EXTENDABLE ENDOMORPHISMS OF FACTORS

By

PANCHUGOPAL BIKRAM

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The Institute of Mathematical Sciences, Chennai

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As members of the Viva Voce Board, we certify that we have read the dissertation prepared by Panchugopal Bikram entitled "Extendable Endomorphisms of Factors" and recommend that it may be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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____ Date:

Member 1 - Vijay Kodiyalam

____ Date:

Member 2 - Krishna Maddaly

____ Date:

Member 3 - Amritanshu Prasad

____ Date:

External Examiner - B V Rajarama Bhat

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DECLARATION

I hereby declare that this thesis entitled " Extendable Endomorphisms of Factors" submitted by me for the degree of Doctor of Philosophy in Mathematics is the record of original work done by me under the supervision of Professor V. S. Sunder during the period 2008 to 2013. The work is original and has not been submitted earlier as a whole or in part of a degree/diploma at this or any other Institution/University.

Panchugopal Bikram

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First of all, I would like to express my deep gratitude to my advisor V. S. Sunder who is more than a teacher to me. He has always encouraged me and helped me with wonderful insights and ideas whenever needed. Without his help this thesis would never have seen the light of the day.

I am extremely grateful to R. Srinivasan who is like my co-advisor from Chennai Mathematical Institute. Right from my beginning of the study of my research topics, he was always there with the best possible suggestion for me with a friendly smile.

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Finally, I greatly appreciate the patience and unconditional love that I have received over the years from my family and this thesis is dedicated to my supernal father whom I have lost during these days.

Panchugopal

LIST OF PUBLICATIONS

a **Published:** *Hilbert von Neumann modules*, with Kunal Mukherjee and R. Srinivasan and V. S. Sunder, Communications in Stochastic Analysis : Volume 6, no. 1, (March 2012).

b **Accepted:** *On extendability of endomorphisms and of E_0-semigroups on factors*, with Masaki Izumi, R. Srinivasan and V.S. Sunder accepted at the Kyushu Journal of Mathematics.

c **Preprint:** *Non-extendable E_0-semigroups on type III factors*, e-print arXiv:1304.4341 [math.OA].

Contents

Synopsis

This thesis is devoted to the study of weak-* continuous semigroups of unital normal $*$-endomorphisms o n arbitrary factors. It is divided into three chapters. The first chapter proposes a rephrasing of the notion of a Hilbert von Neumann module, while the last two chapters are dedicated to a class of endomorphisms and E_0-semigroups on factors which we call *extendable*. These three chapters are more or less the content of the three papers labeled a, b, and c, in my list of publications.

Hilbert von Neumann modules

In the first chapter we rephrase the notion of Hilbert von Neumann modules as spaces of operators between Hilbert space, not unlike [Ske01], but in a seemingly simpler manner and involving far less machinery.

- If A_2 is a von Neumann algebra, a Hilbert von Neumann A_2 - module is a tuple $\mathcal{E} = (E, \mathcal{H}_1, (\pi_2, \mathcal{H}_2))$ where E is a weakly closed subspace of $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ with $E \supset EE^*E$, equipped with a normal isomorphism $\pi_2 : A_2 \to [E^*E]$ [1].

- If A_1, A_2 are von Neumann algebras, a Hilbert von Neumann $A_1 - A_2$ - bimodule is a tuple $\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ comprising a Hilbert von Neumann A_2 - module $(E, \mathcal{H}_1, (\pi_2, \mathcal{H}_2))$ equipped with a normal unital homomorphism $\pi_1 : A_1 \to [EE^*]$.

We then revisit the proof, in our formulation, of the 'Riesz lemma' or what is called 'self-duality' in [Ske01] and establish the analogue of the Stinespring dilation theorem for Hilbert von Neumann bimodules.

We develop our version of 'internal tensor products' (which we refer to as Connes fusion for obvious reasons) in the following way. Suppose $\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ is a

[1] We shall always use the following notation: if $\mathcal{S} \subset \mathcal{H}$ (resp., $S \subset \mathcal{B}(\mathcal{M}, \mathcal{K})$ and $T \subset \mathcal{B}(\mathcal{H}, \mathcal{M})$, then $[\mathcal{S}]$ (resp., $[S]$) is the norm-, equivalently weakly (resp., SOT, equivalently WOT) closed subspace spanned by $\mathcal{S}$ (resp., S); whereas $S^*T = \{x^*y : x \in S, y \in T\}$.

Hilbert von Neumann $A_1 - A_2$ - bimodule and $\mathcal{F} = (F, (\rho_2, \mathcal{K}_2), (\rho_3, \mathcal{K}_3))$ is a Hilbert von Neumann $A_2 - A_3$ - bimodule. We know that the normal representation ρ_2 of A_2 is equivalent to a sub-representation of an infinite ampliation of the faithful normal representation π_2 of A_2; thus there exists an A_2 - linear isometry $u : \mathcal{K}_2 \to \mathcal{H}_2 \otimes \ell^2$: i.e., $u^*u = id_{\mathcal{K}_2}$ and $u\rho_2(x) = (\pi_2(x) \otimes id_{\ell^2})u \ \forall x \in A_2$. It follows that $p = uu^* \in (\pi_2(A_2) \otimes id_{\ell^2})'$. Now, set $p = uu^*$ and then there exists a naturally associated projection $q \in \mathcal{P}(\pi_1(A_1)')$ (by a sort of push-forward construction). Then, if $x \in E, y \in F$, define $x \bigodot y$ to be the composite operator

$$\mathcal{K}_3 \overset{x \bigodot y}{\longrightarrow} q(\mathcal{H}_1 \otimes \ell^2) =: \mathcal{K}_3 \overset{y}{\longrightarrow} \mathcal{K}_2 \overset{u}{\longrightarrow} uu^*(\mathcal{H}_2 \otimes \ell^2) \overset{x \otimes id_{\ell^2}}{\longrightarrow} q(\mathcal{H}_1 \otimes \ell^2) \ ,$$

set $E \bigodot F = [\{x \bigodot y : x \in E, y \in F\}]$; and finally define the Connes fusion of $\mathcal{E}$ and $\mathcal{F}$ to be

$$\mathcal{E} \otimes_{A_2} \mathcal{F} =: (E \bigodot F, (q(\pi_1 \otimes id_{\ell^2})|_{ran\ q}, q(\mathcal{H}_1 \otimes \ell^2)), (\rho_3, \mathcal{K}_3)).$$

Then we verify that the Connes fusion of two Hilbert von Neumann bimodules is again a Hilbert von Neumann bimodule.

In the section of examples of this chapter we relate Jones' basic construction to the Stinespring dilation associated to the conditional expectation onto a finite-index inclusion.

Suppose $(M, \mathcal{H}, J, P)$ is a standard form of M in the sense of [Haa75]. As indicated in [Haa75], there is then a canonical 'implementing' unitary representation

$$Aut(M) \ni \theta \mapsto u_\theta \in \mathcal{L}(\mathcal{H})$$

satisfying $u_\theta x u_\theta^* = \theta(x) \ \forall x \in M$. We have the natural Hilbert von Neumann $M - M$ bimodule given by

$$\mathcal{E}_\theta = (Mu_\theta, (id_M, L^2(M)), (id_M, L^2(M))).$$

Then we prove that

- 'Connes fusion corresponds to composition' in this case, in the sense that if $\theta, \phi \in Aut(M)$, then

$$\mathcal{E}_\theta \otimes_M \mathcal{E}_\phi \cong \mathcal{E}_{\theta\phi} \;\; :$$

and further, that

- If $\theta, \phi \in Aut(M)$, then $\mathcal{E}_\theta \cong \mathcal{E}_\phi$ if and only if θ and ϕ are inner conjugate.

On extendability of endomorphisms and of E_0-semigroups on factors

In this chapter we study what it means to say that certain endomorphisms of a factor (which we call *equi-modular*) are extendable.

Let ϕ be a faithful normal state on a factor M and J be the modular conjugation operator corresponding to the state ϕ. Let θ be a normal unital *-endomorphism which preserves ϕ. The invariance assumption $\phi \circ \theta = \phi$ implies that there exists a unique isometry u_θ on $L^2(M, \phi)$ such that $u_\theta x \widehat{1_M} = \theta(x)\widehat{1_M}$ and equivalently, that $u_\theta x = \theta(x) u_\theta \; \forall x \in M$ and $u_\theta \widehat{1_M} = \widehat{1_M}$. In this chapter we study certain properties of the following endomorphisms.

- If M, ϕ, θ are as above, and if the associated isometry u_θ of $L^2(M, \phi)$ commutes with the modular conjugation operator J, we shall say θ is an **equi-modular** endomorphism of the factorial non-commutative probability space (M, ϕ).

Then we note the following simple consequences of θ being an equi-modular endomorphism.

- The equation $\theta' = j \circ \theta \circ j$ defines a unital normal *-endomorphism of M' which preserves $\phi' = \overline{\phi \circ j}$, where $j = J(\cdot)J$ is a *-preserving conjugate-linear isomorphism of $\mathcal{L}(L^2(M, \phi))$ onto itself; and

- We have an identification

$$\begin{aligned} L^2(M',\phi') &= L^2(M,\phi) \\ \widehat{1_{M'}} &= \widehat{1_M} \\ u_{\theta'} &= u_\theta \end{aligned}$$

- there exists a unique endomorphism $\theta^{(2)}$ of $\mathcal{L}(L^2(M,\phi))$ satisfying

$$\theta^{(2)}(xj(y)) = \theta(x)j(\theta(y))z, \ \forall x,y \in M$$

where $z = \wedge\{p \in (\theta(M)\cup\theta'(M'))'' : ran(p) \supset \{\widehat{\theta(x)} : x \in M\}\}$.

It turns out that equi-modularity of θ forces $\theta(M)\widehat{1_M}$ to be globally invariant under the modular group σ^ϕ and consequently there exists a faithful normal ϕ-preserving conditional expectation of M onto $\theta(M)$. Let $P = \theta(M) \subset M \subset P_1$ be the Jones' basic construction (thus, $P_1 = JP'J$ and the Jones projection is given by $e_\theta = u_\theta u_\theta^*$). Then we prove that the following statements are equivalent for equi-modular endomorphisms.

- there exists a unique unital normal $*$-endomorphism $\theta^{(2)}$ of $\mathcal{L}(L^2(M,\phi))$ such that $\theta^{(2)}(x) = \theta(x)$ and $\theta^{(2)}(j(x)) = j(\theta(x))$ for all $x \in M$.
- $P \vee JPJ$ is a factor; and in this case, it is necessarily a type I factor.
- $(P \vee JPJ)' = P' \cap P_1$ is a factor; and in this case, it is necessarily a type I factor.
- $\{x\widehat{y} : x \in P' \cap P_1, y \in P\}$ is total in $L^2(M,\phi)$.
- $M = (M \cap \theta(M)') \vee \theta(M)$. (Note that the right-hand side is naturally identified with the von Neumann algebra tensor product $(M \cap \theta(M)') \otimes \theta(M)$ in this case.)

An endomorphism of a factor which satisfies the equivalent conditions above will be said to be **extendable**. Then we exhibit some examples of extendable endomorphisms on a McDuff factor.

In the next section of this chapter we discuss extendability of E_0-semigroups on factors. An E_0-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ on a factorial probability space (M, ϕ) is said to be extendable if for every $t \geq 0$, α_t is extendable. We show that extendability of an E_0-semigroup is cocycle conjugacy invariant. Then we make a note that CCR flows on type III factors arising from some quasi-free states are examples of extendable E_0-semigroups while the Clifford flow on the hyperfinite II_1 factor is not extendable. We also observe that free flows on free group factors LF_∞ are not extendable.

Alexis Alevras in [Ale04] proved that Clifford flow and CAR flow on the hyperfinite II_1 factor are cocycle conjugate. We conclude that the CAR flow on the hyperfinite II_1 is not extendable. In the end of this section we point out an error in the authors' claim in [ABS01] that the Clifford flow is extendable. (They state this in terms of a notion they call regularity which is close to our notion of extendability.)

CAR flows on type III factors

This chapter is dedicated to studying extendability of CAR flows on type III factors. Let $\mathcal{H} = L^2(0, \infty) \otimes \mathcal{K}$, where $\mathcal{K}$ is any Hilbert space and $\mathcal{A}$ be the CAR algebra over $\mathcal{H}$. Every positive contraction R on $\mathcal{B}(\mathcal{H})$ determines a so-called quasi-free state ω_R on $\mathcal{A}$. Let $(\pi_R, \mathcal{H}_R, \Omega_R)$ be the GNS triple for the C^*-algebra $\mathcal{A}$ with respect to the state ω_R. We write $M_R = \{\pi_R(\mathcal{A})\}''$, which is always a factor, most often of type III (see [PS70] Theorem 5.1 and Lemma 5.3).

Let $\{s_t\}_{t\geq}$ be the shift semigroup on $\mathcal{H}$. Under the assumption $s_t^* R s_t = R$ for all $t \geq 0$ and by [Arv03] Proposition 13.2.3 and [PS70] Lemma 5.3, there exists an E_0-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ on M_R which is known as **CAR flow of rank dim $\mathcal{K}$** .

Under the following conditions, we prove that CAR flows are not extendable on M_R which is always a type III factor under these conditions (see [PS70] Lemma 5.3).

- Both R and $1 - R$ are invertible; i.e., $\exists \epsilon > 0$ such that $\epsilon \leq R \leq 1 - \epsilon$.

- R is diagonalisable; in fact, there exists an orthonormal basis $\{f_i\}$ for $\mathcal{K}$ with $Rf_i = \lambda_i f_i$ for some $\lambda_i \in [\epsilon, 1-\epsilon] \setminus \{\frac{1}{2}\}$.

- $Rs_t = s_t R \; \forall t \geq 0$. (Clearly then, also the Toeplitz condition $s_t^* R s_t = R$ is met.)

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In the last section of this chapter, we prove that our result together with [MS13] will show that CCR flows and CAR flows on type III factors are not cocycle conjugate.

Introduction to the chapters

Chapter 1: In the first chapter we rephrase the notion of Hilbert von Neumann modules as spaces of operators between Hilbert space, not unlike [Ske01], but in a seemingly simpler manner and involving far less machinery.

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- $(P \vee \mathcal{J}P\mathcal{J})' = P' \cap P_1$ is a factor; and in this case, it is necessarily a type I factor.
- $\{x\widehat{y} : x \in P' \cap P_1, y \in P\}$ is total in $L^2(M, \phi)$.
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case.)

An endomorphism of a factor which satisfies the equivalent conditions above will be said to be **extendable**. Then we exhibit some examples of extendable endomorphisms on a McDuff factor.

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Chapter 3: This chapter is dedicated to studying extendability of CAR flows on type III factors. Let $\mathcal{H} = L^2(0, \infty) \otimes \mathcal{K}$, where $\mathcal{K}$ is any Hilbert space and $\mathcal{A}$ be the CAR algebra over $\mathcal{H}$. Every positive contraction R on $\mathcal{B}(\mathcal{H})$ determines a so-called quasi-free state ω_R on $\mathcal{A}$. Let $(\pi_R, \mathcal{H}_R, \Omega_R)$ be the GNS triple for the C^*-algebra $\mathcal{A}$ with respect to the state ω_R. We write $M_R = \{\pi_R(\mathcal{A})\}''$, which is always a factor, most often of type III (see [PS70] Theorem 5.1 and Lemma 5.3).

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$\mathcal{K}$.

Under the following conditions, we prove that CAR flows are not extendable on M_R which is always a type III factor under these conditions (see [PS70] Lemma 5.3).

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In the last section of this chapter, we prove that our result together with [MS13] will show that CCR flows and CAR flows on type III factors are not cocycle conjugate.

Chapter 1

Hilbert von Neumann Modules

This chapter is devoted to study of Hilbert von Neumann modules. The study of Hilbert von Neumann modules are closely related to the study of Hilbert C^*-modules. Hilbert C^*-module is a module over a C^*-algebra with an inner product taking value in that C^*-algebra fulfilling certain axioms generalizing those of the inner product of a Hilbert space. The notions of C^*-module and von Neumann module have been around for some time now. It is an well established subject (see the monograph [Lan95] for extensive references). Hilbert C^*-modules first appeared in the work of Kaplansky [Kap53]. Then Hilbert modules were studied more or less simultaneously by Paschke [Pas73] and Rieffel [Rie74].

In this chapter we introduce a way of regarding Hilbert von Neumann modules as spaces of operators between Hilbert space, not unlike [Ske01], but in an apparently simpler manner and involving far less machinery. We verify that our definition is equivalent to that of [Ske01], by verifying the 'Riesz lemma' or what is called 'self-duality' in [Ske01]. An advantage with our approach is that we can totally side-step the need to go through C^*-modules and avoid the two stages of completion - first in norm, then in the strong operator topology - involved in the former approach.

We establish the analogue of the Stinespring dilation theorem for Hilbert von Neumann bimodules, and we develop our version of 'internal tensor products' which we refer to as Connes fusion for obvious reasons.

In the section of examples, we examine the bimodules arising from automorphisms of von Neumann algebras, verify that fusion of bimodules corresponds to composition of automorphisms in this case, and that the isomorphism class of such a bimodule depends only on the inner conjugacy class of the automorphism. We also relate Jones' basic construction to the Stinespring dilation associated to the conditional expectation onto a finite-index inclusion (by invoking the uniqueness assertion regarding the latter).

1.1 Preliminaries

The symbols $\mathcal{H}$ and $\mathcal{K}$, possibly anointed with subscripts or other decorations, will always denote complex separable Hilbert spaces, while $\mathcal{L}(\mathcal{H}, \mathcal{K})$ will denote the set of bounded operators from $\mathcal{H}$ to $\mathcal{K}$. For $E \subset \mathcal{L}(\mathcal{H}, \mathcal{K})$, we shall write $[E]$ for the closure, in the weak operator topology (WOT, in the sequel), of the linear subspace of $\mathcal{L}(\mathcal{H}, \mathcal{K})$ spanned by E. Similarly, if $\mathcal{S} \subset \mathcal{H}$ is a set of vectors, we shall write $[\mathcal{S}]$ for the norm-closed subspace of $\mathcal{H}$ spanned by $\mathcal{S}$.

Without explicitly citing it again to justify statements we make, we shall use the fact that a linear subspace of $\mathcal{H}$ (resp., $\mathcal{L}(\mathcal{H}, \mathcal{K})$) is closed in the weak topology (resp., WOT) if and only if it is closed in the strong or norm topology (resp., 'SOT'). (For example, $[E]$ is an algebra if E is.)

If $E \subset \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $F \subset \mathcal{L}(\mathcal{H}_1, \mathcal{H})$, we write

$$EF = \{xy : x \in E, y \in F\} \text{ and } E^* = \{x^* : x \in E\} \ .$$

If $i : \mathcal{H}_0 \hookrightarrow \mathcal{H}$ and $j : \mathcal{K}_0 \hookrightarrow \mathcal{K}$, then we shall think of $\mathcal{L}(\mathcal{H}, \mathcal{K}_0)$ as the subset $j\mathcal{L}(\mathcal{H}, \mathcal{K}_0)i$ of $\mathcal{L}(\mathcal{H}_0, \mathcal{K})$.

Proposition 1.1.1. *For $i = 1, 2$, let e_i denote the projection of $\mathcal{H}_1 \oplus \mathcal{H}_2$ onto $\mathcal{H}_i$. The following conditions on an $E \subset \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ are equivalent:*

1. *There exists a von Neumann algebra $M \subset \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ such that $e_1, e_2 \in M$ and $E = e_1 M e_2$.*

2. *$E = [E] \supset EE^*E$.*

When these equivalent conditions are met, we shall say that $(E, \mathcal{H}_1, \mathcal{H}_2)$ is a **(1,2) von Neumann corner**.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1): Observe that the assumption (2) implies that $[E^*E]$ is a WOT-closed *-subalgebra of $\mathcal{L}(\mathcal{H}_2)$. Let $p_2 = \sup\{p : p \in \mathcal{P}([E^*E]\}$ and define $M_{22} = [E^*E] + \mathbb{C}(e_2 - p_2)$; so M_{22} is a von Neumann subalgebra of $\mathcal{L}(\mathcal{H}_2)$ and $e_2 - p_2$ is a central minimal projection in it.

Similarly, define $M_{11} = [EE^*] + \mathbb{C}(e_1 - p_1)$, where $p_1 = \sup\{p : p \in \mathcal{P}([EE^*]\}$; so M_{11} is a von Neumann subalgebra of $\mathcal{L}(\mathcal{H}_1)$ and $e_1 - p_1$ is a central minimal projection in it.

Finally set $M_{12} = E, M_{21} = E^*$ and $M = \sum_{i,j=1}^{2} M_{ij}$. (Alternatively M is the von Neumann algebra $(E \cup E^*)''$; and it is clear that $E = e_1 M e_2$.

$\square$

Definition 1.1.2. 1. The projection p_1 (resp. p_2) occurring in the proof of Proposition 1.1.1 will be referred to as the **left-support** (resp., **right-support**) **projection** of the (1,2) von Neumann corner E.

2. A (1,2) von Neumann corner $(E, \mathcal{H}_1, \mathcal{H}_2)$ will be said to be **non-degenerate** if its support projections are as large as they can be: i.e., $p_i = (e_i =) 1_{\mathcal{H}_i}, i = 1, 2$.

Remark 1.1.3. *1.* The support projections p_1, p_2 of E have the following equivalent descriptions:

- $ran\ p_1 = [\bigcup\{ran\ x : x \in E\}] = \left(\bigcap\{ker\ x^* : x \in E\}^\perp\right)$; and
- $ran\ p_2 = [\bigcup\{ran\ x^* : x \in E\}] = \left(\bigcap\{ker\ x : x \in E\}^\perp\right)$.

2. A (1,2) von Neumann corner $(E, \mathcal{H}_1, \mathcal{H}_2)$ is non-degenerate precisely when $M_{11}(E) = [EE^*]$ and $M_{22}(E) = [E^*E]$ are unital von Neumann subalgebras of $\mathcal{L}(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$ respectively.

Definition 1.1.4. *1. If A_2 is a von Nemann algebra, a* **Hilbert von Neumann A_2 - module** *is a tuple $\mathcal{E} = (E, \mathcal{H}_1, (\pi_2, \mathcal{H}_2))$ where $(E, \mathcal{H}_1, \mathcal{H}_2)$ is a (1,2) von Neumann corner equipped with a normal* isomorphism $\pi_2 : A_2 \to [E^*E]$.

2. A **submodule** *of a Hilbert von Neumann A_2-module E is a subset $E_1 \subset E$ satisfying*

$$E_1 = [E_1] \supset E_1 E^* E.$$

3. If A_1, A_2 are von Neumann algebras, a **Hilbert von Neumann $A_1 - A_2$ - bimodule** *is a tuple*

$$\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$$

comprising a Hilbert von Neumann A_2 - module $(E, \mathcal{H}_1, (\pi_2, \mathcal{H}_2))$ equipped with a normal unital homomorphism $\pi_1 : A_1 \to [EE^*]$ (where the 'unital requirement' is that $\pi_1(1_{A_1}) = p_1$ is the identity of $[EE^*]$).

Remark 1.1.5. 1. If $E \subset \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ is any (possibly degenerate) (1,2) von Neumann corner, with associated support projections p_1, p_2 (as in Definition 1.1.2), define $\mathcal{K}_i = ran\ p_i, A_1 = [EE^*], A_2 = [E^*E]$ and let π_i denote the identity representation of A_i on $\mathcal{K}_i$; then $(E, (\pi_1, \mathcal{K}_1), (\pi_2, \mathcal{K}_2))$ is seen to be a non-degenerate Hilbert von Neumann $A_1 - A_2$ - bimodule. This is why non-degeneracy is not a serious restriction.

2. A Hilbert von Neumann A_2 - module $(E, \mathcal{H}_1, (\pi_2, \mathcal{H}_2))$ does indeed admit a right-A_2 action and an A_2 - valued inner product thus:

$$x \cdot a_2 = x\pi_2(a_2)\ ;\ \langle x_1, x_2 \rangle_{A_2} = \pi_2^{-1}(x_1^* x_2)$$

(Here and in the sequel, we shall write $\langle \cdot , \cdot \rangle_B$ for the B - valued inner-product on a Hilbert B - module.) Notice, further, that the norm E acquires from this Hilbert A_2 - module structure is nothing but the operator norm on E.

