PROBABILITY IN VON NEUMANN ALGEBRAS

By

MADHUSHREE BASU

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The Institute of Mathematical Sciences, Chennai

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____ Date:

Guide/Convenor/Chairman - V. S. Sunder

____ Date:

Member 1 - Vijay Kodiyalam

____ Date:

Member 2 - Parameswaran Sankaran

____ Date:

Member 3 - Anilesh Mohari

____ Date:

External Examiner - B. V. Rao

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Madhushree Basu

DECLARATION

I hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part of a degree/diploma at this or any other Institution/University.

Madhushree Basu

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LIST OF PUBLICATIONS

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2. *Some explicit computations and models of free products* - Journal of the Ramanujan Mathematical Society: Vol 28, No. 1 (2013), Page 71-89
3. *Continuous minmax theorems*, with V. S. Sunder, e-print arXiv:1210.7581 [math. OA].

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Synopsis

A discussion on the preliminaries required to understand this thesis and brief introductions to the chapters are given here.

Preliminaries

This thesis is based on a few observations on applications and analogues of certain features of Probability theory in *non-commutative W^*-probability spaces*. A non-commutative W^*-probability space is a pair (A, ϕ) of an algebra and a linear functional on it. For us, A is a *finite von Neumann algebra* ([Tak02]) (or *finite W^*-algebra*) with a separable predual. More precisely A is a unital *-subalgebra of the algebra of bounded operators on a separable Hilbert space - closed in the weak* topology (known as the σ-weak topology), with ϕ - a unital, positive, faithful, tracial linear functional on it - continuous with respect to the σ-weak topology; in other words ϕ is a *faithful normal tracial state* ([Tak02]) on A. Probability theory - the branch of Mathematics that analyzes random phenomena - deals with random variables, which are scalar-valued functions on a non-empty set equipped with a σ-algebra and a probability measure on it. In the case of von Neumann algebras, random variables are replaced by elements of non-commutative probability spaces, that is, bounded linear operators on separable Hilbert spaces.

For the sake of simplicity we introduce a common notation here. If S is a subset of a von Neumann algebra A, then $W^*(S)$ denotes the von Neumann algebra generated by S in A.

Spectral theory and the theory of measurable functional calculus ([Sun97]) for a bounded normal operator a on a separable Hilbert Space, assures the existence of a *-algebra isomorphism from $L^\infty(\sigma(a), \nu)$ to $W^*(\{a\})$, for some probability measure ν. Through this *-isomorphism the symbol $f(a)$ - for any $f \in L^\infty(\sigma(a), \nu)$ - makes sense as an element of $W^*(\{a\})$. Conversely, any element of $W^*(\{a\})$ can be written as $f(a)$ for some bounded Borel function f.

Our work is based on observations regarding certain behaviours of such non-commutative random variables. The underlying notion of their probabilistic independence, wherever relevant, is taken to be a well-known non-commutative analogue of the classical independence - known as *free independence* ([VDN92]). We define this notion of independence restricting to the context of von Neumann algebras:

Definition 0.0.1. *$(A_i)_{i\in\mathcal{I}}$ - *-subalgebras of a non-commutative W^* probability space (A,ϕ), are said to be* ***freely independent*** *or simply* ***free*** *if $\phi(a_1a_2\cdots a_n)=0$ whenever*

- *$n\in\mathbb{N}$;*
- *$i_j\in\mathcal{I}$ and $a_j\in A_{i_j}, \forall j=1,\cdots n$;*
- *$\phi(a_j)=0, \forall j=1,\cdots,n$ and*
- *$i_1\neq i_2\neq\cdots\neq i_n$, or in other words $a_1a_2\cdots a_n$ is an alternating product.*

*$(S_i)_{i\in\mathcal{I}}$ - subsets of A are called freely independent if the *-algebras generated by these sets are free in (A,ϕ). In particular, $(a_i)_{i\in\mathcal{I}}$ - elements of A are called freely independent if the *-algebras generated by these elements are free in (A,ϕ).*

The following result, assumed frequently in this work without clarification, is well known ([VDN92]):

Proposition 0.0.2. *If $(A_i)_{i\in\mathcal{I}}$ - *-subalgebras are freely independent in W^*-non-commutative probability space (A,ϕ), then $(W^*(A_i))_{i\in\mathcal{I}}$ are also freely independent in (A,ϕ).*

Introduced by Ching in 1973 ([Chi73]) for II_1 factors (infinite-dimensional finite W^*-algebras), and later in the '80s by Voiculescu for general operator algebras, *Free probability theory* studies non-commutative random variables in various *-algebras and their distributions, with 'freeness' or free independence property as the analogue of the classical notion of independence in Probability theory. This theory gives rise to the notion of *free product* of algebras - a universal product of non-commutative probability spaces, denoted by '$*$'. We define this notion in the context of von Neumann algebras:

Definition 0.0.3. *Given a collection of W^*-non-commutative probability spaces $(A_i, \phi_i)_{i\in\mathcal{I}}$, their* ***free product*** *is defined as a universal W^*-non-commutative probability space (A, ϕ), with the notation $A = *_{i\in\mathcal{I}} A_i$, $\phi = *_{i\in\mathcal{I}}\phi_i$, such that there exist *-algebra homomorphisms $\psi_i : A_i \to A$ with the following properties:*

- $\phi \circ \psi_i = \phi_i$,
- $A = W^*(\cup_{i\in\mathcal{I}} \psi_i(A_i))$ *and*
- $(\psi_i(A_i))_{i\in\mathcal{I}}$ *are freely independent in* (A, ϕ).

Free probability theory is closely related to the theory of *non-commutative distributions*. The natural non-commutative analogue of distributions of random variables in classical Probability theory is given in the following definition:

Definition 0.0.4. *Given a non-commutative probability space (A, ϕ), a self-adjoint element a of A is said to have* ***distribution*** *μ - a compactly supported probability measure on $\mathbb{R}$ with its support contained in $\sigma(a)$, if for all Borel subset E of $\mathbb{R}$, $\phi(1_E(a)) = \mu(E)$.*

The above definition and the previous discussion regarding measurable functional calculus together imply that for $a = a^* \in A$ with distribution μ,

$$W^*(\{a\}) = L^\infty(\sigma(a), \mu).$$

An alternate way of defining the distribution of a self-adjoint operator is through the *moments* of the operator. We now define the moments and *free cumulants* of non-commutative random variables. The definitions of these objects hold for any *-algebra, hence in particular for von Neumann algebras and are discussed in various books and articles, for example in [NS06].

Definition 0.0.5.

- *Given a *-algebra A, a family of functions $(f_n : A^n \to \mathbb{C})_{n\in\mathbb{N}}$ is said to be* ***multiplicatively extended****, when for all $n \in \mathbb{N}$, a new family of* *functions is defined on* A^n*, indexed by* $NC(n)$*= the set of non-crossing partitions on* n*, as follows:*

$$f_\pi(a_1, \cdots, a_n) := \Pi_{V\in\pi}\ f_V[a_1, \cdots, a_n],$$

where for $V = \{i_1, \cdots, i_r\}$ *with* $i_j \in \{1, \cdots, n\}$*,*

$$f_V[a_1, \cdots, a_n] = f_r(a_{i_1}, \cdots, a_{i_r}).$$

In particular $f_{1_n} = f_n, \forall\ n$*, where* 1_n *is the full partition on* $\{1, \cdots, n\}$*, that is,* $\{\{1, \cdots, n\}\}$*.*

- *Given a non-commutative probability space* (A, ϕ)*, where* A *is a *-algebra and* ϕ *is a unital linear functional on it, the multiplicatively extended family obtained from* $(\phi_n)_{n\in\mathbb{N}}$ *is called the* ***family of moment functions****, where*

$$\phi_n(a_1, \cdots, a_n) := \phi(a_1 \cdots a_n).$$

In particular for a single element $a \in A$*,* $\phi_n(a, \cdots, a) = \phi(a^n)$ *is simply called its* n^{th} *moment. For more than one element, the term 'joint moment' is used.*

- *The multiplicatively extended family of functions obtained from* $(\kappa_n)_{n\in\mathbb{N}}$*, where*

$$\kappa_n(a_1, \cdots, a_n) := \sum_{\pi\in NC(n)} \phi_\pi(a_1, \cdots, a_n) \cdot \mu(\pi, 1_n),$$

where $\mu(\cdot,\cdot)$ *is the Mobius function on the lattice* $NC(n) \times NC(n)$*, is called the* ***family of cumulant functions****. Similarly as moments the terms '*n^{th} *cumulant' and 'joint cumulant' are used.*

The moments and cumulants form the base of combinatorial Free probability theory (as developed in [Spe94], [NS96] [NS97], [Spe97], [KS00] etc.). Moments of a self-adjoint bounded operator are closely related to its distribution in a W^*-non-commutative probability space. In fact distribution of a self-adjoint bounded operator a can be defined as the compactly supported probability measure μ, for which

$$\phi(a^n) = \int_{\sigma(a)} t^n \, d\mu(t)^{1}.$$

The study of free cumulants is an equivalent analogue of studying moments since each can be expressed in terms of the other explicitly - as can be observed in the following chapters. In particular, being additive for free non-commutative random variables, free cumulants make various computations easier.

The theory of W^*-free probability can be generalized in an *operator-valued* setup (as introduced in [VDN92], [Spe98], [Shl98], [Shl99], [NSS02] etc.), by replacing (A, ϕ) with $(A, \mathcal{E})$, where A is as above and $\mathcal{E} : A \to B$ is a conditional expectation ([Tak03]) onto a von Neumann subalgebra B of A. In this setup the moments and cumulants take values in B and we use such B-valued free probability in the second chapter of our work. For $B = \mathbb{C}$, the operator-valued free probability theory coincides with the usual free probability.

This thesis is divided in three chapters. The first two chapters describe certain non-commutative probabilistic models in Free Probability theory. The main tools for the discussions in these two chapters are the moments and cumulants of non-commutative random variables. The last chapter proves an analogue of a minmax theorem - characterizing a certain extremal behaviour of sums of eigenvalues of finite dimensional Hermitian matrices - for a bounded self-adjoint operator with continuous spectra, involving its *distribution function* - denoted by F_μ - corresponding to the distribution μ of that operator - as the main tool.

[1]Positivity and unital property of ϕ imply that μ defined as above indeed gives a unique probability measure ([Akh65]). Stone-Weierstrass theorem, Lusin's theorem, measurable fucntional calculus and continuity of ϕ in the σ-weak topology determine μ.

Introduction to the chapters

Chapter 1: In this chapter we compute free products of certain finite dimensional W^*-probability space. The motivation behind the computations done in this chapter comes from trying to understand [Dyk94] and [Dyk93]. Most of the results here were proved in those two papers in much more general context. However as against Dykema's general results, our proofs are elementary and require little prior knowledge beyond basic trigonometry and combinatorics; but they do have the disadvantage of constraining us to direct sums and matrix algebras equipped with the 'uniform trace', and consequently to results, where powers of 2 keep cropping up as certain rational indices of the resulting interpolated free group factors.

The results in this chapter are in the form of isomorphisms between W^*-probability spaces, where the left hand side is always a free product of two W^*-probability spaces. Our idea is to find equivalent models of these two spaces inside the W^*-probability space on the right hand side and prove that

(i) they are free inside the latter, and that

(ii) they generate the latter.

We start with the proof of:

- $(\mathbb{C} \oplus \mathbb{C}) * (\mathbb{C} \oplus \mathbb{C}) \cong M_2(\mathbb{C}) \otimes L\mathbb{Z}$,

For the proof, the suitable choice of models of the spaces on the left hand side, turn out to be the two well-known projections in generic positions on the 2-dimensional Hilbert space.

Using the above, we continue with the following results:

- $(\mathbb{C} \oplus \mathbb{C}) * M_2(\mathbb{C}) \cong M_2(\mathbb{C}) \otimes LF_2$; and
- $M_2(\mathbb{C}) * M_2(\mathbb{C}) \cong M_2(\mathbb{C}) \otimes LF_3$,

where LF_n denotes the free group factor ([MvN43]) corresponding to the free group F_n with n generators.

In the following sections we give further alternate proofs (using similar models) of results in [Dyk94] that extend the above results to:

- $(A_1 \oplus A_2) * (B_1 \oplus B_2) \cong M_2(A_1 * A_2 * B_1 * B_2 * L\mathbb{Z})$;
- $(A_1 \oplus A_2) * M_2(B) \cong M_2(A_1 * A_2 * B * LF_2)$; and
- $M_2(A) * M_2(B) \cong M_2(A * B * LF_3)$.

In the final section of this chapter we portray a few applications of our method of computation. The main proposition of this section enables one to easily compute free products involving certain finite dimensional commutative and non-commutative W^*-spaces as well as certain free group factors:

For $m, n \in \mathbb{N}$, $k, l \in \mathbb{N} \cup \{0\}$ and $LF_0 = \mathbb{C}$,

- $(LF_k)^{2^n} * (LF_l)^{2^m} \cong M_2(LF_{5+\frac{2(k-1)}{2^n-1}+\frac{2(l-1)}{2^m-1}})$. In particular $(L\mathbb{Z})^{2^n} * (L\mathbb{Z})^{2^m} \cong M_2(LF_5)$.
- $M_{2^n}(LF_k) * (LF_l)^{2^m} \cong M_2(LF_{5+\frac{k-1}{4^n-1}+\frac{2(l-1)}{2^m-1}})$. In particular $M_{2^n}(L\mathbb{Z}) * (L\mathbb{Z})^{2^m} \cong M_2(LF_5)$.
- $M_{2^n}(LF_k) * M_{2^m}(LF_l) \cong M_2(LF_{5+\frac{k-1}{4^n-1}+\frac{l-1}{4^m-1}})$. In particular $M_{2^n}(L\mathbb{Z}) * M_{2^m}(L\mathbb{Z}) \cong M_2(LF_5)$.

We conclude this chapter by reproving Dykema's result involving the hyperfinite II_1 factor ([MvN43]), as an application of our computations:

- $R * R \cong LF_2$.

Chapter 2: This chapter is devoted to graphical models for finite von Neumann algebras, associating them to finite weighted graphs ([KS11]). The motivation behind this work comes from trying to understand [KS09] and [KS11], in which V. Kodiyalam and V. S. Sunder introduced this graph-von Neumann algebra association and gave independent proofs of some of the results proved in [GJS10] and [GJS11] by A. Guionnet, V. Jones and D. Shlyakhtenko, from slightly different viewpoints. However, unlike [KS09] and [KS11] we allow non-bipartite graphs that leads us to the pleasant, but not surprising fact that the von Neumann algebra associated to to a 'flower with n petals' (i.e. a graph with a single vertex and n loops) is isomorphic to the group von Neumann algebra of the free group on n generators. In general too, the associated algebra gives a free product of algebras corresponding to subgraphs 'with one edge' (actually a pair of dual edges), with amalgamation over a finite-dimensional abelian subalgebra corresponding to the vertex set. This yields certain natural examples of

(i) a non-commutative random variable with a free Poisson distribution;

(ii) non-commutative random variables with operator-valued circular and operator-valued semicircular distribution.

The intuitive idea of associating a weighted graph $\Gamma = (V, \mathcal{E}, \mu)$ (V is a (finite) set of vertices, $\mathcal{E}$ is a (finite) set of directed edges and $\mu : V \to (0, \infty)$ is a normalized weight or spin function to a finite von Neumann algebra, was to identify a path in the graph - in the form of concatenation of edges from $\mathcal{E}$ - to a suitable product of operators in the algebra. However we discuss two different but isomorphic notions ([KS11]) of the graph-von Neumann algebra association, one of which is equipped with the above notion of product and a complicated tracial state on it, where as the other one is equipped with a complicated product but a simple trace on it. In our work we first consider the latter and later on we bring in its equivalent graph-von Neumann algebra associate that eases the cumulant calculation considerably.

We let $\mathcal{P}_n = \mathcal{P}_n(\Gamma)$ denote the set of paths of length n in Γ ($\mathcal{P}_0 := V$) and let $P_n(\Gamma)$ denote the vector space with basis $\{[\xi] : \xi \in \mathcal{P}_n(\Gamma)\}$. $\xi = \xi_1\xi_2 \cdots \xi_n$ is thought as the 'concatenation product' where ξ_i denotes the i-th edge of ξ.

Then $F(\Gamma) := \oplus_{n \geq 0} P_n(\Gamma)$ - equipped with the a slightly complicated multiplication, but with an easy choice of tracial state on it given as $\mu^2 \circ E$, where $E : F(\Gamma) \to P_0(\Gamma)$, is the natural conditional expectation (μ^2 is the linear extension of the original μ^2 to $P_0(\Gamma)$) - is a *-probability space and the finite von Neumann algebra that we want to associate with Γ, denoted by $M(\Gamma, \mu)$, is chosen as the von Neumann algebra generated by $F(\Gamma)$.

With the above association and the fact that any such finite graph can be thought of as concatenation of the following three types of graphs: $|V| = 1, |\mathcal{E}| = 1$ (a single vertex with a self-adjoint loop), $|V| = 1, |\mathcal{E}| = 2$ (a single vertex with a non-self-adjoint loop and its adjoint) and $|V| = 2, |\mathcal{E}| = 2$ (two vertices with an edge and its adjoint), we first compute the von Neumann algebras associated to these three graphs. In the first two cases the results turn out to be $L\mathbb{Z}$ and LF_2 respectively. In the third case, $M(\Gamma, \mu)$ is shown to be $\mathbb{C} \oplus M_2(L\mathbb{Z})$, the $L\mathbb{Z}$ being the singly generated von Neumann algebra generated by a certain operator that we denote by a. We give an analytic proof of the fact that a follows free Poisson distribution.

In the next section we move on to the alternate concept of graph-von Neumann algebra denoted by $Gr(\Gamma)$. The main result of this section is:

- *The $P_0(\Gamma)$-valued mixed cumulants in $Gr(\Gamma)$ are given as, $\kappa_n([e_1], [e_2], \cdots, [e_n]) = 0$, unless $n = 2$ and $e_2 = \widetilde{e_1}$, in which case with the source and range of e_1 as v and w respectively, $\kappa_2([e_1], [\widetilde{e_1}]) = \frac{\mu(w)}{\mu(v)}[v]$.*

This leads us to the examples of P_0-valued circular and semicircular elements. We end this section with a corollary, stating a slightly more generalized version of the result regarding the graph of a 'a flower with n petals' as mentioned above:

- $Gr(\Gamma, \mu) = *_{P_0(\Gamma)}\{Gr(\Gamma_e, \mu_e) : \{e, \widetilde{e}\} \subset \mathcal{E}\}$, *and hence, also*
- $M(\Gamma, \mu) = *_{P_0(\Gamma)}\{M(\Gamma_e, \mu_e) : \{e, \widetilde{e}\} \subset \mathcal{E}\}$.

