Unique Factorization Of Tensor Products For Kac-Moody Algebras

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DECLARATION

I, hereby declare that the investigtion presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

R. VENKATESH

Abstract

In the first part, we address a fundamental question, unique factorization of tensor products, that arises in representation theory. We consider integrable, category $\mathcal{O}$ modules of indecomposable symmetrizable Kac-Moody algebras. We prove that unique factorization of tensor products of irreducible modules holds in this category, upto twisting by one dimensional modules. This generalizes a fundamental theorem of Rajan for finite dimensional simple Lie algebras over $\mathbb{C}$. Our proof is new even for the finite dimensional case, and uses an interplay of representation theory and combinatorics to analyze the Kac-Weyl character formula.

In the second part, we get a new interpretation of the chromatic polynomials using Kac-Moody theory and derive some of its properties using this new interpretation.

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"Nanri marapathu nanranru"

-Thiruvalluvar.

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Contents

Chapter 1

Introduction

1.1 Unique Factorization Of Tensor Products For Kac-Moody Algebras

In the first part of this thesis, we address a fundamental question, unique factorization of tensor products, that arises in representation theory.

More precisely we prove unique factorization of tensor products in a natural category of representations of Kac-Moody algebras. This is a generalization of Rajan's theorem, [5] where he proved the unique factorization of tensor products for finite dimensional simple Lie algebras. We now give a brief description of our results.

Our base field will be the complex numbers $\mathbb{C}$ throughout. In [5], Rajan proved the following fundamental theorem:

Theorem 1.1.1. *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, and $\mathcal{C}$ be the category of finite dimensional $\mathfrak{g}$-modules. Let n, m be positive integers and $V_1, V_2, \cdots, V_n$ and $W_1, W_2, \cdots, W_m$ be non-trivial irreducible $\mathfrak{g}$-modules in $\mathcal{C}$ such that*

$$V_1 \otimes V_2 \otimes \cdots \otimes V_n \cong W_1 \otimes W_2 \otimes \cdots \otimes W_m.$$

Then $n = m$ and there is a permutation σ of $\{1, 2, \cdots, n\}$ such that $V_i \cong W_{\sigma(i)}$ for $1 \leq i \leq n$.

The following is an equivalent formulation of theorem 1.1.1 in which $n = m$, but with trivial modules allowed.

Theorem 1.1.2. *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, and $\mathcal{C}$ be the category of finite dimensional $\mathfrak{g}$-modules. Let n be a positive integer, and suppose $V_1, V_2, \cdots, V_n$ and $W_1, W_2, \cdots, W_n$ are irreducible $\mathfrak{g}$-modules in $\mathcal{C}$ such that*

$$V_1 \otimes V_2 \otimes \cdots \otimes V_n \cong W_1 \otimes W_2 \otimes \cdots \otimes W_n.$$

Then there is a permutation σ of $\{1, 2, \cdots, n\}$ such that $V_i \cong W_{\sigma(i)}$ for $1 \leq i \leq n$.

It is natural to ask what is the analogous result for Kac-Moody algebras. We will be concerned with this question in the first part of this thesis.

In chapter 3, we give an alternate, elementary proof of theorem 1.1.2, and obtain a generalization to symmetrizable Kac-Moody algebras.

When $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra, a natural category of representations is $\mathcal{O}^{\text{int}}$, whose objects are integrable $\mathfrak{g}$-modules in category $\mathcal{O}$. When the generalized Cartan matrix of $\mathfrak{g}$ is singular (for example, when $\mathfrak{g}$ is affine), we have $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \neq 0$; in other words, there are non-trivial one dimensional $\mathfrak{g}$-modules in $\mathcal{O}^{\text{int}}$. Thus, unique factorization of tensor products fails in general for $(\mathfrak{g}, \mathcal{O}^{\text{int}})$. We show that this is essentially the only obstruction, i.e., uniqueness still holds up to twisting by one-dimensional representations. The following is the statement of our main theorem:

Theorem 1.1.3. *([1]) Let $\mathfrak{g}$ be an indecomposable symmetrizable Kac-Moody algebra. Let n be a positive integer and suppose $V_1, V_2, \cdots, V_n$ and $W_1, W_2, \cdots, W_n$ are irreducible $\mathfrak{g}$-modules in category $\mathcal{O}^{\text{int}}$ such that*

$$V_1 \otimes \cdots \otimes V_n \cong W_1 \otimes \cdots \otimes W_n. \tag{1.1.1}$$

Then there is a permutation σ of the set $\{1, ..., n\}$, and one-dimensional $\mathfrak{g}$-modules Z_i such that $V_i \otimes Z_i \cong W_{\sigma(i)}, 1 \leq i \leq n$.

If $\mathfrak{g}$ is finite dimensional, then (a) $\mathcal{O}^{\text{int}} = \mathcal{C}$, (b) *indecomposable* is the same as *simple* and (c) the only one dimensional $\mathfrak{g}$-module is the trivial one. Thus, theorem 1.1.3 is indeed a generalization of theorem 1.1.2.

Theorem 1.1.3 can be interpreted at the level of characters. Characters play a very important role in the representation theory of Kac-Moody algebras. Any question about representations of Kac-Moody algebras can be interpreted at the level of characters. We prove theorem 1.1.3 by proving an analogous result at the character level.

1.2 New Interpretation Of Chromatic Polynomials Using Kac-Moody Theory

On the other hand, representations also give more information about combinatorial objects. Our analysis of the characters in the proof of theorem 1.1.3 leads to an unexpected connection with chromatic polynomials of graphs.

In chapter 4, we obtain a new interpretation of chromatic polynomials using the representation theory of Kac-Moody algebras. We briefly explain our results here.

Let $\mathcal{G}$ be a connected simple graph with vertex set Π and $|\Pi| = \ell$. There is a natural way of associating a generalized Cartan matix (and hence symmetrizable Kac-Moody algebra) with $\mathcal{G}$. Let $M(\mathcal{G})$ be the symmetric generalized Cartan matrix associated with $\mathcal{G}$, defined as follows:

$$M(\mathcal{G})_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and there is an edge between } i \text{ and } j \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{g} = \mathfrak{g}(\mathcal{G})$ be the symmetrizable Kac-Moody algebra associated with $M(\mathcal{G})$ (or with $\mathcal{G}$). Let us identify the vertex set Π with the set of simple roots of $\mathfrak{g}$. Let Δ be the set of roots, and Δ_+ the set of positive roots of $\mathfrak{g}$. Let W be the Weyl group of $\mathfrak{g}$, generated by the simple reflections $\{s_\alpha : \alpha \in \Pi\}$, and let ε be its sign character. Let $(-,-)$ be the standard nondegenerate, W-invariant symmetric bilinear form on $\mathfrak{h}^*$.

Note that $\mathcal{G}$ is the graph underlying the Dynkin diagram of $\mathfrak{g}$, i.e., $\mathcal{G}$ has vertex set Π, with an edge between two vertices α and β iff $(\alpha, \beta) < 0$.

Let us recall the definition of the chromatic polynomial of $\mathcal{G}$.

Definition 1.2.1. *Let $q \in \mathbb{N}$. A mapping $f : \Pi \to \{1, \cdots, q\}$ is called a q-colouring of $\mathcal{G}$ if $f(\alpha) \neq f(\beta)$ whenever the vertices α and β are adjacent in $\mathcal{G}$. Two q-colourings f and g of $\mathcal{G}$ are regarded as distinct if $f(\alpha) \neq g(\alpha)$ for some vertex α of $\mathcal{G}$.*

The number of distinct q-colourings of $\mathcal{G}$ is denoted by $P_{\mathcal{G}}(q)$. By convention $P_{\mathcal{G}}(0) = 0$. The following proposition is well known.

Proposition 1.2.2. *For $q \in \mathbb{N}$, we have $P_{\mathcal{G}}(q) = \sum\limits_{k \geq 1} c_k(\mathcal{G}) \binom{q}{k}$, where $c_k(\mathcal{G})$ was defined in 3.3.1. It is thus a polynomial in q, known as the chromatic polynomial of $\mathcal{G}$. The coefficients of $P_{\mathcal{G}}(q)$ are alternating in sign.*

We define $\widetilde{P}_{\mathcal{G}}(q) := (-1)^\ell P_{\mathcal{G}}(-q)$, it is easy to see that $\widetilde{P}_{\mathcal{G}}(q) \in \mathbb{N}[q]$.

We prove the following theorem in this thesis.

Theorem 1.2.3. *$\widetilde{P}_{\mathcal{G}}(q) = K(\beta; q)$, where $K(\beta; q)$ is the q-Kostant partition function of $\mathfrak{g}$ and $\beta = \sum\limits_{\alpha \in \Pi} \alpha$.*

Equivalently we have,

Theorem 1.2.4.

$$P_{\mathcal{G}}(q) = (-1)^\ell \sum_{k=1}^{\ell} \frac{(-1)^k}{k!} \left\{ \sum_{\overline{\beta} \in S_k(\mathcal{G})} m_{\beta_1} \dots m_{\beta_k} \right\} q^k,$$

where m_α denotes the multiplicity of α, i.e., the dimension of the root space $\mathfrak{g}_\alpha$ of $\mathfrak{g}$ and $S_k(\mathcal{G}) := \left\{ \overline{\beta} = (\beta_1, \dots, \beta_k) : \beta_i \in \Delta_+ \text{ and } \sum\limits_{i=1}^{k} \beta_i = \sum\limits_{\alpha \in \Pi} \alpha \right\}$.

We prove theorem 1.2.4 by comparing the coefficient of the monomial $\prod_{\alpha\in\Pi} e^{-\alpha}$ in $\left(\sum_{w\in W}\varepsilon(w)e^{w\rho-\rho}\right)^q$ in two different ways, where ρ is the Weyl vector, satisfying $\frac{2(\rho,\alpha)}{(\alpha,\alpha)} = 1$ for all $\alpha \in \Pi$.

This new interpretation throws light on a classical expression for $P_{\mathcal{G}}(q)$ by Birkhoff. To state this, we recall the definition and properties of the bond lattice of $\mathcal{G}$.

We begin with the definition of connected partition of $\mathcal{G}$.

Definition 1.2.5. *Let k be a positive integer. A connected k-partition of the graph $\mathcal{G}$ is an unordered k-tuple $\{S_1,...,S_k\}$ such that (a) the S_i's are non-empty pairwise disjoint subsets of the vertex set Π of $\mathcal{G}$, (b) $\bigcup_{i=1}^{k} S_i = \Pi$, and (c) each S_i is a connected subset of Π, i.e. the subgraph spanned by S_i in $\mathcal{G}$ is connected .*

Let $\mathcal{G}$ be a connected graph with ℓ vertices. Consider the poset $\mathcal{L}_{\mathcal{G}}$ whose elements are connected k-partitions of Π for all k, partially ordered by refinement. $\mathcal{L}_{\mathcal{G}}$ is known as the bond lattice of $\mathcal{G}$. The maximum element is $\hat{1} := \{\Pi\}$ and the minimum element is $\hat{0} := \{\{\alpha_1\},\cdots,\{\alpha_\ell\}\}$, where ℓ is the cardinality of Π. Now we collect some facts about the bond lattice of $\mathcal{G}$ from [2].

