Differential calculus and Yang-Mills functional in Noncommutative Geometry

By

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I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution/University.

Satyajit Guin

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Contents

Synopsis

This thesis is concerned with analysis of spectral triples as indicated in the last chapter of Connes [15] and the Yang-Mills action functional in noncommutative geometry. It is divided in five chapters. The first chapter is devoted to the preliminary notions required to understand the content of this thesis. The second and third chapters deal with the equivalence of two approaches to Yang-Mills for noncommutative n-tori and the quantum Heisenberg manifolds respectively. Fourth chapter is devoted to study the behaviour of Connes' calculus for the quantum double suspension. The final chapter deals with the behaviour of Connes' calculus for the tensor product of spectral triples.

Preliminaries on differential calculus and Yang-Mills functional

In this chapter we discuss preliminary notions required to carry on with the materials of latter chapters.

The study of Yang-Mills functional in noncommutative geometry started with [20], where authors defined it on a C^*-dynamical system. Later the spectral triple approach became more popular in noncommutative geometry and Connes defined notion of Yang-Mills in this set-up. There is a general recipe to produce a "spectral triple" from a C^*-dynamical system but it does not tell whether the resulting object is a true spectral triple, but they are candidates, and one has to verify the relevant conditions on a case by case basis. However, for noncommutative torus and the quantum Heisenberg manifolds indeed one obtains a genuine spectral triple. Then one encounters the natural question whether these two notions of Yang-Mills agree. This is the content of the second and third chapters. Final two chapters deal with differential calculus in noncommutative geometry.

We note down here few definitions for the sake of understanding the contents.

Definition 0.0.1. *A spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *over an associative algebra* $\mathcal{A}$ *with involution* $\star$ *consists of the following things :*

1. *a* $\star$*-representation of* $\mathcal{A}$ *on a Hilbert space* $\mathcal{H}$*,*

2. *an unbounded selfadjoint operator* D*,*

3. D *has compact resolvent and* $[D, a]$ *extends to a bounded operator on* $\mathcal{H}$ *for every* $a \in \mathcal{A}$*.*

Definition 0.0.2 ([15])**.** *Let* $\mathcal{A}$ *be a unital* $\star$*-algebra. A Hermitian structure on a finitely generated projective (f.g.p) right* $\mathcal{A}$*-module* $\mathcal{E}$ *is an* $\mathcal{A}$*-valued positive-definite sesquilinear mapping* $\langle\ ,\ \rangle_{\mathcal{A}}$ *such that*

1. $\langle \xi, \xi' \rangle_{\mathcal{A}}^* = \langle \xi', \xi \rangle_{\mathcal{A}}\,, \ \forall\, \xi, \xi' \in \mathcal{E}\,,$

2. $\langle \xi, \xi'.a \rangle_{\mathcal{A}} = (\langle \xi, \xi' \rangle_{\mathcal{A}}).a\,, \ \forall\, \xi, \xi' \in \mathcal{E}, \ \forall\, a \in \mathcal{A}\,,$

3. *The map* $\xi \longmapsto \Phi_\xi$ *from* $\mathcal{E}$ *to* $\mathcal{E}^*$*, given by* $\Phi_\xi(\eta) = \langle \xi, \eta \rangle_{\mathcal{A}}\,, \forall \eta \in \mathcal{E}\,,$ *gives a conjugate linear* $\mathcal{A}$*-module isomorphism between* $\mathcal{E}$ *and* $\mathcal{E}^*$*. This property is referred as the self-duality of* $\mathcal{E}$*.*

Definition 0.0.3 ([15])**.** *Let* $\Omega^\bullet(\mathcal{A}) = \bigoplus_{k=0}^\infty \Omega^k(\mathcal{A})$ *be the reduced universal differential graded algebra over* $\mathcal{A}$*. Here* $\Omega^k(\mathcal{A}) := \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes k}\,,\ \bar{\mathcal{A}} = \mathcal{A}/\mathbb{C}$*. The graded product is the multilinear extension of the following product on simple tensors, given by*

$$
\begin{aligned}
& (a_0 \otimes \overline{a_1} \otimes \ldots \otimes \overline{a_m})\,.\,\big(b_0 \otimes \overline{b_1} \otimes \ldots \otimes \overline{b_n}\big) \\
:= \ & a_0 \otimes (\otimes_{j=1}^{m-1} \overline{a_j}) \otimes \overline{a_m b_0} \otimes (\otimes_{i=1}^{n} \overline{b_i}) \\
& + \sum_{i=1}^{m-1} (-1)^i a_0 \otimes \overline{a_1} \otimes \ldots \otimes \overline{a_{m-i} a_{m-i+1}} \otimes \ldots \otimes \overline{a_m} \otimes (\otimes_{i=0}^{n} \overline{b_i}) \\
& + (-1)^m a_0 a_1 \otimes (\otimes_{j=2}^{m} \overline{a_j}) \otimes (\otimes_{i=0}^{n} \overline{b_i})\,.
\end{aligned}
$$

for $a_0 \otimes \overline{a_1} \otimes \ldots \otimes \overline{a_m} \in \Omega^m(\mathcal{A})$ *and* $b_0 \otimes \overline{b_1} \otimes \ldots \otimes \overline{b_n} \in \Omega^n(\mathcal{A})$*. There is a differential* d *acting on* $\Omega^\bullet(\mathcal{A})$*, given by*

$$d(a_0 \otimes \overline{a_1} \otimes \ldots \otimes \overline{a_k}) := 1 \otimes \overline{a_0} \otimes \overline{a_1} \otimes \ldots \otimes \overline{a_k} \;\; \forall \, a_j \in \mathcal{A}\,.$$

It satisfies the following relations

1. $d^2\omega = 0 \;\; \forall\, \omega \in \Omega^\bullet(\mathcal{A})$,

2. $d(\omega_1\omega_2) = (d\omega_1)\omega_2 + (-1)^{deg(\omega_1)}\omega_1 d\omega_2\,,\ \forall$ *homogeneous* $\omega_1 \in \Omega^\bullet(\mathcal{A})$.

We have a $\star$*-representation* π *of* $\Omega^\bullet(\mathcal{A})$ *on* $\mathcal{H}$*, given by*

$$\pi(a_0 \otimes \overline{a_1} \otimes \ldots \otimes \overline{a_k}) := a_0[D, a_1] \ldots [D, a_k]\,;\; a_j \in \mathcal{A}.$$

Let $J_0^{(k)} := \{\omega \in \Omega^k : \pi(\omega) = 0\}$ *and* $J' = \bigoplus J_0^{(k)}$*. But* J' *fails to be a differential ideal. We consider* $J^\bullet = \bigoplus J^{(k)}$ *where* $J^{(k)} = J_0^{(k)} + dJ_0^{(k-1)}$*. Then* $J^\bullet$ *becomes a differential graded two-sided ideal and hence the quotient* $\Omega_D^\bullet := \Omega^\bullet / J^\bullet$ *becomes a differential graded algebra, called the Connes' calculus. The representation* π *gives an isomorphism*

$$\Omega_D^k \cong \pi(\Omega^k)/\pi(dJ_0^{k-1})\,.$$

Definition 0.0.4 ([15])**.** *Let* $(\mathcal{A}, \mathcal{H}, D)$ *be a spectral triple and* $\mathcal{E}$ *be a Hermitian, f.g.p module over* $\mathcal{A}$*. A compatible connection on* $\mathcal{E}$ *is a* $\mathbb{C}$*-linear mapping* $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1$ *such that*

1. $\nabla(\xi a) = (\nabla\xi)a + \xi \otimes da, \quad \forall\, \xi \in \mathcal{E}, a \in \mathcal{A}\,,$

2. $\langle\, \xi, \nabla\eta \,\rangle - \langle\, \nabla\xi, \eta \,\rangle = d\langle\, \xi, \eta \,\rangle_{\mathcal{A}} \quad \forall\, \xi, \eta \in \mathcal{E}$ *(Compatibility).*

The Yang-Mills functional is a certain functional defined on the affine space $\widetilde{C}(\mathcal{E})$ of compatible connections.

Definition 0.0.5 ([12])**.** *Let* $(\mathcal{A}, G, \alpha, \tau)$ *be a* C^**-dynamical system with a* G*-invariant, faithful trace* τ *and* $\mathcal{E}$ *be a f.g.p module over* $\mathcal{A}$ *with a Hermitian Structure. A compatible connection on* $\mathcal{E}$ *is a* $\mathbb{C}$*-linear map* $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes Lie(G)^*$ *such that for all* $X \in Lie(G)$ *and* $\xi, \xi' \in \mathcal{E}$ *,* $a \in \mathcal{A}$ *one has*

1. $\nabla_X(\xi \cdot a) = \nabla_X(\xi) \cdot a + \xi \cdot \delta_X(a)$,

2. $\langle \nabla_X \xi , \xi' \rangle_{\mathcal{A}} + \langle \xi , \nabla_X \xi' \rangle_{\mathcal{A}} = \delta_X(\langle \xi, \xi' \rangle_{\mathcal{A}})$ *(Compatibility).*

In this setting also, the Yang-Mills functional is a certain functional defined on the affine space $C(\mathcal{E})$ of compatible connections.

Definition 0.0.6 ([24]). *Let $\mathcal{A}$ be a unital C^*-algebra and l denotes the left shift operator on $\ell^2(\mathbb{N})$, defined by $l(e_n) = e_{n-1}$, $l(e_0) = 0$ on the standard orthonormal basis (e_n). The quantum double suspension of $\mathcal{A}$, denoted by $\Sigma^2\mathcal{A}$, is the C^*-algebra generated by $a \otimes |e_0\rangle\langle e_0|$ and $1 \otimes l$ in $\mathcal{A} \otimes \mathscr{T}$, where $\mathscr{T}$ is the Toeplitz algebra.*

Definition 0.0.7 ([11]). *For any spectral triple $(\mathcal{A}, \mathcal{H}, D)$, $(\Sigma^2\mathcal{A}, \Sigma^2\mathcal{H} := \mathcal{H} \otimes \ell^2(\mathbb{N})$, $\Sigma^2 D := D \otimes I + F \otimes N)$ becomes a spectral triple where F is the sign of the operator D and N is the number operator on $\ell^2(\mathbb{N})$ given by $N(e_n) = ne_n$. This is called the quantum double suspension of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.*

Equivalence of the two approaches to Yang-Mills theory on Noncommutative Tori

In this chapter we prove two theorems, one regarding the structure of finitely generated projective module endowed with a *Hermitian* structure over spectrally invariant subalgebras of C^*-algebras, and the other the equivalence of Yang-Mills approaches on noncommutative n-torus. The equivalence for n=2 case was addressed by Connes in [15] (Proposition 13, page 582). Here we take up the higher dimensional noncommutative torus and show that even in this case it holds.

Theorem 0.0.8. *Let $\mathcal{E}$ be a finitely generated projective $\mathcal{A}$-module with a Hermitian structure. Assume $\mathcal{A}$ is stable under holomorphic function calculus in a C^*-algebra A. Then we can have a self-adjoint idempotent $p \in M_n(\mathcal{A})$ such that $\mathcal{E} = p\mathcal{A}^n$ and $\mathcal{E}$ has the induced canonical Hermitian structure.*

Theorem 0.0.9. *Let $C(\mathcal{E}), \widetilde{C}(\mathcal{E})$ be the affine spaces of compatible connections for the C^*-dynamical system and spectral triple approaches respectively. Then both these are in one-one correspondence through an affine isomorphism, and the value of Yang-Mills functional on corresponding elements of these spaces are same upto a positive scalar factor.*

Yang-Mills on Quantum Heisenberg Manifolds

In this chapter we move one step further to deal with this problem of equivalence of two approaches to Yang-Mills on the quantum Heisenberg manifolds, defined by Rieffel in [35].

Definition 0.0.10. *For any positive integer c, let S^c denote the space of smooth functions $\Phi : \mathbb{R} \times \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ such that*

- $\Phi(x + k, y, p) = exp(2\pi ickpy)\Phi(x, y, p)$ *for all $k \in \mathbb{Z}$,*
- *for every polynomial P on $\mathbb{Z}$ and every partial differential operator $\widetilde{X} = \frac{\partial^{m+n}}{\partial x^m \partial y^n}$ on $\mathbb{R} \times \mathbb{T}$ the function $P(p)(\widetilde{X}\Phi)(x, y, p)$ is bounded on $K \times \mathbb{Z}$ for any compact subset K of $\mathbb{R} \times \mathbb{T}$.*

For each $\hbar, \mu, \nu \in \mathbb{R}, \mu^2 + \nu^2 \neq 0$, let $\mathcal{A}_\hbar^\infty$ denote S^c with product and involution defined by

$$(\Phi \star \Psi)(x, y, p) = \sum_q \Phi(x - \hbar(q-p)\mu, y - \hbar(q-p)\nu, q)\Psi(x - \hbar q\mu, y - \hbar q\nu, p - q)$$

$$\Phi^*(x, y, p) = \bar{\Phi}(x, y, -p).$$

Then $\pi : \mathcal{A}_\hbar^\infty \to \mathcal{B}(L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z}))$ given by

$$(\pi(\Phi)\xi)(x, y, p) = \sum_q \Phi(x - \hbar(q - 2p)\mu, y - \hbar(q - 2p)\nu, q)\xi(x, y, p - q)\,,$$

gives a faithful representation of the involutive algebra $\mathcal{A}_\hbar^\infty$. $\mathcal{A}_{\mu,\nu}^{c,\hbar}$ = norm closure of $\pi(\mathcal{A}_\hbar^\infty)$ is called the Quantum Heisenberg Manifold.

The spectral triple on quantum Heisenberg manifold is constructed in [9] and here we prove the following theorem.

Theorem 0.0.11. *Let $\mathcal{E}$ be a finitely generated projective $\mathcal{A}_\hbar^\infty$ module with a Hermitian structure and $C(\mathcal{E}), \widetilde{C}(\mathcal{E})$ be the affine spaces of compatible connections for the C^*-dynamical system and spectral triple approaches respectively. Then $\Phi : C(\mathcal{E}) \longrightarrow \widetilde{C}(\mathcal{E})$ given by $\Phi(\nabla) = \widetilde{\nabla}$, where $\widetilde{\nabla}(\xi) = i\nabla(\xi)$, is well-defined and*

$$\tfrac{1}{2}(Tr_\omega |D|^{-3})\mathrm{YM}(\nabla) = \mathrm{YM}(\Phi(\nabla))\,.$$

Connes' Calculus for The Quantum Double Suspension

Associated to every spectral triple $(\mathcal{A}, \mathcal{H}, D)$ there is a differential graded algebra (dga) $\Omega_D^\bullet(\mathcal{A})$ [15] defined by Connes. Here in this chapter we set ourselves with the task of understanding this concept. There is no general recipe to compute this calculus and instances of computations are very few available in the literature, only for some particular examples ([15],[10]). We want to compute this dga for a family of spectral triples. For this purpose we compute $\Omega_D^\bullet$ for the quantum double suspended spectral triple $(\varSigma^2\mathcal{A}, \varSigma^2\mathcal{H}, \varSigma^2 D)$, a notion introduced in [11]. At the end of this chapter we study behaviour of compatible connections, curvatures on the quantum double suspended spectral triple. We have the following theorems in this chapter.

Theorem 0.0.12. *For $(\varSigma^2\mathcal{A}, \varSigma^2\mathcal{H}, \varSigma^2 D)$ we have,*

1. $\Omega^1_{\varSigma^2 D}(\varSigma^2\mathcal{A}) \cong \Omega^1_D(\mathcal{A}) \otimes \mathcal{S} \bigoplus \varSigma^2\mathcal{A}\,.$

2. $\Omega^n_{\varSigma^2 D}(\varSigma^2\mathcal{A}) \cong \Omega^n_D(\mathcal{A}) \otimes \mathcal{S}\,,$ *for all* $n \geq 2\,.$

3. *The differential* $\delta^0 : \Sigma^2\mathcal{A} \longrightarrow \Omega^1_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ *is given by,*

$$a \otimes T + f \longmapsto [D, a] \otimes T \bigoplus (a \otimes [N, T] + f') .$$

4. *The differential* $\delta^1 : \Omega^1_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \longrightarrow \Omega^2_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ *is given by,*

$$\delta^1|_{\Omega^1_D(\mathcal{A})\otimes\mathcal{S}} = d^1 \otimes 1 \quad \textit{and} \quad \delta^1|_{\Sigma^2\mathcal{A}} = 0.$$

5. *The differential* $\delta^n : \Omega^n_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \longrightarrow \Omega^{n+1}_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ *is given by,*

$$\delta^n = d^n \otimes 1$$

for all $n \geq 2$.

Here $d : \Omega^\bullet_D(\mathcal{A}) \longrightarrow \Omega^{\bullet+1}_D(\mathcal{A})$ *is the differential of Connes' complex and* $\mathcal{S}$ *denotes subspace of finitely supported matrices in* $\mathcal{B}(\ell^2(\mathbb{N}))$.

Theorem 0.0.13. *Let* $\mathcal{E}$ *be a finitely generated projective module over* $\mathcal{A}$*, equipped with a Hermitian structure* $\langle\ ,\ \rangle_{\mathcal{A}}$. *Assume that* $\mathcal{A}$ *is spectrally invariant. Choose a projection* $p \in M_n(\mathcal{A})$ *such that* $\mathcal{E} = p\mathcal{A}^n$ *and* $\mathcal{E}$ *has the induced canonical Hermitian structure. Let* $\widetilde{\mathcal{E}} = (p \otimes u)(\Sigma^2\mathcal{A})^n$ *and restrict the canonical structure on* $(\Sigma^2\mathcal{A})^n$ *to* $\widetilde{\mathcal{E}}$. *We have an one-one affine morphism* $\widetilde{\phi}_{con} : Con(\mathcal{E}) \longrightarrow Con(\widetilde{\mathcal{E}})$ *which preserves the Grassmannian conection, and an one-one* $\mathbb{C}$*-linear map* $\psi : Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D(\mathcal{A})) \longrightarrow Hom_{\Sigma^2\mathcal{A}}\left(\widetilde{\mathcal{E}}, \widetilde{\mathcal{E}} \otimes_{\Sigma^2\mathcal{A}} \Omega^2_{\Sigma^2 D}(\Sigma^2\mathcal{A})\right)$ *such that the following diagram*

$$\begin{array}{ccc}
Con(\mathcal{E}) & \xrightarrow{\quad\widetilde{\phi}_{con}\quad} & Con(\widetilde{\mathcal{E}}) \\
f\Big\downarrow & & \Big\downarrow f \\
Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D(\mathcal{A})) & \xrightarrow[\psi]{} & Hom_{\Sigma^2\mathcal{A}}\left(\widetilde{\mathcal{E}}, \widetilde{\mathcal{E}} \otimes_{\Sigma^2\mathcal{A}} \Omega^2_{\Sigma^2 D}(\Sigma^2\mathcal{A})\right)
\end{array}$$

commutes. Here 'f' is the map which sends any compatible connection to its associated curvature.

It should be noted that probably this is the first systematic computation of this complex.

Multiplicativity of Connes' Calculus

In this final chapter we study behaviour of $\Omega_D^\bullet$ for the tensor product of *even spectral triples*. This problem was investigated earlier by Kastler-Testard in [25] but their work appears inconclusive. Hence we reinvestigate this problem and propose a category theoretic conclusion. We start with a quadruple $(\mathcal{A}, \mathbb{V}, D, \gamma)$ where $\mathcal{A}$ is a unital, associative algebra over a field $\mathbb{K}$, represented on a $\mathbb{K}$-vector space $\mathbb{V}$, $D \in \mathcal{E}nd(\mathbb{V})$ and $\gamma \in \mathcal{E}nd(\mathbb{V})$ is a $\mathbb{Z}_2$-grading operator which commutes with $\mathcal{A}$ and anticommutes with D. We call such a quadruple an *even algebraic spectral triple*. We prove that the collection of such quadruples forms a category $\widetilde{\mathcal{S}pec}$. It turns out that $\widetilde{\mathcal{S}pec}$ is a monoidal category and $\mathcal{F} : \widetilde{\mathcal{S}pec} \longrightarrow DGA$ given by $(\mathcal{A}, \mathbb{V}, D, \gamma) \longmapsto \Omega_D^\bullet(\mathcal{A})$ is a covariant functor, where DGA denotes the category of dgas over $\mathbb{K}$. We are interested in knowing whether $\mathcal{F}$ is a monoidal functor and the article [25] gives a partial answer to it. Here we consider a subcategory $\widetilde{\mathcal{S}pec_{sub}}$ of $\widetilde{\mathcal{S}pec}$, objects of which are $(\mathcal{A}, \mathbb{V}, D, \gamma)$ with $\gamma \in \pi(\mathcal{A})$. Then $\widetilde{\mathcal{S}pec_{sub}}$ becomes a monoidal subcategory of $\widetilde{\mathcal{S}pec}$. We prove that there is a covariant functor $\mathcal{G} : \widetilde{\mathcal{S}pec} \longrightarrow \widetilde{\mathcal{S}pec_{sub}}$. We show that restricted to the subcategory $\widetilde{\mathcal{S}pec_{sub}}$, the functor $\mathcal{F}$ is a monoidal functor. To show that the functor $\mathcal{F}$ when restricted to $\widetilde{\mathcal{S}pec_{sub}}$ is not trivial we compute $\mathcal{F} \circ \mathcal{G}$ for a canonical spectral triple associated with compact smooth manifolds and the noncommutative torus.

Chapter 1

Preliminaries on differential calculus and Yang-Mills functional

This chapter is devoted to register preliminary notions essential to understand the content of this thesis. Here we discuss differential calculus in noncommutative set-up and the two approaches to the Yang-Mills functional defined by Connes-Rieffel ([20]) and Connes ([15]).

1.1 Spectral triple and differential calculus

A noncommutative differential structure on an associative algebra $\mathcal{A}$ is the specification of a differential graded algebra (dga). This is interpreted as the space of differential forms. Study of differential calculus in noncommutative geometry appears in early 80's through the invention of noncommutative differential geometry ([13]), and to search for its examples ([42],[43]). Since then quite a lot of works have been done involving differential calculus in various noncommutative contexts for e.g. see ([34],[1],[4],[31],[32]) and references therein. In his spectral formulation of the subject Connes unified various treatments in noncommutative geometry in terms of a $\mathcal{K}$-cycle $(\mathcal{A}, \mathcal{H}, D)$, often called spectral triple. He defined a canonical dga $\Omega_D^\bullet$ associated to a $\mathcal{K}$-cycle $(\mathcal{A}, \mathcal{H}, D)$ and extended several classical notions including connection, curvature, Yang-Mills action functional etc. to the noncommutative framework. It is also shown in ([15]) that using this dga one can produce Hochschild cocycles and cyclic cocycles (under certain assumption) for Poincaré dual algebras which establishes $\Omega_D^\bullet$ worth studying. Last but not the least Connes showed that in the case of classical spectral triple associated to a compact Riemannian spin manifold, $\Omega_D^\bullet$ gives back the space of de-Rham forms on the manifold. This establishes $\Omega_D^\bullet$ as a genuine noncommutative generalization of the classical de-Rham complex of a manifold.

Definition 1.1.1. *A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ over an associative algebra $\mathcal{A}$ with involution $\star$ consists of the following things :*

1. *a $\star$-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$.*
2. *an unbounded selfadjoint operator D.*
3. *D has compact resolvent and $[D, \pi(a)]$ extends to a bounded operator on $\mathcal{H}$ for every $a \in \mathcal{A}$.*

We shall assume that $\mathcal{A}$ is unital and the unit $1 \in \mathcal{A}$ acts as the identity on $\mathcal{H}$. If there is a $\mathbb{Z}_2$ grading $\gamma \in \mathcal{B}(\mathcal{H})$ which commutes with every element of $\mathcal{A}$ and anticommutes with D then the spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is called an *even spectral triple.*

Definition 1.1.2 ([15])**.** *Let $\mathcal{H}$ be a Hilbert space and*

$$\mathcal{L}^{(1,\infty)} := \{T \in \mathcal{K}(\mathcal{H}) : \ ||T||_{1,\infty} = \sup_N \frac{\sum_{i=1}^N \mu_i(T)}{log(N)} < \infty\}.$$

Here the numbers $\mu_i(T)$ are the eigenvalues of $|T|$ arranged in decreasing order. Let

$$a_N = \frac{\sum_{i=1}^N \mu_i(T)}{log(N)}.$$

The Dixmier trace $Tr_\omega(T)$ of T is defined for positive operators $T \in \mathcal{L}^{(1,\infty)}$ to be

$$Tr_\omega(T) = \lim_\omega a_N$$

where lim_ω is a scale-invariant positive extension of the usual limit, to all bounded sequences.

The Dixmier trace is a non-normal trace on a space of linear operators on a Hilbert space larger than the space of trace class operators. It enjoys the following properties :

1. $Tr_\omega(T)$ is linear in T.
2. For $T \geq 0$, $Tr_\omega(T) \geq 0$.
3. If S is bounded then $Tr_\omega(TS) = Tr_\omega(ST)$.
4. $Tr_\omega(T)$ is independent of the choice of inner product on $\mathcal{H}$.
5. $Tr_\omega(T)$ is zero for all trace class operators T i,e. it is singular trace.

For proof of these facts and general introduction to Dixmier trace one can look at the excellent article by Sukochev ([39]). In ([16]), Connes showed that Wodzicki's noncommutative residue of a pseudodifferential operator on a manifold is equal to its Dixmier trace.

For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, if $|D|^{-p}$ is in the ideal of Dixmier traceable operators $\mathcal{L}^{(1,\infty)}$ then we say that the *spectral triple* is p-summable. Note that the compact resolvent condition is equivalent to saying that $exp(-tD^2)$ is a compact operator for some (and hence all) $t > 0$. This can be seen by considering the closed ideal $\mathcal{G} = \{f \in C_0(\mathbb{R})\colon f(D) \text{ is compact}\}$ of $C_0(\mathbb{R})$ and thus $e^{-tx^2} \in \mathcal{G}$ iff $\mathcal{G} = C_0(\mathbb{R})$ iff $(x \pm i)^{-1} \in \mathcal{G}$. If $exp(-tD^2)$ is a trace class operator for all $t > 0$ then we call the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ a θ-summable spectral triple. Existence of θ-summable spectral triples has been proved in ([14]). Note that every p-summable spectral triple is θ-summable with bound $Tr(exp(-tD^2)) = O(t^{-p})$ as $t \to 0$.

To every spectral triple $(\mathcal{A}, \mathcal{H}, D)$ Connes associated the following differential graded algebra (dga).

Definition 1.1.3 ([15])**.** *Let* $\Omega^\bullet(\mathcal{A}) = \bigoplus_{k=0}^{\infty} \Omega^k(\mathcal{A})$ *be the reduced universal dga over* $\mathcal{A}$. *Here* $\Omega^k(\mathcal{A}) := \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes k}$, $\bar{\mathcal{A}} = \mathcal{A}/\mathbb{C}$. *The graded product is given by*

$$
\begin{aligned}
&\left(\sum_k a_{0k}\otimes\overline{a_{1k}}\otimes\ldots\otimes\overline{a_{mk}}\right)\cdot\left(\sum_{k'} b_{0k'}\otimes\overline{b_{1k'}}\otimes\ldots\otimes\overline{b_{nk'}}\right)\\
:=\ &\sum_{k,k'} a_{0k}\otimes(\otimes_{j=1}^{m-1}\overline{a_{jk}})\otimes\overline{a_{mk}b_{0k'}}\otimes(\otimes_{i=1}^{n}\overline{b_{ik'}})\\
&\quad+\sum_{i=1}^{m-1}(-1)^i a_{0k}\otimes\overline{a_{1k}}\otimes\ldots\otimes\overline{a_{m-i,k}a_{m-i+1,k}}\otimes\ldots\otimes\overline{a_{mk}}\otimes(\otimes_{i=0}^{n}\overline{b_{ik'}})\\
&\quad+(-1)^m a_{0k}a_{1k}\otimes(\otimes_{j=2}^{m}\overline{a_{jk}})\otimes(\otimes_{i=0}^{n}\overline{b_{ik'}})\,.
\end{aligned}
$$

for $\sum_k a_{0k}\otimes\overline{a_{1k}}\otimes\ldots\otimes\overline{a_{mk}}\in\Omega^m(\mathcal{A})$ *and* $\sum_{k'} b_{0k'}\otimes\overline{b_{1k'}}\otimes\ldots\otimes\overline{b_{nk'}}\in\Omega^n(\mathcal{A})$*. There is a differential* d *acting on* $\Omega^\bullet(\mathcal{A})$*, given by*

$$d(a_0\otimes\overline{a_1}\otimes\ldots\otimes\overline{a_k}):=1\otimes\overline{a_0}\otimes\overline{a_1}\otimes\ldots\otimes\overline{a_k}\ \ \forall\, a_j\in\mathcal{A}\,,$$

and it satisfies the relations

1. $d^2\omega=0,\ \forall\,\omega\in\Omega^\bullet(\mathcal{A})$.
2. $d(\omega_1\omega_2)=(d\omega_1)\omega_2+(-1)^{deg(\omega_1)}\omega_1 d\omega_2\,,\ \forall$ *homogeneous* $\omega_1\in\Omega^\bullet(\mathcal{A})$.

We get a $*$*-representation* π *of* $\Omega^\bullet(\mathcal{A})$ *on* $\mathcal{H}$ *by ,*

$$\pi(a_0\otimes\overline{a_1}\otimes\ldots\otimes\overline{a_k}):=a_0[D,a_1]\ldots[D,a_k]\,;\ a_j\in\mathcal{A}.$$

Let $J_0^{(k)}=\{\omega\in\Omega^k:\pi(\omega)=0\}$ *and* $J'=\bigoplus J_0^{(k)}$*. But* J' *fails to be a differential ideal. We consider* $J^\bullet=\bigoplus J^{(k)}$ *where* $J^{(k)}=J_0^{(k)}+dJ_0^{(k-1)}$*. Then* $J^\bullet$ *becomes a differential graded two-sided ideal and hence the quotient* $\Omega_D^\bullet=\Omega^\bullet/J^\bullet$ *becomes a differential graded algebra. This is called the Connes' calculus.*

The representation π gives the isomorphism $\Omega_D^k\cong\pi(\Omega^k)/\pi(dJ_0^{k-1})$. The abstract differential d induces a differential $\widetilde{d}$ on the complex $\Omega_D^\bullet(\mathcal{A})$ so that we get a chain complex $(\Omega_D^\bullet(\mathcal{A}),\widetilde{d})$ and a chain map $\pi_D:\Omega^\bullet(\mathcal{A})\to\Omega_D^\bullet(\mathcal{A})$ such that the following diagram

$$
\begin{tikzcd}
\Omega^\bullet(\mathcal{A}) \arrow[r, "\pi_D"] \arrow[d, "d"'] & \Omega_D^\bullet(\mathcal{A}) \arrow[d, "\widetilde{d}"] \\
\Omega^{\bullet+1}(\mathcal{A}) \arrow[r, "\pi_D"'] & \Omega_D^{\bullet+1}(\mathcal{A})
\end{tikzcd}
\tag{1.1.1}
$$

commutes.

1.2 Hermitian structure

Definition 1.2.1. *Let $\mathcal{E}$ be a finitely generated projective module over $\mathcal{A}$. A Hermitian structure on $\mathcal{E}$ is an $\mathcal{A}$-valued positive-definite sesquilinear mapping $\langle\ ,\ \rangle_{\mathcal{A}}$ such that,*

1. $\langle \xi, \xi' \rangle_{\mathcal{A}}^* = \langle \xi', \xi \rangle_{\mathcal{A}}\,,\ \ \forall\, \xi, \xi' \in \mathcal{E}$.
2. $\langle \xi, \xi'.a \rangle_{\mathcal{A}} = (\langle \xi, \xi' \rangle_{\mathcal{A}}).a\,,\ \ \forall\, \xi, \xi' \in \mathcal{E},\ \forall\, a \in \mathcal{A}$.
3. *The map $\xi \longmapsto \Phi_\xi$ from $\mathcal{E}$ to $\mathcal{E}^*$, given by $\Phi_\xi(\eta) = \langle \xi, \eta \rangle_{\mathcal{A}}\,, \forall \eta \in \mathcal{E}$, gives conjugate linear $\mathcal{A}$-module isomorphism between $\mathcal{E}$ and $\mathcal{E}^*$. This property will be referred as the self-duality of $\mathcal{E}$.*

Any free $\mathcal{A}$-module $\mathcal{E}_0 = \mathcal{A}^q$ has a *Hermitian* structure on it, given by $\langle\, \xi, \eta\, \rangle_{\mathcal{A}} = \sum_{j=1}^q \xi_j^*\, \eta_j\,, \forall\ \xi = (\xi_1, \ldots, \xi_q)\,, \eta = (\eta_1, \ldots, \eta_q) \in \mathcal{E}_0$. We refer this as the canonical Hermitian structure on $\mathcal{A}^q$. For finitely generated projective module existence of *Hermitian* structure is provided by following Lemma.

Lemma 1.2.2. (a) *A finitely generated projective module of the form $p\mathcal{A}^q$, where $p \in M_q(\mathcal{A})$ a self-adjoint idempotent, has a Hermitian structure on it.*

(b) *Let $\mathcal{A}$ be a C^*-algebra or a dense subalgebra in a C^*-algebra stable under the holomorphic function calculus. Every finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ is* *isomorphic as a f.g.p module with $p\mathcal{A}^q$ where p is a self-adjoint idempotent, that is a projection. Hence $\mathcal{E}$ has a Hermitian structure on it.*

Proof. (a) With respect to the canonical *Hermitian* structure $\langle p^*\xi, \eta \rangle_{\mathcal{A}} = \langle \xi, p\eta \rangle_{\mathcal{A}}$ holds for any $p \in M_q(\mathcal{A})$. Suppose $\mathcal{E} = p\mathcal{A}^q$ be a f.g.p module with p a projection in $M_q(\mathcal{A})$. The canonical structure $\langle \xi, \eta \rangle_{\mathcal{A}} = \sum \xi_j^* \eta_j$ on $\mathcal{A}^q$ will induce a pairing on $\mathcal{E}$. We have to show that $\xi \longmapsto \Phi_\xi$ gives an $\mathcal{A}$-module isomorphism between $\mathcal{E}$ and $\mathcal{E}^*$. It is enough to check only the surjectivity of this map. In order to do so let's take an element $f \in \mathcal{E}^*$. Then $\widetilde{f} = f \circ \pi$ is an element of $(\mathcal{A}^q)^*$ where $\pi : \mathcal{A}^q \longrightarrow p\mathcal{A}^q$ is the map $\xi \longmapsto p\xi$. By definition (1.2.1) there exists $\eta \in \mathcal{A}^q$ s.t $\widetilde{f} = \Phi_\eta$. Consider any element $p\xi \in \mathcal{E}$ with $\xi \in \mathcal{A}^q$. Then,

$$f(p\xi) = f \circ \pi(p\xi) = \widetilde{f}(p\xi) = \langle \eta, p\xi \rangle_{\mathcal{A}} = \langle p^*\eta, p\xi \rangle_{\mathcal{A}} = \langle p\eta, p\xi \rangle_{\mathcal{A}} = \Phi_{p\eta}(p\xi).$$

Hence $f = \Phi_{p\eta}$ with $\eta \in \mathcal{A}^q$. So the induced pairing on $\mathcal{E}$ gives a *Hermitian* structure on it.

(b) Let $\mathcal{E}$ be a f.g.p module over $\mathcal{A}$. There exists an $\mathcal{A}$-module $\mathcal{F}$ such that $\mathcal{E} \bigoplus \mathcal{F} \cong \mathcal{A}^q$ for some natural number q. Once we fix such an $\mathcal{F}$ we let $p : \mathcal{A}^q \longrightarrow \mathcal{A}^q$ given by $p(e+f) = e$ for $e \in \mathcal{E}$ and $f \in \mathcal{F}$. So p is an idempotent in $M_q(\mathcal{A})$ with $\mathcal{E} = p\mathcal{A}^q$. By ([41], page 101) we see that in a C^*-algebra (or $*$-subalgebra of a C^*-algebra which is stable under holomorphic function calculus) every idempotent is similar to a selfadjoint idempotent i.e., a projection and this similarity is witnessed by the invertible element $z = ((2p^* - 1)(2p - 1) + 1)^{1/2}$. Since $\mathcal{A}$ is closed under holomorphic function calculus the invertible element z actually belongs to $M_q(\mathcal{A})$. Hence $\widetilde{p} = zpz^{-1}$ is a projection in $M_q(\mathcal{A})$ and $\widetilde{\mathcal{E}} = \widetilde{p}\mathcal{A}^q \cong p\mathcal{A}^q = \mathcal{E}$. Then one restricts the *Hermitian* structure on $\mathcal{A}^q$ to $\widetilde{\mathcal{E}}$ and endows $\mathcal{E}$ with the *Hermitian* structure obtained via the isomorphism between $\mathcal{E}$ and $\widetilde{\mathcal{E}}$. □

Remark 1.2.3. The concept of *Hermitian* structure can be defined for f.g.p modules over involutive algebras and part (a) of Lemma (1.2.2) still holds. But part (b) requires the more finer property of closure under holomorphic function calculus.

Remark 1.2.4 (Open Question). We do not know whether a finitely generated projective module over an involutive algebra always admits *Hermitian* structure.

1.3 Connection, Curvature for a Dynamical system

Definition 1.3.1. *A C^*-dynamical system with an invariant trace is a quadruple $(\mathcal{A}, G, \alpha, \tau)$ where*

1. *$\mathcal{A}$ is a C^*-algebra;*
2. *G is a Lie group acting strongly on $\mathcal{A}$;*
3. *$\alpha : G \longrightarrow Aut(\mathcal{A})$ is the strong action of G on $\mathcal{A}$;*
4. *τ is a G-invariant faithful trace on $\mathcal{A}$.*

Here strong action means that for any $a \in \mathcal{A}$ the map $g \longmapsto \alpha_g(a)$ is a continuous homomorphism, where $Aut(\mathcal{A})$ denotes the group of $\star$-automorphisms of $\mathcal{A}$. It should be noted that G can be any locally compact group ([33]) but in our case we only work with Lie groups. We say that $a \in \mathcal{A}$ is smooth if the map $g \longmapsto \alpha_g(a)$ from G to the normed space $\mathcal{A}$ is smooth. The involutive algebra $\mathcal{A}^\infty = \{a \in \mathcal{A} : a \text{ is smooth}\}$ is a norm dense subalgebra of $\mathcal{A}$, called the smooth subalgebra. Note that this is unital as well. One crucial property enjoyed by this algebra is that it is closed under the holomorphic function calculus inherited from the ambient C^*-algebra $\mathcal{A}$. Let $Lie(G)$ be the Lie algebra of G. Then we have a representation δ of $Lie(G)$ into the Lie algebra $Der(\mathcal{A})$ of derivations on $\mathcal{A}$ given by

$$\delta_X(a) = \tfrac{d}{dt}|_{t=0}\, \alpha_{exp(tX)}(a)\,,$$

where $exp : Lie(G) \longrightarrow G$ is the exponential map.

Definition 1.3.2. *Let $\mathcal{E}$ be a f.g.p module over $\mathcal{A}$ with a Hermitian structure. A connection (on $\mathcal{E}$) is a $\mathbb{C}$-linear map $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes Lie(G)^*$ such that, for all $X \in Lie(G)$ and $\xi \in \mathcal{E}$, $a \in \mathcal{A}$ one has*

$$\nabla_X(\xi \cdot a) = \nabla_X(\xi) \cdot a + \xi \cdot \delta_X(a)\,.$$

We shall say that ∇ is compatible with respect to the *Hermitian* structure on $\mathcal{E}$ iff :

$$\langle \nabla_X\, \xi\,, \xi' \rangle_{\mathcal{A}} + \langle \xi\,, \nabla_X\, \xi' \rangle_{\mathcal{A}} = \delta_X(\langle\, \xi, \xi'\,\rangle_{\mathcal{A}})\,, \quad \forall\, \xi\,, \xi' \in \mathcal{E},\ \forall\, X \in Lie(G)\,.$$

In ([12]) existence of compatible connection has been discussed. We will denote the set of compatible connections on $\mathcal{E}$ by $C(\mathcal{E})$. The algebra $End(\mathcal{E})$ has a natural involution given by

$$\langle\, T^*\xi, \eta\,\rangle_{\mathcal{A}} = \langle\, \xi, T\eta\,\rangle_{\mathcal{A}} \qquad \forall\, \xi, \eta \in \mathcal{E}\,,\, T \in End(\mathcal{E})\,.$$

For any two compatible connections ∇, ∇' it can be easily checked that $\nabla_X - \nabla'_X$ is a skew-adjoint element of $End(\mathcal{E})$ for each $X \in Lie(G)$. The curvature Θ_∇ of a connection ∇ is the alternating bilinear $End(\mathcal{E})$-valued form on $Lie(G)$ defined by

$$\Theta_\nabla : \wedge^2(Lie(G)) \longrightarrow End(\mathcal{E})$$

$$\Theta_\nabla(X \wedge Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\,,\ \forall\, X, Y \in Lie(G)\,.$$

This measures the extent to which ∇ fails to be a Lie algebra homomorphism. A simple calculation will assure that Θ_∇ takes value in $End(\mathcal{E})$. In fact we can say something more.

Lemma 1.3.3. *The range of Θ_∇ is contained in $End(\mathcal{E})_{skew} := \{T \in End(\mathcal{E}) : T^* = -T\}$.*

Proof. We have to show that $\langle\, \Theta_\nabla(X \wedge Y)(\xi), \eta\,\rangle_{\mathcal{A}} = -\langle\, \xi, \Theta_\nabla(X \wedge Y)(\eta)\,\rangle_{\mathcal{A}}$ for all $\xi, \eta \in \mathcal{E}$. Now,

$$\begin{aligned}
\langle\, \Theta_\nabla(X \wedge Y)(\xi), \eta\,\rangle_{\mathcal{A}} &= \langle\, ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})(\xi), \eta\,\rangle_{\mathcal{A}} \\
&= \langle\, \nabla_X(\nabla_Y(\xi)) - \nabla_Y(\nabla_X(\xi)) - \nabla_{[X,Y]}(\xi), \eta\,\rangle_{\mathcal{A}} \\
&= \langle\, \nabla_X(\nabla_Y(\xi)), \eta\,\rangle_{\mathcal{A}} - \langle\, \nabla_Y(\nabla_X(\xi)), \eta\,\rangle_{\mathcal{A}} - \langle\, \nabla_{[X,Y]}(\xi)), \eta\,\rangle_{\mathcal{A}} \\
&= \delta_X(\langle\, \nabla_Y(\xi), \eta\,\rangle_{\mathcal{A}}) - \langle\, \nabla_Y(\xi), \nabla_X(\eta)\,\rangle_{\mathcal{A}} - \delta_Y(\langle\, \nabla_X(\xi), \eta\,\rangle_{\mathcal{A}}) \\
&\quad + \langle\, \nabla_X(\xi), \nabla_Y(\eta)\,\rangle_{\mathcal{A}} - \delta_{[X,Y]}(\langle \xi, \eta \rangle_{\mathcal{A}}) + \langle\, \xi, \nabla_{[X,Y]}(\eta)\,\rangle_{\mathcal{A}}
\end{aligned}$$

$$
\begin{aligned}
&= \delta_X(\delta_Y(\langle\xi,\eta\rangle_{\mathcal{A}}) - \langle\xi,\nabla_Y(\eta)\rangle_{\mathcal{A}}) - \langle\nabla_Y(\xi),\nabla_X(\eta)\rangle_{\mathcal{A}}) \\
&\quad - \delta_Y(\delta_X(\langle\xi,\eta\rangle_{\mathcal{A}}) - \langle\xi,\nabla_X(\eta)\rangle_{\mathcal{A}}) + \langle\nabla_X(\xi),\nabla_Y(\eta)\rangle_{\mathcal{A}}) \\
&\quad - \delta_{[X,Y]}(\langle\xi,\eta\rangle_{\mathcal{A}}) + \langle\xi,\nabla_{[X,Y]}(\eta)\rangle_{\mathcal{A}} \\
&= [\delta_X,\delta_Y](\langle\xi,\eta\rangle_{\mathcal{A}}) - \delta_{[X,Y]}(\langle\xi,\eta\rangle_{\mathcal{A}}) + \langle\xi,\nabla_{[X,Y]}(\eta)\rangle_{\mathcal{A}} \\
&\quad + \langle\nabla_X(\xi),\nabla_Y(\eta)\rangle_{\mathcal{A}} + \delta_Y(\langle\xi,\nabla_X(\eta)\rangle_{\mathcal{A}}) \\
&\quad - \langle\nabla_Y(\xi),\nabla_X(\eta)\rangle_{\mathcal{A}} - \delta_X(\langle\xi,\nabla_Y(\eta)\rangle_{\mathcal{A}}) \\
&= \langle\xi,\nabla_{[X,Y]}(\eta)\rangle_{\mathcal{A}} - \langle\xi,\nabla_X\nabla_Y(\eta)\rangle_{\mathcal{A}} + \langle\xi,\nabla_Y\nabla_X(\eta)\rangle_{\mathcal{A}} \\
&= \langle\xi,\nabla_{[X,Y]}(\eta)\rangle_{\mathcal{A}} - \langle\xi,[\nabla_X,\nabla_Y](\eta)\rangle_{\mathcal{A}} \\
&= -\langle\xi,([\nabla_X,\nabla_Y] - \nabla_{[X,Y]})(\eta)\rangle_{\mathcal{A}} \\
&= -\langle\,\xi,\Theta_\nabla(X\wedge Y)(\eta)\,\rangle_{\mathcal{A}}\ .
\end{aligned}
$$

and this completes the proof. □

1.4 The Yang-Mills functional for Dynamical system

Initially in noncommutative geometry a C^*-dynamical system was the candidate for a noncommutative space and Connes-Rieffel ([20]) successfully extended the classical notion of Yang-Mills functional in this setting. It is certain functional defined on the affine space $C(\mathcal{E})$ of compatible connections on a *Hermitian* finitely generated projective module $\mathcal{E}$. Later Rieffel studied the critical points of this functional for noncommutative torus ([37]). Recently, Kang ([26]) obtained a family of critical points for a specific module $\mathcal{E}_0$ over a quantum Heisenberg manifold and Lee ([28]) constructed a connection ∇_0 on $\mathcal{E}_0$ which is a minimum of the Yang-Mills functional on the module $\mathcal{E}_0$. Let us now recall the definition of the Yang-Mills functional.

Let $(\mathcal{A},G,\alpha,\tau)$ be a C^*-dynamical system with an invariant trace τ. Fix an inner product on $Lie(G)$ which will remain fixed throughout. Next choose an orthonormal basis $\{Z_1,\ldots,Z_n\}$ of $Lie(G)$. The bilinear form on the space of alternating 2-forms with values in $End(\mathcal{E})$ is given by

$$\{\Phi, \Psi\}_{\mathcal{E}} = \sum_{i<j} \Phi(Z_i \wedge Z_j)\Psi(Z_i \wedge Z_j).$$

Recall that we have a G-invariant faithful trace τ on $\mathcal{A}$. We can extend it to a canonical faithful trace $\widetilde{\tau}$ on $End(\mathcal{E})$ with the help of the following lemma from ([20]).

Lemma 1.4.1. *If $\mathcal{E}$ is f.g.p $\mathcal{A}$-module with a Hermitian structure, then every element of $End(\mathcal{E})$ can be written as a linear combination of elements of the form $\langle\, \xi, \eta \,\rangle_{\mathcal{E}}$ for $\xi, \eta \in \mathcal{E}$, where $\langle\, \xi, \eta \,\rangle_{\mathcal{E}}(\zeta) = \xi \langle\, \eta, \zeta \,\rangle_{\mathcal{A}}$, $\forall\, \zeta \in \mathcal{E}$.*

Proof. Let $\mathcal{E} = p\mathcal{A}^q$ where $p \in M_q(\mathcal{A})$ is an idempotent and $\{e_1, \ldots, e_q\}$ be the standard basis for $\mathcal{A}^q$. For any given $T \in End(\mathcal{E})$ one can write $T = \bigoplus_{i=1}^{q} T_i$, where $T_i = \pi_i \circ T$, π_i denotes the projection onto the i-th component of $\mathcal{A}^q$. Then $T_i(\xi) = \langle \eta_i, \xi \rangle_{\mathcal{A}}$ for some $\eta_i \in \mathcal{E}$, which follows from self duality of $\mathcal{E}$. Then one can show directly that $T = \sum \langle\, pe_i, \eta_i \,\rangle_{\mathcal{E}}$. □

Now, using this lemma, we define a linear functional $\widetilde{\tau}$ on $End(\mathcal{E})$ as

$$\widetilde{\tau} : End(\mathcal{E}) \longrightarrow \mathbb{C}$$

$$\widetilde{\tau}(\langle\, \xi, \eta \,\rangle_{\mathcal{E}}) := \tau(\langle \eta, \xi \rangle_{\mathcal{A}})\,.$$

Lemma 1.4.2. *$\widetilde{\tau}$ defined above, is a trace on $End(\mathcal{E})$.*

Proof. One can easily check that $\langle\, \xi_1, \eta_1 \,\rangle_{\mathcal{E}} \langle\, \xi_2, \eta_2 \,\rangle_{\mathcal{E}} = \langle\, \xi_1 \langle \eta_1, \xi_2 \rangle_{\mathcal{A}}, \eta_2 \,\rangle_{\mathcal{E}}$. Now use the fact that τ is a trace on $\mathcal{A}$. □

Moreover it can be shown that $\widetilde{\tau}$ is faithful (see [20]). Finally, the Yang-Mills functional on $C(\mathcal{E})$ is given by

$$YM(\nabla) := -\widetilde{\tau}(\{\Theta_\nabla, \Theta_\nabla\}_{\mathcal{E}})\,.$$

Since Θ_∇ takes value in $End(\mathcal{E})_{skew}$ (Lemma 1.3.3), this minus sign will force YM to take nonnegative real values.

1.5 From Dynamical systems to Spectral triples

Starting from a C^*-dynamical system there is a way to construct a candidate for spectral triple but there is no guarantee that the resulting object is truely a spectral triple. One has to verify the conditions for being a spectral triple on a case by case basis for each examples. The construction is the following.

