

Orbits of Pairs in Finite Modules over Discrete Valuation Rings and Permutation Representations

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

ANIL KUMAR C P

List Of Publications

- [1] C. P. AnilKumar, D. Bose, R. Krishnan, and S. Madan. On Fuglede's conjecture for three intervals. *Online. J. Anal. Comb.*, Vol. 5, 2010. Art. 1, 24 pages.
- [2] C. P. AnilKumar and A. Prasad. Orbits of pairs in Abelian groups. *Séminaire Lotharingien de Combinatoire*, Vol. 70, 2014. Art. B70h, 24 pages.

*Dedicated to my Parents
and my Sister*

To my School Teachers

*Gurur Brahma Gurur Vishnu, Gurur Devoh Maheswaraha
Gurur Sakshath Param Brahma Tasmaih Sri Gurave Namaha*

Abstract

Let \mathbb{A} be a discrete valuation ring whose maximal ideal is generated by a uniformizing element π , and which has a finite residue field \mathbb{F}_q . Let Λ denote the set of all sequences of symbols of the form

$$(\lambda_1^{\rho_1}, \lambda_2^{\rho_2}, \dots, \lambda_k^{\rho_k}),$$

where $\lambda_1 > \lambda_2 > \dots > \lambda_k$ is a strictly decreasing sequence of positive integers and $\rho_1, \rho_2, \dots, \rho_k$ are positive integers. We allow the case where $k = 0$, resulting in the empty sequence, which we denote by \emptyset . Every finite \mathbb{A} -module $\mathcal{A}_{\underline{\lambda}}$ is, up to isomorphism, of the form

$$\mathcal{A}_{\underline{\lambda}} = (\mathbb{A}/\pi^{\lambda_1} \mathbb{A})^{\oplus \rho_1} \oplus (\mathbb{A}/\pi^{\lambda_2} \mathbb{A})^{\oplus \rho_2} \oplus \dots \oplus (\mathbb{A}/\pi^{\lambda_k} \mathbb{A})^{\oplus \rho_k}$$

for a unique $\underline{\lambda} \in \Lambda$. Let $\mathcal{G}_{\underline{\lambda}}$ denote the automorphism group of $\mathcal{A}_{\underline{\lambda}}$.

Fix a $\underline{\lambda} \in \Lambda$, the corresponding finite \mathbb{A} -module $\mathcal{A}_{\underline{\lambda}}$ and its automorphism group $\mathcal{G}_{\underline{\lambda}}$.

The group $\mathcal{G}_{\underline{\lambda}}$ acts on $\mathcal{A}_{\underline{\lambda}}^n$ by the diagonal action

$$g \cdot (x_1, \dots, x_n) = (g(x_1), \dots, g(x_n)) \text{ for } x_i \in \mathcal{A}_{\underline{\lambda}} \text{ and } g \in \mathcal{G}_{\underline{\lambda}}.$$

In this thesis we study the set of $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}}^n$ under the above action for $n = 2$. We find that the cardinality of each orbit is a polynomial in q with integer coefficients and moreover, given such a polynomial, the number of orbits with that cardinality is a polynomial in q with integer coefficients which does not depend on \mathbb{A} , but only on the cardinality of the residue field of \mathbb{A} . When $q = p$ is a prime, the two possibilities for $\mathcal{A}_{\underline{\lambda}}$ are finite abelian p -groups and finite dimensional primary $\mathbb{F}_p[t]$ -modules (isomorphism classes of which correspond to similarity classes of matrices).

In the case of finite abelian p -groups, \mathbb{A} is the ring of p -adic integers. For general q , \mathbb{A} can either be taken to be the ring of Witt vectors of \mathbb{F}_q or the ring $\mathbb{F}_q[[t]]$ of formal power series with coefficients in \mathbb{F}_q .

On the representation theory side, we use these results to analyze the permutation representation of \mathcal{G}_λ on the vector spaces $\mathbb{C}[\mathcal{O}]$ where \mathcal{O} runs over \mathcal{G}_λ -orbits in \mathcal{A}_λ . So first we get a description of the suborbits in similar orbit of pairs $\mathcal{O} \times \mathcal{O}$ corresponding to a general ideal which is useful to prove that the permutation representation corresponding to any orbit is multiplicity free.

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Chapter 1

Synopsis

1.1 Synopsis Abstract

Let \mathbb{A} be a discrete valuation ring whose maximal ideal is generated by a uniformizing element π , and which has a finite residue field \mathbb{F}_q . Let Λ denote the set of all sequences of symbols of the form

$$(\lambda_1^{\rho_1}, \lambda_2^{\rho_2}, \dots, \lambda_k^{\rho_k}), \quad (1.1)$$

where $\lambda_1 > \lambda_2 > \dots > \lambda_k$ is a strictly decreasing sequence of positive integers and $\rho_1, \rho_2, \dots, \rho_k$ are positive integers. We allow the case where $k = 0$, resulting in the empty sequence, which we denote by \emptyset . Every finite \mathbb{A} -module $\mathcal{A}_{\underline{\lambda}}$ is, up to isomorphism, of the form

$$\mathcal{A}_{\underline{\lambda}} = (\mathbb{A}/\pi^{\lambda_1} \mathbb{A})^{\oplus \rho_1} \oplus (\mathbb{A}/\pi^{\lambda_2} \mathbb{A})^{\oplus \rho_2} \oplus \dots \oplus (\mathbb{A}/\pi^{\lambda_k} \mathbb{A})^{\oplus \rho_k} \quad (1.2)$$

for a unique $\underline{\lambda} \in \Lambda$. Let $\mathcal{G}_{\underline{\lambda}}$ denote the automorphism group of $\mathcal{A}_{\underline{\lambda}}$.

Fix a $\underline{\lambda} \in \Lambda$, the corresponding finite \mathbb{A} -module $\mathcal{A}_{\underline{\lambda}}$ and its automorphism group $\mathcal{G}_{\underline{\lambda}}$.

The group \mathcal{G}_λ acts on \mathcal{A}_λ^n by the diagonal action

$$g \cdot (x_1, \dots, x_n) = (g(x_1), \dots, g(x_n)) \text{ for } x_i \in \mathcal{A}_\lambda \text{ and } g \in \mathcal{G}_\lambda.$$

In this thesis we study the set of \mathcal{G}_λ -orbits in \mathcal{A}_λ^n under the above action for $n = 2$. We find that the cardinality of each orbit is a polynomial in q with integer coefficients and moreover, given such a polynomial, the number of orbits with that cardinality is a polynomial in q with integer coefficients which does not depend on \mathbb{A} , but only on the cardinality of the residue field of \mathbb{A} . When $q = p$ is a prime, the two possibilities for \mathcal{A}_λ are finite abelian p -groups and finite dimensional primary $\mathbb{F}_p[t]$ -modules (isomorphism classes of which correspond to similarity classes of matrices). In the case of finite abelian p -groups, \mathbb{A} is the ring of p -adic integers.

For general q , \mathbb{A} can either be taken to be the ring of Witt vectors of \mathbb{F}_q or the ring $\mathbb{F}_q[[t]]$ of formal power series with coefficients in \mathbb{F}_q .

We use these results to analyze the permutation representation of \mathcal{G}_λ on the vector spaces $\mathbb{C}[\mathcal{O}]$ where \mathcal{O} runs over \mathcal{G}_λ -orbits in \mathcal{A}_λ . We are able to prove that these permutation representations are multiplicity free.

1.2 Background

The monograph of Lynne M. Butler titled *Subgroup Lattices and Symmetric Functions* [5] describes two approaches to studying subgroup lattices of finite abelian p -groups. The first approach is linear-algebraic in nature and yields a combinatorial interpretation of the Betti polynomials of these Cohen-Macaulay posets. The second approach, which employs Hall-Littlewood symmetric functions, exploits properties of Kostka polynomials to obtain enumerative results such as rank-unimodality. However, the theory of automorphism-orbits of subgroups of finite abelian groups seems to be largely unexplored. A classification of orbits of elements in finite abelian groups has been available for a long time (G.A. Miller [11], Birkhoff [4], Dutta-Prasad [8]). Calvert, Dutta and Prasad ([6]) have used the notion of degeneration to give a poset structure to the set of automorphism classes of subgroups and tuples of elements in \mathcal{A}_λ . However, these posets are not very well-understood.

The general representation theory of finite groups, along with the representation theory of symmetric groups was developed by Dedekind, Frobenius, Burnside, Schur, Brauer in the early part of the 20th century. In his 1955 paper, Green [9] computed the characters of the irreducible representations of $GL_n(\mathbb{F}_q)$. In contrast, for the representation theory of groups such as $GL_n(\mathbb{Z}/p^k\mathbb{Z})$, despite the existence of extensive literature, is not well understood. These developments are surveyed in Pooja Singla's PhD thesis [14]. In her thesis, she studies the irreducible complex representations of the general linear groups over principal ideal local rings of length two. Using Clifford theory, she exhibits a bijective correspondence between irreducible representations of $GL_n(\mathcal{O}_2)$ and irreducible representations of centralizers in $GL_n(\mathcal{O}_1)$ of representative elements in various similarity classes of $M_n(\mathcal{O}_1)$. These centralizers turn out to be products of the groups \mathcal{G}_λ in the case where \mathbb{A} is $\mathbb{F}_q[[t]]$. In [12], Uri Onn classifies representations of automorphism groups of finite \mathbb{A} -modules of rank two completely. Dutta and Prasad ([7]) show that the Weil repre-

sentation of the symplectic group associated to a finite abelian group of odd order has multiplicity-free decomposition.

In all the available results, it is observed that methods of construction and the dimensions of irreducible representations of groups of the form $\mathcal{G}_{\underline{\lambda}}$ do not depend on the ring \mathbb{A} but only on the cardinality of its residue field. The results of this thesis are consistent with this trend.

1.3 Main Results

The key result of this thesis is a description of the $\mathcal{G}_{\underline{\lambda}}$ -orbit of pairs in $\mathcal{A}_{\underline{\lambda}} \times \mathcal{A}_{\underline{\lambda}}$. This is achieved by describing the stabilizer $\mathcal{G}_{\underline{\lambda}}^I$ of a representative $e(I) \in \mathcal{A}_{\underline{\lambda}}$ of a $\mathcal{G}_{\underline{\lambda}}$ -orbit in $\mathcal{A}_{\underline{\lambda}}$ (see Dutta and Prasad [8] page 8), where I is an ideal in a poset $J(\mathcal{P})_{\underline{\lambda}}$ which classifies the $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}}$. Decompose $\mathcal{A}_{\underline{\lambda}}$ into a direct sum of two \mathbb{A} -modules (this decomposition depends on I):

$$\mathcal{A}_{\underline{\lambda}} = \mathcal{A}_{\underline{\lambda}'} \oplus \mathcal{A}_{\underline{\lambda}''}, \quad (1.3)$$

where $\mathcal{A}_{\underline{\lambda}'}$ consists of those cyclic summands in the decomposition (3.2) of $\mathcal{A}_{\underline{\lambda}}$ where $e(I)$ has non-zero coordinates, and $\mathcal{A}_{\underline{\lambda}''}$ consists of the remaining cyclic summands. Let the projection of $\mathcal{A}_{\underline{\lambda}}$ to $\mathcal{A}_{\underline{\lambda}'}$ take $e(I)$ to $e(I)' \in \mathcal{A}_{\underline{\lambda}'}$.

Theorem 1.4. *The stabilizer of $e(I)$ in $\mathcal{G}_{\underline{\lambda}}$ consists of matrices of the form*

$$\begin{pmatrix} \text{id}_{\mathcal{A}_{\underline{\lambda}'}} + n & y \\ z & w \end{pmatrix},$$

where $n \in N^{\underline{\lambda}'} \subset \text{End}_{\mathbb{A}}(\mathcal{A}_{\underline{\lambda}'})$, $y \in \mathcal{HOM}(\mathcal{A}_{\underline{\lambda}''}, \mathcal{A}_{\underline{\lambda}'})$ is arbitrary, $z \in \mathcal{M}(\underline{\lambda}', \underline{\lambda}'') \subset \mathcal{HOM}(\mathcal{A}_{\underline{\lambda}'}, \mathcal{A}_{\underline{\lambda}''})$ and $w \in \mathcal{G}_{\underline{\lambda}''}$ is invertible. Here

- $N^{\underline{\lambda}'} = \{n \in \text{End}_{\mathbb{A}}(\mathcal{A}_{\underline{\lambda}'}) \mid n(e(I)') = 0\}$ is a nilpotent ideal in $\text{End}_{\mathbb{A}}(\mathcal{A}_{\underline{\lambda}'})$.
- $\mathcal{M}(\underline{\lambda}', \underline{\lambda}'') = \{z \in \mathcal{HOM}(\mathcal{A}_{\underline{\lambda}'}, \mathcal{A}_{\underline{\lambda}''}) \mid z(e(I)') = 0\}$.

Theorem 4.12 allows us to describe the orbit of m under the action of $\mathcal{G}_{\underline{\lambda}}^I$, which is the same as describing the $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}} \times \mathcal{A}_{\underline{\lambda}}$ whose first component lies in the orbit $\mathcal{A}_{\underline{\lambda}}^{I^*}$ of $e(I)$. Write each element $m \in \mathcal{A}_{\underline{\lambda}}$ as $m = (m', m'')$ with respect to the decomposition (4.9) of $\mathcal{A}_{\underline{\lambda}}$. Also, for any $m' \in \mathcal{A}_{\underline{\lambda}'}$, let \bar{m}' denote the image of m' in $\mathcal{A}_{\underline{\lambda}'} / \mathbb{A}e(I)'$.

Theorem 1.5. *Given m and n in $\mathcal{A}_{\underline{\lambda}}$, l lies in the $\mathcal{G}_{\underline{\lambda}}^I$ -orbit of m in $\mathcal{A}_{\underline{\lambda}}$ if and only if the following conditions hold:*

- $l' \in m' + \mathcal{A}_{\underline{\lambda}'}^{I(\bar{m}') \cup I(m'')}$.
- $l'' \in \mathcal{A}_{\underline{\lambda}''}^{I(m'')^*} + \mathcal{A}_{\underline{\lambda}''}^{I(\bar{m}'')}$.

Here, for each $I \in J(\mathcal{P}_{\underline{\lambda}})$, $\mathcal{A}_{\underline{\lambda}}^I$ denotes the smallest $\mathcal{G}_{\underline{\lambda}}$ -invariant submodule of $\mathcal{A}_{\underline{\lambda}}$ which contains $e(I)$.

Using this it is shown that the cardinality of each orbit of pairs is a monic polynomial in q with integer coefficients, where q is the cardinality of the residue field $\mathbb{A}/\pi\mathbb{A}$ independent of the ring \mathbb{A} :

Theorem 1.6. *Let \mathbb{A} be a discrete valuation ring with residue field of order q . Fix $\underline{\lambda} \in \Lambda$ and take $\mathcal{A}_{\underline{\lambda}}$ as in (3.2). Let $\mathcal{G}_{\underline{\lambda}}$ denote the group of \mathbb{A} -module automorphisms of $\mathcal{A}_{\underline{\lambda}}$. Fix order ideals $I, L \in J(\mathcal{P})_{\underline{\lambda}}$ (and hence $\mathcal{G}_{\underline{\lambda}}$ -orbits $\mathcal{A}_{\underline{\lambda}}^{I^*}$ and $\mathcal{A}_{\underline{\lambda}}^{L^*}$ in $\mathcal{A}_{\underline{\lambda}}$).*

1. *The cardinality of each $\mathcal{G}_{\underline{\lambda}}$ -orbit in $\mathcal{A}_{\underline{\lambda}}^{I^*} \times \mathcal{A}_{\underline{\lambda}}^{L^*}$ is a monic polynomial in q whose coefficients are integers.*
2. *Given a monic polynomial $\beta(q)$ with integer coefficients, the number of $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}}^{I^*} \times \mathcal{A}_{\underline{\lambda}}^{L^*}$ of cardinality $\beta(q)$ is a polynomial in q with coefficients that are integers which do not depend on \mathbb{A} .*
3. *The total number of $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}}^{I^*} \times \mathcal{A}_{\underline{\lambda}}^{L^*}$ depends only on whether ρ_i is 0, 1, or any cardinal greater than 1 (and not on the exact value of ρ_i) for each of the multiplicities ρ_i in (3.1).*

On the representation theory side, first we get a description of the suborbits in similar orbit of pairs corresponding to a general ideal which is useful to prove that the permutation representation corresponding to any orbit is multiplicity free.

Theorem 1.7. *Let $I \in J(\mathcal{P})_{\underline{\lambda}}$ be an ideal then the component of an orbit of pair $\mathcal{O} \subset \mathcal{O}_I \times \mathcal{O}_I \subset \mathcal{A}_{\underline{\lambda}} \times \mathcal{A}_{\underline{\lambda}}$ corresponding to any isotypic part $(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ of $\mathcal{A}_{\underline{\lambda}}$ is a set of ordered pairs having the following description.*

if $(\partial_{\lambda_i} I, \lambda_i) \in \max(I)$

- $\{(a, b) \in (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i} I)+1}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \times (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i} I)+1}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \mid b - ay \in \pi^r(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ for some $r > \partial_{\lambda_i} I$ and for some slope unit $y \in \mathbb{A}^*\}$
- $\{(a, b) \in (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i} I)+1}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \times (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i} I)+1}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \mid b - ay \in \pi^r(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{r+1}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ for some $r \geq \partial_{\lambda_i} I$ and for some slope unit $y \in \mathbb{A}^*$ and $\pi^{-r}(b - ay)(\text{mod } \pi)$ is linearly independent with $\pi^{-\partial_{\lambda_i} I} a(\text{mod } \pi)$ in $(\mathbb{F}_q)^{\rho_i}\}$

if $(\partial_{\lambda_i} I, \lambda_i) \notin \max(I)$

- $\{(a, b) \in \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \times \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}\}$
- $\{(a, b) \in \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \times \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \mid b - ay \in \pi^r(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ for some $r > \partial_{\lambda_i} I$ and for some slope unit $y \in \mathbb{A}^*\}$
- $\{(a, b) \in \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \times \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \mid b - ay \in \pi^r(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{r+1}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ for some $r \geq \partial_{\lambda_i} I$ and for some slope unit $y \in \mathbb{A}^*\}$

Moreover \mathcal{O} is the product of these components of pairs.

Theorem 1.8. *Let $(\underline{\lambda} = \lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \dots > \lambda_k^{\rho_k})$ be a partition. Consider the permutation representation of the automorphism group $\mathcal{G}_{\underline{\lambda}}$ of a finite module $\mathcal{A}_{\underline{\lambda}}$ on any orbit $\mathcal{O}_I \subset \mathcal{A}_{\underline{\lambda}}$ for $I \in J(\mathcal{P})_{\underline{\lambda}}$. Suppose $\mathbb{F}_q \cong \mathbb{A}/\pi\mathbb{A}$ has at least three elements or the multiplicity ρ_i of each part λ_i in $\underline{\lambda}$ is > 2 corresponding to every element $(\partial_{\lambda_i} I, \lambda_i) \in \max(I)$. Then this permutation representation is multiplicity-free.*

1.4 The Organization of the Thesis

The thesis has the following chapters with chapterwise summary.

1. Introduction:

2. Preliminaries:

This chapter contains some important notions that are used in the later chapters. Most important is the combinatorial description of orbits due to Dutta and Prasad [8]. This description of orbits is going to be very useful in the analysis of orbits of pairs in chapter three. It also has a section on some basic observations of permutation representations. Apart from this it also contains sections describing some observations on the additive group structure and the unit group structure of $(\mathbb{A}/\pi^l\mathbb{A})$. The units occur as part of the parameters for similar orbits of pairs which enable us to prove multiplicity-free results for all orbits.

3. Orbit of Pairs:

The chapter contains results on orbits of pairs of elements in \mathcal{A}_λ^2 under the diagonal action of \mathcal{A}_λ . The key result being the description of the \mathcal{G}_λ -orbit of a pair in $\mathcal{A}_\lambda \times \mathcal{A}_\lambda$ (refer theorem (4.13)). Using this it is shown that the cardinality of each orbit of pair is a monic polynomial in q with integer coefficients where q is the cardinality of the residue field $\mathbb{A}/\pi\mathbb{A}$ independent of the ring \mathbb{A} . Moreover it is also shown that the number of orbits of a given cardinality is also a monic polynomial in q with integer coefficients which do not depend on the ring \mathbb{A} (refer Theorem 4.27).

4. Multiplicity-Free Representations:

In this chapter we further analyze similar orbit of pairs of elements in \mathcal{A}_λ^2 and

describe the suborbits in $\mathcal{O}_I \times \mathcal{O}_I$. We prove that the permutation representation of \mathcal{G}_λ on any orbit $\mathcal{O}_I \subset \mathcal{A}_\lambda$ is multiplicity-free.

Chapter 2

Introduction

2.1 The nature of the problem

Let \mathbb{A} be a discrete valuation ring with maximal ideal generated by a uniformizing element π and having a finite residue field \mathbb{F}_q . We study two closely related problems concerning finite torsion modules over \mathbb{A} .

1. Let \mathcal{A}_λ denote a finite \mathbb{A} -module corresponding to a partition $\lambda \in \Lambda$ as given in equations (3.1) and (3.2). Let \mathcal{G}_λ denote its automorphism group. In this thesis we study the set of \mathcal{G}_λ -orbits of \mathcal{A}_λ^n under the diagonal action of \mathcal{G}_λ for $n = 2$. There are integer polynomials in one variable x i.e in the ring $\mathbb{Z}[x]$ which upon evaluation at the cardinality q of the residue field $\mathbb{A}/\pi^1\mathbb{A}$ gives cardinalities of orbits of pairs as well as the number of orbits of pairs for any given pair of orbits. Moreover, given such a polynomial, the number of orbits with that cardinality is an integer polynomial in q which does not depend on \mathbb{A} , but only on the cardinality of the residue field of \mathbb{A} . When $q = p$ is a prime, the two possibilities for \mathcal{A}_λ are finite abelian p -groups and finite dimensional primary $\mathbb{F}_p[t]$ -modules (isomorphism classes of which correspond to similarity classes of matrices). In the case of finite abelian p -groups, \mathbb{A} is the ring of

p -adic integers. For general q , \mathbb{A} can either be taken to be the ring of Witt vectors of \mathbb{F}_q or the ring $\mathbb{F}_q[[t]]$ of formal power series with coefficients in \mathbb{F}_q .

2. We use these results to analyze the permutation representation of \mathcal{G}_λ on the vector spaces $\mathbb{C}[\mathcal{O}]$ where \mathcal{O} runs over \mathcal{G}_λ -orbits in \mathcal{A}_λ . We obtain a description of the suborbits of similar orbit of pairs which we use to prove that the permutation representation on any orbit is multiplicity-free.

The monograph of Lynne M. Butler titled *Subgroup Lattices and Symmetric Functions* [5] describes two approaches to studying subgroup lattices of finite abelian p -groups. The first approach is linear-algebraic in nature and yields a combinatorial interpretation of the Betti polynomials of these Cohen-Macaulay posets. The second approach, which employs Hall-Littlewood symmetric functions, exploits properties of Kostka polynomials to obtain enumerative results such as rank-unimodality. However, the theory of automorphism-orbits of subgroups of finite abelian groups seems to be largely unexplored. A classification of orbits of elements in finite abelian groups has been available for a long time (Miller [11], Birkhoff [4], Dutta-Prasad [8]). Calvert, Dutta and Prasad [6] have used the notion of degeneration to give a poset structure to the set of automorphism classes of subgroups and tuples of elements in \mathcal{A}_λ . However, these posets are not very well-understood.

The general representation theory of finite groups, along with the representation theory of symmetric groups was developed by Dedekind, Frobenius, Burnside, Schur, Brauer in the early part of the 20th century. In his 1955 paper, Green [9] computed the characters of the irreducible representations of $GL_n(\mathbb{F}_q)$. In contrast, for the representation theory of groups such as $GL_n(\mathbb{Z}/p^k\mathbb{Z})$, despite the existence of extensive literature, is not well understood. These developments are surveyed in Pooja Singla's PhD thesis [14] and her paper [13]. In her thesis, she studies the irreducible complex representations of the general linear groups over principal ideal local rings of length two. Using Clifford theory, she exhibits a bijective correspondence between

irreducible representations of $GL_n(\mathcal{O}_2)$ and irreducible representations of centralizers in $GL_n(\mathcal{O}_1)$ of representative elements in various similarity classes of $M_n(\mathcal{O}_1)$. These centralizers turn out to be products of the groups \mathcal{G}_λ in the case where the discrete valuation ring \mathbb{A} is $\mathbb{F}_q[[t]]$.

In [12], Uri Onn classifies representations of automorphism groups of finite \mathbb{A} -modules of rank two completely. Dutta and Prasad [7] show that the Weil representation of the symplectic group associated to a finite abelian group of odd order has multiplicity-free decomposition.

In all the available results, it is observed that methods of construction and the dimensions of irreducible representations of groups of the form \mathcal{G}_λ do not depend on the ring \mathbb{A} but only on the cardinality of its residue field. The results of this thesis are consistent with this trend.

Chapter 3

Preliminaries

3.1 Overview

Here we assume we work with a discrete valuation ring \mathbb{A} with maximal ideal generated by a uniformizing element π . We also assume that the residue field $\mathbb{A}/\pi\mathbb{A} \cong \mathbb{F}_q$ is finite. Typical examples are the following two rings:

- The ring of p -adic integers.
- The power series ring in one variable over a finite field.

3.2 Some Observations on the Finite Modules of Discrete Valuations Rings with Finite Residue Fields

Let Λ denote the set of all sequences of symbols of the form

$$(\lambda_1^{\rho_1}, \lambda_2^{\rho_2}, \dots, \lambda_k^{\rho_k}), \tag{3.1}$$

where $\lambda_1 > \lambda_2 > \dots > \lambda_k$ is a strictly decreasing sequence of positive integers and $\rho_1, \rho_2, \dots, \rho_k$ are positive integers. We allow the case where $k = 0$, resulting in the empty sequence, which we denote by \emptyset . Every finite \mathbb{A} -module \mathcal{A}_λ is, up to

isomorphism, of the form

$$\mathcal{A}_{\underline{\lambda}} = (\mathbb{A}/\pi^{\lambda_1} \mathbb{A})^{\oplus \rho_1} \oplus (\mathbb{A}/\pi^{\lambda_2} \mathbb{A})^{\oplus \rho_2} \oplus \dots \oplus (\mathbb{A}/\pi^{\lambda_k} \mathbb{A})^{\oplus \rho_k} \quad (3.2)$$

for a unique $\underline{\lambda} \in \Lambda$. let $\mathcal{G}_{\underline{\lambda}}$ denote its automorphism group. Then the automorphism group $\mathcal{G}_{\underline{\lambda}}$ can be represented by a matrix of the form

$$\begin{bmatrix} GL_{\rho_1}(\mathbb{A}/\pi^{\lambda_1} \mathbb{A}) & Hom((\mathbb{A}/\pi^{\lambda_2} \mathbb{A})^{\rho_2}, (\mathbb{A}/\pi^{\lambda_1} \mathbb{A})^{\rho_1}) & \dots & Hom((\mathbb{A}/\pi^{\lambda_k} \mathbb{A})^{\rho_k}, (\mathbb{A}/\pi^{\lambda_1} \mathbb{A})^{\rho_1}) \\ Hom((\mathbb{A}/\pi^{\lambda_1} \mathbb{A})^{\rho_1}, (\mathbb{A}/\pi^{\lambda_2} \mathbb{A})^{\rho_2}) & GL_{\rho_2}(\mathbb{A}/\pi^{\lambda_2} \mathbb{A}) & \dots & Hom((\mathbb{A}/\pi^{\lambda_k} \mathbb{A})^{\rho_k}, (\mathbb{A}/\pi^{\lambda_2} \mathbb{A})^{\rho_2}) \\ Hom((\mathbb{A}/\pi^{\lambda_1} \mathbb{A})^{\rho_1}, (\mathbb{A}/\pi^{\lambda_3} \mathbb{A})^{\rho_3}) & Hom((\mathbb{A}/\pi^{\lambda_2} \mathbb{A})^{\rho_2}, (\mathbb{A}/\pi^{\lambda_3} \mathbb{A})^{\rho_3}) & \dots & Hom((\mathbb{A}/\pi^{\lambda_k} \mathbb{A})^{\rho_k}, (\mathbb{A}/\pi^{\lambda_3} \mathbb{A})^{\rho_3}) \\ \vdots & \vdots & \ddots & \vdots \\ Hom((\mathbb{A}/\pi^{\lambda_1} \mathbb{A})^{\rho_1}, (\mathbb{A}/\pi^{\lambda_k} \mathbb{A})^{\rho_k}) & Hom((\mathbb{A}/\pi^{\lambda_2} \mathbb{A})^{\rho_2}, (\mathbb{A}/\pi^{\lambda_k} \mathbb{A})^{\rho_k}) & \dots & GL_{\rho_k}(\mathbb{A}/\pi^{\lambda_k} \mathbb{A}) \end{bmatrix}$$

Each element g of the automorphism group $\mathcal{G}_{\underline{\lambda}}$ is represented by a matrix g_{mat} of the following type:

$$g_{mat} = \begin{pmatrix} A_{11} & A_{12}\pi^{\lambda_1-\lambda_2} & \dots & A_{1k}\pi^{\lambda_1-\lambda_k} \\ A_{21} & A_{22} & \dots & A_{2k}\pi^{\lambda_2-\lambda_k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{pmatrix} \quad (3.3)$$

where each A_{ij} is a $\rho_i \times \rho_j$ matrix of elements from \mathbb{A} and $\det(A_{ii})$ is a unit in \mathbb{A} .

Observation 3.4. *This observation has many parts.*

- $|Hom((\mathbb{A}/\pi^{\lambda} \mathbb{A})^r, (\mathbb{A}/\pi^{\mu} \mathbb{A})^s)| = q^{\min(\mu, \lambda)sr}$.
- In case $\mu \geq \lambda$ and $\phi \in Hom((\mathbb{A}/\pi^{\lambda} \mathbb{A})^r, (\mathbb{A}/\pi^{\mu} \mathbb{A})^s)$ then $Image(\phi) \subset (\pi^{\mu-\lambda} \mathbb{A}/(\pi^{\mu} \mathbb{A}))^s$.
- $|End(\mathcal{A}_{\underline{\lambda}})| = q^{\sum_{i,j} \min(\lambda_i, \lambda_j) \rho_i \rho_j}$.
- $|\mathcal{G}_{\underline{\lambda}}| = \prod_i |GL_{\rho_i}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})| q^{\sum_{i,j \ni i \neq j} \min(\lambda_i, \lambda_j) \rho_i \rho_j}$
- There is an exact sequence as follows

$$0 \longrightarrow M_{\rho}(\mathbb{A}/\pi \mathbb{A}) \longrightarrow GL_{\rho}(\mathbb{A}/\pi^{\lambda} \mathbb{A}) \longrightarrow GL_{\rho}(\mathbb{A}/\pi^{\lambda-1} \mathbb{A}) \longrightarrow 1$$

- $|GL_\rho(\mathbb{A}/\pi^\lambda \mathbb{A})| = q^{(\lambda-1)\rho^2} |GL_\rho(\mathbb{A}/\pi \mathbb{A})|$.
- $|GL_\rho(\mathbb{A}/\pi \mathbb{A})| = (q^\rho - 1)(q^\rho - q)(q^\rho - q^2) \dots (q^\rho - q^{\rho-1})$.
- *There is a filtration of subgroups of $GL_\rho(\mathbb{A}/\pi^\lambda \mathbb{A})$ such that each successive quotient is isomorphic to $M_\rho(\mathbb{A}/\pi \mathbb{A})$ hence elementary abelian and the last successive quotient is isomorphic to $GL_\rho(\mathbb{A}/\pi \mathbb{A}) \cong GL_\rho(\mathbb{F}_q)$. This series can be completed to Jordan-Holder series and its length can be determined.*

Proof. Let us prove the exactness of the sequence

$$0 \longrightarrow M_\rho(\mathbb{A}/\pi \mathbb{A}) \longrightarrow GL_\rho(\mathbb{A}/\pi^\lambda \mathbb{A}) \longrightarrow GL_\rho(\mathbb{A}/\pi^{\lambda-1} \mathbb{A}) \longrightarrow 1$$

Since $\pi^{\lambda-1} \mathbb{A}/\pi^\lambda \mathbb{A}$ is a one dimensional vector space over \mathbb{F}_q , we may fix an additive group isomorphism $t : \mathbb{A}/\pi \mathbb{A} \longrightarrow \pi^{\lambda-1} \mathbb{A}/\pi^\lambda \mathbb{A}$. Then define a map $\phi : M_\rho(\mathbb{A}/\pi \mathbb{A}) \longrightarrow GL_\rho(\mathbb{A}/\pi^\lambda \mathbb{A})$ as follows. $\phi(A) \stackrel{def^n}{=} I + t(A)$. The image of ϕ is independent of the isomorphism t we choose. The exactness at $M_\rho(\mathbb{A}/\pi \mathbb{A})$ and $GL_\rho(\mathbb{A}/\pi^\lambda \mathbb{A})$ follows. Now let us prove that the reduction map $GL_\rho(\mathbb{A}/\pi^\lambda \mathbb{A}) \longrightarrow GL_\rho(\mathbb{A}/\pi^{\lambda-1} \mathbb{A})$ is onto. For any $\alpha, \beta \in GL_\rho(\mathbb{A}/\pi^{\lambda-1} \mathbb{A})$ such that $\alpha\beta = \text{identity}$, there exist $A, B \in M_\rho(\mathbb{A})$ such that $AB = I + \pi^{\lambda-1} \Psi$ for some $\Psi \in M_\rho(\mathbb{A})$. So $AB(I - \pi^{\lambda-1} \Psi) = I - \pi^{2(\lambda-1)} \Psi^2$. Now $\lambda \geq 2$ and we have $A, B' = B(I - \pi^{\lambda-1} \Psi)$ such that $\bar{A}\bar{B}' = \text{identity}$ in $GL_\rho(\mathbb{A}/\pi^\lambda \mathbb{A})$ whose further reduction mod $\pi^{\lambda-1}$ gives α, β . □

Observation 3.5. (*Generators for the automorphism group, see Birkhoff [4]*). The transformations on an element $\underline{x} = (x_1, x_2, \dots, x_r) \in \mathcal{A}_\lambda$ and on an element $g = (R_1^t, R_2^t, \dots, R_r^t)^t = (C_1, C_2, \dots, C_r) \in \mathcal{G}_\lambda$ which effect the following moves on \underline{x}

- $x_i \mapsto x_i + \pi^{\lambda_i - \lambda_j} \alpha x_j$ for some $\alpha \in \mathbb{A}$ and $i < j$
- $x_i \mapsto x_i + \alpha x_j$ for some $\alpha \in \mathbb{A}$ and $j < i$
- $x_i \mapsto \beta x_i$ for some unit $\beta \in \mathbb{A}$

- $x_i \longleftrightarrow x_j$ when $\lambda_i = \lambda_j$ (interchange of coordinates)

and respectively on $g \in \mathcal{G}_{\underline{\lambda}}$ given by

- $R_i \mapsto R_i + \pi^{\lambda_i - \lambda_j} \alpha R_j; C_j \mapsto C_j + \pi^{\lambda_i - \lambda_j} \alpha C_i$ for some $\alpha \in \mathbb{A}$ and $i < j$
- $R_i \mapsto R_i + \alpha R_j; C_j \mapsto C_j + \alpha C_i$ for some $\alpha \in \mathbb{A}$ and $j < i$
- $R_i \mapsto \beta R_i; C_j \mapsto \beta C_j$ for some unit $\beta \in \mathbb{A}$
- $R_i \longleftrightarrow R_j; C_i \longleftrightarrow C_j$ when $\lambda_i = \lambda_j$

generate $\mathcal{G}_{\underline{\lambda}}$. For DVR's like $\mathbb{A} = \mathbb{Z}_p$ we can choose $\alpha = 1$ in the above transformations which effect these moves as generators of $\mathcal{G}_{\underline{\lambda}}$.