(1) An atom, by definition, is an element of $\mathcal{L}_{\mathcal{G}}$ that covers $\hat{0}$. Atoms of $\mathcal{L}_{\mathcal{G}}$ are bijective correspondence with the set of edges of $\mathcal{G}$. The bijection is given as follows: $e \mapsto \{\{\alpha_k\}_{k\neq i,j}, \{\alpha_i,\alpha_j\}\}$, where α_i,α_j are end points of e.

(2) $\mathcal{L}_{\mathcal{G}}$ is a geometric lattice.

(3) $\mathcal{L}_{\mathcal{G}}$ has a rank function: $rk(\pi) := \ell - |\pi|$, where $|\pi|$ is the number of parts in π, i.e. if π is a connected k-partition then $|\pi| = k$.

The Möbius function of $\mathcal{L}_{\mathcal{G}}$, $\mu : \mathcal{L}_{\mathcal{G}} \to \mathbb{Z}$, is defined recursively by :

$$\mu(\pi) = \begin{cases} 1 & \text{if } \pi = \hat{0} \\ -\sum\limits_{\pi'<\pi} \mu(\pi') & \text{if } \pi > \hat{0}, \end{cases}$$

Note that μ is the unique $\mathbb{Z}$-valued function on $\mathcal{L}_{\mathcal{G}}$ such that $\sum_{\pi'\leq\pi} \mu(\pi') = \delta_{\hat{0}\pi}$ (Kronecker delta).

(4) The Möbius function of $\mathcal{L}_{\mathcal{G}}$ strictly alternates in sign: $(-1)^{\ell-|\pi|}\mu(\pi) > 0$, for all $\pi \in \mathcal{L}_{\mathcal{G}}$.

Theorem 1.2.6 (Birkhoff, Whitney, Rota)**.** *For a connected graph $\mathcal{G}$, $P_{\mathcal{G}}(q) = \sum_{\pi\in\mathcal{L}_{\mathcal{G}}} \mu(\pi)q^{|\pi|}$.*

Comparing theorem 1.2.6 with theorem 1.2.4 one is led to expect that the absolute value of the mysterious integers $\mu(\pi)$ occurs as a product of root multiplicities of $\mathfrak{g}(\mathcal{G})$. More precisely, we prove the following result in this thesis.

Theorem 1.2.7. *Let $\mathcal{G}$ be a connected graph with ℓ vertices. Let $\pi = \{S_1,\cdots,S_k\} \in \mathcal{L}_{\mathcal{G}}$ and $m_\pi := m_{\beta_{S_1}}\cdots m_{\beta_{S_k}}$, where $\beta_S := \sum_{\alpha\in S}\alpha$ for all $S \subseteq \Pi$. Then $\mu(\pi) = (-1)^{\ell-k}\, m_\pi$.*

It is known that, $m_{\beta_S} \neq 0$ if and only if the subgraph induced by $S \subseteq \Pi$ is connected (see 4.2.6 and 4.2.7). So, to understand the coefficient of the chromatic polynomials, it is enough to understand m_β's.

Towards this direction, we get a nice recursion formula for the m_β, by applying the Peterson recursion formula [4] to $\beta = \sum\limits_{\alpha \in \Pi} \alpha$. More precisely, we show

$$m_\beta = \sum_{\substack{(\beta', \beta'') \in Q_+ \times Q_+ \\ \beta' + \beta'' = \beta}} \frac{E(\beta', \beta'')}{E(\beta)} m_{\beta'} m_{\beta''}, \tag{1.2.2}$$

where $E(\beta)$ = the number of edges in $\mathcal{G}$ and $E(\beta', \beta'')$ = the number of edges between $graph(\beta')$ and $graph(\beta'')$, where $graph(\beta')$ is the subgraph induced by the support of β' (see 4.2.10).

We also show that for $k \geq 1$,

$$\text{the coefficient of } q^k \text{ in } \widetilde{P}_{\mathcal{G}}(q) = \sum_{p \in P(\mathcal{L}_{\mathcal{G}})} \omega(p), \tag{1.2.3}$$

where (if $\pi' \to \pi''$ denotes the covering relation in $\mathcal{L}_{\mathcal{G}}$) the sum runs over all paths p in $\mathcal{L}_{\mathcal{G}}$ such that $p : \pi \to \pi_1 \to \cdots \to \pi_{\ell-k} \to \hat{0}$, and $\omega(p)$ are certain multiplicative edge weights associated with p and $|\pi| = k$.

Chapter 2

Kac-Moody algebras

V. G. Kac and R. V. Moody independently introduced the Kac-Moody algebras in 1967. They are generalizations of the well known finite dimensional simple Lie algebras. But the theory of Kac-Moody algebras is much broader than the theory of finite dimensional simple Lie algebras. It includes many interesting infinite dimensional examples as well. In recent decades, Kac-Moody algebras have found applications in many areas of mathematics, including group theory, combinatorics, modular forms, differential equations and invariant theory. It has also proved important in mathematical physics. Accordingly, their representations are of great interest as well.

We work over the complex field throughout, although any algebraically closed field of characteristic zero would do equally well. In this chapter, we recall certain well known definitions and results which will be used in the thesis.

2.1 Generalized Cartan Matrices(GCM)

Definition 2.1.1. *An $n \times n$ complex matrix $A = (a_{ij})$ is called a generalized Cartan matrix if it satisfies the following conditions:*

(1) $a_{ii} = 2$ for $i = 1, \cdots, n$

(2) $a_{ij} \in \mathbb{Z}_{\leq 0}$ if $i \neq j$

(3) $a_{ij} = 0$ implies $a_{ji} = 0$.

2.1.1 Realizations of a matrix

A realization of an $n \times n$ complex matrix A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where:

1. $\mathfrak{h}$ is a finite dimensional vector space over $\mathbb{C}$

2. $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a linearly independent subset of $\mathfrak{h}^*$

3. $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \cdots, \alpha_n^\vee\}$ is a linearly independent subset of $\mathfrak{h}$

4. $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ for all i, j

5. $\dim \mathfrak{h} = 2n - \mathrm{rank}(A)$.

Two realizations $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ and $(\mathfrak{h}_2, \Pi_2, \Pi_2^\vee)$ are said to be *isomorphic* if there exists a vector space isomorphism $\phi : \mathfrak{h}_1 \to \mathfrak{h}_2$ such that $\phi(\Pi_1^\vee) = \Pi_2^\vee$ and $\phi^*(\Pi_2) = \Pi_1$. It is well known that realization exists and is unique upto isomorphism.

2.1.2 Auxiliary Lie algebras

Definition 2.1.2. *Let $A = (a_{ij})$ be an $n \times n$ matrix over $\mathbb{C}$, and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A. Define an auxiliary Lie algebra $\tilde{\mathfrak{g}}(A)$ with generators e_i, f_i $(i = 1, \cdots, n)$ and $\mathfrak{h}$, and the following defining relations:*

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee \quad \textit{for all } i, j \tag{2.1.1}$$

$$[h, h'] = 0 \quad \textit{for all } h, h' \in \mathfrak{h} \tag{2.1.2}$$

$$[h, e_i] = \langle \alpha_i, h \rangle e_i \quad \forall\, i, j,\ h \in \mathfrak{h} \tag{2.1.3}$$

$$[h, f_i] = -\langle \alpha_i, h \rangle f_i \quad \forall\, i, j,\ h \in \mathfrak{h} \tag{2.1.4}$$

Denote the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by $e_1, \cdots, e_n$ (resp. $f_1, \cdots, f_n$) by $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$).
The following theorem can be found in [4]:

Theorem 2.1.3. *1.* $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-$ (as direct sum of vector spaces),

2. $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$) *is freely generated by* $e_1, \cdots, e_n$(resp. $f_1, \cdots, f_n$),

3. Among the ideals of $\tilde{\mathfrak{g}}(A)$ intersecting $\mathfrak{h}$ trivially, there exists a unique maximal ideal $\boldsymbol{\tau}$. Furthermore,

$$\boldsymbol{\tau} = (\boldsymbol{\tau} \cap \tilde{\mathfrak{n}}_-) \oplus (\boldsymbol{\tau} \cap \tilde{\mathfrak{n}}_+).$$

2.1.3 The Kac-Moody Lie algebra $\mathfrak{g}(A)$

Definition 2.1.4. *For a given $n \times n$ generalized Cartan matrix A, let $\tilde{\mathfrak{g}}(A)$ be the auxiliary Lie algebra as defined above. By theorem 2.1.3 the natural map $\mathfrak{h} \to \tilde{\mathfrak{g}}(A)$ is an imbedding. Let $\boldsymbol{\tau}$ be the maximal ideal of $\tilde{\mathfrak{g}}(A)$, with $\boldsymbol{\tau} \cap \mathfrak{h} = 0$. Define,*

$$\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\boldsymbol{\tau}$$

The Lie algebra $\mathfrak{g}(A)$ is called the Kac-Moody algebra associated with GCM A, and n is called the rank *of $\mathfrak{g}(A)$.*

The matrix A is said to be *symmetrizable* if there exists an invertible diagonal matrix $D = \mathrm{diag}(d_1, \cdots, d_n)$ such that DA is symmetric.

Definition 2.1.5. *Let $A = (a_{ij})$ be a symmetrizable generalized Cartan matrix. Then the Kac-Moody algebra associated with the matrix A is called the symmetrizable Kac-Moody algebra associated with A.*

We drop A in $\mathfrak{g}(A)$, if the underlying matrix A is understood.

2.1.4 Properties of Kac-Moody algebras

We continue to denote e_i, f_i, h for the images of e_i, f_i, h in $\mathfrak{g}$. This should not lead to confusion as we will subsequently be concentrating on $\mathfrak{g}$ rather than on $\tilde{\mathfrak{g}}$. The e_i and f_i are called Chevalley generators of $\mathfrak{g}$. Let $\mathfrak{h}'$ be the span of $\Pi^\vee$ and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ be the derived subalgebra of $\mathfrak{g}$.

Theorem 2.1.6. *Let $\mathfrak{g}$ be a Kac-Moody algebra. Then,*

1. $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, *where* $\mathfrak{n}_+$, $\mathfrak{n}_-$ *are the Lie subalgebras generated by* e_i *and* f_i $(i = 1, \cdots, n)$ *respectively,*

2. $\mathfrak{g} = \mathfrak{g}' + \mathfrak{h}$ *and* $\mathfrak{g}' \cap \mathfrak{h} = \mathfrak{h}'$,

3. $\mathfrak{h}$ *acts diagonalizably on* $\mathfrak{g}$ *i.e.,*
$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$
where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x, \quad \forall h \in \mathfrak{h}\}$.

The subspace $\mathfrak{h}$ is called the Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ the root space of α. An element $\alpha \in \mathfrak{h}^*$ is said to be a root of $\mathfrak{g}$ if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. The integer $\dim \mathfrak{g}_\alpha$ is called *multiplicity* of α, denoted by m_α. Let Δ denote the set of all roots of $\mathfrak{g}$. Elements of Π are called the simple roots of $\mathfrak{g}$, and elements of $\Pi^\vee$ are called the simple co-roots of $\mathfrak{g}$.

