

where the horizontal rows are exact and the vertical arrows are monomorphisms. If either of $\mathcal{W}'$ or $\mathcal{W}''$ is zero then we are done. So assume that both $\mathcal{W}'$ and $\mathcal{W}''$ are non-zero. Now using that both $\mathcal{V}'$ and $\mathcal{V}''$ are semistable with $\mu(\mathcal{V}') = \mu(\mathcal{V}) = \mu(\mathcal{V}'')$ and Lemma 1.8, we see that $\mathcal{V}$ is semistable. □

Lemma 1.14. *Let $\mathcal{V}, \mathcal{W}$ be two coherent torsion free semistable sheaves on X and let*

$\mu := \mu_H$.

(1) If $\mu(\mathcal{V}) > \mu(\mathcal{W})$, then $Hom(\mathcal{V}, \mathcal{W}) = 0$.

(ii) If $\mu(\mathcal{V}) = \mu(\mathcal{W})$ with at least one of $\mathcal{V}$ and $\mathcal{W}$ is stable and if $\phi : \mathcal{V} \to \mathcal{W}$ is non-trivial then ϕ is either injective or generically surjective.

(iii) If $\mathcal{V}, \mathcal{W}$ are vector bundles and $c_1(\mathcal{V}) = c_1(\mathcal{W}), rank(\mathcal{V}) = rank(\mathcal{W})$ and if at least one of $\mathcal{V}$ and $\mathcal{W}$ is stable, then any non-trivial homomorphism $\phi : \mathcal{V} \to \mathcal{W}$ is an isomorphism.

Proof. ([O-S-S, Chapter 2, Lemma 1.2.8]) (i) Let $\phi : \mathcal{V} \to \mathcal{W}$ be a non-trivial homomorphism and let $E := \mathrm{Im}(\phi)$. Since both $\mathcal{V}$ and $\mathcal{W}$ are semistable we have $\mu(\mathcal{V}) \leq \mu(E) \leq \mu(\mathcal{W})$ contradicting the given assumption.

(ii) Let $E := \mathrm{Im}(\phi)$. Suppose $\mathrm{rank}(E) < \mathrm{rank}(\mathcal{V})$ and $\mathrm{rank}(E) < \mathrm{rank}(\mathcal{W})$. Now if $\mathcal{V}$ is stable then we have $\mu(\mathcal{V}) < \mu(E) \leq \mu(\mathcal{W})$ and if $\mathcal{W}$ is stable then we have $\mu(\mathcal{V}) \leq \mu(E) < \mu(\mathcal{W})$. Both of these contradict the given assumption. Thus we must have either $\mathrm{rank}(E) = \mathrm{rank}(\mathcal{V})$ or $\mathrm{rank}(E) = \mathrm{rank}(\mathcal{W})$. In the first case we have ϕ is a monomorphism and in the second case ϕ is generically surjective.

(iii) By (ii) we have that ϕ is a monomorphism and hence $\det(\phi) : \det(\mathcal{V}) \to \det(\mathcal{W})$ is also a monomorphism. From the given condition it follows that $\det(\phi)$ is actually an isomorphism. Thus so is ϕ. □

Remark 1.15. *If $\mathcal{V} = \mathcal{W}$ then the conclusion of (iii) remains true for any torsion free sheaf $\mathcal{V}$([Friedman, Chapter 4, Proposition 7]).*

Let $\mathcal{V}$ be a coherent torsion free sheaf on X. We say that $\mathcal{V}$ is *simple* if $\mathbf{End}(\mathcal{V}) = \{\lambda \cdot Id : \lambda \in \mathbb{C}\}$. As an application of the above we get the following result.

Corollary 1.16. Let $\mathcal{V}$ be a coherent stable torsion free sheaf on X. Then $\mathcal{V}$ is simple.

Proof. ([Friedman, Chapter 4, Corollary 8]) Let $\phi : \mathcal{V} \to \mathcal{V}$ be a non-zero homomorphism. Then by Lemma 1.14(iii) and Remark 1.15 above, ϕ is an isomorphism. Choose $x \in X$ such that $\mathcal{V}$ is free at x. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\phi(x) : \mathcal{V}(x) \to \mathcal{V}(x)$ where $\mathcal{V}(x)$ is the fiber of $\mathcal{V}$ at x. Now consider the map $(\phi - \lambda \cdot Id) : \mathcal{V} \to \mathcal{V}$. This map is not an isomorphism and hence must be zero. □

Lemma 1.17. *Let $\mathcal{V}$ be a coherent torsion free semistable sheaf on X. If rank($\mathcal{V}$) and degree$_H$($\mathcal{V}$) are co-prime, then $\mathcal{V}$ is stable.*

Proof. If $\mathcal{V}$ is not stable then there is a proper subsheaf $\mathcal{W}$ of $\mathcal{V}$ such that $\mu(\mathcal{V}) = \mu(\mathcal{W})$ and this contradicts that rank($\mathcal{V}$) and degree$_H$($\mathcal{V}$) are co-prime. □

Now we will state Bertini's theorem. Before that we will recall some definitions. Here we will consider varieties defined over an arbitrary algebraically closed field (of any characteristic), not merely over $\mathbb{C}$.

Definition 1.18. Let k be an algebraically closed field. A *group variety* G over k is a variety G over k together with morphisms

$$\sigma : G \times G \to G, \rho : G \to G, e : \mathrm{Spec}(k) \to G$$

satisfying the followings:

(a) *Associativity :* The following diagram

$$\begin{tikzcd} G \times G \times G \arrow[r, "1_G \times \sigma"] \arrow[d, "\sigma \times 1_G"] & G \times G \arrow[d, "\sigma"] \\ G \times G \arrow[r, "\sigma"] & G \times G \end{tikzcd}$$

commutes.

(b) *Law of inverse :* Both the compositions

$$G \xrightarrow{\Delta} G \times G \xrightarrow{1_G \times \rho} G \xrightarrow{\sigma} G$$

and

$$G \xrightarrow{\Delta} G \times G \xrightarrow{\rho \times 1_G} G \xrightarrow{\sigma} G$$

equal to $e \circ \pi$ where $\pi : G \to \mathrm{Spec}(k)$ is the structure morphism and $\Delta : G \to G \times G$ is the diagonal morphism.

(c) *Law of identity :* The compositions

$$G \cong \mathrm{Spec}(k) \times_k G \xrightarrow{e \times 1_G} G \times G \xrightarrow{\sigma} G$$

and

$$G \cong G \times_k \mathrm{Spec}(k) \xrightarrow{1_G \times e} G \times G \xrightarrow{\sigma} G$$

both equal to 1_G. ([MF])

Remark 1.19. *Since k is algebraically closed, to say that G is a group variety is same as saying that the set $G(k)$, of k-rational points of G (which is just the set of all closed points of G) forms a group under the morphisms σ, ρ, e.*

Definition 1.20. (1) Let G be a group variety over k and let X be a variety over k. We say that G *acts* on X is there is a morphism $\theta : G \times X \to X$ which induces a homomorphism $G(k) \to \mathrm{Aut}(X)$ of groups.

(2) A *homogeneous space* is a variety X over k together with a group variety G over k acting on it such that the group $G(k)$ acts transitively on $X(k)$.

Remark 1.21. *(1) Any group variety is a homogeneous space with respect to the action of left multiplication on itself. Also a group variety is necessarily a non-singular variety. ([Har, Chapter III, Section 10])*

(2) The projective space $\mathbb{P}^n_k$ is a homogeneous space for the action of $PGL(n)$.

Proposition 1.22. *(Kleiman) Let X be a homogeneous space with group variety G over an algebraically closed field k of characteristic zero. Let $f : Y \to X$ and $g : Z \to X$ be morphisms of non-singular varieties Y, Z to X. For any $\sigma \in G(k)$, let Y^σ be Y with the morphism $\sigma \circ f$ to X. Then there is a non-empty open subset $V \subseteq G$ such that for every $\sigma \in V(k)$, $Y^\sigma \times_X Z$ is non-singular and either empty or dimension exactly*

$$dim(Y) + dim(Z) - dim(X).$$

Proof. See [Har, Chapter III, Theorem 10.8]. □

As a corollary of the above result we get Bertini's theorem.

Corollary 1.23. (Bertini) Let X be a non-singular projective variety over an algebraically closed field k of characteristic zero. Let ð be a linear system without base points. Then almost every element of ð, considered as a closed subscheme of X, is non-singular (but may be reducible).

Proof. See [Har, Chapter III, Corollary 10.9]. □

Remark 1.24. *The hypothesis "X is projective" is not necessary if we choose ð to be a finite dimensional linear system. If X is projective and if ð is a linear system with base points D, then by considering the base-point-free linear system ð on $X - D$ we get a more general result "a general member of ð can have singularities only at the base points" ([Har, Chapter III, Remark 10.9.2]).*

1.2 Stable and Semistable Bundles on Projective Spaces

Let $\mathbb{P}^n$ denotes the projective n-space over $\mathbb{C}$. Since $H^*(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}[h]/h^{n+1}\mathbb{Z}$, where h is the class of a hyperplane, we can consider the Chern classes of a vector bundle on $\mathbb{P}^n$ as integers.

