

SKEIN THEORIES FOR FINITE DEPTH SUBFACTOR PLANAR ALGEBRAS

By

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I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of publications based on thesis

1. Vijay Kodiyalam and Srikanth Tupurani, *Universal skein theory for finite depth subfactor planar algebras*, **Quantum Topology** 2, No. 2, 157 - 172, (2011).

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Synopsis

Introduction

This thesis deals with mathematical objects known as planar algebras. These were introduced by Vaughan Jones in order to study the so-called ‘standard invariant’ of a II_1 -subfactor and have provided a powerful pictorial viewpoint with which to approach various computations in the theory.

Planar algebras are, at their simplest, a collection of vector spaces along with a (very large) family of maps between their tensor products. The maps are indexed by pictures in the plane called planar tangles and are postulated to be compatible with simple planar operations on the planar tangles.

A fundamental result of Jones in [5] may be considered as ensuring a plentiful supply of planar algebras. According to this result, every finite index II_1 subfactor $N \subseteq M$ yields, in a natural way, a planar algebra $P = P^{N \subseteq M}$. This planar algebra satisfies several niceness conditions, such as, for instance, finite dimensionality of the vector spaces involved. Planar algebras arising from subfactors are called subfactor planar algebras and have an intrinsic definition, independent of subfactors. It is these subfactor planar algebras that we will be interested in.

As in group theory, there is a notion of universal planar algebras akin to that of free groups and any planar algebra is a quotient of a universal planar algebra. A

universal planar algebra is determined by just a set L (which is a graded set and referred to as a set of labels). By imposing relations on a universal planar algebra we may obtain an arbitrary planar algebra. Such a generator-relation approach to a particular planar algebra is called a skein theory for that planar algebra. We will use the term in this thesis only when both generators and relations are finite sets.

Among the subfactor planar algebras, the analytically trivial ones are said to be of finite depth. These planar algebras are determined completely by a finite amount of data. Nevertheless they are interesting enough to include several classes of well studied planar algebras such as those associated to finite groups and subgroups or to finite-dimensional Kac algebras, the Temperley-Lieb planar algebras, the ADE type planar algebras as well as some exotic planar algebras such as the Haagerup and extended Haagerup planar algebras.

In each of these planar algebras, there has been work done to show that they have an interesting skein theory. Some of the papers dealing with skein theories include [11] for group subfactors, [6] for Kac algebra subfactors, [3] for higher exchange relation planar algebras, [1] for the ADE planar algebras and [13] and [2] for the Haagerup type planar algebras. In all these papers, one of the main points is an explicit construction of a skein theory for the planar algebras being considered.

In this thesis, we prove two results that are suggested by the previous work. These appear in Chapters 2 and 3 after a preliminary introduction to planar algebras in Chapter 1. A detailed summary of Chapters 2 and 3 follows.

Finite depth planar algebras have a skein theory

The main theorem of Chapter 2 asserts that a finite depth planar algebra admits a skein theory, i.e., a presentation with finitely many generators and finitely many relations. Thus finite depth planar algebras constitute a subclass of planar algebras

that are analogues of finitely presented groups.

The first section is devoted to showing that for an arbitrary planar algebra P satisfying an analogue of the finite depth condition for subfactor algebras, there is a P_{k-1} - P_{k-1} bimodule isomorphism $P_m \otimes_{P_{k-1}} P_n \rightarrow P_{m+n-(k-1)}$ for all $m, n \geq k$ where k is the ‘depth’ of P .

The next section introduces the technical tools that we use to prove our main result. There are three main notions, two of which are those of ‘templates’ and of ‘consequences’. By definition, a template is simply an ordered pair of planar tangles. Consequences of a set of templates roughly correspond to elements of the set obtained by closing the original set under certain planar operations, along with reflexivity and transitivity. Given a planar algebra P together with a subset B of P , there is a notion of a template ‘being satisfied’ in (P, B) , and this is the third main notion. It is not hard to see from the definitions that if certain templates are satisfied for (P, B) , then so are all their consequences. The main result in this section is a collection of various consequences of a set of templates that we call basic templates.

The third section of this chapter proves our main theorem that subfactor planar algebras of finite depth have a finite skein theory. The approach is to show that the basic templates hold for such a planar algebra (together with the distinguished subset being a basis of P_k where k is the depth) and then use their consequences and the bimodule isomorphism referred to above to deduce the theorem.

The last section gives a very simple proof that finite depth subfactor planar algebras are actually singly generated and further have a skein theory with this single generator. This last result is analogous to a finitely presented group having a finitely generated kernel for any surjective homomorphism of a finitely generated free group onto it, and the proof is also an imitation of that proof.

The contents of this chapter have been published in [9].

Bounding the degree of a generator of a planar algebra

The main problem that we deal with in Chapter 3 takes off from the single generation result proved in Chapter 2. If P is a subfactor planar algebra of depth k , then it is easily seen that it is singly generated by an element of P_{2k} . However the number $2k$ is not the best possible and while we do not settle what is the best, we show that for $k \geq 4$, the degree is bounded by $k + 3$ for k odd and $k + 4$ for k even.

Much of the effort in reducing the bound on the degree of the generator is spent in proving a result about finite dimensional complex semisimple algebras that might be of independent interest. In the first section of this chapter, we show that if S is an anti-automorphism of such an algebra without an $M_2(\mathbb{C})$ summand, then there exists an element a in it such that a and Sa generate it as an algebra. The restriction about not having an $M_2(\mathbb{C})$ summand is necessary. The proof of this result exploits a factorization theorem due to Takagi [4] and some simple considerations about the Zariski topology.

The second section proves the bound asserted. This is a consequence of a graph theoretic lemma applied to the principal graph of the subfactor planar algebra along with the generation result of the earlier section.

The contents of this chapter appear in the preprint [10].

Chapter 1

Planar algebras and presentations

In this chapter we define the main object of interest throughout this thesis - a planar algebra - along with the very special variety that we will consider - a subfactor planar algebra. In §1.1, we start with the definition of a planar tangle and stipulate some conventions which we are going to follow in the sequel. In §1.2, we give the definition of a planar algebra. In §1.3, we talk about some properties of planar algebras. §1.4 defines subfactor planar algebras and states the fundamental theorem of Jones. In §1.5, we talk about universal planar algebras and define a presentation of a planar algebra.

1.1 Planar tangles and conventions

Let $Col = \{0_+, 0_-, 1, 2, \dots\}$ be a set, elements of which we will call ‘colours’. A planar tangle consists of an external disc denoted D_0 and several non-overlapping internal discs denoted D_1, D_2, \dots, D_b . It is possible that $b = 0$. Each disc has an even number of marked points on its boundary circle. This number can be 0 too. Then there are strings of two kinds - open and closed. Open strings have two distinct marked points as end points and closed strings are simple closed curves.

All these strings lie in the interior of the complement of the union of all internal discs in the external disc. Every marked point must be an end point of a string. Strings must not intersect. The strings must be transversal at points of intersection with the boundaries of the discs. For each disc having at least one marked point on its boundary, one of the regions (connected components of the complement of the internal discs and curves) that impinge on its boundary is distinguished and marked with a $*$ placed near its boundary. Regions are endowed with a checkerboard shading (i.e., a black and white shading such that no two adjacent regions have same colour) such that the $*$ -region of any box is shaded white. We define the colour of a disc of a tangle to be half the number of marked points on its boundary when this number is positive. If there are no marked points on the disc then the colour of the disc is defined to be 0_{\pm} according as the region adjacent to the boundary of the disc has shading white or black in the checkerboard shading. The colour of a tangle is the colour of the external disc of the tangle. Two tangles are equivalent if there is an orientation preserving diffeomorphism of R^2 taking one tangle to the other and preserving the numbering of the discs, their $*$ -regions and the shading of the regions.

From now, we follow the following conventions. For convenience, for a tangle T , we pick any element in its isotopy class and again call it by T . We will draw (and refer to) discs as (rectangular) boxes to avoid confusion with closed strings. For a tangle T with b internal boxes of colours k_1, k_2, \dots, k_b we denote the i th box of T by $D_i(T)$. We follow the following labelling conventions. If T is a k_0 -tangle with internal boxes of colours k_1, k_2, \dots, k_b , we denote it by $T_{k_1, k_2, \dots, k_b}^{k_0}$. Figure 1.1 illustrates the use of this notation. With these conventions, we may avoid showing the shading, as shading is then unambiguously determined. We illustrate several important tangles in Figure 1.1; these tangles are given special names. This figure (along with others in this paper) uses the following notational device for convenience in drawing tangles. A strand in a tangle with a non-negative integer, say t , adjacent to it will indicate a t -cable of that strand, i.e., a parallel cable of t strands, in place

of the one. Thus for instance, the tangle equations of Figure 1.2 hold. In the sequel, we will have various integers adjacent to strands in tangles. Another convention we follow is that for planar tangles with only one internal disc we omit labelling the internal disc. This convention is also illustrated in Figure 1.1.

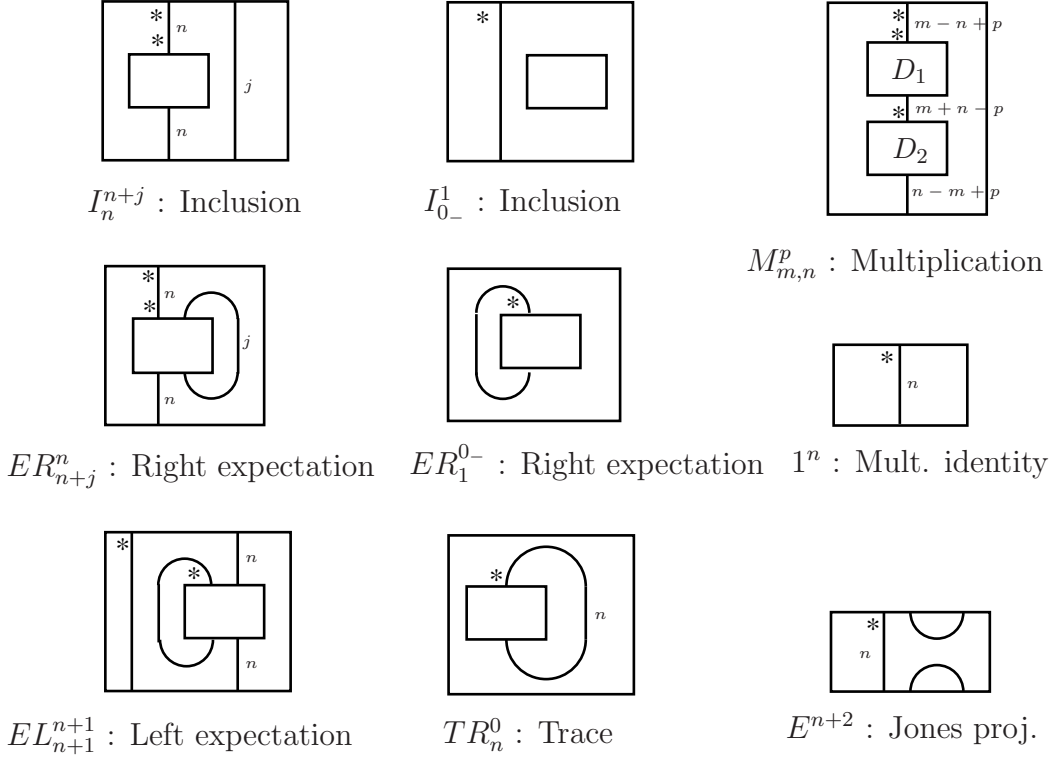


Figure 1.1: Some important tangles ($m, n, j \geq 0, |m - n| \leq p \leq m + n$)

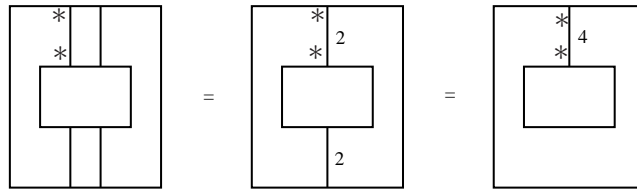


Figure 1.2: Illustration of cabling notation for tangles

If T and S are two coloured tangles such that the colour of S is the same as the colour of some internal disc of T , then we can compose T and S by ‘plugging S appropriately’ into that disc of T , so that the $*$ region of S meets the $*$ region of that disc (and then deleting the boundary of that disc). If $T = T_{k_1, k_2, \dots, k_b}^{k_0}$ is a tangle as described above and $S = S_{l_1, l_2, \dots, l_d}^{l_0}$ with $d > 0$, such that the i th disc of T has same colour as that of S , i.e $k_i = l_0$, then the composite tangle $To_i S$ is a k_0 -tangle with

$b + d - 1$ internal discs with numbering in the following order: for $1 \leq j \leq b + d - 1$, the j -th disc of To_iS is the

- j -th disc of T , if $1 \leq j \leq i - 1$;
- $j - i + 1$ -th disc of S , if $i \leq j \leq i + b - 1$; and the
- $j - b + 1$ -th disc of T , if $i + b \leq j \leq b + d - 1$.