Let
$$P := \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \ \ \forall\, \alpha \in \Pi\},\ Q := \sum_{\alpha \in \Pi} \mathbb{Z}\alpha,$$
$$P^+ := \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}, \ \ \forall\, \alpha \in \Pi\},\ Q^+ := \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0}\alpha$$
be the weight lattice, the root lattice, the sets of dominant weights and non-negative integer linear combinations of simple roots respectively. Then $\Delta = \Delta_+ \cup \Delta_-$(a disjoint union), where $\Delta_+ := \Delta \cap Q_+, \Delta_- := -\Delta_+$ are the sets of positive roots and negative roots respectively. Given $\lambda \in \mathfrak{h}^*$, define $\overline{\lambda} := \lambda\,|_{\mathfrak{h}'}$.

2.1.5 Weyl group of Kac-Moody algebras

Let A be a $n \times n$ generalized Cartan matrix and let $\mathfrak{g}$ be the associated Kac-Moody algebra. For each $\alpha \in \Pi$, define the fundamental reflection s_α by

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$

for $\lambda \in \mathfrak{h}^*$.

The subgroup W of $GL(\mathfrak{h}^*)$ generated by $\{s_\alpha : \alpha \in \Pi\}$ is called the Weyl group of $\mathfrak{g}$. The Weyl group plays an important role in the representation theory of Kac-Moody algebras.

For a symmetrizable Kac-Moody algebra $\mathfrak{g}$ there exists a nondegenerate symmetric bilinear form $(-,-)$ on $\mathfrak{g}$ such that:

(1) $([x,y],z) = (x,[y,z])$ for all $x, y, z \in \mathfrak{g}$, and

(2) the restriction of $(-,-)$ to $\mathfrak{h}^*$ is nondegenerate and W-invariant.

2.2 Representation theory of Kac-Moody algebras

A $\mathfrak{g}$-module V is called $\mathfrak{h}$-diagonalizable if it admits a weight space decomposition $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where $V_\lambda = \{v \in V : h(v) = \lambda(h)v, \ \forall\, h \in \mathfrak{h}\}$. A nonzero vector of V_λ is called a weight vector of weight λ. Let $P(V) := \{\lambda \in \mathfrak{h}^* : V_\lambda \neq 0\}$ denote the set of all weights of V. For $\lambda \in \mathfrak{h}^*$, denote $D(\lambda) := \{\mu \in \mathfrak{h}^* : \mu \leq \lambda\}$.

We shall not consider arbitrary representations, but restrict our attention to those in the category $\mathcal{O}$, which is introduced by Kac for Kac-Moody algebras. Objects of $\mathcal{O}$ are defined as follows:

Definition 2.2.1. A $\mathfrak{g}$-*module* V *is said to be in category* $\mathcal{O}$ *if*

1. *It is* $\mathfrak{h}$-*diagonalizable and with finite dimensional weight spaces, and*

2. *There exists a finite number of elements* $\lambda_1, \cdots, \lambda_m \in \mathfrak{h}^*$ *such that* $P(V) \subseteq \cup_{i=1}^m D(\lambda_i)$.

The morphisms in $\mathcal{O}$ are homomorphisms of $\mathfrak{g}$-modules. The category $\mathcal{O}$ is abelian.

2.2.1 Highest weight modules

Highest weight modules are important examples of objects from the category $\mathcal{O}$. For any Lie algebra $\mathfrak{a}$, we let $U(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$.

Definition 2.2.2. A $\mathfrak{g}$-*module* V *is said to be a highest weight module with highest weight* $\lambda \in \mathfrak{h}^*$ *if there exists a nonzero vector* v_λ *such that*

1. $\mathfrak{n}_+(v_\lambda) = 0$; $h(v_\lambda) = \lambda(h)v_\lambda, \forall\, h \in \mathfrak{h}$; *and*

2. $U(\mathfrak{g}).v_\lambda = V$.

Remark 2.2.3. By conditions (1) and (2) it is easy to see that $U(\mathfrak{n}_-).v_\lambda = V$ and we have $V = \oplus_{\mu \leq \lambda} V_\mu$, $V_\lambda = \mathbb{C}v_\lambda$, $\dim(V_\lambda) < \infty$. Therefore, a highest weight module is an object of category $\mathcal{O}$.

Now, we define an important family of highest weight modules known as Verma modules.

Definition 2.2.4. A $\mathfrak{g}$-*module* $M(\lambda)$ *with highest weight* λ *is called a Verma module if every* $\mathfrak{g}$-*module with highest weight* λ *is a quotient of* $M(\lambda)$.

The following proposition justifies the importance of Verma modules.

Proposition 2.2.5. *(see [4])*

1. *For every* $\lambda \in \mathfrak{h}^*$ *there exists a unique (up to isomorphism) Verma module* $M(\lambda)$,
2. *Viewed as a* $U(\mathfrak{n}_-)$-*module,* $M(\lambda)$ *is a free module of rank 1 generated by the highest weight vector,*
3. $M(\lambda)$ *contains a unique proper maximal submodule* $M'(\lambda)$.

It follows from 3 that for $\lambda \in \mathfrak{h}^*$, there is a unique irreducible module of highest weight λ which we denote by $L(\lambda) := M(\lambda)/M'(\lambda)$. The $\mathfrak{g}$-modules $L(\lambda)$, for $\lambda \in \mathfrak{h}^*$, exhaust all irreducible modules of the category $\mathcal{O}$.

2.2.2 Integrable modules

Definition 2.2.6. A $\mathfrak{g}$-*module* V *is called as integrable if the following holds:*

- *It is* $\mathfrak{h}$-*diagonalizable with finite dimensional weight spaces*
- *The Chevalley generators* e_i *and* f_i $(i = 1, ..., n)$ *are locally nilpotent on* V. *i.e., For any* $v \in V$, $e_i^{n(v)}.v = 0$, $f_i^{m(v)}.v = 0$ *for some* $n(v), m(v) \in \mathbb{Z}_{\geq 0}$.

We will further restrict our attention to the category of integrable modules in category $\mathcal{O}$ denoted as $\mathcal{O}^{\text{int}}(\mathfrak{g})$. We record the following fact from [4].

Proposition 2.2.7. *Let* $\mathfrak{g}$ *be a symmetrizable Kac-Moody algebra and* $L(\lambda)$ *be an irreducible* $\mathfrak{g}$-*module in the category* $\mathcal{O}$. *Then* $L(\lambda)$ *is integrable if and only if* $\lambda \in P^+$.

2.3 Character of a representation

Given a function $f : \mathfrak{h}^* \to \mathbb{Z}$ we define $Supp(f)$, the support of f, to be the set of $\lambda \in \mathfrak{h}^*$ for which $f(\lambda) \neq 0$. Let $\mathcal{E}$ be the set of all functions $f : \mathfrak{h}^* \to \mathbb{Z}$ such that there exists a finite set $\lambda_1, \cdots, \lambda_k \in \mathfrak{h}^*$ with $Supp(f) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_k)$. Given $f, g \in \mathcal{E}$ we define $(f+g)(\lambda) = f(\lambda) + g(\lambda)$ and $(fg)(\lambda) = \sum_{\substack{\mu,\nu \in \mathfrak{h}^* \\ \mu+\nu=\lambda}} f(\mu)g(\nu)$.

It is clear that $f+g, fg \in \mathcal{E}$, thus $\mathcal{E}$ is an algebra over $\mathbb{C}$. For each $\lambda \in \mathfrak{h}^*$ we define $e^{\lambda} \in \mathcal{E}$ to be the characteristic function of λ. Now it is convenient to write $f = \sum_{\lambda \in \mathfrak{h}^*} f(\lambda) e^{\lambda}$, for any $f \in \mathcal{E}$. Let us now define the character of the representation.

Definition 2.3.1. *Let V be a module from the category $\mathcal{O}$ and let $V = \oplus_{\lambda \in \mathfrak{h}^*} V_{\lambda}$ be its weight space decomposition. We define formal character of V by*

$$\operatorname{ch} V := \sum_{\lambda \in \mathfrak{h}^*} (\dim V_{\lambda})\, e^{\lambda}$$

By the definition it is clear that $\operatorname{ch} V \in \mathcal{E}$.

Let us fix an element $\rho \in \mathfrak{h}^*$ such that $\langle \rho, \alpha^{\vee} \rangle = 1$ for all $\alpha \in \Pi$, ρ is called a Weyl vector of $\mathfrak{g}$. Now we are in a position to state the fundamental result of the representation theory of Kac-Moody algebras.

2.3.1 Weyl-Kac character formula

Theorem 2.3.2. *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra, and let $L(\lambda)$ be the irreducible $\mathfrak{g}$-module with highest weight $\lambda \in P^+$. Then*

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w \in W} \varepsilon(w) e^{(w(\lambda+\rho)-\rho)}}{\prod_{\alpha \in \Delta_+} (1-e^{-\alpha})^{m_{\alpha}}}$$

(This is an equality in the ring $\mathcal{E}$.)

Corollary 2.3.3. *(Weyl-Kac denominator identity) For a symmetrizable Kac-Moody algebra we have $\prod_{\alpha \in \Delta_+} (1-e^{-\alpha})^{m_{\alpha}} = \sum_{w \in W} \varepsilon(w) e^{w(\rho)-\rho}$ (this is an equality in the ring $\mathcal{E}$).*

We define the normalized character by $\chi_{\lambda} := e^{-\lambda} \operatorname{ch}(L(\lambda))$, and the normalized Weyl numerator by:

$$U_{\lambda} := e^{-(\lambda+\rho)} \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}. \tag{2.3.5}$$

where ρ is the Weyl vector. The Weyl-Kac character formula gives:

$$\chi_{\lambda} = U_{\lambda}/U_0. \tag{2.3.6}$$

Next, we define the q-Kostant partition function.

2.3.2 q-Kostant partition function of $\mathfrak{g}$

Definition 2.3.4. *Let $\mathfrak{g}$ be a Kac-Moody algebra. Let $\mathfrak{h}, \mathfrak{h}^*, Q$ be its Cartan subalgebra, dual of Cartan subalgebra and root lattice respectively. The q-Kostant partition function K defined on $\mathfrak{h}^*$ by*

$$K(\beta; q) := \text{ the coefficient of } e^{-\beta} \text{ in } \prod_{\alpha \in \Delta_+} \frac{1}{(1 - qe^{-\alpha})^{m_\alpha}}$$

Note that, $K(\beta; q) = 0$ unless $\beta \in Q_+$. For $\beta \in Q_+$, $K(\beta; 1)$ is the usual Kostant partition function, which counts the number of partitions of β into a sum of positive roots, where each root is counted with its multiplicity.

Chapter 3

Unique factorization of tensor products for Kac-Moody algebras

The contents of this chapter have appeared in [1].

3.1 The Main Theorem

Let us recall the statement of our main theorem:

Theorem 3.1.1. *Let $\mathfrak{g}$ be an indecomposable symmetrizable Kac-Moody algebra. Let n be a positive integer and suppose $V_1, V_2, \cdots, V_n$ and $W_1, W_2, \cdots, W_n$ are irreducible $\mathfrak{g}$-modules in category $\mathcal{O}^{\text{int}}$ such that*

$$V_1 \otimes \cdots \otimes V_n \cong W_1 \otimes \cdots \otimes W_n. \tag{3.1.1}$$

Then there is a permutation σ of the set $\{1, ..., n\}$, and one-dimensional $\mathfrak{g}$-modules Z_i such that $V_i \otimes Z_i \cong W_{\sigma(i)}, 1 \leq i \leq n$.

