PERIODIC DIRICHLET SERIES AND TRANSCENDENCE

By

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CERTIFICATE

This is to certify that the Ph.D. thesis entitled **"PERIODIC DIRICHLET SERIES AND TRANSCENDENCE"** submitted by Tapas Chatterjee is a record of bona fide research work done under my supervision. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

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Declaration

I hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part of a degree/diploma at this or any other Institution/University.

This work was done under the guidance of Dr. Sanoli Gun, at The Institute of Mathematical Sciences, Chennai.

Tapas Chatterjee
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Date :
Place :

To My Teachers ....

Gurur Brahma Gurur Vishnu Gurur Devo Maheshwarah
Gurur Sakshat Param Brahma Tasmai Sri Guruve Namaha.

Dedicated To My Parents

and

My Beloved Wife

"This therefore is Mathematics:
She reminds you of the invisible forms of the soul;
She gives life to her own discoveries;
She awakens the mind and purifies the intellect;
She brings light to our intrinsic ideas;
She abolishes oblivion and ignorance which are ours by birth".

Proclus Diadochus

And miles to go before I sleep,
And miles to go before I sleep.

Robert Frost

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Tapas Chatterjee
IMSc

Contents

Notations

$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The ring of rational integers
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{R}^*$	The multiplicative subgroup of real numbers
$\mathbb{R}_+$	The set of positive real numbers
$\overline{\mathbb{Q}}$	The field of algebraic numbers
$\mathbb{C}$	The field of complex numbers
$\Re(s)$	Real part of the complex number s
$\Im(s)$	Imaginary part of the complex number s
p	A rational prime
$\mathbb{F}$	A number field
$\mathcal{O}_{\mathbb{F}}$	The ring of integers of the number field $\mathbb{F}$
$\mathfrak{p}$	A prime ideal in $\mathcal{O}_{\mathbb{F}}$
(a, b)	The greatest commont divisor of a and b
$q \mid n$	q divides n
$q \nmid n$	q does not divide n
ζ_q	A primitive q-th root of unity
Δ	A fundamental discriminant
φ	Euler's phi-function
χ	Dirichlet character

Abstract

The objects which are central to our investigation are periodic Dirichlet series of the form

$$L(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad \Re(s) > 1$$

where f is some periodic arithmetic function. The classical Riemann zeta function and the L-functions associated to Dirichlet characters are the prototypical examples. We also have the Hurwitz zeta function $\zeta(s,x)$ defined by

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

for real $x \in (0,1]$ and s as before. The special values of such Dirichlet series is part of a much larger picture, developed by Deligne, Zagier and others. For instance, the conjectural transcendence of $\zeta(k)$ at odd positive integers k constitutes a central theme in transcendence theory.

However, since these Dirichlet series in general do not have an Euler product, even the existence of zeros in the domain of absolute convergence $\Re(s) > 1$ cannot be ruled out. In 1982, P. Chowla and S. Chowla [12] stated a conjecture which says that $L(2,f) \neq 0$ except when

$$f(1) = f(2) = \cdots = f(p-1) = \frac{f(p)}{1-p^2}.$$

Here they considered only those f which are integer valued.

A little later, Milnor [26] put the conjecture of Chowlas' in a conceptual framework. He interpreted the conjecture of Chowlas' in terms of the values of the linear independence of the Hurwitz zeta function. More precisely, he conjectured that, for a prime p and integer $k > 1$, the $p-1$ real numbers

$$\zeta(k, 1/p), \zeta(k, 2/p), ..., \zeta(k, (p-1)/p)$$

are linearly independent over $\mathbb{Q}$. In fact, Milnor suggested a generalization of this conjecture for arbitrary integer $q > 1$. This conjecture has been investigated in the recent works of Gun, Murty and Rath [19]. Following their convention, for integers $k, q > 1$, let $V_k(q)$ denotes the $\mathbb{Q}$-vector space

generated by the numbers $\zeta(k, a/q)$ where a runs over the co-prime residue classes mod q. Then the conjecture of Milnor states that

$$\mathbb{Q} - \text{dimension of } V_k(q) = \varphi(q).$$

In [19], the authors proved that the above conjecture of Milnor is intimately linked to irrationality of $\zeta(k)$ for odd k. They derived a non-trivial lower bound for the dimension of $V_k(q)$, namely that

$$\mathbb{Q} - \text{dimension of } V_k(q) \geq \varphi(q)/2.$$

They noted that any improvement of the above lower bound will have remarkable consequences. For instance, it will establish the irrationality of $\zeta(k)/\pi^k$ for odd $k > 1$. In this context, they obtained a conditional improvement of this lower bound by exploiting the arithmetic of cyclotomic fields of different moduli.

Further, the investigations carried out by them led them to formulate the following extension of the original conjecture of Milnor. More precisely, they conjecture that, in addition to Milnor's conjecture, *the $\mathbb{Q}$-vector spaces $V_k(q)$ and $\mathbb{Q}$ are linearly disjoint.*

In other words, the $\mathbb{Q}$-vector space $\widehat{V}_k(q)$ generated by 1 and the Hurwitz zeta values $\zeta(k, a/q)$ for $(a, q) = 1$ has $\mathbb{Q}$-dimension $\varphi(q) + 1$.

As before, following Gun, Murty and Rath, we refer to this as the Strong Chowla-Milnor conjecture. In the first part of our thesis, we investigate various ramifications of this generalized conjecture. In many ways, this provides a more natural framework than the original conjecture of Milnor. Further, we formulate and investigate a number field analog of this conjecture.

We now briefly describe some of the result we have managed to derive in our thesis.

To begin, in our work [8], we give an alternate proof of the following theorem which was proved in [19].

Theorem 1. *There exists an integer r such that for all integers q co-prime to r and all odd integers $k > 1$, the dimension of $V_k(q)$ is at least $\varphi(q)/2+1$.*

In [9], we discuss our second set of problems. We prove a non-trivial lower bound of the dimension of the Strong Chowla-Milnor spaces which is the following.

Theorem 2. *Let $k > 1$ and $q > 2$ be two integers. Then*

$$\dim_{\mathbb{Q}} \widehat{V}_k(q) \geq \frac{\varphi(q)}{2} + 1.$$

We also prove a conditional lower bound of the dimension of those spaces which is the following.

Theorem 3. *Let $q, r > 2$ be two co-prime integers. Then for infinitely many odd $k > 1$ either*

$$\dim_{\mathbb{Q}} \widehat{V}_k(q) \geq \frac{\varphi(q)}{2} + 2$$

or

$$\dim_{\mathbb{Q}} \widehat{V}_k(r) \geq \frac{\varphi(r)}{2} + 2.$$

For an integer $k \geq 1$ and complex numbers z with $|z| < 1$, the polylogarithm function $Li_k(z)$ is defined by

$$Li_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

If $k > 1$, then our definition extends to the complex numbers z with $|z| \leq 1$.

We formulate the Strong Polylog conjecture which is a generalization of Baker's theorem about linear forms of logarithms and link this conjecture to the Strong Chowla-Milnor conjecture. In this regard, our theorem is the following:

Theorem 4. *The Strong Polylog conjecture* (1.2.1) *implies the Strong Chowla-Milnor conjecture for all $q > 1$ and $k > 1$.*

Let $l, a_1, \cdots, a_l$ be positive integers with $a_1 > 1$. Then the *multiple zeta values* (MZVs) are defined as

$$\zeta(a_1, \cdots, a_l) := \sum_{n_1 > \ldots > n_l \geq 1} \frac{1}{n_1^{a_1} \ldots n_l^{a_l}}.$$

Our third set of problems is related to a conjecture (1.1.7) (page no. 36) of Gun, Murty and Rath. Assuming this conjecture, we prove the following theorems:

Theorem 5. *Let $k, q > 1$ be two integers and $\mathbb{F}$ be a number field with $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Let d be a positive integer. Then conjecture* (1.1.7) *implies*

$$\dim_{\mathbb{F}} V_{4d+2}(\mathbb{F}) \geq 2,$$

where we define the "generalized Zagier spaces" $V_k(\mathbb{F})$ as the $\mathbb{F}$-linear space defined by

$$V_k(\mathbb{F}) = \mathbb{F} - \text{span of } \{\zeta(a_1, \cdots, a_l) | a_1 + \cdots + a_l = k\}$$

Theorem 6. *Let $\mathbb{F}$ be an algebraic number field and $\mathbb{F}_1 = \mathbb{F}(e^{2\pi i/\varphi(q)})$. Suppose that $\mathbb{F}_1 \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Assume the conjecture* (1.1.7)*. Then for any positive integer k, the values $L(k, \chi)$ as χ ranges over non-trivial Dirichlet characters mod q are linearly independent over $\mathbb{F}_1$.*

In our fourth set of problems, we consider the possible number field extension of Milnor's conjecture. Noting that the arithmetic of the ambient number field is relevant while carrying out such an extension, we formulate the conjecture (1.2.2) (page no. 40). In this direction, we prove the following theorems.

Theorem 7. *The Strong Polylog conjecture* (1.2.1) *implies the conjecture* (1.2.2) *for all number field* $\mathbb{F}$ *with* $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$.

Further more we show the following propositions.

Proposition 8. *Let* $k > 1$ *and* $q > 1$ *be two integers and* $\mathbb{F}$ *be a number field with* $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. *Then*

$$\dim_{\mathbb{F}} \widehat{V}_k(q, \mathbb{F}) \geq \frac{\varphi(q)}{2} + 1$$

where $\widehat{V}_k(q, \mathbb{F})$ be the $\mathbb{F}$-linear space defined by

$$\widehat{V}_k(q, \mathbb{F}) := \mathbb{F} - \text{span of } \{1, \zeta(k, a/q) : 1 \leq a < q,\ (a, q) = 1\}.$$

Proposition 9. *Let* $k > 1$ *be an odd integer and* $\mathbb{F}$ *be a number field with* $\mathbb{F} \cap \mathbb{Q}(i) = \mathbb{Q}$. *Then* $\dim_{\mathbb{F}} \widehat{V}_k(4, \mathbb{F}) = 3$ *for all such* $\mathbb{F}$ *implies* $\zeta(k)$ *is transcendental.*

Proposition 10. *Let* $k > 1$ *be an odd integer and* ω *be a primitive cube root of unity. Let* $\mathbb{F}$ *be a number field with* $\mathbb{F} \cap \mathbb{Q}(\omega) = \mathbb{Q}$. *Then* $\dim_{\mathbb{F}} \widehat{V}_k(3, \mathbb{F}) = 3$ *for all such* $\mathbb{F}$ *implies* $\zeta(k)$ *is transcendental.*

Our main theorem in this regard, is the following.

Theorem 11. *Let* $k > 1$ *be an odd integer and* $q, r > 2$ *be two co-prime integers. Also, let* $\mathbb{F} \subseteq \mathbb{R} \cap \overline{\mathbb{Q}}$ *such that* $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q} = \mathbb{F} \cap \mathbb{Q}(\zeta_r)$ *and* $\mathbb{F}(\zeta_q) \cap \mathbb{F}(\zeta_r) = \mathbb{F}$. *Assume* $\zeta(k) \notin \mathbb{F}$, *then either*

$$\dim_{\mathbb{F}} \widehat{V}_k(q, \mathbb{F}) \geq \frac{\varphi(q)}{2} + 2$$

or

$$\dim_{\mathbb{F}} \widehat{V}_k(r, \mathbb{F}) \geq \frac{\varphi(r)}{2} + 2.$$

The last part of our thesis addresses questions that are of analytic nature. This is motivated by work of Davenport, Heilbronn [14] and Cassels [6]. In [10], we study the zeros of $L(s, f, a)$ in the region $\sigma > 1$. In this direction, we have the following theorems.

Theorem 12. *Let* a *be a positive transcendental number and* f *be a real valued periodic arithmetic function with period* $q \geq 1$. *If* $L(s, f, a)$ *has a pole at* $s = 1$, *then* $L(s, f, a)$ *has infinitely many zeros for* $\sigma > 1$.

Theorem 13. *Let* a *be a positive algebraic irrational number and* f *be a positive valued periodic arithmetic function with period* $q \geq 1$. *Also let*

$$c \ := \ \frac{\max\limits_n \ f(n)}{\min\limits_n \ f(n)} < 1.15.$$

If $L(s, f, a)$ *has a pole at* $s = 1$, *then* $L(s, f, a)$ *has infinitely many zeros for* $\sigma > 1$.

In the other direction, we study zero-free regions for $L(s, f, a)$ when f is a periodic arithmetic function not identically zero and prove the following theorem.

Theorem 14. *Let f be a non-zero periodic arithmetic function with period $q \geq 1$. Also, let a be a positive real number and*

$$c = \max_{1 \leq b \leq q} \{1, \ |f(b)|\}.$$

Then we have $L(s, f, a) \neq 0$ for $\sigma > 1 + c'(a + n_0)$, where n_0 is the smallest positive integer such that $f(n_0) \neq 0$ and $c' = c/|f(n_0)|$.

Finally, as an application of the above theorem, we prove a variant of a conjecture of Erdös. In this direction our theorem is the following:

Theorem 15. *Let f be a non-zero periodic arithmetic function with period $q > 1$ and*

$$f(n) = \begin{cases} \pm\lambda & \text{if } q \nmid n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $L(k, f) \neq 0$ for all integers $k \geq 2$.

List of publications related to this thesis

1. Tapas Chatterjee, *On The Dimension Of Chowla-Milnor Space*, Proc. Indian Acad. Sci. Math. Sci., **122** (3), (2012), 313-317.

2. Tapas Chatterjee, *The Strong Chowla-Milnor spaces and a conjecture of Gun, Murty and Rath*, Int. J. Number Theory, **8**(5), (2012), 1301-1314.

3. Tapas Chatterjee and Sanoli Gun, *Generalization of a problem of Davenport, Heilbronn and Cassels*, submitted.

4. Tapas Chatterjee, Sanoli Gun and Purusottam Rath, *Number field extension of a question of Milnor*, in preparation.

Chapter 1

Introduction

Perhaps the most familiar function in mathematics, especially in number theory is the Riemann zeta function $\zeta(s)$, which is defined by the series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $s \in \mathbb{C}$ with $\Re(s) > 1$. Although we call it the Riemann zeta function, Leonhard Euler [16] was the first to study the function $\zeta(2k)$ for all integers $k \geq 1$. In 1731, he proved that $\zeta(2) = \pi^2/6$. In general for any integer $k \geq 1$, he proved in 1734 that

$$\zeta(2k) = \frac{(-1)^{k-1} B_{2k} (2\pi)^{2k}}{2(2k)!}$$

where B_k is the k-th Bernoulli number given by the generating function

$$\frac{t}{e^t - 1} := \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Euler deduced the above formula for $\zeta(2k)$ by using the identity

$$\sin \pi t = \pi t \prod_{n=1}^{\infty} \left(1 - \frac{t^2}{n^2}\right)$$

and taking logarithmic differentiation.

He also established the Euler product formula,

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}, \text{ for } \Re(s) > 1$$

where p runs through the set of all primes.

In 1859, Riemann [34] showed that the function $\zeta(s)$ can be analytically continued to the whole complex plane except at $s = 1$, where it has a simple pole with residue 1. He also proved the functional equation of the zeta function by establishing the following identity

$$\xi(s) = \xi(1-s),$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\Gamma(s)$ is the gamma function defined by the Hadamard factorization

$$\frac{1}{\Gamma(s)} = se^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-s/n}$$

for $s \in \mathbb{C}$. Here γ is the Euler's constant defined as the limit

$$\gamma := \lim_{k\to\infty}\left(\sum_{n=1}^{k}\frac{1}{n} - \log k\right) = 0.57721566490....$$

We also have the Dirichlet L-function associated to a Dirichlet character χ defined as

$$L(s,\chi) := \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

with $s \in \mathbb{C}$ and $\Re(s) > 1$. In 1837, Dirichlet introduced $L(s,\chi)$ in his celebrated paper [15] to prove that there are infinitely many primes in arithmetic progressions.

Hurwitz studied the "shifted" zeta function, now called the Hurwitz zeta function, which is defined as

$$\zeta(s,x) := \sum_{n=0}^{\infty}\frac{1}{(n+x)^s},$$

where $0 < x \leq 1$ and $s \in \mathbb{C}$ with $\Re(s) > 1$. Note that $\zeta(s,1) = \zeta(s)$, the classical Riemann zeta function. In 1882, he proved that $\zeta(s,x)$ can be extended holomorphically to the entire complex plane except at $s = 1$, where it has a simple pole with residue 1.

For a periodic arithmetic function f with period $q > 1$, the L-function associated to f is defined as

$$L(s,f) := \sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

with $s \in \mathbb{C}$ and $\Re(s) > 1$.

Since f is periodic with period q, the above series can be written as

$$L(s,f) = q^{-s}\sum_{a=1}^{q} f(a)\zeta(s,a/q), \text{ for } \Re(s) > 1. \tag{1.1}$$

This shows that $L(s,f)$ extends holomorphically to the whole complex plane with a possible simple pole at $s = 1$ with residue $q^{-1}\sum_{a=1}^{q} f(a)$. Hence $L(s,f)$ is an entire function if and only if $\sum_{a=1}^{q} f(a) = 0$.

1.1 Historical Background

1.1.1 Chowla's Conjecture

This work is motivated by the following conjecture of Paromita Chowla and Sarvadaman Chowla. In 1982, P. Chowla and S. Chowla [12] made the following conjecture.

Conjecture 1.1.1 (Chowla-Chowla). *Let p be any prime and f be any rational valued periodic function with period p. Then $L(2, f) \neq 0$ except in the case when*

$$f(1) = f(2) = ... = f(p-1) = \frac{f(p)}{1-p^2}.$$

Using equation (1.1) and the identity

$$(p^2 - 1)\zeta(2) = \sum_{a=1}^{p-1} \zeta(2, a/p),$$

it is clear that if

$$f(1) = f(2) = ... = f(p-1) = m$$

and

$$f(p) = m(1-p^2)$$

for some number m, then $L(2, f) = 0$. The conjecture of Chowla-Chowla demands that this is the only obstruction to the non-vanishing of $L(2, f)$.

They also discussed the above conjecture in terms of the values of the linear independence of the Hurwitz zeta function and conjectured the following.

For $p > 3$ an odd prime, the following $\frac{p-1}{2}$ real numbers

$$\zeta(2, 1/p), \zeta(2, 2/p), \cdots, \zeta(2, (p-1)/2p)$$

are linearly independent over $\mathbb{Q}$.

The above conjecture implies that if $1 \leq a, b < p/2$ and $a \neq b$, then all the ratios

$$\zeta(2, a/p)/\zeta(2, b/p)$$

are irrational.

In 1983, John Milnor [26] interpreted the above conjecture in terms of the values of the linear independence of the Hurwitz zeta function and generalized it for all $k > 1$. He used the identity

$$(p^k - 1)\zeta(k) = \sum_{a=1}^{p-1} \zeta(k, a/p) \tag{1.2}$$

in order to translate the question from the non-vanishing of certain L-functions to the $\mathbb{Q}$-linear independence of certain Hurwitz zeta values. He substituted this identity to the above expression (1.1) of $L(s, f)$ to get

$$L(k, f) = p^{-k} \sum_{a=1}^{p-1} \left[f(a) + \frac{f(p)}{(p^k - 1)} \right] \zeta(k, a/p) \tag{1.3}$$

and conjectured the following:

Conjecture 1.1.2 (Milnor). *Let p be a prime. Then for any integer $k > 1$, the real numbers*

$$\zeta(k, 1/p), \zeta(k, 2/p), ..., \zeta(k, (p-1)/p)$$

are linearly independent over $\mathbb{Q}$.

Further, for q not necessarily prime, he suggested the following generalization of the Chowla-Chowla conjecture.

Conjecture 1.1.3 (Chowla-Milnor). *Let $q > 1, k > 1$ be two integers. Then the following $\varphi(q)$ real numbers*

$$\zeta(k, a/q) \text{ with } 1 \leq a < q,\ (a, q) = 1,$$

are linearly independent over $\mathbb{Q}$.

1.1.2 Recent Developments

In 2011, Sanoli Gun, Maruti Ram Murty and Purusottam Rath [19] investigated the above conjecture due to Chowla and Milnor and defined the following $\mathbb{Q}$-linear spaces.

Definition 1.1.1. *For integers $k > 1, q \geq 2$, define the Chowla- Milnor space $V_k(q)$ by*

$$V_k(q) := \mathbb{Q} - \text{span of } \{\zeta(k, a/q) :\ 1 \leq a < q,\ (a, q) = 1\}.$$

The conjecture of Chowla and Milnor is the assertion that the dimension of $V_k(q)$ is equal to $\varphi(q)$, where φ is the Euler's phi-function. As described in [19], the dimension of these spaces are essential in understanding of Riemann zeta values at odd positive integers greater than 1. In relation to $V_k(q)$, they proved the following non-trivial lower bound of the dimension of the Chowla-Milnor spaces $V_k(q)$.

Theorem 1.1.2. *Let $k > 1$ and $q > 2$ be two integers. Then*

$$\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2}.$$

They note that any improvement of the above lower bound of the Chowla-Milnor spaces will lead to irrationality of $\zeta(k)/\pi^k$ for odd positive integers $k > 1$. In relation to this, they proved the following theorem.

Theorem 1.1.3. *Let $k > 1$ be an odd integer and q and r be two co-prime integers > 2. Then either*

$$\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2} + 1$$

or

$$\dim_{\mathbb{Q}} V_k(r) \geq \frac{\varphi(r)}{2} + 1.$$

Thus in particular, there exists a q_0 such that

$$\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2} + 1$$

for any q co-prime to q_0.

Definition 1.1.4. *For an integer $k \geq 1$ and complex numbers z with $|z| < 1$, the k-th polylogarithm function $Li_k(z)$ is defined by*

$$Li_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

If $k > 1$, then our definition extends to the complex numbers z with $|z| \leq 1$.

Note that for $k = 1$, the above series is equal to $-\log(1 - z)$ for $|z| < 1$. Using this polylogarithm function Gun, Murty and Rath [19] formulated the following conjecture which is an analogue to Baker's theorem on linear forms in logarithms.

Conjecture 1.1.4 (Polylog). *Suppose that $\alpha_1, \cdots, \alpha_n$ are algebraic numbers with $|\alpha_i| \leq 1$ for $1 \leq i \leq n$, such that $Li_k(\alpha_1), \cdots, Li_k(\alpha_n)$ are linearly independent over $\mathbb{Q}$. Then they are linearly independent over the field of algebraic numbers $\overline{\mathbb{Q}}$.*

Note that the case $k = 1$ is a special case of Baker's theorem. Gun, Murty and Rath investigated the Chowla-Milnor conjecture in terms of linear independence of polylogarithm functions and proved the following theorem.