Proof. Any element of the automorphism group can be reduced to identity matrix by using transformations of the above type. In the case of \mathbb{Z}_p , we only need transformations with $\alpha = 1$ to get generators for the automorphism group. \square

Observation 3.6. (*Characteristic forms of an element*)

(a) *By applying a sequence of automorphisms of $\mathcal{A}_{\underline{\lambda}}$ we can reduce any element to a unique characteristic form*

$$\left(\underbrace{(\pi^{r_1}(1, 0, 0, \dots, 0))}_{(\rho_1)\text{-tuple}}, \underbrace{(\pi^{r_2}(1, 0, 0, \dots, 0))}_{(\rho_2)\text{-tuple}}, \dots, \underbrace{(\pi^{r_i}(1, 0, 0, \dots, 0))}_{(\rho_i)\text{-tuple}}, \dots, \underbrace{(\pi^{r_k}(1, 0, 0, \dots, 0))}_{(\rho_k)\text{-tuple}} \right)^t$$

such that $r_{i+1} \leq r_i \leq r_{i+1} + \lambda_i - \lambda_{i+1}$ for all i .

(b) *For the abelian group $\mathcal{A}_{\underline{\lambda}}$ we can reduce any element to an alternative characteristic form namely*

$$\left(\underbrace{(\pi^{r_1}, \pi^{r_1}, \pi^{r_1}, \dots, \pi^{r_1})}_{\rho_1 \text{ elements}}, \underbrace{(\pi^{r_2}, \pi^{r_2}, \pi^{r_2}, \dots, \pi^{r_2})}_{\rho_2 \text{ elements}}, \underbrace{(\pi^{r_3}, \pi^{r_3}, \pi^{r_3}, \dots, \pi^{r_3})}_{\rho_3 \text{ elements}}, \dots, \underbrace{(\pi^{r_k}, \pi^{r_k}, \dots, \pi^{r_k})}_{\rho_k \text{ elements}} \right)^t$$

such that $r_{i+1} \leq r_i \leq r_{i+1} + \lambda_i - \lambda_{i+1}$ for all i . (See Birkhoff [4] and Miller [11]).

Also look at Observation 3.21.

3.3 Fundamental Poset and Characteristic Submodules

The group \mathcal{G}_λ acts on \mathcal{A}_λ^n by the diagonal action:

$$g \cdot (x_1, \dots, x_n) = (g(x_1), \dots, g(x_n)) \text{ for } x_i \in \mathcal{A}_\lambda \text{ and } g \in \mathcal{G}_\lambda.$$

For $n = 1$, this is just the action on \mathcal{A}_λ of its automorphism group \mathcal{G}_λ . The \mathcal{G}_λ -orbits in \mathcal{A}_λ have been understood quite well for over a hundred years (see Miller [11], Birkhoff [4]). Some qualitative results concerning \mathcal{G}_λ -orbits in \mathcal{A}_λ^n for general n were obtained by Calvert, Dutta and Prasad [6]. For the present purposes, however, the combinatorial description of orbits due to Dutta and Prasad [8] is the most helpful. This section will be a quick recapitulation of those results. Later in Chapter 2, we describe the set of \mathcal{G}_λ -orbits in \mathcal{A}_λ^2 under the above action for $n = 2$.

It turns out that for any module \mathcal{A}_λ of the form given in equation (3.2), the \mathcal{G}_λ -orbits in \mathcal{A}_λ are in bijective correspondence with a certain class of ideals in a poset \mathcal{P} , which we call the fundamental poset. As a set,

$$\mathcal{P} = \{(v, k) \mid k \in \mathbb{N}, 0 \leq v < k\}.$$

The partial order on \mathcal{P} is defined by setting

$$(v, l) \leq (v', l') \text{ if and only if } v \geq v' \text{ and } l - v \leq l' - v'.$$

The Hasse diagram of the fundamental poset \mathcal{P} is shown in Figure 3.1. Let $J(\mathcal{P})$ denote the lattice of order ideals in \mathcal{P} . A typical element of \mathcal{A}_λ from equation (3.2) is a vector of the form

$$e = (e_{\lambda_i, t_i}),$$

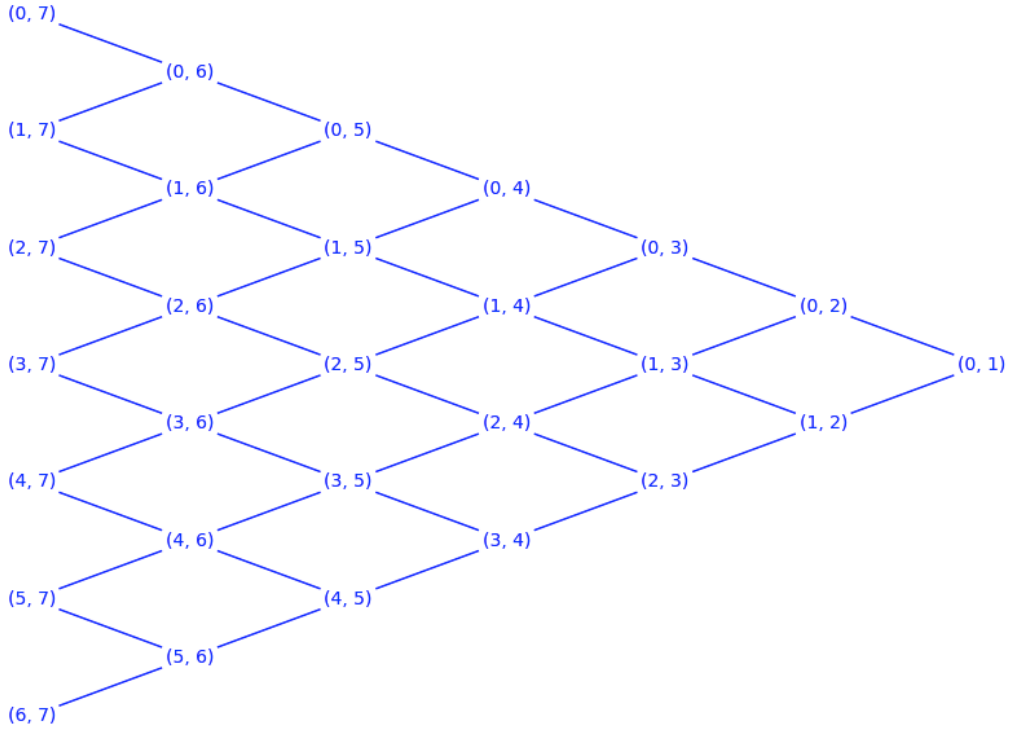


Figure 3.1: The Fundamental Poset

where i runs over the set $\{1, \dots, k\}$, and for each i , t_i runs over the set $\{1, \dots, \rho_i\}$. To $e \in \mathcal{A}_\lambda$ associate the order ideal $I(e) \subset \mathcal{P}$ generated by the elements

$$\left(\min_{t_i \in \{1, \dots, \rho_i\}} v_\pi(e_{\lambda_i, t_i}), \lambda_i \right)$$

for all t_i and for all i such that the coordinate $e_{\lambda_i, t_i} \neq 0$ in $\mathbb{A}/\pi^{\lambda_i} \mathbb{A}$. Here $v_\pi(e_{\lambda_i, t_i})$ is the unique integer j such that $v_\pi(e_{\lambda_i, t_i}) = u\pi^j$ for some unit u in $\mathbb{A}/\pi^{\lambda_i} \mathbb{A}$. In particular, $v(0) = \infty$. Note that the order ideal $I(0)$ equals the empty ideal.

Consider for example, in the finite abelian p -group (or \mathbb{Z}_p -module):

$$\mathcal{A}_\lambda = \mathbb{Z}/p^5\mathbb{Z} \oplus \mathbb{Z}/p^4\mathbb{Z} \oplus \mathbb{Z}/p^4\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^1\mathbb{Z}. \quad (3.7)$$

The order ideal $I(e)$ of $e = (0, up, p^2, vp, 1)$, when u and v are coprime to p is the ideal generated by $\{(1, 4), (1, 2), (0, 1)\}$ represented inside \mathcal{P} by filled-in circles (both grey and black; the significance of the colours will be explained later) in Figure 3.2.

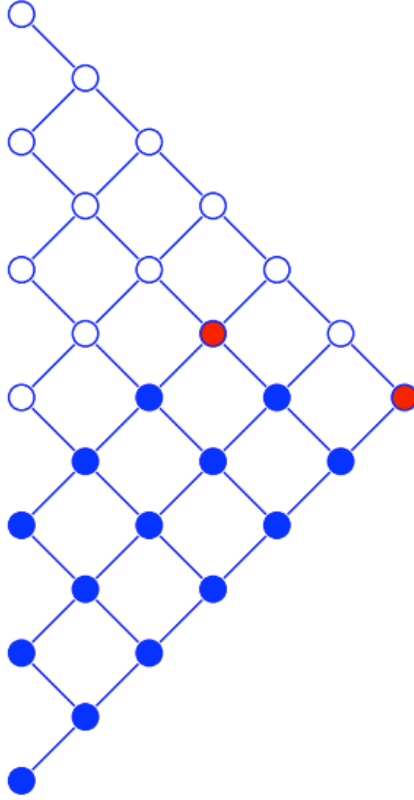


Figure 3.2: The ideal $I(e)$ of $e = (0, up, p^2, vp, 1)$ is generated by $\{(1, 4), (1, 2), (0, 1)\}$

Since the labels of the vertices can be inferred from their positions, they are omitted.

A key observation of Dutta and Prasad [8] is the following theorem:

Theorem 3.8. *Let \mathcal{A}_λ and \mathcal{A}_μ be two finite \mathbb{A} -modules. An element $f \in \mathcal{A}_\mu$ is a homomorphic image of $e \in \mathcal{A}_\lambda$ (in other words, there exists a homomorphism $\phi : \mathcal{A}_\lambda \rightarrow \mathcal{A}_\mu$ such that $\phi(e) = f$) if and only if $I(e) \supset I(f)$.*

It turns out that when the ideals $I(e), I(e')$ corresponding to two elements $e, e' \in \mathcal{A}_\lambda$ are equal then the elements e, e' are automorphic.

Theorem 3.9. *For any $e, e' \in \mathcal{A}_\lambda$, if $I(e) = I(e')$, then e and e' lie in the same \mathcal{G}_λ -orbit.*

This establishes the partial order on the \mathcal{G}_λ -orbits of elements of an abelian group.

Let $J(\mathcal{P})_\lambda$ denote the sublattice of $J(\mathcal{P})$ consisting of ideals such that $\max(I)$ is

contained in the set

$$\mathcal{P}_\lambda = \{(v, l) \in \mathcal{P} \mid l = \lambda_i \text{ for some } 1 \leq i \leq k\}.$$

Then the \mathcal{G}_λ -orbits in \mathcal{A}_λ are in bijective correspondence with the order ideals in $J(\mathcal{P})_\lambda$. For each order ideal $I \subset \mathcal{P}$, we use the notation $\mathcal{A}_\lambda^{I^*}$ or \mathcal{O}_I for the orbit corresponding to I , namely,

$$\mathcal{A}_\lambda^{I^*} = \mathcal{O}_I = \{e \in \mathcal{A}_\lambda \mid I(e) = I\}.$$

A convenient way to think about ideals in \mathcal{P} is in terms of what we call their boundaries: for each positive integer l define the boundary valuation of I at l to be

$$\partial_l I = \min\{v \mid (v, l) \in I\}. \quad (3.10)$$

We denote the sequence $\{\partial_l I\}$ of boundary valuations by ∂I and call it the boundary of I . This is indeed the boundary of the region with colored dots in Figure 3.2.

For each order ideal $I \subset \mathcal{P}$, let $\max(I)$ denote its set of maximal elements. For example, the maximal elements of the ideal in Figure 3.2 are represented by grey circles.

The ideal I is completely determined by $\max(I)$: in fact taking I to $\max(I)$ gives a bijection from the lattice $J(\mathcal{P})_\lambda$ to the set of antichains in \mathcal{P}_λ .

Theorem 3.11. *The orbits $\mathcal{A}_\lambda^{I^*}$ consists of elements $e = (e_{\lambda_i, t_i})$ such that $v((e_{\lambda_i, t_i})) \geq \partial_{\lambda_i} I$ for all λ_i and t_i , and such that $v(m_{\lambda_i, t_i}) = \partial_{\lambda_i} I$ for at least one t_i if $(\partial_{\lambda_i} I, \lambda_i) \in \max(I)$.*

In other words, the elements of $\mathcal{A}_\lambda^{I^*}$ are those elements all of whose coordinates have valuations not less than the corresponding boundary valuation, and at least one coordinate corresponding to each maximal element of I has valuation exactly

equal to the corresponding boundary valuation.

In the running example with \mathcal{A}_λ as given in equation (3.7) and I as in Figure 3.2, the conditions for $e = (e_{5,1}, e_{4,1}, e_{4,2}, e_{2,1}, e_{1,1})$ to be in $\mathcal{A}_\lambda^{I^*}$ are:

1. $v(e_{5,1}) \geq 4$,
2. $\min(v(e_{4,1}), v(e_{4,2})) = 1$,
3. $v(e_{2,1}) \geq 1$,
4. $v(e_{1,1}) = 0$.

For each $I \in J(\mathcal{P})_\lambda$, with

$$\max(I) = \{(v_1, l_1), \dots, (v_s, l_s)\}$$

define an element $e(I)$ of \mathcal{A}_λ whose coordinates are given by

$$e_{\lambda_i, t_i} = \begin{cases} \pi^{v_j} & \text{if } \lambda_i = l_j \text{ and } t_j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, for each element (v_j, l_j) of $\max I$, pick λ_i such that $\lambda_i = l_j$. In the summand $(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\oplus \rho_i}$, set the first coordinate of $e(I)$ to π^{v_j} , and the remaining coordinates to 0.

For example, in the finite abelian p -group of the form given in equation (3.7), and the ideal I of Figure 3.2,

$$e(I) = (0, p, 0, 0, 1) \quad \max(I) = \{(1, 4), (0, 1)\}$$

Theorem 3.12. *Let \mathcal{A}_λ be a finite \mathbb{A} -module of the type given in equation (3.2). The functions $\mathcal{O} \mapsto I(e)$ for any $e \in \mathcal{O}$ and $I \mapsto \mathcal{O}_I$ the orbit of $e(I)$ are mutually*

inverse bijections between the set of $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}}$ and the set of order ideals in $J(\mathcal{P})_{\underline{\lambda}}$.

We shall say that an element of $\mathcal{A}_{\underline{\lambda}}$ is a canonical form if it is equal to $e(I)$ for some order ideal $I \in J(\mathcal{P})_{\underline{\lambda}} = J(\mathcal{P}_{\underline{\lambda}})$.

The set of endomorphic images of elements in this orbit is a $\mathcal{G}_{\underline{\lambda}}$ -invariant submodule of $\mathcal{A}_{\underline{\lambda}}$ which we denote by $\mathcal{A}_{\underline{\lambda}}^I$. We have

$$\mathcal{A}_{\underline{\lambda}}^I = \bigsqcup_{\{J \in J(\mathcal{P})_{\underline{\lambda}} \mid J \subset I\}} \mathcal{A}_{\underline{\lambda}}^{J^*}. \quad (3.13)$$

This submodule is a characteristic submodule as it is a union of $\mathcal{G}_{\underline{\lambda}}$ invariant sets (a submodule of $\mathcal{A}_{\underline{\lambda}}$ is said to be characteristic if it is a $\mathcal{G}_{\underline{\lambda}}$ invariant submodule of $\mathcal{A}_{\underline{\lambda}}$).

The description of $\mathcal{A}_{\underline{\lambda}}^I$ in terms of valuations of coordinates and boundary valuations is very simple:

$$\mathcal{A}_{\underline{\lambda}}^I = \{e = (e_{\lambda_i, t_i}) \mid v(e_{\lambda_i, t_i}) \geq \partial_{\lambda_i} I\}. \quad (3.14)$$

Note that the map $I \mapsto \mathcal{A}_{\underline{\lambda}}^I$ is not injective on $J(\mathcal{P})$. It becomes injective when restricted to $J(\mathcal{P})_{\underline{\lambda}}$. For example, if J is the order ideal in \mathcal{P} generated by $(2, 6)$, $(1, 4)$ and $(0, 1)$, then the ideal J is strictly larger than the ideal I of Figure 3.2, but when $\mathcal{A}_{\underline{\lambda}}$ is as given in equation (3.7), $\mathcal{A}_{\underline{\lambda}}^I = \mathcal{A}_{\underline{\lambda}}^J$.

The $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}}$ are parametrized by the finite distributive lattice $J(\mathcal{P})_{\underline{\lambda}}$. Moreover, each order ideal $I \in J(\mathcal{P})_{\underline{\lambda}}$ gives rise to a $\mathcal{G}_{\underline{\lambda}}$ -invariant submodule $\mathcal{A}_{\underline{\lambda}}^I$ of $\mathcal{A}_{\underline{\lambda}}$. The lattice structure of $J(\mathcal{P})_{\underline{\lambda}}$ gets reflected in the poset structure of the submodules $\mathcal{A}_{\underline{\lambda}}^I$ when they are partially ordered by inclusion:

Theorem 3.15. *Let $\mathcal{A}_{\underline{\lambda}}$ be a finite \mathbb{A} module as given in equation (3.2). The function $I \mapsto \mathcal{A}_{\underline{\lambda}}^I$, with $\mathcal{A}_{\underline{\lambda}}^I$ as in equation (3.13), is a lattice isomorphism between the set of order ideals in $J(\mathcal{P})_{\underline{\lambda}}$ and the set of characteristic submodules of $\mathcal{A}_{\underline{\lambda}}$ of*

the form $\mathcal{A}_{\underline{\lambda}}^I$.

In other words, for ideals $I, J \in J(\mathcal{P})_{\underline{\lambda}}$,

$$\mathcal{A}_{\underline{\lambda}}^{I \cup J} = \mathcal{A}_{\underline{\lambda}}^I + \mathcal{A}_{\underline{\lambda}}^J \text{ and } \mathcal{A}_{\underline{\lambda}}^{I \cap J} = \mathcal{A}_{\underline{\lambda}}^I \cap \mathcal{A}_{\underline{\lambda}}^J.$$

In fact, when $\mathbb{F}_q \cong \mathbb{A}/\pi^1\mathbb{A}$ with \mathbb{F}_q having at least three elements, every $\mathcal{G}_{\underline{\lambda}}$ -invariant submodule is of the form $\mathcal{A}_{\underline{\lambda}}^I$, therefore $J(\mathcal{P})_{\underline{\lambda}}$ is isomorphic to the lattice of $\mathcal{G}_{\underline{\lambda}}$ -invariant submodules (Kerby and Rode [10]). This is not true for $q = 2$. Consider the abelian group $\mathbb{Z}/2^3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and the subgroup $H = \{(0, 0), (2, 1), 2(2, 1) = (4, 0), 3(2, 1) = (6, 1)\}$. This subgroup is characteristic but it does not correspond to any ideal in $J(\mathcal{P})_{\underline{\lambda}}$ where $\underline{\lambda} = (2^3, 1) \in \Lambda$.

The cardinality of the orbit $\mathcal{A}_{\underline{\lambda}}^{I^*}$ is given by

$$|\mathcal{A}_{\underline{\lambda}}^{I^*}| = q^{[I]_{\lambda}} \prod_{(v_i, \lambda_i) \in \max I} (1 - q^{-\rho_i}). \quad (3.16)$$

Here $[I]_{\lambda}$ denotes the number of points in $I \cap \mathcal{P}_{\lambda}$ counted *with multiplicity*:

$$[I]_{\lambda} = \sum_{i=1}^l \sum_{\{v|(v, \lambda_i) \in I\}} \rho_i.$$

In particular, we have:

Theorem 3.17. *Let q denote the cardinality of the residue field of \mathbb{A} . When $\mathcal{A}_{\underline{\lambda}}$ is finite, the cardinality of $\mathcal{A}_{\underline{\lambda}}^{I^*}$ is a monic polynomial in q of degree $[I]_{\lambda}$ whose coefficients are integers which do not depend on \mathbb{A} .*

The formula for the cardinality of the $\mathcal{G}_{\underline{\lambda}}$ -invariant submodule is much simpler:

$$|\mathcal{A}_{\underline{\lambda}}^I| = q^{[I]_{\lambda}}. \quad (3.18)$$

3.3.1 Illustrations with some Examples

For the sake of concreteness, take $\mathbb{A} = \mathbb{Z}_p$ (p -adic integers) for some prime p . Then a finite \mathbb{A} module \mathcal{A}_λ is just a finite abelian p -group. However these examples can be worked out for a general DVR \mathbb{A} with uniformizer π .

Example 1 :

1. $\mathcal{A}_\lambda = \mathbb{Z}/p\mathbb{Z}$ and $\mathcal{G}_\lambda = \text{Aut}(\mathbb{Z}/p\mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})^*$
2. Number of orbits of the group under the automorphism group = 2.
3. Partition $\lambda = (1^1) \in \Lambda$

| Ideal | Orbit $\mathcal{A}_\lambda^{I^*}$ | \mathcal{A}_λ^I |
|-----------------------|-----------------------------------|--------------------------|
| $\max I = \{(0, 1)\}$ | $\{1, 2, 3, \dots, p-1\}$ | $\mathbb{Z}/p\mathbb{Z}$ |
| $\max I = \emptyset$ | $\{0\}$ | 0 |

Example 2 :

1. $\mathcal{A}_\lambda = (\mathbb{Z}/p\mathbb{Z})^n$ and $\mathcal{G}_\lambda = \text{Aut}((\mathbb{Z}/p\mathbb{Z})^n) = GL_n(\mathbb{Z}/p\mathbb{Z})$
2. Number of orbits of the group under the automorphism group = 2.
3. Partition $\lambda = (1^n) \in \Lambda$

| Ideal | Orbit $\mathcal{A}_\lambda^{I^*}$ | \mathcal{A}_λ^I |
|-----------------------|--|------------------------------|
| $\max I = \{(0, 1)\}$ | $\{v \in (\mathbb{Z}/p\mathbb{Z})^n \mid v \neq 0\}$ | $(\mathbb{Z}/p\mathbb{Z})^n$ |
| $\max I = \emptyset$ | $\{0\}$ | 0 |

Example 3 :

1. $\mathcal{A}_\lambda = \mathbb{Z}/p^k\mathbb{Z}$ and $\mathcal{G}_\lambda = \text{Aut}(\mathbb{Z}/p^k\mathbb{Z}) = (\mathbb{Z}/p^k\mathbb{Z})^* = (\mathbb{Z}/p^k\mathbb{Z}) - p(\mathbb{Z}/p^k\mathbb{Z})$
2. Number of orbits of the group under the automorphism group = $k + 1$.

3. Partition $\underline{\lambda} = (k^1) \in \Lambda$

| Ideal | Orbit $\mathcal{A}_{\underline{\lambda}}^{I^*}$ | $\mathcal{A}_{\underline{\lambda}}^I$ |
|-------------------------------------|---|---------------------------------------|
| $\max I = \{(j, k)\}, 0 \leq j < k$ | $p^j(\mathbb{Z}/p^k\mathbb{Z}) - p^{j+1}(\mathbb{Z}/p^k\mathbb{Z})$ | $p^j(\mathbb{Z}/p^k\mathbb{Z})$ |
| $\max I = \emptyset$ | $\{0\}$ | 0 |

Example 4 :

1. $\mathcal{A}_{\underline{\lambda}} = (\mathbb{Z}/p^k\mathbb{Z})^n$ and $\mathcal{G}_{\underline{\lambda}} = \text{Aut}((\mathbb{Z}/p^k\mathbb{Z})^n) = GL_n(\mathbb{Z}/p^k\mathbb{Z})$
2. Number of orbits of the group under the automorphism group = $(k + 1)$.
3. Partition $\underline{\lambda} = (k^n) \in \Lambda$

| Ideal | Orbit $\mathcal{A}_{\underline{\lambda}}^{I^*}$ | $\mathcal{A}_{\underline{\lambda}}^I$ |
|-------------------------------------|---|---------------------------------------|
| $\max I = \{(j, k)\}, 0 \leq j < k$ | $p^j(\mathbb{Z}/p^k\mathbb{Z})^n - p^{j+1}(\mathbb{Z}/p^k\mathbb{Z})^n$ | $p^j(\mathbb{Z}/p^k\mathbb{Z})^n$ |
| $\max I = \emptyset$ | $\{\underline{0}\}$ | 0 |

In the following cases, the orbit description is given below the table in each case.

Example 5 :

1. $\mathcal{A}_{\underline{\lambda}} = \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ and $\mathcal{G}_{\underline{\lambda}} = \text{Aut}(\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z})$
2. Number of orbits of the group under the automorphism group = 4.
3. Partition $\underline{\lambda} = (2, 1) \in \Lambda$

| Ideal | Orbit $\mathcal{A}_{\underline{\lambda}}^{I^*}$ | $\mathcal{A}_{\underline{\lambda}}^I$ |
|-----------------------|---|---|
| $\max I = \{(0, 2)\}$ | \mathcal{O}_3 | $(\mathbb{Z}/p^2\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z})$ |
| $\max I = \{(0, 1)\}$ | \mathcal{O}_2 | $p(\mathbb{Z}/p^2\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z})$ |
| $\max I = \{(1, 2)\}$ | \mathcal{O}_1 | $p(\mathbb{Z}/p^2\mathbb{Z}) \oplus 0$ |
| $\max I = \emptyset$ | $\{\underline{0}\}$ | 0 |

$$\mathcal{O}_3 = \{(y, x) \in \mathcal{A}_\lambda \ni p \nmid y\}$$

$$\mathcal{O}_2 = \{(y, x) \in \mathcal{A}_\lambda \ni p \nmid x \ \& \ p \mid y\}$$

$$\mathcal{O}_1 = \{(y, 0) \in \mathcal{A}_\lambda \ni p \mid y \ \& \ p^2 \nmid y\}$$

Example 6 :

1. $\mathcal{A}_\lambda = (\mathbb{Z}/p^2\mathbb{Z})^i \oplus (\mathbb{Z}/p\mathbb{Z})^j$ and $\mathcal{G}_\lambda = \text{Aut}((\mathbb{Z}/p^2\mathbb{Z})^i \oplus (\mathbb{Z}/p\mathbb{Z})^j)$
2. Number of orbits of the group under the automorphism group = 4.
3. Partition $\underline{\lambda} = (2^i, 1^j) \in \Lambda$

| Ideal | Orbit $\mathcal{A}_\lambda^{I^*}$ | \mathcal{A}_λ^I |
|-----------------------|-----------------------------------|---|
| $\max I = \{(0, 2)\}$ | \mathcal{O}_3 | $(\mathbb{Z}/p^2\mathbb{Z})^i \oplus (\mathbb{Z}/p\mathbb{Z})^j$ |
| $\max I = \{(0, 1)\}$ | \mathcal{O}_2 | $p(\mathbb{Z}/p^2\mathbb{Z})^i \oplus (\mathbb{Z}/p\mathbb{Z})^j$ |
| $\max I = \{(1, 2)\}$ | \mathcal{O}_1 | $p(\mathbb{Z}/p^2\mathbb{Z})^i \oplus 0$ |
| $\max I = \emptyset$ | $\{0\}$ | 0 |

$$\mathcal{O}_3 = \{(\underline{y}, \underline{x}) \in \mathcal{A}_\lambda \ni \underline{y} \in (\mathbb{Z}/p^2\mathbb{Z})^i - p(\mathbb{Z}/p^2\mathbb{Z})^i\}$$

$$\mathcal{O}_2 = \{(\underline{y}, \underline{x}) \in \mathcal{A}_\lambda \ni \underline{x} \in (\mathbb{Z}/p\mathbb{Z})^j - \{0\} \ \& \ \underline{y} \in p(\mathbb{Z}/p^2\mathbb{Z})^i\}$$

$$\mathcal{O}_1 = \{(\underline{y}, 0) \in \mathcal{A}_\lambda \ni \underline{y} \in p(\mathbb{Z}/p^2\mathbb{Z})^i - \{0\}\}$$

Example 7 :

1. $\mathcal{A}_\lambda = \mathbb{Z}/p^{k+1}\mathbb{Z} \oplus \mathbb{Z}/p^k\mathbb{Z}$ and $\mathcal{G}_\lambda = \text{Aut}(\mathbb{Z}/p^{k+1}\mathbb{Z} \oplus \mathbb{Z}/p^k\mathbb{Z})$
2. Number of orbits of the group under the automorphism group = $2k + 2$.
3. Partition $\underline{\lambda} = (k + 1, k) \in \Lambda$

| Ideal | Orbit $\mathcal{A}_\lambda^{I^*}$ | \mathcal{A}_λ^I |
|---|-----------------------------------|--|
| $\max I = \{(j, k + 1)\}, 0 \leq j < k + 1$ | \mathcal{O}_{1j} | $p^j(\mathbb{Z}/p^{k+1}\mathbb{Z}) \oplus p^j(\mathbb{Z}/p^k\mathbb{Z})$ |
| $\max I = \{(i, k)\}, 0 \leq i < k$ | \mathcal{O}_{2i} | $p^{(i+1)}(\mathbb{Z}/p^{k+1}\mathbb{Z}) \oplus p^i(\mathbb{Z}/p^k\mathbb{Z})$ |
| $\max I = \emptyset$ | $\{0\}$ | 0 |

$\mathcal{O}_{1j} = \{(y, x) \in \mathcal{A}_{\underline{\lambda}} \ni p^j \mid y \ \& \ p^j \mid x\}$ (Here $p^\alpha \parallel a$ means p^α exactly divides a)

$\mathcal{O}_{2i} = \{(y, x) \in \mathcal{A}_{\underline{\lambda}} \ni p^i \mid x \ \& \ p^{(i+1)} \mid y\}$

Example 8 :

1. $\mathcal{A}_{\underline{\lambda}} = (\mathbb{Z}/p^{k+1}\mathbb{Z})^r \oplus (\mathbb{Z}/p^k\mathbb{Z})^s$ and $\mathcal{G}_{\underline{\lambda}} = \text{Aut}((\mathbb{Z}/p^{k+1}\mathbb{Z})^r \oplus (\mathbb{Z}/p^k\mathbb{Z})^s)$
2. Number of orbits of the group under the automorphism group = $2k + 2$.
3. Partition $\underline{\lambda} = ((k + 1)^r, k^s) \in \Lambda$

| Ideal | Orbit $\mathcal{A}_{\underline{\lambda}}^{I^*}$ | $\mathcal{A}_{\underline{\lambda}}^I$ |
|---|---|--|
| $\max I = \{(j, k + 1)\}, 0 \leq j < k + 1$ | \mathcal{O}_{1j} | $p^j(\mathbb{Z}/p^{k+1}\mathbb{Z})^r \oplus p^j(\mathbb{Z}/p^k\mathbb{Z})^s$ |
| $\max I = \{(i, k)\}, 0 \leq i < k$ | \mathcal{O}_{2i} | $p^{(i+1)}(\mathbb{Z}/p^{k+1}\mathbb{Z})^r \oplus p^i(\mathbb{Z}/p^k\mathbb{Z})^s$ |
| $\max I = \emptyset$ | $\{\underline{0}\}$ | 0 |

$\mathcal{O}_{1j} = (p^j(\mathbb{Z}/p^{k+1}\mathbb{Z})^r - p^{j+1}(\mathbb{Z}/p^{k+1}\mathbb{Z})^r) \times (p^j(\mathbb{Z}/p^k\mathbb{Z})^s)$

$\mathcal{O}_{2i} = (p^{(i+1)}(\mathbb{Z}/p^{k+1}\mathbb{Z})^r) \times (p^i(\mathbb{Z}/p^k\mathbb{Z})^s - p^{(i+1)}(\mathbb{Z}/p^k\mathbb{Z})^s)$

Example 9 :

1. $\mathcal{A}_{\underline{\lambda}} = \mathbb{Z}/p^l\mathbb{Z} \oplus \mathbb{Z}/p^k\mathbb{Z}$ with $l > k$ and $\mathcal{G}_{\underline{\lambda}} = \text{Aut}(\mathbb{Z}/p^l\mathbb{Z} \oplus \mathbb{Z}/p^k\mathbb{Z})$
2. Number of orbits of the group under the automorphism group = $(l - k + 1)(k + 1)$.
3. Partition $\underline{\lambda} = (l, k) \in \Lambda$

| Ideal | Orbit $\mathcal{A}_{\underline{\lambda}}^{I^*}$ | $\mathcal{A}_{\underline{\lambda}}^I$ |
|---|---|--|
| $\max I = \{(j, l)\}, 0 \leq j < l$ | \mathcal{O}_{1j} | $p^j(\mathbb{Z}/p^l\mathbb{Z}) \oplus p^j(\mathbb{Z}/p^k\mathbb{Z})$ |
| $\max I = \{(i, k)\}, 0 \leq i < k$ | \mathcal{O}_{2i} | $p^{(i+l-k)}(\mathbb{Z}/p^l\mathbb{Z}) \oplus p^i(\mathbb{Z}/p^k\mathbb{Z})$ |
| $\max I = \emptyset$ | $\{\underline{0}\}$ | 0 |
| $\max I = \{(i + r, l), (i, k)\},$ $0 \leq i < k \ \& \ 0 < r < l - k$ | \mathcal{O}_{3ir} | $p^{(i+r)}(\mathbb{Z}/p^l\mathbb{Z}) \oplus p^i(\mathbb{Z}/p^k\mathbb{Z})$ |

$$\mathcal{O}_{1j} = \{(y, x) \in \mathcal{A}_{\underline{\lambda}} \ni p^j \mid y \ \& \ p^j \mid x\}$$

$$\mathcal{O}_{2i} = \{(y, x) \in \mathcal{A}_{\underline{\lambda}} \ni p^i \mid x \ \& \ p^{i+l-k} \mid y\}$$

$$\mathcal{O}_{3ir} = \{(y, x) \in \mathcal{A}_{\underline{\lambda}} \ni p^i \mid x \ \& \ p^{i+r} \mid y\}$$

Here in the case of \mathcal{O}_{3ir} the corresponding ideal contains two elements. The ideal is called a non-principal ideal.