Given a C^*-dynamical system $(\mathcal{A}, G, \alpha)$ with a G-invariant faithful trace τ on $\mathcal{A}$, consider the G.N.S Hilbert space $\widetilde{\mathcal{H}} = L^2(\mathcal{A}, \tau)$. If *dimension* of the Lie group G is m, letting $t = 2^{[m/2]}$, there exist m matrices in $M_t(\mathbb{C})$ denoted by $\gamma_1, \gamma_2, \cdots, \gamma_m$ (called Clifford gamma matrices), such that, $\gamma_r\gamma_s + \gamma_s\gamma_r = 2\delta_{rs}$, $r, s \in \{1, \ldots, m\}$, where δ_{rs} is the Kronecker delta function. Let $\mathcal{H} = \widetilde{\mathcal{H}} \otimes \mathbb{C}^t$ and

$$\widetilde{\delta}_j(a) = \tfrac{d}{dt}|_{t=0}\, \alpha_{exp(tZ_j)}(a)\,.$$

where $\{Z_1, \ldots, Z_m\}$ is the fixed orthonormal basis of $Lie(G)$. We define $D := \sum_{j=1}^m \delta_j \otimes \gamma_j$ where $\delta_j = (-\sqrt{-1})\widetilde{\delta}_j$. Then $(\mathcal{A}, \mathcal{H}, D)$ is a candidate for a spectral triple.

However, there is no general theory to guarantee that D will be self-adjoint and have compact resolvent and one has to check for each example on a case by case basis. It is known that for noncommutative tori ([15],[23]) and the quantum Heisenberg manifolds ([9]) one can obtain a genuine spectral triples through this formalism.

1.6 Connections and Curvature for Spectral triples

Definition 1.6.1 ([15]). *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple and $\mathcal{E}$ be a Hermitian finitely generated projective module over $\mathcal{A}$. A compatible connection on $\mathcal{E}$ is a $\mathbb{C}$-linear mapping $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ such that*

1. $\nabla(\xi a) = (\nabla\xi)a + \xi \otimes da, \quad \forall\, \xi \in \mathcal{E}, a \in \mathcal{A}$;
2. $\langle\, \xi, \nabla\eta\,\rangle - \langle\, \nabla\xi, \eta\,\rangle = d\langle\, \xi, \eta\,\rangle_{\mathcal{A}} \quad \forall\, \xi, \eta \in \mathcal{E}$ *(Compatibility).*

The meaning of the last equality in Ω^1_D is, if $\nabla(\xi) = \sum \xi_j \otimes \omega_j$, with $\xi_j \in \mathcal{E}$, $\omega_j \in \Omega^1_D(\mathcal{A})$, then $\langle \nabla\xi, \eta\rangle = \sum \omega_j^* \langle \xi_j, \eta\rangle_{\mathcal{A}}$. Any finitely generated projective right module has a connection. An example of such is the Grassmannian connection ∇_0 on $\mathcal{E} = p\mathcal{A}^q$, given by $\nabla_0(\xi) = pd\xi$, where $d\xi = (d\xi_1, \ldots, d\xi_q)$. This connection is compatible with the *Hermitian* structure

$$\langle \xi, \eta\rangle_{\mathcal{A}} = \sum_{k=1}^{q} \xi_k^* \, \eta_k \,, \quad \forall\, \xi, \eta \in p\mathcal{A}^q \,.$$

Also, any two compatible connections can only differ by an element of $Hom_{\mathcal{A}}(\mathcal{E}\,, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}))$. That is, the space of all compatible connections on $\mathcal{E}$, which we denote by $\widetilde{C}(\mathcal{E})$, is an affine space with the associated vector space $Hom_{\mathcal{A}}(\mathcal{E}\,, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}))$. The connection ∇ extends to a unique linear map $\widetilde{\nabla} : \mathcal{E} \otimes \Omega^1_D \longrightarrow \mathcal{E} \otimes \Omega^2_D$ such that

$$\widetilde{\nabla}(\xi \otimes \omega) \;=\; (\nabla\xi)\omega + \xi \otimes \widetilde{d}\omega, \;\; \forall\, \xi \in \mathcal{E}, \;\; \omega \in \Omega^1_D \,. \tag{1.6.1}$$

It can be easily checked that $\widetilde{\nabla}$, defined above, satisfies the Leibniz rule, i,e.

$$\widetilde{\nabla}(\eta a) = \widetilde{\nabla}(\eta) a - \eta \widetilde{d}a \,, \;\; \forall\, a \in \mathcal{A}, \eta \in \mathcal{E} \otimes \Omega^1_D \,.$$

A simple calculation shows that $\Theta = \widetilde{\nabla} \circ \nabla$ is an element of $Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D)$ and is called the **curvature** of the connection ∇.

1.7 The Yang-Mills functional for Spectral triple

It has been accepted that the geometry of noncommutative spaces is fully encoded in a spectral triple. A recent deep theorem of Connes strengthens this point ([17]). Because of this realization it was necessary to define the Yang-Mills functional in this setting. Connes sucessfully did this in his book ([15]) using the dga $\Omega^\bullet_D$. Its definition is the following.

Let $(\mathcal{A}, \mathcal{H}, D)$ be a d-summable spectral triple. Recall that $\Omega^2_D \cong \pi(\Omega^2)/\pi(dJ_0^{(1)})$.

Let $\mathcal{H}_2$ be the Hilbert space completion of $\pi(\Omega^2)$ with the inner-product

$$\langle T_1, T_2\rangle \;\;:=\;\; Tr_\omega(T_1^* T_2 |D|^{-d}),\, \forall\, T_1, T_2 \in \pi(\Omega^2). \tag{1.7.1}$$

Let $\widetilde{\mathcal{H}}_2$ be the Hilbert space completion of $\pi(dJ_0^{(1)})$ with the above inner-product. Clearly $\widetilde{\mathcal{H}}_2 \subseteq \mathcal{H}_2$. Let P be the orthogonal projection of $\mathcal{H}_2$ onto the orthogonal complement of the subspace $\pi(dJ_0^{(1)})$. Now define $\langle\, [T_1], [T_2]\,\rangle_{\Omega^2_D} = \langle PT_1, PT_2\rangle$, for all $[T_j] \in \Omega^2_D$. This gives a well defined inner-product on Ω^2_D. Viewing $\mathcal{E} = p\mathcal{A}^q$ we have an embedding

$$Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D) \cong Hom_{\mathcal{A}}(pA^q, p(\Omega^2_D)^q) \subseteq Hom_{\mathcal{A}}(\mathcal{A}^q, (\Omega^2_D)^q)\,.$$

Now for $\phi, \psi \in Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D)$, define $\langle\langle \phi, \psi\rangle\rangle := \sum_k \langle \phi(p\widetilde{e_k}), \psi(p\widetilde{e_k})\rangle_{\Omega^2_D}$ where $\{\widetilde{e_1}, \ldots, \widetilde{e_q}\}$ is the standard basis of $\mathcal{A}^q$ as $\mathcal{A}$-bimodule. Finally, the Yang-Mills functional on $\widetilde{C}(\mathcal{E})$ is defined by, $YM(\nabla) := \langle\langle\, \Theta, \Theta\,\rangle\rangle$.

Proposition 1.7.1. *The definition of Yang-Mills does not depend on the choice of the projection used to describe $\mathcal{E}$.*

Proof. Let $\mathcal{E} \cong p_1\mathcal{A}^q$ and $\mathcal{E} \cong p_2\mathcal{A}^l$ be two isomorphisms. Then there is a unitary $U \in M_{q+l}(\mathcal{A})$ such that $\widetilde{p_2} = U\widetilde{p_1}U^*$ where

$$\widetilde{p_1} = \begin{pmatrix} p_1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad \widetilde{p_2} = \begin{pmatrix} p_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence w.l.o.g. we can assume $l = n = q$ and there is a unitary $U \in M_n(\mathcal{A})$ such that $p_2 = Up_1U^*$. Let $\mathcal{M} = \{A \in M_n(\Omega^2_D) : p_1 A = A\}$ and $\mathcal{M}' = \{B \in M_n(\Omega^2_D) : p_2 B = B\}$. We have obvious bijections $Hom_{\mathcal{A}}(p_1\mathcal{A}^n, p_1\mathcal{A}^n \otimes_{\mathcal{A}} \Omega^2_D) \longrightarrow \mathcal{M}$ and $Hom_{\mathcal{A}}(p_2\mathcal{A}^n, p_2\mathcal{A}^n \otimes_{\mathcal{A}} \Omega^2_D) \longrightarrow \mathcal{M}'$ such that the induced bijection $\Psi : \mathcal{M} \to \mathcal{M}'$ is given by $A \longmapsto UA$. Observe that Ψ makes sense on $M_n(\pi(\Omega^2))$. Using the inner product (1.7.1) we can convert $M_n(\pi(\Omega^2))$ into an inner product space. Clearly Ψ is inner product preserving and respects $\pi(dJ_0^1)$. Recall that $\Omega^2_D \cong \pi(\Omega^2)/\pi(dJ_0^1)$, hence Ψ induces an inner product preserving map on $M_n(\Omega^2_D)$. The equality $\langle\langle\, \Theta, \Theta\,\rangle\rangle = \langle\langle\, \Psi(\Theta), \Psi(\Theta)\,\rangle\rangle$ shows that YM does not depend on the choice of projection p in $\mathcal{E} = p\mathcal{A}^q$. $\square$

Chapter 2

Equivalence of the two approaches to Yang-Mills theory on Noncommutative Tori

In the previous chapter we have encountered two definitions of the Yang-Mills functional, one for C^*-dynamical systems and the other for spectral triples. In this chapter we will answer a natural question of equivalence of these approaches to Yang-Mills functional for the noncommutative n-torus.

Our aim here is to prove two theorems, one regarding the structure of a finitely generated projective module endowed with a *Hermitian* structure over spectrally invariant subalgebras of C^*-algebras, and the other the equivalence of Yang-Mills approaches on noncommutative n-torus. The equivalence for n=2 case was addressed by Connes in ([15], Proposition 13, page 582). Here we take up the higher dimensional noncommutative torus and show that even in this case it holds.

We recall non-commutative n-torus $\mathcal{A}_\Theta$ as defined in ([36]). Let Θ be a $n \times n$ real skew-symmetric matrix. Denote by $\mathcal{A}_\Theta$, the universal C^*-algebra generated by n unitaries $U_1, \ldots, U_n$ satisfying $U_k U_m = e^{2\pi i \Theta_{km}} U_m U_k$, where $k, m \in \{1, \ldots, n\}$. Throughout this chapter i will stand for $\sqrt{-1}$. On the noncommutative n-torus $\mathcal{A}_\Theta$ the Lie group $G = \mathbb{T}^n$

acts as follows :

$$\alpha_{(z_1,\ldots,z_n)}(U_k) = z_k U_k\,;\; k = 1,\ldots,n.$$

The smooth subalgebra of $\mathcal{A}_\Theta$ is given by

$$\mathcal{A}_\Theta^\infty := \{\sum a_{\mathbf{r}}\, U^{\mathbf{r}} : \{a_{\mathbf{r}}\} \in \mathbb{S}(\mathbb{Z}^n)\,,\, \mathbf{r} = (r_1,\ldots,r_n) \in \mathbb{Z}^n\}$$

where $\mathbb{S}(\mathbb{Z}^n)$ denotes vector space of multisequences $(a_{\mathbf{r}})$ that decay faster than the inverse of any polynomial in $\mathbf{r} = (r_1,\ldots,r_n)$.

This subalgebra is equipped with a unique G-invariant tracial state, given by $\tau(a) = a_{\mathbf{0}}$, where $\mathbf{0} = (0,\ldots,0)$. One can further assume that the lattice Λ_Θ generated by columns of Θ is such that $\Lambda_\Theta + \mathbb{Z}^n$ is dense in $\mathbb{R}^n$. The advantage of choosing such a matrix Θ is that $\mathcal{A}_\Theta$ (hence $\mathcal{A}_\Theta^\infty$) becomes simple (see [23], Page 537). But in our case simpleness of $\mathcal{A}_\Theta^\infty$ is not needed and hence we do not require any assumption on Θ except skew-symmetry. The *Hilbert space* obtained by applying the G.N.S. construction to τ can be identified with $l^2(\mathbb{Z}^n)$ ([36]).

2.1 Yang-Mills in the dynamical system approach

From now on we will work with $\mathcal{A}_\Theta^\infty$ only and hence for notational brevity we denote it by $\mathcal{A}_\Theta$. In this case $\mathcal{L} = Lie(G)$ is $\mathbb{R}^n$. Let $\{\nu_1,\nu_2,\ldots,\nu_n\}$ be the standard basis of $\mathbb{R}^n$ and the associated derivations $\delta_{\nu_1},\ldots,\delta_{\nu_n}$. We will denote δ_{ν_j} by $\widetilde{\delta}_j$.
The derivations $\{\widetilde{\delta}_1,\ldots,\widetilde{\delta}_n\}$ on $\mathcal{A}_\Theta$ are given by

$$\widetilde{\delta}_j(\sum_{\mathbf{r}} a_{\mathbf{r}} U^{\mathbf{r}}) = i \sum_{\mathbf{r}} r_j a_{\mathbf{r}} U^{\mathbf{r}}\,. \tag{2.1.1}$$

It can be easily checked that these derivations commute and they are $\star$-derivations of $\mathcal{A}_\Theta$ i,e.

$$(\widetilde{\delta}_j(a))^* = \widetilde{\delta}_j(a^*) \quad ; \quad \widetilde{\delta}_j(ab) = \widetilde{\delta}_j(a)b + a\widetilde{\delta}_j(b).$$

A connection is given by n maps $\nabla_{\widetilde{\delta_j}} : \mathcal{E} \longrightarrow \mathcal{E}$ such that $\nabla_{\widetilde{\delta_j}}(\xi.a) = \nabla_{\widetilde{\delta_j}}(\xi)a + \xi\widetilde{\delta_j}(a)$. So the space of compatible connections ∇ consists of n-tuples of maps $(\nabla_{\widetilde{\delta_1}}, \ldots, \nabla_{\widetilde{\delta_n}})$ such that

$$\nabla(\xi) = \sum_{j=1}^{n} \nabla_{\widetilde{\delta_j}}(\xi) \otimes e_j\,, \tag{2.1.2}$$

and

$$\langle \nabla_{\widetilde{\delta_j}}(\xi), \eta\rangle_{\mathcal{A}_\Theta} + \langle \xi, \nabla_{\widetilde{\delta_j}}(\eta)\rangle_{\mathcal{A}_\Theta} \;=\; \widetilde{\delta_j}\left(\langle \xi, \eta\rangle_{\mathcal{A}_\Theta}\right). \tag{2.1.3}$$

Here $\{e_1, \cdots, e_n\}$ denotes the basis dual to $\{\nu_1, \cdots, \nu_n\}$ of the dual of the Lie algebra $\mathbb{R}^n$. The curvature of a connection ∇ is given by $\Theta_\nabla(\widetilde{\delta_j} \wedge \widetilde{\delta_k}) = [\nabla_{\widetilde{\delta_j}}, \nabla_{\widetilde{\delta_k}}]$, because $[\widetilde{\delta_j}, \widetilde{\delta_k}] = 0$ in this case. We have $[\nabla_{\widetilde{\delta_j}}, \nabla_{\widetilde{\delta_k}}]^* = -[\nabla_{\widetilde{\delta_j}}, \nabla_{\widetilde{\delta_k}}]$ by Lemma (1.3.3). The bilinear form on the space of $End(\mathcal{E})$-valued alternating 2-forms becomes

$$\{\Phi, \Psi\}_{\mathcal{E}} = \sum_{j<k} \Phi(\widetilde{\delta_j} \wedge \widetilde{\delta_k})\Psi(\widetilde{\delta_j} \wedge \widetilde{\delta_k})\,.$$

Finally, the Yang-Mills functional of ∇ is given by

$$\begin{aligned} YM(\nabla) = -\widetilde{\tau}(\{\Theta_\nabla, \Theta_\nabla\}_{\mathcal{E}}) \;&=\; -\widetilde{\tau}\left(\sum_{j<k} [\nabla_{\widetilde{\delta_j}}, \nabla_{\widetilde{\delta_k}}]^2\right) \\ &=\; \widetilde{\tau}\left(\sum_{j<k} [\nabla_{\widetilde{\delta_j}}, \nabla_{\widetilde{\delta_k}}]^* [\nabla_{\widetilde{\delta_j}}, \nabla_{\widetilde{\delta_k}}]\right)\,. \end{aligned}$$

For notational simplicity we write

$$YM(\nabla) = \sum_{j<k} \widetilde{\tau}\left([\nabla_j, \nabla_k]^* [\nabla_j, \nabla_k]\right)\,. \tag{2.1.4}$$

2.2 Finitely generated projective modules with a Hermitian structure

It is almost by definition that a finitely generated projective module over an associative algebra can be embedded in a free module as a complemented submodule. However the situation is different for finitely generated projective modules with a *Hermitian* structure over involutive subalgebras of C^*-algebras. Let $\mathcal{A}$ be an involutive subalgebra of a C^*-algebra A and $\mathcal{E}$ be a finitely generated projective module over $\mathcal{A}$ with a *Hermitian* structure. Note that any free $\mathcal{A}$ module has a canonical *Hermitian* structure and one may ask does there exist an embedding $i : \mathcal{E} \longrightarrow \mathcal{A}^n$ such that (i) there exists a submodule $\mathcal{F}$ of $\mathcal{A}^n$ with the property $i(\mathcal{E}) \oplus \mathcal{F} = \mathcal{A}^n$ and (ii) the *Hermitian* structure of $\mathcal{E}$ is the one induced from $\mathcal{A}^n$. In this section we show that this is indeed the case provided $\mathcal{A}$ is a subalgebra of a C^*-algebra and is closed under holomorphic function calculus. In this result the emphasis is on this condition which is often overlooked. In fact we do not know whether the result is true in general.

Let $\mathcal{A}$ be a unital subalgebra of a C^*-algebra stable under holomorphic functional calculus and represented faithfully on a Hilbert space $\mathcal{H}$. Let $\mathcal{E}$ be a f.g.p (right)module over $\mathcal{A}$ equipped with a *Hermitian* structure on it. There is a right $\mathcal{A}$-module $\mathcal{F}$ such that $\mathcal{E} \bigoplus \mathcal{F} \cong \mathcal{A}^q$ for some q. Since $\mathcal{A}^q$ has a topology, $\mathcal{E}$ inherits the topology from $\mathcal{A}^q$. Also $\mathcal{E}^*$ inherits topology from $\mathcal{A}^q$ because $\mathcal{E}^* \bigoplus \mathcal{F}^* \cong (\mathcal{A}^q)^* \cong \mathcal{A}^q$. As because we have topology now, we can expect the isomorphism between $\mathcal{E}$ and $\mathcal{E}^*$ to be topological, which turns out to be true by the following lemma.

Lemma 2.2.1. *If two finitely generated projective $\mathcal{A}$-modules $\mathcal{E}_1$ and $\mathcal{E}_2$ are algebraically isomorphic then they are also isomorphic as topological vector spaces.*

Proof. Since both the modules are projective, we can find $\mathcal{F}_1$ and $\mathcal{F}_2$ such that, $\mathcal{E}_1 \bigoplus \mathcal{F}_1 \cong \mathcal{A}^k$ and $\mathcal{E}_2 \bigoplus \mathcal{F}_2 \cong \mathcal{A}^l$. Then, $\mathcal{E}_1 \bigoplus \mathcal{F}_1 \bigoplus \mathcal{A}^l \cong \mathcal{A}^{k+l}$ and $\mathcal{E}_2 \bigoplus \mathcal{F}_2 \oplus \mathcal{A}^k \cong \mathcal{A}^{k+l}$. Hence we can write $\mathcal{E}_1 = p_1\mathcal{A}^{k+l}$ and $\mathcal{E}_2 = p_2\mathcal{A}^{k+l}$, where $p_1, p_2 \in M_{k+l}(\mathcal{A})$ are idempotents. Let $u_j : \mathcal{A}^{k+l} \longrightarrow \mathcal{E}_j$ denote the projection maps and $v_j : \mathcal{E}_j \longrightarrow \mathcal{A}^{k+l}$ denote the inclusion maps for $j = 1, 2$. If we denote the isomorphism between $\mathcal{E}_1$ and $\mathcal{E}_2$ by ϕ then considering $f = v_2 \circ \phi \circ u_1$ and $g = v_1 \circ \phi^{-1} \circ u_2$ in $Hom_{\mathcal{A}}(\mathcal{A}^{k+l}, \mathcal{A}^{k+l})$, it is easily seen that $f \circ g = p_2$ and $g \circ f = p_1$. If we choose

$$\widetilde{p}_1 = \begin{pmatrix} p_1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad \widetilde{p}_2 = \begin{pmatrix} p_2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} f & 1 - f \circ g \\ 1 - g \circ f & g \end{pmatrix}$$

then we see that $\widetilde{p}_2 = U \widetilde{p}_1 U^{-1}$. Since f, g both are $\mathcal{A}$-linear maps, U is also an $\mathcal{A}$-linear map from $\mathcal{A}^q$ to $\mathcal{A}^q$ where $q = 2(k + l)$. Since $\mathcal{A}$ is unital, $U \in M_q(\mathcal{A})$. Hence U is bounded and induces a topological isomorphism between $\mathcal{E}_1, \mathcal{E}_2$. $\square$

Lemma 2.2.2. *All Hermitian structures on a free module over $\mathcal{A}$ are isomorphic to each other.*

Proof. The canonical *Hermitian* structure on $\mathcal{A}^q$ was given by $\langle \xi, \eta \rangle_{\mathcal{A}} = \sum_{k=1}^{q} \xi_k^* \eta_k$. We show that any other *Hermitian* structure is isomorphic to this one. Let $\langle \ , \ \rangle' : \mathcal{A}^q \times \mathcal{A}^q \longrightarrow \mathcal{A}$ be another *Hermitian* structure on $\mathcal{A}^q$. Let $\{e_1, \ldots, e_q\}$ be standard basis of $\mathcal{A}^q$. Let $T = ((t_{rs}))$ be given by $t_{sr} = \langle e_r, e_s \rangle'$. Then $\langle \xi, \eta \rangle' = \sum\limits_{r,s} \langle e_r \xi_r, e_s \eta_s \rangle' = \sum\limits_{r,s} \xi_r^* \langle e_r, e_s \rangle' \eta_s$. That is, $\langle \xi, \eta \rangle' = \xi^* T \eta$, where $T \in M_q(\mathcal{A})$ is positive-definite. Hence T is a positive element in the C^*-algebra $M_q(\mathcal{B}(\mathcal{H}))$. Note that for $\xi \in \mathcal{A}^q$, $\xi^* = (\xi_1^*, \ldots, \xi_q^*)$ where $\xi = (\xi_1, \ldots, \xi_q)$. We consider elements of $\mathcal{A}^q$ as column vector, whereas their $*$ will denote row vector. So here ξ^* is a row vector and ξ is a column vector. We denote $\langle \ , \ \rangle'$ by $\langle \ , \ \rangle_T$. Hence, *Hermitian* structures on $\mathcal{A}^q$ are parametrized by such T. We show that T is one to one. Suppose $T\xi = 0$. Then for any $\eta \in \mathcal{E}$, we get $\Phi_\xi(\eta) = \xi^* T \eta = (T\xi)^* \eta = 0$, showing $\Phi_\xi = 0$. Since $\xi \longmapsto \Phi_\xi$ is an isomorphism, we get $\xi = 0$. Hence T is one to one. To see T is onto, we pick any ζ from $\mathcal{A}^q$. Then $\eta \longmapsto \zeta^* \eta$ is a $\mathcal{A}$-linear map on $\mathcal{A}^q$ taking value in $\mathcal{A}$ (we are dealing with right $\mathcal{A}$-module). Hence, there exists ξ in $\mathcal{A}^q$ such that

$$\Phi_\xi(\eta) = \zeta^*\eta = \xi^* T\eta = (T\xi)^*\eta \,.$$

Hence $\zeta = T\xi\,,$ showing T is onto. We define

$$\widetilde{T} : \mathcal{A}^q \longrightarrow \mathcal{A}^q$$

$$\widetilde{T}(T\xi) = \xi$$

To show this map is continuous, let $T\xi_k \to T\xi$ in $\mathcal{A}^q$. Then $\xi_k^* T\eta \to \xi^* T\eta$ for any η because multiplication is continuous with respect to the topology of $\mathcal{A}^q$. Hence $\Phi_{\xi_k} \to \Phi_\xi$. By Lemma (2.2.1) $\xi \longmapsto \Phi_\xi$ is a continuous isomorphism. Hence we get $\xi_k \to \xi$, which shows continuity of $\widetilde{T}$. Thus T has a bounded inverse $\widetilde{T}$ implying spectrum of T is away from zero. Since T is positive, $\sqrt{T}$ is a holomorphic function of T. Now define

$$\Psi : \mathcal{A}^q \longrightarrow \mathcal{A}^q$$

$$\Psi(\xi) = \sqrt{T}\xi$$

Then, $\langle \xi, \eta \rangle_T = \xi^* T\eta = \xi^* \sqrt{T}\sqrt{T}\eta = \langle \Psi(\xi), \Psi(\eta) \rangle$. Since $\mathcal{A}$ is stable under holomorphic functional calculus in $\mathcal{B}(\mathcal{H})$, inverse of T i,e. $\widetilde{T}$ lies in $M_q(\mathcal{A}) \subseteq M_q(\mathcal{B}(\mathcal{H}))$ (see [38]). Invertibility of T in $M_q(\mathcal{A})$ gives invertibility of Ψ. So Ψ gives an isomorphism between the canonical *Hermitian* structure $\langle \, , \rangle_{\mathcal{A}}$ on $\mathcal{A}$ and *Hermitian* structure obtained throught T. Hence we are done. □

Using this lemma we can conclude the following fact about *Hermitian* structures on a f.g.p module which is also important in our computation of the Yang-Mills functional.

Theorem 2.2.3. *Let $\mathcal{E}$ be a f.g.p $\mathcal{A}$-module with a Hermitian structure. Then we can have a self-adjoint idempotent $p \in M_q(\mathcal{A})$ such that $\mathcal{E} = p\mathcal{A}^q$ and $\mathcal{E}$ has the induced Hermitian structure.*

Proof. Let $\mathcal{E}$ be a f.g.p $\mathcal{A}$-module with a *Hermitian* structure $\langle \, , \rangle_{\mathcal{E}}$. Because $\mathcal{E}$ is projective, we can have an $\mathcal{A}$-module $\mathcal{F}$ such that $\mathcal{E} \bigoplus \mathcal{F} \cong \mathcal{A}^q$ for some natural number q. Since $\mathcal{F}$ is also f.g.p $\mathcal{A}$-module, by Lemma (1.2.2) $\mathcal{F}$ has a *Hermitian* structure say $\langle \, , \rangle_{\mathcal{F}}$. Then $\mathcal{E} \bigoplus \mathcal{F}$ posseses a *Hermitian* structure $\langle \, , \rangle$ given by

$$\langle (e_1, f_1), (e_2, f_2) \rangle = \langle e_1, e_2 \rangle_{\mathcal{E}} + \langle f_1, f_2 \rangle_{\mathcal{F}}\,,$$

i,e. we get a *Hermitian* structure on $\mathcal{A}^q$ coming from $\mathcal{E}$ and $\mathcal{F}$. By our previous lemma, this *Hermitian* structure is isomorphic with the canonical one. Note that $\mathcal{E}$ is orthogonal to $\mathcal{F}$ with respect to this *Hermitian* structure. Let p be a projection from $\mathcal{A}^q$ onto $\mathcal{E}$, i,e. $p(e+f) = e$. Then $\mathcal{E} = p\mathcal{A}^q$. Now,

$$\begin{aligned}
\langle p(e_1 + f_1), (e_2 + f_2) \rangle &= \langle e_1, e_2 \rangle_E + \langle e_1, f_2 \rangle_F \\
&= \langle e_1, e_2 \rangle_E \\
&= \langle e_1, p(e_2 + f_2) \rangle_E \\
&= \langle e_1 + f_1, p(e_2 + f_2) \rangle .
\end{aligned}$$

which shows that p is self-adjoint. Once we have a self-adjoint p, we can now restrict the *Hermitian* structure on $\mathcal{A}^q$ to $\mathcal{E}$ (recall the proof of part (a) of Lemma 1.2.2) and hence $\mathcal{E}$ has the induced *Hermitian* structure. $\square$

2.3 Comparison between the two approaches

In this section we work out the Yang-Mills action functional in the second formulation and show that this is same as the one coming from the C^*-dynamical system formulation. In our case of the non-commutative n torus $\mathcal{A}_\Theta$, the Lie group is $\mathbb{T}^n$ and hence we get n Clifford gamma matrices $\gamma_1, \ldots, \gamma_n$. We define $D := \sum_{j=1}^n \delta_j \otimes \gamma_j$ where $\delta_j = (-i)\widetilde{\delta}_j$ (recall the definition of $\widetilde{\delta}_j$ from 2.1.1). Then D becomes self-adjoint on $\mathcal{H} = \widetilde{\mathcal{H}} \otimes \mathbb{C}^N$ with domain $\mathcal{A}_\Theta \otimes \mathbb{C}^N$, $N = 2^{[n/2]}$. Moreover $|D|^{-n}$ lies in $\mathcal{L}^{(1,\infty)}$ with $Tr_\omega(|D|^{-n}) = 2N\pi^{n/2}/(n(2\pi)^n\Gamma(n/2))$ (see [23], Page 545) and $(\mathcal{A}_\Theta, \mathcal{H}, D)$ gives us a (n,∞)-summable spectral triple. Following lemmas and propositions determine the Connes' calculus Ω^1_D, Ω^2_D along with the product $\Omega^1_D \times \Omega^1_D \longrightarrow \Omega^2_D$ and differentials $\widetilde{d}$.

Proposition 2.3.1. $\Omega^1_D \cong \underbrace{\mathcal{A}_\Theta \oplus \ldots \oplus \mathcal{A}_\Theta}_{n\ times}$.

Proof. We know that $\Omega_D^1 \cong \pi(\Omega^1)$. Let $\omega \in \Omega^1$, so $\omega = \sum_j a_j db_j$, $a_j, b_j \in \mathcal{A}_\Theta$. Then,

$$\begin{aligned}\pi(\omega) &= \sum_j (a_j \otimes I)[D, b_j] \\ &= \sum_j \left(\sum_{l=1}^{n} a_j \delta_l(b_j) \otimes \gamma_l \right).\end{aligned}$$

Since $\{\gamma_1, \ldots, \gamma_n\} \subseteq M_N(\mathbb{C})$ is a linearly independent set, their linear span forms a n-dimensional vector space $\mathbb{C}^n$ where we identify γ_l with $\alpha_l = (0, \ldots, 1, \ldots, 0) \in \mathbb{C}^n$ with 1 in the l-th place. $\{\alpha_1, \ldots, \alpha_n\}$ is the canonical basis for $\mathbb{C}^n$. Hence we get $\Omega_D^1 \subseteq \mathcal{A}_\Theta \otimes \mathbb{C}^n$. To see surjectivity notice that for any $a \in \mathcal{A}_\Theta$, we can write $a = aU_l^* U_l = aU_l^* \delta_l(U_l)$ where the element $aU_l^* d(U_l) \in \Omega^1$. $\square$

Remark 2.3.2. Henceforth throughout the chapter $\{\sigma_1, \ldots, \sigma_n\}$ will denote the standard basis of $\mathcal{A}_\Theta^n$ as free $\mathcal{A}_\Theta$-bimodule where $\sigma_k = \underbrace{(0, \ldots, 1, \ldots, 0)}_{n\,tuple}$ with 1 in the k-th place; whereas $\{\widetilde{e}_1, \ldots, \widetilde{e}_q\}$ will stand for the standard basis of $\mathcal{A}_\Theta^q$ where $\widetilde{e}_l = \underbrace{(0, \ldots, 1, \ldots, 0)}_{q\,tuple}$ with 1 in the l-th place. We will reserve this notation for the rest of this chapter. Under the identification in the above proposition, σ_k is identified with $U_k^* \delta_k(U_k) \otimes \gamma_k$ in Ω_D^1 for $k \in \{1, \ldots, n\}$.

Proposition 2.3.3. $\Omega_D^2 \cong \underbrace{\mathcal{A}_\Theta \oplus \ldots\ldots \oplus \mathcal{A}_\Theta}_{n(n-1)/2\ times}$.

Proof. We know that $\Omega_D^2 \cong \pi(\Omega^2)/\pi(dJ_0^{(1)})$. Let $\omega \in \Omega^2$ and write $\omega = \sum_r a_r db_r dc_r$, where $a_r, b_r, c_r \in \mathcal{A}_\Theta$. Then

$$\begin{aligned}\pi(\omega) &= \sum_r (a_r \otimes I)[D, b_r][D, c_r] \\ &= \sum_r (a_r \otimes I) \left(\sum_{j=1}^{n} \delta_j(b_r) \otimes \gamma_j \right) \left(\sum_{k=1}^{n} \delta_k(c_r) \otimes \gamma_k \right) \\ &= \sum_r \left(\sum_{j=1}^{n} a_r \delta_j(b_r) \otimes \gamma_j \right) \left(\sum_{k=1}^{n} \delta_k(c_r) \otimes \gamma_k \right)\end{aligned}$$

$$= \sum_r \left(\left(\sum_{j=1}^n a_r\delta_j(b_r)\delta_j(c_r) \otimes I \right) + \sum_{p<q} (a_r\delta_p(b_r)\delta_q(c_r) - a_r\delta_q(b_r)\delta_p(c_r)) \otimes \gamma_p\gamma_q \right).$$

Since we know that $\gamma_l^2 = I$ and $\gamma_l\gamma_m = -\gamma_m\gamma_l$ for $l \neq m$, we have

$$a_r\delta_p(b_r)\delta_q(c_r) \otimes \gamma_p\gamma_q + a_r\delta_q(b_r)\delta_p(c_r) \otimes \gamma_q\gamma_p$$

$$= (a_r\delta_p(b_r)\delta_q(c_r) - a_r\delta_q(b_r)\delta_p(c_r)) \otimes \gamma_p\gamma_q.$$

Now $\gamma_l\gamma_m$ is independent with all $\gamma_p\gamma_q$ if $l, m \notin \{p, q\}$. Hence, $\pi(\Omega^2) \subseteq \bigoplus_{l=1}^{1+n(n-1)/2} \mathcal{A}_\Theta^{(l)}$ where $\mathcal{A}_\Theta^{(l)} = \mathcal{A}_\Theta \; \forall \, l$, because total number of the elements $(a_r\delta_p(b_r)\delta_q(c_r) \otimes \gamma_p\gamma_q - a_r\delta_q(b_r)\delta_p(c_r) \otimes \gamma_p\gamma_q)$ is $n(n-1)/2$. To show equality we take any non-zero $a \in \mathcal{A}_\Theta$ and $b = U_1$, $c = U_1^*$. Then $adU_1dU_1^* \in \Omega^2$ and $\pi(adU_1dU_1^*) = -a \otimes I$ is a non-zero element of $\pi(\Omega^2)$. Similarly for each p, q we consider $aU_q^*U_p^*d(U_p)d(U_q) \in \Omega^2$. Then $\pi(aU_q^*U_p^*d(U_p)d(U_q)) = a \otimes \gamma_p\gamma_q$. This shows that the above inclusion is an equality.

Now we calculate $\pi(dJ_0^{(1)})$. We have $\omega \in J_0^{(1)}$ implies $\omega = \sum_s a_s db_s$ where $a_s, b_s \in \mathcal{A}_\Theta$, such that $\sum_s (a_s \otimes I)[D, b_s] = 0$. So we get $\sum_s (a_s \otimes I) \left(\sum_{j=1}^n \delta_j(b_s) \otimes \gamma_j \right) = 0$, that is, $\sum_{j=1}^n \left(\sum_s a_s\delta_j(b_s) \right) \otimes \gamma_j = 0$. But $\gamma_1, \ldots, \gamma_n$ being linearly independent we get

$$\sum_s a_s\delta_j(b_s) \otimes \gamma_j = 0, \quad \forall j = 1, \ldots, n. \tag{2.3.1}$$

Now $d\omega = \sum_s da_s db_s$. So,

$$\begin{aligned}
\pi(d\omega) &= \sum_s [D, a_s][D, b_s] \\
&= \sum_s \left(\sum_{j=1}^n \delta_j(a_s) \otimes \gamma_j \right) \left(\sum_{k=1}^n \delta_k(b_s) \otimes \gamma_k \right) \\
&= \sum_s \left((\sum_{j=1}^n \delta_j(a_s)\delta_j(b_s) \otimes I) + \ldots + (\delta_p(a_s)\delta_q(b_s) - \delta_q(a_s)\delta_p(b_s)) \otimes \gamma_p\gamma_q + \ldots \right)
\end{aligned}$$

Now, from equation (2.3.1) we get

$$\sum_s \delta_p(a_s)\delta_q(b_s) \otimes \gamma_p\gamma_q = -\sum_s a_s\delta_p\delta_q(b_s) \otimes \gamma_p\gamma_q \,,$$

and

$$\sum_s \delta_q(a_s)\delta_p(b_s) \otimes \gamma_q\gamma_p = -\sum_s a_s\delta_q\delta_p(b_s) \otimes \gamma_q\gamma_p \,.$$

Hence,

$$\begin{aligned}
(\delta_p(a_s)\delta_q(b_s) - \delta_q(a_s)\delta_p(b_s)) \otimes \gamma_p\gamma_q &= (-a_s\delta_p\delta_q(b_s) + a_s\delta_q\delta_p(b_s)) \otimes \gamma_p\gamma_q \\
&= 0
\end{aligned}$$

because $\delta_p\delta_q = \delta_q\delta_p \,,\ \forall p, q \in \{1, \ldots, n\}$. Hence, $\pi(dJ_0^{(1)}) \subseteq \mathcal{A}_\Theta$ and to show the equality take any $a \in \mathcal{A}_\Theta$. Consider $\omega = a(U_1^* dU_1 - 1/2 \times U_1^{-2} d(U_1^2)) \in \Omega^1$. Then we get $\pi(\omega) = 0$ but $\pi(d\omega) = a \otimes I \neq 0$ (which also shows non-triviality of ω). Hence we conclude $\pi(dJ_0^{(1)}) \cong \pi(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta$. □

Now we want to determine the differential $\widetilde{d} : \pi(\mathcal{A}_\Theta) \longrightarrow \Omega_D^1$ so that $\widetilde{d}(\pi(a)) = \pi(da),\ \forall a \in \mathcal{A}_\Theta$.

Lemma 2.3.4. *$\widetilde{d} : \pi(\mathcal{A}_\Theta) \longrightarrow \Omega_D^1$ is given by, $\pi(a) \longmapsto (\delta_1(a), \ldots, \delta_n(a))$.*

Proof. Pick any element $\pi(a) \in \pi(\mathcal{A}_\Theta)$. Then $da \in \Omega^1$ and hence $\pi(da) = [D, a] = \sum_{j=1}^n \delta_j(a) \otimes \gamma_j$. This is an element in Ω_D^1, which is isomorphic to $\mathcal{A}_\Theta^n$ and under this isomorphism, $\sum_{j=1}^n \delta_j(a) \otimes \gamma_j$ goes to $(\delta_1(a), \ldots, \delta_n(a))$ in $\mathcal{A}_\Theta^n$. Hence the above definition of $\widetilde{d}$ is justified. □

Next we want to determine the differential $\widetilde{d} : \Omega_D^1 \longrightarrow \Omega_D^2$ so that $\widetilde{d}(\pi(\omega)) = \pi(d\omega),\ \forall\, \omega \in \Omega^1$.

Lemma 2.3.5. *$\widetilde{d} : \Omega_D^1 \longrightarrow \Omega_D^2$ is given by,*

$$(0, \ldots, a, \ldots, 0) \longmapsto ((\delta_p(aU_j^*)\delta_q(U_j) - (\delta_q(aU_j^*)\delta_p(U_j)))_{1 \leq p < q \leq n}$$

for a in the j-th place.

Proof. For $(0,\ldots,a,\ldots,0) \in \Omega^1_D$ with a in the j-th place, we have $aU_j^* dU_j \in \Omega^1$, such that $\pi(aU_j^* dU_j)$ is identified with $(0,\ldots,a,\ldots,0)$. Now, $d(aU_j^* dU_j) = d(aU_j^*)dU_j$, an element of Ω^2. Now,

$$\begin{aligned}
\pi(d(aU_j^*)dU_j) &= [D, aU_j^*][D, U_j] \\
&= \left(\sum_{l=1}^{n} \delta_l(aU_j^*) \otimes \gamma_l\right)\left(\sum_{k=1}^{n} \delta_k(U_j) \otimes \gamma_k\right) \\
&= \sum_{l=1}^{n} \delta_l(aU_j^*)\delta_l(U_j) \otimes I + \sum_{p<q} \big(\delta_p(aU_j^*)\delta_q(U_j) - \delta_q(aU_j^*)\delta_p(U_j)\big) \otimes \gamma_p\gamma_q .
\end{aligned}$$

Under the isomorphism $\Omega^2_D \cong \mathcal{A}_\Theta^{n(n-1)/2}$, $\sum_{p<q}(\delta_p(aU_j^*)\delta_q(U_j) - \delta_q(aU_j^*)\delta_p(U_j)) \otimes \gamma_p\gamma_q$ goes to the required element in $\mathcal{A}_\Theta^{n(n-1)/2}$. □

Finally the product map is recognized by the following lemma.

Lemma 2.3.6. *The product map* $\widetilde{\prod} : \Omega^1_D \times \Omega^1_D \longrightarrow \Omega^2_D$ *is given by the following*

$$(a_1,\ldots,a_n).(b_1,\ldots,b_n) := ((a_p b_q - a_q b_p))_{1\le p<q\le n} .$$

Proof. We have a product $\prod : \Omega^1 \times \Omega^1 \longrightarrow \Omega^2$, given by $\prod(a_1 da_2, b_1 db_2) = a_1 da_2 b_1 db_2 = a_1 d(a_2 b_1) db_2 - a_1 a_2 db_1 db_2$. Choose two elements $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ in Ω^1_D. We have seen previously that $\pi(\sum_{m=1}^{n} a_m U_m^* d(U_m))$ in $\pi(\Omega^1)$ is identified with $(a_1,\ldots,a_n)$. Similarly for b_m in place of a_m. Let $\omega = \sum_{m=1}^{n} a_m U_m^* d(U_m)$ and $\omega' = \sum_{m=1}^{n} b_m U_m^* d(U_m)$. Now,

$$\begin{aligned}
\prod(\omega,\omega') &= \left(\sum_{m=1}^{n} a_m U_m^* d(U_m)\right)\left(\sum_{j=1}^{n} b_j U_j^* d(U_j)\right) \\
&= \sum_{m,j=1}^{n} a_m U_m^* d(U_m) b_j U_j^* d(U_j) \\
&= \sum_{m,j=1}^{n} \big(a_m U_m^* d(U_m b_j U_j^*) d(U_j) - a_m d(b_j U_j^*) d(U_j)\big) .
\end{aligned}$$

It is an element of Ω^2. Applying π on it we get

$$
\begin{aligned}
\pi\left(\prod(\omega,\omega')\right) &= \sum_{m,j=1}^{n}\left(a_m U_m^*[D,U_m b_j U_j^*][D,U_j] - a_m[D,b_j U_j^*][D,U_j]\right)\\
&= \sum_{m,j=1}^{n}(a_m U_m^*(\sum_{k=1}^{n}\delta_k(U_m b_j U_j^*)\otimes\gamma_k)(\sum_{l=1}^{n}\delta_l(U_j)\otimes\gamma_l)\\
&\qquad -a_m(\sum_{r=1}^{n}\delta_r(b_j U_j^*)\otimes\gamma_r)(\sum_{s=1}^{n}\delta_s(U_j)\otimes\gamma_s))\\
&= \sum_{p<q}(\sum_{m,j=1}^{n} a_m U_m^*\delta_p(U_m b_j U_j^*)\delta_q(U_j) - \sum_{m,j=1}^{n} a_m U_m^*\delta_q(U_m b_j U_j^*)\delta_p(U_j)\\
&\qquad - \sum_{m,j=1}^{n} a_m\delta_p(b_j U_j^*)\delta_q(U_j) + \sum_{m,j=1}^{n} a_m\delta_q(b_j U_j^*)\delta_p(U_j))\otimes\gamma_p\gamma_q
\end{aligned}
$$

Now for each p and q,

$$
\begin{aligned}
&\sum_{m,j=1}^{n}(a_m U_m^*\delta_p(U_m b_j U_j^*)\delta_q(U_j) - \sum_{m,j=1}^{n} a_m U_m^*\delta_q(U_m b_j U_j^*)\delta_p(U_j)\\
&\quad - \sum_{m,j=1}^{n} a_m\delta_p(b_j U_j^*)\delta_q(U_j) + \sum_{m,j=1}^{n} a_m\delta_q(b_j U_j^*)\delta_p(U_j)\\
=\ & \sum_{m=1}^{n}(a_m U_m^*\delta_p(U_m b_q U_q^*)U_q - a_m\delta_p(b_q U_q^*)U_q + a_m\delta_q(b_p U_p^*)U_p - a_m U_m^*\delta_q(U_m b_p U_p^*)U_p)\\
=\ & \sum_{m=1}^{n}(a_m U_m^*\delta_p(U_m b_q) - a_m\delta_p(b_q) + a_m\delta_q(b_p) - a_m U_m^*\delta_q(U_m b_p))\\
=\ & \sum_{m=1}^{n}(a_m U_m^*\delta_p(U_m)b_q - a_m U_m^*\delta_q(U_m)b_p)\\
=\ & a_p b_q - a_q b_p.
\end{aligned}
$$

Hence for $(a_1,\ldots,a_n),(b_1,\ldots,b_n)\in\Omega_D^1$, we get

$$
\widetilde{\prod}((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = ((a_p b_q - a_q b_p))_{1\le p<q\le n}\,.
$$

□

It can be easily checked that both the $\widetilde{d}$, defined above, are derivations. We first prove the following lemmas which will help us in the computation.

Lemma 2.3.7. *The canonical trace* τ *on* $\mathcal{A}_\Theta$ *equals to* $Tr_\omega(|D|^{-n})^{-1}\int$, *where* Tr_ω *denotes the Dixmier trace and* $\int a := Tr_\omega((a\otimes I)|D|^{-n})$ *for all* $a\in\mathcal{A}_\Theta$.

Proof. We have $\tau(a) = \tau(\alpha_{\mathbf{g}}(a))$, $\forall\,\mathbf{g}\in\mathbb{T}^n$ because τ is G-invariant on $\mathcal{A}_\Theta$. The G.N.S Hilbert space $L^2(\mathcal{A}_\Theta,\tau)$ is identified with $l^2(\mathbb{Z}^n)$. For $\mathbf{g}\in\mathbb{T}^n$, $\alpha_{\mathbf{g}}(U_1^{k_1}\ldots U_n^{k_n}) = \mathbf{g}^{\mathbf{k}}U_1^{k_1}\ldots U_n^{k_n}$. Here $\mathbf{g}=(g_1,\ldots,g_n)\in\mathbb{T}^n$; $\mathbf{g}^{\mathbf{k}} = g_1^{k_1}\ldots g_n^{k_n}$. Define

$$U_{\mathbf{g}}:\, L^2(\mathcal{A}_\Theta,\tau)\longrightarrow L^2(\mathcal{A}_\Theta,\tau)$$

$$a\longmapsto \alpha_{\mathbf{g}}(a)$$

It is easy to check this map is isometry with dense range. Hence extends as unitary on $L^2(\mathcal{A}_\Theta,\tau)$. For $e_{\mathbf{k}}\in l^2(\mathbb{Z}^n)$, $U_{\mathbf{g}}(e_{\mathbf{k}}) = \mathbf{g}^{\mathbf{k}}e_{\mathbf{k}}$. Since $D(e_{\mathbf{k}}\otimes M) = \sum_{j=1}^n k_j e_{\mathbf{k}}\otimes\gamma_j M$ for $M\in M_N(\mathbb{C})$, it follows that $D(U_{\mathbf{g}}\otimes I) = (U_{\mathbf{g}}\otimes I)D$ on $L^2(\mathcal{A}_\Theta\,,\tau)\otimes\mathbb{C}^N$. But $(U_{\mathbf{g}}\otimes I)D(U_{\mathbf{g}}^*\otimes I) = D \Rightarrow (U_{\mathbf{g}}\otimes I)|D|(U_{\mathbf{g}}^*\otimes I) = |D|$. This further implies that $(U_{\mathbf{g}}\otimes I)|D|^{-n}(U_{\mathbf{g}}^*\otimes I) = |D|^{-n}$. Hence,

$$\begin{aligned}
Tr_\omega((U_{\mathbf{g}}aU_{\mathbf{g}}^*\otimes I)|D|^{-n}) &= Tr_\omega((U_{\mathbf{g}}\otimes I)(a\otimes I)|D|^{-n}(U_{\mathbf{g}}^*\otimes I))\\
&= Tr_\omega((a\otimes I)|D|^{-n}).
\end{aligned}$$

This shows that $Tr_\omega(|D|^{-n})^{-1}\int$ is also a G-invariant trace on $\mathcal{A}_\Theta$. Now uniqueness of G-invariant trace on $\mathcal{A}_\Theta$ gives $\tau(a) = Tr_\omega(|D|^{-n})^{-1}Tr_\omega((a\otimes I)|D|^{-n})$, where $Tr_\omega(|D|^{-n})$ is a positive constant. $\square$

Lemma 2.3.8. *If* $\{\gamma_1,\ldots,\gamma_n\}$ *are Clifford gamma matrices in* $M_N(\mathbb{C})$ *then they enjoy the property* $Trace(\gamma_l\gamma_m) = 0$ *for* $l\neq m$.