Theorem 1.1.5. *Assume that the Polylog conjecture is true. Then the Chowla-Milnor conjecture is true for all $q > 1$ and $k > 1$.*

Further the Chowla-Milnor conjecture is linked to a conjecture of Zagier on multiple zeta values (MZVs). Let us recall the definition of MZV's.

Definition 1.1.6. *Let $l, a_1, \cdots, a_l$ be positive integers with $a_1 > 1$. Then the multiple zeta values (MZVs) are defined as*

$$\zeta(a_1, \cdots, a_l) = \sum_{n_1 > \ldots > n_l \geq 1} \frac{1}{n_1^{a_1} \ldots n_l^{a_l}}.$$

Clearly $l = 1$ gives the classical Riemann zeta function. The sum $a_1 + \cdots + a_l$ is called the weight of the multiple zeta value $\zeta(a_1, \cdots, a_l)$ while l is called the depth of $\zeta(a_1, \cdots, a_l)$.

Definition 1.1.7. *Let $k > 1$ be an integer. We define the k-th Zagier space W_k to be the $\mathbb{Q}$-linear space spanned by all $\zeta(a_1, \cdots, a_l)$ with integers $l \geq 1$, $a_1 > 1$ such that $a_1 + \cdots + a_l = k$.*

In 1992, Zagier [43] conjectured, after discussions with Drinfel'd, Kontsevich and Goncharov, that the dimension of W_k satisfies a recurrence relation like Fibonacci recurrence. His conjecture is given below.

Conjecture 1.1.5 (Zagier). *The dimension d_k of the space W_k for integers $k > 2$ is given by the recurrence relation*

$$\delta_k = \delta_{k-2} + \delta_{k-3}$$

with the initial conditions $\delta_0 = 1, \delta_1 = 0$ and $\delta_2 = 1$.

But not a single example of a Zagier space is known with dimension at least 2. In 2001, Goncharov [18] and in 2002, Terasoma [39] independently proved that the dimension d_k of the space W_k is at most δ_k. In [19], Gun, Murty and Rath have proved the following interesting theorem about the dimension of Zagier spaces.

Theorem 1.1.8. *The Chowla-Milnor conjecture implies that the dimension of W_{4d+2} is at least 2 for all $d \geq 1$.*

In the same paper [19], they also formulated a stronger version of the Chowla-Milnor conjecture in the following sence;

Conjecture 1.1.6 (Strong Chowla-Milnor). *For any integers $k > 1$ and $q > 1$, the following $\varphi(q) + 1$ real numbers*

$$1, \zeta(k, a/q) \text{ with } 1 \leq a < q, \ (a, q) = 1$$

are $\mathbb{Q}$-linearly independent.

In relation with the above conjecture, Gun, Murty and Rath defined the following $\mathbb{Q}$-linear spaces.

Definition 1.1.9. *For any two integers $k > 1$ and $q \geq 2$, define the Strong Chowla-Milnor space $\widehat{V}_k(q)$ by*

$$\widehat{V}_k(q) := \mathbb{Q} - \text{span of } \{1, \zeta(k, a/q) : 1 \leq a < q, \ (a, q) = 1\}.$$

The Strong Chowla-Milnor conjecture is the assertion that the dimension of $\widehat{V}_k(q)$ is equal to $\varphi(q)+1$. In relation to $\widehat{V}_k(q)$, they proved the following proposition.

Proposition 1.1.10. *Let $k > 1$ be an odd integer. Then the following statements are equivalent:*

1. Either $\dim_{\mathbb{Q}} \widehat{V}_k(3) = 3$ *or* $\dim_{\mathbb{Q}} \widehat{V}_k(4) = 3$.
2. *The number $\zeta(k)$ is irrational.*

Let $\mathbb{F}$ be a number field. In [20], Gun, Murty and Rath considered a number field extension of the conjecture of Chowla and Milnor. To begin with, they defined the following $\mathbb{F}$-linear spaces;

Definition 1.1.11. *Let $q > 1$ be an integer. For any integers $k > 1$, let $V_k(q, \mathbb{F})$ be the $\mathbb{F}$-linear space defined by*

$$V_k(q, \mathbb{F}) := \mathbb{F} - \text{span of } \{\zeta(k, a/q) : 1 \leq a < q, \ (a,q) = 1\}.$$

Note that the $\mathbb{F}$-dimension of $V_k(q, \mathbb{F})$ for any fixed k and q depends on the arithmetic of the ambient number field. However, if the number field $\mathbb{F}$ is linearly disjoint from the cyclotomic field $\mathbb{Q}(\zeta_q)$, then the story is expected to be similar to the linear independence over $\mathbb{Q}$. In this direction, Gun, Murty and Rath [20] have the following theorem;

Theorem 1.1.12. *Let $q > 1$ be an integer and $\mathbb{F}$ be a number field such that $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then*

$$\dim_{\mathbb{F}} V_k(q, \mathbb{F}) \geq \frac{\varphi(q)}{2}$$

for all integers $k > 1$.

They also showed a conditional improvement of the lower bound of the dimension of $V_k(q, \mathbb{F})$. Their theorem is the following:

Theorem 1.1.13. *Let $k > 1$ be an odd integer and $q, r > 2$ be two co-prime integers. Also, let $\mathbb{F}$ be a subfield of the real numbers such that $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q} = \mathbb{F} \cap \mathbb{Q}(\zeta_r)$ and also $\mathbb{F}(\zeta_q) \cap \mathbb{F}(\zeta_r) = \mathbb{F}$. Then either*

$$\dim_{\mathbb{F}} V_k(q, \mathbb{F}) \geq \frac{\varphi(q)}{2} + 1$$

or

$$\dim_{\mathbb{F}} V_k(r, \mathbb{F}) \geq \frac{\varphi(r)}{2} + 1.$$

In the same paper [20], Gun, Murty and Rath formulated a variant of the Chowla-Milnor conjecture which is the following:

Conjecture 1.1.7. *Let $q > 1$ be an integer and $\mathbb{F}$ be a number field such that $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then $\dim_{\mathbb{F}} V_k(q, \mathbb{F}) = \varphi(q)$ for all integers $k > 1$.*

Clearly, the above conjecture is the assertion that the following $\varphi(q)$ real numbers

$$\zeta(k, a/q) \text{ with } 1 \leq a < q, \ (a, q) = 1,$$

are linearly independent over the number field $\mathbb{F}$ which is linearly disjoint from $\mathbb{Q}(\zeta_q)$. This conjecture is a generalization of the Chowla-Milnor conjecture and has many implications. They showed that this conjecture is also related to the Polylog conjecture.

Theorem 1.1.14. *Assume that the Polylog conjecture is true. Then the conjecture* (1.1.7) *is true.*

1.1.3 Results of Davenport and Heilbronn and recent developments

In a classical paper [14], Davenport and Heilbronn studied the zeros of Hurwitz zeta function in the half plane beyond the line $\Re(s) > 1$. In 1931, they proved the following theorems:

Theorem 1.1.15. *If a is a rational number and $a \neq 1/2$ or 1, $\zeta(s, a)$ has infinitely many zeros for $\Re(s) > 1$.*

Theorem 1.1.16. *For transcendental a, $\zeta(s, a)$ has infinitely many zeros for $\Re(s) > 1$.*

Note that for $a = 1/2$ or 1, $\zeta(s, a)$ has an Euler product since $\zeta(s, 1) = \zeta(s)$ and $\zeta(s, 1/2) = (2^s - 1)\zeta(s)$. Therefore $\zeta(s, 1)$ and $\zeta(s, 1/2)$ can not have zeros for $\Re(s) > 1$. On the other hand, when a is an algebraic irrational, the question becomes rather delicate. In an ingenious paper, Cassels [6] proved the existence of infinitely many zeros of $\zeta(s, a)$ for such an a in the half plane $\Re(s) > 1$.

Let f be a periodic arithmetic function and $L(s, f)$ be the L-function associated to f. Recently in 2009, E. Saias and A. Weingartner [36] showed that $L(s, f)$ has infinitely many zeros for $\Re(s) > 1$, if $L(s, f)$ is not a product of $L(s, \chi)$ and a Dirichlet polynomial.

For the precise statement of their theorem, let us first define some notations. Let $a = (a_n)_{n \geq 1}$ be a periodic sequence of complex numbers and $F_a(s)$ be the meromorphic continuation of $\sum_{n \geq 1} \frac{a_n}{n^s}$. Also let $N_a(\sigma_1, \sigma_2, T)$ (respectively $N'_a(\sigma_1, \sigma_2, T)$) be the number of zeros of $F_a(s)$ in the rectangle $\sigma_1 < \Re(s) < \sigma_2$, $|\Im(s)| \leq T$, counted with their multiplicities (respectively without their multiplicities). Then they proved the following theorem;

Theorem 1.1.17. *Let $a = (a_n)_{n\geq 1}$ be a periodic sequence of complex numbers such that $F_a(s)$ is not of the form $P(s)L(s,\chi)$, where P is a Dirichlet polynomial and $L(s,\chi)$ is the L-function associated with a Dirichlet character χ. Then there exists a positive number η such that, for all real numbers σ_1 and σ_2 with $1/2 < \sigma_1 < \sigma_2 \leq 1+\eta$, there exist positive real numbers c_1, c_2 and T_0 with the property that for all $T \geq T_0$,*

$$c_1 T \leq N'_a(\sigma_1, \sigma_2, T) \leq N_a(\sigma_1, \sigma_2, T) \leq c_2 T.$$

1.2 Main Results

In [8], we give an alternate proof of theorem (1.1.3) by an explicit computation of Hurwitz zeta values in terms of cotangents, avoiding the Fourier expansion of Bernoulli function which was the strategy employed by Gun, Murty and Rath [19].

In [9], we study the Strong Chowla-Milnor conjecture and establish certain consequences of the Strong Chowla-Milnor conjecture. In this direction, we have the following proposition.

Proposition 1.2.1. *Let $k > 1, q > 1$ be two integers and f be a rational valued arithmetic periodic function with period q. Suppose that $f(a) = 0$ for $1 < (a,q) < q$. Then the following statements are equivalent:*

1. *The Strong Chowla-Milnor conjecture is true.*

2. *The L-value $L(k,f)$ is irrational, unless*

$$f(a) = -\frac{f(q)q^{-k}}{\prod\limits_{\substack{p\in \mathrm{P}, \\ p\mid q}} (1-p^{-k})}$$

for $1 \leq a < q, (a,q) = 1$. Here P denotes the set of primes.

Following the Polylog conjecture (1.1.4) of Gun, Murty and Rath, We formulate a stronger conjecture about the polylogarithms, which is a generalization of Baker's theorem about linear forms in logarithms. Our conjecture is given below.

Conjecture 1.2.1 (Strong Polylog). *Suppose $\alpha_1, \cdots, \alpha_n$ are algebraic numbers with $|\alpha_i| \leq 1$ for $1 \leq i \leq n$, such that $Li_m(\alpha_1), \cdots, Li_m(\alpha_n)$ are linearly independent over $\mathbb{Q}$. Then $1, Li_m(\alpha_1), \cdots, Li_m(\alpha_n)$ are linearly independent over $\overline{\mathbb{Q}}$.*

Clearly for $m = 1$ the above conjecture reduces to a consequence of Baker's theorem about linear forms in logarithms. In [9], we establish a link between the Strong Polylog conjecture and the Strong Chowla-Milnor conjecture which is stated as follows.

Theorem 1.2.2. *The Strong Polylog conjecture implies the Strong Chowla-Milnor conjecture for all $q > 1$ and $k > 1$.*

The Strong Chowla-Milnor spaces $\widehat{V}_k(q)$ and their dimensions are of special interest to us. The dimension of these spaces are expected to illuminate our understanding of Riemann zeta values at odd positive integers greater than 1.

Clearly from the definition of the Strong Chowla-Milnor space,

$$\dim_{\mathbb{Q}} \widehat{V}_k(q) \leq \varphi(q) + 1.$$

In relation to $\widehat{V}_k(q)$, we prove the following non-trivial lower bound of the dimension of the Strong Chowla-Milnor spaces $\widehat{V}_k(q)$.

Theorem 1.2.3. *Let $k > 1$ and $q > 2$ be two integers. Then*

$$\dim_{\mathbb{Q}} \widehat{V}_k(q) \geq \frac{\varphi(q)}{2} + 1.$$

One can show that any improvement of the above lower bound of the Strong Chowla-Milnor spaces will imply the irrationality of both the numbers $\zeta(k)$ and $\zeta(k)/\pi^k$ simultaneously for all odd positive integers $k > 1$.

In the theorem (1.1.3), Gun, Murty and Rath proved a conditional improvement of the lower bound of the Chowla-Milnor spaces. One can ask a similar type of question for the dimension of the Strong Chowla-Milnor space i.e., for an odd integer $k > 1$ and two co-prime integers $q, r > 2$, whether or not either

$$\dim_{\mathbb{Q}} \widehat{V}_k(q) \geq \frac{\varphi(q)}{2} + 2$$

or

$$\dim_{\mathbb{Q}} \widehat{V}_k(r) \geq \frac{\varphi(r)}{2} + 2.$$

But if the above statement is true, then clearly the Strong Chowla-Milnor conjecture is true for either $q = 3$ or $q = 4$ i.e. either $\dim_{\mathbb{Q}} \widehat{V}_k(3) = 3$ or $\dim_{\mathbb{Q}} \widehat{V}_k(4) = 3$. Then from the proposition (1.1.10) we get that $\zeta(k)$ is irrational for all odd $k > 1$. In general we do not know for all odd integers k whether $\zeta(k)$ is irrational or not. It is known, thanks to Apery, that $\zeta(3)$ is irrational. On the other hand by a theorem of K. Ball and T. Rivoal (see [35] and [4]), it is known that $\zeta(k)$ is irrational for infinitely many odd $k > 1$. In this regards, we prove the following theorem.

Theorem 1.2.4. *Let $k > 1$ be an odd integer with $\zeta(k)$ irrational and $q, r > 2$ be two co-prime integers. Then either*

$$\dim_{\mathbb{Q}} \widehat{V}_k(q) \geq \frac{\varphi(q)}{2} + 2$$

or

$$\dim_{\mathbb{Q}} \widehat{V}_k(r) \geq \frac{\varphi(r)}{2} + 2.$$

Thus in particular, for infinitely many odd integers $k > 1$ there exists an integer q_0 such that

$$\dim_{\mathbb{Q}} \widehat{V}_k(q) \geq \frac{\varphi(q)}{2} + 2$$

for any integer q co-prime to q_0.

We note that an unconditional improvement of the above lower bound about the dimension of these spaces would imply the irrationality of both $\zeta(2k+1)$ and $\zeta(2k+1)/\pi^{2k+1}$ simultaneously for all $k \geq 1$. A curious corollary of the above theorem is the following:

Corollary 1.2.5. *Either both the numbers given by the infinite series*

$$\frac{1}{1^3} + \frac{1}{4^3} + \frac{1}{7^3} + \cdots$$

and

$$\frac{1}{2^3} + \frac{1}{5^3} + \frac{1}{8^3} + \cdots$$

are irrational or both the numbers given by the infinite series

$$\frac{1}{1^3} + \frac{1}{5^3} + \frac{1}{9^3} + \cdots$$

and

$$\frac{1}{3^3} + \frac{1}{7^3} + \frac{1}{11^3} + \cdots$$

are irrational.

Now if we consider the Strong Chowla-Milnor space over the field of algebraic numbers $\overline{\mathbb{Q}}$, then we can not expect the same bound for the dimensions of these spaces. The following is our proposition about their dimensions.

Proposition 1.2.6. $2 \leq \dim_{\overline{\mathbb{Q}}} \widehat{V}_k(q) \leq \frac{\varphi(q)}{2} + 2$.

We also investigate multiple zeta values over a certain class of algebraic number fields. To begin with, let us define the following linear space of MZVs over certain class of algebraic number fields as follows.

Definition 1.2.7. *Let $q > 1$ be an integer and $\mathbb{F}$ be a number field such that $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. For any integer $k > 1$, we define the generalized k-th Zagier space as the $\mathbb{F}$-linear space $V_k(\mathbb{F})$ defined by*

$$V_k(\mathbb{F}) = \mathbb{F} - \text{span of } \{\zeta(a_1, \cdots, a_l) | a_1 + \cdots + a_l = k\}$$

where l is varying.

We also study the conjecture (1.1.7) and apply this conjecture to prove the following theorem analogous to the theorem (1.1.8) over certain class of number fields.

Theorem 1.2.8. *Let d be a positive integer. Then the conjecture* (1.1.7) *implies*

$$\dim_{\mathbb{F}} V_{4d+2}(\mathbb{F}) \geq 2.$$

One can use the conjecture (1.1.7) to study various interesting problems. For instance, we investigate linear independence of the values of the Dirichlet L-function $L(k, \chi)$, as χ ranges over non-trivial Dirichlet characters modulo some integer $q > 1$, over certain family of algebraic number fields. This is in the spirit of a theorem of Murty and Saradha [28]. Here is our theorem.

Theorem 1.2.9. *Let $\mathbb{F}$ be an algebraic number field and $\mathbb{F}_1 = \mathbb{F}(e^{2\pi i/\varphi(q)})$. Suppose that $\mathbb{F}_1 \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Assume the conjecture* (1.1.7)*. Then for any positive integer k, the values $L(k, \chi)$ as χ ranges over non-trivial Dirichlet characters mod q are linearly independent over $\mathbb{F}_1$.*

In fact for $k > 1$, the above theorem is true for any Dirichlet characters mod q, i.e. one can include the principal character mod q.

Next in [11], we consider the possible number field extension of the Strong Chowla-Milnor conjecture. Noting that the arithmetic of the ambient number field is relevant while carrying out such an extension, we formulate the following conjecture:

Conjecture 1.2.2. *Let $q > 1$ be any integer and ζ_q be a primitive q-th root of unity. Let $\mathbb{F}$ be a number field which is linearly disjoint with $\mathbb{Q}(\zeta_q)$. Then for any integer $k > 1$, the following $\varphi(q) + 1$ real numbers*

$$1, \zeta(k, a/q) \text{ with } 1 \leq a < q, \ (a, q) = 1$$

are linearly independent over $\mathbb{F}$.

This conjecture has many consequence. For example, if this conjecture is true for all number fields $\mathbb{F}$ which are linearly disjoint with $\mathbb{Q}(\zeta_q)$, then both the numbers $\zeta(2k+1)$ and $\zeta(2k+1)/\pi^{2k+1}$ are transcendental simultaneously for all $k \geq 1$. We begin with the following proposition:

Proposition 1.2.10. *Let $k > 1, q > 1$ be two integers and $\mathbb{F}$ be an algebraic number field such that $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Let $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{F}$ with $f(a) = 0$ for $1 < (a, q) < q$. Then the following statements are equivalent:*

1. *The conjecture* (1.2.2) *is true.*
2. *The L-value $L(k, f) \notin \mathbb{F}$, unless*

$$f(a) = -\frac{f(q)q^{-k}}{\prod\limits_{\substack{p \text{ prime,} \\ p|q}} (1 - p^{-k})}$$

for $1 \leq a < q, (a, q) = 1$.

Using the above proposition (1.2.10), we link the Strong Polylog conjecture and the conjecture (1.2.2). Here is the precise statement.

Theorem 1.2.11. *The Strong Polylog conjecture implies the conjecture (1.2.2) for all number field $\mathbb{F}$ with $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$.*

Definition 1.2.12. *For any two integers $k > 1$ and $q > 1$, let $\widehat{V}_k(q, \mathbb{F})$ be the $\mathbb{F}$-linear space defined by*

$$\widehat{V}_k(q, \mathbb{F}) := \mathbb{F} - \text{span of } \{1, \zeta(k, a/q) : 1 \leq a < q, \ (a, q) = 1\}.$$

Then for any number field $\mathbb{F}$ with $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, the conjecture (1.2.2) implies that the dimension of $\widehat{V}_k(q, \mathbb{F})$ is equal to $\varphi(q) + 1$. Without any assumption, we also manage to derive lower bounds for these $\mathbb{F}$-linear spaces analogous to those obtained for the $\mathbb{Q}$-linear spaces $\widehat{V}_k(q)$ by us. Here we have the following theorem:

Theorem 1.2.13. *Let $k > 1$ and $q > 1$ be two integers and $\mathbb{F}$ be a number field with $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then*

$$\dim_{\mathbb{F}} \widehat{V}_k(q, \mathbb{F}) \geq \frac{\varphi(q)}{2} + 1.$$

We note that any improvement of the above lower bound for all number fields $\mathbb{F}$ with $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$ will imply both $\zeta(k)$ and $\zeta(k)/\pi^k$ are transcendental for all integers $k > 1$. In particular, we prove the following two propositions.

Proposition 1.2.14. *Let $k > 1$ be an odd integer and $\mathbb{F}$ be a number field with $\mathbb{F} \cap \mathbb{Q}(i) = \mathbb{Q}$. Then $\dim_{\mathbb{F}} \widehat{V}_k(4, \mathbb{F}) = 3$ for all such $\mathbb{F}$ implies $\zeta(k)$ is transcendental.*

Proposition 1.2.15. *Let $k > 1$ be an odd integer and ω be a primitive cube root of unity. Let $\mathbb{F}$ be a number field with $\mathbb{F} \cap \mathbb{Q}(\omega) = \mathbb{Q}$. Then $\dim_{\mathbb{F}} \widehat{V}_k(3, \mathbb{F}) = 3$ for all such $\mathbb{F}$ implies $\zeta(k)$ is transcendental.*

Our main theorem in this connection, is the following.

Theorem 1.2.16. *Let $k > 1$ be an odd integer and $q, r > 2$ be two co-prime integers. Also, let $\mathbb{F} \subseteq \mathbb{R} \cap \overline{\mathbb{Q}}$ such that $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q} = \mathbb{F} \cap \mathbb{Q}(\zeta_r)$ and $\mathbb{F}(\zeta_q) \cap \mathbb{F}(\zeta_r) = \mathbb{F}$. Assume that $\zeta(k) \notin \mathbb{F}$. Then either*

$$\dim_{\mathbb{F}} \widehat{V}_k(q, \mathbb{F}) \geq \frac{\varphi(q)}{2} + 2$$

or

$$\dim_{\mathbb{F}} \widehat{V}_k(r, \mathbb{F}) \geq \frac{\varphi(r)}{2} + 2.$$

The last part of our thesis address questions that are of analytic nature. This work [10] is motivated by a work of Davenport, Heilbronn [14] and Cassels [6] mentioned before.