Let $\mathcal{V}$ be a torsion free coherent sheaf on $\mathbb{P}^n$ of rank 2. Then there is a uniquely determined integer $k_{\mathcal{V}}$ such that $c_1(\mathcal{V}(k_{\mathcal{V}})) \in \{0, -1\}$. From the Chern class computations it is easy to see that $k_{\mathcal{V}} = -\frac{c_1(\mathcal{V})}{2}$, for $c_1(\mathcal{V})$ even and $k_{\mathcal{V}} = -\frac{c_1(\mathcal{V})+1}{2}$, for $c_1(\mathcal{V})$ odd. Set $\mathcal{V}_{\text{norm}} := \mathcal{V}(k_{\mathcal{V}})$. We say that $\mathcal{V}$ is *normalized* if $\mathcal{V} = \mathcal{V}_{\text{norm}}$. If $\mathcal{V}$ is a normalized rank two coherent torsion free sheaf on $\mathbb{P}^n$, then $\mu(\mathcal{V}) \in \{0, -\frac{1}{2}\}$.

Proposition 1.25. *Let $\mathcal{V}$ be a rank 2 normalized reflexive sheaf on $\mathbb{P}^n$. Then $\mathcal{V}$ is stable if and only if $\mathcal{V}$ has no sections, i.e. $H^0(\mathbb{P}^n, \mathcal{V}) = 0$. If $c_1(\mathcal{V}) = 0$ then $\mathcal{V}$ is semistable if and only if $H^0(\mathbb{P}^n, \mathcal{V}(-1)) = 0$.*

Proof. ([O-S-S, Chapter 2, Lemma 1.2.5]) If $H^0(\mathbb{P}^n, \mathcal{V}) \neq 0$, then we have a monomorphism $\mathcal{O}_{\mathbb{P}^n} \to \mathcal{V}$. Now $\mu(\mathcal{O}_{\mathbb{P}^n}) = 0 \geq \mu(\mathcal{V})$ and hence $\mathcal{V}$ is not stable.

Conversely assume that $H^0(\mathbb{P}^3, \mathcal{V}) = 0$. Let $\mathcal{W}$ be a rank one coherent subsheaf of $\mathcal{V}$ with a torsion free quotient $\mathcal{Q} = \mathcal{V}/\mathcal{W}$. Then by Lemma 1.6, $\mathcal{W}$ is normal. Since it is a subsheaf of a torsion free sheaf, it is also torsion free. Thus by Lemma 1.5, $\mathcal{W}$ is reflexive and hence by Lemma 1.2, $\mathcal{W}$ is locally free. So $\mathcal{W} = \mathcal{O}_{\mathbb{P}^n}(k)$ for some $k \in \mathbb{Z}$. The inclusion $\mathcal{W} \hookrightarrow \mathcal{V}$ defines a non-zero section of $\mathcal{V}(-k)$. By the assumption, $-k > 0$. Hence $\mu(\mathcal{W}) = k < \mu(\mathcal{V})$.

If $c_1(\mathcal{V})$ is even, then one can show in exactly the same way that $\mathcal{V}$ is semistable if and only if $H^0(\mathbb{P}^n, \mathcal{V}(-1)) = 0$. □

If $c_1(\mathcal{V}) = -1$ in the above proposition, then "stable" and "semistable" are same. As an immediate application of the above proposition we have the following result.

Corollary 1.26. Let $\mathcal{V}$ be a rank two vector bundle on $\mathbb{P}^n$ and suppose $c_1(\mathcal{V}) = 2t$, for some $t \in \mathbb{Z}$. Then $\mathcal{V}$ is stable if and only if $H^0(\mathbb{P}^n, \mathcal{V}(-t)) = 0$. Also $\mathcal{V}$ is semistable but not stable if and only if $H^0(\mathbb{P}^n, \mathcal{V}(-t)) \neq 0$ and $H^0(\mathbb{P}^n, \mathcal{V}(-t-1)) \neq 0$.

Proof. Since $c_1(\mathcal{V}) = 2t$, we have, $c_1(\mathcal{V}(-t)) = 0$. Now the conclusion follows from the above proposition. □

Let $\mathcal{V}$ be a rank r torsion free coherent sheaf on $\mathbb{P}^n$. Then there is a unique integer $k_{\mathcal{V}}$ such that $c_1(\mathcal{V}) \in \{0, 1, \cdots, -r+1\}$. Chern class computaion shows that if $c_i(V) = mr + i$ with $m, i \in \mathbb{Z}$ and $0 \leq i \leq r-1$, then $k_{\mathcal{V}} = -\frac{c_1(\mathcal{V})+i}{r}$. We say that $\mathcal{V}$ is *normalized* if $c_1(\mathcal{V}) \in \{0, 1, \cdots, -r+1\}$. For rank 3 normalized reflexive sheaves on $\mathbb{P}^n$ we have the following criterion of stability and semistability.

Lemma 1.27. *Let $\mathcal{V}$ be a normalized rank 3 reflexive sheaf on $\mathbb{P}^n$. Then $\mathcal{V}$ is stable if and only if*

$$H^0(\mathbb{P}^n, \mathcal{V}) = H^0(\mathbb{P}^n, \mathcal{V}^*) = 0, \text{ for } c_1(\mathcal{V}) = 0$$

$$H^0(\mathbb{P}^n, \mathcal{V}) = H^0(\mathbb{P}^n, \mathcal{V}^*(-1)) = 0, \text{ for } c_1(\mathcal{V}) = -1, -2.$$

If $c_1(\mathcal{V}) = 0$ then $\mathcal{V}$ is semistable if and only if

$$H^0(\mathbb{P}^n, \mathcal{V}(-1)) = H^0(\mathbb{P}^n, \mathcal{V}^*(-1)) = 0.$$

Proof. ([O-S-S, Chapter 2, Remark 1.2.6]) This is similar to the case of rank 2. □

We saw before that a stable vector bundle over a smooth irreducible projective variety is simple. For rank two vector bundles over $\mathbb{P}^n$ the converse is also true.

Proposition 1.28. *Let $\mathcal{V}$ be a rank two simple bundle on $\mathbb{P}^n$. Then $\mathcal{V}$ is stable.*

Proof. ([O-S-S, Chapter 2, Theorem 1.2.10]) Without loss of generality we can assume that $\mathcal{V}$ is normalized. Suppose $\mathcal{V}$ is not stable. Then by Proposition 2.18, $H^0(\mathbb{P}^3, \mathcal{V}) \neq 0$. Also $\mathcal{V}^* = \mathcal{V} \otimes \det(\mathcal{V}^*)$. So $H^0(\mathbb{P}^n, \mathcal{V}^*) \neq 0$. Let $s \in H^0(\mathbb{P}^n, \mathcal{V})$ and $t \in H^0(\mathbb{P}^n, \mathcal{V}^*)$ be two non-zero sections. Then $t \otimes s \in H^0(\mathbb{P}^n, \mathcal{V}^* \otimes \mathcal{V})$ defines an endomorphism of $\mathcal{V}$ which in each fiber has rank at most one. Thus it can not be of the form $\lambda \cdot Id$ for some $\lambda \in \mathbb{C}$. This contradicts that $\mathcal{V}$ is simple. □

Remark 1.29. *For vector bundles of rank at least three the above proposition does not hold in general. See [O-S-S, Page 89-90].*

Let $\mathcal{V}$ be a vector bundle on $\mathbb{P}^n$ and let $s \in H^0(\mathbb{P}^n, \mathcal{V})$ be a global section of $\mathcal{V}$. Then s determines a morphism $\mathcal{O}_{\mathbb{P}^n} \to \mathcal{V}$. Taking duals (i.e. applying the functor $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\cdot, \mathcal{O}_{\mathbb{P}^n})$) we get a morphism $\mathcal{V}^* \to \mathcal{O}_{\mathbb{P}^n}$. The image of this morphism is a sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_{\mathbb{P}^n}$. The corresponding closed subscheme Y of $\mathbb{P}^n$ is called the *scheme of zeros of s*, and is denoted by $(s)_0$.

Remark 1.30. *Let $\mathcal{V}$ be a rank two vector bundle on $\mathbb{P}^n, n \geq 2$ and let $s \in H^0(\mathbb{P}^n, \mathcal{V})$ be a non-zero global section. Let Y be the corresponding scheme of zeros of s. If Y is empty then the morphism $\mathcal{V}^* \to \mathcal{O}_{\mathbb{P}^n}$ is a surjection. In this case, $\mathcal{V}$ is a direct sum of two line bundles. If Y has a component D which is of codimension 1, then D is a divisor. In this case, the section s lies in the image of the map $H^0(\mathbb{P}^n, \mathcal{V}(-D)) \to H^0(\mathbb{P}^n, \mathcal{V})$ obtained by taking cohomology long exact sequence corresponding to the short exact sequence $0 \to \mathcal{V}(-D) \to \mathcal{V} \to \mathcal{V}|_D \to 0$. Let $t \in H^0(\mathbb{P}^n, \mathcal{V}(-D))$ be the section whose image is s. Then the scheme of zeros of t has codimension ≥ 2.*

The following result gives a criterion for two distinct sections of a rank two bundle on $\mathbb{P}^3$ to have distinct scheme of zeros.