Proof. This follows immediately from the fact that Clifford gamma matrices satisfy the relation $\gamma_l\gamma_m+\gamma_m\gamma_l = 2\delta_{lm}\ \forall\, l,m$, where δ_{lm} is the Kronecker delta. $\square$

Lemma 2.3.9. *The positive linear functional* $\int : T \longmapsto Tr_\omega(T|D|^{-n})/Tr_\omega(|D|^{-n})$*, for* $T \in M_N(\mathcal{A}_\Theta)$*, equals with* $\tau \otimes Trace$ *, where 'Trace' denotes the ordinary matrix trace (normalized) on* $M_N(\mathbb{C})$*.*

Proof. Since $D^2 = \sum \delta_j^2 \otimes I_N$, $|D|^{-n}$ commutes with $1 \otimes M_N(\mathbb{C})$ it follows that $\int$ is a trace on $M_N(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta \otimes M_N(\mathbb{C})$. Our requirement is now fulfilled because of the fact that $\tau \otimes Trace$ is the unique extention (normalized) of τ on $M_N(\mathcal{A}_\Theta)$. □

Lemma 2.3.10. *If* $l \neq m$ *then any* $a \otimes \gamma_l\gamma_m$ *lies in the range of* P *where* P *is the orthogonal projection onto the orthogonal complement of* $\pi(dJ_0^{(1)}) \subseteq \pi(\Omega^2)$*.*

Proof. Recall that any element of $\pi(dJ_0^{(1)})$ looks like $x \otimes I$. Now $\langle\, a \otimes \gamma_l\gamma_m \,,\, x \otimes I \,\rangle_{\pi(\Omega^2)} = Tr_\omega((a^*x \otimes \gamma_l\gamma_m)|D|^{-n}) = Tr_\omega(|D|^{-n})\tau(a^*x)Trace\,(\gamma_l\gamma_m) = 0$, since $Trace\,(\gamma_l\gamma_m) = 0$ by Lemma (2.3.8). □

Now we are ready to compute the Yang-Mills functional for $\mathcal{A}_\Theta$. Since $\Omega_D^1 \cong \mathcal{A}_\Theta^n$, any compatible connection $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega_D^1$ is given by n-tuple of maps $(\nabla_1, \ldots, \nabla_n)$, where $\nabla_j : \mathcal{E} \longrightarrow \mathcal{E}$ are such that

$$\nabla(\xi) = \sum_{j=1}^{n} \nabla_j(\xi) \otimes \sigma_j\,, \qquad (2.3.2)$$

and

$$\langle \xi, \nabla_j(\eta)\rangle - \langle \nabla_j(\xi), \eta\rangle \;=\; \delta_j(\langle \xi, \eta\rangle_{\mathcal{A}_\Theta}). \qquad (2.3.3)$$

Here $\{\sigma_1, \ldots, \sigma_n\}$ is the standard basis of $\mathcal{A}_\Theta^n$ as free $\mathcal{A}_\Theta$-bimodule. Then

$$\widetilde{\nabla} : \mathcal{E} \otimes \Omega_D^1 \longrightarrow \mathcal{E} \otimes \Omega_D^2$$

is given by

$$\widetilde{\nabla}(\xi \otimes \sigma_m) = \left(\sum_{j=1}^{n} \nabla_j(\xi) \otimes \sigma_j\right) \sigma_m + \xi \otimes \widetilde{d}(\sigma_m)$$

for each $m = 1, \ldots, n$.

Proposition 2.3.11. *The curvature* $\Theta = \widetilde{\nabla} \circ \nabla$ *is given by* $\sum_{m<j} [\nabla_m, \nabla_j](.) \otimes \sigma_m \sigma_j$ *where* $\sigma_m, \sigma_j \in \mathcal{A}_\Theta^n$ *and* $\sigma_m \sigma_j$ *is the element in* $\mathcal{A}_\Theta^{n(n-1)/2}$ *produced by the product map* $\widetilde{\prod}$ *of Lemma (2.3.6).*

Proof. Through direct computation we get

$$
\begin{aligned}
\Theta(\xi) &= \widetilde{\nabla} \circ \nabla(\xi) \\
&= \sum_{m=1}^{n} \widetilde{\nabla}(\nabla_m(\xi) \otimes \sigma_m) \\
&= \sum_{m} \left((\sum_{j} \nabla_j(\nabla_m(\xi)) \otimes \sigma_j)\sigma_m + \nabla_m(\xi) \otimes \widetilde{d}(\sigma_m) \right) \\
&= \sum_{m,j} \nabla_j(\nabla_m(\xi)) \otimes \sigma_j \sigma_m + \nabla_m(\xi) \otimes \widetilde{d}(\sigma_m) \\
&= \sum_{m<j} [\nabla_m, \nabla_j](\xi) \otimes \sigma_m \sigma_j + \sum_{m} \nabla_m(\xi) \otimes \widetilde{d}(\sigma_m).
\end{aligned}
$$

But

$$
\begin{aligned}
\sum_{m} \nabla_m(\xi) \otimes \widetilde{d}(\sigma_m) &= \sum_{m} \nabla_m(\xi) \otimes ((\delta_p(U_m^*)\delta_q(U_m) - \delta_q(U_m^*)\delta_p(U_m))_{1 \le p < q \le n} \\
&= 0
\end{aligned}
$$

because $\delta_j(U_m^*) = -U_m^* \delta_j(U_m) U_m^*$. Hence $\Theta = \sum_{m<j} [\nabla_m, \nabla_j] \otimes \sigma_m \sigma_j$. $\square$

Proposition 2.3.12. $YM(\nabla) = \sum_{m<j} \tau_q([\nabla_m, \nabla_j]^* [\nabla_m, \nabla_j])$ *upto a positive factor where* τ_q *denotes the extended trace* $\tau \otimes Trace$ *on* $M_q(\mathcal{A}_\Theta)$.

Proof. Recall that $\langle\langle \Theta, \Theta \rangle\rangle = \sum_{k=1}^{q} \langle \Theta(p\widetilde{e}_k), \Theta(p\widetilde{e}_k) \rangle_{\Omega_D^2}$ where $\{\widetilde{e}_1, \ldots, \widetilde{e}_q\}$ denotes the standard basis of $\mathcal{A}_\Theta^q$ and $\mathcal{E} = p\mathcal{A}_\Theta^q$. Let

$$
[\nabla_m, \nabla_j](p\widetilde{e}_k) = \eta^{(mjk)} = p\eta^{(mjk)} = (\eta_1^{(mjk)}, \ldots, \eta_q^{(mjk)}) \in \mathcal{A}_\Theta^q .
$$

Then from Proposition (2.3.11) we get

$$\Theta(p\widetilde{e}_k) = \textstyle\sum_{m<j}(\eta_1^{(mjk)}\sigma_m\sigma_j, \ldots, \eta_q^{(mjk)}\sigma_m\sigma_j)\,,$$

an element of $(\Omega_D^2)^q$. It is easily observed that as $\mathbb{C}$-vector spaces $Hom(\mathcal{E}, \mathcal{E}\otimes\Omega_D^2) \cong \bigoplus Hom(\mathcal{E},\mathcal{E})$. We can view $End(\mathcal{E})$ as $pM_q(\mathcal{A}_\Theta)p \subseteq M_q(\mathcal{A}_\Theta)$. We have an inner-product on $\bigoplus M_q(\mathcal{A}_\Theta)$ given by $\langle (A_1,\ldots,A_t),(B_1,\ldots,B_t)\rangle = \sum_{j=1}^{t} \tau_q(A_j^*B_j)$. The following calculation shows that this inner-product becomes same with the one on $Hom(\mathcal{E},\mathcal{E}\otimes\Omega_D^2)$.

$$\begin{aligned}
\langle \Theta(p\widetilde{e}_k), \Theta(p\widetilde{e}_k)\rangle &= \sum_{m<j,l<r} \langle\, (\eta_1^{(mjk)}\sigma_m\sigma_j, \ldots, \eta_q^{(mjk)}\sigma_m\sigma_j)\,,\, (\eta_1^{(lrk)}\sigma_l\sigma_r, \ldots, \eta_q^{(lrk)}\sigma_l\sigma_r)\,\rangle \\
&= \sum_{m<j,l<r}\sum_{s=1}^{q} \langle\, \eta_s^{(mjk)}\sigma_m\sigma_j\,,\, \eta_s^{(lrk)}\sigma_l\sigma_r\,\rangle_{\Omega_D^2} \\
&= \sum_{m<j,l<r}\sum_{s=1}^{q} \langle\, [\eta_s^{(mjk)}\otimes\gamma_m\gamma_j]\,,\, [\eta_s^{(lrk)}\otimes\gamma_l\gamma_r]\,\rangle_{\Omega_D^2} \\
&= \sum_{m<j,l<r}\sum_{s=1}^{q} \langle\, P(\eta_s^{(mjk)}\otimes\gamma_m\gamma_j)\,,\, P(\eta_s^{(lrk)}\otimes\gamma_l\gamma_r)\,\rangle_{\pi(\Omega^2)} \\
&= \sum_{m<j,l<r}\sum_{s=1}^{q} Tr_\omega\left((\eta_s^{(mjk)^*}\eta_s^{(lrk)}\otimes\gamma_j\gamma_m\gamma_l\gamma_r)|D|^{-n}\right).
\end{aligned}$$

The last equality follows from Lemma (2.3.10). Now an application of Lemma (2.3.8) and (2.3.9) shows that

$$\begin{aligned}
\langle\langle \Theta,\Theta\rangle\rangle &= Tr_\omega(|D|^{-n})\sum_{k=1}^{q}\sum_{m<j}\sum_{s=1}^{q}\tau\left(\eta_s^{(mjk)^*}\eta_s^{(mjk)}\right) \\
&= Tr_\omega(|D|^{-n})\sum_{k=1}^{q}\sum_{m<j}\tau\left(\langle\,[\nabla_m,\nabla_j](p\widetilde{e}_k)\,,\,[\nabla_m,\nabla_j](p\widetilde{e}_k)\,\rangle_{\mathcal{A}_\Theta}\right) \\
&= Tr_\omega(|D|^{-n})\sum_{k=1}^{q}\sum_{m<j}\tau\left(\langle\,p\widetilde{e}_k\,,\,[\nabla_m,\nabla_j]^*[\nabla_m,\nabla_j](p\widetilde{e}_k)\,\rangle_{\mathcal{A}_\Theta}\right) \\
&= Tr_\omega(|D|^{-n})\sum_{m<j}\tau_q\left([\nabla_m,\nabla_j]^*[\nabla_m,\nabla_j]\right).
\end{aligned}$$

The last equality follows from the fact that for any $T = ((t_{rs})) \in pM_q(\mathcal{A}_\Theta)p$, where $p \in M_q(\mathcal{A}_\Theta)$ is a projection, $\sum_{k=1}^{q}\langle \widetilde{e}_k, T\widetilde{e}_k\rangle_{\mathcal{A}_\Theta} = \sum_{k=1}^{q}\langle\, p\widetilde{e}_k, Tp\widetilde{e}_k\,\rangle_{\mathcal{A}_\Theta} = \sum_{r=1}^{q} t_{rr}$. Hence follows the proposition. $\square$

Recall that $\{e_1, \ldots, e_n\}$ denotes the standard basis choosen for $\mathbb{R}^n$ and $\{\sigma_1, \ldots, \sigma_n\}$ is the standard basis of Ω^1_D. We have an one to one correspondence between these sets, both being finite sets of same cardinality. The following theorem points out the main result.

Theorem 2.3.13. *Let $C(\mathcal{E})$ and $\widetilde{C}(\mathcal{E})$ denote the affine space of compatible connections for the first and second approaches respectively. Then both these are in one to one correspondence through an affine isomorphism, and the value of Yang-Mills functional on corresponding elements of these spaces are same upto a positive scalar factor. That is to say that the following diagram*

$$\begin{tikzcd}
C(\mathcal{E}) \arrow[rr, "\Phi"] \arrow[dr, "cYM"'] & & \widetilde{C}(\mathcal{E}) \arrow[dl, "YM"] \\
& \mathbb{R}_+ \cup \{0\} &
\end{tikzcd}$$

commutes, where $c = 2N\pi^{n/2}/(n(2\pi)^n\Gamma(n/2))$.

Proof. Recall from equation (2.1.2) for any $\nabla \in C(\mathcal{E})$, $\nabla(\xi) = \sum_{j=1}^n \nabla_j(\xi) \otimes e_j$ where $\nabla_j : \mathcal{E} \longrightarrow \mathcal{E}$. We define $\Phi(\nabla) = \widetilde{\nabla}$ where

$$\widetilde{\nabla}(\xi) = \sum_{j=1}^n (-i)\nabla_j(\xi) \otimes \sigma_j \,.$$

It is easy to see that $\widetilde{\nabla}$ defines a connection. Given compatibility of ∇, we have to check whether $\widetilde{\nabla}$ is compatible with respect to the *Hermitian* structure. This follows from the following computation.

$$\begin{aligned}
\langle\, \xi, \widetilde{\nabla}(\eta)\,\rangle - \langle\, \widetilde{\nabla}(\xi), \eta\,\rangle &= \sum_{j=1}^n (\langle\, \xi, (-i)\nabla_j(\eta) \otimes \sigma_j\,\rangle - \langle\, (-i)\nabla_j(\xi) \otimes \sigma_j, \eta\,\rangle\,) \\
&= \sum_{j=1}^n (\,\langle\, \xi, \nabla_j(\eta)\,\rangle_{\mathcal{A}_\Theta}(-i)\sigma_j - i\sigma_j^*\langle\, \nabla_j(\xi), \eta\,\rangle_{\mathcal{A}_\Theta}) \\
&= \sum_{j=1}^n (\,\langle\, \xi, \nabla_j(\eta)\,\rangle_{\mathcal{A}_\Theta}(-i)\sigma_j - i\sigma_j\langle\, \nabla_j(\xi), \eta\,\rangle_{\mathcal{A}_\Theta})
\end{aligned}$$

$$
\begin{aligned}
&= (-i)(\langle\,\xi,\nabla_1(\eta)\rangle_{\mathcal{A}_\Theta}+\langle\nabla_1(\xi),\eta\,\rangle_{\mathcal{A}_\Theta},\dots,\\
&\qquad\qquad \langle\,\xi,\nabla_n(\eta)\rangle_{\mathcal{A}_\Theta}+\langle\nabla_n(\xi),\eta\,\rangle_{\mathcal{A}_\Theta})\\
&= (-i)(\widetilde{\delta_1}(\,\langle\,\xi,\eta\,\rangle_{\mathcal{A}_\Theta}),\dots,\widetilde{\delta_n}(\,\langle\,\xi,\eta\,\rangle_{\mathcal{A}_\Theta}))\\
&= (\delta_1(\,\langle\,\xi,\eta\,\rangle_{\mathcal{A}_\Theta}),\dots,\delta_n(\,\langle\,\xi,\eta\,\rangle_{\mathcal{A}_\Theta}))\\
&= \widetilde{d}(\,\langle\,\xi,\eta\,\rangle_{\mathcal{A}_\Theta}).
\end{aligned}
$$

This shows the compatibility of $\widetilde{\nabla}$ with respect to the *Hermitian* structure and hence $\widetilde{\nabla}$ belongs to $\widetilde{C}(\mathcal{E})$. Conversely, for given $\widetilde{\nabla}\in\widetilde{C}(\mathcal{E})$ recall from equation (2.3.2) that $\widetilde{\nabla}(\xi)=\sum_{j=1}^{n}\widetilde{\nabla}_j(\xi)\otimes\sigma_j$ where $\widetilde{\nabla}_j:\mathcal{E}\longrightarrow\mathcal{E}$. We define $\Phi^{-1}(\widetilde{\nabla})=\nabla$ where

$$
\nabla(\xi)=\sum_{j=1}^{n}i\widetilde{\nabla}_j(\xi)\otimes e_j\,.
$$

A similar computation shows the compatibility of ∇. So the elements of $C(\mathcal{E})$ and $\widetilde{C}(\mathcal{E})$ are in one-one correspondence. Recall from equation (2.1.4), for the finitely generated projective $\mathcal{A}_\Theta$-module $\mathcal{E}=p\mathcal{A}_\Theta^q$ we obtained for $\nabla\in C(\mathcal{E})$

$$
YM(\nabla)=\sum_{j<k}\widetilde{\tau}([\nabla_j,\nabla_k]^*[\nabla_j,\nabla_k])\,,
$$

where $\widetilde{\tau}$ was the trace on $End(\mathcal{E})$. For $\Phi(\nabla)=\widetilde{\nabla}$ we obtain from Proposition (2.3.12)

$$
YM(\widetilde{\nabla})=Tr_\omega(|D|^{-n})\sum_{j<k}\tau_q([\nabla_j,\nabla_k]^*[\nabla_j,\nabla_k])\,,
$$

where τ_q was the extended trace of τ on $M_q(\mathcal{A}_\Theta)$. Identifying $End(\mathcal{E})$ with $pM_q(\mathcal{A}_\Theta)p\subseteq M_q(\mathcal{A}_\Theta)$ we see that both $\widetilde{\tau}$ and τ_q are equal with $\tau\otimes Trace$. Hence follows the equality of Yang-Mills for both the approaches except for the positive scalar factor $Tr_\omega(|D|^{-n})=2N\pi^{n/2}/(n(2\pi)^n\Gamma(n/2))\,,N=2^{[n/2]}$. $\square$

Chapter 3

Yang-Mills on Quantum Heisenberg Manifolds

In this chapter our objective is to prove that both the Yang-Mills approaches discussed in chapter 1 coincide for the Quantum Heisenberg manifolds.

Quantum Heisenberg manifolds(QHM) were introduced by Rieffel in ([35]) as strict deformation quantization of Heisenberg manifolds. He introduced a parametric family of deformations and for generic parameter values these are simple C^*-algebras carrying an ergodic action of the Heisenberg group of 3×3 upper triangular matrices with ones on the diagonal. They admit a unique invariant trace. Recently Kang ([26]) and Lee ([28]) has studied Yang-Mills for the QHM following the dynamical system approach. It was shown in ([9]) that in the case of QHM the general prescription that produces a candidate for a spectral triple starting from a C^* dynamical system gives rise to an honest spectral triple. Therefore it is natural to ask whether even in this case these two notions of Yang-Mills coincide and this is the content of this chapter. This parallels Theorem (2.3.13) in the previous chapter where we obtained similar result for the noncommutative n-torus.

Notation : For $x \in \mathbb{R}$, $e(x)$ stands for $e^{2\pi i x}$ where $i = \sqrt{-1}$.

Definition 3.0.14. For any positive integer c, let S^c denote the space of smooth functions $\Phi : \mathbb{R} \times \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ such that

- $\Phi(x+k,y,p) = e(ckpy)\Phi(x,y,p)$ for all $k \in \mathbb{Z}$,
- for every polynomial P on $\mathbb{Z}$ and every partial differential operator $\widetilde{X} = \frac{\partial^{m+n}}{\partial x^m \partial y^n}$ on $\mathbb{R} \times \mathbb{T}$ the function $P(p)(\widetilde{X}\Phi)(x,y,p)$ is bounded on $K \times \mathbb{Z}$ for any compact subset K of $\mathbb{R} \times \mathbb{T}$.

For each $\hbar, \mu, \nu \in \mathbb{R}, \mu^2 + \nu^2 \neq 0$, let $\mathcal{A}_\hbar^\infty$ denote S^c with product and involution defined by

$$\begin{aligned} &(\Phi \star \Psi)(x,y,p) \\ := &\sum_q \Phi(x - \hbar(q-p)\mu, y - \hbar(q-p)\nu, q)\Psi(x - \hbar q\mu, y - \hbar q\nu, p-q) \end{aligned} \quad (3.0.1)$$

and

$$\Phi^*(x,y,p) := \bar{\Phi}(x,y,-p). \quad (3.0.2)$$

Then, $\pi : \mathcal{A}_\hbar^\infty \to \mathcal{B}(L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z}))$ given by

$$(\pi(\Phi)\xi)(x,y,p) = \sum_q \Phi(x - \hbar(q-2p)\mu, y - \hbar(q-2p)\nu, q)\xi(x,y,p-q) \quad (3.0.3)$$

gives a faithful representation of the involutive algebra $\mathcal{A}_\hbar^\infty$. $\mathcal{A}_{\mu,\nu}^{c,\hbar}$ = norm closure of $\pi(\mathcal{A}_\hbar^\infty)$ is called the Quantum Heisenberg Manifold.

We will identify $\mathcal{A}_\hbar^\infty$ with $\pi(\mathcal{A}_\hbar^\infty)$ without any mention. Since we are going to work with fixed parameters $c, \mu, \nu, \hbar$ we will drop them altogether and denote $\mathcal{A}_{\mu,\nu}^{c,\hbar}$ simply by $\mathcal{A}_\hbar$. Here the subscript remains merely as a reminiscent of Heisenberg only to distinguish it from a general algebra.

Action of the Heisenberg group : Let c be a positive integer. Let us consider the group structure on $G = \mathbb{R}^3 = \{(r,s,t) : r,s,t \in \mathbb{R}\}$ given by the multiplication

$$(r,s,t)(r',s',t') = (r+r', s+s', t+t'+csr'). \quad (3.0.4)$$

Later we will give an explicit isomorphism between G and H_3, the Heisenberg group of 3×3 upper triangular matrices with real entries and ones on the diagonal. Through this identification we can identify G with the Heisenberg group. For $\Phi \in S^c, (r,s,t) \in \mathbb{R}^3 \equiv G$,

$$(L_{(r,s,t)}\phi)(x,y,p) = e(p(t + cs(x-r)))\phi(x-r, y-s, p) \tag{3.0.5}$$

extends to an ergodic action of the Heisenberg group on $\mathcal{A}^{c,\hbar}_{\mu,\nu}$.

The Trace : The linear functional $\tau : \mathcal{A}^\infty_\hbar \to \mathbb{C}$, given by $\tau(\phi) = \int_0^1 \int_{\mathbb{T}} \phi(x,y,0)dxdy$ is invariant under the Heisenberg group action. So, the group action can be lifted to $L^2(\mathcal{A}^\infty_\hbar)$. We will denote the action at the Hilbert space level by the same symbol.

3.1 Yang-Mills in the dynamical system approach

We recall the definition of Yang-Mills in the setting of C^*-dynamical systems in the context of QHM. Here the dynamics is governed by the Lie group G. We can identify G with H_3 through the isomorphism that identifies $(r,s,t) \in G$ with the matrix $\begin{pmatrix} 1 & cs & t \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$.

Let $\mathfrak{g}$ be the Lie-algebra of G. We can identify $\mathfrak{g}$ with the Lie-algebra of H_3, which is given by 3×3 upper triangular matrices with real entries with zeros on the diagonal. Fix a real number α greater than one. This number will remain fixed throughout and we will comment about it later. In this approach one has to fix an inner product structure on the Lie algebra of the underlying Lie group and in our case we do so by declaring the following basis,

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & c\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.1.1}$$

as orthonormal. Their Lie bracket is given by

$$[X_1, X_3] = [X_2, X_3] = 0\,,\; [X_1, X_2] = -\frac{1}{\alpha}X_3. \tag{3.1.2}$$

The exponential map from $\mathfrak{g}$ to G acts on these elements as follows

$$exp(rX_1) = (r, 0, 0), exp(sX_2) = (0, s, 0),\text{ and } exp(tX_3) = (0, 0, c\alpha t).$$

For $X \in \mathfrak{g}$, let d_X be the derivation of $\mathcal{A}^{\infty}_{\hbar}$ given by $d_X(a) = \frac{d}{dt}\mid_{t=0} L_{exp(tX)}(a)$. Let us denote the d_{X_j}'s, for $j = 1, 2, 3$ by d_j. Then they are given by

$$\begin{aligned} d_1(f) &= -\frac{\partial f}{\partial x}, && (3.1.3)\\ d_2(f) &= 2\pi icpxf(x, y, p) - \frac{\partial f}{\partial y}, && (3.1.4)\\ d_3(f) &= 2\pi ipc\alpha f(x, y, p). && (3.1.5)\end{aligned}$$

Proposition 3.1.1. *Let $\mathcal{E}$ be a finitely generated projective $\mathcal{A}^{\infty}_{\hbar}$ module. Then the space $C(\mathcal{E})$ of compatible connections is given by triples of linear maps $\nabla_j : \mathcal{E} \to \mathcal{E}, j = 1, 2, 3$ such that*

$$\nabla_j(\xi.a) = \nabla_j(\xi).a + \xi.d_j(a), \qquad j = 1, 2, 3 \tag{3.1.6}$$

$$d_j(\langle\, \xi, \xi'\, \rangle_{\mathcal{A}}) = \langle \nabla_j\, \xi\,, \xi'\rangle_{\mathcal{A}} + \langle \xi\,, \nabla_j\, \xi'\rangle_{\mathcal{A}}, \qquad \forall\, \xi\,, \xi' \in \mathcal{E},\; j = 1, 2, 3\,. \tag{3.1.7}$$

Proof. Given a compatible connection ∇ let $\nabla_j = \nabla_{X_j}$ for $j = 1, 2, 3$. The condition (3.1.6 , 3.1.7) holds because of the definition of ∇ (see 1.3). Conversely if (3.1.6 , 3.1.7) holds and we define ∇ by specifying its components on the basis (3.1.1) such that $\nabla_{X_j} = \nabla_j$, then clearly the conditions of a compatible connection are satisfied. □

For the QHM the curvature is given by

$$\Theta_{\nabla}(X_1 \wedge X_3) = [\nabla_{X_1}, \nabla_{X_3}],$$

$$\Theta_\nabla(X_2 \wedge X_3) = [\nabla_{X_2}, \nabla_{X_3}],$$

$$\Theta_\nabla(X_1 \wedge X_2) = [\nabla_{X_1}, \nabla_{X_2}] + \tfrac{1}{\alpha}\nabla_{X_3}.$$

Here the third equality uses the relation $[X_1, X_2] = -\frac{1}{\alpha}X_3$ from (3.1.2).

Definition 3.1.2. Let $\mathcal{E}$ be a finitely generated projective $\mathcal{A}_\hbar^\infty$ module with a *Hermitian* structure. Then the Yang-Mills action functional for a compatible connection $\nabla \in C(\mathcal{E})$ is given by

$$YM(\nabla) = -\widetilde{\tau}(([\nabla_{X_1}, \nabla_{X_3}])^2 + ([\nabla_{X_2}, \nabla_{X_3}])^2 + ([\nabla_{X_1}, \nabla_{X_2}] + \frac{1}{\alpha}\nabla_{X_3})^2)). \tag{3.1.8}$$

3.2 The Equivalence of the two approaches

For our present purpose it is enough to recall the operators $[D, \phi]$ for $\phi \in S^c$. Note that here the dimension of the associated Lie group is three. Let $\sigma_1, \sigma_2, \sigma_3$ be 2×2 self-adjoint trace-less matrices given by

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Then,

$$\sigma_1\sigma_2 = i\sigma_3\,, \ \sigma_2\sigma_3 = i\sigma_1\,, \ \sigma_3\sigma_1 = i\sigma_2\,.$$

Let $\phi \in S^c$, then

$$[D, \phi] = \sum \delta_j(\phi) \otimes \sigma_j \text{ where } \delta_j(\phi) = i d_j(\phi) \tag{3.2.1}$$

and the derivations d_j are given by (3.1.3, 3.1.4, 3.1.5). The δ_j's satisfy the following commutation relations

$$[\delta_1, \delta_3] = [\delta_2, \delta_3] = 0\,, \ [\delta_1, \delta_2] = -\frac{i}{\alpha}\delta_3. \tag{3.2.2}$$

Assumption: Henceforth for the rest of the paper we will only consider generic parameter values namely $1, \hbar\mu, \hbar\nu$ are independent over $\mathbb{Q}$.

The space of forms were computed in ([9]). In the following proposition we recall the description of the space of forms as $\mathcal{A}_\hbar^\infty - \mathcal{A}_\hbar^\infty$-bimodules.

Proposition 3.2.1.

(i) The space of one forms as an $\mathcal{A}_\hbar^\infty - \mathcal{A}_\hbar^\infty$-bimodule is given by

$$\begin{aligned} \Omega_D^1(\mathcal{A}_\hbar^\infty) &= \{\sum a_j \otimes \sigma_j | a_j \in \mathcal{A}_\hbar^\infty, \sigma_j' s \text{ as above } \} \subseteq \mathcal{A}_\hbar^\infty \otimes M_2(\mathbb{C}) \subseteq \mathcal{B}(\mathcal{H}) \\ &\cong \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty. \end{aligned}$$

(ii) $\pi(\Omega^k(\mathcal{A}_\hbar^\infty)) = \mathcal{A}_\hbar^\infty \otimes M_2(\mathbb{C}) = \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty$ for $k \geq 2$.

(iii) $\pi(dJ_0^1) = \mathcal{A}_\hbar^\infty \otimes I_2 \subseteq \mathcal{A}_\hbar^\infty \otimes M_2(\mathbb{C}) \subseteq \mathcal{B}(\mathcal{H})$.

(iv) The space of two forms as an $\mathcal{A}_\hbar^\infty - \mathcal{A}_\hbar^\infty$-bimodule is given by

$$\begin{aligned} \Omega_D^2(\mathcal{A}_\hbar^\infty) &= \{\sum a_j \otimes \sigma_j | a_j \in \mathcal{A}_\hbar^\infty, \sigma_j' s \text{ as above } \} \subseteq \mathcal{A}_\hbar^\infty \otimes M_2(\mathbb{C}) \subseteq \mathcal{B}(\mathcal{H}) \\ &\cong \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty \oplus \mathcal{A}_\hbar^\infty. \end{aligned}$$

(v) The product map from $\Omega_D^1(\mathcal{A}_\hbar^\infty) \times \Omega_D^1(\mathcal{A}_\hbar^\infty)$ to $\Omega_D^2(\mathcal{A}_\hbar^\infty)$ is given by

$$(a \otimes \sigma_j) \cdot (b \otimes \sigma_k) = (1 - \delta_{jk}) ab \otimes \sigma_j \sigma_k, \forall j, k = 1, 2, 3.$$

Here δ_{jk} is the Kronecker delta.

Proof. Only (v) was not mentioned in ([9]). This follows because the space of forms $\Omega_D^1(\mathcal{A}_\hbar^\infty)$, $\Omega_D^2(\mathcal{A}_\hbar^\infty)$ are identified with subspaces of $\mathcal{A}_\hbar^\infty \otimes M_2(\mathbb{C})$ and the multiplication is induced from the multiplication on $\mathcal{A}_\hbar^\infty \otimes M_2(\mathbb{C})$. □

We also recall Proposition 14 from ([9]).

Proposition 3.2.2. *If $1, \hbar\mu, \hbar\nu$ are independent over $\mathbb{Q}$ then the positive linear functional on $\mathcal{A}_\hbar^\infty \otimes M_2(\mathbb{C})$ given by $\tau' : a \mapsto tr_\omega a|D|^{-3}$ coincides with $\frac{1}{2}(tr_\omega |D|^{-3})\tau \otimes tr$ where tr_ω is a Dixmier trace. Thus $\tau' = \frac{1}{2}(tr_\omega|D|^{-3})\widetilde{\tau}$, where $\widetilde{\tau}$, is the trace on $End(\mathcal{E})$ used in definition (3.1.2).*

Proposition 3.2.3. *Assume $\{1, \hbar\mu, \hbar\nu\}$ is rationally independent so that the algebra $\mathcal{A}_\hbar^\infty$ becomes simple.*

(i) The differential $\widetilde{d} : \mathcal{A}_\hbar^\infty \longrightarrow \Omega_D^1(\mathcal{A}_\hbar^\infty)$ satisfies $\widetilde{d}(a) = \sum_{j=1}^3 \delta_j(a) \otimes \sigma_j$.

(ii) The differential $\widetilde{d} : \Omega_D^1(\mathcal{A}_\hbar^\infty) \longrightarrow \Omega_D^2(\mathcal{A}_\hbar^\infty)$ satisfies

$$\widetilde{d}(a \otimes \sigma_1) = \sum_{j=2,3} \delta_j(a) \otimes \sigma_j \sigma_1, \tag{3.2.3}$$

$$\widetilde{d}(a \otimes \sigma_2) = \sum_{j=1,3} \delta_j(a) \otimes \sigma_j \sigma_2, \tag{3.2.4}$$

$$\widetilde{d}(a \otimes \sigma_3) = \delta_1(a) \otimes \sigma_1\sigma_3 + \delta_2(a) \otimes \sigma_2\sigma_3 + \frac{i}{\alpha} a \otimes \sigma_1\sigma_2. \tag{3.2.5}$$

Proof. (i) This follows from $\widetilde{d}(a) = [D, a] = \sum_j \delta_j(a) \otimes \sigma_j$.

(ii) The differential $\widetilde{d} : \Omega_D^1(\mathcal{A}_\hbar^\infty) \longrightarrow \Omega_D^2(\mathcal{A}_\hbar^\infty)$ is defined in such a way that the following diagram

$$\begin{tikzcd}
\Omega^1(\mathcal{A}_\hbar^\infty) \arrow[r, "\pi_D"] \arrow[d, "\delta"'] & \Omega_D^1(\mathcal{A}_\hbar^\infty) \arrow[d, "\widetilde{d}"] \\
\Omega^2(\mathcal{A}_\hbar^\infty) \arrow[r, "\pi_D"'] & \Omega_D^2(\mathcal{A}_\hbar^\infty)
\end{tikzcd}$$

commutes. Therefore to see how it acts on an element of $\Omega_D^1(\mathcal{A}_\hbar^\infty)$ we pick an element and lift that to $\Omega^1(\mathcal{A}_\hbar^\infty)$ and then follow the diagram. Let $\phi_{mn} \in S^c$ be the function

$\phi_{m,n}(x,y,p) = e(mx+ny)\delta_{p0}$. These functions are eigenfunctions for δ_j's and satisfy

$$
\begin{aligned}
&\delta_1(\phi_{10}) = 2\pi\phi_{10}, && \delta_2(\phi_{10}) = 0, && \delta_3(\phi_{10}) = 0,\\
&\delta_1(\phi_{01}) = 0, && \delta_2(\phi_{10}) = 2\pi\phi_{01}, && \delta_3(\phi_{01}) = 0.
\end{aligned}
$$

Let $\widetilde{a} = \frac{1}{2\pi}a\phi_{10}^*\delta(\phi_{10}) \in \Omega^1$. Then,

$$
\begin{aligned}
\pi_D(\widetilde{a}) &= \frac{1}{2\pi}(a\phi_{10}^* \otimes I_2)(\sum_{j=1}^{3}\delta_j(\phi_{10}) \otimes \sigma_j))\\
&= \frac{1}{2\pi}(a\phi_{10}^* 2\pi\phi_{10}) \otimes \sigma_1\\
&= a \otimes \sigma_1.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\widetilde{d}(a \otimes \sigma_1) &= \pi_D(\delta(\widetilde{a}))\\
&= \frac{1}{2\pi}(\sum_{j=1}^{3}\delta_j(a\phi_{10}^*) \otimes \sigma_j)(2\pi\phi_{10} \otimes \sigma_1)\\
&= \sum_{j\neq 1}\delta_j(a) \otimes \sigma_j\sigma_1.
\end{aligned}
$$

Similarly,

$$
\widetilde{d}(a \otimes \sigma_2) = \sum_{j\neq 2}\delta_j(a) \otimes \sigma_j\sigma_2.
$$

To see (3.2.5) observe that

$$
\begin{aligned}
\widetilde{d}(a\delta_3(b) \otimes \sigma_3) &= \widetilde{d}(a\sum_j \delta_j(b) \otimes \sigma_j) - \widetilde{d}(a\delta_1(b) \otimes \sigma_1) - \widetilde{d}(a\delta_2(b) \otimes \sigma_2)\\
&= \widetilde{d}(\pi_D(a\delta(b))) - \widetilde{d}(a\delta_1(b) \otimes \sigma_1) - \widetilde{d}(a\delta_2(b) \otimes \sigma_2)
\end{aligned}
$$

$$
\begin{aligned}
&= \pi_D(\delta(a)\delta(b)) - \widetilde{d}(a\delta_1(b)\otimes\sigma_1) - \widetilde{d}(a\delta_2(b)\otimes\sigma_2) \\
&= \sum(\delta_j(a)\otimes\sigma_j)(\delta_k(a)\otimes\sigma_k) - \widetilde{d}(a\delta_1(b)\otimes\sigma_1) - \widetilde{d}(a\delta_2(b)\otimes\sigma_2) \\
&\quad \bmod \pi(\delta J_1) \\
&= \delta_1(a)\delta_3(b)\otimes\sigma_1\sigma_3 + \delta_2(a)\delta_3(b)\otimes\sigma_2\sigma_3 + a[\delta_2,\delta_1](b)\otimes\sigma_1\sigma_2 \\
&\quad -a\delta_3(\delta_1(b))\otimes\sigma_3\sigma_1 - a\delta_3(\delta_2(b))\otimes\sigma_3\sigma_2 \\
&= \delta_1(a\delta_3(b))\otimes\sigma_1\sigma_3 + \delta_2(a\delta_3(b))\otimes\sigma_2\sigma_3 + a[\delta_2,\delta_1](b)\otimes\sigma_1\sigma_2 \\
&= \delta_1(a\delta_3(b))\otimes\sigma_1\sigma_3 + \delta_2(a\delta_3(b))\otimes\sigma_2\sigma_3 + \frac{i}{\alpha}a\delta_3(b)\otimes\sigma_1\sigma_2 .
\end{aligned}
$$

The last equality uses $[\delta_2,\delta_1] = \frac{i}{\alpha}\delta_3$. Since span of elements of the form $a\delta_3(b)$ forms an ideal in $\mathcal{A}_\hbar^\infty$ and $\mathcal{A}_\hbar^\infty$ is simple, (3.2.5) follows. □

Corollary 3.2.4. $\widetilde{d}(1\otimes\sigma_j) = \begin{cases} 0 & \textit{if } j = 1,2\,; \\ \frac{-1}{\alpha}\otimes\sigma_3 & \textit{if } j = 3\,. \end{cases}$

Now we have all the ingredients to describe the space $\widetilde{C}(\mathcal{E})$ of compatible connections on a finitely generated projective $\mathcal{A}_\hbar^\infty$-module $\mathcal{E}$ with a *Hermitian* structure.

Proposition 3.2.5. *Let $\mathcal{E}$ be a finitely generated projective $\mathcal{A}_\hbar^\infty$ module. Then the space $\widetilde{C}(\mathcal{E})$ of compatible connections for the differential graded algebra $\Omega_D^\bullet(\mathcal{A}_\hbar^\infty)$ is given by triples of linear maps $\widetilde{\nabla}_j : \mathcal{E} \longrightarrow \mathcal{E}, j = 1,2,3$ such that,*

$$\widetilde{\nabla}_j(\xi.a) = \nabla_j(\xi).a + \xi.\delta_j(a), \qquad j = 1,2,3 \tag{3.2.6}$$

$$\delta_j(\langle\,\xi,\xi'\,\rangle_{\mathcal{A}}) = \langle\xi\,,\widetilde{\nabla}_j\,\xi'\rangle - \langle\widetilde{\nabla}_j\,\xi\,,\xi'\rangle\ , \qquad \forall\,\xi\,,\xi'\in\mathcal{E}, j = 1,2,3\,. \tag{3.2.7}$$

Proof. By Proposition (3.2.1) we can identify $\mathcal{E}\otimes_{\mathcal{A}_\hbar^\infty}\Omega_D^1(\mathcal{A}_\hbar^\infty)$ with the subspace $\sum_j \mathcal{E}\otimes\sigma_j \subseteq \mathcal{E}\otimes M_2(\mathbb{C})$. Thus any compatible connection $\widetilde{\nabla}$ is prescribed by three maps $\widetilde{\nabla}_j : \mathcal{E}\longrightarrow\mathcal{E}$ such that

$$\widetilde{\nabla}(\xi) = \textstyle\sum_{j=1}^3 \widetilde{\nabla}_j(\xi)\otimes\sigma_j .$$

Then

$$\begin{aligned}
\widetilde{\nabla}(\xi.a) &= \sum_{j=1}^{3} \widetilde{\nabla}_j(\xi.a) \otimes \sigma_j \\
&= \widetilde{\nabla}(\xi).a + \xi \otimes \widetilde{d}(a) \\
&= \sum_{j=1}^{3} \widetilde{\nabla}_j(\xi).a \otimes \sigma_j + \sum_{j=1}^{3} \xi.\delta_j(a) \otimes \sigma_j.
\end{aligned}$$

Thus comparing coefficients of σ_j we get

$$\widetilde{\nabla}_j(\xi.a) = \nabla_j(\xi).a + \xi.\delta_j(a),\ j = 1, 2, 3.$$

For (3.2.7) note that

$$\begin{aligned}
\sum_{j=1}^{3} \delta_j(\langle \xi, \xi' \rangle) \otimes \sigma_j &= \widetilde{d}\langle \xi, \xi' \rangle) \\
&= (\xi, \widetilde{\nabla}\xi') - (\widetilde{\nabla}\xi, \xi') \\
&= \sum_{j=1}^{3} (\langle \xi, \widetilde{\nabla}_j \xi' \rangle - \langle \widetilde{\nabla}_j \xi\, , \xi' \rangle) \otimes \sigma_j.
\end{aligned}$$

This completes the proof. □

Theorem 3.2.6. Let $\mathcal{E}$ be a finitely generated projective $\mathcal{A}_\hbar^\infty$ module with a *Hermitian* structure. Then $\Phi : C(\mathcal{E}) \longrightarrow \widetilde{C}(\mathcal{E})$ given by $\Phi(\nabla) = \widetilde{\nabla}$, where $\widetilde{\nabla}(\xi) = i\nabla(\xi)$ is well defined and

$$\tfrac{1}{2}(tr_\omega |D|^{-3})\mathrm{YM}(\nabla) = \mathrm{YM}(\Phi(\nabla))\,.$$

Proof. Let $\nabla \in C(\mathcal{E})$ be a compatible connection and $\nabla_j, j = 1, 2, 3$ be its components as defined in the proof of Proposition (3.1.1). If we define $\widetilde{\nabla}_j = i\nabla_j, j = 1, 2, 3$, then $\widetilde{\nabla}_j$'s satisfy (3.2.6) because $\delta_j = i.d_j, j = 1, 2, 3$ and (3.1.6) holds. Similarly (3.2.7) follows from (3.1.7). Thus the triple $\widetilde{\nabla}_j, j = 1, 2, 3$ defines a compatible connection $\widetilde{\nabla} \in \widetilde{C}(\mathcal{E})$. This proves the map Φ is well defined with the given domain and range. In fact it is an isomorphism. Let $\widetilde{\nabla}$ denote the extended connection as defined on (1.6.1). Then using Propositions (3.2.1 , 3.2.3) we get

$$\begin{aligned}
\widetilde{\nabla}(\xi \otimes \sigma_1) &= \sum_{j \neq 1} \widetilde{\nabla}_j(\xi) \otimes \sigma_j \sigma_1 \\
\widetilde{\nabla}(\xi \otimes \sigma_2) &= \sum_{j \neq 2} \widetilde{\nabla}_j(\xi) \otimes \sigma_j \sigma_2 \\
\widetilde{\nabla}(\xi \otimes \sigma_3) &= \sum_{j \neq 3} \widetilde{\nabla}_j(\xi) \otimes \sigma_j \sigma_3 - \frac{1}{\alpha} \xi \otimes \sigma_3.
\end{aligned}$$

The curvature Θ of the connection $\widetilde{\nabla}$ is given by $\Theta = \widetilde{\nabla} \circ \widetilde{\nabla}$, which turns out to be,

$$\Theta(\xi) = i[\widetilde{\nabla}_2, \widetilde{\nabla}_3](\xi) \otimes \sigma_1 + i[\widetilde{\nabla}_3, \widetilde{\nabla}_1](\xi) \otimes \sigma_2 + (i[\widetilde{\nabla}_1, \widetilde{\nabla}_2] - \frac{1}{\alpha} \widetilde{\nabla}_3)(\xi) \otimes \sigma_3.$$

Repeated application of equation (3.2.7) gives,

$$\delta_k(\langle \xi, \widetilde{\nabla}_j(\eta) \rangle = \langle \xi, \widetilde{\nabla}_k(\widetilde{\nabla}_j(\eta)) \rangle - \langle \widetilde{\nabla}_k(\xi), \widetilde{\nabla}_j(\eta) \rangle, \tag{3.2.8}$$

$$\delta_j(\langle \xi, \widetilde{\nabla}_k(\eta) \rangle = \langle \xi, \widetilde{\nabla}_j(\widetilde{\nabla}_k(\eta)) \rangle - \langle \widetilde{\nabla}_j(\xi), \widetilde{\nabla}_k(\eta) \rangle, \tag{3.2.9}$$

$$\delta_j(\langle \widetilde{\nabla}_k(\xi), \eta \rangle = \langle \widetilde{\nabla}_k(\xi), \widetilde{\nabla}_j(\eta) \rangle - \langle \widetilde{\nabla}_j(\widetilde{\nabla}_k(\xi)), \eta \rangle, \tag{3.2.10}$$

$$\delta_k(\langle \widetilde{\nabla}_j(\xi), \eta \rangle = \langle \widetilde{\nabla}_j(\xi), \widetilde{\nabla}_k(\eta) \rangle - \langle \widetilde{\nabla}_k(\widetilde{\nabla}_j(\xi)), \eta \rangle. \tag{3.2.11}$$

Now, (3.2.8) − (3.2.9) + (3.2.10) − (3.2.11) gives,

$$\langle \xi, [\widetilde{\nabla}_k, \widetilde{\nabla}_j](\eta) \rangle - \langle [\widetilde{\nabla}_j, \widetilde{\nabla}_k](\xi), \eta \rangle = [\delta_k, \delta_j] \langle \xi, \eta \rangle). \tag{3.2.12}$$

Combining (3.2.2) and (3.2.12) we get,

$$\begin{aligned}
\langle \xi, [\widetilde{\nabla}_1, \widetilde{\nabla}_3](\eta) \rangle &= \langle [\widetilde{\nabla}_3, \widetilde{\nabla}_1](\xi), \eta \rangle \\
\langle \xi, [\widetilde{\nabla}_2, \widetilde{\nabla}_3](\eta) \rangle &= \langle [\widetilde{\nabla}_3, \widetilde{\nabla}_2](\xi), \eta \rangle \\
\langle \xi, (i[\widetilde{\nabla}_1, \widetilde{\nabla}_2] - \frac{1}{\alpha} \widetilde{\nabla}_3)(\eta) \rangle &= \langle (i[\widetilde{\nabla}_1, \widetilde{\nabla}_2] - \frac{1}{\alpha} \widetilde{\nabla}_3)(\xi), \eta \rangle
\end{aligned}$$

These relations give,

$$\begin{aligned}
YM(\widetilde{\nabla}) &= \langle\langle \Theta, \Theta \rangle\rangle \\
&= \frac{1}{2}(tr_\omega|D|^{-3})\widetilde{\tau}(-([\widetilde{\nabla}_1, \widetilde{\nabla}_3])^2 - ([\widetilde{\nabla}_2, \widetilde{\nabla}_3])^2 + (i[\widetilde{\nabla}_1, \widetilde{\nabla}_2] - \frac{1}{\alpha}\widetilde{\nabla}_3)^2) \\
&= -\frac{1}{2}(tr_\omega|D|^{-3})\widetilde{\tau}(([\nabla_1, \nabla_3])^2 + ([\nabla_2, \nabla_3])^2 + ([\nabla_1, \nabla_2] + \frac{1}{\alpha}\nabla_3)^2)) \\
&= \frac{1}{2}(tr_\omega|D|^{-3})\, YM(\nabla).
\end{aligned}$$

and the proof is completed. □

Remark 3.2.7. *It would be interesting to compute the exact value of the normalizing constant* $\frac{1}{2}(tr_\omega|D|^{-3})$, *however at present we only have an upper bound and a nonzero lower bound.*

Remark 3.2.8. *The computations in this chapter crucially use results of* ([9]). *Unless one assumes* $\{1, \hbar\mu, \hbar\nu\}$ *is rationally independent, the algebra* $\mathcal{A}_\hbar^\infty$ *is not simple and hence the computation of the space of forms* $\Omega_D^\bullet(\mathcal{A}_\hbar^\infty)$ *in* (3.2.1) *and the differentials as executed in Proposition* (3.2.3) *collapses. We believe even in the nonsimple case the result is true but we do not yet have a proof.*

Chapter 4

Connes' Calculus for The Quantum Double Suspension

A calculus or a differential calculus often means a differential graded algebra(dga). Recall the dga $\Omega_D^\bullet$ (1.1.3) defined by Connes. However, outside the works of Connes there are very few instances ([9],[10]) where this calculus has been computed. In view of this scenario, here in this chapter we set ourselves with the task of computation of this calculus for a certain systematic class of examples.

The concept of quantum double suspension(QDS) of an algebra $\mathcal{A}$, denoted by $\Sigma^2\mathcal{A}$, was introduced by Hong-Szymanski in ([24]). Later the quantum double suspension of a spectral triple was introduced by Chakraborty-Sundar ([11]) and a class of examples of regular spectral triples having simple dimension spectrum were constructed, useful in the context of local index formulas of Connes-Moscovici ([19]). Note that by iterating QDS on a manifold one can produce genuine noncommutative spectral triples. Under the following hypotheses

- $[D, a]F - F[D, a]$ is a compact operator for all $a \in \mathcal{A}$, where F is the sign of the operator D,
- $\mathcal{H}^\infty := \bigcap_{k\geq 1} \mathcal{D}om(D^k)$ is a left $\mathcal{A}$-module and $[D, \mathcal{A}] \subseteq \mathcal{A}\otimes\mathcal{E}nd_{\mathcal{A}}(\mathcal{H}^\infty) \subseteq \mathcal{E}nd_{\mathbb{C}}(\mathcal{H}^\infty)$,

on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we compute $\Omega_D^\bullet$ for the quantum double suspended spectral triple $(\Sigma^2\mathcal{A}, \Sigma^2\mathcal{H}, \Sigma^2 D)$. Notable features of these hypotheses are, firstly the spectral triple associated with a first order elliptic differential operator on a manifold will always satisfy them and secondly they are stable under quantum double suspension. Thus our results allows one to compute Connes' calculus for spectral triples obtained by iteratively quantum double suspending spectral triples associated with first order differential operators on smooth compact manifolds. In particular, iterated application of our construction on the spectral triple $\left(C^\infty(\mathbb{T}), L^2(\mathbb{T}), \frac{1}{i}\frac{d}{d\theta}\right)$ imply the computation of the Connes' calculus for odd dimensional quantum spheres. This extends earlier work of ([10]). This computation gives the first systematic computation of Connes' calculus for a large family of spectral triples.

At the end of this chapter we discuss behaviour of geometric notions of compatible connections, curvatures on the quantum double suspended spectral triple. If we take $\mathcal{E}$ to be a *Hermitian* finitely generated projective module over $\mathcal{A}$, and denote the affine space of compatible connections by $Con(\mathcal{E})$, then there is a canonical *Hermitian* finitely generated projective module over $\Sigma^2\mathcal{A}$ which we denote by $\widetilde{\mathcal{E}}$. The affine space of compatible connections on $\widetilde{\mathcal{E}}$ is denoted by $Con(\widetilde{\mathcal{E}})$. We show that there is an affine embedding of $Con(\mathcal{E})$ into $Con(\widetilde{\mathcal{E}})$ which preserves the Grassmannian connection and together with $Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^2(\mathcal{A}))$, the vector space containing the subspace of curvatures, these fit into a commutative diagram.