We conclude this chapter with an alternate (combinatorial) proof of the fact that the operator a that appears in the computation of $M(\Gamma, \mu)$ in the case of a graph with two vertices and two edges, does indeed have free Poisson distribution. This proof involves Narayana polynomials $N_n(T) = \sum_{k=1}^{n} N(n,k)T^k$, with coefficients as Narayana numbers, denoted by $N(n,k)$.

Chapter 3: This chapter extends a well-known minmax characterization by Ky Fan ([Fan49]), of the sum of the k largest eigenvalues of an $n \times n$ Hermitian matrix ($k, n \in \mathbb{N}, k \leq n$), to a statement about a self-adjoint element of an appropriate finite von-Neumann algebra.

We were motivated by Lemma 3.2 of [BV93]) that suggested a possible analogue for self-adjoint operators in finite von Neumann algebras, of the classical Courant-Fischer-Weyl minmax characterization ([CH89]) of eigenvalues of Hermitian matrices:

If $a \in M_n(\mathbb{C})$ *has eigenvalues* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$; $\lambda_j \in \mathbb{R}$, *then*

$$\lambda_j = \min_{\substack{\mathcal{N} \subset \mathbb{C}^n \\ \dim \mathcal{N} = j}} \max_{\substack{\xi \in \mathcal{N} \\ ||\xi|| = 1}} \langle a\xi, \xi \rangle.$$

Ky Fan's result says: *For an operator a as above,*

$$\begin{aligned} \sum_{j=1}^{k} \lambda_j &= \min_{\substack{\mathcal{M}_1 \subset \cdots \subset \mathcal{M}_k \subset \mathbb{C}^n \\ \dim \mathcal{M}_j = j}} \max_{\substack{\xi_j \in \mathcal{M}_j \\ (\xi_j) \text{ orthonormal}}} \sum_{j=1}^{k} \langle a\xi_j, \xi_j \rangle \\ &= \min_{(\xi_j)_{j=1}^{k} \text{ orthonormal in } \mathcal{H}} \sum_{j=1}^{k} \langle a\xi_j, \xi_j \rangle \ . \end{aligned}$$

We extend Ky Fan's theorem - or more precisely Ky Fan's theorem for the case, where the Hermitian matrix a has distinct eigenvalues - to an analogous result for a self-adjoint element of a II_1 factor M equipped with faithful normal tracial state τ, with the help of the quantile function. We use certain properties of the distribution function F_a (or F_{μ_a}) and the quantile function X_a (or X_{μ_a}) corresponding to a, where a is assumed to have a compactly supported probability measure - given by μ_a - with no atoms, as its distribution, i.e. $\mu_a(E) = \tau(1_E(a))$. For us, $F_{\mu_a}(t) := \mu_a(-\infty, t)$, and $X_{\mu_a}(s) := \inf\{t \in \mathbb{R} : F(t) > s\}$, i.e. we follow the same convention as Voiculescu and Bercovici, wherein the distribution function is a left-continuous function. Consequently for us, the quantile function turns out to be right-continuous. The main result of this chapter is:

- *Let a be a self-adjoint element of a finite von Neumann algebra M equipped with a faithful normal tracial state τ. Let A be the von-Neumann algebra generated by a. Then, for all $s \in F_a(\mathbb{R})$,*

$$\begin{aligned} &\inf\{\tau(ap) : p \in \mathcal{P}(M), \tau(p) \geq s\} \\ &= \min\{\tau(ap) : p \in \mathcal{P}(A), \tau(p) \geq s\} \\ &= \int_0^s X_a dm, \end{aligned}$$

  if either (i) ('continuous case') μ_a has no atoms, or (ii) ('finite case') $M = M_n(\mathbb{C})$ and a has spectrum $\{\lambda_1 < \cdots < \lambda_n\}$.

We also give an alternate proof of Ky Fan's original result for Hermitian matrices - not just for the special case stated in the above result - but even for non-distinct eigenvalues.

Chapter 1

Free products of certain finite dimensional algebras

This chapter is dedicated to working out some 'bare hands' computations of a few finite von-Neumann algebras. We start with the most elementary possible free products involving $\mathbb{C}^2$ $(= \mathbb{C} \oplus \mathbb{C})$, with the 'uniform trace' given by $tr(z_1, z_2) = \frac{1}{2}(tr(z_1) + tr(z_2))$, and M_2 $(= M_2(\mathbb{C}))$, with the normalized trace of the matrix. Using these, we identify all free products of the form $C * D$, such that $\{C, D\} \subset \{A_1 \oplus A_2, M_2(B)\}$, where A_i, B are finite von Neumann algebras, as is $A_1 \oplus A_2$, (with the trace $\frac{1}{2}(tr(a_1) + tr(a_2))$ for the element (a_1, a_2)), and $M_2(B)$ $(\cong M_2 \otimes B)$ (with the trace $tr_{M_2} \otimes tr_B$, where tr_{M_2} and tr_B are the 'uniform trace' on M_2 and the tracial state on B respectively). Those results are then used to compute various possible free products involving certain finite dimensional von-Neumann algebras, the free-group von-Neumann algebras and the hyperfinite II_1 factor. In the process we reprove Dykema's result '$R * R \cong LF_2$' ([Dyk94]), and construct some interpolated free group factors with certain rational indices.

1.1 The building blocks

The interval $[0, \pi/2]$ and the unit circle $\mathbb{T}$ in the complex plane are both non-atomic compact Borel spaces with respect to the corresponding Haar measures. Hence $(L^\infty([0, \pi/2], \frac{2}{\pi}\int_0^{\pi/2} \cdot\, dt) \cong$

$L^\infty(\mathbb{T})$, which in turn, as we have seen in Lemma 2.6.5 of [VDN92], is isomorphic to $L\mathbb{Z}$. Often as an interpolated free group factor of the form LF_t appears in our computation with $t \geq 2$, we make use of the fact that $LF_r * LF_s \cong LF_{r+s}$ ([Dyk94]), and work with the model $L^\infty([0, \pi/2], \frac{2}{\pi}\int_0^{\pi/2} \cdot\, dt) * LF_{t-1}$ of LF_t.

Let u and v be Haar unitaries generating the two copies of $L\mathbb{Z}$s, and let $c, s \in L^\infty([0, \pi/2])$ be the functions defined by $c(\theta) = cos\ \theta$ and $s(\theta) = sin\ \theta$ respectively.

Let the trace (as described above) and cumulant on $M_2(LF_t)$ with $t \geq 1$, be denoted by Tr and κ, and the same for LF_t by tr and k respectively.

Let $U = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}$, $V = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$, $W = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$, and let $X = WVW^* = \begin{pmatrix} -cvs & cvc \\ -svs & svc \end{pmatrix}$.

Then U and X are partial isometries satisfying $U^*U + UU^* = 1$ and $X^*X + XX^* = 1$; so each of them generates a copy of M_2.

Let $P = UU^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = XX^* = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} = WPW^*$. Then P, Q are projections of trace $\frac{1}{2}$, each generating a copy of $\mathbb{C}^2$. Our proofs, are strictly restricted to the case where $\mathbb{C}^2$ has the 'uniform trace', whereas [Dyk93] considers the most general (possibly non-uniform) trace.

Also, for finite von-Neumann algebra A, B, we will often use the notations A^k and $M_k(B)$ for $A \oplus A \oplus \cdots (k \text{ times}) \oplus A$ and $M_k(\mathbb{C}) \otimes B$ respectively, with the trace always being taken as the 'uniform trace'.

1.2 $\mathbb{C}^2 * \mathbb{C}^2 \cong M_2(L\mathbb{Z})$

In this section, we compute the free product of the simplest finite dimensional algebra, namely $\mathbb{C}^2$ with itself, depending only on the definition of free independence and basic trigonometry. The two free copies of $\mathbb{C}^2$ here are taken to be generated by projections P and Q. We prove that P and Q freely generate $M_2(L\mathbb{Z})$ as a von-Neumann algebra.

Lemma 1.2.1.

$$W^*(\{P, Q\}) = M_2(L\mathbb{Z})$$

Proof. We have $PQ(1-P) = \begin{pmatrix} 0 & cs \\ 0 & 0 \end{pmatrix}$. Since cs is positive (in $L^\infty([0, \frac{\pi}{2}]), \frac{2}{\pi}\int_0^{\pi/2} \cdot\, dt$) and has no kernel, we see that the polar decomposition of $PQ(1-P)$ is given by $\begin{pmatrix} 0 & cs \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & cs \end{pmatrix}$. Thus $W^*(\{P, Q\})$ contains all the matrix units, viz., $P = e_{11}, 1-P = e_{22}, e_{12}$ and $e_{21} = e_{12}^*$. On the other hand $W^*(\{P, Q\})$ clearly contains $PQP = e_{11} \otimes c^2$, and hence also $e_{11} \otimes W^*(\{c^2\})$; therefore we see that $W^*(\{P, Q\}) \supset e_{11} \otimes L^\infty([0, \frac{\pi}{2}])$; finally, by pre- and post-multiplying by appropriate matrix units, we see that $W^*(\{P, Q\}) \supset e_{ij} \otimes L^\infty([0, \frac{\pi}{2}])$ for all i, j, and the proof of the lemma is complete. □

Lemma 1.2.2. *P and Q are free in $M_2(LF_3)$.*

Proof. Let $P_0 = P - 1/2 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ and $Q_0 = Q - 1/2 = \begin{pmatrix} c^2 - 1/2 & cs \\ cs & s^2 - 1/2 \end{pmatrix}$ be the trace-less versions (i.e., translates with trace 0) of P and Q. We shall find it convenient to write c_n, respectively s_n, for the elements of $L^\infty([0, \frac{\pi}{2}])$ defined by $c_n(\theta) = cos\ n\theta$, respectively, $s_n(\theta) = sin\ n\theta$.

We need to verify that the trace of any alternating product in $2P_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $2Q_0 = \begin{pmatrix} c_2 & s_2 \\ s_2 & -c_2 \end{pmatrix}$ is zero. Since we are working with a trace here and since $(2P_0)^2 = 1 = (2Q_0)^2$, it is enough to prove that $Tr((2P_0.2Q_0)^r) = 0 = Tr((2P_0.2Q_0)^r 2P_0)$.

However,

$$(2P_0.2Q_0)^r = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_2 & s_2 \\ s_2 & -c_2 \end{pmatrix} \right)^r = \left(\begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix} \right)^r = \begin{pmatrix} c_{2r} & s_{2r} \\ -s_{2r} & c_{2r} \end{pmatrix}$$

whereas

$$(2P_0.2Q_0)^r 2P_0 = \begin{pmatrix} c_{2r} & s_{2r} \\ -s_{2r} & c_{2r} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} c_{2r} & -s_{2r} \\ -s_{2r} & -c_{2r} \end{pmatrix},$$

and the desired assertions follows from

$$\begin{aligned} \frac{2}{\pi}\int_0^{\frac{\pi}{2}} c_{2r} d\theta &= \frac{2}{\pi}\frac{1}{2r}(s_{2r}(\pi) - s_{2r}(0)) \\ &= 0 \ \forall r \in \mathbb{N} \end{aligned}$$

□

Now the validity of Proposition 1.2.3 follows immediately from Lemmas 1.2.1 and 1.2.2.

Proposition 1.2.3.

$$\mathbb{C}^2 * \mathbb{C}^2 \cong M_2(L\mathbb{Z})$$

1.3 $M_2 * M_2 \cong M_2(LF_3)$

In this section, we compute the free products of the smallest non-commutative finite dimensional matrix algebra, namely $M_2 = (M_2(\mathbb{C}))$, with $\mathbb{C}^2$ and with itself, using the results of the previous section. The copies of $M_2(\mathbb{C})$ used here are taken to be generated by partial isometries U and X, and the copies of $\mathbb{C}^2$ are taken to be generated by P and Q as above. As in the previous section, we now prove that U and X freely generate $M_2(LF_3)$ (and also that X and P freely generate $M_2(LF_2)$; though their free independence in fact follows from that of X and U).

The two main propositions of this section, along with Proposition 1.2.3 above, are the basic steps for computing free products of von-Neumann algebras of the general form $A_1 \oplus A_2$ and $M_2(B)$.

Proposition 1.3.1.

$$M_2 * \mathbb{C}^2 \cong M_2(LF_2)$$

Proposition 1.3.2.

$$M_2 * M_2 \cong M_2(LF_3)$$

The key for proving Proposition 1.3.1 is contained in the proof of 1.3.2. We start with the following crucial lemma.

Lemma 1.3.3. *U and X are $*$- free in $M_2(LF_3)$.*

Proof. Observe the following simple facts:

1. $U^2 = 0 = (U^*)^2, V^2 = 0 = (V^*)^2$,
2. $P_0^2, Q_0^2 \in \mathbb{C}$,
3. $PU = U, QX = X, UP = 0, XQ = 0$,
4. P, Q are free in $M_2(L^\infty)$.

In view of the above relations, the sets of trace zero words in $W^*(\{U\})$ and $W^*(\{X\})$ are linearly generated by $A = \{U, U^*, 2P_0\}$ and $B = \{X, X^*, 2Q_0\}$ respectively. So we need to check that every alternating product of elements from A and B has trace zero.

We may dispose of the 'trivial case' when Π is an alternating word in only P_0 and Q_0, since that is covered by Lemma 1.2.2.

Consider a typical such product, say Π; we shall prove that $tr(\Pi_{ij}) = 0$ for all i, j (for the non-trivial case). Thus in particular we will have $Tr(\Pi) = \frac{tr(\Pi_{1,1}) + tr(\Pi_{2,2})}{2} = 0$

One can see that each Π_{ij} is a sum of elements of the form

$$\omega = f_0(c,s) w_0 f_1(c,s) w_1 f_2(c,s) \cdots w_{n-1} f_n(c,s) \cdots \tag{1.3.1}$$

where the f_is are (possibly constant) polynomials in $\{c, s\}$ and $w_i \in \{u, u^*, v, v^*\}$. The assumption that we are not dealing with the trivial case (cf. the previous paragraph) implies that there must exist at least one w_i in the string in the right hand side of equation 1.3.1.

Now, fix any such word ω is (of total length m, say) as in equation 1.3.1, which occurs as a summand of Π; then

$$tr(\omega) = \sum_{\sigma \in NC(m)} k_\sigma[f_0(c,s), w_0, f_1(c,s), \cdots, w_{n-1}, f_n(c,s), \cdots] \ .$$

We shall show that $k_\sigma (= k_\sigma[f_0(c,s), w_0, f_(c,s), \cdots, w_{n-1}, f_n(c,s), \cdots]) = 0$ for each $\sigma \in NC(m)$, for each such ω.

But u and v are free Haar unitaries and hence R-diagonal. So following Proposition 15.1 in [NS06], in order for k_σ to be possibly non zero, it must be the case that the blocks of σ must consist of either only $\{u, u^*\}$ in alternate positions, or only $\{v, v^*\}$ in alternate positions (in each block the number of u = number of u^* and same for v, v^*), or only $\{f_i(c,s) : i\}$, all occurring in a non-crossing fashion. Note that in effect ω must have the same number of u and u^* as well as the same number of v and v^*.

Example 1.3.4. $\omega_0 = u(cs)v\,(cs(c^2 - 1/2)s)\,v^*(c^2s)u^* = uf_1(c,s)vf_2(c,s)v^*f_3(c,s)u^*$ - with $m = 7$ - can possibly give non zero cumulant only corresponding to two elements of $NC(7)$, namely

$$((1,7),(3,5),(2),(4),(6)) \ \text{ and } \ ((1,7),(3,5),(2,6),(4)) \ .$$

(Here, $f_1(c,s) = cs$, $f_2(c,s) = cs(c^2 - 1/2)s$, $f_3(c,s) = c^2s$) $\square$

As at least one w_i and necessarily also w_i^* must occur in the string defining ω, and σ is non-crossing, we see that k_σ can hope to be non-zero only if ω has a substring of the form $wf(c,s)w^*$, with $w \in \{u, u^*, v, v^*\}$ and $f(c,s)$ as above, and with w and w^* in the same block of σ; thus ω must contain one of the one of the following four substrings:

1. $uf(c,s)u^*$,

2. $u^*f(c,s)u$,

3. $vf(c,s)v^*$,

4. $v^*f(c,s)v$.

Now we consider the above four cases in the following way:

1. $uf(c,s)u^*$ can occur as a string in some summand ω of Π_{ij} only if USU^* occurs as a substring in the alternating product expression of Π, where $S = (2Q_0 2P_0)^r 2Q_0$. Observe in this case that $USU^* = \begin{pmatrix} -uc_{2r+2}u^* & 0 \\ 0 & 0 \end{pmatrix}$ and that $f(c,s) = -c_{2r+2}$.

2. Similarly $u^*f(c,s)u$ can occur as a string in some summand ω of Π_{ij} only if U^*SU occurs as a substring in the alternating product expression of Π, where $S = (2Q_0 2P_0)^r 2Q_0$. Observe in this case that $U^*SU = \begin{pmatrix} 0 & 0 \\ 0 & u^*c_{2r+2}u \end{pmatrix}$ and that $f(c,s) = c_{2r+2}$.

3. Similarly $vf(c,s)v^*$ can occur as a string in some summand ω of Π_{ij} only if XSX^* occurs as a substring in the alternating product expression of Π, where $S = (2P_0 2Q_0)^r 2P_0$. Observe in this case that $XSX^* = \begin{pmatrix} -cvc_{2r+2}v^*c & cvc_{2r+2}v^*s \\ -svc_{2r+2}v^*c & -svc_{2r+2}v^*s \end{pmatrix}$ and that again, $f(c,s) = \pm c_{2r+2}$.

4. Similarly $v^*f(c,s)v$ can occur as a string in some summand ω of Π_{ij} only if X^*SX occurs as a substring in the alternating product expression of Π, where $S = (2P_0 2Q_0)^r 2P_0$. Observe in this case that $X^*SX = \begin{pmatrix} sv^*c_{2r+2}vs & -sv^*c_{2r+2}vc \\ -cv^*c_{2r+2}vs & cv^*c_{2r+2}vc \end{pmatrix}$ and that again, $f(c,s) = \pm c_{2r+2}$.

Thus, in any case, we find that for any $\sigma \in NC(m)$ for which k_σ can possibly be non-zero, it must be the case that ω must contain a substring of the form $wc_{2r+2}w^*$ with $w \in \{u, u^*, v, v^*\}$ and with w and w^* in the same block of σ. Since $W^*(\{c\})$ and $W^*(\{w\})$

are free, we see that the only way k_σ can be non-zero is for $\{c_{2r+2}\}$ to be a block of σ; but then k_σ has $k(c_{2r+2}) = tr(c_{2r+2}) = 0$ as a factor. Hence, indeed, every $k_\sigma = 0$, as asserted, and the proof is complete. $\square$

Corollary 1.3.5. *P and X are free in $M_2(LF_3)$.*

Proof. This follows from $P \in W^*(\{U\})$. $\square$

Lemma 1.3.6.

$$W^*(\{P, X\}) = M_2(LF_2)$$

Proof. Since $Q = XX^*$, it follows from Lemma 1.2.1 that $W^*(\{P, X\})$ contains each matrix unit e_{ij}. It follows that $W^*(\{P, X\})$ contains $M_2(\mathcal{N})$ where $\mathcal{N}$ is the von Neumann algebra generated by the entries of X. By the last line in the proof of Lemma 1.2.1 shows that $L^\infty([0, \frac{\pi}{2}]) \subset \mathcal{N}$.