We make few remarks before seeing the proof.

Remark 3.1.2. *If $\mathfrak{g}$ is finite dimensional, then (a) $\mathcal{O}^{\text{int}} = \mathcal{C}$, (b)* indecomposable *is the same as* simple *and (c) the only one dimensional $\mathfrak{g}$-module is the trivial one. Thus, theorem 3.1.1 is a generalization of Rajan's theorem 1.1.2.*

Remark 3.1.3. *Unique factorization of tensor products upto twisting by one dimensional modules also appears naturally in the finite dimensional context when $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$, for instance, when considering the Lie algebra $\mathfrak{gl}_n$ instead of $\mathfrak{sl}_n$ [5, theorem 3]. We will not be considering this in this thesis.*

Remark 3.1.4. *Theorem 3.1.1 can be interpreted at the level of characters. For example, the characters of finite dimensional irreducible $\mathfrak{sl}_n$-modules are the Schur functions; so if a symmetric polynomial can be factored as a product of Schur functions, then this factorization is unique* *(cf. [?, proposition 5.1] and [?, theorem 2.6]). Analogously, when $\mathfrak{g}$ is an affine Kac-Moody algebra, the formal characters of irreducible modules in category $\mathcal{O}^{\text{int}}$ form a distinguished basis for the space of theta functions considered as a module over the algebra of holomorphic functions on the upper half plane [4, chapter 13]. In this setting, our main theorem implies that if a theta function has a factorization as a product of irreducible characters, then such a factorization is unique.*

3.1.1 Proof of the main theorem when $\mathfrak{g} = \mathfrak{sl}_2$

The main idea of our proof of theorem 3.1.1 is easily illustrated in the simplest case of $\mathfrak{g} = \mathfrak{sl}_2$. Here, the highest weights of V_i and W_j are indexed by positive integers a_i and b_j. Comparing highest weights of the modules in equation (3.1.1), one obtains $\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j$. Taking formal characters on both sides and simplifying, one gets:

$$\prod_{i=1}^{n}(1 - x^{a_i+1}) = \prod_{j=1}^{n}(1 - x^{b_j+1}) \tag{3.1.2}$$

where $x := e^{-\alpha}$ and α is the positive root of $\mathfrak{sl}_2$. Note that equation (3.1.2) is essentially just the equality of the product of numerators that appear in the Weyl character formula. It is a classical fact that equation (3.1.2) implies the equality of the multisets $\{a_i + 1 : 1 \leq i \leq n\}$ and $\{b_j + 1 : 1 \leq j \leq n\}$. We recall [5, proposition 4] that one way to prove this is by observing that if these multisets are disjoint (which can be ensured by cancelling common terms in equation (3.1.2)), then $x := \exp(2\pi i/K)$, where K is the largest element in the union of these multisets, is a zero of exactly one side of equation (3.1.2).

Alternatively, we can apply the logarithm to equation (3.1.2) to obtain an equality of formal power series:

$$\sum_{i=1}^{n}\sum_{p>0}\frac{x^{p(a_i+1)}}{p} = \sum_{j=1}^{n}\sum_{p>0}\frac{x^{p(b_j+1)}}{p} \tag{3.1.3}$$

Now, letting k denote the minimal element in the union of the (disjoint) multisets as above, we observe that (a) all terms appearing on both sides of equation (3.1.3) involve only x^r for $r \geq k$ and (b) the term x^k appears on exactly one side of the equation (and with coefficient 1). This is the required contradiction.

Our proof of theorem 3.1.1 is based on this latter approach. We reinterpret the given isomorphism of tensor products as an equality of products of (normalized) Weyl numerators. These are now power series in l-variables, where l is the the rank of $\mathfrak{g}$. We then show that the logarithm of a Weyl numerator has a unique monomial of smallest degree containing all variables (propositions 3.2.2, 3.2.3). This is sufficient to establish uniqueness of the irreducible factors in the tensor product, along the same lines as for $\mathfrak{sl}_2$.

We remark that if $\mathfrak{g}$ is a finite dimensional simple Lie algebra, then the Weyl numerator is a priori a polynomial, but its logarithm is in the larger ring of formal power series. When $\mathfrak{g}$ is infinite dimensional, the Weyl numerator is a formal power series to begin with.

This chapter is organized as follows: §3.2 contains the key statements concerning the logarithm of normalized Weyl numerators, and the proof of our main theorem, while §3.3 uses an interplay of combinatorics and representation theory to prove the key propositions of §3.2.

3.2 Proof of the main theorem

Let us start with the definition of graph of $\mathfrak{g}$.

3.2.1 Graph of $\mathfrak{g}$

Let $\mathcal{G}$ be the graph underlying the Dynkin diagram of $\mathfrak{g}$, i.e., $\mathcal{G}$ has vertex set Π, with an edge between two vertices α and β iff $(\alpha, \beta) < 0$. We will refer to $\mathcal{G}$ as the *graph of* $\mathfrak{g}$. Observe that $\mathcal{G}$ does not keep track of the Cartan integers $\frac{2(\alpha,\beta)}{(\beta,\beta)}$; thus for instance the classical series A_n, B_n and C_n all have the same graph.

Now we fix some notations, let $X_\alpha := e^{-\alpha}$, $\alpha \in \Pi$ and consider the algebra of formal power series $\mathcal{A} := \mathbb{C}[[X_\alpha : \alpha \in \Pi]]$. Since $L(\lambda)$ has highest weight λ, it is clear that $\chi_\lambda \in \mathcal{A}$ (see 2.3.1). We also have that $U_\lambda \in \mathcal{A}$, since $(\lambda + \rho) - w(\lambda + \rho) \in Q^+$ for all $w \in W$. Both χ_λ and U_λ have constant term 1.

We call a monomial $\kappa = \prod\limits_{\alpha \in \Pi} X_\alpha^{p_\alpha} \in \mathcal{A}$ *regular* if $p_\alpha \geq 1$ for all $\alpha \in \Pi$. Given $f \in \mathcal{A}$, say $f = \sum_\kappa b_\kappa \, \kappa$ (the sum running over monomials κ), the regular part of f, denoted $f^\#$, is defined to be the sum of only the regular terms in f, i.e., $f^\# := \sum\limits_{\kappa \text{ regular}} b_\kappa \, \kappa$. It is easy to see that $f^\#$ is given by the formula $f^\# = \sum\limits_{J \subset \Pi} (-1)^{|J|} f|_{X_\alpha = 0, \alpha \in J}$ but we will not need this.

Also given $\gamma \in P^+$, define the associated regular monomial $M^\gamma := \prod\limits_{\alpha \in \Pi} X_\alpha^{\langle \gamma + \rho, \alpha^\vee \rangle}$, and let $\deg(\gamma) := \text{degree}(M^\gamma) = \sum\limits_{\alpha \in \Pi} \langle \gamma + \rho, \alpha^\vee \rangle$.

Recall that for $\lambda \in \mathfrak{h}^*$, we denote $\lambda \mid_{\mathfrak{h}'}$ by $\overline{\lambda}$. The following lemma collects together some well-known properties :

Lemma 3.2.1. *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra and $\lambda, \mu \in P^+$. The following statements are equivalent: (a) $\chi_\lambda = \chi_\mu$, (b) $U_\lambda = U_\mu$, (c) $M^\lambda = M^\mu$, (d) $\overline{\lambda} = \overline{\mu}$, (e) $L(\lambda) \cong L(\mu)$ as $\mathfrak{g}'$-modules, (f) $L(\lambda) \otimes Z \cong L(\mu)$ as $\mathfrak{g}$-modules, for some one dimensional $\mathfrak{g}$-module Z.*

Proof. The Weyl character formula (equation (2.3.6)) shows that (a) and (b) are equivalent, while (c) $\Leftrightarrow$ (d) follows from definitions. The equivalence of (d), (e) and (f) can be found in [4, §9.10]. The implication (b) $\Rightarrow$ (d) follows from the observation that the only monomial in U_λ

of the form X_α^n is $-X_\alpha^{\langle\lambda+\rho,\alpha^\vee\rangle}$ (corresponding to $w = s_\alpha$ in equation (2.3.5)). Finally, (d) $\Rightarrow$ (b) because the expression $w(\lambda+\rho)-(\lambda+\rho)$ only depends on the values $\langle\lambda+\rho,\alpha^\vee\rangle$ for $\alpha\in\Pi$ (for instance, this follows from equation (3.3.7) below and induction). $\square$

Next, recall that if $\eta\in\mathcal{A}$ is a formal power series with constant term 1, its logarithm is a well defined formal power series: $\log\eta = -\sum_{p\geq 1}(1-\eta)^p/p$. The next two propositions are the key ingredients in the proof of our main theorem.

Proposition 3.2.2. *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra. Given $\lambda\in P^+$, we have:*

$$(-\log U_\lambda)^\# = c(\mathfrak{g},\lambda)\, M^\lambda + \text{regular monomials of degree} > \deg(\lambda)$$

for some $c(\mathfrak{g},\lambda)\in\mathbb{Z}_{\geq 0}$. Further, $c(\mathfrak{g},\lambda)$ is independent of λ, and only depends on the graph of $\mathfrak{g}$.

Proposition 3.2.3. *Letting $c(\mathfrak{g}) := c(\mathfrak{g},\lambda)$, we have further that $c(\mathfrak{g})\geq 1$ iff $\mathfrak{g}$ is indecomposable, or equivalently, iff the graph of $\mathfrak{g}$ is connected.*

Thus, when $\mathfrak{g}$ is indecomposable, the above propositions imply that M^λ is the unique regular monomial of minimal degree appearing with nonzero coefficient in $\log U_\lambda$. When $\mathfrak{g}$ is a finite dimensional simple Lie algebra, we will in fact show (Corollary 3.3.6) that $c(\mathfrak{g}) = 1$. We defer the proofs of propositions 3.2.2 and 3.2.3 to section 3.3. We first deduce a unique factorization theorem for Weyl numerators (see also [5, theorem 2]), and use this to prove our main theorem.

Theorem 3.2.4. *Let $\mathfrak{g}$ be an indecomposable symmetrizable Kac-Moody algebra. Let n,m be positive integers and suppose $\lambda_1,\cdots,\lambda_n$, $\mu_1,\cdots,\mu_m\in P^+$ are such that the following identity holds in $\mathcal{A}$:*

$$U_{\lambda_1}\cdots U_{\lambda_n} = U_{\mu_1}\cdots U_{\mu_m}. \tag{3.2.4}$$

Then $n=m$, and there is a permutation σ of the set $\{1,2,\cdots,n\}$, such that $U_{\lambda_i} = U_{\mu_{\sigma(i)}}, 1\leq i\leq n$.