Example 10 :

1. $\mathcal{A}_{\underline{\lambda}} = (\mathbb{Z}/p^l\mathbb{Z})^r \oplus (\mathbb{Z}/p^k\mathbb{Z})^s$ with $l > k$ and $\mathcal{G}_{\underline{\lambda}} = \text{Aut}((\mathbb{Z}/p^l\mathbb{Z})^r \oplus (\mathbb{Z}/p^k\mathbb{Z})^s)$
2. Number of orbits of the group under the automorphism group = $(l - k + 1)(k + 1)$.
3. Partition $\underline{\lambda} = (l^r, k^s) \in \Lambda$

| Ideal | Orbit $\mathcal{A}_{\underline{\lambda}}^{I^*}$ | $\mathcal{A}_{\underline{\lambda}}^I$ |
|---|---|--|
| $\max I = \{(j, l)\}, 0 \leq j < l$ | \mathcal{O}_{1j} | $p^j(\mathbb{Z}/p^l\mathbb{Z})^r \oplus p^j(\mathbb{Z}/p^k\mathbb{Z})^s$ |
| $\max I = \{(i, k)\}, 0 \leq i < k$ | \mathcal{O}_{2i} | $p^{(i+l-k)}(\mathbb{Z}/p^l\mathbb{Z})^r \oplus p^i(\mathbb{Z}/p^k\mathbb{Z})^s$ |
| $\max I = \emptyset$ | $\{0\}$ | 0 |
| $\max I = \{(i+r, l), (i, k)\},$ $0 \leq i < k \ \& \ 0 < r < l - k$ | \mathcal{O}_{3ir} | $p^{(i+r)}(\mathbb{Z}/p^l\mathbb{Z})^r \oplus p^i(\mathbb{Z}/p^k\mathbb{Z})^s$ |

$$\mathcal{O}_{1j} = (p^j(\mathbb{Z}/p^l\mathbb{Z})^r - p^{j+1}(\mathbb{Z}/p^l\mathbb{Z})^r) \times (p^j(\mathbb{Z}/p^k\mathbb{Z})^s)$$

$$\mathcal{O}_{2i} = (p^{i+l-k}(\mathbb{Z}/p^l\mathbb{Z})^r) \times (p^i(\mathbb{Z}/p^k\mathbb{Z})^s - p^{(i+1)}(\mathbb{Z}/p^k\mathbb{Z})^s)$$

$$\mathcal{O}_{3ir} = (p^{(i+r)}(\mathbb{Z}/p^l\mathbb{Z})^r - p^{(i+r+1)}(\mathbb{Z}/p^l\mathbb{Z})^r) \times (p^i(\mathbb{Z}/p^k\mathbb{Z})^s - p^{(i+1)}(\mathbb{Z}/p^k\mathbb{Z})^s)$$

Here in the case of \mathcal{O}_{3ir} the corresponding ideal contains two elements. This ideal cannot be generated by a single element.

Observation 3.19. Let $\mathcal{A}_{\underline{\lambda}}$ be a finite \mathbb{A} -module and $\mathcal{G}_{\underline{\lambda}}$ denote its automorphism group. Given an ideal $I \subset \mathcal{P}_{\underline{\lambda}}$, the structure of the orbit \mathcal{O}_I is given by

$$\prod_{(\partial_{\lambda_i I}, \lambda_i) \notin \max(I)} \pi^{(\partial_{\lambda_i I})}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \times \prod_{(\partial_{\lambda_i I}, \lambda_i) \in \max(I)} (\pi^{(\partial_{\lambda_i I})}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i I})+1}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i})$$

and the characteristic submodule $\mathcal{A}_{\underline{\lambda}}^I$ is given by

$$\prod_{(\partial_{\lambda_i I}, \lambda_i)} \pi^{(\partial_{\lambda_i I})}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$$

Observe that each orbit is a product set or a box set in terms of its isotypic components and its cardinality is a polynomial in $q = |\mathbb{F}_q \cong \mathbb{A}/\pi^1 \mathbb{A}|$.

Observation 3.20. (See Miller [11]). Let $\mathcal{A}_{\underline{\lambda}}$ be a finite \mathbb{A} -module of type $\underline{\lambda} \in \Lambda$ as given in equation (3.2). The number of characteristic submodules of $\mathcal{A}_{\underline{\lambda}}$ is given by $\left(\prod_{i=1}^{k-1} (\lambda_i - \lambda_{i+1} + 1) \right) (\lambda_k + 1)$.

Observation 3.21. (See Dutta and Prasad [8], page 8). Given any element $a \in \mathcal{A}_{\underline{\lambda}}$ with ideal $I(a) = I \subset \mathcal{P}_{\lambda}$ such that $\max(I) = \{(r_{i_1}, \lambda_{i_1}), \dots, (r_{i_s}, \lambda_{i_s})\}$ there exists a unique element in the orbit \mathcal{O}_I of the type

$$(0, 0, 0, \dots, \underbrace{\pi^{r_{i_1}}(1, 0, 0, \dots, 0)}_{(\rho_{i_1})\text{-tuple}}, \dots, 0, 0, 0, \dots, \underbrace{\pi^{r_{i_2}}(1, 0, 0, \dots, 0)}_{(\rho_{i_2})\text{-tuple}}, \dots, 0, 0, 0, \dots, \underbrace{\pi^{r_{i_t}}(1, 0, 0, \dots, 0)}_{(\rho_{i_t})\text{-tuple}}, \dots, 0, 0, 0, \dots, \underbrace{\pi^{r_{i_s}}(1, 0, 0, \dots, 0)}_{(\rho_{i_s})\text{-tuple}}, \dots, 0, 0, 0, \dots, 0, 0, 0)^t$$

Observation 3.22. Let $\mathcal{A}_{\underline{\lambda}} = \mathbb{A}/\pi^{\lambda_1} \mathbb{A} \oplus \mathbb{A}/\pi^{\lambda_2} \mathbb{A}$. Let $\pi^{r_1}(\mathbb{A}/\pi^{\lambda_1} \mathbb{A}) \oplus \pi^{r_2}(\mathbb{A}/\pi^{\lambda_2} \mathbb{A})$ be a characteristic submodule. i.e., we have

- $\lambda_1 \geq \lambda_2, r_1 \leq \lambda_1, r_2 \leq \lambda_2$
- $r_1 \geq r_2, \lambda_1 - r_1 \geq \lambda_2 - r_2$

The number of characteristic submodules in $\mathcal{A}_{\underline{\lambda}}$ containing

$\pi^{r_1}(\mathbb{A}/\pi^{\lambda_1} \mathbb{A}) \oplus \pi^{r_2}(\mathbb{A}/\pi^{\lambda_2} \mathbb{A})$ is

- $(r_1 + \lambda_2 - \lambda_1 + 1)(\lambda_1 - \lambda_2 + 1) + \frac{1}{2}((\lambda_1 - \lambda_2) - (r_1 - r_2))((\lambda_1 - \lambda_2) + (r_1 - r_2) + 1)$
if $0 \leq r_1 + \lambda_2 - \lambda_1 < r_2 \Leftrightarrow (r_1 - r_2 < \lambda_1 - \lambda_2 \leq r_1)$
- $(r_2 + 1)(\lambda_1 - \lambda_2 + 1) = (r_2 + 1)(r_1 - r_2 + 1)$
if $0 \leq r_1 + \lambda_2 - \lambda_1 = r_2 \Leftrightarrow (r_1 - r_2 = \lambda_1 - \lambda_2 \leq r_1)$
- $\frac{1}{2}(r_2 + 1)(2r_1 - r_2 + 2)$ if $r_1 + \lambda_2 - \lambda_1 < 0 \Leftrightarrow r_1 < \lambda_1 - \lambda_2$

Proof. Case 1: Suppose $0 \leq r_1 + \lambda_2 - \lambda_1 < r_2 \Leftrightarrow (r_1 - r_2 < \lambda_1 - \lambda_2 \leq r_1)$. Then the position of s_2 has two possibilities.

- Case a: $0 \leq s_2 \leq r_1 + \lambda_2 - \lambda_1$
- Case b: $r_1 + \lambda_2 - \lambda_1 + 1 \leq s_2 \leq r_2$

The possibilities for s_1 are

- Case a: $s_2 \leq s_1 \leq s_2 + \lambda_1 - \lambda_2$
- Case b: $s_2 \leq s_1 \leq r_1$

Hence the total number of characteristic submodules containing $\pi^{r_1}(\mathbb{A}/\pi^{\lambda_1}\mathbb{A}) \oplus \pi^{r_2}(\mathbb{A}/\pi^{\lambda_2}\mathbb{A})$ is $(r_1 + \lambda_2 - \lambda_1 + 1)(\lambda_1 - \lambda_2 + 1) + \sum_{s_2=r_1+\lambda_2-\lambda_1+1}^{r_2} (r_1 - s_2 + 1) = (r_1 + \lambda_2 - \lambda_1 + 1)(\lambda_1 - \lambda_2 + 1) + \frac{1}{2}((\lambda_1 - \lambda_2) - (r_1 - r_2))((\lambda_1 - \lambda_2) + (r_1 - r_2) + 1)$.

Case 2: Suppose $0 \leq r_1 + \lambda_2 - \lambda_1 = r_2 \Leftrightarrow (r_1 - r_2 = \lambda_1 - \lambda_2 \leq r_1)$. Then the position of s_2 has only one possibility namely $0 \leq s_2 \leq r_1 + \lambda_2 - \lambda_1 = r_2$. And the possibilities for s_1 are $s_2 \leq s_1 \leq s_2 + \lambda_1 - \lambda_2$. In this case 2 the total number of characteristic submodules containing $\pi^{r_1}(\mathbb{A}/\pi^{\lambda_1}\mathbb{A}) \oplus \pi^{r_2}(\mathbb{A}/\pi^{\lambda_2}\mathbb{A})$ is

$(r_2 + 1)(\lambda_1 - \lambda_2 + 1) = (r_2 + 1)(r_1 - r_2 + 1)$ which is equal to the number of characteristic submodules in $(\mathbb{A}/\pi^{r_1}\mathbb{A}) \oplus (\mathbb{A}/\pi^{r_2}\mathbb{A}) = \frac{\mathbb{A}/\pi^{\lambda_1}\mathbb{A} \oplus \mathbb{A}/\pi^{\lambda_2}\mathbb{A}}{\pi^{r_1}(\mathbb{A}/\pi^{\lambda_1}\mathbb{A}) \oplus \pi^{r_2}(\mathbb{A}/\pi^{\lambda_2}\mathbb{A})}$. In this case a submodule of $(\mathbb{A}/\pi^{r_1}\mathbb{A}) \oplus (\mathbb{A}/\pi^{r_2}\mathbb{A})$ is characteristic iff its corresponding submodule in $(\mathbb{A}/\pi^{\lambda_1}\mathbb{A}) \oplus (\mathbb{A}/\pi^{\lambda_2}\mathbb{A})$ containing $\pi^{r_1}(\mathbb{A}/\pi^{\lambda_1}\mathbb{A}) \oplus \pi^{r_2}(\mathbb{A}/\pi^{\lambda_2}\mathbb{A})$ is characteristic.

Case 3: Suppose $r_1 + \lambda_2 - \lambda_1 < 0 \Leftrightarrow r_1 < \lambda_1 - \lambda_2$. Then the position of s_2 has only one possibility namely $0 \leq s_2 \leq r_2$. And the possibilities for s_1 are $s_2 \leq s_1 \leq r_1$. In this case 3, the total number of characteristic submodules containing $\pi^{r_1}(\mathbb{A}/\pi^{\lambda_1}\mathbb{A}) \oplus \pi^{r_2}(\mathbb{A}/\pi^{\lambda_2}\mathbb{A})$ is $\sum_{s_2=0}^{r_2} (r_1 - s_2 + 1) = \frac{1}{2}(r_2 + 1)(2r_1 - r_2 + 2)$. \square

Observation 3.23. Let $\mathcal{A}_{\underline{\lambda}} = \mathbb{A}/\pi^{\lambda_1}\mathbb{A} \oplus \mathbb{A}/\pi^{\lambda_2}\mathbb{A} \oplus \mathbb{A}/\pi^{\lambda_3}\mathbb{A} \oplus \dots \oplus \mathbb{A}/\pi^{\lambda_k}\mathbb{A}$. Let $\pi^{r_1}(\mathbb{A}/\pi^{\lambda_1}\mathbb{A}) \oplus \pi^{r_2}(\mathbb{A}/\pi^{\lambda_2}\mathbb{A}) \oplus \pi^{r_3}(\mathbb{A}/\pi^{\lambda_3}\mathbb{A}) \oplus \dots \oplus \pi^{r_k}(\mathbb{A}/\pi^{\lambda_k}\mathbb{A})$ be a characteristic submodule. Suppose we also have $\lambda_1 - r_1 = \lambda_2 - r_2 = \lambda_3 - r_3 = \dots = \lambda_k - r_k$. Then we exhibit that a submodule of $\mathbb{A}/\pi^{r_1}\mathbb{A} \oplus \mathbb{A}/\pi^{r_2}\mathbb{A} \oplus \mathbb{A}/\pi^{r_3}\mathbb{A} \oplus \dots \oplus \mathbb{A}/\pi^{r_k}\mathbb{A}$ is characteristic iff its corresponding submodule in $\mathcal{A}_{\underline{\lambda}}$ containing $\pi^{r_1}(\mathbb{A}/\pi^{\lambda_1}\mathbb{A}) \oplus \pi^{r_2}(\mathbb{A}/\pi^{\lambda_2}\mathbb{A}) \oplus \pi^{r_3}(\mathbb{A}/\pi^{\lambda_3}\mathbb{A}) \oplus \dots \oplus \pi^{r_k}(\mathbb{A}/\pi^{\lambda_k}\mathbb{A})$ is characteristic. Note $\mathbb{A}/\pi^{r_1}\mathbb{A} \oplus \mathbb{A}/\pi^{r_2}\mathbb{A} \oplus \mathbb{A}/\pi^{r_3}\mathbb{A} \oplus \dots \oplus \mathbb{A}/\pi^{r_k}\mathbb{A} = \frac{\mathcal{A}_{\underline{\lambda}}}{\pi^{r_1}(\mathbb{A}/\pi^{\lambda_1}\mathbb{A}) \oplus \pi^{r_2}(\mathbb{A}/\pi^{\lambda_2}\mathbb{A}) \oplus \pi^{r_3}(\mathbb{A}/\pi^{\lambda_3}\mathbb{A}) \oplus \dots \oplus \pi^{r_k}(\mathbb{A}/\pi^{\lambda_k}\mathbb{A})}$

Proof. The proof is omitted here. As it can be extracted from Case 2 in the proof of the Observation 3.22, if one goes through it carefully. \square

3.4 Some Observations on the Additive Group Structure and the Unit Group Structure in the Ring $\mathbb{A}/\pi^l\mathbb{A}$

Let \mathbb{A} be a discrete valuation ring with uniformizing element π with a finite residue field $\mathbb{A}/\pi\mathbb{A} \cong \mathbb{F}_q$. We have the cardinality of $\mathbb{A}/\pi^l\mathbb{A}$ is q^l and the units $(\mathbb{A}/\pi^l\mathbb{A})^*$ has cardinality is $q^l - q^{l-1}$. In general the unit group satisfies the split exact sequence

$$1 \longrightarrow (1 + \pi\mathbb{A}/\pi^l\mathbb{A}) \longrightarrow (\mathbb{A}/\pi^l\mathbb{A})^* \xrightarrow{\leftarrow} (\mathbb{A}/\pi\mathbb{A})^* \cong \mathbb{F}_q^* \longrightarrow 1$$

Here $|(1 + \pi\mathbb{A}/\pi^l\mathbb{A})| = q^{l-1}$. One way to prove that this sequence splits is to produce an element of order $q-1$ in $(\mathbb{A}/\pi^l\mathbb{A})^*$ which is as follows. Since the map to \mathbb{F}_q^* is onto and \mathbb{F}_q^* is cyclic, pick an element $\alpha \in (\mathbb{A}/\pi^l\mathbb{A})^*$ such that it maps to the generator of \mathbb{F}_q^* . Then α^{q-1} has order a power of p say p^r where $\text{char}(\mathbb{F}_q) = p$ a prime. We find α^{p^r} has order $q-1$.

We get a filtration of subgroups using the exact sequence

$$1 \longrightarrow (1 + \pi^{l-1}\mathbb{A}/\pi^l\mathbb{A}) \longrightarrow (\mathbb{A}/\pi^l\mathbb{A})^* \longrightarrow (\mathbb{A}/\pi^{l-1}\mathbb{A})^* \longrightarrow 1$$

as follows. First observe that for any $l > 1$, $\mathbb{F}_q \cong \mathbb{A}/\pi\mathbb{A} \xrightarrow{\cong} (1 + \pi^{l-1}\mathbb{A}/\pi^l\mathbb{A})$. For $i \leq l$ let $\tau_{il} : (\mathbb{A}/\pi^l\mathbb{A})^* \longrightarrow (\mathbb{A}/\pi^i\mathbb{A})^*$ be the reduction modulo π^i map. The filtration is given by

$$0 \subset \tau_{(l-1)l}^{-1}(1) = (1 + \pi^{l-1}\mathbb{A}/\pi^l\mathbb{A}) \subset \tau_{(l-1)l}^{-1}(1 + \pi^{l-2}\mathbb{A}/\pi^{l-1}\mathbb{A}) \subset \tau_{(l-2)l}^{-1}(1 + \pi^{l-3}\mathbb{A}/\pi^{l-2}\mathbb{A}) \subset \dots \subset \tau_{2l}^{-1}(1 + \pi\mathbb{A}/\pi^2\mathbb{A}) \subset \tau_{1l}^{-1}(1) = (1 + \pi\mathbb{A}/\pi^l\mathbb{A}) \subset (\mathbb{A}/\pi^l\mathbb{A})^*$$

Except for the last successive quotient, each one is isomorphic to \mathbb{F}_q and the last one is isomorphic to \mathbb{F}_q^* .

Observation 3.24. *This has two parts. For the first part see Apostol [3, Chapter 10])*

1. *If \mathbb{A} is the ring of p -adic integers \mathbb{Z}_p then we exhibit that $(\mathbb{A}/\pi^l\mathbb{A})^*$ is cyclic of order $p^{l-1}(p-1)$ if p is an odd prime and if $p = 2$ then it is cyclic iff $l \leq 2$.*
2. *If \mathbb{A} is the power series ring in one variable over \mathbb{F}_p then we exhibit that $(\mathbb{A}/\pi^l\mathbb{A})^*$ is cyclic iff $l = 1, 2$.*

Proof. Part 1: Here $(\mathbb{A}/\pi^l\mathbb{A})^* \cong (\mathbb{Z}/p^l\mathbb{Z})^*$. And

$$\begin{cases} (1+p)^{p^{l-1}} \equiv 1 \pmod{p^l} \text{ and } (1+p)^{p^{l-2}} \equiv (1+p^{l-1}) \pmod{p^l} & \text{if } p \text{ is an odd prime} \\ (1+xp)^{p^{l-2}} \equiv 1 \pmod{p^l} \text{ if } p = 2 \text{ for all } x \in (\mathbb{Z}/p^l\mathbb{Z}) \end{cases}$$

So if p is an odd prime then there is an element of order p^{l-1} and there is an element of order $p-1$. Hence $(\mathbb{Z}/p^l\mathbb{Z})^*$ is cyclic. If $p=2$ then any element in $(\mathbb{Z}/p^l\mathbb{Z})^*$ is of the form $(1+xp) \pmod{p^l}$ for some $x \in (\mathbb{Z}/p^l\mathbb{Z})$ and has order dividing p^{l-2} whereas order of $(\mathbb{Z}/p^l\mathbb{Z})^*$ is p^{l-1} . Hence $(\mathbb{Z}/p^l\mathbb{Z})^*$ is not cyclic, unless when $l \leq 2$.

Part 2 : Let $l = qp + r$ with $0 \leq r < p$. Consider the equation $x^p = 1$. We observe that if $(q+2) < l < (q+1)p$ then $(1+xt^{q+1}+yt^{q+2})^p = 1$ for all $x, y \in \mathbb{F}_p$. Hence the number of solutions to the equation $x^p = 1$ is more than p . So the group $(\mathbb{F}_p[[t]]/t^l)^*$ is not cyclic in general. If $q+2 \geq l$ which occurs in the following cases which we examine.

$$\left\{ \begin{array}{l} q = 0, r = l = 1, p \text{ any prime} \\ q = 0, r = l = 2, p \text{ any prime} \\ q = 1, r = 0, l = p = 3 \\ q = 1, r = 0, l = p = 2 \\ q = 1, r = 1, l = 3, p = 2 \\ q = 2, r = 0, l = 4, p = 2 \end{array} \right.$$

Note that the ring $\mathbb{F}_p[[t]]/t^l$ can be embedded in upper triangular matrices in $M_l(\mathbb{F}_p)$ with equal entries on the diagonal and on each of the super diagonals. If $l=1$ then $(\mathbb{F}_p[[t]]/t^l)^* \cong \mathbb{F}_p^*$ is cyclic. If $l=2$ then $(\mathbb{F}_p[[t]]/t^l)^*$ has elements of order p and $p-1$. Order of $1+t$ is p and order of the generator of \mathbb{F}_p^* which is cyclic is $p-1$. So here again $(\mathbb{F}_p[[t]]/t^l)^*$ is cyclic. If $l=3$ and $p \geq 3$ then $(1+xt+yt^2)^p = 1$ for $x, y \in \mathbb{F}_p$. Hence it is not cyclic. If $l=3$ and $p=2$ then again $(\mathbb{F}_p[[t]]/t^l)^* \cong (\mathbb{Z}/p^1\mathbb{Z})^{\oplus 2}$ which is not cyclic. So the only case left is $l=4, p=2$ in this case again $(1+xt^2+yt^3)^p = 1$ for $x, y \in \mathbb{F}_p$. Hence it is not cyclic. This finishes the proof and we conclude that $(\mathbb{F}_p[[t]]/t^l)^*$ is cyclic iff $l=1$ or $l=2$ □

Observation 3.25. *Observe that the abelian group structure upto isomorphism of the successive quotients in the above filtration is elementary abelian and the last successive quotient in the filtration is cyclic.*

Proof. Since this is fairly straight forward the details of the proof are omitted. \square

Observation 3.26. *This has two parts.*

- If $\text{char}(\mathbb{A}) = p$ then the additive group $(\mathbb{A}/\pi^l\mathbb{A})$ is elementary abelian.
- If $\text{char}(\mathbb{A}) = 0$ and let $\hat{\mathbb{A}}$ be the completion of \mathbb{A} with respect to the π -adic filtration and $\hat{\mathbb{A}}$ is finitely generated over the ring of p -adic integers and suppose $\pi^e = p$ for some integer $e > 0$ (called the ramification index) then the additive group structure of $(\mathbb{A}/\pi^l\mathbb{A})$ can be determined and it is given by

$$(\mathbb{A}/\pi^l\mathbb{A}) \cong (\mathbb{Z}/p^{s+1}\mathbb{Z})^{\oplus tr} \oplus (\mathbb{Z}/p^s\mathbb{Z})^{\oplus t(e-r)}$$

Proof. Part 1 : $(\mathbb{A}/\pi^l\mathbb{A})$ is a vector space over \mathbb{F}_p . Hence the claim follows.

Part 2 : Since $\text{char}(\mathbb{A}) = 0$, $\mathbb{Z} \subset \mathbb{A}$ and $(\pi) \cap \mathbb{A}$ is a prime ideal in \mathbb{Z} and we have $(\pi) \cap \mathbb{A} = p\mathbb{Z}$ where p is the characteristic of the finite residue field $\mathbb{A}/\pi\mathbb{A} \cong \mathbb{F}_q$. Since \mathbb{A} is a DVR, $p\mathbb{A} = (\pi^e)$ for some $e > 0$. In the hypothesis it is given that in fact $p = \pi^e$. We use this now. First consider the subring $\mathbb{B} = \mathbb{Z}_{((p))}[\pi] \subset \hat{\mathbb{A}}$. This subring is a complete discrete valuation ring with the same uniformizer π and $\hat{\mathbb{A}}$ is finitely generated over \mathbb{B} , since $\hat{\mathbb{A}}$ is finitely generated over $\mathbb{Z}_{((p))}$. So $\hat{\mathbb{A}}$ is a finitely generated free module over \mathbb{B} as $\hat{\mathbb{A}}$ is torsion free module over a PID \mathbb{B} , hence it is free of finite rank (say = t).

Now consider the unit element $1 \in \hat{\mathbb{A}}/\pi^l\hat{\mathbb{A}}$. Let $l = es + r$ for some $0 \leq r < e$. We have that by adding 1 repeatedly to itself $1 + 1 + 1 + \dots + 1$, p^{s+1} times we get $p^{s+1} = \pi^{es+e} = 0$ since $es + e > l$. Note $(p^{s+1} - 1) = (\pi^{es+e} - 1) = (\pi^{l+(e-r)} - 1) = -1 \neq 0$.

Also note that

- If $r > 0$, $p^s = \pi^{l-r} \neq 0$.
- If $r = 0$, $p^s = \pi^{l-r} = 0$, $p^s - 1 = -1 \neq 0$.

We conclude that $order(1) = p^{s+1}$ if $r > 0$ and $order(1) = p^s$ if $r = 0$. By similar observations we conclude that

- If $0 \leq i < r$ the $order(\pi^i) = p^{(s+1)}$.
- And for $e > i \geq r$ we have $order(\pi^i) = p^s$.

We also observe that the additive group $\mathbb{B}/\pi^l\mathbb{B}$ splits $\langle 1 \rangle \oplus \langle \pi \rangle \oplus \langle \pi^2 \rangle \oplus \dots \oplus \langle \pi^{e-1} \rangle$. Note that $\pi^e = p$ an integer multiple of 1, $\pi^{e+1} = p\pi$ an integer multiple of π and so on. So the additive group $\mathbb{A}/\pi^l\mathbb{A} \cong \hat{\mathbb{A}}/\pi^l\hat{\mathbb{A}}$ is isomorphic to

$$(\mathbb{B}/\pi^l\mathbb{B})^t \cong (\mathbb{Z}/p^{s+1}\mathbb{Z})^{\oplus tr} \oplus (\mathbb{Z}/p^s\mathbb{Z})^{\oplus t(e-r)}$$

This gives the additive group structure of $(\mathbb{A}/\pi^l\mathbb{A})$. □

3.5 Permutation Representations

In this section we develop some preliminaries about the complex permutation representations. Let G, H, \dots denote finite groups and X, Y, \dots denote sets on which they act. The actions give rise to complex permutation representations $\mathbb{C}[X], \mathbb{C}[Y], \dots$ of the finite groups G, H, \dots . We will examine some properties of these permutation representations.

Observation 3.27. *Let $\mathbb{V} = \mathbb{C}[X], \mathbb{W} = \mathbb{C}[Y]$ be two complex vector spaces associated to two permutation representations of G on the sets X, Y . Then $Hom(\mathbb{V}, \mathbb{W})$ is isomorphic to $\mathbb{C}[X \times Y]$ as representations of G .*

Observation 3.28. *Let $\mathbb{V} = \mathbb{C}[X], \mathbb{W} = \mathbb{C}[Y]$ be two complex vector spaces associated to two permutation representations of G on the sets X, Y . Observe that $Hom_G(\mathbb{V}, \mathbb{W})$ is isomorphic to $\mathbb{C}[G \setminus (X \times Y)]$ as representations of G . Let $X = \bigsqcup_i X_i$*

and $Y = \bigsqcup_j Y_j$ is the partition of X and Y into disjoint transitive subsets. Let $x_i \in X_i, y_j \in Y_j$ be a collection of representatives. Let H_i and K_j denote their stabilizers respectively. The dimension of $\text{Hom}_G(\mathbb{V}, \mathbb{W})$ is equal to the number of double cosets of H_i and K_j in G for all i, j .

Observation 3.29. Let X, Y be two transitive G -sets. Let $x \in X$ and $y \in Y$ be two elements with stabilizers G_x and G_y respectively. Then there are natural bijections among the following.

- The set of G orbits in $X \times Y$ under the diagonal action of G .
- The set of G_x orbits in Y
- The set of G_y orbits in X
- The set of double cosets of G_x and G_y in G .

Observation 3.30. Let X, Y be two G -sets. Let $x_i \in X_i$ and $y_j \in Y_j$ be a collection of representatives in their transitive subset-partitions $X = \bigsqcup_i X_i$ and $Y = \bigsqcup_j Y_j$ respectively. Let G_{x_i} and G_{y_j} be their stabilizer subgroups in G respectively. Then there is a natural bijection among the following.

- The set of G orbits in $X \times Y$ under the diagonal action of G which contain x_i in the first coordinate for some element in those orbits with the set of G_{x_i} orbits in Y
- The set of G orbits in $X \times Y$ under the diagonal action of G which contain y_j in the second coordinate for some element in those orbits with the set of G_{y_j} orbits in X
- The set of double cosets of G_{x_i} and G_{y_j} in G with the G -orbits of $X_i \times Y_j$.
- The set of double cosets of G_{x_i} and G_{y_j} in G for all i, j with the G -orbits of $X \times Y$

Observation 3.31. Let X, Y be two transitive G -sets. Let $x \in X$ and $y \in Y$ be two representatives with stabilizers G_x and G_y respectively. Let $O_{x,y}$ be the orbit of $(x, y) \in X \times Y$. Then the following holds.

- $|G| = |G_x||X| = |G_y||Y|$.
- Cardinality of G_x orbit of $y \in Y = |G_x|/|G_x \cap G_y|$
- Cardinality of G_y orbit of $x \in X = |G_y|/|G_x \cap G_y|$
- $|O_{x,y}| = |G/G_x|$ (Cardinality of G_x orbit of $y \in Y$) = $|G/G_y|$ (Cardinality of G_y orbit of $x \in X$) = $|G/(G_x \cap G_y)|$.
- The number of G -orbits in $X \times Y = (1/|G|)(\sum_{g \in G} |X_g||Y_g|)$ where X_g and Y_g are the number of fixed points of g in X and Y respectively.

Observation 3.32. Let $H, K \subset G$ be two finite subgroups of a finite group G . Let C_1, C_2, \dots, C_k be the various conjugacy classes of G . Then the number of double cosets is given by

$$m = \left(\frac{|G|}{|H||K|} \right) \left(\sum_{i=1}^k \frac{|C_i \cap H||C_i \cap G|}{|C_i|} \right)$$

Observation 3.33. Any permutation representation of G on $\mathbb{C}[X]$ contains a trivial subrepresentation. The dimension of $\mathbb{C}[X]^G$ is the number of distinct transitive orbits. And if the action of G is doubly transitive on X then the $\mathbb{C}[X]$ is a direct sum of two irreducible representations one of which is trivial one dimensional.

Observation 3.34. Let $H \subset G$ be a subgroup. Then the $\text{Ind}_H^G 1$ is isomorphic to $\mathbb{C}[G/H]$ the permutation representation of G on the cosets of H in G . When H is trivial it is isomorphic to the left regular representation. If $H = G$ then it is isomorphic to trivial 1-dimensional representation.