For a periodic arithmetic function f with period $q \geq 1$ and $a > 0$, consider the L-function

$$L(s, f, a) := \sum_{n=0}^{\infty} \frac{f(n)}{(n+a)^s},$$

where $s \in \mathbb{C}$ with $\Re(s) = \sigma > 1$. This can be thought of as generalized Hurwitz zeta function. In a recent work, Saias and Weingartner [36] showed that $L(s, f, a)$ has infinitely many zeros for $\sigma > 1$ for $a = 1$ and $L(s, f)$ is not a product of $L(s, \chi)$ and a Dirichlet polynomial. In [10], we study the zeros of $L(s, f, a)$ in the region $\sigma > 1$ for arbitrary positive real number a. In this direction, we have the following theorem.

Theorem 1.2.17. *Let a be a positive transcendental number and f be a real valued periodic arithmetic function with period $q \geq 1$. If $L(s, f, a)$ has a pole at $s = 1$, then $L(s, f, a)$ has infinitely many zeros for $\sigma > 1$.*

Theorem 1.2.18. *Let a be a positive algebraic irrational number and f be a positive valued periodic arithmetic function with period $q \geq 1$. Also let*

$$c \ := \ \frac{\max\limits_{n} f(n)}{\min\limits_{n} f(n)} < 1.15.$$

If $L(s, f, a)$ has a pole at $s = 1$, then $L(s, f, a)$ has infinitely many zeros for $\sigma > 1$.

In particular, we show that for each $\delta > 0$, there is a zero of $L(s, f, a)$ in the region $1 < \Re(s) < 1 + \delta$. Moreover $L(s, f, a)$ has infinitely many zeros on the line $\Re(s) = \sigma$.

In the other direction, we study zero-free regions for $L(s, f, a)$ when f is a periodic arithmetic function not identically zero and prove the following theorem.

Theorem 1.2.19. *Let f be a non-zero periodic arithmetic function with period $q \geq 1$. Also, let a be a positive real number and*

$$c = \max_{1 \leq b \leq q} \{1, \ |f(b)|\}.$$

Then we have $L(s, f, a) \neq 0$ for $\sigma > 1 + c'(a + n_0)$, where n_0 is the smallest positive integer such that $f(n_0) \neq 0$ and $c' = c/|f(n_0)|$.

Finally, as an application of the above theorem, we prove a variant of a conjecture of Erdös (see [25]) about non vanishing of $L(1, f)$, where f belongs to a certain class of rational valued arithmetic functions. In this direction our theorem is the following:

Theorem 1.2.20. *Let f be a non-zero periodic arithmetic function with period $q > 1$ and*

$$f(n) = \begin{cases} \pm\lambda & \text{if } q \nmid n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $L(k, f) \neq 0$ for all integers $k \geq 2$.

1.3 Organization of the Thesis

In this section, we give a concise summary of the topics we discuss in this thesis for the convenience of the reader.

The **first chapter** deals with the history of our research problems and our main results. We briefly discuss a conjecture of Chowla-Chowla and Milnor's generalization of this conjecture. We also discuss about recent developments towards this conjecture by Gun, Murty and Rath. We mention a classical problem of Davenport, Heilbronn and Cassels. Finally we mention all our results link with this thesis.

The **second chapter** recalls various basic definitions, known results etc. To begin with, we define number fields, its discriminants and some theorem about ramified primes. We mention all the theorems which are required for the later chapters.

In the **third chapter**, we discuss our first research problem. We give an alternate proof of theorem (1.1.3) about the lower bound of the Chowla-Milnor spaces. Okada's theorem (see (2.6.2)) is one of the main tools for proving this result. We prove this by an explicit computation of Hurwitz zeta values in terms of cotangents at rational arguments.

In the **fourth chapter**, we study the dimension of the Strong Chowla-Milnor spaces. We prove a conditional lower bound of the dimension of those spaces. We note that an unconditional improvement of the lower bound of the dimension of these spaces would imply the irrationality of both $\zeta(2k+1)$ and $\zeta(2k+1)/\pi^{2k+1}$ simultaneously for all $k \geq 1$. We also establish a relation between the Strong Polylog conjecture (1.2.1) and the Strong Chowla-Milnor conjecture (1.1.6).

In the **fifth chapter**, we investigate the conjecture (1.1.7) of Gun, Murty and Rath. This conjecture is a variant of the Chowla-Milnor conjecture over number fields which are linearly disjoint from the cyclotomic field. We define "generalized Zagier spaces" generated by multiple zeta values of a fixed weight over number fields which are linearly disjoint from the cyclotomic field. Assuming conjecture (1.1.7), we prove that a family of these spaces has dimension at least 2. We also investigate the values of $L(k, \chi)$, as χ ranges over non-trivial Dirichlet characters modulo a positive integer q, over a certain family of algebraic number fields. Assuming conjecture (1.1.7), we prove that this L-values are linearly independent over the said family of number fields.

In the **penultimate chapter**, we consider a possible number field extension of the Strong Chowla-Milnor conjecture. Assuming this conjecture (1.2.2) for all number fields $\mathbb{F}$ with $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, we show that $\zeta(k)$ is transcendental for all odd integers $k > 1$. We also prove that the Strong Polylog conjecture (1.2.1) implies this conjecture for all number fields $\mathbb{F}$ with $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. For a number field $\mathbb{F}$ and integers $k, q > 1$, we define an $\mathbb{F}$-linear space $\widehat{V}_k(q, \mathbb{F})$. For any $k, q > 1$, we prove a lower bound of the dimension of these spaces over $\mathbb{F}$, when $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$ and $\mathbb{F} = \mathbb{Q}(\zeta_q)$ separately. we also establish a conditional improvement of the lower bound of the dimension of these spaces.

In the **last chapter**, we consider a generalization of a problem of Davenport, Heilbronn and Cassels. We study certain generalized Hurwitz zeta functions for periodic arithmetic functions. We show that there exists infinitely many zeros of certain generalized Hurwitz zeta functions in its half plane of absolute convergence. We also give a zero-free region for these functions. Finally, as an application we prove a variant of a conjecture of Erdös.

Chapter 2

Preliminaries

In this chapter we recall some basic definitions, state some known results and fix some notations which will be used throughout the thesis. We also provide proof of some of the preliminary results which have a significant role in proving some of the theorems in this thesis. In the first section, we recall some basic facts from algebraic number theory which we will use in the later chapters. We follow [31], [42] and [22] for most of the material of this section. In the second section, we recall some basic definitions and properties of Dirichlet characters. We follow [1], [27] and [31] for most of the material in this section. In the third section, we recall some definitions related to the Dirichlet series and L-functions. We also note down Dirichlet's theorem about the non-vanishing of $L(1, \chi)$ and establish some important identities. We follow [27] for most of the material in this section. In the fourth section, we define Gauss sums and Kronecker's symbol associated to a fundamental discriminant. We follow [27] and [31] for the mentioned theorem of this section. In the fifth section, we discuss about the Dedekind-Frobenius determinant. We mention a proof of this result. The sixth section is devoted to a theorem of Okada [32]. We give a proof of this theorem following Wang [41]. In the seventh section, we state a theorem of Kronecker and a well known theorem of Rouché from the complex function theory. We also state a crucial lemma due to Cassels [6] and make some remarks about the results. In the penultimate section, we state a fundamental theorem of Baker [2]. We also note some consequence of this theorem. In the last section, we mention a conjecture of Erdös and discuss about all the known results towards this conjecture.

2.1 Some results from Algebraic Number Theory

Definition 2.1.1. *An algebraic number field K is a finite extension of the field of rational numbers.*

By degree of an algebraic number field K, we mean the dimension of K over $\mathbb{Q}$ as a vector space.

Definition 2.1.2. *An element $\alpha \in K$ is said to be an algebraic integer if there is a monic polynomial $P(X) \in \mathbb{Z}[X]$ with $P(\alpha) = 0$. The collection of all such algebraic integers forms a ring, which is called the ring of integers of the number field K and is denoted by $\mathcal{O}_K$.*

Let $K/\mathbb{Q}$ be an algebraic number field and $\mathbf{p} = (p)$ be a prime ideal in $\mathbb{Z}$. The prime ideal $\mathbf{p}$ may not remain prime in $\mathcal{O}_K$. The ideal $\mathbf{p}\mathcal{O}_K$ of $\mathcal{O}_K$ has a factorization as

$$\mathbf{p}\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g},$$

where $\mathfrak{p}_1, \cdots, \mathfrak{p}_g$ are distinct prime ideals in $\mathcal{O}_K$ and $e_1, \cdots, e_g$ are positive integers. The integer e_i is called the ramification index of the prime ideal $\mathfrak{p}_i$ with respect to $\mathbb{Z}$ and is denoted either by $e(\mathfrak{p}_i/\mathbf{p})$ or $e(\mathfrak{p}_i/\mathbb{Z})$. Again we have $\mathfrak{p}_i \cap \mathbb{Z} = \mathbf{p}$ for each i as $\mathbf{p} \subset \mathfrak{p}_\mathbf{i}$ and $\mathbf{p}$ is also a maximal ideal in $\mathbb{Z}$.

Definition 2.1.3. *Let $K/\mathbb{Q}$ be an algebraic number field and $\mathfrak{p}$ be a non zero prime ideal of $\mathcal{O}_K$ such that $\mathfrak{p} \cap \mathbb{Z} = \mathbf{p}$. We say $\mathfrak{p}$ is ramified over $\mathbb{Z}$ if the ramification index $e(\mathfrak{p}/\mathbb{Z})$ is greater than 1. We say the prime ideal $\mathbf{p}$ is ramified in $\mathcal{O}_K$ if $\mathbf{p}\mathcal{O}_K$ is divisible by some ramified prime ideal of $\mathcal{O}_K$. A prime ideal $\mathbf{p}$ is said to be unramified in $\mathcal{O}_K$ if it is not ramified.*

Definition 2.1.4. *Let $K/\mathbb{Q}$ be an algebraic number field and $\alpha \in K$. The relative trace and norm of α over $\mathbb{Q}$ are defined respectively to be the trace and determinant of the endomorphism of the $\mathbb{Q}$-vector space K*

$$T_\alpha : K \to K$$

defined by

$$T_\alpha(x) = \alpha x,$$

i.e.,

$$Tr_{K/\mathbb{Q}}(\alpha) = Tr(T_\alpha), \;\; N_{K/\mathbb{Q}}(\alpha) = \det(T_\alpha).$$

Definition 2.1.5. *Let $K/\mathbb{Q}$ be an algebraic number field with $[K : \mathbb{Q}] = n$ and $\{\alpha_1, \cdots, \alpha_n\}$ be a basis for $\mathcal{O}_K$ over $\mathbb{Z}$. The discriminant of the number field K is defined by*

$$\Delta(\alpha_1, \cdots, \alpha_n) = \det[Tr_{K/\mathbb{Q}}(\alpha_i\alpha_j)].$$

One denotes it by $\Delta(\mathcal{O}_K/\mathbb{Z})$ or Δ_K.

Let d be a square-free integer and $K = \mathbb{Q}(\sqrt{d})$ be a quadratic extension. Then the discriminant of K is called a fundamental discriminant and is equal to

$$\Delta_K = \begin{cases} d & \text{if } d \equiv 1 (\text{mod}) 4 \\ 4d & \text{if } d \equiv 2, 3 (\text{mod}) 4. \end{cases}$$

Hence any fundamental discriminant is congruent to 0 or 1 modulo 4.

Theorem 2.1.6. *Let $K/\mathbb{Q}$ be a number field, and let $\Delta(\mathcal{O}_K/\mathbb{Z})$ be the discriminant of K. A rational prime p ramifies in $\mathcal{O}_K$ if and only if p divides $\Delta(\mathcal{O}_K/\mathbb{Z})$.*

For a proof of the above theorem see theorem 7.3 of [22].

Theorem 2.1.7. *Let $K/\mathbb{Q}$ be a number field in which no rational prime ramifies, then $K = \mathbb{Q}$.*

For a proof of the above theorem see lemma 14.3 of [42].

Let us now discuss about cyclotomic fields. Let p be a rational prime and consider the cyclotomic field $K = \mathbb{Q}(\zeta_{p^n})$, where ζ_{p^n} denote a primitive p^n-th root of unity. The ring of integers of this field is $\mathcal{O}_K = \mathbb{Z}[\zeta_{p^n}]$. The discriminant of $\mathcal{O}_K$ over $\mathbb{Z}$ is

$$\pm p^{p^{n-1}(pn-n-1)}.$$

We will now mention two propositions which we will use in future chapters.

Proposition 2.1.8. *Let m be a positive integer. A rational prime p ramifies in $\mathbb{Q}(\zeta_m)$ if and only if p divides m.*

Proposition 2.1.9. *Let m and n be two positive integers with $(m, n) = 1$. Then $\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$.*

2.2 Dirichlet Characters

An arithmetic function f is a complex-valued function defined on the set of positive integers. We say that f is multiplicative if

$$f(mn) = f(m)f(n)$$

for all co-prime integers m and n. On the other hand if the above equation holds for all integers m and n, then f is completely multiplicative.

Definition 2.2.1. *Let G be a finite group. A character of G is a group homomorphism*

$$\chi : G \to S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

The collection of all characters of a group G forms a group, which is called the character group of G. We denote this character group by $\widehat{G}$.

Definition 2.2.2. *Let q be a positive integer. A Dirichlet Character χ modulo q is a homomorphism*

$$\chi : (\mathbb{Z}/q\mathbb{Z})^* \to S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

One can extend the domain of the definition of Dirichlet characters using periodicity, i.e., for any integer n with $(n, q) = 1$ define

$$\chi(n) = \chi(a),$$

where $n = kq + a$ with $1 \leq a < q, (a, q) = 1$ and define zero otherwise. For a given modulus q, there are $\varphi(q)$ many Dirichlet characters.

The Dirichlet characters are completely multiplicative arithmetic functions. A Dirichlet character is called primitive if it does not factor through any $(\mathbb{Z}/q'\mathbb{Z})^*$, where q' is a proper divisor of q. In other words, it does not arise as the composite

$$(\mathbb{Z}/q\mathbb{Z})^* \longrightarrow (\mathbb{Z}/q'\mathbb{Z})^* \xrightarrow{\chi'} S^1$$

where q' is a proper divisor of q and χ' is a Dirichlet character mod q'.

A Dirichlet character mod q is called principal if it takes value 1 for co-prime residue classes mod q and zero otherwise. We denote the principal Dirichlet character by χ_0. The trivial Dirichlet character is the unique character of modulus 1.

A Dirichlet character χ is said to be an even character if $\chi(-1) = 1$ and is said to be an odd character if $\chi(-1) = -1$.

A quadratic Dirichlet character is a non trivial involution in the group of characters modulo q, that is, it has order 2 in the said group. So a quadratic character takes values 1, 0 and -1 only, with at least one -1. We say a character is real if all its values are real. Consequently a real character can be either the principal character or a quadratic character.

Proposition 2.2.3. *Let q be a positive integer and χ be a Dirichlet character modulo q. Then*

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi(n) = \begin{cases} \varphi(q), & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise.} \end{cases}$$

If $(n, q) = 1$, then

$$\sum_{\chi} \chi(n) = \begin{cases} \varphi(q), & \text{if } n \equiv 1 \pmod{q}, \\ 0 & \text{otherwise,} \end{cases}$$

where χ runs over the $\varphi(q)$ Dirichlet characters modulo q.

Proposition 2.2.4. (Orthogonality). *Let q be a positive integer and χ_1, χ_2 be two Dirichlet characters modulo q. Then*

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi_1(n)\overline{\chi_2}(n) = \begin{cases} \varphi(q), & \text{if } \chi_1 = \chi_2, \\ 0 & \text{otherwise.} \end{cases}$$

Let m and n be two integers with $(n,q)=1$. Then we have

$$\sum_{\chi}\chi(m)\overline{\chi}(n)=\begin{cases}\varphi(q), & \text{if } m\equiv n \pmod q,\\ 0 & \text{otherwise,}\end{cases}$$

where χ runs over the $\varphi(q)$ Dirichlet characters modulo q.

Theorem 2.2.5. (Artin). *Let G be a group and $\chi_1,\cdots,\chi_n$ be distinct characters of G. Then they are linearly independent over $\mathbb{C}$.*

For a proof, see Theorem 4.1 of [24].

2.3 Dirichlet series and L-functions

A Dirichlet series on $\mathbb{C}$ is a complex valued function f of the form

$$f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

where a_n are complex number and $s\in\mathbb{C}$. A prototypical example of such function is the classical Riemann zeta function, which is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

for $s\in\mathbb{C}$ with $\Re(s)>1$.

Let q a positive integer and χ be a Dirichlet character modulo q. The Dirichlet L-function associated with χ is defined by

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

for $s\in\mathbb{C}$ with $\Re(s)>1$. Since χ is completely multiplicative, $L(s,\chi)$ has an Euler product

$$L(s,\chi)=\prod_{p}\left(1-\frac{\chi(p)}{p^s}\right)^{-1} \text{ for } \Re(s)>1,$$

where the product is over all primes p.

Now for principal Dirichlet character χ_0, we have

$$L(s,\chi_0)=\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{1}{n^s}=\zeta(s)\prod_{p|q}(1-p^{-s})$$

for $s\in\mathbb{C}$ with $\Re(s)>1$. Hence $L(s,\chi_0)$ has a simple pole at $s=1$ with residue $\varphi(q)/q$ and can be meromorphically extend to the whole complex plane. If $\chi\neq\chi_0$, then $L(s,\chi)$ can be extend to an entire function.

Theorem 2.3.1. (Dirichlet) *Let q be a positive integer and χ be a Dirichlet character modulo q such that $\chi \neq \chi_0$. Then $L(1,\chi) \neq 0$.*

For a proof of the above theorem see theorem 4.9 in [27].

The Hurwitz zeta function is a complex valued function defined by

$$\zeta(s,x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s},$$

where $0 < x \leq 1$ and $s \in \mathbb{C}$ with $\Re(s) > 1$. Note that $\zeta(s,1) = \zeta(s)$, the classical Riemann zeta function.

For a periodic arithmetic function f with period $q > 1$ and $s \in \mathbb{C}$, the Dirichlet L-function associated to f is defined as

$$L(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

with $s \in \mathbb{C}$ and $\Re(s) > 1$.

Since f is periodic with period q, the above series can be written as

$$L(s,f) = \sum_{a=1}^{q} f(a) \sum_{m=0}^{\infty} \frac{1}{(mq+a)^s} \tag{2.1}$$

$$= q^{-s} \sum_{a=1}^{q} f(a)\zeta(s,a/q) \tag{2.2}$$

for $s \in \mathbb{C}$ with $\Re(s) > 1$.

This shows that $L(s,f)$ extends holomorphically to the whole complex plane with a possible simple pole at $s = 1$ with residue $q^{-1}\sum_{a=1}^{q} f(a)$. Hence $L(s,f)$ is an entire function if and only if $\sum_{a=1}^{q} f(a) = 0$.

Now if we substitute the principal Dirichlet character modulo q for f and integer $k > 1$ for s in the equation (2.2), we get the identity

$$\zeta(k) \prod_{\substack{p \in \mathrm{P}, \\ p \mid q}} (1 - p^{-k}) = q^{-k} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \zeta(k, a/q) \tag{2.3}$$

where P denotes the set of all primes.

For $q = p$, a prime, the above identity (2.3) becomes

$$(p^k - 1)\zeta(k) = \sum_{a=1}^{p-1} \zeta(k, a/p). \tag{2.4}$$

2.4 Gauss sums and Kronecker symbol

Let q be a positive integer and χ be a Dirichlet character modulo q. Let $\zeta_q = e^{\frac{2\pi i}{q}}$ and b be an integer. The Gauss sum associated to χ and b, denoted by $\tau(\chi, b)$, is defined as

$$\tau(\chi, b) = \sum_{a=1}^{q} \chi(a)\zeta_q^{ab}.$$

For $b = 1$, we write the Gauss sum associated to χ by $\tau(\chi) = \tau(\chi, 1)$. For $\chi = \chi_0$, it is the Ramanujan's sum

$$c_q(b) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \zeta_q^{ab}.$$

Proposition 2.4.1. *Let q be a positive integer and χ be a primitive Dirichlet character modulo q. Then*

$$\tau(\chi, b) = \overline{\chi}(b)\tau(\chi) \text{ and } |\tau(\chi)| = \sqrt{q}.$$

For a proof of the above proposition see proposition (2.6) of [31].

Definition 2.4.2. *Let Δ be a fundamental discriminant. We define the Kronecker symbol $(\frac{\Delta}{n})$ by the following relations:*

(i) $(\frac{\Delta}{p}) = 0$ when $p|\Delta$ and p prime,

(ii) $(\frac{\Delta}{2}) = \begin{cases} 1 \text{ when } \Delta \equiv 1 \pmod 8, \\ -1 \text{ when } \Delta \equiv 5 \pmod 8, \end{cases}$

(iii) $(\frac{\Delta}{p}) = (\frac{\Delta}{p})_L$, the Legendre symbol, when $p > 2$,

(iv) $(\frac{\Delta}{-1}) = \begin{cases} 1 \text{ when } \Delta > 0, \\ -1 \text{ when } \Delta < 0, \end{cases}$

(v) $(\frac{\Delta}{n})$ *is a completely multiplicative function of n.*

The following theorem shows that the Kronecker symbol gives rise to a quadratic character. For a proof, see theorem (9.13) of [27].

Theorem 2.4.3. *Let Δ be a fundamental discriminant. Then $\chi_\Delta(n) = (\frac{\Delta}{n})$ is a primitive quadratic character modulo $|\Delta|$ and every primitive quadratic character is given uniquely in this way.*

Clearly from the definition $\chi_\Delta(n)$ is an even character if $\Delta > 0$ and an odd character if $\Delta < 0$.

Proposition 2.4.4. *Let Δ be a fundamental discriminant and $\chi_\Delta(n) = (\frac{\Delta}{n})$ be a primitive quadratic character modulo $|\Delta|$. If $\Delta > 0$, then $\tau(\chi_\Delta) = \sqrt{\Delta}$. If $\Delta < 0$ then $\tau(\chi_\Delta) = i\sqrt{-\Delta}$.*

For a proof of the above proposition, see theorem (9.17) of [27].

2.5 Dedekind-Frobenius determinant

The following proposition due to Dedekind will play an important role in the proofs of some theorems in the later chapters.

Proposition 2.5.1. *Let G be any finite abelian group of order n and $\widehat{G}$ be its character group. Let $F : G \to \mathbb{C}$ be any complex-valued function on G. The determinant of the $n \times n$ matrix given by $(F(xy^{-1}))$ as x, y range over the group elements is called the Dedekind-Frobenius determinant and is equal to*

$$\prod_{\chi} \left(\sum_{x \in G} \chi(x) F(x) \right),$$

where the product is over all $\chi \in \widehat{G}$.