In case $d = 0$, then the composite tangle To_iS is a k_0 -tangle with $b - 1$ internal discs with numbering in the following order: For $1 \leq j \leq b - 1$, the j -th disc of To_iS is the j -th disc of T .

We illustrate composition with one example. Figures 1.3 and 1.4 show 3-tangles T and S respectively with T having 3 internal discs (represented as boxes) and S having 2 internal discs.

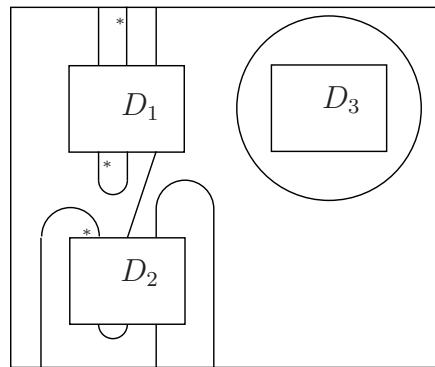


Figure 1.3: An example of a 3-tangle T with three internal boxes

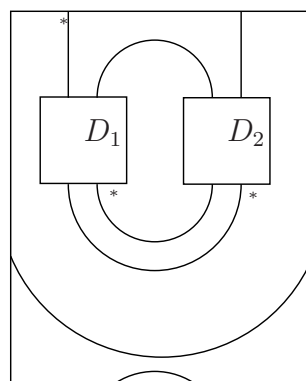


Figure 1.4: An example of a 3-tangle S with two internal boxes

Then, by plugging S in to the internal disc D_2 of tangle T we get the new 3-tangle

$To_{D_2}S$ shown in Fig. 1.5.

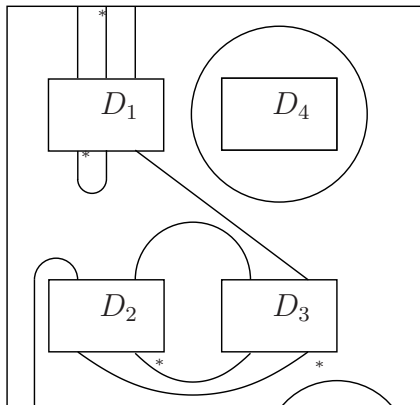


Figure 1.5: The composite tangle $To_{D_2}S$

1.2 Planar algebras

Definition 1. A planar algebra is a collection $\{P_k : k \in Col\}$ of (complex) vector spaces equipped with the following structure: if T is a k_0 tangle with b internal discs of colours $k_1, k_2, k_3, \dots, k_b$, respectively, there is an associated linear map (sometimes called the partition function):

$$Z_T^P : P_{k_1} \otimes P_{k_2} \otimes \dots \otimes P_{k_b} \rightarrow P_{k_0}.$$

If the tangle T has no internal discs then $Z_T^P : \mathbb{C} \rightarrow P_{k_0}$. These tangle maps are to satisfy the following three conditions:

1. *Compatibility under composition:* If $T = T_{k_1, k_2, \dots, k_b}^{k_0}$ and $S = S_{l_1, l_2, \dots, l_d}^{l_0}$ are tangles with $d > 0$ and $k_i = l_0$, then the compatibility condition for tangle maps is that the diagram in Figure 1.6 commute. If $d = 0$ (i.e., if the l_0 tangle S has no internal disc), then, for the compatibility requirement the modified diagram of Figure 1.7 is to commute.

2. *Renumbering compatibility:* Let $T = T_{k_1, k_2, \dots, k_b}^{k_0}$, σ be a permutation on b symbols $1, 2, \dots, b$, and $\sigma^{-1}(T)$ be the tangle which differs from T only in the ordering of discs;

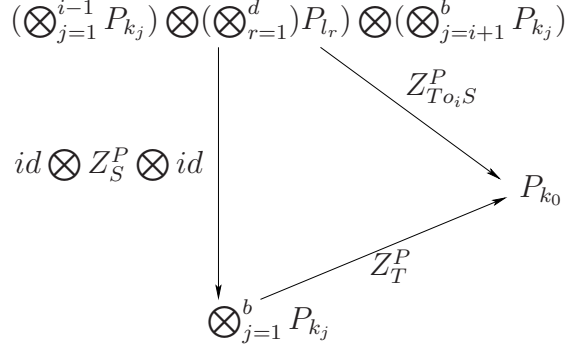


Figure 1.6: Compatibility condition when $d > 0$

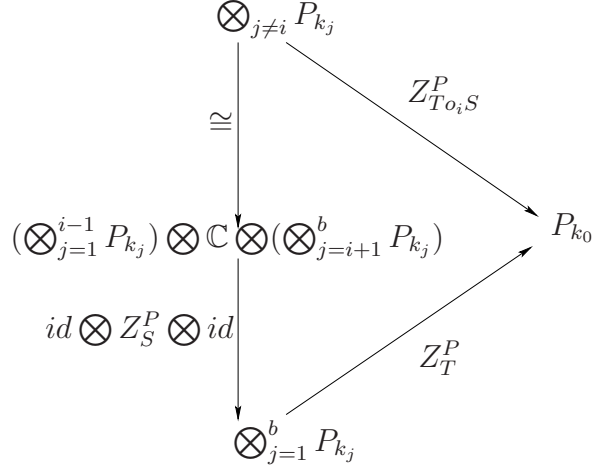
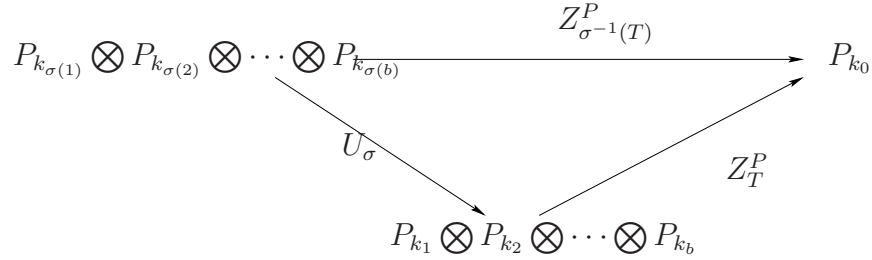


Figure 1.7: Modified compatibility condition when $d = 0$

for $1 \leq i \leq b$, the i -th disc of $\sigma^{-1}(T)$ is the $\sigma(i)$ -th disc of T . Then, the diagram



must commute, where U_σ is the map given by

$$U_\sigma \left(\bigotimes_{j=1}^b x_{\sigma(j)} \right) = \bigotimes_{j=1}^b x_j, \quad \text{for } \bigotimes_{j=1}^b x_j \in \bigotimes_{j=1}^b P_{k_j}.$$

3. Non-degeneracy condition: P_k is spanned by the ranges of the Z_T 's, as T ranges over all k -tangles. It is easily seen that for every k , and for every k -tangle T , we have $I_k^k o_{D_1} T = T$ (for the definition of I_k^k , see Figure 1.1), and hence $Z_{I_k^k} o Z_T = Z_T$.

It follows that $Z_{I_k^k}$ is an idempotent endomorphism of P_k whose range contains the range of Z_T for every k -tangle T . Equivalently the non-degeneracy condition is equivalent to the fact that $Z_{I_k^k} = id_{P_k} \forall k \in Col$.

It is easy to see that if P is a planar algebra, then each P_k is an associative algebra with multiplication defined by $x_1 x_2 = Z_{M^k}(x_1 \otimes x_2)$ where $M^k = M_{k,k}^k$. From pictures we see that $x Z_{1^k}(1) = Z_{1^k}(1) x = x$ for all $x \in P_k$. Clearly $Z_{1^k}(1)$ is the unit for P_k . We denote it by 1_k .

Further, the inclusion tangles I_k^{k+1} yield (not necessarily injective) algebra homomorphisms from P_k to P_{k+1} . Identifying P_k with its image in P_{k+1} , there is a natural algebra structure on $\bigcup_{k \in Col}^\infty P_k$. If $x \in P_m$ and $y \in P_n$ and if $m \geq n$, then $xy = Z_{M_m^m}(x \otimes I_n^m(y))$.

1.3 Some properties of planar algebras

Before describing subfactor planar algebras we need to know the following properties of a planar algebra.

1. *Connectedness*: A planar algebra is connected if $\dim(P_{0_\pm}) = 1$.
2. *Modulus*: A connected planar algebra P is said to have modulus δ if there exists a scalar $\delta \in \mathbb{C}$ such that $Z_{T_{\mp}^\pm}(1) = \delta 1_{\simeq}$. Here, T_{\mp}^\pm are the tangles of Figure 1.8. Note that the single picture in Figure 1.8 actually represents two tangles, depending

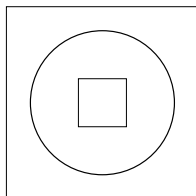


Figure 1.8: The tangles T_{\mp}^\pm

on the shading. (We use \pm and \mp instead of the more accurate 0_\pm and 0_\mp). It must

be noted that if P has modulus δ , then $Z_{E_{k+1}^k} \circ Z_{I_k^{k+1}} = \delta \text{id}_{P_k} \forall k \in \text{Col}$; and thus if $\delta \neq 0$, the inclusion tangles induce injective homomorphisms.

3. *Finite-dimensionality*: A planar algebra P is said to be finite-dimensional if $\dim P_k < \infty \forall k \in \text{Col}$.

4. *Sphericity*: Suppose P is a connected planar algebra and T is a 0_{\pm} -tangle (by which we mean a 0_+ - or a 0_- -tangle). If T has internal discs D_i of colours k_i , and if $x_i \in P_{k_i}$, for $1 \leq i \leq b$, then $Z_T(\bigotimes_{i=1}^{i=b} x_i) \in \mathbb{C}$, where we identify $P_{0_{\pm}}$ with \mathbb{C} . Thus considering the discs of T to be labelled by vectors $x_i \in P_{k_i}$, we can assign a scalar to each labelled 0_{\pm} -tangle. This assignment of scalars to the labelled 0_{\pm} -tangles is known as the partition function associated to the planar algebra P . For a 0_{\pm} -tangle T , by its network, we mean the system of strings and discs of T excluding its outer disc, along with the shading of the regions. The unbounded region of the network of T gets a shading of white or black according as T is a 0_+ - or 0_- -tangle. A planar algebra is said to be spherical if its partition function assigns the same value to any two 0_{\pm} -tangles whose associated networks are isotopic on the 2-sphere.

Another definition we need in order to define a subfactor planar algebra is the notion of adjoint of a tangle. Given a k -tangle T , by reflecting it about any line which lies outside the external disc of T we get the adjoint tangle denoted by T^* . For instance, Figure 1.9 shows a tangle T and its adjoint tangle T^* .

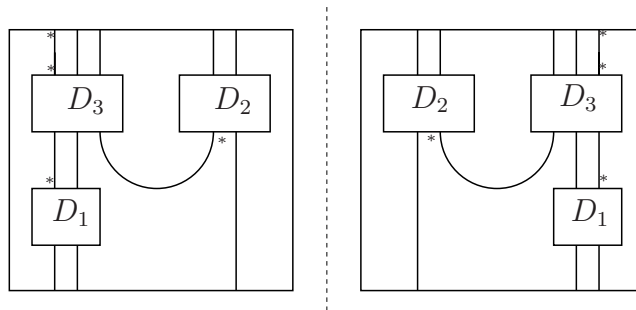


Figure 1.9: T and T^*

1.4 Subfactor planar algebras

Definition 2. A planar algebra P is said to be a subfactor planar algebra if:

- (i) P is connected, finite-dimensional, spherical, and has positive modulus, say δ ,
- (ii) each P_k is a C^* algebra in such a way that, if $T = T_{k_1, k_2, \dots, k_b}^{k_0}$, and if $x_i \in P_{k_i}$, $1 \leq i \leq b$, then $Z_T(x_1 \otimes \cdots \otimes x_b)^* = Z_{T^*}(x_1^* \otimes \cdots \otimes x_b^*)$, and,
- (iii) if we define the pictorial trace on P by $tr_{k+1}(x)1_+ = \delta^{-k-1} Z_{E_1^0} Z_{E_2^1} \cdots Z_{E_{k+1}^k}(x)$ for $x \in P_{k+1}$, then tr_m is a faithful positive trace on P_m for all $m \geq 1$.

Note that the definition of a subfactor planar algebra makes no explicit reference to subfactors. The connection is the content of Jones' theorem. Before we state this, we give a very brief summary of subfactor theory.

A von-Neumann algebra M is said to be a factor if it has trivial centre. Before we talk about II_1 factors, we give some definitions. Let e and f be two projections in a von Neumann algebra M .

Definition 3. 1. $e, f \in M$ are said to be Murray-von Neumann equivalent (denoted $e \sim f$) if there exists an operator $u \in M$ such that $u^*u = e$ and $uu^* = f$.

- 2. e is finite if it is not equivalent to any proper sub-projection of itself.
- 3. e is minimal if, for any sub-projection f of e , either $e = f$ or $f = 0$.