Proposition 1.31. *Let $\mathcal{V}$ be a rank two vector bundle on $\mathbb{P}^3$, and assume that for every non-zero section $s \in H^0(\mathbb{P}^3, \mathcal{V})$, the scheme of zeros $(s)_0$ has codimension 2. Then two non-zero sections $s, s' \in H^0(\mathbb{P}^3, \mathcal{V})$ have the same scheme of zeros if and only if $s = \lambda s'$, for some $\lambda \in \mathbb{C}, \lambda \neq 0$.*

Proof. See [Har78, Proposition 1.3]. □

Lemma 1.32. *Let $\mathcal{V}$ be a rank two vector bundle on $\mathbb{P}^n$ and let $s \in H^0(\mathbb{P}^n, \mathcal{V})$ be a global section of $\mathcal{V}$. Let Y be the scheme of zeros of s and let $\mathcal{I}_Y$ be the ideal sheaf of Y. Then we have an exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{s} \mathcal{V} \to \mathcal{I}_Y(k) \to 0$$

where $k := c_1(\mathcal{V})$.

Proof. See [Barth77, Lemma 3]. □

The next result, analogous to Bertini's theorem for divisors (corollary 1.23), gives a criterion for the scheme of zeros of sections to be non-singular. For this result we will consider vector bundles on projective spaces over an arbitrary algebraically closed field k of any characteristic.

Proposition 1.33. *Let k be an algebraically closed field and let $\mathbb{P}^n_k$ be the projective n-space over k. Let $\mathcal{V}$ be a rank two vector bundle on $\mathbb{P}^n_k$.*

(1) If $\mathcal{V}(-1)$ is generated by global sections, then for all sufficiently general sections $s \in H^0(\mathbb{P}^n, \mathcal{V})$, the scheme of zeros $(s)_0$ is non-singular (but not necessarily connected).

(2) If characteristic of k is zero, then the above conclusion holds for weaker condition that $\mathcal{V}$ is generated by global sections.

(3) If $H^1(\mathbb{P}^n, \mathcal{V}^*) = 0$ and $n \geq 3$, then $(s)_0$ is connected for every $s \in H^0(\mathbb{P}^n, \mathcal{V})$.

Proof. See [Har78, Proposition 1.4]. □

Now we will state a very useful results for computing Chern classes of globally generated rank two vector bundles over $\mathbb{P}^3$. Before that we recall a well known result.

Lemma 1.34. *Let $\mathcal{V}$ be a vector bundle on $\mathbb{P}^n$ which is generated by global sections. If $c_1(\mathcal{V}) = 0$ then $\mathcal{V}$ is trivial.*

Proof. See [O-S-S, Chapter 1, §3]. Also see [LN10, Lemma 3.9] for a proof for vector bundles on non-singular irreducible projective variety over $\mathbb{C}$. □

Lemma 1.35. *Let $\mathcal{V}$ be a non-trivial rank two vector bundle on $\mathbb{P}^3$ together with a surjection*

$$\mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{V}.$$

Then $c_1(\mathcal{V}) = 2m$ and $c_2(\mathcal{V}) = 2m^2$ for some integer $m \geq 1$.

Proof. ([LN10, Lemma 3.5]) The surjection $\mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{V}$ gives rise to an exact sequence of vector bundles on $\mathbb{P}^3$:

$$0 \to S \to \mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{V}.$$

Let $c_1(\mathcal{V}) = a$ and $c_2(\mathcal{V}) = b$. Since $\mathcal{V}$ is generated by global sections and non-trivial, $a > 0$ (Lemma 1.34). Since S is a rank two bundle, we have $c_3(S) = 0$. from this we get that $2ba - a^3 = 0$. Since $a \neq 0, a^2 - 2ab = 0$. From this the conclusion follows. $\square$

2 Null Correlation Bundle on Projective Three Space

In this section, we will define null correlation bundle on $\mathbb{P}^3$, the projective three space over $\mathbb{C}$ and prove the main result: Theorem 0.2.

2.1 Space of Null Correlation Bundles

Let $\Omega^1_{\mathbb{P}^3}$ denotes the cotangent bundle of $\mathbb{P}^3$ and let's consider the projective space bundle $\mathbb{P}(\Omega^1_{\mathbb{P}^3})$. Then we have

$$\mathbb{P}(\Omega^1_{\mathbb{P}^3}) \cong \{(x, H) \in \mathbb{P}^3 \times \mathbb{P}^{3^*} : x \in H\}.$$

Let $p : \mathbb{P}(\Omega^1_{\mathbb{P}^3}) \to \mathbb{P}^3$ be the projection and consider the 4×4 complex matrix

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then for all $x \in \mathbb{C}^4$ we have $\langle Ax, x\rangle = 0$ where $\langle x, y\rangle = \sum_i x_i y_i$. Now choose a homogeneous coordinate (x_0, x_1, x_2, x_3) of $\mathbb{P}^3$ and let $(\xi_0, \xi_1, \xi_2, \xi_3)$ be the dual homogeneous coordinate of $\mathbb{P}^{3^*}$. With respect to these coordinates A defines an isomorphism

$$\Phi : \mathbb{P}^3 \to \mathbb{P}^{3^*}.$$

W. Barth called such an isomorphism as *null-correlation*. The condition $\langle Ax, x\rangle = 0$ implies the existence of a section

$$g : \mathbb{P}^3 \hookrightarrow \mathbb{P}(\Omega^1_{\mathbb{P}^3}), x \mapsto (x, \Phi(x)).$$

We can consider this section as a subbundle of $\Omega^1_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(a) \hookrightarrow \Omega^1_{\mathbb{P}^3}$. It can be proved that $a = -2$. Thus the section g defines a trivial subbundle $\mathcal{O}_{\mathbb{P}^3} \subset \Omega^1_{\mathbb{P}^3}$. By taking dual we get a surjection

$$T_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(1),$$

where $T_{\mathbb{P}^3}$ is the tangent bundle of $\mathbb{P}^3$. The kernel of the above surjection, $\mathcal{V}$ say is a rank two vector bundle on $\mathbb{P}^3$. This bundle is called a null correlation bundle on $\mathbb{P}^3$. Using the short exact sequence

$$0 \to \mathcal{V} \to T_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

we can compute the Chern classes of the bundle $\mathcal{V}$ to get $c_1(\mathcal{V}) = 0, c_2(\mathcal{V}) = 1$. Also the bundle $\mathcal{V}$ is stable but the restriction $\mathcal{V}|_H$ to any hyperplane $H \subset \mathbb{P}^3$ is semistable, not stable ([O-S-S, Chapter2, Theorem 1.3.1]). A null correlation bundle is unique up to translation by automorphisms of $\mathbb{P}^3$ and by tensoring by line bundles as given by the following result:

Theorem 2.1. *([Barth77]) Let $\mathcal{V}$ be some rank two μ-stable bundle on $\mathbb{P}^n$, the complex projective n-space. If $n \geq 4$, there is a non-empty open set $U \subset \mathbb{P}^{n*}$, the dual projective space, such that the restriction of $\mathcal{V}$ to all hyperplanes parametrized by U is again stable. The same holds for $n = 3$, unless $\mathcal{V}$ is Null correlation bundle.*

From the above theorem it is obvious that any rank two vector bundle on $\mathbb{P}^3$ that occurs as a kernel of a surjective morphism

$$T_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(1)$$

is a null correlation bundle on $\mathbb{P}^3$.

Now we will give a short description of the space of all null correlation bundles on $\mathbb{P}^3$.

Let S be a projective variety of over $\mathbb{C}$ and let $c_1, c_2 \cdots, c_r \in \mathbb{Z}$. A family of rank r stable bundles over $\mathbb{P}^n$ with Chern classes $c_1, c_2, \cdots, c_r$ and parametrized by S is a rank r vector bundle $\mathcal{V}$ over $S \times \mathbb{P}^n$ such that for all $s \in S$ the bundle

$$\mathcal{V}_s := \mathcal{V}|_{s \times \mathbb{P}^n}$$

over $\mathbb{P}^n \cong s \times \mathbb{P}^n$ is stable and has Chern classes

$$c_i(\mathcal{V}_s) = c_i.$$

Let $p : S \times \mathbb{P}^n \to S$ be the projection onto the first factor. Two families $\mathcal{V}$ and $\mathcal{W}$ parametrized by S are said to be equivalent if there is a line bundle $\mathcal{L}$ over S such that

$$\mathcal{W} \cong \mathcal{V} \otimes p^* \mathcal{L}.$$

We denote by $\sum := \sum_{c_1, c_2, \cdots, c_r}^{\mathbb{P}^n}(S)$ the set of equivalence classes of families of rank r stable bundles over $\mathbb{P}^n$ with Chern classes $c_1, c_2, \cdots, c_r$ and parametrized by S.

Let $\underline{\text{Var}}$ denotes the category of projective varieties over $\mathbb{C}$ and let $\underline{\text{Sets}}$ denotes the category of sets. Then we have a contravariant functor

$$\sum : \underline{\text{Var}} \to \underline{\text{Sets}}.$$

Definition 2.2. A *fine moduli space* for rank r stable bundles over $\mathbb{P}^n$ with given Chern classes $c_1, c_2, \cdots, c_r \in \mathbb{Z}$ is a scheme $\mathcal{M} := \mathcal{M}_{\mathbb{P}^n}(c_1, c_2, \cdots, c_r)$ over $\mathbb{C}$ together with a bundle $\mathcal{U}$ over $\mathcal{M} \times \mathbb{P}^n$ such that the contravariant functor $\sum = \sum_{c_1, c_2, \cdots, c_r}^{\mathbb{P}^n}$ is represented by the pair $(\mathcal{M}, \mathcal{U})$.

It was proved by Le Potier in [lePotier79] that there is no fine moduli space for rank 2 stable bundles over $\mathbb{P}^2$ with $c_1 = 0$ and c_2 even.