Consider the bounded Borel functions f_n defined on $[0, 1]$ by $f_n(t) = \begin{cases} \frac{1}{t} & t \geq \frac{1}{n} \\ 0 & t < \frac{1}{n} \end{cases}$; observe that $f_n(c)c$ converges strongly to 1 (since c is injective). Since $cvc \in \mathcal{N}$, deduce that $v = lim_n(f_n(c)cvcf_n(c)) \in \mathcal{N}$. Since c and v are $*$-free, and $c, v \in \mathcal{N}$, it follows that $\mathcal{N} \supset LF_2$. Since the entries of P and X all lie in LF_2, the proof of the Lemma is complete. $\square$

Remark 1.3.7. *We can prove the above lemma also by proving $W^*(\{U, Q\}) = M_2(LF_2)$ (where U and Q are free), in an exactly similar way.*

Lemma 1.3.8.

$$W^*(\{U, X\}) = M_2(LF_3)$$

Proof. In view of Lemma 1.3.6, we only need to observe that $\{u, v, c\}'' = LF_3$. $\square$

Finally, Proposition 1.3.1 follows from Lemma 1.3.6 and Corollary 1.3.5, while Proposition 1.3.2 follows from Lemma 1.3.8 and Lemma 1.3.3.

1.4 $(A_1 \oplus A_2) * (B_1 \oplus B_2) \cong M_2(A_1 * A_2 * B_1 * B_2 * L\mathbb{Z})$

Let A_1, A_2 and B_1, B_2 be finite von-Neumann algebras. In this section, we compute the free product of $A_1 \oplus A_2$ and $B_1 \oplus B_2$ (with uniform traces, as mentioned above). We denote the trace and cumulant on matrix algebras over the finite von-Neumann algebras as Tr and κ and those on the finite von-Neumann algebras themselves as tr and k respectively.

Following the steps in the previous proofs, here too we find freely independent copies of $A_1 \oplus A_2$ and $B_1 \oplus B_2$ inside $M_2(A_1 * A_2 * B_1 * B_2 * L\mathbb{Z})$ as subalgebras generating it as a W^*- probability space.

The computations in this section and the next are dependent on the Kreweras compliment ([NS06]) of a non-crossing partition $\pi \in NC(\{1, \cdots, n\})$, i.e. the biggest element, say $\sigma \in NC(\{\bar{1}, \cdots, \bar{n}\})$ such that $\pi \cup \sigma$ is a non-crossing partition over $\{1, \bar{1}, 2, \bar{2}, \cdots, n, \bar{n}\}$. The Kreweras compliment of π is denoted by $K(\pi)$.

For example, if $\pi = \{\{1, 2, 7\}, \{3\}, \{4, 6\}, \{5\}, \{8\}\}$, then $K(\pi) = \{\{\bar{1}\}, \{\bar{2}, \bar{3}, \bar{6}\}, \{\bar{4}, \bar{5}\}, \{\bar{7}, \bar{8}\}\}$, as is clearly seen from the following diagram:

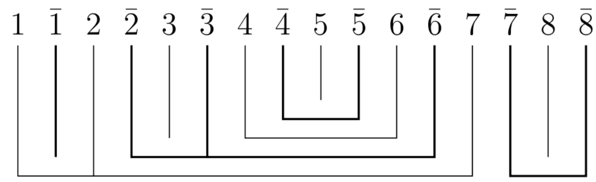

Figure 1.1: π and $K(\pi)$ for $\pi = \{\{1, 2, 7\}, \{3\}, \{4, 6\}, \{5\}, \{8\}\} \in NC(8)$

We start with a simple but useful lemma on $K(\pi)$:

Lemma 1.4.1. *Let $\pi \in NC(n)$ and $1 \sim_\pi n$. Let $V = (k', \cdots, (k+l)')$ be an interval in its Kreweras compliment $K(\pi)$ for $1 \leq k \leq n, 0 \leq l \leq n - k$. Then $k \sim_\pi (k + l + 1)$, where all positive integers are taken modulo n.*

Proof. Note that if (k) is a singloton block then $(k-1)' \sim_{K(\pi)} k'$, a contradiction since V is an interval. Similarly $(k + l + 1)$ cannot be a singleton block. More generally suppose $r_k \in \{1, \cdots, k\}$ and $s_k \in \{k, (k + l + 1), \cdots, n\}$ are minimum positive integers such that $k \sim_\pi r_k$ and $k \sim_\pi s_k$. Suppose $r_{k+l+1} \in \{1, \cdots, k, (k+l+1)\}$ and $s_{k+l+1} \in \{(k+l+1), \cdots, n\}$ are the same for $(k+l+1)$. We already saw that we cannot have $r_k = s_k = k$ or $r_{k+l+1} = s_{k+l+1} = (k+l+1)$. For simplicity, we assume $1 < k, l < (n-k)$. The cases $k = 1$ or $l = n-k$ will follow similarly.

Case $s_k \gneq k$: In this case s_k being minimum in $\{(k+l+1), \cdots, n\}$ such that $k \sim_\pi s_k$ we must have $(s_k - 1)' \sim_{K(\pi)} (k+l)'$, a contradiction unless $k+l+1 = s_k \sim_\pi k$.

Case $s_k = k$: In this case r_k being minimum in $\{1, \cdots, k-1\}$ such that $k \sim_\pi r_k$, unless $r_k = 1$ we must have $(r_k - 1)' \sim_{K(\pi)} k'$, a contradiction. On the other hand if $r_k = 1$, then r_{k+l+1} is forced to be $(k+l+1)$. Thus similarly we must have $s_{k+l+1} = n$, otherwise leading into a contradiction. But $1 \sim_\pi n$. Hence $k \sim_\pi r_k = 1 \sim_\pi n = s_{k+l+1} \sim_\pi (k+l+1)$

□

Proposition 1.4.2.

$$(A_1 \oplus A_2) * (B_1 \oplus B_2) \cong M_2(A_1 * A_2 * B_1 * B_2 * L\mathbb{Z})$$

Proof. Consider the two matrix subalgebras $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ $(\cong A_1 \oplus A_2)$ and $W \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} W^* (\cong B_1 \oplus B_2)$ of $A_1 * A_2 * B_1 * B_2 * L\mathbb{Z}$, where W is the unitary matrix of the previous sections. Following the method used in Lemma 1.3.6 we know that these two subalgebras indeed generate the matrix algebra on the right side. We need to show that these are free.

Note that $W \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} W^* = \begin{pmatrix} cb_1c + sb_2s & cb_1s - sb_2c \\ sb_1c - cb_2s & sb_1s + cb_2c \end{pmatrix}$

Suppose Π is an alternating product of matrices of the form $\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ and $W \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} W^*$, with $a_j \in A_j, b_j \in B_j, tr(a_1 + a_2) = 0 = tr(b_1 + b_2)$.

Instead of directly proving $Tr(\Pi) = 0$, we will prove a stronger statement: $tr(\Pi_{i_1,i_1}) = 0 \forall i_1 \in \{1, 2\}$.

We will need to look at alternating products of the form

$$\Pi = \begin{pmatrix} a_1^1 & 0 \\ 0 & a_2^1 \end{pmatrix} \begin{pmatrix} cb_1^2c + sb_2^2s & cb_1^2s - sb_2^2c \\ sb_1^2c - cb_2^2s & sb_1^2s + cb_2^2c \end{pmatrix} \cdots \begin{pmatrix} a_1^{2r-1} & 0 \\ 0 & a_2^{2r-1} \end{pmatrix}$$
$$\begin{pmatrix} cb_1^{2r}c + sb_2^{2r}s & cb_1^{2r}s - sb_2^{2r}c \\ sb_1^{2r}c - cb_2^{2r}s & sb_1^{2r}s + cb_2^{2r}c \end{pmatrix} \begin{pmatrix} a_1^{2r+1} & 0 \\ 0 & a_2^{2r+1} \end{pmatrix},$$

where $a_j^i \in A_j, b_j^i \in B_j, tr(a_1^i + a_2^i) = 0 = tr(b_1^i + b_2^i) \forall i$ - as well as three other kinds of products (depending on which sort of matrix the product starts or ends with).

Note that by taking $\begin{pmatrix} a_1^1 & 0 \\ 0 & a_2^1 \end{pmatrix}$ or both $\begin{pmatrix} a_1^1 & 0 \\ 0 & a_2^1 \end{pmatrix}$ and $\begin{pmatrix} a_1^{2r+1} & 0 \\ 0 & a_2^{2r+1} \end{pmatrix}$ as $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and using the fact that we are working with traces here, we find that it is sufficient to consider the one special case listed above.

For $i_1 \in \{1,2\}$, the $(i_1, i_1)^{th}$ diagonal entry Π_{i_1,i_1} of Π is a sum of words of the form

$$\omega = a_{i_1}^1 t_{i_1,i_2(\omega)} b_{i_2(\omega)}^2 t'_{i_2(\omega),i_3(\omega)} a_{i_3(\omega)}^3 t_{i_3(\omega),i_4(\omega)} b_{i_4(\omega)}^4 t'_{i_4(\omega),i_5(\omega)} \cdots a_{i_{2r-1}(\omega)}^{2r-1} t_{i_{2r-1}(\omega),i_{2r}(\omega)} b_{i_{2r}(\omega)}^{2r} t'_{i_{2r}(\omega),i_1} a_{i_1}^{2r+1},$$

for $i_2(\omega), \cdots, i_{2r}(\omega) \in \{1,2\}$ and $t_{i_j(\omega),i_{j+1}(\omega)}, t'_{i_j(\omega),i_{j+1}(\omega)} \in \{c,s\}$ (the reason behind using the cumbersome notation $i_j(\omega)$ is to emphasize the dependence of the indices i_j on the particular summand ω).

Now since the a's and b's are free from the t's and t''s, we see from Theorem 14.4, [NS06], that

$$tr(\omega) = \sum_{\pi \in NC(2r+1)} k_\pi(a_{i_1}^1, b_{i_2(\omega)}^2, \cdots, a_{i_1}^{2r+1}) tr_{K(\pi)}(t_{i_1,i_2(\omega)}, t'_{i_2(\omega),i_3(\omega)}, \cdots, t_{i_{2r-1}(\omega),i_{2r}(\omega)}, t'_{i_{2r}(\omega),i_1}, 1).$$

Thus, in order to prove that $tr(\Pi_{i_1,i_1}) = 0$, it is enough to prove that for any $\pi \in NC(2r+1), i_1 \in \{1,2\}$,

$$\sum_{\substack{\omega \text{ a summand of } \Pi_{i_1,i_1} \\ i_2(\omega),\cdots,i_{2r}(\omega)\in\{1,2\}}} k_\pi(a^1_{i_1}, b^2_{i_2(\omega)}, \cdots, a^{2r+1}_{i_1}) tr_{K(\pi)}(t_{i_1,i_2(\omega)}, t'_{i_2(\omega),i_3(\omega)} \cdots, t_{i_{2r-1}(\omega),i_{2r}(\omega)}, t'_{i_{2r}(\omega),i_1}, 1) = 0 \,. \tag{1.4.1}$$

The crux of the proof lies in the following key lemma.

Lemma 1.4.3. *Let $p \in \mathbb{N}, i_1 \in \{1,2\}$. For $a^j_i \in A_i, b^j_i \in B_k$, where $i = 1,2$, $j = 1,2,\cdots,p+1$, write $a^j = \begin{pmatrix} a_1^{2j-1} & 0 \\ 0 & a_2^{2j-1} \end{pmatrix}$ and $b^j = \begin{pmatrix} cb_1^{2j}c + sb_2^{2j}s & cb_1^{2j}s - sb_2^{2j}c \\ sb_1^{2j}c - cb_2^{2j}s & sb_1^{2j}s + cb_2^{2j}c \end{pmatrix}$; and define*

$$\Pi' = a^1b^1a^2\cdots a^pb^pa^{p+1} \quad \textit{and} \quad \Pi'' = b^1a^2\cdots a^pb^p.$$

Then

$$\sum_{\substack{\omega \text{ a summand of } \Pi'_{i_1,i_1} \\ i_2(\omega),\cdots,i_{2p}(\omega)\in\{1,2\}}} k_{0_{2p+1}}(a^1_{i_1}, b^2_{i_2(\omega)}, \cdots, b^{2p}_{i_{2p}(\omega)}, a^{2p+1}_{i_1}) tr_{1_{2p+1}}(t_{i_1,i_2(\omega)}, t'_{i_2(\omega),i_3(\omega)}, \cdots, t_{i_{2p-1}(\omega),i_{2p}(\omega)}, t'_{i_{2p}(\omega),i_1}, 1) = 0 \,. \tag{1.4.2}$$

In particular, when $a^1 = a^{p+1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$$\sum_{\substack{\omega \text{ a summand of } \Pi''_{i_1,i_1} \\ i_2(\omega),\cdots,i_{2p}(\omega)\in\{1,2\}}} k_{0_{2p-1}}(b^2_{i_2(\omega)}, a^3_{i_3(\omega)}, \cdots, b^{2p}_{i_{2p}(\omega)}) tr_{1_{2p}}(t_{i_1,i_2(\omega)}, t'_{i_2(\omega),i_3(\omega)}, \cdots, t_{i_{2p-1}(\omega),i_{2p}(\omega)}, t'_{i_{2p}(\omega),i_1}) = 0 \,. \tag{1.4.3}$$

Proof. In this proof we write $i_j(\omega)$ as i_j for simplicity.

$$\begin{aligned}
&\sum_{\substack{\omega \text{ is a summand of } \Pi'_{i_1,i_1} \\ i_2,\cdots,i_{2p}\in\{1,2\}}} k_{0_{2p+1}}(a^1_{i_1}, b^2_{i_2}, \cdots, a^{2p+1}_{i_1}) tr_{1_{2p+1}}(t_{i_1,i_2}, t'_{i_2,i_3}, \cdots, t_{i_{2p-1},i_{2p}}, t'_{i_{2p},i_1}, 1) \\
&= \sum_{\substack{\omega \text{ is a summand of } \Pi'_{i_1,i_1} \\ i_2,\cdots,i_{2p}\in\{1,2\}}} tr(a^1_{i_1}) tr(b^2_{i_2}) \cdots tr(a^{2p+1}_{i_1}) tr(t_{i_1,i_2} t'_{i_2,i_3} \cdots t_{i_{2p-1},i_{2p}} t'_{i_{2p},i_1}),
\end{aligned}$$

which is the (i_1, i_1) entry of

$$\begin{pmatrix} tr(a^1_1) & 0 \\ 0 & tr(a^1_2) \end{pmatrix} \begin{pmatrix} ctr(b^2_1)c + str(b^2_2)s & ctr(b^2_1)s - str(b^2_2)c \\ str(b^2_1)c - ctr(b^2_2)s & str(b^2_1)s + ctr(b^2_2)c \end{pmatrix} \cdots \begin{pmatrix} tr(a^{2p-1}_1) & 0 \\ 0 & tr(a^{2p-1}_2) \end{pmatrix}$$
$$\begin{pmatrix} ctr(b^{2p}_1)c + str(b^{2p}_2)s & ctr(b^{2p}_1)s - str(b^{2p}_2)c \\ str(b^{2p}_1)c - ctr(b^{2p}_2)s & str(b^{2p}_1)s + ctr(b^{2p}_2)c \end{pmatrix} \begin{pmatrix} tr(a^{2p+1}_1) & 0 \\ 0 & tr(a^{2p+1}_2) \end{pmatrix} = \lambda(2P_0 2Q_o)^p 2P_0,$$

$$\text{where } \lambda \begin{cases} = 0 & \text{if } \exists i : tr(a^i_1) = tr(a^i_2) = 0 \text{ or } tr(b^i_1) = tr(b^i_2) = 0 \\ \in \mathbb{C}\setminus\{0\} & \text{if } \forall i, tr(a^i_1) = -tr(a^i_2) \neq 0, tr(b^i_1) = -tr(b^i_2) \neq 0 \end{cases}$$

Now the proof follows since by the proof of Lemma 1.2.2, each diagonal entry of an alternating product in $2P_0$ and $2Q_0$ has trace zero. □

Now let us fix an arbitrary $\pi \in NC(2r+1)$.

If $\pi = 0_{2r+1}$, then equation 1.4.1 follows from equation 1.4.2 for $p = r$.

If $\pi \neq 0_{2r+1}$ then $\exists m < n \in \{1, \cdots, 2r+1\}$ such that $m \sim_\pi n$.

Consider an interval, say $V = (k, \cdots, l-1)$ in $K(\pi)$ for $1 \leq m \leq k \leq l-1 \leq n \leq (r+1)$. Then by Lemma 1.4.1, $k \sim_\pi l$ (since $m \sim_\pi n$).

Case 1: Suppose $|V|$ is odd. Then either $a^k_{i_k}$ and $b^l_{i_l}$ or $b^k_{i_k}$ and $a^l_{i_l}$ are joined through π (depending on whether k or l is odd), which leads to corresponding k_π being zero, since A_i, B_j are free $\forall i, j$ (Theorem 11.20 [NS06]).

Case 2: Suppose $|V|$ is even, both k and l are odd. Then $a^k_{i_k}$ and $a^l_{i_l}$ are joined through π. Thus $l-1 \gneq k$.

Also note that if $i_k \neq i_l$ then due to A_{i_k} and A_{i_l} being free the corresponding k_π would be zero. So we now assume that $i_k = i_l$.

Let $\pi_0 = \pi \setminus \{(k+1) \vee \cdots \vee (l-1)\}$ and $\pi_1 = K(\pi) \setminus \{(k, \cdots, l-1)\}$.