4.1 Preliminaries on The Quantum Double Suspension

We define Connes calculus $\Omega_D^\bullet$ in a slightly modified way in this chapter and then justify it.

Definition 4.1.1. *Recall from (1.1.3) that $(\Omega^\bullet(\mathcal{A}), d)$ is a differential graded algebra. We have a $*$-representation π of $\Omega^\bullet(\mathcal{A})$ on the Calkin algebra $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$,* *given by*

$$\pi(a_0 \otimes \overline{a_1} \otimes \ldots \otimes \overline{a_k}) := a_0[D, a_1] \ldots [D, a_k] + \mathcal{K}(\mathcal{H})\ ;\ a_j \in \mathcal{A}\,.$$

Let $J_0^{(k)} = \{\omega \in \Omega^k : \pi^k(\omega) = 0\}$ *and* $J' = \bigoplus J_0^{(k)}$. *But* J' *fails to be a differential ideal in* $\Omega^\bullet$. *We consider* $J^\bullet = \bigoplus J^{(k)}$ *where* $J^{(k)} = J_0^{(k)} + dJ_0^{(k-1)}$. *Then* $J^\bullet$ *becomes a differential graded two-sided ideal in* $\Omega^\bullet$ *and hence, the quotient* $\Omega_D^\bullet = \Omega^\bullet / J^\bullet$ *becomes a differential graded algebra.*

Remark 4.1.2. *If we compare the definition of* $\Omega_D^\bullet$ *given in* (1.1.3) *with that in* (4.1.1), *then one sees that the definition given in* (1.1.3) *does not involve the projection map* $\theta : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ *and Connes represented* $\Omega^\bullet(\mathcal{A})$ *on* $\mathcal{B}(\mathcal{H})$ *instead on* $\mathcal{Q}(\mathcal{H})$. *However often, the explicit computation of* $\Omega_D^\bullet$ *is rather difficult, even in the particular cases. In* ([10]), *authors have computed* $\Omega_D^\bullet$ *for the quantum* $SU(2)$ *and the Podleś sphere by replacing* $\mathcal{B}(\mathcal{H})$ *with* $\mathcal{Q}(\mathcal{H})$, *i,e. following the above prescription. Justification for this has been discussed in section* (3) *of* ([10]). *Here in this chapter we follow the above prescription of* $\Omega_D^\bullet$ *given in definition* (4.1.1).

Now we define $\Omega_D^\bullet(\mathcal{A})$ for non-unital algebra $\mathcal{A}$. Notice that elements of Ω^k are linear combination of elements of the form $a_0 da_1 \ldots da_k$. For non-unital algebra $\mathcal{A}$, one first considers the minimal unitization $\widetilde{\mathcal{A}} := \mathcal{A} \oplus \mathbb{C}$ and embeds $\mathcal{A}$ in $\widetilde{\mathcal{A}}$ by the map $a \longmapsto (a, 0)$. This makes $\mathcal{A}$ an ideal in $\widetilde{\mathcal{A}}$. The map $(a, \lambda) \longmapsto \pi(a) + \lambda I$ gives a faithful representation of $\widetilde{\mathcal{A}}$ on $\mathcal{B}(\mathcal{H})$. Now, using the embedding $\mathcal{A} \hookrightarrow \widetilde{\mathcal{A}}$, define elements of $\Omega^k(\mathcal{A})$ as linear combination of elements of the form $(a_0, 0)d(a_1, 0) \ldots d(a_k, 0)$. Observe that $\Omega^n(\mathcal{A}) \subseteq \Omega^n(\widetilde{\mathcal{A}})$ and hence $\frac{\Omega^n(\mathcal{A})}{\Omega^n(\mathcal{A}) \bigcap J^n(\widetilde{\mathcal{A}})} \subseteq \frac{\Omega^n(\widetilde{\mathcal{A}})}{J^n(\widetilde{\mathcal{A}})}$ and subsequently we define $\Omega_D^\bullet(\mathcal{A}) = \frac{\Omega^n(\mathcal{A})}{\Omega^n(\mathcal{A}) \bigcap J^n(\widetilde{\mathcal{A}})}$ for the nonunital case.

Now we define the concept of quantum double suspension(QDS) introduced by Hong-Szymanski in ([24]). Let l denotes the left shift operator on $\ell^2(\mathbb{N})$ defined on the standard orthonormal basis (e_n) by $l(e_n) = e_{n-1}$, $l(e_0) = 0$ and N be the number operator on $\ell^2(\mathbb{N})$ defined as $N(e_n) = ne_n$. We fix the notation 'u' throughout the chapter which denotes the rank one projection $|e_0\rangle\langle e_0| := I - l^*l$. Let $\mathcal{K}$ denotes space of compact operators on $\ell^2(\mathbb{N})$.

Definition 4.1.3. *Let $\mathcal{A}$ be a unital C^*-algebra. The quantum double suspension of $\mathcal{A}$, denoted by $\Sigma^2\mathcal{A}$, is the C^*-algebra generated by $a \otimes u$ and $1 \otimes l$ in $\mathcal{A} \otimes \mathscr{T}$ where $\mathscr{T}$ is the Toeplitz algebra.*

There is a symbol map $\sigma : \mathscr{T} \longrightarrow C(S^1)$ which sends l to the standard unitary generator z of $C(S^1)$ and one has a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathscr{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0\,.$$

If ρ denotes the restriction of $1 \otimes \sigma$ to $\Sigma^2\mathcal{A}$ then one gets the following short exact sequence

$$0 \longrightarrow \mathcal{A} \otimes \mathcal{K} \longrightarrow \Sigma^2\mathcal{A} \xrightarrow{\rho} C(S^1) \longrightarrow 0\,.$$

There is a $\mathbb{C}$-linear splitting map σ' from $C(S^1)$ to $\Sigma^2\mathcal{A}$ which sends z to $1 \otimes l$ and yields the following $\mathbb{C}$-vector spaces isomorphism :

$$\Sigma^2\mathcal{A} \cong (\mathcal{A} \otimes \mathcal{K}) \bigoplus C(S^1)\,.$$

Notice that σ' is injective since it has a left inverse ρ and hence any $f \in C(S^1)$ can be identified with $1 \otimes \sigma'(f) \in \Sigma^2\mathcal{A}$. For $f = \sum_n \lambda_n z^n \in C(S^1)$, we write $\sigma'(f) := \sum_{n\geq 0} \lambda_n l^n + \sum_{n>0} \lambda_{-n} l^{*n}$. The finite subalgebra, denoted by $(\Sigma^2\mathcal{A})_{fin}$, is generated by $a \otimes T$ and $\sum_{0\leq n<\infty} \lambda_n l^n + \sum_{0<n<\infty} \lambda_{-n} l^{*n}$, where $a \in \mathcal{A}$ and $T \in \mathcal{B}(\ell^2(\mathbb{N}))$ is a finitely supported matrix.

Definition 4.1.4 ([11])**.** *For any spectral triple $(\mathcal{A}, \mathcal{H}, D)$, $(\Sigma^2\mathcal{A}, \Sigma^2\mathcal{H} := \mathcal{H} \otimes \ell^2(\mathbb{N})$, $\Sigma^2 D := D \otimes I + F \otimes N)$ becomes a spectral triple where F is the sign of the operator D and N is the number operator on $\ell^2(\mathbb{N})$. It is called the quantum double suspension of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.*

Notice that $(\mathbb{C}[z,z^{-1}],\ell^2(\mathbb{N}),N)$ is also a spectral triple, and for any $f\in\mathbb{C}[z,z^{-1}]$, we have $[\Sigma^2 D, 1\otimes\sigma'(f)] = F\otimes[N,f]$. Here we record two conditions on a spectral triple which will be used later.

Conditions :

(A) $[D,a]F-F[D,a]$ is a compact operator for all $a\in\mathcal{A}$.

(B) $\mathcal{H}^\infty := \bigcap_{k\geq 1}\mathcal{D}om(D^k)$ is a left $\mathcal{A}$-module and $[D,\mathcal{A}]\subseteq\mathcal{A}\otimes\mathcal{E}nd_{\mathcal{A}}(\mathcal{H}^\infty)\subseteq\mathcal{E}nd_{\mathbb{C}}(\mathcal{H}^\infty)$.

Proposition 4.1.5. *These conditions are valid for the classical case where $\mathcal{A}=C^\infty(\mathbb{M})$ and D is a first order differential operator. Moreover, if a spectral triple $(\mathcal{A},\mathcal{H},D)$ satisfies these conditions then the quantum double suspended spectral triple $((\Sigma^2\mathcal{A})_{fin},\Sigma^2\mathcal{H},\Sigma^2 D)$ also satisfies them.*

Proof. When D is of order 1, $[D,a]F-F[D,a]$ has order -1 and hence it is a compact operator. Now suppose $[D,a]F-F[D,a]$ is a compact operator for all $a\in\mathcal{A}$. To check the stability under QDS, note that $(\Sigma^2\mathcal{A})_{fin}=\mathcal{A}\otimes\mathcal{S}\bigoplus\mathbb{C}[z,z^{-1}]$ as linear space and $sign(\Sigma^2 D)=F\otimes 1$. Now

$$\begin{aligned}
& [\Sigma^2 D, a\otimes T+f](F\otimes 1)-(F\otimes 1)[\Sigma^2 D, a\otimes T+f]\\
= \ & [D,a]F\otimes T+FaF\otimes[N,T]+1\otimes[N,f]-F[D,a]\otimes T-a\otimes[N,T]-1\otimes[N,f]\\
= \ & [D,a]F\otimes T-F[D,a]\otimes T+[F,a]F\otimes[N,T].
\end{aligned}$$

This says that $[\Sigma^2 D,(\Sigma^2\mathcal{A})_{fin}](F\otimes 1)-(F\otimes 1)[\Sigma^2 D,(\Sigma^2\mathcal{A})_{fin}]$ is also a compact operator on $\mathcal{H}\otimes\ell^2(\mathbb{N})$. The second condition follows similarly. $\square$

Lemma 4.1.6. *Let σ_1, σ_2 denote any two 2×2 Pauli spin matrices. For a given spectral triple $(\mathcal{A},\mathcal{H},D)$ consider the even spectral triple $(\widetilde{\mathcal{A}},\widetilde{\mathcal{H}},\widetilde{D},\gamma)$ where $\widetilde{\mathcal{H}}=\mathcal{H}\otimes\mathbb{C}^2$, $\widetilde{\mathcal{A}}=\mathcal{A}\otimes I_2$, $\widetilde{D}=D\otimes\sigma_1$, $\gamma=1\otimes\sigma_2$. Then $\Omega^\bullet_{\widetilde{D}}(\widetilde{\mathcal{A}})\cong\Omega^\bullet_D(\mathcal{A})$.*

Proof. First observe that $\sum\widetilde{a_0}\prod_{i=1}^n[\widetilde{D},\widetilde{a_i}]=(\sum a_0\prod_{i=1}^n[D,a_i])\otimes\sigma_1^n$ where $\widetilde{a_i}=a_i\otimes I_2$.

Now

$$\sigma_1^n = \begin{cases} \sigma_1 & for\ n\ odd\,, \\ I_2 & for\ n\ even\,. \end{cases}$$

immediately shows that $\pi(\Omega^n(\widetilde{\mathcal{A}})) \cong \pi(\Omega^n(\mathcal{A}))$ for all $n \geq 1$. Since $\sum a_0 \prod_{i=1}^n [D, a_i] \otimes \sigma_1^n = 0$ implies $\prod_{i=0}^n [D, a_i] \in \pi(dJ_0^n(\mathcal{A}))$, we have $\pi(dJ_0^n(\widetilde{\mathcal{A}})) \cong \pi(dJ_0^n(\mathcal{A}))$ for all $n \geq 1$. This completes the proof. $\square$

Remark 4.1.7. (*a*) *Proposition* (4.1.5) *says that iterating the classical case of spectral triples canonically associated with a first order differential operator on a compact manifold, one gets a lot of examples satisfying our conditions.*
(*b*) *Observe that* $sign(\widetilde{D}) = sign(D) \otimes \sigma_1$ *and hence if* $(\mathcal{A}, \mathcal{H}, D)$ *satisfies our conditions then so does* $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D}, \gamma)$. *For any even spectral triple it is obvious that* $F\mathcal{A} \cap \mathcal{A} = \{0\}$ *where* $F = sign(D)$. *Hence, Lemma* (4.1.6) *will guarantee that in our context, without loss of generality, we can always take* $F\mathcal{A} \cap \mathcal{A} = \{0\}$. *Throughout the chapter we stick to this fact.*

Notation :

1. In this chapter we will work with $(\Sigma^2\mathcal{A})_{fin}$ and denote it by $\Sigma^2\mathcal{A}$ for notational brevity.

2. For all $f \in \mathbb{C}[z, z^{-1}]$, we denote $[N, f]$ by f' for notational brevity.

3. '$\mathcal{S}$' denotes the space of finitely supported matrices in $\mathcal{B}(\ell^2(\mathbb{N}))$.

4. Any $T = (T_{ij}) \in \mathcal{S}$ is said to have order m if $m \geq 1$ is the least natural number such that $T_{ij} = 0$ for all $i, j > m$.

5. (e_{ij}) will denote infinite matrix with 1 at the ij-th place and zero elsewhere. We call it elementary matrix.

4.2 Connes' Calculus for The Quantum Double Suspension

In this section we take a spectral triple satisfying Conditions $(A), (B)$ mentioned in the previous section. Because of Remark (4.1.7) we can assume $F\mathcal{A} \cap \mathcal{A} = \{0\}$. Our goal here is computation of $\Omega^{\bullet}_{\Sigma^2 D}\left((\Sigma^2\mathcal{A})_{fin}\right)$. Note that $(\mathcal{S}, \ell^2(\mathbb{N}), N)$ is a spectral triple but $\mathcal{S}$ is non-unital. We first consider $\Omega^{\bullet}_N(\mathcal{S})$, following the definition of $\Omega^{\bullet}_D$ for non-unital algebras. We need to compute this complex first.

Lemma 4.2.1. $\pi_N\left(\Omega^n(\mathcal{S})\right) = \mathcal{S}$ *for all* $n \geq 0$.

Proof. Let's take $n > 0$. We have to show $\pi_N\left(\Omega^n(\mathcal{S})\right) \supseteq \mathcal{S}$. Choose any $T \in \mathcal{S}$ of order l. If n is even then take $n = 2r$. Observe that

$$T = \textstyle\sum_{j=1}^{l-1} T\left(\left[N, \frac{1}{j-l}(e_{j,l})\right]\left[N, \frac{1}{l-j}(e_{l,j})\right]\right)^r + T\left(\left[N, \frac{1}{l-1}(e_{l,1})\right]\left[N, \frac{1}{1-l}(e_{1,l})\right]\right)^r .$$

For $n = 2r+1$ we can similarly write,

$$\begin{aligned} T &= \sum_{j=1}^{l-1} T(e_{j,l})\left(\left[N, \frac{1}{l-2}(e_{l,2})\right]\left[N, \frac{1}{2-l}(e_{2,l})\right]\right)^r \left[N, \frac{1}{l-j}(e_{l,j})\right] \\ &\qquad + T(e_{l,1})\left[N, \frac{1}{1-l}(e_{1,l})\right]\left(\left[N, \frac{1}{l-1}(e_{l,1})\right]\left[N, \frac{1}{1-l}(e_{1,l})\right]\right)^r \end{aligned}$$

and this completes the proof. □

Lemma 4.2.2. $\pi_N\left(dJ_0^n(\mathcal{S})\right) = \mathcal{S}$ *for all* $n \geq 1$.

Proof. Notice that $\pi_N\left(dJ_0^n(M_l(\mathbb{C}))\right)$ is an ideal in $M_l(\mathbb{C})$ for any l and hence if we can produce one nontrivial element then $\pi_N\left(dJ_0^n(M_l(\mathbb{C}))\right)$ will be equal with $M_l(\mathbb{C})$ for all l. For $n = 1$, choose

$$\zeta = (e_{2,3})d\left(\frac{1}{2}(e_{3,1})\right) - (e_{2,2})d\left((e_{2,1})\right) .$$

For $n = 2$, choose

$$\zeta = \left((e_{2,3})d\left(\frac{1}{2}(e_{3,1})\right) - (e_{2,2})d((e_{2,1}))\right) d\left(\frac{-1}{3}(e_{1,4})\right) .$$

For $n = 3$, choose

$$\zeta = \left((e_{2,3})d\left(\frac{1}{2}(e_{3,1})\right) - (e_{2,2})d((e_{2,1}))\right)d\left(\frac{-1}{3}(e_{1,4})\right)d\left(\frac{1}{2}(e_{4,2})\right).$$

For $n \geq 4$, choose

$$\begin{aligned}\zeta &= \left((e_{2,n+1})d\left(\frac{1}{n}(e_{n+1,1})\right) - (e_{2,2})d((e_{2,1}))\right)d\left(\frac{-1}{3}(e_{1,4})\right) \bullet \\ &\quad \prod_{j=4}^{n} d((-1)(e_{j,j+1}))d(\frac{1}{n-1}(e_{n+1,2})).\end{aligned}$$

It is easy to see that for all $n \geq 1$ these elements lie in $J_0^n(M_l(\mathbb{C}))$. One can verify that $\pi(d\zeta) \neq 0$ in each case. $\square$

Proposition 4.2.3. *For $(\mathcal{S}, \ell^2(\mathbb{N}), N)$ we have*

1. $\Omega_N^n(\mathcal{S}) = \mathcal{S}$ *for* $n = 0, 1$.
2. $\Omega_N^n(\mathcal{S}) = 0$ *for all* $n \geq 2$.

Proof. Combine Lemmas (4.2.1) and (4.2.2). $\square$

Now we are ready for the computation of $\Omega^\bullet_{\Sigma^2 D}$. Note that both $\pi_{\Sigma^2 D}(\Omega^\bullet(\mathcal{A} \otimes \mathcal{S}))$ and $\pi_{\Sigma^2 D}(\Omega^\bullet(\mathbb{C}[z, z^{-1}]))$ are subspaces of $\pi_{\Sigma^2 D}(\Omega^\bullet(\Sigma^2\mathcal{A}))$, because $\Sigma^2\mathcal{A} = \mathcal{A} \otimes \mathcal{S} \bigoplus \mathbb{C}[z, z^{-1}]$ as $\mathbb{C}$-vector spaces. Furthermore, $\pi_{\Sigma^2 D}(\Omega^\bullet(\mathbb{C}[z, z^{-1}])) = F^\bullet \otimes \pi_N(\Omega^\bullet(\mathbb{C}[z, z^{-1}]))$. We always write π instead of $\pi_{\Sigma^2 D}$ and π_N for notational brevity.

Lemma 4.2.4. $\pi(\Omega^1(\Sigma^2\mathcal{A})) = \pi(\Omega^1(\mathcal{A} \otimes \mathcal{S})) + (F \otimes 1)(\mathcal{A} \otimes \mathcal{S} + \pi(\Omega^1(\mathbb{C}[z, z^{-1}])))$.

Proof. Note that $\pi(\Omega^1(\mathcal{A} \otimes \mathcal{S})) \subseteq \pi(\Omega^1(\Sigma^2\mathcal{A}))$. Now for any element $(F \otimes 1)(a \otimes T + f_0 f_1')$ of $(F \otimes 1)(\mathcal{A} \otimes \mathcal{S} + \pi(\Omega^1(\mathbb{C}[z, z^{-1}])))$ we see that,

$$\begin{aligned}(F \otimes 1)(a \otimes T + f_0 f_1') &= (a \otimes T + f_0 f_1')(F \otimes 1) \\ &= (a \otimes T + f_0 f_1')(1 \otimes l)(1 \otimes l^*)(F \otimes 1) \\ &= (a \otimes T + f_0 f_1')(1 \otimes l)\left[\Sigma^2 D, 1 \otimes l^*\right]\end{aligned}$$

This is clearly in $\pi\left(\Omega^1(\Sigma^2\mathcal{A})\right)$. To see the reverse inclusion we start with arbitrary element $\sum_k(a_{0k}\otimes T_{0k}+f_{0k})\left[\sum^2 D, a_{1k}\otimes T_{1k}+f_{1k}\right]$ of $\pi\left(\Omega^1(\Sigma^2\mathcal{A})\right)$. Then,

$$\begin{aligned}
&\sum_k (a_{0k}\otimes T_{0k}+f_{0k})\left[\Sigma^2 D, a_{1k}\otimes T_{1k}+f_{1k}\right]\\
&=\sum_k (a_{0k}\otimes T_{0k})\left[\Sigma^2 D, a_{1k}\otimes T_{1k}\right]+(a_{0k}\otimes T_{0k}+f_{0k})\left[\Sigma^2 D, f_{1k}\right]\\
&\qquad+\left[\Sigma^2 D, a_{1k}\otimes f_{0k}T_{1k}\right]-Fa_{1k}\otimes f'_{0k}T_{1k}
\end{aligned}$$

which is an element of $\pi\left(\Omega^1(\mathcal{A}\otimes\mathcal{S})\right)+(F\otimes 1)\left(\mathcal{A}\otimes\mathcal{S}+\pi(\Omega^1(\mathbb{C}[z,z^{-1}]))\right)$. □

Lemma 4.2.5. $\pi(\Omega^n(\Sigma^2\mathcal{A}))=\sum_{j=1}^n F^{j-1}\pi(\Omega^{n+1-j}(\mathcal{A}\otimes\mathcal{S}))+(F\otimes 1)^n(\mathcal{A}\otimes\mathcal{S}+\pi(\Omega^n(\mathbb{C}[z,z^{-1}])))$ *for all* $n\geq 1$ *.*

Proof. We prove the Lemma by induction. Suppose the statement is true for $n=k$. Any element of $\pi\left(\Omega^{k+1}(\Sigma^2\mathcal{A})\right)$ can be written as $\omega[\Sigma^2 D, a\otimes T+f]$, where ω is in $\pi\left(\Omega^k(\Sigma^2\mathcal{A})\right)$. By assumtion $\omega=\sum_{i=1}^{k+1}\omega_i$ where $\omega_i\in F^{i-1}\pi\left(\Omega^{k+1-i}(\mathcal{A}\otimes\mathcal{S})\right)$ for $1\leqslant i\leqslant k$ and $\omega_{k+1}\in(F\otimes 1)^k\left(\mathcal{A}\otimes\mathcal{S}+\pi(\Omega^k(\mathbb{C}[z,z^{-1}]))\right)$. Hence,

$$\omega[\Sigma^2 D, a\otimes T+f]=\sum_{i=1}^{k+1}\omega_i[\Sigma^2 D, a\otimes T]+\sum_{i=1}^{k+1}\omega_i(F\otimes f').$$

This is an element of

$$\sum_{j=1}^{k+1}F^{j-1}\pi\left(\Omega^{k+2-j}(\mathcal{A}\otimes\mathcal{S})\right)+(F\otimes 1)^{k+1}\left(\mathcal{A}\otimes\mathcal{S}+\pi\left(\Omega^{k+1}(\mathbb{C}[z,z^{-1}])\right)\right).$$

To get the reverse inclusion one can use the same trick used in Lemma (4.2.4). □

Lemma 4.2.6. $\pi(\Omega^1(\mathcal{A}\otimes\mathcal{S}))\bigcap(F\otimes 1)(\mathcal{A}\otimes\mathcal{S}+\pi(\Omega^1(\mathbb{C}[z,z^{-1}])))=(F\otimes 1)(\mathcal{A}\otimes\mathcal{S})$.

Proof. Choose any arbitrary element $\sum_k(a_{0k}\otimes T_{0k})[\Sigma^2 D, a_{1k}\otimes T_{1k}]$ of $\pi\left(\Omega^1(\mathcal{A}\otimes\mathcal{S})\right)$. In terms of elementary matrices $((e_{ij}))$ we can rewrite this element as following ,

$$\begin{aligned}
\sum_k (a_{0k}\otimes T_{0k})\left[\Sigma^2 D, a_{1k}\otimes T_{1k}\right] &= \sum_k\left(\sum_{i,j}a_{0kij}\otimes e_{ij}\right)\left[\Sigma^2 D,\sum_{p,q}a_{1kpq}\otimes e_{pq}\right]\\
&=\sum_{k,i,j,q}a_{0kij}\left([D,a_{1kjq}]+F(j-q)a_{1kjq}\right)\otimes e_{iq}\,.
\end{aligned}$$

Now any element of $(F\otimes 1)(\mathcal{A}\otimes S)+(F\otimes 1)\pi\left(\Omega^1(\mathbb{C}[z,z^{-1}])\right)$ looks like $\sum_{k'} Fa_{k'}\otimes T_{k'} + F\otimes f$ for some $f\in\mathbb{C}[z,z^{-1}]$. The equality of these two elements shows that f has to be a compact operator on $\ell^2(\mathbb{N})$ (Take any linear functional $\int$ on $B(\mathcal{H})$ and hit both elements by $\int\otimes Id$). Hence, if intersection is nontrivial then it must be contained in the ideal $(F\otimes 1)(\mathcal{A}\otimes\mathcal{S})$. We now show that $(F\otimes 1)(\mathcal{A}\otimes\mathcal{S})$ is contained in the intersection. Choose any arbitrary element $\sum_{k'} Fa_{k'}\otimes T_{k'}$. Consider the following equation

$$\sum_{k'} Fa_{k'}\otimes T_{k'} \;=\; \sum_k (a_{0k}\otimes T_{0k})\left[\Sigma^2 D, a_{1k}\otimes T_{1k}\right]. \tag{4.2.1}$$

Choose $a_{1k}=1$ for each k. Then this equation reduces to,

$$\sum_{k'} Fa_{k'}\otimes T_{k'} \;=\; \sum_k Fa_{0k}\otimes T_{0k}[N,T_{1k}]. \tag{4.2.2}$$

Using Lemma (4.2.1) we can write each $T_{k'}$ as $\sum_{m<\infty} T_{0m}^{(k')}[N,T_{1m}^{(k')}]$. Hence, this equation has nontrivial solution, which shows that $(F\otimes 1)(\mathcal{A}\otimes\mathcal{S})\subseteq\pi\left(\Omega^1(\mathcal{A}\otimes\mathcal{S})\right)$. □

Lemma 4.2.7. *$(F\otimes 1)\pi\left(\Omega^n(\mathcal{A}\otimes\mathcal{S})\right)\subseteq\pi\left(\Omega^{n+1}(\mathcal{A}\otimes\mathcal{S})\right)$ for all $n\geq 1$.*

Proof. This follows from the fact that for algebra $\mathcal{B}$

$$\Omega^n(\mathcal{B})=\underbrace{\Omega^1(\mathcal{B})\otimes_{\mathcal{B}}\ldots\otimes_{\mathcal{B}}\Omega^1(\mathcal{B})}_{n\ times}.$$

Now use Lemma (4.2.6) which says that $(F\otimes 1)(\mathcal{A}\otimes\mathcal{S})\subseteq\pi\left(\Omega^1(\mathcal{A}\otimes\mathcal{S})\right)$. □

Proposition 4.2.8. *$\pi\left(\Omega^n(\Sigma^2\mathcal{A})\right)=\pi\left(\Omega^n(\mathcal{A}\otimes\mathcal{S})\right)\bigoplus\pi\left(\Omega^n(\mathbb{C}[z,z^{-1}])\right)$ for all $n\geq 0$.*

Proof. Combine Lemmas (4.2.5, 4.2.6, 4.2.7). □

Recall that $\Omega^n_{\Sigma^2 D}(\Sigma^2\mathcal{A})\cong\pi\left(\Omega^n(\Sigma^2\mathcal{A})\right)/\pi\left(dJ_0^{n-1}(\Sigma^2\mathcal{A})\right)$. Hence our next target is to identify the quotient $\pi\left(dJ_0^{n-1}(\Sigma^2\mathcal{A})\right)$.

Lemma 4.2.9. $\pi\left(dJ_0^1(\Sigma^2\mathcal{A})\right)=\pi\left(dJ_0^1(\mathcal{A}\otimes\mathcal{S})\right)\bigoplus\pi\left(dJ_0^1(\mathbb{C}[z,z^{-1}])\right)$.

Proof. Suppose $\zeta = \sum_k d(a_{0k} \otimes T_{0k} + f_{0k})d(a_{1k} \otimes T_{1k} + f_{1k})$ is an element of $dJ_0^1(\Sigma^2\mathcal{A})$. Then,

$$\sum_k (a_{0k} \otimes T_{0k} + f_{0k})[\Sigma^2 D, a_{1k} \otimes T_{1k} + f_{1k}] = 0$$

Then $\pi(\zeta) = \sum_k [\Sigma^2 D, a_{0k} \otimes T_{0k} + f_{0k}][\Sigma^2 D, a_{1k} \otimes T_{1k} + f_{1k}]$ equals the following sum

$$\begin{aligned}\sum_k \ & [\Sigma^2 D, a_{0k} \otimes T_{0k}][\Sigma^2 D, a_{1k} \otimes T_{1k}] + [\Sigma^2 D, f_{0k}][\Sigma^2 D, f_{1k}] \\ & + [\Sigma^2 D, a_{0k} \otimes T_{0k}][\Sigma^2 D, f_{1k}] + [\Sigma^2 D, f_{0k}][\Sigma^2 D, a_{1k} \otimes T_{1k}].\end{aligned}$$

The term $\sum_k [\Sigma^2 D, f_{0k}][\Sigma^2 D, f_{1k}]$ lies in $\pi\left(dJ_0^1(\mathbb{C}[z, z^{-1}])\right)$. If we can write each term $[\Sigma^2 D, a_{0k} \otimes T_{0k}][\Sigma^2 D, f_{1k}]$ as $\sum_{k'} [\Sigma^2 D, b_{0k'} \otimes S_{0k'}][\Sigma^2 D, b_{1k'} \otimes S_{1k'}]$ in such a way that

$$(a_{0k} \otimes T_{0k})[\Sigma^2 D, f_{1k}] = \textstyle\sum_{k'} (b_{0k'} \otimes S_{0k'})([\Sigma^2 D, b_{1k'} \otimes S_{1k'}])$$

where $S_{0k'}$ and $S_{1k'}$'s are from $\mathcal{S}$ and similarly for the term $[\Sigma^2 D, f_{0k}][\Sigma^2 D, a_{1k} \otimes T_{1k}]$, then we can conclude that $\pi\left(dJ_0^1(\Sigma^2\mathcal{A})\right) \subseteq \pi\left(dJ_0^1(\mathcal{A} \otimes \mathcal{S})\right) \bigoplus \pi\left(dJ_0^1(\mathbb{C}[z, z^{-1}])\right)$. First let $b_{1k'} = 1$ for all k'. Then we have the following equations to solve:

$$\begin{cases} F a_{0k} \otimes T_{0k} f'_{1k} & = \ \sum_{k'} F b_{0k'} \otimes S_{0k'}[N, S_{1k'}] \\ F[D, a_{0k}] \otimes T_{0k} f'_{1k} + a_{0k} \otimes [N, T_{0k}] f'_{1k} & = \ \sum_{k'} F[D, b_{0k'}] \otimes S_{0k'}[N, S_{1k'}] \\ & \qquad + b_{0k'} \otimes [N, S_{0k'}][N, S_{1k'}]\,. \end{cases} \tag{4.2.3}$$

For that it is enough to solve the following equations:

$$\begin{cases} T_{0k} f'_{1k} & = \ \sum_{k'} S_{0k'}[N, S_{1k'}] \\ [N, T_{0k}] f'_{1k} & = \ \sum_{k'} [N, S_{0k'}][N, S_{1k'}]\,. \end{cases} \tag{4.2.4}$$

Note that f'_{1k} is of the form $\sum_{i=1}^n \lambda_i z^i + \sum_{j=1}^m \lambda_{-j}(z^{-1})^j$. Then $\sigma'(f'_{1k}) \in \mathcal{B}\left(\ell^2(\mathbb{N})\right)$ is the following matrix

$$\begin{pmatrix} 0 & \lambda_1 & \lambda_2 & \dots & \dots & \lambda_n & 0 & \dots & \dots\dots \\ \lambda_{-1} & 0 & \lambda_1 & \lambda_2 & \dots & \dots & \lambda_n & 0 & \dots\dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\dots \\ \lambda_{-m} & \lambda_{-m+1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots\dots \\ 0 & \lambda_{-m} & \lambda_{-m+1} & \dots & \dots & \dots & \dots & \dots & \dots\dots \\ 0 & 0 & \lambda_{-m} & \lambda_{-m+1} & \dots & \dots & \dots & \dots & \dots\dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\dots \end{pmatrix}.$$

Each row and column of this matrix has only finitely many non-zero entries and all the diagonal entries are zero. Denote this matrix by $(\beta_{pq})_{p,q}$. Notice that $(e_{ij}).\sigma'(f'_{1k})$ is the matrix whose i-th row consists of j-th row of $(\beta_{pq})_{p,q}$ and zero everywhere else, whereas $\sigma'(f'_{1k}).(e_{ij})$ is the matrix whose j-th column consists of i-th column of $(\beta_{pq})_{p,q}$ and zero everywhere else. Let T_{0k} be denoted by the matrix $(\alpha_{ij})_{ij}$. Since $T_{0k} \in \mathcal{S}$, one can assume that $\alpha_{ij} = 0$ for all $i \geq r+1$ and $j \geq s+1$, for some r, s. Observe that $T_{0k}\sigma'(f'_{1k}) = (\alpha_{ij})_{ij}(\widetilde{\beta_{pq}})_{p,q}$, where

$$\widetilde{\beta_{pq}} = \begin{cases} \beta_{pq} & for\ \ 1 \leq p \leq s\,,\ q \leq n+s\,, \\ 0 & otherwise\,. \end{cases}$$

Hence $(\widetilde{\beta_{pq}})_{p,q} \in \mathcal{S}$ and we have a solution for equation (4.2.4). Similarly one can do for the term $[\Sigma^2 D, f_{0k}][\Sigma^2 D, a_{1k} \otimes T_{1k}]$. □

Proposition 4.2.10. $\pi\,(dJ_0^n(\Sigma^2\mathcal{A})) = \pi\,(dJ_0^n(\mathcal{A}\otimes\mathcal{S})) \bigoplus \pi\,(dJ_0^n(\mathbb{C}[z,z^{-1}]))\ \forall\, n \geq 1\,.$

Proof. For arbitrary 'n' it follows from our observation in the previous Lemma that both $[\Sigma^2 D, a\otimes T][\Sigma^2 D, f]$ and $[\Sigma^2 D, g][\Sigma^2 D, a'\otimes T']$ for $f, g \in \mathbb{C}[z,z^{-1}]$ can be replaced by $[\Sigma^2 D, b\otimes S][\Sigma^2 D, b'\otimes S']$ where T, T', S, S' all lie in $\mathcal{S}$. □

Lemma 4.2.11. *For the spectral triple* $(\mathbb{C}[z,z^{-1}], \ell^2(\mathbb{N}), N)$ *we have* $\pi_N\,(\Omega^n(\mathbb{C}[z,z^{-1}])) = \mathbb{C}[z,z^{-1}]$ *for all* $n \geq 0$.

Proof. Clearly $\pi\,(\Omega^n(\mathbb{C}[z,z^{-1}])) \subseteq \mathbb{C}[z,z^{-1}]$. For the other inclusion consider $\xi, \eta \in \mathbb{C}[z,z^{-1}]$ where $\xi = z, \eta = z^{-1}$. Then $[N,\xi] = \xi$ and $[N,\eta] = -\eta$.

Case 1 : Suppose $n = 2r$ is even. Choose any $\phi \in \mathbb{C}[z, z^{-1}]$ and consider $\omega = \phi \underbrace{(d\xi d\eta) \ldots (d\xi d\eta)}_{r} \in \Omega^n \left(\mathbb{C}[z, z^{-1}]\right)$. Then

$$\pi(\omega) = \phi(\underbrace{[N, \xi][N, \eta]) \ldots ([N, \xi][N, \eta]}_{r}) \,.$$

But $[N, \xi][N, \eta] = -1$. This proves that $\mathbb{C}[z, z^{-1}] \subseteq \pi\left(\Omega^n(\mathbb{C}[z, z^{-1}])\right)$.

Case 2 : Suppose $n = 2r + 1$ is odd. Choose any $\phi \in C(S^1)$. Consider $\omega = \phi \xi d\eta \underbrace{(d\xi d\eta) \ldots (d\xi d\eta)}_{r} \in \Omega^n \left(C(S^1)\right)$ for $r \neq 0$ and $\omega = \phi \xi d\eta$ for $r = 0$. $\square$

Lemma 4.2.12. *For the spectral triple* $(\mathbb{C}[z, z^{-1}], \ell^2(\mathbb{N}), N)$ *we have* $\pi_N \left(dJ_0^n(\mathbb{C}[z, z^{-1}])\right)$ $= \mathbb{C}[z, z^{-1}]$ *for all* $n \geq 1$.

Proof. It is clear from the previous Lemma (4.2.11) that $\pi\left(dJ_0^n(\mathbb{C}[z, z^{-1}])\right) \subseteq \mathbb{C}[z, z^{-1}]$. For the other inclusion notice that for $n \geq 1$, $\pi\left(dJ_0^n(\mathbb{C}[z, z^{-1}])\right)$ is an ideal in $\mathbb{C}[z, z^{-1}]$. We will show that $1 \in \mathbb{C}[z, z^{-1}]$ lies in $\pi\left(dJ_0^n(\mathbb{C}[z, z^{-1}])\right)$. Consider $\xi = z, \eta = z^{-1} \in \mathbb{C}[z, z^{-1}]$.

Case 1 : For $n \geq 3$ odd, consider

$$\omega = \xi d\eta \underbrace{(d\xi d\eta) \ldots (d\xi d\eta)}_{(n-1)/2} + \eta d\xi \underbrace{(d\xi d\eta) \ldots (d\xi d\eta)}_{(n-1)/2} \in \Omega^n \left(\mathbb{C}[z, z^{-1}]\right).$$

Case 2 : For $n \geq 2$ even, consider

$$\omega = -\xi^2 d\eta d\eta \underbrace{(d\xi d\eta) \ldots (d\xi d\eta)}_{(n-2)/2} + \eta^2 d\xi d\xi \underbrace{(d\xi d\eta) \ldots (d\xi d\eta)}_{(n-2)/2} \in \Omega^n \left(\mathbb{C}[z, z^{-1}]\right).$$

Case 3 : For $n = 1$, consider $\omega = \xi d\eta + \eta d\xi$.

One can check that for all $n \geq 1$, $\pi(\omega) = 0$ i,e. $\omega \in J_0^n \left(\mathbb{C}[z, z^{-1}]\right)$. But

$$\pi(d\omega) = \begin{cases} -2 & for \;\; n = 1\,, \\ -2(-1)^{(n-1)/2} & for \;\; n \geq 3 \; odd\,, \\ -4(-1)^{(n-2)/2} & for \;\; n \geq 2 \; even\,. \end{cases}$$

This justifies our claim. □

Proposition 4.2.13. *For the spectral triple* $(\mathbb{C}[z, z^{-1}], \ell^2(\mathbb{N}), N)$*,*

1. $\Omega^n_N\,(\mathbb{C}[z, z^{-1}]) = \mathbb{C}[z, z^{-1}]\,,\;$ *for* $n = 0\,, 1\,.$

2. $\Omega^n_N\,(\mathbb{C}[z, z^{-1}]) = 0\,,\;$ *for* $n \geq 2\,.$

Proof. Combine Lemmas (4.2.11) and (4.2.12). □

Proposition 4.2.14. *For* $(\Sigma^2\mathcal{A}, \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2 D)\,,$

1. $\Omega^1_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \cong \Omega^1_{\Sigma^2 D}(\mathcal{A} \otimes \mathcal{S}) \bigoplus \mathbb{C}[z, z^{-1}]\,,$

2. $\Omega^n_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \cong \Omega^n_{\Sigma^2 D}(\mathcal{A} \otimes \mathcal{S})$ *, for all* $n \geq 2$.

Proof. Use Propositions (4.2.8 , 4.2.10 , 4.2.13). □

Our next goal is to determine $\Omega^n_{\Sigma^2 D}\,(\mathcal{A} \otimes \mathcal{S})$ in terms of $\Omega^n_D(\mathcal{A})$. Note that we are viewing $\mathcal{A} \otimes \mathcal{S}$ inside the unital algebra $\Sigma^2\mathcal{A}$ as an embedded subspace.

Lemma 4.2.15. $\pi\,(\Omega^n(\mathcal{A} \otimes \mathcal{S})) = \sum_{r=0}^n F^r \pi(\Omega^{n-r}(\mathcal{A})) \otimes \mathcal{S}$ *for all* $n \geq 0\,.$

Proof. The inclusion '$\subseteq$' is obvious since one just has to expand the commutators $[\Sigma^2 D, \,.\,]$ involved in LHS. For '$\supseteq$' we show that $F^r \pi\,(\Omega^{n-r}(\mathcal{A})) \otimes \mathcal{S} \subseteq \pi\,(\Omega^n(\mathcal{A} \otimes \mathcal{S}))$, for each $0 \leqslant r \leqslant n$. Consider first $F^r a_0 \prod_{i=1}^{n-r}[D, a_i] \otimes T \in F^r \pi\,(\Omega^{n-r}(\mathcal{A})) \otimes \mathcal{S}$, where $1 \leqslant r \leqslant n-1$. By Lemma (4.2.1) one can write $T = \sum_k T_{0k} \prod_{i=1}^r [N, T_{ik}]$. Let $I_{(0k)}$ be the infinite matrix having an identity block matrix in top left most corner of order same as that of T_{0k} and zero elsewhere. Then,

$$\begin{aligned}
&\sum_k (a_0 \otimes T_{0k}) \left(\prod_{i=1}^{n-r} [\Sigma^2 D, a_i \otimes I_{(0k)}] \right) \left(\prod_{j=1}^{r} [\Sigma^2 D, 1 \otimes T_{jk}] \right) \\
&= \sum_k (a_0 \otimes T_{0k}) \prod_{i=1}^{n-r} \left([D, a_i] \otimes I_{(0k)} \right) \left(F^r \otimes \prod_{j=1}^{r} [N, T_{jk}] \right) \\
&= \sum_k F^r a_0 \left(\prod_{i=1}^{n-r} [D, a_i] \right) \otimes T_{0k} \left(\prod_{j=1}^{r} [N, T_{jk}] \right) \\
&= F^r a_0 \prod_{i=1}^{n-r} [D, a_i] \otimes T.
\end{aligned}$$

For $r = 0$, observe that $a_0 \prod_{i=1}^{n} [D, a_i] \otimes T = (a_0 \otimes T) \prod_{i=1}^{n} [\Sigma^2 D, a_i \otimes I_{(T)}]$, where $I_{(T)}$ denotes the infinite matrix having an identity block matrix in top left most corner of order same as that of T and zero elsewhere. Finally for $r = n$, $F^n a \otimes T = \sum_k (a \otimes T_{0k}) \left(\prod_{i=1}^{n} [\Sigma^2 D, 1 \otimes T_{ik}] \right)$ where $T = \sum_k T_{0k} \prod_{i=1}^{n} [N, T_{ik}]$ (by Lemma 4.2.1). $\square$

Lemma 4.2.16. $\pi \left(dJ_0^1(\mathcal{A} \otimes \mathcal{S}) \right) = \pi \left(dJ_0^1(\mathcal{A}) \right) \otimes \mathcal{S} + F\pi \left(\Omega^1(\mathcal{A}) \right) \otimes \mathcal{S} + \mathcal{A} \otimes \mathcal{S}$.

Proof. In terms of elementary matrices (e_{ij}), arbitrary element of $\pi \left(dJ_0^1(\mathcal{A} \otimes \mathcal{S}) \right)$ looks like

$$\begin{aligned}
& \sum [\Sigma^2 D, a_0 \otimes T_0][\Sigma^2 D, a_1 \otimes T_1] \\
= \; & \sum_{i,s} \sum_j \Big\{ \sum [D, a_{0ij}][D, a_{1js}] + F a_{0ij} [D, a_{1js}](i-j) \\
& \qquad + a_{0ij} a_{1js} (i-j)(j-s) + F[D, a_{0ij}] a_{1js} (j-s) \Big\} \otimes e_{is}
\end{aligned}$$

such that for each i and s we have

$$\sum_j \sum a_{0ij} [D, a_{1js}] + \sum_j F a_{0ij} a_{1js} (j-s) \;=\; 0\,. \tag{4.2.5}$$

Consider the following equations

$$\xi = \sum \sum_{j} a_{0ij}[D, a_{1js}], \tag{4.2.6}$$

$$\eta = \sum \sum_{j} a_{0ij} a_{1js}(j - s). \tag{4.2.7}$$

Hence, $\xi + F\eta = 0$ by equation (4.2.5). Let n be any natural number. For each i and s, consider

$$\begin{cases} a_{0,i,s+n} = -1 \\ a_{1,s+n,s} = a \end{cases} \quad \begin{cases} a_{0,i,s+n+1} = 1 \\ a_{1,s+n+1,s} = a \end{cases} \quad \begin{cases} a_{0,i,s+n+2} = 1 \\ a_{1,s+n+2,s} = a \end{cases} \quad \begin{cases} a_{0,i,s+n+3} = 1 \\ a_{1,s+n+3,s} = -a\,. \end{cases}$$

One easily checks that $\xi = 0$ in equation (4.2.6) and $\eta = 0$ in equation (4.2.7) for these four pairs together and hence these pairs can produce infinitely many solutions to the equation $\xi + F\eta = 0$. We can now conclude that arbitrary $a \otimes e_{is}$ lies in $\pi(dJ_0^1(\mathcal{A} \otimes \mathcal{S}))$, for each i and s and for any $a \in \mathcal{A}$. We will now show that any $Fa[D, b] \otimes e_{is}$ lies in $\pi\,(dJ_0^1(\mathcal{A} \otimes \mathcal{S}))$, for each i and s. For any natural number m and for each i and s, consider

$$\begin{cases} a_{0,i,s+m} = a \\ a_{1,s+m,s} = b \end{cases} \quad \begin{cases} a_{0,i,s+m+1} = -a \\ a_{1,s+m+1,s} = b \end{cases} \quad \begin{cases} a_{0,i,s+m+2} = (1/(m+2))ab \\ a_{1,s+m+2,s} = 1\,. \end{cases}$$

Again one checks that $\xi = 0$ in equations (4.2.6) and $\eta = 0$ in equation (4.2.7) and one gets $2Fa[D, b] \otimes e_{is} + ab(m-1) \otimes e_{is}$ as an element of $\pi\,(dJ_0^1(\mathcal{A} \otimes \mathcal{S}))$. Hence $Fa[D, b] \otimes e_{is}$ lies in $\pi\,(dJ_0^1(\mathcal{A} \otimes \mathcal{S}))$, for each i and s. We will now show that $\pi\,(dJ_0^1(\mathcal{A})) \otimes \mathcal{S} \subseteq \pi\,(dJ_0^1(\mathcal{A} \otimes \mathcal{S}))$. Choose any $\sum_k [D, a_{0k}][D, a_{1k}] \otimes T_k \in \pi\,(dJ_0^1(\mathcal{A})) \otimes \mathcal{S}$. Then $a_{0k}[D, a_{1k}] = 0$ for each k. Let $I_{(k)}$ be the infinite matrix having an identity block matrix in top left most corner of order same as that of T_k and zero elsewhere. Then $[\Sigma^2 D, a_{0k} \otimes I_{(k)}][\Sigma^2 D, a_{1k} \otimes I_{(k)}] \in \pi\,(dJ_0^1(\mathcal{A} \otimes \mathcal{S}))$ for each k and hence,

$$(1 \otimes T_k)[\Sigma^2 D, a_{0k} \otimes I_{(k)}][\Sigma^2 D, a_{1k} \otimes I_{(k)}] \in \pi\,(dJ_0^1(\mathcal{A} \otimes \mathcal{S}))$$

for each k. Now observe that

$$\sum_k [D, a_{0k}][D, a_{1k}] \otimes T_k = \sum_k (1 \otimes T_k)[\Sigma^2 D, a_{0k} \otimes I_{(k)}][\Sigma^2 D, a_{1k} \otimes I_{(k)}] \,.$$

This proves the inclusion '$\supseteq$'. Now Lemma (4.2.15) shows that $\pi\,(dJ_0^1(\mathcal{A} \otimes \mathcal{S})) \subseteq \pi\,(\Omega^2(\mathcal{A})) \otimes \mathcal{S} + F\pi\,(\Omega^1(\mathcal{A})) \otimes \mathcal{S} + \mathcal{A} \otimes \mathcal{S}$. Finally, the fact that $[D, \mathcal{A}] \subseteq \mathcal{A} \otimes \mathcal{E}nd_{\mathcal{A}}(\mathcal{H}^\infty) \subseteq \mathcal{E}nd_{\mathbb{C}}(\mathcal{H}^\infty)$ and $F\mathcal{A} \cap \mathcal{A} = \{0\}$ implies the inclusion '$\subseteq$' by equation (4.2.5). $\square$

Lemma 4.2.17. *For all $n \geq 1$, $F^{n+1}a \otimes e_{ij} \in \pi\,(dJ_0^n(\mathcal{A} \otimes \mathcal{S}))$ for any $a \in \mathcal{A}$ and each i and j.*

Proof. The $n = 1$ case has been addressed in Lemma (4.2.16). Let's take $n \geq 2$. Arbitrary element of $\pi\,(dJ_0^n(\mathcal{A} \otimes \mathcal{S}))$ looks like

$$\sum_{i_1, i_{n+2}} \sum \left(\sum_{i_2, \ldots, i_{n+1}} \prod_{j=1}^{n+1} [D, a_{j i_j i_{j+1}}] + \sum_{t=1}^{n+1} F^t (\prod_{j=1}^{n+1} [D, a_{j i_j i_{j+1}}])^{(t)} \right) \otimes e_{i_1 i_{n+2}} \,, \tag{4.2.8}$$

where $\left(\prod_{j=1}^{n+1} [D, a_{j i_j i_{j+1}}]\right)^{(t)}$ is the following expression

$$\sum_{1 \leq r_1 < r_2 < \ldots < r_t}^{n+1} [D, a_{1 i_1 i_2}] \ldots\ldots [D, a_{r_1 i_{r_1} i_{r_1+1}}\hat{]} \ldots [D, a_{r_2 i_{r_2} i_{r_2+1}}\hat{]}$$
$$\ldots [D, a_{r_t i_{r_t} i_{r_t+1}}\hat{]} \ldots\ldots [D, a_{(n+1) i_{n+1} i_{n+2}}]$$

with $[D, a_{r i_r i_{r+1}}\hat{]} = (i_r - i_{r+1}) a_{r i_r i_{r+1}}$ (the total number of $\hat{}$ appearing in each summand of the summation $\sum_{1 \leq r_1 < r_2 < \ldots < r_t}^{n}$ is exactly t) ; such that

$$\sum \sum_{i_2, \ldots, i_{n+1}} \{ a_{1 i_1 i_2} \prod_{j=2}^{n+1} [D, a_{j i_j i_{j+1}}] + \sum_{t=2,\, t\, even}^{n} a_{1 i_1 i_2} \left(\prod_{j=2}^{n+1} [D, a_{j i_j i_{j+1}}] \right)^{(t)}$$
$$+ \sum_{t=3,\, t\, odd}^{n} F a_{1 i_1 i_2} \left(\prod_{j=2}^{n+1} [D, a_{j i_j i_{j+1}}] \right)^{(t)} \} = 0$$

for each i_1 and i_{n+2}. Here $\left(\prod_{j=2}^{n+1} [D, a_{j i_j i_{j+1}}]\right)^{(t)}$ is the same expression as $\left(\prod_{j=1}^{n+1} [D, a_{j i_j i_{j+1}}]\right)^{(t)}$ except for the fact that there is no r_1 present i,e. the summation will be over $r_2, \ldots, r_t$ and for $t = 1$ this term is zero. Consider

$$\xi \;=\; \sum \sum_{i_2,\ldots,i_{n+1}} a_{1i_1i_2} \prod_{j=2}^{n+1} [D, a_{ji_ji_{j+1}}]\,, \tag{4.2.9}$$

$$\begin{aligned}\eta \;=\; \sum \sum_{i_2,\ldots,i_{n+1}} \{ \sum_{t=2,\, t\, even}^{n} a_{1i_1i_2} (\prod_{j=2}^{n+1} [D, a_{ji_ji_{j+1}}])^{(t)} \\ + \sum_{t=3,\, t\, odd}^{n} F a_{1i_1i_2} (\prod_{j=2}^{n+1} [D, a_{ji_ji_{j+1}}])^{(t)} \}\,.\end{aligned} \tag{4.2.10}$$

Hence, $\xi + \eta = 0$. For each i_1 and i_{n+2} , consider

$$\begin{cases} a_{1,i_1,i_1+m} = a \\ a_{s,i_1+(s-1)m,i_1+sm} = -1/m \;;\; \forall\; 2 \le s \le n \\ a_{n+1,i_1+nm,i_{n+2}} = 1/(i_1 + nm - i_{n+2}) \end{cases}$$

and

$$\begin{cases} a_{1,i_1,i_1+m+1} = -a \\ a_{s,i_1+(s-1)m+1,i_1+sm+1} = -1/m \;;\; \forall\; 2 \le s \le n \\ a_{n+1,i_1+nm+1,i_{n+2}} = 1/(i_1 + nm + 1 - i_{n+2}) \end{cases}$$

Here m is a natural number s.t. $i_1 + nm + 1 - i_{n+2}$ and $i_1 + nm - i_{n+2}$ both are nonzero. Note that infinitely many such m can be found for given i_1, i_{n+2}, n. The term $\sum_{i_2,\ldots,i_{n+1}} a_{1i_1i_2} \prod_{j=2}^{n+1} (a_{ji_ji_{j+1}} (i_j - i_{j+1}))$ becomes zero for above choice and these pairs satisfy $\xi = 0$ in equation (4.2.9) and $\eta = 0$ in equation (4.2.10). The existence of infinitely many natural numbers m gives us infinitely many solutions to the equation $\xi + \eta = 0$. The only surviving term in the expression (4.2.8) for these solution is the term $\sum_{i_2,\ldots,i_{n+1}} F^{n+1} \left(\prod_{k=1}^{n+1} a_{ki_ki_{k+1}}\right) \prod_{j=1}^{n+1} (i_j - i_{j+1})$ (when $t = n+1$), which is equal to $F^{n+1}a$ for each i_1 and i_{n+2}. All the other terms become zero because of the existence of commutators (except for $t = n$, which also vanishes for our choice $a_{1,i_1,i_1+m} = a$, $a_{1,i_1,i_1+m+1} = -a$). This justifies our claim. □

Lemma 4.2.18. $\sum_{j=0}^{n} F^{n+1-j}\pi(\Omega^j(\mathcal{A}))\otimes\mathcal{S}\subseteq\pi\left(dJ_0^n(\mathcal{A}\otimes\mathcal{S})\right)$, *for all* $n\geq 1$.