3. A submodule of a Hilbert von Neumann A_2 - module is a (possibly degenerate) (1,2) von Neumann corner.

4. In a general Hilbert von Neumann A_2 - module $\mathcal{E} = (E, \mathcal{H}_1, (\pi_2, \mathcal{H}_2))$, note that

$$[EE^*] \ni a \mapsto (E \ni x \mapsto a \cdot x =: ax)$$

defines a *-homomorphism of $[EE^*]$ into the space $\mathcal{L}^a(E)$ of bounded adjointable operators on E, since, for instance

$$\begin{aligned} \langle a \cdot x, y \rangle_{A_2} &= (ax)^* y \\ &= x^*(a^* y) \\ &= x^*(a^* \cdot y) \\ &= \langle x, a^* \cdot y \rangle_{A_2} . \end{aligned}$$

5. In the language of (2) above, the 'rank-one operator' $\theta_{x,y}$ is seen to be given by

$$\begin{aligned} \theta_{x,y}(z) &= x \langle y, z \rangle_{A_2} \\ &= xy^* z , \end{aligned}$$

so that the 'rank-one operator' $\theta_{x,y}$ on E is nothing but left multiplication by xy^* on E, for any $x, y \in E$. Let us write $B = [EE^*]$, $C = A_2$ and A for the norm-closure of the linear span of EE^*. Then it is clear that A is a norm-closed ideal in B, and that there is a unique C^* - algebra isomorphism $\alpha : A \to \mathcal{K}(E)$ such that $\alpha(xy^*) = \theta_{x,y}$, $\forall x, y \in E$. If E is non-degenerate, then A is an essential ideal in B and α is injective. It then follows from [Lan] Proposition 2.1, that α extends uniquely to an isomorphism of B onto $\mathcal{L}^a(E)$. (In fact, the reason for introducing the symbols A, B, C above was in order to use exactly the same symbols as in the Proposition 2.1 referred to above.)

6. This remark concerns our requirement, in the definition of a Hilbert von Neumann A_2-module, that $\pi_2 : A_2 \to [E^*E]$ must be an isomorphism. What is really needed is that π_2 is onto. If π_2 is merely surjective but not injective, there must exist a central projection $z \in A_2$ such that $ker\ \pi_2 = (1-z)A_2$ so π_2 would map zA_2 isomorphically onto $[E^*E]$ and the A_2-valued inner product (see item (2) of this remark) would actually take values in zA_2 and we could apply our analysis to zA_2 and think of A_2 as acting via its quotient (and ideal) zA_2.

7. The 'unital requirement' made in the definition of a Hilbert von Neumann bimodule has the consequence that $\pi_1(A_1)E = E$.

Lemma 1.1.6. *Let $(E, \mathcal{H}_1, \mathcal{H}_2)$ be a (1,2) von Neumann corner. Suppose $x \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ has polar decomposition $x = u|x|$. Then the following conditions are equivalent:*

1. $x \in E$.

2. $u \in E$ and $|x| \in [E^*E]$.

3. $u \in E$ and $|x^*| \in [EE^*]$.

Proof. Since (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are obvious, let us prove the reverse implications. So, suppose $x \in E$. Then $x^*x \in E^*E$ (resp., $xx^* \in EE^*$) and as, $|t|$ is uniformly approximable on compact subsets of $\mathbb{R}$ by polynomials with vanishing constant term, it is seen that $|x| \in [E^*E]$ and $|x^*| \in [EE^*]$. Define $f_n \in C_0([0,\infty))$ by

$$f_n(t) = \begin{cases} 0 & \text{if } t < \frac{1}{2n} \\ 2n^2(t - \frac{1}{2n}) & \text{if } \frac{1}{2n} \leq t \leq \frac{1}{n} \\ \frac{1}{t} & \text{if } t \geq \frac{1}{n} \end{cases}$$

Since f_n is uniformly approximable on $sp(|x|)$ by polynomials with vanishing constant term, it is seen that $f_n(|x|) \in [E^*E]$, and hence $xf_n(|x|) \in E$. It follows from the definitions that $|x|f_n(|x|)$ WOT-converges to $1_{(0,\infty)}(|x|) = u^*u$. In particular, $u = u(u^*u) = WOT - lim\ u(|x|f_n(|x|)) = WOT - lim\ xf_n(|x|) \in [E\,[E^*E]] \subset E$.

□

Proposition 1.1.7. *If E_1 is a submodule of a Hilbert von Neumann A_2 - module E, and if $E_1 \neq E$, there exists a non-zero $y \in E$ such that $y^*x = 0\ \forall x \in E_1$.*

Proof. As observed in Remark 1.1.5(3), E_1 is a possibly degenerate (1,2) von Neumann corner in $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$. Let $p_1 = \bigvee\{e : e \in \mathcal{P}([E_1^*E_1])\}$ and $q_1 = \bigvee\{f : f \in \mathcal{P}([E_1E_1^*])\}$ be the right- and left- support projections of E_1. Similarly, let $p = \bigvee\{e : e \in \mathcal{P}([E^*E])\}$ and $q = \bigvee\{f : f \in \mathcal{P}([EE^*])\}$ be the right- and left- support projections of E.

First observe that the hypotheses imply that

$$(E^*E)(E_1^*E_1)(E^*E) = (E_1E^*E)^*(E_1E^*E) \subset E_1^*E_1$$

and hence that $[E_1^*E_1]$ is a WOT-closed ideal in the von Neumann subalgebra $[E^*E]$ of $\mathcal{L}(p\mathcal{H}_2)$; consequently $p_1 = \bigvee\{e : e \in \mathcal{P}([E_1^*E_1])\}$ is a central projection in $[E^*E]$ and $[E_1^*E_1] = [E^*E]p_1$. It follows that if $x_1 \in E_1$ has polar decomposition $x_1 = u_1|x_1|$, then (by Lemma 1.1.6) $u_1 \in E_1$ and $|x_1| \in [E_1^*E_1] = [E^*E]p_1$, and in particular, $x_1p_1 = u_1|x_1|p_1 = u_1|x_1| = x_1$; i.e., $E_1 = E_1p_1$.

Next, by definition, $[\bigcup\{ran\, x_1 : x_1 \in E_1\}] = [\bigcup\{ran\, x_1x_1^* : x_1 \in E_1\}] = [\bigcup\{ran 1_{(0,\infty)}(|x_1^*|) : x_1 \in E_1\}] =\ ran\ q_1$; hence if $x_1 \in E_1$, then $x_1 = q_1x_1$, and we see that $E_1 = q_1E_1$.

Summarising the previous two paragraphs, we have

$$E_1 = q_1E_1 = E_1p_1\ . \tag{1.1.1}$$

(In fact, $x_1 = x_1p_1 = q_1x_1\ \forall\ x_1 \in E_1$.)

We now consider three cases:

Case 1: $p_1 \neq p$

Here $(p - p_1) \neq 0$ and the definition of p implies that there exists a $y \in E$ such that $y = y(p - p_1) \neq 0$. Then, for any $x \in E_1$, we have $x = xp_1$ and hence

$$y^*x = (p - p_1)y^*x = (p - p_1)y^*xp_1 \in (p - p_1)E^*Ep_1 = (p - p_1)p_1E^*E = \{0\} \ .$$

Case 2: $q_1 \neq q$

Here $(q - q_1) \neq 0$ and the definition of q implies that there exists a $y \in E$ such that $y = (q - q_1)y \neq 0$. Then, for any $x \in E_1$, we have $x = q_1x$ and hence

$$y^*x = y^*(q - q_1)x = y^*(q - q_1)q_1x = 0 \ .$$

Case 3: $p_1 = p, q_1 = q$.

We shall show that the hypotheses of this case imply that $E_1 = E$ and hence cannot arise. To see this, begin by noting that the collection of non-zero partial isometries in E_1 is non-empty in view of Lemma 1.1.6. (Otherwise $E_1 = \{0\}, p_1 = q_1 = 0$ and so $E = \{0\} = E_1$.) Hence the family $\mathcal{F}$ of collections $\{u_i : i \in I\}$ of partial isometries in E_1 with pairwise orthogonal ranges, is non-empty. Clearly $\mathcal{F}$ is partially ordered by inclusion, and it is easy to see that Zorn's lemma is applicable to $\mathcal{F}$.

If $\{u_i : i \in I\}$ is a maximal element of $\mathcal{F}$, we assert that $\sum_{i \in I} u_iu_i^* = q$. Indeed, if $(q - \sum_{i \in I} u_iu_i^*) \neq 0$, the assumption $q = q_1$ will imply the existence of an $x_1 \in E_1$ such that $x_1 = (q - \sum_{i \in I} u_iu_i^*)x_1 \neq 0$. Then $x_1 \in [E_1E_1^*E_1] \subset E_1$ and so if $x_1 = v_1|x_1|$ is its polar decomposition, then $v_1 \in E_1 \setminus \{0\}$ and $ran\ v_1 = \overline{ran\ x_1}$ is orthogonal to $ran\ u_i$ for each $i \in I$, thus contradicting the maximality of $\{u_i : i \in I\}$.

Thus, indeed $q = \sum_{i \in I} u_iu_i^*, \ u_i \in E_1$.

Now, if $x \in E$ is arbitrary, then,

$$\begin{aligned} x &= qx \\ &= \sum_{i \in I} u_i u_i^* x \\ &\in [E_1 E_1^* E] \\ &\subset [E_1 E^* E] \\ &\subset E_1 \end{aligned}$$

and so $E = E_1$ in this case, and the proof of the Proposition is complete. $\square$

Given a submodule E_1 of a Hilbert von Neumann module E, as above, we shall write $E_1^\perp$ for the set $\{y \in E : y^* E_1 = \{0\}\}$ and refer to it as the **orthogonal complement of** E_1 **in** E. We now reap the consequences of Proposition 1.1.7 in the following Corollary.

Corollary 1.1.8. *Let E_1 be a submodule of a Hilbert von Neumann A_2 - module. Then,*

1. *$E_1^\perp = (1 - q_1)E$, where q_1 is the left support projection of E_1.*
2. *$E_1^{\perp\perp} = q_1 E$.*
3. *If S is any subset of E, then $S^{\perp\perp} = [S[E^* E]]$.*
4. *If E_1 is a submodule of a Hilbert von Neumann module E, there exists a projection $q_1 \in [EE^*]$ such that $E_1 = E_1^{\perp\perp} = q_1 E$ and $E_1^\perp = (1 - q_1)E$; and in particular E_1 is complemented in the sense that $E = E_1 \oplus E_1^\perp$.*

Proof. It is clear that $y^* x = 0$ if and only if y and x have mutually orthogonal ranges.

(1) The previous sentence and the definition of q_1 imply that

$$y \in E_1^\perp \Leftrightarrow (q_1 y = 0 \text{ and } y \in E) \Leftrightarrow y \in (1 - q_1)E.$$

(2) follows from (1) and the definition of the orthogonal complement.

(3) Let $E_1 = [SE^*E]$. It should be clear that $y \in S^\perp \Leftrightarrow y \in E_1^\perp = q_1 E$, by part (1) of this Corollary, and hence that

$$S^{\perp\perp} = E_1^{\perp\perp} .$$

In view of Remark 1.1.5(1) we may view $S^{\perp\perp}$ as a Hilbert von Neumann bimodule, and regard E_1 as a submodule of $S^{\perp\perp}$. We may then deduce from Proposition 1.1.7 that if E_1 were not equal to $S^{\perp\perp}$, then there would have to exist a non-zero $y \in S^{\perp\perp}$ such that $y^* E_1 = \{0\}$. This would imply that $y \in S^\perp$ and $y \in S^{\perp\perp}$ so that $y^* y = 0$, a contradiction.

(4) follows from the preceding parts of this Corollary. □

That our definitions of Hilbert von Neumann modules and bimodules are consistent with those of [Skei] is a consequence of the following version of Riesz' Lemma, which establishes that our Hilbert von Neumann modules are indeed 'self-dual' which is one of the equivalent conditions for a von Neumann module in the sense of [Skei].

On the other hand, it is clear from [Skei] that any Hilbert von Neumann A_2 - module in the sense of [Skei] is also a Hilbert von Neumann A_2 - module in our sense, and the two formulations are thus equivalent.

Proposition 1.1.9. *(***Riesz lemma***) Suppose $\mathcal{E}$ is a Hilbert von Neumann A_2 - module, and $f : E \to A_2$ is right A_2-linear - meaning $f(x\pi_2(a_2)) = \pi_2^{-1}(f(x)\pi_2(a_2))$ for all $x \in E, a_2 \in A_2$, or equivalently and less clumsily, suppose $f : E \to [E^*E]$ is linear and satisfies $f(xz) = f(x)z$ for all $x \in E,\ z \in [E^*E]$; and suppose f is bounded - meaning $\|f(x)\| \leq K\|x\|$ for all $x \in E$, and some $K > 0$. Then there exists $y \in E$ such that $f(x) = y^*x\ \forall x \in E$.*

Proof. First notice that if $x \in E$ has polar decomposition $x = u|x|$ (so $u \in E, |x| \in$

$[E^*E] = \pi_2(A_2)$, and if $\xi \in \mathcal{H}_2$, then

$$\begin{aligned} \|f(x)\xi\| &= \|f(u)|x|\xi\| \quad \text{(by right } A_2 \text{ - linearity of } f) \\ &\leq \|f(u)\|\||x|\xi\| \\ &\leq K\||x|\xi\| \\ &= K\|u^*x\xi\| \\ &\leq K\|x\xi\| \ . \end{aligned} \tag{1.1.2}$$

Next, find vectors $\xi_n \in \mathcal{H}_2$ such that $\mathcal{H}_2 = \oplus_n[\pi_2(A_2)\xi_n]$ (orthogonal direct sum). It follows that $p_1\mathcal{H}_1 = \oplus_n[E\xi_n]$, where p_1 is the left-support projection of E, because if $n \neq m$ and $x, y \in E$, then

$$\langle x\xi_n, y\xi_m\rangle = \langle \xi_n, x^*y\xi_m\rangle = 0$$

and

$$[\bigcup_n [E\xi_n]] = [\bigcup_n [EE^*E\xi_n]] = [E\mathcal{H}_2] = p_1\mathcal{H}_1 \ .$$

Infer from the above paragraph and equation 1.1.2 that for arbitrary $a_n \in A_2$ with $\sum_n \|\pi_2(a_n)\xi_n)\|^2 < \infty$ and $x \in E$, we have

$$\begin{aligned} \|f(x)(\sum_n \pi_2(a_n)\xi_n)\|^2 &= \|\sum_n (f(x)\pi_2(a_n))\xi_n\|^2 \\ &= \sum_n \|f(x\pi_2(a_n))\xi_n\|^2 \\ &\leq \sum_n K^2\|x\pi_2(a_n)\xi_n\|^2 \quad \text{(by eq. (1.1.2))} \\ &= K^2\|x(\sum_n \pi_2(a_n)\xi_n)\|^2 \ ; \end{aligned}$$

Now deduce that there exists a unique bounded operator $z_f \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ satisfying $z_f = z_f p_1$ and

$$z_f(x\xi) = f(x)\xi \ , \forall x \in E, \xi \in \mathcal{H}_2 \ .$$

The definition of z_f implies that $z_f E \subset [E^*E]$; hence

$$z_f = z_f p_1 \in z_f[EE^*] \subset [z_f EE^*] \subset [[E^*E]E^*] = E^* \ .$$

So $y =: z_f^* \in E$ and we have

$$f(x) = z_f x = y^* x$$

as desired. □

1.2 Standard bimodules and complete positivity

Given an element x of a von Neumann algebra M, et us write $pr(x)$ for the projection onto the range of x. (Thus $pr(x) = 1_{(0,\infty)}(xx^*)$.)

Lemma 1.2.1. *Suppose $\eta : A \to B$ is a normal positive linear map of von Neumann algebras. Let $e_\eta = \bigvee\{u \ pr(\eta(1)) \ u^* : u \in \mathcal{U}(B)\}$ be the (B-)central support of $pr(\eta(1))$. Then the smallest WOT-closed ideal in B which contains $\eta(A)$ (equivalently $\eta(1)$) is $e_\eta B$. (In particular, $\eta(a) = e_\eta \eta(a) \ \forall a \in A$.)*

Proof. If $p \in \mathcal{P}(A)$, then $\eta(p) \leq \eta(1) \ \ \Rightarrow pr(\eta(p)) \leq pr(\eta(1)) \leq e_\eta$. Hence $\eta(p) = e_\eta \eta(p) \in e_\eta B$, so also $B\eta(p)B \subset e_\eta B$. Conclude that $[B\eta(A)B] = [B\eta([\mathcal{P}(A)])B] = [B\eta(\mathcal{P}(A))B] \subset e_\eta B$. Conversely, $[B\eta(A)B] \supset [B\mathcal{U}(B)\eta(1)\mathcal{U}(B)B] \supset [Be_\eta B] = e_\eta B$, and the proof is complete. □

Definition 1.2.2. *A Hilbert von Neumann A_2 - module $\mathcal{E} = (E, \mathcal{H}_1, (\pi_2, \mathcal{H}_2))$ will be called* **standard** *if :*

- *$\mathcal{H}_2 = L^2(A_2, \phi)$ for some faithful normal state ϕ on A_2;*
- *π_2 is the left-regular representation; and*
- *E is non-degenerate.*

A Hilbert von Neumann $A_1 - A_2$ - bimodule will be called standard if it is standard as a Hilbert von Neumann A_2 - module.

Theorem 1.2.3. *If $\eta : A_1 \to A_2$ is a normal completely positive map, there exists a standard Hilbert von Neumann $A_1 - e_\eta A_2$ bimodule $\mathcal{E}_\eta$, with e_η as in Lemma 1.2.1, which is singly generated, (i.e., $E = [\pi_1(A_1)V\pi_2(e_\eta A_2)]$) with a generator $V \in E$ satisfying $V^*\pi_1(a_1)V = \pi_2 \circ \eta(a_1)$.*

Further, such a pair $(\mathcal{E}, V)$ of a standard bimodule and generator is unique in the sense that if $(\widetilde{\mathcal{E}}, \widetilde{V})$ is another such pair, then there exists A_i - linear unitary operators $U_i : \mathcal{H}_i(\eta) \to \widetilde{\mathcal{H}_i}$, $i = 1, 2$ such that $\widetilde{V} = U_1 V U_2^*$ and $\widetilde{\mathcal{E}} = U_1 \mathcal{E} U_2^*$.

Proof. Fix a faithful normal state ϕ on $e_\eta A_2$ and set $\mathcal{H}_2(\eta) = L^2(e_\eta A_2, \phi)$, with π_2 being the left-regular representation of $e_\eta A_2$. We employ the standard notation $\hat{a} = \pi(a)\hat{1}$ where $\hat{1}$ is the canonical cyclic vector for $\pi(A)$ in $L^2(A)$. The Hilbert space $\mathcal{H}_1(\eta)$ is obtained after separation and completion of the algebraic tensor product $A_1 \otimes e_\eta A_2$ with respect to the semi-inner product given by $\langle a_1 \otimes a_2, b_1 \otimes b_2 \rangle = \phi(b_2^* \eta(b_1^* a_1) a_2)$; and $\pi_1 : A_1 \to \mathcal{L}(\mathcal{H}_1(\eta))$ is defined by $\pi_1(a_1)(b_1 \otimes b_2) = a_1 b_1 \otimes b_2$. The verification that π_1 is a normal representation is a fairly routine application of normality of η and ϕ.

Define $V : \mathcal{H}_2(\eta) \to \mathcal{H}_1(\eta)$ to be the unique bounded operator for which $V(e_\eta \hat{a}_2) = 1 \otimes e_\eta a_2$. For arbitrary $a_1 \in A_1, a_2, b_2 \in e_\eta A_2$, note that

$$
\begin{aligned}
\langle V^*\pi_1(a_1)V\hat{a}_2, \hat{b}_2 \rangle &= \langle a_1 \otimes a_2, 1 \otimes b_2 \rangle \\
&= \phi(b_2^* \eta(a_1) a_2 \rangle \\
&= \langle \pi_2(\eta(a_1))\hat{a}_2, \hat{b}_2 \rangle
\end{aligned}
$$

thus showing that indeed $V^*\pi_1(a_1)V = \pi_2(\eta(a_1))$ for all $a_1 \in A_1$.

Set $E = [\pi_1(A_1)V\pi_2(e_\eta A_2)]$ and observe that

$$
\begin{aligned}
[E^*E] &= [\pi_2(e_\eta A_2)V^*\pi_1(A_1)\pi_1(A_1)V\pi_2(e_\eta A_2)] \\
&= [\pi_2(e_\eta A_2)\pi_2(\eta(A_1))\pi_2(e_\eta A_2)] \\
&= [\pi_2(e_\eta A_2\eta(A_1)e_\eta A_2)] \\
&= \pi_2(e_\eta A_2) \ ,
\end{aligned}
$$

by Lemma 1.2.1. Further, if $x = \pi_1(a_1)V\pi_2(e_\eta a_2)$ for $a_i \in A_i$, note that, by definition, we have $x(\hat{e}_\eta) = a_1 \otimes e_\eta a_2$ and hence, $[\bigcup\{ran\ x : x \in E\}] = \mathcal{H}_1(\eta)$. This shows that there exist projections $\{p_i : i \in I\} \subset [EE^*]$ such that $id_{\mathcal{H}_1(\eta)} = WOT - lim_i p_i$. Hence, we see that

$$
\pi_1(A_1) \subset [\bigcup\{\pi_1(A_1)p_i : i \in I\}] \subset [\pi_1(A_1)EE^*] \subset [EE^*] \ ;
$$

and we have verified everything need to see that the tuple $\mathcal{E}_\eta = (E, (\pi_1, \mathcal{H}_1(\eta)), (\pi_2, \mathcal{H}_2(\eta)))$ defines a standard Hilbert von Neumann $A_1 - e_\eta A_2$ - bimodule. As for the uniqueness assertion, if $(\widetilde{\mathcal{E}}, \widetilde{V})$ also works, then $\widetilde{\mathcal{H}_2} = L^2(e_\eta A_2, \widetilde{\phi})$ for some faithful normal state $\widetilde{\phi}$ on $e_\eta A_2$. In view of the 'uniqueness of the standard module of a von Neumann algebra' - see [Haa], for instance - there exists an $e_\eta A_2$ - linear unitary operator $U_2 : \mathcal{H}_2(\eta) \to \widetilde{\mathcal{H}_2}$. Observe next that if $\xi, \zeta \in \mathcal{H}_2$ and $a_1, b_1 \in A_1, a_2, b_2 \in e_\eta A_2$, then

$$
\begin{aligned}
&\langle \pi_1(a_1)V\pi_2(a_2)\xi, \pi_1(b_1)V\pi_2(b_2)\zeta\rangle \\
&= \langle \pi_2(b_2^*)V^*\pi_1(b_1^*a_1)V\pi_2(a_2)\xi, \zeta\rangle \\
&= \langle \pi_2(b_2^*)\pi_2(\eta(b_1^*a_1))\pi_2(a_2)\xi, \zeta\rangle \\
&= \langle \pi_2(b_2^*\eta(b_1^*a_1)a_2)\xi, \zeta\rangle \\
&= \langle U_2\pi_2(b_2^*\eta(b_1^*a_1)a_2)\xi, U_2\zeta\rangle \\
&= \langle \widetilde{\pi}_2(b_2^*\eta(b_1^*a_1)a_2)U_2\xi, U_2\zeta\rangle \\
&= \langle \widetilde{\pi}_2(b_2^*)\widetilde{V^*}\widetilde{\pi}_1(b_1^*a_1)\widetilde{V}\widetilde{\pi}_2(a_2)U_2\xi, U_2\zeta\rangle \\
&= \langle \widetilde{\pi}_1(a_1)\widetilde{V}\widetilde{\pi}_2(a_2)U_2\xi, \widetilde{\pi}_1(b_1)\widetilde{V}\widetilde{\pi}_2(b_2)U_2\zeta\rangle \ .
\end{aligned}
$$

Deduce from the above equation and the assumed non-degeneracy of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ that there is a unique unitary operator $U_1 : \mathcal{H}_1 \to \widetilde{\mathcal{H}_1}$ such that

$$U_1 \left(\pi_1(a_1) V \pi_2(a_2)\xi\right) = \widetilde{\pi}_1(a_1)\widetilde{V}\widetilde{\pi}_2(a_2)U_2\xi \tag{1.2.1}$$

for all $a_1 \in A_1, a_2 \in e_\eta A_2$ and $\xi \in \mathcal{H}_2(\eta)$ It is easy to see from equation (1.2.1) that U_1 is necessarily A_1 - linear, that $U_1 V = \widetilde{V} U_2$ or $\widetilde{V} = U_1 V U_2^*$ and that $\widetilde{\mathcal{E}} = U_1 \mathcal{E} U_2^*$, and the proof of the theorem is complete. □

Remark 1.2.4. *Notice that the irritating e_η above is equal to the 1 of A_2 in some good cases, such as the following:*

- *when η is unital, i.e., $\eta(1) = 1$;*
- *when $\eta(1) \neq 0$ and A_2 is a factor.*

The uniqueness assertion in Theorem 1.2.3 can also be deduced from the following useful criterion for isomorphism of standard bimodules:

Lemma 1.2.5. *Two standard Hilbert von Neumann A_2 bimodules $\mathcal{E}^{(i)} = (E^{(i)}, (\pi_1^{(i)}, \mathcal{H}_1^{(i)}), (\pi_2^{(i)}, \mathcal{H}_2^{(i)})), i = 1, 2$ are isomorphic if and only if there exist $E_0^{(i)} = \{x_j^{(i)} : j \in I\} \subset E^{(i)}$ such that*

1. *$[E_0^{(i)}] = E^{(i)}$, and*
2. *$(\pi_2^{(1)})^{-1}(x_j^{(1)*} x_k^{(1)}) = (\pi_2^{(2)})^{-1}(x_j^{(2)*} x_k^{(2)})\ \forall j, k \in I$*

Proof. The only if implication is clear, as we may choose $E_0^{(i)} = E^{(i)}$ and $x^{(2)} = U_1 x^{(1)} U_2^*$ for all $x^{(1)} \in E^{(1)} (= I)$. Now for the other 'if half'.