Then

$$\sum_{\substack{\omega \text{ is a summand of } \Pi_{i_1,i_1} \\ i_2(\omega),\cdots,i_{2r}(\omega)}} k_\pi(a^1_{i_1}, b^2_{i_2(\omega)}, \cdots, a^{2r+1}_{i_1}) tr_{K(\pi)}(t_{i_1,i_2(\omega)}, \cdots, t'_{i_{2r}(\omega),i_1})$$

$$= \sum_{\substack{\omega \text{ is a summand of } \Pi_{i_1,i_1} \\ i_2(\omega),\cdots,i_k(\omega)(=i_l(\omega)),i_{l+1}(\omega),\cdots,i_{2r}(\omega)}} k_{\pi_0}(a^1_{i_1}, b^2_{i_2}, \cdots, a^k_{i_l}, a^l_{i_l}, b^{l+1}_{i_{l+1}}, \cdots, b^{2r}_{i_{2r}}, a^{2r+1}_{i_1})$$

$$tr_{\pi_1}(t_{i_1,i_2}, \cdots, t'_{i_{k-1},i_k}, t_{i_l,i_{l+1}}, \cdots, t'_{i_{2r},i_1}) \Big(\sum_{i_{k+1}(\omega),\cdots,i_{l-1}(\omega)} k_{0_V}(b^{k+1}_{i_{k+1}}, \cdots, b^{l-1}_{i_{l-1}}) tr_{1_V}(t_{i_k,i_{k+1}}, \cdots, t'_{i_{l-1},i_l}) \Big)$$

$$= \sum_{\substack{\omega \text{ is a summand of } \Pi_{i_1,i_1} \\ i_2(\omega),\cdots,i_k(\omega)(=i_l(\omega)),i_{l+1}(\omega),\cdots,i_{2r}(\omega)}} k_{\pi_0}(a^1_{i_1}, b^2_{i_2}, \cdots, a^k_{i_l}, a^l_{i_l}, b^{l+1}_{i_{l+1}}, \cdots, b^{2r}_{i_{2r}}, a^{2r+1}_{i_1})$$

$$tr_{\pi_1}(t_{i_1,i_2}, \cdots, t'_{i_{k-1},i_k}, t_{i_l,i_{l+1}}, \cdots, t'_{i_{2r},i_1}) \Big(\sum_{i_{k+1}(\omega),\cdots,i_{l-1}(\omega)} k_{0_{l-k-1}}(b^{k+1}_{i_{k+1}}, \cdots, b^{l-1}_{i_{l-1}}) tr_{1_{l-k}}(t_{i_k,i_{k+1}}, \cdots, t'_{i_{l-1},i_k}) \Big) .$$

Write $\Pi = \Pi^{(1,k)} \Pi^{(k+1,l-1)} \Pi^{(l,2r+1)}$,

where $\Pi^{(1,k)} = a^1 b^1 \cdots a^{\frac{k+1}{2}}$, $\Pi^{(k+1,l-1)} = b^{\frac{k+1}{2}} a^{\frac{k+3}{2}} \cdots b^{\frac{l-1}{2}}$, $\Pi^{(l,2r+1)} = a^{\frac{l+1}{2}} \cdots b^{2r} a^{r+1}$.

Further let $\widetilde{\Pi} = \Pi^{(1,k)} \Pi^{(l,2r+1)}$. Note that $\Pi^{(1,k)}$ or $\Pi^{(l,2r+1)}$ can be trivial for the extreme cases.

Then using the fact that $i_k(\omega) = i_l(\omega)$ the above sum may be re-written as

$$\sum_{\substack{\widetilde{\omega} \text{ is a summand of } \widetilde{\Pi}_{i_1,i_1} \\ i_2(\widetilde{\omega}),\cdots,i_k(\widetilde{\omega})(=i_l(\widetilde{\omega})),i_{l+1}(\widetilde{\omega}),\cdots,i_{2r}(\widetilde{\omega})}} k_{\pi_0}(a^1_{i_1}, b^2_{i_2}, \cdots, a^k_{i_l}, a^l_{i_l}, b^{l+1}_{i_{l+1}}, \cdots, b^{2r}_{i_{2r}}, a^{2r+1}_{i_1})$$

$$tr_{\pi_1}(t_{i_1,i_2}, \cdots, t'_{i_{k-1},i_l}, t_{i_l,i_{l+1}}, \cdots, t'_{i_{2r},i_1})\Big(\sum_{\substack{\widetilde{\widetilde{\omega}} \text{ is a summand of } \Pi^{(k+1,l-1)}_{i_k,i_k} \\ i_{k+1}(\widetilde{\widetilde{\omega}}),\cdots,i_{l-1}(\widetilde{\widetilde{\omega}})}} k_{0_{l-k-1}}(b^{k+1}_{i_{k+1}}, \cdots, b^{l-1}_{i_{l-1}})$$

$$tr_{1_{l-k}}(t_{i_k,i_{k+1}}, \cdots, t'_{i_{l-1},i_k})\Big) .$$

Now we put $p = \frac{l-k}{2}, \Pi'' = \Pi^{(k+1,l-1)}_{i_k,i_k}$ in equation 1.4.3 to get for any $i_k(\widetilde{\omega}) = i_k(= i_l = i_l(\widetilde{\omega})) \in \{1, 2\}$,

$$\sum_{\substack{\widetilde{\widetilde{\omega}} \text{ is a summand of } \Pi^{(k+1,l-1)}_{i_k,i_k} \\ i_{k+1}(\widetilde{\widetilde{\omega}}),\cdots,i_{l-1}(\widetilde{\widetilde{\omega}})}} k_{0_{l-k-1}}(b^{k+1}_{i_{k+1}(\widetilde{\widetilde{\omega}})}, \cdots, b^{l-1}_{i_{l-1}(\widetilde{\widetilde{\omega}})})tr_{1_{l-k}}(t_{i_k,i_{k+1}(\widetilde{\widetilde{\omega}})}, \cdots, t'_{i_{l-1}(\widetilde{\widetilde{\omega}}),i_k}) = 0,$$

thus proving 1.4.1, as desired.

Case 3: The case when $|V|$ is even and both k and l are even is proved exactly as in the previous case. □

Corollary 1.4.4.

$$(A_1 \oplus A_2) * L\mathbb{Z} \cong M_2(A_1 * A_2 * LF_3)$$

Proof. Follows from Proposition 1.4.2 as well as by noting the fact that $L\mathbb{Z} \cong L\mathbb{Z} \oplus L\mathbb{Z}$, both being singly generated von Neumann algebras with non atomic distributions □

1.5 $M_2(A) * M_2(B) \cong M_2(A * B * LF_3)$

Let A, B be finite von-Neumann algebras. In this section we compute the free product of $M_2(A)$ and $M_2(B)$ (with 'normalized' traces, as mentioned before). While in the process of computing the above, we also compute the free product of $A_1 \oplus A_2$ and $M_2(B)$. The notations for trace and cumulant remain same as above.

As above, the proofs consist of finding representatives (models) of the free copies of the finite von-Neumann algebras on the left hand side inside the finite von-Neumann algebra on the right hand side, as subalgebras generating it as a W^*-probability space.

Proposition 1.5.1.

$$M_2(A) * M_2(B) \cong M_2(A * B * LF_3)$$

Proof. Set $Y = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$ in $M_2(LF_3)$, where u, v are Haar unitaries as in the introduction, such that $A, B, \{u\}, \{v\}, \{c\}$ are free.

We want to show that $Y^* M_2(A) Y (\cong M_2(A))$ and $WZ^* M_2(B) ZW^* (\cong M_2(B))$ are free in $M_2(A * B * LF_3)$. As before, following the method used in Lemma 1.3.6, we can conclude that these two subalgebras generate the algebra on the right in the proposition.

For $a_{i,j} \in A, b_{i,j} \in B$,

$$\begin{aligned} Y^*(a_{i,j})Y &= \begin{pmatrix} 1 & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \\ &= \begin{pmatrix} a_{1,1} & a_{1,2}u \\ u^* a_{2,1} & u^* a_{2,2} u \end{pmatrix}; \end{aligned}$$

$$\begin{aligned} WZ^*(b_{i,j})ZW^* &= \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v^* \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \\ &= \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2}v \\ v^* b_{2,1} & v^* b_{2,2} v \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \\ &= \begin{pmatrix} cb_{1,1}c - sv^*b_{2,1}c - cb_{1,2}vs + sv^*b_{2,2}vs & cb_{1,1}s - sv^*b_{2,1}s + cb_{1,2}vc - sv^*b_{2,2}vc \\ sb_{1,1}c + cv^*b_{2,1}c - sb_{1,2}vs - cv^*b_{2,2}vs & sb_{1,1}s + cv^*b_{2,1}s + sb_{1,2}vc + cv^*b_{2,2}vc \end{pmatrix}. \end{aligned}$$

As above, to prove freeness, it is enough to check on an alternating product of the form

$$\Pi = \begin{pmatrix} a^1_{1,1} & a^1_{1,2}u \\ u^*a^1_{2,1} & u^*a^1_{2,2}u \end{pmatrix} \left(\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} b^2_{1,1} & b^2_{1,2}v \\ v^*b^2_{2,1} & v^*b^2_{2,2}v \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \right)$$

$$\cdots \left(\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} b^{2r}_{1,1} & b^{2r}_{1,2}v \\ v^*b^{2r}_{2,1} & v^*b^{2r}_{2,2}v \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \right) \begin{pmatrix} a^{2r+1}_{1,1} & a^{2r+1}_{1,2}u \\ u^*a^{2r+1}_{2,1} & u^*a^{2r+1}_{2,2}u \end{pmatrix},$$

for $tr(a^i_{1,1}) + tr(a^i_{2,2}) = 0 = tr(b^j_{1,1}) + tr(b^j_{2,2})$.

Here too we will prove that each diagonal entry of Π has trace zero.

Let ω be a summand of $(i_1, i_1)^{th}$ diagonal entry of the above product. Then ω is an alternating product of the form

$$a^1_{i_1,i_2} w^2_{i_2,i_3} b^3_{i_3,i_4} w^4_{i_4,i_5} \cdots w^{2r-2}_{i_{2r-2},i_{2r-1}} b^{2r-1}_{i_{2r-1},i_{2r}} w^{2r}_{i_{2r},i_{2r+1}} a^{2r+1}_{i_{2r+1},i_1},$$

where $w^k_{i_k,i_{k+1}} \in \pm\{c, sv^*, us, ucv^*, vs, su^*, vcu^*\}$ depending on ω.

As before from [NS06] we can say that

$$tr(\omega) = \sum_{\pi \in NC(r+1)} k_\pi(a^1_{i_1,i_2}, b^3_{i_3,i_4}, \cdots, b^{2r-1}_{i_{2r-1},i_{2r}}, a^{2r+1}_{i_{2r+1},i_1}) tr_{K(\pi)}(w^2_{i_2,i_3}, w^4_{i_4,i_5}, \cdots, w^{2r}_{i_{2r},i_{2r+1}}, 1).$$

Using the fact that $u^*u = v^*v = 1$, as in Lemma 1.4.3, here also

$$\sum_{\omega \text{ is a summand of } \Pi_{i_1,i_1}} k_{0_{r+1}}(a^1_{i_1,i_2}, b^3_{i_3,i_4}, \cdots, b^{2r-1}_{i_{2r-1},i_{2r}}, a^{2r+1}_{i_{2r+1},i_1}) tr_{1_{r+1}}(w^2_{i_2,i_3}, w^4_{i_4,i_5}, \cdots, w^{2r}_{i_{2r},i_{2r+1}}, 1)$$

is the $(i_1, i_1)^{th}$ entry of the matrix

$$
\begin{aligned}
&\Big(tr(a^1_{1,2})U + tr(a^1_{2,1})U^* + tr(a^1_{1,1})2P_0\Big)\Big(W(tr(b^2_{1,2})V + tr(b^2_{2,1})V^* + tr(b^2_{1,1})2P_0)W^*\Big) \\
&\qquad \cdots \Big(W(tr(b^{2r}_{1,2})V + tr(b^{2r}_{2,1})V^* + tr(b^{2r}_{1,1})2P_0)W^*\Big)\Big(tr(a^{2r+1}_{1,2})U + tr(a^{2r+1}_{2,1})U^* + tr(a^{2r+1}_{1,1})2P_0\Big) \\
= &\Big(tr(a^1_{1,2})U + tr(a^1_{2,1})U^* + tr(a^1_{1,1})2P_0\Big)\Big(tr(b^2_{1,2})X + tr(b^2_{2,1})X^* + tr(b^2_{1,1})2Q_0\Big) \\
&\qquad \cdots \Big(tr(b^{2r}_{1,2})X + tr(b^{2r}_{2,1})X^* + tr(b^{2r}_{1,1})2Q_0\Big)\Big(tr(a^{2r+1}_{1,2})U + tr(a^{2r+1}_{2,1})U^* + tr(a^{2r+1}_{1,1})2P_0\Big),
\end{aligned}
$$

where U, V, X are trace zero partial isometries in $M_2(LF_3)$ as defined in the introduction,

But by proof of Lemma 1.3.3, each diagonal entry of an alternating product in $\{U, U^*, 2P_0\}$ and $\{X, X^*, 2Q_0\}$ has trace zero.

Thus exactly as Lemma 1.4.3, for any $i_1 \in \{1, 2\}$,

$$
\sum_{\omega \text{ is a summand of } \Pi_{i_1,i_1}} k_{0_{r+1}}(a^1_{i_1,i_2}, b^3_{i_3,i_4}, \cdots, b^{2r-1}_{i_{2r-1},i_{2r}}, a^{2r+1}_{i_{2r+1},i_1})tr_{1_{r+1}}(w^2_{i_2,i_3}, w^4_{i_4,i_5}, \cdots, w^{2r}_{i_{2r},i_{2r+1}}, 1) = 0.
$$

Now rest of the proof follows similarly as Proposition 1.4.2. □

Proposition 1.5.2.

$$
(A_1 \oplus A_2) * M_2(B) \cong M_2(A_1 * A_2 * B * LF_2)
$$

Proof. Here one needs to prove that $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ $(\cong A_1 \oplus A_2)$ and $WZ^*M_2(B)ZW^*(\cong M_2(B))$ are free and they generate the right hand side. The proof is exactly similar to that of Proposition 1.4.2 or Proposition 1.5.1 using Corollary 1.3.5. □

Remark 1.5.3. *As in Remark 1.3.7 here too we can have an exactly similar alternate proof using* $W \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} W^*$ *and* $YM_2(B)Y^*$ *as model.*

Corollary 1.5.4.

$$M_2(A) * L\mathbb{Z} \cong M_2(A * LF_4)$$

Corollary 1.5.5. *For* $k_1, k_2, l_1, l_2 \in \mathbb{N} \cup \{0\}$*,*

1. $(LF_{k_1} \oplus LF_{k_2}) * (LF_{l_1} \oplus LF_{l_2}) \cong M_2(LF_{k_1+k_2+l_1+l_2+1})$*;*

2. $(LF_{k_1} \oplus LF_{k_2}) * L\mathbb{Z} \cong M_2(LF_{k_1+k_2+3})$*;*

3. $M_2(LF_k) * M_2(LF_l) \cong M_2(LF_{k+l+3})$*;*

4. $(LF_{k_1} \oplus LF_{k_2}) * M_2(LF_l) \cong M_2(LF_{k_1+k_2+l+2})$*;*

5. $L\mathbb{Z} * M_2(LF_l) \cong M_2(LF_{l+4})$*.*

Proof. (1), (3) and (4) are direct consequences of Proposition 1.4.2, Proposition 1.5.1 and Proposition 1.5.2 respectively. (2) and (5) follow from those as well as Corollary 1.4.4 and 1.5.4 (In fact (3) follows directly from Theorem 5.4.1 [VDN92]). □

1.6 Applications

In this section, we use the results proved in the previous sections and compute various possible free products involving the hyperfinite II_1 factor R, $(LF_l)^{2^m}$ and $M_{2^n}(LF_k)$, where $m, n \in \mathbb{N}, k, l \in \mathbb{N} \cup \{0\}$ (LF_0 is considered as $\mathbb{C}$). These results were proved in section 1-3 of [Dyk93] in a much more general context, but with a different approach. The resulting von-Neumann algebras, as we know from [Dyk93], turn out to be certain II_1 factors known as interpolated free group factors ([Dyk94], [Rad94]). In particular, these interpolated free group factors appearing here, are of the form LF_{2+d}, where d is a dyadic rational number ≥ -2.

We now need to use Theorem 5.4.1 of [VDN92]. For our purpose it will be enough to stick to the following 2-dimensional version of the theorem: For $k \geq 2, k \in \mathbb{N}$,

$$LF_k \cong M_2(LF_{(k-1)4+1}).$$

However Theorem 2.4 of [Dyk94] shows that even for k not an integer, the above result holds, i.e. $\forall t \geq 2, t \in \mathbb{R}$,

$$LF_t \cong M_2(LF_{(t-1)4+1}), \tag{1.6.1}$$

and it is this version of the theorem that we will frequently use for the following computations.

Example 1.6.1. $\mathbb{C}^{2^n} * \mathbb{C}^{2^m} \cong M_2(LF_{5-2(\frac{1}{2^{n-1}}+\frac{1}{2^{m-1}})}), n \geq 1$. *In particular,* $\mathbb{C}^{2^n} * \mathbb{C}^{2^n} \cong M_2(LF_{5-\frac{4}{2^{n-1}}}), n \geq 1$.

Proof. We first prove inductively that $\mathbb{C}^{2^n} * \mathbb{C}^{2^n} \cong M_2(LF_{a_n})$ for some $a_n \in [1, \infty)$, for all $n \geq 1$. Then the basic step follows from Proposition 1.2.3. Also $a_1 = 1$ by the same Proposition.

$$\begin{aligned}
\mathbb{C}^{2^{n+1}} * \mathbb{C}^{2^{n+1}} &\cong (\mathbb{C}^{2^n} \oplus \mathbb{C}^{2^n}) * (\mathbb{C}^{2^n} \oplus \mathbb{C}^{2^n}) \\
&\cong M_2(\mathbb{C}^{2^n} * \mathbb{C}^{2^n} * \mathbb{C}^{2^n} * \mathbb{C}^{2^n} * L\mathbb{Z}), \text{by Proposition 1.2.3} \\
&\cong M_2(M_2(LF_{a_n}) * M_2(LF_{a_n}) * L\mathbb{Z}), \text{by induction hypothesis} \\
&\cong M_2(M_2(LF_{2a_n+3}) * L\mathbb{Z}), \text{by Corollary 1.5.5} \\
&\cong M_2(LF_{\frac{2a_n+3-1}{4}+2} * L\mathbb{Z}), \text{by equation 1.6.1} \\
&\cong M_2(LF_{\frac{a_n}{2}+\frac{5}{2}})\ .
\end{aligned}$$

Thus the induction is complete. Moreover we have the recurrence relation $a_{n+1} = \frac{a_n}{2} + \frac{5}{2}$.

Now,

$$\begin{aligned}
a_{n+1} &= \frac{a_n}{2} + \frac{5}{2} = \frac{\frac{a_{n-1}}{2} + \frac{5}{2}}{2} + \frac{5}{2} \\
&= \frac{a_{n-1}}{2^2} + \frac{5}{2.2} + \frac{5}{2} = \frac{a_1}{2^n} + \frac{5}{2}(\frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + 1) \\
&= \frac{1}{2^n} + \frac{5}{2}(\frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + 1) = \frac{1}{2^n} + \frac{5(2^n-1)}{2^n} = 5 - \frac{4}{2^n} .
\end{aligned}$$

From the above calculations and equation 1.6.1,

$$\begin{aligned}
\mathbb{C}^{2^n} * \mathbb{C}^{2^n} &\cong M_2(LF_{5-\frac{4}{2^{n-1}}}), \text{ as required} \\
&\cong LF_{2-\frac{1}{2^{n-1}}}, n \geq 1.
\end{aligned}$$

Without loss of generality we can assume that $n \geq m$. For $m = n = 1$ the proof follows from Proposition 1.2.3.