Definition 2.3. A scheme $\mathcal{M} := \mathcal{M}_{\mathbb{P}^n}(c_1, c_2, \cdots, c_r)$ over $\mathbb{C}$ is a *coarse moduli space* for the contravariant functor $\sum_{c_1, c_2, \cdots, c_r}^{\mathbb{P}^n}$ if the following conditions are satisfied:

(i) there is a natural transformation of contravariant functors

$$\sum \to \mathcal{H}om(-, \mathcal{M}_{\mathbb{P}^n}(c_1, c_2, \cdots, c_r))$$

which is bijective for any (reduced) point x_0.

(ii) for every variety X and every natural transformations $\sum \to \mathcal{H}om(-, X)$ there is a unique morphism

$$\phi : \mathcal{M}_{\mathbb{P}^n}(c_1, c_2, \cdots, c_r) \to X$$

for which the diagram

$$\begin{tikzcd}
\sum^{\mathbb{P}^n}_{c_1,c_2,\cdots,c_r} \arrow[r] \arrow[d] & \mathcal{H}om(-, \mathcal{M}_{\mathbb{P}^n}(c_1, c_2, \cdots, c_r)) \arrow[d, "\phi_*"] \\
\mathcal{H}om(-, X) \arrow[r, "\mathrm{Id}"] & \mathcal{H}om(-, X)
\end{tikzcd}$$

commutes.

The *existence* of the moduli space $\mathcal{M} := \mathcal{M}_{\mathbb{P}^3}(0, 1)$ is of particular interest. Before describing this space, we will recall some results.

Lemma 2.4. *Let $\mathcal{V}$ be a rank two stable bundle on $\mathbb{P}^3$ with Chern classes $c_1(\mathcal{V}) = 0, c_2(\mathcal{V}) = 1$. Suppose $H^1(\mathbb{P}^3, \mathcal{V}(-2)) = 0$. Then $\mathcal{V}$ fits into the exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega^1_{\mathbb{P}^3}(1) \to \mathcal{V} \to 0.$$

In other words, $\mathcal{V}$ is a null correlation bundle on $\mathbb{P}^3$.

Proof. See [O-S-S, Chapter 2, §3, Example 6]. □

The next result shows that the condition $H^1(\mathbb{P}^3, \mathcal{V}(-2)) = 0$ in the above lemma is not necessary.

Lemma 2.5. *Let $\mathcal{V}$ be a rank two stable bundle on $\mathbb{P}^3$ with Chern classes $c_1(\mathcal{V}) = 0, c_2(\mathcal{V}) = 1$. Then*

$$H^1(\mathbb{P}^3, \mathcal{V}(-2)) = 0.$$

Proof. See [O-S-S, Chapter 2, Lemma 4.3.1]. □

Combining above two lemmas we get the following result.

Lemma 2.6. *Every stable rank 2 bundle $\mathcal{V}$ on $\mathbb{P}^3$ with Chern classes $c_1(\mathcal{V}) = 0, c_2(\mathcal{V}) = 1$ is given by an exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega^1_{\mathbb{P}^3}(1) \to \mathcal{V} \to 0.$$

Each of these bundles $\mathcal{V}$ is a null correlation bundle on $\mathbb{P}^3$.

Proof. See [O-S-S, Chapter 2, Lemma 4.3.2]. □

The above lemma shows that the moduli space $\mathcal{M}_{\mathbb{P}^3}(0, 1)$ is the actually the moduli space of null correlation bundles on $\mathbb{P}^3$. The description of this space is given in the following result.

Proposition 2.7. *The moduli space $\mathcal{M}_{\mathbb{P}^3}(0, 1)$ for the null correlation bundles on $\mathbb{P}^3$ is isomorphic to the complement*

$$\mathbb{P}^5 \setminus Gr(2, \mathbb{C}^4)$$

in $\mathbb{P}^5$ of the Grassmannian variety of two dimensional quotients of $\mathbb{C}^4$.

Proof. See [O-S-S, Chapter 2, Theorem 4.3.4]. □

2.2 Main Result

Let $\sigma : \mathbb{P}^3 \to \mathbb{P}^3$ be the automorphism

$$[x_1, x_2, x_3, x_4] \mapsto [-x_1, -x_2, x_3, x_4].$$

Then σ is an involution i.e., $\sigma^2 = id_{\mathbb{P}^3}$. The automorphism σ lifts to an automorphism of the vector space

$$\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$$

with subspace of σ invariants (i.e., $v \in \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)); \sigma(v) = v$)

$$V =< X_1^2, X_2^2, X_3^2, X_4^2, X_1X_2 + X_3X_4, X_1X_2 - X_3X_4 > .$$

Clearly the linear system defined by V is base point free on $\mathbb{P}^3$ and hence

$$[x_1, x_2, x_3, x_4] \mapsto [x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2 + x_3x_4, x_1x_2 - x_3x_4]$$

gives a morphism

$$\phi : \mathbb{P}^3 \to \mathbb{P}^5.$$

Since this morphism is defined by σ invariants, the morphism ϕ factors through $\mathbb{P}^3/< \sigma >$. In other words, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}^3 & \xrightarrow{\phi} & \mathbb{P}^5 \\ \downarrow & \nearrow & \\ \mathbb{P}^3/< \sigma > & & \end{array}$$

Set $[Z_0, Z_1, \dots, Z_5]$ as homogeneous coordinates of $\mathbb{P}^5$. Note that image of $\mathbb{P}^3$ under ϕ is contained in the intersection of two linearly independent quadrics, namely

$$Z_2Z_3 - Z_0Z_1 + Z_4Z_5 \tag{1}$$

and

$$2Z_0Z_1 + 2Z_2Z_3 - Z_4^2 - Z_5^2. \tag{2}$$

Since a non constant morphism from a projective space is always finite, the image of $\mathbb{P}^3$ under ϕ is equal to intersection of two quadrics in $\mathbb{P}^5$. Degree considerations will imply that the morphism ϕ maps $\mathbb{P}^3/ < \sigma >$ in a bijective manner onto the intersection of the quadrics (1) and (2).

Since equation (1) defines the Grassmannian $Gr(2, \mathbb{C}^4)$ in $\mathbb{P}^5$ under the Plüker imbedding, the morphism ϕ can also be thought of as a morphism into the Grassmannian $Gr(2, \mathbb{C}^4)$. Let

$$Q := \phi^*(Q_u),$$

where Q_u is the universal quotient bundle on the Gassmannian $Gr(2, \mathbb{C}^4)$ of rank two.

Chern classes of the rank two vector bundle Q on $\mathbb{P}^3$

First note that the determinant of the universal quotient bundle on the Grassmannian $Gr(2, \mathbb{C}^4)$ is the restriction of $\mathcal{O}_{\mathbb{P}^5}(1)$. Thus we see that the determinant of Q is $\mathcal{O}_{\mathbb{P}^3}(2)$, and hence the first Chern class of Q,

$$c_1(Q) = 2[H], \tag{3}$$

where H is the hyperplane class of $\mathbb{P}^3$.

Next we calculate the second Chern class $c_2(Q)$ of Q on $\mathbb{P}^3$. This can be found by considering the pull back under the morphism ϕ of a two dimensional linear subspace of $\mathbb{P}^5$

contained in $Gr(2, \mathbb{C}^4)$ (see below 2.8). Let's consider a two dimensional linear subspace defined by $Z_4 = Z_2 - Z_0 = Z_1 - Z_3 = 0$ contained in $Gr(2, \mathbb{C}^4)$. Then the pull back of this linear subspace under the morphism ϕ is the scheme $\ell_1 \cup \ell_2$, where $\ell_1 \subset \mathbb{P}^3$(resp. $\ell_2 \subset \mathbb{P}^3$) is a line defined by $X_1 - X_3 = X_2 + X_4 = 0$ (resp. $X_1 + X_3 = X_2 - X_4 = 0$). From this we see that

$$c_2(Q) = 2[H^2]. \tag{4}$$

Remark 2.8. *A rank two vector bundle E on $\mathbb{P}^3$ which is generated globally by four sections induces an exact sequence*

$$0 \to S \to \mathcal{O}^4_{\mathbb{P}^3} \to E \to 0 \tag{5}$$

where S is a rank 2 vector bundle on $\mathbb{P}^3$. From this exact sequence we see that

$$c_2(S) = c_2(E),$$

[LN10, Lemma 3.3]. This Chern class is the class of the pull back, under the morphism of $\mathbb{P}^3$ to $Gr(2, \mathbb{C}^4)$ given by the exact sequence (5), of a two dimensional linear subspace of $\mathbb{P}^5$ contained in $Gr(2, \mathbb{C}^4)$ [GH, p. 757].