In view of the 'uniqueness of the standard module of a von Neumann algebra - see [Haa] -there exists an A_2 - linear unitary operator $U_2 : \mathcal{H}_2^{(1)} \to \mathcal{H}_2^{(2)}$. For arbitrary

$j, k \in I, \xi_1, \xi_2 \in \mathcal{H}_2^{(1)}$, observe that

$$\begin{aligned}
\langle x_j^{(1)}\xi_1, x_k^{(1)}\xi_2\rangle &= \langle \xi_1, x_j^{(1)*}x_k^{(1)}\xi_2\rangle \\
&= \langle U_2\xi_1, U_2\pi_2^{(1)}(\pi_2^{(1)})^{-1}(x_j^{(1)*}x_k^{(1)})\xi_2\rangle \\
&= \langle U_2\xi_1, \pi_2^{(2)}(\pi_2^{(1)})^{-1}(x_j^{(1)*}x_k^{(1)})U_2\xi_2\rangle \\
&= \langle U_2\xi_1, \pi_2^{(2)}(\pi_2^{(2)})^{-1}(x_j^{(2)*}x_k^{(2)})U_2\xi_2\rangle \\
&= \langle x_j^{(2)}U_2\xi_1, x_k^{(2)}U_2\xi_2\rangle \ ;
\end{aligned}$$

deduce from the above equation and the non-degeneracy of the $\mathcal{E}^{(i)}$ that there exists a unique unitary operator $U_1 : \mathcal{H}_1^{(1)} \to \widetilde{\mathcal{H}_1^{(2)}}$ such that $U_1(x_j^{(1)}\xi) = x_j^{(2)}U_2\xi \ \forall j \in I, \xi \in \mathcal{H}_2^{(1)}$. The definitions show that $U_1 x_j^{(1)} = x_j^{(2)}U_2 \ \forall j \in I$ and hence that $U_1E^{(1)} = E^{(2)}U_2$. Thus indeed $E^{(2)} = U_1E^{(1)}U_2^*$ and the proof of the 'if half' is complete.

$\square$

Notice, incidentally, that in the setting of the Lemma above, the equation

$$Tx^{(1)} = U_1x^{(1)}U_2^*$$

defines a WOT-continuous linear bijection $T : E^{(1)} \to E^{(2)}$ satisfying

$$Tx^{(1)}(Ty^{(1)})^*Tz^{(1)} = T(x^{(1)}(y^{(1)})^*z^{(1)})$$

for all $x^{(1)}y^{(1)}, z^{(1)} \in E^{(1)}$.

Remark 1.2.6. *1. The 'generator' V of Theorem 1.2.3 is an isometry precisely when η is unital.*

2. If $\mathcal{E}$ is a singly generated Hilbert von Neumann A_1-A_2 bimodule, then it is generated by a partial isometry (by Lemma 1.1.6). Further, that generator, say V may be used *to define the obviously completely positive map $\eta; A_1 \to A_2$ by*

$$\eta(a_1) = \pi_2^{-1}(V^*\pi_1(a_1)V) \ ;$$

and then $\mathcal{E}$ would be isomorphic to $\mathcal{E}_\eta$ if and only if $\mathcal{E}$ is a standard non-degenerate bimodule.

1.3 Connes fusion

Example 1.3.1. If $\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ is a Hilbert von Neumann $A_1 - A_2$ - bimodule and $\mathcal{K}$ is any Hilbert space, then $\mathcal{E} \otimes id_{\mathcal{K}} = (E \otimes id_{\mathcal{K}}, (\pi_1 \otimes id_{\mathcal{K}}, \mathcal{H}_1 \otimes \mathcal{K}), (\pi_2 \otimes id_{\mathcal{K}}, \mathcal{H}_2 \otimes \mathcal{K}))$ is also a Hilbert von Neumann $A_1 - A_2$ - bimodule, where of course we write $E \otimes id_{\mathcal{K}}$ for $\{x \otimes id_{\mathcal{K}} : x \in E\}$.

Lemma 1.3.2. *Let $\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ be a Hilbert von Neumann $A_1 - A_2$ - bimodule. For a projection $p \in \mathcal{P}(\pi_2(A_2)')$, let q be the projection with range $[\bigcup\{ran(xp) : x \in E\}]$. Then*

1. *$q \in \mathcal{P}(\pi_1(A_1)')$;*
2. *$y \in E \Rightarrow qyp = qy = yp$; and*
3. *$q\mathcal{E}p = (qEp, (q\pi_1(\cdot), q\mathcal{H}_1), (p\pi_2(\cdot), p\mathcal{H}_2))$ satisfies all the requirements for a non-degenerate Hilbert von Neumann $A_1 - A_2$ - bimodule, with the possible exception of injectivity of $p\pi_2(\cdot)$.*

We shall use the suggestive notation $\mathcal{E}_* p = q$ when $q, \mathcal{E}, p$ are so related.

Proof. 1. Since $\pi_1(A_1)E \subset E$, it follows that $ran(q)$ is stable under $\pi_1(A_1)$.

2. For all $y \in E$, $ran(yp) \subset ran(q) \Rightarrow qyp = yp$. Next, if $\xi, \zeta \in \mathcal{H}_2$, and $x, y \in E$,

note that

$$\begin{aligned} \langle xp\xi, y(1-p)\zeta\rangle &= \langle \xi, px^*y(1-p)\zeta\rangle \\ &\in \langle \xi, p\,[E^*E]\,(1-p)\zeta\rangle \\ &= 0\ , \end{aligned}$$

since $[E^*E] = \pi_2(A_2) \subset \{p\}'$; since $\{xp\xi : \xi \in \mathcal{H}_2\}$ is total in $ran(q)$, this says that $qy(1-p) = 0$, as desired.

3.

$$[(qEp)^*(qEp)] = [(Ep)^*(Ep)] = p\,[E^*E]\,p = p\pi_2(A_2) \tag{1.3.1}$$

since $[E^*E] = \pi_2(A) \subset \{p\}'$; while

$$[(qEp)(qEp)^*] = q\,[EE^*]\,q \supset q\pi_1(A_1). \tag{1.3.2}$$

Non-degeneracy of $q\mathcal{E}p$ follows immediately from equations (1.3.1) and (1.3.2). □

Remark 1.3.3. *In general, if $\pi : M \to \mathcal{L}(\mathcal{H})$ is a faithful normal representation, and if $p \in \pi(M)'$, the sub-representation $p\pi(\cdot)$ is faithful if and only if the central support of p is 1 - i.e.,* $\sup\{upu^* : u \in \pi(M)'\} = 1$.

In particular if the $\mathcal{E}$ of Lemma 1.3.2 is actually a Hilbert von Neumann $A_1 - A_2$ - bimodule, and if A_2 happens to be a factor, then the $q\mathcal{E}p$ of Lemma 1.3.2 is actually a Hilbert von Neumann bimodule.

We next lead to our description of what is sometimes termed 'internal tensor product' but which we prefer (in view of this terminology being already in use for tensor products of bimodules over von Neumann algebras) to refer to as the *Connes fusion* of Hilbert von Neumann bimodules. Thus, suppose $\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ is a Hilbert von Neumann $A_1 - A_2$ - bimodule and $\mathcal{F} = (F, (\rho_2, \mathcal{K}_2), (\rho_3, \mathcal{K}_3))$ is a Hilbert von Neumann $A_2 - A_3$ - bimodule. We know that the normal representation ρ_2 of A_2 is equivalent to a sub-representation of an infinite ampliation of the faithful normal representation π_2 of

A_2; thus there exists an A_2 - linear isometry $u : \mathcal{K}_2 \to \mathcal{H}_2 \otimes \ell^2$: i.e., $u^*u = id_{\mathcal{K}_2}$ and $u\rho_2(x) = (\pi_2(x) \otimes id_{\ell^2})u \ \forall x \in A_2$. It follows that $p = uu^* \in (\pi_2(A_2) \otimes id_{\ell^2})'$.

Now, set $p = uu^*$ and let $q = (\mathcal{E} \otimes 1_{\ell^2})_*(p)$ be associated to this p as in Lemma 1.3.2 (applied to $\mathcal{E} \otimes 1_{\ell^2}$).

Finally, if $x \in E, y \in F$, define $x \bigodot y$ to be the composite operator

$$\mathcal{K}_3 \stackrel{x \bigodot y}{\longrightarrow} q(\mathcal{H}_1 \otimes \ell^2) \ = \ \mathcal{K}_3 \stackrel{y}{\longrightarrow} \mathcal{K}_2 \stackrel{u}{\longrightarrow} uu^*(\mathcal{H}_2 \otimes \ell^2) \stackrel{x \otimes id_{\ell^2}}{\longrightarrow} q(\mathcal{H}_1 \otimes \ell^2) \ ,$$

set $E \bigodot F = [\{x \bigodot y : x \in E, y \in F\}]$; and finally define the Connes fusion of $\mathcal{E}$ and $\mathcal{F}$ to be

$$\mathcal{E} \otimes_{A_2} \mathcal{F} = (E \bigodot F, (q(\pi_1 \otimes id_{\ell^2})|_{ran\ q}, q(\mathcal{H}_1 \otimes \ell^2)), (\rho_3, \mathcal{K}_3)) \ . \tag{1.3.3}$$

The justification for our use of 'Connes fusion' for our construction lies (at least for standard bimodules, by Lemma 1.2.5) in the fact that (in the notation defining Connes fusion) the A_3 - valued inner product on $\mathcal{E} \circ \mathcal{F}$ satisfies

$$\begin{aligned}
\langle x_1 \bigodot y_1, x_2 \bigodot y_2 \rangle_{A_3} &= (x_1 \bigodot y_1)^*(x_2 \bigodot y_2) \\
&= ((x_1 \otimes id_{\ell^2})uy_1)^*(x_2 \otimes id_{\ell^2})uy_2 \\
&= y_1^* u^*(x_1^* x_2 \otimes id_{\ell^2})uy_2 \\
&= y_1^*(x_1^* x_2)y_2 \quad \text{(since } u \text{ is an } A_2 \text{ - linear isometry)} \\
&= y_1^* \langle x_1, x_2 \rangle_{A_2} y_2 \\
&= \langle y_1, \langle x_1, x_2 \rangle_{A_2} y_2 \rangle_{A_3} \ .
\end{aligned}$$

Proposition 1.3.4. *The Connes fusion of (non-degenerate) Hilbert von Neumann bimodules is again a (non-degenerate) Hilbert von Neumann bimodule.*

Proof. Clearly $E \bigodot F$ is a WOT-closed linear space of operators between the asserted spaces. Observe next that

$$
\begin{aligned}
\left[(E \bigodot F)(E \bigodot F)^*\right] &= [\{((x_1 \otimes id_{\ell^2})uy_1)((x_2 \otimes id_{\ell^2})uy_2)^* : x_i \in E, y_j \in F\}] \\
&= [\{(x_1 \otimes id_{\ell^2})uy_1y_2^*u^*(x_2 \otimes id_{\ell^2})^* : x_i \in E, y_j \in F\}] \\
&= [\{(x_1 \otimes id_{\ell^2})u\,[FF^*]\,u^*(x_2 \otimes id_{\ell^2})^* : x_i \in E\}] \\
&\supset [\{(x_1 \otimes id_{\ell^2})u\rho_2(A_2)u^*(x_2 \otimes id_{\ell^2})^* : x_i \in E\}] \\
&= [\{(x_1 \otimes id_{\ell^2})(\pi_2(A_2) \otimes id_{\ell^2})uu^*(x_2 \otimes id_{\ell^2})^* : x_i \in E\}] \\
&= [(E \otimes id_{\ell^2})uu^*(E \otimes id_{\ell^2})^*] \quad (\text{since } E\pi_2(A_2) = E) \\
&= q(\pi_1(A_1) \otimes id_{\ell^2})
\end{aligned}
$$

(in particular $q \in [(E \bigodot F)(E \bigodot F)^*]$) and that

$$
\begin{aligned}
\left[(E \bigodot F)^*(E \bigodot F)\right] &= [\{((x_1 \otimes id_{\ell^2})uy_1)^*(x_2 \otimes id_{\ell^2})uy_2) : x_i \in E, y_j \in F\}] \\
&= [\{(y_1^*u^*(x_1^*x_2 \otimes id_{\ell^2}))uy_2) : x_i \in E, y_j \in F\}] \\
&= [\{(y_1^*u^*(\pi_2(A_2) \otimes id_{\ell^2})uy_2) : y_j \in F\}] \\
&= [\{(y_1^*u^*u\,[\rho_2(A_2)]\,y_2) : y_j \in F\}] \\
&= [\{(y_1^*(\rho_2(A_2))y_2) : y_j \in F\}] \\
&= F^*F \quad (*) \\
&= \rho_3(A_3)\ ,
\end{aligned}
$$

where the justification for the step labelled (*) is that $\rho_2(A_2)F = F$ (see Remark 1.1.5 (7)). This completes the verification that $\mathcal{E} \otimes_{A_2} \mathcal{F}$ is indeed a Hilbert von Neumann $A_1 - A_3$ bimodule.

Now, suppose $\mathcal{E}$ and $\mathcal{F}$ are both non-degenerate. Then

$$\begin{aligned}
&\xi \in \bigcap \{ker z : z \ in \ E \bigodot F\} \\
&\quad \Rightarrow \quad (x \otimes id_{\ell^2}) u y \xi = 0 \ \forall x \in E, y \in F \\
&\quad \Rightarrow \quad u y \xi = 0 \ \forall y \in F \quad (\text{as } E \otimes id_{\ell^2} \text{ is non-degenerate}) \\
&\quad \Rightarrow \quad y \xi = 0 \ \forall y \in F \quad (\text{as } u \text{ is isometric}) \\
&\quad \Rightarrow \quad \xi = 0 \quad (\text{as } F \text{ is non-degenerate}) \ ;
\end{aligned}$$

while

$$\begin{aligned}
&\left[\bigcup \{ran((x \otimes id_{\ell^2}) u y) : x \in E, y \in F\}\right] \\
&\quad = \quad \left[\bigcup \{ran((x \otimes id_{\ell^2}) u) : x \in E\}\right] \quad (\text{since } F \text{ is non-degenerate}) \\
&\quad = \quad \left[\bigcup \{ran((x \otimes id_{\ell^2}) u u^*) : x \in E\}\right] \\
&\quad = \quad ran \ q \quad (\text{by } \ \text{definition})
\end{aligned}$$

and hence $E \bigodot F$ is indeed non-degenerate.

□

Before addressing the question of the dependence of the definition of Connes fusion and the seemingly *ad hoc* A_2 - linear partial isometry u, we introduce a necessary definition and the ubiquitous lemma.

Definition 1.3.5. *Two Hilbert von Neumann A_2 modules, say $\mathcal{E}^{(i)} = (E^{(i)}, \mathcal{H}_1^{(i)}, (\pi_2^{(i)}, \mathcal{H}_2^{(i)})), i = 1, 2$ are considered* isomorphic *if there exists unitary operators $w_j : \mathcal{H}_j^{(1)} \to \mathcal{H}_j^{(2)}$, with w_2 being A_2 - linear, such that*

$$E^{(2)} = w_1 E^{(1)} w_2^* \ .$$

If the $\mathcal{E}^{(i)}$ happen to be $A_1 - A_2$ bimodules, they are said to be isomorphic if, in addition to the above, the unitary w_1 happens to be A_1 - linear.

Lemma 1.3.6. *Let* $\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ *be a Hilbert von Neumann* $A_1 - A_2$ *bi-module. Suppose* $w \in \pi_2(A_2)'$ *is a partial isometry with* $w^*w = p, ww^* = \widetilde{p}$. *Let* $q = \mathcal{E}_* p$ *and* $\widetilde{q} = \mathcal{E}_* \widetilde{p}$ *in the notation of Lemma 1.3.2. Then there exists a unique partial isometry* $w_1 \in \pi_1(A_1)'$ *such that* $w_1^* w_1 = q, w_1 w_1^* = \widetilde{q}$.

Proof. We first assert that there is a unique unitary operator $W_1 : q(\mathcal{H}_1) \to \widetilde{q}(\mathcal{H}_1)$ satisfying $WTp = Tw\ \forall T \in E$. This is because:

- $(T_1 w)^*(T_2 w) = w^* T_1^* T_2 w = T_1^* T_2 p = p^* T_1^* T_2 p,\ \forall T_1, T_2 \in E$ and
- $q(\mathcal{H}_1) = [\bigcup\{ran(Tp) : T \in E\}]$ and $\widetilde{q}(\mathcal{H}_1) = [\bigcup\{ran(Tw) : T \in E\}]$ (since $ran\ w = ran\ \widetilde{p}$).

Finally $w_1 = W_1 q$ does the job. $\square$

Remark 1.3.7.

1. We now verify that the definition we gave of $\mathcal{E} \otimes_{A_2} \mathcal{F}$ is really independent of the choice of the isometry u used in that definition. Indeed, suppose $u, \widetilde{u} : \mathcal{K}_2 \to \mathcal{H}_2 \otimes \ell^2$ are two A_2 - linear isometries. If $uu^* = p, \widetilde{u}\widetilde{u}^* = \widetilde{p}$, then $w = \widetilde{u}u^*$ is a partial isometry in $(\pi_2(A_2) \otimes id_{\ell^2})'$ with $w^*w = p, ww^* = \widetilde{p}$. Now apply Lemma 1.3.6 to $\mathcal{E} \otimes id_{\ell^2}$ and $w, p, \widetilde{p}$ to find a $W \in (\pi_1(A_1) \otimes id_{\ell^2})'$ such that $W^*W = q = (\mathcal{E} \otimes id_{\ell^2})_* p$ and $WW^* = \widetilde{q} = (\mathcal{E} \otimes id_{\ell^2})_* \widetilde{p}$. Then, as the proof of Lemma 1.3.6 shows, $W : q(\mathcal{H}_1 \otimes \ell^2) \to \widetilde{q}(\mathcal{H}_1 \otimes \ell^2)$ is a unitary operator satisfying $W(x \otimes id_{\ell^2})p = (x \otimes id_{\ell^2})w\ \forall x \in E$. It is now a routine matter to verify that the unitary operators $W : q(\mathcal{H}_1 \otimes \ell^2) \to \widetilde{q}(\mathcal{H}_1 \otimes \ell^2)$ and $id_{\mathcal{K}_3}$ establish an isomorphism between the models of $\mathcal{E} \otimes_{A_2} \mathcal{F}$ given by u and $\widetilde{u}$ are isomorphic.

2. A not dissimilar reasoning shows that the isomorphism type of the Connes fusion of two standard bimodules depends only on the isomorphism classes of the two 'factors' in the fusion, and is also standard.

3. If $\mathcal{E}$ is only a Hilbert von Neumann A_2-module, and $\mathcal{F}$ is a Hilbert von Neumann $A_2 - A_3$-bimodule, their Connes fusion $\mathcal{E} \otimes_{A_2} \mathcal{F}$ would still make sense as a Hilbert von Neumann A_3-module.

1.4 Examples

We now discuss some examples of Hilbert von Neumann (bi)modules.

1. The simplest (non-degenerate) example is obtained when $A_j = \mathcal{L}(\mathcal{H}_j), \pi_j = id_{A_j}$ for $j = 1, 2$ and $E = \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$; all the verifications reduce just to matrix multiplication.

2. Suppose A_2 is a unital von Neumann subalgebra of A_1, and suppose there exists a faithful normal conditional expectation $\epsilon : A_1 \to A_2$. Let ϕ_2 be a faithful normal state (even semi-finite weight will do). Let $\phi_1 = \phi_2 \circ \epsilon, \mathcal{H}_j = L^2(A_j, \phi_j)$, and let π_j be the left regular representation of A_j on $\mathcal{H}_j$. Write U for the natural isometric identification of $\mathcal{H}_2$ as a subspace of $\mathcal{H}_1$ (so that the 'Jones projection' will be just UU^*). Finally, define

$$\mathcal{E}_{(A_2 \subset A_1)} = (\pi_1(A_1)U, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$$

   In this case, we find that $[EE^*] = [\pi_1(A_1) e \pi_1(A_1)]$, and we find the 'basic construction of Jones appearing naturally in this context.

   Further, it is a consequence of the uniqueness assertion in Theorem 1.2.3 that $\mathcal{E}_\epsilon \cong \mathcal{E}_{(A_2 \subset A_1)}$.