Let $n \geq 2$. Then,

$$\begin{aligned}
\mathbb{C}^{2^n} * \mathbb{C}^{2^m} &\cong M_2(\mathbb{C}^{2^{n-1}} * \mathbb{C}^{2^{n-1}} * \mathbb{C}^{2^{m-1}} * \mathbb{C}^{2^{m-1}} * L\mathbb{Z}) \\
&\cong M_2(LF_{2-\frac{1}{2^{n-2}}} * LF_{2-\frac{1}{2^{m-2}}} * L\mathbb{Z}) \\
&\cong M_2(LF_{5-2(\frac{1}{2^{n-1}}+\frac{1}{2^{m-1}})}) .
\end{aligned}$$

$\square$

Before stating the final result of this section, we compute few more similar examples using Corollary 1.5.5:

Example 1.6.2.

$$\mathbb{C}^{2^n} * LF_k \cong M_2(LF_{4k+1-\frac{2}{2^{n-1}}}), n, k \geq 1.$$

Proof. Let $k = 1$. First let $n = 1$. Then

$$\begin{aligned}
\mathbb{C}^2 * L\mathbb{Z} &\cong M_2(LF_3), \text{by Corollary 1.5.5} \\
&= M_2(LF_{4.1+1-\frac{2}{2^{1-1}}}), \text{thus proving the Proposition for this case.}
\end{aligned}$$

Again,

$$\begin{aligned}
\mathbb{C}^4 * L\mathbb{Z} &\cong (\mathbb{C}^2 \oplus \mathbb{C}^2) * (L\mathbb{Z} \oplus L\mathbb{Z}) \\
&\cong M_2(\mathbb{C}^2 * \mathbb{C}^2 * L\mathbb{Z} * L\mathbb{Z} * L\mathbb{Z}), \text{by Proposition 1.4.2} \\
&\cong M_2(M_2(L\mathbb{Z}) * L\mathbb{Z} * LF_2), \text{by Proposition 1.2.3} \\
&\cong M_2(M_2(LF_5) * LF_2), \text{by Corollary 1.5.5} \\
&\cong M_2(LF_4), \text{by equation 1.6.1} \\
&= M_2(LF_{4.1+1-\frac{2}{2^{2-1}}}) \ .
\end{aligned}$$

Now let $n \geq 3$. Then by Proposition 1.4.2 and equation 1.6.1,

$$\begin{aligned}
\mathbb{C}^{2^n} * L\mathbb{Z} &\cong (\mathbb{C}^{2^{n-1}} \oplus \mathbb{C}^{2^{n-1}}) * (L\mathbb{Z} \oplus L\mathbb{Z}) \\
&\cong M_2(LF_{5-\frac{1}{2^{n-2}}}) \\
&= M_2(LF_{4.1+1-\frac{2}{2^{n-2}}}) \\
&\cong LF_{2-\frac{1}{2^n}} \ .
\end{aligned}$$

Finally let $k \geq 2$. Then

$$\begin{aligned}
\mathbb{C}^{2^n} * LF_k &\cong LF_{2-\frac{1}{2^n}} * LF_{k-1} \\
&\cong LF_{k+1-\frac{1}{2^n}} \\
&\cong M_2(LF_{4k+1-\frac{2}{2^{n-1}}}) \ .
\end{aligned}$$

∎

Example 1.6.3.

$$M_{2^n} * LF_k \cong M_2(LF_{4k+1-\frac{1}{4^{n-1}}}), n, k \geq 1.$$

Proof. We first prove it for $k = 1$.

$$M_2 * L\mathbb{Z} \cong M_2(LF_4), \text{by Corollary 1.5.5}$$

Similarly as before, we inductively prove that for $n \geq 1$, $M_{2^n} * L\mathbb{Z} \cong M_2(LF_{b_n})$, where $b_1 = 4$ by the above calculation.

$$\begin{aligned}
M_{2^{n+1}} * L\mathbb{Z} &\cong M_2(M_{2^n}) * (L\mathbb{Z} \oplus L\mathbb{Z}) \\
&\cong M_2(M_{2^n} * L\mathbb{Z} * L\mathbb{Z} * LF_2) \\
&\cong M_2((M_{2^n} * L\mathbb{Z}) * (L\mathbb{Z} * LF_2)) \\
&\cong M_2(M_2(LF_{b_n}) * LF_3), \text{by induction hypothesis} \\
&\cong M_2(LF_{\frac{b_n-1}{4}+1} * LF_3), \text{by equation 1.6.1} \\
&\cong M_2(LF_{\frac{b_n}{4}+\frac{15}{4}})\ .
\end{aligned}$$

So we have $b_{n+1} = \frac{b_n}{4} + \frac{15}{4}$ $(b_1 = 4)$, solving which, we get $b_n = 5 - \frac{1}{4^{n-1}}$. Hence,

$$M_{2^n} * L\mathbb{Z} \cong M_2(LF_{5-\frac{1}{4^{n-1}}}) \tag{1.6.2}$$

Let $k \geq 2$.

$$\begin{aligned}
M_{2^n} * LF_k &\cong (M_{2^n} * L\mathbb{Z}) * LF_{k-1} \\
&\cong M_2(LF_{5-\frac{1}{4^{n-1}}}) * LF_{k-1} \\
&\cong LF_{2-\frac{1}{4^n}} * LF_{k-1}, \text{by equation 1.6.1} \\
&\cong LF_{k+1-\frac{1}{4^n}} \\
&\cong M_2(LF_{4k+1-\frac{1}{4^{n-1}}}), \text{by equation 1.6.1.}
\end{aligned}$$

$\square$

Remark 1.6.4. *The above result was proved for a more general case in Theorem 5.4.1 of [VDN92], using approximation of semicircular elements by random matrices in distribution. It also uses the fact that a symmetric square matrix with free entries such that the diagonal entries follow standard semicircular distribution and the off diagonal entries follow circular distribution, is free from the matrix algebra over the scalars*[1].

Corollary 1.6.5. $\mathbb{C}^4 * LF_k \cong M_2 * LF_k \cong M_2(LF_{4k}), k \geq 1.$

Example 1.6.6. $M_{2^n} * \mathbb{C}^{2^m} \cong M_2(LF_{5-\frac{1}{4^{n-1}}-\frac{2}{2^{m-1}}}), n, m \geq 1.$

Proof. Note that the case of $n = m = 1$ follows from Proposition 1.3.1. Let $n = 1, m \geq 2$. Then

$$\begin{aligned} M_2 * \mathbb{C}^{2^m} &\cong M_2(\mathbb{C}^{2^{m-1}} * \mathbb{C}^{2^{m-1}} * LF_2), \text{by Proposition 1.5.2} \\ &\cong M_2(\mathbb{C}^{2^{m-1}} * L\mathbb{Z} * \mathbb{C}^{2^{m-1}} * L\mathbb{Z}) \\ &\cong M_2(LF_{2(2-\frac{1}{2^{m-1}})}), \text{by Example 1.6.2 and equation 1.6.1} \\ &\cong M_2(LF_{4-\frac{2}{2^{m-1}}}) \ . \end{aligned}$$

Finally let $n \geq 2$. Then,

$$\begin{aligned} M_{2^n} * \mathbb{C}^{2^m} &\cong M_2(M_{2^{n-1}} * \mathbb{C}^{2^{m-1}} * \mathbb{C}^{2^{m-1}} * LF_2) \\ &\cong M_2(M_{2^{n-1}} * LF_{2-\frac{1}{2^{m-2}}} * LF_2), \text{by Example 1.6.1} \\ &\cong M_2(M_{2^{n-1}} * LF_2 * LF_{2-\frac{1}{2^{m-2}}}) \\ &\cong M_2(M_2(LF_{9-\frac{1}{4^{n-2}}}) * LF_{2-\frac{1}{2^{m-2}}}), \text{by Example 1.6.3} \\ &\cong M_2(LF_{3-\frac{1}{4^{n-1}}+2-\frac{1}{2^{m-2}}}) \\ &\cong M_2(LF_{5-\frac{1}{4^{n-1}}-\frac{1}{2^{m-2}}}) \ . \end{aligned}$$

[1] This can also be argued from Lecture 20 of [NS06], according to which, such a matrix turns out to be R-cyclic, where the value of a mixed cumulant depends only on its size, and thus is free from scalar matrices

In particular,

$$M_{2^n} * \mathbb{C}^{2^n} \cong M_2(LF_{5-\frac{1}{4^{n-1}}-\frac{2}{2^{n-1}}}) \ . \tag{1.6.3}$$

□

Example 1.6.7.

$$M_{2^n} * M_{2^m} \cong M_2(LF_{5-(\frac{1}{4^{n-1}}+\frac{1}{4^{m-1}})}), n, m \geq 1.$$

Proof. Without loss of generality let us assume $n \geq m$. We first note the following facts:

- The case of $n = m = 1$ follows from Proposition 1.3.2.
- $M_{2^n} * M_2 \cong M_2(LF_{4-\frac{1}{4^{n-1}}}), n \geq 2$, which follows from Proposition 1.5.1, Example 1.6.3 and equation 1.6.1.
- For all $n \geq m \geq 1$, $M_{2^n} * M_{2^m} \cong M_2(LF_{e_{n,m}})$ for some $e_{n,m} \in [1, \infty)$.
- $e_{n+1,m+1} = \frac{e_{n,m}}{4} + \frac{15}{4}$, where $e_{n+1-m,1} = 4 - \frac{1}{4^{n+1-m-1}} = 4 - \frac{1}{4^{n-m}}$, considering the free product of M_{2^n} and M_2 as given above.

Solving the above doubly recursive relation, we get $e_{m,n} = 5 - (\frac{1}{4^{n-1}} + \frac{1}{4^{m-1}})$, as required.

In particular,

$$M_{2^n} * M_{2^n} \cong M_2(LF_{5-\frac{2}{4^{n-1}}}), n \geq 1. \tag{1.6.4}$$

□

We are now ready to state the following proposition that summarizes the promised computations of the free products involving certain finite dimensional von-Neumann algebras and the free-group von-Neumann algebras. We will omit the proof since it is a simple exercise of induction using the previous sections, similar to the above examples.

Proposition 1.6.8. *For $m, n \in \mathbb{N}$, $k, l \in \mathbb{N} \cup \{0\}$ and $LF_0 = \mathbb{C}$,*

1. $(LF_k)^{2^n} * (LF_l)^{2^m} \cong M_2(LF_{5+\frac{2(k-1)}{2^n-1}+\frac{2(l-1)}{2^m-1}})$. *In particular* $(L\mathbb{Z})^{2^n} * (L\mathbb{Z})^{2^m} \cong M_2(LF_5)$.

2. $M_{2^n}(LF_k) * (LF_l)^{2^m} \cong M_2(LF_{5+\frac{k-1}{4^n-1}+\frac{2(l-1)}{2^m-1}})$. *In particular* $M_{2^n}(L\mathbb{Z}) * (L\mathbb{Z})^{2^m} \cong M_2(LF_5)$.

3. $M_{2^n}(LF_k) * M_{2^m}(LF_l) \cong M_2(LF_{5+\frac{k-1}{4^n-1}+\frac{l-1}{4^m-1}})$. *In particular* $M_{2^n}(L\mathbb{Z}) * M_{2^m}(L\mathbb{Z}) \cong M_2(LF_5)$.

Equation 1.6.1 shows that these resulting interpolated free group factors are indeed of the form LF_{2+d}, where $d \geq -2$ is a dyadic rational number.

Remark 1.6.9. *There exist explicitly computable functions $f, g : [1, \infty) \to [1, \infty)$, such that whenever finite von-Neumann algebras A, B satisfy $A * B \cong LF_t$ for some $t \in [1, \infty)$, then, for all $n \in \mathbb{N}$, we have*

- $A^{2^n} * B^{2^n} \cong M_2(LF_{f(t)})$;
- $M_{2^n}(A) * M_{2^n}(B) \cong M_2(LF_{g(t)})$.

Remark 1.6.10. *One can obviously extend the above proposition by taking $LF_{k_1} \oplus \cdots \oplus LF_{k_{2^n}}$ for $k_i \geq 0$, instead of $(LF_k)^{2^n}$.*

We know that the **hyperfinite II_1 factor** R can be constructed as an infinite tensor product of type I_{2^n} factors, i.e. scalar matrix algebras of dimension 2^n. Again $L\mathbb{Z} \cong L^\infty([0, \frac{\pi}{2}])$ can be thought of as an infinite tensor product of $\mathbb{C}^{2^n}$.

Proposition 1.6.8 suggests that on 'taking the limit as $m, n \to \infty$' , with $k = l = 0$,

1. $L\mathbb{Z} * L\mathbb{Z} \cong M_2(LF_5)$ (trivially true);
2. $R * L\mathbb{Z} \cong M_2(LF_5)$ (Theorem 5.4.3 [VDN92]);
3. $R * R \cong M_2(LF_5)$.

We could not come up with a matrix model to prove the above statements, approximating $L\mathbb{Z}$ and R as by finite dimensional algebras as $n \to \infty$ in Proposition 1.6.8. But we shall indeed give a rigorous proof for the assertion about $R * R$, as against the 'limiting' heuristics:

In view of the uniqueness of the hyperfinite II_1 factor ([MvN43]), we know that

$$R \cong M_2(R) \tag{1.6.5}$$

Now using Theorem 5.4.3 of [VDN92], i.e.

$$LF_k * R \cong LF_{k+1} \tag{1.6.6}$$

we may deduce the following:

Proposition 1.6.11. *For finite von-Neumann algebra A_1, A_2, B,*

1. *$R * (A_1 \oplus A_2) \cong M_2(A_1 * A_2 * LF_3)$;*
2. *$R * M_2(B) \cong M_2(B * LF_4)$.*

Proof. By above equations and Proposition 1.5.2

$$\begin{aligned} R * (A_1 \oplus A_2) &\cong M_2(R) * (A_1 \oplus A_2) \\ &\cong M_2(A_1 * A_2 * R * L\mathbb{Z}) \\ &\cong M_2(A_1 * A_2 * LF_3) \end{aligned}$$

The other statement follows similarly using Proposition 1.5.1. □

Corollary 1.6.12. $R * R \cong M_2(LF_5)$

Proof. It follows from equation 1.6.5 and the above proposition. □

We now observe that our proof can also be led to the strengthened version Proposition 1.6.13 of Corollary 1.6.12). Following [Dyk94], let us write the left side as $R * \widetilde{R}$ to emphasize the distinction between the two free copies of the hyperfinite II_1 factor. Consider $R \cong M_2(R), \widetilde{R} \cong M_2(\widetilde{R})$. Then we notice that by proof of Proposition 1.5.1, $M_2(R)$ on the left hand side gets mapped into $M_2(R)$ on the right hand side (which is $M_2(R * \widetilde{R} * LF_3) \cong M_2(R * LF_4)$) as conjugated by the unitary matrix Y^* where $Y = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$.

On the other hand, by Proposition 1.5.2 and Theorem 5.4.3 of [VDN92], we have

$$\begin{aligned} M_2(R) * L\mathbb{Z} &\cong M_2(R) * (L\mathbb{Z} \oplus L\mathbb{Z}) \\ &\cong M_2(R * LF_4), \end{aligned}$$

where by Remark 1.5.3, $M_2(R)$ on the left hand side is mapped into $M_2(R)$ on the right hand side in exactly the same manner as above, i.e. as conjugated by the same unitary matrix Y^*.

In fact we note here that for projection $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$,

$$P(M_2(R) * M_2(\widetilde{R}))P \cong PM_2(R)P * LF_4 \cong P(M_2(R) * L\mathbb{Z})P,$$

where the isomorphisms restricted to $PM_2(R)P$ (which is naturally isomorphic to R), in all three cases are the identity maps.

Thus similarly as in Corollary 3.6 of [Dyk94], we can conclude that

Proposition 1.6.13.

$$R * \widetilde{R} \cong R * L\mathbb{Z} \cong M_2(LF_5),$$

where the first isomorphism restricted to R on the left hand side is the identity map to R on the right hand side.

Chapter 2

Graphical models for von-Neumann algebras

In this chapter we discuss the construction of a graphical model for finite von Neumann algebras by associating them to finite weighted graphs ([KS11]). The resulting associated algebra to a given graph gives a free product of algebras corresponding to subgraphs 'with one edge' (actually a pair of dual edges), with amalgamation over a finite-dimensional abelian subalgebra corresponding to the vertex set. This yields certain natural examples of a non-commutative random variables with free Poisson distribution, operator-valued circular and operator-valued semicircular distribution.

2.1 The graph-von-Neumann algebra association

This short section is dedicated to revisiting the construction in [KS11] of the associated W^*-probability space to a weighted graph, without assuming that the graph is bipartite[1].

A weighted graph is a tuple $\Gamma = (V, \mathcal{E}, \mu)$, where

- V is a (finite) set of vertices;

[1]Actually, [KS11] treated only the case of bipartite graphs, and sometimes restricted attention to the case of the Perron-Frobenius weighting; but for the proof of statements made in this section, none of those restrictions is necessary.

- $\mathcal{E}$ is a (finite) set of edges, equipped with 'source' and 'range' maps $s, r : \mathcal{E} \to V$ and '(orientation) reversal' involution map $\mathcal{E} \ni e \mapsto \widetilde{e} \in \mathcal{E}$ with $(s(e), r(e)) = (r(\widetilde{e}), s(\widetilde{e}))$; and

- $\mu : V \to (0, \infty)$ is a 'weight or spin function' so normalized that $\sum_{u \in V} \mu^2(v) = 1$

Let $\mathcal{P}_n = \mathcal{P}_n(\Gamma)$ denote the set of paths of length n in Γ and let $P_n(\Gamma)$ denote the vector space with basis $\{[\xi] : \xi \in \mathcal{P}_n(\Gamma)\}$. $\xi = \xi_1\xi_2\cdots\xi_n$ is thought as the 'concatenation product' where ξ_i denotes the i-th edge of ξ.

With the above notations, $F(\Gamma) := \oplus_{n \geq 0} P_n(\Gamma)$ is equipped with the following slightly complicated multiplication:

If $\xi \in \mathcal{P}_m(\Gamma), \eta \in \mathcal{P}_n(\Gamma)$, then

$$[\xi]\#[\eta] = \sum_{k=0}^{\min(m,n)} [\zeta_k],$$

where $\zeta_k \in \mathcal{P}_{m+n-2k}$ is defined by

$$\zeta_k = \begin{cases} \frac{\mu(v^\xi_m)}{\mu(v^\xi_{m-k})}[\xi_1\xi_2\cdots\xi_{m-k}\eta_{k+1}\eta_{k+2}\cdots\eta_n] & \text{if } \xi_{m-j+1} = \widetilde{\eta_j} \forall 1 \leq j \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Following [KS11], we adopt the convention throughout this chapter that if $\xi \in \mathcal{P}_n$, then $\xi = \xi_1\xi_2\cdots\xi_n$ denotes concatenation product, with $\xi_i \in \mathcal{E}$ and we write $s(\xi_i) = v^\xi_{i-1}$ (so also $r(\xi_i) = s(\xi_{i+1}) = v^\xi_i$).