Stability property of the bundle Q

Note that, by construction, the rank two bundle Q on $\mathbb{P}^3$ comes with an exact sequence

$$0 \to E \to \mathcal{O}^4_{\mathbb{P}^3} \to Q \to 0.$$

In fact this exact sequence is the pull back of the universal exact sequence on $Gr(2, \mathbb{C}^4)$. Since Q is quotient of the trivial bundle of rank four, Q is generated by the images of these four sections. It is easy to see that these are four independent sections of Q. Let V be the four dimensional subspace of $H^0(Q)$ generated by these four sections of Q. Since the morphism

$$\phi : \mathbb{P}^3 \to Gr(2, \mathbb{C}^4)(\subset \mathbb{P}(\wedge^2 V) \simeq \mathbb{P}^5)$$

is given by the linear system

$$\wedge^2 V \subset H^0(\mathbb{P}^3, \det(Q)) = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$$

we see that, for a general element s of V the zero set is a non degenerate (i.e., not contained in a hyperplane) codimension two smooth subscheme of degree two of $\mathbb{P}^3$ and hence disjoint union of two lines. In fact smoothness of the subscheme follows from the general result of Kleiman [Har, Section III, Theorem 10.8]. To see the zero locus is non-degenerate we argue as follows: By Bertini's Theorem [Har, Section III, Corollary 10.9], $\phi^{-1}(H)$ is a smooth quadric in $\mathbb{P}^3$, where H is a general hyperplane of $\mathbb{P}^5$. Restriction of ϕ to such a quadric will pull back the class of a linear plane contained in $Gr(2, \mathbb{C}^4)(\subset \mathbb{P}(\wedge^2 V) \simeq \mathbb{P}^5)$ to a multiple of the class of one of the rulings of the quadric.

Let s be a general global section of Q. By previous observation

$$(s)_0 = \{p \in \mathbb{P}^3 : s(p) = 0\} = \ell_1 \cup \ell_2,$$

where ℓ_1 and ℓ_2 are two disjoint lines in $\mathbb{P}^3$. The section s gives rise to an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to Q \to \mathcal{I}_{\ell_1 \cup \ell_2}(2) \to 0 \tag{6}$$

[Barth77, Lemma 3]. Tensoring the exact sequence (6) by $\mathcal{O}_{\mathbb{P}^3}(-1)$ and using the fact that $\ell_1 \cup \ell_2$ is not contained in a hyperplane, we conclude that $H^0(\mathbb{P}^3, Q(-1)) = 0$. Since $c_1(Q) = 2$, using Corollary 1.26 we have the following:

Lemma 2.9. *The bundle Q on $\mathbb{P}^3$ is stable.*

Restriction of the bundle Q to hyperplanes of $\mathbb{P}^3$

The aim of this section is to prove the following

Lemma 2.10. *The bundle $Q|_H$ is semi-stable but not stable, where H is any hyper-plane of $\mathbb{P}^3$.*

Proof: Let H be a hyper-plane of $\mathbb{P}^3$. Since $c_1(Q) = 2$ and $H \simeq \mathbb{P}^2$, using Corollary 1.26 we see that to prove the lemma, we only need to show

$$H^0(H, Q|_H(-1)) \neq 0 \text{ and } H^0(H, Q|_H(-2)) = 0.$$

The restriction of the exact sequence (6) to H gives an exact sequence on H:

$$0 \to \mathcal{O}_H \to Q|_H \to \mathcal{I}_{\ell_1 \cup \ell_2}(2) \otimes \mathcal{O}_H \to 0. \tag{7}$$

Suppose that the lines ℓ_1 and ℓ_2 are not contained in H. Then

$$H \cap (\ell_1 \cup \ell_2) = \{p_1, p_2\},$$

where p_1 and p_2 are two distinct points of H. In this case the exact sequence (7) reduced to the following exact sequence

$$0 \to \mathcal{O}_H \to Q|_H \to \mathcal{I}_{\{p_1,p_2\}}(2) \to 0. \tag{8}$$

Tensoring the exact sequence (8) by $\mathcal{O}_H(-1)$ (resp. by $\mathcal{O}_H(-2)$) and taking the cohomology, we see that $H^0(H, Q|_H(-1)) \neq 0$ (respectively, $H^0(H, Q|_H(-2)) = 0$).

Since ℓ_1 and ℓ_2 are disjoint, a hyper-plane H of $\mathbb{P}^3$ can contain at most one of these lines. Suppose a hyper-plane H contains one of these lines, say ℓ_1. Then H intersects ℓ_2 at exactly one point, say at p. So, the restriction of the exact sequence (6) to H give rise to an exact sequence

$$0 \to \mathcal{O}_H(1) \to Q|_H \to \mathcal{I}_p(1) \to 0. \tag{9}$$

Tensoring the exact sequence (9) by $\mathcal{O}_H(-1)$ (resp. by $\mathcal{O}_H(-2)$) and taking the cohomology, we see that $H^0(H, Q|_H(-1)) \neq 0$ (respectively, $H^0(H, Q|_H(-2)) = 0$). This completes the proof the lemma.

Note that, in [Barth77], a Null correlation bundle is defined to be a rank two bundle on $\mathbb{P}^3$ with the first Chern class $c_1 = 0$ and the second Chern class $c_2 = 1$, which is stable but when restricted to a general hyper plane is not stable, only semi-stable. It is easy to see that the bundle $Q(-1)$, where Q is the rank two bundle on $\mathbb{P}^3$ that we defined above, is a Null correlation bundle. Since tensoring by line bundle does not change the stability properties of a bundle, any bundle which is obtained from Null-correlation bundle by tensoring by a line bundle will also be called Null correlation bundle. Using above two lemmas we get the main result of this section.

Theorem 2.11. *The rank two vector bundle Q is a null correlation bundle on $\mathbb{P}^3$.*

Remark 2.12. *According to Barth [Barth77], a Null-correlation bundle is homogeneous for the symplectic group and is unique up to translation by automorphisms of $\mathbb{P}^3$.*

If we replace the involution σ in our construction by a conjugate involution $g\sigma g^{-1}$ the bundle we get is $g^*(Q)$, where g is an automorphism of $\mathbb{P}^3$. Since a Null-correlation bundle is a unique bundle (up to translation by automorphisms) which is stable but the restriction of it to any hyper-plane H is semi-stable, not stable [Barth77], we conclude that the bundle Q, that we constructed above, is a Null-correlation bundle on $\mathbb{P}^3$.

3 Semistability of Secant Bundles on Second Symmetric Power of Curves

Let C be a smooth irreducible projective curve of genus g over $\mathbb{C}$, the field of complex numbers. Consider the variety $C^n := C \times C \times \cdots \times C$ (n times), the n-fold cartesian product of C. Let S_n denotes the symmetric group of n elements. Then there is a natural action of S_n on C^n, given by $(\sigma, (x_1, x_2, \cdots, x_n)) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, \cdots x_{\sigma(n)})$. The quotient $S^n(C) := C^n/S_n$ is called the *n-th symmetric power of* C. It is an irreducible smooth projective variety of dimension n. A point in $S^n(C)$ can be thought as an effective divisor of degree n on C and $S^n(C)$ can be regarded as the set of all effective divisors of degree n on C. With respect to this identification, we can write an element of $S^n(C)$ as $\sum_{i=1}^n x_i, x_i \in C$. Given a vector bundle E on C, there is a naturally associated vector bundle on $S^n(C)$. In this section we will recall some properties of this bundle and also prove the main results: Theorems 0.5, 0.6, 0.7.

3.1 Secant Bundles on Symmetric Power of Curves

Consider the variety $S^n(C) \times C$. Set $\Delta_n := \{(D, p) \in S^n(C) \times C : p \in D\}$. Then Δ_n is a divisor on $S^n(C) \times C$, called *the universal divisor of degree n on C*. Let $q_1 : S^n(C) \times C \to S^n(C)$ and $q_2 : S^n(C) \times C \to C$ be the projections. Let $q := q_1|_{\Delta_n} : \Delta_n \to S^n(C)$ be the restriction of q_1. Then q is a finite morphism of degree n. Let E be a vector bundle of rank r on C. Then we have the following exact sequence of $S^n(C) \times C$ modules:

$$0 \to q_2^*E \otimes \mathcal{O}_{S^n(C)\times C}(-\Delta_n) \to q_2^*E \to q_2^*E \otimes \mathcal{O}_{\Delta_n} \to 0.$$

Set $\mathcal{F}_n(E) := (q_2)_*(q_1^*E \otimes \mathcal{O}_{\Delta_n}) = q_*(q_1^*E|_{\Delta_n})$. Then from the above exact sequence we get an exact sequence on $S^n(C)$:

$$0 \to (q_2)_*(q_1^*E \otimes \mathcal{O}_{S^n(C)\times C}(-\Delta_n)) \to (q_2)_*(q_1^*E) \to \mathcal{F}_n(E) \to R^1(q_2)_*(q_1^*E \otimes \mathcal{O}_{S^n(C)\times C}(-\Delta_n)).$$

The coherent sheaf $\mathcal{F}_n(E)$ is actually a vector bundle on $S^n(C)$ of rank nr. This bundle is called the *secant bundle* on $S^n(C)$. This was first introduced by R. Schwarzenberger ([S64]).

Let $\pi_i : C^n \to C$ be the i-th coordinate projections and let $\mathcal{E}(n) := \oplus_{i=1}^n \pi_i^* E$. Then $\mathcal{E}(n)$ is a locally free sheaf of rank nr on C^n. Let $\pi : C^n \to S^n(C)$ be the natural quotient morphism. This is a Galois covering morphism with the Galois group S_n. Then the action of S_n on C^n induces an action on $\mathcal{E}(n)$. Any element of S_n permutes the fiber of $\mathcal{E}(n)$ the same way as it permutes the elements of C^n. The action of S_n on $\mathcal{E}(n)$ induces an action on $\pi_*\mathcal{E}(n)$. Let $\mathcal{V}_n(E) := (\pi_*\mathcal{E}(n))^{S_n} \subset \pi_*\mathcal{E}(n)$ be the invariant subsheaf defined as follows: Over any open set U the sections are the S_n-invariant sections of $\pi_*\mathcal{E}(n)$, i.e. $\Gamma(U, \mathcal{V}_n(E)) := \Gamma(U, \pi_*\mathcal{E}(n))^{S_n}$ Since π is finite proper surjective morphism of smooth projective varieties, $\pi_*\mathcal{E}(n)$ is locally free. The coherent sheaf $\mathcal{V}_n(E)$ is also locally free because locally it is a direct summand of $\pi_*\mathcal{E}(n)$.