Proof. Let $a := \min(\{\deg(\lambda_i) : 1\leq i\leq n\}\cup\{\deg(\mu_j) : 1\leq j\leq m\})$. We can assume without loss of generality that $\deg(\lambda_1) = a$. Now apply the operator $-\log$ to equation (3.2.4) and consider the regular monomials on both sides :

$$\sum_{i=1}^n(-\log U_{\lambda_i})^\# = \sum_{j=1}^m(-\log U_{\mu_j})^\#. \tag{3.2.5}$$

By propositions 3.2.2 and 3.2.3, it is clear that M^{λ_1} occurs on the left hand side of equation (3.2.5) with nonzero coefficient. Since all μ_j's have degree $\geq a$, there must exist $1\leq j\leq m$ for which $M^{\mu_j} = M^{\lambda_1}$. By lemma 3.2.1, $U_{\lambda_1} = U_{\mu_j}$. Cancelling these terms and proceeding by induction, we obtain the desired conclusion. $\square$

We now complete the proof of theorem 3.1.1. Given irreducible $\mathfrak{g}$-modules V_i, W_j as in equation (3.1.1), we let λ_i, μ_j be dominant integral weights such that $V_i = L(\lambda_i)$ and $W_j = L(\mu_j)$ for $1 \leq i, j \leq n$. Observe that (a) all weights of the module $\bigotimes_{i=1}^{n} V_i$ are $\leq \sum_{i=1}^{n} \lambda_i$ where $\leq$ is the usual partial order on the weight lattice, and (b) $\sum_{i=1}^{n} \lambda_i$ is a weight of this module. Thus, comparing highest weights of the modules $\otimes_i V_i$ and $\otimes_j W_j$, we conclude $\sum_{i=1}^{n} \lambda_i = \sum_{j=1}^{n} \mu_j =: \beta$ (say). Taking formal characters of the modules in equation (3.1.1) one obtains:

$$\prod_{i=1}^{n} \operatorname{ch}(L(\lambda_i)) = \prod_{j=1}^{n} \operatorname{ch}(L(\mu_j))$$

Multiplying both sides by $e^{-\beta} U_0^n$ and using the Weyl-Kac character formula (equation (2.3.6)), we get $U_{\lambda_1} \cdots U_{\lambda_n} = U_{\mu_1} \cdots U_{\mu_n}$. Theorem 3.2.4 and lemma 3.2.1 now complete the proof. $\square$

3.3 Proof of propositions 3.2.2 and 3.2.3

Throughout this section, we fix a dominant integral weight λ of $\mathfrak{g}$.

Let $a_\alpha := \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{>0}$ for each $\alpha \in \Pi$; thus $M^\lambda := \prod_{\alpha \in \Pi} X_\alpha^{a_\alpha}$. We write

$$\lambda + \rho - w(\lambda + \rho) = \sum_{\alpha \in \Pi} c_\alpha(w) \alpha \tag{3.3.6}$$

where $c_\alpha(w) \in \mathbb{Z}_{\geq 0}$, and define $X(w) := \prod_{\alpha \in \Pi} X_\alpha^{c_\alpha(w)} = e^{w(\lambda+\rho)-(\lambda+\rho)}$.

For $w \in W$, let $\mathbf{w}$ denote a reduced word for w. We define $I(w) := \{\alpha \in \Pi : s_\alpha \text{ appears in } \mathbf{w}\}$; this is a well defined subset of Π, since $I(w)$ is independent of the reduced word chosen [3]. A non-empty subset $K \subset \Pi$ is said to be *totally disconnected* if $(\alpha, \beta) = 0$ for all distinct $\alpha, \beta \in K$, i.e., there are no edges in $\mathcal{G}$ between vertices of K. Let $\mathcal{I} := \{w \in W \backslash \{e\} : I(w) \text{ is totally disconnected}\}$. Given a totally disconnected subset K of Π, there is a unique element $w(K) \in \mathcal{I}$ with $I(w(K)) = K$; it is clear that $w(K)$ is just the product of the commuting simple reflections $\{s_\alpha : \alpha \in K\}$. Thus, $\mathcal{I}$ is in natural bijection with the set of all totally disconnected subsets of Π. Note that the elements of $\mathcal{I}$ are involutions in W. We now have the following key lemma.

Lemma 3.3.1. *Let $w \in W$. Then*

(a) $I(w) = \{\alpha \in \Pi : c_\alpha(w) \neq 0\}$, i.e., $X(w) = \prod_{\alpha \in I(w)} X_\alpha^{c_\alpha(w)}$.

(b) $c_\alpha(w) \geq a_\alpha$ for all $\alpha \in I(w)$.

(c) If $w \in \mathcal{I}$, then $c_\alpha(w) = a_\alpha$ for all $\alpha \in I(w)$.

(d) If $w \notin \mathcal{I} \cup \{e\}$, then there exists $\beta \in I(w)$ such that $c_\beta(w) > a_\beta$.

Proof. We set $\gamma := \lambda + \rho$. First, observe that (c) follows immediately from definitions. Further, equation (3.3.6) shows that $c_\alpha(w) = 0$ for all $\alpha \notin I(w)$. Thus (a) follows from (b). We now prove (b) by induction on $l(w)$. If $w = e$, then (b) is trivially true. Suppose that $l(w) \geq 1$, write $w = \sigma s_\alpha$ with $l(\sigma) = l(w) - 1$ and $\alpha \in \Pi$. This implies $\sigma(\alpha) \in \Delta_+$. Now

$$\gamma - w\gamma = (\gamma - \sigma\gamma) + \sigma(\gamma - s_\alpha\gamma) = (\gamma - \sigma\gamma) + a_\alpha\, \sigma\alpha \tag{3.3.7}$$

Now, either (i) $I(w) = I(\sigma)$ or (ii) $I(w) = I(\sigma) \sqcup \{\alpha\}$. In case (i), we are done by the induction hypothesis. If (ii) holds, observe that $\sigma\alpha = \alpha + \alpha'$ for some α' in the $\mathbb{Z}_{\geq 0}$-span of $I(\sigma)$. Equation (3.3.7) and the induction hypothesis now complete the proof of (b).

The proof of (d) is along similar lines, by induction on $l(w)$. Observe $l(w) \geq 2$ since $w \notin \mathcal{I} \cup \{e\}$. Write $w = \sigma s_\alpha$ as above. If $\sigma \notin \mathcal{I}$, then the result follows by the induction hypothesis and the fact that $I(\sigma) \subset I(w)$. If $\sigma \in \mathcal{I}$, then clearly $I(w) \neq I(\sigma)$ and so $I(w) = I(\sigma) \sqcup \{\alpha\}$. Since $I(w)$ is not totally disconnected, we must have $\sigma\alpha \neq \alpha$, i.e., $\sigma\alpha = \alpha + \alpha'$ for some *non-zero* $\alpha' \in \mathbb{Z}_{\geq 0}$-span of $I(\sigma)$. We are again done by (c) and equation (3.3.7). □

3.3.1 The $c(\mathcal{G})$

We now make the following useful definition.

Definition 3.3.2. *Let k be a positive integer. A k-partition of the graph $\mathcal{G}$ is an ordered k-tuple $(J_1, ..., J_k)$ such that (a) the J_i's are non-empty pairwise disjoint subsets of the vertex set Π of $\mathcal{G}$, (b) $\bigcup_{i=1}^{k} J_i = \Pi$, and (c) each J_i is a totally disconnected subset of Π.*

We let $P_k(\mathcal{G})$ denote the set of k-partitions of $\mathcal{G}$ and $c_k(\mathcal{G}) :=\mid P_k(\mathcal{G}) \mid$. We also define

$$c(\mathcal{G}) := (-1)^l \sum_{k=1}^{l} (-1)^k \frac{c_k(\mathcal{G})}{k} \tag{3.3.8}$$

where $l = |\Pi|$ is the cardinality of the vertex set of $\mathcal{G}$. Finally, given $\mathcal{J} := (J_1, \cdots, J_k)$ in $P_k(\mathcal{G})$, define $w(\mathcal{J}) := w(J_1)\, w(J_2) \cdots w(J_k)$ (this is in fact a Coxeter element of W, i.e., product of all generators in some order).

We now proceed to analyze $(-\log U_\lambda)^{\#}$. Write $U_\lambda = 1 - \xi$, where

$$\xi := - \sum_{w \in W \setminus \{e\}} \varepsilon(w) X(w) = \xi_1 + \xi_2$$

with $\xi_1 := - \sum_{w \in \mathcal{I}} \varepsilon(w) X(w)$ and $\xi_2 := - \sum_{w \notin \mathcal{I} \cup \{e\}} \varepsilon(w) X(w)$.

Since $-\log U_\lambda = \xi + \xi^2/2 + ... + \xi^k/k + \cdots$, lemma 3.3.1 clearly implies that any *regular* monomial $\kappa = \prod_{\alpha \in \Pi} X_\alpha^{p_\alpha}$ that occurs in $-\log U_\lambda$ must satisfy $p_\alpha \geq a_\alpha$ for all $\alpha \in \Pi$. It further implies that there is no contribution of ξ_2 to the coefficient of the regular monomial

$M^\lambda = \prod_{\alpha\in\Pi} X_\alpha^{a_\alpha}$, i.e., M^λ occurs with the same coefficient in $-\log(1-\xi)$ and in $-\log(1-\xi_1)$. Thus, the coefficient of M^λ in $-\log(U_\lambda)$ is:

$$\sum_{k\geq 1} \sum_{\mathcal{J}\in P_k(\mathcal{G})} \frac{(-1)^k}{k}\,\varepsilon(w(\mathcal{J}))$$

Since $\varepsilon(w(\mathcal{J})) = (-1)^l$ for all $\mathcal{J} \in P_k(\mathcal{G})$ and all $k \geq 1$, we deduce that this coefficient is equal to $c(\mathcal{G})$. Thus $c(\mathfrak{g}, \lambda) = c(\mathcal{G})$ only depends on the graph $\mathcal{G}$ of $\mathfrak{g}$. This establishes all assertions of proposition 3.2.2, except for the non-negative integrality of the coefficient $c(\mathcal{G})$. This will be established in proposition 3.3.3 below.

3.3.2 Characterization Of $c(\mathcal{G})$

In this section, we obtain another characterization of $c(\mathcal{G})$. Since $c(\mathcal{G}) = c(\mathfrak{g}, \lambda)$ is independent of λ, we can take $\lambda = 0$. Thus, $c(\mathcal{G})$ is the coefficient of M^0 in $-\log U_0$. Now, by the Weyl-Kac denominator identity, we have

$$U_0 = \sum_{w\in W} \varepsilon(w) e^{w\rho-\rho} = \prod_{\beta\in\Delta_+} (1-e^{-\beta})^{\operatorname{mult}\beta}$$

where $\operatorname{mult}\beta$ is the root multiplicity of β. So

$$-\log U_0 = \sum_{\beta\in\Delta_+} \operatorname{mult}\beta \sum_{k\geq 1} \frac{e^{-k\beta}}{k}$$

Since $M^0 = \prod_{\alpha\in\Pi} X_\alpha = e^{-\sum_{\alpha\in\Pi}\alpha}$, we have thus proved:

Proposition 3.3.3. *$c(\mathcal{G})$ is the multiplicity of the root $\sum_{\alpha\in\Pi} \alpha$ in $\mathfrak{g}$. Thus $c(\mathcal{G}) \in \mathbb{Z}_{\geq 0}$.*

The following statement about roots is well-known, but is included for completeness sake.