3. Suppose $(M, \mathcal{H}, \mathcal{J}, P)$ is a standard form of M in the sense of [Haa75]. As indicated in [Haa], there is a canonical 'implementing' unitary representation

$$Aut(M) \ni \theta \mapsto u_\theta \in \mathcal{L}(\mathcal{H})$$

   satisfying $u_\theta x u_\theta^* = \theta(x) \; \forall x \in M$. We have the natural Hilbert von Neumann $M-M$ bimodule given by

$$\mathcal{E}_\theta = (M u_\theta, (id_M, L^2(M)), (id_M, L^2(M)))$$

4. If $\theta, \phi \in Aut(M), M$ are as in the previous example, we see now that 'Connes fusion corresponds to composition' in this case:

$$\mathcal{E}_\theta \otimes_M \mathcal{E}_\phi \cong \mathcal{E}_{\theta\phi}$$

(*Reason:* The 'u' in the definition of Connes fusion is just id_M, while

$$Mu_\theta M u_\phi = M\theta(M)u_\theta u_\phi = Mu_{\theta\phi} \text{ .)}$$

Proposition 1.4.1. *If $\theta, \phi \in Aut(M)$ are as in Example (4) above, then $\mathcal{E}_\theta \cong \mathcal{E}_\phi$ if and only if θ and ϕ are inner conjugate.*

Proof. First, note that any M-linear unitary operator on $L^2(M)$ has the form $\mathcal{J}v^*\mathcal{J}$ for some unitary $v \in M$, where of course $\mathcal{J}$ denotes the modular conjugation operator. Observe next that each u_θ commutes with $\mathcal{J}$ since θ is a *-preserving map, and hence, for any $x \in M$, we have

$$u_\theta \mathcal{J} v^* \mathcal{J} = \mathcal{J}\theta(v^*)\mathcal{J}u_\theta \tag{1.4.1}$$

If $\mathcal{E}_\theta$ is isomorphic to $\mathcal{E}_\phi$, there must exist unitary $v_1, v_2 \in M$ such that

$$\begin{aligned} Mu_\phi &= \mathcal{J}v_1^*\mathcal{J}Mu_\theta\mathcal{J}v_2\mathcal{J} \\ &= M\mathcal{J}v_1^*\mathcal{J}u_\theta\mathcal{J}v_2\mathcal{J} \\ &= M\mathcal{J}v_1^*\mathcal{J}\mathcal{J}\theta(v_2)\mathcal{J}u_\theta \\ &= M\mathcal{J}v_1^*\theta(v_2)\mathcal{J}u_\theta \ ; \end{aligned}$$

in particular, there must exist a $y \in M$ such that

$$u_\phi = y\mathcal{J}v_1^*\theta(v_2)\mathcal{J}u_\theta \ .$$

We find that y is necessarily unitary and hence, writing u for y and v for $v_1^*\theta(v_2)$, we see that there must be a unitary $u \in M$ such that

$$\begin{aligned} \phi(x) &= u_\phi x u_\phi^* \\ &= u\mathcal{J}v\mathcal{J}u_\theta x u_\theta^* \mathcal{J}v^*\mathcal{J}u^* \\ &= u\mathcal{J}v\mathcal{J}\theta(x)\mathcal{J}v^*\mathcal{J}u^* \\ &= u\theta(x)u^* \ . \end{aligned}$$

In other words, ϕ and θ are indeed inner conjugate.

Conversely, if $\phi(\cdot) = u\theta(\cdot)u^*$ for some unitary $u \in M$, we see that $u_\phi = u\mathcal{J}u\mathcal{J}u_\theta = uu_\theta\mathcal{J}\theta^{-1}(u)\mathcal{J}$; so we find that $w_1 = id_M$ and $w_2 = \mathcal{J}\theta^{-1}(u)^*\mathcal{J}$ define M - linear unitary operators on $L^2(M)$ such that $Mu_\phi = Muu_\theta w_2^* = w_1 M u_\theta w_2^*$, thereby establishing that $\mathcal{E}_\theta \cong \mathcal{E}_\phi$. □

Chapter 2

On Extendability of endomorphisms and of E_0-semigroups on factors

In this chapter we begin with a von Neumann algebraic version of the elementary but extremely useful fact about being able to extend inner-product preserving maps from a total set of the domain Hilbert space to an isometry defined on the entire domain. This leads us to the notion of when a well-behaved (equi-modular, as we term it) endomorphism of a factorial probability space (M, ϕ) admits a natural extension to an endomorphism of $L^2(M, \phi)$. After deriving some equivalent conditions under which an endomorphism is extendable, we exhibit examples of such extendable endomorphisms.

We then pass to E_0-semigroups $\alpha = \{\alpha_t : t \geq 0\}$ of factors, and observe that extendability of this semigroup (i.e., extendability of each α_t) is a cocycle-conjugacy invariant of the semigroup. We conclude by giving examples of extendable E_0-semigroups, and by showing that the Clifford flow on the hyperfinite II_1 factor is not extendable, neither is the free flow.

Our notion of extendable E_0-semigroups is related to a notion called 'regular semigroups' in [ABS01], where they erroneously claim to prove that the Clifford flow is extendable.

2.1 An existence result

We start by setting up some notation. For any index set I, we write $I^* = \bigcup_{n=0}^{\infty} I^n$ where $I^0 = \emptyset$, and $\mathbb{I} \vee \mathbb{J} = (i_1, \cdots, i_m) \vee (j_1, \cdots, j_n) = (i_1, \cdots, i_m, j_1, \cdots, j_n)$ whenever $\mathbb{I} = (i_1, \cdots, i_m), \mathbb{J} = (j_1, \cdots, j_n) \in I^*$.

By a von Neumann probability space, we shall mean a pair (M, ϕ) consisting of a von Neumann algebra and a normal state. For such an (M, ϕ), and an $x \in M$, we shall write $\hat{x} = \lambda_M(x)\widehat{1_M}$ and $\widehat{1_M}$ for the cyclic vector for $\lambda_M(M)$ in $L^2(M, \phi)$, where λ_M is the left regular representation of M.

Recall that the central support of the normal state ϕ is the central projection $z(=: z_\phi)$ such that $ker(\lambda_M) = M(1 - z)$. Clearly $z_\phi = 1_M$ if M is a factor.

Finally, if $\{x_i : i \in I\} \subset M$, and $\mathbb{I} = (i_1, \cdots, i_n) \in I^n$, we shall write $x_{\mathbb{I}} = x_{i_1} x_{i_2} \cdots x_{i_n}$. We also use $[S]$ either to denote the norm (respectively strong) closure of the span, for $S \subseteq \mathcal{H}$ (respectively $S \subseteq \mathcal{L}(\mathcal{H})$), for any Hilbert space $\mathcal{H}$.

Proposition 2.1.1. *Let $(M_i, \phi_i), i = 1, 2$ be von Neumann probability spaces with $z_i = z_{\phi_i}$. Suppose $S^{(j)} = \{x_i^{(j)} : i \in I\}$ is a set of self-adjoint elements which generates M_j as a von Neumann algebra, for $j = 1, 2$. (Note the crucial assumption that both the $S^{(j)}$ are indexed by the same set.) Suppose*

$$\phi_1(x_{\mathbb{I}}^{(1)}) = \phi_2(x_{\mathbb{I}}^{(2)}) \ \forall \mathbb{I} \in I^* \ . \tag{2.1.1}$$

Then there exists a unique isomorphism $\theta : M_1 z_1 \to M_2 z_2$ such that $\phi_2 \circ \theta|_{M_1 z_1} = \phi_1|_{M_1 z_1}$ and $\theta(x_i^{(1)} z_1) = x_i^{(2)} z_2 \ \forall i \in I$.

Proof. The hypothesis implies that, for $j = 1, 2$, the set $\{x_{\mathbb{I}}^{(j)} : \mathbb{I} \in I^*\}$ linearly spans a *-subalgebra which is necessarily σ-weakly dense in M_j. Since $\langle \widehat{x_{\mathbb{I}}^{(1)}}, \widehat{x_{\mathbb{J}}^{(1)}} \rangle = \langle \widehat{x_{\mathbb{I}}^{(2)}}, \widehat{x_{\mathbb{J}}^{(2)}} \rangle \ \forall \mathbb{I}, \mathbb{J} \in I^*$, there exists a unique unitary operator $u : L^2(M_1, \phi_1) \to L^2(M_2, \phi_2)$ such that $u\widehat{x_{\mathbb{I}}^{(1)}} = \widehat{x_{\mathbb{I}}^{(2)}} \ \forall \mathbb{I} \in I^*$.

Now observe that

$$\begin{aligned} u\lambda_{M_1}(x_{\mathbb{I}}^{(1)})u^*\widehat{x_{\mathbb{J}}^{(2)}} &= u\lambda_{M_1}(x_{\mathbb{I}}^{(1)})\widehat{x_{\mathbb{J}}^{(1)}} \\ &= u\widehat{x_{\mathbb{I}\vee\mathbb{J}}^{(1)}} \\ &= \widehat{x_{\mathbb{I}\vee\mathbb{J}}^{(2)}} \\ &= \lambda_{M_2}(x_{\mathbb{I}}^{(2)})\widehat{x_{\mathbb{J}}^{(2)}}\ ; \end{aligned}$$

and hence that $u\lambda_{M_1}(x_{\mathbb{I}}^{(1)})u^* = \lambda_{M_2}(x_{\mathbb{I}}^{(2)})\ \forall \mathbb{I} \in I$.

On the other hand, $\{x \in M_1 : u\lambda_{M_1}(x)u^* \in \lambda_{M_2}(M_2)\}$ is clearly a von Neumann subalgebra of M_1; since this has been shown to contain each $x_{\mathbb{I}}^{(1)}$, we may deduce that this must be all of M_1. Now notice that $L^2(M_j, \phi_j) = L^2(M_j z_j, \phi_j|_{M_j z_j})$, that $\lambda_{M_j}(x) = \lambda_{M_j z_j}(x z_j)\ \forall x \in M_j$, and that $\lambda_{M_j z_j}$ maps $M_j z_j$ isomorphically onto its image.

The proof is completed by defining

$$\theta(x) = \lambda_{M_2 z_2}^{-1}(u\lambda_{M_1}(x)u^*)\ \forall x \in M_1 z_1.$$

□

Remark 2.1.2. *1. In the proposition, even if it is the case that $N := \{x_i^{(2)} : i \in I\}'' \subsetneq M_2$, we can still apply the result to $(N, \phi_2|_N)$ in place of (M_2, ϕ_2) and deduce the existence of a normal homomorphism of M_1 into M_2 which sends $x_i^{(1)}$ to $x_i^{(2)} z$ for each i (and 1_{M_1} to the projection $z = z_{\phi_2|_N} \in N$).*

2. In the special case that the N of the last paragraph is a factor, the z there is nothing but id_{M_2} and in particular, Proposition 2.1.1 can be strengthened as follows:

Let $(M_j, \phi_j), j = 1, 2$ be von Neumann probability spaces. Suppose $S^{(j)} = \{x_i^{(j)} : i \in I\} \subset M_j$ is a set of self-adjoint elements such that $S^{(1)''} = M_1$ and $S^{(2)''}$ is a factor $N \subset M_2$. Suppose

$$\phi_1(x_{\mathbb{I}}^{(1)}) = \phi_2(x_{\mathbb{I}}^{(2)})\ \forall \mathbb{I} \in I^*\ . \tag{2.1.2}$$

Then there exists a unique normal $*$-homomorphism $\theta : M_1 \to N \subset M_2$ such that $\theta(x_i^{(1)}) = x_i^{(2)}$ for all $i \in I$.

Corollary 2.1.3. *1. If θ_i is a ϕ_i-preserving unital endomorphism of a von Neumann probability space (M_i, ϕ_i), for $i \in \Lambda$, then there exists:*

(a) a unique unital endomorphism $\otimes_{i\in\Lambda}\theta_i$ of the tensor product $(\otimes_{i\in\Lambda}M_i, \otimes_{i\in\Lambda}\phi_i)$ such that

$$(\otimes_{i\in\Lambda}\theta_i)(\otimes_{i\in\Lambda}x_i) = z(\otimes_{i\in\Lambda}\theta_i(x_i)) \ \forall x_i = x_i^* \in M_i;$$

(b) a unique unital endomorphism $*_{i\in\Lambda}\theta_i$ of the free product (see [VDN92]) $(*_{i\in\Lambda}M_i, *_{i\in\Lambda}\phi_i)$ such that

$$(*_{i\in\Lambda}\theta_i)(\lambda(x_j)) = z\lambda(\theta_j(x_j)) \ \forall x_j \in M_j$$

where we simply write λ for each 'left-creation representation' $\lambda : M_j \to \mathcal{L}(*_{i\in\Lambda}L^2(M_i, \phi_i))$ for every $j \in I$.

In the above existence assertions, the symbol z represents an appropriate projection (= image of the identity of the domain of the endomorphism in question).

2. If each M_i above is a factor, then (the z in the above statement can be ignored, as it is the identity of the appropriate algebra) and all endomorphisms above are unital monomorphisms.

Proof. It is not hard to see that Remark 2.1.2(1) is applicable to $S^{(1)} = \{\otimes_i x_i : x_i = x_i^* \in M_i, x_i = 1_{M_i}$ for all but finitely many $i\}$ and $S^{(2)} = \{\otimes_i\theta_i(x_i) : x_i = x_i^* \in M_i, x_i = 1_{M_i}$ for all but finitely many $i\}$ (resp., $S^{(1)} = \{\lambda(x_i) : i \in \Lambda, x_i = x_i^* \in M_i, \phi_i(x_i) = 0\}$ and $S^{(2)} = \{\lambda(\theta_i(x_i)) : i \in \Lambda, x_i = x_i^* \in M_i, \phi_i(x_i) = 0\}$.

The second fact follows from Remark 2.1.2(2) because normal endomorphisms of factors are unital isomorphisms onto their images, and the tensor (resp., free) product of factors is a factor. $\square$

For later reference, the next lemma identifies the central support z_ϕ of a normal state ϕ on a von Neumann algebra in the simple special case when ϕ is a vector-state.

Lemma 2.1.4. *Suppose $N \subset \mathcal{L}(\mathcal{H})$ is a von Neumann algebra, $\xi \in \mathcal{H}$ is a unit vector, and ϕ is the vector state defined on N by $\phi(x) = \langle x\xi, \xi\rangle$. If $\mathcal{H}_0 = \overline{N\xi}$, then a candidate for 'the GNS triple for (N, ϕ)' is given by $(\mathcal{H}_0, id_N|_{\mathcal{H}_0}, \xi)$. In particular, the central support of ϕ is given by the projection $z = \wedge\{p \in N : ran\ p \supset \mathcal{H}_0\}$ and $ran\ z = [N'N\xi]$.*

Proof. It is clear that ξ is a cyclic vector for $N|_{\mathcal{H}_0}$ and the assertion regarding GNS triples follows. Hence if $z \in \mathcal{P}(Z(N))$ is such that $N(1-z) = ker\ id_N|_{\mathcal{H}_0}$, then $z = \wedge\{p \in \mathcal{P}(N) : p|_{\mathcal{H}_0} = (1_N)|_{\mathcal{H}_0}\}$, i.e., $z = \wedge\{p \in \mathcal{P}(N) : ran\ p \supset \mathcal{H}_0\}$. As z is the smallest projection in $(N \cap N')$ whose range contains $N\xi$, or equivalently the smallest subspace containing $[N\xi]$ which is invariant under $(N \cap N')'$, equivalently invariant under $N'N$, the last assertion follows. $\square$

2.2 Extendable endomorphisms

For the remainder of this paper, we make the standing assumption that ϕ is a faithful normal state on a factor M. We identify $x \in M$ with $\lambda_M(x)$, and simply write $\mathcal{J}$ and Δ for the modular conjugation operator $\mathcal{J}_\phi$ and the modular operator Δ_ϕ respectively. Recall, thanks to the Tomita-Takesaki theorem that $j = \mathcal{J}(\cdot)\mathcal{J}$ is a *-preserving conjugate-linear isomorphism of $\mathcal{L}(L^2(M,\phi))$ onto itself, which maps M and M' onto one another, and that $\widehat{1_M}$ is also a cyclic and separating vector for M'. We shall assume that θ is a normal unital *-endomorphism which preserves ϕ. The invariance assumption $\phi \circ \theta = \phi$ implies that there exists a unique isometry u_θ on $L^2(M,\phi)$ such that $u_\theta x\widehat{1_M} = \theta(x)\widehat{1_M}$ and equivalently, that $u_\theta x = \theta(x)u_\theta\ \forall x \in M$ and $u_\theta\widehat{1_M} = \widehat{1_M}$.

Definition 2.2.1. *If M, ϕ, θ are as above, and if the associated isometry u_θ of $L^2(M,\phi)$ commutes with the modular conjugation operator $\mathcal{J}(=\mathcal{J}_\phi)$, we shall simply say θ is a* **equi-modular** *(as this is related to endomorphisms commuting with the modular au-*

tomorphism group) endomorphism of the factorial non-commutative probability space (M, ϕ).

Remark 2.2.2. *It is true that if θ is an equi-modular endomorphism of a factor M as above, then there always exists a ϕ-preserving faithful normal conditional expectation $E : M \to \theta(M)$, and in fact $u_\theta u_\theta^*$ is the Jones projection associated to this conditional expectation. For this, notice to start with, that as θ is a *-homomorphism, u_θ commutes with the conjugate-linear Tomita operator S (which has $M\widehat{1_M}$ as a core and maps $x\widehat{1_M}$ to $x^*\widehat{1_M}$ for $x \in M$). More precisely, we have $Su_\theta \supset u_\theta S$, meaning that whenever ξ is in the domain of S, so is $u_\theta \xi$ and $Su_\theta \xi = u_\theta S\xi$ holds. Since u_θ commutes with $\mathcal{J}$ and $S = \mathcal{J}\Delta^{1/2}$, we have $\Delta^{1/2} u_\theta \supset u_\theta \Delta^{1/2}$, and so Δ^{it} commutes with u_θ for any $t \in \mathbb{R}$. Hence, conclude that for $x \in M$ and $t \in \mathbb{R}$, we have*

$$\begin{aligned} \theta(\sigma_t^\phi(x))\widehat{1_M} &= u_\theta \Delta^{it} x \Delta^{-it} \widehat{1_M} = u_\theta \Delta^{it} x \widehat{1_M} \\ &= \Delta^{it} u_\theta x \widehat{1_M} = \Delta^{it} \theta(x) \widehat{1_M} \\ &= \sigma_t^\phi(\theta(x))\widehat{1_M} \ . \end{aligned}$$

As $\widehat{1_M}$ is a separating vector for M, deduce that

$$\theta(\sigma_t^\phi(x)) = \sigma_t^\phi(\theta(x)) \ \forall x \in M, t \in \mathbb{R} \ .$$

Hence $\sigma_t^\phi(\theta(M)) = \theta(M) \ \forall t \in \mathbb{R}$ and it follows from Takesaki's theorem (see [Tak72, Section 4]) that there exists a unique ϕ-preserving conditional expectation E of M onto the subfactor $P = \theta(M)$. It is true, as the definition shows, that $e_\theta = u_\theta (u_\theta)^*$ is the orthogonal projection onto $[P\widehat{1_M}]$ and $E(x)e_\theta = e_\theta x e_\theta \ \forall x \in M$.

Theorem 2.2.3. *Suppose θ is an equi-modular endomorphism of a factorial non-commutative probability space (M, ϕ). Then,*

1. *The equation $\theta' = j \circ \theta \circ j$ defines a unital normal *-endomorphism of M' which preserves $\phi' = \overline{\phi \circ j}$; and*

2. *We have an identification*

$$\begin{aligned} L^2(M',\phi') &= L^2(M,\phi) \\ \widehat{1_{M'}} &= \widehat{1_M} \\ u_{\theta'} &= u_\theta \end{aligned}$$

3. *there exists a unique endomorphism* $\theta^{(2)}$ *of* $\mathcal{L}(L^2(M,\phi))$ *satisfying*

$$\theta^{(2)}(xj(y)) = \theta(x)j(\theta(y))z, \ \forall x,y \in M$$

where $z = \wedge\{p \in (\theta(M) \cup \theta'(M'))'' : ran(p) \supset \{\widehat{\theta(x)} : x \in M\}\}$.

Proof. 1. It is clear that $\theta' = j \circ \theta \circ j$ is a unital normal linear *-endomorphism of M' and that

$$\overline{\phi' \circ \theta'} = \overline{\phi'} \circ \theta' = (\phi \circ j) \circ (j \circ \theta \circ j) = (\phi \circ \theta) \circ j = \phi \circ j = \overline{\phi'} \ ,$$

thereby proving (1).

2. This follows from the facts that $\widehat{1_M}$ is a cyclic and separating vector for M and

hence also for M', the definition of ϕ' which guarantees that

$$\begin{aligned}
\langle j(x)\widehat{1_{M'}}, j(y)\widehat{1_{M'}}\rangle &= \phi'(j(y)^*j(x)) \\
&= \phi'(j(y^*x)) \\
&= \overline{\phi(y * x)} \\
&= \phi(x^*y) \\
&= \langle y\widehat{1_M}, x\widehat{1_M}\rangle \\
&= \langle \mathcal{J}x\widehat{1_M}, \mathcal{J}y\widehat{1_M}\rangle \\
&= \langle \mathcal{J}x\mathcal{J}\widehat{1_M}, \mathcal{J}y\mathcal{J}\widehat{1_M}\rangle \\
&= \langle j(x)\widehat{1_M}, j(y)\widehat{1_M}\rangle
\end{aligned}$$

and the definitions of the 'implementing isometries', which show that

$$\begin{aligned}
u_{\theta'}(j(x)\widehat{1_{M'}}) &= \theta'(j(x))\widehat{1_{M'}} \\
&= j(\theta(x))\widehat{1_{M'}} \\
&= \mathcal{J}\theta(x)\mathcal{J}\widehat{1_M} \\
&= \mathcal{J}\theta(x)\widehat{1_M} \\
&= \mathcal{J}u_\theta x\widehat{1_M} \\
&= u_\theta\mathcal{J}x\widehat{1_M} \\
&= u_\theta\mathcal{J}x\mathcal{J}\widehat{1_M} \\
&= u_\theta j(x)\widehat{1_{M'}} \; .
\end{aligned}$$

3. Notice that if $x, y \in M$, then

$$\begin{aligned}
\langle \theta(x)\mathcal{J}\theta(y)\mathcal{J}\widehat{1_M}, \widehat{1_M}\rangle &= \langle \theta(x)\mathcal{J}\theta(y)\widehat{1_M}, \widehat{1_M}\rangle \\
&= \langle \theta(x)\mathcal{J}u_\theta y\widehat{1_M}, \widehat{1_M}\rangle \\
&= \langle \theta(x)u_\theta\mathcal{J}y\widehat{1_M}, \widehat{1_M}\rangle \\
&= \langle u_\theta x\mathcal{J}y\widehat{1_M}, \widehat{1_M}\rangle \\
&= \langle u_\theta x\mathcal{J}y\widehat{1_M}, u_\theta\widehat{1_M}\rangle \\
&= \langle x\mathcal{J}y\mathcal{J}\widehat{1_M}, \widehat{1_M}\rangle \ ,
\end{aligned}$$

where we have used the fact that θ is equi-modular.

Set $S^1 = \{xj(y) : x = x^*, y = y^*, x, y \in M\}$, and $S^{(2)} = \{\theta(x)j(\theta(y)) : xj(y) \in S^{(1)}\}$, and deduce from the factoriality of M that $S^{(1)''} = \mathcal{L}(L^2(M, \phi))$.

Now we wish to apply Remark 2.1.2(1) with $N = S^{(2)''} = \theta(M) \vee j(\theta(M))$ (where, both here and in the sequel, we write $A \vee B = (A \cup B)''$ for the von Neumann algebra generated by von Neumann algebras A and B) and $\phi_1 = \phi_2 = \langle(\cdot)\hat{1}_M, \hat{1}_M\rangle$. For this, deduce from Lemma 2.1.4 that

$$\begin{aligned}
z &= \wedge\{p \in \mathcal{P}(N) : ran\ p \supset N\hat{1_M}\} \\
&= \wedge\{p \in \mathcal{P}(N) : ran\ p \supset \{\theta(x)\hat{1_M}, \theta'(j(x))\hat{1_M} : x \in M\}\} \\
&= \wedge\{p \in \mathcal{P}(N) : ran\ p \supset \{\widehat{\theta(x)} : x \in M\}\}
\end{aligned}$$

and the proof of the Theorem is complete.

□

Remark 2.2.4. *It must be observed that the projection z of Theorem 2.2.3 is nothing but the central support of the projection $e_\theta = u_\theta u_\theta^*$ in $P' \cap P_1$ where $P = \theta(M) \subset M \subset P_1$ is*

Jones' basic construction (thus, $P_1 = \mathcal{J}P'\mathcal{J}$) since, by Lemma 2.1.4, we have:

$$ran\ z = [(P \vee \mathcal{J}P\mathcal{J})'(P \vee \mathcal{J}P\mathcal{J})\widehat{1_M}] = [(P' \cap P_1)e_\theta L^2(M, \phi)] \ .$$

This is because

$$\begin{aligned} [(P \vee \mathcal{J}P\mathcal{J})\widehat{1_M}] &= [P\mathcal{J}P\mathcal{J}\widehat{1_M}] = [P\mathcal{J}P\widehat{1_M}] \\ &= [P\mathcal{J}u_\theta M\widehat{1_M}] = [Pu_\theta \mathcal{J}M\widehat{1_M}] \\ &= [Pu_\theta M\widehat{1_M}] \ = [P\widehat{1_M}] \ . \end{aligned}$$

In particular, since e_θ is a minimal projection in $P' \cap P_1$, its central support z in $P' \cap P_1$ is 1 if and only if $P' \cap P_1$ is a type I factor. In the following corollary, we continue to use the symbols P and P_1 with the meaning attributed to them here.