Proof. We use induction. We have seen that $F\pi(\Omega^1(\mathcal{A}))\otimes\mathcal{S}+\mathcal{A}\otimes\mathcal{S}\subseteq\pi(dJ_0^1(\mathcal{A}\otimes\mathcal{S}))$ (Lemma 4.2.16). Assume this is true at the k-th stage. To prove for the $(k+1)$-th stage we use Lemma (4.2.17). Let $\xi=\sum_{j=0}^{k+1}F^{k+2-j}\xi_j\in\sum_{j=0}^{k+1}F^{k+2-j}\pi(\Omega^j(\mathcal{A}))\otimes\mathcal{S}$. Lemma (4.2.17) shows that $F^{k+2}\xi_0\in\pi\left(dJ_0^{k+1}(\mathcal{A}\otimes\mathcal{S})\right)$. To prove $\xi-F^{k+2}\xi_0\in\pi\left(dJ_0^{k+1}(\mathcal{A}\otimes\mathcal{S})\right)$, it is enough to show that for each i and j , if $\pi(\omega)\otimes e_{ij}\in\pi(dJ_0^k(\mathcal{A}\otimes\mathcal{S}))$ then $\pi(\omega)[D,a]\otimes e_{ij}\in\pi(dJ_0^{k+1}(\mathcal{A}\otimes\mathcal{S}))$ for any $a\in\mathcal{A}$. Let

$$\pi(\omega)\otimes e_{ij} = \sum\prod_{m=0}^{k}[\Sigma^2 D, x_m]$$

such that

$$\sum x_0\prod_{m=1}^{k}[\Sigma^2 D, x_m] = 0. \tag{4.2.11}$$

for all $x_m\in\mathcal{A}\otimes\mathcal{S}$. Now

$$\begin{aligned}(\pi(\omega)\otimes e_{ij})([D,a]\otimes e_{jj}) &= \left(\sum\prod_{m=0}^{k}[\Sigma^2 D, x_m]\right)([D,a]\otimes e_{jj})\\ &= \sum\prod_{m=0}^{k}[\Sigma^2 D, x_m][\Sigma^2 D, a\otimes e_{jj}]\,,\end{aligned}$$

since $[D,a]\otimes e_{jj}=[\Sigma^2 D, a\otimes e_{jj}]$. If $\pi(\omega)[D,a]\otimes e_{ij}$ has to be in $\pi(dJ_0^{k+1}(\mathcal{A}\otimes\mathcal{S}))$ then

$$\sum x_0\prod_{m=1}^{k}[\Sigma^2 D, x_m][\Sigma^2 D, a\otimes e_{jj}] = 0$$

should hold. But this is clear from equation (4.2.11). ◻

Lemma 4.2.19. *For all* $n\geq 1$, $\pi\left(dJ_0^n(\mathcal{A})\right)\otimes\mathcal{S}\subseteq\pi\left(dJ_0^n(\mathcal{A}\otimes\mathcal{S})\right)$.

Proof. The $n=1$ case has been addressed in Lemma (4.2.16). Let's take $n\geq 2$. Now choose any $\sum_k\prod_{j=0}^{n}[D,a_{jk}]\otimes T_k\in\pi\left(dJ_0^n(\mathcal{A})\right)\otimes\mathcal{S}$. Then $a_{0k}\prod_{j=1}^{n}[D,a_{jk}]=0$ for each

k. Let $I_{(k)}$ be the infinite matrix having an identity block matrix in top left-most corner of order same as that of T_k and zero elsewhere. Then $\prod_{j=0}^{n}[\Sigma^2 D, a_{jk}\otimes I_{(k)}] \in \pi\left(dJ_0^n(\mathcal{A}\otimes\mathcal{S})\right)$ for each k and hence, $(1\otimes T_k)\prod_{j=0}^{n}[\Sigma^2 D, a_{jk}\otimes I_{(k)}] \in \pi\left(dJ_0^n(\mathcal{A}\otimes\mathcal{S})\right)$ for each k. Now observe that

$$\sum_k \prod_{j=0}^{n}[D, a_{jk}]\otimes T_k = \sum_k (1\otimes T_k)\prod_{j=0}^{n}[\Sigma^2 D, a_{jk}\otimes I_{(k)}]\,,$$

which completes the proof. □

Lemma 4.2.20. $\pi\left(dJ_0^n(\mathcal{A}\otimes\mathcal{S})\right) = \pi\left(dJ_0^n(\mathcal{A})\right)\otimes\mathcal{S} + \sum_{r=0}^{n} F^{n+1-r}\pi\left(\Omega^r(\mathcal{A})\right)\otimes\mathcal{S}\;\forall\, n\geq 1$.

Proof. Since $\pi\left(dJ_0^n(\mathcal{A}\otimes\mathcal{S})\right)\subseteq\pi\left(\Omega^{n+1}(\mathcal{A}\otimes\mathcal{S})\right)$, Lemma (4.2.15) says the following:

$$\pi\left(dJ_0^n(\mathcal{A}\otimes\mathcal{S})\right)\subseteq\pi\left(\Omega^{n+1}(\mathcal{A})\right)\otimes\mathcal{S} + \sum_{r=0}^{n} F^{n+1-r}\pi\left(\Omega^r(\mathcal{A})\right)\otimes\mathcal{S}.$$

Now Lemmas (4.2.17, 4.2.18) and (4.2.19) show that

$$\pi\left(dJ_0^n(\mathcal{A})\right)\otimes\mathcal{S} + \sum_{r=0}^{n} F^{n+1-r}\pi\left(\Omega^r(\mathcal{A})\right)\otimes\mathcal{S}\subseteq\pi\left(dJ_0^n(\mathcal{A}\otimes\mathcal{S})\right).$$

We need to show that

$$\pi\left(dJ_0^n(\mathcal{A}\otimes\mathcal{S})\right)\subseteq\pi\left(dJ_0^n(\mathcal{A})\right)\otimes\mathcal{S} + \sum_{r=0}^{n} F^{n+1-r}\pi\left(\Omega^r(\mathcal{A})\right)\otimes\mathcal{S}.$$

We use induction on n. Lemma (4.2.16) gives the basis step of the induction and suppose that

$$\pi\left(dJ_0^{n-1}(\mathcal{A}\otimes\mathcal{S})\right) = \pi\left(dJ_0^{n-1}(\mathcal{A})\right)\otimes\mathcal{S} + \textstyle\sum_{r=0}^{n-1} F^{n-r}\pi\left(\Omega^r(\mathcal{A})\right)\otimes\mathcal{S}.$$

Recall that for the algebra $\mathcal{B}$ and for all $n\geq 1$,

$$\begin{aligned}\Omega^n(\mathcal{B}) &= \underbrace{\Omega^1(\mathcal{B})\otimes_{\mathcal{B}}\ldots\ldots\otimes_{\mathcal{B}}\Omega^1(\mathcal{B})}_{n\ times}\\ &= \Omega^1(\mathcal{B})\otimes_{\mathcal{B}}\Omega^{n-1}(\mathcal{B})\,.\end{aligned}$$

Hence $J_0^n = J_0^1 \otimes \Omega^{n-1} + \Omega^1 \otimes J_0^{n-1}$. Since d satisfies the graded Leibniz rule, we have

$$dJ_0^n \subseteq (dJ_0^1).\, \Omega^{n-1} + J_0^1.\, (d\Omega^{n-1}) + (d\Omega^1).\, J_0^{n-1} + \Omega^1.\, (dJ_0^{n-1})$$

(recall the graded product on $\Omega^\bullet$) and hence applying the algebra homomorphism π we get $\pi(dJ_0^n) \subseteq \pi(dJ_0^1)\pi(\Omega^{n-1}) + \pi(\Omega^1)\pi(dJ_0^{n-1})$. Since $J^\bullet$ is a graded ideal in $\Omega^\bullet$ we have

$$\begin{aligned}
& \pi\left(dJ_0^n(\mathcal{A}\otimes\mathcal{S})\right) \\
\subseteq\ & \pi\left(dJ_0^1(\mathcal{A}\otimes\mathcal{S})\right)\pi\left(\Omega^{n-1}(\mathcal{A}\otimes\mathcal{S})\right) + \pi\left(\Omega^1(\mathcal{A}\otimes\mathcal{S})\right)\pi\left(dJ_0^{n-1}(\mathcal{A}\otimes\mathcal{S})\right) \\
=\ & \left(\pi\left(dJ_0^1(\mathcal{A})\right)\otimes\mathcal{S} + F\pi\left(\Omega^1(\mathcal{A})\right)\otimes\mathcal{S} + \mathcal{A}\otimes\mathcal{S}\right)\left(\sum_{r=0}^{n-1} F^r\pi(\Omega^{n-1-r}(\mathcal{A}))\otimes\mathcal{S}\right) \\
& + \left(\pi(\Omega^1(\mathcal{A}))\otimes\mathcal{S} + F\mathcal{A}\otimes\mathcal{S}\right)\left(\pi\left(dJ_0^{n-1}(\mathcal{A})\right)\otimes\mathcal{S} + \sum_{r=0}^{n-1} F^{n-r}\pi(\Omega^r(\mathcal{A}))\otimes\mathcal{S}\right) \\
=\ & \sum_{r=0}^{n-1} F^r\pi(dJ_0^{n-r}(\mathcal{A}))\otimes\mathcal{S} + \sum_{r=0}^{n-1} F^{r+1}\pi(\Omega^{n-r}(\mathcal{A}))\otimes\mathcal{S} \\
& + \sum_{r=0}^{n-1} F^r\pi(\Omega^{n-1-r}(\mathcal{A}))\otimes\mathcal{S} + \pi\left(dJ_0^n(\mathcal{A})\right)\otimes\mathcal{S} + F\pi\left(dJ_0^{n-1}(\mathcal{A})\right)\otimes\mathcal{S} \\
& + \sum_{r=0}^{n-1} F^{n+1-r}\pi\left(\Omega^r(\mathcal{A})\right)\otimes\mathcal{S} + \sum_{r=1}^{n} F^{n+1-r}\pi\left(\Omega^r(\mathcal{A})\right)\otimes\mathcal{S} \\
=\ & \pi\left(dJ_0^n(\mathcal{A})\right)\otimes\mathcal{S} + \sum_{r=0}^{n} F^{n+1-r}\pi\left(\Omega^r(\mathcal{A})\right)\otimes\mathcal{S}.
\end{aligned}$$

Here the first equality follows from Lemmas (4.2.16 , 4.2.15) and the induction hypothesis.

□

Remark 4.2.21. *Note that the second condition, i,e.* $[D, \mathcal{A}] \subseteq \mathcal{A}\otimes\mathcal{E}nd_{\mathcal{A}}(\mathcal{H}^\infty)$, *is needed only for Lemmas* (4.2.16 , 4.2.20).

Theorem 4.2.22. *For the spectral triple* $(\Sigma^2\mathcal{A}, \Sigma^2\mathcal{H}, \Sigma^2 D)$, *we have*

1. $\Omega^1_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \cong \Omega^1_D(\mathcal{A})\otimes\mathcal{S}\bigoplus\Sigma^2\mathcal{A}$.

2. $\Omega^n_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \cong \Omega^n_D(\mathcal{A})\otimes\mathcal{S}$, *for all* $n \geq 2$.

3. *The differential* $\delta^0 : \Sigma^2\mathcal{A} \longrightarrow \Omega^1_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ *is given by,*

$$a \otimes T + f \longmapsto [D,a] \otimes T \bigoplus (a \otimes [N,T] + f').$$

4. *The differential* $\delta^1 : \Omega^1_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \longrightarrow \Omega^2_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ *is given by,*

$$\delta^1|_{\Omega^1_D(\mathcal{A})\otimes\mathcal{S}} = d^1 \otimes 1 \quad and \quad \delta^1|_{\Sigma^2\mathcal{A}} = 0.$$

5. *The differential* $\delta^n : \Omega^n_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \longrightarrow \Omega^{n+1}_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ *is given by,*

$$\delta^n = d^n \otimes 1$$

for all $n \geq 2$ *.*

Here $d : \Omega^\bullet_D(\mathcal{A}) \longrightarrow \Omega^{\bullet+1}_D(\mathcal{A})$ *is the differential of the Connes' complex.*

Proof. 1. Recall from Lemma (4.2.15) that $\Omega^1_{\Sigma^2 D}(\mathcal{A}\otimes\mathcal{S}) \cong \Omega^1_D(\mathcal{A})\otimes\mathcal{S} + F\mathcal{A}\otimes\mathcal{S}$. Since $\mathcal{A}\otimes\mathcal{E}nd_{\mathcal{A}}(\mathcal{H}^\infty) \subseteq \mathcal{E}nd_{\mathbb{C}}(\mathcal{H}^\infty)$ by the map $a\otimes g \longmapsto a\circ g$, $F\mathcal{A}$ can be embedded in $F\mathcal{A}\otimes\mathcal{E}nd_{\mathcal{A}}(\mathcal{H}^\infty) \subseteq \mathcal{E}nd_{\mathbb{C}}(\mathcal{H}^\infty)$ by the map $Fa \longmapsto Fa\otimes I$. Now $[D,\mathcal{A}] \subseteq \mathcal{E}nd_{\mathbb{C}}(\mathcal{H}^\infty)$ and $F\mathcal{A}\cap\mathcal{A} = \{0\}$ gives the direct sum. Finally, use the fact that $F\mathcal{A}\otimes\mathcal{S} \cong \mathcal{A}\otimes\mathcal{S}$ and Proposition (4.2.14) to conclude Part (1).

2. For all $n \geq 2$, we have

$$\begin{aligned}
& \Omega^n_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \\
\cong\; & \frac{\pi(\Omega^n(\Sigma^2\mathcal{A}))}{\pi\left(dJ_0^{n-1}(\Sigma^2\mathcal{A})\right)} \\
\cong\; & \frac{\pi(\Omega^n(\mathcal{A}\otimes\mathcal{S}))}{\pi\left(dJ_0^{n-1}(\mathcal{A}\otimes\mathcal{S})\right)} && \textit{by Proposition } (4.2.14) \\
\cong\; & \frac{\sum_{r=0}^{n} F^r\pi(\Omega^{n-r}(\mathcal{A}))\otimes\mathcal{S}}{\pi\left(dJ_0^{n-1}(\mathcal{A})\right)\otimes\mathcal{S} + \sum_{r=0}^{n-1} F^{n-r}\pi(\Omega^r(\mathcal{A}))\otimes\mathcal{S}} && \textit{by Lemmas } (4.2.15\,, 4.2.20) \\
\cong\; & \frac{\pi(\Omega^n(\mathcal{A}))\otimes\mathcal{S} + \sum_{r=1}^{n} F^r\pi(\Omega^{n-r}(\mathcal{A}))\otimes\mathcal{S}}{\pi\left(dJ_0^{n-1}(\mathcal{A})\right)\otimes\mathcal{S} + \sum_{r=0}^{n-1} F^{n-r}\pi(\Omega^r(\mathcal{A}))\otimes\mathcal{S}} \\
\cong\; & \frac{\pi(\Omega^n(\mathcal{A}))\otimes\mathcal{S} + \sum_{r=0}^{n-1} F^{n-r}\pi(\Omega^{r}(\mathcal{A}))\otimes\mathcal{S}}{\pi\left(dJ_0^{n-1}(\mathcal{A})\right)\otimes\mathcal{S} + \sum_{r=0}^{n-1} F^{n-r}\pi(\Omega^r(\mathcal{A}))\otimes\mathcal{S}} \\
\cong\; & \frac{\pi(\Omega^n(\mathcal{A}))\otimes\mathcal{S}}{\pi\left(dJ_0^{n-1}(\mathcal{A})\right)\otimes\mathcal{S}} \\
\cong\; & \Omega^n_D(\mathcal{A})\otimes\mathcal{S}
\end{aligned}$$

3. Obvious since $[\Sigma^2 D, a \otimes T + f] = [D,a] \otimes T + Fa \otimes [N,T] + f'$.

4. Take arbitrary $(a_0[D,a_1] \otimes T\,,\, b \otimes S + f) \in \Omega^1_D(\mathcal{A}) \otimes \mathcal{S} \bigoplus \Sigma^2\mathcal{A}$. Using Lemma (4.2.1) we have $S = \sum S_0[N,S_1]$ and Proposition (4.2.13) implies $f = \sum f_0 f_1'$. Now, as an element of $\Omega^1_{\Sigma^2 D}(\Sigma^2\mathcal{A})$

$$\begin{aligned}
& (a_0[D,a_1] \otimes T\,,\, b \otimes S + f) \\
= \;& (a_0 \otimes T)[\Sigma^2 D, a_1 \otimes I_{(T)}] + \sum (b \otimes S_0)[\Sigma^2 D, 1 \otimes S_1] \\
& + \sum (1 \otimes f_0)[\Sigma^2 D, 1 \otimes f_1]\,,
\end{aligned}$$

where $I_{(T)}$ is the identity block matrix of order same as that of T. Hence,

$$\begin{aligned}
& \delta^1((a_0[D,a_1] \otimes T\,,\, b \otimes S + f)) \\
= \;& \Big([\Sigma^2 D, a_0 \otimes T][\Sigma^2 D, a_1 \otimes I_{(T)}] + \sum [\Sigma^2 D, b \otimes S_0][\Sigma^2 D, 1 \otimes S_1] \\
& + \sum [\Sigma^2 D, 1 \otimes f_0][\Sigma^2 D, 1 \otimes f_1]\Big) + \pi(dJ_0^1(\Sigma^2\mathcal{A})),
\end{aligned}$$

as an element of $\Omega^2_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \cong \frac{\pi(\Omega^2(\Sigma^2\mathcal{A}))}{\pi(dJ_0^1(\Sigma^2\mathcal{A}))}$. Now, by Part (2), we finally get

$$\delta^1((a_0[D,a_1] \otimes T\,,\, b \otimes S + f)) \;=\; \big([D,a_0][D,a_1] + \pi(dJ_0^1(\mathcal{A}))\big) \otimes T,$$

as an element of $\Omega^2_D(\mathcal{A}) \otimes \mathcal{S}$.

5. Follows in a similar fashion to Part (4).

$\square$

Now we want to iterate this Theorem and Proposition (4.1.5) guarantees that we are allowed to do so. Let $k \geq 1$ and $\Sigma^{2k}\mathcal{A} = \Sigma^2(\Sigma^{2(k-1)}\mathcal{A})$. We put the convention $\Sigma^0\mathcal{A} = \mathcal{A}$ and $\Sigma^0 D = D$. Let $F^{(k)}$ be the sign of the operator $\Sigma^{2(k-1)}D$, acting on the Hilbert space $\mathcal{H} \otimes \ell^2(\mathbb{N})^{\otimes k-1}$. Then $\Sigma^{2k}D = \Sigma^{2(k-1)}D \otimes I + F^{(k)} \otimes N$ and $F^{(k)} = F \otimes 1^{\otimes k-1}$. Any element $\Sigma^{2k}a$ of $\Sigma^{2k}\mathcal{A}$ is of the form $\Sigma^{2(k-1)}a \otimes T_{(k-1)} + f_{(k-1)}$ where $\Sigma^{2(k-1)}a \in$

$\Sigma^{2(k-1)}\mathcal{A}$, $T_{(k-1)} \in \mathcal{S}$ and $f_{(k-1)} \in \mathbb{C}[z, z^{-1}]$. Using this functional equation one can write $\Sigma^{2k}a$ in terms of elements only from $\mathcal{A}$, $\mathcal{S}$ and $\mathbb{C}[z, z^{-1}]$.

Corollary 4.2.23. *For the spectral triple* $\left(\Sigma^{2k}\mathcal{A}, \Sigma^{2k}\mathcal{H}, \Sigma^{2k}D\right)$, $k \geq 1$, *we have*

1. $\Omega^1_{\Sigma^{2k}D}\left(\Sigma^{2k}\mathcal{A}\right) \cong \Omega^1_D(\mathcal{A}) \otimes \mathcal{S}^{\otimes k} \bigoplus \bigoplus_{j=1}^{k}\left(\Sigma^{2j}\mathcal{A} \otimes \mathcal{S}^{\otimes(k-j)}\right)$.

2. $\Omega^n_{\Sigma^{2k}D}\left(\Sigma^{2k}\mathcal{A}\right) \cong \Omega^n_D(\mathcal{A}) \otimes \mathcal{S}^{\otimes k}$, *for all* $n \geq 2$.

3. *The differential* $\delta^0 : \Sigma^{2k}\mathcal{A} \longrightarrow \Omega^1_{\Sigma^{2k}D}\left(\Sigma^{2k}\mathcal{A}\right)$ *is given by,*

$$\begin{aligned}\Sigma^{2(k-1)}a \otimes T_{(k-1)} + f_{(k-1)} \longmapsto\ & [D, a] \otimes T_{(0)} \otimes T_{(1)} \otimes \ldots \otimes T_{(k-1)} \bigoplus \\ & \left(\Sigma^{2(k-1)}a \otimes [N, T_{(k-1)}] + f'_{(k-1)}\right) \bigoplus \\ & \left(\bigoplus_{j=2}^{k}\left(\Sigma^{2(k-j)}a \otimes [N, T_{(k-j)}] + f'_{(k-j)}\right) \otimes Q_j\right)\end{aligned}$$

where $Q_j := T_{(k-(j-1))} \otimes T_{(k-(j-2))} \otimes \ldots \otimes T_{(k-1)} \in \mathcal{S}^{\otimes(j-1)}$.

4. *The differential* $\delta^1 : \Omega^1_{\Sigma^{2k}D}\left(\Sigma^{2k}\mathcal{A}\right) \longrightarrow \Omega^2_{\Sigma^{2k}D}\left(\Sigma^{2k}\mathcal{A}\right)$ *is given by,*

$$\delta^1|_{\Omega^1_D(\mathcal{A})\otimes\mathcal{S}^{\otimes k}} = d^1 \otimes 1^{\otimes k} \quad \textit{and} \quad \delta^1|_{\bigoplus_{j=1}^{k}\left(\Sigma^{2j}\mathcal{A}\otimes\mathcal{S}^{\otimes(k-j)}\right)} = 0.$$

5. *The differential* $\delta^n : \Omega^n_{\Sigma^{2k}D}\left(\Sigma^{2k}\mathcal{A}\right) \longrightarrow \Omega^{n+1}_{\Sigma^{2k}D}\left(\Sigma^{2k}\mathcal{A}\right)$ *is given by,*

$$\delta^n = d^n \otimes 1^{\otimes k}$$

for all $n \geq 2$.

Here $d : \Omega^\bullet_D(\mathcal{A}) \longrightarrow \Omega^{\bullet+1}_D(\mathcal{A})$ *is the differential of the Connes' complex at the* $k = 0$ *level.*

Proof. Note that $F^{(k)}\Sigma^{2(k-1)}\mathcal{A} \otimes \mathcal{S} \cong \Sigma^{2(k-1)}\mathcal{A} \otimes \mathcal{S}$ for all $k \geq 1$. Now the proof follows easily by induction on k where Theorem (4.2.22) is the basis step of the induction. $\square$

4.3 Connection and Curvature for the Quantum Double Suspension

Recall the notions of Connection, Curvature from (1.6). These notions are meaningful whenever one has a spectral triple. In this section, we discuss these notions on quantum double suspended spectral triple.

Assumption : Henceforth throughout the section we assume that $\mathcal{A}$ is spectrally invariant subalgebra in a C^*-algebra A.

Throughout this section 'u' will stand for the rank one projection $|e_0\rangle\langle e_0| = I - l^*l$ in $\mathcal{B}(\ell^2(\mathbb{N}))$. The map $\phi : a \longmapsto a \otimes u$ gives an algebra embedding of $\mathcal{A}$ in $\Sigma^2\mathcal{A}$ and hence extends to the map

$$\widetilde{\phi} : M_q(\mathcal{A}) \longrightarrow M_q(\Sigma^2\mathcal{A})$$

$$\mathbf{a} = (a_{ij}) \longmapsto (a_{ij} \otimes u)_{ij}$$

By definition of projective module, let $\mathcal{E} = p\mathcal{A}^n$ for some natural number n and an idempotent $p \in M_n(\mathcal{A})$. For $p = (p_{ij})_{ij}$, if we denote the matrix $(p_{ij} \otimes u)_{ij}$ by $p \otimes u$, then $\widetilde{\phi}$ gives a f.g.p right $\Sigma^2\mathcal{A}$-module $\widetilde{\mathcal{E}} = (p \otimes u)(\Sigma^2\mathcal{A})^n$. However, note that $\widetilde{\mathcal{E}} = (p \otimes u)(\Sigma^2\mathcal{A})^n$ is same as $(p \otimes u)(\mathcal{A} \otimes \mathcal{S})^n$ because u is a rank one projection operator. For reader's convenience we recall Theorem (2.2.3) here.

Theorem 4.3.1 (2.2.3)**.** *Let $\mathcal{E}$ be a f.g.p $\mathcal{A}$-module with a Hermitian structure where $\mathcal{A}$ is spectrally invariant subalgebra in a C^*-algebra. Then we can have a self-adjoint idempotent $p \in M_n(\mathcal{A})$ such that $\mathcal{E} = p\mathcal{A}^n$ and $\mathcal{E}$ has the induced canonical Hermitian structure.*

The goal of this section is to prove the following theorem.

Theorem 4.3.2. *Let $\mathcal{E}$ be a f.g.p module over $\mathcal{A}$ equipped with a Hermitian structure $\langle\ ,\ \rangle_{\mathcal{A}}$. Choose a projection $p \in M_n(\mathcal{A})$ such that $\mathcal{E} = p\mathcal{A}^n$ and $\mathcal{E}$ has the induced canonical Hermitian structure. Let $\widetilde{\mathcal{E}} = (p \otimes u)(\Sigma^2\mathcal{A})^n$ and restrict the canonical* *structure on $(\Sigma^2\mathcal{A})^n$ to $\widetilde{\mathcal{E}}$. We have an one-one affine morphism $\widetilde{\phi}_{con} : Con(\mathcal{E}) \longrightarrow Con(\widetilde{\mathcal{E}})$ which preserves the Grassmannian conections, and an one-one $\mathbb{C}$-linear map $\psi : Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D(\mathcal{A})) \longrightarrow Hom_{\Sigma^2\mathcal{A}}\left(\widetilde{\mathcal{E}}, \widetilde{\mathcal{E}} \otimes_{\Sigma^2\mathcal{A}} \Omega^2_{\Sigma^2 D}(\Sigma^2\mathcal{A})\right)$ such that the following diagram*

$$\begin{tikzcd}
Con(\mathcal{E}) \arrow[r, "\widetilde{\phi}_{con}"] \arrow[d, "f"'] & Con(\widetilde{\mathcal{E}}) \arrow[d, "f"] \\
Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D(\mathcal{A})) \arrow[r, "\psi"'] & Hom_{\Sigma^2\mathcal{A}}\left(\widetilde{\mathcal{E}}, \widetilde{\mathcal{E}} \otimes_{\Sigma^2\mathcal{A}} \Omega^2_{\Sigma^2 D}(\Sigma^2\mathcal{A})\right)
\end{tikzcd}$$

commutes. Here f is the map which sends any compatible connection to its associated curvature.

Remark 4.3.3. *The Choice of such a projection $p \in M_n(\mathcal{A})$ of Theorem (4.3.1), such that $\mathcal{E} = p\mathcal{A}^n$, has the advantage that now we have to deal with the canonical Hermitian structure, which is much easier to tackle than an arbitrary Hermitian structure. This is one of the main reasons for our assumption that $\mathcal{A}$ is spectrally invariant subalgebra in a C^*-algebra because this assumption is crucial for Theorem (4.3.1) to hold.*

We break the proof of this theorem into several lemmas and propositions to make it transparent and then combine them together at the end.

Lemma 4.3.4. *As right $\Sigma^2\mathcal{A}$ module,*

$$(p \otimes u)(\Omega^1_D \otimes \mathcal{S})^n \cong p(\Omega^1_D)^n \otimes u\mathcal{S}$$

Proof. We define

$$\Phi : p(\Omega^1_D)^n \otimes u\mathcal{S} \longrightarrow (p \otimes u)(\Omega^1_D \otimes \mathcal{S})^n$$

$$p(\omega_1, \ldots, \omega_n) \otimes uT \longmapsto (p \otimes u)(\omega_1 \otimes T, \ldots, \omega_n \otimes T)$$

and

$$\Psi : (p \otimes u)(\Omega^1_D \otimes \mathcal{S})^n \longrightarrow p(\Omega^1_D)^n \otimes u\mathcal{S}$$

$$(p\otimes u)(\omega_1\otimes T_1,\ldots,\omega_n\otimes T_n)\longmapsto\sum_{i=1}^{n}p(0,\ldots,\omega_i,\ldots,0)\otimes uT_i\,.$$

The proof is now routine verification. □

Lemma 4.3.5. *$(p\otimes u)(\mathcal{A}\otimes\mathcal{S})^n\cong p\mathcal{A}^n\otimes u\mathcal{S}$ as right $\Sigma^2\mathcal{A}$-module.*

Proof. Exact similar description of Φ,Ψ of previous Lemma (4.3.4) gives the isomorphism.

□

Notation : Henceforth throughout this chapter we write $\delta(T)=[N,T]$ for all $T\in\mathcal{S}$ and $\underbrace{(0,\ldots,a_i,\ldots,0)}_{n\ tuple}$ will denote the element of $\mathcal{A}^n$ with $a_i\in\mathcal{A}$ at the i-th co-ordinate and zero elsewhere.

Proposition 4.3.6. *Let $\nabla:\mathcal{E}\longrightarrow\mathcal{E}\otimes_{\mathcal{A}}\Omega^1_D(\mathcal{A})$ be a connection where $\mathcal{E}=p\mathcal{A}^n$. Define,*

$$\widetilde{\nabla}:(p\otimes u)(\mathcal{A}\otimes\mathcal{S})^n\longrightarrow(p\otimes u)(\mathcal{A}\otimes\mathcal{S})^n\otimes_{\Sigma^2\mathcal{A}}\Omega^1_{\Sigma^2D}(\Sigma^2\mathcal{A})$$

by the rule,

$$\begin{aligned}(p\otimes u)(a_1\otimes T_1,\ldots,a_n\otimes T_n)\longmapsto\quad&\sum_{i=1}^{n}\nabla(p(0,\ldots,a_i,\ldots,0))\otimes uT_i\\&+(p\otimes u)\left(a_1\otimes\delta(uT_1),\ldots,a_n\otimes\delta(uT_n)\right)\end{aligned}$$

where $\delta(T)=[N,T]$. Then $\widetilde{\nabla}$ defines a connection on $\widetilde{\mathcal{E}}=(p\otimes u)(\Sigma^2\mathcal{A})^n$.

Proof. Well-definedness is easy to check. Now consider any $a\otimes T+f\in\Sigma^2\mathcal{A}$. Then,

$$\begin{aligned}&(p\otimes u)(a_1\otimes T_1,\ldots,a_n\otimes T_n)(a\otimes T+f)\\=\;&(p\otimes u)(a_1a\otimes T_1T,\ldots,a_na\otimes T_nT)+(p\otimes u)(a_1\otimes T_1f,\ldots,a_n\otimes T_nf).\end{aligned}$$

The image of this element under $\widetilde{\nabla}$ is,

$$\sum_{i=1}^{n} \nabla(p(0,\ldots,a_i a,\ldots,0)) \otimes uT_iT + (p\otimes u)(a_1 a \otimes \delta(uT_1T),\ldots,a_n a\otimes \delta(uT_nT))$$
$$+\sum_{i=1}^{n} \nabla(p(0,\ldots,a_i,\ldots,0)) \otimes uT_if + (p\otimes u)(a_1 \otimes \delta(uT_1f),\ldots,a_n \otimes \delta(uT_nf))$$

Now,

$$\begin{aligned}
& \widetilde{\nabla}\left((p\otimes u)(a_1\otimes T_1,\ldots,a_n\otimes T_n)\right).(a\otimes T+f) \\
& +(p\otimes u)(a_1\otimes T_1,\ldots,a_n\otimes T_n)\otimes \widetilde{d}(a\otimes T+f) \\
= \; & \sum_{i=1}^{n}\nabla(p(0,\ldots,a_i,\ldots,0))a\otimes uT_iT + \sum_{i=1}^{n}\nabla(p(0,\ldots,a_i,\ldots,0))\otimes uT_if \\
& +(p\otimes u)(a_1\otimes\delta(uT_1),\ldots,a_n\otimes\delta(uT_n)).(a\otimes T+f) \\
& +(p\otimes u)(a_1\otimes T_1,\ldots,a_n\otimes T_n)\otimes(da\otimes T + a\otimes\delta T + 1\otimes\delta f) \\
= \; & \sum_{i=1}^{n}\{\nabla(p(0,\ldots,a_i,\ldots,0))a\otimes uT_iT + (p(0,\ldots,a_i,\ldots,0)\otimes da)\otimes uT_iT\} \\
& +(p\otimes u)\{(a_1a\otimes\delta(uT_1)T,\ldots,a_na\otimes\delta(uT_n)T) + (a_1a\otimes T_1\delta T,\ldots,a_na\otimes T_n\delta T)\} \\
& +(p\otimes u)\{(a_1\otimes\delta(uT_1)f,\ldots,a_n\otimes\delta(uT_n)f) + (a_1\otimes T_1\delta f,\ldots,a_n\otimes T_n\delta f)\} \\
& +\sum_{i=1}^{n}\nabla(p(0,\ldots,a_i,\ldots,0))\otimes uT_if \\
= \; & \sum_{i=1}^{n}\{\nabla(p(0,\ldots,a_ia,\ldots,0))\otimes uT_iT + \sum_{i=1}^{n}\nabla(p(0,\ldots,a_i,\ldots,0))\otimes uT_if \\
& +(p\otimes u)(a_1a\otimes\delta(uT_1T),\ldots,a_na\otimes\delta(uT_nT)) \\
& +(p\otimes u)(a_1\otimes\delta(uT_1f),\ldots,a_n\otimes\delta(uT_nf))\,.
\end{aligned}$$

This shows that,

$$\begin{aligned}
& \widetilde{\nabla}\left((p\otimes u)(a_1\otimes T_1,\ldots,a_n\otimes T_n).(a\otimes T+f)\right) \\
= \; & \widetilde{\nabla}\left((p\otimes u)(a_1\otimes T_1,\ldots,a_n\otimes T_n)\right).(a\otimes T+f) \\
& +(p\otimes u)(a_1\otimes T_1,\ldots,a_n\otimes T_n)\otimes\widetilde{d}(a\otimes T+f)\,,
\end{aligned}$$

i,e. $\widetilde{\nabla}$ is a connection on $\widetilde{\mathcal{E}}$. $\square$

Lemma 4.3.7. *If ∇ is the Grassmannian connection on $\mathcal{E}$, then $\widetilde{\nabla}$ is the Grassmannian connection on $\widetilde{\mathcal{E}}$.*

Proof. Let $\nabla = \nabla_0(\mathcal{E})$ denotes the Grassmannian connection on $\mathcal{E}$. Then

$$\begin{aligned}
& \widetilde{\nabla_0}((p\otimes u)(a_1\otimes T_1,\ldots,a_n\otimes T_n)) \\
= & \sum_{i=1}^{n} \nabla_0(\mathcal{E})(p(0,\ldots,a_i,\ldots,0))\otimes uT_i + (p\otimes u)(a_1\otimes \delta(uT_1),\ldots,a_n\otimes\delta(uT_n)) \\
= & \sum_{i=1}^{n} \nabla_0(\mathcal{E})((p_{1i}a_i,\ldots,p_{ni}a_i))\otimes uT_i + (p\otimes u)(a_1\otimes \delta(uT_1),\ldots,a_n\otimes\delta(uT_n)) \\
= & \sum_{i=1}^{n} p(d(p_{1i}a_i),\ldots,d(p_{ni}a_i))\otimes uT_i + (p\otimes u)(a_1\otimes \delta(uT_1),\ldots,a_n\otimes\delta(uT_n))\,.
\end{aligned}$$

Now if $\nabla_0(\widetilde{\mathcal{E}})$ denotes the Grassmannian connection on $\widetilde{\mathcal{E}}$, then

$$\begin{aligned}
& \nabla_0(\widetilde{\mathcal{E}})((p\otimes u)(a_1\otimes T_1,\ldots,a_n\otimes T_n)) \\
= & \nabla_0(\widetilde{\mathcal{E}})(\sum_{j=1}^{n} p_{1j}a_j\otimes uT_j,\ldots,\sum_{j=1}^{n} p_{nj}a_j\otimes uT_j) \\
= & \sum_{j=1}^{n}(p\otimes u)(\widetilde{d}(p_{1j}a_j\otimes uT_j),\ldots,\widetilde{d}(p_{1j}a_j\otimes uT_j)) \\
= & \sum_{j=1}^{n}(p\otimes u)(d(p_{1j}a_j)\otimes uT_j + p_{1j}a_j\otimes\delta(uT_j),\ldots,d(p_{nj}a_j)\otimes uT_j + p_{nj}a_j\otimes\delta(uT_j))\,.
\end{aligned}$$

Here $\widetilde{d} : \Sigma^2\mathcal{A} \longrightarrow \Omega^1_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ is the differential of Part (3), Proposition (4.2.22). Notice that

$\sum_{j=1}^{n}(p\otimes u)(p_{1j}a_j\otimes\delta(uT_j),\ldots,p_{nj}a_j\otimes\delta(uT_j)) = (p\otimes u)(a_1\otimes\delta(uT_1),\ldots,a_n\otimes\delta(uT_n))$,

and this completes the proof. $\square$

Proposition 4.3.8. *Let $\Theta : p\mathcal{A}^n \longrightarrow p\mathcal{A}^n\otimes_{\mathcal{A}}\Omega^2_D(\mathcal{A})$ be the curvature of the connection ∇ on $\mathcal{E} = p\mathcal{A}^n$ and let $\widetilde{\Theta}$ denotes the curvature of the connection $\widetilde{\nabla}$ of Proposition (4.3.6). Then*

$$\widetilde{\Theta} : (p \otimes u)(\mathcal{A} \otimes \mathcal{S})^n \longrightarrow (p \otimes u)(\mathcal{A} \otimes \mathcal{S})^n \otimes_{\Sigma^2 \mathcal{A}} \Omega^2_{\Sigma^2 D}(\Sigma^2 \mathcal{A})$$

is the map given by

$$(p \otimes u)(a_1 \otimes T_1, \ldots, a_n \otimes T_n) \longmapsto \textstyle\sum_{i=1}^n \Theta(p(0, \ldots, a_i, \ldots, 0)) \otimes uT_i \,.$$

Proof. Let $\nabla : p\mathcal{A}^n \longrightarrow p\mathcal{A}^n \otimes_{\mathcal{A}} \Omega^1_D$ be a connection for which Θ is the curvature and $\widetilde{\nabla}$ denotes the connection in Proposition (4.3.6). We let $\widetilde{\nabla}'$ be the extended map,

$$\widetilde{\nabla}' : \widetilde{\mathcal{E}} \otimes_{\Sigma^2 \mathcal{A}} \Omega^1_{\Sigma^2 D}(\Sigma^2 \mathcal{A}) \longrightarrow \widetilde{\mathcal{E}} \otimes_{\Sigma^2 \mathcal{A}} \Omega^2_{\Sigma^2 D}(\Sigma^2 \mathcal{A}).$$

Then $\widetilde{\Theta} = \widetilde{\nabla}' \circ \widetilde{\nabla}$. Now,

$$\begin{aligned}
& \widetilde{\nabla}' \left(\sum_{i=1}^n \nabla(p(0, \ldots, a_i, \ldots, 0)) \otimes uT_i \right) \\
= \;& \widetilde{\nabla}' \left(\sum_{i=1}^n p(\omega_1^{(i)}, \ldots, \omega_n^{(i)}) \otimes uT_i \right) \\
= \;& \widetilde{\nabla}' \left(\sum_{i=1}^n (p \otimes u)(\omega_1^{(i)} \otimes uT_i, \ldots, \omega_n^{(i)} \otimes uT_i) \right) \\
= \;& \widetilde{\nabla}' \left(\sum_{i=1}^n \sum_{j=1}^n (p \otimes u)(0, \ldots, \underbrace{1 \otimes u}_{j-th\, place}, \ldots, 0) \otimes (\omega_j^{(i)} \otimes uT_i) \right) \\
= \;& \sum_{i=1}^n \sum_{j=1}^n \widetilde{\nabla} \left((p \otimes u)(0, \ldots, \underbrace{1 \otimes u}_{j-th\, place}, \ldots, 0) \right) .(\omega_j^{(i)} \otimes uT_i) \\
& + (p \otimes u)(0, \ldots, \underbrace{1 \otimes u}_{j-th\, place}, \ldots, 0) \otimes (d\omega_j^{(i)} \otimes uT_i) \\
= \;& \sum_{i=1}^n \sum_{j=1}^n \nabla(p(0, \ldots, \underbrace{1}_{j-th\, place}, \ldots, 0))\omega_j^{(i)} \otimes uT_i + (p \otimes u)(0, \ldots, \underbrace{d\omega_j^{(i)} \otimes uT_i}_{j-th\, place}, \ldots, 0)
\end{aligned}$$

In the last equality use the fact that $\delta(u) = [N, u] = 0$. Also,

$$\begin{aligned}
& \widetilde{\nabla}'((p \otimes u)(a_1 \otimes \delta(uT_1), \dots, a_n \otimes \delta(uT_n))) \\
= \; & \widetilde{\nabla}'(\sum_{i=1}^{n}(p \otimes u)(0, \dots, a_i \otimes u, \dots, 0) \otimes (1 \otimes \delta(uT_i))) \\
= \; & \sum_{i=1}^{n} \widetilde{\nabla}((p \otimes u)(0, \dots, a_i \otimes u, \dots, 0))(1 \otimes \delta(uT_i)) \\
& \quad + (p \otimes u)(0, \dots, a_i \otimes u, \dots, 0) \otimes d(1 \otimes \delta(uT_i)) \\
= \; & \sum_{i=1}^{n} \{\nabla(p(0, \dots, a_i, \dots, 0)) \otimes u + (p \otimes u)(0, \dots, a_i \otimes \delta(u), \dots, 0)\}(1 \otimes \delta(uT_i)) \\
= \; & \sum_{i=1}^{n} \{\nabla(p(0, \dots, a_i, \dots, 0)) \otimes u\}(1 \otimes \delta(uT_i)) \\
= \; & 0 . \quad (\textit{see Theorem } 4.2.22)
\end{aligned}$$

Finally,

$$\begin{aligned}
& \widetilde{\Theta}((p \otimes u)(a_1 \otimes T_1, \dots, a_n \otimes T_n)) \\
= \; & \widetilde{\nabla}'\left(\sum_{i=1}^{n} \nabla(p(0, \dots, a_i, \dots, 0)) \otimes uT_i\right) + \widetilde{\nabla}'((p \otimes u)(a_1 \otimes \delta(uT_1), \dots, a_n \otimes \delta(uT_n))) \\
= \; & \sum_{i=1}^{n} \sum_{j=1}^{n} \nabla(p(0, \dots, \underbrace{1}_{j-th\, place}, \dots, 0))\omega_j^{(i)} \otimes uT_i + (p \otimes u)(0, \dots, \underbrace{d\omega_j^{(i)} \otimes uT_i}_{j-th\, place}, \dots, 0) \\
= \; & \sum_{i=1}^{n} \sum_{j=1}^{n} (\nabla(p(0, \dots, \underbrace{1}_{j-th\, place}, \dots, 0))\omega_j^{(i)} + p(0, \dots, \underbrace{1}_{j-th\, place}, \dots, 0) \otimes d\omega_j^{(i)}) \otimes uT_i \\
= \; & \sum_{i=1}^{n} \sum_{j=1}^{n} \nabla'(p(0, \dots, \underbrace{1}_{j-th\, place}, \dots, 0) \otimes \omega_j^{(i)}) \otimes uT_i \\
= \; & \sum_{i=1}^{n} \sum_{j=1}^{n} \nabla'(p(0, \dots, \underbrace{\omega_j^{(i)}}_{j-th\, place}, \dots, 0)) \otimes uT_i \\
= \; & \sum_{i=1}^{n} \Theta(p(0, \dots, a_i, \dots, 0)) \otimes uT_i
\end{aligned}$$

and this completes the proof. ◻

Lemma 4.3.9. *Let* $\widetilde{\xi}, \widetilde{\eta} \in (p \otimes u)(\mathcal{A} \otimes \mathcal{S})^n$ *and* $\Psi(\widetilde{\xi}) = \sum_k \xi_k \otimes uT_k$, $\Psi(\widetilde{\eta}) = \sum_k \eta_k \otimes uS_k$

where

$$\Psi : (p \otimes u)(\mathcal{A} \otimes \mathcal{S})^n \longrightarrow p\mathcal{A}^n \otimes u\mathcal{S}$$

$$(p \otimes u)(a_1 \otimes T_1, \ldots, a_n \otimes T_n) \longmapsto \sum_{i=1}^{n} p(0, \ldots, a_i, \ldots, 0) \otimes uT_i$$

is the isomorphism of Lemma (4.3.5). Then the induced canonical Hermitian structure on $(p \otimes u)(\mathcal{A} \otimes \mathcal{S})^n$ *has the following form:*

$$\langle \widetilde{\xi}, \widetilde{\eta} \rangle_{\Sigma^2\mathcal{A}} = \sum_{k,k'} \langle \xi_k, \eta_{k'} \rangle_{\mathcal{A}} \otimes (uT_k)^*(uS_{k'})\,.$$

Proof. Let $\widetilde{\xi} = (p \otimes u)(a_1 \otimes T_1, \ldots, a_n \otimes T_n)$ and $\widetilde{\eta} = (p \otimes u)(a'_1 \otimes T'_1, \ldots, a'_n \otimes T'_n)$. Then

$$\begin{aligned}
& \langle \widetilde{\xi}, \widetilde{\eta} \rangle_{\Sigma^2\mathcal{A}} \\
= \; & \langle \, (\sum_{j=1}^{n} p_{1j} a_j \otimes uT_j, \ldots, \sum_{j=1}^{n} p_{nj} a_j \otimes uT_j) \, , (\sum_{j=1}^{n} p_{1j} a'_j \otimes uT'_j, \ldots, \sum_{j=1}^{n} p_{nj} a'_j \otimes uT'_j) \, \rangle \\
= \; & \langle \, \sum_{j=1}^{n} (p_{1j} a_j \otimes uT_j, \ldots, p_{nj} a_j \otimes uT_j) \, , \sum_{j=1}^{n} (p_{1j} a'_j \otimes uT'_j, \ldots, p_{nj} a'_j \otimes uT'_j) \, \rangle \\
= \; & \sum_{j,l=1}^{n} \langle \, (p_{1j} a_j \otimes uT_j, \ldots, p_{nj} a_j \otimes uT_j) \, , \, (p_{1l} a'_l \otimes uT'_l, \ldots, p_{nl} a'_l \otimes uT'_l) \, \rangle \\
= \; & \sum_{j,l=1}^{n} \sum_{k=1}^{n} (p_{kj} a_j \otimes uT_j)^* (p_{kl} a'_l \otimes uT'_l)
\end{aligned}$$

Now $\Psi(\widetilde{\xi}) = \sum_{j=1}^{n} p(0, \ldots, a_j, \ldots, 0) \otimes uT_j$ and $\Psi(\widetilde{\eta}) = \sum_{j=1}^{n} p(0, \ldots, a'_j, \ldots, 0) \otimes uT'_j$. Let $\xi_j = p(0, \ldots, a_j, \ldots, 0)$ and $\eta_j = p(0, \ldots, a'_j, \ldots, 0)$. Now,

$$\begin{aligned}
\langle \xi_r, \eta_s \rangle_{\mathcal{A}} &= \langle p(0, \ldots, a_r, \ldots, 0), p(0, \ldots, a'_s, \ldots, 0) \rangle_{\mathcal{A}} \\
&= \langle \, (p_{1r} a_r, \ldots, p_{nr} a_r), (p_{1s} a'_s, \ldots, p_{ns} a'_s) \rangle_{\mathcal{A}} \\
&= \sum_{k=1}^{n} (p_{kr} a_r)^* p_{ks} a'_s \\
&= \sum_{k=1}^{n} a_r^* p_{kr} p_{ks} a'_s
\end{aligned}$$

Hence, $\sum_{r,s} \langle \xi_r, \eta_s \rangle_{\mathcal{A}} \otimes (uT_r)^*(uT'_s) = \sum_{r,s} \left(\sum_{i=1}^{n} (p_{ir} a_r \otimes uT_r)^* (p_{is} a'_s \otimes uT'_s) \right)$. $\square$

Lemma 4.3.10. *For $\xi, \eta \in \mathcal{E}$ we have*

(a) $\langle \xi \otimes uT, \nabla\eta \otimes uS\rangle = \langle \xi, \nabla\eta\rangle \otimes (uT)^* uS$,

(b) $\langle \nabla\xi \otimes uT, \eta \otimes uS\rangle = \langle \nabla\xi, \eta\rangle \otimes (uT)^* uS$.