Proof. Let V be the space of all complex valued functions on G. Then V is a vector space over $\mathbb{C}$ of dimension equal to n which has two natural bases. First, the collection of all characters of G and secondly the functions $\{\delta_y : y \in G\}$, where

$$\delta_y(z) = \begin{cases} 1 & \text{if } z = y, \\ 0 & \text{otherwise.} \end{cases}$$

Now for each $x \in G$, define

$$T_x : V \to V$$

by,

$$(T_x f)(z) = f(xz),$$

where $z \in G$. Then for any character $\chi \in \widehat{G}$, we have

$$(T_x \chi)(z) = \chi(xz) = \chi(x)\chi(z)$$

i.e.,

$$T_x \chi = \chi(x)\chi.$$

Hence χ is a eigen vector of T_x with eigen value $\chi(x)$.

Now consider the map $T : V \to V$ by

$$Tf = \sum_{x \in G} F(x) T_x f.$$

We now show that T is a linear map. For that observe that

$$\begin{aligned} T(f+g)(z) &= \sum_{x\in G} F(x)T_x(f+g)(z) \\ &= \sum_{x\in G} F(x)(f+g)(xz) \\ &= \sum_{x\in G} F(x)f(xz) + \sum_{x\in G} F(x)g(xz) \\ &= \sum_{x\in G} F(x)(T_x f)(z) + \sum_{x\in G} F(x)(T_x g)(z) \\ &= (Tf+Tg)(z) \end{aligned}$$

and for any $c \in \mathbb{C}$

$$T(cf) = cTf.$$

This shows that T is a linear map. Now we compute the determinant of T with respect to the two bases mentioned above.

For each $\chi \in \widehat{G}$, we have

$$\begin{aligned} T\chi &= \sum_{x\in G} F(x)T_x\chi \\ &= \left(\sum_{x\in G} F(x)\chi(x)\right)\chi. \end{aligned}$$

Hence each χ is an eigen vector of T with eigen value $(\sum_{x\in G}\chi(x)F(x))$. So the determinant of T is

$$\det T = \prod_{\chi}\left(\sum_{x\in G}\chi(x)F(x)\right).$$

Again for each $x, y \in G$, we have

$$(T_x\delta_y)(z) = \delta_y(xz) = \delta_{x^{-1}y}(z)$$

and hence

$$\begin{aligned} T\delta_y &= \sum_{x\in G} F(x)T_x\delta_y \\ &= \sum_{x\in G} F(x)\delta_{x^{-1}y} \\ &= \sum_{x\in G} F(yx^{-1})\delta_x. \end{aligned}$$

This shows that $T = (F(xy^{-1}))$ and hence we have

$$\det(F(xy^{-1})) = \prod_{\chi}\left(\sum_{x\in G}\chi(x)F(x)\right).$$

This completes the proof of the proposition.

□

2.6 Okada's theorem

In 1970, S. Chowla [13] proved the following theorem about the linear independence of co-tangent values at rational arguments:

Theorem 2.6.1. *Let p be a prime. Then the $\frac{1}{2}(p-1)$ real numbers $\cot(\pi a/p)$, $a = 1, \cdots, \frac{1}{2}(p-1)$, are linearly independent over the field of rational numbers $\mathbb{Q}$.*

In 1981, T. Okada [32] generalized Chowla's theorem for all derivatives of cotangent values. This theorem plays a significant role in proving some of the theorems in this thesis.

Theorem 2.6.2. *Let k and q be positive integers with $k \geq 1$ and $q > 2$. Let T be a set of $\varphi(q)/2$ representations mod q such that the union $T \cup (-T)$ constitutes a complete set of co-prime residue classes mod q. Then the set of real numbers*

$$\frac{d^{k-1}}{dz^{k-1}} \cot(\pi z)|_{z=a/q}, \ a \in T$$

is linearly independent over $\mathbb{Q}$.

Immediately after that, K. Wang [41] gave another proof of Okada's theorem. Here we present a proof basically by Wang and modified by P. Rath.

Proof. Let ζ_q be a primitive q-th root of unity and $K = \mathbb{Q}(\zeta_q)$ be the cyclotomic extension of $\mathbb{Q}$ with the Galois group G isomorphic to $(\mathbb{Z}/q\mathbb{Z})^*$ via the map $a \mapsto \sigma_a$ where $\sigma_a(\zeta_q) = \zeta_q^a$. Let us define the following sets

$$K^+ = K \cap \mathbb{R} \text{ and } K^- = K \cap \mathbb{R}i.$$

Note that K^+ and K^- are $\mathbb{Q}$ vector spaces of dimension $\varphi(q)/2$ and are invariant under every Galois action. Moreover, $K^\pm$ is the eigen space of K corresponding to the eigen value ± 1 of the involution σ_{-1}. That is, $K^\pm = \{\alpha \in K | \sigma_{-1}(\alpha) = \pm\alpha\}$ and $K = K^+ \oplus K^-$. Clearly K^+ is a subfield of K, but K^- is not a subfield of K.

We first prove the following assertions.

1. Let $\alpha \in K^-$ and suppose that the sum

$$\sum_{a \in T} \chi(a)\sigma_a(\alpha)$$

does not vanish for all odd Dirichlet characters mod q. Then the set of numbers $\{\sigma_a(\alpha), \ a \in T\}$ forms a $\mathbb{Q}$ basis for K^-.

2.Let $\alpha \in K^+$ and suppose that the sum

$$\sum_{a \in T} \chi(a)\sigma_a(\alpha)$$

does not vanish for all even Dirichlet characters mod q. Then the set of numbers $\{\sigma_a(\alpha),\ a \in T\}$ forms a $\mathbb{Q}$ basis for K^+.

Let us now prove these assertions. We begin by noting that, since $\{1, -1\}$ is a subgroup of $G = (\mathbb{Z}/q\mathbb{Z})^*$, there exists a subgroup H of G such that the quotient $G/H \simeq \{\pm 1\}$ and hence

$$G = (\mathbb{Z}/q\mathbb{Z})^* = H \cup -H.$$

Further note that restrictions of the $\varphi(q)/2$ even Dirichlet characters mod q to H gives all the distinct characters of H. Also every character ψ of H extends to an odd Dirichlet character $\overline{\psi}$ mod q by defining $\overline{\psi}(-h) = -\psi(h)$ for $h \in H$.

Now suppose that $\alpha \in K^-$ and

$$\sum_{a \in T} r_a \sigma_a(\alpha) = 0$$

for some $r_a \in \mathbb{Q}$. Let us define the following function $F : H \to \{\pm 1\}$ where

$$F(h) = \begin{cases} 1 & \text{if } h \in T, \\ -1 & \text{if } h \notin T. \end{cases}$$

Then we have

$$\sum_{a \in T} r_a \sigma_a(\alpha) = \sum_{a \in H} R_a \sigma_a(\alpha) = 0$$

where $R_a = F(a) r_{F(a)a} \in \mathbb{Q}$. Now applying the automorphism $\sigma_{h^{-1}}$, $h \in H$, we get

$$\sum_{a \in H} R_a \sigma_{ah^{-1}}(\alpha) = 0$$

for all $h \in H$. Thus we are done if the determinant of the matrix $M = (\sigma_{ah^{-1}})_{a,h \in H}$ is non-zero. Now by the evaluation of the Dedekind-Frobenius determinant (2.5.1) on the group H, we get the determinant of the matrix M given by

$$\det(M) = \prod_{\psi \in \widehat{H}} \sum_{h \in H} \psi(h) \sigma_h(\alpha).$$

As remarked before, every character ψ of H extends to an odd Dirichlet character $\overline{\psi}$ of G by defining $\overline{\psi}(-h) = -\psi(h)$ for $h \in H$. Again note that

$$\sum_{h \in H} \psi(h) \sigma_h(\alpha) = \sum_{h \in T} \overline{\psi}(h) \sigma_h(\alpha).$$

But the later sum is non-zero by hypothesis and hence $\det(M) \neq 0$.

Now in the second case, we have

$$\sum_{a \in T} r_a \sigma_a(\alpha) = \sum_{a \in H} R_a \sigma_a(\alpha) = 0$$

where $R_a = r_{F(a)a} \in \mathbb{Q}$. Now as before applying the automorphism $\sigma_{h^{-1}}$, $h \in H$, we get

$$\sum_{a \in H} R_a \sigma_{ah^{-1}}(\alpha) = 0$$

for all $h \in H$. Thus as before we are done if the determinant of the matrix $M = (\sigma_{ah^{-1}})_{a,h \in H}$ is non-zero and this determinant is given by

$$\det(M) = \prod_{\psi \in \widehat{H}} \sum_{h \in H} \psi(h)\sigma_h(\alpha).$$

Now since, the restrictions of the $\varphi(q)/2$ even Dirichlet characters mod q to H gives all the distinct characters of H, we have $\sum_{h \in H} \psi(h)\sigma_h(\alpha) = \sum_{h \in T} \psi(h)\sigma_h(\alpha)$, and hence we see that

$$\prod_{\psi \in \widehat{H}} \sum_{h \in H} \psi(h)\sigma_h(\alpha) = \prod_{\substack{\chi \in \widehat{G} \\ \chi \text{ even}}} \sum_{h \in T} \chi(h)\sigma_h(\alpha).$$

Again by the hypothesis, we get $\det(M) \neq 0$.

Now go back to the proof of the theorem. Let $q > 2$ be an integer and T be a subset of $G = (\mathbb{Z}/q\mathbb{Z})^*$ such that T has cardinality $\varphi(q)/2$ and $T \cup (-T) = G$. For simplicity, we denote $\frac{d^{k-1}}{dz^{k-1}} \cot(\pi z)|_{z=a/q}$ by $\cot^{k-1}(\pi a/q)$. Define

$$G(k, a/q) = \zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q).$$

Now for a Dirichlet character χ mod q having the same parity with k, we have

$$\chi(q - a) = (-1)^k \chi(a)$$

and hence, we get

$$\begin{aligned}
\sum_{a \in T} \chi(a) G(k, a/q) &= \sum_{a \in T} \chi(a)[\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q)] \\
&= \sum_{a \in T} \chi(a)\zeta(k, a/q) + \sum_{a \in T} \chi(q - a)\zeta(k, 1 - a/q) \\
&= \sum_{a \in G} \chi(a)\zeta(k, a/q) \\
&= q^k L(k, \chi).
\end{aligned}$$

Again note the following:

1. $i \cot \pi/q = \frac{1+\zeta_q}{1-\zeta_q} \in K^-$.

2. For any $a \in T$, we have

$$\begin{aligned}
\sigma_a(i \cot \pi/q) &= \sigma_a \left(\frac{1 + \zeta_q}{1 - \zeta_q} \right) \\
&= \frac{1 + \sigma_a(\zeta_q)}{1 - \sigma_a(\zeta_q)} \\
&= i \cot(\pi a/q).
\end{aligned}$$

3. $\sigma_a(i(\frac{i}{\pi})^{k-1}\cot^{k-1}(\pi/q)) = i(\frac{i}{\pi})^{k-1}\cot^{k-1}(\pi a/q)$ and

$$i(\frac{i}{\pi})^{k-1}\cot^{k-1}(\pi a/q) \in K^{\pm}$$

according as k is even or odd.

Thus to prove the theorem we need to show

$$\sum_{a\in T}\chi(a)\cot^{k-1}(\pi a/q) \neq 0$$

whenever k and χ have the same parity.

Now since for a non-integral $z \in \mathbb{C}$,

$$\pi\cot(\pi z) = \sum_{n\in\mathbb{Z}}\frac{1}{z+n},$$

we have the following identity (for a proof see lemma (3.2.3))

$$G(k, a/q) = \frac{(-1)^{k-1}}{(k-1)!}(\pi\cot^{k-1}\pi a/q).$$

And hence we get

$$i(\frac{i}{\pi})^{k-1}\cot^{k-1}(\pi a/q) = (-1)^{k-1}(\frac{i}{\pi})^{k}(k-1)!G(k, a/q).$$

Again we see that

$$\begin{aligned}\sum_{a\in T}\chi(a)\sigma_a(i(\frac{i}{\pi})^{k-1}\cot^{k-1}(\pi/q)) &= \sum_{a\in T}\chi(a)(i(\frac{i}{\pi})^{k-1}\cot^{k-1}(\pi a/q)) \\ &= (-1)^{k-1}(\frac{i}{\pi})^{k}(k-1)!\sum_{a\in T}\chi(a)G(k, a/q) \\ &= (-1)^{k-1}(\frac{i}{\pi})^{k}(k-1)!q^{k}L(k,\chi).\end{aligned}$$

Since $L(k,\chi) \neq 0$ for all $k \geq 1$, we deduce that the set of numbers

$$i\left(\frac{i}{\pi}\right)^{k-1}\cot^{k-1}(\pi/q), \qquad a \in T$$

is linearly independent over $\mathbb{Q}$ and hence $\cot^{k-1}(\pi/q)$, $a \in T$ are linearly independent over $\mathbb{Q}$ for all $k \geq 1$. This completes the proof of the theorem. $\square$

In 2009, Murty and Saradha [28] showed that the Okada's theorem can be generalized over number fields which are linearly disjoint from the cyclotomic field $\mathbb{Q}(\zeta_q)$. Their theorem is the following:

Lemma 2.6.3. *Let k and q be positive integers with $k > 0$ and $q > 2$. Let T be a set of $\varphi(q)/2$ representations mod q such that the union $T \cup (-T)$ constitutes a complete set of co-prime residue classes mod q. Let $\mathbb{F}$ be a number field which is linearly disjoint with the cyclotomic field $\mathbb{Q}(\zeta_q)$. Then the set of real numbers*

$$\frac{d^{k-1}}{dz^{k-1}} \cot(\pi z)|_{z=a/q}, \ \ a \in T$$

is linearly independent over $\mathbb{F}$.

2.7 Kronecker's theorem, Rouché's theorem and Cassels lemma

The following theorem due to Kronecker will play an important role in proving some of the theorems in the later chapters.

Theorem 2.7.1. (Kronecker). *Let $a_1, \cdots, a_N$ be real numbers which are linearly independent over the integers and $b_1, \cdots, b_N$ be arbitrary real numbers. Then for any real number T and $\epsilon > 0$, there exists a real number $t > T$ and integers $x_1, \cdots, x_N$ such that*

$$|ta_n - b_n - x_n| < \epsilon$$

for all $n = 1, \cdots, N$.

For a proof of the above theorem see Theorem 8 of [7].

We will need a well known theorem due to Rouché to prove some theorems in the later chapters. Several versions of this theorem are available in the literature. We mention a version, which is useful in the later chapters.

Theorem 2.7.2. (Rouché). *Let G be a domain (i.e. open and connected) in $\mathbb{C}$ and K be a compact subset of G which is connected and simply connected. Again let f and g be two holomorphic functions in G such that $|f(z) - g(z)| < |f(z)|$ for every point z in the boundary of K. Then f and g have the same number of zeros in the interior of K, taking into account multiplicities.*

Lemma 2.7.3. (Cassels). *Let a be a real algebraic irrational number and $K = \mathbb{Q}(a)$. Also let $\mathfrak{b}$ be an integral ideal such that $\mathfrak{b}a\mathcal{O}_K$ is an integral ideal. Then there exists an $N_0 > 10^6$ satisfying the following property;*

for any $N > N_0$ and $M = [10^{-6}N]$, there are at least $51M/100$ integers n in $N < n \leq N + M$ such that $(n + a)\mathfrak{b}$ is divisible by a prime ideal $\mathfrak{p}_n$ for which

$$\mathfrak{p}_n \nmid \prod_{\substack{m \leq N+M \\ m \neq n}} (m + a)\mathfrak{b}.$$

Remarks

1. The hypothesis of the Rouché theorem implies that both the functions f and g do not vanish on the boundary of K. The theorem says approximately, the number of zeros of a holomorphic function, in a compact set is stable under small perturbations of the function on the boundary, provided it has no zero on the boundary of the compact set.

2. The proof of the lemma of Cassels is intricate involving some deep tools from algebraic number theory. It is not difficult to see that the proof of Cassels yields at least $27M/50$ integers n in $N < n \leq N + M$ with the above property. We will make use of this fact in proving some theorems.

2.8 Baker's theorem

In 1966, Alan Baker proved a fundamental theorem about linear forms in logarithms. This theorem is a strong motivation for us to formulate the Strong Polylog conjecture (1.2.1). The following are the theorems of Baker about linear forms in logarithms.

Theorem 2.8.1. *If $\alpha_1, \cdots, \alpha_n$ are non-zero algebraic numbers such that $\log \alpha_1, \cdots, \log \alpha_n$ are linearly independent over the field of rational numbers, then $1, \log \alpha_1, \cdots, \log \alpha_n$ are linearly independent over the field of algebraic numbers.*

Theorem 2.8.2. *Any non-vanishing linear combination of logarithms of non-zero algebraic numbers with algebraic coefficients is transcendental.*

For proofs of the above theorems, see Theorems 2.1 and 2.2 of [2].

From the above theorems, we see that for any non-zero algebraic numbers $\alpha_1, \cdots, \alpha_n$ and any algebraic numbers $\beta_1, \cdots, \beta_n$

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental. It is transcendental if $\log \alpha_1, \cdots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$ and $\beta_1, \cdots, \beta_n$ are not all zero.

2.9 Erdös conjecture

Erdös made the following conjecture (see [25]):

Conjecture 2.9.1. (Erdös). *If f is a periodic arithmetic function with period $q > 1$ and*

$$f(n) = \begin{cases} \pm 1 & \text{if } q \nmid n, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

In 1973, Baker, Birch and Wirsing, using Baker's theory of linear forms in logarithms, proved a theorem (see Theorem 1 of [3]) which settles the above conjecture for prime number q. In 1982, Okada[33] proved that the above conjecture is true if $2\varphi(q) + 1 > q$. Hence, if q is a prime power or a product of two distinct primes, the conjecture is true. In 2002, R. Tijdeman [40] proved that the conjecture is true for periodic completely multiplicative function f (see Theorems 9 and 10 of [33]). In the next year, Saradha and Tijdeman [38] (see Corollary 2) proved that the conjecture is also true if f is periodic and multiplicative with $|f(p^k)| < p - 1$ for every divisor p of q and every positive integer k.

In 2007, Murty and Saradha [29] proved that if q is odd and f is a odd valued odd periodic function then the above conjecture is true. In 2010, they further proved that the Erdös conjecture is true if $q \equiv 3 \pmod 4$ (see Theorem 7 of [30]). Again we know that

$$L(1, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n}$$

exists if and only if $\sum_{n=1}^{q} f(n) = 0$. Now if q is even and f takes values ± 1 with $f(q) = 0$, then $\sum_{n=1}^{q} f(n) \neq 0$. Hence this conjecture is trivially true. Thus, the only case of the Erdös conjecture is open when $q \equiv 1 \pmod 4$.

Chapter 3

The Chowla-Milnor spaces

3.1 Introduction

In this chapter, we discuss our first research problem. In 1982, P. Chowla and S. Chowla [12] asked a seemingly innocent question about the non-vanishing of $L(2, f)$ for some period arithmetic function f. They formulated the conjecture (1.1.1), which says that $L(2, f) \neq 0$ except when

$$f(1) = f(2) = \cdots = f(p-1) = \frac{f(p)}{1-p^2}.$$

In the next year, J. Milnor put the conjecture of Chowla and Chowla in a conceptual framework. He interpreted the conjecture of Chowla and Chowla in terms of the linear independence of Hurwitz zeta values and formulated the conjecture (1.1.2). The raison de'tre for his formulation rests on the following identity (studied by D. Kubert [23])

$$f(x) = q^{s-1} \sum_{k=0}^{q-1} f\left(\frac{x+k}{q}\right), \qquad (*_s)$$

valid for complex valued functions f over $\mathbb{R}/\mathbb{Z}$ and positive integer q. Inspired by this basic identity, Milnor conjectured that every $\mathbb{Q}$-linear relation between the real numbers $\zeta(2, x)$, for $x \in \mathbb{Q} \cap (0, 1)$ is a consequence of the above Kubert relations for the case $s = -1$. More generally, for q not necessarily prime, he suggested the conjecture (1.1.3). This conjecture says that for any two integers $k > 1$ and $q > 1$ the following $\varphi(q)$ real numbers

$$\zeta(k, a/q) \text{ with } 1 \leq a < q,\ (a, q) = 1, \qquad (3.1)$$

are linearly independent over the field of rational numbers. Note that the co-primality condition between a and q is necessary. For instances, we have the following $\mathbb{Q}$-linear relation

$$\zeta(k, 1/4) + \zeta(k, 3/4) = 2^k \zeta(k, 1/2).$$

In 2011, Gun, Murty and Rath [19] investigated the Chowla-Milnor conjecture and defined a family of $\mathbb{Q}$-linear spaces $V_k(q)$ for integers $k > 1$ and $q > 1$. In relation to the dimension of these spaces, they proved Theorem (1.1.3) using the expansion of Bernoulli polynomials. In this chapter, we give another proof of this theorem (1.1.3) by an explicit evaluation of co-tangent derivatives.

3.2 Basic lemmas

We begin with some basic lemmas in order to prove Theorem (1.1.3), which recall is the following assertion:

Theorem 3.2.1. *Let $k > 1$ be an odd integer and q and r be two co-prime integers > 2. Then either*

$$\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2} + 1$$

or

$$\dim_{\mathbb{Q}} V_k(r) \geq \frac{\varphi(r)}{2} + 1.$$

Thus in particular, there exists a q_0 such that

$$\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2} + 1$$

for any q co-prime to q_0.

Lemma 3.2.2. *For an integer $k \geq 1$,*

$$D^{k-1}(\pi \cot \pi z) = \pi^k \times \mathbb{Z} \text{ linear combination of } (\csc \pi z)^{2l} (\cot \pi z)^{k-2l},$$

for some non-negative integer l. Here $D^{k-1} = \frac{d^{k-1}}{dz^{k-1}}$.