A factor M is said to be a II_1 factor, if it has a non-zero finite projection and does not have a non-zero minimal projection.

Definition 4. A functional ϕ on M is said to be

- 1. normal if it is σ -weakly continuous.
- 2. positive if $\phi(x^*x) \geq 0$.
- 3. a state if it is positive and $\phi(1) = 1$.

4. faithful if $\phi(x^*x) > 0 \forall x \neq 0$.
5. tracial if $\phi(xy) = \phi(yx) \forall x, y \in M$

Theorem 5. *If M is a II_1 -factor, then there exists a unique faithful normal tracial state tr_M on M .*

Let M denote a II_1 factor. We write $tr = tr_M$. Let $\mathcal{H} = \mathcal{H}_1 = L^2(M, tr)$ - the Hilbert space completion of M for the inner product given by $\langle x|y \rangle = tr(y^*x)$. Let $\mathcal{H}_\infty = \mathcal{H}_1 \otimes l^2(\mathbb{N})$. Let $Mat_\infty(M)$ be the set of all bounded operators x on \mathcal{H}_∞ which are given by infinite matrices with entries from M , i.e., $x = (x(m, n))_{m, n \in \mathbb{N}}$ for $x(m, n) \in M$. Then \mathcal{H}_∞ is an $M - Mat_\infty(M)$ bimodule such that $\pi_l(M)' = \pi_r[Mat_\infty(M)]$, where π_l and π_r stand for the natural left and right regular representations.

Theorem 6. *Let \mathcal{H} be any separable M -module, where M is a II_1 -factor. Then there exists a projection $p \in Mat_\infty(M)$ such that $\mathcal{H} \cong \mathcal{H}_\infty p$. Also, the projection p is uniquely determined by $Tr(p)$, where Tr is the faithful normal semifinite trace on the II_∞ -factor $Mat_\infty(M)$ defined by $Tr((p_{ij})) = \sum_i tr(p_{ii})$.*

Definition 7. If M is a II_1 factor and \mathcal{H} is a separable M module, then $dim_M \mathcal{H} = Tr(p)$.

Definition 8. A subfactor of a factor M is a unital subalgebra $N \subseteq M$ such that N is also a factor of the same type. The subfactor $N \subseteq M$ is said to be irreducible if $N' \cap M = \mathbb{C}$. If N is a subfactor of a II_1 factor M , we define the index of N in M by the expression $[M : N] = dim_N L^2(M)$.

Definition 9. Given a II_1 -subfactor $N \subseteq M$, there is a canonically associated tower of factors associated to it called the basic construction tower. This tower is obtained by adjoining a sequence of projections e_2, e_3, \dots called the Jones projections.

Now we state Jones' theorem.

Theorem 10. [5] *Let*

$$(M_0 =)N \subset M(= M_1) \subset^{e_2} M_2 \subset \cdots \subset^{e_n} M_n \subset^{e_{n+1}} \cdots$$

be the tower of the basic construction associated to an extremal subfactor with index $[M : N] = \delta^2 < \infty$. Then there exists a unique subfactor planar algebra $P = P^{N \subset M}$ of modulus δ satisfying the following conditions:

(0) $P_n^{N \subset M} = N' \cap M_n \ \forall n \geq 0$ - *where this is regarded as an equality of $*$ -algebras which is consistent with the inclusions on the two sides;*

$$(1) \ Z_{E^{n+1}}(1) = \delta \ e_{n+1} \ \forall n \geq 1;$$

$$(2) \ Z_{EL(1)_{n+1}^{n+1}}(x) = \delta \ E_{M' \cap M_{n+1}}(x) \ \forall x \in N' \cap M_{n+1}, \ \forall n \geq 0;$$

(3) $Z_{ER_{n+1}^n}(x) = \delta \ E_{N' \cap M_n}(x) \ \forall x \in N' \cap M_{n+1}$; *and this (suitably interpreted for $n = 0_{\pm}$) is required to hold for all $n \in Col$.*

Conversely, any subfactor planar algebra P with modulus δ arises from an extremal subfactor of index δ^2 in this fashion.

Definition 11. Let $P = \{P_k : k \in Col\}$ be a planar algebra and $Q_k \subseteq P_k$ be a subspace of P_k for each $k \in Col$. Then $Q = \{Q_k : k \in Col\}$ is said to be a planar subalgebra of P if for any k_0 -tangle T with internal discs of colour's k_1, k_2, \dots, k_b , $Z_T(x_1 \otimes x_2 \otimes \cdots \otimes x_b) \in Q_{k_0}, \forall x_i \in Q_{k_i}$ for $1 \leq i \leq b$.

Definition 12. Let P and Q be two planar algebras. A planar algebra morphism from P to Q is a collection $\phi = \{\phi_k : k \in col\}$ of linear maps $\phi_k : P_k \rightarrow Q_k$ which commutes with all the tangle maps, i.e., if T is a k_0 -tangle with b internal boxes of colours k_1, k_2, \dots, k_b , then

$$\phi_{k_0} \circ Z_T^P = Z_T^Q \circ (\bigotimes_{i=1}^b \phi_{k_i}), \text{ if } b > 0; \text{ and}$$

$$\phi_{k_0} \circ Z_T^P = Z_T^Q, \text{ if } b = 0.$$

ϕ is said to be a planar algebra isomorphism if the maps ϕ_k are all linear isomor-

phisms.

1.5 Universal planar algebras and presentations

Given a label set $L = \coprod_{k \in Col} L_k$, an L -labelled tangle is a tangle T equipped with a labelling of every internal box of colour k by an element from L_k . The universal planar algebra on L , denoted by $P(L) = \{P(L)_k : k \in Col\}$ is defined by requiring that $P(L)_k$ is the k -vector space with basis consisting of all L -labelled k -tangles. This collection admits an obvious planar algebra structure.

Before defining a presentation we should know the definition of a planar ideal.

Definition 13. A planar ideal of a planar algebra P is a set $I = \{I_k : k \in Col\}$ with the property that

- (i) each I_k is a subspace of P_k , and
- (ii) for any k_0 -tangle T with internal discs of colours $k_1, k_2, k_3, \dots, k_b$, respectively, $Z_T(\bigotimes_{i=1}^b x_i) \in I_{k_0}$ whenever $x_i \in I_{k_i}$ for at least one i .

This definition generalises the usual definition of an ideal in an algebra.

Given a planar ideal I in a planar algebra P , there is a natural planar algebra structure on the quotient $P/I = \{P_k/I_k : k \in Col\}$. Given any subset $R = \{R_k : k \in Col\}$ of P (meaning $R_k \subset P_k, \forall k \in Col$), there is a smallest planar ideal $I(R) = \{I(R)_k : k \in Col\}$ such that $R_k \subset I(R)_k$ for all $k \in Col$. Given a label set L as above, and any subset R of the universal planar algebra $P(L)$, the quotient $P(L)/I(R)$ is said to be the planar algebra with generators L and relations R . A presentation of a planar algebra is an expression of the planar algebra in terms of generators and relations.

Chapter 2

A skein theory for a finite depth subfactor planar algebra

A skein theory for a planar algebra is an expression for the planar algebra in terms of finitely many generators and relations. Equivalently it is a finite presentation of a planar algebra.

Skein theories have been studied for group subfactor planar algebras by Landau [11]; for irreducible depth two subfactor planar algebras by Kodiyalam, Sunder and Landau [6]; for irreducible depth two (not necessarily subfactor) planar algebras by Kodiyalam and Sunder [8]; for the D_{2n} planar algebras by Morrison, Peters and Snyder [12]; for the ADE planar algebras by Bigelow [1]; for the Haagerup subfactor planar algebra by Peters [13] and for the extended Haagerup subfactor planar algebra by Bigelow, Morrison, Peters and Snyder [2].

The skein theory developed in this chapter is for an arbitrary finite depth subfactor planar algebra. This skein theory is not as nice or compact as the ones mentioned above that are valid for special subfactor planar algebras or families of such. Nevertheless, the focus of our result is that it holds for an arbitrary subfactor planar algebra of finite depth. The most important point is that all such planar algebras

have a skein theory, or equivalently, a finite presentation.

We now summarise the rest of the chapter. In §2.1, we review and prove some well-known facts about finite depth subfactor planar algebras. In §2.2, we define the notion of a ‘template’ as well as that of a ‘consequence’ which is a certain relationship between templates. The long §2.3 derives several consequences of a basic set of templates. In §2.4, we apply the results of the previous section to derive a skein theory for a finite depth subfactor planar algebra. In §2.5, we make a couple of simple observations including the single generation of a finite depth subfactor planar algebra.

2.1 On the finite depth condition

In this chapter, we follow all the conventions for tangles given in §1.1. In addition, we omit the external box of a tangle and consider the region in the top left corner to be the $*$ -region. This convention is illustrated in Figure 2.1.

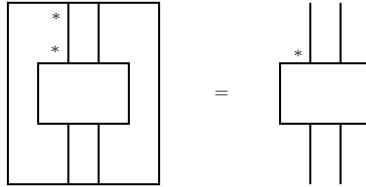


Figure 2.1: Convention regarding omission of external box

Any subfactor planar algebra P (of modulus δ) contains the distinguished Jones projections $e_n \in P_n$ for $n \geq 2$ defined by $e_n = \delta^{-1}Z_{E^n}^P(1)$ (see Figure 1.1 for the tangles E^n) and their non-normalised versions $E_n = Z_{E^n}^P(1)$. A subfactor planar algebra P is said to have finite depth if there is a positive integer k such that $P_k E_{k+1} P_k = P_{k+1}$ and the smallest such k is said to be the depth of P .

We begin with the following lemma - see Lemma 5.7 of [6] - which we reprove for convenience.

Lemma 14. *In any planar algebra P , we have*

- (a) $E_{k+1}P_k = E_{k+1}P_{k+1}$, and $P_kE_{k+1} = P_{k+1}E_{k+1}$ for any $k \geq 1$, and
- (b) $P_kE_{k+1}P_k$ is a two-sided ideal in P_{k+1} for any $k \geq 1$.
- (c) If $P_kE_{k+1}P_k = P_{k+1}$ for some $k \geq 1$, then,
 - (i) $P_lE_{l+1}P_l = P_{l+1}$ for all $l \geq k$, and
 - (ii) $P_kE_{k+1}E_{k+2} \cdots E_{l+1}P_l = P_{l+1} \forall l \geq k$.

Proof. (a) Clearly, $E_{k+1}P_k \subseteq E_{k+1}P_{k+1}$. Next, consider $E_{k+1}z$ where $z \in P_{k+1}$. A pleasant pictorial verification shows that $E_{k+1}z = E_{k+1}Z_{E_{k+1}^k}(E_{k+1}z)$. However, $Z_{E_{k+1}^k}(E_{k+1}z) \in P_k$ thus proving the other inclusion. Similarly, we can show that $P_kE_{k+1} = P_{k+1}E_{k+1}$.

(b) This is an immediate consequence of (a).

(c) We will prove (i) and (ii) by induction on l .

(i) The assertion is clearly valid for $l = k$. Suppose the assertion is true for $l \geq k$.

Then by inductive hypothesis, we can find $a_i, b_i \in P_l$ where $1 \leq i \leq s$ such that $1_{P_{l+1}} = \sum_{i=1}^s a_i E_{l+1} b_i$. Then $1_{P_{l+2}} (= 1_{P_{l+1}}) = \sum_{i=1}^s a_i E_{l+1} b_i = \sum_{i=1}^s a_i E_{l+1} E_{l+2} E_{l+1} b_i \in P_{l+1} E_{l+2} P_{l+1}$. Hence, $P_{l+1} E_{l+2} P_{l+1} = P_{l+2}$.

(ii) For $l = k$, the assertion is clearly valid. Suppose the assertion is valid for some $l \geq k$. Then $P_k E_{k+1} E_{k+2} \cdots E_{l+1} P_l = P_{l+1}$. From the previous result and by inductive hypothesis, $P_{l+2} = P_{l+1} E_{l+2} P_{l+1} = P_k E_{k+1} E_{k+2} \cdots E_{l+1} P_l E_{l+2} P_{l+1} = P_k E_{k+1} E_{k+2} \cdots E_{l+1} E_{l+2} P_l P_{l+1} = P_k E_{k+1} E_{k+2} \cdots E_{l+1} E_{l+2} P_{l+1}$. \square

Before stating the next proposition, we introduce a family of tangles denoted T^n for $n \in Col$, which will be used in its proof. These tangles also depend on a positive integer k but we suppress this dependence. The tangles T^n are shown in Figure 2.2, for the three cases $n \geq k, 0 \leq n < k$ and $n = 0_-$. Note that for $n \geq k$ the tangle T^n has $n - k + 1$ internal boxes all of colour k , while for $n < k$, $T^n = ER_k^n$. In particular $T^k = I_k^k$. Proposition 15 is the main result of this section. It should be noted that it applies to very general planar algebras, for instance, without assumptions on

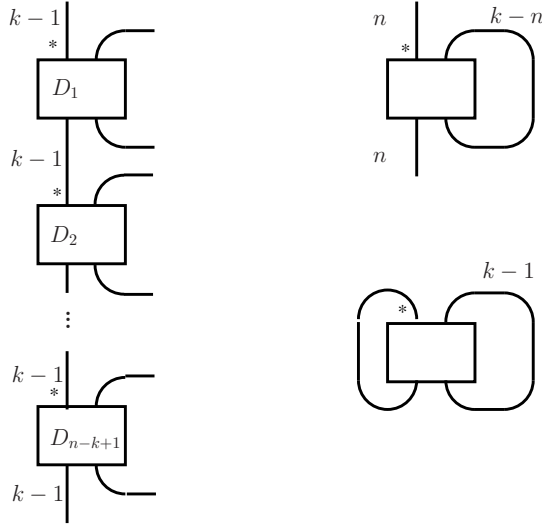


Figure 2.2: The tangles T^n for $n \geq k$, $0 \leq n < k$ and $n = 0$.

modulus, finite-dimensionality etc.