Observation 3.35. Let $N \subset G$ be a normal subgroup. Then the permutation representation $\mathbb{C}[G/N] \cong \text{Ind}_N^G 1$ of G gives rise to the left regular representation of G/N on $\mathbb{C}[G/N]$.

Observation 3.36. *The representation of a finite group G on a vector space V is multiplicity free if and only if $\text{End}_G(V)$ is commutative.*

Observation 3.37. *The left regular representation of G is multiplicity free if and only if the G is abelian. The permutation representation of G on the cosets of the commutator subgroup is multiplicity free.*

Observation 3.38. *The character table of the cyclic group is the Discrete Fourier Transform matrix upto a constant. We can express the character table of a finite abelian group as a Kronecker product of the character table of the cyclic groups using the structure theorem of finite abelian groups.*

Chapter 4

Polynomials for Orbits of Pairs

Let $\mathcal{G}_{\underline{\lambda}}$ be the automorphism group of a finite \mathbb{A} module $\mathcal{A}_{\underline{\lambda}}$. Consider two orbits \mathcal{O}_I and \mathcal{O}_L corresponding to two ideals I and L in the poset $J(\mathcal{P})_{\underline{\lambda}}$. In this chapter we describe the structure of the decomposition of $\mathcal{O}_I \times \mathcal{O}_L$ into orbits of pairs and the structure of orbit of pairs.

We prove that there are integer polynomials in one variable x i.e in the ring $\mathbb{Z}[x]$ which upon evaluation at the cardinality q of the residue field $\mathbb{A}/\pi^1\mathbb{A}$ gives cardinalities of orbits of pairs as well as the number of orbits of pairs for any given set of two orbits $\mathcal{O}_I \subset \mathcal{A}_{\underline{\lambda}}$ and $\mathcal{O}_L \subset \mathcal{A}_{\underline{\lambda}}$.

4.1 A Lattice Homomorphism

Theorem 4.1. *Let $\mathcal{A}_{\underline{\lambda}}$ and $\mathcal{A}_{\underline{\mu}}$ be two finite modules over the discrete valuation ring \mathbb{A} of type $\underline{\lambda}, \underline{\mu} \in \Lambda$ (refer equation (3.1)) and suppose $\mathcal{A}_{\underline{\lambda}}^I \subset \mathcal{A}_{\underline{\lambda}}$ be a characteristic submodule of $\mathcal{A}_{\underline{\lambda}}$ corresponding to ideal $I \in J(\mathcal{P})_{\underline{\lambda}}$. Let $\partial I = \{(\partial_{\lambda_l} I, \lambda_l) = (\nu_l, \lambda_l) \mid 1 \leq l \leq k\}$. Define*

$$\mathcal{HOM}(\mathcal{A}_{\underline{\lambda}}, \mathcal{A}_{\underline{\mu}})\mathcal{A}_{\underline{\lambda}}^I = \left(\sum_{\phi \in \mathcal{HOM}(\mathcal{A}_{\underline{\lambda}}, \mathcal{A}_{\underline{\mu}})} \phi(\mathcal{A}_{\underline{\lambda}}^I) \right) \subset \mathcal{A}_{\underline{\mu}}$$

Then

- $\mathcal{HOM}(\mathcal{A}_{\underline{\lambda}}, \mathcal{A}_{\underline{\mu}})$ is a lattice homomorphism from the lattice of characteristic submodules of $\mathcal{A}_{\underline{\lambda}}$ to the lattice of characteristic submodules of $\mathcal{A}_{\underline{\mu}}$.

- $\mathcal{HOM}(\mathcal{A}_{\underline{\lambda}}, \mathcal{A}_{\underline{\mu}})\mathcal{A}_{\underline{\lambda}}^I = \mathcal{A}_{\underline{\mu}}^J$ where $\mathcal{A}_{\underline{\mu}}^J$ is a characteristic submodule of $\mathcal{A}_{\underline{\mu}}$ such that $\partial J = \{(\partial_{\mu_l} J, \mu_l) = (\sigma_l, \mu_l) \mid 1 \leq l \leq m\}$ where the determination of boundary valuations of $\underline{\sigma}$ is mentioned below.

$$\sigma_i = \begin{cases} \min(\nu_{j_i-1}, \mu_i - \lambda_{j_i} + \nu_{j_i}), & \text{if there exists } j_i \text{ such that } \lambda_{j_i-1} \geq \mu_i > \lambda_{j_i} \\ \nu_k, & \text{if } \lambda_k \geq \mu_i \\ \mu_i - \lambda_1 + \nu_1, & \text{if } \mu_i > \lambda_1 \end{cases}$$

- The ideal $I \in J(\mathcal{P})_{\underline{\lambda}}$ generates an ideal in the fundamental poset $J(\mathcal{P})$ which when restricted to $J(\mathcal{P})_{\underline{\mu}}$ gives the ideal $J \in J(\mathcal{P})_{\underline{\mu}}$.

Proof. -

- It is clear that $\mathcal{HOM}(\mathcal{A}_{\underline{\lambda}}, \mathcal{A}_{\underline{\mu}})$ maps characteristic submodules to characteristic submodules and it is a lattice homomorphism.
- Now that $\underline{\lambda} = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_k)$, $\underline{\mu} = (\mu_1 \geq \mu_2 \geq \mu_3 \geq \dots \geq \mu_l)$ and $\underline{\nu} \subset \underline{\lambda}$ is given by $\underline{\nu} = (\nu_1 \geq \nu_2 \geq \nu_3 \geq \dots \geq \nu_k)$ and $\underline{\lambda} - \underline{\nu} = (\lambda_1 - \nu_1 \geq \lambda_2 - \nu_2 \geq \dots \geq \lambda_k - \nu_k)$, define $\underline{\sigma} = (\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_l)$ as follows. Define σ_i as in the Theorem 4.1.

Claim: $\underline{\sigma} = (\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_l)$ and $\underline{\mu} - \underline{\sigma} = (\mu_1 - \sigma_1 \geq \mu_2 - \sigma_2 \geq \dots \geq \mu_k - \sigma_k)$.

Let us prove that $\sigma_2 \leq \sigma_1$. Rest is similar. $\nu_{j_1-1} \geq \nu_{j_2-1} \geq \sigma_2$. If $j_1 = j_2$ then $\sigma_2 \leq \mu_2 - \lambda_{j_2} + \nu_{j_2} \leq \mu_1 - \lambda_{j_1} + \nu_{j_1}$. If $j_1 \neq j_2$ then $j_1 \leq j_2 - 1$ and so $\sigma_2 \leq \nu_{j_2-1} \leq \nu_{j_1} \leq \mu_1 - \lambda_{j_1} + \nu_{j_1}$. So $\sigma_2 \leq \min(\nu_{j_1-1}, \mu_1 - \lambda_{j_1} + \nu_{j_1})$.

Now let us prove that $\mu_1 - \sigma_1 \geq \mu_2 - \sigma_2$. i.e $\mu_1 - \min(\nu_{j_1-1}, \mu_1 - \lambda_{j_1} + \nu_{j_1}) \geq \mu_2 - \min(\nu_{j_2-1}, \mu_2 - \lambda_{j_2} + \nu_{j_2})$. Rest is again similar. We will show that $\max(\mu_1 - \nu_{j_1-1}, \lambda_{j_1} - \nu_{j_1}) \geq \max(\mu_2 - \nu_{j_2-1}, \lambda_{j_2} - \nu_{j_2})$. $\lambda_{j_2} - \nu_{j_2} \leq \lambda_{j_1} - \nu_{j_1}$ so $\lambda_{j_2} - \nu_{j_2} \leq \max(\mu_1 -$

$\nu_{j_1-1}, \lambda_{j_1} - \nu_{j_1}$). If $j_1 = j_2$ then $\mu_2 - \nu_{j_2-1} \leq \mu_1 - \nu_{j_1-1} \leq \max(\mu_1 - \nu_{j_1-1}, \lambda_{j_1} - \nu_{j_1})$.
if $j_1 \leq j_2 - 1$ then $\mu_2 - \nu_{j_2-1} \leq \lambda_{j_2-1} - \nu_{j_2-1} \leq \lambda_{j_1} - \nu_{j_1} \leq \max(\mu_1 - \nu_{j_1-1}, \lambda_{j_1} - \nu_{j_1})$.
Therefore $\max(\mu_1 - \nu_{j_1-1}, \lambda_{j_1} - \nu_{j_1}) \geq \max(\mu_2 - \nu_{j_2-1}, \lambda_{j_2} - \nu_{j_2})$ follows.

So this implies that ideal J is well determined in $J(\mathcal{P})_{\underline{\mu}}$ and $\mathcal{A}_{\underline{\mu}}^J \subset \mathcal{A}_{\underline{\mu}}$ is a characteristic submodule by the correspondence Theorem 3.15 and also $\mathcal{HOM}(\mathcal{A}_{\underline{\lambda}}, \mathcal{A}_{\underline{\mu}})\mathcal{A}_{\underline{\lambda}}^I = \mathcal{A}_{\underline{\mu}}^J$.

- We also observe that $\sigma_i \leq \nu_{j_i-1}, \nu_{j_i} \leq \nu_{j_i-1}, \nu_{j_i} \leq \mu_i - \lambda_{j_i} + \nu_{j_i}$. Hence $\nu_{j_i} \leq \sigma_i \leq \nu_{j_i-1}$. Similarly for $\mu_i - \sigma_i = \max(\mu_i - \nu_{j_i-1}, \lambda_{j_i} - \nu_{j_i})$. $\lambda_{j_i-1} - \nu_{j_i-1} \geq \lambda_{j_i} - \nu_{j_i}$ and $\lambda_{j_i-1} - \nu_{j_i-1} \geq \mu_i - \nu_{j_i-1}$ hence $\lambda_{j_i-1} - \nu_{j_i-1} \geq \mu_i - \sigma_i \geq \lambda_{j_i} - \nu_{j_i}$.

So we have the following chain of inequalities.

$$\nu_1 \geq \nu_2 \geq \nu_3 \geq \dots \geq \nu_{j_1-1} \geq \sigma_1 \geq \nu_{j_1} \geq \dots \geq \nu_{j_2-1} \geq \sigma_2 \geq \nu_{j_2} \geq \dots \geq \nu_{j_3-1} \geq \sigma_3 \geq \nu_{j_3} \geq \dots$$

and similarly

$$\lambda_1 - \nu_1 \geq \lambda_2 - \nu_2 \geq \lambda_3 - \nu_3 \geq \dots \geq \lambda_{j_1-1} - \nu_{j_1-1} \geq \mu_1 - \sigma_1 \geq \lambda_{j_1} - \nu_{j_1} \geq \dots \geq \lambda_{j_2-1} - \nu_{j_2-1} \geq \mu_2 - \sigma_2 \geq \lambda_{j_2} - \nu_{j_2} \geq \dots \geq \lambda_{j_3-1} - \nu_{j_3-1} \geq \mu_3 - \sigma_3 \geq \lambda_{j_3} - \nu_{j_3} \geq \dots$$

Now the Theorem 4.1 follows. □

Lemma 4.2. *Let $\mathcal{A}_{\underline{\lambda}}$ and $\mathcal{A}_{\underline{\mu}}$ be two finite modules over the discrete valuation ring \mathbb{A} . The image of the lattice homomorphism $\mathcal{HOM}(\mathcal{A}_{\underline{\lambda}}, \mathcal{A}_{\underline{\mu}})$ is a sublattice of characteristic submodules of $\mathcal{A}_{\underline{\mu}}$ which is isomorphic to characteristic submodule lattice of $\mathcal{A}_{\underline{\nu}}$ which is $J(\mathcal{P})_{\underline{\nu}}$ for some partition ν .*

Proof. Let $\underline{\lambda} = (\lambda_1^{\rho_1}, \dots, \lambda_k^{\rho_k})$ and $\underline{\mu} = (\mu_1^{\sigma_1}, \dots, \mu_l^{\sigma_l})$ be the partitions corresponding to $\underline{\lambda}$ and $\underline{\mu}$. And suppose $Bor(\underline{\lambda}, \underline{\mu}) = (\lambda_{i_1} > \lambda_{i_2} > \dots > \lambda_{i_r})$ are the border parts of the partition $\underline{\lambda}$ with respect to partition $\underline{\mu}$. i.e parts of the partition $\underline{\lambda}$ that are adjacent to parts of $\underline{\mu}$ on the number line as shown in the example figure above. Then it is clear that given an ideal $I \in J(\mathcal{P})_{\underline{\lambda}}$ it gives rise to an ideal

Border parts of Lambda w.r.t. Mu

... $\lambda_{i_1} > \mu_1 > \mu_2 > \mu_3 > \lambda_{i_2} > \dots > \lambda_{i_3} > \mu_4 > \mu_5 > \lambda_{i_4} > \dots > \lambda_{i_5} > \mu_6 > \mu_7 > \lambda_{i_6} > \mu_8 > \lambda_{i_7} > \dots$

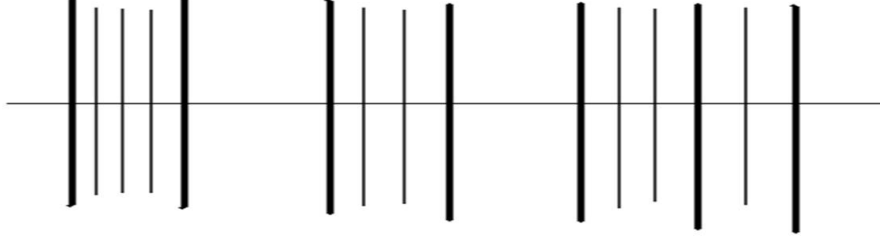


Figure 4.1: mu-values in between lambda-values

$I \cap \mathcal{P}_{\text{Border}(\underline{\lambda}, \underline{\mu})} \in J(\mathcal{P})_{\text{Border}(\underline{\lambda}, \underline{\mu})}$. This map $I \rightarrow I \cap \mathcal{P}_{\text{Border}(\underline{\lambda}, \underline{\mu})}$ is not injective in general but it is onto set of all ideals in $J(\mathcal{P})_{\text{Border}(\underline{\lambda}, \underline{\mu})}$. However ideals in $J(\mathcal{P})_{\text{Border}(\underline{\lambda}, \underline{\mu})}$ bijectively correspond to characteristic submodules of $\mathcal{A}_{\underline{\mu}}$ which lie in the image of the lattice homomorphism $\mathcal{HOM}(\mathcal{A}_{\underline{\lambda}}, \mathcal{A}_{\underline{\mu}})$ which maps characteristic submodules of $\mathcal{A}_{\underline{\lambda}}$ into characteristic submodules of $\mathcal{A}_{\underline{\mu}}$. Now the lemma follows. \square

4.2 Sum of Orbits

This section proves a combinatorial lemma on the sum of two $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}}$ which will be needed in Section 4.3. Given order ideals $I, J \subset J(\mathcal{P})_{\underline{\lambda}}$, the set

$$\mathcal{A}_{\underline{\lambda}}^{I^*} + \mathcal{A}_{\underline{\lambda}}^{J^*} = \{e + f \mid e \in \mathcal{A}_{\underline{\lambda}}^{I^*} \text{ and } f \in \mathcal{A}_{\underline{\lambda}}^{J^*}\}.$$

This set is clearly $\mathcal{G}_{\underline{\lambda}}$ -invariant, and therefore a union of $\mathcal{G}_{\underline{\lambda}}$ -orbits. In this section, we determine exactly which $\mathcal{G}_{\underline{\lambda}}$ -orbits occur in $\mathcal{A}_{\underline{\lambda}}^{I^*} + \mathcal{A}_{\underline{\lambda}}^{J^*}$.

Lemma 4.3. *For $I, J \in J(\mathcal{P})_{\underline{\lambda}}$, every element (e_{λ_i, r_i}) of $\mathcal{A}_{\underline{\lambda}}^{I^*} + \mathcal{A}_{\underline{\lambda}}^{J^*}$ satisfies the conditions*

1. $v(e_{\lambda_i, r_i}) \geq \max(\partial_{\lambda_i} I, \partial_{\lambda_i} J)$.
2. If $(\partial_{\lambda_i} I, \lambda_i) \in \max I - J$, then $\min_{r_i} v(e_{\lambda_i, r_i}) = \partial_{\lambda_i} I$.
3. If $(\partial_{\lambda_i} J, \lambda_i) \in \max J - I$, then $\min_{r_i} v(e_{\lambda_i, r_i}) = \partial_{\lambda_i} J$.

If the residue field of \mathbb{A} has at least three elements, then every element of $\mathcal{A}_{\underline{\lambda}}$ satisfying these three conditions is in $\mathcal{A}_{\underline{\lambda}}^{I^*} + \mathcal{A}_{\underline{\lambda}}^{J^*}$.

To see why the condition on the residue field is necessary consider the case where $\mathcal{A}_{\underline{\lambda}} = \mathbb{Z}/2\mathbb{Z}$, and $\mathcal{A}_{\underline{\lambda}}^{I^*}$ is the non-zero orbit (corresponding to the ideal I in \mathcal{P} generated $(0, 1)$), $\mathcal{A}_{\underline{\lambda}}^{I^*} + \mathcal{A}_{\underline{\lambda}}^{I^*}$ consists only of 0. If, on the other hand, the residue field has at least three elements, this phenomenon does not occur.

Proof of the lemma. Let $\mathcal{A}_{\underline{\lambda}_i}$ denote the summand $(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\oplus \rho_i}$ of $\mathcal{A}_{\underline{\lambda}}$ in the decomposition given in the equation (3.2). Let $\mathcal{A}_{\underline{\lambda}_i}^* = \mathcal{A}_{\underline{\lambda}_i} - \pi \mathcal{A}_{\underline{\lambda}_i}$. By Theorem 3.11 it suffices to show that

$$\pi^k \mathcal{A}_{\underline{\lambda}_i}^* + \pi^l \mathcal{A}_{\underline{\lambda}_i}^* = \begin{cases} \pi^{\min(k, l)} \mathcal{A}_{\underline{\lambda}_i}^* & \text{if } k \neq l, \\ \pi^k \mathcal{A}_{\underline{\lambda}_i} & \text{if } k = l \text{ and } |\mathbb{A}/\pi^1 \mathbb{A}| \geq 3. \end{cases} \quad (4.4)$$

which follows from the well-known non-Archimedean inequality

$$v(x + y) \geq \min(v(x), v(y)),$$

and the fact that strict inequality is possible only if $v(x) = v(y)$. □

Together with Theorem 3.11, the above lemma gives the following description of the set of orbits which occur in $\mathcal{A}_{\underline{\lambda}}^{I^*} + \mathcal{A}_{\underline{\lambda}}^{J^*}$:

Theorem 4.5. *Assume that the residue field of \mathbb{A} has at least three elements. For ideals $I, J \in J(\mathcal{P})_{\underline{\lambda}}$,*

$$\mathcal{A}_{\underline{\lambda}}^{I^*} + \mathcal{A}_{\underline{\lambda}}^{J^*} = \bigsqcup_{K \subset I \cup J, \max K \supset (\max I - J) \cup (\max J - I)} \mathcal{A}_{\underline{\lambda}}^{K^*}.$$

In the following lemma the restriction on the residue field of \mathbb{A} in Lemma 4.3 is not needed:

Lemma 4.6. *For ideals I and J in $J(\mathcal{P})_{\underline{\lambda}}$, an element (e_{λ_i, r_i}) is in $\mathcal{A}_{\underline{\lambda}}^{I^*} + \mathcal{A}_{\underline{\lambda}}^J$ if and only if the following conditions are satisfied:*

1. $v(e_{\lambda_i, r_i}) \geq \min(\partial_{\lambda_i} I, \partial_{\lambda_i} J)$.
2. If $(\partial_{\lambda_i} I, \lambda_i) \in \max I - J$, then $\min_{r_i} v(e_{\lambda_i, r_i}) = \partial_{\lambda_i} I$.

Proof. The proof is similar to that of Lemma 4.3, except that instead of the equation (4.4), we use:

$$\pi^k \mathcal{A}_{\underline{\lambda}_i} + \pi^l \mathcal{A}_{\underline{\lambda}_i}^* = \begin{cases} \pi^k \mathcal{A}_{\underline{\lambda}_i} & \text{if } k \leq l, \\ \pi^l \mathcal{A}_{\underline{\lambda}_i}^* & \text{if } k > l. \end{cases}$$

□

The above lemma allows us to describe the sum of an orbit and a characteristic submodule:

Theorem 4.7. *For ideals $I, J \in J(\mathcal{P})_{\underline{\lambda}}$,*

$$\mathcal{A}_{\underline{\lambda}}^{I^*} + \mathcal{A}_{\underline{\lambda}}^J = \bigsqcup_{K \subset I \cup J, \max K \supset \max I - J} \mathcal{A}_{\underline{\lambda}}^{K^*}. \quad (4.8)$$

4.3 Stabilizer of Canonical Forms

By Theorem 3.12, every $\mathcal{G}_{\underline{\lambda}}$ -orbit of pairs of elements $(a, b) \in \mathcal{A}_{\underline{\lambda}} \times \mathcal{A}_{\underline{\lambda}}$ contains a pair of the form $(e(I), f)$, for some $I \in J(\mathcal{P})_{\underline{\lambda}}$. Now fix an ideal $I \in J(\mathcal{P})_{\underline{\lambda}}$. Let $\mathcal{G}_{\underline{\lambda}}^I$ denote the stabilizer in $\mathcal{G}_{\underline{\lambda}}$ of $e(I)$. Then the $\mathcal{G}_{\underline{\lambda}}$ -orbits of pairs $(a, b) \in \mathcal{A}_{\underline{\lambda}} \times \mathcal{A}_{\underline{\lambda}}$ which contain an element of the form $(e(I), f)$ are in bijective correspondence with $\mathcal{G}_{\underline{\lambda}}^I$ -orbits in $\mathcal{A}_{\underline{\lambda}}$. In this section, we give a description of $\mathcal{G}_{\underline{\lambda}}^I$ which facilitates the classification of $\mathcal{G}_{\underline{\lambda}}^I$ -orbits in $\mathcal{A}_{\underline{\lambda}}$.

The main idea here is to decompose $\mathcal{A}_{\underline{\lambda}}$ into a direct sum of two \mathbb{A} -modules (and this decomposition depends on I):

$$\mathcal{A}_{\underline{\lambda}} = \mathcal{A}_{\underline{\lambda}'} \oplus \mathcal{A}_{\underline{\lambda}''}, \quad (4.9)$$

where $\mathcal{A}_{\underline{\lambda}'}$ consists of those cyclic summands in the decomposition given in the equation (3.2) of $\mathcal{A}_{\underline{\lambda}}$ where $e(I)$ has non-zero coordinates, and $\mathcal{A}_{\underline{\lambda}''}$ consists of the remaining cyclic summands. In the example mentioned above, $\mathcal{A}_{\underline{\lambda}}$ is given by equation (3.7), and the ideal I is given in the Figure 3.2. We have

$$\mathcal{A}_{\underline{\lambda}'} = \mathbb{Z}/p^4\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}, \quad \mathcal{A}_{\underline{\lambda}''} = \mathbb{Z}/p^5\mathbb{Z} \oplus \mathbb{Z}/p^4\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}.$$

Note that $e(I)' \in \mathcal{A}_{\underline{\lambda}'}$. The reason for introducing this decomposition is that the description of the stabilizer of $e(I)'$ in the automorphism group of $\mathcal{A}_{\underline{\lambda}'}$ is quite nice:

Lemma 4.10. *The stabilizer of $e(I)' \neq 0$ in $\mathcal{G}_{\underline{\lambda}'}$ is*

$$\mathcal{G}_{\underline{\lambda}'}^I = \{\text{id}_{\mathcal{A}_{\underline{\lambda}'}} + n \mid n \in \text{End}_{\mathbb{A}}(\mathcal{A}_{\underline{\lambda}'}) \text{ satisfies } n(e(I)') = 0\}.$$

Proof. Obviously, the elements of $\mathcal{G}_{\underline{\lambda}'}^I$ are all the elements of $\text{End}_{\mathbb{A}}(\mathcal{A}_{\underline{\lambda}'})$ which map $e(I)'$ to itself. The only thing to check is that they are all invertible. For this, it suffices to show that if $n(e(I)') = 0$, then n is nilpotent, which will follow from

Lemma 4.11 below. □

Lemma 4.11. *For any \mathbb{A} -module of the form*

$$\mathcal{A}_{\underline{\mu}} = \mathbb{A}/\pi^{\mu_1}\mathbb{A} \oplus \cdots \oplus \mathbb{A}/\pi^{\mu_m}\mathbb{A},$$

with $\mu_1 > \cdots > \mu_m$, and $x = (\pi^{v_1}, \dots, \pi^{v_m}) \in \mathcal{A}_{\underline{\mu}}$ such that the set

$$(v_1, \mu_1), \dots, (v_m, \mu_m)$$

is an antichain in \mathcal{P} , if $n \in \text{End}_{\mathbb{A}}\mathcal{A}_{\underline{\mu}}$ is such that $n(x) = 0$, then n is nilpotent.

Proof. Case: $m = 1$. In this case $n = (n_{11}) : \mathbb{A}/\pi^{\mu_1}\mathbb{A} \rightarrow \mathbb{A}/\pi^{\mu_1}\mathbb{A}$ and $n(\pi^{v_1}) = n_{11}\pi^{v_1} = 0$ with $v_1 < \mu_1$. So $n_{11} \in (\pi)$. Hence n is nilpotent.

Case: $m > 1$. Write n as a matrix (n_{ij}) , where $n_{ij} : \mathbb{A}/\pi^{\mu_j}\mathbb{A} \rightarrow \mathbb{A}/\pi^{\mu_i}\mathbb{A}$. We have

$$n(\pi^{v_1}, \dots, \pi^{v_m})_i = n_{ii}(\pi^{v_i}) + \sum_{j \neq i} n_{ij}(\pi^{v_j}) = 0,$$

for $1 \leq i \leq m$. If $n_{ii}1$ is a unit, then $n_{ii}\pi^{v_i}$ has valuation v_i , hence at least one of the summands $n_{ij}\pi^{v_j}$ must have valuation v_i or less. It follows from Theorem 3.8 (applied to $\mathbb{A}/\pi^{\mu_j}\mathbb{A}$ and $\mathbb{A}/\pi^{\mu_i}\mathbb{A}$) that $(v_i, \mu_i) \leq (v_j, \mu_j)$ contradicting the antichain hypothesis. Thus, for each i , $n_{ii}(1) \in \pi(\mathbb{A}/\pi^{\mu_i}\mathbb{A})$. It follows that n is nilpotent. □

Every endomorphism of $\mathcal{A}_{\underline{\lambda}}$ can be written as a matrix $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$, where $x : \mathcal{A}_{\underline{\lambda}'} \rightarrow \mathcal{A}_{\underline{\lambda}'}$, $y : \mathcal{A}_{\underline{\lambda}''} \rightarrow \mathcal{A}_{\underline{\lambda}'}$, $z : \mathcal{A}_{\underline{\lambda}'} \rightarrow \mathcal{A}_{\underline{\lambda}''}$ and $w : \mathcal{A}_{\underline{\lambda}''} \rightarrow \mathcal{A}_{\underline{\lambda}''}$ are homomorphisms.

We are now ready to describe the stabilizer of $e(I)$ in $\mathcal{G}_{\underline{\lambda}}$:

- Let $N^{\underline{\lambda}'} = \{n \in \text{End}_{\mathbb{A}}(\mathcal{A}_{\underline{\lambda}'}) \mid n(e(I)') = 0\}$ is a nilpotent ideal in $\text{End}_{\mathbb{A}}(\mathcal{A}_{\underline{\lambda}'})$.
- And $\mathcal{M}(\underline{\lambda}', \underline{\lambda}'') = \{z \in \mathcal{HOM}(\mathcal{A}_{\underline{\lambda}'}, \mathcal{A}_{\underline{\lambda}''}) \mid z(e(I)') = 0\}$.

Theorem 4.12. *The stabilizer of $e(I)$ in $\mathcal{G}_{\underline{\lambda}}$ consists of matrices of the form*

$$\begin{pmatrix} \text{id}_{\mathcal{A}_{\underline{\lambda}'}} + n & y \\ z & w \end{pmatrix},$$

where $n \in N_{\underline{\lambda}'} \subset \text{End}_{\mathbb{A}}(\mathcal{A}_{\underline{\lambda}'})$, $y \in \mathcal{HOM}(\mathcal{A}_{\underline{\lambda}''}, \mathcal{A}_{\underline{\lambda}'})$ is arbitrary, $z \in \mathcal{M}(\underline{\lambda}', \underline{\lambda}'') \subset \mathcal{HOM}(\mathcal{A}_{\underline{\lambda}'}, \mathcal{A}_{\underline{\lambda}''})$, and $w \in \mathcal{G}_{\underline{\lambda}''}$ is invertible.

Proof. Clearly, all the endomorphisms of $\mathcal{A}_{\underline{\lambda}}$ which fix $e(I)$ are of the form stated in the theorem, except that w need not be invertible. We need to show that the invertibility of such an endomorphism is equivalent to the invertibility of w .