Proof. We will prove this by induction on k. For $k = 1$, we have $D^{k-1}(\pi \cot(\pi z)) = \pi \cot(\pi z)$. Assume that the statement is true for $k - 1$, i.e.

$$D^{k-2}(\pi \cot(\pi z)) = \pi^{k-1} \sum a_i (\csc \pi z)^{2l_i} (\cot \pi z)^{(k-1)-2l_i}$$

where a_i's are integers.
Differentiating both sides with respect to z we get,

$$\begin{aligned} D^{k-1}(\pi \cot \pi z) &= \pi^k \sum \Big[b_i (\csc \pi z)^{2l_i} (\cot \pi z)^{k-2l_i} \\ &\quad + c_i (\csc \pi z)^{2l_i+2} (\cot \pi z)^{k-(2l_i+2)} \Big], \end{aligned}$$

where b_i, c_i's are integers. This completes the proof of the lemma. ∎

Lemma 3.2.3. *For an integer $k \geq 2$,*

$$\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) = \frac{(-1)^{k-1}}{(k-1)!} D^{k-1}(\pi \cot \pi z)|_{z=a/q}.$$

Proof.

$$\begin{aligned}
\text{L.H.S.} &= \zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) \\
&= \sum_{n\geq 0}^{\infty} \frac{1}{(n + a/q)^k} + (-1)^k \sum_{n\geq 0}^{\infty} \frac{1}{(n + 1 - a/q)^k} \\
&= \sum_{n\geq 0}^{\infty} \frac{1}{(n + a/q)^k} + (-1)^k \sum_{n=1}^{\infty} \frac{1}{(n - a/q)^k} \\
&= \sum_{n\geq 0}^{\infty} \frac{1}{(n + a/q)^k} + (-1)^{2k} \sum_{n=1}^{\infty} \frac{1}{(-n + a/q)^k} \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{(n + a/q)^k}.
\end{aligned}$$

Again we know that for $z \notin \mathbb{Z}$,

$$\pi \cot \pi z = \sum_{n\in\mathbb{Z}} \frac{1}{z + n}.$$

This implies that

$$D^{k-1}(\pi \cot \pi z) = (-1)^{k-1}(k-1)! \sum_{n\in\mathbb{Z}} \frac{1}{(z + n)^k}.$$

So,

$$\frac{(-1)^{k-1}}{(k-1)!} D^{k-1}(\pi \cot \pi z)|_{z=a/q} = \sum_{n\in\mathbb{Z}} \frac{1}{(n + a/q)^k},$$

which completes the proof of the lemma. □

Finally, we have the identity (2.3), which is the following:

$$\zeta(k) \prod_{\substack{p\in \mathrm{P}, \\ p|q}} (1 - p^{-k}) = q^{-k} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \zeta(k, a/q)$$

where P denotes the set of all primes.

Now we are ready to give an alternative proof of the theorem (3.2.1).

3.3 Proof of theorem (3.2.1)

Proof. First note that the space $V_k(q)$ is also spanned by the following sets of real numbers:

$$\{\zeta(k, a/q) + \zeta(k, 1 - a/q) | (a, q) = 1,\ 1 \leq a < q/2\},$$

$$\{\zeta(k, a/q) - \zeta(k, 1 - a/q) | (a, q) = 1,\ 1 \leq a < q/2\}.$$

Now from the lemma (3.2.3), we have the following

$$\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) = \frac{(-1)^{k-1}}{(k-1)!} D^{k-1}(\pi \cot \pi z)|_{z=a/q}.$$

Applying the Okada's theorem (2.6.2), we see that

$$\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2}.$$

Now from lemma (3.2.2) and lemma (3.2.3) for an odd integer $k > 1$, we have

$$\begin{aligned}
&\frac{\zeta(k, a/q) - \zeta(k, 1 - a/q)}{(2\pi i)^k} \\
&= \frac{i}{2^k} \times \mathbb{Q} \text{ linear combinations of } (\csc \pi a/q)^{2l} (\cot \pi a/q)^{k-2l}.
\end{aligned}$$

We note that

$$i \cot(\pi a/q) = \frac{1 + \zeta_q^a}{1 - \zeta_q^a}$$

belongs to $\mathbb{Q}(\zeta_q)$ and hence so do the numbers $\csc(\pi a/q)^{2l}$ and $\cot(\pi a/q)^{2l}$. Since k is odd, we have

$$\frac{\zeta(k, a/q) - \zeta(k, 1 - a/q)}{(2\pi i)^k} \in \mathbb{Q}(\zeta_q) \tag{3.2}$$

Now we go back to the main part of the proof. Let q and r be two co-prime integers. Suppose that

$$\dim_{\mathbb{Q}} V_k(q) = \frac{\varphi(q)}{2}.$$

Then the numbers

$$\zeta(k, a/q) - \zeta(k, 1 - a/q), \text{ where } (a, q) = 1, \ 1 \leq a < q/2$$

generate $V_k(q)$. Now from the identity (2.3), we get

$$\zeta(k) \prod_{p|q} (1 - p^{-k}) \ = \ q^{-k} \sum_{\substack{a=1, \\ (a,q)=1}}^{q-1} \zeta(k, a/q) \in V_k(q).$$

and hence

$$\zeta(k) = \sum_{\substack{(a,q)=1 \\ 1 \leq a < q/2}} \lambda_a [\zeta(k, a/q) - \zeta(k, 1 - a/q)], \ \lambda_a \in \mathbb{Q}$$

so that

$$\frac{\zeta(k)}{(2\pi i)^k} = \sum_{\substack{(a,q)=1 \\ 1 \leq a < q/2}} \frac{\lambda_a [\zeta(k, a/q) - \zeta(k, 1 - a/q)]}{(2\pi i)^k}.$$

Thus by (3.2)

$$\frac{\zeta(k)}{i\pi^k} \in \mathbb{Q}(\zeta_q).$$

Similarly , if

$$\dim_{\mathbb{Q}} V_k(r) = \frac{\varphi(r)}{2},$$

then

$$\frac{\zeta(k)}{i\pi^k} \in \mathbb{Q}(\zeta_r)$$

and hence

$$\frac{\zeta(k)}{i\pi^k} \in \mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r).$$

Since any non-trivial finite extension of $\mathbb{Q}$ is ramified, if $\mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r) \neq \mathbb{Q}$ then there exists a prime which is ramified in $\mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r)$, hence both in $\mathbb{Q}(\zeta_q)$ and $\mathbb{Q}(\zeta_r)$. Note a prime which ramifies in this intersection must necessarily divide both q and r. This is impossible because $(q, r) = 1$. So $\mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r) = \mathbb{Q}$. Hence we arrive at a contradiction as $\frac{\zeta(k)}{\pi^k}$ is a real number. Thus

$$\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2} + 1 \quad \text{or} \quad \dim_{\mathbb{Q}} V_k(r) \geq \frac{\varphi(r)}{2} + 1.$$

This completes the proof of the theorem.

□

3.4 Concluding Remarks

The conjecture of Chowla and Milnor has many interesting consequences. In [19], Gun, Murty and Rath mentioned some of them. For example,

1. If the Chowla-Milnor conjecture is true for $q = 4$, then one can show that $\zeta(2k + 1)/\pi^{2k+1}$ is irrational for all $k \geq 1$.

2. If the Chowla-Milnor conjecture is true for $q = 12$ and $k = 2$ then the real number
$$\alpha := \frac{1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + ...}{1^{-2} - 2^{-2} + 4^{-2} - 5^{-2} + ...}$$
is irrational. This number is mentioned in the paper of P. Chowla and S. Chowla [12]. A. Borel, Lichtenstein, Milnor and Thurston have raised question about the irrationality of this number.

3. Suppose that Chowla-Milnor conjecture is true. Then for any rational valued periodic function f with prime period, $L(s, f)$ is holomorphic at $s = 1$ implies that it does not vanish at all integers $k > 1$.

Here we want to add the following proposition in the above list.

Proposition 3.4.1. *Let $k > 1$ be an odd integer and suppose that the Chowla-Milnor conjecture is true for $q = 3$, i.e. $\dim_{\mathbb{Q}} V_k(3) = 2$. Then $\zeta(k)/\pi^k \notin \mathbb{Q}(\sqrt{3})$.*

Proof. We know that $V_k(3)$ is generated by $\zeta(k, 1/3)$ and $\zeta(k, 2/3)$ and we have the identity

$$\zeta(k, 1/3) + \zeta(k, 2/3) = (3^k - 1)\zeta(k).$$

Since k is odd, from lemma (3.2.3) we get

$$\zeta(k, 1/3) - \zeta(k, 2/3) = \frac{1}{(k-1)!} D^{k-1}(\pi \cot \pi z)|_{z=1/3}.$$

But $D^{k-1}(\pi \cot \pi z)|_{z=1/3}$ is a rational multiple of $\sqrt{3}\pi^k$. Again we know that the following set

$$\{\zeta(k, 1/3) + \zeta(k, 2/3), \zeta(k, 1/3) - \zeta(k, 2/3)\}$$

generates $V_k(3)$. So the dimension of $V_k(3)$ is equal to 2 implies that the ratio

$$\frac{\zeta(k, 1/3) + \zeta(k, 2/3)}{\zeta(k, 1/3) - \zeta(k, 2/3)} = \frac{\zeta(k)}{\sqrt{3}\pi^k}$$

is an irrational number. Hence $\zeta(k)/\pi^k \notin \mathbb{Q}(\sqrt{3})$. This completes the proof of the proposition. $\square$

Chapter 4

The Strong Chowla-Milnor spaces

4.1 Introduction

In this chapter, we discuss our second set of research problems. In 2011, Gun, Murty and Rath [19] formulated a stronger version the Chowla-Milnor conjecture (1.1.3) which they called the Strong Chowla-Milnor conjecture. This conjecture (1.1.6) states that for all $q > 1, k > 1$, the following $\varphi(q) + 1$ real numbers

$$1,\ \zeta(k, a/q) \text{ with } 1 \leq a < q \text{ and } (a, q) = 1$$

are linearly independent over $\mathbb{Q}$. In relation to their conjecture (1.1.6), Gun, Murty and Rath defined the Strong Chowla-Milnor space $\widehat{V}_k(q)$, for any integer $k > 1$ and $q > 2$, by

$$\widehat{V}_k(q) := \mathbb{Q} - \text{span of } \{1, \zeta(k, a/q) : 1 \leq a < q,\ (a, q) = 1\}$$

Clearly, the Strong Chowla-Milnor conjecture implies that $\dim_{\mathbb{Q}} \widehat{V}_k(q) = \varphi(q) + 1$. We prove a non-trivial lower bound about the dimension of the Strong Chowla-Milnor spaces which is theorem (1.2.3). In the last section, we note that any improvement of the above lower bound of the Strong Chowla-Milnor spaces will imply simultaneously both the numbers $\zeta(k)$ and $\zeta(k)/\pi^k$ are irrational for all odd positive integers $k > 1$. We also establish a conditional improvement of the lower bound of the Strong Chowla-Milnor spaces $\widehat{V}_k(q)$ in terms of the theorem (1.2.4).

4.2 Dimension of the Strong Chowla-Milnor Spaces

In this section, we give proofs of the theorems (1.2.3) and (1.2.4). We also establish the corollary (1.2.5) as a consequence of the theorem (1.2.4). Let us first recall the theorem (1.2.3).

Theorem 4.2.1. *Let $k > 1$ and $q > 2$ be two integers. Then*

$$\dim_{\mathbb{Q}} \widehat{V}_k(q) \geq \frac{\varphi(q)}{2} + 1.$$

4.2.1 Proof of theorem (4.2.1)

Proof. Notice that the space $\widehat{V}_k(q)$ is also spanned by the following sets of real numbers:

$$1, \{\zeta(k, a/q) + \zeta(k, 1 - a/q) | (a, q) = 1,\ 1 \leq a < q/2\},$$

$$\{\zeta(k, a/q) - \zeta(k, 1 - a/q) | (a, q) = 1,\ 1 \leq a < q/2\}.$$

Then by lemma (3.2.3), for $k \geq 2$ we have the following

$$\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z)|_{z=a/q}.$$

Now using Okada's theorem (2.6.2), we get

$$\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q)$$

for $1 \leq a < q/2, (a, q) = 1$, are linearly independent over $\mathbb{Q}$. Again by lemma (3.2.2), we have

$$\frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z) = \pi^k \times \mathbb{Z} \text{ linear combination of } (\csc \pi z)^{2l} (\cot \pi z)^{k-2l},$$

for some non-negative integer l and for an integer $k \geq 1$. Since $\csc \pi z$ and $\cot \pi z$ are algebraic at rationals, we have all the numbers $\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q)$ are transcendental for any k as they are algebraic multiple of π. Hence $\dim_{\mathbb{Q}} \widehat{V}_k(q) \geq \frac{\varphi(q)}{2} + 1$ as 1 and π are linearly independent over $\mathbb{Q}$. This completes the proof of the theorem.

□

Now we give a proof of theorem (1.2.4) which is the following:

Theorem 4.2.2. *Let $k > 1$ be an odd integer with $\zeta(k)$ irrational and $q, r > 2$ be two co-prime integers. Then either*

$$\dim_{\mathbb{Q}} \widehat{V}_k(q) \geq \frac{\varphi(q)}{2} + 2$$

or

$$\dim_{\mathbb{Q}} \widehat{V}_k(r) \geq \frac{\varphi(r)}{2} + 2.$$

Thus in particular, for infinitely many odd integers $k > 1$ there exists an integer q_0 such that

$$\dim_{\mathbb{Q}} \widehat{V}_k(q) \geq \frac{\varphi(q)}{2} + 2$$

for any integer q co-prime to q_0.

4.2.2 Proof of theorem (4.2.2)

Proof. Suppose not, then we have

$$\dim_{\mathbb{Q}} \widehat{V}_k(q) = \frac{\varphi(q)}{2} + 1.$$

This gives that the numbers

$$1, \zeta(k, a/q) - \zeta(k, 1 - a/q), \text{ where } (a, q) = 1, \ 1 \le a < q/2$$

generate $\widehat{V}_k(q)$.

Since k is odd, we have the equation (3.2) (See also Hecke [21] and paper 41 of E. Hecke, Mathematische Werke, Dritte Auflage, Vandenhoeck und Rupertecht, Gottingen, 1983),

$$\frac{\zeta(k, a/q) - \zeta(k, 1 - a/q)}{(2\pi i)^k} \in \mathbb{Q}(\zeta_q). \tag{4.1}$$

Again we have the identity (2.3)

$$\zeta(k) \prod_{\substack{p \in \mathrm{P}, \\ p|q}} (1 - p^{-k}) \ = \ q^{-k} \sum_{\substack{a=1, \\ (a,q)=1}}^{q-1} \zeta(k, a/q) \in \widehat{V}_k(q),$$

where P is the set of primes.
Thus $\zeta(k) \in \widehat{V}_k(q)$ and hence we have

$$\zeta(k) = q_1 + \sum_{\substack{(a,q)=1 \\ 1 \le a < q/2}} \lambda_a [\zeta(k, a/q) - \zeta(k, 1 - a/q)] \text{ for some } q_1, \ \lambda_a \in \mathbb{Q}$$

so that

$$\frac{\zeta(k) - q_1}{(2\pi i)^k} = \sum_{\substack{(a,q)=1 \\ 1 \le a < q/2}} \frac{\lambda_a [\zeta(k, a/q) - \zeta(k, 1 - a/q)]}{(2\pi i)^k}.$$

Thus by (4.1)

$$\frac{\zeta(k) - q_1}{i\pi^k} = a_1(\text{say}) \in \mathbb{Q}(\zeta_q).$$

Similarly, if

$$\dim_{\mathbb{Q}} \widehat{V}_k(r) = \frac{\varphi(r)}{2} + 1,$$

then

$$\frac{\zeta(k) - q_2}{i\pi^k} = a_2(\text{say}) \in \mathbb{Q}(\zeta_r), \quad \text{with } q_2 \in \mathbb{Q}.$$

So we have

$$a_1 i\pi^k + q_1 = a_2 i\pi^k + q_2$$

which implies

$$(a_1 - a_2) i\pi^k = q_2 - q_1.$$

The L.H.S of the above equation is algebraic number times transcendental number hence transcendental and the R.H.S is a rational number. Hence we get that $q_1 = q_2$ and $a_1 = a_2$.

Thus we have

$$\frac{\zeta(k) - q_1}{i\pi^k} \in \mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r) = \mathbb{Q}(\text{ see theorem (2.1.9)}).$$

Let

$$\frac{\zeta(k) - q_1}{i\pi^k} = a \in \mathbb{Q}.$$

Since L.H.S of the above equation is purely imaginary and R.H.S is rational, we have $a = 0$ and $\zeta(k) = q_1$, a rational number. This is a contradiction to the irrationality of $\zeta(k)$. Thus either

$$\dim_{\mathbb{Q}} \widehat{V}_k(q) \geq \frac{\varphi(q)}{2} + 2$$

or

$$\dim_{\mathbb{Q}} \widehat{V}_k(r) \geq \frac{\varphi(r)}{2} + 2.$$

This completes the proof of the theorem. □

As a consequence of the above theorem, we prove the corollary (1.2.5).

Proof. (**Proof of the corollary** (1.2.5)) We know, from the Apery's theorem, that $\zeta(3)$ is irrational. Now using the above theorem for $q = 3$ and $r = 4$, we get that either

$$\dim_{\mathbb{Q}} \widehat{V}_3(3) \geq 3$$

or

$$\dim_{\mathbb{Q}} \widehat{V}_3(4) \geq 3.$$

But both the $\dim_{\mathbb{Q}} \widehat{V}_3(3)$ and $\dim_{\mathbb{Q}} \widehat{V}_3(4)$ are less than equal to 3. So we get either $\dim_{\mathbb{Q}} \widehat{V}_3(3) = 3$ or $\dim_{\mathbb{Q}} \widehat{V}_3(4) = 3$. This implies that either both the numbers $\zeta(3, 1/3)$ and $\zeta(3, 2/3)$ are irrational or both the numbers $\zeta(3, 1/4)$ and $\zeta(3, 3/4)$ are irrational. But we have

$$\zeta(3, 1/3) = 27\left(\frac{1}{1^3} + \frac{1}{4^3} + \frac{1}{7^3} + \cdots\right),$$

$$\zeta(3, 2/3) = 27\left(\frac{1}{2^3} + \frac{1}{5^3} + \frac{1}{8^3} + \cdots\right),$$

$$\zeta(3, 1/4) = 64\left(\frac{1}{1^3} + \frac{1}{5^3} + \frac{1}{9^3} + \cdots\right),$$

and

$$\zeta(3, 3/4) = 64\left(\frac{1}{3^3} + \frac{1}{7^3} + \frac{1}{11^3} + \cdots\right).$$

Hence we get either both the numbers given by the infinite series

$$\frac{1}{1^3} + \frac{1}{4^3} + \frac{1}{7^3} + \cdots$$

and

$$\frac{1}{2^3} + \frac{1}{5^3} + \frac{1}{8^3} + \cdots$$

are irrational or both the numbers given by the infinite series

$$\frac{1}{1^3} + \frac{1}{5^3} + \frac{1}{9^3} + \cdots$$

and

$$\frac{1}{3^3} + \frac{1}{7^3} + \frac{1}{11^3} + \cdots$$

are irrational. This completes the proof of the corollary.

□

4.3 Relation with Strong Polylog conjecture

In this section, we discuss the relationship between the Strong Polylog conjecture (1.2.1) and the Strong Chowla-Milnor conjecture (1.1.6). We formulated the Strong Polylog conjecture (1.2.1), following the Polylog conjecture (1.1.4) of Gun, Murty and Rath, which is a stronger conjecture about the polylogarithms. This conjecture is a generalization of Baker's theorem about linear forms in logarithms. In relation to this conjecture we prove the theorem (1.2.2), which says that the Strong Polylog conjecture implies the Strong Chowla-Milnor conjecture. First we give a proof of proposition (1.2.1) which is the following:

Proposition 4.3.1. *Let $k > 1, q > 1$ be two integers and f be a rational valued arithmetic periodic function with period q. Suppose that $f(a) = 0$ for $1 < (a, q) < q$. Then the following statements are equivalent:*

1. *The Strong Chowla-Milnor conjecture is true.*
2. *The L-value $L(k, f)$ is irrational, unless*

$$f(a) = -\frac{f(q)q^{-k}}{\prod\limits_{\substack{p\in\mathrm{P},\\ p|q}} (1-p^{-k})}$$

for $1 \le a < q, (a, q) = 1$. Here P denotes the set of primes.

Proof. Consider the identity (2.3)

$$\zeta(k) \prod_{\substack{p\in\mathrm{P},\\ p|q}} (1-p^{-k}) = q^{-k} \sum_{\substack{a=1\\ (a,q)=1}}^{q-1} \zeta(k, a/q).$$

Substituting this in the expression (1.1) for $L(s, f)$ and using $f(a) = 0$ for $1 < (a, q) < q$ and $s = k$, we get

$$L(k, f) = q^{-k} \sum_{\substack{a=1\\ (a,q)=1}}^{q-1} \left[f(a) + \frac{f(q)q^{-k}}{\prod\limits_{p|q}(1-p^{-k})} \right] \zeta(k, a/q). \tag{4.2}$$

If $L(k, f)$ is rational, then the above equation shows that $1, \zeta(k, a/q)$ for $1 \le a < q, (a, q) = 1$ are linearly dependent over the rationals since f is a rational-valued function.

Conversely, if $1, \zeta(k, a/q)$ with $1 \leq a < q, (a,q) = 1$ are linearly dependent over the rationals, then there are rational numbers $c_0, c_a, 1 \leq a < q, (a,q) = 1$, not all zero, such that

$$c_0 + \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} c_a \zeta(k, a/q) = 0.$$

Now define the following rational-valued periodic function f with period q. Set $f(a) = 0$ for $1 < a \leq q, (a,q) > 1$ and $f(a) = c_a$ for $1 \leq a < q, (a,q) = 1$. Then, our identity shows that $q^k L(k, f) = -c_0$ so that $L(k, f)$ is rational. This completes the proof of the proposition. □

Now we are ready to prove the theorem (1.2.2), which is the following:

Theorem 4.3.2. *The Strong Polylog conjecture implies the Strong Chowla-Milnor conjecture for all $q > 1$ and $k > 1$.*

4.3.1 Proof of theorem (4.3.2)

Proof. Let $k > 1$ and $q > 1$. Let f be a rational valued period function with period q satisfying $f(a) = 0$ for $1 < (a,q) < q$. Suppose that $L(k, f) = r$ is a rational number. Then we have

$$L(k, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^k} = r.$$

As f is a periodic function, we have the Fourier transformation of f given by

$$\hat{f}(n) = \frac{1}{q} \sum_{a=1}^{q} f(a) \zeta_q^{-an}$$

where $\zeta_q = e^{\frac{2\pi i}{q}}$ and hence we have the Fourier inversion formula

$$f(n) = \sum_{a=1}^{q} \hat{f}(a) \zeta_q^{an}.$$

Then we have

$$L(k, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^k} = \sum_{n=1}^{\infty} \frac{1}{n^k} \sum_{a=1}^{q} \hat{f}(a) \zeta_q^{an} = r$$

and hence

$$\sum_{a=1}^{q} \hat{f}(a) Li_k(\zeta_q^a) - r = 0.$$

Let $Li_k(\alpha_1), ..., Li_k(\alpha_t)$ be a maximal linearly independent subset of

$$\{Li_k(\zeta_q^a) | 1 \leq a < q\}$$

over $\mathbb{Q}$.