Proposition 15. *Let P be any planar algebra and suppose that for some positive integer k , $1_{k+1} \in P_k E_{k+1} P_k$. For all $m, n \geq k$, there is an isomorphism of $P_{k-1} - P_{k-1}$ -bimodules, $P_m \otimes_{P_{k-1}} P_n \cong P_{m+n-(k-1)}$, which is implemented by the tangle $M = M_{m,n}^{m+n-(k-1)}$.*

Proof. Since $1_{k+1} \in P_k E_{k+1} P_k$, we have by Lemma 14(c)(ii) that for $n \geq k$, $P_{n+1} = P_k E_{k+1} E_{k+2} \cdots E_{n+1} P_n$. We then get by induction that,

$$P_{n+1} = P_k E_{k+1} E_{k+2} \cdots E_{n+1} P_k E_{k+1} E_{k+2} \cdots E_n P_k \cdots \cdots P_k E_{k+1} P_k.$$

Observe now that for $x_1, x_2, \cdots, x_{n-k+2} \in P_k$, we have

$$Z_{T^{n+1}}(x_1, x_2, \cdots, x_{n-k+2}) = x_1 E_{k+1} E_{k+2} \cdots E_{n+1} x_2 E_{k+1} E_{k+2} \cdots E_n x_3 \cdots \cdots x_{n-k+1} E_{k+1} x_{n-k+2}.$$

This (along with the fact that $T^k = I_k^k$) yields the surjectivity of $Z_{T^n}^P$ for all $n \geq k$.

Now consider the tangle $M = M_{m,n}^{m+n-(k-1)}$. Thus $Z_M^P : P_m \otimes P_n \rightarrow P_{m+n-(k-1)}$ and a little thought shows that this is a $P_{k-1} - P_{k-1}$ -bimodule map that factors

through $P_m \otimes_{P_{k-1}} P_n$. Surjectivity of this map follows from the tangle equation $M \circ_{(D_1, D_2)} (T^m, T^n) = T^{m+n-(k-1)}$.

Next, we show injectivity. This proof uses the tangles $W = W_{n, 2n-k+1}^n$ and W^* shown in Figure 2.3.

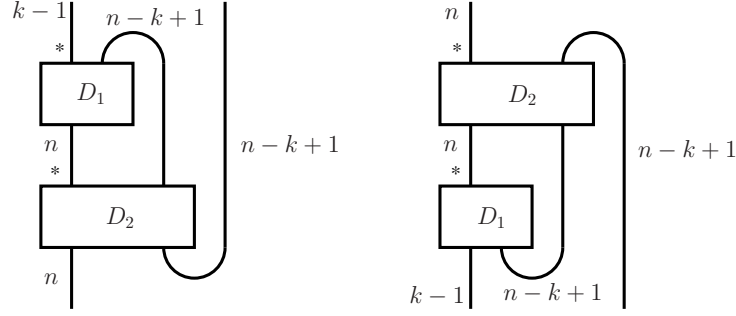


Figure 2.3: The tangles W and W^*

First use the surjectivity above for $m = n$ to conclude that there exist $x_i, y_i \in P_n$, for $i \in I$ - a finite set - such that $1_{2n-(k-1)} = \sum_{i \in I} Z_M^P(x_i \otimes y_i)$. Hence, for any $v \in P_n$, $Z_W^P(v, 1_{2n-(k-1)}) = \sum_{i \in I} Z_{W \circ_{D_2} M}^P(v \otimes x_i \otimes y_i)$. Equivalently, for all $v \in P_n$, we have $v = \sum_{i \in I} Z_{ER_n^{k-1}}(vx_i)y_i$.

Now, we claim that if $\sum_{j \in J} u_j \otimes v_j \in \ker(Z_M^P)$, then,

$$\sum_{j \in J} u_j \otimes v_j = \left(\sum_{i \in I, j \in J} u_j \otimes Z_{ER_n^{k-1}}(v_j x_i) y_i \right) - \left(\sum_{i \in I, j \in J} u_j Z_{ER_n^{k-1}}(v_j x_i) \otimes y_i \right).$$

In fact, the left hand side equals the first term on the right hand side while the second term on the right vanishes since for each $i \in I$, the sum $\sum_{j \in J} u_j Z_{ER_n^{k-1}}(v_j x_i)$ is of the form $Z_{W^* \circ_{D_2} M}^P(x_i \otimes \sum_{j \in J} u_j \otimes v_j) = Z_{W^*}^P(x_i \otimes Z_M^P(\sum_{j \in J} u_j \otimes v_j)) = 0$. The displayed equation above expresses $\sum_{j \in J} u_j \otimes v_j$ as an element in the kernel of the natural map $P_m \otimes P_n \rightarrow P_m \otimes_{P_{k-1}} P_n$ and concludes the proof. \square

We will need the following corollary whose proof follows easily by induction using Proposition 15.

Corollary 16. *Let P be any planar algebra and suppose that for some positive integer k , $1_{k+1} \in P_k E_{k+1} P_k$. Then, for all $n \geq k$ there is a $P_{k-1} - P_{k-1}$ -bimodule isomorphism*

$$P_k \otimes_{P_{k-1}} P_k \otimes_{P_{k-1}} \cdots \otimes_{P_{k-1}} P_k \cong P_n,$$

where there are $n - k + 1$ P_k 's on the left, which is implemented by the tangle T^n .

2.2 Templates and consequences

This section introduces the main technical notions used in the proof of our main result - templates, consequences and the notion of a template holding for (P, B) where P is a planar algebra and $B \subseteq P$.

Definition 17. A template is an ordered pair of tangles (S, T) of the same colour but will be written as a tangle implication $S \Rightarrow T$.

Given any set of templates, we will be interested in their consequences which are by definition those that can be obtained from them using (i) ‘reflexivity’ (ii) ‘transitivity’ and (iii) ‘composition on the outside’. We state this formally in the following definition.

Definition 18. If \mathcal{S} is a set of templates the set $\mathcal{C}(\mathcal{S})$ of consequences of \mathcal{S} is the smallest set of templates containing \mathcal{S} and such that (i) all $T \Rightarrow T$ are in $\mathcal{C}(\mathcal{S})$ (ii) if $S \Rightarrow T$ and $T \Rightarrow V$ are in $\mathcal{C}(\mathcal{S})$, then so is $S \Rightarrow V$, and (iii) if W is an arbitrary $(n_0; n_1, \dots, n_b)$ tangle and $S_i \Rightarrow T_i$ are in $\mathcal{C}(\mathcal{S})$ with colour n_i , then, $W \circ_{(D_1, \dots, D_b)} (S_1, \dots, S_b) \Rightarrow W \circ_{(D_1, \dots, D_b)} (T_1, \dots, T_b)$ is also in $\mathcal{C}(\mathcal{S})$.

In this section, planar algebras will play no role. But the motivation for the definition of consequences comes from the following. Let P be a planar algebra and $B \subseteq P$, i.e., $B = \coprod_{n \in Col} B_n$ where $B_n \subseteq P_n$ for all $n \in Col$. Given the pair (P, B) ,

each $(n_0; n_1, \dots, n_b)$ -tangle T then determines a certain subspace $R_{(P,B)}(T) \subseteq P_{n_0}$ defined to be (i) the span of all $Z_T^P(x_1 \otimes \dots \otimes x_b)$ for $x_i \in B_{n_i}$ if $b > 0$ or (ii) the span of $Z_T^P(1)$ if $b = 0$. Intuitively, $R_{(P,B)}(T)$ is the span of all elements obtained by substituting elements of B into the boxes of T in the planar algebra P .

Definition 19. A template $S \Rightarrow T$ is said to hold for the pair (P, B) if $R_{(P,B)}(S) \subseteq R_{(P,B)}(T)$.

It is now easy to see that if a set of templates holds for (P, B) then so do all their consequences.

For this section we need a particular collection of templates shown in Figure 2.4 which we will refer to as the basic templates. Here k is a fixed positive integer. Note that Figure 2.4 names each of the templates, shows them as tangle implications, and in the process, defines some tangles. We begin with a simple but very useful lemma which we will refer to later as ‘removing loops’.

Lemma 20. *Let $S \Rightarrow T$ be any template such that the tangle S has a contractible loop somewhere in it and let \tilde{S} be S with the loop removed. The modulus templates together with $S \Rightarrow T$ have as consequence $\tilde{S} \Rightarrow T$.*

Proof. Suppose that the contractible loop of S lies in a white region. Let W be the tangle obtained from S by replacing the contractible loop with a 0_+ box numbered $b + 1$, where S has b internal boxes. Then it is clear that $S = W \circ_{D_{b+1}} (C^{0+})$ while $\tilde{S} = W \circ_{D_{b+1}} (1^{0+})$. Since the modulus tangle gives $1^{0+} \Rightarrow C^{0+}$, by composing on the outside with W , we get $\tilde{S} \Rightarrow S$ and so by transitivity $\tilde{S} \Rightarrow T$. A similar proof applies when the loop lies in a black region. \square

The main result of this section is an omnibus theorem listing various consequences of the templates of Figure 2.4. While all the consequences are written as tangle implications, we emphasise that the proofs are purely pictorial. Recall the tangles T^n defined for $n \in Col$ in Figure 2.2.

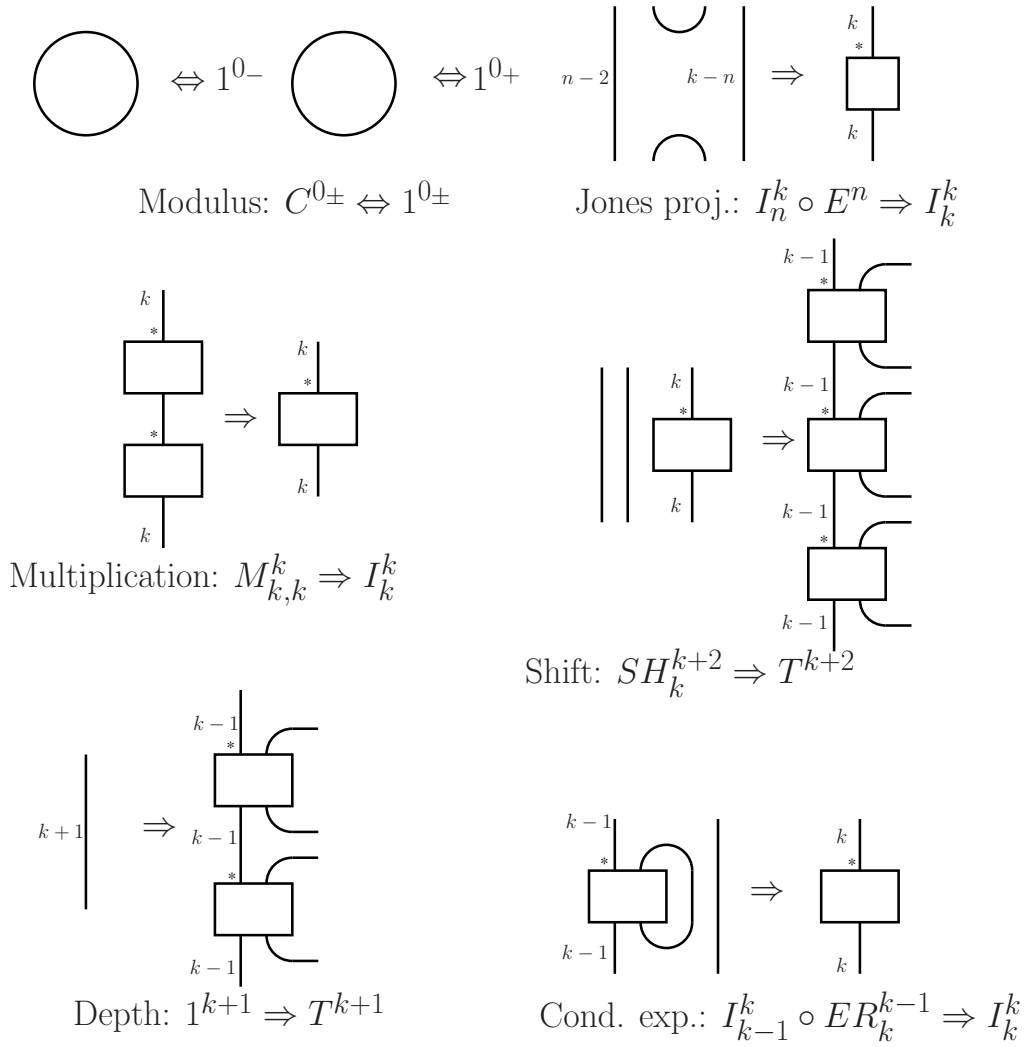


Figure 2.4: The basic templates ($2 \leq n \leq k$ for the Jones projections)

Theorem 21. *The following templates are all consequences of the basic templates of Figure 2.4.*

1. $1^k \Rightarrow T^k$.
2. $I_k^{k+1} \Rightarrow T^{k+1}$.
3. For all $n \in Col$, $ER_{n+1}^n \circ T^{n+1} \Rightarrow T^n$.
4. For any $n \geq k$, $I_n^{n+1} \circ T^n \Rightarrow T^{n+1}$.
5. For any $n \geq k$, $I_k^n \Rightarrow T^n$ and $1^n \Rightarrow T^n$.