Proof. Let $\xi = p(a_1, \ldots, a_n) \in p\mathcal{A}^n$, $\nabla\eta = \sum_i p(b_{1i}, \ldots, b_{ni}) \otimes \omega_i \in p\mathcal{A}^n \otimes \Omega^1_D(\mathcal{A})$. Then,

$$\begin{aligned} \xi \otimes uT &= p(a_1, \ldots, a_n) \otimes uT \\ &= (p \otimes u)(a_1 \otimes T, \ldots, a_n \otimes T) \end{aligned}$$

and

$$\begin{aligned} \nabla\eta \otimes uS &= \sum_i (p(b_{1i}, \ldots, b_{ni}) \otimes \omega_i) \otimes uS \\ &= \sum_i p(b_{1i}\omega_i, \ldots, b_{ni}\omega_i) \otimes uS \\ &= \sum_i (p \otimes u)(b_{1i}\omega_i \otimes S, \ldots, b_{ni}\omega_i \otimes S) \\ &= \sum_i (p \otimes u)(b_{1i} \otimes u, \ldots, b_{ni} \otimes u) \otimes (\omega_i \otimes uS) \end{aligned}$$

Hence,

$$\begin{aligned} & \langle \xi \otimes uT, \nabla\eta \otimes uS\rangle \\ = & \sum_i \langle (p \otimes u)(a_1 \otimes T, \ldots, a_n \otimes T), (p \otimes u)(b_{1i} \otimes u, \ldots, b_{ni} \otimes u)\rangle_{\Sigma^2\mathcal{A}} (\omega_i \otimes uS) \\ = & \sum_i \langle p(a_1, \ldots, a_n) \otimes uT, p(b_{1i}, \ldots, b_{ni}) \otimes u\rangle_{\Sigma^2\mathcal{A}} (\omega_i \otimes uS) \\ = & \sum_i \left(\langle p(a_1, \ldots, a_n), p(b_{1i}, \ldots, b_{ni})\rangle_{\mathcal{A}} \otimes (uT)^* u\right) (\omega_i \otimes uS) \\ = & \sum_i (\langle p(a_1, \ldots, a_n), p(b_{1i}, \ldots, b_{ni})\rangle_{\mathcal{A}} \omega_i) \otimes (uT)^* uS \\ = & \sum_i \langle p(a_1, \ldots, a_n), p(b_{1i}, \ldots, b_{ni}) \otimes \omega_i\rangle \otimes (uT)^* uS \\ = & \langle \xi, \nabla\eta\rangle \otimes (uT)^* uS \end{aligned}$$

This proves part (a) and part (b) follows similarly. $\square$

Lemma 4.3.11. *The connection $\widetilde{\nabla}$ of Proposition (4.3.6) is compatible with the Hermitian structure $\langle\ ,\ \rangle_{\Sigma^2\mathcal{A}}$ on $\widetilde{\mathcal{E}}$, if ∇ is compatible with respect to $\langle\ ,\ \rangle_{\mathcal{A}}$ on $\mathcal{E}$.*

Proof. For $\widetilde{\xi}, \widetilde{\eta} \in (p \otimes u)(\mathcal{A} \otimes \mathcal{S})^n$, we have isomorphic elements $\sum \xi \otimes uT, \sum \eta \otimes uS \in p\mathcal{A}^n \otimes u\mathcal{S}$ respectively. Let $\xi = p(a_1, \ldots, a_n)$ and $\eta = p(b_1, \ldots, b_n)$. It is easy to see that,

$$\widetilde{\nabla}(\widetilde{\xi}) = \nabla(\xi) \otimes uT + (p \otimes u)(a_1 \otimes \delta(uT), \ldots, a_n \otimes \delta(uT)).$$

$$\widetilde{\nabla}(\widetilde{\eta}) = \nabla(\eta) \otimes uS + (p \otimes u)(b_1 \otimes \delta(uS), \ldots, b_n \otimes \delta(uS)).$$

Now,

$$\begin{aligned}
& \langle \widetilde{\xi}, \widetilde{\nabla}\widetilde{\eta} \rangle_{\Sigma^2\mathcal{A}} - \langle \widetilde{\nabla}\widetilde{\xi}, \widetilde{\eta} \rangle_{\Sigma^2\mathcal{A}} \\
= \ & \langle \xi \otimes uT, \nabla\eta \otimes uS \rangle_{\Sigma^2\mathcal{A}} - \langle \nabla\xi \otimes uT, \eta \otimes uS \rangle_{\Sigma^2\mathcal{A}} \\
& + \langle \xi \otimes uT, (p \otimes u)(b_1 \otimes \delta(uS), \ldots, b_n \otimes \delta(uS)) \rangle_{\Sigma^2\mathcal{A}} \\
& - \langle (p \otimes u)(a_1 \otimes \delta(uT), \ldots, a_n \otimes \delta(uT), \eta \otimes uS \rangle_{\Sigma^2\mathcal{A}} \\
= \ & (\langle \xi, \nabla\eta \rangle_{\mathcal{A}} - \langle \nabla\xi, \eta \rangle_{\mathcal{A}}) \otimes (uT)^*(uS) \\
& + \langle \xi \otimes uT, \eta \otimes u\delta(uS) \rangle_{\Sigma^2\mathcal{A}} - \langle \xi \otimes u\delta(uT), \eta \otimes uS \rangle_{\Sigma^2\mathcal{A}} \quad (\textit{by Lemma } 4.3.10) \\
= \ & d(\langle \xi, \eta \rangle_{\mathcal{A}}) \otimes (uT)^*(uS) + \langle \xi, \eta \rangle_{\mathcal{A}} \otimes (uT)^* u\delta(uS) - \langle \xi, \eta \rangle_{\mathcal{A}} \otimes (u\delta(uT))^* uS
\end{aligned}$$

Finally,

$$\begin{aligned}
\widetilde{d}(\langle \widetilde{\xi}, \widetilde{\eta} \rangle_{\Sigma^2\mathcal{A}}) &= \widetilde{d}(\langle \xi \otimes uT, \eta \otimes uS \rangle_{\Sigma^2\mathcal{A}} \\
&= \widetilde{d}(\langle \xi, \eta \rangle_{\mathcal{A}} \otimes (uT)^* uS) \\
&= d(\langle \xi, \eta \rangle_{\mathcal{A}}) \otimes (uT)^* uS + \langle \xi, \eta \rangle_{\mathcal{A}} \otimes \delta((uT)^* uS) \\
&= d(\langle \xi, \eta \rangle_{\mathcal{A}}) \otimes (uT)^* uS + \langle \xi, \eta \rangle_{\mathcal{A}} \otimes ((uT)^* \delta(uS) + \delta((uT)^*) uS) \\
&= d(\langle \xi, \eta \rangle_{\mathcal{A}}) \otimes (uT)^* uS + \langle \xi, \eta \rangle_{\mathcal{A}} \otimes ((uT)^* \delta(uS) - (\delta(uT))^* uS)
\end{aligned}$$

This shows compatibility of $\widetilde{\nabla}$. $\square$

Proof of the Theorem (4.3.2) : Define

$$\widetilde{\phi}_{con} : Con(\mathcal{E}) \longrightarrow Con(\widetilde{\mathcal{E}})$$

$$\nabla \longmapsto \widetilde{\nabla}$$

where $\widetilde{\nabla}$ is as defined in Proposition (4.3.6). Lemma (4.3.11) proves that $\mathcal{R}an(\widetilde{\phi}_{con}) \subseteq Con(\widetilde{\mathcal{E}})$ and Lemma (4.3.7) proves preservation of the Grassmannian connection. It is easy to check that $\widetilde{\phi}_{con}$ is an affine morphism between $Con(\mathcal{E})$ and $Con(\widetilde{\mathcal{E}})$. To see injectivity, let $\widetilde{\phi}_{con}(\nabla_1) = \widetilde{\phi}_{con}(\nabla_2)$ and choose any $\xi = p(a_1, \ldots, a_n) \in \mathcal{E}$. Then $\widetilde{\xi} = (p \otimes u)(a_1 \otimes u, \ldots, a_n \otimes u) \in \widetilde{\mathcal{E}}$. Then it follows that $\nabla_1(\xi) \otimes u = \nabla_2(\xi) \otimes u$ (use Lemma 4.3.5) i,e. $\nabla_1(\xi) = \nabla_2(\xi)$. Now define

$$\psi : Hom_{\mathcal{A}}\left(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D(\mathcal{A})\right) \longrightarrow Hom_{\Sigma^2\mathcal{A}}\left(\widetilde{\mathcal{E}}, \widetilde{\mathcal{E}} \otimes_{\Sigma^2\mathcal{A}} \Omega^2_{\Sigma^2 D}(\Sigma^2\mathcal{A})\right)$$

$$\psi(g)\left((p \otimes u)(a_1 \otimes T_1, \ldots, a_n \otimes T_n)\right) := \sum_{i=1}^n g\left(p(0, \ldots, a_i, \ldots, 0)\right) \otimes uT_i\,.$$

It is easy to see that ψ is a well-defined linear map because $\widetilde{\mathcal{E}} \otimes_{\Sigma^2\mathcal{A}} \Omega^2_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \cong p\left(\Omega^2_D(\mathcal{A})\right)^n \otimes u\mathcal{S}$ as right $\Sigma^2\mathcal{A}$-module (the proof of this fact goes on the same route as described in Lemma 4.3.4). Injectivity follows similarly as before. Finally, in view of Proposition (4.3.8), we see that the diagram commutes and this completes the proof. $\square$

Chapter 5

Multiplicativity of Connes' Calculus

In this final chapter we study behaviour of $\Omega_D^\bullet$ for the tensor product of *even spectral triples*. More precisely, we are interested in knowing whether $\Omega_D^\bullet(\mathcal{A}_1 \otimes \mathcal{A}_2) \cong \Omega_{D_1}^\bullet(\mathcal{A}_1) \otimes \Omega_{D_2}^\bullet(\mathcal{A}_2)$ for two even spectral triples $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2)$. Here $\Omega_{D_1}^\bullet \otimes \Omega_{D_2}^\bullet$ denotes the tensor product of two differential graded algebras. We call it multiplicativity property of $\Omega_D^\bullet$.

This question has been studied in ([25]), but because of the partial nature of their result we reinvestigate it here. Note that to define $\Omega_D^\bullet$ one does not use self-adjointness and compactness of the resolvent of D. We cast Connes' definition in a slightly more general algebraic framework. We consider the quadruple $(\mathcal{A}, \mathbb{V}, D, \gamma)$ where $\mathcal{A}$ is an associative, unital algebra over $\mathbb{K}$, represented on a vector space $\mathbb{V}$, $D \in \mathcal{E}nd(\mathbb{V})$, $\gamma \in \mathcal{E}nd(\mathbb{V})$ is a $\mathbb{Z}_2$-grading operator which commutes with $\mathcal{A}$ and anticommutes with D. We show that the collection of such quadruple $(\mathcal{A}, \mathbb{V}, D, \gamma)$ is a monoidal category and denote it by $\widetilde{\mathcal{S}pec}$. We identify a smaller subcategory $\widetilde{\mathcal{S}pec_{sub}}$ and show that there is a covariant functor $\mathcal{G} : \widetilde{\mathcal{S}pec} \longrightarrow \widetilde{\mathcal{S}pec_{sub}}$. Moreover, $\widetilde{\mathcal{S}pec_{sub}}$ becomes a monoidal subcategory of $\widetilde{\mathcal{S}pec}$. Next we consider the category DGA of differential graded algebras over a field $\mathbb{K}$ and show that the association $\mathcal{F} : (\mathcal{A}, \mathbb{V}, D, \gamma) \longmapsto \Omega_D^\bullet(\mathcal{A})$ gives a covariant functor from $\widetilde{\mathcal{S}pec}$ to DGA. We show that restricted to $\widetilde{\mathcal{S}pec_{sub}}$, $\mathcal{F}$ becomes a monoidal functor. To validate the nontriviality of this functor, i,e. to show that the associated dga $\Omega_D^\bullet$ is not trivial, we explicitly compute $\mathcal{F}\circ\mathcal{G}$ for a canonical spectral triple associated with compact smooth manifolds and for the noncommutative torus. We also compute the associated cohomologies in each case and it turns out that the resulting dga $\Omega_D^\bullet$ is cohomologically also not trivial.

5.1 Multiplicativity of Connes' Calculus

Definition 5.1.1. *A quadruple $(\mathcal{A}, \mathbb{V}, D, \gamma)$ is called an even algebraic spectral triple if*

1. *$\mathcal{A}$ is a unital, associative algebra over a field $\mathbb{K}$,*
2. *π is a representation of $\mathcal{A}$ on the $\mathbb{K}$-vector space $\mathbb{V}$,*
3. *$D \in \mathcal{E}nd(\mathbb{V})$,*
4. *$\gamma \in \mathcal{E}nd(\mathbb{V})$ is a $\mathbb{Z}_2$-grading operator which commutes with $\pi(\mathcal{A})$ and anticommutes with D.*

Let $(\mathcal{A}_1, \mathbb{V}_1, D_1, \gamma_1)$ and $(\mathcal{A}_2, \mathbb{V}_2, D_2, \gamma_2)$ be two even algebraic spectral triples. The product of these is given by the following even algebraic spectral triple

$$(\mathcal{A}_1, \mathbb{V}_1, D_1, \gamma_1) \otimes (\mathcal{A}_2, \mathbb{V}_2, D_2, \gamma_2) := (\mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{V}_1 \otimes \mathbb{V}_2, D_1 \otimes 1 + \gamma_1 \otimes D_2, \gamma_1 \otimes \gamma_2) .$$

One can consider two dgas $\Omega^\bullet_{D_1}(\mathcal{A}_1)$ and $\Omega^\bullet_{D_2}(\mathcal{A}_2)$. The product of these two dgas is given by

$$\Omega^\bullet_{D_1}(\mathcal{A}_1) \otimes \Omega^\bullet_{D_2}(\mathcal{A}_2) := \bigoplus_{n\geq 0} \bigoplus_{i+j=n} \Omega^i_{D_1}(\mathcal{A}_1) \otimes \Omega^j_{D_2}(\mathcal{A}_2) .$$

It is natural to ask how $\Omega^\bullet_D$ behaves under this multiplication, i,e. whether

$$\Omega^n_D(\mathcal{A}_1 \otimes \mathcal{A}_2) \cong \bigoplus_{i+j=n} \Omega^i_{D_1}(\mathcal{A}_1) \otimes \Omega^j_{D_2}(\mathcal{A}_2) \ \ \forall\, n \geq 0 .$$

The article ([25]) deals with this investigation and does not lead to a final conclusion. However, using the universality of $\Omega^\bullet(\mathcal{A}_1 \otimes \mathcal{A}_2)$, a useful outcome is that for all $n \geq 0$

$$\Omega^n_D(\mathcal{A}_1 \otimes \mathcal{A}_2) \cong \widetilde{\Omega}^n_D(\mathcal{A}_1, \mathcal{A}_2) , \tag{5.1.1}$$

where $\widetilde{\Omega}^\bullet_D(\mathcal{A}_1, \mathcal{A}_2)$ is defined as follows.

Definition 5.1.2. *Consider the reduced universal dgas* $(\Omega^\bullet(\mathcal{A}_1), d_1)$ *and* $(\Omega^\bullet(\mathcal{A}_2), d_2)$, *associated with the algebraic spectral triples* $(\mathcal{A}_1, \mathbb{V}_1, D_1, \gamma_1)$ *and* $(\mathcal{A}_2, \mathbb{V}_2, D_2, \gamma_2)$ *respectively. Consider the product dga* $\left(\Omega^\bullet(\mathcal{A}_1) \otimes \Omega^\bullet(\mathcal{A}_2)\, , \widetilde{d}\right)$ *where*

$$(\omega_i \otimes u_j).(\omega_p \otimes u_q) := (-1)^{jp}\omega_i\omega_p \otimes u_j u_q\,, \tag{5.1.2}$$

$$\widetilde{d}(\omega_i \otimes u_j) := d_1(\omega_i) \otimes u_j + (-1)^i \omega_i \otimes d_2(u_j)\,, \tag{5.1.3}$$

for $\omega_\bullet \in \Omega^\bullet(\mathcal{A}_1)$ *and* $u_\bullet \in \Omega^\bullet(\mathcal{A}_2)$. *One can define a representation* $\widetilde{\pi}$ *of* $\Omega^\bullet(\mathcal{A}_1) \otimes \Omega^\bullet(\mathcal{A}_2)$ *by*

$$\widetilde{\pi}(\omega_i \otimes u_j) := \pi_1(\omega_i)\gamma_1^j \otimes \pi_2(u_j)\,. \tag{5.1.4}$$

Let

$$\widetilde{J}_0^k := Ker\left\{\widetilde{\pi} : \bigoplus_{i+j=k} \Omega^i(\mathcal{A}_1) \otimes \Omega^j(\mathcal{A}_2) \longrightarrow \mathcal{E}nd(\mathbb{V}_1 \otimes \mathbb{V}_2)\right\}\,, \tag{5.1.5}$$

and $\widetilde{J}^n = \widetilde{J}_0^n + \widetilde{d}\widetilde{J}_0^{n-1}$. *Define* $\widetilde{\Omega}^n_D(\mathcal{A}_1, \mathcal{A}_2) := \frac{\bigoplus_{i+j=n} \Omega^i(\mathcal{A}_1) \otimes \Omega^j(\mathcal{A}_2)}{\widetilde{J}^n(\mathcal{A}_1, \mathcal{A}_2)}\,, \forall\, n \geq 0.$

At this point we would like to register one interesting observation about the tensored complex $\Omega^\bullet(\mathcal{A}_1) \otimes \Omega^\bullet(\mathcal{A}_2)$. Recall that $\Omega^1(\mathcal{A}_1 \otimes \mathcal{A}_2)$ enjoys the following universal property :

For any $\mathcal{A}_1 \otimes \mathcal{A}_2$-bimodule M and any derivation $\delta : \mathcal{A}_1 \otimes \mathcal{A}_2 \longrightarrow M$, there exists a unique $\mathcal{A}_1 \otimes \mathcal{A}_2$-bimodule morphism $\Phi : \Omega^1(\mathcal{A}_1 \otimes \mathcal{A}_2) \longrightarrow M$ such that the following diagram

$$\begin{tikzcd} \mathcal{A}_1 \otimes \mathcal{A}_2 \arrow[rr, "d"] \arrow[dr, "\delta"'] & & \Omega^1(\mathcal{A}_1 \otimes \mathcal{A}_2) \arrow[dl, "\Phi"] \\ & M & \end{tikzcd}$$

commutes.

The $\mathcal{A}_1 \otimes \mathcal{A}_2$-bimodule $\widetilde{\Omega}^1(\mathcal{A}_1, \mathcal{A}_2) := (\Omega^1(\mathcal{A}_1) \otimes \mathcal{A}_2) \bigoplus (\mathcal{A}_1 \otimes \Omega^1(\mathcal{A}_2))$ enjoys the following universal property.

Proposition 5.1.3. *For any $\mathcal{A}_1 \otimes \mathcal{A}_2$-bimodule M and any derivation $\delta : \mathcal{A}_1 \otimes \mathcal{A}_2 \longrightarrow M$ satisfying,*

1. *$\delta(a \otimes 1)$ commutes with $1 \otimes b$,*

2. *$\delta(1 \otimes b)$ commutes with $a \otimes 1$,*

for all $a \in \mathcal{A}_1, b \in \mathcal{A}_2$; there exists a unique $\mathcal{A}_1 \otimes \mathcal{A}_2$-bimodule morphism

$$\Psi : \widetilde{\Omega}^1(\mathcal{A}_1, \mathcal{A}_2) \longrightarrow M$$

such that the following diagram

$$\begin{tikzcd} \mathcal{A}_1 \otimes \mathcal{A}_2 \arrow[rr, "\widetilde{d}"] \arrow[dr, "\delta"'] & & \widetilde{\Omega}^1(\mathcal{A}_1, \mathcal{A}_2) \arrow[dl, "\Psi"] \\ & M & \end{tikzcd}$$

commutes.

Proof. Define $\Psi : \widetilde{\Omega}^1(\mathcal{A}_1, \mathcal{A}_2) \longrightarrow M$ by the following map

$$a_1 d_1 a_2 \otimes b_1 \longmapsto (a_1 \otimes 1)\delta(a_2 \otimes 1)(1 \otimes b_1)$$

$$a_1 \otimes b_1 d_2 b_2 \longmapsto (a_1 \otimes 1)(1 \otimes b_1)\delta(1 \otimes b_2)\,.$$

Checking left module homomorphism is trivial. We will check that Ψ is a right module homomorphism. For any simple tensor $x \otimes y \in \mathcal{A}_1 \otimes \mathcal{A}_2$,

$$\begin{aligned}(a_1 d_1 a_2 \otimes b_1 + \widetilde{a}_1 \otimes \widetilde{b}_1 d_2 \widetilde{b}_2)(x \otimes y) &= a_1 d_1(a_2 x) \otimes b_1 y + \widetilde{a}_1 x \otimes \widetilde{b}_1 d_2(\widetilde{b}_2 y) \\ &\quad - a_1 a_2 d_1 x \otimes b_1 y - \widetilde{a}_1 x \otimes \widetilde{b}_1 \widetilde{b}_2 d_2 y\end{aligned}$$

Hence,

$$
\begin{aligned}
& \Psi\left((a_1 d_1 a_2 \otimes b_1 + \widetilde{a}_1 \otimes \widetilde{b}_1 d_2 \widetilde{b}_2)(x \otimes y)\right) \\
= \; & (a_1 \otimes 1)\delta(a_2 x \otimes 1)(1 \otimes b_1 y) - (a_1 a_2 \otimes 1)\delta(x \otimes 1)(1 \otimes b_1 y) \\
& + (\widetilde{a}_1 x \otimes \widetilde{b}_1)\delta(1 \otimes \widetilde{b}_2 y) - (\widetilde{a}_1 x \otimes \widetilde{b}_1 \widetilde{b}_2)\delta(1 \otimes y) \\
= \; & (a_1 \otimes 1)\delta(a_2 \otimes 1)(1 \otimes b_1)(x \otimes y) + (\widetilde{a}_1 \otimes \widetilde{b}_1)\delta(1 \otimes \widetilde{b}_2 y)(x \otimes 1) \\
& - (\widetilde{a}_1 x \otimes \widetilde{b}_1 \widetilde{b}_2)\delta(1 \otimes y) \\
= \; & (a_1 \otimes 1)\delta(a_2 \otimes 1)(1 \otimes b_1)(x \otimes y) + (\widetilde{a}_1 \otimes \widetilde{b}_1)\delta(1 \otimes \widetilde{b}_2)(x \otimes y) \\
& + (\widetilde{a}_1 \otimes \widetilde{b}_1)(1 \otimes \widetilde{b}_2)\delta(1 \otimes y)(x \otimes 1) - (\widetilde{a}_1 x \otimes \widetilde{b}_1 \widetilde{b}_2)\delta(1 \otimes y) \\
= \; & \left((a_1 \otimes 1)\delta(a_2 \otimes 1)(1 \otimes b_1) + (\widetilde{a}_1 \otimes \widetilde{b}_1)\delta(1 \otimes \widetilde{b}_2)\right)(x \otimes y) \\
= \; & \left(\Psi(a_1 d_1 a_2 \otimes b_1 + \widetilde{a}_1 \otimes \widetilde{b}_1 d_2 \widetilde{b}_2)\right)(x \otimes y)
\end{aligned}
$$

Suppose there exists another $\mathcal{A}_1 \otimes \mathcal{A}_2$ bimodule morphism Ψ' such that the diagram commutes. For any $a_1 d_1 a_2 \otimes b_1 \in \Omega^1(\mathcal{A}_1) \otimes \mathcal{A}_2$, observe that it is same as $(a_1 \otimes 1)\widetilde{d}(a_2 \otimes 1)(1 \otimes b_1)$. Then,

$$
\begin{aligned}
\Psi'(a_1 d_1 a_2 \otimes b_1) &= (a_1 \otimes 1)\Psi'\left(\widetilde{d}(a_2 \otimes 1)\right)(1 \otimes b_1) \\
&= (a_1 \otimes 1)\delta(a_2 \otimes 1)(1 \otimes b_1) \\
&= \Psi(a_1 d_1 a_2 \otimes b_1)
\end{aligned}
$$

One can prove a similar fact for $a \otimes b_1 d_2 b_2$. This proves the uniqueness of Ψ. □

Note that $\widetilde{\Omega}^n_D(\mathcal{A}_1, \mathcal{A}_2) \cong \bigoplus_{i+j=n} \Omega^i_{D_1}(\mathcal{A}_1) \otimes \Omega^j_{D_2}(\mathcal{A}_2)$ if and only if

$$
\widetilde{J}^n(\mathcal{A}_1, \mathcal{A}_2) \cong \bigoplus_{i+j=n} J^i(\mathcal{A}_1) \otimes \Omega^j(\mathcal{A}_2) + \Omega^i(\mathcal{A}_1) \otimes J^j(\mathcal{A}_2)\,. \tag{5.1.6}
$$

But it is in general not true. This is the point of investigation in this chapter. We propose a subcategory of the category of even algebraic spectral triples, which satisfies (5.1.6).

Also there is a canonical projector onto this subcategory.

Definition 5.1.4. *The objects of the category* $\widetilde{\mathcal{S}pec}$ *are even algebraic spectral triples* $(\mathcal{A}, \mathbb{V}, D, \gamma)$*. Given two such objects* $(\mathcal{A}_i, \mathbb{V}_i, D_i, \gamma_i)$*, with* $i = 1, 2$*, a morphism between them is a pair* (ϕ, Φ) *where* $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ *is unital algebra morphism between the algebras* $\mathcal{A}_1, \mathcal{A}_2$ *and* $\Phi \in \mathcal{E}nd(\mathbb{V}_1, \mathbb{V}_2)$ *is surjective which intertwines the representations* $\pi_1, \pi_2 \circ \phi$ *and the operators* D_1, D_2 *or equivalently, for which the following diagrams commute for every* $x \in \mathcal{A}_1$ *:*

$$\begin{tikzcd} \mathbb{V}_1 \arrow[r, "\Phi"] \arrow[d, "D_1"'] & \mathbb{V}_2 \arrow[d, "D_2"] \\ \mathbb{V}_1 \arrow[r, "\Phi"'] & \mathbb{V}_2 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{V}_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1(x)"'] & \mathbb{V}_2 \arrow[d, "\pi_2 \circ \phi(x)"] \\ \mathbb{V}_1 \arrow[r, "\Phi"'] & \mathbb{V}_2 \end{tikzcd}$$

and Φ *also intertwines the grading operators* γ_1, γ_2*,*

$$\begin{tikzcd} \mathbb{V}_1 \arrow[r, "\Phi"] \arrow[d, "\gamma_1"'] & \mathbb{V}_2 \arrow[d, "\gamma_2"] \\ \mathbb{V}_1 \arrow[r, "\Phi"'] & \mathbb{V}_2 \end{tikzcd}$$

Remark 5.1.5. *This definition is essentially from* ([2]). *However, our requirement demands the extra condition on surjectivity of* Φ *. This is in line with* ([22], [40]).

Proposition 5.1.6. *The category* $\widetilde{\mathcal{S}pec}$ *is a monoidal category.*

Proof. Define the identity object '1' of monoidal category as follows

$$1 := (\mathbb{K}, \mathbb{K}, 0, 1).$$

Define the functor tensor product '$\otimes$' on objects as

$$\begin{aligned} & (\mathcal{A}_1, \mathbb{V}_1, D_1, \gamma_1) \otimes (\mathcal{A}_2, \mathbb{V}_2, D_2, \gamma_2) \\ := \;& (\mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{V}_1 \otimes \mathbb{V}_2, D_1 \otimes 1 + \gamma_1 \otimes D_2, \gamma_1 \otimes \gamma_2) \end{aligned}$$

and on morphisms

$$(\phi\,,\Phi):(\mathcal{A},\mathbb{V},D,\gamma)\longmapsto(\widetilde{\mathcal{A}},\widetilde{\mathbb{V}},\widetilde{D},\widetilde{\gamma})$$

$$(\phi'\,,\Phi'):(\mathcal{A}',\mathbb{V}',D',\gamma')\longmapsto(\widetilde{\mathcal{A}}',\widetilde{\mathbb{V}}',\widetilde{D}',\widetilde{\gamma}')$$

by $(\phi\otimes\phi',\Phi\otimes\Phi')$, where $\phi\otimes\phi'$ is the usual tensor product of two algebra morphisms and $\Phi\otimes\Phi'$ is the usual tensor product of two linear maps. Now one can easily verify all the conditions of a monoidal category. $\square$

Let DGA be the category of differential graded algebras over field $\mathbb{K}$. We will only consider nonnegatively graded algebras in this chapter.

Lemma 5.1.7. *There is a covariant funtor* $\mathcal{F}:\widetilde{\mathcal{S}pec}\longrightarrow DGA$ *given by* $(\mathcal{A},\mathbb{V},D,\gamma)\longmapsto\Omega^\bullet_D(\mathcal{A})$.

Proof. Consider two objects $(\mathcal{A}_1,\mathbb{V}_1,D_1,\gamma_1),(\mathcal{A}_2,\mathbb{V}_2,D_2,\gamma_2)\in\mathcal{O}b(\widetilde{\mathcal{S}pec})$ and suppose there is a morphism $(\phi\,,\Phi):(\mathcal{A}_1,\mathbb{V}_1,D_1,\gamma_1)\longrightarrow(\mathcal{A}_2,\mathbb{V}_2,D_2,\gamma_2)$. Define

$$\Psi:\Omega^\bullet_{D_1}(\mathcal{A}_1)\longrightarrow\Omega^\bullet_{D_2}(\mathcal{A}_2)$$

$$[\textstyle\sum a_0\prod_{i=1}^n[D_1,a_i]]\longmapsto[\sum\phi(a_0)\prod_{i=1}^n[D_2,\phi(a_i)]]$$

for all $a_j\in\mathcal{A}_j$, $n\geq 0$. To show Ψ is well-defined we must show that $\Psi(\pi(d_1J_0^m))\subseteq\pi(d_2J_0^m)$ for all $m\geq 1$, where d_1,d_2 are the universal differentials for $\Omega^\bullet(\mathcal{A}_1),\Omega^\bullet(\mathcal{A}_2)$ respectively. Observe that

$$\Phi\circ\left(\sum a_0\prod_{i=1}^n[D_1,a_i]\right)=\left(\sum\phi(a_0)\prod_{i=1}^n[D_2,\phi(a_i)]\right)\circ\Phi\,. \tag{5.1.7}$$

Consider an arbitrary element $\xi\in\pi(d_1J_0^n)$. By definition, $\xi=\sum\prod_{i=0}^n[D_1,a_i]\in\pi(d_1J_0^n)$ such that $\sum a_0\prod_{i=1}^n[D_1,a_i]=0$. Now using equation (5.1.7) and surjectivity of Φ we have

$$\textstyle\sum\phi(a_0)\prod_{i=1}^n[D_2,\phi(a_i)]=0\,.$$

This shows the well-definedness of Ψ. Now it is easy to check that Ψ is a differential graded algebra morphism. $\square$

Remark 5.1.8. *This is the only place where we need the stronger assumption on surjectivity of the map* Φ *and because of this reason we differ from* ([2]).

Now consider $(\mathcal{A}, \mathbb{V}, D, \gamma) \in \mathcal{O}b(\widetilde{\mathcal{S}pec})$ such that $\gamma \in \pi(\mathcal{A})$. Let $\widetilde{\mathcal{S}pec_{sub}}$ be the subcategory of $\widetilde{\mathcal{S}pec}$, objects of which are $(\mathcal{A}, \mathbb{V}, D, \gamma)$ with $\gamma \in \pi(\mathcal{A})$. Clearly $\widetilde{\mathcal{S}pec_{sub}}$ is a monoidal subcategory of $\widetilde{\mathcal{S}pec}$. Now suppose $(\mathcal{A}, \mathbb{V}, D, \gamma) \in \mathcal{O}b(\widetilde{\mathcal{S}pec})$ and $\gamma \notin \pi(\mathcal{A})$. Consider the vector space $\mathcal{A} \oplus \mathcal{A}$ with the product rule

$$(a, b) \star (\bar{a}, \bar{b}) := (a\bar{a} + b\bar{b}\,,\, a\bar{b} + b\bar{a})\,.$$

The algebra $(\mathcal{A} \oplus \mathcal{A}\,,\, \star)$ becomes unital with unit $(1, 0)$. The map $(a, b) \longmapsto (a+b, a-b)$ gives a unital algebra isomorphism between the algebra $(\mathcal{A} \oplus \mathcal{A}\,,\, \star)$ and the direct sum algebra $\mathcal{A} \oplus \mathcal{A}$ where the multiplication is defined as co-ordinatewise. Now the map

$$(a, b) \longmapsto \pi(a) + \gamma\pi(b) \in \mathcal{E}nd(\mathbb{V})$$

gives a representation of the unital algebra $(\mathcal{A} \oplus \mathcal{A}\,,\, \star)$ on the vector space $\mathbb{V}$. Since $(0, 1) \longmapsto \gamma \in \mathcal{E}nd(\mathbb{V})$ we have $\gamma \in \pi((\mathcal{A} \oplus \mathcal{A}\,, \star))$ and hence $((\mathcal{A} \oplus \mathcal{A}\,,\, \star), \mathbb{V}, D, \gamma) \in \mathcal{O}b(\widetilde{\mathcal{S}pec_{sub}})$.

Proposition 5.1.9. *The association* $\mathcal{G} : (\mathcal{A}, \mathbb{V}, D, \gamma) \longmapsto ((\mathcal{A} \oplus \mathcal{A}\,,\, \star), \mathbb{V}, D, \gamma)$ *gives a covariant functor from* $\widetilde{\mathcal{S}pec}$ *to* $\widetilde{\mathcal{S}pec_{sub}}$.

Proof. For a morphism $(\phi\,, \Phi) : (\mathcal{A}, \mathbb{V}, D, \gamma) \longrightarrow (\mathcal{A}', \mathbb{V}', D', \gamma')$, define

$$(\widetilde{\phi}\,, \widetilde{\Phi}) : ((\mathcal{A} \oplus \mathcal{A}\,, \star), \mathbb{V}, D, \gamma) \longrightarrow ((\mathcal{A}' \oplus \mathcal{A}'\,, \star), \mathbb{V}', D', \gamma')$$

by taking $\widetilde{\Phi} := \Phi$ and

$$\widetilde{\phi} : \mathcal{A} \oplus \mathcal{A} \longrightarrow \mathcal{A}' \oplus \mathcal{A}'$$

$$(a, b) \longmapsto (\phi(a), \phi(b))\,.$$

It is easy to check that $(\widetilde{\phi}\,, \widetilde{\Phi})$ defines a morphism in $\widetilde{\mathcal{S}pec_{sub}}$. $\square$

In the next two sections we will see that the funtor $\mathcal{F}\circ\mathcal{G}$ is not trivial. Throughout the rest of this chapter the notation $\mathcal{F}$ and $\mathcal{G}$ will be reserved to mean the functors in Lemma (5.1.7) and Proposition (5.1.9) respectively.

Lemma 5.1.10. *The category DGA is monoidal.*

Proof. For two dgas $(C=\oplus_{n\geq 0}C^n, d)$ and $(E=\oplus_{n\geq 0}E^n,\delta)$, define the tensor product functor '$\otimes$' by $(C\otimes E := \oplus_{n\geq 0}\oplus_{i+j=n} C^i\otimes E^j, d\otimes\delta)$ where,

$$d\otimes\delta(c_i\otimes e_j) := d(c_i)\otimes e_j + (-1)^i c_i\otimes\delta(e_j)$$

for all $c_i\in C^i, e_j\in E^j$. For two morphisms

$$f:(C,d)\longrightarrow(E,\delta)\quad ;\quad g:(\widetilde{C},\widetilde{d})\longrightarrow(\widetilde{E},\widetilde{\delta})$$

define $f\otimes g:(C,d)\otimes(\widetilde{C},\widetilde{d})\longrightarrow(E,\delta)\otimes(\widetilde{E},\widetilde{\delta})$ by $f\otimes g=\oplus_{n\geq 0}\oplus_{i+j=n} f^i\otimes g^j$. The identity object '1' is the dga $(A=\oplus_{n\geq 0}A^n, 0)$ where $A^0=\mathbb{C}$ and $A^n=0$ for all $n\geq 1$. The rest of the results can be routinely checked. $\square$

Theorem 5.1.11. *Restricted to the monoidal subcategory $\widetilde{\mathcal{S}pec_{sub}}$ of $\widetilde{\mathcal{S}pec}$, the covariant functor $\mathcal{F}:\widetilde{\mathcal{S}pec_{sub}}\longrightarrow DGA$ defined in Lemma (5.1.7) is a monoidal functor.*

Proof. Only nontrivial part is to prove that

$$\Omega^n_D(\mathcal{A}_1\otimes\mathcal{A}_2)\cong\bigoplus_{i+j=n}\Omega^i_{D_1}(\mathcal{A}_1)\otimes\Omega^j_{D_2}(\mathcal{A}_2)$$

where $D=D_1\otimes 1+\gamma_1\otimes D_2$. We break the proof into two lemmas.

Lemma 5.1.12. *For any $a\in\mathcal{A}$, $[D^2,a]\in\pi(dJ^1_0)$.*

Proof. Consider $p=(1+\gamma)/2$ and $q=(1-\gamma)/2$. Then $pq=0$ and $pDp=qDq=0$. Consider $a\in\mathcal{A}$ and $\eta\in\mathcal{A}$ be such that $\pi(\eta)=\gamma$. Now consider

$$\omega=\tfrac{1}{4}(1+\eta)d(a)(1+\eta)+\tfrac{1}{4}(1-\eta)d(a)(1-\eta)\in\Omega^1(\mathcal{A}).$$

Then,

$$
\begin{aligned}
\pi(\omega) &= p[D,ap]-pa[D,p]+q[D,aq]-qa[D,q] \\
&= pDap - papD - paDp + papD + qDaq - qaqD - qaDq + qaqD \\
&= 0\,; \quad \mathit{since}\;\; pap = pa = ap\,;\; qaq = qa = aq.
\end{aligned}
$$

This shows that $\omega \in J_0^1(\mathcal{A}) \subseteq \Omega^1(\mathcal{A})$. Now,

$$
\begin{aligned}
\pi(d\omega) &= [D,p][D,ap]-[D,pa][D,p]+[D,q][D,aq]-[D,qa][D,q] \\
&= (Dp-pD)(Dap-apD) - Dpa[D,p] + paD[D,p] \\
&\quad +(Dq-qD)(Daq-aqD) - Dqa[D,q] + qaD[D,q] \\
&= DpDap - DpapD - pD^2ap + pDapD - DpaDp + DpapD \\
&\quad +paD^2p - paDpD + DqDaq - DqaqD - qD^2aq \\
&\quad +qDaqD - DqaDq + DqaqD + qaD^2q - qaDqD \\
&= 0 - DpaD - pD^2ap + 0 - 0 + DpaD + paD^2p - 0 \\
&\quad +0 - DqaD - qD^2aq + 0 - 0 + DqaD + qaD^2q - 0 \\
&= -[pD^2p, pa] - [qD^2q, qa]
\end{aligned}
$$

Now observe that $pD^2q = pD(p+q)Dq = 0$ and $qD^2p = 0$. Hence,

$$
\begin{aligned}
[D^2, a] &= [(p+q)D^2(p+q), (p+q)a] \\
&= [pD^2p + qD^2q, pa + qa] \\
&= [pD^2p, pa] + [qD^2q, qa] \\
&= \pi(d\omega)
\end{aligned}
$$

This proves that $[D^2, a] \in \pi(dJ_0^1(\mathcal{A}))$. $\square$

Lemma 5.1.13. *We have*

$$\widetilde{J}^n(\mathcal{A}_1,\mathcal{A}_2) \;=\; \bigoplus_{i+j=n} J^i(\mathcal{A}_1)\otimes\Omega^j(\mathcal{A}_2)+\Omega^i(\mathcal{A}_1)\otimes J^j(\mathcal{A}_2)\,,$$

where definition of $\widetilde{J}^n$ is provided in definition (5.1.2).

Proof. Let $\omega=\widetilde{\pi}(\widetilde{d}\omega')$ where $\omega'\in\widetilde{J}_0^{n-1}$. Suppose $\omega'=\sum_k\bigoplus_{i+j=n-1}v^i_{1,k}\otimes v^j_{2,k}\,,$ where $v^i_{1,k}\in\Omega^i(\mathcal{A}_1)$ and $v^j_{2,k}\in\Omega^j(\mathcal{A}_2)$. Hence we have the following equation,

$$\sum_k\sum_{i+j=n-1}\pi_1(v^i_{1,k})\gamma_1^j\otimes\pi_2(v^j_{2,k})=0\,. \tag{5.1.8}$$

Let

$$v^i_{1,k}=\sum a^{(i)}_{0,k}\prod_{r=1}^{i}d_1(a^{(i)}_{r,k})$$

$$v^j_{2,k}=\sum b^{(j)}_{0,k}\prod_{s=1}^{j}d_2(b^{(j)}_{s,k})$$

for $a^{(i)}_{r,k}\in\mathcal{A}_1$ and $b^{(j)}_{s,k}\in\mathcal{A}_2$. Then equation (5.1.8) becomes

$$\sum_k\sum_{i+j=n-1}\sum\left(a^{(i)}_{0,k}\prod_{r=1}^{i}[D_1,a^{(i)}_{r,k}]\gamma_1^j\right)\otimes\left(b^{(j)}_{0,k}\prod_{s=1}^{j}[D_2,b^{(j)}_{s,k}]\right)=0\,. \tag{5.1.9}$$

Now since $\omega'=\sum_k\bigoplus_{i+j=n-1}v^i_{1,k}\otimes v^j_{2,k}\,,$

$$\widetilde{d}(\omega')=\sum_k\sum_{i+j=n-1}d_1(v^i_{1,k})\otimes v^j_{2,k}+(-1)^i v^i_{1,k}\otimes d_2(v^j_{2,k})$$

and hence,

$$\widetilde{\pi}(\widetilde{d}\omega') \;=\; \sum_k\sum_{i+j=n-1}\pi_1(d_1(v^i_{1,k}))\gamma^j\otimes\pi_2(v^j_{2,k})+(-1)^i\pi_1(v^i_{1,k})\gamma^{j+1}\otimes\pi_2(d_2(v^j_{2,k})).$$

Using equation (5.1.9) we get,

$$\begin{aligned}&\sum_k\sum_{i+j=n-1}\pi_1(d_1(v^i_{1,k}))\gamma^j\otimes\pi_2(v^j_{2,k})\\ =\;&\sum_k\sum_{i+j=n-1}\sum\left(-a^{(i)}_{0,k}D_1\prod_{r=1}^{i}[D_1,a^{(i)}_{r,k}]\gamma_1^j\right)\otimes\left(b^{(j)}_{0,k}\prod_{s=1}^{j}[D_2,b^{(j)}_{s,k}]\right)\end{aligned}$$

$$= -\sum_k \sum_{i+j=n-1} \sum \Big\{ \sum_{r=1}^{i} \Big((-1)^{r+1} a_{0,k}^{(i)} [D_1, a_{1,k}^{(i)}] \dots [D_1^2, a_{r,k}^{(i)}] \dots [D_1, a_{i,k}^{(i)}] \gamma_1^j \Big)$$
$$\otimes \left(b_{0,k}^{(j)} \prod_{s=1}^{j} [D_2, b_{s,k}^{(j)}] \right) - \left((-1)^i a_{0,k}^{(i)} \prod_{t=1}^{i} [D_1, a_{t,k}^{(i)}] D_1 \gamma_1^j \right) \otimes \left(b_{0,k}^{(j)} \prod_{s=1}^{j} [D_2, b_{s,k}^{(j)}] \right) \Big\} .$$

This term is contained in $\sum_{i+j=n} \pi_1(J^i)\gamma_1^j \otimes \pi_2(\Omega^j)$, since we have seen that $[D_1^2, a_{r,k}^{(i)}]$ is in $\pi_1(J^2)$ for each $1 \leqslant r \leqslant i$ (Lemma 5.1.12). Finally,

$$\sum_k \sum_{i+j=n-1} (-1)^i \pi_1(v_{1,k}^i)\gamma^{j+1} \otimes \pi_2(d_2(v_{2,k}^j))$$
$$= \sum_k \sum_{i+j=n-1} \sum \left[\gamma_1 \otimes D_2 \, , \, \left(a_{0,k}^{(i)} \prod_{t=1}^{i} [D_1, a_{t,k}^{(i)}] \gamma_1^j \right) \otimes \left(b_{0,k}^{(j)} \prod_{s=1}^{j} [D_2, b_{s,k}^{(j)}] \right) \right]$$
$$+ \sum_{r=1}^{j} (-1)^{r+1} \left(a_{0,k}^{(i)} \prod_{t=1}^{i} [D_1, a_{t,k}^{(i)}] \gamma_1^{j+1} \right) \otimes \left(b_{0,k}^{(j)} [D_2, b_{1,k}^{(j)}] \dots [D_2^2, b_{r,k}^{(j)}] \dots [D_2, b_{j,k}^{(j)}] \right) .$$

This term is in $\sum_{i+j=n} \pi_1(\Omega^i)\gamma_1^j \otimes \pi_2(J^j)$, since $[D_2^2, b_{r,k}^{(j)}] \in \pi_2(J^2)$ for each $1 \leqslant r \leqslant j$ (5.1.12). So we get

$$\widetilde{\pi}(\widetilde{J}^n) \subseteq \textstyle\sum_{i+j=n} \pi_1(\Omega^i)\gamma_1^j \otimes \pi_2(J^j) + \pi_1(J^i)\gamma_1^j \otimes \pi_2(\Omega^j) \, ,$$

i,e. $\widetilde{\pi}(\widetilde{J}^n) \subseteq \widetilde{\pi}\left(\bigoplus_{i+j=n} (J^i \otimes \Omega^j + \Omega^i \otimes J^j) \right)$. The converse is trivial. Hence,

$$\frac{\widetilde{J}^n}{\bigoplus_{i+j=n}(J^i \otimes \Omega^j + \Omega^i \otimes J^j)} \cong \frac{\widetilde{\pi}(\widetilde{J}^n)}{\widetilde{\pi}\left(\bigoplus_{i+j=n}(J^i \otimes \Omega^j + \Omega^i \otimes J^j)\right)} = \{0\}$$

and our claim has been justified. $\square$

Proof of Theorem (5.1.11) : Lemma (5.1.13) proves that the isomorphism in equation (5.1.6) holds i,e.

$$\widetilde{J}^n(\mathcal{A}_1 \, , \, \mathcal{A}_2) \cong \textstyle\bigoplus_{i+j=n} J^i(\mathcal{A}_1) \otimes \Omega^j(\mathcal{A}_2) + \Omega^i(\mathcal{A}_1) \otimes J^j(\mathcal{A}_2) \, ,$$

when we restrict ourselves to the subcategory $\widetilde{\mathcal{S}pec_{sub}}$. Hence the proof follows from the fact that $\Omega_D^n(\mathcal{B}_1 \otimes \mathcal{B}_2) \cong \widetilde{\Omega}_D^n(\mathcal{B}_1, \mathcal{B}_2)$ for all $n \geq 0$ and for any unital algebras $\mathcal{B}_1, \mathcal{B}_2$ (see the isomorphism in 5.1.1). $\square$

Corollary 5.1.14. $\mathcal{F}(\mathcal{G}(\mathcal{A}_1) \otimes \mathcal{G}(\mathcal{A}_2)) \cong \mathcal{F} \circ \mathcal{G}(\mathcal{A}_1) \otimes \mathcal{F} \circ \mathcal{G}(\mathcal{A}_2)$.