Then

$$Li_k(\zeta_q^a) = \sum_{b=1}^{t} C_{ab} Li_k(\alpha_b)$$

for some $C_{ab} \in \mathbb{Q}$. So we have

$$\sum_{b=1}^{t} \beta_b Li_k(\alpha_b) - r = 0$$

where

$$\beta_b = \sum_{a=1}^{q} \hat{f}(a) C_{ab}.$$

Since f is rational valued, $\hat{f}$ is algebraic valued. So by the Strong Polylog conjecture (1.2.1), we have

$$r = 0 \text{ and } \beta_b = \sum_{a=1}^{q} \hat{f}(a) C_{ab} = 0, \ 1 \leq b \leq t.$$

Now for any automorphism σ of the field $\overline{\mathbb{Q}}$ over $\mathbb{Q}$, we have

$$\sum_{a=1}^{q} \sigma(\hat{f}(a)) C_{ab} = 0, \ 1 \leq b \leq t,$$

and hence

$$\sum_{a=1}^{q} \sigma(\hat{f}(a)) Li_k(\zeta_q^a) = 0.$$

In particular, if for $1 \leq h < q, (h, q) = 1$, σ_h is the element of the Galois group of $\mathbb{Q}(\zeta_q)$ over $\mathbb{Q}$ such that

$$\sigma_h(\zeta_q) = \zeta_q^h,$$

then we have,

$$\sigma_h(\hat{f}(n)) = \hat{f}_h(n)$$

where

$$f_h(n) = f(nh^{-1}).$$

Thus, we have

$$\begin{aligned} L(k, f_h) &= \sum_{n=1}^{\infty} \frac{f_h(n)}{n^k} \\ &= \sum_{a=1}^{q} \hat{f}_h(a) Li_k(\zeta_q^a) \\ &= \sum_{a=1}^{q} \sigma_h(\hat{f}(a)) Li_k(\zeta_q^a) = 0 \end{aligned}$$

for all $1 \leq h < q, (h, q) = 1$. Thus by equation (4.2), we get

$$L(k, f_h) = q^{-k} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \left[f_h(a) + \frac{f_h(q) q^{-k}}{\prod_{p|q} (1 - p^{-k})} \right] \zeta(k, a/q) = 0$$

for all $1 \leq h < q, (h,q) = 1$.

Now, putting $ah^{-1} = b$ and noting that $f_h(q) = f(q)$, we have

$$L(k, f_h) = q^{-k} \sum_{\substack{b=1 \\ (b,q)=1}}^{q-1} \left[f(b) + \frac{f(q)q^{-k}}{\prod\limits_{p|q}(1-p^{-k})} \right] \zeta(k, bh/q) = 0 \qquad (4.3)$$

for all $1 \leq h < q, (h,q) = 1$.

Thus we get a matrix equation with M being the $\varphi(q) \times \varphi(q)$ matrix whose (b,h)-th entry is given by $\zeta(k, bh/q)$. Then by the evaluation of the Dedekind-Frobenius determinant as in proposition (2.5.1), we get

$$Det(M) = \pm \prod_{\chi} q^k L(k, \chi) \neq 0.$$

Thus the matrix M is invertible and hence by the equation (4.3), we have

$$f(a) + \frac{f(q)q^{-k}}{\prod\limits_{p|q}(1-p^{-k})} = 0, \ 1 \leq a < q, \ (a,q) = 1$$

and hence

$$f(a) = -\frac{f(q)q^{-k}}{\prod\limits_{p|q}(1-p^{-k})}$$

for all $1 \leq a < q, (a,q) = 1$. This completes the proof of the theorem. □

4.4 Concluding Remarks

Let us now investigate the linear independence of the real numbers $1, \zeta(k, a/q)$ with $1 \leq a < q$ and $(a,q) = 1$, over the field of algebraic numbers. In this case, one can not expect result similar to the theorem (1.2.3). In this direction, we prove proposition (1.2.6), which is the following:

Proposition 4.4.1. $2 \leq \dim_{\overline{\mathbb{Q}}} \widehat{V}_k(q) \leq \frac{\varphi(q)}{2} + 2$.

Proof. We know that $\zeta(k, a/q) + (-1)^k \zeta(k, 1-a/q) \in \pi^k \overline{\mathbb{Q}}^*$ for $1 \leq a < q/2, (a,q) = 1$. As 1 and π are linearly independent over $\overline{\mathbb{Q}}$, we have $\dim_{\overline{\mathbb{Q}}} \widehat{V}_k(q) \geq 2$.

Now for k even, the $\varphi(q)/2$ real numbers $\zeta(k, a/q) + \zeta(k, 1-a/q)$ for $1 \leq a < q/2, (a,q) = 1$, are linearly dependent over $\overline{\mathbb{Q}}$, and for k odd $\zeta(k, a/q) - \zeta(k, 1-a/q)$ for $1 \leq a < q/2, (a,q) = 1$, are linearly dependent over $\overline{\mathbb{Q}}$. So in any case whether k is even or odd, this $\varphi(q)/2$ real numbers contribute at most 1 in the dimension. Hence we have $\dim_{\overline{\mathbb{Q}}} \widehat{V}_k(q) \leq \frac{\varphi(q)}{2} + 2$. This completes the proof of the proposition. □

Chapter 5

The Chowla-Milnor conjecture over number fields

5.1 Introduction

In this chapter, we discuss our third set of research problems. In 2012, Gun, Murty and Rath [20] formulated the conjecture (1.1.7) which is a generalization of Chowla-Milnor conjecture over number fields which are linearly disjoint from the cyclotomic extension $\mathbb{Q}(\zeta_q)$. This conjecture says that the following $\varphi(q)$ real numbers

$$\zeta(k, a/q) \text{ with } 1 \leq a < q, \ (a, q) = 1, \tag{5.1}$$

are linearly independent over the number field $\mathbb{F}$ which is linearly disjoint from $\mathbb{Q}(\zeta_q)$. This is a natural generalization of the Chowla-Milnor conjecture. For that

$$\frac{\zeta(k, a/q) - \zeta(k, 1 - a/q)}{(2\pi i)^k} \in \mathbb{Q}(\zeta_q),$$

and hence one can expect that if $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, then the above set of real numbers is linearly independent over $\mathbb{F}$.

5.2 Generalized Zagier spaces

In [19], Gun, Murty and Rath provided an infinite family of Zagier spaces W_k having $\mathbb{Q}$ dimension at least 2, assuming the Chowla-Milnor conjecture. In this section, we investigate a similar kind of question over certain family of number fields. For that, we define the generalized k-th Zagier space $V_k(\mathbb{F})$ as $\mathbb{F}$ –span of the multiple zeta values $\zeta(a_1, \cdots, a_l)$ with $a_1 + \cdots + a_l = k$, where $\mathbb{F}$ is a number field linearly disjoint from the cyclotomic field $\mathbb{Q}(\zeta_q)$. Assuming the conjecture (5.1), we prove the theorem (1.2.8), which we recall here:

Theorem 5.2.1. *Let d be a positive integer. Then the conjecture* (5.1) *implies*

$$\dim_{\mathbb{F}} V_{4d+2}(\mathbb{F}) \geq 2.$$

5.2.1 Proof of theorem (5.2.1)

To prove theorem (5.2.1), we need two lemmas. We first state and prove the lemmas, then proceed to the proof of the theorem (5.2.1).

Lemma 5.2.2. *Let $\mathbb{F}$ be an algebraic number field. Then $\left[\frac{\zeta(2d+1)}{\pi^{2d+1}}\right]^2 \notin \mathbb{F}$ implies* $\dim_{\mathbb{F}} V_{4d+2}(\mathbb{F}) \geq 2$.

Proof. To begin with, let us multiply $\zeta(s_1)$ with $\zeta(s_2)$ and using the definition of multiple zeta values we get

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).$$

Now substituting $s_1 = s_2 = 2d + 1$ in the above equation, we get

$$\zeta(2d + 1)^2 = 2\zeta(2d + 1, 2d + 1) + \zeta(4d + 2).$$

Dividing both sides by π^{4d+2}, to get

$$\left[\frac{\zeta(2d+1)}{\pi^{2d+1}}\right]^2 = 2\frac{\zeta(2d+1, 2d+1)}{\pi^{4d+2}} + \frac{\zeta(4d+2)}{\pi^{4d+2}}.$$

Since

$$\left[\frac{\zeta(2d+1)}{\pi^{2d+1}}\right]^2 \notin \mathbb{F} \textit{ and } \frac{\zeta(4d+2)}{\pi^{4d+2}} \in \mathbb{Q} \subset \mathbb{F}$$

it follows that $\zeta(2d + 1, 2d + 1)$ is not in the $\mathbb{F}$- span of $\zeta(4d + 2)$ and hence the $\mathbb{F}$-dimension of the space $V_{4d+2}(\mathbb{F}) \geq 2$. $\square$

Lemma 5.2.3. *Suppose the conjecture* (5.1) *is true. Then*

$$\left[\frac{\zeta(2d+1)}{\pi^{2d+1}}\right]^2 \notin \mathbb{F},$$

for all $d \geq 1$.

Proof. Let $\Delta < 0$ be a fundamental discriminant. Then from the theorem (2.4.3), we know that the Kronecker symbol $\chi_\Delta(n) = (\frac{\Delta}{n})$ is an odd, primitive, quadratic character modulo $|\Delta|$. Let $q = |\Delta|$.

Now from the theory of Gauss sums we know that the Gauss sum $\tau(\chi_\Delta)$ associated with the Kronecker symbol χ_Δ (see theorem (2.4.4)) is given by

$$\tau(\chi_\Delta) = \sum_{a=1}^{q} \chi_\Delta(a)\zeta_q^a = i\sqrt{q}.$$

Again using the primitivity of χ_Δ, we have

$$\tau(\chi_\Delta, b) = \sum_{a=1}^{q} \chi_\Delta(a)\zeta_q^{ab} = \overline{\chi}_\Delta(b)i\sqrt{q}.$$

Since χ_Δ is an odd character, we have

$$\sum_{a=1}^{q/2} \chi_\Delta(a)(\zeta_q^{ab} - \zeta_q^{-ab}) = \overline{\chi}_\Delta(b) i\sqrt{q}.$$

Let $B_l(x)$ be the lth Bernoulli polynomial. Multiplying both sides of the above equation by $B_{2d+1}(b/q)$ and taking sum over $b = 1$ to q we get,

$$\sum_{a=1}^{q/2} \chi_\Delta(a) \sum_{b=1}^{q} (\zeta_q^{ab} - \zeta_q^{-ab}) B_{2d+1}(b/q) = i\sqrt{q} \sum_{b=1}^{q} \overline{\chi}_\Delta(b) B_{2d+1}(b/q).$$

Let $k = 2d+1$. Then from proposition 1 of [19], we have

$$\frac{\zeta(k, a/q) - \zeta(k, 1-a/q)}{(2\pi i)^k} = \frac{q^{k-1}}{2k!} \sum_{b=1}^{q} (\zeta_q^{ab} - \zeta_q^{-ab}) B_k(b/q)$$

for any $(a, q) = 1$ and $1 \le a < q/2$. As χ_Δ is a quadratic character we get that the number $i\sqrt{q}$ lies in the $\mathbb{F}$-linear space generated by the real numbers

$$\frac{\zeta(k, a/q) - \zeta(k, 1-a/q)}{(2\pi i)^k}$$

with $(a, q) = 1$ and $1 \le a < q/2$.

Again from the identity (2.3), we know that

$$\zeta(k) \prod_{\substack{p \in \mathrm{P}, \\ p | q}} (1 - p^{-k}) = q^{-k} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \zeta(k, a/q)$$

where P be the set of primes. So that

$$\zeta(k) \prod_{\substack{p \in \mathrm{P}, \\ p | q}} (1 - p^{-k}) = q^{-k} \sum_{\substack{a=1 \\ (a,q)=1}}^{q/2} [\zeta(k, a/q) + \zeta(k, 1-a/q)].$$

Hence $\zeta(k)/(2\pi i)^k$ lies in the $\mathbb{F}$-linear space generated by the real numbers

$$\frac{\zeta(k, a/q) + \zeta(k, 1-a/q)}{(2\pi i)^k}$$

with $(a, q) = 1$ and $1 \le a < q/2$.

Thus the conjecture (5.1) for the modulus q implies that $i\sqrt{q}$ and $\zeta(k)/(2\pi i)^k$ lie in two disjoint $\mathbb{F}$-spaces. Hence for any such q, we have

$$\frac{\zeta(2d+1)}{\pi^{2d+1}\sqrt{q}} \notin \mathbb{F}.$$

Thus, if the conjecture (5.1) is true for all modulus, then

$$\left[\frac{\zeta(2d+1)}{\pi^{2d+1}}\right]^2 \notin \mathbb{F}$$

for all $d \ge 1$.

$\square$

Now we prove theorem (5.2.1) using the above two lemmas.

Proof. **(Proof of theorem** (5.2.1)**)** Suppose the conjecture (5.1) is true. Then the lemma (5.2.3) implies

$$\left[\frac{\zeta(2d+1)}{\pi^{2d+1}}\right]^2 \notin \mathbb{F}.$$

Hence from the lemma (5.2.2), we get

$$\dim_{\mathbb{F}} V_{4d+2}(\mathbb{F}) \geq 2.$$

This completes the proof of the theorem. □

5.3 Linear independence of L-functions

In this section, we discuss the linear independence of the values of the Dirichlet L-functions over a certain family of number fields. Let $\mathbb{F}$ be a number field and q be a positive integer. Set $\mathbb{F}_1 = \mathbb{F}(e^{2\pi i/\varphi(q)})$. Suppose that $\mathbb{F}_1 \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. In [28], Murty and Saradha showed that the values $L(1,\chi)$, as χ ranges over non-trivial primitive characters mod q, are linearly independent over $\mathbb{F}_1$. Their proof involves a fundamental lemma of Baker, Birch and Wirsing [3]. Further for any integer $k > 1$, they proved that the L-values $L(k,\chi)$ as χ ranges over Dirichlet characters mod q with the same parity as k are linearly independent over $\mathbb{F}_1$. In the spirit of their theorem we prove theorem (1.2.9), which is the following:

Theorem 5.3.1. *Let $\mathbb{F}$ be an algebraic number field and $\mathbb{F}_1 = \mathbb{F}(e^{2\pi i/\varphi(q)})$. Suppose that $\mathbb{F}_1 \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Assume the conjecture* (5.1)*. Then for any positive integer k, the values $L(k,\chi)$ as χ ranges over non-trivial Dirichlet characters mod q are linearly independent over $\mathbb{F}_1$.*

Proof. The case $k = 1$ is a theorem of M. Ram Murty and N. Saradha [28]. Let $k > 1$ and assume that

$$\sum_{\chi} c_\chi L(k,\chi) = 0,\ \ c_\chi \in \mathbb{F}_1$$

where the summation is over all Dirichlet characters mod q.

Again we know that

$$L(k,\chi) = q^{-k} \sum_{\substack{1\leq a<q \\ (a,q)=1}} \chi(a)\zeta(k, a/q).$$

So from the above equation we get

$$q^{-k} \sum_{\chi} c_\chi \sum_{\substack{1\leq a<q \\ (a,q)=1}} \chi(a)\zeta(k, a/q) = 0,$$

and hence we have
$$\sum_{\substack{1\le a<q\\(a,q)=1}} \zeta(k,a/q)\sum_{\chi} c_\chi \chi(a) = 0.$$
The values of χ lie in the field $\mathbb{F}_1$. Therefore the sum $\sum_{\chi} c_\chi \chi(a) \in \mathbb{F}_1$ which is disjoint from $\mathbb{Q}(\zeta_q)$. Hence using conjecture (5.1), we get
$$\sum_{\chi} c_\chi \chi(a) = 0$$
for all $a \in (\mathbb{Z}/qZ)^*$. Hence we see that
$$\sum_{\chi} c_\chi \chi = 0$$
where the sum is over all the Dirichlet characters modulo q.

Again from Artin's Theorem 2.2.5, we know that the Characters of a group are linearly independent over the field of complex numbers. Hence we have $c_\chi = 0$ for all Dirichlet characters χ modulo q. This completes the proof of the theorem. $\square$

5.4 Concluding Remarks

Let $\mathbb{F}$ be a number field and q be a positive integer such that $\mathbb{F}_1 = \mathbb{F}(e^{2\pi i/\varphi(q)})$. Suppose that
$$\Gamma = \sum_{\chi} c_\chi L(k,\chi) \tag{5.2}$$
is any non-trivial linear combination of $L(k,\chi)$ over $\mathbb{F}_1$, where χ ranges over non-trivial Dirichlet characters mod q. Then Theorem 5.3.1 implies that Γ is non zero.

Now for each χ, consider the Fourier transformation of χ given by
$$\hat{\chi}(n) = \frac{1}{q}\sum_{a=1}^{q} \chi(a)\zeta_q^{-an}.$$
So we have the Fourier inversion formula given by
$$\chi(n) = \sum_{a=1}^{q} \hat{\chi}(a)\zeta_q^{an}.$$
Again we have
$$L(k,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^k}.$$
Substituting the Fourier inversion formula in the above equation, we get
$$L(k,\chi) = \sum_{a=1}^{q} \hat{\chi}(a) Li_k(\zeta_q^a). \tag{5.3}$$

Now taking a maximal $\mathbb{Q}$-linearly independent subset of $\{Li_k(\zeta_q^a) | 1 \leq a < q\}$, we can write the above equation (5.3) as

$$L(k, \chi) = \sum_{a=1}^{t} c_a Li_k(\alpha_a)$$

for some $c_a \in \overline{\mathbb{Q}}$ and positive integer t.

Finally substituting the above equation of the $L(k, \chi)$ for each χ in the equation (5.2), we get that Γ can be expressed as an algebraic linear combination of polylogarithms of algebraic numbers satisfying the hypothesis of the Strong Polylog conjecture (1.2.1). Hence by the Strong Polylog conjecture, we deduce that Γ is transcendental.

Chapter 6

The Strong Chowla-Milnor conjecture over number fields

6.1 Introduction

In this chapter, we discuss our fourth set of research problems. We consider the number field extension of the Strong Chowla-Milnor conjecture (1.1.6) formulated before. We recall this conjecture for the sake of completion. Let q be a positive integer and $\mathbb{F}$ be a number field which is linearly disjoint from the cyclotomic field $\mathbb{Q}(\zeta_q)$. Then conjecture (1.2.2) says that for any integer $k > 1$, the following $\varphi(q) + 1$ real numbers

$$1, \zeta(k, a/q) \text{ with } 1 \leq a < q, \ (a, q) = 1 \tag{6.1}$$

are linearly independent over $\mathbb{F}$. We show that if the above conjecture is true for all number fields $\mathbb{F}$ which are linearly disjoint with $\mathbb{Q}(\zeta_q)$, then both the numbers $\zeta(2k+1)$ and $\zeta(2k+1)/\pi^{2k+1}$ are transcendental simultaneously for all integers $k \geq 1$. In connection to the conjecture (6.1), we define the $\mathbb{F}$ linear space $\widehat{V}_k(q, \mathbb{F})$ spanned by the $\varphi(q) + 1$ real numbers $1, \zeta(k, a/q)$ with $1 \leq a < q$, and $(a, q) = 1$. In the next section, we study the dimension of these spaces.

6.2 Dimension of $\widehat{V}_k(q, \mathbb{F})$

We investigate the dimension of $\widehat{V}_k(q, \mathbb{F})$ in two separate cases. First we assume that $\mathbb{F}$ is linearly disjoint with the cyclotomic field $\mathbb{Q}(\zeta_q)$. With this assumption we prove theorem (1.2.13) and theorem (1.2.16). Next we discuss about case when $\mathbb{F} = \mathbb{Q}(\zeta_q)$.

6.2.1 The case $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$

In this case, our main tool for proving the theorems (1.2.13) and (1.2.16) is lemma (2.6.3). This is a generalization of theorem (2.6.2) of Okada due to Murty and

Saradha [28] about linear independence of cotangent values at rational arguments. We first recall the theorem (1.2.13).

Theorem 6.2.1. *Let $k > 1$ and $q > 1$ be two integers and $\mathbb{F}$ be a number field with $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then*

$$\dim_{\mathbb{F}} \widehat{V}_k(q, \mathbb{F}) \geq \frac{\varphi(q)}{2} + 1.$$

Proof. The space $\widehat{V}_k(q, F)$ is also spanned by the following sets of real numbers:

$$1, \{\zeta(k, a/q) + \zeta(k, 1 - a/q) | (a, q) = 1,\ 1 \leq a < q/2\},$$
$$\{\zeta(k, a/q) - \zeta(k, 1 - a/q) | (a, q) = 1,\ 1 \leq a < q/2\}$$

over $\mathbb{F}$.

Now from the lemma (3.2.3), we get for $k > 1$

$$\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z)|_{z=a/q}.$$

Using the lemma (2.6.3) of Murty and Saradha [28], we get

$$\frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z)|_{z=a/q}$$

for $1 \leq a < q/2, (a, q) = 1$, are linearly independent over $\mathbb{F}$. Then from the above identity, we get that the following $\varphi(q)/2$ real numbers

$$\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q)$$

for $1 \leq a < q/2, (a, q) = 1$, are linearly independent over $\mathbb{F}$.

Again by the lemma (3.2.2), we have

$$\frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z) = \pi^k \times \mathbb{Z} \text{ linear combination of } (\csc \pi z)^{2l} (\cot \pi z)^{k-2l},$$

for some non-negative integer l and for an integer $k \geq 1$. Since $\csc \pi z$ and $\cot \pi z$ are algebraic at rationals, all these numbers $\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q)$ are transcendental for any $k > 1$, being algebraic multiple of π. Since $\pi \notin \mathbb{F}$, $\dim_{\mathbb{F}} \widehat{V}_k(q, \mathbb{F}) \geq \frac{\varphi(q)}{2} + 1$. This completes the proof of the theorem. □

Now let us recall the theorem (1.2.16).

Theorem 6.2.2. *Let $k > 1$ be an odd integer and $q, r > 2$ be two co-prime integers. Also, let $\mathbb{F} \subseteq \mathbb{R} \cap \overline{\mathbb{Q}}$ such that $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q} = \mathbb{F} \cap \mathbb{Q}(\zeta_r)$ and $\mathbb{F}(\zeta_q) \cap \mathbb{F}(\zeta_r) = \mathbb{F}$. Assume that $\zeta(k) \notin \mathbb{F}$. Then either*

$$\dim_{\mathbb{F}} \widehat{V}_k(q, \mathbb{F}) \geq \frac{\varphi(q)}{2} + 2$$

or

$$\dim_{\mathbb{F}} \widehat{V}_k(r, \mathbb{F}) \geq \frac{\varphi(r)}{2} + 2.$$

Proof. On the contrary, we assume first that both

$$\dim_{\mathbb{F}} \widehat{V}_k(q,\mathbb{F}) = \frac{\varphi(q)}{2} + 1$$

and

$$\dim_{\mathbb{F}} \widehat{V}_k(r,\mathbb{F}) = \frac{\varphi(r)}{2} + 1.$$

Now for the first case, the numbers

$$1, \zeta(k,a/q) - \zeta(k,1-a/q), \text{ where } (a,q)=1,\ 1 \leq a < q/2$$

generate $\widehat{V}_k(q,\mathbb{F})$ over $\mathbb{F}$.