To begin with, consider the case where $\mathcal{A}_{\underline{\lambda}} = (\mathbb{A}/\pi^k \mathbb{A})^n$ for some positive integers k, n . Then, if $e(I) \neq 0$ (the case $e(I) = 0$ is trivial), then $\mathcal{A}_{\underline{\lambda}'} = (\mathbb{A}/\pi^k \mathbb{A})$, and $\mathcal{A}_{\underline{\lambda}''} = (\mathbb{A}/\pi^k \mathbb{A})^{n-1}$. The endomorphisms which fix $e(I)$ are all of the form

$$\begin{pmatrix} 1 + n & y \\ z & w \end{pmatrix},$$

where $n, z \in (\pi)$. Such endomorphisms, being block upper-triangular modulo π , are invertible if and only if w is invertible, proving the claim when $\mathcal{A}_{\underline{\lambda}} = (\mathbb{A}/\pi^k \mathbb{A})^n$. In general, $\mathcal{A}_{\underline{\lambda}}$ is a sum of such modules, and an endomorphism of $\mathcal{A}_{\underline{\lambda}}$ is invertible if and only if its diagonal block corresponding to each of these summands is invertible. Therefore the claim follows in general as well. \square

To elaborate more on the above proof consider the following example. Let

$$\mathcal{A}_{\underline{\lambda}} = (\mathbb{A}/\pi^{\lambda_1} \mathbb{A})^{\rho_1} \oplus (\mathbb{A}/\pi^{\lambda_2} \mathbb{A})^{\rho_2} \oplus (\mathbb{A}/\pi^{\lambda_3} \mathbb{A})^{\rho_3}$$

with

$$\underline{\lambda} = (\lambda_1^{\rho_1}, \lambda_2^{\rho_2}, \lambda_3^{\rho_3}) \in \Lambda$$

(refer equation (3.1)) with $\rho_i \geq 2$ for all i . Consider the non-principal ideal I generated by an antichain of the form $\{(v_1, \lambda_1), (v_3, \lambda_3)\}$ in \mathcal{P} . Let $v_2 = \partial_{\lambda_2} I$, and suppose that $(v_2, \lambda_2) \notin \max(I)$. We have $\lambda_1 - v_1 > \lambda_2 - v_2$, $v_2 > v_3$ and

$$\mathcal{A}_{\underline{\lambda}'} = \mathbb{A}/\pi^{\lambda_1} \mathbb{A} \oplus \mathbb{A}/\pi^{\lambda_3} \mathbb{A}, \quad \mathcal{A}_{\underline{\lambda}''} = (\mathbb{A}/\pi^{\lambda_1} \mathbb{A})^{\oplus(\rho_1-1)} \oplus (\mathbb{A}/\pi^{\lambda_2} \mathbb{A})^{\oplus\rho_2} \oplus (\mathbb{A}/\pi^{\lambda_3} \mathbb{A})^{\oplus(\rho_3-1)}.$$

Note that $e(I)' = (\pi^{v_1}, \pi^{v_3}) \in \mathcal{A}_{\underline{\lambda}'}$ and $e(I) = (\underbrace{\pi^{v_1}, 0, \dots, 0}_{\rho_1\text{-tuple}}, \underbrace{0, 0, \dots, 0}_{\rho_2\text{-tuple}}, \underbrace{\pi^{v_3}, 0, \dots, 0}_{\rho_3\text{-tuple}}) \in \mathcal{A}_{\underline{\lambda}}$. A typical element g of $\mathcal{G}_{\underline{\lambda}}$ is of the form

$$\begin{aligned} \bullet g_{11} &= \begin{bmatrix} (m_{1,1})_{1 \times 1} & (m_{1,2})_{(1 \times (\rho_1 - 1))} \\ (m_{2,1})_{((\rho_1 - 1) \times 1)} & (m_{2,2})_{((\rho_1 - 1) \times (\rho_1 - 1))} \end{bmatrix}_{(\rho_1 \times \rho_1)} \\ \bullet g_{12} &= \begin{bmatrix} m_{1,3} & m_{1,4} \\ m_{2,3} & m_{2,4} \end{bmatrix}_{(\rho_1 \times \rho_2)} \pi^{\lambda_1 - \lambda_2} \\ \bullet g_{13} &= \begin{bmatrix} (m_{1,5})_{(1 \times 1)} & (m_{1,6})_{(1 \times (\rho_3 - 1))} \\ (m_{2,5})_{((\rho_1 - 1) \times 1)} & (m_{2,6})_{((\rho_1 - 1) \times (\rho_3 - 1))} \end{bmatrix}_{(\rho_1 \times \rho_3)} \pi^{\lambda_1 - \lambda_3} \\ \bullet g_{21} &= \begin{bmatrix} (m_{3,1})_{(1 \times 1)} & (m_{3,2})_{(1 \times (\rho_1 - 1))} \\ (m_{4,1})_{((\rho_2 - 1) \times 1)} & (m_{4,2})_{((\rho_2 - 1) \times (\rho_1 - 1))} \end{bmatrix}_{(\rho_2 \times \rho_1)} \\ \bullet g_{22} &= \begin{bmatrix} m_{3,3} & m_{3,4} \\ m_{4,3} & m_{4,4} \end{bmatrix}_{(\rho_2 \times \rho_2)} \\ \bullet g_{23} &= \begin{bmatrix} (m_{3,5})_{(1 \times 1)} & (m_{3,6})_{(1 \times (\rho_3 - 1))} \\ (m_{4,5})_{((\rho_2 - 1) \times 1)} & (m_{4,6})_{((\rho_2 - 1) \times (\rho_3 - 1))} \end{bmatrix}_{(\rho_2 \times \rho_3)} \pi^{\lambda_2 - \lambda_3} \\ \bullet g_{31} &= \begin{bmatrix} (m_{5,1})_{(1 \times 1)} & (m_{5,2})_{(1 \times (\rho_1 - 1))} \\ (m_{6,1})_{((\rho_3 - 1) \times 1)} & (m_{6,2})_{((\rho_3 - 1) \times (\rho_1 - 1))} \end{bmatrix}_{(\rho_3 \times \rho_1)} \end{aligned}$$

$$\bullet g_{32} = \begin{bmatrix} m_{5,3} & m_{5,4} \\ m_{6,3} & m_{6,4} \end{bmatrix}_{(\rho_3 \times \rho_2)}$$

$$\bullet g_{33} = \begin{bmatrix} (m_{5,5})_{(1 \times 1)} & (m_{5,6})_{(1 \times (\rho_3 - 1))} \\ (m_{6,5})_{((\rho_3 - 1) \times 1)} & (m_{6,6})_{((\rho_3 - 1) \times (\rho_3 - 1))} \end{bmatrix}_{(\rho_3 \times \rho_3)}$$

Bringing together the coordinates corresponding to $\mathcal{A}_{\underline{\lambda}'}$ (namely the first and fifth coordinates) and the coordinates corresponding to $\mathcal{A}_{\underline{\lambda}''}$ (the remaining coordinates), we get that with respect to the decomposition $\mathcal{A}_{\underline{\lambda}} = \mathcal{A}_{\underline{\lambda}'} \oplus \mathcal{A}_{\underline{\lambda}''}$ we conclude that g is in the following form

$$g_{11} = \begin{bmatrix} (m_{1,1})_{(1 \times 1)} & (m_{1,5})_{(1 \times 1)} \pi^{\lambda_1 - \lambda_3} \\ (m_{5,1})_{(1 \times 1)} & (m_{5,5})_{(1 \times 1)} \end{bmatrix}_{(2 \times 2)}$$

$$g_{12} = \begin{bmatrix} (m_{1,2})_{(1 \times (\rho_1 - 1))} & \begin{bmatrix} m_{1,3} & m_{1,4} \end{bmatrix}_{(1 \times \rho_2)} \pi^{\lambda_1 - \lambda_2} & (m_{1,6})_{(1 \times (\rho_3 - 1))} \pi^{\lambda_1 - \lambda_3} \\ (m_{5,2})_{(1 \times (\rho_1 - 1))} & \begin{bmatrix} m_{5,3} & m_{5,4} \end{bmatrix}_{(1 \times \rho_2)} & (m_{5,6})_{(1 \times (\rho_3 - 1))} \end{bmatrix}$$

$$g_{21} = \begin{bmatrix} (m_{2,1})_{((\rho_1 - 1) \times 1)} & (m_{2,5})_{((\rho_1 - 1) \times 1)} \pi^{\lambda_1 - \lambda_3} \\ \begin{bmatrix} (m_{3,1})_{(1 \times 1)} \\ (m_{4,1})_{((\rho_2 - 1) \times 1)} \end{bmatrix} & \begin{bmatrix} (m_{3,5})_{(1 \times 1)} \\ (m_{4,5})_{((\rho_2 - 1) \times 1)} \end{bmatrix} \pi^{\lambda_2 - \lambda_3} \\ (m_{6,1})_{((\rho_3 - 1) \times 1)} & (m_{6,5})_{((\rho_3 - 1) \times 1)} \end{bmatrix}$$

$$g_{22} = \begin{bmatrix} (m_{2,2})_{((\rho_1 - 1) \times (\rho_1 - 1))} & [m_{2,3} \ m_{2,4}]_{((\rho_1 - 1) \times \rho_2)} \pi^{\lambda_1 - \lambda_2} & (m_{2,6})_{((\rho_1 - 1) \times (\rho_3 - 1))} \pi^{\lambda_1 - \lambda_3} \\ \begin{bmatrix} (m_{3,2})_{(1 \times (\rho_1 - 1))} \\ (m_{4,2})_{((\rho_2 - 1) \times (\rho_1 - 1))} \end{bmatrix} & \begin{bmatrix} m_{3,3} & m_{3,4} \\ m_{4,3} & m_{4,4} \end{bmatrix}_{(\rho_2 \times \rho_2)} & \begin{bmatrix} (m_{3,6})_{(1 \times (\rho_3 - 1))} \\ (m_{4,6})_{((\rho_2 - 1) \times (\rho_3 - 1))} \end{bmatrix} \pi^{\lambda_2 - \lambda_3} \\ (m_{6,2})_{((\rho_3 - 1) \times (\rho_1 - 1))} & [m_{6,3} \ m_{6,4}]_{((\rho_3 - 1) \times \rho_2)} & (m_{6,6})_{((\rho_3 - 1) \times (\rho_3 - 1))} \end{bmatrix}$$

The invertibility of g is equivalent to invertibility of the following diagonal block matrices in g given in

$$\begin{bmatrix} (m_{1,1})_{(1 \times 1)} & (m_{1,2})_{(1 \times (\rho_1 - 1))} \\ (m_{2,1})_{((\rho_1 - 1) \times 1)} & (m_{2,2})_{((\rho_1 - 1) \times (\rho_1 - 1))} \end{bmatrix}_{(\rho_1 \times \rho_1)}, \begin{bmatrix} m_{3,3} & m_{3,4} \\ m_{4,3} & m_{4,4} \end{bmatrix}_{(\rho_2 \times \rho_2)}, \\ \begin{bmatrix} (m_{5,5})_{(1 \times 1)} & (m_{5,6})_{(1 \times (\rho_3 - 1))} \\ (m_{6,5})_{((\rho_3 - 1) \times 1)} & (m_{6,6})_{((\rho_3 - 1) \times (\rho_3 - 1))} \end{bmatrix}_{(\rho_3 \times \rho_3)}$$

Since g is in the stabilizer \mathcal{G}_λ^I of $e(I)$, the diagonal blocks corresponding to one and three ($\max(I) = \{(v_1, \lambda_1), (v_3, \lambda_3)\}$) co-ordinates mentioned above have the following property.

- $(m_{1,1})_{(1 \times 1)}$ and $(m_{5,5})_{(1 \times 1)}$ are units
- $(m_{2,1})_{((\rho_1 - 1) \times 1)}$ and $(m_{6,5})_{((\rho_3 - 1) \times 1)}$ are 0 mod π .

Hence invertibility of g is equivalent to invertibility of the following matrices.

$$(m_{2,2})_{((\rho_1 - 1) \times (\rho_1 - 1))}, \begin{bmatrix} m_{3,3} & m_{3,4} \\ m_{4,3} & m_{4,4} \end{bmatrix}_{(\rho_2 \times \rho_2)}, (m_{6,6})_{((\rho_3 - 1) \times (\rho_3 - 1))}$$

Observe that this is equivalent to invertibility of the following matrix because w is triangular mod π .

$$\begin{bmatrix} (m_{2,2})_{((\rho_1 - 1) \times (\rho_1 - 1))} & [m_{2,3} \ m_{2,4}]_{(\rho_1 - 1 \times \rho_2)} \pi^{\lambda_1 - \lambda_2} & (m_{2,6})_{((\rho_1 - 1) \times (\rho_3 - 1))} \pi^{\lambda_1 - \lambda_3} \\ \begin{bmatrix} (m_{3,2})_{(1 \times (\rho_1 - 1))} \\ (m_{4,2})_{((\rho_2 - 1) \times (\rho_1 - 1))} \\ (m_{6,2})_{((\rho_3 - 1) \times (\rho_1 - 1))} \end{bmatrix} & \begin{bmatrix} m_{3,3} & m_{3,4} \\ m_{4,3} & m_{4,4} \end{bmatrix}_{(\rho_2 \times \rho_2)} & \begin{bmatrix} (m_{3,6})_{(1 \times (\rho_3 - 1))} \\ (m_{4,6})_{((\rho_2 - 1) \times (\rho_3 - 1))} \\ (m_{6,6})_{((\rho_3 - 1) \times (\rho_3 - 1))} \end{bmatrix} \pi^{\lambda_2 - \lambda_3} \end{bmatrix}$$

Now the claim about invertibility follows.

4.4 Cardinality of the Stabilizer

Here we compute the cardinality of the stabilizing group \mathcal{G}_λ^I . This is going to be polynomial in q . Let λ'/I denote the partition corresponding to the isomorphism

class of $\mathcal{A}_{\underline{\lambda}'} / \mathbb{A}e(I)'$. The partition $\underline{\lambda}'/I$ is completely determined by the partition $\underline{\lambda}'$ and the ideal $I \in J(\mathcal{P})_{\underline{\lambda}}$ where $\underline{\lambda}'$ and $\underline{\lambda}''$ arise as partitions with respect to the decomposition 4.9 of $\mathcal{A}_{\underline{\lambda}}$ with respect to I . We will show later that $\underline{\lambda}'/I$ is completely independent of \mathbb{A} (refer Lemma 4.26) for the structure of $\underline{\lambda}'/I$. By using Theorem 4.12 and Observation 3.4 we compute the cardinality of $\mathcal{G}_{\underline{\lambda}}^I$. We have

- $|N^{\underline{\lambda}'}| = |\mathcal{HOM}(\mathcal{A}_{\underline{\lambda}'/I}, \mathcal{A}_{\underline{\lambda}'})|$
- $|\mathcal{M}(\underline{\lambda}', \underline{\lambda}'')| = |\mathcal{HOM}(\mathcal{A}_{\underline{\lambda}'/I}, \mathcal{A}_{\underline{\lambda}''})|$
- $|\mathcal{HOM}(\mathcal{A}_{\underline{\alpha}}, \mathcal{A}_{\underline{\beta}})| = |\mathcal{HOM}(\bigoplus_{i=1}^k (\mathbb{A}/\pi^{\alpha_i} \mathbb{A})^{\rho_i}, \bigoplus_{j=1}^l (\mathbb{A}/\pi^{\beta_j} \mathbb{A})^{\sigma_j})| = q^{\sum_{i=1}^k \sum_{j=1}^l \min(\alpha_i, \beta_j) \rho_i \sigma_j}$
- $|\mathcal{G}_{\underline{\alpha}}| = q^{\sum_{i=1}^k \sum_{j=1}^k \min(\alpha_i, \alpha_j) \rho_i \rho_j} \prod_{i=1}^k q^{(\alpha_i - 1) \rho_i^2} |GL_{\rho_i}(\mathbb{F}_q)|$

Here $\alpha = (\alpha_1^{\rho_1}, \alpha_2^{\rho_2}, \dots, \alpha_k^{\rho_k})$ and $\beta = (\beta_1^{\sigma_1}, \beta_2^{\sigma_2}, \dots, \beta_l^{\sigma_l})$.

So by knowing the partitions $\underline{\lambda}, \underline{\lambda}', \underline{\lambda}'', \underline{\lambda}'/I$ the cardinality of $\mathcal{G}_{\underline{\lambda}}^I$ is given by

$$|\mathcal{G}_{\underline{\lambda}}^I| = |\mathcal{HOM}(\mathcal{A}_{\underline{\lambda}'/I}, \mathcal{A}_{\underline{\lambda}'})| \times |\mathcal{HOM}(\mathcal{A}_{\underline{\lambda}''}, \mathcal{A}_{\underline{\lambda}'})| \times |\mathcal{HOM}(\mathcal{A}_{\underline{\lambda}'/I}, \mathcal{A}_{\underline{\lambda}''})| \times |\mathcal{G}_{\underline{\lambda}''}|$$

4.5 The Stabilizer Orbit of an Element

Let $\mathcal{G}_{\underline{\lambda}}^I$ denote the stabilizer of $e(I) \in \mathcal{A}_{\underline{\lambda}}$. Write each element $m \in \mathcal{A}_{\underline{\lambda}}$ as $m = (m', m'')$ with respect to the decomposition given in the equation (4.9) of $\mathcal{A}_{\underline{\lambda}}$. Also, for any $m' \in \mathcal{A}_{\underline{\lambda}'}$, let \bar{m}' denote the image of m' in $\mathcal{A}_{\underline{\lambda}'}/\mathbb{A}e(I)'$.

Theorem 4.12 allows us to describe the orbits of m under the action of $\mathcal{G}_{\underline{\lambda}}^I$, which is the same as describing the $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}} \times \mathcal{A}_{\underline{\lambda}}$ whose first component lies in the orbit $\mathcal{A}_{\underline{\lambda}}^{I*}$ of $e(I)$.

Theorem 4.13. *Given l and m in $\mathcal{A}_{\underline{\lambda}}$, l lies in the $\mathcal{G}_{\underline{\lambda}}^I$ -orbit of m in $\mathcal{A}_{\underline{\lambda}}$ if and only if the following conditions hold:*

- $l' \in m' + \mathcal{A}_{\underline{\lambda}'}^{I(\bar{m}') \cup I(m'')}$.
- $l'' \in \mathcal{A}_{\underline{\lambda}''}^{I(m'')^*} + \mathcal{A}_{\underline{\lambda}''}^{I(\bar{m}')}$.

Proof. By Theorem 4.12, $l = (l', l'')$ lies in the $\mathcal{G}_{\underline{\lambda}}^I$ -orbit of m if and only if

$$l' = m' + \bar{n}(\bar{m}') + y(m'') \text{ and } l'' = \bar{z}(\bar{m}') + w(m'')$$

for homomorphisms $\bar{n} \in \text{Hom}_{\mathbb{A}}(\mathcal{A}_{\underline{\lambda}'} / \mathbb{A}e(I)', \mathcal{A}_{\underline{\lambda}'}), y \in \text{Hom}_{\mathbb{A}}(\mathcal{A}_{\underline{\lambda}''}, \mathcal{A}_{\underline{\lambda}'}),$
 $\bar{z} \in \text{Hom}_{\mathbb{A}}(\mathcal{A}_{\underline{\lambda}'} / \mathbb{A}e(I)', \mathcal{A}_{\underline{\lambda}''})$ and $w \in \text{Aut}_{\mathbb{A}}(\mathcal{A}_{\underline{\lambda}''})$. By Theorems 3.8 and 3.9, this means

$$l' \in m' + \mathcal{A}_{\underline{\lambda}'}^{I(\bar{m}')} + \mathcal{A}_{\underline{\lambda}'}^{I(m'')} \text{ and } l'' \in \mathcal{A}_{\underline{\lambda}''}^{I(\bar{m}')} + \mathcal{A}_{\underline{\lambda}''}^{I(m'')^*}.$$

By the remark following Theorem 3.15, $\mathcal{A}_{\underline{\lambda}'}^{I(\bar{m}')} + \mathcal{A}_{\underline{\lambda}'}^{I(m'')} = \mathcal{A}_{\underline{\lambda}'}^{I(\bar{m}') \cup I(m'')}$, giving the conditions in the lemma. \square

Given $m = (m', m'') \in \mathcal{A}_{\underline{\lambda}}$, the ideals $I(\bar{m}')$ and $I(m'')$ may be regarded as combinatorial invariants of m . Now suppose that the residue field k of \mathbb{A} is finite of order q . We can show that, having fixed these combinatorial invariants, the cardinality of the orbit of m is a polynomial in q whose coefficients are integers which do not depend on \mathbb{A} . Also, the number of elements of $\mathcal{A}_{\underline{\lambda}}$ having these combinatorial invariants is a polynomial in q whose coefficients are integers which do not depend on \mathbb{A} . Using these observations, we will be able to conclude that the number of orbits of pairs in $\mathcal{A}_{\underline{\lambda}}$ is a polynomial in q whose coefficients are integers which do not depend on \mathbb{A} .

Theorem 4.14. *Fix $J \in J(\mathcal{P})_{\underline{\lambda}'/I}$, $K \in J(\mathcal{P}_{\lambda''})$. Then the cardinality of the $\mathcal{G}_{\underline{\lambda}}^I$ -orbit of any element $m = (m', m'')$ such that $I(\bar{m}') = J$ and $I(m'') = K$ is given by*

$$\alpha_{I,J,K} = |\mathcal{A}_{\underline{\lambda}'}^{J \cup K}| \left(\sum_{K' \subset J \cup K, \max K' \supset \max K - J} |\mathcal{A}_{\underline{\lambda}''}^{K'^*}| \right). \quad (4.15)$$

Proof. This is a direct consequence of Theorems 4.7 and 4.13. \square

Applying Theorem 3.17 and 3.18 to Theorem 4.14 gives:

Theorem 4.16. *Given the module \mathcal{A}_λ and q denoting the cardinality of the residue field of \mathbb{A} , the cardinality of every \mathcal{G}_λ^I -orbit in \mathcal{A}_λ is of the form $\alpha_{I,J,K} = \hat{\alpha}_{I,J,K}(q)$ for some $J \in J(\mathcal{P})_{\lambda'/I}$ and some $K \in J(\mathcal{P})_{\lambda''}$. Each $\hat{\alpha}_{I,J,K}(q)$ is a monic polynomial in q of degree $[J \cup K]_\lambda$ whose coefficients are integers which are independent of the ring \mathbb{A} .*

If the sets

$$X_{I,J,K} = \{(m', m'') \in \mathcal{A}_\lambda \mid I(\bar{m}') = J \text{ and } I(\bar{m}'') = K\}$$

were \mathcal{G}_λ^I -stable, we could have concluded that $X_{I,J,K}$ consists of

$$\frac{|X_{I,J,K}|}{\alpha_{I,J,K}}$$

many orbits, each of cardinality $\alpha_{I,J,K}$. However, $X_{I,J,K}$ is not, in general, \mathcal{G}_λ^I -stable (this can be seen easily by viewing the condition 4.13 in the context of Theorem 4.7).

Note that this condition involves many ideals instead of just one single ideal K .

However they do give a partition of \mathcal{A}_λ i.e $\mathcal{A}_\lambda = \bigsqcup_{J \in J(\mathcal{P})_{\lambda'/I}, K \in J(\mathcal{P})_{\lambda''}} X_{I,J,K}$. The

dependence of $X_{I,J,K}$ on I comes via the homomorphism $\mathcal{A}_{\lambda'} \rightarrow \mathcal{A}_{\lambda'}/\mathbb{A}e(I)'$. The

following lemma gives us a way to work around the non- \mathcal{G}_λ^I -stability of $X_{I,J,K}$:

Lemma 4.17. *Let S be a finite set with a partition $S = \bigsqcup_{i=1}^N S_i$ (for the application we have in mind, these will be the \mathcal{G}_λ^I -orbits in \mathcal{A}_λ). Suppose that S has another partition $S = \bigsqcup_{j=1}^Q T_j$, such that there exist positive integers n_1, n_2, \dots, n_Q for which, if $x \in T_j \cap S_i$, then $|S_i| = n_j$ (in our case, the T_j 's will be the sets $X_{I,J,K}$). Then the number of $i \in \{1, \dots, N\}$ such that $|S_i| = n$ is given by*

$$\frac{1}{n} \sum_{\{j \mid n_j = n\}} |T_j|.$$

Proof. Note that

$$\bigsqcup_{\{j|n_j=n\}} |T_j|$$

is the union of all the S_i 's for which $|S_i| = n$. □

Taking S to be the set \mathcal{A}_λ , the S_i 's to be the \mathcal{G}_λ^I -orbits in \mathcal{A}_λ , and T_j 's to be the sets $X_{I,J,K}$ in Lemma 4.17 gives:

Theorem 4.18. *Let $\alpha(q)$ be a monic polynomial in q with integer coefficients. Then the number of \mathcal{G}_λ^I -orbits in \mathcal{A}_λ with cardinality $\alpha(q)$ is*

$$N_\alpha(q) = \frac{1}{\alpha(q)} \sum_{\{(I,J,K)|\alpha_{I,J,K}(q)=\alpha(q)\}} |X_{I,J,K}|.$$

Since $\alpha(q)$ and $|X_{I,J,K}|$ are polynomials in q , the number $N_\alpha(q)$ of \mathcal{G}_λ^I -orbits in \mathcal{A}_λ of cardinality $\alpha(q)$ is a rational function in q . The following lemma will show that it is in fact a polynomial in q with integer coefficients:

Lemma 4.19. *Let $r(q)$ and $s(q)$ be polynomials in q with integer coefficients. Suppose that $r(q)/s(q)$ takes integer values for infinitely many values of q . Then $r(q)/s(q)$ is a polynomial in q with rational coefficients. If, in addition $s(q)$ is primitive i.e has content 1 (for example $s(q)$ is monic) , then $r(q)/s(q)$ has integer coefficients.*

Proof. Suppose we could write $r(q) = s(q)t(q) + u(q)$ where $t(q), u(q) \neq 0$ are polynomials with rational coefficients with $\deg u(q) < \deg s(q)$. So $r(q)/s(q) = t(q) + u(q)/s(q)$. Now multiply both sides by a positive integer $n > 0$ such that $nt(q)$ is a polynomial in q with integer coefficients. We have $nr(q)/s(q) = nt(q) + nu(q)/s(q)$ and $nr(q)/s(q)$ takes integer values for infinitely many values of q by hypothesis. So it follows that $nu(q)/s(q)$ takes integer values for infinitely many values of q . However $\deg u(q) < \deg s(q)$ and therefore $nu(q)/s(q) \rightarrow 0$ as $q \rightarrow \infty$ which is a contradiction. So $u(q) = 0$. Hence $r(q)/s(q) = t(q)$ a polynomial with rational coefficients.

Now if in addition $s(q)$ is primitive then choose a positive integer $n > 0$ such that $nt(q)$ is a polynomial in q with integer coefficients with content 1 i.e primitive. Then $nr(q) = nt(q)s(q)$. Here we observe that $RHS = nt(q)s(q)$ has content 1 i.e. primitive being a product of two primitive polynomials whereas $LHS = nr(q)$ is certainly not primitive unless $n = 1$ in which case $t(q)$ is a polynomial with integer coefficients. \square

Example 4.20. Consider an arbitrary $\underline{\lambda} \in \Lambda$, and take I to be the maximal ideal in $J(\mathcal{P})_{\underline{\lambda}}$ (this is the ideal in \mathcal{P} generated by $\mathcal{P}_{\underline{\lambda}}$) itself. Then, in the notation of equation (3.1),

$$\underline{\lambda}' = (\lambda_1), \quad \underline{\lambda}'' = (\lambda_1^{\rho_1-1}, \lambda_2^{\rho_2}, \dots, \lambda_l^{\rho_l}).$$

The element $e(I)'$ is a generator of $\mathcal{A}_{\underline{\lambda}'}$, and so $\mathcal{A}_{\underline{\lambda}'}/\mathbb{A}e(I)' = 0$. It follows that the only possibility for the ideal $J \in J(\mathcal{P})_{\underline{\lambda}'/I}$ is $J = \emptyset$. As a result, the only combinatorial invariant of $\mathcal{G}_{\underline{\lambda}}^I$ -orbits in $\mathcal{A}_{\underline{\lambda}}$ is $K \in J(\mathcal{P})_{\underline{\lambda}''}$. We have

$$\alpha_{I, \emptyset, K}(q) = |\mathcal{A}_{\underline{\lambda}'}^{K}| |\mathcal{A}_{\underline{\lambda}''}^{K^*}|.$$

On the other hand,

$$|X_{I, \emptyset, K}| = q^{\lambda_1} |\mathcal{A}_{\underline{\lambda}''}^{K^*}|.$$

Therefore, given a polynomial $\alpha(q)$, the number of $\mathcal{G}_{\underline{\lambda}}^I$ -orbits of cardinality $\alpha(q)$ is

$$\sum_{\{K \in J(\mathcal{P})_{\underline{\lambda}''} \mid \alpha_{I, \emptyset, K} = \alpha(q)\}} \frac{q^{\lambda_1}}{|\mathcal{A}_{\underline{\lambda}'}^{K}|}.$$

For example, if $\underline{\lambda} = (2, 1^{\rho_2})$, then the number of $\mathcal{G}_{\underline{\lambda}}^I$ -orbits in $\mathcal{A}_{\underline{\lambda}}$ is $q^2 + q$, and if $\underline{\lambda} = (2^{\rho_1}, 1^{\rho_2})$ with $\rho_1 > 1$, then the number of $\mathcal{G}_{\underline{\lambda}}^I$ -orbits in $\mathcal{A}_{\underline{\lambda}}$ is $q^2 + 2q + 1$.

Example 4.21. Now consider the case where $\underline{\lambda} = (5, 4, 4, 2, 1)$ and I is the ideal of Figure 3.2. Then the first column of Table 4.1 gives all the possible cardinalities for $\mathcal{G}_{\underline{\lambda}}^I$ -orbits in $\mathcal{A}_{\underline{\lambda}}$.

| Cardinality | Number of Orbits |
|-----------------|------------------|
| 1 | q^3 |
| $(q-1)q^7$ | $(q-1)q$ |
| $(q-1)q^{12}$ | $(q-1)$ |
| q^4 | $(q-1)q^2$ |
| $(q-1)^2q^{11}$ | 1 |
| $(q-1)^2q^8$ | q |
| $(q-1)^2q^{10}$ | 1 |
| $(q-1)q^2$ | q^2 |
| $(q-1)^2q^6$ | q |
| $(q-1)^2q^3$ | q^2 |
| $(q-1)^2q^5$ | q |
| $(q-1)$ | q^3 |
| $(q-1)q^{15}$ | 1 |
| $(q-1)q^5$ | q |
| q^9 | $(q-1)q$ |
| $(q-1)q^8$ | q |
| $(q-1)q^{14}$ | 1 |
| $(q-1)q^{11}$ | $(q-1)$ |
| $(q-1)q^6$ | q^2 |
| $(q-1)q^4$ | $(q-1)q^2$ |
| $(q-1)q^3$ | $2q^2$ |
| $(q-1)q^9$ | q^2 |
| $(q-1)q^{10}$ | q |

Table 4.1: Cardinalities and numbers of stabilizer orbits

The corresponding element of the second column is the number of orbits with that cardinality. The total number of \mathcal{G}_λ^I -orbits in \mathcal{A}_λ is given by the polynomial

$$4q^3 + 6q^2 + 6q + 2.$$

This data was generated using a computer program written in sage. In general the total number of \mathcal{G}_λ^I -orbits in \mathcal{A}_λ need not be a polynomial with positive integer coefficients, for example, take $\lambda = (2)$ ($\mathbb{A} = \mathbb{Z}_p$ so $\mathcal{A}_\lambda = \mathbb{Z}/p^2\mathbb{Z}$) and I is the ideal generated by $(1, 2)$ (the corresponding orbit in \mathcal{A}_λ contains p). The total number of \mathcal{G}_λ^I -orbits in \mathcal{A}_λ is $2p - 1$.

The above results can be summarized to give the following Theorem:

Theorem 4.22. *Let \mathbb{A} be a discrete valuation ring with finite residue field. Fix $\underline{\lambda} \in \Lambda$ and take \mathcal{G}_λ as given in equation (3.2). Let \mathcal{G}_λ denote the group of \mathbb{A} -module automorphisms of \mathcal{A}_λ . Fix an order ideal $I \in J(\mathcal{P})_\lambda$ (and hence the \mathcal{G}_λ -orbit $\mathcal{A}_\lambda^{I^*}$ in \mathcal{A}_λ).*

1. *The cardinality of each \mathcal{G}_λ -orbit in $\mathcal{A}_\lambda^{I^*} \times \mathcal{A}_\lambda$ is a monic polynomial in q whose coefficients are integers.*
2. *Given a monic polynomial $\beta(q)$ with integer coefficients, the number of \mathcal{G}_λ -orbits in $\mathcal{A}_\lambda^{I^*} \times \mathcal{A}_\lambda$ of cardinality $\beta(q)$ is a polynomial in q with coefficients that are integers which do not depend on \mathbb{A} .*
3. *The total number of \mathcal{G}_λ -orbits in $\mathcal{A}_\lambda^{I^*} \times \mathcal{A}_\lambda$ depends only on whether ρ_i is 0, 1, or any cardinal greater than 1 (and not on the exact value of ρ_i) for each of the multiplicities ρ_i in equation (3.1).*

For part three of the above Theorem 4.22 we request the reader to refer Corollary 4.28.

For the total number of orbits in $\mathcal{A}_{\underline{\lambda}} \times \mathcal{A}_{\underline{\lambda}}$, we have:

Theorem 4.23. *Let \mathbb{A} be a discrete valuation ring with finite residue field of order q . Fix $\underline{\lambda} \in \Lambda$ and take $\mathcal{A}_{\underline{\lambda}}$ as in equation (3.2). Let $\mathcal{G}_{\underline{\lambda}}$ denote the group of \mathbb{A} -module automorphisms of $\mathcal{A}_{\underline{\lambda}}$. Then there exists a monic polynomial $n_{\underline{\lambda}}(q)$ of degree λ_1 with integer coefficients (which do not depend on \mathbb{A} or q) such that the number of $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}} \times \mathcal{A}_{\underline{\lambda}}$ is $n_{\underline{\lambda}}(q)$.*

Proof. The only thing that remains to be proved is the assertion about the degree of $n_{\underline{\lambda}}(q)$. By Theorem 4.18,

$$\deg n_{\underline{\lambda}}(q) = \max_{I,J,K} (\deg |X_{I,J,K}| - \deg \alpha_{I,J,K}(q)).$$

Recalling the definitions of $X_{I,J,K}$ and $\alpha_{I,J,K}(q)$, we find that we need to show that

$$[J \cup K]_{\underline{\lambda}'/I} + \log_q |\mathbb{A}e(I)'| + [K]_{\underline{\lambda}''} \leq \lambda_1 + [J \cup K]_{\underline{\lambda}}.$$

Observe that $[J \cup K]_{\underline{\lambda}} = [J \cup K]_{\underline{\lambda}'} + [J \cup K]_{\underline{\lambda}''}$, and $[K]_{\underline{\lambda}''} \leq [J \cup K]_{\underline{\lambda}''}$. Moreover, it turns out that $[J \cup K]_{\underline{\lambda}'/I} \leq [J \cup K]_{\underline{\lambda}'}$ (see Lemma 4.24 below). Therefore, the inequality to be proved reduces to $\log_q |\mathbb{A}e(I)'| \leq \lambda_1$, which is obviously true. Furthermore, if equality holds, then $|\mathbb{A}e(I)'| = q^{\lambda_1}$, which is only possible if I is the maximal ideal in $J(\mathcal{P})_{\underline{\lambda}}$, which was considered in Example 4.20, where a monic polynomial of degree 0 was obtained. \square

Lemma 4.24. *For any ideal $J \in J(\mathcal{P})$,*

$$[J]_{\underline{\lambda}'/I} \leq [J]_{\underline{\lambda}'}$$

Proof. The partition $\underline{\lambda}'/I$ is described in Lemma 4.26. Observe that

$$k_1 \geq v_1 + k_2 - v_2 \geq k_2 \geq v_2 + k_3 - v_3 \geq \cdots \geq v_{s-1} + k_s - v_s \geq v_s,$$

In other words, the parts of $\underline{\lambda'}/I$ alternate with the parts of $\underline{\lambda'}$. For each ideal $J \in J(\mathcal{P})_{\underline{\lambda'}/I}$, the contribution of J to $[J]_{\underline{\lambda'}/I}$ in a given chain $(*, v_i + k_{i+1} - v_{i+1}) \subset \mathcal{P}_{\underline{\lambda'}/I}$ (or $(*, v_s) \subset \mathcal{P}_{\underline{\lambda'}/I}$) is less than equal to its contribution to $[J]_{\underline{\lambda'}}$ in the chain $(*, k_i) \subset \mathcal{P}_{\underline{\lambda'}}$ (resp. $(*, k_s) \subset \mathcal{P}_{\underline{\lambda'}}$). It follows that $[J]_{\underline{\lambda'}} \geq [J]_{\underline{\lambda'}/I}$. \square

4.6 Product of Two Orbits I and L

In order to refine Theorem 4.22 to the enumeration of $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}}^{I*} \times \mathcal{A}_{\underline{\lambda}}^{L*}$ for a pair of order ideals $(I, L) \in J(\mathcal{P})_{\underline{\lambda}}^2$, we need to repeat the calculations in section 4.5 with $X_{I,J,K}$ replaced by its subset

$$X_{I,J,K,L} = \{m \in X_{I,J,K} \mid m \in \mathcal{A}_{\underline{\lambda}}^{L*}\}.$$

Thus our goal is to show that $|X_{I,J,K,L}|$ is a polynomial in q whose coefficients are integers which do not depend on \mathbb{A} . By using Moebius inversion on the lattice $J(\mathcal{P})_{\underline{\lambda}}$, it suffices to show that

$$Y_{I,J,K,L} = \{m \in X_{I,J,K} \mid m \in \mathcal{A}_{\underline{\lambda}}^L\}$$

has cardinality polynomial in q whose coefficients are integers which do not depend on \mathbb{A} . This is easier, because $m = (m', m'') \in \mathcal{A}_{\underline{\lambda}}^L$ if and only if $m' \in \mathcal{A}_{\underline{\lambda}'}^L$ and $m'' \in \mathcal{A}_{\underline{\lambda}''}^L$. If $(m', m'') \in Y_{I,J,K,L}$, we also have that $m'' \in \mathcal{A}_{\underline{\lambda}''}^{K*}$. Thus $Y_{I,J,K,L} = \emptyset$ unless $K \subset L$, in which case

$$|Y_{I,J,K,L}| = \#\{m' \in \mathcal{A}_{\underline{\lambda}'}^L \mid I(\bar{m}') = J\} |\mathcal{A}_{\underline{\lambda}''}^{K*}|.$$

Therefore, we are reduced to proving the following lemma:

Lemma 4.25. *The cardinality of the set*

$$\{m' \in \mathcal{A}_{\underline{\lambda}'} \mid I(m') \subset L \text{ and } I(\bar{m}') = J\}$$

is a polynomial in q whose coefficients are integers which do not depend on \mathbb{A} .