In particular, $\mathcal{P}_0(\Gamma) = \{v : v \in V\}$, and if $v = s(\xi), w = r(\xi)$ for some $\xi \in \mathcal{P}_n$, and if $u_1, u_2 \in V$, then $[u_1][\xi][u_2] = \delta_{u_1,v}\delta_{u_2,w}[\xi]$; and less trivially, if $\xi \in \mathcal{P}_1$ and $\eta \in \mathcal{P}_m, m \geq 1$, then

$$[\xi]\#[\eta] = \begin{cases} 0 & \text{if } r(\xi) \neq s(\eta) \\ [\xi\eta_1\cdots\eta_m] & \text{if } r(\xi) = s(\eta) \text{ but } \xi \neq \widetilde{\eta_1} \\ [\xi\eta_1\cdots\eta_m] + \frac{\mu(r(\xi))}{\mu(s(\xi))}[\eta_2\cdots\eta_m] & \text{if } \xi = \widetilde{\eta_1}. \end{cases}$$

However the definition of the trace τ on $F(\Gamma)$ is fairly simple -

$$\tau := \mu^2 \circ E,$$

where $E : F(\Gamma) \to P_0$, such that for $\xi \in P_n$, $E([\xi]) = \begin{cases} 0 & \text{if } n > 0 \\ [\xi] & \text{if } n = 0, \end{cases}$

and μ^2 is simply the linear extension to $P_0(\Gamma)$ that agrees with μ^2 on the basis $\mathcal{P}_0(\Gamma)$.

It was shown in [KS11] that $(F(\Gamma), \tau)$ is a tracial non-commutative *-probability space, with $e^* = \widetilde{e}$, that the mapping $y \mapsto xy$ extends to a $*$-algebra representation $F(\Gamma) \to \mathcal{B}(L^2(F(\Gamma)), \tau)$ and that $M(\Gamma, \mu) := \lambda(F\Gamma))'' \subset \mathcal{B}(L^2(F(\Gamma)), \tau)$ is in standard form.

Then for $\xi, \eta \in \cup_n \mathcal{P}_n(\Gamma)$,

$$\tau([\xi]\#[\eta]^*) = \delta_{\xi,\eta}\mu(r(\xi))\mu(s(\xi)),$$

and hence, if $\{\xi\} = (\mu(s(\xi))\mu(r(\xi)))^{-\frac{1}{2}}[\xi]$, then $\{\{\xi\} : \xi \in \cup_{n\geq 0}\mathcal{P}_n(\Gamma)\}$ is an orthonormal basis for $\mathcal{H}(\Gamma) = L^2(F(\Gamma), \tau)$.

2.2 The building blocks

In this section we tend to observe just how $M(\Gamma, \mu)$ depends on (Γ, μ). We begin by spelling out some simple examples, which turn out to be building blocks for the general case.

Example 2.2.1. *1. Suppose $|V| = |\mathcal{E}| = 1$, say $V = \{v\}$ and $\mathcal{E} = \{e\}$. Then we must have $e = \widetilde{e}, s(e) = r(e) = v, \mu(v) = 1, \mathcal{P}_n = \{e^n\}$ and $\{\xi(n) = \{e^n\} : n \geq 0\}$ (where $\{e^0\} = \{v\}$) is an orthonormal basis for $\mathcal{H}(\Gamma)$; and the definitions show that $x = \lambda(e)$ satisfies $x\xi_n = \xi(n+1) + \xi(n-1)$. Thus x is a semicircular element and $M(\Gamma) = \{x\}'' \cong L\mathbb{Z}$.*

2. Suppose $|V| = 1, |\mathcal{E}| = 2$, say $V = \{v\}$ and $\mathcal{E} = \{e_1, e_2\}$ suppose $e_2 = \widetilde{e_1}$. Then we must have $s(e_j) = r(e_j) = v, \mu(v) = 1$. Further $\{\{e_1\}, \{e_2\}\}$ is an orthonormal basis for $\mathcal{H}_2 = P_1(\Gamma)$, and $P_n(\Gamma)$ is isomorphic to $\otimes^n \mathcal{H}_2$. Thus $\mathcal{H}(\Gamma)$ may be identified with the full Fock space $\mathcal{F}(\mathcal{H}_2)$ and the definitions show that $x_1 = \lambda(e_1)$ may be *identified as $x_1 = l_1 + l_2^*$, where the l_j denote the standard creation operators. It follows that x_1 is a circular element and $M(\Gamma) = \{x_1\}'' \cong LF_2$.*

3. *Suppose $|V| = 2, |\mathcal{E}| = 2$, say $V = \{v, w\}$ and $\mathcal{E} = \{e, \widetilde{e}\}$ and suppose $s(e) = v, r(e) = w$ and $\mu(w) \leq \mu(v)$. Write $\rho = \frac{\mu(v)}{\mu(w)}(\geq 1)$. If we let $p_v = \lambda([v]), p_w = \lambda([w])$, it follows that $\mathcal{H}_v = ran\ p_v$ (respectively, $\mathcal{H}_w = ran\ p_w$) has an orthonormal basis given by $\{\{\eta(n)\} : n \geq 0\}$ (respectively, $\{\{\xi(n)\} : n \geq 0\}$ where $\eta(n) \in \mathcal{P}_n$ (respectively, $\xi(n) \in \mathcal{P}_n$) and $\eta(n)_k = e$ or $\widetilde{e}$ (respectively., $\xi(n)_k = \widetilde{e}$ or e according as k is odd or even).*

   Writing $x = \lambda(e)$, we see that with respect to the decomposition $\mathcal{H}(\Gamma) = \mathcal{H}_v \oplus \mathcal{H}_w$, the operator x has a matrix decomposition of the form

$$x = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

   where $t \in \mathcal{B}(\mathcal{H}_w, \mathcal{H}_v)$ is seen to be given by

$$\begin{aligned} t[\xi(n)] &= x[\xi(n)] \\ & [e]\#[\widetilde{e}e\widetilde{e}e \cdots (n \text{ terms})] \\ & [\eta(n+1)] + \rho^{-1}[\eta(n-1)]; \end{aligned}$$

   and hence,

$$\begin{aligned} t\{\xi(n)\} &= (\mu(s(\xi(n))\mu(r(\xi(n)))^{-\frac{1}{2}} t[\xi(n)] \\ &= (\mu(w)\mu(r(\xi(n)))^{-\frac{1}{2}} \left([\eta(n+1)] + \rho^{-1}[\eta(n-1)]\right) \\ &= (\rho^{-1}\mu(v)\mu(r(\eta(n \pm 1)))^{-\frac{1}{2}} \left([\eta(n+1)] + \rho^{-1}[\eta(n-1)]\right) \\ &= \rho^{\frac{1}{2}}\{\eta(n+1)\} + \rho^{-\frac{1}{2}}\{\eta(n-1)\}. \end{aligned}$$

   It is a fact - see Proposition 2.2.2 - that t^*t has has absolutely continuous spectrum.

This fact has two consequences:

(i) if $t = u|t|$ is the polar decomposition of t, then u maps $\mathcal{H}_w$ isometrically onto the subspace $\mathcal{M} = \overline{ran\ t}$ of $\mathcal{H}_v$, and if z is the projection onto $\mathcal{H}_v \ominus \mathcal{M}$ then $\tau(z) = \mu^2(v) - \mu^2(w)$; and

(ii) $W^*(|t|) \cong L\mathbb{Z}$.

Since $p_v + p_w = 1$ and $z \leq p_v$, the definitions show that $M(\Gamma, \mu)$ is isomorphic to $\mathbb{C} \oplus M_2(L\mathbb{Z})$ via the unique isomorphism which maps p_v, p_w, z, u and $|t|$, respectively, to $(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})$, $(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$, $(1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})$, $(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$, and $(0, \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix})$ for some positive a with absolutely continuous spectrum that generates $L\mathbb{Z}$ as a von Neumann algebra. (This must be compared with Lemma 17 of [GJS11], bearing in mind that their μ is our μ^2.)

□

We now analyze the operator t^*t by calculating its spectrum and its distribution so as to testify the claim made above.

Proposition 2.2.2. *Let $\ell^2(\mathbb{N}_0)$ have its standard orthonormal basis $\{\delta_n : n \in \mathbb{N}_0\}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\ell\delta_n = \delta_{n+1}$ denote the creation operator (or unilateral shift), with $\ell^*\delta_n = \delta_{n-1}$ (where $\delta_{-1} := 0$). Let $\rho > 1$ and $t = \rho^{\frac{1}{2}}\ell + \rho^{-\frac{1}{2}}\ell^*$. Then,*

1. *t^*t leaves the subspace $\ell^2(2\mathbb{N}_0)$ invariant;*
2. *δ_0 is a cyclic vector for $a_\rho :=$ the restriction of t^*t on $\ell^2(2\mathbb{N}_0)$; and*
3. *the (scalar) spectral measure of a_ρ associated to δ_0 is absolutely continuous with respect to Lebesgue measure; in fact a_ρ has a free Poisson distribution.*

Proof. A little algebra shows that

$$\begin{aligned} t^*t &= (\rho^{\frac{1}{2}}\ell^* + \rho^{-\frac{1}{2}}\ell)(\rho^{\frac{1}{2}}\ell + \rho^{-\frac{1}{2}}\ell^*) \\ &= \ell^2 + \ell^{*2} + (\rho + \rho^{-1}) - \rho^{-1}p_0, \end{aligned}$$

where p_0 is the rank-one projection onto $\mathbb{C}\delta_0$. It is seen that this operator leaves both subspaces $\ell^2(2\mathbb{N})$ and $\ell^2(2\mathbb{N}+1)$ invariant, with its restrictions to these subspaces being unitarily equivalent to $\ell+\ell^*+(\rho+\rho^{-1})-\rho^{-1}p_0$ and $\ell+\ell^*$ respectively. Since the spectral type does not change under scalar translation, we may assume without loss of generality that $a_\rho = \ell+\ell^* - \rho^{-1}p_0$ and establish that a_0 has absolutely continuous scalar spectral measure corresponding to δ_0.

Write $a_0 = \ell+\ell^*$ so that $a_\rho = a_0 - \rho^{-1}p_0$. Let the scalar spectral measures of a_0 and a_ρ be denoted by μ and μ_ρ respectively, and consider their Cauchy transforms given by

$$F_\lambda(z) = \langle (a_\lambda - z)^{-1}\delta_0, \delta_0\rangle = \int_{\mathbb{R}} \frac{d\mu_\lambda(x)}{x-z}$$

for $\lambda \in \{0,\rho\}$ and $z \in \mathbb{C}^+ = \{\zeta \in \mathbb{C} : \Im(\zeta) > 0\}$.

It follows from the resolvent equation that

$$\begin{aligned} F_\rho(z) &= \langle (a_\rho - z)^{-1}\delta_0, \delta_0\rangle \\ &= \langle (a_0 - z)^{-1}\delta_0, \delta_0\rangle + \langle (a_\rho - z)^{-1}\rho^{-1}p_0(a_\lambda - z)^{-1}\delta_0, \delta_0\rangle \\ &= F_0(z) + \rho^{-1}F_\rho(z)F_0(z)\ ; \end{aligned}$$

Hence

$$F_\rho(z) = \frac{F_0(z)}{1-\rho^{-1}F_0(z)} = \frac{\rho F_0(z)}{\rho - F_0(z)} \tag{2.2.1}$$

It is seen from Lemma 2.21 of [NS06] - after noting that the G of that lemma is the negative of the F_0 here - that $F_0(z) = \frac{z-\sqrt{z^2-4}}{2}$, where $\sqrt{z^2-4}$ is a branch of that square root such that $\sqrt{z^2-4} = \sqrt{z+2}\sqrt{z-2}$ where the two individual factors are respectively defined by using the branch-cuts $\{\mp 2 - it : t \in (0,\infty)$. (This choice ensures that $lim_{|z|\to\infty} F_0(z) = 0$, which is clearly necessary.) It follows that F_0, which is holomorphic in $\mathbb{C}^+$, actually extends to a continuous function on $\mathbb{C}^+ \cup \mathbb{R}$, and that if we write $f_0(a) = \lim_{b\downarrow 0} F_0(a+ib)$, then we have

$$2f_0(t) = \begin{cases} -t + \sqrt{t^2 - 4}\text{if } t \geq 2 \\ -t + i\sqrt{4 - t^2}\text{if } t \in [-2, 2] \\ -t - \sqrt{t^2 - 4}\text{if } t \leq -2. \end{cases} \tag{2.2.2}$$

It is easy to check that f_0 is strictly increasing in $(-\infty, -2)$, as well as in in $(2, \infty)$, has non-zero imaginary part in $(-2, 2)$, and satisfies $f(\mathbb{R} \setminus (-2, 2)) = [-1, 0) \cup (0, 1]$. Since $\rho > 1$, we may deduce that $F_0(z) \neq \rho \; \forall z \in \mathbb{C}^+ \cup \mathbb{R}$, and hence F_ρ extends to a continuous function on $\mathbb{C}^+ \cup \mathbb{R}$, with equation 2.2.1 continuing to hold for all $z \in \mathbb{C}^+ \cup \mathbb{R}$. Writing $f_\lambda(t) = F_\lambda(t + i \cdot 0)$ for $\lambda \in \{0, \rho\}$, we find that

$$f_\rho(t) = \frac{\rho f_0(t)}{\rho - f_0(t)} = \frac{1}{f_0(t)^{-1} - \rho^{-1}},$$

and hence

$$\begin{aligned} \Im(f_\rho(t)) &= -\frac{\Im(f_0(t)^{-1})}{|f_0(t)^{-1} - \rho^{-1}|^2} \\ &= \frac{\Im(f_0(t))}{|1 - f_0(t)\rho^{-1}|^2} \\ &= \rho^2 \frac{\Im(f_0(t))}{|f_0(t) - \rho|^2} \\ &= 1_{[-2,2]}(t) \frac{\rho^2 \sqrt{4 - t^2}}{2|f_0(t) - \rho|^2} \; . \end{aligned}$$

Now, for $t \in [-2, 2]$,

$$\begin{aligned} |f_0(t) - \rho|^2 &= |\frac{-t + i\sqrt{4 - t^2}}{2} - \rho|^2 \\ &= \frac{1}{4}\left((t + 2\rho)^2 + 4 - t^2\right) \\ &= \rho^2 + \rho t + 1 \; . \end{aligned}$$

It follows from Stieltjes's inversion formula that our a_ρ has absolutely continuous scalar spectral measure μ_ρ, with density given by

$$\begin{aligned} g_\rho(t) &= \frac{1}{\pi}\Im f_\rho(t) \\ &= 1_{[-2,2]}(t)\frac{\rho^2\sqrt{4-t^2}}{2\pi(\rho^2+\rho t+1)} \ . \end{aligned}$$

Hence the operator $t^*t = a_\rho + (\rho+\rho^{-1})1$ has absolutely continuous scalar spectral measure, with density given by

$$\begin{aligned} g(t) &= g_\rho(t-(\rho+\rho^{-1})) \\ &= 1_{[(\rho+\rho^{-1})-2,(\rho+\rho^{-1})+2]}(t)\frac{\rho^2\sqrt{4-(t-(\rho+\rho^{-1})}^2}{2\pi\rho^{-2}(\rho^2+\rho(t-\rho-\rho^{-1})+1)} \\ &= 1_{[(\rho+\rho^{-1})-2,(\rho+\rho^{-1})+2]}(t)\frac{\rho^2\sqrt{4-(t-(\rho+\rho^{-1})}^2}{2\pi\rho^{-1}t} \end{aligned}$$

If we write $\lambda = \rho^2$ and $\alpha = \rho^{-1}$ ($\Rightarrow \rho+\rho^{-1} = \alpha(1+\lambda)$), then by comparing with equation 12.15 of [NS06] we see that not only does t^*t have absolutely continuous spectrum, but it actually has a free Poisson distribution, with rate ρ^2 and jump size ρ^{-1}. However, we actually discovered this fact about t^*t having a free Poisson distribution with the stated λ and α by a cumulant computation that we present in the final section of this chapter, both for giving a combinatorial rather than analytic proof of the above proposition, and because we came across that proof first. □

2.3 Free cumulants on an equivalent alternative graphical model

Before proceeding with further study of a general (Γ, μ), we need an alternative but equivalent description of $M(\Gamma, \tau)$, since the calculation of cumulants turns out to be much simpler with respect to the trace that appears in this other description.

Let $Gr(\Gamma) := \oplus_{n\geq 0} P_n(\Gamma)$ be equipped with a $*$-algebra structure, where $[\xi]\circ[\eta] = [\xi\eta]$

and $[\xi]^* = [\widetilde{\xi}] = [\widetilde{\xi_n} \cdots \widetilde{\xi_1}]$ for $\xi = \widetilde{\xi_1} \cdots \widetilde{\xi_n} \in \mathcal{P}_n, \eta \in \mathcal{P}_m$. It was proved in [KS11][2] that $Gr(\Gamma)$ and $F(\Gamma)$ are isomorphic as $*$-algebras. While the multiplication is simpler in $Gr(\Gamma)$, the trace on it, i.e. τ on $F(\Gamma)$ transported by the above isomorphism, is given by a slightly more complicated formula. (It is what has been called *the Voiculescu trace* by Jones et al.) We shall write tr for this transported trace on $Gr(\Gamma)$, and E for the tr-preserving conditional expectation of $M(\Gamma, \mu)(= \lambda(Gr(\Gamma))'')$ onto $P_0(\Gamma)$. We shall use the same letter E to denote restrictions to subalgebras which contain $P_0(\Gamma)$.

Consider $(Gr(\Gamma), E)$ as an operator-valued non-commutative probability space over $P_0(\Gamma)$, our first order of business being the determination of the $P_0(\Gamma)$-valued mixed cumulants in $Gr(\Gamma)$.

Proposition 2.3.1. *The $P_0(\Gamma)$-valued mixed cumulants in $Gr(\Gamma)$ are given as*

$\kappa_n([e_1], [e_2], \cdots, [e_n]) = 0$, unless $n = 2$ and $e_2 = \widetilde{e_1}$; in which case, with $s(e_1) = v, r(e_1) = w$, $\kappa_2([e_1], [\widetilde{e_1}]) = \frac{\mu(w)}{\mu(v)}[v]$.

Proof. The proof depends on the 'moment-cumulant' relations, that completely determine one another.