The following corollary is an immediate consequence of Lemma 2.1.4, Theorem 2.2.3 and Remark 2.2.4.

Corollary 2.2.5. *Let θ be a equi-modular endomorphism of a factorial non-commutative probability space (M, ϕ) in standard form (i.e., viewed as embedded in $\mathcal{L}(L^2(M, \phi))$ as above). The following conditions on θ are equivalent:*

1. *there exists a unique unital normal $*$-endomorphism $\theta^{(2)}$ of $\mathcal{L}(L^2(M, \phi))$ such that $\theta^{(2)}(x) = \theta(x)$ and $\theta^{(2)}(j(x)) = j(\theta(x))$ for all $x \in M$.*
2. *$P \vee \mathcal{J}P\mathcal{J}$ is a factor; and in this case, it is necessarily a type I factor.*
3. *$(P \vee \mathcal{J}P\mathcal{J})' = P' \cap P_1$ is a factor; and in this case, it is necessarily a type I factor.*
4. *$\{x\widehat{y} : x \in P' \cap P_1, y \in P\}$ is total in $L^2(M, \phi)$.*

An endomorphism of a factor which satisfies the equivalent conditions above will be said to be **extendable***.*

Remark 2.2.6. *It should be noted that extendability is not a property of just an endomorphism θ but it is also dependent on a state which is not only left invariant under the endomorphism but must also satisfy the requirement we have called equi-modular. Strictly speaking, we should probably talk of ϕ-extendability, but shall not do so in the interest of notational convenience.*

Theorem 2.2.7. *Let the notation be as above. Then the following conditions are equivalent:*

1. *θ is extendable.*

2. *$M = (M \cap \theta(M)') \vee \theta(M)$. (Note that the right-hand side is naturally identified with the von Neumann algebra tensor product $(M \cap \theta(M)') \otimes \theta(M)$ in this case.)*

Proof. Recall that $P = \theta(M)$ is globally preserved by the modular automorphism group $\{\sigma_t^\phi\}_{t\in\mathbb{R}}$ and there exists a ϕ-preserving faithful normal conditional expectation E from M onto P. Thus $(M \cap P') \vee P$ is naturally identified with the von Neumann algebra tensor product $(M \cap P') \otimes P$ (see [Tak72, Corollary 1]). If we assume the second condition in the statement, the basic construction for $P \subset M$ essentially comes from that of $\mathbb{C} \subset (M \cap P')$, and so $P' \cap P_1$ is a type I factor. This means that θ is extendable.

Assume that θ is extendable now. We will show that $Q := (M \cap P') \vee P$ coincides with M. For this, it suffices to show $[Q\widehat{1_M}] = L^2(M, \phi)$. Indeed, since P is globally preserved by σ_t^ϕ, so is Q, and there exists a ϕ-preserving faithful normal conditional expectation from M onto Q thanks to Takesaki's theorem. Thus if Q were a proper subalgebra of M, $[Q\widehat{1_M}]$ would be a proper subspace of $L^2(M, \phi)$.

Let $\hat{E}$ be the dual operator valued weight from P_1 to M (see [Kos86] for the definition of $\hat{E}$ and its properties). Since $E \circ \hat{E}(e_\theta) = 1 < \infty$ and $P' \cap P_1$ is a factor, the restriction of $E \circ \hat{E}$ to the type I factor $P' \cap P_1$ is a faithful normal semifinite weight (see [ILP98, Lemma 2.5]). Thus there exists a (not necessarily bounded) non-singular positive operator

ρ affiliated to $P' \cap P_1$ satisfying $\sigma_t^{E\circ\hat{E}} = \mathrm{Ad}\rho^{it}$ and

$$E \circ \hat{E}(a) = \lim_{n\to\infty} \mathrm{Tr}(\rho(1 + \frac{1}{n}\rho)^{-1}a), \quad \forall a \in (P' \cap P_1)_+,$$

where $\{\sigma_t^{E\circ\hat{E}}\}_{t\in\mathbb{R}}$ is the relative modular automorphism group (the restriction of $\{\sigma_t^{\phi\circ\hat{E}}\}_{t\in\mathbb{R}}$ to $P' \cap P_1$). Note that the trace Tr makes sense as $P' \cap P_1$ is a type I factor.

From the above argument we see that there exists a partition of unity $\{e_i\}_{i\in I}$ consisting of minimal projections $e_i \in P' \cap P_1$ with $E \circ \hat{E}(e_i) < \infty$. Since e_θ is a minimal projection in $P' \cap P_1$ satisfying $\sigma_t^{E\circ\hat{E}}(e_\theta) = e_\theta$ and $E \circ \hat{E}(e_\theta) = 1$, we may assume $0 \in I$ and $e_0 = e_\theta$. Let $\{e_{ij}\}_{i,j\in I}$ be a system of matrix units in $P' \cap P_1$ satisfying $e_{ii} = e_i$. Then we can apply the push down lemma [ILP98, Proposition 2.2] to e_{0i}, and we have $e_{0i} = e_\theta q_i$, where $q_i = \hat{E}(e_{0i}) \in P' \cap M$. Now for any $x \in M$, we have

$$x\widehat{1_M} = \sum_{i\in I} e_{ii}x\widehat{1_M} = \sum_{i\in I} q_i^* e_\theta q_i x\widehat{1_M} = \sum_{i\in I} q_i^* E(q_i x)\widehat{1_M},$$

which shows $[Q\widehat{1_M}] = L^2(M, \phi)$. □

2.3 Examples of Extendable Endomorphisms

Note that any automorphism on a factor is extendable, since the conditions in Corollary 2.2.5 are satisfied.

Let $\mathcal{R}$ denote the hyperfinite II_1 factor and M be any II_1 factor which is also a McDuff factor; i.e., $M \otimes \mathcal{R} \cong M$. Let $\alpha : M \otimes \mathcal{R} \mapsto M$ be an isomorphism and $\beta : M \mapsto M \otimes \mathcal{R}$ be the monomorphism defined by $\beta(m) = m \otimes 1$, for $m \in M$. Let us write $\theta = \beta \circ \alpha$. so θ is an endomorphism of $M \otimes \mathcal{R}$ such that $\theta(M \otimes \mathcal{R}) = M \otimes 1$. As $M \otimes \mathcal{R}$ is a II_1 factor, the endomorphism θ is necessarily equi-modular. Now by corollary 2.2.5, showing that θ is extendable is equivalent to showing that $\{\theta(M \otimes \mathcal{R}) \vee \mathcal{J}\theta(M \otimes \mathcal{R})\mathcal{J}\}$ is a type I factor, where $\mathcal{J}$ is the modular conjugation of $M \otimes \mathcal{R}$, which, of course, is $\mathcal{J}_M \otimes \mathcal{J}_R$.

Note that

$$\begin{aligned}\{\theta(M\otimes\mathcal{R})\vee\mathcal{J}\theta(M\otimes\mathcal{R})\mathcal{J}\} &= \{M\otimes 1\vee\mathcal{J}(M\otimes 1)\mathcal{J}\}\\ &= \{M\otimes 1\vee\mathcal{J}_M M\mathcal{J}_M\otimes 1)\}\\ &= \mathcal{L}(L^2(M)\otimes 1\end{aligned}$$

So $\{\theta(M\otimes\mathcal{R})\vee\mathcal{J}\theta(M\otimes\mathcal{R})\mathcal{J}\}$ is a type I factor. That is θ is extendable.

2.4 Extendability for E_0-semigroups

Definition 2.4.1. *$\{\alpha_t : t\geq 0\}$ is said to be an E_0-semigroup on a von Neumann probability space (M,ϕ) if:*

1. *α_t is a ϕ-preserving normal unital *-homomorphism of M for each $t\geq 0$;*
2. *$\alpha_0 = id_M$ and $\alpha_s\circ\alpha_t = \alpha_{s+t}$; and*
3. *$[0,\infty)\ni t\mapsto\rho(\alpha_t(x))$ is continuous for each $x\in M,\rho\in M_*$.*

Suppose α_t is (equi-modular and) extendable for each t, then we say that the E_0-semigroup α is extendable.

Remark 2.4.2. *A remark along the lines of Remark 2.2.6, with endomorphism replaced by E_0-semigroup, is in place here.*

Proposition 2.4.3. *Suppose $\alpha=\{\alpha_t : t\geq 0\}$ is an E_0-semigroup on a factorial non-commutative probability space (M,ϕ).*

1. *The equation $\alpha'_t(x') = j(\alpha_t(j(x'))$ defines an E_0-semigroup on (M',ϕ'), where $\phi'(x') = \omega_{\widehat{1_M}}(x') = \langle x'\widehat{1_M},\widehat{1_M}\rangle$;*
2. *If α is extendable for each t, then there exists a unique E_0-semigroup $\{\alpha_t^{(2)} : t\geq 0\}$ on $(\mathcal{L}(L^2(M,\phi)),\omega_{\widehat{1_M}})$ such that $\alpha_t^{(2)}(xx') = \alpha_t(x)\alpha'_t(x')\ \forall x\in M, x'\in M'$.*

Proof. Existence of the endomorphisms α'_t and $\alpha^{(2)}_t$ is guaranteed by Corollary 2.2.5, The equation $\alpha'_t = j \circ \alpha_t \circ j$ shows that $\{\alpha'_t : t \geq 0\}$ inherits the property of being an E_0-semigroup from that of $\{\alpha_t : t \geq 0\}$. The corresponding property for $\{\alpha^{(2)}_t : t \geq 0\}$ is now seen to follow easily from the uniqueness assertion in Corollary 2.2.5(1). □

By using standard arguments from the theory of E_0-semigroups on type I factors, we can strengthen Corollary 2.2.5 in the case of E_0-semigroups thus:

Proposition 2.4.4. *Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup on a factorial non-commutative probability space (M, ϕ), and suppose α_t is equi-modular for each t. Suppose M is acting standardly on $\mathcal{H} = L^2(M)$. Consistent with the notation of Remark 2.2.4, we shall write $P(t) = \alpha_t(M) \subset M \subset P_1(t)$ for Jones' basic construction.*

The following conditions on α are equivalent.

1. *α is extendable.*

2. *$P'(t) \cap P_1(t)$ is either: (i) a factor of type I_1 (i.e., is isomorphic to $\mathbb{C}$) and α_t is an automorphism for all t; or (ii) a factor of type I_∞ for all t and no α_t is an automorphism.*

Proof. (1) ⇒ (2) If each α_t is extendable, then $P'(t) \cap P_1(t)$ is a factor of type I_{n_t}, say, by Corollary 2.2.5. The first fact to be noted is that if $\{\mathcal{E}_{\alpha^{(2)}}(t) : t \geq 0\}$ is the product system associated to the E_0-semigroup $\alpha^{(2)}$ on $\mathcal{L}(L^2(M))$, then n_t is the dimension of the Hilbert space $\mathcal{E}_{\alpha^{(2)}}(t)$. Hence $\{n_t : t \geq 0\}$ is a multiplicative semi-group of integers. Hence either n_t is constant in t (identically 1 or identically infinity).

(2) ⇒ (1) follows from Corollary 2.2.5. □

Remark 2.4.5. *Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup on a factorial non-commutative probability space (M, ϕ), and suppose α_t is equi-modular for each t. If α_t is an extendable endomorphism for some $t > 0$, then α_s is also extendable for all $0 \leq s < t$. Indeed, α_t being extendable means that M as a $P(t) - P(t)$ bimodule is a direct sum of copies of*

$P(t)$, and hence $P(t-s)$ as a $P(t)-P(t)$ bimodule is also a direct sum of copies of $P(t)$. This means that $\alpha_{t-s}(M)$ is generated by $\alpha_{t-s}(M)\cap\alpha_t(M)'$ and $\alpha_t(M)$, which means α_s is extendable. Now, since the compositions of extendable endomorphisms are extendable, the E_0-semigroup α itself is extendable.

Now let us consider the following spaces;

$$E^{\alpha_t} = \{T \in \mathcal{L}(L^2(M)) : \alpha_t(x)T = Tx,\ \text{forall } x \in M\};$$

$$E^{\alpha'_t} = \{T \in \mathcal{L}(L^2(M)) : \alpha'_t(x')T = Tx',\ \text{forall } x' \in M'\}.$$

For every $t \geq 0$, we write $H(t) = E^{\alpha_t} \cap E^{\alpha'_t}$. Then we have the following Lemma.

Proposition 2.4.6. *Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup on a factorial non-commutative probability space (M,ϕ) and suppose α_t is equi-modular for each t. If α is extendable then*

$$H = \{(t,T) : t \in (0,\infty), T \in H(t)\}$$

is a product system (in the sense of [Arv03]) with the family of unitary maps $u_{st} : H(t)\otimes H(s) \mapsto H(s+t)$, given by

$$u_{st}(T\otimes S) = TS\ \forall T \in H(t), S \in H(s).$$

Proof. Let $\alpha^{(2)} = \{\alpha_t^{(2)} : t > 0\}$ be the extension of α on $\mathcal{L}(L^2(M))$. For $t > 0$, consider

$$\mathcal{E}(t) = \{T \in \mathcal{L}(L^2(M)) : \alpha_t^{(2)}(x)T = Tx,\ \text{for all } x \in \mathcal{L}(L^2(M))\}.$$

We shall write $\mathcal{E} = \{(t,T) : T \in \mathcal{E}(t)\}$; then $\mathcal{E}$ is a product system (see [Arv03]), and $H(t) = \mathcal{E}(t)$ for every $t > 0$. Indeed, if $T \in H(t) = E^{\alpha_t} \cap E^{\alpha'_t}$, then $\alpha_t(m)T = Tm$ for all $m \in M$ and $\alpha'_t(m')T = Tm'$ for all $m' \in M'$. So it is clear that $\alpha_t^{(2)}(x)T = Tx$ for all $x \in M \cup M'$ and hence also for all $x \in (M \vee M') = \mathcal{L}(L^2(M))$. So, $T \in \mathcal{E}(t)$, and $H(t) \subset \mathcal{E}(t)$. The reverse inclusion is immediate from the definition $\alpha_t^{(2)}$. So we have

$H(t) = \mathcal{E}(t)$ and clearly H is a product system. $\square$

Now recall that an E_0-semigroup $\{\beta_t : t \geq 0\}$ of a von Neumann probability space (M, ϕ) is said to be a **cocycle perturbation** of an E_0-semigroup $\{\alpha_t : t \geq 0\}$ if there exists a weakly continuous family $\{u_t : t \geq 0\}$ of unitary elements of M such that

1. $u_{t+s} = u_s \alpha_s(u_t)$; and
2. $\beta_t(x) = u_t \alpha_t(x) u_t^*$ for all $x \in M$ and $s, t \geq 0$.

In such a case, we shall simply write

$$\{u_t : t \geq 0\} : \{\alpha_t : t \geq 0\} \simeq \{\beta_t : t \geq 0\}.$$

Proposition 2.4.7. *Suppose $\beta = \{\beta_t : t \geq 0\}$ is an E_0 semigroup on a factorial probability space (M, ϕ), which is a cocycle perturbation of another E_0 semigroup $\alpha = \{\alpha_t : t \geq 0\}$ on (M, ϕ) with $\{u_t : t \geq 0\} : \{\alpha_t : t \geq 0\} \simeq \{\beta_t : t \geq 0\}$. Then*

1. $\{j(u_t) : t \geq 0\} : \{\alpha'_t : t \geq 0\} \simeq \{\beta'_t : t \geq 0\}$.
2. *If each α_t is extendable, as is each β_t, then*

$$\{u_t j(u_t) : t \geq 0\} : \{\alpha_t^{(2)} : t \geq 0\} \simeq \{\beta_t^{(2)} : t \geq 0\}.$$

Proof. The verifications are elementary and a routine computation. For example, once (1) has been verified, the verification of (2) involves such straightforward computations as: if we let $U_t = u_t u'_t$, where we write $u'_t = j(u_t)$, and if $x \in M, x' \in M'$, then

$$\begin{aligned}
U_t \alpha_t(x) U_t^* &= u_t \alpha_t(x) u_t^* = \beta_t(x) \\
U_t \alpha'_t(x') U_t^* &= u'_t \alpha'_t(x') u'^*_t = \beta'_t(x') \\
\beta_t(M) \vee \beta'_t(M') &= U_t(\alpha_t(M) \vee \alpha'_t(M')) U_t^* \\
U_{s+t} &= u_{s+t} u'_{s+t} \\
&= u_s \alpha_s(u_t) u'_s \alpha'_s(u'_t) \\
&= U_s \alpha_s^{(2)}(U_t)
\end{aligned}$$

and

$$\begin{aligned}
\beta_t^{(2)}(xx') &= \beta_t(x) \beta'_t(x') \\
&= u_t \alpha_t(x) u_t^* u'_t \alpha'_t(x') u'^*_t \\
&= U_t \alpha_t^{(2)}(xx') U_t^*.
\end{aligned}$$

□

Recall that two E_0-semigroups $\{\alpha_t : t \geq 0\}$ and $\{\beta_t : t \geq 0\}$ on a von Neumann algebra M are said to be **conjugate** if there exists an automorphism θ of M such that $\beta_t \circ \theta = \theta \circ \alpha_t \; \forall t$, while they are said to be **cocycle conjugate** if each is conjugate to a cocycle perturbation of the other.

Remark 2.4.8. *While the index of E_0-semigroups of type I_∞ factors has been well-defined, we may now define the index of an extendable E_0 semigroup α of an arbitrary factor as the index of $\alpha^{(2)}$; and we may infer from Proposition 2.4.7 that the index of an extendable E_0-semigroup of an arbitrary factor is invariant under cocycle conjugacy - in the restricted sense that cocycle conjugate extendable E_0-semigroups have the same index. (One has to exercise some caution here in that there is a problem with invariance* *of equi-modularity under cocycle conjugacy!) It is to be noted from Corollary 2.2.5 that the extendability of an E_0-semigroup, each of whose endomorphisms is equi-modular, is a property which is invariant under cocycle conjugacy within the class of such E_0-semigroups.*

Proposition 2.4.9. *If $\alpha = \{\alpha_t : t \geq 0\}$ (resp., $\beta = \{\beta_t : t \geq 0\}$) is an extendable E_0-semigroup of a factor M (resp., N), then $\alpha \otimes \beta = \{\alpha_t \otimes \beta_t : t \geq 0\}$ is an extendable E_0-semigroup of the factor $M \otimes N$, and in fact,*

$$(\alpha \otimes \beta)^{(2)} = \alpha^{(2)} \otimes \beta^{(2)}.$$

Proof. The hypothesis is that $\alpha_t(M) \vee \mathcal{J}\alpha_t(M)\mathcal{J}$ and $\beta_t(N) \vee \mathcal{J}_N\beta_t(N)\mathcal{J}_N$ are factors, for each $t \geq 0$, while the conclusions follow from the definition of $\alpha \otimes \beta$. □

2.5 Examples

First we give examples of extendable E_0-semigroups. Throughout this section, let $\mathcal{H} = L^2(0,\infty) \otimes \mathcal{K}$, be the real Hilbert space of square integrable functions taking values in a real Hilbert space $\mathcal{K}$. We always denote by $(\cdot)_{\mathbb{C}}$ the complexification of $(\cdot)$. Let s_t be the shift semigroup on $\mathcal{H}_{\mathbb{C}}$ defined by

$$\begin{aligned} (s_t f)(s) &= 0, \quad p < t, \\ &= f(p-t), \quad p \geq t. \end{aligned}$$

Thus $(s_t : t \geq 0)$ is a semigroup of isometries, and we denote its restriction to $\mathcal{H}$ also by $\{s_t\}$.

For the first set of examples, given by 'canonical commutation relations', we only need complex Hilbert spaces. Let $R \geq 1$ be a complex linear operator on $\mathcal{H}_{\mathbb{C}}$ such that $T = \frac{1}{2}(R-1)$ is injective. Consider the the quasi free state on the CCR algebra over $\mathcal{H}_{\mathbb{C}}$

given by

$$\varphi_R(W(f)) = e^{-\frac{1}{2}\langle Rf,f\rangle} = e^{-\frac{1}{2}\|\sqrt{1+2T}f\|^2} \ \ \forall \ \ f \in \mathcal{H}_{\mathbb{C}}.$$

The space underlying the corresponding GNS representation may be identified with $\Gamma_s(\mathcal{H}_{\mathbb{C}}) \otimes \Gamma_s(\mathcal{H}_{\mathbb{C}})$, with the GNS representation being described by

$$\pi(W(f)) = W_0(\sqrt{1+T}f) \otimes W_0(q\sqrt{T}f) \ \ \forall \ \ f \in \mathcal{H}_{\mathbb{C}},$$

where $\Gamma_s(\cdot)$ is the symmetric Fock space, $W_0(\cdot)$ is the Weyl operator on $\Gamma_s(\mathcal{H}_{\mathbb{C}})$ and q is an anti-unitary on $\mathcal{H}_{\mathbb{C}}$ induced by an anti-unitary operator on $\mathcal{K}_{\mathbb{C}}$. The vacuum vector $\Omega \otimes \Omega \in \Gamma_s(\mathcal{H}_{\mathbb{C}}) \otimes \Gamma_s(\mathcal{H}_{\mathbb{C}})$ is the cyclic and separating vector for $M_R = \{\pi(W(f)) : f \in \mathcal{H}_{\mathbb{C}}\}''$ (see [AW63]).

Example 2.5.1. *Let $R = \frac{1+\lambda}{1-\lambda}$ with $\lambda \in (0,1)$, then it is well-known that $M_\lambda = M_R$ is a type III_λ factor. There exists a unique E_0-semigroup β_t^λ on M_λ satisfying*

$$\beta_t^\lambda(\pi(W(f))) = \pi(W(s_t f)) \ \ \forall \ f \in \mathcal{H}_{\mathbb{C}}.$$

Further, $\{\beta_t^\gamma; t \geq 0\}$ is equi-modular and the relative commutant is given by

$$\beta_t^\lambda(M_\lambda)' \cap M_\lambda = \big\{\pi(W(f)) : f \in \big(L^2(0,t) \otimes \mathcal{K}\big)_{\mathbb{C}}\big\}''.$$

Now theorem 2.2.7 imply that all these E_0-semigroups on type III_λ factors are extendable. (See [MS13], where these examples are discussed in more detail.)

We will write $\mathcal{F}(\mathcal{H})$ for the anisymmetric Fock space; thus

$$\mathcal{F}(\mathcal{H}_{\mathbb{C}}) = \mathbb{C}\Omega \oplus \mathcal{H}_{\mathbb{C}} \oplus (\mathcal{H}_{\mathbb{C}} \wedge \mathcal{H}_{\mathbb{C}}) \oplus (\mathcal{H}_{\mathbb{C}} \wedge \mathcal{H}_{\mathbb{C}} \wedge \mathcal{H}_{\mathbb{C}}) \oplus \cdots,$$

where Ω is a fixed complex number with modulus 1.

Recall the left creation operator on $\mathcal{F}(\mathcal{H})$ (corresponding to $f \in \mathcal{H}_{\mathbb{C}}$ given by

$$\begin{aligned} a(f)\Omega &= f \\ a(f)(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n) &= f \wedge \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \ , \ \xi_i \in \mathcal{H}_{\mathbb{C}} \ . \end{aligned}$$

These operators obey the *canonical anticommutation relations*:

$$a(f)a(g) + a(g)a(f) = 0 \ , \ a(f)a(g)^* + a(g)^*a(f) = \langle f \ , \ g\rangle id_{\mathcal{F}(\mathcal{H})}$$

for all $f \ , \ g \in \mathcal{H}_{\mathbb{C}}$.

For any $f \in \mathcal{H}$, let $u(f) = a(f) + a(f)^*$. It is well-known that the von Neumann algebra

$$\{u(f) : f \in \mathcal{H}\}'' \subseteq \mathcal{L}(\mathcal{F}(\mathcal{H}_{\mathbb{C}}))$$

is the hyperfinite II_1 factor $\mathcal{R}$ with cyclic and separating (trace) vector Ω.