However, we do not know whether $\mathcal{F}\circ\mathcal{G}(\mathcal{A}_1\otimes\mathcal{A}_2)\cong\mathcal{F}\circ\mathcal{G}(\mathcal{A}_1)\otimes\mathcal{F}\circ\mathcal{G}(\mathcal{A}_2)$.

5.2 Computation for Compact Manifold

In this section our objective is to show that there exists a contravariant functor $\mathcal{P}$ from the category of manifolds $\mathcal{M}$ with embeddings as morphisms to the category $\widetilde{\mathcal{S}pec}$ such that the functor $\mathcal{F}\circ\mathcal{G}\circ\mathcal{P}$ is not trivial.

Let $\mathbb{M}$ be a compact manifold of dimension n with atlas $\{U_i,\phi_i\}_{i=1}^k$. Consider the complexified exterior bundle $\wedge_{\mathbb{C}}^{\bullet}T^*\mathbb{M}$ over $\mathbb{M}$ and let $(x^1,\ldots,x^n)$ denote the local co-ordinates. Let d be the exterior differentiation. To construct the contravariant functor $\mathcal{P}$ consider the following object

$$\left(C^\infty(\mathbb{M})\,,\,\Gamma(\wedge_{\mathbb{C}}^{\bullet}T^*\mathbb{M})\cong\Gamma(\wedge_{\mathbb{C}}^{even}T^*\mathbb{M})\oplus\Gamma(\wedge_{\mathbb{C}}^{odd}T^*\mathbb{M})\,,\,D:=\begin{pmatrix}0&d\\d&0\end{pmatrix}\,,\,\gamma:=parity\right)$$

in $\widetilde{\mathcal{S}pec}$, where 'parity' means the odd-even parity of a form in $\Gamma(\wedge_{\mathbb{C}}^{\bullet}T^*\mathbb{M})$. Now for an embedding $\phi:\mathbb{M}\hookrightarrow\mathbb{N}$, we have

$$\widetilde{\phi}:\left(C^\infty(\mathbb{N})\,,\,\Gamma(\wedge_{\mathbb{C}}^{\bullet}T^*\mathbb{N})\,,\,\begin{pmatrix}0&d\\d&0\end{pmatrix}\,,\,\gamma\right)\longrightarrow\left(C^\infty(\mathbb{M})\,,\,\Gamma(\wedge_{\mathbb{C}}^{\bullet}T^*\mathbb{M})\,,\,\begin{pmatrix}0&d\\d&0\end{pmatrix}\,,\,\gamma\right),$$

a morphism in $\widetilde{\mathcal{S}pec}$. Moreover, the following commutative diagram

$$\begin{array}{ccc}\Gamma(\wedge_{\mathbb{C}}^{k}T^*\mathbb{N}) & \xrightarrow{\quad d\quad} & \Gamma(\wedge_{\mathbb{C}}^{k+1}T^*\mathbb{N})\\ {\scriptstyle\phi^*}\big\downarrow & & \big\downarrow{\scriptstyle\phi^*}\\ \Gamma(\wedge_{\mathbb{C}}^{k}T^*\mathbb{M}) & \xrightarrow[\quad d\quad]{} & \Gamma(\wedge_{\mathbb{C}}^{k+1}T^*\mathbb{M})\end{array}$$

where ϕ^* is the pullback of ϕ, explains that the quadruple $(C^\infty(\mathbb{M}),\Gamma(\wedge_{\mathbb{C}}^{\bullet}T^*\mathbb{M}),D,\gamma)$ is natural. Henceforth we will be dealing with $(C^\infty(\mathbb{M}),\Gamma(\wedge_{\mathbb{C}}^{\bullet}T^*\mathbb{M}),D,\gamma)\in\mathcal{O}b\left(\widetilde{\mathcal{S}pec}\right)$ in this section, where $D=\begin{pmatrix}0&d\\d&0\end{pmatrix}$ and $\gamma=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Notice that $D^2=0$. Since

$\gamma \notin \pi(C^\infty(\mathbb{M}))$ we first apply the functor $\mathcal{G}$ of Proposition (5.1.9) and then compute $\mathcal{F} \circ \mathcal{G}$ along with the associated cohomologies.

Notation : $\widetilde{C^\infty(\mathbb{M})} := \mathcal{G}\left(C^\infty(\mathbb{M}), \Gamma(\wedge^\bullet_{\mathbb{C}} T^*\mathbb{M}), D, \gamma\right)$ where $\mathcal{G}$ is defined in Proposition (5.1.9) and $dim(\mathbb{M}) = n$ throughout this section.

Lemma 5.2.1. $\Omega_D^m\left(\widetilde{C^\infty(\mathbb{M})}\right) \cong \pi\left(\Omega^m(\widetilde{C^\infty(\mathbb{M})})\right)\ \forall\, m \geq 0.$

Proof. Observe that $J_0^0\left(\widetilde{C^\infty(\mathbb{M})}\right) = \{0\}$ in this case. We show that $\pi\left(dJ_0^m(\widetilde{C^\infty(\mathbb{M})})\right) = \{0\}\ \forall\, m \geq 1$. Note that

$$\begin{aligned}
\pi(dJ_0^m) &= \{\sum \prod_{i=0}^m [D, x_i] : x_i \in \widetilde{C^\infty(\mathbb{M})}\ ;\ \sum x_0 \prod_{i=1}^m [D, x_i] = 0\} \\
&= \{-\sum x_0 D \prod_{i=1}^m [D, x_i] : x_i \in \widetilde{C^\infty(\mathbb{M})}\ ;\ \sum x_0 \prod_{i=1}^m [D, x_i] = 0\}
\end{aligned}$$

Now,

$$\begin{aligned}
\sum x_0 D \prod_{i=1}^m [D, x_i] &= \sum x_0 D \prod_{i=1}^m (Dx_i - x_i D) \\
&= -\sum x_0 D x_1 D \prod_{i=2}^m (Dx_i - x_i D) \\
&= (-1)^m \sum x_0 \prod_{i=1}^m D x_i D \\
&= (-1)^m \left(\sum x_0 \prod_{i=1}^{m-1} [D, x_i]\right) D x_m D \\
&= (-1)^m \left(\sum x_0 \prod_{i=1}^m [D, x_i]\right) D
\end{aligned}$$

But $\sum x_0 \prod_{i=1}^m [D, x_i] = 0$ by assumption and hence we are done. $\square$

Let $1 \leq m \leq n$, where $n = dim(\mathbb{M})$. We define the following linear operator

$$\begin{aligned}
T_{a_0, \ldots, a_m} : \Gamma(\wedge^\bullet_{\mathbb{C}} T^*\mathbb{M}) &\longrightarrow \Gamma(\wedge^\bullet_{\mathbb{C}} T^*\mathbb{M}) \\
\omega &\longmapsto a_0 da_1 \wedge \ldots \wedge d(a_m \omega)
\end{aligned}$$

where $a_i \in C^\infty(\mathbb{M})$. Let

$$\mathcal{M}_m = span\{T_{a_0,\dots,a_m} : \Gamma(\wedge^\bullet_{\mathbb{C}} T^*\mathbb{M}) \longrightarrow \Gamma(\wedge^\bullet_{\mathbb{C}} T^*\mathbb{M}) : a_i \in C^\infty(\mathbb{M})\}\,.$$

Then $\mathcal{M}_m$ is a $\mathbb{C}$-vector space. Note that for $a, b \in C^\infty(\mathbb{M})$

$$\left[D, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right] = \begin{pmatrix} 0 & T_{1,b} - T_{a,1} \\ T_{1,a} - T_{b,1} & 0 \end{pmatrix}.$$

Since elements of $\pi\left(\Omega^m(\widetilde{C^\infty(\mathbb{M})})\right)$ are of the form

$$\sum \begin{pmatrix} a_0 & 0 \\ 0 & b_0 \end{pmatrix} \prod_{i=1}^m \left[D, \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix}\right] \;;\; a_j, b_j \in C^\infty(\mathbb{M})\,,$$

it is easy to observe that $\pi\left(\Omega^m(\widetilde{C^\infty(\mathbb{M})})\right)$ is a subspace of $\mathcal{M}_m \oplus \mathcal{M}_m$. Moreover, using the equality

$$\sum_k \begin{pmatrix} a_{0k} & 0 \\ 0 & a'_{0k} \end{pmatrix} \left[D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} -a'_{1k} & 0 \\ 0 & a_{1k} \end{pmatrix} = \sum_k \begin{pmatrix} 0 & T_{a_{0k}\,,\,a_{1k}} \\ T_{a'_{0k}\,,\,a'_{1k}} & 0 \end{pmatrix} \quad (5.2.1)$$

we see that for $m \geq 3$ odd

$$\begin{aligned}
&\sum \begin{pmatrix} 0 & T_{a_0,a_1,\dots,a_m} \\ T_{a'_0,a'_1,\dots,a'_m} & 0 \end{pmatrix} \\
= &\sum \begin{pmatrix} 0 & T_{a_0,a_1} \\ T_{a'_0,a'_1} & 0 \end{pmatrix} \prod_{i=2, i\ even}^{m} \left(\begin{pmatrix} 0 & T_{1,a'_i} \\ T_{1,a_i} & 0 \end{pmatrix} \begin{pmatrix} 0 & T_{1,a_{i+1}} \\ T_{1,a'_{i+1}} & 0 \end{pmatrix} \right) \\
= &\sum \begin{pmatrix} a_0 & 0 \\ 0 & a'_0 \end{pmatrix} \left[D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} -a'_1 & 0 \\ 0 & a_1 \end{pmatrix} \bullet \\
&\prod_{i=2, i\ even}^{m} \left(\left\{ \left[D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} -a_i & 0 \\ 0 & a'_i \end{pmatrix} \right\} \left\{ \left[D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} -a'_{i+1} & 0 \\ 0 & a_{i+1} \end{pmatrix} \right\} \right)
\end{aligned}$$

and similarly for $m \geq 2$ even. Hence we conclude that $\pi\left(\Omega^m\left(\widetilde{C^\infty(\mathbb{M})}\right)\right) = \mathcal{M}_m \oplus \mathcal{M}_m$.

Lemma 5.2.2. *Let $\mathbb{V}$ be the vector space of all linear endomorphisms acting on $\Gamma(\wedge^{\bullet}_{\mathbb{C}} T^*\mathbb{M})$. We have the following subspaces of $\mathbb{V}$*

$$\mathcal{M}_m^{(1)} := \{M_{\omega_{m-1}} \circ d : \Gamma(\wedge^{\bullet}_{\mathbb{C}} T^*\mathbb{M}) \longrightarrow \Gamma(\wedge^{\bullet}_{\mathbb{C}} T^*\mathbb{M}) : \, \omega_{m-1} \in \Gamma(\wedge^{m-1}_{\mathbb{C}} T^*\mathbb{M})\} \,,$$

$$\mathcal{M}_m^{(2)} := \{M_{\omega_m} : \Gamma(\wedge^{\bullet}_{\mathbb{C}} T^*\mathbb{M}) \longrightarrow \Gamma(\wedge^{\bullet}_{\mathbb{C}} T^*\mathbb{M}) : \, \omega_m \in \Gamma(\wedge^{m}_{\mathbb{C}} T^*\mathbb{M})\} \,.$$

where M_ξ denotes multiplication by ξ. Then $\mathcal{M}_m^{(1)} \bigcap \mathcal{M}_m^{(2)} = \{0\}$ and $\mathcal{M}_m \subseteq \mathcal{M}_m^{(1)} \oplus \mathcal{M}_m^{(2)} \subseteq \mathbb{V}$.

Proof. Observe that $T_{a_0,\ldots,a_m}(\omega) = (M_{a_0 a_m da_1 \wedge \ldots \wedge da_{m-1}} \circ d + M_{a_0 da_1 \wedge \ldots \wedge da_m})(\omega) \,, \forall\, \omega \in \Gamma(\wedge^{\bullet}_{\mathbb{C}} T^*\mathbb{M})$. Since $d(1) = 0$ and $\wedge(1) = 1$, we have the direct sum. $\square$

Lemma 5.2.3. *For $1 \leq m \leq n$ define*

$$\widetilde{\Phi} : \mathcal{M}_m \longrightarrow \Omega^{m-1}\mathbb{M} \oplus \Omega^m \mathbb{M}$$

$$T_{a_0,\ldots,a_m} \longmapsto (a_0 a_m da_1 \wedge \ldots \wedge da_{m-1} \,,\, a_0 da_1 \wedge \ldots \wedge da_m)$$

where $\Omega^k\mathbb{M} := \Gamma(\wedge^k_{\mathbb{C}} T^*\mathbb{M})$ denotes the space of k-forms on $\mathbb{M}$. Then

$$\Phi = (\widetilde{\Phi}, \widetilde{\Phi}) : \mathcal{M}_m \oplus \mathcal{M}_m \longrightarrow \Omega^{m-1}\mathbb{M} \oplus \Omega^m\mathbb{M} \oplus \Omega^{m-1}\mathbb{M} \oplus \Omega^m\mathbb{M}$$

is a linear bijection.

Proof. Observe that to prove well-definedness of $\widetilde{\Phi}$, in view of Lemma (5.2.2), we only need to show that for $0 \leq k \leq n-1$, if $M_{\omega_k} \circ d$ is zero then $\omega_k = 0$. In a co-ordinate neighbourhood around a point $p \in \mathbb{M}$, suppose $\omega_k = \sum_{i_1 < \ldots < i_k} f_{i_1,\ldots,i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$. Since $k \leq n-1$, there always exist $j \notin \{i_1, \ldots, i_k\}$ and we have $\omega_k \wedge dx^j = 0$ i,e. $\sum_{i_1 < \ldots < i_k} f_{i_1,\ldots,i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^j = 0$ at each point of the co-ordinate neighbourhood around $p \in \mathbb{M}$. This will show that each $f_{i_1,\ldots,i_k}$ is zero showing $\omega_k = 0$. Injectivity of $\widetilde{\Phi}$ easily follows from Lemma (5.2.2). To see surjectivity, choose $(\omega_{m-1}, \omega_m) \in \Omega^{m-1}\mathbb{M} \oplus \Omega^m\mathbb{M}$. Let in a co-ordinate neighbourhood

$$\omega_m = \textstyle\sum_{i_1 < \ldots < i_m} f_{i_1 \ldots i_m} dx^{i_1} \wedge \ldots \wedge dx^{i_m}$$

$$\omega_{m-1} = \sum_{j_1<\ldots<j_{m-1}} g_{j_1\cdots j_{m-1}} dx^{j_1} \wedge \ldots \wedge dx^{j_{m-1}}$$

with support of $f_{i_1\ldots i_m}$, $g_{j_1\ldots j_{m-1}}$ in that neighbourhood. Then

$$\sum T_{g_{j_1\cdots j_{m-1}},x^{j_1},\ldots,x^{j_{m-1}},1} \longmapsto (\omega_{m-1},0)$$

$$\sum T_{f_{i_1\ldots i_m},x^{i_1},\ldots,x^{i_m}} \longmapsto (\sum f_{i_1\ldots i_m} x^{i_m} dx^{i_1} \wedge \ldots \wedge dx^{i_{m-1}}, \omega_m)$$

This shows that

$$\begin{aligned}\widetilde{\Phi}^{-1}(\omega_{m-1},\omega_m) &= \sum T_{f_{i_1\ldots i_m},x^{i_1},\ldots,x^{i_m}} + \sum T_{g_{j_1\cdots j_{m-1}},x^{j_1},\ldots,x^{j_{m-1}},1} \\ &\quad - \sum T_{f_{i_1\ldots i_m}x^{i_m},x^{i_1},\ldots,x^{i_{m-1}},1}\end{aligned}$$

and containment of support of the functions $f_{i_1\ldots i_m}$ and $g_{j_1\ldots j_{m-1}}$ in the co-ordinate neighbourhood fulfills our claim. $\square$

Lemma 5.2.4. *For all $m \geq n+1$, where $n = dim(\mathbb{M})$, $\mathcal{M}_m = \{0\}$.*

Proof. Note that for any $\omega \in \Gamma(\wedge^{\bullet}_{\mathbb{C}} T^*\mathbb{M})$,

$$T_{a_0,\ldots,a_m}(\omega) := \left(M_{a_0 a_m da_1\wedge\ldots\wedge da_{m-1}} \circ d + M_{a_0 da_1\wedge\ldots\wedge da_m}\right)(\omega)\,.$$

Since $m \geq n+1$, it follows that $\mathcal{M}_m = \{0\}$ because $\Omega^k\mathbb{M} = \Gamma(\wedge^k_{\mathbb{C}} T^*\mathbb{M}) = \{0\}$ for all $k > n$. $\square$

Proposition 5.2.5. $\Omega^{m-1}\mathbb{M} \oplus \Omega^m\mathbb{M} \oplus \Omega^{m-1}\mathbb{M} \oplus \Omega^m\mathbb{M}$ *has a $\widetilde{C^\infty(\mathbb{M})}$-bimodule structure given by,*

$$\begin{pmatrix}\phi & 0\\ 0 & \psi\end{pmatrix} . (\omega_{m-1},\omega_m,\widetilde{\omega}_{m-1},\widetilde{\omega}_m) \quad := \quad (\phi\omega_{m-1},\phi\omega_m,\psi\widetilde{\omega}_{m-1},\psi\widetilde{\omega}_m)$$

$$(\omega_{m-1},\omega_m,\widetilde{\omega}_{m-1},\widetilde{\omega}_m)\,.\begin{pmatrix}\phi & 0\\ 0 & \psi\end{pmatrix}$$

$$:=\begin{cases}(\phi\omega_{m-1},\phi\omega_m-d\phi\wedge\omega_{m-1},\psi\widetilde{\omega}_{m-1},\psi\widetilde{\omega}_m-d\psi\wedge\widetilde{\omega}_{m-1})\ ;\ \ if\ m\ even\\ (\psi\omega_{m-1},\psi\omega_m+d\psi\wedge\omega_{m-1},\phi\widetilde{\omega}_{m-1},\phi\widetilde{\omega}_m+d\phi\wedge\widetilde{\omega}_{m-1})\ ;\ \ if\ m\ odd\end{cases}$$

Proof. In the co-ordinate chart,

$$\omega_{m-1}=\sum\nolimits_{j_1<\ldots<j_{m-1}} g_{j_1\ldots j_{m-1}}dx^{j_1}\wedge\ldots\wedge dx^{j_{m-1}}$$

$$\omega_m=\sum\nolimits_{i_1<\ldots<i_m} f_{i_1\ldots i_m}dx^{i_1}\wedge\ldots\wedge dx^{i_m}$$

$$\widetilde{\omega}_{m-1}=\sum\nolimits_{j_1<\ldots<j_{m-1}} \widetilde{g_{j_1\ldots j_{m-1}}}dx^{j_1}\wedge\ldots\wedge dx^{j_{m-1}}$$

$$\widetilde{\omega}_m=\sum\nolimits_{i_1<\ldots<i_m} \widetilde{f_{i_1\ldots i_m}}dx^{i_1}\wedge\ldots\wedge dx^{i_m}\,.$$

Alos let

$$\begin{aligned}\xi\ &=\ \sum T_{f_{i_1\ldots i_m},x^{i_1},\ldots,x^{i_m}}+\sum T_{g_{j_1\cdots j_{m-1}},x^{j_1},\ldots,x^{j_{m-1}},1}\\ &\quad-\sum T_{f_{i_1\ldots i_m}x^{i_m},x^{i_1},\ldots,x^{i_{m-1}},1}\end{aligned}$$

and

$$\begin{aligned}\widetilde{\xi}\ &=\ \sum T_{\widetilde{f_{i_1\ldots i_m}},x^{i_1},\ldots,x^{i_m}}+\sum T_{\widetilde{g_{j_1\cdots j_{m-1}}},x^{j_1}\ldots,x^{j_{m-1}},1}\\ &\quad-\sum T_{\widetilde{f_{i_1\ldots i_m}}x^{i_m},x^{i_1},\ldots,x^{i_{m-1}},1}\end{aligned}$$

Define,

$$\begin{aligned}&\begin{pmatrix}\phi & 0\\ 0 & \psi\end{pmatrix}\,.\,(\omega_{m-1},\omega_m,\widetilde{\omega}_{m-1},\widetilde{\omega}_m)\\ :=\ \ &\Phi\left(\begin{pmatrix}\phi & 0\\ 0 & \psi\end{pmatrix}\,.\,\Phi^{-1}(\omega_{m-1},\omega_m,\widetilde{\omega}_{m-1},\widetilde{\omega}_m)\right)\end{aligned}$$

$$
= \begin{cases} \Phi\left(\begin{pmatrix}\phi & 0\\ 0 & \psi\end{pmatrix} . \begin{pmatrix}\xi & 0\\ 0 & \widetilde{\xi}\end{pmatrix}\right) & ; if\ m\ even \\ \Phi\left(\begin{pmatrix}\phi & 0\\ 0 & \psi\end{pmatrix} . \begin{pmatrix}0 & \xi\\ \widetilde{\xi} & 0\end{pmatrix}\right) & ; if\ m\ odd \end{cases}
$$

$$
= (\phi\omega_{m-1}, \phi\omega_m, \psi\widetilde{\omega}_{m-1}, \psi\widetilde{\omega}_m)\ ;\ for\ both\ even\ and\ odd\ m\,.
$$

and

$$
\begin{aligned}
&(\omega_{m-1}, \omega_m, \widetilde{\omega}_{m-1}, \widetilde{\omega}_m) . \begin{pmatrix}\phi & 0\\ 0 & \psi\end{pmatrix} \\
:= \ & \Phi\left(\Phi^{-1}(\omega_{m-1}, \omega_m, \widetilde{\omega}_{m-1}, \widetilde{\omega}_m) . \begin{pmatrix}\phi & 0\\ 0 & \psi\end{pmatrix}\right) \\
= \ & \begin{cases} \Phi\left(\begin{pmatrix}\xi & 0\\ 0 & \widetilde{\xi}\end{pmatrix} . \begin{pmatrix}\phi & 0\\ 0 & \psi\end{pmatrix}\right) & ; if\ m\ even \\ \Phi\left(\begin{pmatrix}0 & \xi\\ \widetilde{\xi} & 0\end{pmatrix} . \begin{pmatrix}\phi & 0\\ 0 & \psi\end{pmatrix}\right) & ; if\ m\ odd \end{cases} \\
= \ & \begin{cases} \Phi\left(\begin{pmatrix}\xi\phi & 0\\ 0 & \widetilde{\xi}\psi\end{pmatrix}\right) & ; if\ m\ even \\ \Phi\left(\begin{pmatrix}0 & \xi\psi\\ \widetilde{\xi}\phi & 0\end{pmatrix}\right) & ; if\ m\ odd \end{cases}
\end{aligned}
$$

where Φ is the map defined in Lemma (5.2.3). Now

$$
\begin{aligned}
\xi\phi \ = \ & \sum T_{f_{i_1 \cdots i_m}, x^{i_1}, \ldots, x^{i_m}\phi} + \sum T_{g_{j_1 \cdots j_{m-1}}, x^{j_1}, \ldots, x^{j_{m-1}}, \phi} \\
& - \sum T_{f_{i_1 \ldots i_m} x^{i_m}, x^{i_1}, \ldots, x^{i_{m-1}}, \phi}\,,
\end{aligned}
$$

and

$$\begin{aligned}\widetilde{\xi}\psi &= \sum T_{\widetilde{f_{i_1\ldots i_m}},x^{i_1},\ldots,x^{i_m}\psi} + \sum T_{\widetilde{g_{j_1\ldots j_{m-1}}},x^{j_1},\ldots,x^{j_{m-1}},\psi} \\ &\quad - \sum T_{\widetilde{f_{i_1\ldots i_m}}x^{i_m},x^{i_1},\ldots,x^{i_{m-1}},\psi}\,.\end{aligned}$$

So,

$$\begin{aligned}&\Phi\left(\begin{pmatrix}\xi\phi & 0\\ 0 & \widetilde{\xi}\psi\end{pmatrix}\right)\\
=\;& \Big(\sum f_{i_1\ldots i_m}x^{i_m}\phi dx^{i_1}\wedge\ldots\wedge dx^{i_{m-1}} - f_{i_1\ldots i_m}x^{i_m}\phi dx^{i_1}\wedge\ldots\wedge dx^{i_{m-1}}\\
&+\phi\omega_{m-1}\ ,\sum f_{i_1\ldots i_m}dx^{i_1}\wedge\ldots dx^{i_{m-1}}\wedge d(x^{i_m}\phi)+\omega_{m-1}\wedge d\phi\\
&-f_{i_1\ldots i_m}x^{i_m}dx^{i_1}\wedge\ldots dx^{i_{m-1}}\wedge d\phi\ ,\sum \widetilde{f_{i_1\ldots i_m}}x^{i_m}\psi dx^{i_1}\wedge\ldots\wedge dx^{i_{m-1}}\\
&-\widetilde{f_{i_1\ldots i_m}}x^{i_m}\psi dx^{i_1}\wedge\ldots\wedge dx^{i_{m-1}}+\psi\widetilde{\omega}_{m-1}\ ,\\
&\sum \widetilde{f_{i_1\ldots i_m}}dx^{i_1}\wedge\ldots\wedge dx^{i_{m-1}}\wedge d(x^{i_m}\psi)+\widetilde{\omega}_{m-1}\wedge d\psi\\
&-\widetilde{f_{i_1\ldots i_m}}x^{i_m}\psi dx^{i_1}\wedge\ldots\wedge dx^{i_{m-1}}\wedge d\psi\,\Big)\\
=\;& (\phi\omega_{m-1},\phi\omega_m-d\phi\wedge\omega_{m-1},\psi\widetilde{\omega}_{m-1},\psi\widetilde{\omega}_m-d\psi\wedge\widetilde{\omega}_{m-1})\end{aligned}$$

Similarly one can prove that

$$\begin{aligned}&\Phi\left(\begin{pmatrix}0 & \xi\psi\\ \widetilde{\xi}\phi & 0\end{pmatrix}\right)\\
=\;&(\psi\omega_{m-1},\psi\omega_m+\omega_{m-1}\wedge d\psi,\phi\widetilde{\omega}_{m-1},\phi\widetilde{\omega}_m+\widetilde{\omega}_{m-1}\wedge d\phi)\\
=\;&(\psi\omega_{m-1},\psi\omega_m+d\psi\wedge\omega_{m-1},\phi\widetilde{\omega}_{m-1},\phi\widetilde{\omega}_m+d\phi\wedge\widetilde{\omega}_{m-1})\end{aligned}$$

This is clearly a bimodule structure since it is induced by that on $\Omega^m_D\left(\widetilde{C^\infty(\mathbb{M})}\right)$. $\square$

Notation : $\widetilde{\Omega^m_D} := \Omega^{m-1}\mathbb{M}\oplus\Omega^m\mathbb{M}\oplus\Omega^{m-1}\mathbb{M}\oplus\Omega^m\mathbb{M}$, $1\le m\le n$, untill the end this section, where $\Omega^\bullet\mathbb{M}$ denotes the space of forms on $\mathbb{M}$.

Theorem 5.2.6. *$\Omega_D^m\left(\widetilde{C^\infty(\mathbb{M})}\right) \cong \widetilde{\Omega_D^m}$, for all $1 \le m \le n$, and $\Omega_D^m\left(\widetilde{C^\infty(\mathbb{M})}\right) = \{0\}$ for $m > n$. This isomorphism is a $\widetilde{C^\infty(\mathbb{M})}$-bimodule isomorphism.*

Proof. We have for all $1 \le m \le n$,

$$\begin{aligned}\Omega_D^m\left(\widetilde{C^\infty(\mathbb{M})}\right) &\cong \pi\left(\Omega^m(\widetilde{C^\infty(\mathbb{M})}\right) && (\textit{by Lemma } 5.2.1)\\ &\cong \Omega^{m-1}\mathbb{M}\oplus\Omega^m\mathbb{M}\oplus\Omega^{m-1}\mathbb{M}\oplus\Omega^m\mathbb{M} && (\textit{by Lemma } 5.2.3).\end{aligned}$$

Lemma (5.2.4) proves that $\Omega_D^m\left(\widetilde{C^\infty(\mathbb{M})}\right) = \{0\}$ for $m > n$. Finally, Proposition (5.2.5) proves that this isomorphism is $\widetilde{C^\infty(\mathbb{M})}$-bimodule isomorphism for all $1 \le m \le n$. $\square$

Now we will turn $\widetilde{\Omega_D^\bullet}$ into a chain complex.

Lemma 5.2.7. *The differential $\widetilde{d} : \Omega_D^m\left(\widetilde{C^\infty(\mathbb{M})}\right) \longrightarrow \Omega_D^{m+1}\left(\widetilde{C^\infty(\mathbb{M})}\right)$ of diagram (1.1.1) has the following action :*

1. *For $m \ge 1$ odd ,*

$$\widetilde{d} : \begin{pmatrix} 0 & T_{a_0,\dots,a_m} \\ T_{a'_0,\dots,a'_m} & 0 \end{pmatrix} \longmapsto \begin{pmatrix} T_{1,a'_0,\dots,a'_m} + T_{a_0,\dots,a_m,1} & 0 \\ 0 & T_{1,a_0,\dots,a_m} + T_{a'_0,\dots,a'_m,1} \end{pmatrix}$$

2. *For $m \ge 2$ even,*

$$\widetilde{d} : \begin{pmatrix} T_{a_0,\dots,a_m} & 0 \\ 0 & T_{a'_0,\dots,a'_m} \end{pmatrix} \longmapsto \begin{pmatrix} 0 & T_{1,a'_0,\dots,a'_m} - T_{a_0,\dots,a_m,1} \\ T_{1,a_0,\dots,a_m} - T_{a'_0,\dots,a'_m,1} & 0 \end{pmatrix}$$

Proof. We first note that

$$\left[D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 0 & T_{1,1} \\ -T_{1,1} & 0 \end{pmatrix} \tag{5.2.2}$$

$$\begin{pmatrix} 0 & T_{1,1} \\ -T_{1,1} & 0 \end{pmatrix}\begin{pmatrix} -a'_1 & 0 \\ 0 & a_1 \end{pmatrix} = \begin{pmatrix} 0 & T_{1,a_1} \\ T_{1,a'_1} & 0 \end{pmatrix} \tag{5.2.3}$$

$$\begin{pmatrix} a_0 & 0 \\ 0 & a'_0 \end{pmatrix} \begin{pmatrix} 0 & T_{1,a_1} \\ T_{1,a'_1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & T_{a_0,a_1} \\ T_{a'_0,a'_1} & 0 \end{pmatrix} \tag{5.2.4}$$

Hence combining these three we get,

$$\begin{pmatrix} a_0 & 0 \\ 0 & a'_0 \end{pmatrix} \left[D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} -a'_1 & 0 \\ 0 & a_1 \end{pmatrix} = \begin{pmatrix} 0 & T_{a_0,a_1} \\ T_{a'_0,a'_1} & 0 \end{pmatrix} \tag{5.2.5}$$

<u>Case 1 :</u> Let $m \geq 3$ be odd. Observe that

$$\begin{aligned}
&\begin{pmatrix} 0 & T_{a_0,a_1,\ldots,a_m} \\ T_{a'_0,a'_1,\ldots,a'_m} & 0 \end{pmatrix} \\
= &\begin{pmatrix} 0 & T_{a_0,a_1} \\ T_{a'_0,a'_1} & 0 \end{pmatrix} \prod_{i=2,i\ even}^{m} \left(\begin{pmatrix} 0 & T_{1,a'_i} \\ T_{1,a_i} & 0 \end{pmatrix} \begin{pmatrix} 0 & T_{1,a_{i+1}} \\ T_{1,a'_{i+1}} & 0 \end{pmatrix} \right) \\
= &\begin{pmatrix} a_0 & 0 \\ 0 & a'_0 \end{pmatrix} \left[D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} -a'_1 & 0 \\ 0 & a_1 \end{pmatrix} \bullet \\
&\prod_{i=2,i\ even}^{m} \left(\left\{ \left[D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} -a_i & 0 \\ 0 & a'_i \end{pmatrix} \right\} \left\{ \left[D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} -a'_{i+1} & 0 \\ 0 & a_{i+1} \end{pmatrix} \right\} \right)
\end{aligned}$$

Consider the expression $\eta = x_0 \prod_{i=1}^{m} \left(\bar{d}(b) x_i \right)$ where,

$$x_0 = \begin{pmatrix} a_0 & 0 \\ 0 & a'_0 \end{pmatrix} \ ; \ b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ ; \ \bar{d}(y) = \left[D, \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \right]$$

$$x_i = \begin{pmatrix} -a'_i & 0 \\ 0 & a_i \end{pmatrix} \ for\ 1 \leq i \leq m\,, odd \ ; \ x_j = \begin{pmatrix} -a_j & 0 \\ 0 & a'_j \end{pmatrix} \ for\ 1 \leq j \leq m\,, even.$$

One should note that $\bar{d} \circ \bar{d}(b) = 0$ because $d^2 = 0$, d being the exterior differentiation.

Now for the differential $\widetilde{d}:\Omega_D^m\left(\widetilde{C^\infty(\mathbb{M})}\right)\longrightarrow\Omega_D^{m+1}\left(\widetilde{C^\infty(\mathbb{M})}\right)$ of diagram 1.1.1 we get,

$$
\begin{aligned}
\widetilde{d}\eta &= \bar{d}(x_0)\prod_{i=1}^{m}\{\bar{d}(b)x_i\}+x_0\widetilde{d}\left(\bar{d}(b)x_1\prod_{i=2}^{m}\{\bar{d}(b)x_i\}\right)\\
&= \bar{d}(x_0)\prod_{i=1}^{m}\{\bar{d}(b)x_i\}+\sum_{k=2}^{m}(-1)^{k-1}\prod_{j=0}^{k-2}\{x_j\bar{d}(b)\}\bar{d}(x_{k-1})\left(\prod_{i=k}^{m}\{\bar{d}(b)x_i\}\right)\\
&\quad +(-1)^m\left(\prod_{i=0}^{m-1}\{x_i\bar{d}(b)\}\right)\bar{d}(x_m)\\
&= \left[D,\begin{pmatrix} a_0 & 0\\ 0 & a_0'\end{pmatrix}\right]\left[D,\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\right]\begin{pmatrix} -a_1' & 0\\ 0 & a_1\end{pmatrix}\prod_{i=2,i\ even}^{m}\left(\begin{pmatrix} 0 & T_{1,a_i'}\\ T_{1,a_i} & 0\end{pmatrix}\begin{pmatrix} 0 & T_{1,a_{i+1}}\\ T_{1,a_{i+1}'} & 0\end{pmatrix}\right)\\
&\quad +\sum_{k=2}^{m}(-1)^{k-1}\prod_{j=0}^{k-2}\{x_j\bar{d}(b)\}\bar{d}(x_{k-1})\left(\prod_{i=k}^{m}\{\bar{d}(b)x_i\}\right)\\
&\quad +(-1)^m\begin{pmatrix} a_0 & 0\\ 0 & a_0'\end{pmatrix}\left[D,\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\right]\prod_{i=1,i\ odd}^{m-1}\boldsymbol{\{}\begin{pmatrix} -a_i' & 0\\ 0 & a_i\end{pmatrix}\left[D,\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\right]\\
&\quad \begin{pmatrix} -a_{i+1} & 0\\ 0 & a_{i+1}'\end{pmatrix}\left[D,\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\right]\boldsymbol{\}}\left[D,\begin{pmatrix} -a_m' & 0\\ 0 & a_m\end{pmatrix}\right]
\end{aligned}
$$

Now it is straightforward computation to observe that

$$
\begin{aligned}
\prod_{i=k}^{m}\{\bar{d}(b)x_i\} &= \prod_{i=k}^{m}\left(\begin{pmatrix} 0 & T_{1,1}\\ -T_{1,1} & 0\end{pmatrix}x_i\right)\\
&= \begin{cases}\begin{pmatrix} T_{1,a_k',\ldots,a_m'} & 0\\ 0 & T_{1,a_k,\ldots,a_m}\end{pmatrix} & ;\ if\ k\ even\\ \begin{pmatrix} 0 & T_{1,a_k,\ldots,a_m}\\ T_{1,a_k',\ldots,a_m'} & 0\end{pmatrix} & ;\ if\ k\ odd\end{cases}
\end{aligned}
$$

and

$$\prod_{j=0}^{k-2}\{x_j\bar{d}(b)\}\bar{d}(x_{k-1}) \;=\; \begin{cases} \begin{pmatrix} -T_{a_0,\ldots,a_{k-1},1} & 0 \\ 0 & -T_{a'_0,\ldots,a'_{k-1},1} \end{pmatrix} & ;\ if\ k\ even \\ \begin{pmatrix} 0 & -T_{a_0,\ldots,a_{k-1},1} \\ -T_{a'_0,\ldots,a'_{k-1},1} & 0 \end{pmatrix} & ;\ if\ k\ odd \end{cases}$$

The fact that $d^2 = 0$ will now ensure that only the first and last term in the expression for $\widetilde{d}\eta$ survive. Hence,

$$\begin{aligned} \widetilde{d}\eta &= \begin{pmatrix} (T_{1,a'_0} - T_{a_0,1})T_{1,a'_1,\ldots,a'_m} & 0 \\ 0 & (T_{1,a_0} - T_{a'_0,1})T_{1,a_1,\ldots,a_m} \end{pmatrix} + (-1)^m \begin{pmatrix} -T_{a_0,\ldots,a_m,1} & 0 \\ 0 & -T_{a'_0,\ldots,a'_m,1} \end{pmatrix} \\ &= \begin{pmatrix} T_{1,a'_0,\ldots,a'_m} + T_{a_0,\ldots,a_m,1} & 0 \\ 0 & T_{1,a_0,\ldots,a_m} + T_{a'_0,\ldots,a'_m,1} \end{pmatrix}. \end{aligned}$$

<u>Case 2 :</u> Let m be even.

One can prove this in exactly the same way as in the 'odd' case. The only difference in this case is a negative sign and it appears because of the presence of $(-1)^m$ at the last term in the expression for $\widetilde{d}\eta$.

<u>Case 3 :</u> Let $m = 1$. Recall from equation (5.2.5),

$$\begin{pmatrix} 0 & T_{a_0,a_1} \\ T_{a'_0,a'_1} & 0 \end{pmatrix} = \begin{pmatrix} a_0 & 0 \\ 0 & a'_0 \end{pmatrix} \left[D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} -a'_1 & 0 \\ 0 & a_1 \end{pmatrix}$$

and hence,

$$\widetilde{d} : \begin{pmatrix} 0 & T_{a_0,a_1} \\ T_{a'_0,a'_1} & 0 \end{pmatrix} \longmapsto \begin{pmatrix} T_{1,a'_0,a'_1} + T_{a_0,a_1,1} & 0 \\ 0 & T_{1,a_0,a_1} + T_{a'_0,a'_1,1} \end{pmatrix}$$

$\square$

Using the isomorphism in Theorem (5.2.6) we can transfer Connes' differential $\widetilde{d} : \Omega_D^\bullet\left(\widetilde{C^\infty(\mathbb{M})}\right) \longrightarrow \Omega_D^{\bullet+1}\left(\widetilde{C^\infty(\mathbb{M})}\right)$ to the differential $\delta : \widetilde{\Omega_D^\bullet} \longrightarrow \widetilde{\Omega_D^{\bullet+1}}$. This will turn

$\widetilde{\Omega_D^\bullet}$ into a chain complex and then we will be able to compute the cohomologies of the complex $\left(\widetilde{\Omega_D^\bullet}\,,\,\delta\right)$.

Proposition 5.2.8. *For $1 \leq m \leq n$, the map*

$$\delta : \widetilde{\Omega_D^m} \longrightarrow \widetilde{\Omega_D^{m+1}}$$

$$(\omega_{m-1}, \omega_m, \widetilde{\omega}_{m-1}, \widetilde{\omega}_m) \longmapsto$$
$$(d\widetilde{\omega}_{m-1} + (-1)^m(\widetilde{\omega}_m - \omega_m)\,,\, d\widetilde{\omega}_m\,,\, d\omega_{m-1} + (-1)^m(\omega_m - \widetilde{\omega}_m)\,,\, d\omega_m)$$

makes the following diagram

$$\begin{array}{ccc} \Omega_D^m\left(\widetilde{C^\infty(\mathbb{M})}\right) & \xrightarrow{\widetilde{d}} & \Omega_D^{m+1}\left(\widetilde{C^\infty(\mathbb{M})}\right) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \widetilde{\Omega_D^m} & \xrightarrow[\delta]{} & \widetilde{\Omega_D^{m+1}} \end{array}$$

commutative.

Proof. For $1 \leq m \leq n$ take $(\omega_{m-1}, \omega_m, \widetilde{\omega}_{m-1}, \widetilde{\omega}_m) \in \widetilde{\Omega_D^m}$. In terms of local co-ordinates

$$\omega_{m-1} = \sum_{j_1<\ldots<j_{m-1}} g_{j_1\ldots j_{m-1}} dx^{j_1} \wedge \ldots \wedge dx^{j_{m-1}}$$

$$\omega_m = \sum_{i_1<\ldots<i_m} f_{i_1\ldots i_m} dx^{i_1} \wedge \ldots \wedge dx^{i_m}$$

$$\widetilde{\omega}_{m-1} = \sum_{j_1<\ldots<j_{m-1}} \widetilde{g_{j_1\ldots j_{m-1}}} dx^{j_1} \wedge \ldots \wedge dx^{j_{m-1}}$$

$$\widetilde{\omega}_m = \sum_{i_1<\ldots<i_m} \widetilde{f_{i_1\ldots i_m}} dx^{i_1} \wedge \ldots \wedge dx^{i_m}$$

Using Lemma (5.2.3) we see that, isomorphic image of this element in $\Omega_D^m\left(\widetilde{C^\infty(\mathbb{M})}\right)$ is

$$\begin{cases} \begin{pmatrix} \xi & 0 \\ 0 & \widetilde{\xi} \end{pmatrix} & ; \, if \;\, m \; even \\ \begin{pmatrix} 0 & \xi \\ \widetilde{\xi} & 0 \end{pmatrix} & ; \, if \;\, m \; odd \end{cases}$$

where,

$$\begin{aligned}\xi &= \sum T_{f_{i_1\cdots i_m},x^{i_1},\ldots,x^{i_m}} - T_{f_{i_1\cdots i_m}x^{i_m},x^{i_1},\ldots,x^{i_{m-1}},1} \\ &\quad + \sum T_{g_{j_1\cdots j_{m-1}},x^{j_1},\ldots,x^{j_{m-1}},1}\end{aligned}$$

and

$$\begin{aligned}\widetilde{\xi} &= \sum T_{\widetilde{f_{i_1\cdots i_m}},x^{i_1},\ldots,x^{i_m}} - T_{\widetilde{f_{i_1\cdots i_m}}x^{i_m},x^{i_1},\ldots,x^{i_{m-1}},1} \\ &\quad + \sum T_{\widetilde{g_{j_1\cdots j_{m-1}}},x^{j_1},\ldots,x^{j_{m-1}},1}\end{aligned}$$

By Lemma (5.2.7) we see that the differential $\widetilde{d} : \Omega_D^m\left(\widetilde{C^\infty(\mathbb{M})}\right) \longrightarrow \Omega_D^{m+1}\left(\widetilde{C^\infty(\mathbb{M})}\right)$ sends this element to

$$\begin{cases} \begin{pmatrix} d_{\widetilde{\xi}} + \xi\, T_{1,1} & 0 \\ 0 & d_\xi + \widetilde{\xi}\, T_{1,1} \end{pmatrix} & ;\, if\ m\ odd \\ \begin{pmatrix} 0 & d_{\widetilde{\xi}} - \xi\, T_{1,1} \\ d_\xi - \widetilde{\xi}\, T_{1,1} & 0 \end{pmatrix} & ;\, if\ m\ even \end{cases}$$

where

$$\begin{aligned}d_\xi &= \sum T_{1,f_{i_1\cdots i_m},x^{i_1},\ldots,x^{i_m}} - T_{1,f_{i_1\cdots i_m}x^{i_m},x^{i_1},\ldots,x^{i_{m-1}},1} \\ &\quad + \sum T_{1,g_{j_1\cdots j_{m-1}},x^{j_1},\ldots,x^{j_{m-1}},1}\end{aligned}$$

and

$$\begin{aligned}d_{\widetilde{\xi}} &= \sum T_{1,\widetilde{f_{i_1\cdots i_m}},x^{i_1},\ldots,x^{i_m}} - T_{1,\widetilde{f_{i_1\cdots i_m}}x^{i_m},x^{i_1},\ldots,x^{i_{m-1}},1} \\ &\quad + \sum T_{1,\widetilde{g_{j_1\cdots j_{m-1}}},x^{j_1},\ldots,x^{j_{m-1}},1}\end{aligned}$$

The isomophic image of this element in $\widetilde{\Omega_D^{m+1}}$, under the map Φ of Lemma 5.2.3, is

$$\begin{cases} (d\widetilde{\omega}_{m-1} - \widetilde{\omega}_m + \omega_m\,,\, d\widetilde{\omega}_m\,,\, d\omega_{m-1} - \omega_m + \widetilde{\omega}_m\,,\, d\omega_m)\ ;\ if\ m\ odd \\ (d\widetilde{\omega}_{m-1} + \widetilde{\omega}_m - \omega_m\,,\, d\widetilde{\omega}_m\,,\, d\omega_{m-1} + \omega_m - \widetilde{\omega}_m\,,\, d\omega_m)\ ;\ if\ m\ even \end{cases}$$

i,e. $(d\widetilde{\omega}_{m-1} + (-1)^m(\widetilde{\omega}_m - \omega_m)\,,\, d\widetilde{\omega}_m\,,\, d\omega_{m-1} + (-1)^m(\omega_m - \widetilde{\omega}_m)\,,\, d\omega_m)\,.$ □

Remark 5.2.9. *Notice that* $\delta = \Phi \circ \widetilde{d} \circ \Phi^{-1}$, *and hence* $\delta^2 = 0$. *Thus* $\left(\widetilde{\Omega^\bullet_D}\,,\delta\right)$ *is a chain complex. Furthermore, the graded algebra structure on* $\Omega^\bullet_D\left(\widetilde{C^\infty(\mathbb{M})}\right)$ *will induce the same on* $\left(\widetilde{\Omega^\bullet_D}\,,\delta\right)$ *through the commutative diagram of Proposition* (5.2.8)*. So we get* $\left(\Omega^\bullet_D(\widetilde{C^\infty(\mathbb{M})}), \widetilde{d}\right) \cong \left(\widetilde{\Omega^\bullet_D}\,,\delta\right)$ *as differential graded algebras and Theorem* (5.2.6) *gives* $\widetilde{C^\infty(\mathbb{M})}$*-bimodule isomorphism at each term of these chain complexes.*

Theorem 5.2.10. *The cohomologies* $\widetilde{H^\bullet(\mathbb{M})}$ *of the chain complex* $\left(\widetilde{\Omega^\bullet_D}\,,\delta\right)$ *are given by*

$$\widetilde{H^m(\mathbb{M})} \cong H^{m-1}(\mathbb{M}) \oplus H^m(\mathbb{M})\,;\ for\ 0 \leq m \leq dim(\mathbb{M})\,,$$

where $H^\bullet(\mathbb{M})$ *denotes the de-Rham cohomology of* $\mathbb{M}$.

Proof. (1) Let $m = 0$. Recall that for $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \in \widetilde{C^\infty(\mathbb{M})}$,

$$\left[D, \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}\right] = \begin{pmatrix} 0 & T_{1,g} - T_{f,1} \\ T_{1,f} - T_{g,1} & 0 \end{pmatrix}.$$

The isomorphism of Lemma (5.2.3) sends this element to $(g - f, dg, f - g, df)$. Hence

$$\begin{aligned} \widetilde{H^0(\mathbb{M})} &= \left\{\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} : df = 0\,,\ f \in C^\infty(\mathbb{M})\right\} \\ &\cong H^0(\mathbb{M})\,. \end{aligned}$$

(2) Let $1 \leq m \leq dim(\mathbb{M})$. Consider $\delta^{m-1} : \widetilde{\Omega^{m-1}_D} \longrightarrow \widetilde{\Omega^m_D}$ and $\delta^m : \widetilde{\Omega^m_D} \longrightarrow \widetilde{\Omega^{m+1}_D}$. Then,

$$\begin{aligned} \delta^{m-1}(v_{m-2}, v_{m-1}, \widetilde{v_{m-2}}, \widetilde{v_{m-1}}) &= (d\widetilde{v_{m-2}} + (-1)^{m-1}(\widetilde{v_{m-1}} - v_{m-1}), d\widetilde{v_{m-1}}, \\ &\quad dv_{m-2} + (-1)^{m-1}(v_{m-1} - \widetilde{v_{m-1}}), dv_{m-1}) \quad (5.2.6) \end{aligned}$$

for all $(v_{m-2}, v_{m-1}, \widetilde{v_{m-2}}, \widetilde{v_{m-1}}) \in \widetilde{\Omega_D^{m-1}}$. Let $\zeta = (w_{m-1}, w_m, \widetilde{w_{m-1}}, \widetilde{w_m}) \in Ker(\delta^m)$. Then we have the following

$$\begin{cases} d(w_m) = 0 & \\ & ; \\ d(\widetilde{w_m}) = 0 & \end{cases} \quad \begin{array}{l} d(\widetilde{w_{m-1}}) + (-1)^m(\widetilde{w_m} - w_m) = 0 \\ \\ d(w_{m-1}) + (-1)^m(w_m - \widetilde{w_m}) = 0 \end{array} \tag{5.2.7}$$

Define

$$\Psi : \frac{Ker(\delta^m)}{Im(\delta^{m-1})} \longrightarrow H^m(\mathbb{M}) \oplus H^{m-1}(\mathbb{M})$$

$$[\zeta] \longmapsto \left([w_m + \widetilde{w_m}], [w_{m-1} + \widetilde{w_{m-1}}]\right).$$

This map is well-defined (because of equation 5.2.7) and linear. Now define

$$\Phi : H^m(\mathbb{M}) \oplus H^{m-1}(\mathbb{M}) \longrightarrow \frac{Ker(\delta^m)}{Im(\delta^{m-1})}$$

$$([v_m], [v_{m-1}]) \longmapsto \left[\left(\tfrac{1}{2}v_{m-1}, \tfrac{1}{2}v_m, \tfrac{1}{2}v_{m-1}, \tfrac{1}{2}v_m\right)\right].$$

Using equation (5.2.6) one can check that Φ is well-defined and linear. Now observe that $\Psi \circ \Phi = Id$, and

$$\Phi \circ \Psi([\zeta]) = \left[\left(\tfrac{1}{2}(w_{m-1} + \widetilde{w_{m-1}}), \tfrac{1}{2}(w_m + \widetilde{w_m}), \tfrac{1}{2}(w_{m-1} + \widetilde{w_{m-1}}), \tfrac{1}{2}(w_m + \widetilde{w_m})\right)\right].$$

If we can show that

$$\xi = \left(\tfrac{1}{2}(\widetilde{w_{m-1}} - w_{m-1}), \tfrac{1}{2}(\widetilde{w_m} - w_m), \tfrac{1}{2}(w_{m-1} - \widetilde{w_{m-1}}), \tfrac{1}{2}(w_m - \widetilde{w_m})\right) \in Im(\delta^{m-1}),$$

then $\Phi \circ \Psi$ will also be the identity. Observe that

$$\delta^{m-1}\left(0, \tfrac{(-1)^{m+1}}{4}(w_{m-1} - \widetilde{w_{m-1}}), 0, \tfrac{(-1)^{m+1}}{4}(\widetilde{w_{m-1}} - w_{m-1})\right) = \xi$$

using equation (5.2.7), and hence (2) follows. $\square$

5.3 Computation for the Noncommutative Torus

In this section our objective is to show that the functor $\mathcal{F} \circ \mathcal{G}$ is not trivial for the case of noncommutative torus, one of the most fundamental and widely studied example in noncommutative geometry. Recall the general definition of the noncommutative torus from ([36]). However, for our purpose we consider the following definition of the noncommutative torus.