(a) We first define for all $n \in \mathbb{N}_0$, $\kappa_n : (Gr(\Gamma))^n \to P_0(\Gamma)$ to be the unique multilinear map tuples of paths as arguments, as asserted in the proposition. Then it is easy to see that these κ_n is

- 'balanced' over $P_0(\Gamma)$ in the sense that

  $\kappa_n(x_1, \cdots, x_{i-1}b, x_i, \cdots, x_n) = \kappa_n(x_1, \cdots, x_{i-1}, bx_i, \cdots, x_n)$, $\forall x_j \in Gr(\Gamma), b \in P_0(\Gamma)$ and $1 < i \leq n$; and

- $P_0(\Gamma)$-bilinear in the sense that

  $\kappa_n(bx_1, x_2, \cdots, x_{n-1}, x_n b') = b\kappa_n(x_1, x_2, \cdots, x_{n-1}, x_n)b' \; \forall x_j \in Gr(\Gamma), b, b' \in P_0(\Gamma)$;

(b) we inductively define the 'multiplicative extensions' $\kappa_\pi : (Gr(\Gamma))^n \to P_0(\Gamma)$ for $\pi \in NC(n)$ by requiring that if $[k, l]$ is an interval constituting a class of π. If we write

[2] The remark made in an earlier footnote, concerning assumptions regarding bipartiteness of Γ, applies here as well.

σ for the element of $NC(n-l+k-1)$ given by the restriction of π to $\{1,\cdots,k-1,l+1,\cdots,n\}$, i.e. $\pi=\sigma\bigvee 1_{[k,l]}$ following the notation of Lecture 9, [NS06], then

$$\begin{aligned}\kappa_\pi(x_1,\cdots,x_n)&=\kappa_\sigma(x_1,\cdots,x_{k-1}\kappa_{l-k+1}(x_k,\cdots,x_l),x_{l+1},\cdots,x_n)\\&=\kappa_\sigma(x_1,\cdots,x_{k-1},\kappa_{l-k+1}(x_k,\cdots,x_l)x_{l+1},\cdots,x_n);\end{aligned}$$

(c) finally we verify that for any $e_1,\cdots,e_n\in\mathcal{P}_1(\Gamma)$,

$$E([e_1]\cdots[e_n])=\sum_{\pi\in NC(n)}\kappa_\pi([e_1],[e_2],\cdots,[e_n]). \tag{2.3.1}$$

For this verification, we first assert that

if $e_1,e_2,\cdots,e_n\in\mathcal{E}$ and $\pi\in NC(n)$, then $\kappa_\pi([e_1],[e_2],\cdots,[e_n])$ (yielded by the unique 'multiplicative extension' of the κ_n's as in (b) above) can be non-zero only if

(i) $e_1e_2\cdots e_n$ is a meaningfully defined loop based at $s(e_1)$, meaning $f(e_i)=s(e_{i+1})$ for $1\le i\le n$, with e_{n+1} being interpreted as e_1;

(ii) n is even and $\pi\in NC_2(n)$, i.e. π is a pair partition of n, such that $\{i,j\}\in\pi\Leftrightarrow e_j=\widetilde{e_i}$;

and if that is the case, then,

$$\kappa_\pi([e_1],[e_2],\cdots,[e_n])=\left(\prod_{\substack{\{i,j\}\in\pi\\ i<j}}\frac{\mu(r(e_i)}{\mu(r(e_j)}\right)[s(e_1)]. \tag{2.3.2}$$

We prove this assertion by induction on n. This is trivial for $n=1$ since $\kappa_1\equiv 0$. By the inductive definition of the multiplicative extension, it is clear that if $\kappa_\pi([e_1],[e_2],\cdots,[e_n])$ is to be non-zero, π must contain an interval class of the form $\{k,k+1\}$ such that

$e_{k+1} = \widetilde{e_k}$; if σ denotes $\pi|_{\{1,2,\cdots,k-1,k+2,\cdots n\}}$ we must have

$$\begin{aligned}\kappa_\pi([e_1],\cdots,[e_n]) &= \frac{\mu(r(e_k))}{\mu(r(e_{k+1}))}\kappa_\sigma([e_1],\cdots,[e_{k-1}][s(e_k)],[e_{k+2}],\cdots,[e_n]) \\ &= \frac{\mu(r(e_k))}{\mu(r(e_{k+1}))}\kappa_\sigma([e_1],\cdots,[e_{k-1}],[s(e_k)][e_{k+2}],\cdots,[e_n]) \\ &= \frac{\mu(r(e_k))}{\mu(r(e_{k+1}))}\kappa_\sigma([e_1],\cdots,[e_{k-1}][r(e_{k+1})],[e_{k+2}],\cdots,[e_n]);\end{aligned}$$

and for this to be non-zero, we must have $r(e_{k-1}) = s(e_k) = r(e_{k+1}) = s(e_{k+2})$, in which case we would have

$$\kappa_\pi([e_1],\cdots,[e_n]) = \frac{\mu(r(e_k))}{\mu(r(e_{k+1}))}\kappa_\sigma([e_1],\cdots,[e_{k-1}],[e_{k+2}],\cdots,[e_n]).$$

The requirement that $\kappa_\sigma([e_1],\cdots,[e_{k-1}],[e_{k+2}],\cdots,[e_n])$ be non-zero, along with the induction hypothesis, finally completes the proof of the assertion.

Now, in order to verify equation 2.3.1, it suffices to check that for any $v \in V$, we have

$$tr([e_1][e_2]\cdots[e_n][v]) = \sum_{\pi \in NC(n)} tr(\kappa_\pi([e_1],[e_2],\cdots,[e_n])[v]). \tag{2.3.3}$$

First observe that both sides of equation 2.3.3 vanish unless $e_1 \cdots e_n$ is a meaningfully defined path with both source and range equal to v (since tr is a trace and $[v]$ is idempotent). In view of our description above of the multiplicative extension κ_π, we thus need to verify that for such a loop, we have

$$tr([e_1\cdots e_n]) = \sum_{\pi \in NC_2(n)} \left(\prod_{\substack{\{i,j\}\in\pi \\ i<j}} \delta_{e_j,\widetilde{e_i}} \frac{\mu(r(e_i))}{\mu(r(e_j))} \right) \mu^2(s(e_1)),$$

but that is indeed the case ([KS09]). $\square$

In order to derive the true import of Proposition 2.3.1, we should first introduce some notation:

For each dual pair $e, \widetilde{e}$ of edges with, say - $s(e) = v, r(e) = w$, we shall write $\Gamma_e = (V_e, \mathcal{E}_e, \mu_e)$ where $V_e = V, \mu_e = \mu$ and $\mathcal{E}_e = \{e, \widetilde{e}\}$ (with source, range and reversal in $\mathcal{E}_e$ as in $\mathcal{E}$). If $e = \widetilde{e}$, the above definitions are to be suitably interpreted. Now for 'the true import of Proposition 2.3.1':

Corollary 2.3.2. *With the foregoing notation, we have:*

$$Gr(\Gamma, \mu) = *_{P_0(\Gamma)}\{Gr(\Gamma_e, \mu_e) : \{e, \widetilde{e}\} \subset \mathcal{E}\},$$

and hence, also

$$M(\Gamma, \mu) = *_{P_0(\Gamma)}\{M(\Gamma_e, \mu_e) : \{e, \widetilde{e}\} \subset \mathcal{E}\}.$$

Proof. An operator valued analogue of Proposition 2.5.5 of [VDN92] (see Proposition 3.3.3 in [Spe98] for validity of this analogue) combined with Theorem 11.16 of [NS06] show that if $A \stackrel{E}{\to} B$ is a 'non-commutative probability space over B', if $\{A_i : i \in I\}$ is a family of subalgebras of A containing B, such that $\{A_i : i \in I\}$ generates A, and if G_i is a set of generators of the algebra A_i, then A is the free product with amalgamation over B of $\{A_i : i \in I\}$ if and only if the mixed B-valued cumulants $\kappa_n(x_1, \cdots, x_n)$ vanish whenever $x_1, \cdots, \cdots x_n \in \cup_i G_i$, unless all the x_i belong to the same G_k for some k. The desired assertion then follows from Proposition 2.3.1. □

The following corollary is an immediate consequence of Corollary 2.3.2 and Examples 2.2.1 (1) and (2).

Corollary 2.3.3. *If Γ_n denotes the 'flower with n petals' (thus $|V| = 1, |\mathcal{E}| = n$), then $M(\Gamma) \cong L\mathbb{F}_n$, independent of the reversal map on $\mathcal{E}$.*

Remark 2.3.4. *We may deduce from Proposition 2.3.1 that the $x = \lambda(e)$ of Example 2.2.1 (3) is a $P_0(\Gamma)$-valued circular operator, in the sense of Definition 4.1 of [Dyk05], with covariance (α, β) where $\alpha(b) = E(x^*bx)$ and $\beta(b) = E(xbx^*)$ for all $b \in \mathcal{P}_0$ are the* *completely positive self-maps of* $P_0(\Gamma)(=\mathbb{C}p_v \oplus \mathbb{C}p_w)$ *induced by the matrices*

$$\alpha = \begin{pmatrix} 0 & \rho^{-1} \\ 0 & 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 0 & 0 \\ \rho & 0 \end{pmatrix}.$$

It follows that $s := x + x^*$ *is a* $P_0(\Gamma)$*-valued semicircular element, since* $\kappa_n(sb_1, sb_2, \cdots sb_{n-1}, s) = 0$ *unless* $n = 2$*, and* $\kappa_2(sb, s) = \eta(b)$ *where* η *is the (completely) positive self-map of* $\mathbb{C} \oplus \mathbb{C}$ *induced by the matrix*

$$\eta = \begin{pmatrix} 0 & \rho^{-1} \\ \rho & 0 \end{pmatrix}.$$

2.4 Narayana numbers

The Narayana numbers $N(n,k)$ are defined for all $n, k \in \mathbb{N}$ with $1 \leq k \leq n$ by

$$N(n,k) = |\{\pi \in NC(n) : |\pi| = k\}|.$$

The associated polynomials N_n are defined by

$$N_n(T) = \sum_{k=1}^{n} N(n,k) T^k.$$

From [NS06] we know that a random variable in a non-commutative probability space (A, τ) is said to be free Poisson with rate λ and jump size α if its free cumulants are given by $\kappa_n = \lambda \alpha^n$ for all $n \in \mathbb{N}$. An easy application of the moment-cumulant relations shows that an equivalent condition for a random variable to be free Poisson with rate λ and jump size α is that its moments are given by $\mu_n = \alpha^n N_n(\lambda)$ for all $n \in \mathbb{N}$.

We now illustrate an application of this characterization of a free Poisson variable in the situation of Example 2.2.1 (3). There, $x = \lambda(e)$ has a matrix decomposition involving $t \in \mathcal{L}(\mathcal{H}_w, \mathcal{H}_v)$ where t^*t was shown to have a free Poisson distribution. We will verify below by a cumulant computation that t^*t is free Poisson with rate ρ^2 and jump size

ρ^{-1} in the non-commutative probability space $p_w M(\Gamma, \mu) p_w$. Let us denote the trace on $p_w M(\Gamma, \mu) p_w$ by tr_w.

We begin by observing that $x^* x$ has a non-zero entry only in the w-corner and that this entry is $t^* t$. Thus the trace in $M(\Gamma, \mu)$ of $(x^* x)^n$ is $\mu^2(w) tr_w((t^* t)^n)$. We now compute $tr((x^* x)^n) = tr(([e]^*[e])^n)$.

We first apply the moment-cumulant relations and Proposition 2.3.1 to conclude that

$$E(([e]^*[e])^n) = \sum_{\pi \in NC(2n)} \kappa_\pi([e]^*, [e], \cdots, [e]^*, [e]).$$

While this sum ranges over all $\pi \in NC(2n)$, Proposition 2.3.1 enables us to conclude that unless π is a non-crossing pair partition, its contribution vanishes. Thus we have:

$$E(([e]^*[e])^n) = \sum_{\pi \in NC_2(2n)} \kappa_\pi([e]^*, [e], \cdots, [e]^*, [e]).$$

Now we use the well-known bijection between non-crossing pair partitions (or equivalently, Temperley-Lieb diagrams) on $2n$ points and all non-crossing partitions on n points. We denote this bijection as $\pi \in NC_2(2n) \leftrightarrow \widetilde{\pi} \in NC(n)$. This is illustrated by example in the following figure for $\pi = \{\{1,8\}, \{2,5\}, \{3,4\}, \{6,7\}, \{9,12\}, \{10,11\}\}$ and may be summarized by saying that the black regions of the Temperley-Lieb diagram for $\pi \in NC_2(2n)$ correspond to the classes of $\widetilde{\pi} \in NC(n)$. Note that in the figure the numbers above refer to the vertices while those below refer to the black segments.

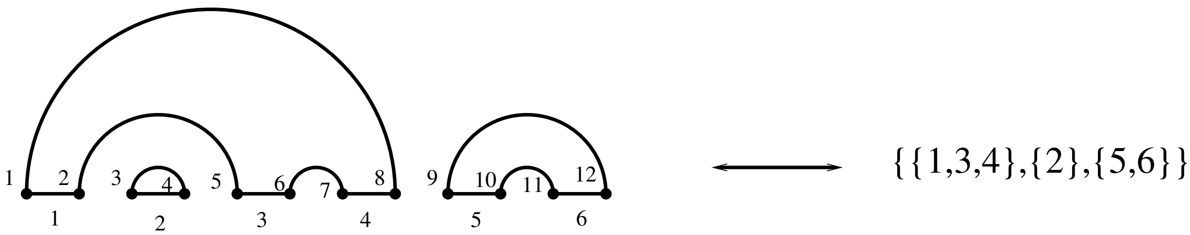

Figure 2.1: $\pi \in NC_2(12) \leftrightarrow \widetilde{\pi} \in NC(6)$

It follows from Proposition 2.3.1 that for any $\pi \in NC_2(2n)$, the term $\kappa_\pi([e]^*, [e], \cdots, [e]^*, [e])$

is a scalar multiple of p_w, where the scalar is given by a product of n terms each of which is $\rho = \frac{\mu(v)}{\mu(w)}$ or $\rho^{-1} = \frac{\mu(w)}{\mu(v)}$. Classes of π for which the smaller element is odd give ρ, while those for which the smaller element is even give ρ^{-1}. Thus $\kappa_\pi([e]^*, [e], \cdots, [e]^*, [e])$ evaluates to $\rho^{(|\pi|_{odd}-|\pi|_{even})}p_w = \rho^{(2|\pi|_{odd}-n)}p_w$, where, of course, $|\pi|_{odd}$ (respectively, $|\pi|_{even}$) denotes the number of classes of π whose smaller element is odd (respectively, even).

Our main combinatorial observation is contained in the following simple lemma.

Lemma 2.4.1. *For any $\pi \in NC_2(2n)$, $|\pi|_{odd} = |\widetilde{\pi}|$.*

Proof. We induce on n with the basis case $n = 1$ having only one π with $|\pi|_{odd} = |\widetilde{\pi}| = 1$. For larger n, consider a class of π of the form $\{i, i+1\}$, and remove it to get $\rho \in NC_2(2n-2)$. A moment's thought shows that if i is odd then $|\pi|_{odd} = |\rho|_{odd} + 1 = |\widetilde{\rho}| + 1 = |\widetilde{\pi}|$, while if i is even then $|\pi|_{odd} = |\rho|_{odd} = |\widetilde{\rho}| = |\widetilde{\pi}|$. $\square$

Thus:

$$\begin{aligned}
E(([e]^*[e])^n) &= \sum_{\pi \in NC_2(2n)} \rho^{(2|\pi|_{odd}-n)} p_w \\
&= \sum_{\widetilde{\pi} \in NC(n)} \rho^{(2|\widetilde{\pi}|-n)} p_w \\
&= \sum_{k=1}^{n} \sum_{\{\widetilde{\pi} \in NC(n):|\widetilde{\pi}|=k\}} \rho^{2k-n} p_w \\
&= \sum_{k=1}^{n} N(n,k) \rho^{2k-n} p_w
\end{aligned}$$

Hence $tr(([e]^*[e])^n) = \sum_{k=1}^n N(n,k)\rho^{2k-n}\mu^2(w)$ and thus $tr_w((t^*t)^n) = \sum_{k=1}^n N(n,k)\rho^{2k-n}$. Now the characterization of free Poisson elements in terms of their moments shows that t^*t is free Poisson with rate ρ^2 and jump size ρ^{-1}.

Remark 2.4.2. 1. Thus, for $t = \rho^{\frac{1}{2}}\ell + \rho^{-\frac{1}{2}}\ell^*$(where $\rho > 1$), we have shown that t^*t is a free Poisson element with rate ρ^2 and jump size ρ^{-1}. By scaling with an appropriate constant, we can similarly obtain such simple Fock-type models of free Poisson elements with arbitrary jump size and rate.

2. Similar scaling, and the fact that $e^{i\theta}\ell$ is unitarily equivalent to ℓ (by a unitary operator which fixes δ_0) show that, in fact, if $t = a\ell + b\ell^*$ for any $a, b \in \mathbb{C}$, then t^*t is a free Poisson element.

Chapter 3

Continuous minmax theorems

This chapter extends a well-known result by Ky Fan, regarding a 'minmax statement' about the sum of first k eigenvalues of an $n \times n$ matrix ($k, n \in \mathbb{N}, k \leq n$), for a suitable self-adjoint operator in a finite von-Neumann algebra[1]. We were motivated by [BV93] describing an extremal characterization of the distribution of a self-adjoint operator affiliated to a finite von Neumann algebra - suggesting a possible analogue of the classical Courant-Fischer-Weyl minmax theorem for a Hermitian matrix to the case of a self-adjoint operator in a finite von Neumann algebra.

3.1 The building blocks

In order to describe our result, we re-prove the well-known fact that any monotonic function with appropriate one-sided continuity is the distribution function of a random variable X - which can in fact be assumed to be defined on the familiar Lebesgue space $[0, 1)$ equipped with the Borel σ-algebra and Lebesgue measure. (We adopt the convention of [BV93] that the distribution function F_μ of a compactly supported probability measure[2] μ defined on the σ-algebra $\mathcal{B}_{\mathbb{R}}$ of Borel sets in $\mathbb{R}$, is *left-continuous*; thus

[1]The only von Neumann algebras considered here have separable pre-duals.

[2]Actually Bercovici and Voiculescu considered possibly unbounded self-adjoint operators affiliated to M, so as to also be able to handle probability measures which are not necessarily compactly supported, but we shall be content with the case of bounded $a \in M$, having a compactly supported probability measure as its distribution.

$F_\mu(x) = \mu((-\infty, x))$.

Proposition 3.1.1. *If $F : \mathbb{R} \to [0,1]$ is monotonically non-decreasing and left continuous and if there exists $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ such that*

$$F(t) = 0, \text{ for } t \leq \alpha \text{ and } F(t) = 1 \text{ for } t \geq \beta, \tag{3.1.1}$$

then there exists a monotonically non-decreasing right-continuous function $X : [0,1) \to \mathbb{R}$ such that F is the distribution function of X, i.e., $F(t) = m(\{s : X(s) < t\})$, where m denotes the Lebesgue measure on $[0,1)$. Moreover range$(X) \subset [\alpha, \beta]$.