Example 2.5.2. *For every $t \geq 0$ there exist a unique normal, unital $*$-endomorphism $\alpha_t : \mathcal{R} \mapsto \mathcal{R}$ satisfying*

$$\alpha_t(u(f)) = u(s_t f) \ \forall f \in \mathcal{H}_{\mathbb{C}} \ .$$

(Although this is a well-known fact, we remark that this is in fact a consequence of Remark 2.1.2 (2).) Then $\alpha = \{\alpha_t : t \geq 0\}$ is an E_0-semigroup on $\mathcal{R}$, called the **Clifford flow of rank dim** $\mathcal{K}$*.*

It is known from [Ale04] that

$$\alpha_t(M)' \cap M = \{u(f)u(g) : spt(f), spt(g) \subset [0,t]\}'' \ . \tag{2.5.1}$$

It follows from equation 2.5.1 that if $spt(f) \subset [0,t]$, then $u(f)\Omega \perp \{(\alpha_t(\mathcal{R})' \cap \mathcal{R})\Omega \cup \alpha_t(\mathcal{R})\Omega\}$; (in fact the same assertion holds for any $a(f_1)\cdots a(f_{2n+1})\Omega$ for

any n and any $f_1, \cdots, f_{2n+1}$ with support in $[0,t]$.) Consequently, in view of Ω being a separating vector for $\mathcal{R}$, it is an easy consequence of Theorem 2.2.7 that the **Clifford**

flow on $\mathcal{R}$ (of any rank) is not extendable.

The Clifford flows of the hyperfinite II_1 factor are closely related to another family of E_0-semigroups, called the CAR flows. (We should remember these are CAR flows on type II_1 factors, not to be confused with the usual CAR flows on the type I factor of all bounded operators on the antisymmetric Fock space.) We recall the definition of CAR algebra and some facts regarding the GNS representations of CAR algebras given by quasi-free states.

For a complex Hilbert space K, the associated CAR algebra $CAR(K)$ is the universal C^*-algebra generated by a unit 1 and elements $\{a(f): f \in K\}$, subject to the following relations

(i) $a(\lambda f) = \lambda a(f)$,

(ii) $a(f)a(g) + a(g)a(f) = 0$,

(iii) $a(f)a^*(g) + a^*(g)a(f) = \langle f, g\rangle 1$,

for all $\lambda \in \mathbb{C}$, $f, g \in K$, where $a^*(f) = a(f)^*$.

Given a positive contraction R on K, there exists a unique quasi-free state ω_R on $CAR(K)$ satisfying

$$\omega_R(a(x_n)\cdots a(x_1)a(y_1)^*\cdots a(y_m)^*) = \delta_{n,m} det(\langle Rx_i, y_j\rangle),$$

where $det(\cdot)$ denotes the determinant of a matrix. Let (H_R, π_R, Ω_R) be the corresponding GNS triple. Then $M_R = \pi_R(CAR(K))''$ is a factor.

Here onwards we fix the contraction with $R = \frac{1}{2}$, then $M_R = \mathcal{R}$ is the hyperfinite type II_1 factor and ω_R is a tracial state. We define the CAR flow on $\mathcal{R}$ as follows.

Now let $K = \mathcal{H}_{\mathbb{C}}$. Then there exists a unique E_0-semigroup $\{\alpha_t\}$ on $\mathcal{R}$ satisfying

$$\alpha_t(\pi(a(f))) = \pi(a(s_t f)) \;\; \forall\, f \in \mathcal{H}_{\mathbb{C}}.$$

This α is called as the CAR flow of index $\mathbf{dim}\mathcal{K}$ on $\mathcal{R}$.

We recall the following proposition from [Ale04] (see proposition 2.6).

Proposition 2.5.3. *The CAR flow of rank n on $\mathcal{R}$ is conjugate to the Clifford flow of rank $2n$.*

We point out an error in [ABS01] in the following remark.

Remark 2.5.4. *In section 5, [ABS01], it is claimed that CAR flows of any given rank are extendable. In fact a 'proof' is given, for any $\lambda \in (0, \frac{1}{2}]$ with $R = \lambda$, that the corresponding E_0-semigroup on M_R is extendable. (When $\lambda \neq \frac{1}{2}$ they are type III factors.) But we have seen that Clifford flows are not extendable. This consequently implies, thanks to proposition 2.5.3, and the invariance of extendability of E_0-semigroups of II_1 factors (where equi-modularity - with respect to the trace - comes for free), that CAR flows on the hyperfinite type II_1 factor $\mathcal{R}$ are not extendable.*

In fact it has been proved in [Bik13] that CAR flows flows on any type III_λ factors (considered in [ABS01]) are also not extendable.)

Let $\Gamma_f(\mathcal{H}_\mathbb{C})$ be the full Fock space associated with a Hilbert space K. For $f \in \mathcal{H}$, define $s(f) = \frac{l(f)+l(f)^*}{2}$ where

$$l(f)\xi = \left\{ \begin{array}{ll} f & \text{if } \xi = \Omega, \\ f \otimes \xi & \text{if } \langle \xi, \Omega \rangle = 0. \end{array} \right\}.$$

The von Neumann algebra $\Phi(\mathcal{K}) = \{s(f) : f \in \mathcal{H}\}''$, is isomorphic to the free group factor $L(F_\infty)$ and the vacuum is cyclic and separating with $\langle \Omega, x\Omega \rangle = \tau(x)$ (see [VDN92]) a tracial state on $\Phi(\mathcal{K})$.

Example 2.5.5. *There exists a unique E_0-semigroup γ on $\Phi(\mathcal{K})$ satisfying*

$$\gamma_t((s(f)) := s(s_t f) \qquad (f \in \mathcal{H},\ t \geq 0).$$

This is called the free flow of rank $dim(\mathcal{K})$.

Let γ be a free flow of any rank. It is known - see [Pop83] - that $\gamma_t(\Phi(\mathcal{K}))' \cap \Phi(\mathcal{K}) = \mathbb{C}1$. *So it follows from Theorem 2.2.7 that free flow is not extendable.*

It is proved in [MS12] that

$$H(t) = (E^{\gamma_t} \cap E^{\gamma_t'}) = \mathbb{C}s_t.$$

So $H = \{(t, \eta) : \eta \in H(t)\}$ *is a product system. This means that free flows provide examples to show that the converse of the Corollary 2.4.6 is not true: for free flows, the family* $\{(t, (E^{\gamma_t} \cap E^{\gamma_t'})\ ) : t \geq 0\}$ *forms a product system, but still they are not extendable.*

Chapter 3

CAR flows on type III factors

E_0-semigroups on type I factors have received much attention (see the monograph [Arv03] for an extensive list of references). The study of E_0-semigroups on type II_1 factors was initiated by Powers in 1998 (see [Pow88]). There was little progress on E_0-semigroups on type II_1 factors until the results independently obtained recently in [Ale04] and [MS12]). On the other hand, E_0-semigroups on type III factors have not received much attention. In the second chapter we have attempted to study E_0-semigroups on arbitrary factors and studied a certain class of endomorphisms and E_0-semigroups which we call extendable. In the section of examples 2.5, we have already discussed CAR flows on hyperfinite type II_1 factor arising from the quasi-free state of $A = \frac{1}{2}$.

In this chapter we discuss CAR flows in type III factors arising from the quasi-free states of $A \neq \frac{1}{2}$. We prove that CAR flows arising from some class of quasi-free states are not extendable. Also we point out an error in [ABS01]. In the last section we study the relations between CCR and CAR flows on type III factors.

3.1 CAR Flow

Let $\mathcal{H} = L^2(0,\infty) \otimes \mathcal{K}$, where $\mathcal{K}$ is any Hilbert space. Let $\mathcal{F}_-(\mathcal{H})$ denote the anti-symmetric Fock space. For given $f \in \mathcal{H}$, let $a(f)$ be the creation operator in $\mathcal{B}(\mathcal{F}_-(\mathcal{H}))$;

thus:

1. $\mathcal{H} \ni f \mapsto a(f)$ is $\mathbb{C}$-linear,

2. **(CAR)**

   $a(f)a(g) + a(g)a(f) = 0$ and $a(f)a(g)^* + a(g)^*a(f) = \langle f, g\rangle 1$,

where $f, g \in \mathcal{H}$. Let $\mathcal{A}$ be the unital C^*-algebra generated by $\{a(f) : f \in \mathcal{H}\}$ in $\mathcal{B}(\mathcal{F}_-(\mathcal{H}))$. We note that $||a(f)|| = ||f||$ for $f \in \mathcal{H}$. Now suppose $R \in \mathcal{B}(\mathcal{H})$ satisfies $0 \leq R \leq 1$, where of course 1 is the identity operator $id_{\mathcal{H}}$. The operator R determines the so-called quasi-free state ω_R on $\mathcal{A}$ which satisfies the condition:

$$\omega_R(a^*(f_m)\cdots a^*(f_1)a(g_1)\cdots a(g_n)) = \delta_{mn}\det(\langle f_i, Rg_j\rangle).$$

It is known - see [BR81], [Amo01] - that there exists a representation π_R of the C^*-algebra $\mathcal{A}$ on the Hilbert space $\mathcal{H}_R = \mathcal{F}_-(\mathcal{H}) \otimes \mathcal{F}_-(\mathcal{H})$ defined by the formulae

$$\begin{aligned}
&\pi_R(a(f)) = a((1-R)^{1/2}f) \otimes \Gamma + 1 \otimes a^*(qR^{1/2}f),\\
&\pi_R(a^*(f)) = a^*((1-R)^{1/2}f) \otimes \Gamma + 1 \otimes a(qR^{1/2}f),\\
&\pi_R(1) = 1,
\end{aligned}$$

where $f \in \mathcal{H}$. Here Ω is the 'vacuum vector' for the antisymmetric Fock space $\mathcal{F}_-(\mathcal{H})$, q is an anti-unitary operator on $\mathcal{H}$ with $q^2 = 1$, and Γ is the unique unitary operator on $\mathcal{F}_-(\mathcal{H})$ satisfying the conditions $\Gamma a(f) = -a(f)\Gamma, f \in \mathcal{H}$, and $\Gamma\Omega = \Omega$. In this representation, the state ω_R becomes the vector state

$$\omega_R(x) = \langle \Omega \otimes \Omega, \pi_R(x)\Omega \otimes \Omega\rangle,$$

for $x \in \mathcal{A}$, and $\mathcal{H}_R = \mathcal{F}_-(\mathcal{H}) \otimes \mathcal{F}_-(\mathcal{H}) = \overline{\pi_R(\mathcal{A})\Omega \otimes \Omega}$ becomes the GNS Hilbert space, under the assumption that both R and $1 - R$ are injective (and hence also have dense range). So $(\pi_R, \mathcal{H}_R, \Omega \otimes \Omega)$ is the GNS triple for the C^*-algebra $\mathcal{A}$ with respect to the state ω_R. We write $M_R = \{\pi_R(\mathcal{A})\}''$, which is always a factor, most often of type III (see [PS70] Theorem 5.1 and Lemma 5.3).

Let $\{s_t\}_{t\geq}$ be the shift semigroup on $\mathcal{H}$. Assume $s_t^* R s_t = R$ for all $t \geq 0$. Then, by [Arv03] Proposition 13.2.3 and [PS70] Lemma 5.3, there exists an E_0-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ on M_R, where α_t is uniquely determined by the following condition:

$$\alpha_t(\pi_R(a(f)) = \pi_R(a(s_t f)),$$

for all $f \in \mathcal{H}, t \geq 0$. This E_0-semigroup is called the **CAR flow of rank dim $\mathcal{K}$** (on M_R).

3.2 Non-Extendability of CAR flows

For the remainder of this section, we shall assume the following:

1. $q s_t = s_t q$ for all $t \geq 0$. (Such a q always exists.)

2. We write $a_R(f)$ for $\pi_R(a(f))$ whenever $f \in \mathcal{H}$, and write $\mathcal{J}$ for the modular conjugation operator of M_R.

3. Both R and $1 - R$ are invertible; i.e., $\exists \epsilon > 0$ such that $\epsilon \leq R \leq 1 - \epsilon$.

4. R is diagonalisable; in fact, there exists an orthonormal basis $\{f_i\}$ for $\mathcal{K}$ with $R f_i = \lambda_i f_i$ for some $\lambda_i \in [\epsilon, 1-\epsilon] \setminus \{\frac{1}{2}\}$.

5. $R s_t = s_t R \ \forall t \geq 0$. (Clearly then, also the Toeplitz condition $s_t^* R s_t = R$ is met.)

As we are unaware of whether, and if so where, these details may be found in the literature, we shall explicitly determine the modular operators in this case, and eventually ascertain (in Remark 3.2.6) the equi-modularity of the CAR flow.

For any (usually orthonormal)set $\{w_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$, we shall use the following notation for the rest of the paper: if $I = (i_1, i_2, \cdots, i_n)$ and $J = (j_1, j_2, \cdots, j_m)$ are ordered subsets of $\mathbb{N}$, then

1. $w_I = w_{i_1} \wedge \cdots \wedge w_{i_n}$,

2. $w_{IJ} = w_{i_1} \wedge \cdots \wedge w_{i_n} \wedge w_{j_1} \wedge \cdots \wedge w_{j_m}$,

3. $Tw_I = Tw_{i_1} \wedge \cdots \wedge Tw_{i_n}$ for any operator $T \in \mathcal{B}(\mathcal{H})$;

4. $\widetilde{I} = \{i_n, \cdots, i_1\}$ so $w_{\widetilde{I}} = w_{i_n} \wedge \cdots \wedge w_{i_1}$;

5. $a_R(w_I) = a_R(w_{i_1}) \cdots a_R(w_{i_n})$,

6. $a_R^*(f) = (a_R(f))^*$, so $a_R^*(w_{\widetilde{I}}) =: a_R^*(w_{i_n}) \cdots a_R^*(w_{i_1}) =: (a_R(w_I))^*$,

For a while, to simplify the notations, we write $A = (1-R)^{1/2},\ B = qR^{1/2}$ and notice that

$$\begin{aligned} \langle Bh_i, Bh_j\rangle &= \langle qR^{1/2}h_i, qR^{1/2}h_j\rangle \\ &= \langle R^{1/2}h_j, R^{1/2}h_i\rangle \quad \text{since } q \text{ is anti-unitary} \\ &= \langle Rh_j, h_i\rangle \\ &= \delta_{i,j}\lambda_i\ . \end{aligned}$$

Lemma 3.2.1. *Let $L = \{l_1 < \cdots < l_p\}$ be an ordered subset*[1] *of $\mathbb{N}$. Then we have*

$$a_R(h_L)a_R^*(h_{\widetilde{L}})\Omega \otimes \Omega = \sum c(L_1)Ah_{L_1} \otimes Bh_{L_1}, \tag{3.2.1}$$

where the summation is taken over all ordered (possibly empty) subsets L_1 of L and the $c(L_1)$ are all non-zero real numbers - with $Ah_\emptyset$ and $Bh_\emptyset$ being interpreted as Ω.

Proof. We will use induction on the cardinality of L. If $|L| = 1$, i.e $L = \{l\}$, then observe

[1]For us, an ordered subset of $\mathbb{N}$ will always mean a finite subset of $\mathbb{N}$ with elements ordered in increasing order

that

$$\begin{aligned} a_R(h_l)a_R^*(h_l)\Omega\otimes\Omega &= a_R(h_l)\Omega\otimes Bh_l \\ &= -Ah_l\otimes Bh_l + \|Bh_l\|^2\Omega\otimes\Omega \end{aligned}$$

Thus our lemma is true if $|L| = 1$. Suppose the result is true for ordered sets with n elements. Let $l \in \mathbb{N}$ such that $l \notin L = \{l_1 < \cdots < l_n\}$ and $l_n < l$. Then we have,

$$\begin{aligned} & a_R(h_L)a_R(h_l)a_R^*(h_{\widetilde{L}})a_R^*(h_l)\Omega\otimes\Omega \\ &= (-1)^{3|L|}a_R(h_l)a_R^*(h_l)a_R(h_L)a_R^*(h_{\widetilde{L}})\Omega\otimes\Omega \\ &= (-1)^{|L|}a_R(h_l)a_R^*(h_l)\sum c(L_1)Ah_{L_1}\otimes Bh_{L_1} \\ &= (-1)^{|L|}a_R(h_l)\sum c(L_1)Ah_{L_1}\otimes Bh_l\wedge Bh_{L_1} \\ &= \sum(-1)^{|L|+1+|L_1|}c(L_1)Ah_l\wedge Ah_{L_1}\otimes Bh_l\wedge Bh_{L_1} \\ &\quad + \sum(-1)^{|L|}\lambda_l c(L_1)Ah_{L_1}\otimes Bh_{L_1} \\ &= \sum(-1)^{|L|+1+|L_1|}c(L_1)Ah_{L_1}\wedge Ah_l\otimes Bh_{L_1}\wedge Bh_l \\ &\quad + \sum(-1)^{|L|}c(L_1)\lambda_l Ah_{L_1}\otimes Bh_{L_1} \end{aligned}$$

and the induction step is complete. □

Corollary 3.2.2. *Let $L = \{l_1 < \cdots < l_p\}$ so that, by Lemma 3.2.1, equation 3.2.1 is satisfied. Then we have*

$$\begin{aligned} (i) \quad & a_R(s_th_L)a_R^*(s_th_{\widetilde{L}})\Omega\otimes\Omega = \sum c(L_1)As_th_{L_1}\otimes Bs_th_{L_1}, \forall t\geq 0; \\ (ii) \quad & a_R(h_I)a_R(h_L)a_R^*(h_{\widetilde{L}})a_R^*(h_{\widetilde{J}})\Omega\otimes\Omega \\ = \quad & \sum(-1)^{|I||J|+|L_1|(|I|+|J|)}c(L_1)Ah_I\wedge Ah_{L_1}\otimes Bh_{L_1}\wedge Bh_J \\ (iii) \quad & a_R(s_th_I)a_R(s_th_L)a_R^*(s_th_{\widetilde{L}})a_R^*(s_th_{\widetilde{J}})\Omega\otimes\Omega \\ = \quad & \sum(-1)^{|I||J|+|L_1|(|I|+|J|)}c(L_1)As_th_I\wedge As_th_{L_1}\otimes Bs_th_{L_1}\wedge Bh_J,\ \forall t\geq 0 \end{aligned}$$

where I and J are finite ordered subsets of $\mathbb{N}$ with $I \cap J = I \cap L = L \cap J = \phi$, and the summation is taken over all ordered subsets L_1 of L .

Proof. 1. For the first part we use induction on cardinality of L. If $|L| = 1$, then $L = \{l\}$ and

$$a_R(h_l)a_R^*(h_l)\Omega \otimes \Omega = -Ah_l \otimes Bh_l + \lambda_l \Omega \otimes \Omega.$$

Our strong Toeplitz assumption (that s_t commutes with R and hence also with A and B) then guarantees that

$$a_R(s_t h_l)a_R^*(s_t h_l)\Omega \otimes \Omega = -As_t h_l \otimes Bs_t h_l + \lambda_l \Omega \otimes \Omega$$

and part one of the corollary is true for $|L| = 1$. Suppose the result is true for ordered sets with n elements. Let $l \in \mathbb{N}$ such that $l \notin L = \{l_1 < \cdots < l_n\}$ and $l_n < l$. Then the proof of Lemma 3.2.1 shows that

$$\begin{aligned}
&a_R(h_L)a_R(h_l)a_R^*(h_{\tilde{L}})a_R^*(h_l)\Omega \otimes \Omega \\
&\quad = \sum (-1)^{|L|+1+|L_1|} c(L_1)(Ah_l \wedge Ah_{L_1} \otimes Bh_l \wedge Bh_{L_1} \\
&\quad\quad + (-1)^{|L|} \sum \lambda_l Ah_{L_1} \otimes Bh_{L_1})
\end{aligned}$$

On the other hand, arguing as in the proof of Lemma 3.2.1 and appealing to the Toeplitz condition, we observe that

$$\begin{aligned}
&a_R(s_t h_L)a_R(s_t h_l)a_R^*(s_t h_{\tilde{L}})a_R^*(s_t h_l)\Omega \otimes \Omega \\
&\quad = \sum (-1)^{|L|+1+|L_1|} c(L_1)(As_t h_l \wedge As_t h_{L_1} \otimes Bs_t h_l \wedge Bs_t h_{L_1} \\
&\quad\quad + (-1)^{|L|} \sum \lambda_l As_t h_{L_1} \otimes Bs_t h_{L_1}).
\end{aligned}$$

This completes the inductive step in the proof of (i).

2. This easily follows from:

$$
\begin{aligned}
& a_R(h_I)a_R(h_L)a_R^*(h_{\tilde{L}})a_R^*(h_{\tilde{J}})\Omega\otimes\Omega \\
&\quad = a_R(h_I)a_R^*(h_{\tilde{J}})a_R(h_L)a_R^*(h_{\tilde{L}})\Omega\otimes\Omega \\
&\quad = a_R(h_I)a_R^*(h_{\tilde{J}})\sum c(L_1)Ah_{L_1}\otimes Bh_{L_1} \\
&\quad = a_R(h_I)\sum c(L_1)Ah_L\otimes Bh_J\wedge Bh_L \\
&\quad = \sum c(L_1)(-1)^{|I|(|J|+|L_1|)}Ah_I\wedge Ah_{L_1}\otimes Bh_J\wedge Bh_{L_1} \\
&\quad = \sum c(L_1)(-1)^{|I|(|J|+|L_1|)}(-1)^{|J||L_1|}Ah_I\wedge Ah_{L_1}\otimes Bh_J\wedge Bh_{L_1} \\
&\quad = \sum(-1)^{|I||J|+|L_1|(|I|+|J|)}c(L_1)Ah_I\wedge Ah_{L_1}\otimes Bh_{L_1}\wedge Bh_J
\end{aligned}
$$

3. Part (iii) of the Corollary follows from part (i) of the Corollary, exactly as part (ii) of the Corollary follows from equation 3.2.1.

□

Remark 3.2.3. *1. In the above corollary observe that the coefficients in the expansion of $a_R(h_I)a_R(h_L)a_R^*(h_{\tilde{L}})a_R^*(h_{\tilde{J}})\Omega\otimes\Omega$ are symmetric in I and J, that is if we interchange I and J, then the corresponding coefficient will not be changed.*

2. Since we have assumed that both R and $1-R$ are injective (and have dense range), we may deduce from Corollary3.2.2(ii) and the fact that every coefficient $c(L_1)$ is non-vanishing, that $\mathcal{H}_R=\overline{\pi_R(\mathcal{A})\Omega\otimes\Omega}=\mathcal{F}_-(\mathcal{H})\otimes\mathcal{F}_-(\mathcal{H})$.

Now the following lemma describes the action of the modular conjugation $\mathcal{J}$ and the commutant of M_R.

Lemma 3.2.4. *With the above notation,*

$$(i)\quad \mathcal{J}(h_I \wedge h_L \otimes qh_L \wedge qh_J) = h_{\tilde{J}} \wedge h_L \otimes qh_L \wedge qh_{\tilde{I}}$$

$$(ii)\quad \mathcal{J}M_R\mathcal{J} = M_R' = \{\Gamma \otimes \Gamma b_R(h_i), b_R^*(h_j)\Gamma \otimes \Gamma : i, j \in \mathbb{N}\}''$$

$$(iii)\quad \mathcal{J}a_R(h_l)\mathcal{J} = \Gamma \otimes \Gamma b_R^*(h_l)$$

where $\quad b_R(h) = a(R^{1/2}h) \otimes \Gamma - 1 \otimes a^*(q(1-R)^{1/2}h).$

Proof. Recall the definition of the anti-linear (Tomita) operator S, given by $Sx\Omega \otimes \Omega = x^*\Omega \otimes \Omega$, $x \in M_R$. We want to show the following expression for S:

$$\begin{aligned} &S(Ah_I \wedge Ah_L \otimes Bh_L \wedge Bh_J) \\ &\qquad = Ah_{\tilde{J}} \wedge Ah_L \otimes Bh_L \wedge Bh_{\tilde{I}} \end{aligned} \tag{3.2.2}$$

The proof is again by induction on the cardinality of L. For $|L| = 0$, the above assertion follows from

$$\begin{aligned} &S((1-R)^{1/2}h_I \otimes qR^{1/2}h_J) \\ &\qquad = Sa_R(h_I)a_R^*(h_{\tilde{J}})(\Omega \otimes \Omega) \\ &\qquad = a_R(h_{\tilde{J}})a_R^*(h_I)(\Omega \otimes \Omega) \\ &\qquad = (1-R)^{1/2}h_{\tilde{J}} \otimes qR^{1/2}h_{\tilde{I}} \end{aligned}$$

Assume now that $|L| = n$ and that we know the validity of equation 3.2.2 whenever $|L| < 1$.