Since k is odd, from equation (3.2) we have

$$\frac{\zeta(k,a/q) - \zeta(k,1-a/q)}{(2\pi i)^k} \in \mathbb{Q}(\zeta_q) \subseteq \mathbb{F}(\zeta_q). \tag{6.2}$$

Now consider the identity (2.3)

$$\zeta(k) \prod_{\substack{p \text{ prime,} \\ p|q}} (1-p^{-k}) \ = \ q^{-k} \sum_{\substack{a=1, \\ (a,q)=1}}^{q-1} \zeta(k,a/q) \in \widehat{V}_k(q,\mathbb{F}).$$

Thus $\zeta(k) \in \widehat{V}_k(q,\mathbb{F})$ and hence we have

$$\zeta(k) = q_1 + \sum_{\substack{(a,q)=1 \\ 1 \leq a < q/2}} \lambda_a[\zeta(k,a/q) - \zeta(k,1-a/q)] \text{ for some } q_1,\ \lambda_a \in \mathbb{F}.$$

So we get

$$\frac{\zeta(k) - q_1}{(2\pi i)^k} = \sum_{\substack{(a,q)=1 \\ 1 \leq a < q/2}} \frac{\lambda_a[\zeta(k,a/q) - \zeta(k,1-a/q)]}{(2\pi i)^k}.$$

Thus by (6.2)

$$\frac{\zeta(k) - q_1}{i\pi^k} = a_1(\text{say}) \in \mathbb{F}(\zeta_q). \tag{6.3}$$

Similarly, for the second case, i.e. when

$$\dim_{\mathbb{F}} \widehat{V}_k(r,\mathbb{F}) = \frac{\varphi(r)}{2} + 1,$$

we get

$$\frac{\zeta(k) - q_2}{i\pi^k} = a_2(\text{say}) \in \mathbb{F}(\zeta_r), \quad \text{with } q_2 \in \mathbb{F}. \tag{6.4}$$

Hence from the equations (6.3) and (6.4), we get

$$a_1 i\pi^k + q_1 = a_2 i\pi^k + q_2$$

which implies

$$(a_1 - a_2) i\pi^k = q_2 - q_1.$$

The L.H.S of the above equation is algebraic number times transcendental number hence transcendental and the R.H.S is an algebraic number belonging to $\mathbb{F}$. Hence we deduce that $q_1 = q_2$ and $a_1 = a_2$.

Thus we have

$$\frac{\zeta(k) - q_1}{i\pi^k} \in \mathbb{F}(\zeta_q) \cap \mathbb{F}(\zeta_r) = \mathbb{F}.$$

Let

$$\frac{\zeta(k) - q_1}{i\pi^k} = a \in \mathbb{F}.$$

Clearly the L.H.S of the above equation is purely imaginary and the R.H.S belongs to $\mathbb{F}$, hence a real number as $\mathbb{F} \subset \mathbb{R}$. So we have $a = 0$ and $\zeta(k) = q_1 \in \mathbb{F}$. This is a contradiction to the fact that $\zeta(k) \notin \mathbb{F}$. Thus either

$$\dim_{\mathbb{F}} \widehat{V}_k(q, \mathbb{F}) \geq \frac{\varphi(q)}{2} + 2$$

or

$$\dim_{\mathbb{F}} \widehat{V}_k(r, \mathbb{F}) \geq \frac{\varphi(r)}{2} + 2.$$

This completes the proof of the theorem. □

6.2.2 The case $\mathbb{F} = \mathbb{Q}(\zeta_q)$

For $\mathbb{F} = \mathbb{Q}(\zeta_q)$, we do not have the same result as in theorem (6.2.1). We show that there are some linear relations between the generators over $\mathbb{Q}(\zeta_q)$. In this direction, we prove the following propositions.

Proposition 6.2.3. $2 \leq \dim_{\mathbb{F}} \widehat{V}_k(q, \mathbb{F}) \leq \frac{\varphi(q)}{2} + 2$.

Proof. From (3.2.2) and (3.2.3), we have

$\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) = \pi^k \times \mathbb{Q}$ linear combination of $(\csc \pi z)^{2l} (\cot \pi z)^{k-2l}$,

at $z = a/q$ and for some non-negative integer l.

Now note that

$$i \cot(\pi a/q) = \frac{1 + \zeta_q^a}{1 - \zeta_q^a}$$

belongs to $\mathbb{Q}(\zeta_q)$ and hence so do the numbers $\csc(\pi a/q)^{2l}$ and $\cot(\pi a/q)^{2l}$. Hence from the above equation, we deduce that

$$\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) \in i^k \pi^k \mathbb{Q}(\zeta_q) \tag{6.5}$$

for all $1 \leq a < q/2$ with $(a, q) = 1$.

Clearly for k even, the $\varphi(q)/2$ real numbers

$$\zeta(k, a/q) + \zeta(k, 1 - a/q)$$

for $1 \leq a < q/2, (a, q) = 1$ are linearly dependent over $\mathbb{Q}(\zeta_q)$, and for k odd $\zeta(k, a/q) - \zeta(k, 1 - a/q)$ for $1 \leq a < q/2, (a, q) = 1$, are linearly dependent over $\mathbb{Q}(\zeta_q)$. So in any case whether k is even or odd, this $\varphi(q)/2$ real numbers contribute at most 1 in the dimension. Hence we have $\dim_{\mathbb{F}} \widehat{V}_k(q, \mathbb{F}) \leq \frac{\varphi(q)}{2} + 2$. Again since 1 and π are linearly independent over $\mathbb{Q}(\zeta_q)$, we have $\dim_{\mathbb{F}} \widehat{V}_k(q, \mathbb{F}) \geq 2$. This completes the proof of the proposition. □

Proposition 6.2.4. *Let $k > 1$ be an odd integer such that $\zeta(k)$ is irrational. Then there exists an integer $q_0 > 1$ such that for all integers $q > 2$ with $(q_0, q) = 1$, the dimension of the space $\widehat{V}_k(q, \mathbb{Q}(\zeta_q))$ is at least 3.*

Proof. On the contrary, suppose that this is not true. Then for any two coprime integers q and r, we have

$$\dim_{\mathbb{Q}(\zeta_q)} \widehat{V}_k(q, \mathbb{Q}(\zeta_q)) = 2 \text{ and } \dim_{\mathbb{Q}(\zeta_r)} \widehat{V}_k(q, \mathbb{Q}(\zeta_r)) = 2.$$

As k is an odd integer, we get from (6.5)

$$\zeta(k, a/q) - \zeta(k, 1 - a/q) \in i\pi^k \mathbb{Q}(\zeta_q)$$

for all $1 \leq a < q/2$ with $(a, q) = 1$ and

$$\zeta(k, a/r) - \zeta(k, 1 - a/r) \in i\pi^k \mathbb{Q}(\zeta_r)$$

for all $1 \leq a < r/2$ with $(a, r) = 1$.

Hence both the spaces $\widehat{V}_k(q, \mathbb{Q}(\zeta_q))$ and $\widehat{V}_k(r, \mathbb{Q}(\zeta_r))$ are generated by 1 and $i\pi^k$ over $\mathbb{Q}(\zeta_q)$ and $\mathbb{Q}(\zeta_r)$ respectively.

Again we know that $\zeta(k)$ belongs to both the spaces $\widehat{V}_k(q, \mathbb{Q}(\zeta_q))$ and $\widehat{V}_k(r, \mathbb{Q}(\zeta_r))$. Hence $\zeta(k)$ can be written as

$$\zeta(k) = q_1 + q_2 i\pi^k = r_1 + r_2 i\pi^k \tag{6.6}$$

for some $q_1, q_2 \in \mathbb{Q}(\zeta_q)$ and $r_1, r_2 \in \mathbb{Q}(\zeta_r)$.

From (6.6), we get

$$(q_2 - r_2)i\pi^k = r_1 - q_1.$$

The L.H.S of the above equation is transcendental while the R.H.S is an algebraic number. Hence we deduce that $q_1 = r_1$ and $q_2 = r_2$. As $\mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r) = \mathbb{Q}$, we see that both q_1, q_2 are rational numbers. Since $\zeta(k)$ is irrational, from (6.6) we get $q_2 \neq 0$. Again, from (6.6) we have

$$\frac{\zeta(k) - q_1}{q_2} = i\pi^k.$$

Note that L.H.S of the above equation is a non zero real number. But the R.H.S of the above equation is a purely imaginary number. Hence we get a contradiction. Thus either

$$\dim_{\mathbb{Q}(\zeta_q)} \widehat{V}_k(q, \mathbb{Q}(\zeta_q)) \geq 3$$

or

$$\dim_{\mathbb{Q}(\zeta_r)} \widehat{V}_k(q, \mathbb{Q}(\zeta_r)) \geq 3.$$

In particular, for infinitely many odd integers $k > 1$ there exists an integer q_0 such that

$$\dim_{\mathbb{Q}(\zeta_q)} \widehat{V}_k(q, \mathbb{Q}(\zeta_q)) \geq 3$$

for any integer q coprime to q_0. $\square$

6.3 Transcendence of $\zeta(k)$

We prove that if the conjecture (6.1) holds for all number fields $\mathbb{F}$ which are linearly disjoint from the cyclotomic field $\mathbb{Q}(\zeta_q)$, then $\zeta(k)$ is transcendental for all odd integers $k > 1$. In particular we prove the following two propositions.

Proposition 6.3.1. *Let $k > 1$ be an odd integer. If $\dim_{\mathbb{F}} \widehat{V}_k(4, \mathbb{F}) = 3$ for all number fields $\mathbb{F}$ such that $\mathbb{F} \cap \mathbb{Q}(i) = \mathbb{Q}$, then $\zeta(k)$ is transcendental.*

Proof. Assume that $\dim_{\mathbb{F}} \widehat{V}_k(4, \mathbb{F}) = 3$ for all such number fields $\mathbb{F}$. We have

$$\zeta(k, 1/4) + \zeta(k, 3/4) = (4^k - 2^k)\zeta(k).$$

Suppose that $\zeta(k) \in \overline{\mathbb{Q}}$. Then $\zeta(k) \in \mathbb{R} \cap \overline{\mathbb{Q}}$ since $\zeta(k)$ is real for real k. Consider the number field $\mathbb{F} = \mathbb{Q}(\zeta(k))$. Then $\mathbb{F} \cap \mathbb{Q}(i) = \mathbb{Q}$ as $\mathbb{F}$ is subfield of real numbers. Now from the above equation we get $\dim_{\mathbb{F}} \widehat{V}_k(4, \mathbb{F}) < 3$ for $\mathbb{F} = \mathbb{Q}(\zeta(k))$, Which is a contradiction. Hence $\zeta(k)$ is transcendental. $\square$

Proposition 6.3.2. *Let $k > 1$ be an odd integer and ω be a primitive cube root of unity. If $\dim_{\mathbb{F}} \widehat{V}_k(3, \mathbb{F}) = 3$ for all number fields $\mathbb{F}$ such that $\mathbb{F} \cap \mathbb{Q}(\omega) = \mathbb{Q}$, then $\zeta(k)$ is transcendental.*

Proof. Assume that $\dim_{\mathbb{F}} \widehat{V}_k(3, \mathbb{F}) = 3$ for all such number fields $\mathbb{F}$. We have

$$\zeta(k, 1/3) + \zeta(k, 2/3) = (3^k - 1)\zeta(k).$$

Suppose that $\zeta(k) \in \overline{\mathbb{Q}}$. Then $\zeta(k) \in \mathbb{R} \cap \overline{\mathbb{Q}}$. Consider the number field $\mathbb{F} = \mathbb{Q}(\zeta(k))$. Then $\mathbb{F} \cap \mathbb{Q}(\omega) = \mathbb{Q}$ as $\mathbb{F}$ is a subfield of real numbers. Now from the above equation, we get that $\dim_{\mathbb{F}} \widehat{V}_k(3, \mathbb{F}) < 3$ for $\mathbb{F} = \mathbb{Q}(\zeta(k))$ which is a contradiction. Hence $\zeta(k)$ is transcendental. $\square$

6.4 Relation with the Strong Polylog conjecture

In this section, we discuss the relationship between the Strong Polylog conjecture (1.2.1) and the conjecture (6.1). Let q be a positive integer and $\mathbb{F}$ be a number field which is linearly disjoint from the cyclotomic field $\mathbb{Q}(\zeta_q)$. We prove that the Strong Polylog conjecture (1.2.1) implies conjecture (6.1) for all such number fields. We first prove the proposition (1.2.10), which is the following:

Proposition 6.4.1. *Let $k > 1, q > 1$ be two integers and $\mathbb{F}$ be an algebraic number field such that $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Let $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{F}$ with $f(a) = 0$ for $1 < (a, q) < q$. Then the following statements are equivalent:*

1. *The conjecture* (1.2.2) *is true.*

2. The L-value $L(k,f) \notin \mathbb{F}$, unless

$$f(a) = -\frac{f(q)q^{-k}}{\prod\limits_{\substack{p \text{ prime}, \\ p|q}} (1 - p^{-k})}$$

for $1 \le a < q, (a,q) = 1.$

Proof. To prove the proposition, let us Consider the identity (2.3)

$$\zeta(k) \prod_{\substack{p \text{ prime}, \\ p|q}} (1 - p^{-k}) = q^{-k} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \zeta(k, a/q).$$

Substituting this in the expression (1.1) for $L(s,f)$ and using $f(a) = 0$ for $1 < (a,q) < q$, we get

$$L(k,f) = q^{-k} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \left[f(a) + \frac{f(q)q^{-k}}{\prod\limits_{p|q}(1 - p^{-k})} \right] \zeta(k, a/q). \tag{6.7}$$

Now if $L(k,f) \in \mathbb{F}$, then the above equation shows that $1, \zeta(k, a/q)$ for $1 \le a < q, (a,q) = 1$ are linearly dependent over $\mathbb{F}$ since f is an $\mathbb{F}$-valued function.

Conversely, assume that $1, \zeta(k, a/q)$ with $1 \le a < q, (a,q) = 1$ are linearly dependent over $\mathbb{F}$. Then there are numbers $\beta_0, \beta_a \in \mathbb{F}, 1 \le a < q, (a,q) = 1$, not all zero, such that

$$\beta_0 + \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \beta_a \zeta(k, a/q) = 0.$$

Now define the following $\mathbb{F}$-valued periodic function f with period q. Set $f(a) = 0$ for $1 < a \le q, (a,q) > 1$ and $f(a) = \beta_a$ for $1 \le a < q, (a,q) = 1$. Then using the above identity we get $q^k L(k,f) = -\beta_0$ so that $L(k,f) \in \mathbb{F}$. This completes the proof of the proposition. □

6.4.1 Proof of theorem (1.2.11)

In this subsection we give a proof of theorem (1.2.11) using the above proposition. Let us first recall theorem (1.2.11).

Theorem 6.4.2. *The Strong Polylog conjecture implies the conjecture* (6.1) *for all number field* $\mathbb{F}$ *with* $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$.

Proof. Let $k > 1$ and $q > 1$ be two integers and $\mathbb{F}$ be an algebraic number field such that $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Consider a map

$$f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{F}$$

satisfying $f(a) = 0$ for $1 < (a, q) < q$. Then f can be thought of as an $\mathbb{F}$-valued period function with period q.

On the contrary, let us assume that $L(k, f) = \alpha \in \mathbb{F}$. Then we have

$$L(k, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^k} = \alpha.$$

As f is a periodic function, we have the Fourier transformation of f given by

$$\hat{f}(n) = \frac{1}{q} \sum_{a=1}^{q} f(a)\zeta_q^{-an}.$$

Hence we have the Fourier inversion formula

$$f(n) = \sum_{a=1}^{q} \hat{f}(a)\zeta_q^{an}.$$

Using the Fourier inversion formula we get

$$\begin{aligned} L(k, f) &= \sum_{n=1}^{\infty} \frac{f(n)}{n^k} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^k} \sum_{a=1}^{q} \hat{f}(a)\zeta_q^{an} \end{aligned}$$

and hence

$$\sum_{a=1}^{q} \hat{f}(a) Li_k(\zeta_q^a) - \alpha = 0.$$

Let $Li_k(\alpha_1), ..., Li_k(\alpha_t)$ be a maximal linearly independent subset of

$$\{Li_k(\zeta_q^a) | 1 \leq a < q\}$$

over $\mathbb{F}$.

Then

$$Li_k(\zeta_q^a) = \sum_{b=1}^{t} C_{ab} Li_k(\alpha_b)$$

for some $C_{ab} \in \mathbb{F}$. So we have

$$\sum_{b=1}^{t} \beta_b Li_k(\alpha_b) - \alpha = 0$$

where

$$\beta_b = \sum_{a=1}^{q} \hat{f}(a) C_{ab}.$$

Since f is an $\mathbb{F}$-valued function, $\hat{f}$ is algebraic valued. So by the Strong Polylog conjecture (1.2.1), we have

$$\alpha = 0 \text{ and } \beta_b = \sum_{a=1}^{q} \hat{f}(a) C_{ab} = 0, \; 1 \leq b \leq t.$$

Now for any automorphism σ of the field extension $\mathbb{F}(\zeta_q)$ over $\mathbb{F}$, we have

$$\sum_{a=1}^{q} \sigma(\hat{f}(a))C_{ab} = 0, \ 1 \le b \le t,$$

so we get that

$$\sum_{a=1}^{q} \sigma(\hat{f}(a))Li_k(\zeta_q^a) = 0.$$

In particular, for $1 \le h < q, (h,q) = 1$, let $\sigma_h \in \mathrm{Gal}(\mathbb{F}(\zeta_q)/\mathbb{F})$ be such that

$$\sigma_h(\zeta_q) = \zeta_q^h.$$

Then we have,

$$\sigma_h(\hat{f}(n)) = \hat{f}_h(n)$$

where

$$f_h(n) = f(nh^{-1}).$$

Now we have

$$\begin{aligned} L(k, f_h) &= \sum_{n=1}^{\infty} \frac{f_h(n)}{n^k} \\ &= \sum_{a=1}^{q} \hat{f}_h(a)Li_k(\zeta_q^a) \\ &= \sum_{a=1}^{q} \sigma_h(\hat{f}(a))Li_k(\zeta_q^a) = 0 \end{aligned}$$

for all $1 \le h < q$ with $(h,q) = 1$. Thus by equation (6.7), we get

$$L(k, f_h) = q^{-k} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \left[f_h(a) + \frac{f_h(q)q^{-k}}{\prod_{p|q}(1-p^{-k})} \right] \zeta(k, a/q) = 0$$

for all $1 \le h < q$ with $(h,q) = 1$.

Now, putting $ah^{-1} = b$ and noting that $f_h(q) = f(q)$, we have

$$L(k, f_h) = q^{-k} \sum_{\substack{b=1 \\ (b,q)=1}}^{q-1} \left[f(b) + \frac{f(q)q^{-k}}{\prod_{p|q}(1-p^{-k})} \right] \zeta(k, bh/q) = 0 \qquad (6.8)$$

for all $1 \le h < q$ with $(h,q) = 1$.

Thus we get a matrix equation with M being the $\varphi(q) \times \varphi(q)$ matrix whose (b,h)-th entry is given by $\zeta(k, bh/q)$. Then by the evaluation of the Dedekind-Frobenius determinant as in proposition (2.5.1), we get

$$Det(M) = \pm \prod_{\chi} q^k L(k, \chi) \ne 0.$$

Thus the matrix M is invertible and hence by the equation (6.8), we have

$$f(a) + \frac{f(q)q^{-k}}{\prod\limits_{p|q}(1-p^{-k})} = 0,\ 1 \le a < q,\ (a,q) = 1$$

and hence

$$f(a) = -\frac{f(q)q^{-k}}{\prod\limits_{p|q}(1-p^{-k})}$$

for all $1 \le a < q$ with $(a,q) = 1$. This completes the proof of the theorem. $\square$

Chapter 7

Generalized Hurwitz zeta functions

7.1 Introduction

In this chapter we discuss the last set of research problems. This work is motivated by works of Davenport, Heilbronn [14] and Cassels [6]. In 1936, Davenport and Heilbronn [14] proved that if $a \neq 1/2, 1$ is a rational or transcendental number, then $\zeta(s,a)$ has infinitely many zeros for $\sigma > 1$. Note that $\zeta(s,1) = \zeta(s)$ and $\zeta(s,1/2) = (2^s - 1)\zeta(s)$. Hence $\zeta(s,1)$ or $\zeta(s,1/2)$ can not have zeros for $\sigma > 1$. In 1961, Cassels [6] showed the existence of infinitely many zeros of $\zeta(s,a)$ for $\sigma > 1$, when a is an algebraic irrational number.

For a periodic arithmetic function f with period $q \geq 1$ and $a > 0$, consider the generalized Hurwitz zeta function $L(s,f,a)$ associated to f

$$L(s,f,a) := \sum_{n=0}^{\infty} \frac{f(n)}{(n+a)^s},$$

where $s \in \mathbb{C}$ with $\Re(s) = \sigma > 1$. In 2009, Saias and Weingartner [36] showed that $L(s,f,a)$ has infinitely many zeros for $\sigma > 1$ provided $a = 1$ and $L(s,f,1)$ is not a product of $L(s,\chi)$ and a Dirichlet polynomial, where χ is a Dirichlet character.

Since f is periodic with period $q \geq 1$, the generalized Hurwitz zeta function $L(s,f,a)$ can be written as

$$L(s,f,a) = q^{-s} \sum_{b=1}^{q} f(b)\zeta(s,(a+b)/q),$$

for $s \in \mathbb{C}$ with $\sigma > 1$. This shows that $L(s,f,a)$ extends meromorphically to the whole complex plane with a possible simple pole at $s = 1$ with residue $q^{-1}\sum_{b=1}^{q} f(b)$. Hence $L(s,f,a)$ has a simple pole at $s = 1$ if and only if $\sum_{b=1}^{q} f(b) \neq 0$. In the later section, we prove theorems (1.2.17) and (1.2.18), which recall are the following:

Theorem 7.1.1. *Let a be a positive transcendental number and f be a real valued periodic arithmetic function with period $q \geq 1$. If $L(s,f,a)$ has a pole at $s = 1$, then $L(s,f,a)$ has infinitely many zeros for $\sigma > 1$.*

Theorem 7.1.2. *Let a be a positive algebraic irrational number and f be a positive valued periodic arithmetic function with period $q \geq 1$. Also let*

$$c \;:=\; \frac{\max\limits_{n}\, f(n)}{\min\limits_{n}\, f(n)} < 1.15.$$

If $L(s, f, a)$ has a pole at $s = 1$, then $L(s, f, a)$ has infinitely many zeros for $\sigma > 1$.