6. For any $n \geq k$, $M_{n,n}^n \circ_{(D_1, D_2)} (T^n, T^n) \Rightarrow T^n$.

7. $1^{0\pm} \Rightarrow T^{0\pm}$ and for any $n \geq 2$, $E^n \Rightarrow T^n$.

8. For any $n \geq k$ and any Temperley-Lieb tangle Q^n , $Q^n \Rightarrow T^n$.

9. For any $n \geq k$, $SH_n^{n+2} \circ T^n \Rightarrow T^{n+2}$.

10. For any $n \geq 1$, $EL_n^n \circ T^n \Rightarrow T^n$.

11. For all $n \in Col$, $I_n^{n+1} \circ T^n \Rightarrow T^{n+1}$.

12. For all $n \in Col$, $M_{n,n}^n \circ_{(D_1, D_2)} (T^n, T^n) \Rightarrow T^n$.

Proof. (1) According to the depth template $1^{k+1} \Rightarrow T^{k+1}$. Applying ER_{k+1}^k on both sides yields $ER_{k+1}^k \circ 1^{k+1} \Rightarrow ER_{k+1}^k \circ T^{k+1} = M_{k,k}^k$.

Since $ER_{k+1}^k \circ 1^{k+1}$ is 1^k with a contractible loop on the right, we may remove this loop by Lemma 20 and conclude that $1^k \Rightarrow I_k^k = T^k$.

(2) Since $1^{k+1} \Rightarrow T^{k+1}$ and $I_k^{k+1} \Rightarrow I_k^{k+1}$ we may apply the multiplication tangle $M_{k+1, k+1}^{k+1}$ to the outside to get

$$M_{k+1, k+1}^{k+1} \circ_{(D_1, D_2)} (1^{k+1}, I_k^{k+1}) \Rightarrow M_{k+1, k+1}^{k+1} \circ_{(D_1, D_2)} (T^{k+1}, I_k^{k+1}).$$

This may also be written as $I_k^{k+1} \Rightarrow T^{k+1} \circ_{D_2} M_{k,k}^k$. Since $M_{k,k}^k \Rightarrow I_k^k$, we have $T^{k+1} \circ_{D_2} M_{k,k}^k \Rightarrow T^{k+1} \circ_{D_2} I_k^k = T^{k+1}$. Now appeal to transitivity.

(3) Suppose that $n < k$. Then $ER_{n+1}^n \circ T^{n+1} = T^n$, so the asserted result is clear by reflexivity. If $n \geq k$, there are two cases depending on the parity of $n - k$. These cases are shown on the left in Figure 2.5.

We see that each is obtained by inserting a k -tangle into a box of T^n and using the multiplication and conditional expectation templates, this k -tangle, in each case, implies I_k^k .

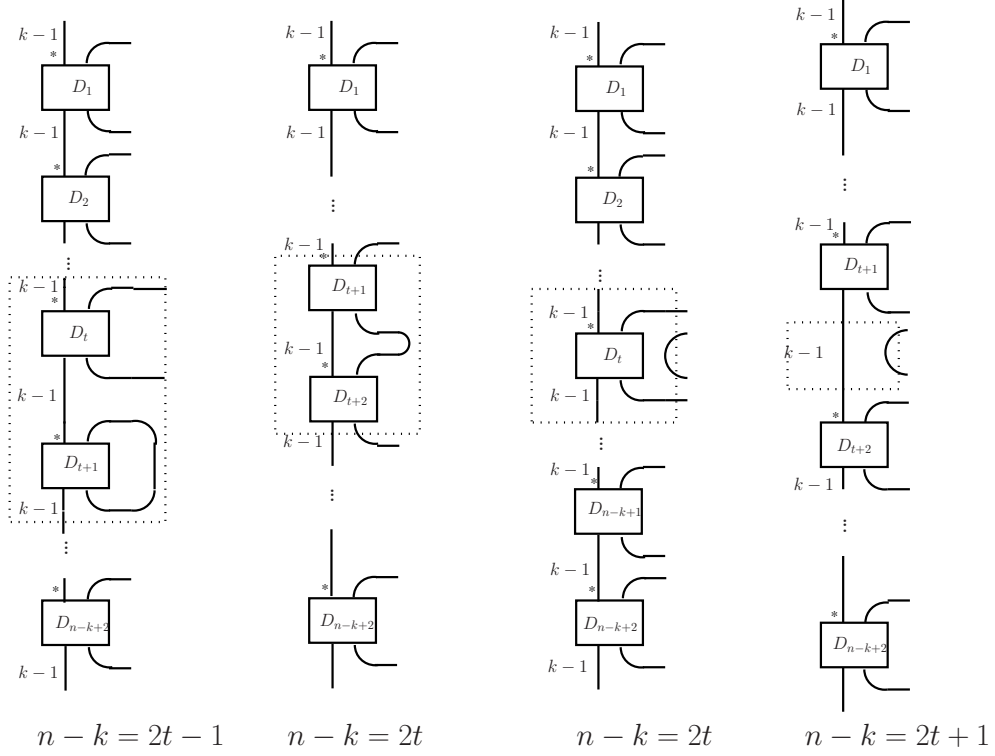


Figure 2.5: $ER_{n+1}^n \circ T^{n+1}$ and $I_n^{n+1} \circ T^n$

(4) Again, there are two cases according to the parity of $n - k$ which are shown on the right in Figure 2.5.

If $n - k = 2t$, we see that $I_n^{n+1} \circ T^n = W \circ I_k^{k+1}$ for a suitable tangle W (where W has a $k + 1$ -box indicated by the dotted line and the rest of it looking like T^n). Note that by (2), $I_k^{k+1} \Rightarrow T^{k+1}$ and therefore $W \circ I_k^{k+1} \Rightarrow W \circ T^{k+1}$. It remains only to note that $W \circ T^{k+1} = T^{n+1}$ and use transitivity to complete the proof in this case. The case $n - k = 2t + 1$ is even easier. Here $I_n^{n+1} \circ T^n = T^{n+1} \circ_{D_{t+2}} 1^k$. Since $1^k \Rightarrow I_k^k$, we get $I_n^{n+1} \circ T^n = T^{n+1} \circ_{D_{t+2}} 1^k \Rightarrow T^{n+1} \circ_{D_{t+2}} I_k^k = T^{n+1}$.

(5) We have by reflexivity that $I_k^k \Rightarrow T^k$. Applying (4) inductively shows that for all $n \geq k$, $I_k^n \Rightarrow T^n$. A similar proof beginning with (1) shows that $1^n \Rightarrow T^n$.

(6) For $n = k$, this is just the multiplication template. For $n > k$, a little doodling should convince the reader that $M_{n,n}^n \circ_{(D_1, D_2)} (T^n, T^n) = ER_{2n-k+1}^n \circ T^{2n-k+1}$. Transitivity, (3) and induction finish the proof.

(7) Begin with the identity template $1^k \Rightarrow I_k^k$ and apply $ER_k^{0\pm}$ to both sides to get $ER_k^{0\pm} \circ 1^k \Rightarrow ER_k^{0\pm} \circ I_k^k = ER_k^{0\pm} = T^{0\pm}$. The left side of this implication is a 0^\pm -tangle which is a collection of loops which may be removed by Lemma 20 to yield $1^{0\pm} \Rightarrow T^{0\pm}$. A very similar proof beginning with the Jones projection templates gives $E^n \Rightarrow T^n$ for $2 \leq n \leq k$. To show that $E^n \Rightarrow T^n$ for $n > k$, consider the following chain of implications.

$$\begin{aligned}
E^n &= ER_{2n-k-1}^n \circ M_{n-1, n-1}^{2n-k-1} \circ_{(D_1, D_2)} (1^{n-1}, 1^{n-1}) \\
&\Rightarrow ER_{2n-k-1}^n \circ M_{n-1, n-1}^{2n-k-1} \circ_{(D_1, D_2)} (T^{n-1}, T^{n-1}) \\
&= ER_{2n-k-1}^n \circ T^{2n-k-1} \\
&\Rightarrow T^n,
\end{aligned}$$

where the first implication is a consequence of (5) and the second, of (3) and induction.

(8) This is an easy corollary of (5), (6) and (7).

(9) Induce on n , with the basis case being asserted by the shift template. For $n > k$,

$$\begin{aligned}
SH_n^{n+2} \circ T^n &= M_{n+1, k+2}^{n+2} \circ_{(D_1, D_2)} (SH_{n-1}^{n+1} \circ T^{n-1}, SH_k^{k+2}) \\
&\Rightarrow M_{n+1, k+2}^{n+2} \circ_{(D_1, D_2)} (T^{n+1}, T^{k+2}) \\
&\Rightarrow T^{n+2},
\end{aligned}$$

where the last implication uses the multiplication and conditional expectation templates together with a suitable outside composition.

(10) First suppose that $n \geq k$. Begin with the conclusion $SH_n^{n+2} \circ T^n \Rightarrow T^{n+2}$ in (9). Let Q^{n+2} and Q^{*n+2} be the Temperley-Lieb tangles shown in Figure 2.6, so that, by (8), $Q^{n+2} \Rightarrow T^{n+2}$ and $Q^{*n+2} \Rightarrow T^{n+2}$. Then, with $M = M_{n+2, n+2, n+2}^{n+2}$

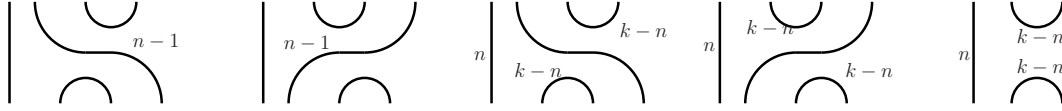


Figure 2.6: The tangles Q^{n+2} , Q^{*n+2} , K^{2k-n+1} , $K^{*2k-n+1}$ and L^{2k-n}

denoting the iterated multiplication tangle we have,

$$M \circ (Q^{n+2}, SH_n^{n+2} \circ T^n, Q^{*n+2}) \Rightarrow M \circ (T^{n+2}, T^{n+2}, T^{n+2}) \Rightarrow T^{n+2}.$$

(For typographical convenience, we have omitted the subscripts to \circ). Hence $ER_{n+2}^n \circ M \circ (Q^{n+2}, SH_n^{n+2}, Q^{*n+2}) \Rightarrow ER_{n+2}^n \circ T^{n+2} \Rightarrow T^n$. The left hand side of this chain of implications is $EL_n^n \circ T^n$ with a loop at its right; therefore, using Lemma 20, we get the desired result. For $1 \leq n < k$, merely apply ER_k^n to both sides of $EL_k^k \circ T^k \Rightarrow T^k$.

(11) In view of (4), we only need consider the case $n < k$. If $n = 0_-$, this is just the case $n = 1$ of (10). So suppose that $0 \leq n < k$. Let $t = 2k - n + 1$. Start with $I_k^t \Rightarrow T^t$ deduced inductively from (4). Let K^t and K^{*t} be the Temperley-Lieb tangles in Figure 2.6 so that by (8), $K^t \Rightarrow T^t$ and $K^{*t} \Rightarrow T^t$. Now, with $M = M_{t,t,t}^t$, $M \circ (K^t, I_k^t, K^{*t}) \Rightarrow M \circ (T^t, T^t, T^t) \Rightarrow T^t$. Applying ER_t^{n+1} to both sides of this and removing the $k - n$ loops that arise on the left hand side, we get the desired conclusion using (3).