The point to be noticed is that Corollary 3.2.2(ii) may be re-written - in view of (i) each $c(L_1)$ (and $c(L)$ in particular) being non-zero, and (ii) Remark 3.2.3(i) - as:

$$Ah_I \wedge Ah_L \otimes Bh_L \wedge Bh_J \tag{3.2.3}$$

$$= da_R(h_I)a_R(h_L)a_R^*(h_{\widetilde{L}})a_R^*(h_{\widetilde{J}})\Omega \otimes \Omega \tag{3.2.4}$$

$$+ \sum_{L_1 \subsetneq L} d(L_1)Ah_I \wedge Ah_{L_1} \otimes \ Bh_{L_1} \wedge Bh_J, \tag{3.2.5}$$

where the constants $d, d(L_1)$ are all real and remain unchanged under changing (I, J) to $(\widetilde{J}, \widetilde{I})$.

Now apply S to both sides of the above equation. Then the two terms on the right side get replaced by the terms obtained by replacing (I, J) by $(\widetilde{J}, \widetilde{I})$ (3.2.4 by definition of S and 3.2.5 by the induction hypothesis regarding 3.2.2), thereby completing the proof of equation 3.2.2.

Equation (3.2.2) clearly implies that

$$S(h_I \otimes qh_J) = \left((1-R)R^{-1}\right)^{\frac{1}{2}} h_{\widetilde{J}} \otimes q \left((R(1-R)^{-1}\right)^{\frac{1}{2}} h_{\widetilde{I}} \tag{3.2.6}$$

(even if $I \cap J \neq \emptyset$; consideration of their intersections was needed essentially in order to establish Lemma 3.2.1 and thereby deduce the foregoing conclusions.)

Let $\mathcal{D}$ be the linear subspace spanned by $\{h_I \otimes qh_J : |I|, |J| \geq 0\}$. Thus $\mathcal{D}$ is an obviously dense subspace of $\mathcal{H}_R$ which is contained in the domain of the Tomita conjugation operator S, where its action is given by equation 3.2.6. We now wish to show that $\mathcal{D}$ is also contained in $dom(S^*)$ and that $S^*|_{\mathcal{D}}$ is the operator F defined by the equation

$$F(h_I \otimes qh_J) = \left((R(1-R)^{-1}\right)^{\frac{1}{2}} h_{\widetilde{J}} \otimes q \left((1-R)R^{-1}\right)^{\frac{1}{2}} h_{\widetilde{I}} \tag{3.2.7}$$

Indeed, notice that

$$\begin{aligned}
&\langle S(h_I \otimes qh_J, h_{I'} \otimes qh_{J'}\rangle \\
&= \langle \left((1-R)R^{-1}\right)^{\frac{1}{2}} h_{\tilde{J}} \otimes q \left(R(1-R)^{-1}\right)^{\frac{1}{2}} h_{\tilde{I}}, h_{I'} \otimes qh_{J'}\rangle \\
&= \langle \left((1-R)R^{-1}\right)^{\frac{1}{2}} h_{\tilde{J}}, h_{I'}\rangle \langle q \left(R(1-R)^{-1}\right)^{\frac{1}{2}} h_{\tilde{I}}, qh_{J'}\rangle \\
&= \langle \left((1-R)R^{-1}\right)^{\frac{1}{2}} h_J, h_{\tilde{I'}}\rangle \langle h_{\tilde{J'}}, \left(R(1-R)^{-1}\right)^{\frac{1}{2}} h_I\rangle \\
&= \langle \left(R(1-R)^{-1}\right)^{\frac{1}{2}} h_{\tilde{J'}}, h_I\rangle \langle q \left((1-R)R^{-1}\right)^{\frac{1}{2}} h_{\tilde{I'}}, qh_J\rangle \\
&= \langle \left(R(1-R)^{-1}\right)^{\frac{1}{2}} h_{\tilde{J'}} \otimes q \left((1-R)R^{-1}\right)^{\frac{1}{2}} h_{\tilde{I'}}, h_I \otimes qh_J\rangle \\
&= \langle F(h_{I'} \otimes qh_{J'}), h_I \otimes qh_J\rangle
\end{aligned}$$

Then, as S and F leave $\mathcal{D}$ invariant, we see that

$$\begin{aligned}
&FS(h_I \otimes qh_J) \\
&\quad = F(\left((1-R)R^{-1}\right)^{\frac{1}{2}} h_{\tilde{J}}) \otimes q \left(R(1-R)^{-1}\right)^{\frac{1}{2}} h_{\tilde{I}}) \\
&\quad = R(1-R)^{-1} h_I \otimes q(1-R)R^{-1} h_J
\end{aligned}$$

If $S = \mathcal{J}\Delta^{1/2}$ is its polar decomposition, with $\mathcal{J}$ the modular conjugation and Δ the modular operator for M_R, the action of $\mathcal{J}$ and Δ on $\mathcal{D}$ are thus seen to be given by the following rules respectively:

$$\mathcal{J}(h_I \wedge h_L \otimes qh_L \wedge qh_J) = h_{\tilde{J}} \wedge h_L \otimes qh_L \wedge qh_{\tilde{I}}$$

$$\mathcal{J}(\Omega \otimes \Omega) = \Omega \otimes \Omega = \Delta(\Omega \otimes \Omega)$$

and

$$\begin{aligned}&\Delta(h_I \wedge h_L \otimes qh_L \wedge qh_J)\\&\quad = R(1-R)^{-1}h_I \wedge R(1-R)^{-1}h_L \otimes q(1-R)R^{-1})h_L \wedge q(1-R)R^{-1}h_J\end{aligned}$$

This proves part (i) of the Lemma, while the proof of parts (ii) and (iii) only involve of the following facts:

1. Lemma 3.2.1 and Corollary 3.2.2 imply that $M_R(\Omega \otimes \Omega)$ is dense in $\mathcal{F}_-(\mathcal{H}) \otimes \mathcal{F}_-(\mathcal{H})$;
2. Lemma 3.2.4 (i) implies that $\mathcal{J}(\mathcal{D}) = \mathcal{D}$
3. A painful but not difficult case-by-case computation reveals that

$$\mathcal{J} a_R(f) \mathcal{J} = (\Gamma \otimes \Gamma) b_R^*(f) \in M_R' \ \forall f$$

□

The fact that s_t commutes with R is seen to imply that the state ω_R is preserved by the CAR flow $\{\alpha_t : t \geq 0\}$ and hence there exists a canonical semi-group $\{S_t : t \geq 0\}$of isometries on $\mathcal{H}_R$ such that

$$S_t(x(\Omega \otimes \Omega) = \alpha_t(x)(\Omega \otimes \Omega) \ \forall x \in M_R.$$

The next lemma relates this semigroup $\{S_t : t \geq 0\}$ of isometries on $\mathcal{H}_R$ and the shift semigroup $\{s_t : t \geq 0\}$ of isometries on $\mathcal{H}$.

Lemma 3.2.5. *Let $\{h_i\}_{i \in \mathbb{N}}$ be the orthonormal basis of $\mathcal{H}$ as above. Then for every $t \geq 0$, we have,*

$$S_t(h_L \wedge h_I \otimes qh_L \wedge qh_J) = s_t h_L \wedge s_t h_I \otimes q s_t h_L \wedge q s_t h_J,$$

where I, J, and L are ordered subsets of $\mathbb{N}$ with $I \cap J = I \cap L = L \cap J = \phi$.

Proof. From the proof of the Lemma 3.2.4, we have

$$
\begin{aligned}
&Ah_I \wedge Ah_L \otimes Bh_L \wedge Bh_J \\
&\quad = da_R(h_L)a_R(h_I)a_R^*(h_{\tilde{L}})a_R^*(h_{\tilde{J}})\Omega \otimes \Omega \\
&\quad + \sum d_1(L_1)Ah_I \wedge Ah_{L_1} \otimes \ Bh_{L_1} \wedge Bh_J,
\end{aligned}
$$

where d, d_1 are non-zero real numbers. Again from of the corollary 3.2.2(iii), we also have,

$$
\begin{aligned}
&As_th_I \wedge As_th_L \otimes Bs_th_L \wedge Bs_th_J \\
&\quad = da_R(s_th_L)a_R(s_th_I)a_R^*(s_th_{\tilde{L}})a_R^*(s_th_{\tilde{J}})\Omega \otimes \Omega \\
&\quad + \sum d_1(L_1)As_th_I \wedge As_th_{L_1} \otimes \ Bs_th_{L_1} \wedge Bh_J.
\end{aligned}
$$

So finally to prove our Lemma, again we use induction on the cardinality of L. For $|L| = 0$ or 1, it is a easy calculation. suppose $|L| = n$ and our Lemma is true for all ordered subset L_1 of L with $|L_1| < |L|$. Then observe that,

$$
\begin{aligned}
&As_th_I \wedge As_th_L \otimes Bs_th_L \wedge Bs_th_J \\
&\quad = da_R(s_th_L)a_R(s_th_I)a_R^*(s_th_{\tilde{L}})a_R^*(s_th_{\tilde{J}})\Omega \otimes \Omega \\
&\quad + \sum d_1(L_1)As_th_I \wedge As_th_{L_1} \otimes \ Bs_th_{L_1} \wedge Bh_J. \\
&\quad = S_tda_R(h_L)a_R(h_I)a_R^*(h_{\tilde{L}})a_R^*(h_{\tilde{J}})\Omega \otimes \Omega, (\text{ Recall } S_t) \\
&\quad\ \ + S_t(\sum d_1(L_1)Ah_I \wedge Ah_{L_1} \otimes \ Bh_{L_1} \wedge Bh_J) \\
&\quad = S_t[da_R(h_L)a_R(h_I)a_R^*(h_{\tilde{L}})a_R^*(h_{\tilde{J}})\Omega \otimes \Omega \\
&\quad\ \ + \sum d_1(L_1)Ah_I \wedge Ah_{L_1} \otimes \ Bh_{L_1} \wedge Bh_J] \\
&\quad = S_t(Ah_I \wedge Ah_L \otimes Bh_L \wedge Bh_J)
\end{aligned}
$$

So we have

$$S_t(Ah_I \wedge Ah_L \otimes Bh_L \wedge Bh_J) = As_th_I \wedge As_th_L \otimes Bs_th_L \wedge Bs_th_J.$$

As we have $s_tR = Rs_t$, clearly above implies,

$$S_t(h_L \wedge h_I \otimes qh_L \wedge qh_J) = s_th_L \wedge s_th_I \otimes qs_th_L \wedge qs_th_J,$$

□

Remark 3.2.6. *Using the definition of S_t and $\mathcal{J}$, it easily follows that $S_t\mathcal{J} = \mathcal{J}S_t$ for all $t \geq 0$, which implies that α_t is equi-modular endomorphism for every $t \geq 0$. So now we are in the perfect situation to talk about the extendability of the* **CAR flow** *and under the above assumptions on R, we prove that* **CAR flows** *are not extendable.*

Now our aim is to explicitly determine $(\alpha_t(M_R)' \cap M_R)(\Omega \otimes \Omega)$ for the CAR flow $\alpha = \{\alpha_t : t \geq 0\}$.

Let $\mathcal{P}$ and $\mathcal{F}$ denote copies of $\mathbb{N}$ - where we wish to think of $\mathcal{F}$ and $\mathcal{P}$ as signifying the future and past respectively. Let us write $f_i = s_th_i$, so $\{f_j\}_{j\in\mathcal{F}}$ is an orthonormal basis for $L^2(t,\infty) \otimes \mathcal{K}$. Also consider an orthonormal basis $\{e_i\}_{i\in\mathcal{P}}$ of $L^2(0,t) \otimes \mathcal{K}$. Then clearly $\{e_i\}_{i\in\mathcal{P}} \cup \{f_j\}_{j\in\mathcal{F}}$ is an orthonormal basis for $L^2(0,\infty) \otimes \mathcal{K}$.

Let $F(\mathcal{F})$ and $F(\mathcal{P})$ denote the collections of all finite ordered subsets of $\mathcal{F}$ and $\mathcal{P}$ respectively. Then $\mathcal{L} = \{v_{I_1J_1} \otimes qv_{I_2J_2} : I_1, I_2 \in F(\mathcal{P}), J_1, J_2 \in F(\mathcal{F})\}$ is an orthonormal basis for $\mathcal{F}_-(\mathcal{H}) \otimes \mathcal{F}_-(\mathcal{H})$, where $v_{IJ} = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} \wedge f_{j_1} \wedge f_{j_2} \wedge \cdots \wedge f_{j_m}$, with $I = \{i_1 < i_2 \cdots < i_n\} \subset \mathcal{P}$ and $J = \{j_1 < j_2 \cdots j_m \subset \mathcal{F}\}$.

Now if $T \in \mathcal{B}(\mathcal{H}_R)$, we will be working with the expansion of $T(\Omega \otimes \Omega)$ with respect to above orthonormal basis. Let us fix an $l \in \mathcal{F}$. We shall write $T(\Omega\otimes\Omega)$ in the following fashion, paying special attention to the occurrence or not of l in the first and/or second tensor factor:

$$
\begin{aligned}
T(\Omega \otimes \Omega) = &\sum (p_{00}(I_1J_1, I_2J_2)\ v_{I_1J_1} \otimes qv_{I_2J_2} \\
&+ p_{11}(I_1J_1, I_2J_2)\ f_l \wedge v_{I_1J_1} \otimes qf_l \wedge qv_{I_2J_2}) \\
&+ \sum u_{00}(I_1J_1, I_2J_2)\ v_{I_1J_1} \otimes qv_{I_2J_2} \\
&+ \sum u_{10}(I_1J_1, I_2J_2)\ f_l \wedge v_{I_1J_1} \otimes qv_{I_2J_2} \\
&+ \sum u_{01}(I_1J_1, I_2J_2)\ v_{I_1J_1} \otimes qf_l \wedge qv_{I_2J_2} \\
&+ \sum u_{11}(I_1J_1, I_2J_2)\ f_l \wedge v_{I_1J_1} \otimes qf_l \wedge qv_{I_2J_2}. \qquad (3.2.8)
\end{aligned}
$$

Here and in the sequel, it will be tacitly assumed that the sums range over $((I_1J_1), (I_2, J_2)) \in (F(\mathcal{P}) \times F_l(\mathcal{F}))^2$ - where we write $F_l(\mathcal{F}) = F(\mathcal{F} \setminus \{l\})$ - and $p_{mn}, u_{mn} : \{(I_1J_1, I_2J_2) : I_k \in F(\mathcal{P}), J_l \in F_l(\mathcal{F})\} \to \mathbb{C}, m, n \in \{0, 1\}$ where it is demanded that $spt(p_{00}) = spt(p_{11})$ and that $spt(p_{11})$, $spt(u_{00})$,$spt(u_{10})$, $spt(u_{01})$ and $spt(u_{11})$ are all disjoint sets - where we write $spt(f)$ for the subset of its domain where the function f is non-zero. When necessary to show their dependence on the index l, we shall anoint these functions with an appropriate superscript, as in: $p^l_{11}(IJ, I'J')$.

The letters p and u are meant to signify ‘paired’ and ‘unpaired’. Thus, suppose $l \in \mathcal{F}$, $I, L \in F(\mathcal{P})$, and $J, K \in F_l(\mathcal{F})$. If both $v_{IJ} \otimes qv_{LK}$ and $f_l \wedge v_{IJ} \otimes qf_l \wedge qv_{LK}$ appear in the representation of $T(\Omega \otimes \Omega)$ with non-zero coefficients, then we shall think of (IJ, KL) as being *an l-paired ordered pair.* Thus $spt(p_{00}) = spt(p_{11})$ is the collection of l-paired ordered pairs, while $\cup^1_{m,n=0} spt(u_{mn})$ is the collection of l-unpaired ordered pairs.

Note that in such an expression of $T(\Omega \otimes \Omega)$ with respect to different l, the type of a summand may change but the coefficients remain the same up to sign, since two vectors anti-commute under wedge product. We also note that $T(\Omega \otimes \Omega)$ has been written with respect to the basis $\mathcal{L}' = \{v_{I_1J_1} \otimes qv_{I_2J_2}, f_l \wedge v_{I_1J_1} \otimes qv_{I_2J_2}, v_{I_1J_1} \otimes qf_l \wedge qv_{I_2J_2}, f_l \wedge v_{I_1J_1} \otimes qf_l \wedge qv_{I_2J_2} : I_i \in F(\mathcal{P}), J_r \in F_l(\mathcal{F})\}$. There are five types of sums in the representation of $T(\Omega \otimes \Omega)$. For simplicity of notation, let us write:

$$\begin{aligned}
(i) \quad & \xi_T(p) = \sum (p_{00}(I_1J_1, I_2J_2)\ v_{I_1J_1} \otimes q v_{I_2J_2} \\
& + p_{11}(I_1J_1, I_2J_2)\ f_l \wedge v_{I_1J_1} \otimes q f_l \wedge q v_{I_2J_2}) \\
(ii) \quad & \xi_T(u_{00}) = \sum u_{00}(I_1J_1, I_2J_2)\ v_{I_1J_1} \otimes q v_{I_2J_2} \\
(iii) \quad & \xi_T(u_{10}) = \sum u_{10}(I_1J_1, I_2J_2)\ f_l \wedge v_{I_1J_1} \otimes q v_{I_2J_2} \\
(iv) \quad & \xi_T(u_{01}) = \sum u_{01}(I_1J_1, I_2J_2)\ v_{I_1J_1} \otimes q f_l \wedge q v_{I_2J_2} \\
(v) \quad & \xi_T(u_{11}) = \sum u_{11}(I_1J_1, I_2J_2)\ f_l \wedge v_{I_1J_1} \otimes q f_l \wedge q v_{I_2J_2},
\end{aligned}$$

and $\mathcal{S} = \{p, u_{00}, u_{10}, u_{01}, u_{11}\}$. So we have:

$$T(\Omega \otimes \Omega) = \sum_{x \in \mathcal{S}} \xi_T(x).$$

We also write:

$$\begin{aligned}
(i) \quad & A_1 = \frac{1}{(1-\lambda_l)^{1/2}} a_R(f_l), \\
(ii) \quad & A_2 = \frac{-1}{\lambda_l{}^{1/2}} \Gamma \otimes \Gamma b_R(f_l) \\
(iii) \quad & B_1 = \frac{1}{\lambda_l^{1/2}} a_R^*(f_l), \\
(iv) \quad & B_2 = \frac{-1}{(1-\lambda_l)^{1/2}} b_R^*(f_l)(\Gamma \otimes \Gamma)
\end{aligned}$$

There is an implicit dependence in the definition of the A_i's and B_i's of the preceding equations on the arbitrarily chosen $l \in \mathcal{F}$. When we wish to make this dependence explicit (as in Theorem 3.2.7 below), we shall adopt the following notational device: $\mathcal{A}_l = \{A_1, A_2\}$ and $\mathcal{B}_l = \{B_1, B_2\}$. We shall frequently use the following facts in the sequel:

1. $R^{1/2} f_l = R^{1/2} s_t h_l = \lambda_l{}^{1/2} f_l$;

2. $(1-R)^{1/2} f_l = (1-R)^{1/2} s_t h_l = (1-\lambda_l)^{1/2} f_l$; and

3. $f_l \otimes \Omega, \Omega \otimes f_l \in ran(S_t) \; \forall \; l \in \mathcal{F}$.

Theorem 3.2.7. *If $T \in \mathcal{B}(\mathcal{H}_R)$ satisfies $A_1 T(\Omega \otimes \Omega) = A_2 T(\Omega \otimes \Omega)$ and $B_1 T(\Omega \otimes \Omega) = B_2 T(\Omega \otimes \Omega)$ for $A_1, A_2 \in \mathcal{A}_l, B_1, B_2 \in \mathcal{B}_l$ and for all $l \in \mathcal{F}$, then*

$$T\Omega \otimes \Omega \subset [\{v_{I_1} \otimes q v_{I_2} : I_1, I_2 \in F(\mathcal{P}), (-1)^{|I_1|} = (-1)^{|I_2|}\}],$$

where [ ] denotes span closure.

We start with a $T \in \mathcal{B}(\mathcal{H}_R)$, which satisfies the hypothesis of the above Theorem 3.2.7 and write $T(\Omega \otimes \Omega)$ as in 3.2.8 , for an arbitrary choice of index l. Then we go through the following Lemmas and prove that the coefficient functions $p_{00}, p_{11}, u_{10}, u_{01}, u_{11}$ are identically zero, while the support of u_{00} is contained in the set $\{(I_1J_1, I_2J_2) : J_1 \cup J_2 = \emptyset, (-1)^{|I_1|} = (-1)^{|I_2|}\}$. The truth of this assertion for all choices of l will prove our Theorem 3.2.7. We start with a Lemma regarding the representation of $T(\Omega \otimes \Omega)$.

Lemma 3.2.8. *Let $\eta(x)$ (resp., $\eta(y)$) be a summand[2] of the sum $\xi_T(x)$ (resp, $\eta(y)$), where $x, y \in \mathcal{S}$. Then $\langle \eta(x), \eta(y) \rangle = 0$ implies that $\langle X\eta(x), Y\eta(y) \rangle = 0$, for all $x, y \in \mathcal{S}$ and $X, Y \in \mathcal{A}$ or $X, Y \in \mathcal{B}$.*

Proof. This follows from (i) the assumptions that $spt(p_{00}) = spt(p_{11})$, (ii) $sp(p_{11})$, $spt(u_{00})$,$spt(u_{10})$, $spt(u_{01})$ and $spt(u_{11})$ are all disjoint sets and (iii) the definition of the action of X, Y on $\eta(x)$. □

Lemma 3.2.9. *If $A_1 T(\Omega \otimes \Omega) = A_2 T(\Omega \otimes \Omega)$, then $A_1 \xi_T(x) = A_2 \xi_T(x)$ for all $x \in \mathcal{S}$. Similarly if $B_1 T(\Omega \otimes \Omega) = B_2 T(\Omega \otimes \Omega)$, then $B_1 \xi_T(x) = B_2 \xi_T(x)$ for all $x \in \mathcal{S}$.*

Proof. This follows from

$$\|(A_1 - A_2) T(\Omega \otimes \Omega)\|^2 = \sum_{x \in \mathcal{S}} \|(A_1 - A_2)\xi_T(x)\|^2 \; ,$$

[2]By a summand of $\xi_T(p)$ we shall mean a 'paired term' of the form $(p_{00}(I_1J_1, I_2J_2)\; v_{I_1J_1} \otimes q v_{I_2J_2} + \; p_{11}(I_1J_1, I_2J_2)\; f_l \wedge v_{I_1J_1} \otimes q f_l \wedge q v_{I_2J_2})$ rather than an individual term of such a pair

which is a consequence of Lemma 3.2.8. ☐

Now onwards we assume that T satisfies the hypothesis of the Theorem 3.2.7 and with the foregoing notations we have the following Lemma regarding the coefficients of the representation of $T(\Omega \otimes \Omega)$.