Proof. Let $\bar{\mathcal{A}}_{\underline{\lambda}'}$ denote the quotient $\mathcal{A}_{\underline{\lambda}'}/\mathbb{A}e(I)'$ (so $\bar{\mathcal{A}}_{\underline{\lambda}'}$ is isomorphic to $\mathcal{A}_{\underline{\lambda}'/\underline{\mathcal{I}}}$ in the notation of section 4.5). Suppose that $\max I = \{(v_1, k_1), \dots, (v_s, k_s)\}$. Then

$$\mathcal{A}_{\underline{\lambda}'} = \mathbb{A}/\pi^{k_1}\mathbb{A} \oplus \dots \oplus \mathbb{A}/\pi^{k_s}\mathbb{A}.$$

Lemma 4.26. *Let $\underline{\lambda'}/I$ be the partition given by*

$$\underline{\lambda'}/I = (v_1 + k_2 - v_2, v_2 + k_3 - v_3, \dots, v_{s-1} + k_s - v_s, v_s).$$

and $\mathcal{A}_{\underline{\lambda}'/\underline{\mathcal{I}}}$ be the corresponding \mathbb{A} -module as given by equation (3.2) with all multiplicities equal to 1. If $Q \in SL_s(\mathbb{A})$ is the matrix

$$Q = \begin{pmatrix} 1 & \pi^{v_1-v_2} & \pi^{v_1-v_3} & \dots & \pi^{v_1-v_s} \\ 0 & 1 & \pi^{v_2-v_3} & \dots & \pi^{v_2-v_s} \\ 0 & 0 & 1 & \dots & \pi^{v_3-v_s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

then the isomorphism $\mathbb{A}^s \rightarrow \mathbb{A}^s$ whose matrix is Q descends to a homomorphism $\bar{Q} : \mathcal{A}_{\underline{\lambda}'} \rightarrow \mathcal{A}_{\underline{\lambda}'/\underline{\mathcal{I}}}$ such that $\ker \bar{Q} \supset \mathcal{A}_{\underline{\lambda}'}e(I)'$. The induced homomorphism $\mathcal{A}_{\underline{\lambda}'}/\mathbb{A}e(I)' \rightarrow \mathcal{A}_{\underline{\lambda}'/\underline{\mathcal{I}}}$ is an isomorphism of \mathbb{A} -modules.

Proof. Let e_1, e_2, \dots, e_s denote the standard basis of \mathbb{A}^s , and $f_1 = Q(e_1), f_2 = Q(e_2), \dots, f_s = Q(e_s)$ denote another basis of \mathbb{A}^s which is the image of the standard basis under the isomorphism Q . Let $\bar{e}_1, \dots, \bar{e}_s$ denote the generators of $\mathcal{A}_{\underline{\lambda}'}$,

and $\bar{f}_1, \dots, \bar{f}_s$ denote the generators of $\mathcal{A}_{\underline{\lambda}'/\underline{\mathcal{I}'}}$. We have

$$Qe_j = \begin{cases} f_1 & \text{for } j = 1, \\ -\pi^{v_{j-1}-v_j} f_{j-1} + f_j & \text{for } 1 < j \leq n. \end{cases}$$

By using the inequalities $\lambda_j > v_j + \lambda_{j+1} - v_{j+1}$ for $1 \leq j < s$ and $\lambda_s \geq v_s$, one easily verifies that $Q(\pi^{\lambda_j} e_j)$ is 0 in $\mathcal{A}_{\underline{\lambda}'/\underline{\mathcal{I}'}}$. Therefore Q induces a well-defined \mathbb{A} -module homomorphism $\bar{Q} : \mathcal{A}_{\underline{\lambda}'} \rightarrow \mathcal{A}_{\underline{\lambda}'/\underline{\mathcal{I}'}}$. Now

$$\begin{aligned} \bar{Q}(e(I)) &= \bar{Q}\left(\sum \pi^{v_j} \bar{e}_j\right) \\ &= \pi^{v_1} \bar{f}_1 + (-\pi^{v_2+v_1-v_2} \bar{f}_1 + \pi^{v_2} \bar{f}_2) + (-\pi^{v_3+v_2-v_3} \bar{f}_2 + \pi^{v_3} \bar{f}_3) + \\ &\quad \dots + (-\pi^{v_s+v_{s-1}-v_s} \bar{f}_{s-1} + \pi^{v_s} \bar{f}_s) \\ &= 0. \end{aligned}$$

Therefore \bar{Q} induces a homomorphism $\mathcal{A}_{\underline{\lambda}'}/\mathbb{A}e(I)' \rightarrow \mathcal{A}_{\underline{\lambda}'/\underline{\mathcal{I}'}}$. Because $Q \in SL_s(\mathbb{A})$, \bar{Q} is onto. When the residue field of \mathbb{A} is finite, one easily verifies that $|\mathbb{A}e(I)'| |\mathcal{A}_{\underline{\lambda}'/\underline{\mathcal{I}'}}| = |\mathcal{A}_{\underline{\lambda}'}|$, whereby \bar{Q} is an isomorphism. Indeed, $|\mathbb{A}e(I)'| = q^{\lambda_1 - v_1}$, $|\mathcal{A}_{\underline{\lambda}'}| = q^{|\underline{\lambda}'|}$ and $|\mathcal{A}_{\underline{\lambda}'/\underline{\mathcal{I}'}}| = q^{|\mathcal{A}_{\underline{\lambda}'/\underline{\mathcal{I}'}}|} = q^{v_1 + \lambda_2 + \dots + \lambda_s}$. In general, this argument using cardinalities can be easily replaced by an argument using the lengths of modules of \mathbb{A} . \square

We now return to the proof of Lemma 4.25. Using Moebius inversion on the lattice $J(\mathcal{P})_{\underline{\lambda}'/\underline{\mathcal{I}'}}$, in order to prove Lemma 4.25, it suffices to show that the cardinality of the set

$$S = \{m' \in \mathcal{A}_{\underline{\lambda}'} \mid m' \in \mathcal{A}_{\underline{\lambda}'}^L \text{ and } \bar{m}' \in \mathcal{A}_{\underline{\lambda}'/\underline{\mathcal{I}'}}^J\}$$

is a polynomial in q whose coefficients are integers which do not depend on \mathbb{A} . Write $m' \in \mathcal{A}_{\underline{\lambda}'}$ as $m'_1 e_1 + \dots + m'_s e_s$, and $n \in \mathcal{A}_{\underline{\lambda}'/\underline{\mathcal{I}'}}$ as $n_1 f_1 + \dots + n_s f_s$. By equation (3.14)

and Lemma 4.26, S consists of elements $m' \in \mathcal{A}_{\underline{\lambda}'}$ such that

$$\begin{aligned} v(m'_i) &\geq \partial_{k_i} L && \text{for } i = 1, \dots, s, \\ v(\bar{Q}(m')_i) &\geq \partial_{v_i+k_{i+1}-v_{i+1}} J && \text{for } i = 1, \dots, s-1, \text{ and} \\ v(\bar{Q}(m')_s) &\geq \partial_{v_s} J, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} v(m'_i) &\geq \partial_{k_i} L && \text{for } i = 1, \dots, s, \\ v(m'_i - \pi^{v_i-v_{i+1}} m'_{i+1}) &\geq \partial_{v_i+k_{i+1}-v_{i+1}} J && \text{for } i = 1, \dots, s-1, \text{ and} \\ v(m'_s) &\geq \partial_{v_s} J. \end{aligned}$$

Therefore we are free to choose for m_s any element of $\mathbb{A}/\pi^{k_s}\mathbb{A}$ which satisfies

$$v(m'_s) \geq \max(\partial_{k_s} L, \partial_{v_s} J).$$

Thus the number of possible choices of m'_s of any given valuation is a polynomial in q with coefficients that are integers which do not depend on \mathbb{A} . Having fixed m'_s , we are free to choose m'_{s-1} satisfying

$$\begin{aligned} v(m'_{s-1}) &\geq \partial_{k_{s-1}} L \\ v(m'_{s-1} + \pi^{v_{s-1}-v_s} m'_s) &\geq \partial_{v_{s-1}+k_s-v_s} J. \end{aligned}$$

Note that for any $x, y \in \mathbb{A}/\pi^k\mathbb{A}$ and non-negative integers u, v , the cardinality of the set

$$\{x \mid v(x+y) \geq v \text{ and } v(x) = u\}$$

is a polynomial in q with coefficients that are integers which do not depend on \mathbb{A} . This shows that for each fixed valuation of m'_s , the number of possible choices for

m'_{s-1} of a fixed valuation is again a polynomial in q whose coefficients are integers that do not depend on \mathbb{A} . Continuing in this manner, we find that the cardinality of S is a polynomial in q whose coefficients are integers which do not depend on \mathbb{A} . \square

Proceeding exactly as in the proof of the Theorem 4.22 we can obtain the following refinement:

Theorem 4.27 (Main Theorem). *Let \mathbb{A} be a discrete valuation ring with finite residue field. Fix $\underline{\lambda} \in \Lambda$ and take $\mathcal{A}_{\underline{\lambda}}$ as given in equation (3.2). Let $\mathcal{G}_{\underline{\lambda}}$ denote the group of \mathbb{A} -module automorphisms of $\mathcal{A}_{\underline{\lambda}}$. Fix order ideals $I, L \in J(\mathcal{P})_{\underline{\lambda}}$ (and hence $\mathcal{G}_{\underline{\lambda}}$ -orbits $\mathcal{A}_{\underline{\lambda}}^{I^*}$ and $\mathcal{A}_{\underline{\lambda}}^{L^*}$ in $\mathcal{A}_{\underline{\lambda}}$).*

1. *The cardinality of each $\mathcal{G}_{\underline{\lambda}}$ -orbit in $\mathcal{A}_{\underline{\lambda}}^{I^*} \times \mathcal{A}_{\underline{\lambda}}^{L^*}$ is a monic polynomial in q whose coefficients are integers.*
2. *Given a monic polynomial $\beta(q)$ with integer coefficients, the number of $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}}^{I^*} \times \mathcal{A}_{\underline{\lambda}}^{L^*}$ of cardinality $\beta(q)$ is a polynomial in q with coefficients that are integers which do not depend on \mathbb{A} .*
3. *The total number of $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}}^{I^*} \times \mathcal{A}_{\underline{\lambda}}^{L^*}$ depends only on whether ρ_i is 0, 1, or any cardinal greater than 1 (and not on the exact value of ρ_i) for each of the multiplicities ρ_i in equation (3.1).*

Proof. If we are only interested in the number of orbits, Corollary 4.28 below allows us to reduce any $\underline{\lambda} \in \Lambda$ to $\underline{\lambda}^{(2)} \in \Lambda$. Here we give a different proof of part 3 of Theorem 4.27. Once we fix an I , the partitions $\underline{\lambda}, \underline{\lambda}', \underline{\lambda}''$ and $\underline{\lambda}'/I$ are all well determined. Let $L \in J(\mathcal{P})_{\underline{\lambda}}$ be as given in Theorem 4.27. Let $J \in J(\mathcal{P})_{\underline{\lambda}'/I}$ and $K \in J(\mathcal{P})_{\underline{\lambda}''}$. The ideal corresponding to the characteristic submodule $\mathcal{HOM}(\mathcal{A}_{\underline{\lambda}'/I}, \mathcal{A}_{\underline{\lambda}'})(\mathcal{A}_{\underline{\lambda}'/I}^J) + \mathcal{HOM}(\mathcal{A}_{\underline{\lambda}'}, \mathcal{A}_{\underline{\lambda}'}) (\mathcal{A}_{\underline{\lambda}''}^K)$ of $\mathcal{A}_{\underline{\lambda}}$ is $[J \cup K]_{\underline{\lambda}'}$ the ideal generated by J and K in $\mathcal{P}_{\underline{\lambda}'}$. Let an element $m' \in \mathcal{A}_{\underline{\lambda}'}$ be such that $I(\bar{m}') = J$. Then they determine the following $\mathcal{G}_{\underline{\lambda}}^I$ orbit $\mathcal{O}_{m', J, K}$

- $m' + \mathcal{A}_{\underline{\lambda}'}^{J \cup K}$.
- $\mathcal{A}_{\underline{\lambda}''}^{K^*} + \mathcal{A}_{\underline{\lambda}''}^J$.

So we have a surjection onto the $\mathcal{G}_{\underline{\lambda}}^I$ -orbits in $\mathcal{A}_{\underline{\lambda}}^{L^*}$ from the space $S = \{(m', J, K) \mid J \in J(\mathcal{P})_{\underline{\lambda}'/I}, K \in J(\mathcal{P})_{\underline{\lambda}''}, m' \in \mathcal{A}_{\underline{\lambda}'}$ with $I(\bar{m}') = J$ and $\mathcal{O}_{m', J, K} \subset \mathcal{A}_{\underline{\lambda}}^{L^*}\}$ to the stabilizer suborbits in $\mathcal{A}_{\underline{\lambda}}^{L^*}$ or to the orbits under the diagonal action of $\mathcal{G}_{\underline{\lambda}}$ in $\mathcal{A}_{\underline{\lambda}}^{I^*} \times \mathcal{A}_{\underline{\lambda}}^{L^*}$.

Now we immediately see that (m', J_1, K_1) and (n', J_2, K_2) gives rise to the same $\mathcal{G}_{\underline{\lambda}}^I$ orbit if and only if

- $[J_1]_{\underline{\lambda}'} \cup [K_1]_{\underline{\lambda}'} = [J_2]_{\underline{\lambda}'} \cup [K_2]_{\underline{\lambda}'}$
- $[J_1]_{\underline{\lambda}''} \cup [K_1]_{\underline{\lambda}''} = [J_2]_{\underline{\lambda}''} \cup [K_2]_{\underline{\lambda}''}$
- $\max([K_1]_{\underline{\lambda}''}) - [J_1]_{\underline{\lambda}''} = \max([K_2]_{\underline{\lambda}''}) - [J_2]_{\underline{\lambda}''}$
- $m' - n' \in \mathcal{A}_{\underline{\lambda}'}^{J_1 \cup K_1} = \mathcal{A}_{\underline{\lambda}'}^{J_2 \cup K_2}$

We observe that $\underline{\lambda}, \underline{\lambda}', \underline{\lambda}''$ and $\underline{\lambda}'/I$ and the above conditions do not depend on the multiplicities ρ_i when we change them when they are already greater than 1. Hence the proof of 3 in Theorem 4.27 follows. \square

Corollary 4.28 (Independence of multiplicities larger than two). *Consider the partition $\underline{\lambda}^{(m)}$ derived from $\underline{\lambda}$ by:*

$$\underline{\lambda}^{(m)} = (\lambda_1^{\min(m_1, m)}, \lambda_2^{\min(m_2, m)}, \dots, \lambda_l^{\min(m_l, m)}).$$

Let $\mathcal{A}_{\underline{\lambda}^{(m)}}$ denote the \mathbb{A} -module corresponding to $\underline{\lambda}^{(m)}$, with automorphism group $\mathcal{G}_{\underline{\lambda}^{(m)}}$. Then the standard inclusion map $\mathcal{A}_{\underline{\lambda}^{(2)}} \hookrightarrow \mathcal{A}_{\underline{\lambda}}$ induces a bijection

$$\mathcal{G}_{\underline{\lambda}^{(2)}} \backslash \mathcal{A}_{\underline{\lambda}^{(2)}} \times \mathcal{A}_{\underline{\lambda}^{(2)}} \xrightarrow{\sim} \mathcal{G}_{\underline{\lambda}} \backslash \mathcal{A}_{\underline{\lambda}} \times \mathcal{A}_{\underline{\lambda}}. \quad (4.29)$$

Proof. We shall use the fact that the canonical forms $e(I)$ of Theorem 3.12 lie in $\mathcal{A}_{\underline{\lambda}^{(1)}} \subset \mathcal{A}_{\underline{\lambda}}$. Thus given a pair $(x, y) \in \mathcal{A}_{\underline{\lambda}} \times \mathcal{A}_{\underline{\lambda}}$, we can reduce x to $e(I) \in \mathcal{A}_{\underline{\lambda}^{(1)}}$ using automorphisms of $\mathcal{A}_{\underline{\lambda}}$. Theorem 4.12 shows that, while preserving $e(I)$, automorphisms of $\mathcal{A}_{\underline{\lambda}}$ can be used to further reduce y to an element of $\mathcal{A}_{\underline{\lambda}'} \oplus \mathcal{A}_{\underline{\lambda}''^{(1)}} \subset \mathcal{A}_{\underline{\lambda}^{(2)}}$. This proves the surjectivity of the map in (4.29).

To see injectivity, suppose that two pairs (x_1, y_1) and (x_2, y_2) in $\mathcal{A}_{\underline{\lambda}^{(2)}} \times \mathcal{A}_{\underline{\lambda}^{(2)}}$ lie in the same $\mathcal{G}_{\underline{\lambda}}$ -orbit. Since $\mathcal{A}_{\underline{\lambda}^{(2)}}$ is a direct summand of $\mathcal{A}_{\underline{\lambda}}$, we can write $\mathcal{A}_{\underline{\lambda}} = \mathcal{A}_{\underline{\lambda}^{(2)}} \oplus \mathcal{B}$.

If $g \in \mathcal{G}_{\underline{\lambda}}$ has matrix $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ with respect to this decomposition, then $g_{11} \in \mathcal{G}_{\underline{\lambda}^{(2)}}$ also maps $(x_1, y_1) \in \mathcal{A}_{\underline{\lambda}^{(2)}} \times \mathcal{A}_{\underline{\lambda}^{(2)}}$ to $(x_2, y_2) \in \mathcal{A}_{\underline{\lambda}^{(2)}} \times \mathcal{A}_{\underline{\lambda}^{(2)}}$. \square

Remark 4.30. Corollary 4.28 and its proof remain valid if we restrict ourselves to $\mathcal{G}_{\underline{\lambda}}$ -orbits in $\mathcal{A}_{\underline{\lambda}}^{I^*} \times \mathcal{A}_{\underline{\lambda}}^{J^*}$ for order ideals $I, J \in J(\mathcal{P})_{\underline{\lambda}}$.

Chapter 5

Multiplicity-Free

In this chapter we consider the permutation representation of the automorphism group \mathcal{G}_λ of a module \mathcal{A}_λ on any particular orbit \mathcal{O}_I . We prove that $\mathbb{C}[\mathcal{O}_I]$ decomposes as a direct sum of distinct irreducible representations of the automorphism group \mathcal{G}_λ by showing that the endomorphism algebra $\mathcal{E}\mathcal{N}\mathcal{D}_{\mathcal{G}_\lambda}(\mathbb{C}[\mathcal{O}_I])$ is commutative. We use both the notations \mathcal{O}_I and $\mathcal{A}_\lambda^{I^*}$ for the orbit in \mathcal{A}_λ corresponding to an ideal I .

5.1 Two Simple Cases

Here are two simple cases:

Theorem 5.1. *For $\underline{\lambda} = (n) \in \Lambda$, the permutation representation of \mathcal{G}_λ on any \mathcal{G}_λ -orbit \mathcal{O}_I in \mathcal{A}_λ is multiplicity-free.*

Proof. In this case we have that the number of orbits of the group under the automorphism group is $(n + 1)$.

| Ideal I | Orbit $\mathcal{A}_\lambda^{I^*} = \mathcal{O}_I$ | \mathcal{A}_λ^I |
|--|---|-------------------------------------|
| $\max I = \{(j, n)\}$ such that $0 \leq j < n$ | $\pi^j(\mathbb{A}/\pi^n\mathbb{A}) - \pi^{j+1}(\mathbb{A}/\pi^n\mathbb{A})$ | $\pi^j(\mathbb{A}/\pi^n\mathbb{A})$ |
| $\max I = \emptyset$ | $\{\underline{0}\}$ | 0 |

Consider the orbit \mathcal{O}_I with $\max(I) = \{(j, n)\}$. For each $y \in (\mathbb{A}/\pi^{n-j}\mathbb{A})^*$, let

$$\mathcal{O}_I^y = \{(a, ya) \mid \text{for all } a \in \mathcal{O}_I\}.$$

The partition of $\mathcal{O}_I \times \mathcal{O}_I$ into suborbits under the diagonal action of \mathcal{G}_λ is given by

$$\mathcal{O}_I \times \mathcal{O}_I = \bigsqcup_{y \in (\mathbb{A}/\pi^{n-j}\mathbb{A})^*} \mathcal{O}_I^y.$$

Let I_y denote the indicator function of \mathcal{O}_I^y . Then we have $I_{y_1} * I_{y_2} = I_{y_1 y_2}$ for all $y_1, y_2 \in (\mathbb{A}/\pi^{n-j}\mathbb{A})^*$ which is obvious in this case (convolution of two lines in the plane corresponding to \mathcal{O}_{y_1} and \mathcal{O}_{y_2} passing through the origin with slopes not in the set $\{0, \infty\}$ correspond to multiplication of their slopes y_1, y_2). So the endomorphism algebra $\mathcal{E}\mathcal{N}\mathcal{D}_{\mathcal{G}_\lambda}(\mathbb{C}[\mathcal{O}_I])$ is commutative. The permutation representation on the zero orbit is the trivial representation. Hence the permutation representation on any orbit in the case $\lambda = (n) \in \Lambda$ is multiplicity-free. \square

Theorem 5.2. *For $\lambda = (n^k) \in \Lambda$, the permutation representation of \mathcal{G}_λ on any \mathcal{G}_λ -orbit \mathcal{O}_I in \mathcal{A}_λ is multiplicity-free.*

Proof. Here also the number of orbits of the group under the automorphism group is $(n+1)$.

| Ideal I | Orbit $\mathcal{A}_\lambda^{I^*} = \mathcal{O}_I$ | \mathcal{A}_λ^I |
|-------------------------------------|---|---------------------------------------|
| $\max I = \{(j, n)\}, 0 \leq j < n$ | $\pi^j(\mathbb{A}/\pi^n\mathbb{A})^k - \pi^{j+1}(\mathbb{A}/\pi^n\mathbb{A})^k$ | $\pi^j(\mathbb{A}/\pi^n\mathbb{A})^k$ |
| $\max I = \emptyset$ | $\{0\}$ | 0 |

Consider the orbit \mathcal{O}_I with $\max(I) = \{(j, k)\}$. Again for each $y \in (\mathbb{A}/\pi^{n-j}\mathbb{A})^*$ we have a suborbit \mathcal{O}_I^y of $\mathcal{O}_I \times \mathcal{O}_I$ defined as $\mathcal{O}_I^y = \{(a, ya) \mid a \in \mathcal{O}_I\}$. The complement of $\bigcup_{y \in (\mathbb{A}/\pi^{n-j}\mathbb{A})^*} \mathcal{O}_I^y$ in $\mathcal{O}_I \times \mathcal{O}_I$ is also a suborbit, which we denote by \mathcal{O}_g . Thus the decomposition of $(\mathcal{O}_I \times \mathcal{O}_I)$ into suborbits under the action of \mathcal{G}_λ is given by

$$(\mathcal{O}_I \times \mathcal{O}_I) = \mathcal{O}_g \bigsqcup \left(\bigsqcup_{y \in (\mathbb{A}/\pi^{n-j}\mathbb{A})^*} \mathcal{O}_I^y \right).$$

Note that here the suborbits are parametrized by the set $(\mathbb{A}/\pi^{n-j}\mathbb{A})^* \cup \{g\}$. Let I_y denote the indicator function of \mathcal{O}_I^y . Then we have $I_{y_1} * I_{y_2} = I_{y_1 y_2}$ for all $y_1, y_2 \in$

$(\mathbb{A}/\pi^{n-j}\mathbb{A})^*$. Let I_g denote the indicator function of \mathcal{O}_g then I_g commutes with I_y because the indicator function of the whole set $\mathcal{O}_I \times \mathcal{O}_I$ commutes with I_y for each y . So the endomorphism algebra $\mathcal{E}\mathcal{N}\mathcal{D}_{\mathcal{G}_{\underline{\lambda}}}(\mathbb{C}[\mathcal{O}_I])$ is commutative. The permutation representation on the zero orbit is trivial and 1-dimensional. Hence the permutation representation on any orbit in the case $\lambda = (n^k) \in \Lambda$ is multiplicity-free. \square

5.2 Description Model

Each orbit $\mathcal{O}_I \subset \mathcal{A}_{\underline{\lambda}}$ intersects any isotypic part $(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}$ of $\mathcal{A}_{\underline{\lambda}}$ in a set of the form

1. $\pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i}I)+1}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}$ if $(\partial_{\lambda_i}I, \lambda_i) \in \max(I)$.
2. $\pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}$ if $(\partial_{\lambda_i}I, \lambda_i) \notin \max(I)$.

and moreover \mathcal{O}_I is the product of these intersections. We are going to show in Section 5.3 that each orbit of pair $\mathcal{O} \subset \mathcal{O}_I \times \mathcal{O}_I \subset \mathcal{A}_{\underline{\lambda}} \times \mathcal{A}_{\underline{\lambda}}$ intersects any isotypic part $(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} \times (\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}$ of $\mathcal{A}_{\underline{\lambda}}^2$ in a set of ordered pairs having the following description given below. Moreover \mathcal{O} is the product of these intersections of ordered pairs.

In case 1, if $(\partial_{\lambda_i}I, \lambda_i) \in \max(I)$

- $\{(a, b) \in (\pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i}I)+1}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}) \times (\pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i}I)+1}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}) \mid b - ay \in \pi^r(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}$ for some $r > \partial_{\lambda_i}I$ and for some slope unit $y \in \mathbb{A}^*\}$

OR

- $\{(a, b) \in (\pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i}I)+1}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}) \times (\pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i}I)+1}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i})\}$

| $b - ay \in \pi^r(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{r+1}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}$ for some $r \geq \partial_{\lambda_i}I$

and for some slope unit $y \in \mathbb{A}^*$ and

$\pi^{-r}(b - ay)(\text{mod } \pi)$ is linearly independent with $\pi^{-\partial_{\lambda_i}I}a(\text{mod } \pi)$ in $(\mathbb{F}_q)^{\rho_i}$

In case 2, if $(\partial_{\lambda_i}I, \lambda_i) \notin \max(I)$

- $\{(a, b) \in \pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} \times \pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}\}$

OR

- $\{(a, b) \in \pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} \times \pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} \mid b - ay \in \pi^r(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}$

for some $r > \partial_{\lambda_i}I$ and for some slope unit $y \in \mathbb{A}^*\}$

OR

- $\{(a, b) \in \pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} \times \pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} \mid$

$b - ay \in \pi^r(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{r+1}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}$

for some $r \geq \partial_{\lambda_i}I$ and for some slope unit $y \in \mathbb{A}^*\}$

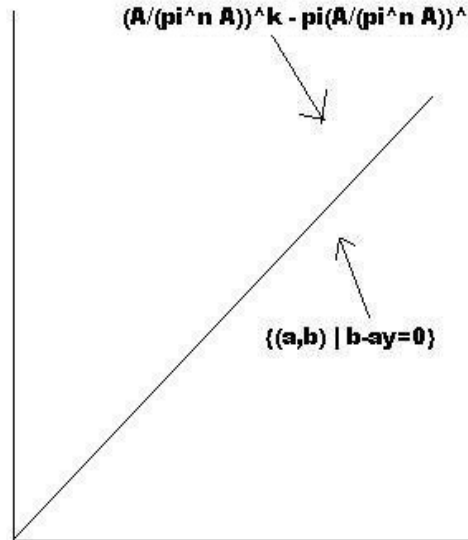
Note that unrestricted condition does not arise in Case 1 and linear independence condition does not arise in Case 2.

As we have seen in the single component case in Theorem 5.2, the Figure 5.1 below gives a pictorial description of the suborbits in $((\mathbb{A}/\pi^n\mathbb{A})^k - \pi(\mathbb{A}/\pi^n\mathbb{A})^k) \times ((\mathbb{A}/\pi^n\mathbb{A})^k - \pi(\mathbb{A}/\pi^n\mathbb{A})^k)$.

General Description based on this component model

Valuation Set
 $(A/(pi^n A))^k - pi(A/(pi^n A))^k$
The orbit 0

A suborbit of 0×0 corresponding to slope parameter y in
 $(A/(pi^n A))^k - pi(A/(pi^n A))^k$



$(A/(pi^n A))^k - pi(A/(pi^n A))^k$
The orbit 0
Valuation Set

There is also another suborbit corresponding to the linear independent condition which has maximum cardinality.

Figure 5.1: Component Model

5.3 Description of Orbit of Pairs for an Ideal

Let $I \in J(\mathcal{P})_{\underline{\lambda}}$ be an ideal. Let the orbit \mathcal{O}_I corresponding to the ideal $I \in J(\mathcal{P})_{\underline{\lambda}}$ be the box set of the form:

$$\prod_{(\partial_{\lambda_l I, \lambda_l}) \notin \max(I)} \pi^{(\partial_{\lambda_l I})} (\mathbb{A}/\pi^{\lambda_l} \mathbb{A})^{\rho_l} \times \prod_{j=1}^t (\pi^{s_{i_j}} (\mathbb{A}/\pi^{\lambda_{i_j}} \mathbb{A})^{\rho_{i_j}} - \pi^{s_{i_j}+1} (\mathbb{A}/\pi^{\lambda_{i_j}} \mathbb{A})^{\rho_{i_j}}) \quad (5.3)$$

Theorem 5.4. *Let $I \in J(\mathcal{P})_{\underline{\lambda}}$ be an ideal with*

$$\max I = \{(s_{i_j}, \lambda_{i_j}) \mid j = 1 \text{ to } t\}$$

Let $\underline{\lambda}'$ and $\underline{\lambda}''$ be the partitions associated to the finite modules which arise in the decomposition (4.9) of $\mathcal{A}_{\underline{\lambda}}$ with respect to the ideal I . Let $\mathcal{O}_{m', J, K} = \{(l', l'') \in \mathcal{A}_{\underline{\lambda}'} \oplus \mathcal{A}_{\underline{\lambda}''} = \mathcal{A}_{\underline{\lambda}} \mid l' \in m' + \mathcal{A}_{\underline{\lambda}'}^{J \cup K}, l'' \in \mathcal{A}_{\underline{\lambda}''}^{K^*} + \mathcal{A}_{\underline{\lambda}''}^J, I(\bar{m}') = J\} \subset \mathcal{O}_I \subset \mathcal{A}_{\underline{\lambda}}$ be any $\mathcal{G}_{\underline{\lambda}}^I$ -suborbit. Then there exists a vector made up of units in \mathbb{A} (need not be unique) with $\underline{y} = (y_{i_1}, y_{i_2}, y_{i_3}, \dots, y_{i_t}) \in (\mathbb{A}^*)^t$ such that

1. If we take x to be the characteristic element of $\mathcal{O}_{[J \cup K]_{\underline{\lambda}''}} \subset \mathcal{A}_{\underline{\lambda}''}$: (refer part (a) of Observation 3.6)

$$x = \left(\underbrace{(\pi^{r_1}(1, 0, 0, \dots, 0))}_{(\rho_1)\text{-tuple}}, \underbrace{(\pi^{r_2}(1, 0, 0, \dots, 0))}_{(\rho_2)\text{-tuple}}, \dots, \underbrace{(\pi^{r_k}(1, 0, 0, \dots, 0))}_{(\rho_k)\text{-tuple}} \right)^t \in \mathcal{O}_{[J \cup K]_{\underline{\lambda}''}} \subset \mathcal{A}_{\underline{\lambda}''}$$

here x excludes the coordinates that occur in $\mathcal{A}_{\underline{\lambda}'}$ and if $\rho_i > 1$ then $r_i = \partial_{\lambda_i}([J \cup K]_{\underline{\lambda}''})$.

2. And if we take

$$f = ((\pi^{s_{i_1}} y_{i_1}, \pi^{s_{i_2}} y_{i_2}, \dots, \pi^{s_{i_t}} y_{i_t}), x)$$

then $f \in \mathcal{O}_{m', J, K} \subset \mathcal{O}_I$ and the suborbit \mathcal{O} of $\mathcal{O}_I \times \mathcal{O}_I$ corresponding to $\mathcal{O}_{m', J, K}$ (refer

Observation 3.30) is

$$\mathcal{O} = \{(a, b) = (ge, gf) \in \mathcal{O}_I \times \mathcal{O}_I \subset \mathcal{A}_{\underline{\lambda}} \times \mathcal{A}_{\underline{\lambda}} \mid g \in \mathcal{G}_{\underline{\lambda}}\}$$

where a and b satisfy:

For any given $i \in \{1, 2, \dots, k\}$

- If $i = i_l \in S = \{i_1, i_2, \dots, i_t\}$, then
 $a_{i_l}, b_{i_l} \in (\pi^{s_{i_l}}(\mathbb{A}/\pi^{\lambda_{i_l}}\mathbb{A})^{\rho_{i_l}} - \pi^{s_{i_l}+1}(\mathbb{A}/\pi^{\lambda_{i_l}}\mathbb{A})^{\rho_{i_l}})$
- if we define

$$m_{i_l} = \min \left(\min_{j < i_l, j \in S} (s_j + \text{val}(y_j - y_{i_l})), \right. \\ \min_{j > i_l, j \in S} (\lambda_{i_l} - \lambda_j + s_j + \text{val}(y_j - y_{i_l})), \\ \min_{j < i_l, j \in S, \rho_j > 1} (r_j), \min_{j < i_l, j \notin S} (r_j), \\ \left. \min_{j > i_l, j \in S, \rho_j > 1} (\lambda_{i_l} - \lambda_j + r_j), \min_{j > i_l, j \notin S} (\lambda_{i_l} - \lambda_j + r_j) \right)$$

then we have $m_{i_l} > s_{i_l}$ and exactly one of the following holds.