Proposition 3.1. *The two vector bundles $\mathcal{V}_n(E)$ and $\mathcal{F}_n(E)$ on $S^n(C)$ as defined above are canonically isomorphic.*

Proof. See [BL11, Proposition 2.1]. □

Assume $g \geq 2$. Let $\mathrm{Pic}^n(C)$ denotes the the variety parametrizing all degree n line bundles on C and let $\nu : S^n(C) \to \mathrm{Pic}^n(C)$ be map defined by $\{x_1, x_2, \ldots, x_n\} \mapsto \sum_{i=1}^n x_i$. (Here $\{x_1, x_2, \cdots, x_n\}$ is an unordered collection of points $x_i \in C$, considered as an element of $S^n(C)$ and we are identifying the divisor $\sum_{i=1}^n x_i$ on C and the corresponding line bundle on C.) Let $\theta \in H^2(S^n(C), \mathbb{Z})$ be the pull back of the class of theta divisor in $H^2(\mathrm{Pic}^n(C), \mathbb{Z})$

under the map ν. Using Künneth formula we can write

$$H^2(S^n(C) \times C, \mathbb{Z}) =$$
$$(H^2(S^n(C), \mathbb{Z}) \otimes H^0(C, \mathbb{Z})) \oplus (H^1(S^n(C), \mathbb{Z}) \otimes H^1(C, \mathbb{Z})) \oplus (H^0(S^n(C), \mathbb{Z}) \otimes H^2(C, \mathbb{Z}))$$

Let δ be the class of the universal divisor Δ_n. Then using above we can decompose δ as $\delta = \delta^{2,0} + \delta^{1,1} + \delta^{0,2}$, where $\delta^{i,j} \in H^i(S^n(C), \mathbb{Z}) \otimes H^j(C, \mathbb{Z})$. Let's denote the Künneth component $\delta^{2,0}$ by x.

Lemma 3.2. *Let E be a line bundle on C and let the degree of E is d, then the Chern polynomial of $\mathcal{F}_n(E)$ is*

$$c_t(\mathcal{F}_n(E)) = (1 - xt)^{n-d+g-1} e^{t\theta/(1-xt)}.$$

Proof. See [ACGH, Chapter VIII, Lemma 2.5]. □

Remark 3.3. *Let $n = 2$ and E be a line bundle on C of degree d. Then from the above Lemma we get that the first Chern class*

$$c_1(\mathcal{F}_2(E)) = (d - g - 1)x + \theta$$

and the second Chern class

$$c_2(\mathcal{F}_2(E)) = \binom{d-g}{2} x^2 + (d-g)x \cdot \theta + \frac{\theta^2}{2}.$$

The cohomology group $H^4(S^2(C), \mathbb{Z})$ is canonically isomorphic to $\mathbb{Z}$. Also we have $x^2 = 1, x \cdot \theta = g, \theta^2 = g(g-1)$. Using this we get $c_2(\mathcal{F}_2(E)) = \binom{d}{2}$. (See [ELN11])

Now assume E is any rank r vector bundle on C. Choose a filtration of E such that the successive quotients are line bundles. This filtration induces a filtration of subbundles of

$\mathcal{F}_n(E)$. Also note that if $\mathcal{V} = \oplus_k L_k$, where each L_k is a line bundle on C, then $\mathcal{F}_n(\mathcal{V}) = \oplus_k \mathcal{F}_n(L_k)$. Using this, we see that, the Chern character of $\mathcal{F}_n(E)$ is ([BL11])

$$\mathrm{ch}(\mathcal{F}_n(E)) = \mathrm{degree}(E)(1 - e^{-x}) - r(g-1) + r(n+g-1+\theta)e^{-x}.$$

In particular, when $n = 2$ the first Chern class of $\mathcal{F}_2(E)$ is given by

$$c_1(\mathcal{F}_2(E)) = (d - r(g+1))x + r\theta, \tag{10}$$

where $d = \mathrm{degree}(E)$.

Let C be a smooth irreducible projective curve of genus g over $\mathbb{C}$. Let E be a line bundle on C of degree d. Consider the associated vector bundle $\mathcal{F}_2(E)$ of rank two on $S^2(C)$, the second symmetric power of C. Let $x \in C$ and consider the divisor $H' = x + C$ on $S^2(C)$. Then H' is an ample divisor on $S^2(C)$. The following results gives criterion for $\mathcal{F}_2(E)$ to be stable and semistable with respect to H'.

Theorem 3.4. *Let E be a very ample line bundle on C. Then the vector bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is semistable with respect to the ample divisor $x + C$.*

Proof. See [ELN11, Theorem 4.3]. □

Theorem 3.5. *Let E be a very ample line bundle on C. Assume that genus of C is greater than zero or genus of C is zero and degree(E) is greater than one. Then the vector bundle $\mathcal{F}_2(E)$ is stable with respect to the ample divisor $x + C$.*

Proof. See [ELN11, Theorem 4.6]. □

Remark 3.6. *Let $C = \mathbb{P}^1$ and $E = \mathcal{O}_{\mathbb{P}^1}(1)$ then $S^2(C) = \mathbb{P}^2$ and $\mathcal{F}_2(E) = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$. Hence $\mathcal{F}_2(E)$ is not stable. We will see later that the condition on degree of E in Theorem 3.5 is a necessary and sufficient one.*

We have a further generalization of the above theorems.

Theorem 3.7. *Let E be a non-trivial line bundle on C.*

(1) The vector bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is semistable with respect to the ample divisor $x+C$.

(2) If E and E' are two line bundles on C such that $\mathcal{F}_2(E) \cong \mathcal{F}_2(E')$, then $E \cong E'$. In fact, every isomorphism of $\mathcal{F}_2(E)$ and $\mathcal{F}_2(E')$ induces an isomorphism of E and E'.

(3) The vector bundle $\mathcal{F}_2(E)$ is stable with respect to the ample divisor $x + C$, unless $E \cong \mathcal{O}_C(x)$ or $E \cong \mathcal{O}_C(-x)$ for some $x \in C$.

Proof. See [BN13, Theorem 1.1]. □

Theorem 3.7(2) is a special case of the following result.

Theorem 3.8. *Assume genus of C is greater than one. If E and E' are two semistable bundles on C such that the vector bundles $\mathcal{F}_n(E)$ and $\mathcal{F}_n(E')$ on $S^n(C)$ are isomorphic for a fixed n, then E is isomorphic to E'.*

Proof. See [BN12, Theorem 1.1]. □

Remark 3.9. *The condition on genus of C in the above theorem can be relaxed with an additional assumption that $\mu(E) > n-1$ and $\mu(E') > n-1$. See [BP11, Theorem 2.1, 3.1].*

Let $x \in C$. Then $x + S^{n-1}(C)$ is an ample divisor on $S^n(C)$. The following result gives a criterion for the vector bundle $\mathcal{F}_n(E)$ on $S^n(C)$ to be stable when E is a line bundle on C.

Theorem 3.10. *Let E be a line bundle on C of degree $d \geq n$. Then the vector bundle $\mathcal{F}_n(E)$ on $S^n(C)$ is stable with respect to the ample divisor $x + S^{n-1}(C)$.*

Proof. See [M, Proposition 4.3.6]. □

3.2 Main Results

Let C be a smooth irreducible projective curve of genus g over $\mathbb{C}$. Let E be a rank 2 vector bundle on C. and let $\mathcal{F}_2(E)$ be the associated secant bundle on $S^2(C)$. Define a morphism

$f : C \times C \to S^2(C) \times C$ by $(x, y) \mapsto (x + y, y)$. Then this morphism gives an isomorphism of $C \times C$ onto the universal divisor Δ_2. Let $\pi : C \times C \to S^2(C)$ be the quotient map and let $q : \Delta_2 \to S^2(C)$ be the map obtained by restricting the projection $q_1 : S^2(C) \times C \to S^2(C)$ to Δ_2. Then we have $q \circ f = \pi$. Let $p_i : C \times C \to C$ be the i-th coordinate projections, $i = 1, 2$. Then we have $\mathcal{F}_2(E) = \pi_* p_2^* E$. Fix an element $x \in C$. Then the image of $x \times C$ in $S^2(C)$ (under the morphism π) is an ample divisor. We denote this divisor by $x + C$. We prove the following result:

Theorem 3.11. *Let E be a rank two stable vector bundle of even degree $d \geq 2$ on C such that $\mathcal{F}_2(E)$ is globally generated. Then the bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is $\mu_{H'}$-semistable with respect to the ample class $H' = x + C$.*

Remark 3.12. *If E is an even degree vector bundle which is a quotient of direct sum of very ample line bundles, i.e. if there is a surjection $\oplus L_i \to E$ where each L_i is a very ample line bundle on C, then E satisfies the property of Theorem 3.11.*

We recall some well known results.