Proposition 3.3.4. *$\sum_{\alpha\in\Pi} \alpha$ is a root of $\mathfrak{g} \Leftrightarrow \mathfrak{g}$ is indecomposable.*

Proof. One half ($\Rightarrow$) follows from [4, Lemma 1.6]. For the converse, observe that the connectedness of $\mathcal{G}$ allows us to order the set Π of simple roots as $(\alpha_1, \alpha_2, \cdots, \alpha_l)$ such that the partial sums $\beta_j := \sum_{i=1}^{j} \alpha_i$ satisfy $(\beta_j, \alpha_{j+1}) < 0$ for $1 \leq j < l = |\Pi|$. Since $(\beta_j - \alpha_{j+1}) \notin \Delta$, a standard $\mathfrak{sl}_2$ argument proves that $\beta_{j+1} \in \Delta$ if $\beta_j \in \Delta$. Since $\beta_1 \in \Delta$, we conclude $\beta_l = \sum_{\alpha\in\Pi} \alpha$ is also a root. $\square$

Finally, observe that propositions 3.3.3 and 3.3.4 prove proposition 3.2.3.

3.3.3 An Algorithm to compute $c(\mathcal{G})$

In this subsection, we obtain an algorithm for the computation of $c(\mathcal{G})$, and give an alternate proof of proposition 3.2.3.

We note that the definitions of §3.3.1 only need $\mathcal{G}$ to be an abstract graph. The main results of this subsection can be viewed as statements about abstract graphs.

Proposition 3.3.5. *Let $\mathcal{G}$ be a graph containing at least two vertices, and p a vertex of $\mathcal{G}$ that is adjacent to a unique vertex of $\mathcal{G}$. Let $\mathcal{G}'$ be the graph which is obtained from $\mathcal{G}$ by deleting the vertex p. Then $c(\mathcal{G}) = c(\mathcal{G}')$.*

Proof. Let q denote the unique vertex adjacent to p. Consider $\mathcal{J} = (J_1, \cdots, J_k) \in P_k(\mathcal{G})$. Then, there are only two (mutually exclusive) possibilities: (a) $J_i = \{p\}$ for some i, or (b) $p \in J_i$ for some i for which $|J_i| \geq 2$. We enumerate the number of k-partitions of each type. If $\mathcal{J}$ is of type (a), then removing J_i gives us a $(k-1)$-partition of $\mathcal{G}'$. Thus, the number of $\mathcal{J}$'s of type (a) is precisely $kc_{k-1}(\mathcal{G}')$, since there are k possibilities for i. Next, if $\mathcal{J}$ is of type (b), deleting p from the part in which it occurs leaves us with a k-partition $\mathcal{J}'$ of $\mathcal{G}'$. Conversely given $\mathcal{J}' \in P_k(\mathcal{G}')$, the vertex p can be inserted into any of the $k-1$ parts of $\mathcal{J}'$ which do not contain q. Thus the number of $\mathcal{J}$'s of type (b) is $(k-1)c_k(\mathcal{G}')$. Putting these together, we obtain for all $k \geq 1$:

$$c_k(\mathcal{G}) = kc_{k-1}(\mathcal{G}') + (k-1)c_k(\mathcal{G}')$$

where $c_0(\mathcal{G}') := 0$. Plugging this into equation (3.3.8), we obtain

$$c(\mathcal{G}) - c(\mathcal{G}') = (-1)^l \sum_{k \geq 1} (-1)^k (c_k(\mathcal{G}') + c_{k-1}(\mathcal{G}'))$$

where l is the number of vertices in $\mathcal{G}$. Since $c_0(\mathcal{G}') = 0$ and $c_k(\mathcal{G}') = 0$ for all $k \geq l$, the telescoping sum on the right evaluates to 0. □

Corollary 3.3.6. (1) *Let $\mathcal{G}$ be a tree. Then $c(\mathcal{G}) = 1$.* (2) *Let $\mathfrak{g}$ be an indecomposable Kac-Moody algebra of finite or affine type with $\mathfrak{g} \neq A_n^{(1)} (n \geq 2)$ (in the notation of Kac [4]). Then $c(\mathfrak{g}) = 1$.*

Proof. The first part immediately follows from the above theorem by induction on the number of vertices of $\mathcal{G}$. The second follows from the first since for such $\mathfrak{g}$, the associated graph is a tree. □

Next, let $\mathcal{G}$ be a graph and e be an edge in $\mathcal{G}$. Define $\mathcal{G}_e^\dagger$ to be the graph obtained from $\mathcal{G}$ by deleting the edge e alone (keeping all vertices, and edges other than e intact). Let $\mathcal{G}_e$ be the graph which is obtained from $\mathcal{G}$ by contraction of the edge e [2, §1.7], in other words, letting p, q denote the vertices at the two ends of e, $\mathcal{G}_e$ is constructed in two steps as follows: (i) delete the vertices p, q of $\mathcal{G}$ and all edges incident on them; call this graph Γ, (ii) create a new vertex r; for each vertex s in Γ, draw an edge between r and s iff s was adjacent to either p or q (or both) in $\mathcal{G}$.

Proposition 3.3.7. *With notation as above, $c(\mathcal{G}) = c(\mathcal{G}_e^{\dagger}) + c(\mathcal{G}_e)$.*

Proof. Let l be the number of vertices in $\mathcal{G}$. Suppose $\mathcal{J} = (J_1, \cdots, J_k)$ is a k-partition of $\mathcal{G}_e^{\dagger}$. Then either (a) p and q occur in different J_i's, or (b) they occur in the same J_i. In case (a), $\mathcal{J}$ is also a k-partition of $\mathcal{G}$. In case (b), observe that if we delete p and q from J_i and add r to J_i, keeping the remaining J_p's the same, we obtain a k-partition of $\mathcal{G}_e$. So we get $c_k(\mathcal{G}) + c_k(\mathcal{G}_e) = c_k(\mathcal{G}_e^{\dagger})$ for all $k \geq 1$. From equation (3.3.8), we get

$$(-1)^l \sum_{k\geq 1} (-1)^k \frac{c_k(\mathcal{G})}{k} = (-1)^l \sum_{k\geq 1} (-1)^k \frac{c_k(\mathcal{G}_e^{\dagger})}{k} + (-1)^{l-1} \sum_{k\geq 1} (-1)^k \frac{c_k(\mathcal{G}_e)}{k}$$

Since the number of vertices in $\mathcal{G}_e$ is $l-1$, this proves the proposition. □

Corollary 3.3.8. *Let $\mathfrak{g}$ be the affine Kac-Moody algebra of type $A_n^{(1)}, n \geq 2$. Then $c(\mathfrak{g}) = n$.*

Proof. This follows from the above theorem since the graph of $\mathfrak{g}$ is an $(n+1)$-cycle. Alternatively, this also follows from proposition 3.3.3 since $\sum_{\alpha\in\Pi} \alpha$ is just the null root of this affine root system, which has multiplicity n. □

Next, we give a purely combinatorial proof of proposition 3.2.3 and proposition 3.3.4.

Proposition 3.3.9. *Let $\mathcal{G}$ be a graph. (i) If $\mathcal{G}$ is connected, then $c(\mathcal{G}) > 0$, and (ii) if $\mathcal{G}$ is disconnected, then $c(\mathcal{G}) = 0$.*

Proof. Suppose $\mathcal{G}$ is a tree then we are done, since $c(\mathcal{G}) = 1$. So assume that $\mathcal{G}$ contains a cycle, and pick an edge e of this cycle. Then, $\mathcal{G}_e^{\dagger}$ remains connected. It is easy to see that $\mathcal{G}_e$ is also connected. Both $\mathcal{G}_e^{\dagger}$ and $\mathcal{G}_e$ have strictly fewer edges than $\mathcal{G}$. Thus, proposition 3.3.7 together with an induction on the number of edges of $\mathcal{G}$ proves (i). For (ii), suppose there is no edge in $\mathcal{G}$, then $\mathcal{G}$ has at least two vertices. Let v be a vertex in $\mathcal{G}$. Then since $c_k(\mathcal{G}) = k(c_k(\mathcal{G}-v) + c_{k-1}(\mathcal{G}-v))$, equation (3.3.8) gives $c(\mathcal{G}) = 0$. So assume that $\mathcal{G}$ contains an edge, and let e be a choosen edge. Then, both $\mathcal{G}_e$ and $\mathcal{G}_e^{\dagger}$ remain disconnected and have strictly fewer edges than $\mathcal{G}$. Thus, proposition 3.3.7 together with an induction on the number of edges of $\mathcal{G}$ proves the result. □

Remark 3.3.10. *We note that proposition 3.3.7 gives a recursive algorithm to compute $c(\mathcal{G})$. Since both $\mathcal{G}_e^{\dagger}$ and $\mathcal{G}_e$ have fewer edges than $\mathcal{G}$, this process terminates in at most p steps, where p is the number of edges in $\mathcal{G}$. In practice, it is even better, terminating as soon as the resulting graphs are trees.*

Finally, putting together the two points of view on $c(\mathcal{G})$, we deduce the following corollary concerning multiplicities of certain roots of symmetrizable Kac-Moody algebras.

Corollary 3.3.11. *Let $\mathfrak{g}$ be an indecomposable symmetrizable Kac-Moody algebra and let $\alpha(\mathfrak{g})$ denote the sum of the simple roots of $\mathfrak{g}$; recall that $\alpha(\mathfrak{g})$ is a root of $\mathfrak{g}$. Let $A = (a_{ij})$ be the generalized Cartan matrix of $\mathfrak{g}$.*

- *The root multiplicity of $\alpha(\mathfrak{g})$ only depends on the graph $\mathcal{G}$ of $\mathfrak{g}$. In other words,* $\operatorname{mult}\alpha(\mathfrak{g})$ *only depends on the set $\{(i,j) : a_{ij} \neq 0\}$ and not on the actual values of the a_{ij}.*
- *If $\mathcal{G}_e^\dagger$ and $\mathcal{G}_e$ are as in proposition 3.3.7 and if $\mathfrak{g}_e^\dagger$ and $\mathfrak{g}_e$ are symmetrizable Kac-Moody algebras with graphs $\mathcal{G}_e^\dagger$ and $\mathcal{G}_e$ respectively, then* $\operatorname{mult}\alpha(\mathfrak{g}) = \operatorname{mult}\alpha(\mathfrak{g}_e^\dagger) + \operatorname{mult}\alpha(\mathfrak{g}_e)$ $\square$

Chapter 4

New interpretation of chromatic polynomials using Kac-Moody theory

In the last chapter, we have seen some applications of combinatorics in representation theory. On the other hand, representations also give more information about combinatorial objects. Our analysis of the characters in the proof of theorem 3.1.1 leads to an unexpected connection with chromatic polynomials of graphs.

4.1 Kac-Moody algebra associated with the given graph $\mathcal{G}$

Let $\mathcal{G}$ be a connected simple graph with vertex set Π and $|\Pi| = \ell$. There is a natural way of associating a generalized Cartan matix (and hence symmetrizable Kac-Moody algebra) with $\mathcal{G}$. Let $M(\mathcal{G})$ be a symmetric generalized Cartan matrix associated with $\mathcal{G}$, defined as follows:

$$M(\mathcal{G})_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and there is an edge between } i \text{ and } j \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{g}(\mathcal{G})$ be the symmetrizable Kac-Moody algebra associated with $M(\mathcal{G})$ (or with $\mathcal{G}$). We will drop $\mathcal{G}$ in $\mathfrak{g}(\mathcal{G})$, if the underlying graph is understood. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. Let us identify the vertex set Π with the set of simple roots of $\mathfrak{g}$. Let Δ be the set of roots, and Δ_+ the set of positive roots of $\mathfrak{g}$. For $\alpha \in \Pi$, let $\alpha^\vee$ denote the corresponding simple coroot. Let P, Q, P^+, Q^+ be the weight lattice, the root lattice and the sets of dominant weights and non-negative integer linear combinations of simple roots respectively. Let W be the Weyl group of $\mathfrak{g}$, generated by the simple reflections $\{s_\alpha : \alpha \in \Pi\}$, and let ε be its sign character. Let $(-,-)$ be the standard nondegenerate, W-invariant symmetric bilinear form on

$\mathfrak{h}^*$ (see [4, Chapter 2]). Let us fix an element $\rho \in \mathfrak{h}^*$ such that $(\rho, \alpha^\vee) = 1$ for all $\alpha \in \Pi$, ρ is called the Weyl vector of $\mathfrak{g}$.