- A. $(b_{i_l} - y_{i_l} a_{i_l}) \in \pi^{m_{i_l}}(\mathbb{A}/\pi^{\lambda_{i_l}}\mathbb{A})^{\rho_{i_l}}$ if $m_{i_l} \leq r_{i_l}$, $\rho_{i_l} > 1$ or if $\rho_{i_l} = 1$.
- B. $(b_{i_l} - y_{i_l} a_{i_l}) \in \pi^{r_{i_l}}(\mathbb{A}/\pi^{\lambda_{i_l}}\mathbb{A})^{\rho_{i_l}}$ if $m_{i_l} > r_{i_l}$, $\rho_{i_l} > 1$ and $\pi^{-r_{i_l}}(b_{i_l} - y_{i_l} a_{i_l})(\text{mod } \pi)$ is linearly independent with $\pi^{-s_{i_l}} a_{i_l}(\text{mod } \pi)$ in $\mathbb{F}_q^{\rho_{i_l}}$.

- If $i \notin S$, there exists an $i_l \in S$, $\partial_{\lambda_i} I = s_{i_l}$ or $\partial_{\lambda_i} I = \lambda_i - \lambda_{i_l} + s_{i_l}$ such that
 $a_i, b_i \in \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}$

- if we define

$$m_i = \min \left(\min_{j < i, j \in S} (s_j + \text{val}(y_j - y_i)), \right. \\ \min_{j > i, j \in S} (\lambda_i - \lambda_j + s_j + \text{val}(y_j - y_i)), \\ \min_{j < i, j \in S, \rho_j > 1} (r_j), \min_{j < i, j \notin S} (r_j), \\ \left. \min_{j > i, j \in S, \rho_j > 1} (\lambda_i - \lambda_j + r_j), \min_{j > i, j \notin S} (\lambda_i - \lambda_j + r_j) \right)$$

then we have $m_i \geq \partial_{\lambda_i} I$ and exactly one of the following holds.

- $b_i - y_{i_1} a_i \in \pi^{m_i} (\mathbb{A} / \pi^{\lambda_i} \mathbb{A})^{\rho_i}$ if $r_i \geq m_i > \partial_{\lambda_i} I$
- $b_i - y_{i_1} a_i \in (\pi^{\partial_{\lambda_i} ([J]_{\lambda'} \cup [K]_{\lambda'})} (\mathbb{A} / \pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{\partial_{\lambda_i} ([J]_{\lambda'} \cup [K]_{\lambda'}) + 1} (\mathbb{A} / \pi^{\lambda_i} \mathbb{A})^{\rho_i})$ with $m_i > r_i = \partial_{\lambda_i} ([J]_{\lambda'} \cup [K]_{\lambda'}) \geq \partial_{\lambda_i} I$
- $(a_i, b_i) \in \pi^{\partial_{\lambda_i} I} (\mathbb{A} / \pi^{\lambda_i} \mathbb{A})^{\rho_i} \oplus \pi^{\partial_{\lambda_i} I} (\mathbb{A} / \pi^{\lambda_i} \mathbb{A})^{\rho_i}$ can be any element.

Proof. Consider the suborbit $\mathcal{O}_{m', J, K} \subset \mathcal{O}_I$. Let $m' = (m'_{i_1}, m'_{i_2}, \dots, m'_{i_t}) = (\pi^{n_{i_1}} y'_{i_1}, \pi^{n_{i_2}} y'_{i_2}, \dots, \pi^{n_{i_t}} y'_{i_t})$ where $\text{val}(m'_{i_t}) = n_{i_t}$. We modify m' as follows. First of all, using Theorems 4.12 and 4.13, we note that m'_{i_t} can be changed to any new element \tilde{m}'_{i_t} in the coset $m'_{i_t} + \pi^{\partial_{\lambda_{i_t}} ([J]_{\lambda'} \cup [K]_{\lambda'})} \mathbb{A} / \pi^{\lambda_{i_t}} \mathbb{A}$. If $n_{i_t} > s_{i_t}$ then $\rho_{i_t} > 1$, $s_{i_t} = r_{i_t} = \partial_{\lambda_{i_t}} ([J]_{\lambda'} \cup [K]_{\lambda'}) = \partial_{\lambda_{i_t}} ([J]_{\lambda'} \cup [K]_{\lambda'})$. So we modify m'_{i_t} in this coset to get a new element \tilde{m}'_{i_t} with valuation s_{i_t} . So $\tilde{m}' = (\tilde{m}'_{i_1}, \tilde{m}'_{i_2}, \dots, \tilde{m}'_{i_t}) = (\pi^{s_{i_1}} y_{i_1}, \pi^{s_{i_2}} y_{i_2}, \dots, \pi^{s_{i_t}} y_{i_t})$ for some unit vector $\underline{y} = (y_{i_1}, y_{i_2}, y_{i_3}, \dots, y_{i_t}) \in (\mathbb{A}^*)^t$. So if we can take $f = \tilde{m}' \oplus x \in \mathcal{A}_{\lambda'} \oplus \mathcal{A}_{\lambda''} = \mathcal{A}_{\lambda}$ then $f \in \mathcal{O}_{m', J, K} \subset \mathcal{O}_I$ and $\mathcal{O} = \{(a, b) = (ge, gf) \in \mathcal{O}_I \times \mathcal{O}_I \subset \mathcal{A}_{\lambda} \times \mathcal{A}_{\lambda} \mid g \in \mathcal{G}_{\lambda}\}$ is the suborbit corresponding to the \mathcal{G}_{λ}^I -suborbit $\mathcal{O}_{m', J, K}$ by Observation 3.30. Now we describe the orbit of pairs containing (e, f) .

Let $g \in \mathcal{G}_{\lambda}$ be an element as described in equation (3.3). Then we get the following

equations. For $1 \leq l \leq t$

$$\begin{aligned}
a_{i_l} &= \sum_{j < i_l, j \in S} A_{i_l j} \pi^{s_j} \frac{(1, 0, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}} + A_{i_l i_l} \pi^{s_{i_l}} \frac{(1, 0, 0, \dots, 0)^t}{(\rho_{i_l})\text{-tuple}} \\
&+ \sum_{j > i_l, j \in S} A_{i_l j} \pi^{\lambda_{i_l} - \lambda_j + s_j} \frac{(1, 0, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}}
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
b_{i_l} &= \sum_{j < i_l, j \in S} A_{i_l j} \pi^{s_j} \frac{(y_j, \pi^{r_j - s_j}, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}} + A_{i_l i_l} \pi^{s_{i_l}} \frac{(y_{i_l}, \pi^{r_{i_l} - s_{i_l}}, 0, \dots, 0)^t}{(\rho_{i_l})\text{-tuple}} \\
&+ \sum_{j > i_l, j \in S} A_{i_l j} \pi^{\lambda_{i_l} - \lambda_j + s_j} \frac{(y_j, \pi^{r_j - s_j}, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}} \\
&+ \sum_{j < i_l, j \notin S} A_{i_l j} \pi^{r_j} \frac{(1, 0, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}} + \sum_{j > i_l, j \notin S} A_{i_l j} \pi^{\lambda_{i_l} - \lambda_j + r_j} \frac{(1, 0, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}}
\end{aligned} \tag{5.6}$$

Note that because $f \in \mathcal{O}_I$ and from the structure of the box set of \mathcal{O}_I in equation (5.3) we get that for any $i \in S = \{i_1, i_2, \dots, i_t\}$

$$s_{i_1} > s_{i_2} > \dots > s_{i_t}$$

$$\lambda_{i_1} - s_{i_1} > \lambda_{i_2} - s_{i_2} > \dots > \lambda_{i_t} - s_{i_t}$$

$$r_j \geq s_j > s_i \text{ for all } j < i, j \notin S \text{ and for all } j < i, j \in S \text{ if } \rho_j > 1$$

$$\lambda_i - \lambda_j + r_j \geq \lambda_i - \lambda_j + s_j > s_i \text{ for all } j > i, j \notin S \text{ and for all } j > i, j \in S \text{ if } \rho_j > 1 \tag{5.7}$$

Hence from equations (5.5) and (5.6) we have that $a_{i_l}, b_{i_l} \in \pi^{s_{i_l}} ((\mathbb{A}/\pi^{\lambda_{i_l}} \mathbb{A})^{\rho_{i_l}})^*$ (which automatically holds because $\underline{a}, \underline{b} \in \mathcal{O}_I$). In addition we also have

$$\begin{aligned}
b_{i_l} - y_{i_l} a_{i_l} &= \sum_{j < i_l, j \in S} A_{i_l j} \pi^{s_j} \frac{(y_j - y_{i_l}, \pi^{r_j - s_j}, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}} \\
&+ A_{i_l i_l} \pi^{s_{i_l}} \frac{(0, \pi^{r_{i_l} - s_{i_l}}, 0, \dots, 0)^t}{(\rho_{i_l})\text{-tuple}} \\
&+ \sum_{j > i_l, j \in S} A_{i_l j} \pi^{\lambda_{i_l} - \lambda_j + s_j} \frac{(y_j - y_{i_l}, \pi^{r_j - s_j}, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}} \\
&+ \sum_{j < i_l, j \notin S} A_{i_l j} \pi^{r_j} \frac{(1, 0, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}} + \sum_{j > i_l, j \notin S} A_{i_l j} \pi^{\lambda_{i_l} - \lambda_j + r_j} \frac{(1, 0, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}}
\end{aligned} \tag{5.8}$$

If we define m_{i_l} as in Theorem 5.4 we conclude that from inequalities (5.7), $m_{i_l} > s_{i_l}$ and also exactly one of the following holds.

- $b_{i_l} - y_{i_l} a_{i_l} \in \pi^{m_{i_l}} (\mathbb{A}/\pi^{\lambda_{i_l}} \mathbb{A})^{\rho_{i_l}}$ if $m_{i_l} \leq r_{i_l}$, $\rho_{i_l} > 1$ or if $\rho_{i_l} = 1$.
- $b_{i_l} - y_{i_l} a_{i_l} \in \pi^{r_{i_l}} (\mathbb{A}/\pi^{\lambda_{i_l}} \mathbb{A})^{\rho_{i_l}}$ if $m_{i_l} > r_{i_l}$, $\rho_{i_l} > 1$ and $\pi^{-r_{i_l}} (b_{i_l} - y_{i_l} a_{i_l}) \pmod{\pi}$ is linearly independent with $\pi^{-s_{i_l}} a_{i_l} \pmod{\pi}$ in $\mathbb{F}_q^{\rho_{i_l}}$.

For $i \notin S = \{i_1 < i_2 < \dots < i_t\} \subset \{1, 2, 3, \dots, k\}$

$$a_i = \sum_{j < i, j \in S} A_{ij} \pi^{s_j} \frac{(1, 0, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}} + \sum_{j > i, j \in S} A_{ij} \pi^{\lambda_i - \lambda_j + s_j} \frac{(1, 0, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}} \quad (5.9)$$

$$\begin{aligned} b_i &= \sum_{j < i, j \in S} A_{ij} \pi^{s_j} \frac{(y_j, \pi^{r_j - s_j}, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}} \\ &+ \sum_{j > i, j \in S} A_{ij} \pi^{\lambda_i - \lambda_j + s_j} \frac{(y_j, \pi^{r_j - s_j}, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}} \\ &+ \sum_{j < i, j \notin S} A_{ij} \pi^{r_j} \frac{(1, 0, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}} + A_{ii} \pi^{r_i} \frac{(1, 0, 0, \dots, 0)^t}{(\rho_i)\text{-tuple}} + \\ &\sum_{j > i, j \notin S} A_{ij} \pi^{\lambda_i - \lambda_j + r_j} \frac{(1, 0, 0, \dots, 0)^t}{(\rho_j)\text{-tuple}} \end{aligned} \quad (5.10)$$

Suppose $i_l < i < i_{l+1}$ if such an i_l and i_{l+1} exist (otherwise either $i < i_{l+1} = i_l$ or $i_l = i_t < i$). Then we have the following cases.

1. $s_{i_l} < \lambda_i - \lambda_{i_{l+1}} + s_{i_{l+1}}$, $\partial_{\lambda_i} I = s_{i_l}$
2. $s_{i_l} > \lambda_i - \lambda_{i_{l+1}} + s_{i_{l+1}}$, $\partial_{\lambda_i} I = \lambda_i - \lambda_{i_{l+1}} + s_{i_{l+1}}$
3. if $i < i_1 = i_{l+1}$, $\partial_{\lambda_i} I = \lambda_i - \lambda_{i_{l+1}} + s_{i_{l+1}}$
4. if $i > i_t = i_l$, $\partial_{\lambda_i} I = s_{i_l}$
5. $s_{i_l} = \lambda_i - \lambda_{i_{l+1}} + s_{i_{l+1}} = \partial_{\lambda_i} I$

First we note that $a_i, b_i \in \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$. Let m_i be as defined in the Theorem 5.4.

Cases 1, 2, 3, 4 : $\partial_{\lambda_i} I = s_{i_l}$ or $\partial_{\lambda_i} I = \lambda_i - \lambda_{i_{l+1}} + s_{i_{l+1}}$

Let l_0 be any element having the following property.

- $l_0 \leq i$ and $l_0 \notin S$ such that $r_{l_0} = \partial_{\lambda_i} I$
- $l_0 < i$ and $l_0 \in S$ with $\rho_{l_0} > 1$ such that $r_{l_0} = \partial_{\lambda_i} I$
- $l_0 > i$ and $l_0 \notin S$ such that $\lambda_i - \lambda_{l_0} + r_{l_0} = \partial_{\lambda_i} I$
- $l_0 > i$ and $l_0 \in S$ with $\rho_{l_0} > 1$ such that $\lambda_i - \lambda_{l_0} + r_{l_0} = \partial_{\lambda_i} I$

1. If there does not exist any such l_0 then

- we conclude that $m_i > \partial_{\lambda_i} I$
- $b_i - y_{i_l} a_i \in \pi^{m_i}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ if $r_i \geq m_i$
- $b_i - y_{i_l} a_i \in \pi^{r_i}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{r_i+1}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ if $r_i < m_i$

2. If there exist unique such l_0 then

- If $l_0 = i$ and $r_{l_0} = r_i$ then
 - $m_i > \partial_{\lambda_i} I = r_{l_0} = r_i$.
 - $b_i - y_{i_l} a_i \in \pi^{r_i}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{r_i+1}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$
- If $l_0 \neq i$ then (a_i, b_i) can be any element of $\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \oplus \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$.

3. If there exist more than one l_0 then (a_i, b_i) can be any element of $\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \oplus \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$.

Case 5 : $s_{i_l} = \lambda_i - \lambda_{i_{l+1}} + s_{i_{l+1}} = \partial_{\lambda_i} I$

In this case (a_i, b_i) can be any element of $\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \oplus \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$.

Now we look at the converse. Let I be the ideal with its corresponding orbit \mathcal{O}_I , and associated partitions $\underline{\lambda}', \underline{\lambda}''$. Let $\mathcal{O}_{m', J, K}$ be the $\mathcal{G}_{\underline{\lambda}}^I$ -suborbit and $\underline{y} \in (\mathbb{A}^*)^k$ be

the unit vector. These are all given as defined in Theorem 5.4. Let $(e, f), m_i, i \in \{1, 2, \dots, k\}$ be also as defined in the Theorem 5.4. Let $(\underline{a}, \underline{b}) \in \mathcal{O}_I \times \mathcal{O}_I$ such that for each $1 \leq i \leq k$, similar to $(e_i, f_i), (a_i, b_i)$ also satisfies the same one of the cases with the conditions given in the hypothesis of these cases then we observe that there exists a $g \in \mathcal{G}_\lambda$ such that $(ge, gf) = (\underline{a}, \underline{b})$. The construction of an invertible matrix $g \in \mathcal{G}_\lambda$ is done in each block row. The conditions are such that we can perform this construction independently in each block row using appropriate valuations and linearly independent conditions.

Let $g \in \mathcal{G}_\lambda$ be an element as described in equation (3.3).

Suppose there exists $i_l \in S$ such that $s_{i_l} < m_{i_l} \leq r_{i_l}$ and the minimum is attained at $s_j + \text{val}(y_j - y_{i_l})$ for some $j < i_l, j \in S$. Also suppose (a, b) satisfies the hypothesis of this condition i.e. $b_{i_l} - a_{i_l}y_{i_l} \in \pi^{m_{i_l}}(\mathbb{A}/\pi^{\lambda_{i_l}}\mathbb{A})^{\rho_{i_l}}$. This occurs in Case A of Theorem 5.4. We determine the i_l^{th} block row of $g \in \mathcal{G}_\lambda$ as follows. Set $A_{i_l\epsilon} = 0$ for $\epsilon \neq i_l$ and $\epsilon \neq j$. To determine A_{i_lj} and $A_{i_li_l}$ we solve the equations.

$$A_{i_lj} \frac{(\pi^{s_j}, 0, 0, 0, \dots, 0)^t}{\rho_{i_l}\text{-tuple}} + A_{i_li_l} \frac{(\pi^{s_{i_l}}, 0, 0, 0, \dots, 0)^t}{\rho_{i_l}\text{-tuple}} = a_{i_l}.$$

$$A_{i_lj} \frac{(y_j \pi^{s_j}, \pi^{r_j}, 0, 0, \dots, 0)^t}{\rho_{i_l}\text{-tuple}} + A_{i_li_l} \frac{(y_{i_l} \pi^{s_{i_l}}, \pi^{r_{i_l}}, 0, 0, \dots, 0)^t}{\rho_{i_l}\text{-tuple}} = b_{i_l}.$$

$$A_{i_lj} \frac{((y_j - y_{i_l}) \pi^{s_j}, \pi^{r_j}, 0, 0, \dots, 0)^t}{\rho_{i_l}\text{-tuple}} + A_{i_li_l} \frac{(0, \pi^{r_{i_l}}, 0, 0, \dots, 0)^t}{\rho_{i_l}\text{-tuple}} = b_{i_l} - a_{i_l}y_{i_l} = \pi^{m_{i_l}}C$$

for some column vector $C \in (\mathbb{A}/\pi^{\lambda_{i_l}}\mathbb{A})^{\rho_{i_l}}$.

Let $(y_j - y_{i_l})\pi^{s_j} = \pi^{m_{i_l}}y'$ for some unit $y' \in \mathbb{A}$. Let $C_{i_lj}^1, C_{i_lj}^2, C_{i_li_l}^1, C_{i_li_l}^2$ denote the first and second columns of $A_{i_lj}, A_{i_li_l}$ respectively. Choose $C_{i_li_l}^2$ to be any vector in $(\mathbb{A}/\pi^{\lambda_{i_l}}\mathbb{A})^{\rho_{i_l}}$ such that $C_{i_li_l}^2(\text{mod } \pi)$ is linearly independent from $\pi^{-s_{i_l}}a_{i_l}(\text{mod } \pi)$.

Choose $C_{ij}^p = 0$ for $p > 2$. We get the following equations for the columns.

$$\begin{aligned}\pi^{s_j} C_{ij}^1 + C_{i_i i_l}^1 &= a_{i_l}. \\ y' \pi^{m_{i_l}} C_{ij}^1 + \pi^{r_j} C_{ij}^2 + \pi^{r_{i_l}} C_{i_l i_l}^2 &= \pi^{m_{i_l}} C.\end{aligned}$$

Choose $C_{ij}^2 = 0$ and solving for C_{ij}^1 and then for $C_{i_l i_l}^1$ we get $C_{ij}^1 = (y')^{-1}(C - \pi^{r_{i_l} - m_{i_l}} C_{i_l i_l}^2)$ and $C_{i_l i_l}^1 = a_{i_l} - \pi^{s_j} C_{ij}^1$. Since $s_j > s_{i_l}$, $\pi^{-s_{i_l}} C_{i_l i_l}^1 \equiv \pi^{-s_{i_l}} a_{i_l} \pmod{\pi}$ and is linearly independent from $C_{i_l i_l}^2 \pmod{\pi}$. Now extend the columns of $A_{i_l i_l}$ to a matrix such that $A_{i_l i_l} \pmod{\pi}$ is invertible.

Suppose there exists $i_l \in S$ such that $m_{i_l} > r_{i_l} \geq s_{i_l}$. Also suppose (a, b) satisfies the hypothesis of this condition i.e. $b_{i_l} - a_{i_l} y_{i_l} \in \pi^{r_{i_l}} (\mathbb{A}/\pi^{\lambda_{i_l}} \mathbb{A})^{\rho_{i_l}}$ and $\pi^{-r_{i_l}} (b_{i_l} - y_{i_l} a_{i_l}) \pmod{\pi}$ is linearly independent with $\pi^{-s_{i_l}} a_{i_l} \pmod{\pi}$. This occurs in Case B of Theorem 5.4. We determine the i_l^{th} block row of $g \in \mathcal{G}_\lambda$ as follows. Set $A_{i_l \epsilon} = 0$ for $\epsilon \neq i_l$. To determine $A_{i_l i_l}$ we solve the equations.

$$\begin{aligned}A_{i_l i_l} \underbrace{(\pi^{s_{i_l}}, 0, 0, 0, \dots, 0)^t}_{\rho_{i_l}\text{-tuple}} &= a_{i_l}. \\ A_{i_l i_l} \underbrace{(y_{i_l} \pi^{s_{i_l}}, \pi^{r_{i_l}}, 0, 0, \dots, 0)^t}_{\rho_{i_l}\text{-tuple}} &= b_{i_l}. \\ A_{i_l i_l} \underbrace{(0, \pi^{r_{i_l}}, 0, 0, \dots, 0)^t}_{\rho_{i_l}\text{-tuple}} &= b_{i_l} - a_{i_l} y_{i_l} = \pi^{r_{i_l}} C\end{aligned}$$

for some column vector $C \in (\mathbb{A}/\pi^{\lambda_{i_l}} \mathbb{A})^{\rho_{i_l}} - \pi(\mathbb{A}/\pi^{\lambda_{i_l}} \mathbb{A})^{\rho_{i_l}}$.

Let $C_{i_l i_l}^1, C_{i_l i_l}^2$ denote the first and second columns of $A_{i_l i_l}$ respectively. Choose $C_{i_l i_l}^1$ to be $\pi^{-s_{i_l}} a_{i_l}$ and $C_{i_l i_l}^2$ to be the vector $\pi^{-r_{i_l}} (b_{i_l} - y_{i_l} a_{i_l}) \in (\mathbb{A}/\pi^{\lambda_{i_l}} \mathbb{A})^{\rho_{i_l}}$. Then we have the linearly independent condition satisfied for $A_{i_l i_l}$ and extend the columns of $A_{i_l i_l}$ such that the matrix $A_{i_l i_l} \pmod{\pi}$ is invertible.

Now suppose there exists an $i \notin S$ and $i_l < i < i_{l+1}$ such that $r_i \geq m_i > \partial_{\lambda_i} I$ and the minimum is attained at $s_j + \text{val}(y_j - y_{i_l})$ for some $j < i_l, j \in S$. Also suppose (a, b) satisfies the hypothesis of this condition i.e. $b_i - a_i y_i \in \pi^{m_i} (\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$. This occurs

in Case *a* of Theorem 5.4. We determine the i^{th} block row of $g \in \mathcal{G}_\lambda$ as follows. Set $A_{i\epsilon} = 0$ for $\epsilon \neq i_l, j, i$. To determine A_{ij}, A_{ii_l}, A_{ii} we solve the equations.

$$\begin{aligned} & \frac{A_{ij}(\pi^{s_j}, 0, 0, 0, \dots, 0)^t}{\rho_j\text{-tuple}} + \frac{A_{ii_l}(\pi^{s_{i_l}}, 0, 0, 0, \dots, 0)^t}{\rho_{i_l}\text{-tuple}} = a_i. \\ & \frac{A_{ij}(y_j \pi^{s_j}, \pi^{r_j}, 0, 0, \dots, 0)^t}{\rho_j\text{-tuple}} + \frac{A_{ii_l}(y_{i_l} \pi^{s_{i_l}}, \pi^{r_{i_l}}, 0, 0, \dots, 0)^t}{\rho_{i_l}\text{-tuple}} + \frac{A_{ii}(\pi^{r_i}, 0, 0, \dots, 0)^t}{\rho_i\text{-tuple}} = b_i. \\ & \frac{A_{ij}(\pi^{s_j}(y_j - y_{i_l}), \pi^{r_j}, 0, 0, \dots, 0)^t}{\rho_j\text{-tuple}} + \frac{A_{ii_l}(0, \pi^{r_{i_l}}, 0, 0, 0, \dots, 0)^t}{\rho_{i_l}\text{-tuple}} + \frac{A_{ii}(\pi^{r_i}, 0, 0, 0, \dots, 0)^t}{\rho_i\text{-tuple}} \\ & = b_i - a_i y_{i_l} = \pi^{m_i} C \text{ for some column vector } C \in (\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}. \end{aligned}$$

Let $(y_j - y_{i_l})\pi^{s_j} = \pi^{m_i} y'$ for some unit $y' \in \mathbb{A}$. Let $C_{ii_l}^1, C_{ii_l}^2, C_{ij}^1, C_{ij}^2, C_{ii}^1$ denote the first and second columns of A_{ii_l}, A_{ij} and first column of A_{ii} respectively. Set the columns $C_{ij}^\epsilon = C_{ii_l}^\epsilon = 0$ for $\epsilon > 2$. Choose C_{ii}^1 to be any vector in $(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$. Now extend the columns of A_{ii} to a matrix such that $A_{ii} \pmod{\pi}$ is invertible. We get the following equations for the columns.

$$\begin{aligned} & \pi^{s_j} C_{ij}^1 + \pi^{s_{i_l}} C_{ii_l}^1 = a_i. \\ & y' \pi^{m_i} C_{ij}^1 + \pi^{r_j} C_{ij}^2 + \pi^{r_{i_l}} C_{ii_l}^2 + \pi^{r_i} C_{ii}^1 = \pi^{m_i} C. \end{aligned}$$

Choose $C_{ij}^2 = C_{ii_l}^2 = 0$ and solving for C_{ij}^1 and then for $C_{ii_l}^1$ we get $C_{ij}^1 = (y')^{-1}(C - \pi^{r_i - m_i} C_{ii}^1)$ and $C_{ii_l}^1 = a_i - \pi^{s_j - s_{i_l}} C_{ij}^1$.

The rest of the cases are similar. □

Here is a worked out example that describes the orbit of pairs in the two component case. Consider the finite module $\mathbb{A}/\pi^l \mathbb{A} \oplus \mathbb{A}/\pi^k \mathbb{A}$ with $k < l$ and without multiplicity corresponding to the partition $\lambda = (l^1, k^1) \in \Lambda$. Consider the orbit $\mathcal{O}_I \subset \mathbb{A}/\pi^l \mathbb{A} \oplus \mathbb{A}/\pi^k \mathbb{A}$ of the non-principal ideal I with $\max(I) = \{(s+r, l), (s, k)\} \in \mathcal{P}$. Hence $\mathcal{O}_I = (\pi^{s+r}(\mathbb{A}/\pi^l \mathbb{A}) - \pi^{s+r+1}(\mathbb{A}/\pi^l \mathbb{A})) \times (\pi^s(\mathbb{A}/\pi^k \mathbb{A}) - \pi^{s+1}(\mathbb{A}/\pi^k \mathbb{A}))$ with $0 < r < l - k$. The suborbits of $\mathcal{O}_I \times \mathcal{O}_I$ under the diagonal action \mathcal{G}_λ is given as follows. Given two units y and x in \mathbb{A}^* , the suborbit $I_{y,x} = \{((\pi^{r+s}u, \pi^s w), (\pi^{r+s}u', \pi^s w')) \mid$

u, u', w, w' are units & $(\pi^{r+s}u' - \pi^{r+s}uy, \pi^s w' - \pi^s wx) \in \pi^{l-k+s+t}(\mathbb{A}/\pi^l\mathbb{A}) \oplus \pi^{r+s+t}(\mathbb{A}/\pi^k\mathbb{A})$ where $t \parallel (x - y)$. Similarly the unit pair (y, x) parameter group is $(\mathbb{A}/\pi^{l-k-r+t}\mathbb{A})^* \oplus (\mathbb{A}/\pi^{r+t}\mathbb{A})^*$ which is independent of the shift parameter s .

5.4 Commutativity

For $j = 1, 2$ let $\mathcal{O}_j \subset \mathcal{O}_I \times \mathcal{O}_I$ denote two suborbits. The multiplication in the endomorphism algebra $\mathcal{E}\mathcal{N}\mathcal{D}_{\mathcal{G}_\lambda}(\mathbb{C}[\mathcal{O}_I])$ is given by convolution. Let $\mathcal{I}_{\mathcal{O}_1}$ and $\mathcal{I}_{\mathcal{O}_2}$ denote the indicator functions of these two orbits. Suppose $(\alpha, \beta) \in \mathcal{A}_\lambda^2$ be an element. Then $\mathcal{I}_{\mathcal{O}_1} * \mathcal{I}_{\mathcal{O}_2}(\alpha, \beta) = \sum_{\gamma \in \mathcal{A}_\lambda} \mathcal{I}_{\mathcal{O}_1}(\alpha, \gamma)\mathcal{I}_{\mathcal{O}_2}(\gamma, \beta)$ and $\mathcal{I}_{\mathcal{O}_2} * \mathcal{I}_{\mathcal{O}_1}(\alpha, \beta) = \sum_{\delta \in \mathcal{A}_\lambda} \mathcal{I}_{\mathcal{O}_2}(\alpha, \delta)\mathcal{I}_{\mathcal{O}_1}(\delta, \beta)$. To prove commutativity we need to prove that the existence of an element $\gamma \in \mathcal{A}_\lambda$ such that $\mathcal{I}_{\mathcal{O}_1}(\alpha, \gamma) = 1 = \mathcal{I}_{\mathcal{O}_2}(\gamma, \beta)$ is equivalent to the existence of an element $\delta \in \mathcal{A}_\lambda$ such that $\mathcal{I}_{\mathcal{O}_2}(\alpha, \delta) = 1 = \mathcal{I}_{\mathcal{O}_1}(\delta, \beta)$ and that the number of solutions for γ to the equations

$$\begin{aligned} \mathcal{I}_{\mathcal{O}_1}(\alpha, \gamma) &= 1 \\ \mathcal{I}_{\mathcal{O}_2}(\gamma, \beta) &= 1 \end{aligned} \tag{5.11}$$

is equal to the number of solutions for δ to the equations

$$\begin{aligned} \mathcal{I}_{\mathcal{O}_2}(\alpha, \delta) &= 1 \\ \mathcal{I}_{\mathcal{O}_1}(\delta, \beta) &= 1 \end{aligned} \tag{5.12}$$

We prove this componentwise for $\mathcal{I}_{\mathcal{O}_1}$ and $\mathcal{I}_{\mathcal{O}_2}$ corresponding to each isotypic component $(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}$ of \mathcal{A}_λ .

First we prove a simple lemma on counting number of solutions to certain congruences with certain conditions.