Lemma 3.2.10. *$p_{00},\ p_{11},\ u_{00},\ u_{10},\ u_{01},\ u_{11}$ satisfy the following equations:*

$$\begin{aligned}
(i) \quad & \sigma(I_2, J_2)p_{00}(I_1J_1, I_2J_2) + p_{11}(I_1J_1, I_2J_2)\frac{\lambda_l^{1/2}}{(1-\lambda_l)^{1/2}} \\
= \quad & \sigma(I_1, J_1)p_{00}(I_1J_1, I_2J_2) + \rho(I_1J_1, I_2J_2)p_{11}(I_1J_1, I_2J_2)\frac{(1-\lambda_l)^{1/2}}{\lambda_l^{1/2}}, \\
(ii) \quad & \sigma(I_2, J_2)u_{00}(I_1J_1, I_2J_2) = \sigma(I_1, J_1)u_{00}(I_1J_1, I_2J_2), \\
(iii) \quad & u_{01}(I_1J_1, I_2J_2)\frac{\lambda_l^{1/2}}{(1-\lambda_l)^{1/2}} = \star u_{01}(I_1J_1, I_2J_2)\frac{(1-\lambda_l^{1/2}}{\lambda_l)^{1/2}}, \\
(iv) \quad & \star u_{11}(I_1J_1, I_2J_2)\frac{\lambda_l^{1/2}}{(1-\lambda_l)^{1/2}} = \star u_{11}(I_1J_1, I_2J_2)\frac{(1-\lambda_l)^{1/2}}{\lambda_l{}^{1/2}}, \\
(v) \quad & \star\frac{(1-\lambda_l)^{1/2}}{\lambda_l^{1/2}}u_{10}(I_1J_1, I_2J_2) = \star\frac{\lambda_l^{1/2}}{(1-\lambda_l)^{1/2}}u_{10}(I_1J_1, I_2J_2)
\end{aligned}$$

where $\sigma : F(\mathcal{P}) \times F(\mathcal{F}\} \to \{1, -1\}$ is defined by $\sigma(I, J) = (-1)^{|I|+|J|}$, $\rho : \{(I_1J_1, I_2J_2) : I_k \in F(\mathcal{P}), J_l \in F(\mathcal{F})\} \to \{1, -1\}$, defined by $\rho(I_1J_1, I_2J_2) = (-1)^{|I_1|+|J_1|+|I_2|+|J_2|}$, and $\star = \pm 1$.

Proof. T satisfies $A_1(T\Omega \otimes \Omega) = A_2T(\Omega \otimes \Omega)$. So from the Lemma 3.2.9, we have $A_1\xi_T(x) = A_2\xi_T(x)$ foll $x \in \mathcal{S}$. Now for every $x \in \mathcal{S}$, we separately compute $A_1\xi_T(x)$ and $A_2\xi_T(x)$ and compare their coefficients.

(i) If $A_1\xi_T(p) = A_2\xi_T(pi)$, observe that

$$
\begin{aligned}
A_1\xi_T(p) &= \frac{1}{(1-\lambda_l)^{1/2}} a_R(f_l)\xi_T(p) \\
&= \sum\big(\frac{p_{00}(I_1J_1, I_2J_2)}{(1-\lambda_l)^{1/2}} a_R(f_l) v_{I_1J_1} \otimes q v_{I_2J_2} \\
&+ \frac{p_{11}(I_1J_1, I_2J_2)}{(1-\lambda_l)^{1/2}} a_R(f_l) f_l \wedge v_{I_1J_1} \otimes q f_l \wedge q v_{I_2J_2}\big) \\
&= \sum\big(\sigma(I_2, J_2) p_{00}(I_1J_1, I_2J_2 \\
&\quad + \ p_{11}(I_1J_1, I_2J_2) \frac{\lambda_l^{1/2}}{(1-\lambda_l)^{1/2}}\big) f_l \wedge v_{I_1J_1} \otimes q v_{I_2J_2}
\end{aligned}
$$

while

$$
\begin{aligned}
&A_2\xi_T(p) \\
&= \frac{-1}{\lambda_l{}^{1/2}} \Gamma \otimes \Gamma b(f_l)\xi_T(p) \\
&= \sum\big(p_{00}(I_1J_1, I_2J_2) \frac{-1}{\lambda_l{}^{1/2}} \Gamma \otimes \Gamma b(f_l) v_{I_1J_1} \otimes q v_{I_2J_2} \\
&+ p_{11}(I_1J_1, I_2J_2) \frac{-1}{\lambda_l{}^{1/2}} \Gamma \otimes \Gamma b(f_l) f_l \wedge v_{I_1J_1} \otimes q f_l \wedge q v_{I_2J_2} \\
&= \sum\big(\sigma(I_1, J_1) p_{00}(I_1J_1, I_2J_2) \\
&\quad + \rho(I_1J_1, I_2J_2) p_{11}(I_1J_1, I_2J_2) \frac{(1-\lambda_l)^{1/2}}{\lambda_l^{1/2}}\big) f_l \wedge v_{I_1J_1} \otimes q v_{I_2J_2}
\end{aligned}
$$

and (i) follows upon comparing coefficients in the two equations above.

(ii) If $A_1\xi_T(u_{00}) = A_2\xi_T(u_{00})$, observe that

$$\begin{aligned}
A_1\xi_T(u_{00}) &= \frac{1}{(1-\lambda_l)^{1/2}} a_R(f_l)\xi_T(u_{00}) \\
&= \sum \frac{u_{00}(I_1J_1, I_2J_2)}{(1-\lambda_l)^{1/2}} a_R(f_l) v_{I_1J_1} \otimes q v_{I_2J_2} \\
&= \sum \sigma(I_2, J_2) u_{00}(I_1J_1, I_2J_2) f_l \wedge v_{I_1J_1} \otimes q v_{I_2J_2}
\end{aligned}$$

while

$$\begin{aligned}
A_2\xi_T(u_{00}) &= \frac{-1}{\lambda_l^{1/2}} \Gamma \otimes \Gamma b(f_l)\xi_T(u_{00}) \\
&= \sum u_{00}(I_1J_1, I_2J_2) \frac{-1}{\lambda_l^{1/2}} \Gamma \otimes \Gamma b(f_l) v_{I_1J_1} \otimes q v_{I_2J_2} \\
&= \sum \sigma(I_1, J_1) u_{00}(I_1J_1, I_2J_2) f_l \wedge v_{I_1J_1} \otimes q v_{I_2J_2}.
\end{aligned}$$

and (ii) follows upon comparing coefficients in the two equations above.

Equations (iii) and (iv) are proved by arguing exactly as for (ii) above.

As for (v), we also have $B_1T(\Omega \otimes \Omega) = B_2T(\Omega \otimes \Omega)$. So from Lemma 3.2.9, we have $B_1\xi_T(x) = B_2\xi_T(x)$ for all $x \in \mathcal{S}$. In particular we have $B_1\xi_T(u_{10}) = B_2\xi_T(u_{10})$ We compute $B_1\xi_T(u_{10})$ and $B_2\xi_T(u_{10})$:

$$\begin{aligned}
&B_1\xi_T(u_{10}) \\
&= \frac{1}{\lambda_l^{1/2}} a_R^*(f_l) \sum u_{10}(I_1J_1, I_2J_2) f_l \wedge v_{I_1J_1} \otimes q v_{I_2J_2} \\
&= \sum (\star \frac{(1-\lambda_l)^{1/2}}{\lambda_l^{1/2}} u_{10}(I_1J_1, I_2J_2) v_{I_1J_1} \otimes q v_{I_2J_2} + v_{I_1J_1} \otimes q f_l \wedge q v_{I_2J_2})
\end{aligned}$$

while

$$
\begin{aligned}
&B_2\xi_T(u_{10}) \\
&= \frac{-1}{(1-\lambda_l)^{1/2}} b_R^*(f_l)(\Gamma\otimes\Gamma)\sum u_{10}(I_1J_1,I_2J_2)f_l\wedge v_{I_1J_1}\otimes qv_{I_2J_2} \\
&= \sum(\star\frac{\lambda_l^{1/2}}{(1-\lambda_l)^{1/2}}u_{10}(I_1J_1,I_2J_2)v_{I_1J_1}\otimes qv_{I_2J_2}+\star v_{I_1J_1}\otimes qf_l\wedge qv_{I_2J_2})
\end{aligned}
$$

where $\star\in\{+,-\}$. So by comparing coeficients in the expansion of $B_1\xi_T(u_{10})$ and $B_2\xi_T(u_{10})$, we find equation (v) of Lemma 3.2.10. □

With foregoing notation and the assumptions on T, we have the following Corollary.

Corollary 3.2.11. *If we represent $T(\Omega\otimes\Omega)$ las in eqn. 3.2.8, then*

$$u_{01}=u_{11}=u_{10}=0.$$

That is the functions u_{01}, u_{11} and u_{10} are identically zero.

Proof. This is an immediate consequence of Lemma 3.2.10(iii), 3.2.10(iv) and 3.2.10(v) and the assumption that $\lambda_l\neq 1/2\ \forall l$. □

We continue to assume that an operator $T\in\mathcal{B}(\mathcal{H}_R)$ satisfies the hypothesis of the Theorem 3.2.7 and proceed to analyse the representation of $T(\Omega\otimes\Omega)$ as in eqn. (3.2.8).

Remark 3.2.12. 1. *Lemma 3.2.10(i) implies that if (IJ,KL) are l-paired, then $\sigma(I,J)\neq\sigma(K,L)$. (*Reason: *Otherwise, since $\rho(IJ,KL)=\pm1$, and $|p_{11}(IJ.KL)|\neq 0$, we must have $\lambda_l=\frac{1}{2}$.)*

2. *Lemma 3.2.10(ii) implies that if $u_{00}(I_1J_1,I_2J_2)\neq 0$, then $\sigma(I_2,J_2)=\sigma(I_1,J_1)$, i.e. $(-1)^{|I_1|+|J_1|}=(-1)^{|I_2|+|J_2|}$.*

Now we wish to compare the representations of $T(\Omega\otimes\Omega)$ for different l's.

Lemma 3.2.13. *Let $I, K \in F(\mathcal{P})$ and $J, L \in F(\mathcal{F})$. If a term of the form $v_{IJ} \otimes qv_{KL}$ appears in $T(\Omega \otimes \Omega)$ with non-zero coefficient, then (IJ, KL) can be w-**paired** for at most finitely many $w \in \mathcal{F}$ with $w \notin J \cup L$.*

Proof. Suppose, if possible, that $\{l_n : n \in \mathcal{F}\}$ is an infinite sequence of distinct indices such that (IJ, KL) is l_n-paired for each $n \in \mathbb{N}$. Then we may, by Remark 3.2.12(1), conclude that $\{\sigma(I, J), \sigma(K, L)\} = \{1, -1\}$.

Deduce now from Lemma 3.2.10(i) that

$$\begin{aligned}
&\sigma(K, L)p_{00}(IJ, LK) + \star p_{11}^{l_n}(IJ, Kl)\frac{\lambda_{l_n}^{1/2}}{(1-\lambda_{l_n})^{1/2}} \\
&= \sigma(I, J)p_{00}(IJ, LK) + \star p_{11}^{l_n}(IJ, KL)\frac{(1-\lambda_{l_n})^{1/2}}{\lambda_{l_n}^{1/2}}
\end{aligned} \tag{3.2.9}$$

where $\star \in \{+, -\}$. Since $\lambda_{l_n} \in (\epsilon, 1-\epsilon) \setminus \{1/2\}$ for all n, we see that $\{\frac{\lambda_{l_n}^{1/2}}{(1-\lambda_{l_n})^{1/2}} : n \in \mathbb{N}\}$ and $\{\frac{(1-\lambda_{l_n})^{1/2}}{\lambda_{l_n}^{1/2}} : n \in \mathbb{N}\}$ are bounded sequences. As $p_{11}^{l_n}(IJ, LK)$ are Fourier coefficients, the sequence $\{p_{11}^{l_n}(IJ, LK)\}$ converges to 0, as $n \to \infty$. Clearly then $\{\frac{\lambda_{l_n}^{1/2}}{(1-\lambda_{l_n})^{1/2}} p_{11}^{l_n}(IJ, LK) : n \in \mathbb{N}\}$ and $\{\frac{(1-\lambda_{l_n})^{1/2}}{\lambda_{l_n}^{1/2}} p_{11}^{l_n}(IJ, LK) : n \in \mathbb{N}\}$ are sequences converges to 0, as $n \to \infty$. So from the above equation we get $p_{00}(IJ, LK) = 0$. But we had assumed that $p_{00}(IJ, LK)$ is non-zero. Hence $v_{IJ} \otimes qv_{KL}$ can not be l-paired for infinitely many $l \in \mathcal{F}$ with $l \notin J \cup L$.

□

Lemma 3.2.14. *Let $I, K \in F(\mathcal{P})$ and $J, L \in F(\mathcal{F})$ with $l \notin J \cup L$. Suppose an element of the form $v_{IJ} \otimes qv_{KL}$, appearing in $T\Omega \otimes \Omega$ with a non-zero coefficient. Then we have $(-1)^{|I|+|J|} = (-1)^{|K|+|L|}$.*

Proof. From Lemma 3.2.13, we can find a $l_0 \in \mathcal{F}$ such that $l_0 \notin J \cup L$ and $v_{IJ} \otimes qv_{KL}$ is not l_0-paired. If we write $T\Omega \otimes \Omega$ with respect to l_0, we see that $v_{IJ} \otimes qv_{KL}$ appears with exactly the same coefficient as in the third type of sum. So by observing the Remark 3.2.12 with respect to l_0, see that $(-1)^{|I|+|J|} = (-1)^{|K|+|L|}$. □

Again with the foregoing notations, we have the following Lemma about the coefficients of the representation of $T(\Omega \otimes \Omega)$.

Lemma 3.2.15. *$p_{00} = p_{11} = 0$.*

Proof. Recall the equation 3.2.10(i) from Lemma 3.2.10:

$$\sigma(I_2, J_2)p_{00}(I_1J_1, I_2J_2) + p_{11}(I_1J_1, I_2J_2)\frac{\lambda_l^{1/2}}{(1-\lambda_l)^{1/2}}$$
$$= \sigma(I_1, J_1)p_{00}(I_1J_1, I_2J_2) + \rho(I_1J_1, I_2J_2)p_{11}(I_1J_1, I_2J_2)\frac{(1-\lambda_l)^{1/2}}{\lambda_l^{1/2}}$$

where $\sigma(I_2, J_2) = (-1)^{|I_2|+|J_2|}$ and $\sigma(I_1, J_1) = (-1)^{|I_1|+|J_1|}$. But from 3.2.14 we have $(-1)^{|I_2|+|J_2|} = (-1)^{|I_1|+|J_1|}$. Since $\lambda_l \neq 1/2$, from the above equation we get $p_{11}(I_1J_1, I_2J_2) = 0$, which implies that $p_{00}(I_1J_1, I_2J_2) = 0$, since $spt(P_{00}) = spt(p_{11})$, i.e they have the same support. $\square$

Lemma 3.2.16. *$T\Omega \otimes \Omega = \sum x(I_1I_2)\ v_{I_1} \otimes qv_{I_2}$, where the summation is taken over $I_1, I_2 \in F(\mathcal{P})$ with $(-1)^{|I_1|} = (-1)^{|I_2|}$.*

Proof. So we started with a representation of $T\Omega\otimes\Omega$ like 3.2.8 and by using the Corollary 3.2.11 and Lemma 3.2.15 ended up with the following conclusions;

$$p_{00} = p_{11} = u_{10} = u_{01} = u_{11} = 0.$$

Thus finally the representation of $T\Omega \otimes \Omega$ will be of the form

$$T(\Omega \otimes \Omega) = \sum u_{00}(I_1J_1, I_2J_2)\ v_{I_1J_1} \otimes qv_{I_2J_2},$$

where the summation is taken over $I_1, I_2 \in F(\mathcal{P})$, $J_1, J_2 \in F(\mathcal{F})$ with $(-1)^{|I_2|+|J_2|} = (-1)^{|I_1|+|J_1|}$ and $l \notin J_1 \cup J_2$, for $l \in \mathcal{F}$. Since this is true for all $l \in \mathcal{F}$, J_1, J_2 are empty sets, i.e.

$$T(\Omega \otimes \Omega) = \sum x(I_1, I_2)\ v_{I_1} \otimes qv_{I_2},$$

where the summation is taken over $I_1, I_2 \in F(\mathcal{P})$ with $(-1)^{|I_1|} = (-1)^{|I_2|}$ and $x(I_1, I_2) = u_{00}(I_1\emptyset, I_2\emptyset)$ are complex numbers. $\square$

So finally the above Lemma 3.2.16 proves our theorem 3.2.7.

Theorem 3.2.17. *Let* $T \in \alpha_t(M_R)' \cap M_R$*, then*

$$T(\Omega \otimes \Omega) \subset [\{v_{I_1} \otimes qv_{I_2} : I_1, I_2 \in F(\mathcal{P}), (-1)^{|I_1|} = (-1)^{|I_2|}\}],$$

Proof. It is enough to prove that $A_1T(\Omega\otimes\Omega) = A_2T(\Omega\otimes\Omega)$ and $B_1T(\Omega\otimes\Omega) = B_2T(\Omega\otimes\Omega)$, then it follows from the Theorem 3.2.17.

Observe that,

$$\begin{aligned}
&A_1T(\Omega \otimes \Omega) \\
&\quad = \frac{1}{(1-\lambda_l)^{1/2}} a_R(f_l) T\Omega \otimes \Omega \\
&\quad = \frac{1}{(1-\lambda_l)^{1/2}} T a_R(f_l)\Omega \otimes \Omega \quad \text{since } T \in \alpha_t(M_R)' \\
&\quad = T f_l \otimes \Omega \\
&\quad = \frac{-1}{{\lambda_l}^{1/2}} T\Gamma \otimes \Gamma b_R(f_l)\Omega \otimes \Omega \\
&\quad = \frac{-1}{{\lambda_l}^{1/2}} \Gamma \otimes \Gamma b_R(f_l) T\Omega \otimes \Omega \quad \text{since } T \in M_R \text{ and } \Gamma \otimes \Gamma b_R(f_l) \in M_R' \\
&\quad = A_2T(\Omega \otimes \Omega).
\end{aligned}$$

So we have $A_1T(\Omega\otimes\Omega)=A_2T(\Omega\otimes\Omega)$. Again observe that,

$$\begin{aligned}
&B_1T(\Omega\otimes\Omega)\\
&=\frac{1}{\lambda_l{}^{1/2}}a_R^*(f_l)T\Omega\otimes\Omega\\
&=\frac{1}{\lambda_l{}^{1/2}}Ta_R^*(f_l)\Omega\otimes\Omega \ \text{ since } T\in\alpha_t(M_R)'\\
&=T\Omega\otimes f_l\\
&=\frac{-1}{(1-\lambda_l)^{1/2}}Tb_R^*(f_l)\Gamma\otimes\Gamma\Omega\otimes\Omega\\
&=\frac{-1}{(1-\lambda_l)^{1/2}}b_R^*(f_l)\Gamma\otimes\Gamma T\Omega\otimes\Omega,\ \text{ since } T\in M_R \text{ and } b_R^*(f_l)\Gamma\otimes\Gamma\in M_R'\\
&=B_2T(\Omega\otimes\Omega).
\end{aligned}$$

That is $B_1T(\Omega\otimes\Omega)=B_2T(\Omega\otimes\Omega)$.

□

Theorem 3.2.18. *CAR flow $\alpha=\{\alpha_t:t\geq 0\}$ is not extendable.*

Proof. It is enough to show that for some $t>0$, $\alpha_t:M_R\mapsto M_R$ is not extendable. To prove α_t is not extendable, we use the Theorem 2.2.7. We observe that

$$\begin{aligned}
&[\{y\alpha_t(x)\Omega\otimes\Omega : x\in M_R, y\in\alpha_t(M_R)'\cap M_R\}]\\
&\quad=\ [\{\alpha_t(x)y\Omega\otimes\Omega : x\in M_R, y\in\alpha_t(M_R)'\cap M_R\}]\\
&\quad=\ [\{(a_R(f_J)a_R^*(f_L))^*T\Omega\otimes\Omega : J,L\in F(\mathcal{F}), T\in\alpha_t(M_R)'\cap M_R\}].
\end{aligned}$$

Now if $T\in\alpha_t(M_R)'\cap M_R$, then from the Theorem 3.2.17 we have, $T(\Omega\otimes\Omega)\in\{v_{I_1}\otimes qv_{I_2} : I_1,I_2\in F(\mathcal{P}),(-1)^{|I_1|}=(-1)^{|I_2|}\}$. If $g\in\mathcal{P}$, we notice that

$$\begin{aligned}
&\langle(a_R(f_J)a_R^*(f_L))^*T\Omega\otimes\Omega,\ e_g\otimes\Omega\rangle\\
&\qquad=\langle T\Omega\otimes\Omega,\ a_R(f_I)a_R^*(f_L)e_g\otimes\Omega\rangle\\
&\qquad=0.
\end{aligned}$$

So from the above, we conclude that $\{y\alpha_t(x)\Omega \otimes \Omega : x \in M_R, y \in \alpha_t(M_R)' \cap M_R\}$ is orthogonal to the vector $e_g \otimes \Omega$, i.e. $\{y\alpha_t(x) : x \in M_R, y \in \alpha_t(M_R)' \cap M_R\}$ can not be weakly total in M_R, so by the Theorem 2.2.7 α_t can not be extendable. $\square$

Remark 3.2.19. *It has been proved in section 5[ABS01] that CAR flows, arising from quasi-free states for scalars $\neq \frac{1}{2}$, on type III factors, are extendable. But we have shown that CAR arising from quasi-free states for diagonalisable positive contractions (in particular scalars) are not extendable. So our result shows that there is some error in section 5 [ABS01] regarding the conclusion of extendability of CAR flows. In-fact we think that there is a mistake in the theorem 4 of section 5 [ABS01] and for that their conclusion regarding the extendability of CAR flows went wrong.*

3.3 CCR and CAR flows

We have already described CAR flows on type III factors. Recall the CCR algebra and its GNS representation with respect to quasi-free states from the section of examples 2.5. We recall the von Neumann algebra $M_A = \{\pi(W(f)) : f \in \mathcal{H}_{\mathbb{C}}\}''$.

Let $\{s_t\}_{t\geq}$ be the shift semigroup on $\mathcal{H}_{\mathbb{C}}$ and suppose that A commutes with s_t for all $t \geq 0$. Then M_A is a type III factor (see [Hol71]) and the CCR flow [Arv03] restricts to an E_0-semigroup on M_A, $\beta^A = \{\beta_t^A : t \geq 0\}$ uniquely determined by the following condition:

$$\beta_t^A(\pi(W(f)) = \pi(W(s_t f)),$$

for all $f \in \mathcal{H}_{\mathbb{C}}, t \geq 0$. This E_0-semigroup is called **CCR flow of rank dim $\mathcal{K}$** .

Note that if $A = \frac{1+\lambda}{1-\lambda}$ with $\lambda \in (0,1)$, then it is well-known that $M_\lambda = M_A$ is a type III_λ factor. Further, we have also mentioned in the section of examples 2.5 that $\{\beta_t^\lambda; t \geq 0\}$ is equi-modular and all these E_0-semigroups on type III_λ factors are extendable.

Remark 3.3.1. *Type III factors arising from quasi-free representations of CCR and*

CAR algebras with respect to the quasi-free states are always hyperfinite (see [AW69]). In the case of CAR algebra we note that if $A = \lambda$, for $\lambda(0,1) \setminus \{\frac{1}{2}\}$ then $M_A = M_\lambda$ is type III_λ factor. So in both the case we find the hyperfinite III_λ factor for $\lambda \in (0,1) \setminus \{\frac{1}{2}\}$. Since the hyperfinite III_λ factor is unique for every $\lambda \in (0,1) \setminus \{\frac{1}{2}\}$, we thus have two families of E_0-semigroups namely CAR flows and CCR flows on the hyperfinite type III_λ factor.

Now we have the following Corollary to the Theorem 3.2.18 regarding the cocycle conjugacy of CAR flows and CCR flows.

Corollary 3.3.2. *The CAR and CCR flows arising from quasi-free states are not cocycle conjugate.*

Proof. Srinivasan and Margetts have proved in [MS12] that CCR flows arising from these quasi-free states are extendable. By Theorem 3.2.18, CAR flows arising from quasi-free states are not extendable. But extendability of E_0-semigroup is a cocycle conjugacy invariant, so the result follows. □

Remark 3.3.3. *This result is surprising, since on the type I factor CCR and CAR flows of the same Arveson index are cocycle conjugate([Arv03]).*

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