Definition 5.3.1. *Let θ be a real number. The noncommutative torus, denoted by $\mathcal{A}_\Theta$, is the universal $\star$-algebra generated by U, V subject to the following relations*

$$UU^* = U^*U = 1 \quad , \quad VV^* = V^*V = 1\,,$$
$$UV = e^{-2\pi i\theta} VU\,.$$

There is a nondegenerate sesquilinear pairing on $\mathcal{A}_\Theta$ given by

$$\langle \sum a_{r_1,r_2} U^{r_1}V^{r_2}, \sum b_{s_1,s_2} U^{s_1}V^{s_2} \rangle := \sum_{r_1=s_1, r_2=s_2} \overline{a_{r_1,r_2}} b_{s_1,s_2}.$$

We have the following derivations acting on $\mathcal{A}_\Theta$,

$$\widetilde{\delta}_j \left(\sum_{r_1,r_2} a_{r_1,r_2} U^{r_1} V^{r_2} \right) := \sqrt{-1} \sum_{r_1,r_2} r_j a_{r_1,r_2} U^{r_1} V^{r_2} \quad for\ j = 1, 2\,.$$

Let $\delta_j := -\sqrt{-1}\,\widetilde{\delta}_j\,, j = 1, 2$. Throughout this section i will stand for $\sqrt{-1}$.
Our candidate for the *even algebraic spectral triple* is the following quadruple

$$\mathcal{E} := \left(\mathcal{A}_\Theta\,, \mathcal{A}_\Theta \otimes \mathbb{C}^2\,, D := \begin{pmatrix} 0 & \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 & 0 \end{pmatrix}, \gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right).$$

Here

$$\pi : \mathcal{A}_\Theta \longrightarrow \mathcal{E}nd(\mathcal{A}_\Theta \otimes \mathbb{C}^2)$$
$$a \longmapsto \begin{pmatrix} M_a & 0 \\ 0 & M_a \end{pmatrix}$$

where M_a denotes multiplication by a. Since $\gamma \notin \pi(\mathcal{A}_\Theta)$ we first apply the functor $\mathcal{G}$ of Proposition (5.1.9) and then compute $\mathcal{F} \circ \mathcal{G}$ along with the associated cohomologies. Note that

$$\delta_1(U) = U\,,\, \delta_1(V) = 0\,,\, \delta_2(U) = 0\,,\, \delta_2(V) = V\,.$$

We denote $d := \delta_1 - i\delta_2$ and $d^* := \delta_1 + i\delta_2$. Hence,

$$d(U) = U\,,\, d^*(U) = U\,,\, d(V) = -iV\,,\, d^*(V) = iV\,,$$

$$d(U^*) = -U^*\,,\, d^*(U^*) = -U^*\,,\, d(V^*) = iV^*\,,\, d^*(V^*) = -iV^*.$$

Notation : $\widetilde{\mathcal{A}_\Theta} = \mathcal{G}(\mathcal{E})$ throughout this section where $\mathcal{G}$ is as defined in Proposition (5.1.9).

Note that $J_0^0\left(\widetilde{\mathcal{A}_\Theta}\right) = \{0\}$ in this case. Conisder the following linear operators

$$T_{a,b} : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta$$

$$c \longmapsto ad(bc)$$

$$\widetilde{T_{a,b}} : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta$$

$$c \longmapsto ad^*(bc).$$

Now observe that

$$\left[D, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right] = \begin{pmatrix} 0 & T_{1,b} - T_{a,1} \\ \widetilde{T_{1,a}} - \widetilde{T_{b,1}} & 0 \end{pmatrix}, \tag{5.3.1}$$

and hence each element of $\pi\left(\Omega^1(\widetilde{\mathcal{A}_\Theta})\right)$ is the linear span of following elements :

$$\begin{pmatrix} 0 & T_{c,e} \\ \widetilde{T_{c',e'}} & 0 \end{pmatrix} \quad \text{such that } c, e, c', e' \in \mathcal{A}_\Theta.$$

Let $\mathcal{M}_1 := span\{T_{c,e} : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta \colon c, e \in \mathcal{A}_\Theta\}$ and $\widetilde{\mathcal{M}_1} := span\{\widetilde{T_{c',e'}} : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta \colon c', e' \in \mathcal{A}_\Theta\}$. Then $\mathcal{M}_1$ and $\widetilde{\mathcal{M}_1}$ are $\mathbb{C}$-vector spaces and using equation (5.3.1) we see that $\pi\left(\Omega^1(\widetilde{\mathcal{A}_\Theta})\right) \subseteq \mathcal{M}_1 \oplus \widetilde{\mathcal{M}_1}$. Now the following equality

$$\begin{pmatrix} 0 & T_{c,e} \\ \widetilde{T_{c',e'}} & 0 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & -c' \end{pmatrix} \left[D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} e' & 0 \\ 0 & e \end{pmatrix}$$

proves that $\pi\left(\Omega^1(\widetilde{\mathcal{A}_\Theta})\right) = \mathcal{M}_1 \oplus \widetilde{\mathcal{M}_1}$.

Lemma 5.3.2. *Let $\mathbb{V}$ be the vector space of linear endomorphisms acting on $\mathcal{A}_\Theta$. Let M_ξ denotes multiplication by ξ. The vector subspaces $\{M_{\sum c_i d(b_i)} : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta : c_i, b_i \in \mathcal{A}_\Theta\}$ and $\{M_e \circ d : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta : e \in \mathcal{A}_\Theta\}$ of $\mathbb{V}$ has trivial intersection and $\mathcal{M}_1 \subseteq \{M_{cd(b)} : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta\} \bigoplus \{M_{cb} \circ d : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta\}$.*

Proof. Observe that $T_{c,b}(e) = (M_{cd(b)} + M_{cb} \circ d)(e)$ for any $e \in \mathcal{A}_\Theta$. Since $d(1) = 0$ we have the direct sum. □

Lemma 5.3.3. *Let $\mathbb{V}$ be the vector space of linear endomorphisms acting on $\mathcal{A}_\Theta$. Let M_ξ denotes multiplication by ξ. The vector subspaces $\{M_{\sum c_i d^*(b_i)} : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta : c_i, b_i \in \mathcal{A}_\Theta\}$ and $\{M_e \circ d : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta : e \in \mathcal{A}_\Theta\}$ of $\mathbb{V}$ has trivial intersection and $\widetilde{\mathcal{M}_1} \subseteq \{M_{cd^*(b)} : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta\} \bigoplus \{M_{cb} \circ d : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta\}$.*

Proof. Observe that $\widetilde{T_{c,b}}(e) = (M_{cd^*(b)} + M_{cb} \circ d^*)(e)$ for any $e \in \mathcal{A}_\Theta$. Since $d^*(1) = 0$ we have the direct sum. □

Define

$$\Phi : \pi\left(\Omega^1(\widetilde{\mathcal{A}_\Theta})\right) \longrightarrow \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta$$

$$\begin{pmatrix} 0 & T_{a,b} \\ \widetilde{T_{a',b'}} & 0 \end{pmatrix} \longmapsto (ad(b)\,,\, ab\,,\, a'd^*(b')\,,\, a'b').$$

Lemma 5.3.4. Φ *is a linear bijection.*

Proof. To prove Φ is well-defined, let $\sum T_{a_i,b_i} = 0$. Acting it on $1 \in \mathcal{A}_\Theta$ and $U \in \mathcal{A}_\Theta$ respectively, we see that both $\sum a_i d(b_i)$ and $\sum a_i b_i$ are zero. Similarly for the case of $\sum \widetilde{T_{a'_i,b'_i}} = 0$. This proves well-definedness and Lemmas (5.3.2 , 5.3.3) proves injectivity. To see surjectivity, observe that

$$\begin{pmatrix} 0 & T_{aU^*,U} + T_{b,1} - T_{a,1} \\ \widetilde{T_{a'U^*,U}} + \widetilde{T_{b',1}} - \widetilde{T_{a',1}} & 0 \end{pmatrix} \xrightarrow{\Phi} (a, b, a', b').$$

□

Proposition 5.3.5. *$\mathcal{A}_\Theta \otimes \mathbb{C}^4$ is a $\widetilde{\mathcal{A}_\Theta}$-bimodule where the module action is specified by*

$$\begin{aligned} &\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} .(a, b, a', b'). \begin{pmatrix} f' & 0 \\ 0 & g' \end{pmatrix} \\ := \; &(fag' + fbd(g'), fbg', ga'f' + gb'd^*(f'), gb'f') \,. \end{aligned}$$

Proof. If we define

$$\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} .(a, b, a', b') := \Phi\left(\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} .\Phi^{-1}(a, b, a', b')\right) ,$$

where Φ is in Lemma (5.3.4), then it is clearly a left module structure induced by that on $\Omega^1_D(\widetilde{\mathcal{A}_\Theta})$. Now one can check that

$$\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} .(a, b, a', b') = (fa, fb, ga', gb')$$

Similarly for the right module structure, we define

$$(a, b, a', b'). \begin{pmatrix} f' & 0 \\ 0 & g' \end{pmatrix} := \Phi\left(\Phi^{-1}(a, b, a', b'). \begin{pmatrix} f' & 0 \\ 0 & g' \end{pmatrix}\right)$$

and check that it is equal to $(ag' + bd(g'), bg', a'f' + b'd^*(f'), b'f')$. □

Proposition 5.3.6. *$\Omega^1_D(\widetilde{\mathcal{A}_\Theta}) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^4$ as $\widetilde{\mathcal{A}_\Theta}$-bimodule.*

Proof. The $\widetilde{\mathcal{A}_\Theta}$-bimodule action on the right hand side is given by Proposition (5.3.5) and Φ of Lemma (5.3.4) becomes a bimodule isomorphism under this action. $\square$

Now consider the following linear operators

$$T_{a,b,c} : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta$$

$$e \longmapsto ad\,(bd^*(ce))\ ,$$

$$\widetilde{T_{a',b',c'}} : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta$$

$$e \longmapsto a'd^*\,(b'd(c'e))\ .$$

Then,

$$\begin{cases} T_{a,b,c} & \equiv\ M_{ad(b)d^*(c)+abdd^*(c)} + M_{abd^*(c)} \circ d + M_{ad(bc)} \circ d^* + M_{abc} \circ d \circ d^* \\ \widetilde{T_{a',b',c'}} & \equiv\ M_{a'd^*(b')d(c')+a'b'd^*d(c')} + M_{a'b'd(c')} \circ d^* + M_{a'd^*(b'c')} \circ d + M_{a'b'c'} \circ d^* \circ d\,, \end{cases} \tag{5.3.2}$$

where M_ξ denotes multiplication by ξ. Since elements of $\pi\left(\Omega^2(\widetilde{\mathcal{A}_\Theta})\right)$ are linear sums of

$$\begin{pmatrix} a_0 & 0 \\ 0 & b_0 \end{pmatrix} \left[D, \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}\right] \left[D, \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}\right],$$

they are of the form $\sum \begin{pmatrix} T_{a,b,c} & 0 \\ 0 & \widetilde{T_{a',b',c'}} \end{pmatrix}$ for $a, b, a', b' \in \mathcal{A}_\Theta$. This shows that $\pi\left(\Omega^2(\widetilde{\mathcal{A}_\Theta})\right) \subseteq \mathcal{M}_2 \oplus \widetilde{\mathcal{M}_2}$, where

$$\mathcal{M}_2 := span\{T_{a,b,c} : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta\}\,,$$

$$\widetilde{\mathcal{M}_2} := span\{\widetilde{T_{a',b',c'}} : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta\}\,.$$

To see equality use equation (5.3.1) and observe that

$$\begin{aligned}
&\sum \begin{pmatrix} T_{a,b,c} & 0 \\ 0 & \widetilde{T_{a',b',c'}} \end{pmatrix} \\
= &\sum \begin{pmatrix} a & 0 \\ 0 & -a' \end{pmatrix} \left[D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} b' & 0 \\ 0 & b \end{pmatrix} \left[D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} -c & 0 \\ 0 & c' \end{pmatrix}.
\end{aligned}$$

Lemma 5.3.7. $\{M_f \circ d : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta\} \bigcap \{M_g \circ d^* : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta\} = \{0\}$.

Proof. Let $e_k = U^{k_1} V^{k_2}$ for $k = (k_1, k_2) \in \mathbb{Z}^2$. Any element in the intersection must satisfy

$$\begin{aligned}
& \langle e_\alpha \,,\, M_f \circ d(e_\beta) \rangle = \langle e_\alpha \,,\, M_g \circ d^*(e_\beta) \rangle \quad \forall\, \alpha, \beta \in \mathbb{Z}^2 . \\
\Rightarrow \; & \langle \sum_k \widehat{f_k^*} e_{k+\alpha} \,,\, d(e_\beta) \rangle = \langle \sum_k \widehat{g_k^*} e_{k+\alpha} \,,\, d^*(e_\beta) \rangle . \\
\Rightarrow \; & \langle \sum_k \widehat{f_k^*} e_{k+\alpha} \,,\, e_\beta \rangle (\beta_1 - i\beta_2) = \langle \sum_k \widehat{g_k^*} e_{k+\alpha} \,,\, e_\beta \rangle (\beta_1 + i\beta_2).
\end{aligned}$$

So,

$$\widehat{f^*_{\beta-\alpha}}(\beta_1 - i\beta_2) = \widehat{g^*_{\beta-\alpha}}(\beta_1 + i\beta_2)$$

for all $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{Z}^2$, i,e.

$$\widehat{f_\gamma^*}(\alpha_1 + \gamma_1 - i\alpha_2 - i\gamma_2) \;=\; \widehat{g_\gamma^*}(\alpha_1 + \gamma_1 + i\alpha_2 + i\gamma_2) \tag{5.3.3}$$

where $\beta - \alpha = \gamma \in \mathbb{Z}^2$. In order to have nontrivial intersection, equation (5.3.3) must have nontrivial solution for all $\alpha, \gamma \in \mathbb{Z}^2$. Let $\widehat{f_\gamma^*} = x$ and $\widehat{g_\gamma^*} = y$. We get

$$x(1 + \gamma_1 - i - i\gamma_2) \;=\; y(1 + \gamma_1 + i + i\gamma_2), \tag{5.3.4}$$

$$x(2 + \gamma_1 - 2i - i\gamma_2) \;=\; y(2 + \gamma_1 + 2i + i\gamma_2). \tag{5.3.5}$$

Now (5.3.5) − (5.3.4) implies

$$x(1-i) \;=\; y(1+i). \tag{5.3.6}$$

Again equation (5.3.3) gives

$$x(1+\gamma_1-i\gamma_2) \;=\; y(1+\gamma_1+i\gamma_2). \tag{5.3.7}$$

Equations (5.3.4) and (5.3.7) together impliy $x=-y$. Hence, from equation (5.3.6) we get $x=y=0$, i,e. $\widehat{f_\gamma^*}=0$ for all γ, which proves the triviality of the intersection. $\square$

Lemma 5.3.8. $\{M_a+M_b\circ d+M_c\circ d^*:\mathcal{A}_\Theta\longrightarrow\mathcal{A}_\Theta\}\bigcap\{M_f\circ dd^*:\mathcal{A}_\Theta\longrightarrow\mathcal{A}_\Theta\}=\{0\}$.

Proof. As before, any element in the intersection must satisfy

$$\begin{aligned}
&\langle e_\alpha\,,\,(M_a+M_b\circ d+M_c\circ d^*)(e_\beta)\,\rangle=\langle e_\alpha\,,\,M_f\circ dd^*(e_\beta)\,\rangle\quad\forall\,\alpha,\beta\in\mathbb{Z}^2\,.\\
\Rightarrow\;&\langle\sum_k\widehat{a_k^*}e_{k+\alpha}\,,\,e_\beta)\,\rangle+\langle\sum_k\widehat{b_k^*}e_{k+\alpha}\,,\,e_\beta)\,\rangle(\beta_1-i\beta_2)+\\
&\langle\sum_k\widehat{c_k^*}e_{k+\alpha}\,,\,e_\beta)\,\rangle(\beta_1+i\beta_2)=\langle\sum_k\widehat{f_k^*}e_{k+\alpha}\,,\,e_\beta\,\rangle(\beta_1^2+\beta_2^2).
\end{aligned}$$

So,

$$\widehat{a^*_{\beta-\alpha}}+\widehat{b^*_{\beta-\alpha}}(\beta_1-i\beta_2)+\widehat{c^*_{\beta-\alpha}}(\beta_1+i\beta_2)=\widehat{f^*_{\beta-\alpha}}(\beta_1^2+\beta_2^2)$$

for all $\alpha=(\alpha_1,\alpha_2),\beta=(\beta_1,\beta_2)\in\mathbb{Z}^2$, i,e.

$$\begin{aligned}
&\widehat{a_\gamma^*}+\widehat{b_\gamma^*}(\alpha_1+\gamma_1-i\alpha_2-i\gamma_2)+\widehat{c_\gamma^*}(\alpha_1+\gamma_1+i\alpha_2+i\gamma_2)\\
=\;&\widehat{f_\gamma^*}((\alpha_1+\gamma_1)^2+(\alpha_2+\gamma_2)^2)
\end{aligned}\tag{5.3.8}$$

where $\beta-\alpha=\gamma\in\mathbb{Z}^2$. In order to have a nontrivial intersection, equation (5.3.8) must have nontrivial solution for all $\alpha,\gamma\in\mathbb{Z}^2$. Let $\widehat{a_\gamma^*}=w$, $\widehat{b_\gamma^*}=x$, $\widehat{c_\gamma^*}=y$, and $\widehat{f_\gamma^*}=z$. So equation (5.3.8) turns to

$$\begin{aligned} & w + x(\alpha_1 + \gamma_1 - i\alpha_2 - i\gamma_2) + y(\alpha_1 + \gamma_1 + i\alpha_2 + i\gamma_2) \\ = \ & z((\alpha_1 + \gamma_1)^2 + (\alpha_2 + \gamma_2)^2). \end{aligned} \tag{5.3.9}$$

From equation (5.3.9) we get

$$w + x(1 + \gamma_1 - i - i\gamma_2) + y(1 + \gamma_1 + i + i\gamma_2) = z((1 + \gamma_1)^2 + (1 + \gamma_2)^2), \tag{5.3.10}$$

$$w + x(2 + \gamma_1 - 2i - i\gamma_2) + y(2 + \gamma_1 + 2i + i\gamma_2) = z((2 + \gamma_1)^2 + (2 + \gamma_2)^2). \tag{5.3.11}$$

Now (5.3.11) – (5.3.10) gives

$$x(1 - i) + y(1 + i) \tag{5.3.12}$$

$$= z((2 + \gamma_1)^2 + (2 + \gamma_2)^2 - (1 + \gamma_1)^2 - (1 + \gamma_2)^2),$$

and equation (5.3.9) gives

$$w + x(\gamma_1 - i\gamma_2) + y(\gamma_1 + i\gamma_2) = z(\gamma_1^2 + \gamma_2^2). \tag{5.3.13}$$

Now (5.3.10) – (5.3.13) gives

$$x(1 - i) + y(1 + i) = z((1 + \gamma_1)^2 + (1 + \gamma_2)^2 - \gamma_1^2 - \gamma_2^2). \tag{5.3.14}$$

Finally, (5.3.12) – (5.3.14) gives $z = 0$. Hence $\widehat{f_\gamma^*} = 0$ for all γ i,e. intersection is trivial.

□

Lemma 5.3.9. $\{M_a \circ d + M_b \circ d^* : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta\} \bigcap \{M_f : \mathcal{A}_\Theta \longrightarrow \mathcal{A}_\Theta\} = \{0\}$.

Proof. Since $d(1) = d^*(1) = 0$ for $1 \in \mathcal{A}_\Theta$, this follows trivially. □

Proposition 5.3.10. *The following map*

$$\Phi : \pi\left(\Omega^2(\widetilde{\mathcal{A}_\Theta})\right) \longrightarrow \mathcal{A}_\Theta \otimes \mathbb{C}^8$$

$$\Phi = (\widetilde{\Phi}\,,\,\widetilde{\Phi}')$$

where

$$\widetilde{\Phi} : T_{a,b,c} \longmapsto (ad(b)d^*(c) + abdd^*(c), abd^*(c), ad(bc), abc)\,,$$

and

$$\widetilde{\Phi}' : \widetilde{T_{a',b',c'}} \longmapsto (a'd^*(b')d(c') + a'b'd^*d(c'), a'b'd(c'), a'd^*(b'c'), a'b'c')\,,$$

is a linear bijection.

Proof. Since $d(U) = d^*(U) = U$ and $UU^* = U^*U = I$, Lemmas (5.3.7 , 5.3.8 , 5.3.9) prove well-definedness as well as injectivity of Φ . To see surjectivity observe that

$$\begin{aligned} &T_{aU^*,1,U} - T_{aU^*,U,1} - T_{-ia,V^*,V} - T_{ia,1,1} + T_{-ib,V^*,V} - T_{-ib,1,1} \\ &+T_{cU^*,U,1} - T_{c,1,1} + T_{e,1,1} \xrightarrow{\widetilde{\Phi}} (a,b,c,e) \in \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \end{aligned}$$

and

$$\begin{aligned} &T_{a'U^*,1,U} - T_{a'U^*,U,1} - T_{-ia',V,V^*} - T_{ia',1,1} + T_{-ib',V,V^*} - T_{-ib',1,1} \\ &+T_{c'U^*,U,1} - T_{c',1,1} + T_{e',1,1} \xrightarrow{\widetilde{\Phi}'} (a',b',c',e') \in \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \end{aligned}$$

This completes the proof. ∎

Proposition 5.3.11. $\pi\left(dJ_0^1(\widetilde{\mathcal{A}_\Theta})\right) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^6$.

Proof. Elements of $\pi\left(dJ_0^1(\widetilde{\mathcal{A}_\Theta})\right)$ looks like

$$\sum [D, pa + qb][D, pe + qf] \quad \text{where} \quad \sum (pa + qb)[D, pe + qf] = 0\,,$$

where $p = (1+\gamma)/2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q = (1-\gamma)/2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are the projections onto the eigenspaces of γ. Expanding the commutators and simplifying we get

$$\sum \begin{pmatrix} T_{ae,1,1} - T_{a,1,e} & 0 \\ 0 & \widetilde{T_{bf,1,1}} - \widetilde{T_{b,1,f}} \end{pmatrix} \quad s.t. \quad \begin{cases} \sum T_{a,f} = \sum T_{ae,1} \\ \sum \widetilde{T_{b,e}} = \sum \widetilde{T_{bf,1}} \end{cases} . \tag{5.3.15}$$

The bijection of Proposition (5.3.10) gives

$$\widetilde{\Phi}\,(T_{ae,1,1} - T_{a,1,e}) = (-add^*(e), -ad^*(e), -ad(e), 0)$$
$$\widetilde{\Phi}'(\widetilde{T_{bf,1,1}} - \widetilde{T_{b,1,f}}) = (-bd^*d(f), -bd(f), -bd^*(f), 0).$$

To fullfil our claim it is enough to show that elements of the form

$$(add^*(e), ad^*(e), ad(e), bd^*d(f), bd(f), bd^*(f))$$

can generate $\mathcal{A}_\Theta \otimes \mathbb{C}^6$, where the conditions in equation (5.3.15) hold. Choose any arbitrary element $(a_1, a_2, a_3, a_1', a_2', a_3') \in \mathcal{A}_\Theta \otimes \mathbb{C}^6$. Observe that

$$\begin{aligned} & (a_1V^*dd^*(V), a_1V^*d^*(V), a_1V^*d(V), a_1'V^*d^*d(V), a_1'V^*d(V), a_1'V^*d^*(V)) \\ + \; & (a_1Vdd^*(V^*), a_1Vd^*(V^*), a_1Vd(V^*), a_1'Vd^*d(V^*), a_1'Vd(V^*), a_1'Vd^*(V^*)) \\ = \; & (2a_1, 0, 0, 2a_1', 0, 0) \end{aligned}$$

and the conditions of equation (5.3.15) also satisfied. Hence $(a_1, 0, 0, a_1', 0, 0) \in \pi\left(dJ_0^1(\widetilde{\mathcal{A}_\Theta})\right)$. Now,

$$\begin{aligned} & (ia_3Udd^*(U^*), ia_3Ud^*(U^*), ia_3Ud(U^*), 0, 0, 0) \\ + \; & (a_3V^*dd^*(V), a_3V^*d^*(V), a_3V^*d(V), 0, 0, 0) \\ + \; & (-\frac{1}{2}(a_3 + ia_3)V^*dd^*(V), -\frac{1}{2}(a_3 + ia_3)V^*d^*(V), -\frac{1}{2}(a_3 + ia_3)V^*d(V), 0, 0, 0) \\ + \; & (-\frac{1}{2}(a_3 + ia_3)Vdd^*(V^*), -\frac{1}{2}(a_3 + ia_3)Vd^*(V^*), -\frac{1}{2}(a_3 + ia_3)Vd(V^*), 0, 0, 0) \end{aligned}$$

$$= \quad (0,0,-2ia_3,0,0,0)$$

and conditions of equation (5.3.15) also satisfied. Hence, $(0,0,a_3,0,0,0) \in \pi\left(dJ_0^1(\widetilde{\mathcal{A}_\Theta})\right)$. Finally,

$$\begin{aligned}
& (0,0,0,ia_2'Ud^*d(U^*),ia_2'Ud(U^*),ia_2'Ud^*(U^*)) \\
+ \; & (0,0,0,a_2'V^*d^*d(V),a_2'V^*d(V),a_2'V^*d^*(V)) \\
+ \; & (0,0,0,-\frac{1}{2}(a_2'+ia_2')V^*d^*d(V),-\frac{1}{2}(a_2'+ia_2')V^*d(V),-\frac{1}{2}(a_2'+ia_2')V^*d^*(V)) \\
+ \; & (0,0,0,-\frac{1}{2}(a_2'+ia_2')Vd^*d(V^*),-\frac{1}{2}(a_2'+ia_2')Vd(V^*),-\frac{1}{2}(a_2'+ia_2')Vd^*(V^*)) \\
= \; & (0,0,0,0,-2ia_2',0)
\end{aligned}$$

and conditions of equation (5.3.15) also satisfied. Hence, $(0,0,0,0,a_2',0) \in \pi\left(dJ_0^1(\widetilde{\mathcal{A}_\Theta})\right)$. Thus we get $(0,0,a_3,0,a_2',0) \in \pi\left(dJ_0^1(\widetilde{\mathcal{A}_\Theta})\right)$. Similarly one can show that $(0,a_2,0,0,0,a_3') \in \pi\left(dJ_0^1(\widetilde{\mathcal{A}_\Theta})\right)$ and this completes the proof. □

Proposition 5.3.12. *The following action*

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.(a_1,a_2) := (xa_1, ya_2)$$

$$(a_1,a_2).\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} := (a_1y, a_2x).$$

defines an $\widetilde{\mathcal{A}_\Theta}$*-bimodule structure on* $\frac{\pi\left(\Omega^2(\widetilde{\mathcal{A}_\Theta})\right)}{\pi\left(dJ_0^1(\widetilde{\mathcal{A}_\Theta})\right)} \cong \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta$.

Proof. If we define

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.(a_1,a_2) := \Phi\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.\Phi^{-1}(a_1,a_2)\right)$$

$$(a_1,a_2).\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} := \Phi\left(\Phi^{-1}(a_1,a_2).\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right)$$

where Φ is as defined in Proposition (5.3.10), then clearly it is a bimodule action induced by that on $\Omega_D^2(\widetilde{\mathcal{A}_\Theta})$. One can verify that these actions are precisely the actions defined in question. $\square$

Theorem 5.3.13. *For the noncommutative torus we have,*

1. $\Omega_D^1(\widetilde{\mathcal{A}_\Theta}) \cong \mathcal{A}_\Theta \bigoplus \mathcal{A}_\Theta \bigoplus \mathcal{A}_\Theta \bigoplus \mathcal{A}_\Theta$ *, as $\widetilde{\mathcal{A}_\Theta}$-bimodule.*

2. $\Omega_D^n(\widetilde{\mathcal{A}_\Theta}) \cong \mathcal{A}_\Theta \bigoplus \mathcal{A}_\Theta$ *, for all $n \geq 2$ as $\widetilde{\mathcal{A}_\Theta}$-bimodule.*

Proof. Proposition (5.3.6) gives part (1). Proposition (5.3.10) and (5.3.11) proves part (2) for $n = 2$. The fact that the isomorphisms in Propositions (5.3.10 , 5.3.11) are not only $\mathbb{C}$-linear but also $\widetilde{\mathcal{A}_\Theta}$-bimodule isomorphisms follows from the defining property of the bimodule action in Proposition (5.3.12).

We need to prove part (2) for $n \geq 3$. For that purpose first note that

$$\left\{\begin{array}{l} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array}\right. \tag{5.3.16}$$

These matrices plays a key role to compute $\Omega_D^n(\widetilde{\mathcal{A}_\Theta})$ for all $n \geq 3$. Now for any unital algebra $\mathcal{A}$,

$$\Omega^n(\mathcal{A}) = \underbrace{\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \ldots\ldots \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})}_{n \text{ times}} . \tag{5.3.17}$$

By Lemma (5.3.4) we have

$$\pi\left(\Omega^1(\widetilde{\mathcal{A}_\Theta})\right) = (\mathcal{A}_\Theta \oplus \mathcal{A}_\Theta) \otimes_{\mathbb{C}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (\mathcal{A}_\Theta \oplus \mathcal{A}_\Theta) \otimes_{\mathbb{C}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

In view of Proposition (5.3.5) and using (5.3.16 , 5.3.17) we get $\pi\left(\Omega^n(\widetilde{\mathcal{A}_\Theta})\right) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^{2n} \bigoplus \mathcal{A}_\Theta \otimes \mathbb{C}^{2n}$ for all $n \geq 3$ (actually true for all $n \geq 1$ by part (1) and Proposition 5.3.10). We will show the following

$$\pi\left(dJ_0^n(\widetilde{\mathcal{A}_\Theta})\right) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^{2n+1} \bigoplus \mathcal{A}_\Theta \otimes \mathbb{C}^{2n+1}\,, \quad \forall\, n \geq 2\,.$$

Recall from Lemma (5.1.12), $[D^2, a] \in \pi(dJ_0^1)$. One can easily prove that

$$\pi(dJ_0^n) = \textstyle\sum_{i=0}^{n-1} \pi\left(\Omega^i \otimes_{\mathcal{A}} J^2 \otimes_{\mathcal{A}} \Omega^{n-1-i}\right) \quad \forall\, n \geq 2$$

by writing down any arbitrary element of $\pi(dJ_0^n)$ and then passing D through the commutators from left to right. Hence, for all $n \geq 2$ odd

$$\begin{aligned}
\pi\left(dJ_0^n(\widetilde{\mathcal{A}_\Theta})\right) &= \sum_{i=0\,,\,i\,even}^{n-1} \pi\left(\Omega^i \otimes J^2 \otimes \Omega^{n-1-i}\right) + \sum_{i=1\,,\,i\,odd}^{n-1} \pi\left(\Omega^i \otimes J^2 \otimes \Omega^{n-1-i}\right) \\
&= \sum_{i=0\,,\,i\,even}^{n-1} \left(\mathcal{A}_\Theta \otimes \mathbb{C}^{2i} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{A}_\Theta \otimes \mathbb{C}^{2i} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&\qquad \left(\mathcal{A}_\Theta \otimes \mathbb{C}^{3} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{A}_\Theta \otimes \mathbb{C}^{3} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&\qquad \left(\mathcal{A}_\Theta \otimes \mathbb{C}^{2(n-1-i)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{A}_\Theta \otimes \mathbb{C}^{2(n-1-i)} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&\quad + \sum_{i=1\,,\,i\,odd}^{n-1} \left(\mathcal{A}_\Theta \otimes \mathbb{C}^{2i} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathcal{A}_\Theta \otimes \mathbb{C}^{2i} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) \\
&\qquad \left(\mathcal{A}_\Theta \otimes \mathbb{C}^{3} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{A}_\Theta \otimes \mathbb{C}^{3} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&\qquad \left(\mathcal{A}_\Theta \otimes \mathbb{C}^{2(n-1-i)} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathcal{A}_\Theta \otimes \mathbb{C}^{2(n-1-i)} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) \\
&= \mathcal{A}_\Theta \otimes \mathbb{C}^{2n+1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{A}_\Theta \otimes \mathbb{C}^{2n+1} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&\cong \mathcal{A}_\Theta \otimes \mathbb{C}^{2n+1} \bigoplus \mathcal{A}_\Theta \otimes \mathbb{C}^{2n+1}\,.
\end{aligned}$$

Here the last equality uses (5.3.16) frequently. One can give a similar proof for all even $n \geq 2$. Hence, for all $n \geq 3$, we have $\Omega_D^n(\widetilde{\mathcal{A}_\Theta}) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^2$. $\square$

Remark 5.3.14. *One can also consider*

$$D = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \quad \text{or} \quad D = \begin{pmatrix} 0 & d^* \\ d^* & 0 \end{pmatrix}.$$

However, in that case one will get same answer as in Theorem (5.3.13). Since in non-commutative geometry it is customary to take $D = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix}$, *we provide computations with this value for* D.

Notation : $\widetilde{\Omega_D^1} := \mathcal{A}_\Theta \otimes \mathbb{C}^4$ and $\widetilde{\Omega_D^2} := \mathcal{A}_\Theta \otimes \mathbb{C}^2$ untill the end of this chapter.

Now we want to show that $\Omega_D^\bullet(\widetilde{\mathcal{A}_\Theta})$ is cohomologically not trivial. For that purpose we use the isomorphism in Theorem (5.3.13) to compute the differentials on $\widetilde{\Omega_D^1}$ and $\widetilde{\Omega_D^2}$.

Proposition 5.3.15. *Under the isomorphism in Theorem (5.3.13) the differentials* $\widetilde{d}$: $\widetilde{\mathcal{A}_\Theta} \longrightarrow \Omega_D^1(\widetilde{\mathcal{A}_\Theta})$ *and* $\widetilde{d} : \Omega_D^1(\widetilde{\mathcal{A}_\Theta}) \longrightarrow \Omega_D^2(\widetilde{\mathcal{A}_\Theta})$ *of Connes' complex* $\Omega_D^\bullet(\widetilde{\mathcal{A}_\Theta})$ *are given by*

$$\delta : \widetilde{\mathcal{A}_\Theta} \longrightarrow \widetilde{\Omega_D^1}$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \longmapsto (d(b), b-a, d^*(a), a-b)$$

and

$$\delta : \widetilde{\Omega_D^1} \longrightarrow \widetilde{\Omega_D^2}$$

$$(a, b, c, e) \longmapsto (b+e, b+e)$$

i,e. the following diagrams

$$\begin{array}{ccc} \widetilde{\mathcal{A}_\Theta} & \xrightarrow{\widetilde{d}} & \Omega_D^1(\widetilde{\mathcal{A}_\Theta}) \\ {\scriptstyle id}\downarrow & & \downarrow\cong \\ \widetilde{\mathcal{A}_\Theta} & \xrightarrow[\delta]{} & \widetilde{\Omega_D^1} \end{array} \qquad \begin{array}{ccc} \Omega_D^1(\widetilde{\mathcal{A}_\Theta}) & \xrightarrow{\widetilde{d}} & \Omega_D^2(\widetilde{\mathcal{A}_\Theta}) \\ \cong\downarrow & & \downarrow\cong \\ \widetilde{\Omega_D^1} & \xrightarrow[\delta]{} & \widetilde{\Omega_D^2} \end{array}$$

commute.

Proof. Use Lemma (5.3.4) to see commutativity of the first diagram. For the second, take any $(a,b,c,e) \in \widetilde{\Omega^1_D}$ and use Φ^{-1} of Lemma (5.3.4) to get the following element in $\pi\left(\Omega^1(\widetilde{\mathcal{A}_\Theta})\right)$

$$\begin{pmatrix} 0 & T_{aU^*,U} + T_{b,1} - T_{a,1} \\ \widetilde{T_{cU^*,U}} + \widetilde{T_{e,1}} - \widetilde{T_{c,1}} & 0 \end{pmatrix}. \tag{5.3.18}$$

Use the fact

$$\begin{pmatrix} -U^* & 0 \\ 0 & 0 \end{pmatrix}\left[D, \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}\right] + \begin{pmatrix} 0 & 0 \\ 0 & -U^* \end{pmatrix}\left[D, \begin{pmatrix} 0 & 0 \\ 0 & U \end{pmatrix}\right] = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix}$$

to observe that (5.3.18) can be re-written as

$$\begin{pmatrix} -aU^*U^* & 0 \\ 0 & 0 \end{pmatrix}\left[D, \begin{pmatrix} U^2 & 0 \\ 0 & 0 \end{pmatrix}\right] + \begin{pmatrix} aU^* & 0 \\ 0 & 0 \end{pmatrix}\left[D, \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\right] +$$
$$\begin{pmatrix} 0 & 0 \\ 0 & -cU^*U^* \end{pmatrix}\left[D, \begin{pmatrix} 0 & 0 \\ 0 & U^2 \end{pmatrix}\right] + \begin{pmatrix} 0 & 0 \\ 0 & cU^* \end{pmatrix}\left[D, \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\right] +$$
$$\begin{pmatrix} (a-b)U^* & 0 \\ 0 & 0 \end{pmatrix}\left[D, \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}\right] + \begin{pmatrix} 0 & 0 \\ 0 & (c-e)U^* \end{pmatrix}\left[D, \begin{pmatrix} 0 & 0 \\ 0 & U \end{pmatrix}\right].$$

Applying $\Phi \circ \widetilde{d}$ (Φ of Proposition 5.3.10 and $\widetilde{d} : \Omega^1_D \longrightarrow \Omega^2_D$), we get the following element

$$\big(d(c) + 2c + 2a + b\,,\; b + c\,,\; d(e) + a + b\,,\; b + e\,,\; d^*(a) + 2a + 2c + e\,,\; e + a\,,$$
$$d^*(b) + c + e\,,\; e + b\big) \;+\; \pi\left(dJ^1_0(\widetilde{\mathcal{A}_\Theta})\right)$$

of $\Omega^2_D(\widetilde{\mathcal{A}_\Theta})$. This element is equal to $(b+e\,,\, b+e) \in \Omega^2_D(\widetilde{\mathcal{A}_\Theta})$ by Theorem (5.3.13). $\square$

Before we proceed to show that $\Omega^\bullet_D(\widetilde{\mathcal{A}_\Theta})$ is cohomologically not trivial we first compute the cohomologies for $(\Omega^\bullet_D(\mathcal{A}_\Theta)\,, d)$, in order to notice the similarity. To do so recall Proposition 13, in the last chapter of ([15]).

Proposition 5.3.16 ([15]). *For the noncommutative torus $\mathcal{A}_\Theta$, we have*

1. $\Omega^1_D(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta$,

2. $\Omega^2_D(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta$,

3. *The differentials* $\widetilde{d} : \mathcal{A}_\Theta \longrightarrow \Omega^1_D(\mathcal{A}_\Theta)$ *and* $\widetilde{d} : \Omega^1_D(\mathcal{A}_\Theta) \longrightarrow \Omega^2_D(\mathcal{A}_\Theta)$ *are given by*

$$\widetilde{d} : a \longmapsto (\delta_1 a \,,\, \delta_2 a)$$

$$\widetilde{d} : (a_1, a_2) \longmapsto \delta_2(a_1) - \delta_1(a_2)$$

Remark 5.3.17. *For $n \geq 3$, the space of higher forms $\Omega^n_D(\mathcal{A}_\Theta)$ vanish. To see this first observe that $[D, a] = \delta_1(a) \otimes \sigma_1 + \delta_2(a) \otimes \sigma_2$ where σ_1, σ_2 are the spin matrices satisfing $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$. The isomorphism $\Omega^1_D(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^2$ is obtained using the linear independence of σ_1, σ_2 in $M_2(\mathbb{C})$. The isomorphism $\pi(\Omega^2(\mathcal{A}_\Theta)) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^2$ is obtained using the linear independence of I_2 and $\sigma_1\sigma_2$ in $M_2(\mathbb{C})$ and in this way one will obtain that $\pi(dJ^1_0(\mathcal{A}_\Theta)) \cong \mathcal{A}_\Theta \otimes I_2$. Because of this reason $\Omega^2_D(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta$. Observe that $\pi(\Omega^3(\mathcal{A}_\Theta)) = \mathcal{A}_\Theta \otimes \sigma_1 + \mathcal{A}_\Theta \otimes \sigma_2$ and hence $\pi(\Omega^3(\mathcal{A}_\Theta)) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^2$. Now recall that $J^\bullet$ is a graded ideal in $\Omega^\bullet$ and hence we have*

$$\pi\left(\Omega^1(\mathcal{A}_\Theta)J^2(\mathcal{A}_\Theta)\right) \subseteq \pi\left(J^3(\mathcal{A}_\Theta)\right) \subseteq \pi\left(\Omega^3(\mathcal{A}_\Theta)\right).$$

This shows that $\pi\left(J^3(\mathcal{A}_\Theta)\right) = \pi\left(dJ^2_0(\mathcal{A}_\Theta)\right) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^2$, i,e. $\Omega^3_D(\mathcal{A}_\Theta) = \{0\}$. Now note that $\Omega^n(\mathcal{A}) = \underbrace{\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \ldots\ldots \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})}_{n \text{ times}}$ for any unital algebra $\mathcal{A}$. Hence, $\pi\left(\Omega^n(\mathcal{A}_\Theta)\right) \cong \mathcal{A}_\Theta \otimes \mathbb{C}^2$ for all $n \geq 4$. Finally, the inclusion

$$\pi\left(\Omega^{n-2}(\mathcal{A}_\Theta)J^2(\mathcal{A}_\Theta)\right) \subseteq \pi\left(J^n(\mathcal{A}_\Theta)\right) \subseteq \pi\left(\Omega^n(\mathcal{A}_\Theta)\right)$$

proves that $\Omega^n_D(\mathcal{A}_\Theta) = \{0\}$ for all $n \geq 4$. This is needed in the next Lemma.

Lemma 5.3.18. *The cohomologies $H^\bullet(\mathcal{A}_\Theta)$ are given by ,*

1. $H^0(\mathcal{A}_\Theta) \cong \mathbb{C}$,

2. $H^1(\mathcal{A}_\Theta) \cong \mathbb{C} \oplus \mathbb{C}$,

3. $H^2(\mathcal{A}_\Theta) \cong \mathbb{C}$.

Proof. 1. We have

$$\begin{aligned} H^0(\mathcal{A}_\Theta) &= \{a \in \mathcal{A}_\Theta : \delta_1(a) = \delta_2(a) = 0\} \\ &\cong \mathbb{C} \end{aligned}$$

2. We have

$$H^1(\mathcal{A}_\Theta) = \frac{\{(a,b) : a, b \in \mathcal{A}_\Theta; \delta_2(a) = \delta_1(b)\}}{\{(\delta_1(a), \delta_2(a)) : a \in \mathcal{A}_\Theta\}}$$

Let

$$a = \textstyle\sum_{m,n} \alpha_{m,n} U^m V^n - \alpha_{0,0} \;,\; b = \sum_{p,q} \beta_{p,q} U^p V^q - \beta_{0,0}$$

i,e. $a, b \notin \mathbb{C}1$. Then $\delta_2(a) = \delta_1(b)$ will imply

$$\sum_{m\neq 0, n\neq 0} n\alpha_{m,n} U^m V^n + \sum_{n \neq 0} n\alpha_{0,n} V^n = \sum_{p\neq 0, q\neq 0} p\beta_{p,q} U^p V^q + \sum_{p\neq 0} p\beta_{p,0} U^p$$

Let $e_{mn} = U^m V^n$, $m, n \in \mathbb{Z}$. Then we get

$$\beta_{p,0} = 0 \;\forall\, p \neq 0 \;;\; \alpha_{0n} = 0 \;\forall\, n \neq 0 \;;\; n\alpha_{mn} = m\beta_{mn} \;\forall\, m \neq 0, n \neq 0.$$

Let

$$c = \sum_{m\neq 0, n\neq 0} \frac{\gamma_{m,n}}{mn} U^m V^n + \sum_{m\neq 0} \frac{\alpha_{m,0}}{m} U^m + \sum_{n \neq 0} \frac{\beta_{0,n}}{n} V^n$$

For $m \neq 0, n \neq 0$, if we choose $\gamma_{m,n} = n\alpha_{mn}$ then we get $\delta_1(c) = a$ and $\delta_2(c) = b$ which proves our claim.

3. Finally,

$$H^2(\mathcal{A}_\Theta) = \frac{\mathcal{A}_\Theta}{\{\delta_2(a) - \delta_1(b) : a, b \in \mathcal{A}_\Theta\}}$$

Let $a \in \mathcal{A}_\Theta$ be s.t. $a \notin \mathbb{C}1$. Let $a = \sum_{m\neq 0 \ or \ n\neq 0} \alpha_{m,n} U^m V^n$. Then

$$a = \sum_{m\in\mathbb{Z}, n\in\mathbb{Z}-\{0\}} \alpha_{m,n} U^m V^n + \sum_{m\in\mathbb{Z}-\{0\}} \alpha_{m,0} U^m$$

Consider $b = \sum_{m\in\mathbb{Z}, n\in\mathbb{Z}-\{0\}} \frac{\alpha_{m,n}}{n} U^m V^n$ and $c = -\sum_{m\in\mathbb{Z}-\{0\}} \frac{\alpha_{m,0}}{m} U^m$. Then $\delta_2(b) - \delta_1(c) = a$, which proves our claim.

□

Theorem 5.3.19. *If $\widetilde{H^\bullet(\mathcal{A}_\Theta)}$ denotes the cohomology groups of the chain complex $\left(\Omega_D^\bullet(\widetilde{\mathcal{A}_\Theta}), \delta\right)$ then we have*

1. $\widetilde{H^0(\mathcal{A}_\Theta)} \cong \mathbb{C}$,
2. $\widetilde{H^1(\mathcal{A}_\Theta)} \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathcal{A}_\Theta/\mathbb{C}$.

Proof. 1. We have

$$\begin{aligned}
\widetilde{H^0(\mathcal{A}_\Theta)} &= \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : d(b) = 0, d^*(a) = 0, a = b\} \\
&= \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : \delta_1(a) = \delta_2(a) = 0\} \\
&\cong \mathbb{C}.
\end{aligned}$$

2. We have

$$\widetilde{H^1(\mathcal{A}_\Theta)} = \frac{\{(a, b, c, e) : b + e = 0\}}{\{(d(f), f - g, d^*(g), g - f)\}}$$

Let $\mathcal{M} = \{(a, b, c, -b) : a, b, c \in \mathcal{A}_\Theta\}$ and $\mathcal{N} = \{(d(f), f - g, d^*(g), g - f) : f, g \in \mathcal{A}_\Theta\}$. Clearly $\mathcal{M} \cong \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{A}_\Theta$. Now define

$$\psi : \mathcal{N} \bigoplus \mathbb{C} \bigoplus \mathcal{A}_\Theta/\mathbb{C} \bigoplus \mathbb{C} \longrightarrow \mathcal{M}$$

$$(d(f), f-g, d^*(g), g-f, \lambda_1, a, \lambda_2) \longmapsto (d(f)+\lambda_1, f-g+a, d^*(g)+\lambda_2)$$

This map is $\mathbb{C}$-linear and one-one. To see surjectivity take any $(a,b,c) \in \mathcal{A}_\Theta^3$. Suppose $a = \sum \alpha_{m,n} U^m V^n$. If we choose

$$f = \textstyle\sum_{m\neq 0\, or\, n\neq 0} \frac{1}{m-in} \alpha_{m,n} U^m V^n$$

then $d(f) = a - \alpha_{0,0}$ and we see that

$$(d(f), f, 0, -f, \alpha_{0,0}, -f, 0) \longmapsto (a, 0, 0)$$

Now suppose $b = \sum \beta_{m,n} U^m V^n$. If we choose $f = \beta_{0,0}, g = 0$ then

$$(0, \beta_{0,0}, 0, -\beta_{0,0}, 0, b - \beta_{0,0}, 0) \longmapsto (0, b, 0)$$

Finally let $c = \sum \gamma_{m,n} U^m V^n$ and choose

$$g = \textstyle\sum_{m\neq 0\, or\, n\neq 0} \frac{1}{m+in} \gamma_{m,n} U^m V^n$$

then we see that

$$(0, -g, c - \gamma_{0,0}, g, 0, g, \gamma_{0,0}) \longmapsto (0, 0, c)$$

This shows that ψ is a linear isomorphism with $\psi(\mathcal{N}) = \mathcal{N}$ and hence our claim has been justified.

This shows that the complex $\Omega_D^\bullet(\widetilde{\mathcal{A}_\Theta})$ is cohomologically not trivial. $\square$

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