(12) In view of (6), we may assume that $n < k$. We first deal with the case $n \neq 0_-$. Let $u = 2k - n$ and $M = M_{u,u,u,u}^u$. Then, with L^u as in Figure 2.6,

$$M \circ (L^u, I_k^u, L^u, I_k^u, L^u) \Rightarrow M \circ (T^u, T^u, T^u, T^u, T^u) \Rightarrow T^u.$$

As in (11), applying ER_u^n to both sides and removing the $k - n$ loops gives the desired conclusion. The case $n = 0_-$ is a little more complicated. Here, let $u = 2k + 1$ and $M = M_{u,u,u,u}^u$. Then, with L^u, L^{*u}, W^u as in Figure 2.7,

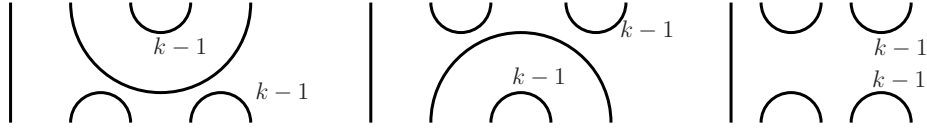


Figure 2.7: The tangles L^{2k+1} , L^{*2k+1} and W^{2k+1}

$$M \circ (L^u, I_{k+2}^u \circ SH_k^{k+2}, W^u, I_{k+2}^u \circ SH_k^{k+2}, L^{*u}) \Rightarrow M \circ (T^u, T^u, T^u, T^u, T^u) \Rightarrow T^u.$$

Now apply ER_u^{0-} to both sides and remove the $k + 1$ loops to get the desired conclusion. \square

2.3 The main theorem

Recall from §2.2, the notion of template holding for a pair (P, B) where P is a planar algebra and $B \subseteq P$. We claim that if P is a subfactor planar algebra of finite depth at most k , and $B = B_k$ is a basis of P_k , then all templates of Figure 2.5 hold for (P, B) . Modulus templates obviously hold, as the planar algebra has non-zero modulus. Jones projection, conditional expectation and multiplication templates hold since their right sides are all the identity tangles I_k^k . For the depth and shift templates, the tangles T^{k+1} and T^{k+2} on their right surject on to their range.

Now, we consider the universal planar algebra, which is defined in §1.5. For the label set $L = \coprod_{n \in Col} L_n$, we denote the universal planar algebra on L by $P(L)$. Thus, $P(L)_n$ is the vector space spanned by all L -labelled n -tangles. For a subset $R \subseteq P(L)$, there is a planar algebra $P(L, R)$, which is the quotient planar algebra $P(L)/I(R)$, where $I(R)$ is the planar ideal generated by the subset $R \subseteq P(L)$.

Let B be the basis of P_k for a subfactor planar algebra P of depth at most k . Let $B^{\times b}$ be the cartesian product of b copies of B for $b > 0$ and $\{1\}$ for $b = 0$. Consider the label set $L = \coprod_{n \in Col} L_n$, where the only non-empty L_n is $L_k = B$. Let $P(L)$ be the universal planar algebra and consider the templates of Figure 2.5. We now

specify a subset $R \subseteq P(L)$. Suppose $S \Rightarrow T$ is a template which holds for (P, B) and that S has b internal boxes and T has c internal boxes. Here the colour of each internal boxes (if any) of S and T is k . Given $(x_1, x_2, \dots, x_b) \in B^{\times b}$ we write

$$Z_S^P(x_1 \otimes \cdots \otimes x_b) = \sum_{\{(y_1, \dots, y_c) \in B^{\times c}\}} \lambda^{(y_1, \dots, y_c)} Z_T^P(y_1 \otimes \cdots \otimes y_c),$$

where $\lambda^{(y_1, \dots, y_c)} \in \mathbb{C}$. This is possible as $S \Rightarrow T$ holds for (P, B) .

Now, we consider the following element of $P(L)$:

$$S(x_1, \dots, x_b) - \sum_{\{(y_1, \dots, y_c) \in B^{\times c}\}} \lambda^{(y_1, \dots, y_c)} T(y_1, \dots, y_c),$$

where $S(x_1, \dots, x_b)$ denotes the tangle S with boxes labelled x_1, \dots, x_b etc. For each $(x_1, x_2, \dots, x_b) \in B^{\times b}$, we get one such element for a template $S \Rightarrow T$ (the value $\lambda^{(y_1, \dots, y_c)} \in \mathbb{C}$ may not be uniquely determined). Consider the collection consisting of one such element of $P(L)$ for each $(x_1, \dots, x_b) \in B^{\times b}$ and take the union of these collections over all templates $S \Rightarrow T$ of Figure 2.4. This (clearly finite) subset of $P(L)$ is what we will call R . Note that R is not a uniquely determined set but depends on choices. We will call this a set of relations determined by the templates of Figure 2.4.

Theorem 22. *Let P be a subfactor planar algebra of finite depth at most k . Let B be a fixed basis of P_k . Consider the labelling set $L = \coprod_{n \in \text{Col}} L_n$ where the only non-empty L_n is $L_k = B$. Let R be any (necessarily finite) set of relations in $P(L)$ determined by the templates in Figure 2.4. Then, the quotient planar algebra $P(L, R) \cong P$.*

Proof. Consider the natural surjective planar algebra morphism from the universal planar algebra $P(L)$ to P defined uniquely by taking a labelled k -box to itself regarded as an element of P . Equivalently, under this morphism, for any tangle

S all of whose internal boxes are of colour k , $S(x_1, \dots, x_b) \mapsto Z_S^P(x_1 \otimes \dots \otimes x_b)$. Since the relations R were chosen to hold in P , this morphism factors through the quotient planar algebra $P(L, R)$ thus yielding a surjective planar algebra morphism $P(L, R) \rightarrow P$. We wish to see that this is an isomorphism.

For $n \in Col$, let Q_n be the subspace of $P(L, R)_n$ spanned by all $Z_{T^n}^{P(L, R)}(x_1 \otimes x_2 \otimes \dots \otimes x_{n-k+1})$ for $x_1, \dots, x_{n-k+1} \in B$, if $n \geq k$ or the subspace spanned by all $Z_{T^n}^{P(L, R)}(x)$ for $x \in B$, if $n < k$. Let \mathcal{T} be the set of all $(n_0; n_1, \dots, n_b)$ tangles T such that (i) if $b > 0$, then $Z_T(Q_{n_1} \otimes \dots \otimes Q_{n_b}) \subseteq Q_{n_0}$, and (ii) if $b = 0$, then $Z_T(1) \in Q_{n_0}$. Chasing definitions shows that \mathcal{T} may be equivalently described as the set of $(n_0; n_1, \dots, n_b)$ -tangles T for which $T \circ_{(D_1, \dots, D_b)} (T^{n_1}, \dots, T^{n_b}) \Rightarrow T^{n_0}$ holds for $(P(L, R), B)$. We will show that \mathcal{T} consists of all tangles, or equivalently, that Q is a planar subalgebra of $P(L, R)$.

For this, we appeal to the main result of [7] which states that if \mathcal{T} is a collection of tangles that is closed under composition (whenever it makes sense) and contains the tangles $1^{0\pm}$, E^n for $n \geq 2$, ER_{n+1}^n , $M_{n,n}^n$, I_n^{n+1} for all $n \in Col$ and EL_n^n for all $n \geq 1$, then \mathcal{T} contains all tangles.

To verify the hypotheses for our \mathcal{T} , observe first that by definition if $T \in \mathcal{T}$ is a $(n_0; n_1, \dots, n_b)$ tangle and $S \in \mathcal{T}$ is any n_i -tangle for $i > 0$, then, $T \circ_{D_i} S \in \mathcal{T}$. Thus \mathcal{T} is closed under composition. That the other hypotheses hold for \mathcal{T} follows from the observation that the templates of Figure 2.4 hold for $(P(L, R), B)$ by construction of R and therefore their consequences (3),(7),(10),(11),(12) of Theorem 21 also hold.

It follows that Q is a planar subalgebra of $P(L, R)$. Since it contains all generators of $P(L, R)$, it is the whole of $P(L, R)$. In particular, $P(L, R)_k$ which maps onto P_k equals Q_k which is spanned by B and so $P(L, R)_k$ maps isomorphically onto P_k . It easily follows that the map $P(L, R)_n \rightarrow P_n$ is an isomorphism for $n \leq k$.

For $n \geq k$, observe that Corollary 16 applies to $P(L, R)$ since the depth template

holds for $(P(L, R), B)$. Hence we have an isomorphism of $P(L, R)_{k-1} - P(L, R)_{k-1}$ -bimodules

$$P(L, R)_k \otimes_{P(L, R)_{k-1}} P(L, R)_k \otimes_{P(L, R)_{k-1}} \cdots \otimes_{P(L, R)_{k-1}} P(L, R)_k \rightarrow P(L, R)_n,$$

and therefore an isomorphism of $P_{k-1} - P_{k-1}$ -bimodules

$$P_k \otimes_{P_{k-1}} P_k \otimes_{P_{k-1}} \cdots \otimes_{P_{k-1}} P_k \rightarrow P(L, R)_n.$$

Since the left side is, by Corollary 16 applied to P , isomorphic to P_n while the right side maps onto P_n , it follows that $P(L, R)_n$ maps isomorphically to P_n also for all $n \geq k$. □

2.4 On single generation

Rather surprisingly, the fact that finite depth subfactor planar algebras are singly generated has a simple proof.

Proposition 23. *Let P be a subfactor planar algebra of finite depth at most k . Then P is generated by a single $2k$ -box.*

Proof. As a planar algebra, P is generated by P_k . Since P_k is a finite-dimensional C^* -algebra, it is singly generated, by say $x \in P_k$. By adding a multiple of 1_k to x , we may assume without loss of generality that $\tau(x) \neq 0$ (recall that $\tau(\cdot)$ is the normalised picture trace on P). Thus the planar algebra generated by x and x^* which certainly contains the subalgebra of P_k generated by them, contains P_k , and so must be the whole of P . Now consider the element $z \in P_{2k}$ defined by Figure 2.8. It should be clear that applying suitable annular tangles to z yields non-zero (since $\tau(x) \neq 0$) multiples of x and x^* . Hence the planar subalgebra of P generated by z contains both x and x^* and consequently is P . □

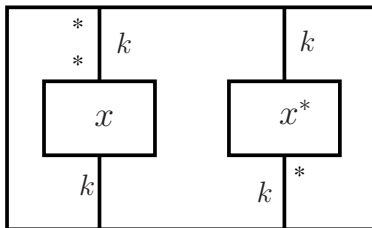


Figure 2.8: Definition of $z \in P_{2k}$

It is natural to ask whether, when a finite depth planar algebra P is presented as a quotient of a singly generated planar algebra as above, the kernel is a finitely generated planar ideal. A standard proof shows that this is indeed so.

Proposition 24. *Suppose P be a planar algebra and let L and \tilde{L} be finite label sets. If $\pi : P(L) \rightarrow P$ and $\tilde{\pi} : P(\tilde{L}) \rightarrow P$ are surjective planar algebra maps, then the ideal $I = \ker(\pi)$ is a finitely generated planar ideal of $P(L)$ if and only if $\tilde{I} = \ker(\tilde{\pi})$ is a finitely generated planar ideal of $P(\tilde{L})$.*

Proof. First note that universality of $P(L)$ and $P(\tilde{L})$ yield (possibly non-unique) planar algebra maps $\phi : P(L) \rightarrow P(\tilde{L})$ and $\tilde{\phi} : P(\tilde{L}) \rightarrow P(L)$ that satisfy $\tilde{\pi} \circ \phi = \pi$ and $\pi \circ \tilde{\phi} = \tilde{\pi}$.

By symmetry, it suffices to prove one implication. Suppose that $\tilde{I} = I(\tilde{R})$ for a finite subset $\tilde{R} \subseteq P(\tilde{L})$. Let $R = \tilde{\phi}(\tilde{R}) \cup \{x - \tilde{\phi}\phi(x) : x \in L\}$, which is clearly a finite subset of $P(L)$. We claim that $I = I(R)$.

Clearly $R \subseteq I$ and so $I(R) \subseteq I$. The other inclusion needs a little work. First observe that $\{x - \tilde{\phi}\phi(x) : x \in L\} \subseteq R$ implies that for all $z \in P(L)$, $z - \tilde{\phi}\phi(z) \in I(R)$. To see this we may reduce easily to the case that $z = T(x_1, \dots, x_b)$ where T is a $(n_0; n_1, \dots, n_b)$ -tangle and $x_i \in L_{n_i}$. Then

$$z - \tilde{\phi}\phi(z) = Z_T^{P(L)}(x_1 \otimes \dots \otimes x_b) - Z_T^{P(L)}(\tilde{\phi}\phi(x_1) \otimes \dots \otimes \tilde{\phi}\phi(x_b)).$$

This may be expressed as a telescoping sum of b terms indexed by $k = 1, 2, \dots, b$

where the k^{th} term is given by

$$Z_T^{P(L)}(\tilde{\phi}\phi(x_1) \otimes \cdots \otimes \tilde{\phi}\phi(x_{k-1}) \otimes (x_k - \tilde{\phi}\phi(x_k)) \otimes x_{k+1} \otimes \cdots \otimes x_b)$$

Each of these terms is clearly in the planar ideal generated by $\{x - \tilde{\phi}\phi(x) : x \in L\}$ and hence in $I(R)$. Therefore $z - \tilde{\phi}\phi(z) \in I(R)$.