Lemma 3.13. *Let $f : X \longrightarrow Y$ be a finite surjective morphism of non-singular surfaces, F be a vector bundle on Y, and H be an ample divisor on Y. Assume $f^*(F)$ is $\mu_{f^*(H)}$-semistable (respectively, $\mu_{f^*(H)}$-stable). Then F is μ_H-semistable (respectively, μ_H-stable).*

Proof. ([ELN11, Lemma 4.4]). If F is not μ_H-semistable (respectively, μ_H-stable) then there is a torsion free coherent proper subsheaf M of F such that $\mu_H(M) > \mu_H(F)$ (respectively, $\mu_H(M) \geq \mu_H(F)$). Then we have $\mu_{f^*(H)}(f^*(M)) > \mu_{f^*(H)}(f^*(F))$ (respectively, $\mu_{f^*(H)}(f^*(M)) \geq \mu_{f^*(H)}(f^*(F))$). This contradicts that $f^*(F)$ is $\mu_{f^*(H)}$-semistable (respectively, $\mu_{f^*(H)}$-stable). $\square$

Note that $\pi : C \times C \to S^2(C)$ is a finite surjective morphism between two non-singular surfaces. So by Lemma 3.13, to prove Theorem 3.11 it is sufficient to prove that the vector bundle $\pi^*\mathcal{F}_2(E)$ is semistable on $C \times C$ with respect to the ample divisor $H := \pi^*(H') = [x \times C + C \times x]$.

The first Chern class of $\mathcal{F}_2(E)$ is given by $c_1(\mathcal{F}_2(E)) = (d-2(g+1))x + 2\theta$ (see equation (10)). Thus $c_1(\pi^*\mathcal{F}_2(E)) = \pi^* c_1(\mathcal{F}_2(E)) = (d-2(g+1))[x \times C + C \times x] + 2\pi^*(\theta)$. To find an expression for $\pi^*(\theta)$ we use the following result:

Lemma 3.14. *Let C be a smooth irreducible curve of genus $g \geq 1$ and let K_C be the canonical bundle of C. Let $J^{g-1}(C)$ be the variety of line bundles of degree $g-1$ on C, and let Θ be the divisor on $J^{g-1}(C)$ consisting of line bundles with non-zero sections. Let ξ be a line bundle on C of degree $g-3$ and*

$$\nu_\xi : C \times C \longrightarrow J^{g-1}(C)$$

be the morphism $(x, y) \mapsto \mathcal{O}_{C\times C}(x+y) \otimes \xi$. Then

$$\nu_\xi^*(\Theta) \cong p_1^*(K_C \otimes \xi^*) \otimes p_2^*(K_C \otimes \xi^*) \otimes \mathcal{O}_{C\times C}(-\Delta)$$

where Δ is the diagonal of $C \times C$ and $p_i : C \times C \longrightarrow C$ is the i-th coordinate projection.

Proof. ([ELN11, Lemma 4.5]) For each $x \in C$ we have

$$\nu_\xi|_{x\times C}(\Theta) = p_2^*(K_C \otimes \xi^*(-x))|_{x\times C}$$

and

$$\nu_\xi|_{C\times x}(\Theta) = p_1^*(K_C \otimes \xi^*(-x))|_{C\times x}.$$

From this we can conclude the result. □

Using this we get $\pi^*(\theta) = (g+1)[x \times C + C \times x] - \Delta$. Thus

$$c_1(\pi^*(\mathcal{F}_2(E))) = d[x \times C + C \times x] - 2\Delta,$$

and

$$\mu_H(\pi^*(\mathcal{F}_2(E))) = \frac{d-2}{2}.$$

Now Δ is canonically isomorphic to C and hence any vector bundle on C can be consider as a vector bundle on Δ in a canonical way. Also note that $p_i^*(E)|_\Delta = E$. The vector bundle $\pi^*(\mathcal{F}_2(E))$ fits in the following short exact sequence on $C \times C$:

$$0 \to \pi^*(\mathcal{F}_2(E)) \to p_1^*(E) \oplus p_2^*(E) \xrightarrow{\sigma} E = p_1^*(E)|_\Delta = p_2^*(E)|_\Delta \to 0 \tag{11}$$

where the map σ is given by $\sigma : (u, v) \mapsto u|_\Delta - v|_\Delta$. Let $\phi_i : p_1^*(E) \oplus p_2^*(E) \longrightarrow p_i^*(E)$ be the restriction of the projection $p_1^*(E) \oplus p_2^*(E) \longrightarrow p_i^*(E)$ to $\pi^*(\mathcal{F}_2(E)) \subset p_1^*(E) \oplus p_2^*(E)$. Then from the exact sequence (11), we get the following two exact sequences:

$$0 \to p_1^*(E) \otimes \mathcal{O}_{C\times C}(-\Delta) \to \pi^*(\mathcal{F}_2(E)) \xrightarrow{\phi_1} p_2^*(E) \to 0, \tag{12}$$

and

$$0 \to p_2^*(E) \otimes \mathcal{O}_{C\times C}(-\Delta) \to \pi^*(\mathcal{F}_2(E)) \xrightarrow{\phi_2} p_1^*(E) \to 0 \tag{13}$$

[BN12, Section 3].

Now we prove the following result.

Lemma 3.15. *$p_i^*(E)$ is μ_H-stable, $\forall i = 1, 2$.*

Proof. Due to symmetry, we will do it only for ${p_2}^*E$. Since over a smooth irreducible projective surface double dual of a coherent torsion free sheaf is free, by taking double dual if necessary, we see that to prove stability or semistability it is enough to consider subsheaves which are line bundles. Let L be a line bundle on $C \times C$ which is a subsheaf of ${p_2}^*E$ such that the quotient, M say, is torsion free. We have an exact sequence

$$0 \longrightarrow L \longrightarrow {p_2}^*E \longrightarrow M \longrightarrow 0.$$

We restrict this exact sequence to $x \times C$ and $C \times x$, respectively, to obtain the following exact sequences

$$0 \longrightarrow L|_{x\times C} \longrightarrow E \longrightarrow M|_{x\times C} \longrightarrow 0,$$

and

$$0 \longrightarrow L|_{C\times x} \longrightarrow \mathcal{O}_C \oplus \mathcal{O}_C \longrightarrow M|_{C\times x} \longrightarrow 0.$$

From the first exact sequence we get, $\deg(L|_{x\times C}) = c_1(L).[x \times C] < \mu(E) = \frac{d}{2}$, since E is stable. And from the second exact sequence we get $\deg(L|_{C\times x}) = c_1(L).[C \times x] \leq 0$. Thus $\deg(L) = c_1(L).[x \times C + C \times x] < \frac{d}{2} = \mu_H(p_2^*E)$, proving the Lemma. $\square$

Proof. (of Theorem 3.11):

Let L be a line bundle which is a subsheaf of $\pi^*(\mathcal{F}_2(E))$ such that the quotient is torsion free. Suppose there is a non-zero homomorphism from L to $p_1^*(E)(-\Delta) := p_1^*(E) \otimes \mathcal{O}_{C\times C}(-\Delta)$. Then $\mu_H(L) < \mu_H(p_1^*(E)(-\Delta)) = \frac{d-4}{2} < \frac{d-2}{2}$. So assume that there is no non-zero map from L to $p_1^*(E)(-\Delta)$. Thus there is an injection $L \to p_2^*(E)$ so that $\mu_H(L) < \mu_H(p_2^*(E)) = \frac{d}{2}$. Since d is even, $\mu_H(L) \leq \frac{d}{2} - 1 = \frac{d-2}{2}$.

Now let F be a rank two coherent subsheaf of $\pi^*(\mathcal{F}_2(E))$ such that quotient is torsion-free. Then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & p_1^*(E)(-\Delta) \arrow[r] & \pi^*(\mathcal{F}_2(E)) \arrow[r] & p_2^*(E) \arrow[r] & 0 \\
0 \arrow[r] & F' \arrow[r] \arrow[u] & F \arrow[r] \arrow[u] & F'' \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where the vertical arrows are injections. Suppose that both F' and F'' are non-zero. These two are rank 1 coherent sheaf. So we have, $\deg(F') = \mu_H(F') < \mu_H(p_1^*(E)(-\Delta)) = \frac{d-4}{2}$ and $\deg(F'') = \mu_H(F'') < \mu_H(p_2^*(E)) = \frac{d}{2}$. Thus $\mu_H(F) = \frac{1}{2}(\deg(F') + \deg(F'')) < \frac{d-2}{2}$. Now assume at least one of F' and F'' is zero. First let F'' be zero. Then we have an injection $F \to p_1^*(E)(-\Delta)$ and the cokernel is a torsion sheaf. If the cokernel is supported at only finitely many points, then $\mu_H(F) = \mu_H(p_1^*(E)(-\Delta)) < \frac{d-2}{2}$. If the cokernel is supported at a co-dimension 1 subscheme, then $\mu_H(F) < \mu_H(p_1^*(E)(-\Delta)) < \frac{d-2}{2}$. Now let F' is zero. So we have an injection $F \to p_2^*(E)$ and the cokernel is a torsion sheaf. If the cokernel is supported at a subscheme of co-dimension 1, then $\mu_H(F) < \mu_H(p_2^*(E)) = \frac{d}{2}$ so that $\mu_H(F) \leq \frac{d-1}{2}$. If $\mu_H(F) = \frac{d-1}{2}$, then the cokernel is supported on a divisor of degree one. Now an effective divisor of degree one on $C \times C$ is of the form $x \times C$ or $C \times x$, for some $x \in C$. Thus $c_1(F)$ is of the form $c_1(p_2^*(E)) + [-x \times C]$ or $c_1(p_2^*(E)) + [-C \times x]$. But $c_1(\pi^*(\mathcal{F}_2(E)) = d[C \times x + x \times C] - 2\Delta$, therefore $c_1((\pi^*(\mathcal{F}_2(E)/F)) = (d+1)[x \times C] - 2\Delta$ or $d[x \times C] + [C \times x] - 2\Delta$. In both the cases the torsion free sheaf $\pi^*(\mathcal{F}_2(E)/F$ restricted to any curve of the form $x \times C$ has negative degree. This gives a contradiction to the fact that $\pi^*(\mathcal{F}_2(E)$ is generated by sections. Thus we have, $\mu_H(F) \leq \frac{d-2}{2}$. If the cokernel is supported only at finitely many points then $\mu_H(F) = \mu_H(p_2^*(E)) = \frac{d}{2}$. In this case, F is a rank two stable sheaf and hence it is isomorphic to $p_2^*(E)$. So the exact sequence (2) splits, i.e., $\pi^*(\mathcal{F}_2(E)) \cong p_1^*(E)(-\Delta) \oplus p_2^*(E)$. Since $p_1^*(E)|_{x \times C}$ is trivial, $\deg(p_1^*(E)(-\Delta)|_{x \times C}) < 0$. This contradicts the fact that $\mathcal{F}_2(E)$ and hence $\pi^*(\mathcal{F}_2(E))$ is globally generated.