7.2 Some propositions

In order to prove Theorem 7.1.1 and Theorem 7.1.2, we need the following propositions.

Proposition 7.2.1. *Let $a > 0$ be any transcendental number and $N \geq 1$ be an integer. Also, let $g_0, \cdots, g_N$ be a sequence of complex numbers having absolute value 1. Then for any real number T and $\epsilon > 0$, there exists a real number $t > T$ such that*

$$|(n+a)^{-it} - g_n| < \epsilon$$

for all $0 \leq n \leq N$.

Proof. Since a is transcendental, the numbers $\log(n+a)$ are linearly independent over $\mathbb{Q}$. Otherwise there will be integers q_n, not all zero, with the relation

$$\sum_{n=0}^{N} q_n \log(n+a) = 0$$

and this implies

$$\prod_{n=0}^{N} (n+a)^{q_n} = 1.$$

Hence we get a contradiction.

Now we write $g_n = e^{-i\alpha_n}$, where α_n's are real numbers. Let $\delta > 0$ be arbitrary. Then by Kronecker's theorem, there exists a real number $t > T$ and integers $x_1, \cdots, x_N$ such that

$$\left| t\,\frac{\log(n+a)}{2\pi} - \frac{\alpha_n}{2\pi} - x_n \right| < \frac{\delta}{2\pi},$$

for all $0 \leq n \leq N$. Multiplying both sides by 2π, we get

$$|t \log(n+a) - \alpha_n - 2\pi x_n| < \delta,$$

for all $0 \leq n \leq N$. Hence using the continuity of the function e^{-ix}, we get

$$|e^{-it\log(n+a)} - e^{-i\alpha_n}| < \epsilon,$$

which implies

$$|(n+a)^{-it} - g_n| < \epsilon,$$

for all $0 \leq n \leq N$. This completes the proof of the proposition. $\square$

Our next proposition shows that with a little modification in the properties of g_n, one can get a similar result as above when $a \in \mathbb{R}_+$ is an algebraic number. In this case, consider the number field $K = \mathbb{Q}(a)$. For each prime ideal $\mathfrak{p}$ in the ring of integers $\mathcal{O}_K$, let $\chi(\mathfrak{p})$ be a complex number with $|\chi(\mathfrak{p})| = 1$. We extend χ to the elements γ of the number field K by setting

$$\chi(\gamma) = \prod_{\mathfrak{p}} \chi(\mathfrak{p})^{\nu_{\mathfrak{p}}} \qquad \text{if} \quad \gamma\mathcal{O}_K = \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_{\mathfrak{p}}}.$$

Here, we have the following proposition.

Proposition 7.2.2. *Let $N \in \mathbb{N}$, $a \in \overline{\mathbb{Q}} \cap \mathbb{R}_+$. Also let $K = \mathbb{Q}(a)$ and χ be as before. Then for any real number T and $\epsilon > 0$, there exists a real number $t > T$ such that*

$$|(n+a)^{-it} - \chi(n+a)| < \epsilon$$

for all $0 \leq n \leq N$.

Proof. Consider the multiplicative subgroup A of $\mathbb{R}^*$ generated by

$$\{n + a \mid 0 \leq n \leq N\}.$$

One can choose a $\mathbb{Z}$-basis B $:= \{s_j \mid 1 \leq j \leq l\}$ of A. Then for any $0 \leq n \leq N$, there exists an integer $M > 0$ such that

$$n + a = \prod_{j=1}^{l} s_j^{u_j},$$

where $u_j \in \mathbb{Z}$ with $|u_j| \leq M$. This implies that

$$\chi(n+a) = \prod_{j=1}^{l} \chi(s_j)^{u_j}. \tag{7.1}$$

By Kronecker's theorem, for any $\epsilon > 0$, there exists a real number $t > T$ such that

$$|s_j^{-it} - \chi(s_j)| < \epsilon, \tag{7.2}$$

for all $s_j \in$ B. Now for any $n + a \in$ A and $\epsilon > 0$, we have by (7.1) and (7.2) that

$$\begin{aligned} & |(n+a)^{-it} - \chi(n+a)| \\ = & \left| \prod_{j=1}^{l} s_j{}^{-itu_j} - \prod_{j=1}^{l} \chi(s_j)^{u_j} \right|, \text{ where } u_j \in \mathbb{Z} \text{ and } |u_j| \leq M \\ \leq & \; M \sum_{j=1}^{l} \left| s_j{}^{-it} - \chi(s_j) \right| \; < \epsilon. \end{aligned}$$

This completes the proof of the proposition. □

Now we prove another proposition which will play an important role to prove Theorem 7.1.2.

Proposition 7.2.3. *Let $0 < r_1 \le r_2 \le \cdots \le r_n$ be real numbers. Then the set*

$$\Delta_n := \{c_1 r_1 + \cdots + c_n r_n \mid |c_i| = 1,\ c_i \in \mathbb{C}\}$$

for $n \ge 1$ is a closed annulus with outer radius $R_n := r_1 + \cdots + r_n$ and inner radius

$$T_n := \begin{cases} r_n - R_{n-1} & \text{if } R_{n-1} \le r_n, \\ 0 & \text{otherwise.} \end{cases}$$

Here $R_0 := 0$.

Proof. Note that the set Δ_n is compact, connected and invariant under rotation around the origin. Hence Δ_n is a closed annulus with outer radius $R_n = r_1 + \cdots + r_n$ and inner radius, say T_n for any n.

We will prove by induction on n that T_n has the desired form. For $n = 1$, it is trivially true. Suppose that it is true for T_{n-1}. In order to prove that the result is true for T_n, we consider two cases: either $r_n \ge R_{n-1}$ or $r_n \le R_{n-1}$. In the first case, T_n clearly has the desired form. When $r_n \le R_{n-1}$ and $T_{n-1} = 0$, then $r_n \in \Delta_{n-1}$ and hence $0 \in \Delta_n$. If $T_{n-1} \ne 0$, then there is an element $r \in \Delta_{n-1}$ with $|r| \le r_{n-1}$. Since $r_{n-1} \le r_n \le R_{n-1}$, once again $0 \in \Delta_n$. This completes the proof. □

Before we state our next proposition, we shall formulate a hypothesis which is integral to our proofs.

Hypothesis: *For any $\delta > 0$, there exists a function $F(s)$ analytic in the region $\Re(s) > 1$ satisfying the following properties;*

1. *There exists a σ_0 with $F(\sigma_0) = 0$ and $1 < \sigma_0 < 1 + \delta$.*
2. *For any real number T and any real numbers $\epsilon, \theta > 0$, there exists a real number $t > T$ such that*

$$|L(s + it, f, a) - F(s)| < \epsilon \qquad \text{for all} \quad \sigma > 1 + \theta.$$

In this context, we have the following proposition.

Proposition 7.2.4. *Let a be a positive real number and f be as before. Assume the previous hypothesis. Then $L(s, f, a)$ has infinitely many zeros for $\sigma > 1$.*

Proof. Let $T, \beta > 0$ be real numbers. We will show that there exists a zero s_1 of $L(s, f, a)$ with $1 < \Re(s_1) < 1 + \beta$ and $\Im(s_1) > T$.

Let $F(s)$ be the function corresponding to β in the hypothesis. By property (1) of $F(s)$, it has a zero σ_0 with $1 < \sigma_0 < 1 + \beta$. Since $F(s)$ is an analytic function, one can choose $\beta_1 > 0$ such that $1 + \beta_1 < 1 + \beta$ and $1 + \beta_1 < \sigma_0$ with $F(s) \ne 0$ for $|s - \sigma_0| = \beta_1$. Set

$$\epsilon := \min_{|s-\sigma_0| = \beta_1} |F(s)| \quad \text{and} \quad \theta < \sigma_0 - \beta_1 - 1.$$

Then $\sigma_0 - \beta_1 > 1 + \theta$ and hence by the property (2) of $F(s)$, there exists a real number $t > T$ such that

$$|L(s + it, f, a) - F(s)| < |F(s)|$$

on $|s - \sigma_0| = \beta_1$. Thus by Rouché's theorem, the function $L(s + it, f, a)$ has a zero s_1 which gives a zero $s_1 + it$ of $L(s, f, a)$. □

7.3 Proofs of Theorem 7.1.1 and Theorem 7.1.2

In view of the proposition (7.2.4), our task is reduced to constructing functions $F(s)$ as described in the above hypothesis.

7.3.1 Proof of the Theorem 7.1.1

Proof. Let a be a positive transcendental number. In this case, replacing f by $-f$ if needed, we can assume that the residue $\frac{1}{q}\sum_{b=1}^{q} f(b)$ of $L(s,f,a)$ is a positive real number.

Since $L(s,f,a)$ converges absolutely for $\sigma > 1$, for any $\delta > 0$, one can choose an integer m such that

$$\sum_{n=0}^{m} \frac{f(n)}{(n+a)^{1+\delta}} > \sum_{n=m+1}^{\infty} \frac{f(n)}{(n+a)^{1+\delta}}. \tag{7.3}$$

Define

$$F(s) := \sum_{n=0}^{\infty} \frac{f(n)\alpha(n)}{(n+a)^s} \qquad \text{for } \Re(s) > 1,$$

where α is an arithmetic function defined by

$$\alpha(n) := \begin{cases} -1 & \text{if } n > m, \\ 1 & \text{otherwise.} \end{cases}$$

By (7.3), it is clear that $F(1+\delta) > 0$. On the other hand, since $L(s,f,a)$ has a pole at $s = 1$, $F(s) \to -\infty$ as $s \to 1^+$. Since $F(s)$ is a real valued continuous function when s is real and $s > 1$, it follows that $F(s)$ has a zero in the interval $(1, 1+\delta)$. Thus $F(s)$ is an analytic function satisfying (1). It follows from Proposition 7.2.1 that given any real numbers $T, \epsilon > 0$, there exists a real number $t > T$ such that

$$|L(s+it,f,a) - F(s)| < \epsilon \text{ for } \Re(s) > 1.$$

Since $F(s)$ is a function satisfying properties (1) and (2), Theorem 7.1.1 follows from Proposition 7.2.4. □

7.3.2 Proof of the Theorem 7.1.2

Proof. Suppose that a is a positive algebraic irrational number and f is a positive real valued periodic function satisfying the conditions of Theorem 7.1.2. Let $\delta > 0$ be fixed. We shall define for $\Re(s) > 1$, a function

$$F(s) = \sum_{n=0}^{\infty} \frac{f(n)\chi(n+a)}{(n+a)^s},$$

where χ is a suitable character on the group of fractional ideals of $K = \mathbb{Q}(a)$. Here $\chi(n+a)$ is the value of this character on the principal ideal $(n+a)\mathcal{O}_K$.

Clearly, such a function is holomorphic in $\Re(s) > 1$. Furthermore, it follows from Proposition 7.2.2 that $F(s)$ satisfies property (2).

We shall show that it is possible to define χ suitably to ensure the existence of a σ with $1 < \sigma < \min(1+\delta, 2)$ satisfying

$$F(\sigma) = \sum_{n=0}^{\infty} \frac{f(n)\chi(n+a)}{(n+a)^\sigma} = 0. \tag{7.4}$$

Thus this function also satisfies property (1) and hence will establish the theorem.

We first begin by setting $N_1 = [\max(N_0, 10^7, 10^7 a)]$, where $[x]$ is the integral part of a real number x. Then since $L(s, f, a)$ has a pole at $s = 1$, there exists a σ such that

$$\sum_{n=0}^{N_1} \frac{f(n)}{(n+a)^\sigma} < 10^{-2} \sum_{n=N_1+1}^{\infty} \frac{f(n)}{(n+a)^\sigma} \tag{7.5}$$

and $1 < \sigma < \min(1+\delta, 2)$.

We now define an infinite sequence of integer pairs N_j, M_j for $j \geq 1$ by

$$M_j := \left[\frac{N_j}{10^6}\right] \text{ and } N_{j+1} := N_j + M_j.$$

To prove (7.4), it is sufficient to show that we can construct a character χ such that

$$\left| \sum_{n=0}^{N_j} \frac{f(n)\chi(n+a)}{(n+a)^\sigma} \right| < 10^{-2} \sum_{n=N_j+1}^{\infty} \frac{f(n)}{(n+a)^\sigma} \tag{7.6}$$

for all j.

Let $\mathfrak{b}$ be the ideal denominator of a so $\mathfrak{b}(a\mathcal{O}_K)$ is an integral ideal for every integer n. If $\mathfrak{p}|\mathfrak{b}$ or $\mathfrak{p}|(n+a)\mathfrak{b}$ for $n \leq N_1$, we choose $\chi(\mathfrak{p}) := 1$. Then by (7.5), we see that (7.6) is true for $j = 1$.

Suppose that (7.6) is true for all integers $\leq j$. Define two sets of integers as follows;

$$P_1 := \left\{ N_j < n \leq N_{j+1} \mid \exists \mathfrak{p}_n | (n+a)\mathfrak{b} \text{ but } \mathfrak{p}_n \nmid \prod_{\substack{m \leq N_{j+1} \\ m \neq n}} (m+a)\mathfrak{b} \right\}$$

and

$$P_2 := \left\{ N_j < n \leq N_{j+1} \mid n \notin P_1 \right\}.$$

By Cassels's lemma and the choice of N_j, we have $|P_1| \geq 27M_j/50$.

Note that if $\mathfrak{p}|\prod_{m \leq N_{j+1}}(m+a)\mathfrak{b}$ then either $\mathfrak{p}|\prod_{m \leq N_j}(m+a)\mathfrak{b}$ or $\mathfrak{p} = \mathfrak{p}_n$ for some $n \in P_1$ or $\mathfrak{p}$ is not in either of these sets. By induction hypothesis, we already know the values of $\chi(\mathfrak{p})$ for $\mathfrak{p}$ in the first set and we define $\chi(\mathfrak{p}) := 1$ for $\mathfrak{p}$ in the third set. Now we will define the value of $\chi(\mathfrak{p})$ for $\mathfrak{p}$ in the second set in such a way that (7.6) is true for $j+1$.

By the hypothesis of the theorem, we have

$$\frac{f(n)}{f(m)} \leq 1.15 \quad \text{for all } n, m \in \mathbb{N},$$

and hence

$$\frac{f(n)}{(n+a)^\sigma} < 4\frac{f(m)}{(m+a)^\sigma}$$

for any $n, m \in P_1$. Since $|P_1| > 5$, it follows from Proposition 7.2.3 that

$$\sum_{n \in P_1} \frac{f(n)\chi(n+a)}{(n+a)^\sigma}$$

can take any value x with

$$|x| \le \sum_{n \in P_1} \frac{f(n)}{(n+a)^\sigma} = A_1, \text{ say.}$$

Write

$$\Gamma := \sum_{n \le N_j} \frac{f(n)\chi(n+a)}{(n+a)^\sigma} + \sum_{n \in P_2} \frac{f(n)\chi(n+a)}{(n+a)^\sigma}$$

and set

$$\begin{aligned} A_2 : &= \left| \sum_{n \le N_j} \frac{f(n)\chi(n+a)}{(n+a)^\sigma} \right| \\ A_3 : &= \sum_{n \in P_2} \frac{f(n)}{(n+a)^\sigma}. \end{aligned}$$

Also set

$$x := \begin{cases} -\Gamma & \text{if } 0 < |\Gamma| \le A_1, \\ -A_1\Gamma/|\Gamma| & \text{if } |\Gamma| > A_1, \\ 0 & \text{if } \Gamma = 0. \end{cases}$$

Then by appropriate choice of $\chi(n+a)$ for $n \in P_1$, we have

$$\left| \sum_{n \le N_{j+1}} \frac{f(n)\chi(n+a)}{(n+a)^\sigma} \right| \le \max\{0, \ A_2 + A_3 - A_1\}.$$

Since $|P_2| \le 23M_j/50$ and $|P_1| \ge 27M_j/50$, we have

$$\begin{aligned} \frac{A_1}{A_3} &\ge \frac{27}{23c} \frac{(N_j+a)^\sigma}{(N_{j+1}+a)^\sigma} \\ &> \frac{27 \times 10^7 \times 10^7}{23(10^7+11)^2 \times 1.15} > \frac{101}{99}. \end{aligned}$$

This implies that

$$100(A_1 - A_3) > A_1 + A_3. \tag{7.7}$$

Set

$$A_4 := \sum_{n > N_{j+1}} \frac{f(n)}{(n+a)^\sigma}.$$

By induction,

$$A_2 < 10^{-2}(A_1 + A_3 + A_4).$$

Thus by (7.7), we have

$$A_2 + A_3 - A_1 < 10^{-2} A_4.$$

This proves (7.6) by induction and hence we have (7.4). This completes the proof of the theorem. □

7.4 Zero-free region for $L(s, f, a)$

In the other direction, we establish zero-free regions for $L(s, f, a)$ for an arbitrary periodic arithmetic function f not identically zero and prove Theorem 1.2.19 which is the following:

Theorem 7.4.1. *Let f be a non-zero periodic arithmetic function with period $q \geq 1$. Also, let a be any positive real number and*

$$c = \max_{1 \leq b \leq q} \{1, \ |f(b)|\}.$$

Then we have $L(s, f, a) \neq 0$ for $\sigma > 1 + c'(a + n_0)$, where n_0 is the smallest integer such that $f(n_0) \neq 0$ and $c' = c/|f(n_0)|$.

Proof. Let f be an arbitrary non-zero periodic arithmetic function with period $q \geq 1$. Note that for $\Re(s) = \sigma > 1$, one has

$$\begin{aligned}
|L(s, f, a)| &= \left| \sum_{n=0}^{\infty} \frac{f(n)}{(n+a)^s} \right| \\
&\geq \frac{|f(n_0)|}{(n_0+a)^\sigma} - \frac{c}{(n_0+a+1)^\sigma} - c \sum_{n=2}^{\infty} \frac{1}{(n_0+a+n)^\sigma} \\
&\geq \frac{|f(n_0)|}{(n_0+a)^\sigma} - \frac{c}{(n_0+a+1)^\sigma} - c \int_1^\infty \frac{dx}{(n_0+a+x)^\sigma} \\
&\geq \frac{|f(n_0)|}{(n_0+a)^\sigma} - \frac{c}{(n_0+a+1)^\sigma} - c \frac{(n_0+a+1)^{1-\sigma}}{\sigma - 1}.
\end{aligned}$$

In order to prove that $|L(s, f, a)| \neq 0$ for $\sigma > 1 + c'(n_0 + a)$, it is sufficient to prove that

$$\frac{|f(n_0)|}{(n_0+a)^\sigma} > c \left[\frac{1}{(n_0+a+1)^\sigma} + \frac{(n_0+a+1)^{1-\sigma}}{\sigma - 1} \right]$$

$$\text{i.e.} \qquad \frac{(n_0+a+1)^\sigma}{(n_0+a)^\sigma} > c' \left[1 + \frac{n_0+a+1}{\sigma - 1} \right]$$

$$\text{i.e.} \qquad \left(1 + \frac{1}{n_0+a} \right)^\sigma > c' \left[1 + \frac{n_0+a+1}{\sigma - 1} \right]$$

in that region. Here $c' = c/|f(n_0)|$. To prove this, we consider two functions

$$\begin{aligned}
h_1(\sigma) &:= 1 + \frac{\sigma}{n_0+a} \\
\text{and} \quad h_2(\sigma) &:= c' \left[1 + \frac{n_0+a+1}{\sigma - 1} \right]
\end{aligned}$$

for $\sigma > 1$. After evaluating the value of above two functions at $\sigma = 1 + c'(n_0 + a)$, we get

$$h_1(1 + c'(n_0+a)) = h_2(1 + c'(n_0+a)) = 1 + c' + \frac{1}{n_0+a}.$$

Again note that h_1 is an increasing function of σ whereas h_2 is a decreasing function of σ and they are equal where $\sigma = 1 + c'(n_0 + a)$. Hence if $\sigma > 1 + c'(n_0 + a)$, we have

$$h_1(\sigma) > h_2(\sigma).$$

which implies

$$\left(1 + \frac{\sigma}{n_0 + a}\right) > c'\left[1 + \frac{n_0 + a + 1}{\sigma - 1}\right].$$

Thus we have

$$\left(1 + \frac{1}{n_0 + a}\right)^{\sigma} > \left(1 + \frac{\sigma}{n_0 + a}\right) > c'\left[1 + \frac{n_0 + a + 1}{\sigma - 1}\right].$$

Hence we get $|L(s; a, f)| > 0$. This completes the proof of the theorem. □

7.5 Application

As a curious application of the above theorem, we prove a variant of a conjecture of Erdös (see also [25]) about non vanishing of $L(1, f)$, where f belongs to a certain class of rational valued arithmetic functions. In this direction we prove the Theorem 1.2.20, which is the following:

Theorem 7.5.1. *Let f be a non-zero periodic arithmetic function with period $q > 1$ and*

$$f(n) = \begin{cases} \pm\lambda & \text{if } q \nmid n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $L(k, f) \neq 0$ for all integers $k \geq 2$.

Proof. We first assume that

$$f(n) = \begin{cases} \pm 1 & \text{if } q \nmid n, \\ 0 & \text{otherwise .} \end{cases}$$

In this case we have $a = 1$, $n_0 = 1$, $c = 1$ and $c' = 1$. Then by Theorem 7.4.1, we see that $L(k, f) \neq 0$ for all $k > 3$.

Now if $f(1) = 1$, then for $k = 2$, we note that

$$L(2, f) > 1 - \{\zeta(2) - 1\} = 2 - \zeta(2) > 0$$

and for $k = 3$, we note that

$$L(3, f) > 1 - \{\zeta(3) - 1\} = 2 - \zeta(3) > 0.$$

Again if $f(1) = -1$, then for $k = 2$, we note that

$$L(2, f) < -1 + \{\zeta(2) - 1\} = \zeta(2) - 2 < 0$$

and for $k = 3$, we note that

$$L(3, f) < -1 + \{\zeta(3) - 1\} = \zeta(3) - 2 < 0.$$

Hence $L(k, f) \neq 0$ for all integers $k \geq 2$.

In the general case write $g(n) = \lambda f(n)$. Then $L(k, g) = \lambda L(k, f)$ and hence $L(k, g) = 0$ if and only if $L(k, f) = 0$. This completes the proof. □

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