Note that $\mathcal{G}$ is the graph underlying the Dynkin diagram of $\mathfrak{g}$, i.e., $\mathcal{G}$ has vertex set Π, with an edge between two vertices α and β iff $(\alpha, \beta) < 0$.

4.1.1 Chromatic Polynomial of $\mathcal{G}$

We recall the definition of chromatic polynomial of $\mathcal{G}$ here.

Definition 4.1.1. *Let $q \in \mathbb{N}$. A mapping $f : \Pi \to \{1, \cdots, q\}$ is called a q-colouring of $\mathcal{G}$ if $f(\alpha) \neq f(\beta)$ whenever the vertices α and β are adjacent in $\mathcal{G}$. Two q-colourings f and g of $\mathcal{G}$ are regarded as distinct if $f(\alpha) \neq g(\alpha)$ for some vertex α of $\mathcal{G}$.*

The number of distinct q-colourings of $\mathcal{G}$ is denoted by $P_{\mathcal{G}}(q)$. By convention $P_{\mathcal{G}}(0) = 0$. The following proposition is well known.

Proposition 4.1.2. *For $q \in \mathbb{N}$, we have $P_{\mathcal{G}}(q) = \sum\limits_{k \geq 1} c_k(\mathcal{G}) \binom{q}{k}$, where $c_k(\mathcal{G})$ was defined in 3.3.1. It is thus a polynomial in q and is known as the chromatic polynomial of $\mathcal{G}$. The coefficients of $P_{\mathcal{G}}(q)$ are alternating in sign.*

We define $\widetilde{P}_{\mathcal{G}}(q) := (-1)^\ell P_{\mathcal{G}}(-q)$, it is easy to see that $\widetilde{P}_{\mathcal{G}}(q) \in \mathbb{N}[q]$.

4.2 Chromatic polynomials and some of its properties using Kac-Moody theory

The notations in this section are from 3.2.1 and 3.3.1.

Lemma 4.2.1. $P_{\mathcal{G}}(q) = (-1)^\ell$. *the coefficient of M^0 in U_0^q.*

Proof. Write $U_0 = 1 - \xi_0$, where

$$\xi_0 := -\sum_{w \in W \setminus \{e\}} \varepsilon(w) X(w) = \xi_1 + \xi_2$$

with $\xi_1 := -\sum\limits_{w \in \mathcal{I}} \varepsilon(w) X(w)$ and $\xi_2 := -\sum\limits_{w \notin \mathcal{I} \cup \{e\}} \varepsilon(w) X(w)$.

Since $U_0^q = \sum\limits_{k \geq 0} (-1)^k \binom{q}{k} \xi_0^k$, lemma 3.3.1 clearly implies that there is no contribution of ξ_2 to the coefficient of the regular monomial $M^0 = \prod\limits_{\alpha \in \Pi} X_\alpha$, i.e., M^0 occurs with the same coefficient in $(1 - \xi_0)^q$ and in $(1 - \xi_1)^q$. Thus, the coefficient of M^0 in U_0^q is:

$$\sum_{k \geq 1} (-1)^k \binom{q}{k} \left\{ \sum_{\mathcal{J} \in P_k(\mathcal{G})} (-1)^k \varepsilon(w(\mathcal{J})) \right\}$$

Since $\varepsilon(w(\mathcal{J})) = (-1)^\ell$ for all $\mathcal{J} \in P_k(\mathcal{G})$ and all $k \geq 1$, we deduce that this coefficient is equal to:

$$(-1)^\ell \sum_{k\geq 1} \binom{q}{k} c_k(\mathcal{G}) = (-1)^\ell P_{\mathcal{G}}(q)$$

This completes the proof. □

Now, we have $U_o^q = \exp(q \log U_0) = 1 + \frac{q}{1!} \log U_0 + \frac{q^2}{2!} (\log U_0)^2 + \ldots$, and the Weyl-Kac denominator identity states that $U_0 = \prod_{\alpha\in\Delta_+} (1 - e^{-\alpha})^{m_\alpha}$. Thus we have,

$$(-1)^k (\log U_0)^k = (-\log U_0)^k = \sum_{\beta\in\Delta_+} m_\beta \{e^{-\beta} + \frac{e^{-2\beta}}{2} + \ldots\}.$$

So the coefficient of $M^0 = \prod\limits_{\alpha\in\Pi} X_\alpha$ in $(-1)^k (\log U_0)^k$ is:

$$\sum_{\overline{\beta}\in S_k(\mathcal{G})} m_{\beta_1} m_{\beta_2} ... m_{\beta_k},$$

where $S_k(\mathcal{G}) = \left\{ \overline{\beta} = (\beta_1, \ldots, \beta_k) : \beta_i \in \Delta_+ \text{ and } \sum\limits_{i=1}^{k} \beta_i = \sum\limits_{\alpha\in\Pi} \alpha \right\}$.

So the coefficient of $\prod\limits_{\alpha\in\Pi} X_\alpha$ in U_0^q is:

$$\sum_{k=1}^{\ell} \frac{(-1)^k}{k!} \left\{ \sum_{\overline{\beta}\in S_k(\mathcal{G})} m_{\beta_1} \ldots m_{\beta_k} \right\} q^k.$$

Thus by comparing the coefficients of $\prod\limits_{\alpha\in\Pi} X_\alpha$ in U_0^q in two different ways we get:

$$(-1)^\ell \sum_{k\geq 1} \binom{q}{k} c_k(\mathcal{G}) = \sum_{k=1}^{\ell} \frac{(-1)^k}{k!} \left\{ \sum_{\overline{\beta}\in S_k(\mathcal{G})} m_{\beta_1} \ldots m_{\beta_k} \right\} q^k.$$

Thus we have proved,

Theorem 4.2.2.

$$P_{\mathcal{G}}(q) = (-1)^\ell \sum_{k=1}^{\ell} \frac{(-1)^k}{k!} \left\{ \sum_{\overline{\beta}\in S_k(\mathcal{G})} m_{\beta_1} \ldots m_{\beta_k} \right\} q^k.$$

Corollary 4.2.3. *Let $\ell \geq 2$ and $\beta = \sum\limits_{\alpha\in\Pi} \alpha$. Then*

$$m_\beta = \sum_{k=2}^{\ell} \frac{(-1)^k}{k!} \left\{ \sum_{\overline{\beta}\in S_k(\mathcal{G})} m_{\beta_1} \ldots m_{\beta_k} \right\}.$$

Proof. For any connected graph $\mathcal{G}$ with atleast two vertices, we have $P_{\mathcal{G}}(1) = 0$. $\square$

Corollary 4.2.4. *$\widetilde{P}_{\mathcal{G}}(q) = K(\beta; q)$, where $K(\beta; q)$ is the q-Kostant partition function of $\mathfrak{g}$ and $\beta = \sum\limits_{\alpha \in \Pi} \alpha$.*

Proof. This directly follows from the definition of q-Kostant partition function 2.3.4. $\square$

Corollary 4.2.5. *Let $S \subseteq \Pi$, $\beta_S = \sum\limits_{\alpha \in S} \alpha$ and $\mathcal{G}_S$ be subgraph of $\mathcal{G}$ induced by S. The coefficient of $M_S^0 = \prod\limits_{\alpha \in S} X_\alpha$ in U_0^q is $(-1)^{|S|} P_{\mathcal{G}_S}(q)$. Hence we have*

$$P_{\mathcal{G}_S}(q) = (-1)^{|S|} \sum_{k=1}^{\ell} \frac{(-1)^k}{k!} \left\{ \sum_{\overline{\beta} \in S_k(\beta_S)} m_{\beta_1} \dots m_{\beta_k} \right\} q^k.$$

where $S_k(\beta_S) = \left\{ \overline{\beta} = (\beta_1, \dots, \beta_k) : \beta_i \in \Delta_+ \text{ and } \sum\limits_{i=1}^{k} \beta_i = \sum\limits_{\alpha \in S} \alpha \right\}$.

Proof. The coefficient of $M_S^0 = \prod\limits_{\alpha \in S} X_\alpha$ in $U_0^q = \exp(q \log U_0)$ is $\sum\limits_{k=1}^{|S|} \frac{(-1)^k}{k!} \left\{ \sum\limits_{\overline{\beta} \in S_k(\beta_S)} m_{\beta_1} \dots m_{\beta_k} \right\} q^k$. On the other hand, the coefficient of $M_S^0 = \prod\limits_{\alpha \in S} X_\alpha$ in $U_0^q = \sum\limits_{k \geq 0} (-1)^k \binom{q}{k} \xi_0^k$ is $(-1)^{|S|} \sum\limits_{k \geq 1} \binom{q}{k} c_k(\mathcal{G}_S)$. This completes the proof. $\square$

The proofs of the following facts follow from the arguments used in the proof of Theorem 3.1.1. So we omit them here.

Proposition 4.2.6. *Let $S \subseteq \Pi$ and $\beta_S = \sum\limits_{\alpha \in S} \alpha$. Then m_{β_S} is the coefficient of $M_S^0 = \prod\limits_{\alpha \in S} X_\alpha$ in $-\log U_0$.*

Proposition 4.2.7. *Let $S \subseteq \Pi$ and $\beta_S = \sum\limits_{\alpha \in S} \alpha$. Then $\beta_S \in \Delta_+$ if and only if S is connected, i.e., the subgraph induced by S in $\mathcal{G}$ is connected.*

Now we use these results to get some properties of chromatic polynomials.

Proposition 4.2.8. *$\frac{d}{dq} P_{\mathcal{G}}(q) = \sum\limits_{S \subseteq \Pi} (-1)^{|S|+1} m_{\beta_S} P_{\mathcal{G}_{S'}}(q)$, where $S' = \Pi \backslash S$.*

Proof. It is easy to see that $\frac{\partial}{\partial q} U_0^q = (\log U_0) U_0^q$. Now compare the coefficient of M^0 on both sides and use corollary 4.2.5 and proposition 4.2.6 to get desired result. $\square$

The following proposition is well known and can be proved by using the definition of chromatic polynomials. We give an alternate proof using our interpretation of chromaitc polynomials.