Let F be a rank 3 coherent subsheaf of $\pi^*(\mathcal{F}_2(E))$ such that the quotient is torsion free. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p_1^*(E)(-\Delta) & \longrightarrow & \pi^*(\mathcal{F}_2(E)) & \longrightarrow & p_2^*(E) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F' & \longrightarrow & \mathrm{F} & \longrightarrow & \mathrm{F''} & \longrightarrow & 0
\end{array}$$

where the vertical arrows are injections. We have two possibilities: (I) $\mathrm{rank} F' = 2$ and $\mathrm{rank} F'' = 1$; (II) $\mathrm{rank} F' = 1$ and $\mathrm{rank} F'' = 2$. Suppose that $\mathrm{rank} F' = 2$ and $\mathrm{rank} F'' = 1$. By the arguments above, we have, $\mu_H(F') \leq \frac{d-4}{2}$ and $\mu_H(F'') < \frac{d}{2}$. So

$$\mu_H(F) < \frac{3d-8}{6} < \frac{d-2}{2}.$$

Now assume that $\mathrm{rank} F' = 1$ and $\mathrm{rank} F'' = 2$. In this case, we have, $\mu_H(F') < \frac{d-4}{2}$ and

$\mu_H(F'') \leq \frac{d}{2}$. If d is even, $\mu_H(F') \leq \frac{d-4}{2} - 1$, hence $\mu_H(F) \leq \frac{3d-6}{6} = \frac{d-2}{2}$. $\square$

Now we prove the semistablity of $\mathcal{F}_2(E)$ for degree E odd. First we recall some definitions.

Definition 3.16. Let E be a non-zero vector bundle on C and $k \in \mathbb{Z}$, we denote by $\mu_k(E)$ the rational number

$$\mu_k(E) := \frac{\text{degree}(E) + k}{\text{rank}(E)}.$$

We say that the vector bundle E is (k, l)-stable (resp. (k, l)-semistable) if, for every proper subbundle F of E we have

$$\mu_k(F) < \mu_{-l}(E/F)(resp.\mu_k(F) \leq \mu_{-l}(E/F)).$$

Note that usual Mumford stability is equivalent to $(0, 0)$-stability. If $g \geq 3$, then there always exists a $(0, 1)$-stable bundle and if $g \geq 4$, then the set of $(0, 1)$-stable bundles form a dense open subset of the moduli space of stable bundles over C of rank 2 and degree d. [NR78, Section 5]

Theorem 3.17. *Assume the genus of C greater than 2. Let E be a rank two $(0, 1)$-stable bundle of odd degree $d \geq 1$ on C such that $\mathcal{F}_2(E)$ is globally generated. Then the bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is $\mu_{H'}$-semistable with respect to the ample class $H' = x + C$.*

Proof. Let L be a line bundle which is a subsheaf of $\pi^*(\mathcal{F}_2(E))$ such that the quotient is torsion free. Suppose there is a non-zero homomorphism from L to $p_1^*(E)(-\Delta) := p_1^*(E) \otimes \mathcal{O}_{C\times C}(-\Delta)$. Then $\mu_H(L) < \mu_H(p_1^*(E)(-\Delta)) = \frac{d-4}{2} < \frac{d-2}{2}$. So assume that there is no non-zero map from L to $p_1^*(E)(-\Delta)$. Thus there is an injection $L \to p_2^*(E)$. Now consider the exact sequence,

$$0 \longrightarrow L \longrightarrow p_2^*(E) \longrightarrow M \longrightarrow 0, \tag{14}$$

where M is a sheaf of rank 1. Restricting the above exact sequence to $C \times x$, we see that, $c_1(L).[C \times x] \leq 0$. On the other hand, restricting the above exact sequence to $C \times x$ and using that E is (0.1)-stable, we get that $c_1(L).[C \times x] < \frac{d-1}{2}$. Since L is a line bundle, $c_1(L).[C \times x] \leq \frac{d-3}{2}$. So we have $\mu_H(L) \leq \frac{d-3}{2} < \frac{d-2}{2}$.

Let's assume F be a rank two coherent subsheaf of $\pi^*(\mathcal{F}_2(E))$ such that quotient is torsion-free. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_1^*(E)(-\Delta) & \longrightarrow & \pi^*(\mathcal{F}_2(E)) & \longrightarrow & p_2^*(E) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \end{array}$$

where the vertical arrows are injections. We need to consider three different cases: (I) rank $F' = 1 =$ rank F''; (II) $F'' = 0$; (III) $F' = 0$. In each of these cases, we can argue exactly as in the case of even degree to conclude that $\mu_H(F) \leq \frac{d-2}{2} = \mu_H(\pi^*\mathcal{F}_2(E))$.

Now assume F is subsheaf of $\pi^*\mathcal{F}_2(E)$ rank 3. Then again we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_1^*(E)(-\Delta) & \longrightarrow & \pi^*(\mathcal{F}_2(E)) & \longrightarrow & p_2^*(E) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & F' & \longrightarrow & \mathrm{F} & \longrightarrow & \mathrm{F}" & \longrightarrow & 0 \end{array}$$

where the vertical arrows are injections. We have two possibilities: (I) rank$F' = 2$ and rank$F'' = 1$; (II) rank$F' = 1$ and rank$F'' = 2$. Using the same argument as in Theorem 3.11, we can show that in the case of (I), $\mu_H(F) < \frac{d-2}{2}$. Now consider the case (II). In this case, restricting the exact sequence $0 \to F' \to p_1^*(E)(-\Delta)$ to $x \times C$ and $C \times x$, we get that

$$0 \to F'|_{x \times C} \to \mathcal{O}_C(-x)$$

and

$$0 \to F'|_{C \times x} \to E \otimes \mathcal{O}_C(-x).$$

From these two exact sequences and using the fact that E is $(0,1)$-stable we see that $\mu_H(F') < \frac{d-4}{2}$ and hence $\mu_H(F') \leq \frac{d-6}{2}$. Also using the same argument as above, we have, in any case, $\mu_H(F'') \leq \frac{d}{2}$. Combining all these, we get that $\mu_H(F) < \frac{d-2}{2}$. □

Now we will investigate the restriction of $\mathcal{F}_2(E)$ to the curves of the form $x + C$ where $x + C$ is the reduced divisor of $S^2(C)$ whose support equals to $\{x + c : c \in C\}$. For this we have the following theorem.

Theorem 3.18. *Let C be a smooth irreducible projective curve over $\mathbb{C}$ of genus g and let E be a rank two vector bundle on C of degree $d \geq 3$. Then for any $x \in C, \mathcal{F}_2(E)|_{x+C}$ is not semistable.*

Proof. First note that, since E is locally free, p_2^*E is flat over $S^2(C)$ and using the base change formula we get

$$\mathcal{F}_2(E)|_{x+C} = \pi_*(p_2^*E|_{\pi^{-1}(x+C)}).$$

Also we have the following exact sequence

$$0 \to p_2^*E|_{\pi^{-1}(x+C)} \to p_2^*E|_{x\times C} \oplus p_2^*E|_{C\times x} \to E|_{(x,x)} \to 0.$$

From this exact sequence and using the fact that $\pi|_{x\times C} : x\times C \to x+C$ and $\pi|_{C\times x} : C\times x \to x+C$ are isomorphisms and $p_2^*E|_{x\times C} = E$ and $p_2^*E|_{C\times x} = E_x \otimes \mathcal{O}_C$, we get an injective map

$$0 \to E \otimes \mathcal{O}_C(-x) \to \mathcal{F}_2(E)|_{x+C}.$$

Now the degree of $E \otimes \mathcal{O}_C(-x) = d - 2$ and that of $\mathcal{F}_2(E)|_{x+C} = d - 2$. So the cokernel is rank 2 coherent sheaf of degree zero. If it is torsion free then clearly $\mathcal{F}_2(E)|_{x+C}$ is not semistable. If the cokernel has torsion, then there is an effective divisor D such that the above map factors through $E \otimes \mathcal{O}_C(-x) \otimes \mathcal{O}_C(D)$ and the cokernel will be again torsion free. But in this case the degree of the cokernel will be of negative degree. So in this case $\mathcal{F}_2(E)|_{x+C}$ has a torsion free quotient of negative degree and hence not semistable. □

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