Proof. Define $X : [0,1) \to \mathbb{R}$ by

$$\begin{aligned} X(s) &= \inf\{t : F(t) > s\} \qquad (3.1.2) \\ &= \inf\{t : t \in E_s\} \ , \end{aligned}$$

where $E_s = \{t \in \mathbb{R} : F(t) > s\}$ $\forall s \in [0,1)$. (The hypothesis 3.1.1 is needed to ensure that E_s is a non-empty bounded set for every $s \in [0,1)$ so that, indeed $X(s) \in \mathbb{R}$.)

First deduce from the monotonicity of F that

$$\begin{aligned} s_1 \leq s_2 &\Rightarrow E_{s_2} \subset E_{s_1} \\ &\Rightarrow X(s_1) \leq X(s_2) \end{aligned}$$

and hence X is indeed monotonically non-decreasing.

The definition of X and the fact that F is monotonically non-increasing and left continuous are easily seen to imply that $E_s = (X(s), \infty)$, and hence, it is seen that

$$\begin{aligned} X(s) < t &\Leftrightarrow \exists t_0 < t \text{ such that } F(t_0) > s \\ &\Leftrightarrow F(t) > s \text{ (since } F \text{ is left-continuous)} \qquad (3.1.3) \end{aligned}$$

Hence, if $t \in \mathbb{R}$

$$m(\{s \in [0,1) : X(s) < t\}) = m([0, F(t)) = F(t), \text{ proving the required statement.} \tag{3.1.4}$$

Moreover, if for any $s \in [0,1)$, $X(s) < \alpha$, then by definition of X, $\exists\, t' < \alpha$ such that $F(t') > s \geq 0$, a contradiction to the first hypothesis in 3.1.1. On the other hand, if for any $s \in [0,1)$, $X(s) > \beta$, then by 3.1.3, $s \geq F(\beta)$ =1 (by the second hypothesis in 3.1.1) - a contradiction. Hence indeed $range(X) \subset [\alpha, \beta]$. $\square$

This function X is known as quantile function[3] of the distribution F. If $F = F_\mu$ for a probability measure μ on $\mathbb{R}$, then X is denoted as X_μ. X can also be thought of as an element of $L^\infty(\mathbb{R}, \mu)$, where μ is a compactly supported probability measure on $\mathbb{R}$ such that $\mu = m \circ X^{-1}$ and $supp\ \mu \subset [\alpha, \beta]$. We will elaborate on this later in Proposition 3.2.1.

In the von-Neumann algebra setting, given a a bounded self-adjoint element a in a von Neumann algebra M and a (usually faithful normal) tracial state τ on M, define

$$\mu_a(E) := \tau(1_E(a))$$

(for the associated scalar spectral measure) to be the distribution of a. Since τ is positivity preserving, μ_a indeed turns out to be a probability measure on $\mathbb{R}$.

For simplicity, we write F_a, X_a for the distribution and the quantile function corresponding to a instead of F_{μ_a}, X_{μ_a}, and also avoid indicating the dependence on (M, τ), which is usually clear from the context. Note that only the abelian von Neumann subalgebra A generated by a and $\tau|_A$ are relevant for the definition of F_a and X_a.

[3]This function acts as the inverse of the distribution function at every point that is not an atom of the probability measure μ.

For M, a, τ as above, it was shown[4] in [BV93] that

$$1 - F_a(t) = \max\{\tau(p) : p \in \mathcal{P}(M), pap \geq ta\}. \tag{3.1.5}$$

Example 3.1.2. *Let $M = M_n(\mathbb{C})$ with τ as the uniform normalized tracial state on this M If $a = a^* \in M$ has distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, then $F_a(t) = \frac{1}{n}|\{j : \lambda_j < t\}| = \sum_{j=1}^{n} \frac{j}{n} 1_{(\lambda_j, \lambda_{j+1}]}$. We see that the distinct numbers less than 1 in the range of F_a are attained at the n distinct eigenvalues of a, and further that equation 3.1.5 for $t = \lambda_j$ says that $n - j + 1$ is the largest possible dimension of a subspace W of $\mathbb{C}^n$ such that $\langle a\xi, \xi\rangle \geq \lambda_j$ for every unit vector $\xi \in W$. In other words equation 3.1.5 suggests a possible extension of the classical Courant-Fischer minmax theorem for a self-adjoint operator in a von Neumann algebra, involving its distribution (see [BS12]).*

It is also true and not hard to see that the right side of equation 3.1.5 is indeed a maximum (and not just a supremum), and is in fact attained at a spectral projection of a; i.e., the two sides of equation 3.1.5 are also equal to $\max\{\tau(p) : p \in \mathcal{P}(A), pap \geq ta\}$, where $A = \{a\}''$.

3.2 The main result

We now proceed towards obtaining non-commutative counterparts of the classical Ky Fan's minmax theorem formulated for appropriate self-adjoint elements of appropriate finite von Neumann algebras.

Proposition 3.2.1. *Let $(\Omega, \mathcal{B}, P)$ be a probability measure, and suppose $Y : \Omega \to \mathbb{R}$ is a random variable. Let $\sigma(Y) = \{Y^{-1}(E) : E \in \mathcal{B}_{\mathbb{R}}\}$ and let $\mu = P \circ Y^{-1}$ be the distribution*

[4] Actually Bercovici and Voiculescu consider possibly unbounded self-adjoint operators affiliated to M, so as to also be able to handle probability measures which are not necessarily compactly supported, but we shall be content with the case of $a \in M$, i.e. when the distribution of a is indeed compactly supported.

of Y. Then, for any $s_0 \in F_\mu(\mathbb{R})$, we have

$$\begin{aligned}
&\inf\{\int_{\Omega_0} Y dP : \Omega_0 \in \sigma(Y), P(\Omega_0) = s_0\} \\
&= \inf\{\int_E f_0 d\mu : E \in \mathcal{B}_{\mathbb{R}}, \mu(E) = s_0\} \\
&= \inf\{\int_G X_\mu dm : G \in \sigma(X_\mu), m(G) = s_0\} \\
&= \int_0^{s_0} X_\mu dm, \qquad (3.2.1)
\end{aligned}$$

where $f_0 = id_{\mathbb{R}}$ and m denotes Lebesgue measure on [0,1).

Proof. The version of the change of variable theorem we need says that if $(\Omega_i, \mathcal{B}_i, P_i), i = 1, 2$ are probability spaces and $T : \Omega_1 \to \Omega_2$ is a measurable function such that $P_2 = P_1 \circ T^{-1}$, then

$$\int_{\Omega_2} g dP_2 = \int_{\Omega_1} g \circ T dP_1 \ , \qquad (3.2.2)$$

for every bounded measurable function $g : \Omega_2 \to \mathbb{R}$.

For every $\Omega_0 \in \sigma(Y)$, which is of the form $Y^{-1}(E)$ for some $E \in \mathcal{B}_{\mathbb{R}}$, set $G = X_\mu^{-1}(E)$. Notice, from equations 3.1.3 and 3.1.4 that

$$\begin{aligned}
m \circ X_\mu^{-1}(-\infty, t) &= \mu(\{s \in [0,1) : X_\mu(s) < t\}) \\
&= \mu(\{s \in [0,1) : s < F_\mu(t)\}) \\
&= F_\mu(t) \\
&= \mu(-\infty, t) \ ;
\end{aligned}$$

i.e. $m \circ X_\mu^{-1} = \mu = P \circ Y^{-1}$. Now, set $g = 1_E \cdot f_0$. Since $g \circ Y = 1_E \circ Y \cdot Y = 1_{Y^{-1}(E)} Y = 1_{\Omega_0} Y$, and (similarly) $g \circ X_\mu = 1_G X_\mu$, we see that the first two equalities in 3.2.1 are immediate consequences of two applications of the version stated in equation 3.2.2 above, of the 'change of variable' theorem.

As for the last, if $G \in \mathcal{B}_{[0,1)}$ with $m(G) \geq s_0$, then write $I = G \cap [0, s_0), J = [0, s_0) \setminus I, K = G \setminus I$ and note that $G = I \coprod K, [0, s_0) = I \coprod J$ (where $\coprod$ denotes disjoint union, and $K = G \setminus [0,1) \subset [s_0, 1)$. So we may deduce that

$$\begin{aligned}\int_G X_\mu dm - \int_0^{s_0} X_\mu dm &= \int_K X_\mu dm - \int_J X_\mu dm \\ &\geq X_\mu(s_0) m(K) - X_\mu(s_0) m(J) \\ &\geq 0 \ ,\end{aligned}$$

since $s_1 \in J, s_2 \in K \Rightarrow s_1 \leq s_0 \leq s_2 \Rightarrow X_\mu(s_1) \leq X_\mu(s_0) \leq X_\mu(s_2)$ (by the monotonicity of X_μ), and $m(K) \geq m(J)$. Thus, we see that

$$\inf\{\int_G X_\mu dm : G \in \sigma(X_\mu), m(G) \geq s_0\} \geq \int_0^{s_0} X_\mu dm \ ,$$

while conversely,

$$\inf\{\int_G X_\mu dm : G \in \mathcal{B}_\mathbb{R}, m(G) \geq s_0\} \leq \int_{[0,s_0)} X_\mu dm = \int_0^{s_0} X_\mu dm \ ,$$

thereby establishing the last equality in 3.2.1. □

Theorem 3.2.2. *Let a be a self-adjoint element of a von Neumann algebra M equipped with a faithful normal tracial state τ. Let A be the von-Neumann algebra generated by a. Then, for all $s \in F_a(\mathbb{R})$,*

$$\begin{aligned}&\inf\{\tau(ap) : p \in \mathcal{P}(M), \tau(p) \geq s\} \\ &= \inf\{\tau(ap) : p \in \mathcal{P}(A), \tau(p) \geq s\} \\ &= \int_0^s X_a dm \end{aligned} \tag{3.2.3}$$

(hence the infima are attained and are actually minima),

if either

1. *('continuous case') μ_a has no atoms, or*

2. *('finite case') $M = M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ and a has spectrum $\{\lambda_1 < \cdots < \lambda_n\}$.*

Proof. We begin by noting that in both the cases, the last equality in 3.2.3 is an immediate consequence of Proposition 3.2.1. Moreover the set $\{\tau(ap) : p \in \mathcal{P}(A), \tau(p) \geq s\}$ being contained in $\{\tau(ap) : p \in \mathcal{P}(M), \tau(p) \geq s\}$, it is clear that

$$\inf\{\tau(ap) : p \in \mathcal{P}(A), \tau(p) \geq s\} \geq \inf\{\tau(ap) : p \in \mathcal{P}(M), \tau(p) \geq s\}.$$

So we just need to prove that

$$\inf\{\tau(ap) : p \in \mathcal{P}(A), \tau(p) \geq s\} \leq \inf\{\tau(ap) : p \in \mathcal{P}(M), \tau(p) \geq s\}. \tag{3.2.4}$$

1. (the continuous case) Due to the assumption of μ_a being compactly supported and having no atoms, it is clear that F_a is continuous and that $F_a(\mathbb{R}) = [0, 1]$.

   Under the standing assumption of separability of pre-duals of our von Neumann algebras, the hypothesis of this case implies the existence of a probability space $(\Omega, \mathcal{B}, P)$ and a map $\pi : A \to L^\infty(\Omega, \mathcal{B}, P)$ such that $\int \pi(x) dP = \tau(x)$ $\forall x \in A$, $Y := \pi(a)$ is a random variable and π is an isomorphism onto $L^\infty(\Omega, \sigma(Y), P)$.

   We shall establish the first equality of 3.2.3 by showing that if $p_0 \in \mathcal{P}(M)$ and $\tau(p_0) = s$, then $\tau(ap_0) \geq \min\{\tau(ap) : p \in \mathcal{P}(A), \tau(p) \geq s\}$. For this, first note that since τ is a faithful normal tracial state on M, there exists a τ-preserving conditional expectation $\mathcal{E} : M \to A$. Then

$$\tau(ap_0) = \tau(a\mathcal{E}(p_0)) = \int YZdP,$$

   where $Z = \pi(\mathcal{E}(p_0))$. Since $\mathcal{E}$ is linear and positive, it is clear that $0 \leq Z \leq 1$, P-a.e. So it is enough to prove that

$$\inf\{\int_\Omega YZdP : 0 \leq Z \leq 1, \int ZdP \geq s\} = \inf\{\int_E YdP : E \in \mathcal{B}, P(E) \geq s\}.$$

   For this, it is enough, thanks to the Krein-Milman theorem (see, e.g. [KM40]), to note that $K = \{Z \in L^\infty(\Omega, \mathcal{B}, P) : 0 \leq Z \leq 1, \int Z dP \geq s\}$ is a convex set which is compact in the weak* topology inherited from $L^1(\Omega, \mathcal{B}, P)$, and prove that the set $\partial_e(K)$ of its extreme points is $\{1_E : P(E) \geq s\}$.

For this, suppose $Z \in K$ is not a projection, Clearly then $P(\{Z \in (0,1)\}) > 0$, so there exists $\epsilon > 0$ such that $P(\{\epsilon < Z < 1 - \epsilon\}) > 0$. Since μ_a, and hence P has no atoms, we may find disjoint Borel subsets $E_1, E_2 \subset \{Z \in (\epsilon, 1-\epsilon)\}$ such that $P(E_1) = P(E_2) > 0$. If we now set $Z_1 = Z + \epsilon(1_{E_1} - 1_{E_2})$ and $Z_2 = Z + \epsilon(1_{E_2} - 1_{E_1})$, it is not hard to see that $Z_1, Z_2 \in K, Z_1 \neq Z_2$ and $Z = \frac{1}{2}(Z_1 + Z_2)$ showing that $Z \notin \partial_e(K)$, thereby proving 3.2.4.

2. (the finite case) Since a has distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, A is a maximal abelian self-adjoint subalgebra of $M_n(\mathbb{C})$. Recall that in this case, $F_a(t) = \frac{1}{n}|\{j : \lambda_j < t\}| = \sum_{j=1}^n \frac{j}{n} 1_{(\lambda_j, \lambda_{j+1}]}$. It then follows that $F_a(\mathbb{R}) = \{\frac{j}{n} : 0 \leq j \leq n\}$ and that $X_a = \sum_{j=1}^n \lambda_j 1_{[\frac{j-1}{n}, \frac{j}{n})}$ and 3.2.3 is then (after multiplying by n) precisely the statement of Ky Fan's theorem (in the case of self-adjoint matrices with distinct eigenvalues):

For $1 \leq j \leq n$,

$$\begin{aligned}
&\inf\{\tau(ap) : p \in \mathcal{P}(M_n(\mathbb{C})), rank(p) \geq j\} \\
&= \inf\{\tau(ap) : p \in \mathcal{P}(A), rank(p) \geq j\} = \frac{1}{n}\sum_{i=1}^{j} \lambda_i = \int_0^{\frac{j}{n}} X_a(s)ds.
\end{aligned}$$

It suffices to prove the following:

$$\inf\{\tau(ap) : p \in \mathcal{P}(A), rank(p) \geq j\} \leq \inf\{\tau(ap) : p \in \mathcal{P}(M_n(\mathbb{C})), rank(p) \geq j\}.$$

For this, begin by deducing from the compactness of $\mathcal{P}(M_n(\mathbb{C}))$ that there exists a $p_0 \in \mathcal{P}(M_n(\mathbb{C}))$ with $rank(p_0) \geq j$ such that $\tau(ap_0) \leq \tau(ap)\ \forall p \in \mathcal{P}(M_n(\mathbb{C}))$ with $rank(p) \geq j$. We assert that any such minimizing p_0 must belong to A. The assumption that A is a masa means we only need to prove that $p_0 a = ap_0$. For this pick any self-adjoint

$x \in M_n(\mathbb{C})$, and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = \tau(e^{itx}p_0e^{-itx}a)$. Since clearly $e^{itx}p_0e^{-itx} \in \mathcal{P}(M)$ and $rank(e^{itx}p_0e^{-itx}) = rank(p_0) \geq j$, for all $t \in \mathbb{R}$, we find that $f(t) \geq f(0)\ \forall t$. As f is clearly differentiable, we may conclude that $f'(0) = 0$. Hence,

$$0 = \tau(ixp_0a - ip_0xa) = i(\tau(xp_0a) - \tau(p_0xa)) = i(\tau(xp_0a) - \tau(xap_0)),$$

so that $\tau(x(p_0a - ap_0)) = 0$ for all $x = x^* \in M$, and indeed $ap_0 = p_0a$ as desired.

□

Case 1 of Theorem 3.2.2 is our continuous formulation of Ky Fan's result while Case 2 only captures the classical Ky Fan's theorem for the case of distinct eigenvalues. However the general case of non-distinct eigenvalues can also be deduced from our proof, as we show in the following corollary:

Corollary 3.2.3. *Let a be a Hermitian matrix in $M_n(\mathbb{C})$ with spectrum $\{\lambda_1 \leq \cdots \leq \lambda_n\}$, where not all λ_js are necessarily distinct. Then for all $j \in \{1, \cdots, n\}$,*

$$\min\{\tau(ap) : p \in \mathcal{P}(M_n(\mathbb{C})),\ rank(p) \geq j\} = \frac{1}{n}\sum_{i=1}^{j}\lambda_i.$$

Proof. We may assume that a is diagonal. Let A_1 be the set of all diagonal matrices, so that $A \subsetneq A_1$. Pick $a^{(m)} = diag(\lambda_1^{(m)}, \lambda_2^{(m)}, \cdots, \lambda_n^{(m)}) \in A_1$ such that $\lambda_j^{(m)}$s are all distinct and $\lim_{m\to\infty} \lambda_j^{(m)} = \lambda_j\ \forall 1 \leq j \leq n$. Then the already established case of Theorem 3.2.2 in the case of distinct eigenvalues shows that for all $p \in \mathcal{P}(M_n(\mathbb{C}))$ with $rank(p) \geq j$,

$$\begin{aligned}
\tau(ap) &= \lim_{m\to\infty} \tau(a^{(m)}p) \\
&\geq \lim_{m\to\infty} \frac{1}{n}\sum_{i=1}^{j}\lambda_i^{(m)} \\
&= \frac{1}{n}\sum_{i=1}^{j}\lambda_i\ .
\end{aligned}$$

The above, along with the fact that $\tau(ap_j) = \frac{1}{n}\sum_{i=1}^{j} \lambda_i$, where p_j is the obvious diagonal projection, completes our proof of Ky Fan's theorem for Hermitian matrices in full generality. $\square$

Remark 3.2.4. *It is not difficult to see that equation 3.2.3 holds even if we replace the inequality $\tau(p) \geq s$ with equality.*

Remark 3.2.5. *Notice that the hypothesis and hence the conclusion, of the 'continuous case' of Theorem 3.2.2 are satisfied by any self-adjoint generator of a masa in a II_1 factor.*

Chapter 4

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