Proposition 4.2.9. *$P_{\mathcal{G}}(q_1+\cdots+q_n)=\sum P_{\mathcal{G}_{S_1}}(q_1)\cdots P_{\mathcal{G}_{S_n}}(q_n)$, where the sum runs over all ordered set partitions $(S_1,\cdots,S_n)$ of Π, and we use the convention that $P_{\mathcal{G}_S}(q)=1$ if S is empty.*

Proof. Use the identity $U_0^{q_1+\cdots+q_n}=U_0^{q_1}\cdots U_0^{q_n}$ and compare the coefficient of M^0 using corollary 4.2.5. □

Remark 4.2.10. *(1) Since $\sum\limits_{i=1}^{k}\beta_i=\sum\limits_{\alpha\in\Pi}\alpha$ and $\beta_i\in\Delta_+$, for each i there exists a connected subset $S_i\subseteq\Pi$ such that $\beta_i=\beta_{S_i}$ and $(S_1,\cdots,S_k)$ form a connected k-partition of Π. Define support of β_i to be S_i, and it is denoted by $supp(\beta_i)$.*

(2) Thus $S_k(\mathcal{G})$ is naturally in bijective correcpondence with the set of all ordered connected k-partitions of $\mathcal{G}$, see definition 4.3.1 below. The bijection is given by $(\beta_1,\cdots,\beta_k)\mapsto(supp(\beta_1),\cdots,supp(\beta_k))$. Often we identify $(\beta_1,\cdots,\beta_k)$ with $(supp(\beta_1),\cdots,supp(\beta_k))$ using this bijection, similarly we identify the unordered tuple $\{\beta_1,\cdots,\beta_k\}$ with $\{supp(\beta_1),\cdots,supp(\beta_k)\}$.

4.3 Bond lattice of $\mathcal{G}$

This new interpretation throws light on a classical expression for $P_{\mathcal{G}}(q)$ by Birkhoff. To state this, first we recall the definition and properties of the bond lattice of $\mathcal{G}$.

We begin with the definition of connected partition of $\mathcal{G}$.

Definition 4.3.1. *Let k be a positive integer. A connected k-partition of the graph $\mathcal{G}$ is an unordered k-tuple $\{S_1,...,S_k\}$ such that (a) the S_i's are non-empty pairwise disjoint subsets of the vertex set Π of $\mathcal{G}$, (b) $\bigcup\limits_{i=1}^{k}S_i=\Pi$, and (c) each S_i is a connected subset of Π, i.e. the subgraph induced by S_i in $\mathcal{G}$ is connected .*

Let $\mathcal{G}$ be a connected graph with ℓ vertices. Consider the poset $\mathcal{L}_{\mathcal{G}}$ whose elements are connected k-partitions of Π for all k, partially ordered by refinement. $\mathcal{L}_{\mathcal{G}}$ is known as the bond lattice of $\mathcal{G}$. The maximum element is $\hat{1}:=\{\Pi\}$ and minimum element is $\hat{0}:=\{\{\alpha_1\},\cdots,\{\alpha_\ell\}\}$, where ℓ is the cardinality of Π. Now we collect some facts about the bond lattice of $\mathcal{G}$ from [2].

(1) An atom, by definition, is an element of $\mathcal{L}_{\mathcal{G}}$ that covers $\hat{0}$. Atoms of $\mathcal{L}_{\mathcal{G}}$ are bijective correspondence with the set of edges of $\mathcal{G}$. The bijection is given as follows : $e\mapsto\{\{\alpha_k\}_{k\neq i,j},\{\alpha_i,\alpha_j\}\}$, where α_i,α_j are end points of e.

(2) $\mathcal{L}_{\mathcal{G}}$ is a geometric lattice.

(3) $\mathcal{L}_{\mathcal{G}}$ has a rank function: $rk(\pi):=\ell-|\pi|$, where $|\pi|$ is the number of parts in π, i.e. if π is a connected k-partition then $|\pi|=k$.

The Möbius function of $\mathcal{L}_{\mathcal{G}}$, $\mu:\mathcal{L}_{\mathcal{G}}\to\mathbb{Z}$, is defined recursively by :

$$\mu(\pi) = \begin{cases} 1 & \text{if } \pi = \hat{0} \\ -\sum\limits_{\pi' < \pi} \mu(\pi') & \text{if } \pi > \hat{0}, \end{cases}$$

Note that μ is the unique $\mathbb{Z}$-valued function on $\mathcal{L}_{\mathcal{G}}$ such that $\sum\limits_{\pi' \leq \pi} \mu(\pi') = \delta_{\hat{0}\pi}$ (Kronecker delta).

(4) The Möbius function of $\mathcal{L}_{\mathcal{G}}$ strictly alternates in sign: $(-1)^{\ell - |\pi|}\mu(\pi) > 0$, for all $\pi \in \mathcal{L}_{\mathcal{G}}$.

Theorem 4.3.2 (Birkhoff, Whitney, Rota)**.** *For a connected graph $\mathcal{G}$, $P_{\mathcal{G}}(q) = \sum\limits_{\pi \in \mathcal{L}_{\mathcal{G}}} \mu(\pi) q^{|\pi|}$.*

Comparing theorem 4.3.2 with theorem 4.2.2 one is led to expect that the absolute value of the mysterious integers $\mu(\pi)$ occurs as a product of root multiplicities of $\mathfrak{g}(\mathcal{G})$. More precisely, we have the following result.

Theorem 4.3.3. *Let $\mathcal{G}$ be a connected graph with ℓ vertices. Let $\pi = \{S_1, \cdots, S_k\} \in \mathcal{L}_{\mathcal{G}}$ and $m_\pi := m_{\beta_{S_1}} \cdots m_{\beta_{S_k}}$, where $\beta_S := \sum\limits_{\alpha \in S} \alpha$ for all $S \subseteq \Pi$. Then $\mu(\pi) = (-1)^{\ell - k}\, m_\pi$.*

Proof. If $\ell = 1$, then the result is clear. So we assume that $\ell \geq 2$.

It is clear that $m_{\hat{0}} = 1$. Now we claim that $\sum\limits_{\pi' \leq \pi} (-1)^{\ell - |\pi'|} m_{\pi'} = 0$, for all $\hat{0} \neq \pi \in \mathcal{L}_{\mathcal{G}}$.

If $\pi = \hat{1}$, then $\sum\limits_{\pi' \leq \hat{1}} (-1)^{\ell - |\pi'|} m_{\pi'} = P_{\mathcal{G}}(1) = 0$, since $\mathcal{G}$ has at least two vertices (since we may now assume that $\hat{1} \neq \hat{0}$).

Assume that $\hat{0} < \pi < \hat{1}$. Let $\pi = (S_1, \cdots, S_k)$ and $\mathcal{G}_i =$ the subgraph induced by S_i in $\mathcal{G}$.

Then it is easy to see that,

$$\sum_{\pi' \leq \pi} (-1)^{\ell - |\pi'|} m_{\pi'} = \prod_{i=1}^{k} \left\{ \sum_{\pi'(\mathcal{G}_i) \leq \hat{1}(\mathcal{G}_i)} (-1)^{\ell - |\pi'(\mathcal{G}_i)|} m_{\pi'(\mathcal{G}_i)} \right\} = \prod_{i=1}^{k} P_{\mathcal{G}_i}(1)$$

Now, since $\hat{0} < \pi$, there exists i such that $\hat{1}(\mathcal{G}_i) \neq \hat{0}(\mathcal{G}_i)$, i.e. $\mathcal{G}_i$ has atleast two vertices and so $P_{\mathcal{G}_i}(1) = 0$. This completes the proof. □

4.4 Peterson recursion formula

It is known that, $m_{\beta_S} \neq 0$ if and only if the subgraph generated by $S \subseteq \Pi$ is connected, see 3.3.4. So, to understand the coefficient of the chromatic polynomials, it is enough to understand m_β's.

Towards this direction, we get a nice recursion formula for the mysterious number m_β, by applying the Peterson recursion formula (§Chapter 11, [4]) to $\beta = \sum\limits_{\alpha \in \Pi} \alpha$.

For $\beta \in Q_+$, set $c_\beta = \sum\limits_{n \geq 1} n^{-1} m_{(\beta/n)}$. Then Peterson recursion formula says

$$(\beta, \beta - 2\rho)c_\beta = \sum_{\substack{(\beta',\beta'')\in Q_+\times Q_+ \\ \beta'+\beta''=\beta}} (\beta', \beta'')c_{\beta'}c_{\beta''}$$

Now we apply this to $\beta = \sum\limits_{\alpha\in\Pi} \alpha$. It is easy to see that, in this particular case, Peterson recursion formula becomes

$$(\beta, \beta - 2\rho)m_\beta = \sum_{\substack{(\beta',\beta'')\in Q_+\times Q_+ \\ \beta'+\beta''=\beta}} (\beta', \beta'')m_{\beta'}m_{\beta''} \tag{4.4.1}$$

But $(\beta, \beta - 2\rho) = 2$(the number of edges in $\mathcal{G}$) and
$(\beta', \beta'') =$ the number of edges between $supp(\beta')$ and $supp(\beta'')$.

Denote $E(\beta) =$ the number of edges in $\mathcal{G}$ and $E(\beta', \beta'') =$ the number of edges between $supp(\beta')$ and $supp(\beta'')$. Then (5) becomes

$$m_\beta = \sum_{\substack{(\beta',\beta'')\in Q_+\times Q_+ \\ \beta'+\beta''=\beta}} \frac{E(\beta', \beta'')}{E(\beta)} m_{\beta'}m_{\beta''}. \tag{4.4.2}$$

Given $\pi = \{\beta_1, \cdots, \beta_k\} \in \mathcal{L}_\mathcal{G}$, let $\pi^+ = \{\beta \in \pi : |supp(\beta)| > 1\}$. We write $\pi \to \pi'$ to denote the covering relations, i.e., there exists only one β_i in π such that $\pi' = \{\beta_1, \cdots, \beta_{i-1}, \beta', \beta'', \beta_{i+1}, \cdots, \beta_k\}$ with $\beta' + \beta'' = \beta$. For such pair π and π' we define

$$\omega(\pi, \pi') = \frac{1}{|\pi^+|}\frac{E(\beta', \beta'')}{E(\beta_i)}.$$

With the notations above, we have the following theorem.

Theorem 4.4.1. *For all $\hat{0} \neq \pi \in \mathcal{L}_\mathcal{G}$, we have*

$$m_\pi = \sum_{\substack{\pi'\in\mathcal{L}_\mathcal{G} \\ \pi\to\pi'}} \omega(\pi, \pi')m_{\pi'}$$

Proof. Using Peterson formula one can easily see that,

$$m_\pi = \frac{1}{|\pi^+|} \sum_{\beta_i\in\pi^+} \sum_{\beta'+\beta''=\beta_i} \frac{E(\beta', \beta'')}{E(\beta_i)} m_{\beta_1} \cdots m_{\beta_{i-1}} m_{\beta'} m_{\beta''} m_{\beta_{i+1}} \cdots m_{\beta_k}.$$

Now rewrite this using above notations to get desired result. □

Theorem 4.4.2. *For $k \geq 1$, the coefficient of q^k in $\widetilde{P}_\mathcal{G}(q) = \sum\limits_{p\in P(\mathcal{L}_\mathcal{G})} \omega(p)$, where sum runs over all paths p such that $p : \pi \to \pi_1 \to \cdots \to \pi_{\ell-k} \to \hat{0}$, and $\omega(p) = \omega(\pi, \pi_1)\omega(\pi_1, \pi_2) \cdots \omega(\pi_{\ell-k}, \hat{0})$.*

Proof. This immediately follows from theorem 3. □

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