Say $z \in I$, so that $\pi(z) = 0$. Then $\phi(z) \in \ker(\tilde{\pi}) = \tilde{I} = I(\tilde{R})$, i.e., $\phi(z)$ is in the planar ideal generated by \tilde{R} . It follows that $\tilde{\phi}\phi(z)$ is in the planar ideal generated by $\tilde{\phi}(\tilde{R})$ and therefore in $I(R)$. Since $z - \tilde{\phi}\phi(z) \in I(R)$, we also have $z \in I(R)$ and the proof is finished. \square

A direct consequence of Theorem 22 and Propositions 23 and 24 is the following corollary.

Corollary 25. *If P is a subfactor planar algebra of finite depth at most k , then P is generated by a single $2k$ -box subject to finitely many relations.* \square

Chapter 3

On the single generation of a finite depth subfactor planar algebra

In the previous chapter, we have seen that a subfactor planar algebra P of depth k can be generated as a planar algebra by a single element in P_{2k} . In this chapter we will show that numbers less than $2k$ work. More precisely, let $2t$ be the even number of $k + 3$ and $k + 4$. We prove that P is generated as a planar algebra by a single element of P_s where $s = \min\{2k, 2t\}$.

3.1 Generation of complex semisimple algebras

The main fact which we require is the following proposition about finite-dimensional complex semisimple algebras. Algebra automorphisms and anti-automorphisms will always refer to unital \mathbb{C} -algebra automorphisms and anti-automorphisms.

Recall that by Wedderburn's theorem any finite-dimensional complex semisimple algebra is a direct sum of matrix algebras over \mathbb{C} .

Proposition 26. *Let A be a finite-dimensional complex semisimple algebra without*

an $M_2(\mathbb{C})$ summand and let $S : A \rightarrow A$ be an involutive algebra anti-automorphism. Then there exists $a \in A$ such that a and Sa generate A as an algebra.

Before taking up the proof of Proposition 26, we remark that the condition of not having an $M_2(\mathbb{C})$ summand is necessary. For, if S is the involutive algebra anti-automorphism of $M_2(\mathbb{C})$ given by $Sa = \text{adj}(a)$, then, for no $a \in M_2(\mathbb{C})$ do a and Sa generate it, since a and $\text{adj}(a)$ always commute. The proof of Proposition 26 relies on a series of auxiliary results which we will first prove.

Lemma 27. *Let S be an involutive algebra anti-automorphism of $M_n(\mathbb{C})$. Then there is an algebra automorphism of $M_n(\mathbb{C})$ under which S is identified with either (i) the transpose map or (ii) the map $a \mapsto Ja^T J^{-1}$, where T is the transpose map and*

$$J = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix},$$

where I_k is the identity matrix in $M_k(\mathbb{C})$. The second case may arise only when $n = 2k$ is even.

Proof. With T denoting the transpose map on $M_n(\mathbb{C})$, as above, note that TS is an algebra automorphism of $M_n(\mathbb{C})$. Hence it is given by conjugation by an invertible matrix, say u . Thus $Sx = (uxu^{-1})^T$. Involutivity of S implies that $(u((uxu^{-1})^T)u^{-1})^T = x$ for all x in $M_n(\mathbb{C})$ and therefore $xu^{-1}u^T = u^{-1}u^T x$ for all x in $M_n(\mathbb{C})$. It follows that $u^{-1}u^T$ is a scalar matrix, say $u^{-1}u^T = \lambda I_n$. From this we get $u^T = \lambda u$ and $\lambda = \pm 1$. Therefore $u^T = u$ or $u^T = -u$. So u is either symmetric or skew-symmetric. By Takagi's factorization (see p204 and p217 of [4]), u is of the form $v^T v$ if it is symmetric and of the form $v^T J v$ if it is skew-symmetric, for some invertible v . In the symmetric case the algebra automorphism of $M_n(\mathbb{C})$ gets identified with the transpose map via the automorphism ad_v (where $ad_v(a) = vav^{-1}$)

as shown below.

$$\begin{array}{ccc} M_n(\mathbb{C}) & \xrightarrow{S} & M_n(\mathbb{C}) \\ \downarrow ad_v & & \downarrow ad_v \\ M_n(\mathbb{C}) & \xrightarrow{T} & M_n(\mathbb{C}) \end{array}$$

If u is skew-symmetric then $Sx = (v^T J v x (v^T J v)^{-1})^T$ and $v^T J^T v Sx = x^T v^T J^T v$. In this case S can be identified with $a \mapsto J a^T J^{-1}$ as shown below.

$$\begin{array}{ccc} M_n(\mathbb{C}) & \xrightarrow{S} & M_n(\mathbb{C}) \\ \downarrow ad_v & & \downarrow ad_v \\ M_n(\mathbb{C}) & \xrightarrow{a \mapsto J a^T J^{-1}} & M_n(\mathbb{C}) \end{array}$$

□

Before we prove the next proposition, we will need a couple of facts. One is a simple corollary of the finite dimensional case of von Neumann's double commutant theorem, which we will just state, and the other is a certain Zariski density result, which we supply a proof of.

Proposition 28. *Suppose $x_1, x_2, \dots, x_t \in M_n(\mathbb{C})$. Then, the unital $*$ -algebra generated by these is $M_n(\mathbb{C})$ if and only if the only elements that commute with $x_1, \dots, x_t, x_1^*, \dots, x_t^*$ are the scalar matrices.*

Proposition 29. *The set $D = \{(z_1, z_2, \dots, z_n, w_1, w_2, \dots, w_n) \in \mathbb{C}^{2n} : \bar{z}_i = w_i\}$ is Zariski dense in \mathbb{C}^{2n} .*

The proof of Proposition 29 appeals to the following simple lemma which we first prove.

Lemma 30. *Suppose that $f \in \mathbb{C}[z_1, z_2, \dots, z_n, w_1, w_2, \dots, w_n]$ and that*

$$f(z_1, z_2, \dots, z_n, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_n) = 0$$

for all $z_1, \dots, z_n \in \mathbb{C}$. Then $f = 0$ identically.

Proof. We first prove by induction on n that if $f \in \mathbb{C}[z_1, z_2, \dots, z_n]$ vanishes on \mathbb{R}^n , then $f = 0$ identically. Let $n = 1$. If $f(z)$ is a polynomial with complex coefficients that vanishes on \mathbb{R} then $f = 0$ identically, as f is an analytic function and it vanishes on a set containing a limit point. Suppose we assume that the statement holds for $n = k$. Now suppose $n = k + 1$, and that $f \in \mathbb{C}[z_1, z_2, \dots, z_{k+1}]$ vanishes on \mathbb{R}^{k+1} . Write $f(z_1, z_2, \dots, z_{k+1}) = z_{k+1}^d g_d(z_1, \dots, z_k) + z_{k+1}^{d-1} g_{d-1}(z_1, \dots, z_k) + \dots + g_0(z_1, \dots, z_k)$, where $g_j(z_1, \dots, z_k) \in \mathbb{C}[z_1, z_2, \dots, z_k]$.

For a fixed value of (z_1, \dots, z_k) in \mathbb{R}^k , f vanishes for all $z_{k+1} \in \mathbb{R}$. This implies that each g_j vanishes on \mathbb{R}^k . By the inductive hypothesis, each $g_j = 0$ identically. From this it follows that $f = 0$ identically.

Now suppose that $f \in \mathbb{C}[z_1, \dots, z_n, w_1, \dots, w_n]$ and that f vanishes whenever $w_i = \bar{z}_i$ for $1 \leq i \leq n$. Then let

$$p(u_1, \dots, u_n, v_1, \dots, v_n) = f(u_1 + iv_1, \dots, u_n + iv_n, u_1 - iv_1, \dots, u_n - iv_n)$$

so that

$$f(z_1, \dots, z_n, w_1, \dots, w_n) = p\left(\frac{z_1 + w_1}{2}, \dots, \frac{z_n + w_n}{2}, \frac{z_1 - w_1}{2i}, \dots, \frac{z_n - w_n}{2i}\right).$$

The assumption on f implies that p vanishes on \mathbb{R}^{2n} . So $p = 0$ identically, which implies $f = 0$ identically. \square

Proof of Proposition 29: Suppose, if possible, that there is a non-empty Zariski open subset U of \mathbb{C}^{2n} which does not intersect D . Then the complement of U contains D . Since U^c is Zariski closed in \mathbb{C}^{2n} and not the whole of \mathbb{C}^{2n} , there is a non-zero polynomial $f \in \mathbb{C}[z_1, \dots, z_n, w_1, \dots, w_n]$ which vanishes on U^c and hence on D . But this is impossible by Lemma 30. \square

Proposition 31. *Let S be an involutive algebra anti-automorphism of $M_n(\mathbb{C})$, where $n \neq 2$. Then there exists an invertible $x \in M_n(\mathbb{C})$ such that the algebra generated by x and Sx is $M_n(\mathbb{C})$.*

Proof. It suffices, by Lemma 27, to check that when S is the transpose map or the J -conjugate of the transpose map, some x and Sx generate $M_n(\mathbb{C})$. When S is the transpose map, then, if we consider the matrix $x = I + N$, where N is the nilpotent matrix with all 1's on the superdiagonal and 0's elsewhere, it is easy to see that that x and x^T generate $M_n(\mathbb{C})$ as a unital algebra. We note for future reference that this is valid even when $n = 2$.

Next we consider the case when S is $a \mapsto Ja^TJ^{-1}$. In this case, $n = 2k$ is even and, by assumption, $k > 1$. Define the following subset S of $M_n(\mathbb{C})$.

$$S = \left\{ \begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix} : P, Q \in M_k(\mathbb{C}) \text{ and } \left\{ \begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix}, \begin{bmatrix} 0 & P^T \\ Q^T & 0 \end{bmatrix} \right\}' = \mathbb{C}I_n \right\}.$$

The notation $'$ in the definition above stands for the commutant as usual.

We claim that S is a Zariski open non-empty subset of \mathbb{C}^{2k^2} . To show that S is open, consider its complement which consists of all $\begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix}$ such that there exists

a non-scalar matrix $\begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$ satisfying the equations

$$\begin{aligned} \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix} &= \begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix} \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}, \\ \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \begin{bmatrix} 0 & P^T \\ Q^T & 0 \end{bmatrix} &= \begin{bmatrix} 0 & P^T \\ Q^T & 0 \end{bmatrix} \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}. \end{aligned}$$

These matrix equations are equivalent to the following set of matrix equations for X, Y, Z, W .

$$\begin{aligned} YQ &= PZ & YQ^T &= P^T Z \\ XP &= PW & XP^T &= P^T W \\ WQ &= QX & WQ^T &= Q^T X \\ ZP &= QY & ZP^T &= Q^T Y \end{aligned}$$

We may regard these as a system of $8k^2$ homogeneous equations in the $4k^2$ variables that are the entries of the matrices X, Y, Z and W . The coefficient matrix for this homogeneous system is a $8k^2 \times 4k^2$ matrix with entries from the matrices P, Q .

The condition that there exist a non-scalar matrix $\begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$ satisfying these equations is equivalent to the coefficient matrix having nullity at least 2 or equivalently rank at most $4k^2 - 2$. This is clearly a Zariski closed condition in the entries of P and Q .

Next, we prove that S is non empty. Consider the matrix $\begin{bmatrix} 0 & I \\ Q & 0 \end{bmatrix}$ where Q is invertible and Q and Q^T generate $M_k(\mathbb{C})$ as an algebra. For instance, we may choose $Q = I + N$, as before. The conditions that $\begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$ commutes with $\begin{bmatrix} 0 & I \\ Q & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & I \\ Q^T & 0 \end{bmatrix}$ are equivalent to the following set of equations

$$\begin{aligned} YQ &= Z & YQ^T &= Z \\ X &= W & X &= W \\ WQ &= QX & WQ^T &= Q^T X \\ Z &= QY & Z &= Q^T Y. \end{aligned}$$

Since Q and Q^T commute with Y and generate $M_k(\mathbb{C})$, it follows that $Y = \lambda I$ where λ is a complex number. Then $Z = \lambda Q = \lambda Q^T$. Now observe that since $k > 1$ and Q and Q^T generate $M_k(\mathbb{C})$, they cannot be equal and so $\lambda = 0$. Hence $Y = Z = 0$. As $X = W$ and X commutes with both Q and Q^T , so $X = W = \mu I$, where μ is a complex number. Thus $\begin{bmatrix} 0 & I \\ Q & 0 \end{bmatrix}$ is in S so that S is non-empty. Further, since the invertible matrices in $M_k(\mathbb{C})$ also form a non-empty and Zariski open subset, we see that

$$\left\{ \begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix} \in S : P, Q \in M_k(\mathbb{C}) \text{ invertible} \right\}$$

is also a non-empty and Zariski open subset of \mathbb{C}^{2k^2} .