Lemma 5.13. *Let $y, y' \in \mathcal{A}_{n,k} = (\mathbb{A}/\pi^n\mathbb{A})^k$ and $r, r' \in \{0, 1, 2, \dots, n\}$. Let $\mathcal{A}_{n,k}^*$ denote the set $\mathcal{A}_{n,k} - \pi\mathcal{A}_{n,k}$. Then*

$$1. |(y + \pi^r \mathcal{A}_{n,k}^*) \cap (y' + \pi^{r'} \mathcal{A}_{n,k}^*)| = |(y' + \pi^r \mathcal{A}_{n,k}^*) \cap (y + \pi^{r'} \mathcal{A}_{n,k}^*)|$$

- For $r > r'$ if $((y + \pi^r \mathcal{A}_{n,k}^*) \cap (y' + \pi^{r'} \mathcal{A}_{n,k}^*)) \neq \emptyset$ then

$$|((y + \pi^r \mathcal{A}_{n,k}^*) \cap (y' + \pi^{r'} \mathcal{A}_{n,k}^*))| = q^{(n-r-1)k}(q^k - 1).$$

- For $r = r'$ if $((y + \pi^r \mathcal{A}_{n,k}^*) \cap (y' + \pi^{r'} \mathcal{A}_{n,k}^*)) \neq \emptyset$ then

$$|((y + \pi^r \mathcal{A}_{n,k}^*) \cap (y' + \pi^{r'} \mathcal{A}_{n,k}^*))| = \begin{cases} q^{(n-r-1)k}(q^k - 1) & \text{if } y - y' \in \pi^{(r+1)} \mathcal{A}_{n,k} \\ q^{(n-r-1)k}(q^k - 2) & \text{if } y - y' \in \pi^r \mathcal{A}_{n,k} \end{cases}$$

$$2. |(y + \pi^r \mathcal{A}_{n,k}) \cap (y' + \pi^{r'} \mathcal{A}_{n,k})| = |(y' + \pi^r \mathcal{A}_{n,k}) \cap (y + \pi^{r'} \mathcal{A}_{n,k})|$$

- For $r \geq r'$ if $((y + \pi^r \mathcal{A}_{n,k}) \cap (y' + \pi^{r'} \mathcal{A}_{n,k})) \neq \emptyset$ then

$$|((y + \pi^r \mathcal{A}_{n,k}) \cap (y' + \pi^{r'} \mathcal{A}_{n,k}))| = q^{(n-r)k}.$$

$$3. |(y + \pi^r \mathcal{A}_{n,k}) \cap (y' + \pi^{r'} \mathcal{A}_{n,k}^*)| = |(y' + \pi^r \mathcal{A}_{n,k}) \cap (y + \pi^{r'} \mathcal{A}_{n,k}^*)|$$

- For $r > r'$ if $((y + \pi^r \mathcal{A}_{n,k}) \cap (y' + \pi^{r'} \mathcal{A}_{n,k}^*)) \neq \emptyset$ then

$$|((y + \pi^r \mathcal{A}_{n,k}) \cap (y' + \pi^{r'} \mathcal{A}_{n,k}^*))| = q^{(n-r)k}.$$

- For $r \leq r'$ if $((y + \pi^r \mathcal{A}_{n,k}) \cap (y' + \pi^{r'} \mathcal{A}_{n,k}^*)) \neq \emptyset$ then

$$|((y + \pi^r \mathcal{A}_{n,k}) \cap (y' + \pi^{r'} \mathcal{A}_{n,k}^*))| = q^{(n-r'-1)k}(q^k - 1).$$

Proof. Suppose $r \geq r'$ we produce a bijection between the sets $(y + \pi^r \mathcal{A}_{n,k}^*) \cap (y' + \pi^{r'} \mathcal{A}_{n,k}^*)$ and $(y' + \pi^r \mathcal{A}_{n,k}^*) \cap (y + \pi^{r'} \mathcal{A}_{n,k}^*)$ as follows. Let

$$\begin{aligned} \frac{(y + \pi^r \mathcal{A}_{n,k}^*) \cap (y' + \pi^{r'} \mathcal{A}_{n,k}^*)}{x=y+\pi^r a_1=y'+\pi^{r'} a_2} &\longleftrightarrow \frac{B \subset \mathcal{A}_{n-r,k}^* \times \mathcal{A}_{n-r',k}^*}{(a_1, a_2)} \longleftarrow \\ \longrightarrow \frac{B' \subset \mathcal{A}_{n-r,k}^* \times \mathcal{A}_{n-r',k}^*}{(b_1, b_2)=(a_1, -a_2+2a_1\pi^{r-r'})} &\longleftrightarrow \frac{(y' + \pi^r \mathcal{A}_{n,k}^*) \cap (y + \pi^{r'} \mathcal{A}_{n,k}^*)}{z=y'+\pi^r b_1=y+\pi^{r'} b_2} \end{aligned}$$

Note that the middle bijection extends to an automorphism

$$\begin{pmatrix} I_k & 0_k \\ 2\pi^{r-r'} I_k & -I_k \end{pmatrix}$$

of the finite torsion module $\mathcal{A}_{n-r,k} \times \mathcal{A}_{n-r',k} = \mathcal{A}_{\underline{\lambda}}$ for $\underline{\lambda} = ((n-r)^k, (n-r')^k) \in \Lambda$.

This proves the equality of the cardinality of sets in case 1.

Now we note that the existence of a solution to the congruences

$$\begin{aligned} x &\equiv y \pmod{\pi^r} \\ x &\equiv y' \pmod{\pi^{r'}} \end{aligned}$$

with conditions

$$\begin{aligned} x - y &\in \pi^r \mathcal{A}_{n,k}^* \\ x - y' &\in \pi^{r'} \mathcal{A}_{n,k}^* \end{aligned}$$

is equivalent to existence of a solution to the congruences

$$\begin{aligned} x &\equiv y \pmod{\pi^{r'}} \\ x &\equiv y' \pmod{\pi^r} \end{aligned}$$

with conditions

$$\begin{aligned} x - y &\in \pi^{r'} \mathcal{A}_{n,k}^* \\ x - y' &\in \pi^r \mathcal{A}_{n,k}^* \end{aligned}$$

And to exactly find the cardinality of the number of solutions, we use standard ideal filtration of $\mathbb{A}/\pi^n \mathbb{A}$ and deduce that the number of solutions to both these sets of equations with the respective given conditions is

$$\left\{ \begin{array}{ll} q^{(n-r-1)k}(q^k - 1) & \text{if } r > r' \\ q^{(n-r'-1)k}(q^k - 1) & \text{if } r' > r \\ q^{(n-r-1)k}(q^k - 1) & \text{if } r = r' \text{ and } y - y' \in \pi^{(r+1)} \mathcal{A}_{n,k} \\ q^{(n-r-1)k}(q^k - 2) & \text{if } r = r' \text{ and } y - y' \in \pi^r \mathcal{A}_{n,k}^* \end{array} \right.$$

The other cases 2 and 3 are similar. □

Lemma 5.14. *Let x be a nonzero vector in a finite dimensional vector space \mathbb{F}_q^k . Let \mathbb{A} be a discrete valuation ring with a uniformizing parameter π such that the residue*

field is \mathbb{F}_q . Let $S \subset (\mathbb{A}/\pi^n\mathbb{A})^k$ be the set consisting of k -tuples such that $S \pmod{\pi}$ is a set of vectors in \mathbb{F}_q^k which are linearly independent to x . Let $a, b \in (\mathbb{A}/\pi^n\mathbb{A})^k$ be two elements such that $a \equiv b \pmod{\pi^r}$ where $0 \leq r < n$. Consider the equations

$$\begin{aligned} e &\equiv a \pmod{\pi^r} \\ e &\equiv b \pmod{\pi^r} \end{aligned} \tag{5.15}$$

with conditions

$$\begin{aligned} \pi^{-r}(e - a) &\in S \\ \pi^{-r}(e - b) &\in S \end{aligned} \tag{5.16}$$

If there exists a solution to the equations (5.15) satisfying conditions (5.16), then the total number of such solutions is $\begin{cases} q^{k(n-r-1)}(q^k - 2q) & \text{if } \pi^{-r}(a - b) \in S \\ q^{k(n-r-1)}(q^k - q) & \text{if } \pi^{-r}(a - b) \notin S \end{cases}$

Proof. Since $\mathbb{A}/\pi\mathbb{A} \cong_{\phi} \mathbb{F}_q$. Let $t^k : \mathbb{A}^k \rightarrow \mathbb{F}_q^k$ with $t^k = \phi^k \circ pr^k$ where $pr : \mathbb{A} \rightarrow (\mathbb{A}/\pi\mathbb{A})$ be the quotient map. Let $s^k : \mathbb{F}_q^k \rightarrow \mathbb{A}^k$ be any section. Then given any element $c \in (\mathbb{A}/\pi^n\mathbb{A})^k$ there exists a unique set $\{c_0, c_1, c_2, \dots, c_{n-1}\}$ of vectors in \mathbb{F}_q^k such that

$$c = s^k(c_0) + s^k(c_1)\pi + s^k(c_2)\pi^2 + \dots + s^k(c_{n-1})\pi^{n-1} \tag{5.17}$$

with condition

$$t^k(s^k(c_i)) = c_i \text{ for all } i = 0, 1, 2, \dots, (n-1) \tag{5.18}$$

Let

$$\begin{aligned} a &= s^k(a_0) + s^k(a_1)\pi + s^k(a_2)\pi^2 + \dots + s^k(a_{n-1})\pi^{n-1} \\ b &= s^k(b_0) + s^k(b_1)\pi + s^k(b_2)\pi^2 + \dots + s^k(b_{n-1})\pi^{n-1} \end{aligned} \tag{5.19}$$

Since a solution to the equations (5.15) satisfying conditions (5.16) exists, we have

$$a \equiv b \pmod{\pi^r} \text{ and hence}$$

$$a_i = b_i \in \mathbb{F}_q^k \text{ for all } i = 0, 1, 2, \dots, (r-1)$$

The conditions (5.16) implies that we need to count the number of solutions $e \in (\mathbb{A}/\pi^n \mathbb{A})^k$ such that the vectors

$$e_i = a_i = b_i \text{ for } 0 \leq i \leq (r-1).$$

$e_r - a_r, e_r - b_r$ are both linearly independent with x .

e_i can be any element in \mathbb{F}_q^k for $r+1 \leq i < n$.

Suppose $a_r - b_r \notin S$. Then $\{a_r + \epsilon x \mid \epsilon \in \mathbb{F}_q\} = \{b_r + \epsilon x \mid \epsilon \in \mathbb{F}_q\}$. So the number of such solutions $e \in (\mathbb{A}/\pi^n \mathbb{A})^k$ in this case is $q^{(n-r-1)k}(q^k - q)$. Suppose $a_r - b_r \in S$. Then $\{a_r + \epsilon x \mid \epsilon \in \mathbb{F}_q\} \cap \{b_r + \epsilon x \mid \epsilon \in \mathbb{F}_q\} = \emptyset$. So the number of such solutions $e \in (\mathbb{A}/\pi^n \mathbb{A})^k$ in this case is $q^{(n-r-1)k}(q^k - 2q)$. \square

Lemma 5.20. *Let s, r_1, r_2 be nonnegative integers such that $s \leq r_1, s \leq r_2, s \leq \lambda$. Let $a, b \in \pi^s(\mathbb{A}/\pi^\lambda \mathbb{A})^\rho - \pi^{s+1}(\mathbb{A}/\pi^\lambda \mathbb{A})^\rho$. Let y_1, y_2 be two units in \mathbb{A}^* . Suppose the residue field $\mathbb{F}_q \cong \mathbb{A}/\pi \mathbb{A}$ has at least three elements or the multiplicity ρ is > 2 . Then the number of solutions for $e \in \pi^s(\mathbb{A}/\pi^\lambda \mathbb{A})^\rho - \pi^{s+1}(\mathbb{A}/\pi^\lambda \mathbb{A})^\rho$ to the congruences*

$$\begin{aligned} b &\equiv ey_2 \pmod{\pi^{r_2}} \\ e &\equiv ay_1 \pmod{\pi^{r_1}} \end{aligned} \tag{5.21}$$

with conditions

$$\begin{aligned} \{\pi^{-r_2}(b - ey_2) \pmod{\pi}, \pi^{-s}e \pmod{\pi}\} &\text{ are linearly independent in } \mathbb{F}_q^\rho \\ \{\pi^{-r_1}(e - ay_1) \pmod{\pi}, \pi^{-s}a \pmod{\pi}\} &\text{ are linearly independent in } \mathbb{F}_q^\rho \end{aligned} \tag{5.22}$$

is the same as the number of solutions for $e \in \pi^s(\mathbb{A}/\pi^\lambda \mathbb{A})^\rho - \pi^{s+1}(\mathbb{A}/\pi^\lambda \mathbb{A})^\rho$ to the congruences

$$\begin{aligned} b &\equiv ey_1 \pmod{\pi^{r_1}} \\ e &\equiv ay_2 \pmod{\pi^{r_2}} \end{aligned} \tag{5.23}$$

with conditions

$$\begin{aligned} \{\pi^{-r_1}(b - ey_1) \pmod{\pi}, \pi^{-s}e \pmod{\pi}\} & \text{ are linearly independent in } \mathbb{F}_q^\rho \\ \{\pi^{-r_2}(e - ay_2) \pmod{\pi}, \pi^{-s}a \pmod{\pi}\} & \text{ are linearly independent in } \mathbb{F}_q^\rho \end{aligned} \quad (5.24)$$

Also the number of solutions for $e \in \pi^s(\mathbb{A}/\pi^\lambda\mathbb{A})^\rho - \pi^{s+1}(\mathbb{A}/\pi^\lambda\mathbb{A})^\rho$ to the congruences

$$\begin{aligned} b &\equiv ey_2 \pmod{\pi^{r_2}} \\ e &\equiv ay_1 \pmod{\pi^{r_1}} \end{aligned} \quad (5.25)$$

with conditions

$$\{\pi^{-r_2}(b - ey_2) \pmod{\pi}, \pi^{-s}e \pmod{\pi}\} \text{ are linearly independent in } \mathbb{F}_q^\rho \quad (5.26)$$

is the same as the number of solutions for $e \in \pi^s(\mathbb{A}/\pi^\lambda\mathbb{A})^\rho - \pi^{s+1}(\mathbb{A}/\pi^\lambda\mathbb{A})^\rho$ to the congruences

$$\begin{aligned} b &\equiv ey_1 \pmod{\pi^{r_1}} \\ e &\equiv ay_2 \pmod{\pi^{r_2}} \end{aligned} \quad (5.27)$$

with conditions

$$\{\pi^{-r_2}(e - ay_2) \pmod{\pi}, \pi^{-s}a \pmod{\pi}\} \text{ are linearly independent in } \mathbb{F}_q^\rho \quad (5.28)$$

Proof. First let us look at the congruence equations (5.21) with conditions (5.22) and congruence equations (5.23) with conditions (5.24). Without loss of generality, let $s \leq r_1 \leq r_2$. Existence of such a solution \underline{e} in any of the equations implies

$$b \equiv ay_1y_2 \pmod{\pi^{r_1}} \quad (5.29)$$

and if $r_1 < r_2$ then we also have

$$\{\pi^{-r_1}(b - ay_1y_2) \pmod{\pi}, \pi^{-s}e \pmod{\pi}\} \text{ are linearly independent in } \mathbb{F}_q^\rho \quad (5.30)$$

If there exists an element \underline{e} satisfying equation (5.21) and condition (5.22) then we choose $\tilde{e} = ay_2 + \pi^{r_2}\alpha \pmod{\pi^\lambda}$ for some $\alpha \in \mathbb{A}^\rho$ such that

- $\alpha \pmod{\pi}$ is linearly independent with $\pi^{-s}a \pmod{\pi}$ in \mathbb{F}_q^ρ .
- $(\pi^{-r_1}(b - ay_1y_2) - \pi^{r_2-r_1}(y_1\alpha) \pmod{\pi^\lambda}) \pmod{\pi}$ is linearly independent with $\pi^{-s}a \pmod{\pi}$ in \mathbb{F}_q^ρ

Existence of such an α in the case when

- $r_1 = r_2$
- $\pi^{-r_1}(b - ay_1y_2)$ is linearly independent with $\pi^{-s}a \pmod{\pi}$ in \mathbb{F}_q^ρ
- The residue field $\mathbb{A}/\pi\mathbb{A} \cong \mathbb{F}_q$ has exactly two elements

requires that the multiplicity ρ must be > 2 . This element \tilde{e} gives rise to a solution to the equation (5.23) satisfying the condition (5.24).

Conversely if there exists an element \underline{e} satisfying equation (5.23) and condition (5.24) then we choose $\tilde{e} = by_2^{-1} + \pi^{r_2}y_2^{-1}\alpha \pmod{\pi^\lambda}$ for some $\alpha \in \mathbb{A}^\rho$ such that

- $\alpha \pmod{\pi}$ is linearly independent with $\pi^{-s}a \pmod{\pi}$ in \mathbb{F}_q^ρ .
- $(\pi^{-r_1}(by_2^{-1} - ay_1) + \pi^{r_2-r_1}(y_2^{-1}\alpha) \pmod{\pi^\lambda}) \pmod{\pi}$ is linearly independent with $\pi^{-s}a \pmod{\pi}$ in \mathbb{F}_q^ρ

Again existence of such an α in the case when

- $r_1 = r_2$

- $\pi^{-r_1}(by_2^{-1} - ay_1)$ is linearly independent with $\pi^{-s}a \pmod{\pi}$ in \mathbb{F}_q^ρ
- The residue field $\mathbb{A}/\pi\mathbb{A} \cong \mathbb{F}_q$ has exactly two elements

requires that the multiplicity ρ must be > 2 .

This element \tilde{e} gives rise to a solution to the equation (5.21) satisfying the condition (5.22).

And to exactly count the cardinality of the number of solutions, we use standard ideal filtration of $\mathbb{A}/\pi^n\mathbb{A}$ and deduce that the number of solutions $e \in (\mathbb{A}/\pi^\lambda\mathbb{A})^\rho$ to the set of equations (5.21) satisfying the conditions (5.22) is same as the number of solutions $e \in (\mathbb{A}/\pi^\lambda\mathbb{A})^\rho$ to the set of equations (5.23) satisfying the conditions (5.24) and it is given by

- $q^{(\rho)(\lambda-r_2-1)}(q^\rho - q)$ if $s \leq r_1 < r_2$
- $|(ay_1y_2 + \pi^{r_1=r_2}S) \cap (b + \pi^{r_1=r_2}S)|$ if $s < r_1 = r_2$ where $S \subset (\mathbb{A}/\pi^\lambda\mathbb{A})^\rho$ is a set such that $S \pmod{\pi}$ is a set of vectors linearly independent to $\pi^{-s}a \pmod{\pi}$ in \mathbb{F}_q^ρ . This cardinality can be easily calculated and it is

$$\begin{cases} q^{(\rho)(\lambda-r-1)}(q^\rho - 2q) & \text{if } (ay_1y_2 - b) \in \pi^r S \text{ where } r = r_1 = r_2. \\ q^{(\rho)(\lambda-r-1)}(q^\rho - q) & \text{if } (ay_1y_2 - b) \notin \pi^r S \text{ where } r = r_1 = r_2. \end{cases}$$
- Cardinality of the set $\{e \in \pi^s(\mathbb{A}/\pi^\lambda\mathbb{A})^\rho \mid \pi^{-s}e \pmod{\pi} \text{ is linearly independent to both } \pi^{-s}a \pmod{\pi} \text{ and } \pi^{-s}b \pmod{\pi} \text{ in } \mathbb{F}_q^\rho\}$ if $s = r_1 = r_2$. Again this cardinality can be easily calculated and it is $q^{(\rho)(\lambda-s-1)}(q^\rho - 2q + 1)$.

The proof of the cardinalities of the number of solutions is similar to the one given in Lemma 5.14.

Now let us look at the congruence equations (5.25) with conditions (5.26) and congruence equations (5.27) with conditions (5.28). Existence of such a solution \underline{e} in any of these congruence equations implies

- If $r_1 \leq r_2$ then

$$b \equiv ay_1y_2 \pmod{\pi^{r_1}} \quad (5.31)$$

- If $r_1 > r_2$ then

$$b \equiv ay_1y_2 \pmod{\pi^{r_2}} \quad (5.32)$$

and

$$\{\pi^{-r_2}(b - ay_1y_2) \pmod{\pi}, \pi^{-s}a \pmod{\pi}\} \text{ are linearly independent in } \mathbb{F}_q^\rho \quad (5.33)$$

Suppose $r_1 \leq r_2$. If there exists an element \underline{e} satisfying equation (5.25) and condition (5.26) then we choose $\tilde{\underline{e}} = ay_2 + \pi^{r_2}\alpha \pmod{\pi^\lambda}$ for some $\alpha \in \mathbb{A}^\rho$ such that

- $\alpha \pmod{\pi}$ is linearly independent with $\pi^{-s}a \pmod{\pi}$ in \mathbb{F}_q^ρ .

This element $\tilde{\underline{e}}$ gives rise to a solution to the equation (5.27) satisfying the condition (5.28).

Conversely if there exists an element \underline{e} satisfying equation (5.27) and condition (5.28) then we choose $\tilde{\underline{e}} = by_2^{-1} + \pi^{r_2}y_2^{-1}\alpha \pmod{\pi^\lambda}$ for some $\alpha \in \mathbb{A}^\rho$ such that

- $\alpha \pmod{\pi}$ is linearly independent with $\pi^{-s}a \pmod{\pi}$ in \mathbb{F}_q^ρ .

This element $\tilde{\underline{e}}$ gives rise to a solution to the equation (5.25) satisfying the condition (5.26).

Suppose $r_1 > r_2$. If there exists an element \underline{e} satisfying equation (5.25) and condition (5.26) then choose $\tilde{\underline{e}} = by_1^{-1}$. This element $\tilde{\underline{e}}$ gives rise to a solution to the equation (5.27) satisfying the condition (5.28).

Conversely if there exists an element \underline{e} satisfying equation (5.27) and condition (5.28) then we choose $\tilde{e} = ay_1$. This element \tilde{e} gives rise to a solution to the equation (5.25) satisfying the condition (5.26).

And to exactly count the cardinality of the number of solutions, we use standard ideal filtration of $\mathbb{A}/\pi^n\mathbb{A}$ and deduce that the number of solutions $e \in (\mathbb{A}/\pi^\lambda\mathbb{A})^\rho$ to the set of equations (5.25) satisfying the conditions (5.26) is same as the number of solutions $e \in (\mathbb{A}/\pi^\lambda\mathbb{A})^\rho$ to the set of equations (5.27) satisfying the conditions (5.28) and it is given by

- $q^{(\rho)(\lambda-r_2-1)}(q^\rho - q)$ if $s \leq r_1 < r_2$
- $q^{(\rho)(\lambda-r_1)}$ if $s \leq r_2 < r_1$
- $|(ay_1y_2 + \pi^{r_1=r_2}(\mathbb{A}/\pi^\lambda\mathbb{A})^\rho) \cap (b + \pi^{r_1=r_2}S)| = |(ay_1y_2 + \pi^{r_1=r_2}S \cap (b + \pi^{r_1=r_2}(\mathbb{A}/\pi^\lambda\mathbb{A})^\rho))|$ if $s < r_1 = r_2$ where $S \subset (\mathbb{A}/\pi^\lambda\mathbb{A})^\rho$ is a set such that $S \pmod{\pi}$ is a set of vectors linearly independent to $\pi^{-s}a \pmod{\pi}$ in \mathbb{F}_q^ρ . This cardinality can be easily calculated and it is $q^{(\rho)(\lambda-r-1)}(q^\rho - q)$ where $r = r_1 = r_2$.
- Cardinality of the set $\{e \in \pi^s(\mathbb{A}/\pi^\lambda\mathbb{A})^\rho \mid \pi^{-s}e \pmod{\pi} \text{ is linearly independent to } \pi^{-s}a \pmod{\pi} \text{ in } \mathbb{F}_q^\rho\} = \text{Cardinality of the set } \{e \in \pi^s(\mathbb{A}/\pi^\lambda\mathbb{A})^\rho \mid \pi^{-s}e \pmod{\pi} \text{ is linearly independent to } \pi^{-s}b \pmod{\pi} \text{ in } \mathbb{F}_q^\rho\}$ if $s = r_1 = r_2$. Again this cardinality can be easily calculated and it is $q^{(\rho)(\lambda-s-1)}(q^\rho - q)$.

Again the proof of the cardinalities of the number of solutions is similar to the one given in Lemma 5.14. □

Theorem 5.34. *Let $(\underline{\lambda} = \lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \dots > \lambda_k^{\rho_k})$ be a partition. Let $\mathcal{A}_{\underline{\lambda}}$ be the corresponding finite \mathbb{A} -module. Let $\mathcal{G}_{\underline{\lambda}}$ be its automorphism group. Let I be an*

ideal in $J(\mathcal{P})_{\underline{\lambda}}$. Suppose $\mathbb{F}_q \cong \mathbb{A}/\pi\mathbb{A}$ has at least three elements or the multiplicity ρ_i of each part λ_i in $\underline{\lambda}$ is > 2 corresponding to every element $(\partial_{\lambda_i}I, \lambda_i) \in \max(I)$. Then the endomorphism algebra $\mathcal{EN}\mathcal{D}_{\mathbb{G}_{\underline{\lambda}}}(\mathbb{C}[\mathcal{O}_I])$ is commutative.

Proof. For $j = 1, 2$ let $\mathcal{O}_j \subset \mathcal{O}_I \times \mathcal{O}_I$ denote two suborbits. Let $(a, b) \in \mathcal{A}_{\underline{\lambda}}^2$. Then $\mathcal{I}_{\mathcal{O}_1} * \mathcal{I}_{\mathcal{O}_2}(a, b) = 0 = \mathcal{I}_{\mathcal{O}_2} * \mathcal{I}_{\mathcal{O}_1}(a, b)$ if $(a, b) \notin \mathcal{O}_I \times \mathcal{O}_I$. So assume $(a, b) \in \mathcal{O}_I \times \mathcal{O}_I$.

Now suppose $(\partial_{\lambda_i}I, \lambda_i) \in \max(I)$. Then the i^{th} -component of orbit of pair corresponding to the i^{th} -component $\pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i}I+1)}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}$ of the orbit \mathcal{O}_I is given by

$$\begin{aligned} \text{A. } (\mathcal{O}_1)_{\lambda_i} = & \{(a_i, b_i) \in (\pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i}I+1)}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}) \times \\ & (\pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i}I+1)}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}) \\ & | b_i - a_i y_1 \in \pi^{r_{i1}}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} \text{ for some } r_{i1} > \partial_{\lambda_i}I \text{ and for some slope unit } y_1 \in \\ & \mathbb{A}^* \} \end{aligned}$$

OR

$$\begin{aligned} \text{B. } (\mathcal{O}_1)_{\lambda_i} = & \{(a_i, b_i) \in (\pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i}I+1)}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}) \times \\ & (\pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i}I+1)}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}) \\ & | b_i - a_i y_1 \in \pi^{r_{i1}}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{r_{i1}+1}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} \text{ for some } r_{i1} \geq \partial_{\lambda_i}I \\ & \text{and for some slope unit } y_1 \in \mathbb{A}^* \text{ and} \\ & \pi^{-r_{i1}}(b_i - a_i y_1) \pmod{\pi} \text{ is linearly independent with } \pi^{-\partial_{\lambda_i}I} a_i \pmod{\pi} \text{ in } (\mathbb{F}_q)^{\rho_i} \} \end{aligned}$$

and

$$\begin{aligned} \text{a. } (\mathcal{O}_2)_{\lambda_i} = & \{(a_i, b_i) \in (\pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i}I+1)}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}) \times \\ & (\pi^{\partial_{\lambda_i}I}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i}I+1)}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i}) \\ & | b_i - a_i y_2 \in \pi^{r_{i2}}(\mathbb{A}/\pi^{\lambda_i}\mathbb{A})^{\rho_i} \text{ for some } r_{i2} > \partial_{\lambda_i}I \text{ and for some slope unit } y_2 \in \\ & \mathbb{A}^* \} \end{aligned}$$

OR

- b. $(\mathcal{O}_2)_{\lambda_i} = \{(a_i, b_i) \in (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i} I+1)}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \times$
 $(\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i} I+1)}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i})$
 $| b_i - a_i y_2 \in \pi^{r_{i2}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{r_{i2}+1}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \text{ for some } r_{i2} \geq \partial_{\lambda_i} I$
and for some slope unit $y_2 \in \mathbb{A}^*$ and
 $\pi^{-r_{i2}}(b_i - a_i y_2)(\text{mod } \pi)$ is linearly independent with $\pi^{-\partial_{\lambda_i} I} a_i(\text{mod } \pi)$ in $(\mathbb{F}_q)^{\rho_i}$

Consider case A and case a . From the Lemma 5.13 the number of solutions $e_i \in \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i} I+1)}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ such that $e_i - a_i y_1 \in \pi^{r_{i1}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ and $b_i - e_i y_2 \in \pi^{r_{i2}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ is the same as the number of solutions $e_i \in \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i} I+1)}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ such that $e_i - a_i y_2 \in \pi^{r_{i2}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ and $b_i - e_i y_1 \in \pi^{r_{i1}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$.

Consider case B and case a . From the Lemma 5.20 the number of solutions $e_i \in \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i} I+1)}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ such that $e_i - a_i y_1 \in \pi^{r_{i1}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$, $\pi^{-r_{i1}}(e_i - a_i y_1)(\text{mod } \pi)$ is linearly independent with $\pi^{-\partial_{\lambda_i} I} a_i(\text{mod } \pi)$ in $(\mathbb{F}_q)^{\rho_i}$ and $b_i - e_i y_2 \in \pi^{r_{i2}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ is the same as the number of solutions $e_i \in \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i} I+1)}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ such that $e_i - a_i y_2 \in \pi^{r_{i2}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ and $b_i - e_i y_1 \in \pi^{r_{i1}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$, $\pi^{-r_{i1}}(b_i - e_i y_1)(\text{mod } \pi)$ is linearly independent with $\pi^{-\partial_{\lambda_i} I} e_i(\text{mod } \pi)$ in $(\mathbb{F}_q)^{\rho_i}$.

Consider case A and case b . This case is similar to the above case B and case a .

Consider case B and case b . Again from the Lemma 5.20 the number of solutions $e_i \in \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i} I+1)}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ such that $e_i - a_i y_1 \in \pi^{r_{i1}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$, $\pi^{-r_{i1}}(e_i - a_i y_1)(\text{mod } \pi)$ is linearly independent with $\pi^{-\partial_{\lambda_i} I} a_i(\text{mod } \pi)$ in $(\mathbb{F}_q)^{\rho_i}$ and $b_i - e_i y_2 \in \pi^{r_{i2}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$, $\pi^{-r_{i2}}(b_i - e_i y_2)(\text{mod } \pi)$ is linearly independent with $\pi^{-\partial_{\lambda_i} I} e_i(\text{mod } \pi)$ in $(\mathbb{F}_q)^{\rho_i}$ is the same as the number of solutions $e_i \in \pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{(\partial_{\lambda_i} I+1)}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ such that $e_i - a_i y_2 \in \pi^{r_{i2}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$, $\pi^{-r_{i2}}(e_i - a_i y_2)(\text{mod } \pi)$ is linearly independent with $\pi^{-\partial_{\lambda_i} I} a_i(\text{mod } \pi)$ in $(\mathbb{F}_q)^{\rho_i}$ and $b_i - e_i y_1 \in \pi^{r_{i1}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$, $\pi^{-r_{i1}}(b_i - e_i y_1)(\text{mod } \pi)$ is linearly independent with $\pi^{-\partial_{\lambda_i} I} e_i(\text{mod } \pi)$ in $(\mathbb{F}_q)^{\rho_i}$.

Now suppose $(\partial_{\lambda_i} I, \lambda_i) \notin \max(I)$. Then the i^{th} -component of orbit of pair corresponding to the i^{th} -component $\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}$ of the orbit \mathcal{O}_I is given by

$$\text{A. } (\mathcal{O}_1)_{\lambda_i} = \{(a_i, b_i) \in (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \times (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i})\}$$

OR

$$\text{B. } (\mathcal{O}_1)_{\lambda_i} = \{(a_i, b_i) \in (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \times (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \mid$$

$$b_i - a_i y_1 \in \pi^{r_{i1}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \text{ for some } r_{i1} > \partial_{\lambda_i} I \text{ and for some slope unit } y_1 \in \mathbb{A}^*\}$$

OR

$$\text{C. } (\mathcal{O}_1)_{\lambda_i} = \{(a_i, b_i) \in (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \times (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \mid$$

$$b_i - a_i y_1 \in \pi^{r_{i1}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{r_{i1}+1}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \text{ for some } r_{i1} \geq \partial_{\lambda_i} I$$

and for some slope unit $y_1 \in \mathbb{A}^*\}$

and

$$\text{a. } (\mathcal{O}_2)_{\lambda_i} = \{(a_i, b_i) \in (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \times (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i})\}$$

OR

$$\text{b. } (\mathcal{O}_2)_{\lambda_i} = \{(a_i, b_i) \in (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \times (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \mid$$

$$b_i - a_i y_2 \in \pi^{r_{i2}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \text{ for some } r_{i2} > \partial_{\lambda_i} I \text{ and for some slope unit } y_2 \in \mathbb{A}^*\}$$

OR

$$\text{c. } (\mathcal{O}_1)_{\lambda_i} = \{(a_i, b_i) \in (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \times (\pi^{\partial_{\lambda_i} I}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i}) \mid$$

$$b_i - a_i y_2 \in \pi^{r_{i2}}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} - \pi^{r_{i2}+1}(\mathbb{A}/\pi^{\lambda_i} \mathbb{A})^{\rho_i} \text{ for some } r_{i2} \geq \partial_{\lambda_i} I$$

and for some slope unit $y_2 \in \mathbb{A}^*\}$

Here in all pairs of the cases $\{A, B, C\} \times \{a, b, c\}$, we have from the Lemma 5.13, the number of solutions e_i in both way convolutions agree for each pair.

Hence $\mathcal{I}_{\mathcal{O}_1} * \mathcal{I}_{\mathcal{O}_2}(a, b) = \mathcal{I}_{\mathcal{O}_2} * \mathcal{I}_{\mathcal{O}_1}(a, b)$ for all $(a, b) \in \mathcal{A}_{\underline{\lambda}}^2$ and the endomorphism algebra $\mathcal{E}\mathcal{N}\mathcal{D}_{\mathcal{G}_{\underline{\lambda}}}(\mathbb{C}[\mathcal{O}_I])$ is commutative. \square

Theorem 5.35. *Let $(\underline{\lambda} = \lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \dots > \lambda_k^{\rho_k})$ be a partition. Let $\mathcal{A}_{\underline{\lambda}}$ be the corresponding finite \mathbb{A} -module. Let $\mathcal{G}_{\underline{\lambda}}$ be its automorphism group. Let I be an ideal in $J(\mathcal{P})_{\underline{\lambda}}$. Suppose $\mathbb{F}_q \cong \mathbb{A}/\pi\mathbb{A}$ has at least three elements or the multiplicity ρ_i of each part λ_i in $\underline{\lambda}$ is > 2 corresponding to every element $(\partial_{\lambda_i} I, \lambda_i) \in \max(I)$. Then the permutation representation $\mathbb{C}[\mathcal{O}_I]$ of $\mathcal{G}_{\underline{\lambda}}$ is multiplicity-free.*

Proof. From Theorem 5.34, we observe that the endomorphism algebra $\mathcal{E}\mathcal{N}\mathcal{D}_{\mathcal{G}_{\underline{\lambda}}}(\mathbb{C}[\mathcal{O}_I])$ is commutative. So it follows that the permutation representation $\mathbb{C}[\mathcal{O}_I]$ of $\mathcal{G}_{\underline{\lambda}}$ is multiplicity-free.

Note that even though we have the conditions that are given in the Theorem 5.35 under which the Theorem 5.35 holds, here in this thesis we are mainly interested in the case where the residue field \mathbb{F}_q has atleast three elements in which case we have a lattice isomorphism between the set of order ideals in $J(\mathcal{P})_{\underline{\lambda}}$ and the set of characteristic submodules of $\mathcal{A}_{\underline{\lambda}}$. \square

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