We now appeal to Proposition 29 to conclude that there exists an invertible $P \in M_k(\mathbb{C})$ such that $x = \begin{bmatrix} 0 & P \\ \overline{P} & 0 \end{bmatrix}$ is in S . Clearly x is invertible and

$$Sx = Jx^T J^{-1} = \begin{bmatrix} 0 & -P^T \\ -\overline{P}^T & 0 \end{bmatrix} = -x^*.$$

Thus the unital algebra generated by x and Sx is a $*$ -subalgebra of $M_n(\mathbb{C})$ and, by definition of the set S , has commutant just the scalar matrices. By Proposition 28, the unital algebra generated by x and Sx is the whole of $M_n(\mathbb{C})$. Finally, note that as x is invertible, the algebra generated by x contains the unit and therefore the algebra generated by x and Sx is $M_n(\mathbb{C})$, as desired. \square

We will next proceed towards proving an analogue of Proposition 31 for $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$. This needs two preparatory lemmas. The first is an analogue of Proposition 27.

Lemma 32. *Let S be an involutive algebra anti-automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ that interchanges the two minimal central projections. There is an algebra automor-*

phism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ fixing the minimal central projections under which S is identified with the map $x \oplus y \rightarrow y^T \oplus x^T$.

Proof. The map $x \oplus y \rightarrow S(y^T \oplus x^T)$ is an algebra automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ fixing the minimal central projections and is therefore given by $x \oplus y \rightarrow u x u^{-1} \oplus v y v^{-1}$ for invertible u, v (determined upto non-zero scalar multiplication). Hence $S(x \oplus y) = u y^T u^{-1} \oplus v x^T v^{-1}$. By involutivity of S , we get $S^2(x \oplus y) = S(u y^T u^{-1} \oplus v x^T v^{-1}) = u(v x^T v^{-1})^T u^{-1} \oplus v(u y^T u^{-1})^T v^{-1} = x \oplus y$. From this we get $u(v^{-1})^T = \lambda I$ for some non-zero $\lambda \in \mathbb{C}$, which implies $u = \lambda v^T$. Since u and v are determined only upto scaling, we may assume that $u = v^T$ and therefore $S(x \oplus y) = u y^T u^{-1} \oplus u^T x^T (u^T)^{-1}$. It is now easy to check that under the algebra automorphism $ad_{u^{-1}} \otimes id$ of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ given by $x \oplus y \rightarrow u^{-1} x u \oplus y$, S is identified with $x \oplus y \rightarrow y^T \oplus x^T$,

$$\begin{array}{ccc} M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) & \xrightarrow{S} & M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \\ \downarrow ad_{u^{-1}} \otimes id & & \downarrow ad_{u^{-1}} \otimes id \\ M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) & \xrightarrow{x \oplus y \rightarrow y^T \oplus x^T} & M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \end{array}$$

as seen by the commutativity of the diagram above. \square

Lemma 33. *Let A and B be finite-dimensional complex unital algebras and let $a \in A$ and $b \in B$ be invertible. Then, for all but finitely many $\lambda \in \mathbb{C}$, the algebra generated by $a \oplus \lambda b \in A \oplus B$ contains both a and b .*

Proof. We may assume that $\lambda \neq 0$ and then it suffices to see that a is expressible as a polynomial in $a \oplus \lambda b$. Note that since $a \oplus \lambda b$ is invertible and $A \oplus B$ is finite dimensional, the algebra generated by $a \oplus \lambda b$ is actually unital. In particular, it makes sense to evaluate any complex univariate polynomial on $a \oplus \lambda b$. Let $p(x)$ and $q(x)$ be the minimal polynomials of a and b respectively. By invertibility of a and b , neither p nor q has 0 as a root. The minimal polynomial of λb is $\tilde{q}(x) = \lambda^{\deg(q)} q(\frac{x}{\lambda})$. Unless λ is the quotient of a root of p and a root of q , $p(x)$ and $\tilde{q}(x)$ will not have a common root and will hence be coprime. So there exist polynomials $a(x)$ and $b(x)$

such that $1 = a(x)p(x) + b(x)\tilde{q}(x)$. Let $r(x) = 1 - a(x)p(x)$. Then $r(a \oplus \lambda b) = a \oplus 0$.

Hence the lemma is proved. \square

The next proposition is the desired analogue of Proposition 31.

Proposition 34. *Let S be an involutive algebra anti-automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ that interchanges the two minimal central projections. Then, there is an invertible $x \oplus y \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ which together with $S(x \oplus y)$ generates $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ as an algebra.*

Proof. By Proposition 32, it suffices to check that some invertible $x \oplus y$ and $y^T \oplus x^T$ generate $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ as an algebra. Note that from the proof of Proposition 31, there is an invertible $x \in M_n(\mathbb{C})$ such that x and x^T generate $M_n(\mathbb{C})$ as an algebra. By Lemma 33, for all but finitely many $\lambda \in \mathbb{C}$, the algebra generated by $x \oplus \lambda x$ contains $x \oplus 0$ and $0 \oplus x$ and similarly the algebra generated by $\lambda x^T \oplus x^T$ contains $x^T \oplus 0$ and $0 \oplus x^T$. Thus the algebra generated by $x \oplus \lambda x$ and $\lambda x^T \oplus x^T$ is the whole of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$. \square

Lemma 35. *Suppose A is a finite dimensional complex semisimple algebra. Let S be an algebra anti-automorphism of A . Let $L_x : A \rightarrow A$ be the \mathbb{C} -linear map such that $L_x(a) = xa$. Similarly let $R_x : A \rightarrow A$ be the \mathbb{C} -linear map such that $R_x(a) = ax$. Then $\text{trace}(L_x) = \text{trace}(R_{Sx})$.*

Proof. Let f_1, f_2, \dots, f_p be a basis of A as a vector space over \mathbb{C} . Let $L_x : A \rightarrow A$ be the map such that $L_x(a) = xa$. Similarly we have a map $R_x : A \rightarrow A$ such that $R_x(a) = ax$. Clearly both L_x, R_x are \mathbb{C} -linear maps. Let $M = [a_{ij}]_{1 \leq i, j \leq p}$ be the matrix for the linear map L_x with respect to the basis f_1, f_2, \dots, f_p . So $L_x(f_i) = x f_i = \sum_{j=1}^p a_{ji} f_j$ for $1 \leq i \leq p$. Now $S(x f_i) = S(f_i) S(x) = \sum_{j=1}^p S(a_{ji} f_j) = \sum_{j=1}^p a_{ji} S(f_j)$ for $1 \leq i \leq p$. For this it follows that the matrix for the linear transformation R_{Sx} with respect to the basis $S f_1, S f_2, \dots, S f_p$ is M . So we get $\text{trace}(L_x) = \text{trace}(R_{Sx})$. \square

Proof of Proposition 26. Let \hat{A} be the finite set of all distinct irreducible representations of A . For $\pi \in \hat{A}$, let d_π denote its dimension, so that, by assumption, no $d_\pi = 2$. Thus, $A \cong \bigoplus_{\pi \in \hat{A}} M_{d_\pi}(\mathbb{C})$. Let $\{e_{ij}^\pi\}_{\pi \in \hat{A}, 1 \leq i, j \leq d_\pi}$ be a set of matrix units for A . Computing with respect to this basis of A , it is clear that the linear maps $L_{e_{ij}^\pi}$ and $R_{e_{ij}^\pi}$ both have trace 0 if $i \neq j$ and d_π if $i = j$.

Let e^π be the minimal central projection corresponding to $\pi \in \hat{A}$, so that $e^\pi = \sum_{i=1}^{d_\pi} e_{ii}^\pi$. Then the traces of both L_{e^π} and R_{e^π} are equal to d_π^2 . Since S is an involutive anti-automorphism, it acts as an involution on the set of minimal central projections of A and on the set of matrix summands of A . Suppose that $S(e^\pi) = e^{\pi'}$. From Lemma 35 it follows that $\text{trace}(L_{e^\pi}) = \text{trace}(R_{e^{\pi'}})$. Thus $d_\pi = d_{\pi'}$ and $S(M_{d_\pi}(\mathbb{C})) = M_{d_{\pi'}}(\mathbb{C})$.

We now conclude that there exist subsets \hat{A}_1 and \hat{A}_2 of \hat{A} and an identification

$$A \rightarrow \bigoplus_{\pi \in \hat{A}_1} M_{d_\pi}(\mathbb{C}) \oplus \bigoplus_{\pi \in \hat{A}_2} (M_{d_\pi}(\mathbb{C}) \oplus M_{d_\pi}(\mathbb{C}))$$

such that each summand is S -stable. Now, by Proposition 31 and Corollary 34, in each summand of the above decomposition, either $M_{d_\pi}(\mathbb{C})$ or $M_{d_\pi}(\mathbb{C}) \oplus M_{d_\pi}(\mathbb{C})$, there is an invertible element which together with its image under S generates that summand as an algebra. Finally, an inductive application of Lemma 33 shows that there exists an element $a \in A$ such that a and Sa generate A as an algebra. This completes the proof of Proposition 26. \square

3.2 The main theorem

Before we prove the main result of this chapter, we will need a result about connected pointed bipartite graphs. Recall that a bipartite graph has its vertex set partitioned into two sets normally called the ‘even’ and ‘odd’ vertices and is such that all edges in

the graph connect an even to an odd vertex. It is said to be pointed if a certain even vertex, normally denoted by $*$, is distinguished. The depth of a pointed bipartite graph is the largest distance of a vertex from $*$.

Proposition 36. *Suppose Γ be a connected pointed bipartite graph of depth $k \geq 3$. For any vertex v of Γ , let t be the one of $k + 3, k + 4$ with the same parity as v . The number of paths of length t from $*$ to v is at least 3.*

Proof. We now analyse three cases depending on the distance of v from $*$.

Case I: If $v = *$, note that $t \geq 6$ is even. To show that there are at least 3 paths of length t from $*$ to $*$, it suffices to show that there are at least 3 paths of length 6 from $*$ to $*$. Since $k \geq 3$, choose any vertex at distance 2 from $*$ and a path from $*$ to the chosen vertex. It is easy to see that there are at least 3 paths of length 6 from $*$ to $*$ supported on the edges of this path.

Case II: If v is at distance 1 from $*$, then $t \geq 7$ is odd. As observed in Case I, there are at least 3 paths of length 6 from $*$ to $*$ and consequently at least 3 paths of length 7 from $*$ to v .

Case III: Suppose v is at a distance n from $*$, where $n > 1$. Observe that if n and k have the same parity, then $n \leq k$ while in the other case, $n \leq k - 1$. Choose a path $\xi_1 \xi_2 \xi_3 \cdots \xi_n$ from $*$ to v . Then there exists an ξ_i such that $\xi_{i+1} \neq \overline{\xi_i}$, where $\overline{\xi_i}$ is the corresponding edge in the opposite direction. Then we have three paths $\xi_1 \xi_2 \cdots \xi_i \overline{\xi_i} \xi_i \overline{\xi_i} \xi_i \xi_{i+1} \cdots \xi_n$, $\xi_1 \xi_2 \cdots \xi_i \xi_{i+1} \overline{\xi_{i+1}} \xi_{i+1} \overline{\xi_{i+1}} \xi_{i+1} \cdots \xi_n$, and $\xi_1 \xi_2 \cdots \xi_i \overline{\xi_i} \xi_i \xi_{i+1} \overline{\xi_{i+1}} \xi_{i+1} \cdots \xi_n$ of length $n + 4$ from $*$ to v . Thus if n and k have the same parity, so that $t = k + 4$, then there exist at least 3 paths of length t from $*$ to v . If n and k have opposite parity then $t = k + 3$ and since $n \leq k - 1$ in this case, again there exist at least 3 paths of length $n + 4 = t$ from $*$ to v . \square

Now we prove the main theorem of this chapter.

Theorem 37. *Let P be a subfactor planar algebra of finite depth k . Let s be the*

even number in $\{k+3, k+4\}$. Let $t = \min\{2k, s\}$. Then P is generated by a single t -box.

Proof. Case I: If $k \leq 3$, $t = 2k$. Then by Proposition 23, P is generated by a single t box.

Case II: If $k > 3$, let Γ be the principal graph of the subfactor planar algebra of depth k . Then from Proposition 36, the number of paths of length s from the $*$ - vertex to any even vertex v in Γ is at least 3. So P_s does not have an $M_2(\mathbb{C})$ summand. Consider the $\frac{s}{2}$ th power of the s -rotation tangle, say X . This tangle changes the position of $*$ on an s -box from the top left to the bottom right position. Clearly $Z_X^P : P_s \rightarrow P_s$ is an involutive algebra anti-automorphism. From Theorem 26, there exists an element a such that a and $R_s(a)$ generate P_s as a unital algebra. Since $s \geq k$, the planar algebra generated by P_s contains P_k and thus is the whole of P . Hence the single s -box containing a generates the planar algebra P . \square

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